学位論文

Analytical Investigation into Electromagnetic Response of Quantum Fields in de Sitter Spacetime

（ド・ジッター時空における量子場の電磁的応答の解析的研究）

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Abstract

The response of the quantum fields in 4-dimensional de Sitter spacetime to the electromagnetic background fields is studied in the cosmological context. We analytically calculate the expectation value of the bosonic (scalar) and fermionic (Dirac) current induced by the homogeneous and constant electric background configuration. The adiabatic subtraction and point-splitting renormalization are employed, and dependence of the result on the renormalization condition is discussed. We find that quantum electrodynamics in de Sitter spacetime causes anti-screening. The physical origin of the counterintuitive anti-screening effect is explained by analogy with Hawking radiation. The electro-thermodynamical description of the anomalous charge transportation associated with the anti-screening effect is also given in terms of the informatical action of Maxwell's demon on the horizon. The implication of the findings contains a new scenario of the spontaneous generation of the primordial electromagnetic fluctuation without traditional gauge field enhancing mechanisms which intentionally break the conformal invariance. We redevelop the gauge-invariant effective field theoretical approach as a feasible approximation method for quantum electrodynamics in curved spacetime following the explanation of the non-perturbative renormalization group technique established initially in the flat spacetime. We also propose a new renormalization condition which supplements the want of experimental knowledge about curved spacetime quantum field theory with additional requirements on the asymptotic behavior of the physical quantity. The new renormalization condition yields predictions compatible with our physical insights and the results of the semiclassical calculation.

Contents

I DEDICATE THIS THESIS TO AKARI MITANI, MY FIANCÉ WHO SUPPORTED AND GRACED MY ACADEMIC LIFE; *amare et sapere vix deo conceditur.*

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The flow of the river is incessant, and yet its water is never the same, while along the still pools foam floats, now vanishing, now forming, never staying long: So it is with men and women and all their dwelling places here on earth.

Kamo no Chomei, *An Account of My Hut*

1 Introduction

1.1 Background of the research

The creation of matter and its evolution in a dynamic spacetime lie inside the domain of modern cosmology. The former is described as the quantum mechanical generation of the primordial fluctuations while the latter is the main subject of the cosmological perturbation theory. Inflation [1] is the most promising paradigm of the earliest stages of the universal evolution giving a mechanism for the generation of the primordial fluctuations which evolve themselves to form every structure in the later universe. In this mechanism, quantum field theory (QFT) in curved spacetime [2, 3, 4] plays an essential role. Quantum field theory in curved spacetime, especially de Sitter spacetime which approximates the inflationary spacetime, reveals that the cosmological expansion of the inflationary spacetime can stretch the quantum fluctuations of the matter fields. At the moment when the wavelength of a fluctuation mode becomes long enough, the mode is supposed to be classicalized. After the classicalization, we can treat the quantum mode as a classical perturbation, and the classical dynamics becomes capable of tracing the later evolution of the primordial fluctuations. Though the precise justification of the classicalization is still a controversial problem [5, 6], it is widely accepted as an assumption which connects the quantum physics with the classical physics avoiding intricate problems about quantum observation. Thus, quantum field theory in de Sitter spacetime provides us with the solid basis of discussion and the concrete results. One of the most important predictions of the QFT in de Sitter spacetime is the almost flat spectrum of the primordial fluctuations, which is observationally confirmed by CMB experiments, see, e.g., [7, 8].

It should be pointed out that interactions between quantum fields in de Sitter space has not sufficiently been considered despite the inflationary theory has been successful and has given us profound knowledge about the universe. The problem of primordial magnetogenesis theory illustrates this point well. In the context of the primordial magnetogenesis, which seeks the origin of the observationally inferred large-scale magnetic fields [9, 10, 11, 12, 13] in a physical process during inflation [14, 15, 16], strong electric fields naturally occur in the primordial universe, which hinders us proceeding with negligible interactions. The observations of the galactic and extra-galactic magnetic fields encourage people to study the origin of the magnetic fields, and inflationary magnetogenesis is considered as one of the promising candidates as a way to achieve long enough coherent length. For instance, the kinetic coupling model (or f^2FF model [17, 18, 19, 20]), which is a well-studied model of inflationary magnetogenesis, predicts that very strong electric fields are inevitably produced during inflation, if it is assumed to generate the magnetic fields which are strong enough to leave observable signatures. Usually, it is considered by many authors that the overproduced electric fields would spoil the inflationary spacetime background evolution [21], so it violates the observational limit from the cosmic microwave background radiation [22, 23, 24, 25]. Nevertheless, only a limited number of works has been based on models which properly take the quantum electrodynamic interactions into account regardless of its potential to change the dynamics drastically.

Quantum electrodynamics in de Sitter spacetime (dS-QED) is a straight way to treat the interactions between the charged particles and the electromagnetic fields in the inflationary universe. The concrete form of the dS-QED is obtained from the flat-spacetime version of it by some modifications required to adapt to the principal of the general spacetime background. As we will see in the following chapter, the description of dS-QED is somewhat simple. However, it is also admittedly challenging to address the fully interacting theory.

Fortunately, there are a couple of ways to deal with the analytical complexity of interacting theories. Most preferably, we could find the exact solutions in some cases, understandably not always. The perturbation technique dating back to the Feynman's work can be implemented in every situation. However, we should notice that some of the physical effects which are the interest of the present dissertation cannot be obtained by the perturbative methods. For instance, particle-antiparticle production process from a strong electric background field as known as Schwinger effect [26, 27] in flat spacetime is one of the consequences of the nonperturbative nature of the quantum field theory. There are no reasons that we should expect the phenomena becomes the perturbative effect in curved spacetime.

A popular nonperturbative method is the lattice calculation which is also extended to the case of curved spacetime background [28]. The lattice technique is attracting because it is numerically possible to study the nonperturbative effects. The drawbacks are that we have the sign problem [29] in case of the time-dependent electric background field, and that we are not sure whether the zero separation limit of the lattice QED theory recovers the continuous theory since QED is not an asymptotically-free theory unlike the QCD. Therefore, the lattice calculation as an approach to nonperturbative QED effect is difficult, and even if we overcome the sign problem, the result would be doubtful.

None of the methods above can be perfect, but, with careful treatment, the results obtained could offer physical insights. Before explaining the scope of the present dissertation, we will review the achievements in the field up to the present in the next section.

1.2 Research trend of the field

The inflationary cosmology was first proposed as a solution to fundamental problems of the Big bang theory, the horizon and flatness problem [30, 31]. The inflation, or the exponential expansion of the spacetime, at the beginning of the universe aptly explains the emergence of the extremely flat and homogeneous background. Furthermore, the inflation theory predicts the generation of the primordial perturbations as we mentioned. The tensorial perturbation (primordial gravitational wave) was analyzed with amazing foresight in [32]. After that, the generation of the scalar perturbation was discussed in [33, 34]. In contrast to the scalar and tensor perturbations, the generation of the vector modes is insignificant since vector fields have the conformal invariance which prevents the vector quantum fluctuations from growing even in the inflationary spacetime. Thus the theories which explicitly break the conformal symmetry have been considered after the pioneering work [17]. Some of the conformal symmetry-violating theories are in possible descent from developing higher energy theories. For example, the dynamic coupling in the context of the f^2FF model can be a direct consequence of the dilaton field in the theories with compactified extra dimensions. Therefore, the cosmological magnetic fields can be a natural probe of the inflationary physics. Note that the electric fields, if generated during the inflationary period, would be erased by the screening effect caused by charged particles created in the later universe. Since the existence of the large-scale magnetic fields has observationally been implied, the inflationary magnetogenesis scenarios have another advantage that it is possible to generate coherent magnetic fields on large scales, especially, on Mpc scale. Although the f^2FF model is attractive, it has been recognized that the model suffers from either the backreaction problem (excess of the electric fields over the magnetic fields) or the strong coupling problem (the electromagnetic coupling becomes extremely large at early times whenever we avoid the backreaction problem) [21].

We might be possible to neatly get around both of the problems of the inflationary magnetogenesis theory. However, it is also a means of reaching a consistent magnetogenesis scenario to attack the problems squarely dealing with dS-QED. The simplest and analytical set up is the QED system with constant, homogeneous electric background fields in de Sitter spacetime.

A number of studies on this system have appeared [35, 36, 37, 38, 39, 40, 41, 42]. These research have the wide variety of the quantum physical motivations from false vacuum decay and bubble nucleation to a thermal interpretation of particle production or cosmological consequences including magnetogenesis. Schwinger effect and its induced current in de Sitter spacetime for a *scalar* charged particle have been investigated both in the $1+1$ dimension case [37], in the $1+2$ dimension case [43] and the $1+3$ case [39, 40]. In those works, it is found that the scalar induced current is strongly enhanced for the small mass field and weak electric field regime. This phenomenon was called IR hyperconductivity and found in [37] for the first time. In [39, 40], the authors reported a negative current which flows in a direction against the electric field in addition to the IR hyperconductivity. In the $1+3$ dimensional case, it was also found the terms which are not suppressed by the exponential factor $\exp(-\pi m^2/eE)$ or $\exp(-2\pi m/H)$ appear in the massive field limit. These suppression factors are naturally expected from the semiclassical approximation. Thus it suggests the breakdown of the semiclassical description in the massive limit. The physical implication of the negative current and the incorrect semiclassical limit has not been understood yet. Until very recently, the *spinor* induced current has been calculated only in the case of $1+1$ dimensional de Sitter spacetime [41], in which the authors have shown that there is neither the IR hyperconductivity nor the negativity of the current. The fermionic induced current in $1+3$ dimensional de Sitter spacetime [42] will be the subject of Chapter 3 of the present thesis.

Historically, the electromagnetic response of quantum fields has been a central problem in a field of condensed matter physics. The methodology established in the context of condensed matter physics is to some extent similar to the one in the cosmological context. Both consider the response of the quantum fields on nontrivial backgrounds (e.g., gravity, interactions inside matters) to the test electromagnetic fields. Through the research, multifarious phenomena have been found. A noteworthy recent example is the negative electric capacitance (anti-screening effect) in a ferroelectric system [44, 45, 46] possibly corresponds to the negative current we mentioned above.

Another perspective of electromagnetic response is to regard the induced current as a transportation phenomenon. The transportation theories can naturally involve thermodynamical concepts such as heat conveyance. Meanwhile, it is also known gravitational systems with event horizon bear thermodynamical properties. However, thermal aspects of transport phenomena in the gravitational background have not gained much attention so far.

1.3 Structure of the present dissertation

This dissertation addresses the following research questions:

- Does dS-QED have curious phenomena?
- If so, what is the physics behind the phenomena?
- What is their cosmological consequence?

In other words, we aim to explore new effects of dS-QED and their consequence. Our focus is at first on the spinor QED in $1+3$ dimensional de Sitter spacetime where we will discover the negativity of the current like the scalar QED case. We will then argue a problem with renormalization condition employed to obtain the finite expectation value of the current operator and propose an alternative (we call it maximal subtraction) to the conventional renormalization condition called minimal subtraction. While we lack the experimental information about the renormalization point of QFT in curved spacetime, theoretical evidence will be presented to persuade ourselves of the appropriateness of the maximal subtraction scheme. We will also advocate the use of a variant of renormalization group technology as an approximation method aspiring after the extension of the analytical result of the relatively simpler system.

With these purposes in mind, the following contents will be discussed in each of the succeeding chapters. In chapter 2, the fundamentals of the quantum field theory in curved spacetime required for the later part of the dissertation will be introduced. In chapter 3, analysis of the electric response of the Dirac field will be shown. We will, then, discuss the relation between the thermodynamical aspect and microphysical QED process near de Sitter horizon. The calculation of the scalar QED based on another renormalization scheme, point-splitting renormalization, will be implemented in chapter 4 to examine the validity of the conventional renormalization scheme (the minimal adiabatic subtraction). Finally, in chapter 5, we will summarize the results obtained and discuss the cosmological consequence. Appendices A, B, C, and D show analytical details. Appendix E will be dedicated to developing a more flexible analytical method which applies to more complex systems. The effective field theoretical approach combined with nonperturbative renormalization group technique will be employed, and a germinal result will be shown at last.

Not only does God play dice but... he sometimes throws them where they cannot be seen.

Stephen Hawking

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Fundamentals of Quantum Field Theory in Curved Spacetime

Harmonizing two major fundamental low of physics, theory of general relativity and quantum field theory, is arguably one of the most recondite problems in physics.

2.1 Physics in curved spacetime

The ultimate goal of fundamental science is supposed to be an unattained quantum gravity theory which both matter and gravity are quantized. Nevertheless, it is practical and meaningful to consider the theory of *quantized* matter with *classical* gravity. This type of theories is usually called quantum field theory in curved spacetime. The classical treatment of the gravity part is validated unless we consider the Planck scale physics. Sub-Planck scale curvature is typically assumed in the context of the modern cosmology. Accordingly, the quantum field theory in curved spacetime can be a basis of the inflationary cosmology.

The theory of general relativity tells us that properties of a curved spacetime are encoded in a metric tensor $g_{\mu\nu}$. This metric gives the infinitesimal proper distance between two points in the spacetime $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. Here, $\mathrm{d}x^\mu$ is the coordinate difference between the two points. Minkowski space, or the flat spacetime is characterized by $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2$ and the flat spacetime metric is written as $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Physics in the flat spacetime can be elevated to the one in the curved spacetime by a few replacements required by a principle of general relativity, i.e., general covariance. Only three changes are practically necessitated:

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu}, \quad \mathrm{d}^4x \longrightarrow \sqrt{-g}\mathrm{d}^4x, \quad \partial_\mu \longrightarrow \nabla_\mu. \tag{2.1}$$

Firstly, Minkowski metric is replaced by a general metric. The second replacement makes the 4-volume d^4x covariant. $g = \det g_{\mu\nu}$ is the determinant of the metric tensor. Finally, the covariant derivative which is transformed as a vector under the general coordinate transformation is used instead of the partial derivative. The action of the covariant derivative differs depending on the spin of the field on which it operates.

It is possible to consider modifications beyond the minimal replacement shown above while abiding by the general covariance. An example is a non-minimal coupling $\xi R\phi^2$ between a scalar field ϕ and Ricci scalar R (ξ is the coupling constant). Another example is electromagnetic fields $F_{\mu\nu}$ coupled to gravity in the form of $RF_{\mu\nu}F^{\mu\nu}$, $R^{\mu\nu}F_{\mu\lambda}F_\nu^\lambda, \cdots$ and so on. At the same time, we can stay on the field theory minimally coupled to gravity to enjoy the theoretical beauty and simplicity. In this dissertation, our focus will be limited to this most straightforward but pithy extension of the flat spacetime physics.

In the rest of this section, the quantization procedures of the scalar field, the Dirac field, and the treatment of the gauge field theory in curved spacetime are demonstrated in turn with further clarifying our convention and notation. This chapter is heavily based on the standard textbooks [3, 4].

2.2 Scalar field

Let us begin with the simplest case: a free complex scalar field ϕ. The classical action is given by

$$S[\phi, \phi^\dagger] = \int \sqrt{-g(x)}\mathrm{d}^4x \left(-g^{\mu\nu}(x)\phi^\dagger_{;\mu}(x)\phi_{;\nu}(x) - m^2\phi^\dagger(x)\phi(x)\right), \tag{2.2}$$

where the semicolon $\phi_{;\mu}$ in super/subscript denotes the covariant derivative ∇_μ. The partial derivative ∂_μ is, in contrast, abbreviated to the comma notation $\phi_{,\mu}$. Both of these have an identical action on a scalar field, i.e.

$$\phi_{;\mu} = \phi_{,\mu}, \tag{2.3}$$

while the covariant derivative on a vector field V^μ is given by

$$V^\mu{}_{;\nu} = V^\mu{}_{,\nu} + \Gamma^\mu_{\nu\lambda} V^\lambda, \quad V_{\mu;\nu} = V_{\mu,\nu} - \Gamma^\lambda_{\mu\nu} V_\lambda. \tag{2.4}$$

Here Christoffel symbol $\Gamma^\mu_{\nu\lambda}$ consists of the first derivatives of the metric:

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\lambda} (g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}). \tag{2.5}$$

The equation of motion for the scalar field ϕ is readily obtained by functional differentiation of the action with respect to $\phi^\dagger$. We obtain the following Klein-Gordon (KG) type equation of motion

$$\frac{1}{\sqrt{-g(x)}} \frac{\delta S[\phi, \phi^\dagger]}{\delta \phi^\dagger(x)} = \left(\Box_x - m^2\right) \phi(x) = 0. \tag{2.6}$$

The definition of $\Box$ is given by $\Box\phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = \frac{1}{\sqrt{-g}} (\sqrt{-g} g^{\mu\nu} \phi_{,\nu})_{,\mu}$.

Once a normalized time-like vector n^μ is fixed, the scalar product (KG inner product) $(\cdot, \cdot)$ is defined as follows

$$(\phi_1, \phi_2) = i \int_\Sigma \sqrt{-g_\Sigma} \mathrm{d}\Sigma \; n^\mu \phi_2^* \overleftrightarrow{\partial_\mu} \phi_1, \tag{2.7}$$

where g_Σ is an induced metric on a space-like hypersurface Σ orthogonal to the time-like vector n^μ and $\mathrm{d}\Sigma$ is the volume element of the hypersurface. This expression is independent of the choice of Σ. This is a consequence of Gauss's theorem and the equation of motion (2.6). In a specific spacetime coordinate $(t, \boldsymbol{x})$, quantization of the scalar field is done by imposing the canonical commutation relation between ϕ and its canonical conjugate

$$\pi_\phi(t, \boldsymbol{x}) = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \dot{\phi}(t, \boldsymbol{x})}. \tag{2.8}$$

The canonical commutation relation is given by

$$[\hat{\phi}(t, \boldsymbol{x}), \hat{\pi}_\phi(t, \boldsymbol{x}')] = i\delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{2.9}$$

The Dirac's delta function is defined as $\int \mathrm{d}\Sigma \delta(\boldsymbol{x} - \boldsymbol{x}') = 1$.

A complete set of the mode solutions $u_i(x)$ of (2.6) yields the decomposition of the quantized field

$$\hat{\phi}(x) = \sum_i \left(\hat{a}_i u_i(x) + \hat{b}_i^\dagger u_i^*(x) \right). \tag{2.10}$$

The second quantization is performed by imposing the commutation relations between the creation-annihilation operators:

$$[\hat{a}_i, \hat{a}_j^\dagger] = [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad (\text{others}) = 0. \tag{2.11}$$

Orthogonality of these modes is given, in terms of KG product, by

$$(u_i, u_j) = -(u_i^*, u_j^*) = \delta_{ij}, \quad (u_i, u_j^*) = 0. \tag{2.12}$$

The index i schematically labels the mode functions, and it can be discrete or continuous. The vacuum state $|0\rangle$ is defined as a state that satisfies the relation $\hat{a}_i |0\rangle = \hat{b}_i |0\rangle = 0$ for all i.

What matters here is how to select a natural complete set of modes. It is an a priori procedure to decompose the flat spacetime into space and time. Conversely, in a general spacetime, we cannot determine a unique set of physically relevant modes without assistance of any particular symmetry. This fact leads to the vagueness of the definition of the vacuum state and vague concept of particles in a general background spacetime. More importantly, even when we can find a physical vacuum at some point of time t_i, This is not generally the same as the one at another time t_f. In brief, the notion of the vacuum state can change in time affected by background gravity. This is the mechanism of gravitational particle production.

To demonstrate this point more concretely, let us consider two different complete sets of mode functions u_i and $\tilde{u}_i$. The two complete sets respectively determine the associated creation-annihilation operators expressed as

$$\hat{\phi}(x) = \sum_i \left(\hat{a}_i u_i(x) + \hat{b}_i^\dagger u_i^*(x) \right) = \sum_i \left(\hat{\tilde{a}}_i \tilde{u}_i(x) + \hat{\tilde{b}}_i^\dagger \tilde{u}_i^*(x) \right). \tag{2.13}$$

The vacuum states $|0\rangle$ and $|\tilde{0}\rangle$ accompanied by these operators are also defined. Owing to the completeness, there must exist a linear relation between the sets u_i and $\tilde{u}_i$, which takes the following forms

$$u_i = \alpha_{ij} \tilde{u}_j + \beta_{ij} \tilde{u}_j^*, \quad \tilde{u}_i = \alpha_{jk}^* u_k - \beta_{jk} u_k^*. \tag{2.14}$$

The relation is called Bogoliubov transformation. The coefficients α_{ij}, β_{ij} are called Bogoliubov coefficients, which satisfy the following conditions

$$\alpha_{ij}\alpha^*_{jk} - \beta_{ij}\beta^*_{jk} = \delta_{ik}, \quad \alpha_{ij}\beta_{jk} - \beta_{ij}\alpha_{jk} = 0. \tag{2.15}$$

Combining these relations, the following linear relations between the creation-annihilation operators can be immediately obtained

$$\hat{a}_i = \alpha^*_{ji}\hat{\tilde{a}}_j - \beta^*_{ji}\hat{\tilde{b}}^\dagger_j, \quad \hat{b}^\dagger_i = -\beta_{ji}\hat{\tilde{a}}_j + \alpha_{ji}\hat{\tilde{b}}^\dagger_j \ . \tag{2.16}$$

When the expectation value of the number operator of u_i mode particles is calculated by the $\tilde{u}_i$ vacuum $|\tilde{0}\rangle$, it yields non-zero expectation values

$$\langle\tilde{0}|\hat{N}_i|\tilde{0}\rangle \equiv \langle\tilde{0}|\hat{a}^\dagger_i\hat{a}_i|\tilde{0}\rangle = \langle\tilde{0}|\hat{b}^\dagger_i\hat{b}_i|\tilde{0}\rangle = \sum_j \beta_{ij}\beta^*_{ji}. \tag{2.17}$$

In other words, $|\tilde{0}\rangle$ vacuum contains $\sum_j \beta_{ij}\beta^*_{ji}$ particles of u_i modes.

2.3 Dirac field

Treatment of Dirac spinor in curved spacetime is explained in this section. The spin of a field in the flat spacetime is determined by transformation property under Lorentz transformations. We need to extend the consideration of the spin to curved spacetime. Needless to say, there is no global Lorentz symmetry in a general curved spacetime, but local Lorentz transformations are still possible to be considered. We employ with this intention the tetrad formalism, in which a set of four normalized, orthogonal vector fields $e^a_\mu(x)$ is defined. The four vectors span the tangent space at each spacetime point x. Hereafter, the Latin indices ($a, b, c \cdots = 0, 1, 2, 3$) are associated with the tetrads, and the Greek letters ($\mu, \nu \cdots$) are the indices for spacetime.

The metric of curved spacetime is related to the Minkowski metric using a tetrad as an intermediary

$$g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab}. \tag{2.18}$$

The dual of a tetrad $e^a_\mu(x)$ is written as $e^\mu_a(x)$ and determined by $e^a_\mu(x)e^\mu_b(x) = \delta^a_b$. A tetrad $e^a_\mu(x)$ behaves as a covariant vector in the case of general coordinate transformations. Besides, it is also possible to consider a local Lorentz transformation $\Lambda_a{}^b(x)$ which changes the local coordinate at each spacetime point yielding

$$e^a_\mu(x) \longrightarrow \Lambda_b{}^a(x) e^b_\mu(x). \tag{2.19}$$

Contraction of a tetrad and an arbitrary covariant vector field $V_\mu(x)$ makes the vector an object $V_a(x) = e^\mu_a(x)V_\mu(x)$ that behaves as a collection of four scalars under general coordinate transformations. This produces a local Lorentz vector instead.

It is enough to consider an infinitesimal local Lorentz transformation $\Lambda_a{}^b = \delta_a{}^b + \omega_a{}^b$ for the purpose of revealing transformation properties of fields with spin. The generator Σ^{ab} of local Lorentz transformations obeys the algebra

$$[\Sigma^{ab}, \Sigma^{cd}] = (\eta^{ac}\Sigma^{db} - \eta^{bc}\Sigma^{da}) - (\eta^{ad}\Sigma^{cb} - \eta^{bd}\Sigma^{ca}). \tag{2.20}$$

For a Dirac spinor ψ, which is a reducible $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of local Lorentz transformations, the generator Σ^{ab} has an explicit expression

$$\Sigma^{ab} = -\frac{1}{4}[\gamma^a, \gamma^b], \tag{2.21}$$

in terms of gamma matrices γ^a defined on the tangent space. The gamma matrices obey the anti-commutation relation (Clifford algebra)

$$\{\gamma^a, \gamma^b\} = -2\eta^{ab}. \tag{2.22}$$

The covariant derivative ∇_μ acting on a Dirac field is written in terms of the spinor connection Γ_μ as

$$\nabla_\mu(x) \equiv \partial_\mu + \Gamma_\mu(x), \tag{2.23}$$

which ensures the general covariance. We can explicitly write down the spinor connection as

$$\Gamma_\mu = \frac{1}{2} e^\nu_a e_{b\nu;\mu} \Sigma^{ab}. \tag{2.24}$$

To see the transformation property of the covariant derivative, let us begin with examining the local Lorentz transformation of the spin connection. The infinitesimal transformation of Γ_μ by

$\Lambda_a^{\ b} = \delta_a^{\ b} + \omega_a^{\ b}$ up to linear order of $\omega_a^{\ b}$ is given by,

$$\begin{aligned}\Gamma_\mu(x) \to \bar{\Gamma}_\mu &= \frac{1}{2}\Lambda_a^{\ c} e_c^{\ \nu} (\Lambda_b^{\ d} e_{d\nu})_{;\mu} \Sigma^{ab} \\ &= \Gamma_\mu + \frac{1}{2}\omega_{ab}[\Sigma^{ab}, \Gamma_\mu] - \frac{1}{2}\omega_{ab,\mu}\Sigma^{ab}.\end{aligned} \tag{2.25}$$

Equation (2.25) is also written as

$$\Gamma_\mu \to \bar{\Gamma}_\mu = D(\Lambda)\Gamma_\mu D^{-1}(\Lambda) - (\partial_\mu D(\Lambda))D^{-1}(\Lambda), \tag{2.26}$$

with $D(\Lambda) = 1 + \frac{1}{2}\omega_{ab}\Sigma^{ab}$ being a representation of the infinitesimal transformation Λ. The latter term absorbs the difference of Lorentz transformations at different points. As a result, we obtain the required transformation nature of the covariant derivative on a Dirac field $e_a^{\ \mu}\nabla_\mu\psi$ as a Lorentz scalar

$$\begin{aligned}e_a^{\ \mu}(\partial_\mu + \Gamma_\mu)\psi \to \Lambda_a^{\ b} e_b^{\ \mu} \left(\partial_\mu + D(\Lambda)\Gamma_\mu D^{-1}(\Lambda) - (\partial_\mu D(\Lambda))D^{-1}(\Lambda)\right) D(\Lambda)\psi \\ = \Lambda_a^{\ b} e_b^{\ \mu} D(\Lambda)\left((\partial_\mu + \Gamma_\mu)\psi\right).\end{aligned} \tag{2.27}$$

The product $\psi^\dagger\psi$ does not behave as a Lorentz scalar as is the case with the flat spacetime. This is because $\psi^\dagger$ is transformed as $\psi^\dagger \to \psi^\dagger D(\Lambda)^\dagger$, and $D(\Lambda)^\dagger = \gamma^0 D(\Lambda)^{-1}\gamma^0$ Instead we can make the quantity $\bar{\psi}\psi$ a Lorentz scalar where $\bar{\psi} = \psi^\dagger\gamma^0$ is Dirac conjugate of the field ψ. [1]

Finally, the action of a free Dirac field ψ in curved spacetime is available,

$$\begin{aligned}S[\psi, \bar{\psi}] &= \int \sqrt{-g}\, \mathrm{d}^4x \left\{ \frac{i}{2}\left(\bar{\psi} e_a^\mu \gamma^a \nabla_\mu \psi - (\nabla_\mu \bar{\psi}) e_a^\mu \gamma^a \psi\right) - m\bar{\psi}\psi \right\} \\ &= \int \sqrt{-g}\, \mathrm{d}^4x\, \bar{\psi}(i e_a^\mu \gamma^a \nabla_\mu - m)\psi.\end{aligned} \tag{2.28}$$

Dirac equation, namely, the equation of motion for the Dirac field ψ is obtained by differentiating the action with respect to the conjugate $\bar{\psi}$

$$\frac{1}{\sqrt{-g(x)}} \frac{\delta S[\psi, \bar{\psi}]}{\delta \bar{\psi}(x)} = \left(i e_a^\mu \gamma^a \nabla_\mu - m\right)\psi(x) = 0. \tag{2.29}$$

[1] Note that Dirac conjugate $\bar{\psi}$ is defined by the gamma matrix on the tangent space, $\gamma^0 = \delta_a^0 \gamma^a$ even in a curved spacetime. See also Appendix A for more details.

For the Dirac conjugate field $\bar{\psi}$, equation of motion is given by

$$i(\partial_\mu \bar{\psi} - \bar{\psi}\Gamma_\mu)e^\mu_a \gamma^a + m\bar{\psi} = 0. \tag{2.30}$$

The property of the hermite conjugate of the spin connection $\Gamma^\dagger_\mu = -\gamma^0\Gamma_\mu\gamma^0$ is useful to find the equation of $\bar{\psi}$ from (2.29).

In the quantization procedure, we impose the anti-commutation relation $\{\psi(t,\boldsymbol{x}), \pi_\psi(t,\boldsymbol{x}')\} = i\delta(\boldsymbol{x}-\boldsymbol{x}')$ instead of the commutation relation. The canonical conjugate momentum π_ψ is given by

$$\pi_\psi(t,\boldsymbol{x}) = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\dot{\psi}(t,\boldsymbol{x})}. \tag{2.31}$$

Mode decomposition of a Dirac field takes the following form

$$\hat{\psi}(x) = \sum_i \left(\hat{b}_i u_i(x) + \hat{d}^\dagger_i v_i(x)\right), \tag{2.32}$$

where u_i and v_i are four spinors which independently satisfy the Dirac equation (2.29). The relation between the u spinor and the v spinor will be discussed in the next chapter.

2.4 Gauge field

The action for a complex scalar field (2.2) and for a Dirac field (2.28) each has a global $U(1)$ symmetry which is invariant under simultaneous rotations of the phase of fields by α

$$(\phi, \phi^\dagger) \longrightarrow (\mathrm{e}^{-i\alpha}\phi, \mathrm{e}^{i\alpha}\phi^\dagger), \tag{2.33}$$

$$(\psi, \psi^\dagger) \longrightarrow (\mathrm{e}^{-i\alpha}\psi, \mathrm{e}^{i\alpha}\psi^\dagger), \tag{2.34}$$

where α is a constant in spacetime.

In $U(1)$ gauge theory, the global $U(1)$ symmetry is upgraded to the local symmetry by the replacement of the parameter α by an arbitrary function $e\alpha(x)$ dependent on the spacetime point. e is a constant parameter. This prescription requires that we take a gauge vector field $A_\mu(x)$ into account to compensate an excess term appears in the transformation of the derivatives, $\nabla_\mu\phi$, and $\nabla_\mu\psi$.

As transformation of the gauge field is determined by

$$A_\mu(x) \to A_\mu(x) + \alpha_{,\mu}(x), \tag{2.35}$$

we can recover the correct gauge transformation rules for the objects such as

$$D_\mu\phi \equiv (\nabla_\mu + ieA_\mu)\phi \to \mathrm{e}^{-i\alpha(x)} D_\mu\phi, \tag{2.36}$$

and similarly,

$$D_\mu\psi \to \mathrm{e}^{-i\alpha(x)} D_\mu\psi. \tag{2.37}$$

As a result, the combinations $(D_\mu\phi)^\dagger D_\nu\phi$ and $\bar{\psi}\gamma^\mu D_\mu\psi$ become gauge invariant quantities. The operator we have defined $D_\mu \equiv \nabla_\mu + ieA_\mu$ is also called (gauge) covariant derivative. Moreover, the kinetic term of the gauge field is given by Maxwell Lagrangian with $\eta^{\mu\nu}$ replaced by the general curved spacetime metric $g^{\mu\nu}$.

2.5 Quantum electrodynamics

The descriptions above in all gives manifestly covariant and gauge invariant action in curved spacetime for spinor QED:

$$S = S_{\text{gauge}}[A_\mu] + S_{\text{fermion}}[\psi, \bar{\psi}, A_\mu] = \int \mathrm{d}^4x\sqrt{-g}\left\{-\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi\right\}, \tag{2.38}$$

and for scalar QED:

$$S = S_{\text{gauge}}[A_\mu] + S_{\text{boson}}[\phi, \phi^\dagger, A_\mu] = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} - g^{\mu\nu}(D_\mu\phi)^\dagger D_\nu\phi - m^2\phi^\dagger\phi\right). \tag{2.39}$$

The equation of motion for the gauge field, i.e. Maxwell equation is given by

$$F^{\mu\nu}_{\ \ ;\nu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{\mu\nu})_{,\nu} = J^\mu, \tag{2.40}$$

where the source term J^μ is defined by the differentiation of the matter action $S_{\text{fermion/boson}}$ with respect to the gauge field $A_\mu(x)$. The explicit forms for the fermionic/bosonic current are, respectively, given by

$$J^\mu(x) = \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{fermion}}[\psi, \bar{\psi}, A_\mu]}{\delta A_\mu(x)} = -e\bar{\psi}\gamma^\mu\psi, \tag{2.41}$$

$$\begin{aligned} J^\mu(x) = \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{boson}}[\phi, \phi^\dagger, A_\mu]}{\delta A_\mu(x)} &= ie\hat{\phi}^\dagger(x)\overleftrightarrow{D^\mu}\hat{\phi}(x) \\ &= ie\{\hat{\phi}^\dagger(x)D^\mu\hat{\phi}(x) - (D^\mu\hat{\phi}(x))^\dagger\hat{\phi}(x)\}. \end{aligned} \tag{2.42}$$

By taking covariant derivative of (2.40), we can explicitly confirm the conservation law of the current

$$\nabla_\mu J^\mu = F^{\mu\nu}{}_{;\nu;\mu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{\mu\nu})_{,\nu\mu} = 0. \tag{2.43}$$

The current conservation is also proved by direct calculation with the aid of the equations of motion for the Dirac field ψ [2]

$$(i\gamma^\mu D_\mu - m)\psi = (i\gamma^\mu\nabla_\mu - eA_\mu\gamma^\mu - m)\psi = 0, \tag{2.44}$$

and for the scalar field ϕ

$$(g^{\mu\nu}D_\mu D_\nu - m^2)\phi = (g^{\mu\nu}(\nabla_\mu + ieA_\mu)(\nabla_\nu + ieA_\nu) - m^2)\phi = 0. \tag{2.45}$$

The fermionic current (2.41) can be expressed similarly to the bosonic current (2.42),

$$\begin{aligned} -e\bar{\psi}\gamma^\mu\psi &= \frac{ie}{2m}\left(\bar{\psi}\overleftrightarrow{D^\mu}\psi - \frac{1}{2}\nabla_\nu\left(\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi\right)\right) \\ &\equiv \frac{ie}{2m}\left(\bar{\psi}D^\mu\psi - \overline{(D^\mu\psi)}\psi - \frac{1}{2}\nabla_\nu\left(\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi\right)\right). \end{aligned} \tag{2.46}$$

Here, the first term $\bar{\psi}\overleftrightarrow{D^\mu}\psi$, which has the same structure as the bosonic current, corresponds to the free charge and current. The remaining part, the divergence of an anti-symmetric tensor $\mathcal{M}^{\mu\nu} \equiv -(ie/4m)\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi$, corresponds to the bound charge and current. The tensor $\mathcal{M}^{\mu\nu}$ is called polarization-magnetization tensor. We can define two 3-vectors P^i (polarization) and M^i

[2] Note that the sign in front of the charge e differs in case of the field equation for the Dirac conjugate field $\bar{\psi}$, which is given by $i\overline{D_\mu\psi}\gamma^\mu + m\bar{\psi} = i(\nabla_\mu - ieA_\mu)\bar{\psi}\gamma^\mu + m\bar{\psi} = 0$.

(magnetization) as

$$P^i = \mathcal{M}^{i0} = -\mathcal{M}^{0i}, \quad M^i = \frac{1}{2}\epsilon^{ijk}\mathcal{M}_{jk}. \tag{2.47}$$

In terms of these 3-vectors, the bounded charge and current are given by

$$\rho_b = -\frac{1}{\sqrt{-g}}(\sqrt{-g}P^i)_{,i}, \quad j_b^i = \frac{1}{\sqrt{-g}}(\sqrt{-g}P^i)_{,0}, + \frac{1}{\sqrt{-g}}(\sqrt{-g}\epsilon^{ijk}M_k)_{,j} \tag{2.48}$$

both of which are the curved spacetime extension of the familiar results $\rho_b = -\nabla \cdot \boldsymbol{P}$ and $\boldsymbol{j}_b = \dot{\boldsymbol{P}} + \nabla \times \boldsymbol{M}$ in the flat spacetime.

The canonical quantization procedure of the gauge field involves a subtle complication stems from the gauge invariance. This is usually cured by the gauge fixing technique used with the path integral quantization scheme. We will come back to this point in Appendix E.

Daemon irrepit callidus,
Allicit cor honoribus.

Anonymous Goliyard

3

Fermionic Induced Current in de Sitter Spacetime

The success of the inflation theory drives us to explore the quantum nature of fields in curved spacetime. Even before Ratra pioneered the possibility of the inflationary magnetogenesis [17], studies on the quantum electrodynamics in de Sitter spacetime have gained great attention. Strong electromagnetic fields on large scales naturally arise in the context of the magnetogenesis theory, which requires the utmost care of the interaction between the electromagnetic fields and charged fields in the presence of gravity. However, it is a challenging task to analyze the quantum interaction in curved spacetime due to both technical complexities and conceptual difficulties, which quantum field theory in curved spacetime possesses.

In this chapter, our focus will be on the fermionic current induced by the electric field in 4-dimensional de Sitter spacetime rather than the particle production rate. We will also discuss the stability of the background electric field considering the backreaction from the induced current. Finally, we will try to understand the physics of the complex result exploiting thermodynamical concepts including Maxwell's demon.

3.1 Induced current

At the early stage of the research on this topic, the particle production from the electric field known as Schwinger effect was the subject of discussion. Those works were done with the Bogoliubov calculation, which necessarily depends on the particle concept and involves with the ambiguity as we have remarked in the previous chapter. Regarding Schwinger effect in de Sitter spacetime, two nontrivial backgrounds, electric field and gravitational field, are involved. Both fields are responsible for the particle production from the quantum vacuum. The combination of the two different production sources makes the problem challenging and interesting.

Moving through the electric field, the produced particles induce the electric current in the universe and cause the backreaction to the electric field. The induced electric current which characterizes the size of the backreaction is quantitatively evaluated by the vacuum expectation value of the current operator $\hat{J}^{\mu}$. Notably, the induced electric current is a well-defined physical quantity since it appears in the semiclassical Maxwell equation as the source term in contrast to the particle production rate, which has no connection to basic equations.

In the calculation of the induced electric current, one has to deal with the divergence originating from vacuum contribution. In [39, 41], the adiabatic subtraction method [47, 48] was employed. It is known that the WKB (adiabatic) expansion for a fermionic field cannot satisfy the equation of motion while satisfying the normalization condition to all orders of the expanding parameter $\hbar$. Nevertheless, the equivalence of the adiabatic and the DeWitt-Schwinger renormalization schemes was shown in [49], and there appeared applications of the adiabatic regularization of fermionic fields [50, 51].

The rest of the present chapter is organized as follows. In Sec. 3.2, we introduce the Dirac equation in de Sitter spacetime and solve it in a background electric field. In Sec. 3.3, the evaluation of the induced current is described. The properties of the renormalized current are investigated in Sec. 3.4. Finally, an electro-thermodynamical point of view is introduced to gain the understanding of the result in Section 3.5. Some of the technical details can be found in appendices B, C, and D.

3.2 Dirac equation in background field and solution

In a spatially flat Friedman-Lemaître-Robertson-Walker spacetime

$$\mathrm{d}s^2 = a(\eta)^2(-\mathrm{d}\eta^2 + \mathrm{d}\boldsymbol{x}^2), \tag{3.1}$$

the action (2.38) reduces to

$$S = \int \mathrm{d}^4x \left\{ -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \bar{\xi}\left(i\gamma^a\partial_a - \delta^\mu_a eA_\mu\gamma^a - ma\right)\xi \right\}, \tag{3.2}$$

where $\xi(\eta, \boldsymbol{x})$ is the canonical Dirac field $\xi = a^{3/2}\psi$, and we have used the following equations held in the FLRW universe:

$$e_a^{\ \mu} = \frac{1}{a}\delta_a^{\ \mu}, \quad \gamma^a e_a^{\ \mu}\Gamma_\mu = \frac{3a'}{2a^2}\gamma^0, \quad \gamma^a e_a^{\ \mu}(eA_\mu) = \frac{1}{a}\delta^\mu_a eA_\mu\gamma^a, \tag{3.3}$$

where the prime denotes the derivative with respect to the conformal time η. It is clearly shown that the conformal symmetry is recovered for massless fermion case, $m = 0$.

The equation of motion for ξ field (the Dirac equation) is given by

$$(i\gamma^a\partial_a - eA_a\gamma^a - ma)\xi(\eta, \boldsymbol{x}) = 0. \tag{3.4}$$

Substituting $\xi = (i\gamma^a\partial_a - \delta^\mu_a eA_\mu\gamma^a + ma)\zeta$ into Eq. (3.4), we obtain the quadratic Dirac equation,

$$\left\{(\partial_a + ieA_a)^2 - m^2a^2 + i\left(ma'\gamma^0 - \frac{e}{2}\gamma^a\gamma^b F_{\mu\nu}\delta^\mu_a\delta^\nu_b\right)\right\}\zeta(\eta, \boldsymbol{x}) = 0. \tag{3.5}$$

Hereafter, we consider a homogeneous electric background field $A_\mu(x) = (0, 0, 0, A_z(\eta))$ in de Sitter spacetime, which is determined by the constant Hubble parameter $a'/a^2 = H = const.$. Explicit time dependence of the background field A_z is given by

$$A_z = -\frac{E}{H}(a-1) = -\frac{E}{H}\left(\frac{1}{1-H\eta} - 1\right), \tag{3.6}$$

where E is a constant represents electric field strength. The scale factor is taken as $a = 1/(1 - H\eta)$ and the offset -1 is introduced in $(a - 1)$ so that we can properly take Minkowski limit

($H \to 0$) resulting in a gauge field which represents the homogeneous constant electric field in Minkowski spacetime,

$$A_z \xrightarrow{H\to 0} -Et. \tag{3.7}$$

Note here that $\eta = (1 - \mathrm{e}^{-Ht})/H \xrightarrow{H\to 0} t$ and $\eta \in (-\infty, 1/H)$ in our notation. The physical strength of electric field in de Sitter spacetime is given by $-a^{-2}\partial_\eta A_z = E$. Therefore the background gauge (3.6) has a constant electromagnetic energy density.

Let us further manipulate Eq. (3.5). It can be shown that

$$i\left(ma'\gamma^0 - \frac{e}{2}\gamma^a\gamma^b F_{ab}\right) = iH^2a^2\left(\frac{m}{H}\gamma^0 + \frac{eE}{H^2}\gamma^0\gamma^3\right), \tag{3.8}$$

and

$$\left(\frac{m}{H}\gamma^0 + \frac{eE}{H^2}\gamma^0\gamma^3\right)^2 = \left(M\gamma^0 + L\gamma^0\gamma^3\right)^2 = \left(M^2 + L^2\right)\mathbf{1}, \tag{3.9}$$

hold. We have introduced two dimensionless parameters,

$$M \equiv \frac{m}{H}, \qquad L \equiv \frac{eE}{H^2}. \tag{3.10}$$

M is the mass and L is the electric field strength normalized by the Hubble parameter H.

A matrix $B \equiv r^{-1}(M\gamma^0 + L\gamma^0\gamma^3)$ with $r = \sqrt{M^2 + L^2}$ plays an important role in this problem and behaves nicely. First, $B^2 = \mathbf{1}$ as is described in (3.9). In addition, B is traceless $\mathrm{tr}B = 0$ due to trace properties of the gamma matrices, $\mathrm{tr}(\gamma^a) = 0$ and $\mathrm{tr}(\gamma^a\gamma^b) = 4\eta^{ab}$. Unitarity and hermiticity $B = B^\dagger = B^{-1}$ also follow from the Hermitian/anti-Hermitian property of the gamma matrices $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ immediately.

From these properties, we can conclude that there exist four time-independent eigenvectors w_s $(s = 1, 2, 3, 4)$ of the matrix B,

$$Bw_s = \lambda_s w_s, \tag{3.11}$$

which have eigenvalues $\lambda_s = +1$ (for $s = 1, 2$) and $\lambda_s = -1$ (for $s = 3, 4$), respectively. The normalization and the completeness conditions are respectively given by

$$w_s^\dagger w_{s'} = \delta_{ss'}, \qquad \sum_{s=1}^{4} w_s w_s^\dagger = \mathbf{1}. \tag{3.12}$$

Now we introduce charge conjugation operator $\mathcal{C}$ whose operation is defined by

$$\mathcal{C}\,{}^t\gamma^\mu\mathcal{C}^{-1} = -\gamma^\mu, \tag{3.13}$$

where ${}^t\gamma^\mu$ denotes the transpose of the gamma matrices. It is possible to choose $\mathcal{C}$ to be an operator which satisfies $\mathcal{C} = -\mathcal{C}^{-1} = \mathcal{C}^* = -\mathcal{C}^\dagger$. We then find $\mathcal{C}\,B^*\mathcal{C}^{-1} = \mathcal{C}\,{}^tB\mathcal{C}^{-1} = -B$. Thus, $\mathcal{C}w_s^*$ is an eigen vector of the matrix B corresponding to the eigenvalue $-\lambda_s$, that is,

$$B(\mathcal{C}w_s^*) = -\lambda_s(\mathcal{C}w_s^*). \tag{3.14}$$

For later convenience, we specify the relations among ω_s as follows

$$w_1 = -\mathcal{C}w_3^*,\; w_3 = \mathcal{C}w_1^*,\; w_2 = -\mathcal{C}w_4^*,\; w_4 = \mathcal{C}w_2^*. \tag{3.15}$$

With the aid of the spatial homogeneity of the system we consider, we introduce the following decomposition of the solution for the Dirac equation:

$$\zeta(\eta, \boldsymbol{x}) = \mathrm{e}^{-iHLz}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\zeta_{\boldsymbol{k},\,s}(\eta)w_s, \tag{3.16}$$

where we multiplied a gauge fixing phase factor e^{-iHLz} to simplify equations below. The Schrödinger type equation for the mode function $\zeta_{\boldsymbol{k},\,s}(\eta)$ becomes

$$\left(\partial_\eta^2 + \omega_k^2(\eta) - i\lambda_s\sigma(\eta)\right)\zeta_{\boldsymbol{k},\,s}(\eta) = 0, \tag{3.17}$$

where

$$\omega_k^2(\eta) \equiv k^2 - 2aHLk_z + a^2H^2r^2, \quad \sigma \equiv a^2H^2r, \quad k \equiv \sqrt{k_x^2 + k_y^2 + k_z^2}. \tag{3.18}$$

Two independent solutions for Eq. (3.17) are obtained in terms of the Whittaker functions $M_{\kappa,\mu^\pm}(z)$ and $W_{\kappa,\mu^\pm}(z)$. The parameters are given by

$$\kappa = -iL\frac{k_z}{k},\;\; \mu^+ = \frac{1}{2} + ir,\;\; z = -2i\frac{k}{aH}, \tag{3.19}$$

for $s = 1, 2$ and

$$\kappa = -iL\frac{k_z}{k}, \quad \mu^- = \frac{1}{2} - ir, \quad z = -2i\frac{k}{aH}, \tag{3.20}$$

for $s = 3, 4$ respectively. To determine the positive frequency mode in the in-region ($\eta \to -\infty$), we can make use of an asymptotic formula of the Whittaker function [52] $W_{\kappa,\mu^\pm}(z) \sim \mathrm{e}^{-z/2} z^\kappa$, and the positive frequency mode is given by

$$\zeta^+_{\boldsymbol{k},\, s=1,2} = \frac{\mathrm{e}^{\pi i\kappa/2}}{\sqrt{2k}}\sqrt{\frac{r}{r - i\kappa}} W_{\kappa,\mu^+}(z) \xrightarrow[z\to -i\infty]{\eta\to-\infty,\, a\to 0} \frac{1}{\sqrt{2k}}\sqrt{\frac{r}{r-i\kappa}}\mathrm{e}^{-ik\eta}(-2k\eta)^\kappa, \tag{3.21}$$

where the normalization is chosen to satisfy the canonical quantization condition as seen below. We can also find the negative frequency mode as

$$\zeta^-_{\boldsymbol{k},\, s=1,2} = \frac{\mathrm{e}^{\pi i\kappa/2}}{\sqrt{2k}}\sqrt{\frac{r}{r + i\kappa}} W_{-\kappa,\mu^+}(-z) \xrightarrow[z\to -i\infty]{\eta\to-\infty,\, a\to 0} \frac{1}{\sqrt{2k}}\sqrt{\frac{r}{r+i\kappa}}\mathrm{e}^{+ik\eta}(-2k\eta)^{-\kappa}. \tag{3.22}$$

The mode functions for $s = 3, 4$ is given by complex conjugation as follows

$$(\zeta^\pm_{\boldsymbol{k},\, s=1,2})^* = \zeta^\mp_{\boldsymbol{k},\, s=3,4}. \tag{3.23}$$

This choice of the mode functions corresponds to the so-called Bunch-Davies vacuum.

From Eqs. (3.15) and (3.23), we can obtain four independent solutions for the Dirac equation Eq. (3.4) and construct the mode expansion for the quantized Dirac field $\hat{\xi}$ as

$$\hat{\xi}(\eta, \boldsymbol{x}) = \mathrm{e}^{-iHLz}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{s=1,\,2}\left[\hat{b}_{\boldsymbol{k},\,s}u_{\boldsymbol{k},\,s}(\eta)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{d}^\dagger_{\boldsymbol{k},\,s}v_{\boldsymbol{k},\,s}(\eta)\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right], \tag{3.24}$$

where only $s = 1, 2$ components are exploited. From the first order Dirac equation (3.4), u-spinor and v-spinor satisfy the equations of motion below

$$\hat{\mathcal{D}}_- u_{\boldsymbol{k},\,s} = \left(i\partial_\eta - k_i\gamma^0\gamma^i - aHr\gamma^0 B\gamma^0\right)u_{\boldsymbol{k},\,s} = 0, \tag{3.25}$$

$$\hat{\mathcal{D}}_+ v_{\boldsymbol{k},\,s} = \left(i\partial_\eta + k_i\gamma^0\gamma^i - aHr\gamma^0 B\gamma^0\right)v_{\boldsymbol{k},\,s} = 0. \tag{3.26}$$

Importantly, v-spinor can be constructed from u-spinor. If $u_{\boldsymbol{k},\,s}$ satisfies its equation of motion, then a spinor $\mathcal{C}u^*_{-\boldsymbol{k},\,s}$ automatically satisfies the equation of motion for $v_{\boldsymbol{k},\,s}$ because of an equation $\mathcal{C}\hat{\mathcal{D}}^*_\pm\mathcal{C}^\dagger = -\hat{\mathcal{D}}_\pm$. Hence we can identify $v_{\boldsymbol{k},\,s}$ with $\mathcal{C}u^*_{-\boldsymbol{k},\,s}$.

Finally, u and v spinors are given in terms of the mode function and the eigen vector by

$$u_{\boldsymbol{k},s} = \gamma^0 \hat{\mathcal{D}} \zeta^{+}_{\boldsymbol{k},s} w_s, \quad v_{\boldsymbol{k},s} = \mathcal{C} u^{*}_{-\boldsymbol{k},s}, \quad \hat{\mathcal{D}} = \left(i\partial_\eta - k_i \gamma^0 \gamma^i + aHrB\right). \tag{3.27}$$

The anti-commutation relations

$$\{\hat{b}_{\boldsymbol{k},s},\, \hat{b}^{\dagger}_{\boldsymbol{k}',s'}\} = \{\hat{d}_{\boldsymbol{k},s},\, \hat{d}^{\dagger}_{\boldsymbol{k}',s'}\} = (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')\delta_{s,s'}, \quad \text{others} = 0, \tag{3.28}$$

are imposed as usual. The conjugate momentum of the canonical Dirac field $\hat{\xi}$ is given by $\hat{\pi}(\eta, \boldsymbol{x}) = \dfrac{\delta S}{\delta \hat{\xi}'} = i\hat{\xi}^{\dagger}$. Therefore we obtain the conventional canonical quantization condition

$$\{\hat{\xi}(\eta,\, \boldsymbol{x}),\, \hat{\pi}(\eta,\, \boldsymbol{y})\} = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}). \tag{3.29}$$

For the detailed explanation on the consistency of the normalization conditions, namely, for the mode functions (3.21) and (3.22), for the creation-annihilation operators (3.28), and for the canonical anti-commutation relation (3.29), see Appendix B.

3.3 Evaluation of induced current

3.3.1 Vacuum expectation value of spinor current

The in-vacuum state $|0\rangle$ is defined as a state that satisfies the condition $b_{\boldsymbol{k},s}|0\rangle = d_{\boldsymbol{k},s}|0\rangle = 0$ for all $\boldsymbol{k}$ and $s = 1, 2$. Then, using the mode decomposition (3.24) and the anti-commutation relation (3.28), the expectation value of the spinor current operator $\hat{J}^3$ (along z-axis) is expressed as [1]

$$\langle \hat{J}^3 \rangle = -e \left\langle 0 \left| \hat{\bar{\xi}} \gamma^3 \hat{\xi} \right| 0 \right\rangle = -e \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{s=1,2} v^{\dagger}_{\boldsymbol{k},s} \gamma^0 \gamma^3 v_{\boldsymbol{k},s}. \tag{3.30}$$

[1] Other components of the vacuum expectation value of the current operator $\hat{J}^a = -e\hat{\bar{\xi}}\gamma^a\hat{\xi}$ vanish dew to the cylindrical symmetry of the spacetime and electric field configuration.

The spinors $v_{\boldsymbol{k},s}$ are defined in Eq. (3.27). Since the matrix B can be regarded as $\mathbf{1}$ on w_s or $-\mathbf{1}$ on $\mathcal{C}w_s^*$ for $s = 1,\, 2$, the vacuum expectation value of the induced current can be computed as

$$
\begin{aligned}
&-e\left\langle 0\left|\hat{\bar{\xi}}\gamma^3\hat{\xi}\right|0\right\rangle \\
&=\frac{-2eL}{r}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left\{\zeta^{+\prime}\zeta^{+*\prime}+i\left(\gamma k_z-F_k\right)\left(\zeta^{+}\zeta^{+*\prime}-\zeta^{+\prime}\zeta^{+*}\right)+\left(2F_k^2-\omega_k^2-2\gamma F_k k_z\right)|\zeta^+|^2\right\}\\
&=\frac{-2eL}{r}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left\{1+i\gamma k_z(\zeta^{+}\zeta^{+*\prime}-\zeta^{+\prime}\zeta^{+*})+2\left(F_k^2-\omega_k^2-\gamma F_k k_z\right)|\zeta^+|^2\right\},
\end{aligned}
\tag{3.31}
$$

where the parameters are given by

$$
\gamma\equiv\frac{r}{L}-\frac{L}{r},\qquad F_k\equiv\frac{\omega_k\omega_k'}{\sigma}=aHr-\frac{Lk_z}{r}, \tag{3.32}
$$

and we have used the normalization condition (B.6) (see Appendix B) in the last line. This integral diverges in the ultraviolet (UV) region ($k\to\infty$), so some renormalization procedure is required. Here, we will apply the adiabatic subtraction method.

3.3.2 Adiabatic subtraction

The adiabatic subtraction is a renormalization scheme with which one subtracts the lower-order parts in the adiabatic (WKB) expansion of a quantity from its unrenormalized calculation result. The leading term in the adiabatic expansion of the expectation value of the current $-e\,\langle 0|\hat{\bar{\xi}}\gamma^3\hat{\xi}|0\rangle\,\big|^{(A)}$ imitates the divergence(s) in the UV (large k) region in the momentum space. Here, the adiabatic part $-e\,\langle 0|\hat{\bar{\xi}}\gamma^3\hat{\xi}|0\rangle\,\big|^{(A)}$ is obtained by replacing the mode function ζ^+ by the adiabatically (WKB) expanded counterpart $\zeta^+\big|^{(A)}$. Subtracting this quantity from the formally divergent expectation value $-e\,\langle 0|\hat{\bar{\xi}}\gamma^3\hat{\xi}|0\rangle$, one obtains the renormalized expectation value of the current operator. We perform these calculations in this subsection.

In this subsection, Planck constant $\hbar$ is temporally visualized to clearly count the adiabatic order. The equation of motion for $\zeta=\zeta^+_{\boldsymbol{k},\,s=1,2}$ is, again with $\hbar$, given by

$$
\left(\hbar^2\partial_\eta^2+\omega_k^2(\eta)-i\hbar\sigma(\eta)\,\right)\zeta(\eta)=0. \tag{3.33}
$$

Because the $-i\sigma$ term in the equation above comes from a first-order derivative term, we have to assign $\hbar^1$ in front of it. This odd order term does not appear in the scalar case. Thus, the usual

WKB ansatz, which is valid for the scalar mode function,

$$\zeta \overset{!}{=} \frac{1}{\sqrt{\Omega_k(\eta)}} \mathrm{e}^{-\frac{i}{\hbar}\int \mathrm{d}\eta' \Omega_k(\eta')}, \tag{3.34}$$

is inappropriate (Ω_k is a function supposed to be determined as a power series of $\hbar$). Instead, the WKB ansatz for the spinor positive mode function should take the following form [49, 50, 51, 53] (see Appendix C for the derivation),

$$\zeta = \sqrt{\frac{\sigma}{2\omega^2(\sigma+\omega')}} (1 + \hbar F^{(1)} + \hbar^2 F^{(2)} + \cdots)\, \mathrm{e}^{-i/\hbar \int \mathrm{d}\eta' (\omega + \hbar\omega^{(1)} + \hbar^2 \omega^{(2)} + \cdots)}, \tag{3.35}$$

where $F^{(i)}$s and $\omega^{(i)}$s are (real) unknown functions to be determined by the equation of motion (3.33) and the normalization condition [2]

$$\hbar^2 \zeta'(\zeta^*)' - i\hbar F_k(\zeta(\zeta^*)' - \zeta'\zeta^*) + \omega_k^2 |\zeta|^2 = 1, \tag{3.36}$$

where we have recovered $\hbar$.

Note that this normalization condition can be satisfied only perturbatively (in an order-by-order manner). More detailed explanation on this ansatz is shown in Appendix C. It is easy to find that all the odd order terms vanish, that is, $F^{(1)} = F^{(3)} = \cdots = 0$ and $\omega^{(1)} = \omega^{(3)} = \cdots = 0$. We can express $F^{(i)}$ and $\omega^{(i)}$ in terms of ω and σ. For example, at the second order they read

$$\omega^{(2)} = -\frac{\sigma+\omega'}{8\sigma} \frac{\sigma^2 + 2\omega\sigma' - 5\omega'\sigma}{\omega^3}, \qquad F^{(2)} = -\frac{\sigma+\omega'}{16\sigma} \frac{5\omega'\sigma - 2\omega\sigma'}{\omega^4}. \tag{3.37}$$

With the adiabatic subtraction method, the renormalized current is given by

$$\left\langle 0 \middle| \hat{J}^3 \middle| 0 \right\rangle_{\mathrm{ren}} = \left\langle 0 \middle| \hat{J}^3 \middle| 0 \right\rangle - \left\langle 0 \middle| \hat{J}^3 \middle| 0 \right\rangle \Big|^{(2)}, \tag{3.38}$$

where $|^{(2)}$ means that the second term in the right hand side includes the contribution up to adiabatic order two, i.e order of $\mathcal{O}(\hbar^2)$.

The calculation of the momentum integral takes the following procedure. First, we introduce a momentum cutoff Λ to control the divergences. Second, the momentum integrals of the exact

[2] This condition is valid only for $s = 1, 2$ mode functions. For $s = 3, 4$ case, the normalization condition becomes $\hbar^2 \zeta'(\zeta^*)' + i\hbar F_k(\zeta(\zeta^*)' - \zeta'\zeta^*) + \omega_k^2 |\zeta|^2 = 1$. Of course, this is obtained by a replacement $\zeta \leftrightarrow \zeta^*$ in (3.36).

part (the first term in (3.38)) and the adiabatic part (the second term) are computed separately. Third, the subtraction is implemented while the momentum cutoff Λ is kept finite. Finally, the limit $\Lambda \to \infty$ is taken, and we obtain a finite result.

The detailed calculation of the integrals can be found in Appendix D, and here we show the final results. The first term of (3.38) is given by

$$\begin{aligned}\langle \hat{J}^3 \rangle = &-2eL(aH)^3 \lim_{\Lambda\to\infty}\Bigg[\frac{1}{6\pi^2}\left(\frac{\Lambda}{aH}\right)^2 - \frac{1}{6\pi^2}\ln\left(\frac{2\Lambda}{aH}\right)\\
&+\frac{7}{72\pi^2} - \frac{L^2}{15\pi^2} - \frac{M^2}{12\pi^2} - \frac{3rM^2}{8\pi^2L^2} - \frac{3M^2r}{16\pi^2L^3}\log\left(\frac{r-L}{r+L}\right)\\
&-\frac{r\mathrm{csch}(2\pi r)}{48\pi^5L^2}\left\{(45-\pi^2(11-12L^2+8r^2))\cosh(2\pi L) - (45-\pi^2(11-72L^2+8r^2))\frac{\sinh(2\pi L)}{2\pi L}\right\}\\
&+\frac{3rM^2\mathrm{csch}(2\pi r)}{32\pi^2L^3}\sum_{s=\pm} s\mathrm{e}^{2\pi rs}(\mathrm{Ei}(2\pi s(r+L)) - \mathrm{Ei}(2\pi s(r-L)))\\
&+\frac{\mathrm{csch}(2\pi r)}{16\pi^2}\Re[\int_{-1}^{1}\mathrm{d}x(1+r^2-(1+3L^2+3r^2)x^2+5L^2x^4)\sum_{s=\pm}s(\mathrm{e}^{2\pi Lx}-\mathrm{e}^{-2\pi rs})\psi(i(Lx+rs))]\Bigg],\end{aligned} \tag{3.39}$$

where $\mathrm{Ei}(z)$ is the exponential integral function defined as $\mathrm{Ei}(z) = -\mathcal{P}\int_{-z}^{\infty}\mathrm{d}t\mathrm{e}^{-t}/t$ ($\mathcal{P}$ denotes Cauchy's principal value) and $\psi(z) = (\ln\Gamma(z))'$ is the digamma function. We can see there are the quadratic and the logarithmic divergences in momentum cutoff Λ.

The second term of (3.38) (the subtraction term) is directly calculated and the result is given by

$$\left\langle 0 \,\middle|\, \hat{J}^3 \,\middle|\, 0 \right\rangle\Bigg|^{(2)} = -\lim_{\Lambda\to\infty} 2eL\left\{\left(\frac{aH\Lambda^2}{6\pi^2} - \frac{(aH)^3(2L^2+5M^2)}{60\pi^2}\right)\hbar^0 + \frac{(aH)^3}{18\pi^2}\left(4-3\ln\left(\frac{2\Lambda}{aHM}\right)\right)\hbar^2\right\}, \tag{3.40}$$

where $\hbar$ is a constant which is taken to be a small expansion parameter in the adiabatic expansion and set to be unity after the truncation. It should be noted that the divergent parts of (3.39) and (3.40) are the same. Therefore, after the subtraction, we obtain the renormalized expectation value of the induced current as follows

$$
\begin{aligned}
\left\langle \hat{J}^3 \right\rangle_{\text{ren}} = \frac{eL(aH)^3}{4\pi^2}\Bigg[&1 + \frac{4L^2}{15} + \frac{4}{3}\log M + \frac{3M^2}{L^2}\left(1 + \frac{r}{2L}\log\left(\frac{r-L}{r+L}\right)\right) \\
&+ \frac{r\text{csch}(2\pi r)}{6\pi^3 L^2}\left\{(45 - \pi^2(11 - 12L^2 + 8r^2))\cosh(2\pi L) - (45 - \pi^2(11 - 72L^2 + 8r^2))\frac{\sinh(2\pi L)}{2\pi L}\right\} \\
&- \frac{3rM^2\text{csch}(2\pi r)}{4L^3}\sum_{s=\pm} s\mathrm{e}^{2\pi rs}(\text{Ei}(2\pi s(r+L)) - \text{Ei}(2\pi s(r-L))) \\
&- \frac{\text{csch}(2\pi r)}{2}\int_{-1}^{1}\mathrm{d}x(1 + r^2 - (1 + 3L^2 + 3r^2)x^2 + 5L^2x^4)\sum_{s=\pm} s(\mathrm{e}^{2\pi Lx} - \mathrm{e}^{-2\pi rs})\Re\psi(i(Lx + rs))\Bigg].
\end{aligned}
\tag{3.41}
$$

This expression is a bit lengthy and it requires farther effort to extract relevant physical implications from this result. We will examine it in detail in the next section.

3.4 Result

3.4.1 General properties

We will investigate the renormalized current (3.41) in this section. To this end, we introduce a dimensionless quantity $\mathcal{J}$ which is a function of the two dimensionless parameters $L = eE/H^2$ (electric field strength) and $M = m/H$ (spinor mass),

$$
\mathcal{J}(L, M) \equiv \frac{|\langle \hat{J}^3 \rangle_{\text{ren}}|}{ea^3H^3}. \tag{3.42}
$$

Behavior of the spinor current $\mathcal{J}(L, M)$ is shown in Fig. 3.1.

The renormalized current $\mathcal{J}(L, M)$ has some remarkable properties. The most intelligible one is an antisymmetry $\mathcal{J}(L, M) = -\mathcal{J}(-L, M)$ and its consequence $\mathcal{J}(0, M) = 0$. This means that the renormalized current always vanishes at $L = 0$ as expected. Besides, $\mathcal{J}(L, M)$ also becomes zero at a certain positive L depending on M, namely $L_*(M) > 0$, for any value of the mass parameter M. The spinor renormalized current is positive for $L > L_*$ (represented by solid lines in Fig. 3.1), and is negative for $0 < L < L_*$ (by dashed lines). The negative spinor current is always (for any value of M) observed in the weak electric field regime unlike the bosonic case [39] which shows the negative current only in the small mass regime $M_{\text{scalar}} \lesssim 0.003$.

Another striking difference between the bosonic and the fermionic current is the absence of the IR hyperconductivity which is the rapid growth of $|\mathcal{J}|$ for smaller L. The hyperconductivity

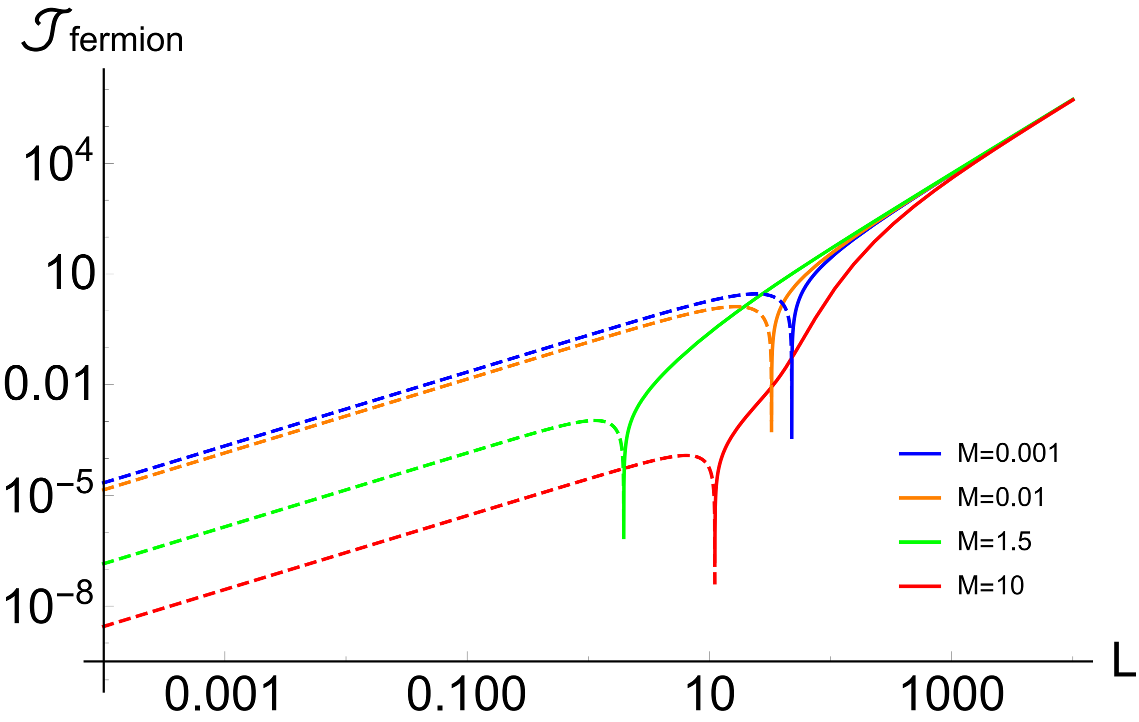

Figure 3.1: The renormalized spinor current (3.41).
The current induced by the homogeneous, constant energy density electric field in $1+3$ de Sitter spacetime given by (3.6) is plotted. The horizontal axis denotes the strength of the electric field $L \equiv eE/H^2$. The solid (dashed) line indicates that the sign of the renormalized current is positive (negative). The mass parameter of the charged particles is $M \equiv m/H = 10^{-3}$ (blue), 10^{-2} (orange), 1.5 (green) and 10 (red). The sign flips around $L \sim 10$ in the spinor case with any mass.

happens only in IR regime ($L < 1$ and $M < 1$) of the bosonic current. It was first found and discussed for the bosonic current in the $1+1$ dimensional de Sitter spacetime [37] and that in the $1+3$ dimensional de Sitter spacetime [39]. The absence of the hyperconductivity of the fermionic current in the $1+1$ dimensional de Sitter spacetime is also reported in [41]. We will come back to this point again in the next chapter.

3.4.2 Strong and weak field limits

In the strong electric field limit, $L \gg 1,\ M$, the second line of (3.41) dominates the expectation value and we obtain (for $L > 0$)

$$\mathcal{J} \simeq \frac{L^2}{6\pi^3} \mathrm{e}^{-\frac{\pi M^2}{L}} = H^{-4} \frac{(eE)^2}{6\pi^3} \mathrm{e}^{-\frac{\pi m^2}{eE}}, \tag{3.43}$$

where the famous suppression factor of Schwinger effect in Minkowski spacetime $\exp(-\pi m^2/eE)$ is reproduced. This factor comes from terms like

$$\lim_{L\to\infty} 2(\sqrt{L^2+M^2}-L) = \frac{M^2}{L}. \tag{3.44}$$

We see the quadratic behavior ($\mathcal{J} \sim L^2$) of the renormalized current. This strong electric field limit corresponds to the Minkowski (weak curvature) limit $H \to 0$. Thus, $\langle \hat{J} \rangle_{\rm ren}$ has H^{-1} divergence in this limit. This can be regarded as a consequence of the lack of cosmic dilution in the flat spacetime limit. The particles produced at the initial time $t = -\infty$ can contribute to the electric current forever, which results in the diverging result. If we regulate the H^{-1} divergence by the cosmic time interval $(t - t_0)$ with t_0 being the turn-on time of the electric field, we obtain $\langle \hat{J} \rangle_{\rm ren} \sim e^3 E^2 (t-t_0) \exp(-\pi m^2/eE)$. This replacement makes sense for two reasons. First, this linear growth in time is consistent with the study on Schwinger effect in Minkowski spacetime shown in a previous work [54]. Second, it is natural to recover the flat spacetime physics in the strong field regime because microscale effects dominate global effects and the spacetime curvature becomes negligible.

In contrast to the intuitive behavior in the strong field limit, the strange negativity of the renormalized current appears in the weak electric field regime $L \ll 1$. In this limit, (3.41) becomes

$$\mathcal{J} \simeq \frac{L}{3\pi^2}\left[\log M - \Re\psi(iM) - \frac{\pi M(4M^2+1)}{3\sinh(2\pi M)}\right]. \tag{3.45}$$

We can define a dimensionless conductivity in the weak electric field limit as

$$\sigma(M) \equiv \left.\frac{\mathcal{J}(L,M)}{L}\right|_{L\to 0} = \frac{1}{3\pi^2}\left[\log M - \Re\psi(iM) - \frac{\pi M(4M^2+1)}{3\sinh(2\pi M)}\right], \tag{3.46}$$

which is negative for all M. The massless limit of the conductivity is given by

$$\sigma(M) \xrightarrow{M\to 0} \frac{1}{3\pi^2}\log M + \frac{6\gamma_E - 1}{18\pi^2} + \mathcal{O}(M^2), \tag{3.47}$$

where γ_E is Euler constant. There is no power-law IR enhancement but the logarithmic divergence in the spinor conductivity. Meanwhile, the massive limit is given by

$$\sigma(M) \xrightarrow{M\to\infty} \left(-\frac{1}{36\pi^2 M^2} + \mathcal{O}(M^{-4})\right) - \frac{2}{9\pi}\mathrm{e}^{-2\pi M} M(4M^2+1). \tag{3.48}$$

As naively expected, $\sigma(M)$ is suppressed in the massive limit. Schwinger mechanism cannot produce massive fermions effectively due to the suppression factor $\exp(-\pi m^2/eE)$. Also the gravitational particle production is suppressed by the factor $(\exp(2\pi M)+1)^{-1} \sim \exp(-2\pi M)$. Thus we might be able to identify the latter term in (3.48) as the effect of the gravitational particle production. Moreover, the terms in the parenthesis in (3.48) do not have exponential suppression factors and cannot merely be attributed to either of Schwinger or the gravitational particle production. Their origins are still unidentified. Here, we note that the higher order adiabatic subtraction can remove some of these terms with the power-law dependence on M. For instance, the first term with M^{-2} in (3.48) can be removed by the adiabatic subtraction of order $\hbar^4$, while it adds a new $\mathcal{O}(L^3/M^2)$ term to the induced current and changes the IR behavior of the current. Nevertheless, the higher order ($\mathcal{O}(M^{-4})$) terms which are not suppressed by the exponential factor in (3.48) still remain even in this case.

3.4.3 Negativity of the induced current

One may feel it questionable whether we should take the strange negativity of the current seriously. The range of wavelength which is short enough to verify the adiabatic subtraction depends on the particle mass, and only the modes with $(k/aH)^2 + M^2 \gg 1$ can imitate the correct behavior of the exact mode function. Thus, the adiabatic approximation is not necessarily correct for the long wavelength modes when $m \ll H$. A possible criticism is that the adiabatic expansion is inappropriate for fields with tiny masses and the adiabatic subtraction scheme becomes invalid for the modes with $m/H \ll k/(aH) \ll 1$ though they are in the UV regime. However, it has been confirmed that the point-splitting renormalization scheme is in perfect accord with the adiabatic subtraction for the scalar current. This will be the subject of the next chapter. This implies that the strange behaviors we have found in the previous section have nothing to do with the accuracy of the WKB expansion in infrared regime. Therefore, it is worthwhile to investigate physical consequences of the result (3.41) in this section.

The semiclassical equation of motion for the gauge field (Maxwell equation) is given by $F^{\mu\nu}{}_{,\nu} = \langle \hat{J}^\mu \rangle_{\rm ren}$ in our convention. For the electric background field $E_z = -A_z'$, the equation of motion reduces to

$$E_z' = -\langle \hat{J}^3 \rangle_{\rm ren}\,, \tag{3.49}$$

which can be regarded as an equation for a feedback system. It is easy to figure out the stability of the electric field-current system by looking at the signature of the renormalized current. The positive current reduces the background electric field while the negative current enhances it. The zeros of the current correspond to either a stable point or a saddle (unstable) point.

Surprisingly enough, the trivial zero $L = 0$ (i.e., zero electric fields) is not a stable point but a saddle point [3]. Another zero $L = L_* > 0$ is always a stable point. We plot L_* as a function of spinor mass $M = m/H$ in Fig. 3.2. We have numerically confirmed that $L_*(M) \sim M$ for the massive particles, $M \gg 1$, and $L_*(M) \sim -\log M$ for $M \ll 1$. This figure can be seen as a phase diagram of the system. The negative current occurs in the shaded region below the red line. A similar diagram for the scalar current is discussed also in the next chapter. Note that the negativity of the induced current is not past redemption even though it indicates the instability of the system. This is not a bottomless instability since the current becomes positive for a sufficiently strong electric field due to the high Schwinger particle production rate.

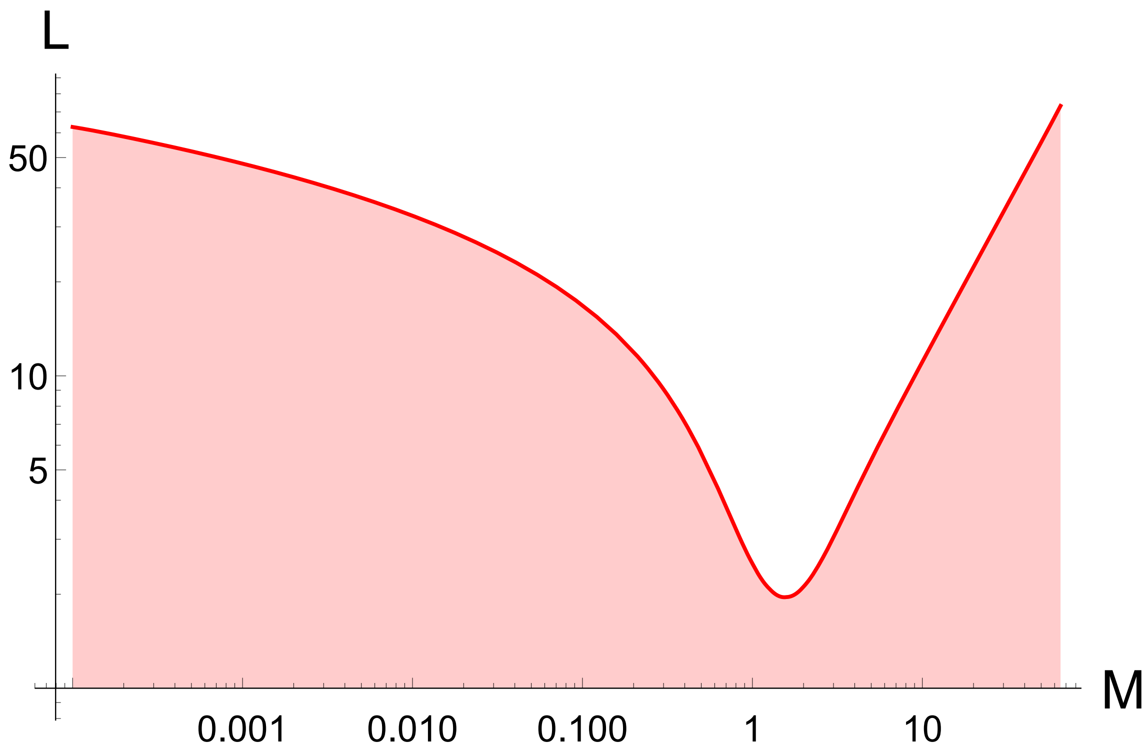

Figure 3.2: Zeros of the renormalized current $\mathcal{J}(L, M)$ in L-M plain. The white (unshaded) region corresponds to the positive current $\mathcal{J} > 0$ (negative feedback) and the shaded region corresponds to the negative current $\mathcal{J} < 0$ (positive feedback). The line shows positions of the stable points $L = L_*(M)$ of the electric field-current system. The trivial zero $L = 0$ is not shown as this is a logarithmic plot. The values of L_* are numerically evaluated.

[3] This situation is opposite of the case of the scalar current.

The remaining problem – what is the physical mechanism of the negative current? – will be addressed in the next section.

3.5 Maxwell's demon lives on a horizon

3.5.1 Mystery of the negative current

As we have discussed in the previous section, de Sitter QED can cause electrical instability. Put another way, the anti-screening effect occurs in QED theory, which is well known as a screening theory in flat space. It is just enough to consider a capacitor exploiting high school-level physics in order to demonstrate the screening in electromagnetism. When a voltage (or equivalently, an electric field) is applied, a simple capacitor consists of a dielectric sandwiched by two metallic plates can accumulate charges in it by producing an amount of polarization inside the dielectric. Usually, the polarization is in the same direction as the electric field and produces an oppositely directed electric field. The strength of the applied electric field between the plates is consequently screened. Even if the electric field is strong enough to cause electrical breakdown, electric current will be generated and screens the electric field. In any event, there seems no room for the anti-screening effect, which necessitate us to seek for a physical interpretation of this effect.

Fortunately, we have very recently reached an understanding that the anti-screening effect can be regarded as a consequence of a counter-intuitive thermodynamical phenomenon: Maxwell's demon. There are a couple of circumstantial evidence that he is responsible for it. The fact that the quantity $\boldsymbol{j} \cdot \boldsymbol{E}$ is the Joule heating and is actually negative indicates the current should convey thermal heat to the horizon of de Sitter spacetime in the first place. Second, it is well known that spacetime horizons can have entropy S_{horizon} identical to its surface area A_{horizon}. Third, the weak limit result (3.45) can be rewritten as

$$\left\langle \hat{j}^3 \right\rangle \sim -ea^3 \left| \sigma \left(\frac{m}{H} \right) \right| \times \left(\frac{eE}{H^2} \right) \times H^3 \propto -A_{\text{horizon}} \times V_{\text{horizon}}^{-1}, \tag{3.50}$$

where $V_{\text{horizon}} \propto H^{-3}$ is the 3-volume of de Sitter horizon patch and should naturally appear in expressions for the electric current *density* $\langle \hat{j}^\mu \rangle$. Then, the total electric current is proportional to the surface area of the horizon A_{horizon} [4]. This suggests that the electromotive force $\mathcal{E}$ is in

[4] If Compton wavelength of the Dirac field m^{-1} is properly scaled with the horizon radius H^{-1}.

connection with other thermal quantities such as entropy. The electromotive force $\mathcal{E}$ which is a source of kinetic motion of charged particles generates the electric current in the following manner

$$\boldsymbol{j} = \sigma_c \boldsymbol{E} - \nabla \mathcal{E}, \tag{3.51}$$

where σ_c denotes the local conductivity. The first term $\sigma_c \boldsymbol{E}$ is the usual linear response to the electric field $\boldsymbol{E}$ while the second term describes effects from other types of electromotive force. As long as we respect the second law of thermodynamics, the local conductivity becomes positive, $\sigma_c > 0$, and semiclassically proportional to the local charge number density.

Let us neglect the local conductivity σ_c keeping in mind that we are considering the weak electric field regime where the particle production is insignificant. Now we have

$$\boldsymbol{j} = -\nabla \mathcal{E} \propto -A_{\mathrm{horizon}} V_{\mathrm{horizon}}^{-1} \propto -H. \tag{3.52}$$

We also introduce the central concept of thermodynamics, namely, temperature T. It has been commoditized to exploit de Sitter temperature $T = H/2\pi$ which is an analog to the Hawking temperature in Schwarzschild spacetime [5] [55, 56, 57]. Therefore the electromotive force phenomenologically has a natural relation with the spacetime temperature, $-\nabla \mathcal{E} \propto -T$. Since the system we consider is homogeneous, the relation

$$\mathcal{E} \propto zT, \tag{3.53}$$

follows from our assumption that the electromotive force $\mathcal{E}$ must vanish when the minkowski limit $T = H/2\pi \to 0$ is taken.

For an observer who stays in de Sitter spacetime, the negative current necessarily creates charge distribution on the horizon. This is because, when seen by an observer, the homogeneous current together with the horizon has a cross-section and has nonzero divergence $\nabla \cdot \boldsymbol{j} \neq 0$ at any point on the horizon. From the local current conservation $\nabla_\mu j^\mu = 0$, the observer must also see charge distribution associated with the nonzero divergence of the current. The surface charge density Σ on the horizon must be negative for $z > 0$ and positive for $z < 0$. This distribution is compatible with the sign of the electromotive force $\mathcal{E}$ we have derived above. Because, in our

[5] In the case of de Sitter spacetime, the temperature T is a global constant as the spacetime is homogeneous. The application of the thermodynamical concepts, however, should always be local in space and also in time.

convention, positively charged particles move in a direction in which $\mathcal{E}$ decreases, and vice versa for negative charges.

The first law of (electro-)thermodynamics including the electromotive force is given in terms of Helmholtz free energy by

$$\mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V + \mathcal{E}\mathrm{d}e. \tag{3.54}$$

Then Maxwell's relation automatically holds true,

$$\left(\frac{\partial \mathcal{E}}{\partial T}\right)_e = -\left(\frac{\partial S}{\partial e}\right)_T. \tag{3.55}$$

Combining this with (3.53) and the argument about the sign of the charge distribution Σ on the horizon, we can conclude that the negative current is a normal entropy-increasing process despite the fact that it generates the ununiform charge distribution from an observer's perspective. For example, at a point on the horizon with $z > 0$, the electromotive force $\mathcal{E}$ is positive and the current generates the negative charge $\Delta e < 0$ in a short time interval. Then the entropy change during the interval become positive, $\Delta S > 0$. This is also true in the region with $z < 0$. As the uniform distribution maximizes the informatical entropy, processes such as the negative current which augments the distributional inhomogeneity are prohibited unless the system throws away more entropy than it loses. Otherwise, such a process would violate the second law of thermodynamics. Of course, in a gravitational system, horizon serves as an entropy reservoir.

What must be remembered, information and entropy are not observer-independent concepts. More precisely, it makes a difference whether a particular physical degree of freedom is accessible to an observer or not. If some degrees of freedom are inaccessible, an observer should count the information about those degrees of freedom as thermodynamically coarse-grained quantities. The quantity in problem (3.31) is calculated based on the assumption that we have perfect access to all the region of the spacetime determined by (3.1), which permits the integration from $k = 0$ to $k = \infty$. However, this is in contradiction to the perspective of a non-accelerated observer who stays in a rest frame. The observer sees the horizon at a distance of H^{-1}, and thus, in principle, has no access to the information about the region outside the horizon.

Here, a couple of questions are in order: How can we reconcile these two perspectives? What is the logical link between the omniscient divine view and the half-blinded human view? To answer the questions, we need to look further into the physical mechanisms of this phenomena and to unveil the role of the horizon as Maxwell's demon.

3.5.2 Demoniac horizon capability

The previous subsection dictates the macroscopic aspect of the anomalous transportation of the charges in de Sitter spacetime. One may wonder what the microphysical mechanism which enables the horizon to winnow appropriate particles from the vacuum fluctuation so as to cause anti-screening is. This subsection will describe the microphysical process corresponds to the macroscopic description we have seen in the previous section.

Let us consider the quantum vacuum fluctuations in the vicinity of the horizon after the model of Hawking [58]. By analogy with the famous explanation for the Hawking radiation around a black hole, we can understand how the horizon can distinguish the particles. Fundamentals of quantum field theory tell us that microscopic, virtual creation-annihilation processes of particles and antiparticles continuously occur in the quantum vacuum. These processes can be interrupted by the horizon because in the event that one of a particle pair drops to the horizon, there will no longer be any chance for the other particle to find the partner again to annihilate. Therefore the particle production from the horizon is caused as a consequence of the missing counterpart of the virtual pair particles.

The exposition of the gravitational particle production and the Hawking radiation above has been widely accepted. As one could see, it alone is not sufficient to elucidate the unexplained mechanism of the electric current generation. With the aid of the background electric field which supplies the system with asymmetry, however, one can reach a satisfactory understanding. The electric fields slightly affect the polarization of the virtual particle pairs. In the region with $z > 0$, positive charges are more likely to exit from the horizon as their motion towards the horizon is enhanced by the electric field. As a result, an observer sees that the negative charge production exceeds the positive charge. Similarly, the observer sees the positive charge distribution on the horizon with $z < 0$. In total, an amount of electric polarization production is observed and simultaneously the negative electric current flows. This is consistent with the result (3.41) and the consideration in the previous subsection. The situation is schematically summarized in Fig. 3.3.

We can also put the same situation in a different way. The observer gains the information of the ununiform charge distribution on the horizon in exchange for the information of the spacetime region outside the horizon. This argument makes sense at least qualitatively. It might be

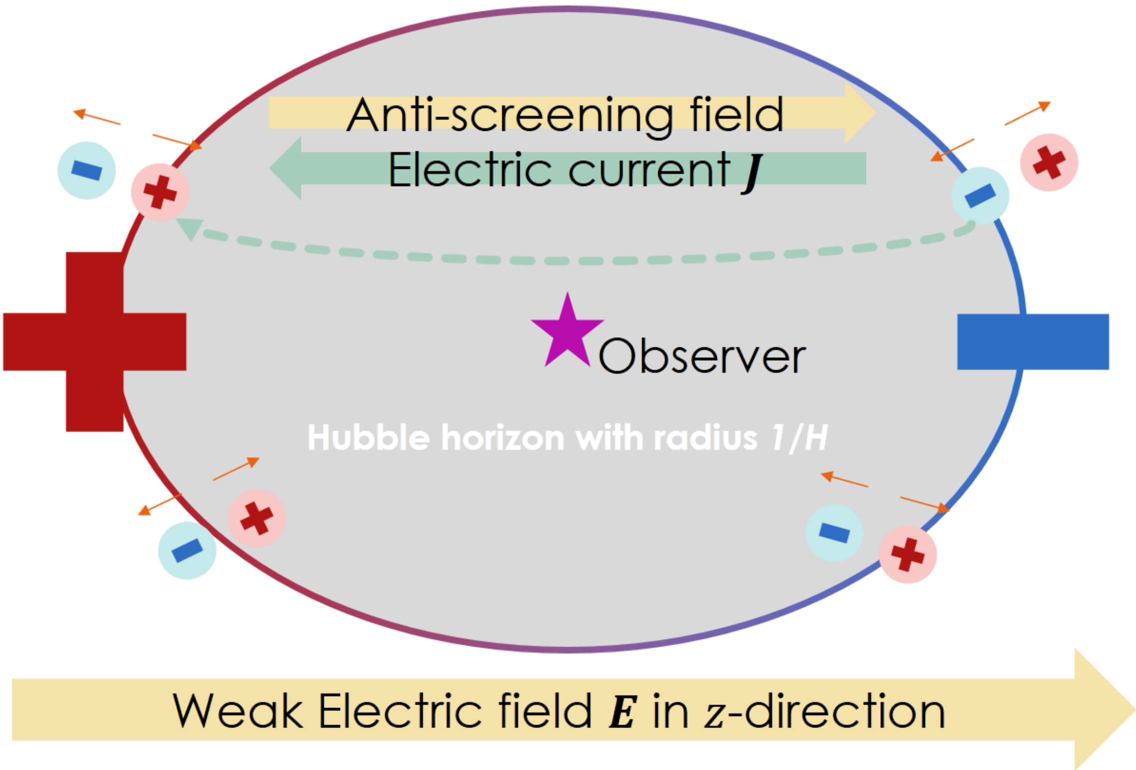

Figure 3.3: Schematic description of the anti-screening in de Sitter spacetime. An observer sees the de Sitter horizon with radius H^{-1} and the gravitational particle production caused by it. Due to the weak large-scale electric field, more positive (negative) charges are escaping from the horizon in the region $z > 0$ ($z < 0$), respectively. Consequently, the observer sees the electric polarization in the counter-intuitive direction. The generation of the charges and polarization require the flow of the charges, the electric current shown in green dashed line whose endpoints are on the horizon. It amounts to the homogeneous negative electric current inside the horizon (green arrow) and causes the anti-screening effect.

possible to conjecture the perfect equivalence between the two pictures, one based on the knowledge of all the spacetime region and another based on the effect of the horizon to an observer. If all the exterior information is encoded on its boundary, i.e., the horizon, the system we consider is a concrete example of the realization of the Holographic principle [59, 60, 61] originally proposed by Gerard 't Hooft. It seems an exciting topic to pursue the relation of this kind of problem to one of the most challenging hypotheses in the modern theoretical physics.

To summarize, we have to consider Maxwell's demon who can deprive charged particles of entropy and transfer it to the horizon. In an informatical sense, he accesses information to generate the directed motion of charges, namely, the electric current. If he can distinguish the charge and momentum of a coming particle, he can also harness that information to make a directed motion of charged particles by controlling an ideal shutter without any consumption of energy. [6] In our set up, the Hubble horizon plays this role in cooperation with the background electric field.

Note again that the anti-screening effect and enhancement of the electric field will come to a halt when the electric field strength increases. In this case, particle production caused by the electric field is no more negligible compared to the gravitational particle production. Thus the local conductivity (the first term of (3.51)) becomes larger than the second term.

Finally, we should mention that all the argument above is valid with the reservation that the particle picture is correct. The condition $M = mH^{-1} \gg 1$ ensures this point. When $M < 1$, the Compton wavelength of the Dirac field m^{-1} is larger than the horizon radius H^{-1}. This means there exist over-horizon scale correlations of the Dirac field, so the particle picture cannot be used. For the cases where the long-range correlation is significant, the description in this section is not applicable and we need to take over-horizon quantum correlations into account. This may also be an exciting challenge.

[6] A microscale device which realizes Maxwell's demon who extracts the electric energy from the thermal motion of particles has very recently been developed [62].

One of the advantages of being disorganized is that one is always having surprising discoveries.

A.A. Milne

4

Gauge Invariance and Renormalization of Scalar QED

HOW CAN WE JUSTIFY THE NON-GAUGE-INVARIANT RENORMALIZATION SCHEME APPLIED TO A GAUGE THEORY? As the explicit violation of the gauge invariance occurs while employing the momentum UV cutoff, our treatment in the previous section is not gauge invariant. Even though this does not always indicate that the result is also not gauge invariant, there is a theoretical motivation to validate the calculation especially if we lack the experimental/observational counterpart for comparison. The simplest way to check the gauge invariance of the result is to compare it with another result obtained by a gauge-invariant renormalization scheme.

In this chapter, we explicitly show the gauge invariant point-splitting renormalization procedure and concordance between the adiabatic regularization and the point-splitting renormalization in the case of the scalar QED for simplicity. After that, we also make a comparison between the fermionic and bosonic induced current.

4.1 Scalar QED

The action for the scalar QED is already given in (2.39). Similar to the fermionic case, we treat the gauge field A_μ as a background field giving rise to a constant electric field. That is, we adopt the gauge field configuration (3.6) again. This system also becomes solvable like the fermionic case.

The mode decomposition of the quantized scalar field is given by

$$\hat{\phi}(x) = \frac{1}{a}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left(\chi_{\boldsymbol{k}}(\eta)\hat{a}_{\boldsymbol{k}} + \chi^*_{\boldsymbol{k}}(\eta)\hat{b}^\dagger_{-\boldsymbol{k}}\right), \tag{4.1}$$

where $\chi_{\boldsymbol{k}}(\eta)$ is the canonical mode function.

From the Klein-Gordon equation (2.45), the mode function $\chi_{\boldsymbol{k}}(\eta)$ satisfies the following mode equation

$$\left[\frac{\partial^2}{\partial\eta^2} + \left(M^2 + L^2 - 2\right)a^2H^2 - 2aHL\left(k_z + HL\right) + k^2 + 2HLk_z + H^2L^2\right]\chi_{\boldsymbol{k}}(\eta) = 0, \tag{4.2}$$

which has the following solution in terms of the Whittaker function $W_{\kappa,\mu}(z)$ [52] as

$$\chi_{\boldsymbol{k}}(\eta) = \frac{\mathrm{e}^{i\pi\kappa/2}}{\sqrt{2p}}W_{\kappa,\mu}(z), \tag{4.3}$$

with the parameters defined as

$$z \equiv -2i\frac{p}{aH},\ \kappa \equiv -iL\frac{p_z}{p},\ \mu \equiv \sqrt{\frac{9}{4} - L^2 - M^2},\ L \equiv \frac{eE}{H^2},\ M \equiv \frac{m}{H}. \tag{4.4}$$

Here we have introduced shifted momentum $\boldsymbol{p} = (k_z, k_y, k_z + HL)$. The choice of the positive frequency mode function (4.3) is corresponding to the Bunch-Davies vacuum.

The creation and annihilation operators $a_{\boldsymbol{k}}$, $b_{\boldsymbol{k}}$, $a^\dagger_{\boldsymbol{k}}$, $b^\dagger_{\boldsymbol{k}}$ satisfy the canonical commutation relations $[a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}] = [b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{k}'}] = (2\pi)^3\delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')$ and $(\text{others}) = 0$.

4.2 Gauge invariant two point current

The local electric current operator is defined by

$$\hat{J}_\mu(x) \equiv ie\hat{\phi}^\dagger(x)\overleftrightarrow{D_\mu}\hat{\phi}(x) = ie\{\hat{\phi}^\dagger(x)D_\mu\hat{\phi}(x) - (D_\mu\hat{\phi}(x))^\dagger\hat{\phi}(x)\}. \tag{4.5}$$

As the vacuum expectation value of the current operator is divergent, we adopt the symmetric point separation $x^\mu \to x^\mu \pm \epsilon^\mu$ to control the divergence and renormalize it in a gauge-invariant manner.

The gauge-invariant two-point current operator with the symmetric point separation is given by

$$\hat{J}_\mu(x;\epsilon) \equiv ie \exp\left[-ie\int_{x-\epsilon}^{x+\epsilon} \mathrm{d}x^\mu A_\mu\right] \hat{\phi}^\dagger(x+\epsilon)\overleftrightarrow{D_\mu}\hat{\phi}(x-\epsilon), \tag{4.6}$$

which is invariant under the gauge transformation with an arbitrary function $\Gamma(x)$,

$$\hat{\phi}(x) \to \mathrm{e}^{-ie\Gamma(x)}\hat{\phi}(x), \quad \hat{\phi}(x)^\dagger \to \mathrm{e}^{+ie\Gamma(x)}\hat{\phi}^\dagger(x), \quad A_\mu(x) \to A_\mu + \Gamma_{,\mu}(x). \tag{4.7}$$

Note that the transformation of the covariant derivative is given by $D_\mu\hat{\phi}(x) \to \mathrm{e}^{-ie\Gamma(x)}D_\mu\hat{\phi}(x)$, and it changes the overall phase. This is canceled by the prefactor $\exp[-ie\int_{x-\epsilon}^{x+\epsilon}\mathrm{d}x^\mu A_\mu]$ which becomes unity when the coincidence limit $\epsilon \to 0$ is taken. Of course, we can recover the locality of the current operator in the coincidence limit,

$$\hat{J}(x) = \lim_{\epsilon\to 0}\hat{J}(x;\epsilon) \tag{4.8}$$

We can also separate the vacuum expectation value $\lim_{\epsilon\to 0}\langle\hat{J}(x;\epsilon)\rangle$ into the ϵ-dependent divergent terms and the ϵ-independent finite terms as we will see below. This fact means that the divergence comes from the ultraviolet (UV) physics. Therefore this can be absorbed by renormalization of the charge e and the field redefinition.

If a straight line is selected as the integration contour in (4.6), we obtain

$$\langle\hat{J}_z(x;\epsilon)\rangle = -2e\left(\frac{a_+}{a_-}\right)^{-iL\frac{\Delta z}{\Delta\eta}}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{-i\boldsymbol{p}\cdot\Delta x}}{a_+a_-}(p_z - \bar{a}HL)\chi_{\boldsymbol{k}}\left(\eta+\frac{\Delta\eta}{2}\right)\chi^*_{\boldsymbol{k}}\left(\eta-\frac{\Delta\eta}{2}\right), \tag{4.9}$$

where we have used $\epsilon^\mu = (\Delta\eta/2, \Delta\boldsymbol{x}/2)$ and introduced $a_\pm = [1 - H(\eta \pm \Delta\eta/2)]^{-1}$ and $\bar{a} = (a_+ + a_-)/2$. We can make use of the Mellin-Barnes representation for the Whittaker function [52] to evaluate this quantity, [1]

$$W_{\kappa,\mu}(z) = \int_{C_s}\frac{\mathrm{d}s}{2\pi i}z^s\mathrm{e}^{-z/2}\frac{\Gamma(s-\kappa)\Gamma(-s-\mu+\frac{1}{2})\Gamma(-s+\mu+\frac{1}{2})}{\Gamma(\frac{1}{2}-\kappa-\mu)\Gamma(\frac{1}{2}-\kappa+\mu)}, \tag{4.10}$$

[1]See also Appendix D for the integration of the Whittaker function.

where the integration contour C_s runs from $-i\infty$ to $i\infty$ and is taken to separate the poles of $\Gamma(s-\kappa)$ ($s=\kappa-n, n=0,1,2,\cdots$) from those of $\Gamma(-s-\kappa-\mu+\frac{1}{2})\Gamma(-s-\kappa+\mu+\frac{1}{2})$. After substituting (4.10), the expectation value reads

$$
\begin{aligned}
\langle \hat{J}_z(x;\epsilon)\rangle &= -\frac{e}{(2\pi)^3 a_+ a_-}\left(\frac{a_+}{a_-}\right)^{-iL\frac{\Delta z}{\Delta\eta}} \int_0^\infty \mathrm{d}p \int_{-1}^{1}\mathrm{d}\xi \int_0^{2\pi}\mathrm{d}\varphi\, \mathrm{e}^{-ip\eta - i\boldsymbol{p}\cdot\Delta x} \int_{C_s}\frac{\mathrm{d}s}{2\pi i}\int_{C_t}\frac{\mathrm{d}t}{2\pi i} \\
&\times p(p\xi - \bar{a}HL)\mathrm{e}^{\pi L\xi}\mathrm{e}^{\pi i(t-s)/2}\left(\frac{2p}{H}\right)^{s+t} a_+^{-s}a_-^{-t} \\
&\times \frac{\Gamma(s+iL\xi)\Gamma(-s-\mu+1/2)\Gamma(-s+\mu+1/2)\Gamma(t-iL\xi)\Gamma(-t-\mu+1/2)\Gamma(-t+\mu+1/2)}{\Gamma(1/2-iL\xi-\mu)\Gamma(1/2-iL\xi+\mu)\Gamma(1/2+iL\xi-\mu)\Gamma(1/2+iL\xi+\mu)}.
\end{aligned}
\tag{4.11}
$$

We are now ready to perform the p-integral with a tiny shift in p axis ($p\to p+i\varepsilon$). The residue theorem and perturbative ordering by the point separation ϵ give an analytic expression for the expectation value,

$$
\begin{aligned}
\langle \hat{J}_z(x;\epsilon)\rangle &= \frac{eaH^3}{4\pi^2}\Bigg[-\frac{L}{3}(\log\epsilon + \log H + \frac{3}{2} + \gamma_E) - \frac{2}{15}L^3 \\
&+ \frac{\mu}{12\pi^3 L\sin(2\pi\mu)}\Big\{(45+4\pi^2(-2+3L^2+2\mu^2))\cosh(2\pi L) \\
&\qquad - (45+8\pi^2(-1+9L^2+\mu^2))\frac{\sinh(2\pi L)}{2\pi L}\Big\} \\
&+ \Re\Big[\frac{iL}{16\sin(2\pi\mu)}\int_{-1}^{1}\mathrm{d}\xi(1-4\mu^2+(-7-12L^2+12\mu^2)\xi^2+20L^2\xi^4) \\
&\times\left\{(\mathrm{e}^{2\pi L\xi}+\mathrm{e}^{-2i\pi\mu})\psi\left(\frac{1}{2}-iL\xi+\mu\right) - (\mathrm{e}^{2\pi L\xi}+\mathrm{e}^{2i\pi\mu})\psi\left(\frac{1}{2}-iL\xi-\mu\right)\right\}\Big] + \mathcal{O}(\epsilon^1)\Bigg],
\end{aligned}
\tag{4.12}
$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ denotes the digamma function and γ_E is the Euler-Mascheroni constant. The covariant separation is expressed as $\epsilon^2 \equiv \epsilon^\mu\epsilon_\mu = a^2(-\Delta\eta^2+|\Delta\boldsymbol{x}|^2)/4$. Note that the separation in scale factor ($a_\pm \neq a$) must be preserved during calculation. Otherwise, it would lead to an incorrect result. We have only a logarithmic divergence of the separation to be absorbed by the renormalization of the charge and the gauge field. In fact, direction dependent divergent terms such as $(-\Delta\eta^2+|\Delta\boldsymbol{x}|^2)^{-2}\Delta z$ appear in the calculation. We eliminate them by adopting a rule that the limit $\Delta z\to 0$ must always be taken in advance when the coincidence limit $\epsilon\to 0$ is taken.

4.3 Point splitting renormalization

The vacuum expectation value of the current operator is placed at the right-hand side of the semi-classical Maxwell equation $F^{\mu\nu}{}_{;\nu}(x) = \langle \hat{J}^\mu(x) \rangle$.

Renormalization prescription is required to deal with the divergence. In our set up, only the z-component is relevant. We can use the usual ansatz for renormalized field $A_{\mathrm{R}\,z}$ and charge e_{R} involving a divergent coefficient C such as

$$A_{\mathrm{R}\,z} = C A_z, \quad e_{\mathrm{R}} = C^{-1} e, \tag{4.13}$$

or instead of renormalizing A_z we can introduce renormalized electric field strength $E_{\mathrm{R}} = CE$. Note that the combination of the charge and the field is unchanged $e_{\mathrm{R}} E_{\mathrm{R}} = eE$, and $\langle J_z(e, E) \rangle = C \langle J_z(e_{\mathrm{R}}, E_{\mathrm{R}}) \rangle$. The minimal choice for the renormalization factor C is found to be

$$C^2 = 1 - \frac{e^2}{24\pi^2} \log \epsilon. \tag{4.14}$$

This choice eliminates the $\log \epsilon$ term from (4.12). We can further subtract the terms proportional to L from the large parenthesis $[\cdots]$ in (4.12) in addition to it. So, the form of C is given by

$$C^2 = 1 - \frac{e^2}{24\pi^2} \left\{ \log \epsilon + (\text{finite terms}) \right\}. \tag{4.15}$$

A physical condition is required to determine the finite part in (4.15). We adopt the requirement that the renormalized current must vanish in massive scalar limit ($m^2 \gg E,\ H^2$),

$$\lim_{M \to \infty} \langle \hat{J}_z \rangle_{\mathrm{ren}} = 0. \tag{4.16}$$

The asymptotic behavior of the digamma function $\psi(z) \sim \log(z) - 1/(2z) + \mathcal{O}(z^{-2})$ is useful to find non-vanishing terms in the massive limit

$$\langle \hat{J}_z \rangle \xrightarrow{M \gg 1,\, L \ll 1} - \lim_{\epsilon \to 0} \frac{eaH^3}{4\pi^2} \frac{L}{3} \left(\log \epsilon + \log m + \gamma_E + 3/2 \right). \tag{4.17}$$

This tells us the minimal form of the finite terms in (4.15). It is still possible to subtract decaying terms $\mathcal{O}(M^{-1})$, however, we stay in the minimal subtraction at this moment. We finally obtain the following expression for the renormalized current

$$
\begin{aligned}
&\langle \hat{J}_z(x)\rangle \\
&= \frac{eaH^3}{4\pi^2}\Bigg[\frac{L}{3}\log M - \frac{2}{15}L^3 + \frac{\mu}{12\pi^3 L\sin(2\pi\mu)}\Big\{(45+4\pi^2(-2+3L^2+2\mu^2))\cosh(2\pi L) \\
&\qquad\qquad - (45+8\pi^2(-1+9L^2+\mu^2))\frac{\sinh(2\pi L)}{2\pi L}\Big\} \\
&+ \Re\Big[\frac{iL}{16\sin(2\pi\mu)}\int_{-1}^{1} \mathrm{d}\xi(1-4\mu^2+(-7-12L^2+12\mu^2)\xi^2+20L^2\xi^4) \\
&\qquad \times\left\{(\mathrm{e}^{2\pi L\xi}+\mathrm{e}^{-2i\pi\mu})\psi\left(\frac{1}{2}-iL\xi+\mu\right)-(\mathrm{e}^{2\pi L\xi}+\mathrm{e}^{2i\pi\mu})\psi\left(\frac{1}{2}-iL\xi-\mu\right)\right\}\Big]\Bigg],
\end{aligned}
\tag{4.18}
$$

Thus we have reached the exactly same expression as obtained by Kobayashi and Afshordi [39] by using the adiabatic regularization up to the second order.

4.4 Result

It is remarkable that our result perfectly agrees with the previous one [39], and worthwhile to list its physical significance. Note that the dimensionless current defined as

$$
\mathcal{J} = \mathcal{J}(L, M) \equiv \frac{\langle \hat{J}_z\rangle}{eaH^3}, \tag{4.19}
$$

is a function of only L and M. The plot of $\mathcal{J}(L, M)$ as a function of the electric field strength L is shown in Fig. 4.1 for different values of the mass parameter M.

4.4.1 Weak electric field limit

First, let us consider various limiting behaviors.

The renormalized current in the weak electric field regime $eE \ll m^2, H^2$ is expressed as

$$
\mathcal{J} \xrightarrow{L\to 0} \frac{L}{12\pi^2}\left\{\log M - \frac{1}{2}\left[\psi\left(\frac{1}{2}+\mu_0\right)+\psi\left(\frac{1}{2}-\mu_0\right)\right]+\frac{8\pi}{3}\frac{\mu_0(\mu_0^2-1)}{\sin(2\pi\mu_0)}\right\}, \tag{4.20}
$$

where $\mu_0 = \sqrt{9/4 - M^2}$.

We can define the dimensionless conductivity $\sigma(M) = \mathcal{J}/L|_{L\to 0}$ as

$$
\sigma(M) = \frac{1}{12\pi^2}\left\{\log M - \frac{1}{2}\left[\psi\left(\frac{1}{2}+\mu_0\right)+\psi\left(\frac{1}{2}-\mu_0\right)\right]+\frac{8\pi}{3}\frac{\mu_0(\mu_0^2-1)}{\sin(2\pi\mu_0)}\right\}. \tag{4.21}
$$

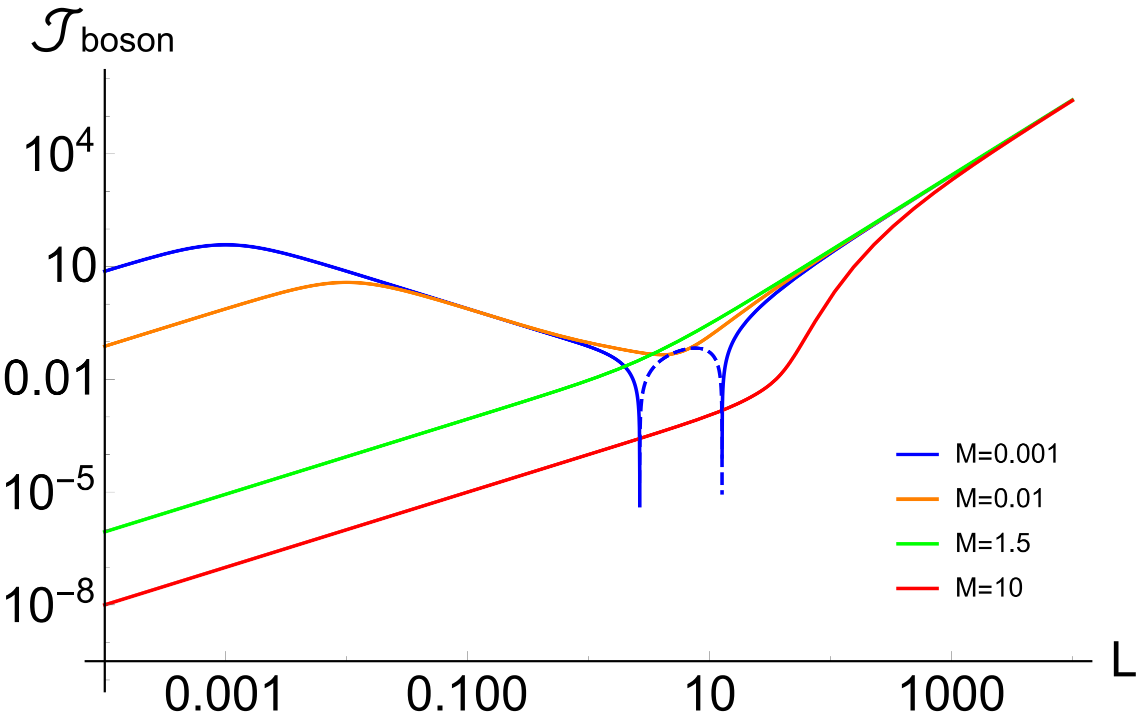

Figure 4.1: Absolute value of the renormalized current $\mathcal{J} = \langle J \rangle / eaH^3$ is shown as a function of $L = eE/H^2$. Each line corresponds to different mass parameter $M = m/H$. Negative current is observed in $L = 1 \sim 10$ for $M = 0.001$ case.

The asymptotic behaviors of $\sigma(M)$ are given by

$$\sigma(M) \to \begin{cases} \dfrac{3}{4\pi^2 M^2} & (M \ll 1) \\ \left(\dfrac{7}{72\pi^2 M^2} + \mathcal{O}(M^{-4})\right) - \mathrm{e}^{-2\pi M}\left(\dfrac{4}{9\pi} M^3 + \mathcal{O}(M^1)\right) & (M \gg 1) \end{cases} . \tag{4.22}$$

First of all, it is easy to find that the conductivity in this limit is always positive $\sigma(M) > 0$. This forms a striking contrast to the fermionic conductivity in the weak field limit, which is always negative for any value of M. Additionally, the strong M^{-2} enhancement for the small scalar mass is observed. This is the four-dimensional analog of the two dimensional IR hyperconductivity reported first in [37]. The physics of the IR hyperconductivity is still unknown.

The exponentially suppressed term ($\propto \mathrm{e}^{-2\pi M}$) in the massive regime must exist naturally because the standard Bogoliubov calculation gives the number density of the scalar particles in dS spacetime $n \sim H^4(\mathrm{e}^{2\pi M} - 1)^{-1}$ which means the exponential suppression of heavy particle production. Nevertheless, we also have inexplicable terms, which are not protected by the exponential factor in the conductivity.

As we mentioned, there is room for changing the renormalization fixing without breaking the condition $\langle J_z \rangle_{\rm ren} \to 0$ for $M \to \infty$. If one naively tried to remove the M^{-2} term in (4.22), it would cause a huge IR correction and even worse negativity to the renormalized current. Furthermore, if all the unprotected terms should be subtracted from the current $\mathcal{J}$, i.e. if we *impose* the exponential damping of the induced current in the massive limit, then the last term in (4.21) will be left and the maximally-subtracted bosonic conductivity is given by

$$\sigma_{boson}(M)|_{\rm maximal} = \frac{2}{9\pi} \frac{\mu_0(\mu_0^2 - 1)}{\sin(2\pi\mu_0)}. \tag{4.23}$$

In Fig.4.2, we exhibit the behaviors of the minimally and maximally-subtracted bosonic conductivity. The fermionic conductivities are also plotted for comparison.

The maximally-subtracted bosonic conductivity has a discontinuity at $M = \sqrt{2}$. This mass parameter corresponds to the conformal coupling $\xi R \phi^2$ in dS spacetime, with the parameters being $\xi = 1/6$ and $R = 12H^2$. Thus this is conformally equivalent to the case of a massless scalar field in Minkowski spacetime. For this reason, this discontinuity (or divergence) at $M = \sqrt{2}$ might be physically reasonable. It is also obvious that $\sigma|_{\rm maximal}$ is positive for $M < \sqrt{2}$ (the blue solid line in Fig.4.2) and negative for $M > \sqrt{2}$ (the blue dashed line in the same figure) in this treatment. We will examine the consequences of the maximal subtraction in the next section.

Note also that the $7/(72\pi^2 M^2)$ term in (4.22) corresponds to the fourth order adiabatic term. The terms in (4.17) correspond to the zeroth and second order adiabatic subtraction terms. We can expect that the formal infinite order adiabatic subtraction of the terms proportional to L (WKB is an asymptotic expansion) will result in the removal of the exponentially unprotected behavior in the massive limit.

4.4.2 Strong electric field or weak curvature regime

In the case of the strong electric field limit $eE \gg m^2$, H^2, or $H^2 \ll m^2$, eE, the L^3 term in the first line of (4.18) and the integration of the digamma functions cancel each other. We find

$$\mathcal{J} \simeq \frac{L^2}{12\pi^3} {\rm sgn}(L) {\rm e}^{-\frac{\pi M^2}{|L|}}, \tag{4.24}$$

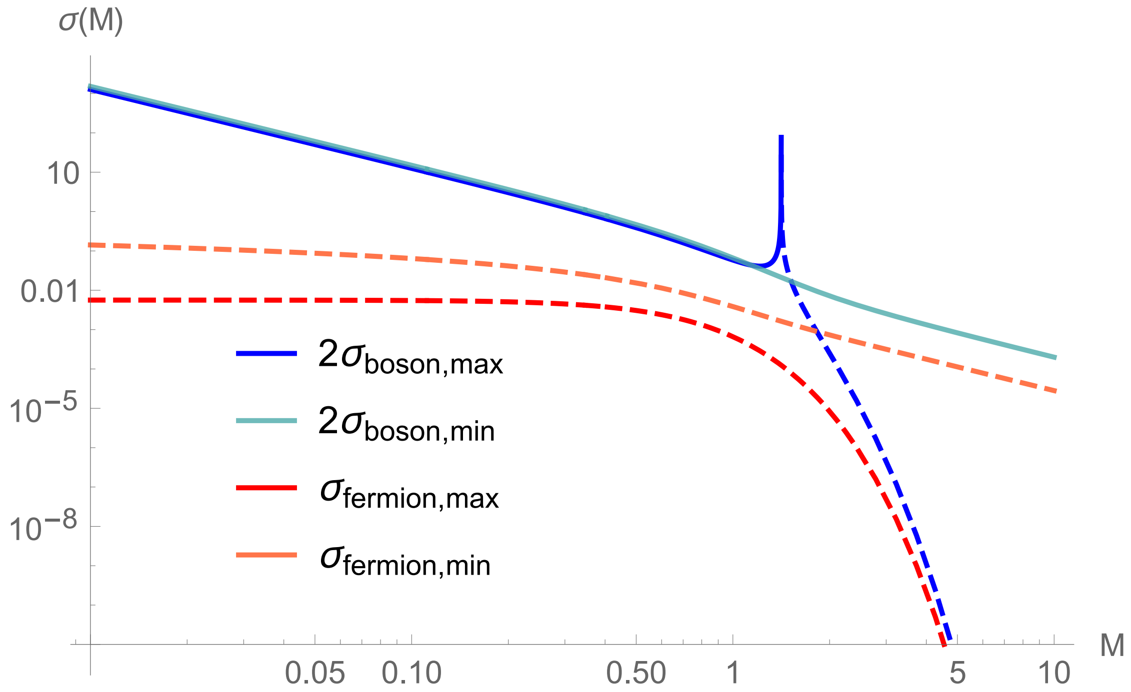

Figure 4.2: The conductivities $\sigma_{boson,\max}$, $\sigma_{boson,\min}$, $\sigma_{fermion,\max}$, and $\sigma_{fermion,\min}$ respectively given in (4.23) (blue, maximally-subtracted boson), (4.21) (light green, minimally-subtracted boson), (4.27) (red, maximally-subtracted fermion), and (3.46) (orange, minimally-subtracted fermion). The solid and dashed lines indicate that the sign of the conductivity is positive and negative, respectively. Both of the maximally-subtracted conductivities show the exponential damping in the massive region while the minimally-subtracted conductivities drop in the power. Moreover, the divergence and sign flipping occur for the maximally-subtracted bosonic conductivity at $M = \sqrt{2}$. Factor 2 is introduced for the bosonic conductivity to compensate the difference in the spin degrees of freedom.

and recover the Schwinger's famous suppression factor $\exp(-\pi m^2/eE)$. This is equivalent to the strong field limit of the fermionic current (3.43) except for the factor of 2. The factor comes from the excess spin degrees of freedom of the Dirac field. The mass dependence of the current for $L \to \infty$ disappears, indeed, all the lines in Fig. 4.1 converge at infinity. Furthermore, the strong limit form is expressed in terms of the dimensionful quantities as

$$\langle J_z \rangle_{\text{ren}} \xrightarrow{H \to 0} \frac{e}{12\pi^3}(eE)^2 \frac{1}{H} e^{-\frac{\pi m^2}{eE}}, \tag{4.25}$$

where asymptotic analysis reveals that no $\mathcal{O}(H^0)$ term appears. Similar to the fermionic current, the divergence H^{-1} is due to the lack of the cosmic dilution in the Minkowski spacetime limit. The particles produced at $t = -\infty$ contributes to the current expectation value forever, so H^{-1} must be replaced by some regulator such as $(t - t_0)$ with t_0 being the turn-on time of the electric field. This prescription is also justified because the differentiation $\frac{\mathrm{d}}{\mathrm{d}\eta}\langle J_z\rangle$ is finite when $H \to 0$. Note that the conformal time η is identical to the cosmic time t in this limit as we are taking the scale factor $a = (1 - H\eta)^{-1}$. This behavior corresponds to the result obtained in Minkowski space. This linear growth of the current in time was previously shown in [54].

4.4.3 Stability analysis

Noteworthy is the existence of the negative current $\mathcal{J} < 0$ around $L \sim \mathcal{O}(1)$ for small mass regime $M \lesssim 10^{-3}$. The precisely no electric field state $L = 0$ is a trivial zero of the bosonic current $\mathcal{J}(L, M)$. However, two more zeros of the current appear in $L > 0$ and $\mathcal{J}$ becomes negative between them.

The positive current causes negative backreaction to the background electric field as expected. The negative current conversely enhances the background electric field. The current-electric field system can be seen as a sort of feedback system, and the stability analysis is easily implemented. The first (nontrivial) zero of the current corresponds to the so-called diverging point. No backreaction occurs at this point. However, a small deviation from this point induces positive feedback which enhances the deviation. The second zero and also the trivial zero are stable points of the system. A small deviation will be pulled back to this point.

Note that negativity of the induced current does not mean fatal instability of the system, as it occurs only in a finite range of the electric field. Actually, the induced current recovers its positivity as L increases. The position of the second zero is numerically given by $L = -\frac{1}{3}\log M$ for $M \ll 1$.

In Fig. 4.3, we show the position of the zeros of $\mathcal{J}(L, M)$ in the L-M plane, which also serves as a phase diagram of the current versus electric field. Each line represents the zeros of the current. The lower (upper) line corresponds to the first (second) zero of the current $\mathcal{J}(L, M)$. The negativity happens in the shaded region between the two lines. There exists a critical mass $M_c \sim 0.0033$, that is, the maximum value of the scalar mass which can cause the negativity of the current. We also emphasize that the negativity be usually mild compared to the IR hyperconductivity which occurs coincidently and the $\mathcal{J} \propto L^2$ behavior in the strong field regime as we mentioned.

So far, our discussion has been mainly based on the minimal subtraction scheme. In the next section, the maximal subtraction scheme will be investigated.

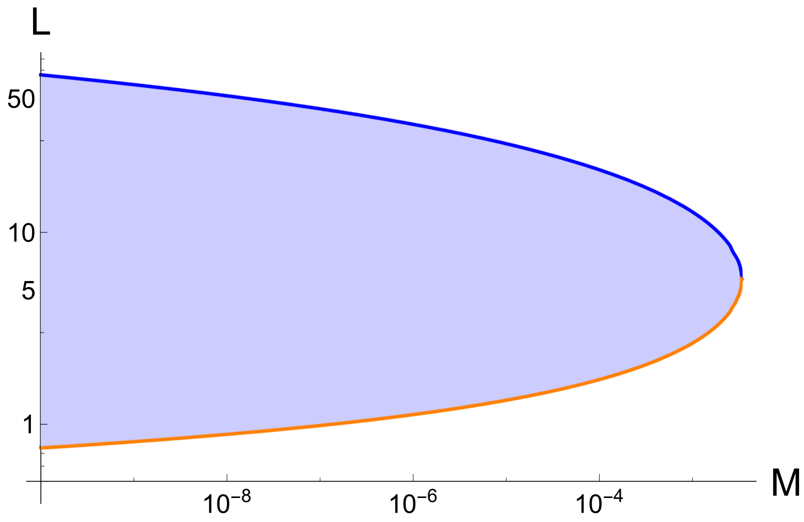

Figure 4.3: Linear stability analysis of the current-electric field system. The upper (blue) line represents the trajectory of stable points, and the lower (orange) line represents that of diverging points. There is a critical point $(L_c, M_c) = (5.7, 0.0033)$.

4.5 Minimal and maximal subtraction

We have introduced two different renormalization conditions, the minimal and maximal subtractions. The minimal subtraction requires that the subtraction term should be minimal, i.e., all the term disappear for $m \to \infty$ must be left in the renormalized current. The maximal subtraction imposes the exponential mass damping on the renormalized current. As a result, we need to remove all the power-law contributions of $\mathcal{O}(M^{-1}), \mathcal{O}(M^{-2}), \cdots$ from the minimally-subtracted current. Note that the idea of the maximal subtraction can naturally arise guided by the fact that some asymptotic terms in $M \to \infty$ lack the expected exponential suppression factor. In this section, the difference between the two renormalization condition will be shown and their physical consequences will be discussed.

4.5.1 Flaw in the minimal subtraction

One might discard the possibility of the maximal subtraction, in which we impose the exponential damping of the physical result in the massive limit, for the reason that it brings about the divergence and sign flipping to the bosonic conductivity. Although the aversion to the analytical discontinuity may be reasonable, it is also true that there is no good excuse to simply accept the minimal subtraction scheme without due deliberation. Besides, there are a couple of facts which motivate us to prefer the maximal subtraction over the minimal subtraction.

The first point, which has been already mentioned, is the appearance of the power-low damping of the currents in the massive limit shown in (3.48) (spinor) and (4.22) (scalar). From the semiclassical point of view, we naively expect the exponential damping factor $\exp(-2\pi m)$ of the quantities which are effectively proportional to the created particle number density. The maximal subtraction can immediately remove these counterintuitive terms from the current, or equivalently, the conductivity. This argument entirely fixes the form of the subtraction term. We need to subtract the term

$$\frac{L}{3\pi^2}\left[\log M - \Re\psi(iM)\right], \tag{4.26}$$

from the minimally-subtracted fermionic current (3.41), yielding the maximally-subtracted fermionic conductivity

$$\sigma_{fermion}(M)|_{\text{maximal}} = -\frac{M(4M^2+1)}{9\pi\sinh(2\pi M)}. \tag{4.27}$$

For the bosonic current, the subtraction term is given by

$$\frac{L}{12\pi^2}\left\{\log M - \frac{1}{2}\left[\psi\left(\frac{1}{2}+\mu_0\right)+\psi\left(\frac{1}{2}-\mu_0\right)\right]\right\}, \tag{4.28}$$

with $\mu_0 = \sqrt{9/4 - M^2}$.

The result of the maximal subtraction is shown in Fig. 4.4 (fermion) and Fig. 4.5 (boson), see the minimally-subtracted results Fig. 3.1 and Fig. 4.1 for comparison. Figure 4.6 is the phase diagram of the maximally-subtracted results, which is the counterpart of the minimally-subtracted results Fig. 3.2 and Fig. 4.3.

The second motivation is the disagreement of the bosonic and fermionic current in the semiclassical limit. This is because the charged fields can be treated as a classical particle without spin in this limit. The particle picture, in general, is validated in the semiclassical limit, which in our case is given by

$$L^2 + M^2 \gg 1. \tag{4.29}$$

In case of the strong field $L \gg 1$, the two currents unerringly correspond to each other bar the factor 2 related to the spin degrees of freedom. However, if we take the other end of the semiclassical limit, viz. considering the massive fields $M \gg 1$ while keeping the electric field relatively weaker $L \lesssim 1$, then the (minimally-subtracted) bosonic and fermionic current behave quite differently. This is unfavoured also because the bosonic current becomes positive in this limit despite the physically acceptable explication for the negativity of the current or the anti-screening effect we gave in the previous chapter, which is arguably independent of the spin of charged particles.

We will see, in the next subsection, how the flaws in the minimal subtraction are cured by the maximal subtraction.

4.5.2 Effects of the maximal subtraction

There are four striking features of the maximally-subtracted induced currents plotted in Fig. 4.4 and Fig. 4.5:

1. Exponential damping of the current in the massive particle limit.
2. Boson/Fermion agreement in the semiclassical limit.
3. Removal of the IR hyperconductivity in the bosonic case.
4. Finite behavior of the massless fermionic current $M = 0$.

The first point is rather trivial since we have imposed it as the renormalization condition. The second point is understood by observing the boson-fermion agreement in the conductivities in the massive limit shown in Fig.4.2 (the blue and red lines). Note that the maximal subtraction does not change the UV (strong electric field) behavior of the current and the correspondence of the currents in the strong field limit $L \to \infty$ remains unchanged. The third point, unlike the 2 $(= 1+1)$ dimensional case, comes in exchange for the enhancement of the (bosonic) conductivity in the $M \to \sqrt{2}$ limit from above $\sigma_{boson,\max} \propto (M - \sqrt{2})^{-1}$. The final point is similar to the 2 dimensional fermion in de Sitter spacetime.

Now we have a much clearer sight of the phenomenon: the maximal subtraction gives the ”physical” results in this case. As we lack the reference to the experimental or observable renormalization point information in curved spacetime, we need some physical conditions to fix the renormalization. The assumption on the asymptotic behavior in the massive limit was employed in this problem. The implementation of the subtraction was possible because we could separate the perturbative part from the nonperturbative part which has the physical origin. Our regulariza-

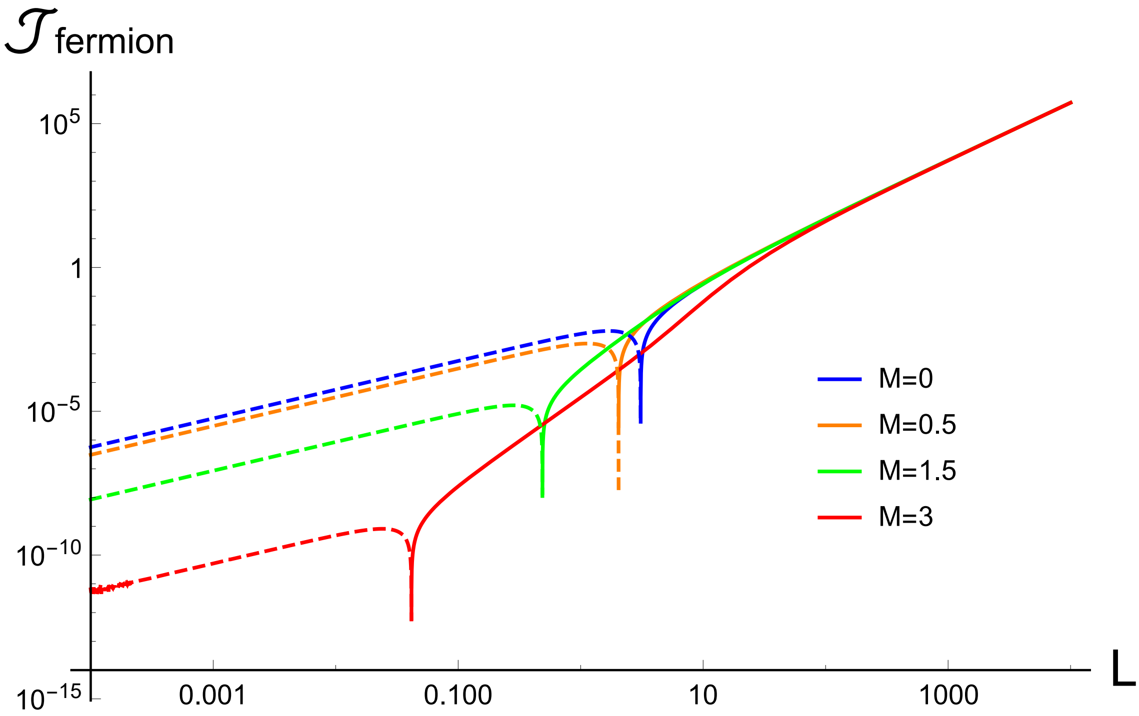

Figure 4.4: The maximally-subtracted fermionic current. The finite behavior of the massless fermion can be seen. Though the negativity of the current is suppressed, it exists for sufficiently small electric field $L = eE/H^2$.

tion scheme is schematically written as

$$\begin{aligned} \langle(\text{Observable})\rangle &\xrightarrow{\text{asymptotic expansion}} (\text{perturbative part}) + (\text{nonperturbative part}) \\ &\xrightarrow{\text{regularization}} (\text{nonperturbative part}) \\ &= \langle(\text{Observable})\rangle_{\text{reg}} \,. \end{aligned} \tag{4.30}$$

In the present problem, (perturbative part) denotes the power series in $\mathcal{O}(M^{-n})$, and (nonperturbative part) represents the exponentially suppressed term in (3.48) and (4.22).

We would like to put a few comments with regard to the correspondence between the adiabatic subtraction and the point-splitting renormalization. The removal of the perturbative part may also be achieved by the adiabatic subtraction as we mentioned. However, the perturbative part as a power series in M^{-1} is not a convergent series. It is rather a formal divergent series since the digamma function has an asymptotic expansion for large z, $\psi(z) \sim \ln z - \Sigma c_{2n} z^{-2n}$ with the rapidly increasing coefficients behaves $c_{2n} \sim (2n-1)!$ for large n. Therefore, we should formally define the meaning of the higher-order (or infinite order) adiabatic subtraction

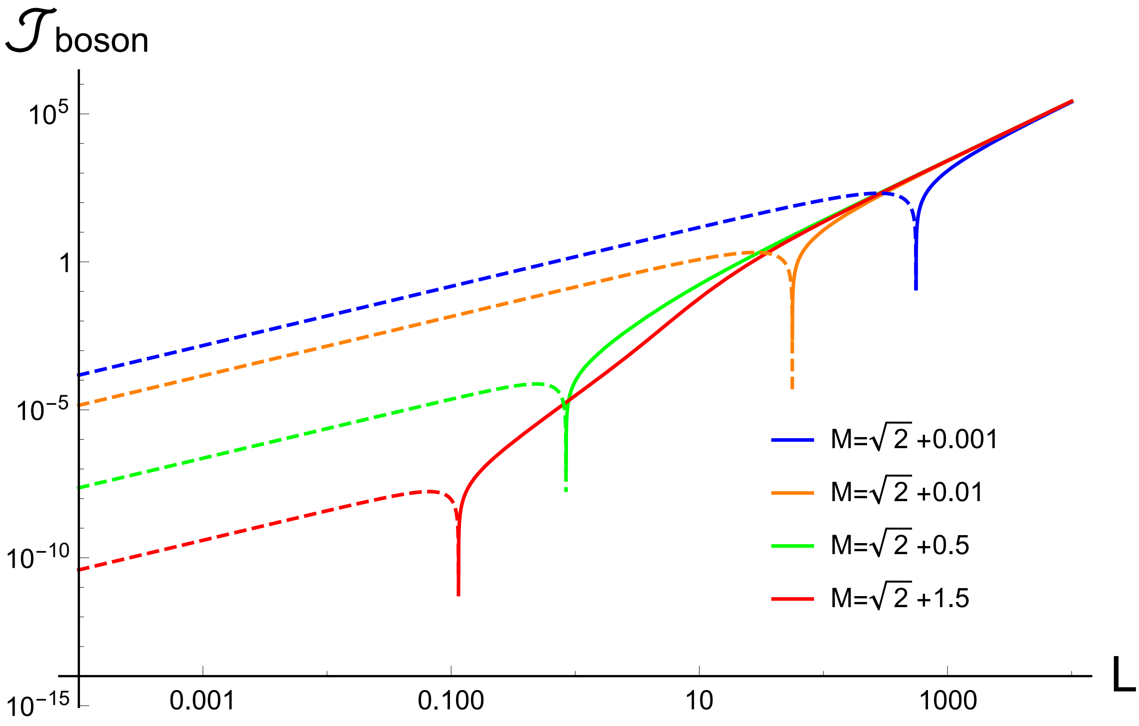

Figure 4.5: The maximally-subtracted bosonic current. The positivity/negativity of the bosonic current now looks similar to the fermionic current, i.e., negative for the weaker electric field and positive for the stronger electric field. The IR hyperconductivity observed in the minimal subtraction is removed. Instead, the enhancement of the negative current in the conformally massless limit $M \to \sqrt{2}$ is seen.

as the adiabatic expansion is also an asymptotic expansion. It involves analytical challenges, and we have not yet had any specific result. Nevertheless, we expect that an establishing tool called complete WKB, which makes diverging WKB expansion meaningful in a nonperturbative way, can shed light on the renormalization of quantum theory in curved spacetime.

4.5.3 Semiclassical calculation

To support our arguments in this section and Sec.3.5, we explicitly perform the semiclassical calculation of the surface charge density induced on the horizon. Disappointingly, the Bogoliubov coefficients for the Dirac field in the 4-dimensional case are not available, but we can find the result for the scalar QED case in [39]. The particle number density of the positive (negative)

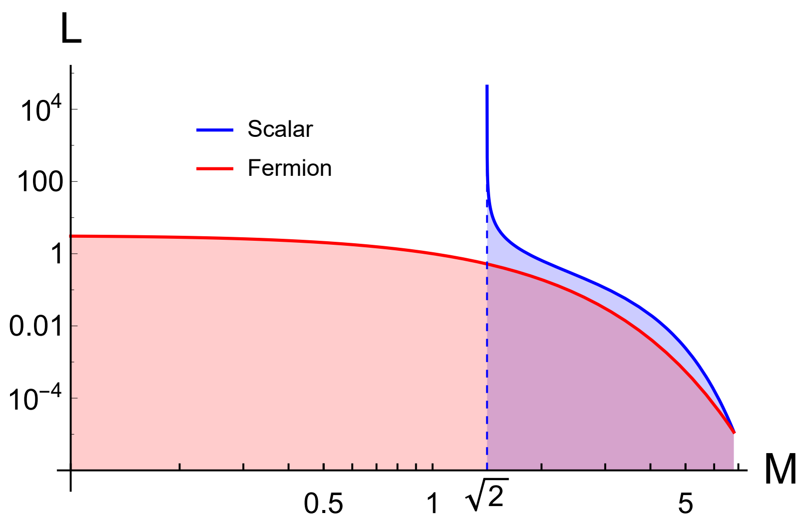

Figure 4.6: The maximal subtraction version of the phase diagram for the spinor (red) and scalar (blue) QED in de Sitter spacetime. The shaded regions correspond to the negative current phase, or the anti-screening phase, while the regions above the curves correspond to the phase with the positive current and the standard screening effect. The mass of the scalar field $M = m/H$ is taken larger than the value $\sqrt{2}$ conformally corresponding to the massless scalar field in the flat spacetime. See also the minimal subtraction counterparts, Fig. 3.2 and Fig. 4.3.

charged χ-particles [2] $n_{\boldsymbol{k}}^{+}$ ($n_{\boldsymbol{k}}^{-}$), which is determined by the Bogoliubov calculation, is given by

$$n_{\boldsymbol{k}}^{\pm} = |\beta_{\boldsymbol{k}}^{\pm}|^2 = \frac{\mathrm{e}^{\pm 2\pi L \cos\theta} + \mathrm{e}^{-2\pi|\mu|}}{\mathrm{e}^{2\pi|\mu|} - \mathrm{e}^{-2\pi|\mu|}}, \tag{4.31}$$

where the adiabatic condition $|\mu| = \sqrt{L^2 + M^2 - 9/4} > 0$ is assumed. The charge density induced on the horizon is then given by

$$\begin{aligned} \rho(\boldsymbol{x}) &= e \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(|\beta_{\boldsymbol{k}}^{+}|^2 - |\beta_{\boldsymbol{k}}^{-}|^2 \right) \\ &= e \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{e}^{2\pi L \cos\theta} - \mathrm{e}^{-2\pi L \cos\theta}}{\mathrm{e}^{2\pi|\mu|} - \mathrm{e}^{-2\pi|\mu|}}, \end{aligned} \tag{4.32}$$

In the massive ($M \gg 1$) and weak field ($L \ll 1$) limit, this expression becomes

$$\rho(\boldsymbol{x}) = 4\pi e L \mathrm{e}^{-2\pi M} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \cos\theta. \tag{4.33}$$

[2]See the definition of the canonical field (4.1) again.
Note that the electric current of the χ-field $J_{\chi\,\mu}$ is related to ϕ-current J_μ in (4.5) by $J_{\chi\,\mu} = a^2 J_\mu$. The index of the χ-current is raised by the Minkowski metric $\eta^{\mu\nu}$, so the relation between the ϕ and χ currents with upper indeces is given by $J_\chi^\mu = a^2 \eta^{\mu\nu} J_\nu = a^4 J^\mu$.

Because of the symmetry of the integration, we immediately obtain $\rho = 0$ for all the spacetime points $\boldsymbol{x} \notin (\text{horizon})$. This is compatible with the charge conservation law. However, the existence of the de Sitter horizon breaks the symmetry as we mentioned in Sec.3.5. To realize the concept of Maxwell's demon on the horizon, all we need to do is to restrict the region of the momentum integration to the inward direction $\boldsymbol{k} \cdot \boldsymbol{x} < 0$. Note also that the momentum UV cutoff $k \sim am$ can be safely introduced since we are considering the massive particle limit. Therefore, we obtain the charge density on the horizon

$$\begin{aligned} \rho(\boldsymbol{x})\big|_{\boldsymbol{x}\in(\text{horizon})} &= 4\pi e L \mathrm{e}^{-2\pi M} \int_0^{am} \frac{k^2 \mathrm{d}k}{(2\pi)} \iint_{\boldsymbol{x}\cdot\boldsymbol{k}<0} \frac{\mathrm{d}(\cos\theta')}{(2\pi)} \frac{\mathrm{d}\phi'}{(2\pi)} \cos\theta' \\ &= -\frac{e}{6\pi} L M^3 \mathrm{e}^{-2\pi M} \cos\theta (aH)^3. \end{aligned} \tag{4.34}$$

Using the delta function [3], the charge density is rewritten as

$$\rho(\boldsymbol{x}) = -\frac{e}{6\pi} L M^3 \mathrm{e}^{-2\pi M} \cos\theta (aH)^3 \left\{ r\delta\left(r - \frac{1}{aH}\right)\right\}. \tag{4.36}$$

From the local charge conservation, we find, with multiplying the Heaviside's step function and the homogeneous induced current in z- direction together,

$$\begin{aligned} \partial_z \left(J^z_\chi \Theta\left(\frac{1}{aH} - r\right)\right) &= -J^z_\chi \cos\theta \delta\left(r - \frac{1}{aH}\right) \\ &= -\frac{\mathrm{d}\rho}{\mathrm{d}\eta} \\ &= \frac{2e}{3\pi} L M^3 \mathrm{e}^{-2\pi M} \cos\theta (aH)^3 \delta\left(r - \frac{1}{aH}\right). \end{aligned} \tag{4.37}$$

Then, we obtain the semiclassical approximation for the induced current in the massive particle limit,

$$J^z_\chi|_{\text{semiclassical}} = -\frac{2e}{3\pi} L M^3 \mathrm{e}^{-2\pi M} (aH)^3. \tag{4.38}$$

[3] The delta function $\delta(x)$ has dimension $[x]^{-1}$, so $r\delta(r - r_0)$ is dimensionless. The surface charge density on the horizon is given by

$$\Sigma|_{\text{horizon}} = -\frac{e}{6\pi} L M^3 \mathrm{e}^{-2\pi M} \cos\theta (aH)^2. \tag{4.35}$$

This estimation corresponds to the massive limit of the maximally subtracted scalar current (see the massive limit of the renormalized dimensionless conductivity (4.22)) which is given by

$$J_\chi^z = a^4 J^z \xrightarrow{M \gg 1} -\frac{4e}{9\pi} L M^3 \mathrm{e}^{-2\pi M} (aH)^3, \tag{4.39}$$

except for the numerical factor $\frac{2}{3}$.

The (approximate) correspondance between the semiclassical estimation (4.38) and the maximally-subtarcted current (4.39) in the massive particle limit, $m^2 \gg H^2, eE$, shows the physical preference of the maximal subtraction over the traditional minimal adiabatic subtraction, which has been employed in many literature, e.g., [39, 47, 48, 50, 51, 63, 64]. Remarkably, some authors reported the unphysical consequences of the minimal subtraction scheme such as the negative value of the renormalized primordial power spectrum [65, 66].

Nobility is the master of talent and talent is the servant of nobility. Otherwise it's as though the family had no head but is ruled by menials. Before long, nasty goblins will be storming madly around.

Hong Zicheng, *Caigentan*

5

Conclusion

We discussed the analytic aspects of dS-QED in the previous part of the dissertation. We focused primarily on the system consists of the scalar or spinor charged field and homogeneous and constant energy-density electric background field (3.6) in 4-dimensional de Sitter spacetime.

In this chapter, the presented results are summarized, and then their consequences in the context of the cosmology are discussed. Finally, we close the present dissertation with concluding remarks.

5.1 Summaries

5.1.1 Summary of chapter 3

We have investigated the fermionic current induced by the electric field in $1 + 3$ dimensional de Sitter spacetime. Using the adiabatic subtraction, we obtained the renormalized expectation value of the current operator for the charged fermion. The analytic result (3.41) which is plotted in Fig. 3.1 was studied in detail. The similarity to and difference from the bosonic (scalar) case was discussed.

The analytic result (3.41) yields information about the behaviors of the induced current in both the weak and strong electric field limits. In the strong field limit, we obtained (3.43) which coincides with the behavior of the bosonic current (4.18) evaluated in the next chapter as well as that of the current in flat spacetime found in [54]. Since the contribution to the induced current mainly comes from Schwinger pair production in this limit, the induced current carries the mass suppression factor for the Schwinger effect $\exp(-\pi m^2/eE)$ as expected. Meanwhile, in the weak electric field regime, we have found two remarkable features, namely, the absence of the IR hyperconductivity and the negativity of the induced current. These two features have already been observed in the $1+1$ dimensional fermionic case [41] and the $1+3$ dimensional bosonic case, respectively.

The negative current occurs for the electric field smaller than a certain value $L_*(M)$ which is determined by the spinor mass $M = m/H$. The plot of $L_*(M)$ shown in Fig. 3.2 can be seen as the phase diagram where the positive and negative current phase are separated by the curve of L_*. Although the negativity of the current indicates the positive feedback which leads to the counterintuitive enhancement of the background electric field (anti-screening), it does not mean an unbounded instability. The system is stable for the electric field stronger than $L_*(m/H)$, and thus the electric field is not enhanced beyond L_*.

We also found the terms which do not carry any exponential mass suppression factor in the massive limit of the fermionic conductivity (3.48). If the particle is sufficiently heavy, the semi-classical description must be precise, and it suggests that the exponential mass suppression factors such as $\exp(-\pi m^2/eE)$ or $\exp(-2\pi m/H)$ should appear. Thus, the lack of the exponential suppression factor guided us in a physical choice of the renormalization point in chapter 4.

We have reached the physical understanding of the negative current. The thermodynamical (macroscopic) aspect and the microscale physics of the anti-screening phenomenon were explained. Employing the fact that an observer stays at an inertial system in de Sitter spacetime will not have access to the information outside the horizon, we argued that the observer will see the surface charge distribution Σ on the horizon, which seemingly reduces the informatical entropy of the observer. We, however, showed that the transportation of the charges itself is a healthy entropy-increasing process abiding by the second law of thermodynamics based on the electro-thermodynamical equation (3.55). We pointed out that the de Sitter horizon plays the role of Maxwell's demon who can separate the charged particles according to their momentum to generate the directed transportation of the charge. The horizon can be functioning as Maxwell's demon with the electric field which breaks the spatial symmetry in its direction. This is graphically summarized in Fig. 3.3. Besides, the possible connection between the gravity and thermodynamics through the informatical arguments was also implied.

5.1.2 Summary of chapter 4

The point-splitting regularization scheme in a covariant and gauge-invariant manner was performed for the scalar QED case. The only divergence we have encountered is the logarithmic divergence which can be absorbed into the kinetic term of the gauge field in a conventional fashion. In a previous calculation done with the momentum cutoff technique and the adiabatic subtraction up to the second order [39], there was also a quadratic divergence which was an obstacle to the gauge-invariant renormalization. We have imposed the renormalization condition (4.16) instead of employing the adiabatic subtraction. Remarkably, the result of the minimal subtraction eliminating the terms appearing in (4.17) and that with the second-order adiabatic subtraction show the perfect agreement.

We have also investigated the properties and the consequences of the renormalized current (4.18) shown in Fig.4.1. We found the negativity of the current and the exponentially unprotected terms in the massive limit again in the bosonic case as well as the fermionic case. The negativity of the renormalized current which shows up in tiny mass regime $m \lesssim 0.003H$, which can be observed in the phase diagram Fig. 4.3.

Due to the lacking experimental knowledge of the renormalization point in curved spacetime, we had been adopting the minimal subtraction scheme in both the adiabatic and point-splitting renormalization for the time being. However, as we have seen, the minimal subtraction scheme is not physically acceptable for the following reasons. Firstly, the massive charged particle limit of the bosonic current does not correspond to the fermionic current, though their behavior must be the same as that of the classical particles, which is independent of the particle spin. Secondly, there appear the terms without exponential suppression factor $\exp(-2\pi m/H)$ in the massive limit of the bosonic and fermionic currents. Lastly, the bosonic current shows the negativity relatively restricted parameter region unlike the fermionic current which shows the negativity for sufficiently weak electric field regime regardless of the particle mass. Importantly, the physics behind the counterintuitive negative current in the case of the weaker electric fields is already understood, and the explanation does not depend on the particle spin.

To fix the problems with the minimal subtraction, we proposed the use of the maximal subtraction which is defined by the further subtraction of the terms in (4.26) (fermion) or (4.28) (scalar) from the minimally-subtracted currents. The resulting currents are visualized in Fig. 4.4 (fermion) and Fig. 4.5 (boson). The comparison of the behaviors of the conductivities (boson/fermion, minimal/maximal) is made in Fig. 4.2, which explicitly shows the boson-fermion agreement and the negativity of the maximally subtracted conductivity. The revised phase diagram is also exhibited in Fig. 4.6. We have checked that the behavior of the maximally-subtracted currents in the massive region corresponds to that of the semiclassical calculation (4.38) without an irrelevant $\mathcal{O}(1)$ numerical factor. Note that the change in the renormalization condition does not affect the behavior of the flat spacetime or the strong electric field limit.

5.1.3 Summary of Appendix E

This part advocates using the effective field theoretical method in combination with the nonperturbative renormalization group (NPRG) technique. We found the governing equation (E.41) (Wetterich equation, or flow equation) reviewing the original derivation in the flat spacetime. We have derived the detailed functional dependence of the flow equation for the scalar QED in de Sitter spacetime in the form of (E.47), which, in principle, describes the dynamics of the scalar field and also the gauge field. We could not found the analytical solution to the full equation. Instead, we gave the functional flow equation for the effective scalar potential (E.63) under the existence of the background electric fields to explain the capability of the method as a numerically friendly way to deal with the quantum field theory which is apparently free of the UV and IR divergence.

What is remarkable about the method is that the flow equation (E.41) is exact, and even after the truncation like (E.43), it still contains nonpertabative contributions. Another advantage of the method together with the (modified) Ward-Takahashi identity (E.38) is the maintained gauge symmetry. This enables us to greatly reduce the number of the relevant terms which should be considered.

The work shown in this appendix is still inconclusive, however, we, as a first step, obtained the result (E.63) which describes the flow of the effective potential affected by the background electric field. The solution of the flow equation (E.63) shows how the shape of the effective scalar potential is deformed by the applied electric fields, which is considered to give further physical insights about the systems with electromagnetic interaction in future studies.

5.2 Discussion on cosmological consequences

The cosmological consequences of our analyses summarised in the previous section are discussed in this section. The implications of the anti-screening phenomena we have found in dS-QED are first considered. Then, possible impacts of the proposed renormalization condition are discussed. In addition, we explain the possible impacts of the present work on other research fields.

5.2.1 Possible scenario of spontaneous electromagnetic fields generation

The anti-screening effect of QED theory as a physically relevant process with an acceptable interpretation is not only theoretically interesting but also cosmologically crucial. As the gauge field naturally couples to the charged fields, this effect totally changes the original program of the inflationary magnetogenesis scenario: to produce the primordial electromagnetic fields by breaking the conformal invariance and consider the subsequent QED interactions caused by the overproduced electric fields.

With the justification of the anti-screening effect, we point out that spontaneous electromagnetic field generation could be realized even without modifying the QED theory. The homogeneous and constant electric field assumed in our analyses can be naturally produced in the inflationary spacetime where the physical length of a quantum mode is stretched by the cosmic expansion, and eventually the mode exits from the horizon. Needless to say, the produced over-horizon scale electric field is insignificant.

A curious scenario is that the strength of the over-horizon electric field will grow in time due to the instability caused by Maxwell's demon aided by the symmetry-breaking electric field. The growth of the electric field strength continues until it reaches the value of the equilibrium (zero induced current) indicated by the lines in Fig. 4.6. [1] Importantly, the induced current which dy-

[1] To validate the use of the present result for this argument, it is needed to show that the time scale of the backreaction is sufficiently longer than that of the cosmological dynamics, or, Hubble time H^{-1}. From (2.40), we can obtain the backreaction equation for the proper electric field strength in z-direction $E_z \equiv a^{-2}A_z'(\eta)$,

$$(a^2E_z)' = a^2H\frac{\mathrm{d}}{\mathrm{d}a}(a^2E_z) = -ea^3H^3\mathcal{J}_{\mathrm{boson/fermion}}(M, L = \frac{eE_z}{H^2}). \tag{5.1}$$

In the weak electric field regime, the induced current is given by $\mathcal{J} = \sigma_{\mathrm{boson/frmion}}(M)L$. We, then, readily obtain

$$\frac{1}{a^2E_z}\frac{\mathrm{d}}{\mathrm{d}a}(a^2E_z) = -e^2\sigma\frac{1}{a}, \tag{5.2}$$

namically appears during the process towards the equilibrium state can source *magnetic* fields by which the horizon is encircled. We can conjecture that, in the realistic inflatinary model with slowly varying Hubble parameter and the transition to the reheating era, the anti-screening effect of the QED adiabatically produces the electromagnetic fields at the horizon scale in each horizon patch and results in the primordial electromagnetic fluctuations with randomly distributed directions at a certain coherent length.

Since our result is limited to the pure de Sitter, homogeneous and constant background electric field, and neglected effects of and to the magnetic fields, further investigations are required to establish the spontaneous generation scenario of the primordial (electro)magnetic fields by the QED instability. It seems that the understanding of the negative current by Maxwell's demon explained in Sec. 3.5, the semiclassical calculation shown in Sec. 4.5.3, and the approximation method proposed in Appendix E are useful for future research. The correct dynamical evolution of the electromagnetic fields during the inflationary era itself is interesting enough to be studied. The basic treatment of it is shown in [67]. The consideration of the magnetic fields can also be done analytically in the light of [68].

Another application of the anti-screening effect is considered in the context of the anisotropic inflation models [69, 70, 71] where a vector field is coupled to the inflaton field and causes anisotropic expansion of the universe. Very recently, Schwinger effect caused by a vector field background configuration which appears in an anisotropic inflation model was studied in [72].

5.2.2 Physical renormalization scheme dependance

We reasoned out that the renormalization condition (the maximal subtraction scheme) proposed in Sec. 4.5 gives physically more decent predictions than the traditionally employed condition. This, however, could be incompatible with the adiabatic subtraction scheme as we explained in the last part of Sec. 4.5.2. The adiabatic subtraction scheme is also often used in many studies involving QFT in curved spacetime because of its simplicity and wide applicability.

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whose solution is given by $a^2 E_z \sim a^{-e^2\sigma} = \mathrm{e}^{-e^2\sigma H t}$. Therefore, the typical time scale of the backreaction is given by

$$t_{br} = \frac{1}{e^2|\sigma(M)|}\frac{1}{H}. \tag{5.3}$$

The backreaction dynamics can be much slower than the cosmic expansion if (i) the coupling is sufficiently weak, $e \ll 1$, or (ii) the conductivity $\sigma(M)$ is small, $|\sigma| \ll 1$. These conditions are the limitation of our analysis but there exist some parameter regions in which the condition $|\sigma(M)| \ll 1$ is satisfied, see Fig. 4.2.

In [73], it was discussed that cosmologically observable quantities are insensitive to the precise renormalization condition. In contrast to this statement, we revealed that a physical process during the inflation can be affected qualitatively as well as quantitatively by choice of the renormalization condition. There is no doubt at the end of the day that the correct renormalization condition is obtained only experimentally or observationally. Notwithstanding, it also seems true that matching the asymptotic behavior of a physical quantity to its semiclassical estimation should be imposed.

We would like to emphasize that the consideration of the gauge interaction was essential to our understanding about the renormalization. Though want of observations cannot perfectly be satisfied, we can elicit novel phenomena from quantum field theories in curved spacetime commensurate with their complexity when we step into interacting theories.

5.2.3 Correspondance to other fields of research

The search for yet incompletely understood aspect of the theory of quantum gravity and thermodynamics is one of the hottest topics in fundamental physics. As we pointed out in Sec. 3.5, the negative induced current has the thermodynamical interpretation. Therefore, the system we considered serves as an analytical example of the study of the Holography. We considered the massive particle limit in Sec. 4.5.3 which corresponds to vanishing quantum correlation length. It is interesting to take the quantum correlation into account so as to obtain the physical quantities such as the electric current. This will enable us to examine the effect of the correlation between the interior and exterior of the event horizon in an explicit way.

At last, the possible connection between our findings and physics of the scales much closer to ours, i.e., condensed matter physics, should be remarked upon. Actually the negative current and negative Joule heating can be seen in many places such as Peltier device (a heat conveying device) and an osmosis membrane [74]. The negative capacitance [44, 45, 46] is also regarded as a static version of the negative current. All the phenomena mentioned above increases inhomogeneity by consuming energy or dropping entropy. The only difference between dS-QED and desktop devices is the experimental verifiability. The fluid dynamical analog of curved spacetime QFT system proposed, e.g., in [75, 76] has potential to realize the experimental study of the anti-screening effect of dS-QED we found.

5.3 Concluding remark

We analytically investigated dS-QED in background electric fields in response to the research questions we set:

- Does dS-QED have curious phenomena?
- If so, what is the physics behind the phenomena?
- What is their cosmological consequence?

As a result of theoretical exploration, we reached the new prediction of the anti-screening effect in dS-QED explained as a consequence of Maxwell's demon lives on the de Sitter horizon. This leads us to a scenario of the spontaneous electromagnetic fields generation in the inflationary spacetime, which, in principle, can be studied with the effective field theoretical method we explicitly wrote down for dS-QED case. We also proposed a new renormalization condition which imposes the nonperturbative behavior expected by the semiclassical argument on the physical quantity, which directs questions to the validity of the traditional adiabatic regularization scheme and necessitates reconsideration. Not only the cosmological significance of the present work, but also the relation to other topics such as thermodynamics and gravity, quantum correlations, and condensed matter physics were pointed out.

A

Dirac Field in Curved Spacetime

The details about Dirac field in a curved spacetime is described here while exhibiting some usefull formulae.

Commutator of the spin connection and the gamma matrices is directly calculated and reads

$$[\Gamma_\nu, \gamma^\mu] = -\nabla_\nu \gamma^\mu = -(\partial_\nu \gamma^\mu + \Gamma^\mu_{\nu\lambda}\gamma^\lambda), \tag{A.1}$$

which is equivalent to the following condition

$$\nabla'_\nu \gamma^\mu \equiv \partial_\nu \gamma^\mu + \Gamma^\mu_{\nu\lambda}\gamma^\lambda + [\Gamma_\nu, \gamma^\mu] = 0. \tag{A.2}$$

Besides, since $\bar{\psi}\psi$ is a scalar, Leibnitz rule reads $\partial_\mu(\bar{\psi}\psi) = (\nabla_\mu\bar{\psi})\psi + \bar{\psi}\nabla_\mu\psi$, which yields

$$\{\bar{\psi}, \Gamma_\mu\} = \bar{\psi}\Gamma_\mu + \Gamma_\mu\bar{\psi} = 0. \tag{A.3}$$

These are used to show the partial integration rule

$$-\int \sqrt{-g}\,\mathrm{d}^4x \ \left((\nabla_\mu\bar{\psi})e^\mu_a\gamma^a\psi\right) = \int \sqrt{-g}\,\mathrm{d}^4x \ \left(\bar{\psi}e^\mu_a\gamma^a\nabla_\mu\psi\right). \tag{A.4}$$

The probability four current is defined by $j^\mu = \bar{\psi}\gamma^\mu\psi$ and its conservation law holds in the sense of tensorial covariant derivative, i.e.

$$\begin{aligned}
\nabla_\mu j^\mu = j^\mu_{,\mu} + \Gamma^\mu_{\lambda\mu} j^\lambda &= \bar{\psi}_{,\mu}\gamma^\mu\psi + \bar{\psi}\gamma^\mu\psi_{,\mu} + \bar{\psi}\gamma^\mu_{;\mu}\psi \\
&= (\bar{\psi}_{,\mu} - \bar{\psi}\Gamma_\mu)\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\psi_{,\mu} + \Gamma_\mu\psi) \\
&= im\bar{\psi}\gamma^\mu\psi - im\bar{\psi}\gamma^\mu\psi \\
&= 0,
\end{aligned} \tag{A.5}$$

where we have used (A.1) and the equations of motion for the Dirac field (2.29) and (2.30).

B

Spinor Calculation

We need the so-called spin sum formula to determine the normalization condition compatible with the anti-commutation relation $\{\hat{\xi}(\eta,\, \boldsymbol{x}),\, \hat{\pi}(\eta,\, \boldsymbol{y})\}$. Target quantity is

$$\sum_{s=1,\,2} (u_{\boldsymbol{k},\,s} u^{\dagger}_{\boldsymbol{k},\,s} + v_{-\boldsymbol{k},\,s} v^{\dagger}_{-\boldsymbol{k},\,s}). \tag{B.1}$$

Let X be a matrix constructed from the eigenvectors w_s of B defined in (3.11)

$$X \equiv \sum_{s=1,\,2} w_s w_s^{\dagger}. \tag{B.2}$$

X is hermitian, $X^{\dagger} = X$. The completeness condition of the eigenvectors w_s reads $X + \mathcal{C} X^* \mathcal{C}^{\dagger} = \mathbf{1}$. X also satisfies a condition $BX = X$. One can show that the unique representation for X in terms of the gamma matrices is

$$X = \frac{1}{2} + \frac{M}{2r}\gamma^0 + \frac{L}{2r}\gamma^0\gamma^3 = \frac{1}{2}(1 + B). \tag{B.3}$$

The definition of the charge conjugate operator (3.13) leads to $\mathcal{C} X^* \mathcal{C}^{\dagger} = \frac{1}{2}(1 - B)$, and one can again check that the completeness condition is manifestly satisfied.

After some algebra, we reach the following expression,

$$\sum_{s=1,\,2} (u_{\boldsymbol{k},\,s} u^\dagger_{\boldsymbol{k},\,s} + v_{-\boldsymbol{k},\,s} v^\dagger_{-\boldsymbol{k},\,s}) = \left[\zeta'(\zeta^*)' - iF_k(\zeta(\zeta^*)' - \zeta'\zeta^*) + \omega_k^2|\zeta|^2\right] \mathbf{1}, \tag{B.4}$$

where ζ is a shorthand notation for $\zeta^+_{\boldsymbol{k},\,s=1,2}$. Indeed, the suffix s is verbose since $\zeta^+_{\boldsymbol{k},\,s=1} = \zeta^+_{\boldsymbol{k},\,s=2}$. Also F_k is defined as $F_k \equiv \dfrac{\omega_k \omega_k'}{\sigma}$. Using the equation of motion (3.17) and the normalization (3.21) of the mode function, it is straightforward to prove that the quantity inside the large parenthesis [ ] in (B.4) is a constant in time and equals to unity. Therefore we obtain the spin sum formula,

$$\sum_{s=1,\,2} (u_{\boldsymbol{k},\,s} u^\dagger_{\boldsymbol{k},\,s} + v_{-\boldsymbol{k},\,s} v^\dagger_{-\boldsymbol{k},\,s}) = \mathbf{1}, \tag{B.5}$$

which guarantees that the cononical quantization condition (3.29) is compatible with the normalization conditions (3.21), (3.22), and (3.28).

We can express this normalization condition for the mode function ζ in a simpler way by introducing an auxiliary function $\tilde{\zeta} \equiv i(\omega_k^2 - F_k^2)^{-1/2}(\partial_\eta - iF_k)\zeta$ as

$$|\zeta|^2 + |\tilde{\zeta}|^2 = \frac{1}{\omega_k^2 - F_k^2}, \tag{B.6}$$

where $\omega_k^2 - F_k^2 = k^2 - \dfrac{L^2}{r^2} k_z^2$ is a time independent constant. It is straightforward to confirm that $\tilde{\zeta}$ satisfies the equation of motion for $\zeta_{\boldsymbol{k},s=3,4}$ by using $F_k' = \sigma$. The inverse of this transformation is given by $\zeta = i(\omega_k^2 - F_k^2)^{-1/2}(\partial_\eta + iF_k)\tilde{\zeta}$. By making comparison between the asymptotic behaviors of the mode functions, we found

$$\tilde{\zeta}^\pm_{\boldsymbol{k},s=1,2} = \zeta^\pm_{\boldsymbol{k},s=3,4}. \tag{B.7}$$

This result can also be proved explicitly by recurrence relations for the Whittaker's functions.

In our convention, the normalization condition for u, v spinors is expressed as

$$u^\dagger_{\boldsymbol{k},\,s} u_{\boldsymbol{k},\,s'} = v^\dagger_{-\boldsymbol{k},\,s} v_{-\boldsymbol{k},\,s'} = \delta_{s,\,s'}, \tag{B.8}$$

and we can also check the orthogonality condition for $s,\ s' = 1,\ 2$

$$u^\dagger_{\boldsymbol{k},\,s} v_{-\boldsymbol{k},\,s'} = v^\dagger_{-\boldsymbol{k},\,s} u_{\boldsymbol{k},\,s'} = 0. \tag{B.9}$$

Let us also write down a useful formula which is needed in calculation of the expectation value of the current operator (3.31), for $s, s' = 1,\, 2$,

$$w_s^\dagger \gamma^0 \gamma^i w_{s'} = w_s^\dagger \frac{B\gamma^0\gamma^i + \gamma^0\gamma^i B}{2} w_s = \frac{L}{r} \eta^{3i} \delta_{s,\, s'}, \tag{B.10}$$

where we have used $Bw_s = w_s$ for $s = 1,\, 2$ and $B^\dagger = B$.

C

Consistent Adiabatic Expansion for Spinor Mode Function

In order to find the consistent WKB expansion (3.35) for the equation of motion of the spinor mode function (3.33), we should begin with the most primitive WKB-type ansatz given by

$$\zeta = \exp\left(\pm\hbar^{-1}\int^{\eta} \mathrm{d}\eta'(X(\eta') + iY(\eta'))\right), \tag{C.1}$$

where X and Y are real functions to be determined by the equation of motion. Substituting this ansatz into (3.33), we obtain two conditions

$$X^2 - Y^2 \pm \hbar X' = -\omega^2, \quad 2XY \pm \hbar Y' = \hbar\sigma. \tag{C.2}$$

Note that X (Y) has only odd (even) order terms in the power series expansion by $\hbar$, respectively. The latter of (C.2) reads $\hbar^{-1}X = \mp\frac{1}{2}(\ln Y)' + \frac{\sigma}{2Y}$. This enables us to eliminate X from (C.1), which yields

$$\zeta = \frac{\mathcal{N}}{\sqrt{Y}}\exp\left(\pm\frac{i}{\hbar}\int^{\eta} \mathrm{d}\eta'\left(Y - i\hbar\frac{\sigma}{2Y}\right)\right), \tag{C.3}$$

where we have introduced a normalization factor $\mathcal{N}$.

Unfortunately, this expression does not satisfy the normalization condition (3.36) in a nonperturbative way. However, the normalization condition can still be satisfied at each order of the $\hbar$ expansion.

At the zeroth order, the solution is found to be $Y = \omega$. The relation $F_k' = (\frac{\omega\omega'}{\sigma})' = \sigma$ can be used to find an integral $\frac{\sigma}{\omega} = (\ln(\omega + F_k))'$ and then we find that the normalized positive frequency mode is given by

$$\zeta^+|^{(0)} = \sqrt{\frac{\sigma}{2\omega^2(\sigma + \omega')}} \mathrm{e}^{-\frac{i}{\hbar}\int \mathrm{d}\eta' \omega}. \tag{C.4}$$

If we write Y in a power series of $\hbar$ as $Y = \sum_{n=0}^{\infty} \hbar^n \omega^{(n)}$ with $\omega^{(0)} = \omega (= \omega_k(\eta))$ and also rewrite $\dfrac{\sigma}{2Y}$ term in the exponential as $\sqrt{\dfrac{\sigma}{2\omega^2(\sigma+\omega')}} \sum_{n=0}^{\infty} \hbar^n F^{(n)}$ with $F^{(0)} = 1$, we finally obtain the consistent expansion (3.35). This expression gives correct asymptotic behavior of the exact solution (Whittaker functions) as described in (3.21). Moreover, we can find that the zeroth order ansatz for the negative frequency mode function $\zeta^-|^{(0)}$ is given by

$$\zeta^-|^{(0)} = \sqrt{\frac{\sigma}{2\omega^2(\sigma - \omega')}} \mathrm{e}^{+\frac{i}{\hbar}\int \mathrm{d}\eta' \omega}, \tag{C.5}$$

which is slightly different from the positive counterpart.

D

Integration of Whittaker Function

Here we describe the procedure to evaluate the integral (3.31),

$$\langle J^3\rangle = \frac{-2eL}{r}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\Big\{1+i\gamma k_z(\zeta^+\zeta^{+*\prime}-\zeta^{+\prime}\zeta^{+*})-2\left(\omega_k^2-F_k^2+\gamma F_k k_z\right)|\zeta^+|^2\Big\}, \quad \text{(D.1)}$$

with a cutoff Λ in the momentum integral $\int_0^\infty \mathrm{d}k \to \lim_{\Lambda\to\infty}\int_0^\Lambda \mathrm{d}k$. The parameters are given, again, by

$$\gamma \equiv \frac{r}{L}-\frac{L}{r}, \quad F_k \equiv \frac{\omega_k\omega_k'}{\sigma} = aHr - \frac{Lk_z}{r}. \quad \text{(D.2)}$$

The procedure is similar to the previous works [37, 39, 41] but involves much more complexity.

In the cylindrical coordinates $(k, x=\cos\theta, \phi)$, the equation above reads

$$\begin{aligned}&\frac{-2eL}{r}\lim_{\Lambda\to\infty}\int_0^\Lambda\frac{\mathrm{d}k}{2\pi}k^2\int_{-1}^1\frac{\mathrm{d}x}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\\&\quad\Big\{1+i\gamma kx(\zeta^+\zeta^{+*\prime}-\zeta^{+\prime}\zeta^{+*})-2\left((1-x^2)k^2+aHr\gamma kx\right)|\zeta^+|^2\Big\},\end{aligned} \quad \text{(D.3)}$$

and the first trivial term gives a divergent contribution

$$\frac{-2eL}{r}\lim_{\Lambda\to\infty}\int_0^\Lambda\frac{\mathrm{d}k}{2\pi}\int_{-1}^{1}\frac{\mathrm{d}x}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}k^2=\lim_{\Lambda\to\infty}\frac{-2eL\Lambda^3}{6\pi^2 r}. \tag{D.4}$$

The positive frequency mode function ζ^+ defined in (3.21) is, again, given by

$$\zeta_{\boldsymbol{k}}^+(\eta)=\frac{\mathrm{e}^{\frac{\pi}{2}Lx}}{\sqrt{2k}}\sqrt{\frac{r}{r-Lx}}W_{-iLx,\,\frac{1}{2}+ir}\left(-2i\frac{k}{aH}\right). \tag{D.5}$$

For the remaining part, we exploit the Mellin-Barnes type integral representation of the Whittaker function $W_{\kappa,\mu}(z)$:

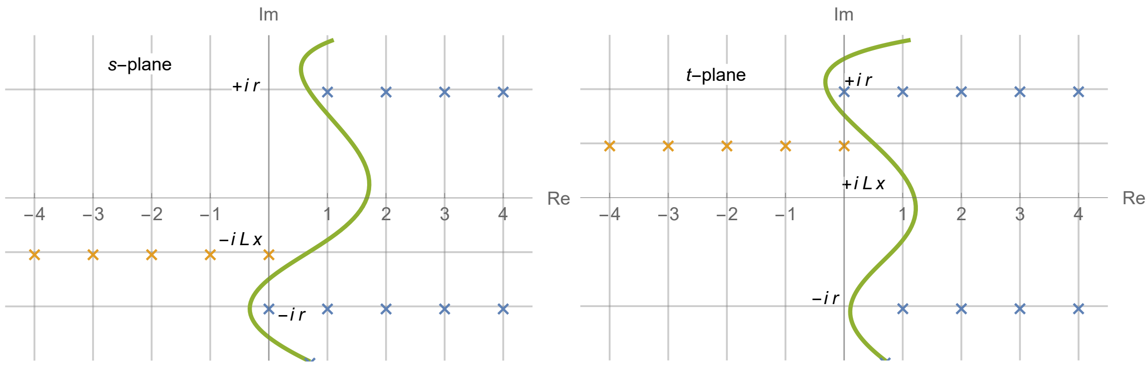
$$W_{\kappa,\mu}(z)=\int_{C_s}\frac{\mathrm{d}s}{2\pi i}z^s\mathrm{e}^{-z/2}\frac{\Gamma(s-\kappa)\Gamma(-s-\mu+\frac{1}{2})\Gamma(-s+\mu+\frac{1}{2})}{\Gamma(\frac{1}{2}-\kappa-\mu)\Gamma(\frac{1}{2}-\kappa+\mu)}, \tag{D.6}$$

where the contour C_s runs from $-i\infty$ to $i\infty$ and is taken to separate the poles of the gamma function $\Gamma(s-\kappa)$, which is located at $s=\kappa-n$ $(n=0,1,2,\cdots)$, from the ones of $\Gamma(-s-\kappa-\mu+\frac{1}{2})\Gamma(-s-\kappa+\mu+\frac{1}{2})$ at $s=n-\kappa\pm\mu+\frac{1}{2}$. Using the complex conjugation nature $(W_{\kappa,\mu}(z))^*=W_{\kappa^*,\mu^*}(z^*)$, a derivative rule $\frac{\mathrm{d}}{\mathrm{d}z}W_{\kappa,\lambda}(z)=\left(\frac{1}{2}-\frac{\kappa}{z}\right)W_{\kappa,\lambda}(z)-\frac{1}{z}W_{1+\kappa,\lambda}(z)$, and the reflection formula for gamma function, the integral is rewritten as

$$\begin{aligned}&-2eL\lim_{\Lambda\to\infty}\int_0^\Lambda\mathrm{d}k\int_{-1}^{1}\mathrm{d}x\int_{C_s}\frac{\mathrm{d}s}{2\pi i}\int_{C_t}\frac{\mathrm{d}t}{2\pi i}\frac{\mathrm{e}^{\pi Lx}}{4\pi^2}\,\mathrm{e}^{\frac{\pi i}{2}(t-s)}\Gamma(s+iLx)\Gamma(-s-ir)\Gamma(-s+ir+1)\\&\times\Gamma(t-iLx)\Gamma(-t+ir)\Gamma(-t-ir+1)\frac{\sinh\pi(r-Lx)\sinh\pi(r+Lx)}{\pi^2(r+Lx)}\left(\frac{2k}{aH}\right)^{s+t}\\&\times\left\{(x^2+\gamma x-1)k^2-aH\gamma(r+Lx)(1+\frac{1}{2}\left(\frac{1+i(r-Lx)}{s+iLx-1}+\frac{1-i(r-Lx)}{t-iLx-1}\right))kx\right\}.\end{aligned} \tag{D.7}$$

The integration contours C_s and C_t run from $-i\infty$ to $+i\infty$. C_s sees the poles at $s=-iLx-n$ $(n=0,1,2,\cdots)$ on the left and the ones at $s=-ir+n$, $ir+1+n$ on the right. C_t sees the poles at $t=+iLx-n$ on the left and the ones at $t=ir+n$, $-ir+1+n$ on the right. See Fig. D.1.

Since C_s and C_t can be taken to ensure $\Re(s+t)>0$, we can perform the k-integral explicitly to find $\mathcal{O}(\Lambda^{(s+t+2)})$ and $\mathcal{O}(\Lambda^{(s+t+3)})$ terms. At this moment, the k-integration produces other poles on complex t-plane at $t=-s-2,-s-3$. We then perform the t-integral with closing the integration path positively (by a counterclockwise path). The residues from $t=iLx-m$, $(m=$



C_s C_t

Figure D.1: Graphs of the contours C_s and C_t (green lines) on the complex s, t-plane. Additional poles $s+t=-2$ and $s+t=-3$ appear when the k-integration is done. As one can see, the contours can be deformed to meet the condition $\Re(s+t)>0$. This condition ensures that the poles at $t=-s-2,-s-3$ are on the left side of the contour C_t and contributing to the t-integration.

$0,1,2,\cdots)$ and $t=-s-2,-s-3$ can contribute to the integral. Note that the contributions from $m\geq 4$ poles will vanish after taking the limit $\Lambda\to\infty$.

The s-integral can be similarly done for the contributions from the ploes at $t=iLx-m$ with $m=0,1,2,3$, and remains only the residues from the poles at $s=-iLx-n$ with $n=0,1,2,3$. The non-vanishing contribution is calculated as

$$-2eL\Big(-\frac{\Lambda^3}{6\pi^2 r}+\frac{aH}{6\pi^2}\Lambda^2+0\times(aH)^2\Lambda^1-\frac{(aH)^3}{6\pi^2}\ln\left(\frac{2\Lambda}{aH}\right)+(aH)^3\mathcal{O}(\Lambda^0)\Big). \tag{D.8}$$

We see the cancelation of the Λ^3 divergence here and in the trivial part (D.4). The finite part ($\mathcal{O}(\Lambda^0)$ in (D.8)) is given by

$$\begin{aligned}\mathcal{O}(\Lambda^0)=&\frac{\gamma_E}{6\pi^2}-\frac{23}{144\pi^2}-\frac{7L^2}{120\pi^2}+\frac{L^4}{420\pi^2}-\frac{23M^2}{192\pi^2}-\frac{3M^2}{4\pi^2L^2}+\frac{L^2M^2}{1440\pi^2}-\frac{5M^4}{576\pi^2}\\&-\frac{3M^2r}{8\pi^2L^3}\log\left(\frac{r-L}{r+L}\right)+i\left(\frac{1}{12\pi}-\frac{1}{2\pi^2r}+\frac{121L^2}{216\pi^2r}-\frac{91L^4}{1440\pi^2r}+\frac{L^6}{2016\pi^2r}-\frac{65r}{144\pi^2}\right.\\&\left.+\frac{89rL^2}{1440\pi^2}-\frac{rL^4}{1120\pi^2}-\frac{41rM^2}{576\pi^2}-\frac{rM^2L^2}{1440\pi^2}-\frac{rM^4}{576\pi^2}\right)\\&+\frac{1}{8\pi^2}\int_{-1}^{1}\mathrm{d}x(1+r^2-(1+3L^2+3r^2)x^2+5L^2x^4)(\psi(iLx-ir)+\psi(iLx+ir)),\end{aligned} \tag{D.9}$$

where $\psi(z)=(\ln\Gamma(z))'$ denotes the digamma function and γ_E is the Euler constant. The x-integral cannot be expressed in terms of simpler functions, but it is real since the imaginary part of the digamma function is given by $2\Im\psi(iy) = 1/y + \pi\coth(\pi y)$.

The remining part (residues from $t = -s-2, -s-3$) is calculated as

$$-2eL(aH)^3 \int_{-1}^{1} \mathrm{d}x \int_{C_s} \frac{\mathrm{d}s}{2\pi i} \frac{\mathrm{e}^{-i\pi s}\mathrm{e}^{\pi Lx} \sinh(\pi(r-Lx))\sinh(\pi(r+Lx))}{\sin(\pi(s+iLx))\sin(\pi(s-ir))\sin(\pi(s+ir))} f(s), \tag{D.10}$$

where $f(s)$ is meromorphic, and has only single poles located at $s = -iLx+1, -iLx, -iLx-1, -iLx-2, -iLx-3$. We further introduce a function

$$\begin{aligned} g(s) = {} & b_3(s+iLx)^3 + b_2(s+iLx)^2 + b_1(s+iLx) \\ & + \frac{c_0}{s+iLx} + \frac{c_1}{s+iLx+1} + \frac{c_2}{s+iLx+2} + \frac{c_3}{s+iLx+3} \end{aligned} \tag{D.11}$$

to express $f(s)$ as

$$f(s) = g(s) - g(s+1) + \frac{d}{s+iLx-1}. \tag{D.12}$$

Of course, all the coefficients b_i, c_i, and d have no s-dependence.

The shift of the contour $s \to s-1$ does not change the coefficient in front of $f(s)$ in (D.10), then we find that the $(g(s) - g(s-1))$ part of (D.10)

$$\int_{C_s} \frac{\mathrm{d}s}{2\pi i} \cdots (g(s) - g(s-1)) = \left(\int_{C_s} - \int_{C_{s-1}} \right) \frac{\mathrm{d}s}{2\pi i} \cdots g(s), \tag{D.13}$$

is given by the sum of the residues of the poles between C_s and C_{s-1}, i.e. $s = -ir-1$, $s = ir$, and $s = -iLx+1$.

The contributing poles of the d-term of (D.10),

$$\int_{C_s} \frac{\mathrm{d}s}{2\pi i} \frac{\mathrm{e}^{-i\pi s}\mathrm{e}^{\pi Lx} \sinh(\pi(r-Lx))\sinh(\pi(r+Lx))}{\sin(\pi(s+iLx))\sin(\pi(s-ir))\sin(\pi(s+ir))} \frac{d}{s+iLx-1}, \tag{D.14}$$

are $s = -iLx+2+n$, $s = ir+n+1$, and $s = -ir+n$ ($n = 0, 1, 2, \cdots$). These poles are negatively encircled by the integration path (C_s+(semicircle on the right half plane)), and the residue theorem gives

$$\begin{aligned} -2eL(aH)^3 \int_{-1}^{1} \mathrm{d}x \frac{d}{\pi} \sum_{n=0}^{\infty} \Bigg\{ & -\frac{1}{1+n} + \frac{\mathrm{e}^{\pi(r+Lx)}}{n+i(r+Lx)} \frac{\sinh(\pi(r-Lx))}{\sinh(2\pi r)} \\ & + \frac{\mathrm{e}^{-\pi(r-Lx)}}{n-1-i(r-Lx)} \frac{\sinh(\pi(r+Lx))}{\sinh(2\pi r)} \Bigg\}. \end{aligned} \tag{D.15}$$

Each of the sums $\sum_{n=0}^{\infty} \frac{1}{n+\alpha}$ $(\alpha \neq 0)$ seems divergent, however, they are indeed finite. Because, using a series formula for the digamma function, one can show

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left\{ -\frac{1}{1+n} + \frac{\mathrm{e}^{\pi(r+Lx)}}{n+i(r+Lx)} \frac{\sinh(\pi(r-Lx))}{\sinh(2\pi r)} + \frac{\mathrm{e}^{-\pi(r-Lx)}}{n-1-i(r-Lx)} \frac{\sinh(\pi(r+Lx))}{\sinh(2\pi r)} \right\} \\
&= \sum_{n=0}^{\infty} \left\{ \mathrm{e}^{\pi(r+Lx)} \frac{\sinh(\pi(r-Lx))}{\sinh(2\pi r)} \left(\frac{1}{n+i(r+Lx)} - \frac{1}{n+1} \right) \right. \\
&\qquad\qquad \left. + \mathrm{e}^{-\pi(r-Lx)} \frac{\sinh(\pi(r+Lx))}{\sinh(2\pi r)} \left(\frac{1}{n-1-i(r-Lx)} - \frac{1}{n+1} \right) \right\} \\
&= \mathrm{e}^{\pi(r+Lx)} \frac{\sinh(\pi(r-Lx))}{\sinh(2\pi r)} (-\gamma_E - \psi(ir+iLx)) \\
&\qquad\qquad + \mathrm{e}^{-\pi(r-Lx)} \frac{\sinh(\pi(r+Lx))}{\sinh(2\pi r)} (-\gamma_E - \psi(-1-ir+iLx)).
\end{aligned} \tag{D.16}$$

This result is also achieved by just applying the (Hurwitz's type) ζ-function regularization technique to each of the sum.

The contribution of (D.10) is then given by $-2eL(aH)^3$ times

$$\begin{aligned}
&-\frac{\gamma_E}{6\pi^2} + \frac{37}{144\pi^2} - \frac{L^2}{120\pi^2} - \frac{L^4}{420\pi^2} + \frac{7M^2}{192\pi^2} + \frac{3M^2}{8\pi^2 L^2} - \frac{L^2 M^2}{1440\pi^2} + \frac{5M^4}{576\pi^2} \\
&+ \frac{3M^2 r}{16\pi^2 L^3} \log\left(\frac{r-L}{r+L}\right) - i\big(\text{imaginary part of (D.9)}\big) \\
&- \frac{r}{48\pi^5 L^2 \sinh(2\pi r)} \Big\{ (45 - \pi^2(11 - 12L^2 + 8r^2)) \cosh(2\pi L) \\
&\qquad\qquad - (45 - \pi^2(11 - 72L^2 + 8r^2)) \frac{\sinh(2\pi L)}{2\pi L} \Big\} \\
&+ \frac{3rM^2}{32\pi^2 L^3 \sinh(2\pi r)} \sum_{s=\pm} s\mathrm{e}^{2\pi r s} (\mathrm{Ei}(2\pi s(r+L)) - \mathrm{Ei}(2\pi s(r-L))) \\
&- \Re\Big[\int_{-1}^{1} \mathrm{d}x \frac{(1 + r^2 - (1 + 3L^2 + 3r^2)x^2 + 5L^2 x^4)}{16\pi^2 \sinh(2\pi r)} \\
&\qquad \times ((\mathrm{e}^{2\pi Lx} - \mathrm{e}^{2\pi r})\psi(i(Lx+r)) - (\mathrm{e}^{2\pi Lx} - \mathrm{e}^{-2\pi r})\psi(i(Lx-r)))\Big].
\end{aligned} \tag{D.17}$$

Finally, (D.8), (D.9) and (D.17) together yield (3.39).

E

Attempt to Effective Field Theoretical Approach

THE EXACT ANALYSIS IN THE PREVIOUS CHAPTERS HELPED US MAKE STRIDES IN THE STUDY OF THE QUANTUM PHYSICS IN CURVED SPACETIME. To extend our analysis to more complicated systems, it is to be hoped that we establish an approximation method which is capable to correctly capture the nonperturbative effects of the field theory. There are many possible candidates, and in this appendix, we will advocate the use of the nonperturbative renormalization group (NPRG) technique [77, 78, 79] combined with the effective field theoretical approach as a truncation method. We will review the basics of the method, which was originally developed in the flat spacetime, in the general curved spacetime employing the scalar QED as one of the simplest examples of the gauge theories.

In Sec. E.1, we will clarify our purpose and show the outline of the method. In Sec. E.2, we will follow the derivation of the main equation, called the flow equation, while amending details necessitated by the general curved spacetime background. Finally, we will put the method to trial employing a simple example so as to obtain physical results in Sec. E.3.

E.1 Overview of the strategy

Before the detailed explanation, we will explain the general view of the NPRG method in this section.

E.1.1 Main goal

All the relevant information of a quantum field theory is encoded in effective action Γ, which is a functional of the classical field variables. As we have already understood, flat spacetime physics can describe the strong field regime, so our focus will mainly be on the weak field regime where curvature effect is significant. Since we treated the gauge field as a background, the method we are going to establish is desired to be flexible enough to include the general electromagnetic field configurations and its dynamics.

Our purpose is, hence, to find the effective action $\Gamma[\phi, \phi^\dagger; A_\mu]$ of the scalar QED in de Sitter spacetime with a general but weak background electromagnetic field A_μ. More precisely, we aim to investigate the electromagnetic response of the charged scalar field in inflationary universe and to calculate electric conductivity σ and magnetic susceptibility χ as functions of the strength of electromagnetic field and mass of the charged carriers.

E.1.2 Sketch of central idea

We will exploit the NPRG technique in which an IR cutoff κ is used to control the IR behavior. An IR-cutoff effective potential $\Gamma_\kappa[\phi, \phi^\dagger; A_\mu]$ and its flow equation,

$$\partial_\kappa \Gamma_\kappa = \partial_\kappa \mathrm{Tr} \log(\Gamma_\kappa^{(2)} + \Gamma_{\mathrm{gf}} + R_\kappa), \tag{E.1}$$

originally found by Wetterich, play the leading part. Here, $\Gamma_\kappa^{(2)}$ is the second functional derivatives of Γ_κ itself with respect to the fields $\psi = (\phi, \phi^\dagger, A_\mu)$, Γ_{gf} is a gauge fixing term in the action of the theory and R_κ term introduces the momentum dependent masses for the IR modes of the fields. The momentum dependent mass term R_κ vanishes for the full quantum limit $\kappa \to 0$. The cutoff effective action reaches the full effective action in this limit,

$$\Gamma[\phi, \phi^\dagger; A_\mu] = \lim_{\kappa \to 0} \Gamma_\kappa[\phi, \phi^\dagger; A_\mu]. \tag{E.2}$$

Initial condition for the Wetterich equation (E.1) is given by the classical action of the theory S, so we have a condition $\lim_{\kappa\to\infty}\Gamma_\kappa \to S$. Further clarification including the meaning of the formulae and quantities is given in the next section.

What is important is the functional flow equation (Wetterich equation) (E.1) is *exact* at this moment. Thus, all the quantum information (about the momentum scale larger than the cutoff $p > \kappa$) is contained in the solution Γ_κ. Moreover, the functional trace Tr appears in the flow equation is involved with the momentum integration. The cutoff term R_κ can be chosen to make the momentum integration finite in both UV/IR region. Accordingly, we can at least numerically solve the flow equation starting from the classical action S with the bare parameters to obtain the quantum action Γ.

E.1.3 Procedure

Our strategy is summarized as follows:

1. Derive the Wetterich equation (E.1) again in general curved spacetime by following previous works.

2. Pose an ansatz for the form of the cutoff effective action Γ_κ by means of the symmetries such as the general covariance and gauge invariance. We also neglect higher derivatives of the field because we only consider nearly constant and homogeneous fields. We will use the following ansatz,

$$\begin{aligned}\Gamma_\kappa = \int \mathrm{d}^D x\sqrt{-g}\Big\{ &-\frac{\mathcal{Z}_{F,\kappa}}{4}F^{\mu\nu}F_{\mu\nu} - \mathcal{Z}_{\phi,\kappa}g^{\mu\nu}(D_\mu\phi)^\dagger(D_\nu\phi) \\ &- \mathcal{U}_\kappa(\phi^\dagger\phi) + \frac{1}{2}F_{\mu\nu}\mathcal{M}^{\mu\nu}_\kappa(\phi^\dagger\phi)\Big\},\end{aligned} \tag{E.3}$$

   where $D_\mu = \nabla_\mu + ieA_\mu$ is the covariant derivative. $\mathcal{Z}$ factors are the usual field renormalizations but can depend on the cutoff κ. $\mathcal{U}_\kappa$ is the potential of the scalar field. $\mathcal{M}^{\mu\nu}_\kappa$ is the magnetization-polarization tensor which can also be depend on κ. The last term $F_{\mu\nu}M^{\mu\nu}$ is considered to describe the amount of deformation of the effective scalar potential.

3. Put the ansatz (E.3) into the flow equation (E.1) and solve it with an appropriate parametrization of the potential $\mathcal{U}_\kappa$ and the magnetization-polarization tensor $\mathcal{M}^{\mu\nu}_\kappa$ to find the effective action $\Gamma = \Gamma_{\kappa\to 0}$. The components of the magnetization-polarization tensor $\mathcal{M}^{\mu\nu} = \mathcal{M}^{\mu\nu}_\kappa|_{\kappa\to 0}$ correspond to the polarization $P^i = -\mathcal{M}^{0i} = \mathcal{M}^{i0}$ and the magnetization $M^i = \frac{1}{2}\epsilon^{ijk}\mathcal{M}_{jk}$.

4. Find potential minimum $\phi = \phi_0$ by solving the equation of motion

$$\frac{1}{\sqrt{-g}} \frac{\delta \Gamma}{\delta \phi} = 0. \tag{E.4}$$

5. Finally, we obtain the electric current J^μ and electromagnetic displacement tensor given by $\mathcal{D}^{\mu\nu}$

$$J^\mu = \frac{1}{\sqrt{-g}} \frac{\delta \Gamma}{\delta A_\mu}\bigg|_{\phi=\phi_0}, \quad \mathcal{D}^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta \Gamma}{\delta F_{\mu\nu}}\bigg|_{\phi=\phi_0}, \tag{E.5}$$

where components of the electromagnetic displacement tensor represent the electric displacement vector $D^i = E^i + P^i = \mathcal{D}^{0i} = -\mathcal{D}^{i0}$ and the magnetic field $H^i = B^i - M^i = \frac{1}{2}\epsilon^{ijk}\mathcal{D}_{jk}$. In our convention, E and B fields are given by $E^i = F^{0i} = -F^{i0}$ and $B^i = \epsilon^{ijk} F_{jk}$, respectively.

The quantities obtained by the last step carry information of the electromagnetic response of the system. For linear response regime (both E and B fields are weak), we introduce the electric conductivity $\sigma^i{}_j$, the electric permittivity $\varepsilon^i{}_j$ and the permeability $\mu^i{}_j$ which relate E and B fields to D and H fields,

$$j^i = \sigma^i{}_j E^j, \quad D^i = \varepsilon^i{}_j E^j, \quad B^i = \mu^i{}_j H^j. \tag{E.6}$$

E.2 Derivation of Wetterich equation in curved spacetime

In this section, we review the derivation of the renormalization group flow equation or Wetterich equation (E.1) following the early-stage works on Wetterich equation for Yang-Mills theory [77, 78, 79] and the recent work [80] on the pure scalar field theory in de Sitter spacetime in which closed time path (in-in) formalism [81] was employed to extend. See also [82] for a review of the subject. We combine these works to derive Wetterich equation for scalar quantum electrodynamics (QED) in de Sitter spacetime.

E.2.1 Effective action

The metric of $D = d + 1$ dimensional de Sitter spacetime is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a(\eta)^2(-\mathrm{d}\eta^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_d^2), \tag{E.7}$$

where η is the conformal time, defined as $a\mathrm{d}\eta = \mathrm{d}t$, and $a(\eta) = \mathrm{e}^{Ht} = -(H\eta)^{-1}$ is the scale factor. The scalar QED action S is given in (2.39). This action is explicitly invariant under the gauge transformation of scalar fields $\phi \to \mathrm{e}^{-ie\alpha}\phi$, $\phi^\dagger \to \mathrm{e}^{ie\alpha}\phi^\dagger$ and gauge field $A_\mu \to A_\mu + \alpha_{,\mu}$.

We will briefly summarize the definition of the effective action and related concepts clarifying our convention and notation. The generating functional Z and 1-PI connected generated functional W are defined by

$$
\begin{aligned}
Z[J^\dagger, J, K^\mu] &= \mathrm{e}^{iW[J^\dagger, J, K^\mu]} \\
&= \int \mathcal{D}\phi\mathcal{D}\phi^\dagger\mathcal{D}A \, \exp\left\{iS[\phi,\phi^\dagger,A_\mu] + iS_{\mathrm{gf}}[A_\mu]\right. \\
&\qquad \left. + i\int \mathrm{d}^D x\sqrt{-g}(J^\dagger\phi + \phi^\dagger J + K^\mu A_\mu)\right\},
\end{aligned}
\tag{E.8}
$$

where $J, J^\dagger, K^\mu$ are the source functions and $S_{\mathrm{gf}}[A_\mu] = \frac{1}{2\alpha}\int \mathrm{d}^D x\sqrt{-g}(A_\mu^{;\mu})^2$ denotes the gauge fixing term. We hereafter use slim notations for the spacetime integrals, $\int_x = \int \mathrm{d}^D x\sqrt{-g(x)}$, $\int_{xy} = \int \mathrm{d}^D x \mathrm{d}^D y\sqrt{-g(x)}\sqrt{-g(y)}$ and so on.

The effective action Γ is defined by the following Legendre transformation,

$$
\Gamma[\phi_{\mathrm{cl}}, \phi_{\mathrm{cl}}^\dagger, A_{\mathrm{cl}\,\mu}] = W[J^\dagger, J, K^\mu] - \int_x (J^\dagger\phi_{\mathrm{cl}} + \phi_{\mathrm{cl}}^\dagger J + K^\mu A_{\mathrm{cl}\,\mu}).
\tag{E.9}
$$

The classical fields are defined by functional derivatives of W,

$$
\begin{aligned}
&\frac{1}{\sqrt{-g(x)}}\frac{\delta W}{\delta J^\dagger(x)} = \phi_{\mathrm{cl}}(x), \quad \frac{1}{\sqrt{-g(x)}}\frac{\delta W}{\delta J(x)} = \phi_{\mathrm{cl}}^\dagger(x), \\
&\frac{1}{\sqrt{-g(x)}}\frac{\delta W}{\delta K^\mu(x)} = A_{\mathrm{cl}\,\mu}(x).
\end{aligned}
\tag{E.10}
$$

Functional derivatives of the effective action Γ with respect to the classical fields give the sources, for example,

$$
\begin{aligned}
&\frac{1}{\sqrt{-g(x)}}\frac{\delta\Gamma}{\delta\phi_{\mathrm{cl}}(x)} \\
&= \int_y \frac{1}{\sqrt{-g(y)}}\frac{1}{\sqrt{-g(x)}}\left(\frac{\delta W}{\delta J^\dagger(y)}\frac{\delta J^\dagger(y)}{\delta\phi_{\mathrm{cl}}(x)} + \frac{\delta W}{\delta J(y)}\frac{\delta J(y)}{\delta\phi_{\mathrm{cl}}(x)} + \frac{\delta W}{\delta K^\mu(y)}\frac{\delta K^\mu(y)}{\delta\phi_{\mathrm{cl}}(x)}\right) \\
&- J^\dagger(x) - \int_y \frac{1}{\sqrt{-g(x)}}\left(\frac{\delta J^\dagger(y)}{\delta\phi_{\mathrm{cl}}(x)}\phi_{\mathrm{cl}}(y) + \frac{\delta J(y)}{\delta\phi_{\mathrm{cl}}(x)}\phi_{\mathrm{cl}}^\dagger(y) + \frac{\delta K^\mu(y)}{\delta\phi_{\mathrm{cl}}(x)}A_{\mathrm{cl}\,\mu}(y)\right) \\
&= -J^\dagger(x),
\end{aligned}
\tag{E.11}
$$

where we have used the definitions (E.10) to see the first term and the third term cancel each other. Similarly, we obtain

$$\frac{1}{\sqrt{-g(x)}}\frac{\delta\Gamma}{\delta\phi_{\rm cl}(x)^\dagger} = -J(x), \quad \frac{1}{\sqrt{-g(x)}}\frac{\delta\Gamma}{\delta A_{{\rm cl}\,\mu}(x)} = -K^\mu(x). \tag{E.12}$$

E.2.2 Closed time path formalism

The path integral in (E.8) corresponds to in-out formalism if the integration path on the exponential factor is straightly taken from the remote past $t = -\infty$ (or, in terms of the conformal time, $\eta = -\infty$) to the infinite past $t = \infty$ ($\eta = 0$). Resulting expectation values obtained by functional derivatives of the generating function Z will be elements such as $\langle 0_{\rm out}|\hat{\mathcal{O}}|0_{\rm in}\rangle$. These states $|0_{\rm in}\rangle$ and $|0_{\rm out}\rangle$ are the vacuum states at the initial ($\eta = -\infty$) and final ($\eta = 0$) time, respectively. These vacuum states are not identical to each other in general.

A simple technique to obtain the in-in values like $\langle 0_{\rm in}|\hat{\mathcal{O}}|0_{\rm in}\rangle$ is to bend the path of the time integration. In the closed time path formalism [81], the time contour is taken on $C = C_+ \cup C_- = \{\eta + i\epsilon | \eta \in (-\infty, 0)\} \cup \{\eta - i\epsilon | \eta \in (-\infty, 0)\}$. η runs from $-\infty$ to 0 on C_+ but it turns back from 0 to $-\infty$ on C_-. As a result, the contour C wraps the negative real axis of the complex η-plane.

We always employ the closed time path without any notice in the rest of the present appendix. Any special functions (e.g., delta function, step function) involved with time ordering are defined along the contour C.

E.2.3 Effective average action

Here, we introduce the infrared momentum cut off κ by adding cutoff terms $\Delta_\kappa S_S$ and $\Delta_\kappa S_G$ which serve as momentum dependent masses of the scalar and gauge fields. These terms give larger masses for IR modes of the quantum fluctuations and do not affect the UV modes. We also introduce the background formalism. The gauge field A_μ is divided into background part $\bar{A}_\mu$ and the fluctuation part a_μ. The generated functionals with IR cutoff and background gauge field are then defined as

$$\begin{aligned} Z_\kappa[J^\dagger, J, K^\mu; \bar{A}_\mu] &= {\rm e}^{iW_\kappa[J^\dagger, J, K^\mu; \bar{A}_\mu]} \\ &= \int \mathcal{D}\phi\mathcal{D}\phi^\dagger\mathcal{D}a\ \exp\Big\{iS[\phi,\phi^\dagger,\bar{A}_\mu + a_\mu] + iS_{\rm gf}[a_\mu] \\ &\qquad + i\Delta_\kappa S_S[\phi,\phi^\dagger;\bar{A}_\mu] + i\Delta_\kappa S_G[a_\mu] + i\int_x (J^\dagger\phi + \phi^\dagger J + K^\mu a_\mu)\Big\}. \end{aligned} \tag{E.13}$$

The definition of the effective action will also be involved with the cutoff terms. The effective average action Γ_κ is defined by

$$\begin{aligned}\Gamma_\kappa[\phi_{\rm cl},\phi^\dagger_{\rm cl},a_{{\rm cl}\,\mu};\bar{A}_\mu] = W_\kappa[J^\dagger,J,K^\mu;\bar{A}_\mu] - \int_x (J^\dagger\phi_{\rm cl}+\phi^\dagger_{\rm cl}J+K^\mu a_{{\rm cl}\,\mu}) \\ - \Delta_\kappa S_S[\phi_{\rm cl},\phi^\dagger_{\rm cl};\bar{A}_\mu] - \Delta_\kappa S_G[a_{{\rm cl}\,\mu}],\end{aligned} \tag{E.14}$$

here the classical functions are defined by functional variations of W_κ instead of W in (E.10),

$$\begin{aligned}&\frac{1}{\sqrt{-g(x)}}\frac{\delta W_\kappa}{\delta J^\dagger(x)} = \phi_{\rm cl}(x), \quad \frac{1}{\sqrt{-g(x)}}\frac{\delta W_\kappa}{\delta J(x)} = \phi^\dagger_{\rm cl}(x), \\ &\frac{1}{\sqrt{-g(x)}}\frac{\delta W_\kappa}{\delta K^\mu(x)} = a_{{\rm cl}\,\mu}(x).\end{aligned} \tag{E.15}$$

Note that the arguments inside the cutoff functions in (E.14) are the classical fields.

Though we do not specify the particular form of the cutoff terms, we still impose some conditions on them. First of all, the cutoff terms must vanish as cutoff κ reaches to 0,

$$\lim_{\kappa\to 0}\Delta_\kappa S_{S,G} = 0, \tag{E.16}$$

so that we can recover the normal effective action in the $\kappa \to 0$ limit, or $\Gamma_{\kappa\to 0}|_{\bar{A}=0} = \Gamma$. The second condition is that the cutoff terms should suppress the quantum fluctuations of the IR modes with physical momentum $p \ll \kappa$ and leave the modes with $p \gg \kappa$ unaffected. This is done by giving momentum-dependent masses to the scalar and the gauge fields. The cutoff terms are chosen to be quadratic. The third requirement is concerning about the UV limit ($\kappa \to \infty$). The cutoff terms are supposed to remove all contributions to the path integral except for that from classical trajectory. Therefore we impose a condition

$$\Gamma_{\kappa\to\infty}[\phi_{\rm cl},\phi^\dagger_{\rm cl},a_{{\rm cl}\,\mu};\bar{A}_\mu] = S[\phi_{\rm cl},\phi^\dagger_{\rm cl},\bar{A}_\mu + a_{{\rm cl}\,\mu}]. \tag{E.17}$$

This condition provides the initial condition of the flow equation as we mentioned.

We further introduce collective field notation. We define a collective field $\psi_m = (\phi,\phi^\dagger,a_\mu)$ and a collective source $\mathcal{J}^m = (J^\dagger,J,K^\mu)$. The standard effective action is expressed as

$$\tilde{\Gamma}_\kappa[\psi_{{\rm cl}\,m};\bar{A}_\mu] = W_\kappa[\mathcal{J}^m;\bar{A}_\mu] - \int_x \psi_{{\rm cl}\,m}\mathcal{J}^m. \tag{E.18}$$

Similar to (E.10), functional derivative of W_κ with respect to the source gives the classical field, we can write it as

$$\frac{1}{\sqrt{-g(x)}}\frac{\delta W_\kappa}{\delta \mathcal{J}^m(x)} = \psi_{\mathrm{cl},\,m}(x). \tag{E.19}$$

It is straightforward to show

$$\frac{1}{\sqrt{-g(x)}}\frac{\delta \tilde{\Gamma}_\kappa}{\delta \psi_{\mathrm{cl},\,m}(x)} = -\mathcal{J}^m(x). \tag{E.20}$$

The cutoff terms can be taken as follows

$$\begin{aligned} &\Delta_\kappa S_S[\phi_{\mathrm{cl}}, \phi_{\mathrm{cl}}^\dagger; \bar{A}_\mu] + \Delta_\kappa S_G[a_{\mathrm{cl}\,\mu}] \\ &\qquad = \frac{1}{2}\int_{x,\,y} \psi_{\mathrm{cl},\,m}(x) R_\kappa^{mn}(x-y;\bar{A}_\mu)\psi_{\mathrm{cl},\,n}(y), \end{aligned} \tag{E.21}$$

where $R_\kappa^{mn}(x-y;\bar{A}_\mu)$ is an arbitrary regulator function which can depend on the cutoff scale κ and background gauge $\bar{A}_\mu$.

A notion of the gauge transformation is in order. In our version of the gauge transformation, the background field $\bar{A}_\mu$ transforms while the fluctuation $a_{\mathrm{cl}\,\mu}$ is unchanged. It assures the invariance of the resulting effective action $\Gamma_\kappa[\phi_{\mathrm{cl}}, \phi_{\mathrm{cl}}^\dagger, a_{\mathrm{cl}\,\mu}; \bar{A}_\mu]$ at $\kappa = 0$. A new gauge field defined by $A_\mu \equiv a_{\mathrm{cl}\,\mu} + \bar{A}_\mu$ also transforms under the same gauge transformation. We define an average effective action depending on two gauge fields A_μ and $\bar{A}_\mu$

$$\Gamma_\kappa[\phi_{\mathrm{cl}}, \phi_{\mathrm{cl}}^\dagger, A_\mu, \bar{A}_\mu] = \Gamma_\kappa[\phi_{\mathrm{cl}}, \phi_{\mathrm{cl}}^\dagger, a_{\mathrm{cl}\,\mu}; \bar{A}_\mu], \tag{E.22}$$

which is invariant under the simultaneous transformations of the scalar field and both gauge field A_μ and $\bar{A}_\mu$ at $\kappa = 0$. If gauge invariant cutoff functions $\Delta_\kappa S_{S,G} = 0$ are chosen, the effective action will also be invariant for all κ.

It is not necessary to select gauge invariant regulators because it does not affect the final result at $\kappa = 0$. However, careful and appropriate choice of the cutoff functions makes the problem easier and simpler. We will come back to this point later.

E.2.4 Flow equation

We derive the flow equation for the effective average action Γ_κ in this section. It is straightforward to calculate the flow of the effective average action $\dot{\Gamma}_\kappa \equiv \kappa\partial_\kappa\Gamma$ using the definitions (E.13), (E.14), (E.21) and connected Green's function defined by

$$G_{\kappa,\,mn}(x,y) = \frac{-i}{\sqrt{-g(x)}\sqrt{-g(y)}}\frac{\delta^2 W_\kappa}{\delta\mathcal{J}^m(x)\delta\mathcal{J}^n(y)}. \tag{E.23}$$

We also make use of the following equation

$$\langle\psi_m(x)\psi_n(y)\rangle = G_{\kappa,\,mn}(x,y) + \psi_{\mathrm{cl},\,m}(x)\psi_{\mathrm{cl},\,n}(y). \tag{E.24}$$

We obtain

$$\begin{aligned}
\dot{\Gamma}_\kappa &= \dot{W}_\kappa + \int_x \frac{1}{\sqrt{-g}}\frac{\delta W_\kappa}{\delta\mathcal{J}^m}\dot{\mathcal{J}}^m - \int_x \psi_{\mathrm{cl}\,m}\dot{\mathcal{J}}^m \\
&\quad - \frac{1}{2}\int_{x,\,y}\psi_{\mathrm{cl},\,m}(x)\dot{R}^{mn}_\kappa(x-y;\bar{A}_\mu)\psi_{\mathrm{cl},\,n}(y), \\
&= \frac{1}{2}\int_{x,\,y}\dot{R}^{mn}_\kappa(x-y;\bar{A}_\mu)\left\{\langle\psi_m(x)\psi_n(y)\rangle - \psi_{\mathrm{cl},\,m}(x)\psi_{\mathrm{cl},\,n}(y)\right\} \\
&= \frac{1}{2}\int_{x,\,y}\dot{R}^{mn}_\kappa(x-y;\bar{A}_\mu)G_{\kappa,\,mn}(x,y) \\
&= \frac{1}{2}\mathrm{Tr}\{\dot{R}_\kappa G_\kappa\},
\end{aligned} \tag{E.25}$$

in the last line, Tr runs over all the indices and spacetime. The inverse of the connected Green's function can be found as follows

$$\begin{aligned}
\delta^m_n\frac{\delta^{(D)}(x-y)}{\sqrt{-g(x)}} &= \frac{1}{\sqrt{-g(x)}}\frac{\delta\psi_{\mathrm{cl},\,n}(y)}{\delta\psi_{\mathrm{cl},\,m}(x)} \\
&= \int_z \frac{1}{\sqrt{-g(x)}\sqrt{-g(z)}}\frac{\delta\psi_{\mathrm{cl},\,n}(y)}{\delta\mathcal{J}^l(z)}\frac{\delta\mathcal{J}^l(z)}{\delta\psi_{\mathrm{cl},\,m}(x)} \\
&= \int_z \frac{1}{\sqrt{-g(x)}^2\sqrt{-g(z)}^2}\frac{\delta W_\kappa}{\delta\mathcal{J}^l(z)\delta\mathcal{J}^n(y)}\frac{\delta\tilde{\Gamma}_\kappa}{\delta\psi_{\mathrm{cl},\,m}(x)\delta\psi_{\mathrm{cl},\,l}(z)} \\
&= i\int_z \left\{(\Gamma^{(2)}_\kappa)^{ml}(x,z) + R^{ml}_\kappa(x-z)\right\}G_{\kappa,\,ln}(z,y),
\end{aligned} \tag{E.26}$$

where $\Gamma^{(2)}_\kappa$ denotes the second functional derivatives of the effective average action,

$$(\Gamma^{(2)}_\kappa)^{ml}(x,z) \equiv \frac{1}{\sqrt{-g(x)}\sqrt{-g(z)}}\frac{\delta^2\Gamma_\kappa}{\delta\psi_{\mathrm{cl},\,m}(x)\delta\psi_{\mathrm{cl},\,l}(z)}. \tag{E.27}$$

We can rewrite (E.25) as

$$\dot{\Gamma}_\kappa[\phi_{\rm cl},\phi_{\rm cl}^\dagger,A_\mu,\bar{A}_\mu]=-\frac{i}{2}{\rm Tr}\left\{\frac{\dot{R}_\kappa}{\Gamma_\kappa^{(2)}+R_\kappa}\right\}=-\frac{i}{2}{\rm Tr}\left\{\kappa\partial_\kappa\ln(\Gamma_\kappa^{(2)}+R_\kappa)\right\}, \tag{E.28}$$

here, we use the rule that $\kappa\partial_\kappa$ acts only on R_κ. It is much easier to see that the right hand side of (E.28) will give a 1-loop result for the effective action if Γ_κ inside the ${\rm Tr}\ln$ is replaced by the classical action $S+S_{\rm gf}$.

To obtain an effective action which depends only on one gauge field A_μ, we identify $\bar{A}_\mu$ with A_μ. At first, we split the effective average action

$$\Gamma_\kappa[\phi_{\rm cl},\phi_{\rm cl}^\dagger,A_\mu,\bar{A}_\mu]=\Gamma_\kappa^{(\rm inv)}[\phi_{\rm cl},\phi_{\rm cl}^\dagger,A_\mu]+\Gamma_\kappa^{(\rm gauge)}[\phi_{\rm cl},\phi_{\rm cl}^\dagger,A_\mu,\bar{A}_\mu], \tag{E.29}$$

where $\Gamma_\kappa^{(\rm inv)}[\phi_{\rm cl},\phi_{\rm cl}^\dagger,A_\mu]\equiv\Gamma_\kappa[\phi_{\rm cl},\phi_{\rm cl}^\dagger,A_\mu,\bar{A}_\mu=A_\mu]$ and as a result $\Gamma_\kappa^{(\rm gauge)}$ must vanish when we set $\bar{A}_\mu=A_\mu$. This $\Gamma_\kappa^{(\rm gauge)}$ term is a generalized gauge fixing term and is needed to obtain a well-behaved propagator. (We can prove that $\Gamma_\kappa^{(\rm gauge)}$ vanishes when the longitudinal mode of the fluctuation vanishes $a_{{\rm L,cl}\,\mu}=0$.)

E.2.5 Symmetry

Symmetry structure is particularly important since it helps us reduce the number of the terms to elaborate. It would be admittedly sufficient to focus only on gauge-invariant quantities if there is an explicit gauge symmetry, whereas it has been lost due to the gauge fixing procedure and the introduced IR cutoff terms. Even though the theory is not invariant under the original version of the gauge transformation which involves ϕ, $\phi^\dagger$ and A_μ, there is an explicit invariance under simultaneous transformations of ϕ, $\phi^\dagger$, A_μ and also $\bar{A}_\mu$. Let $\mathcal{G}$ be a generator of the infinitesimal gauge transformation. This is a combination of infinitesimal field transformations

$$\mathcal{G}=\mathcal{G}_\phi+\mathcal{G}_{\phi^\dagger}+\mathcal{G}_A, \tag{E.30}$$

where each term on the right hand side is given by

$$\begin{gathered}\mathcal{G}_\phi=-ie\phi(x)\frac{1}{\sqrt{-g(x)}}\frac{\delta}{\delta\phi(x)},\quad \mathcal{G}_{\phi^\dagger}=ie\phi^\dagger(x)\frac{1}{\sqrt{-g(x)}}\frac{\delta}{\delta\phi^\dagger(x)},\\ \mathcal{G}_A=\partial_\mu\left(\frac{1}{\sqrt{-g(x)}}\frac{\delta}{\delta A_\mu(x)}\right).\end{gathered} \tag{E.31}$$

The variations of the fields are given by

$$\delta\Phi(x) = \int_y \alpha(y)\mathcal{G}(x)\Phi(y), \tag{E.32}$$

where α is the parameter of the transformation and Φ represents ϕ, $\phi^\dagger$ or A_μ. In addition to this, we define a generator $\bar{\mathcal{G}}$ of the gauge transformation of $\bar{A}_\mu$.

The theory considered in (E.13) is, again, explicitly invariant under $\mathcal{G} + \bar{\mathcal{G}}$. Therefore we observe

$$(\mathcal{G} + \bar{\mathcal{G}})\Gamma_\kappa[\phi_{\text{cl}}, \phi^\dagger_{\text{cl}}, A_\mu, \bar{A}_\mu] = 0, \tag{E.33}$$

setting $\bar{A}_\mu = A_\mu$, we obtain

$$\mathcal{G}\Gamma^{(\text{inv})}_\kappa[\phi_{\text{cl}}, \phi^\dagger_{\text{cl}}, A_\mu] = 0, \tag{E.34}$$

consequently, it is shown in (E.29) that the effective average action is split into the gauge invariant part $\Gamma^{(\text{inv})}_\kappa$ and the gauge non-invariant remainder $\Gamma^{(\text{gauge})}_\kappa$. Though the latter does not have the explicit invariance under $\mathcal{G}$, it is invariant under $\mathcal{G} + \bar{\mathcal{G}}$.

In the following subsections, we exploit the symmetry structures of the theory to gain an insight of the gauge dependence of the flow equation for non-zero κ. At first, we will see the Ward-Takahashi identity which describes the symmetry of the theory in quantum level. We, then, will use it to make an appropriate truncation for the flow equation above.

E.2.6 Ward-Takahashi identity

The behavior of the effective action under the gauge transformation is directly derived from the gauge invariance of the generating functional. From (E.8), we observe

$$\begin{aligned} 0 &= \frac{1}{Z}\int \mathcal{D}\psi_m\, \mathcal{G}\left(iS + iS_{\text{gf}} + i\int_x \psi_m \mathcal{J}^m\right) \mathrm{e}^{iS + iS_{\text{gf}} + i\int_x \psi_m \mathcal{J}^m} \\ &= i\left\langle \mathcal{G}S\right\rangle + i\left\langle \mathcal{G}S_{\text{gf}}\right\rangle + i\mathrm{e}^{-iW}\int_x \mathcal{J}^m(\mathcal{G}\psi_m)\Big|_{\psi_m = \frac{-i}{\sqrt{-g}}\frac{\delta}{\delta\mathcal{J}^m}} \mathrm{e}^{iW}. \end{aligned} \tag{E.35}$$

After the Legendre transformation, we immediately obtain

$$0 = \langle \mathcal{G}S\rangle + \langle \mathcal{G}S_{\text{gf}}\rangle - \int_x \frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta\psi_{\text{cl},\,m}}(\mathcal{G}\psi_{\text{cl},\,m}) \tag{E.36}$$

where the third term is nothing but $\mathcal{G}\Gamma$. Therefore transformation property of the effective action is given by the following Ward-Takahashi identity

$$\mathcal{G}\Gamma[\psi_{\mathrm{cl},\,m}] = \langle \mathcal{G}S \rangle + \langle \mathcal{G}S_{\mathrm{gf}} \rangle \,. \tag{E.37}$$

This manifests, before the gauge fixing, the effective action Γ is gauge invariant provided that the classical action S is gauge invariant. It is not difficult to find the Ward-Takahashi identity for the effective average action Γ_κ,

$$\mathcal{G}\Gamma_\kappa[\psi_{\mathrm{cl},\,m}] = \langle \mathcal{G}S \rangle + \langle \mathcal{G}S_{\mathrm{gf}} \rangle + \langle \mathcal{G}\Delta_\kappa S \rangle - \mathcal{G}\Delta_\kappa S, \tag{E.38}$$

here the extra terms from the cutoff function have appeared. The last two terms do not cancel each other in general. The modification shows the deviation of the effective average action from the gauge invariant surface in the theory space. However, modified version of Ward-Takahashi identity (E.38) comes down to the normal one (E.37) for vanishing cutoff limit $\kappa \to 0$. Therefore, we do not necessarily choose a gauge invariant cutoff function $\Delta_\kappa S$ to obtain a gauge invariant final result $\Gamma = \Gamma_0$. In any case, as long as the cutoff function disappears for $\kappa \to 0$, we will obtain a gauge invariant result.

An advantage of the modified Ward-Takahashi identity is revealed by writing down the flow of a quantity $\mathcal{W}_\kappa \equiv \mathcal{G}\Gamma_\kappa[\psi_{\mathrm{cl},\,m}] - (\langle \mathcal{G}S \rangle + \langle \mathcal{G}S_{\mathrm{gf}} \rangle + \langle \mathcal{G}\Delta_\kappa S \rangle - \mathcal{G}\Delta_\kappa S)$. It is known that the flow of $\mathcal{W}_\kappa$ is proportional to $\mathcal{W}_\kappa$ itself

$$\dot{\mathcal{W}}_\kappa = -\frac{1}{2}\mathrm{Tr}\left[G\dot{R}G\left(\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\psi}\right)\otimes\left(\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\psi}\right)\right]\mathcal{W}_\kappa. \tag{E.39}$$

Suppose Γ_κ is a solution of the modified Ward-Takahashi identity $\mathcal{W}_\kappa = 0$ at some scale $\kappa = \Lambda$, we then automatically obtain Γ_κ which satisfies $\mathcal{W}_\kappa = 0$ for all κ by integrating the flow equation (E.28). Therefore, the modified Ward-Takahashi identity $\mathcal{W}_\kappa = 0$ is compatible with the flow equation for the effective average action.

In a word, a solution obtained by solving the flow equation (E.28) and the modified Ward-Takahashi identity $\mathcal{W}_\kappa = 0$ at the same time will show the physical gauge invariance at $\kappa = 0$, i.e. $\mathcal{G}\Gamma_0^{(\mathrm{inv})} = 0$.

E.2.7 Truncation

Exact solutions for the flow equation (E.28) are not generally available. Thus, truncation technique is required to solve the flow equation. To implement this, an ansatz for the effective average action is employed. The symmetry structure we have observed is particularly useful to build the ansatz. For the gauge invariant part of the effective average action $\Gamma_\kappa^{(\mathrm{inv})}$, we only have to pick up the gauge-invariant quantities and implement the truncation procedure as usual. Moreover, the modified Ward-Takahashi identity (E.38) fixes the truncated form of the gauge non-invariant reminder $\Gamma_\kappa^{(\mathrm{gauge})}$.

Our ansatz for $\Gamma_\kappa^{(\mathrm{inv})}$ is given in (E.3). The reminder part $\Gamma_\kappa^{(\mathrm{gauge})}$ depending on the two gauge fields A_μ and $\bar{A}_\mu$ has an invariance under $\mathcal{G} + \bar{\mathcal{G}}$ and must vanish after identifying $\bar{A}_\mu$ with A_μ. We can expand $\Gamma_\kappa^{(\mathrm{gauge})}$ with these in mind as follows

$$\Gamma_\kappa^{(\mathrm{gauge})} = \int_x (A_\mu - \bar{A}_\mu) M^{\mu\nu} (A_\nu - \bar{A}_\nu) + \mathcal{O}((A_\mu - \bar{A}_\mu)^3). \tag{E.40}$$

From the lowest order of the modified Ward-Takahashi identity (E.38) (neglect the loop contribution in it), we find $M^{\mu\nu} = 1/(2\alpha)\nabla^\mu\nabla^\nu$. Thus, at tree level, $\Gamma_\kappa^{(\mathrm{gauge})}$ is identical to the gauge fixing term Γ_{gf} and does not depend on the cutoff κ. Hereafter, we neglect contributions from higher-loop reminder $\hat{\Gamma}_\kappa^{(\mathrm{gauge})} = \Gamma_\kappa^{(\mathrm{gauge})} - \Gamma_{\mathrm{gf}}$.

E.3 Solution of the flow equation

In the previous section, we discussed the gauge invariant flow equation for the effective average action. The approximated flow equation for the gauge invariant effective action is obtained by setting $\bar{A}_\mu = A_\mu$ and replacing gauge non-invariant part of the effective average action (E.28) by Γ_{gf},

$$\begin{aligned} \dot{\Gamma}_\kappa^{(\mathrm{inv})}[\phi_{\mathrm{cl}}, \phi_{\mathrm{cl}}^\dagger, A_\mu] &= -\frac{i}{2}\mathrm{Tr}\left\{ \frac{\dot{R}_\kappa}{\left(\Gamma_\kappa^{(\mathrm{inv})}\right)^{(2)} + \Gamma_{\mathrm{gf}}^{(2)} + R_\kappa} \right\} \\ &= -\frac{i}{2}\mathrm{Tr}\left\{ \kappa\partial_\kappa \ln\left(\left(\Gamma_\kappa^{(\mathrm{inv})}\right)^{(2)} + \Gamma_{\mathrm{gf}}^{(2)} + R_\kappa \right) \right\}, \end{aligned} \tag{E.41}$$

in the last line, $\kappa\partial_\kappa$ acts only on R_κ as we mentioned. Note also that even the generalized gauge fixing term Γ_{gf} disappears in the coincidence limit $\bar{A}_\mu = A_\mu$, the second functional derivative

$\Gamma^{(2)}_{\rm gf}$ can have contribution.

We adopt (E.3) as the ansatz for the gauge invariant part of the effective action $\Gamma^{(\rm inv)}_\kappa$. The truncation is justified based on the following assumptions:

1. Only gauge invariant building blocks can appear in the expression.
2. We consider the weak and slowly-varying electromagnetic fields.
3. The local interaction and negligible higher-derivative terms.

It is convenient to use a set of new (real) field variables $(\chi(x), \theta(x), \mathcal{A}_\mu(x))$ defined as

$$\phi(x) = \frac{1}{\sqrt{2}}\chi(x)\mathrm{e}^{ie\theta(x)}, \quad \phi^\dagger(x) = \frac{1}{\sqrt{2}}\chi(x)\mathrm{e}^{-ie\theta(x)}, \quad A_\mu(x) = \mathcal{A}_\mu(x) - \theta_{,\mu}(x), \tag{E.42}$$

so as to obviate the troubles with the gauge redundancy. In fact, the ansatz (E.3) becomes independent of the phase field $\theta(x)$, and the ansatz is given in terms of the new variables by

$$\begin{aligned}\Gamma^{(\rm inv)}_\kappa[\chi, \mathcal{A}_\mu] = \int \mathrm{d}^D x\sqrt{-g}\bigg\{ -\frac{\mathcal{Z}_{F,\kappa}}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\mathcal{Z}_{\phi,\kappa}}{2}g^{\mu\nu}(\chi_{,\mu}\chi_{,\nu} + e^2\chi^2\mathcal{A}_\mu\mathcal{A}_\nu) \\ - \mathcal{U}_\kappa(\chi^2) + \frac{1}{2}F_{\mu\nu}\mathcal{M}^{\mu\nu}_\kappa(\chi^2)\bigg\},\end{aligned} \tag{E.43}$$

where the gauge field strength is gauge invariant, i.e. $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} = \mathcal{A}_{\nu,\mu} - \mathcal{A}_{\mu,\nu}$.

The second functional variation of the effective action is directly obtained,

$$\begin{aligned}&\delta^{(2)}\Gamma^{(\rm inv)}_\kappa[\chi, \mathcal{A}_\mu] \\ &= \int \mathrm{d}^D x \mathrm{d}^D y\bigg[\frac{1}{2}\delta\mathcal{A}_\mu(x)\delta\mathcal{A}_\nu(y)\left\{\mathcal{Z}_F\overset{x}{\partial}_\beta(\sqrt{}_x g^{\mu[\nu}(x)g^{\alpha]\beta}(x)\overset{x}{\partial}_\alpha\delta_{xy}) - \mathcal{Z}_\phi e^2\sqrt{}_x g^{\mu\nu}\chi^2\delta_{xy}\right\} \\ &\quad + \frac{1}{2}\chi(x)\delta\chi(y)\bigg\{\mathcal{Z}_\phi\overset{x}{\partial}_\mu(\sqrt{}_x g^{\mu\nu}(x)\overset{x}{\partial}_\nu\delta_{xy}) - \mathcal{Z}_\phi e^2\sqrt{}_x\mathcal{A}_\mu\mathcal{A}_\nu\delta_{xy} \\ &\qquad - 2\sqrt{}_x\delta_{xy}\left((2\chi^2\mathcal{U}''_\kappa + \mathcal{U}'_\kappa) - \frac{1}{2}F_{\mu\nu}(2\chi^2\mathcal{M}''^{\mu\nu}_\kappa + \mathcal{M}'^{\mu\nu}_\kappa)\right)\bigg\} \\ &\quad + \delta\mathcal{A}_\mu(x)\delta\chi(y)\left\{-2\mathcal{Z}_\phi e^2\sqrt{}_x g^{\mu\nu}\chi\mathcal{A}_\nu\delta_{xy} + 2\overset{x}{\partial}_\nu\left(\sqrt{}_x\chi\mathcal{M}'^{\mu\nu}_\kappa\delta_{xy}\right)\right\}\bigg],\end{aligned} \tag{E.44}$$

where we have introduced abbreviated notations,

$$\sqrt{}_x = \sqrt{-g(x)}, \quad \delta_{xy} = \delta^{(D)}(x-y), \quad g^{\mu[\nu}g^{\alpha]\beta} = g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}. \tag{E.45}$$

The prime $'$ on the functions $\mathcal{U}_\kappa$ and $\mathcal{M}^{\mu\nu}_\kappa$ denotes the derivative with respect to their arguments χ^2, e.g. $\mathcal{U}'_\kappa = \partial \mathcal{U}_\kappa(\chi^2)/\partial\chi^2$.

In the form of operators,

$$\begin{aligned}\left(\Gamma^{(\mathrm{inv})}_\kappa\right)^{(2)} &= \left(\Gamma^{(\mathrm{inv})}_\kappa\right)^{(2)ml}(x,z) = \frac{1}{\sqrt{-g(x)}\sqrt{-g(z)}}\frac{\delta^2\Gamma^{(\mathrm{inv})}_\kappa}{\delta\psi_{\mathrm{cl},\,m}(x)\delta\psi_{\mathrm{cl},\,n}(z)} \\ &= \begin{pmatrix} \Gamma^{\chi\chi}_\kappa(x,z) & \Gamma^{\chi\mathcal{A}_\mu}_\kappa(x,z) \\ \Gamma^{\mathcal{A}_\mu\chi}_\kappa(x,z) & \Gamma^{\mathcal{A}_\mu\mathcal{A}_\nu}_\kappa(x,z)\end{pmatrix},\end{aligned} \tag{E.46}$$

the functional derivative in the conformally flat spacetime $g^{\mu\nu} = a^2(x)\eta^{\mu\nu}$ is given by

$$\begin{aligned}\Gamma^{\chi\chi}_\kappa(x,z) &= \frac{\mathcal{Z}_\phi\eta^{\mu\nu}}{a^D(x)a^D(z)}\overset{x}{\partial}_\mu\left(a^{D-2}(x)\overset{x}{\partial}_\nu\delta_{xz}\right) \\ &\quad + \frac{\delta_{xz}}{a^D(x)}\left(-\frac{\mathcal{Z}_\phi e^2}{a^2(x)}\eta^{\mu\nu}\mathcal{A}_\mu\mathcal{A}_\nu - 2(\chi^2\mathcal{U}''_\kappa + \mathcal{U}'_\kappa) + F_{\mu\nu}(2\chi^2\mathcal{M}''^{\mu\nu}_\kappa + \mathcal{M}'^{\mu\nu}_\kappa)\right), \\ \Gamma^{\chi\mathcal{A}_\mu}_\kappa(x,z) &= -2\mathcal{Z}_\phi e^2\frac{\delta_{xz}}{a^D}g^{\mu\nu}\chi\mathcal{A}_\nu - 2\chi(x)\mathcal{M}'^{\mu\nu}_\kappa(x)\frac{\overset{x}{\partial}_\nu\delta_{xz}}{a^D(z)}, \\ \Gamma^{\mathcal{A}_\mu\chi}_\kappa(x,z) &= -2\mathcal{Z}_\phi e^2\frac{\delta_{xz}}{a^D}g^{\mu\nu}\chi\mathcal{A}_\nu + 2\chi(z)\mathcal{M}'^{\mu\nu}_\kappa(z)\frac{\overset{x}{\partial}_\nu\delta_{xz}}{a^D(x)}, \\ \Gamma^{\mathcal{A}_\mu\mathcal{A}_\nu}_\kappa(x,z) &= \frac{\mathcal{Z}_F}{a^D(x)a^D(z)}\overset{x}{\partial}_\beta\left(a^{D-4}\eta^{\mu[\nu}\eta^{\alpha]\beta}\delta_{xz}\right) - \frac{\mathcal{Z}_\phi e^2\delta_{xz}}{a^{D+2}}\eta^{\mu\nu}\chi^2.\end{aligned} \tag{E.47}$$

Also, the second derivative of the gauge fixing term is given by

$$\Gamma^{\mathcal{A}_\mu\mathcal{A}_\nu}_{\mathrm{gf}}(x,z) = \frac{1}{\alpha\sqrt{-g(z)}}\overset{x\mu}{\partial}\left(\frac{1}{\sqrt{-g(x)}}\overset{x}{\partial}_\alpha\left(\sqrt{-g(x)}g^{\alpha\nu}\delta_{xz}\right)\right), \tag{E.48}$$

and other components are zero.

In the case of de Sitter QED, we can take advantage of the Fourier transformation due to the homogeneity of the spacetime. It is easier to consider the explicit form of the regulator function R_κ in the Fourier space. A special form of the regulator called Litim's regulator is often employed,

$$\tilde{R}_\kappa(k) = \mathcal{Z}_\phi H^2\Theta\left(\kappa^2 - \left(\frac{k}{aH}\right)^2\right)\left(\kappa^2 - \left(\frac{k}{aH}\right)^2\right), \tag{E.49}$$

where the Θ function is Heaviside step function defined on the contour C of the in-in formalism. It is obvious that the Litim's regulator gives momentum dependent mass to the IR modes with the physical momentum $p < \kappa$ while leaving UV modes $p \geq \kappa$ unaffected. When we take the vanishing cutoff limit $\kappa \to 0$, the regulator also vanishes. Besides, all the quantum modes will be suppressed in the UV limit $\kappa \to \infty$ so that only the classical contribution exists.

Substituting these expressions into (E.26), the set of the equations for the exact Green's functions is obtained. Unfortunately, we have not found any way to solve the fully dynamical equations nor to approximate the solution. Instead, we will treat the gauge field as a background at the outset in the next section.

E.4 Background gauge analysis

It seems a challenging task to obtain the full Green's functions which are solutions of the equation (E.26). However, it is still possible to find the analytical solution if the gauge field is treated as a background field. In this case, the only equation to be addressed is,

$$\left[\frac{\mathcal{Z}_\phi}{a^2}\{\partial_\eta^2 + (D-2)\frac{a'}{a}\partial_\eta - \partial_i^2 + \eta^{\mu\nu}e^2\mathcal{A}_\mu\mathcal{A}_\nu\} + \partial_\chi^2(\mathcal{U}_\kappa - \frac{1}{2}F_{\mu\nu}\mathcal{M}^{\mu\nu}) + R_\kappa(x,y)\right] G_{\chi\chi}(x,y) = i\frac{\delta^{(D)}(x-y)}{a^D(x)}. \tag{E.50}$$

From the translation symmetry of the system, Fourier transformation of the Green's function is given by

$$G_{\chi\chi}(x,y) = \int \frac{\mathrm{d}^{D-1}k}{(2\pi)^{D-1}} \mathrm{e}^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\tilde{G}_\kappa(k,\eta_x,\eta_y). \tag{E.51}$$

The canonical form of the Green's function is given by

$$\tilde{G}_\kappa(k,\eta_x,\eta_y) = (a_x a_y)^{-\frac{(D-2)}{2}}\frac{\hat{G}_\kappa(p_x,p_y)}{k}, \tag{E.52}$$

where $p = k/(a(\eta)H)$ is the physical momentum. The regulator function transforms as

$$R_\kappa(x,y) \int \frac{\mathrm{d}^{D-1}k}{(2\pi)^{D-1}} \mathrm{e}^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\tilde{R}_\kappa(k), \tag{E.53}$$

where the form of $\tilde{R}_\kappa$ is chosen to be the Litim's regulator (E.49).

For the solvable configuration

$$\mathcal{A}_\mu = (0,0,\cdots,-\frac{E}{H}a^2), \tag{E.54}$$

where only the electric background exists, the equation for the Green's function $\hat{G}_\kappa$ becomes

$$\left[\partial_{p_x}^2 + 1 - \frac{(L^2 + \frac{D(D-2)}{4} - \Theta(\kappa^2 - p_x^2)(\kappa^2 - p_x^2) - \mathcal{Z}_\phi^{-1}\partial_\chi^2(\mathcal{U}_\kappa + L\Pi_\kappa))}{p_x^2}\right] \hat{G}_\kappa(p_x, p_y) = -i\frac{\delta(p_x - p_y)}{\mathcal{Z}_\phi}. \tag{E.55}$$

Here, $L = (eE)/H^2$ is the dimensionless electric field, Π_κ is the dimensionless polarization in the Dth-direction which gives the electric energy when multiplied by L. [1] Using the homogeneous solution of the equation $u_\kappa(p)$, we can construct the Green's function as follows,

$$\begin{aligned}\hat{G}_\kappa(p_x, p_y) &= \mathcal{T}_C[u_\kappa, u_\kappa^*](p_x, p_y) \\ &= \Theta(p_x - p_y)u_\kappa(p_x)u_\kappa^*(p_y) + \Theta(p_y - p_x)u_\kappa(p_y)u_\kappa^*(p_x),\end{aligned} \tag{E.56}$$

where we have defined the time ordering operator $\mathcal{T}_C$ along with the time contour C of the in-in formalism.

In the local potential approximation, $\mathcal{Z}_\phi = 1$, the homogeneous equation for $u_\kappa(p)$ is given by

$$\begin{cases} \left(\partial_p^2 - \dfrac{\nu_\kappa^2 - \kappa^2 - 1/4}{p^2}\right) u_\kappa(p) = 0 \quad (p < \kappa) \\ \left(\partial_p^2 + 1 - \dfrac{\nu_\kappa^2 - 1/4}{p^2}\right) u_\kappa(p) = 0 \quad (p > \kappa) \end{cases}, \tag{E.57}$$

where the parameter ν_κ is defined by

$$\nu_\kappa = \frac{(D-1)^2}{4} - \partial_\chi^2\left(\mathcal{U}_\kappa + L\Pi_\kappa\right) + L^2. \tag{E.58}$$

The solution corresponds to Bunch-Davies vacuum is immediately obtained,

$$u_\kappa(p) = \begin{cases} \sqrt{\dfrac{\pi p}{4}} \mathrm{e}^{i\frac{\pi}{2}(\nu_\kappa + \frac{1}{2})} \left[c_\kappa^+ \left(\dfrac{p}{\kappa}\right)^{\bar{\nu}_\kappa} + c_\kappa^- \left(\dfrac{p}{\kappa}\right)^{-\bar{\nu}_\kappa}\right] & (p < \kappa) \\ \sqrt{\dfrac{\pi p}{4}} \mathrm{e}^{i\frac{\pi}{2}(\nu_\kappa + \frac{1}{2})} H_{\nu_\kappa}^{(1)}(p) & (p > \kappa) \end{cases}, \tag{E.59}$$

where we have defined $\bar{\nu}_\kappa^2 = \nu_\kappa^2 - \kappa^2$. $H_{\nu_\kappa}^{(1)}(p)$ is the Hankel function of the first kind. The

[1] Hereafter, Π_κ, instead of $\mathcal{M}^{\mu\nu}$, is supposed to be determined as a function of χ^2.

connection condition, continuity of the solution and its first derivative, at $p = \kappa$ yields

$$c_\kappa^\pm = \frac{1}{2}\left[H_{\nu_\kappa}^{(1)}(\kappa) \pm \frac{\kappa}{\bar{\nu}_\kappa} H_{\nu_\kappa}^{(1)\prime}(\kappa)\right]. \tag{E.60}$$

Now is the time for examining the flow equation (E.41). We obtain

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}[\dot{R}_\kappa G_\kappa] &= -\frac{1}{2}\int \mathrm{d}^D x\sqrt{-g(x)}\int \frac{\mathrm{d}^{D-1}p}{(2\pi)^{D-1}} H^{D-2}\dot{R}_\kappa(p)\frac{\hat{G}_\kappa(p,p)}{p} \\ &= -\frac{1}{2}\Omega_4 H^{D-2}\int \frac{\mathrm{d}^{D-1}p}{(2\pi)^{D-1}}\left\{2\kappa\Theta(\kappa^2-p^2)\right\}\frac{|u_\kappa(p)|^2}{p},\end{aligned} \tag{E.61}$$

where Ω_4 is the spacetime 4-volume factor. The momentum integration is trivial because of the vanishing behavior of $\dot{R}_\kappa$ for $p > \kappa$ and the simple functional form of $u_\kappa(p)$. We also obtain

$$\dot{\Gamma}_\kappa^{(\mathrm{inv})} = -\Omega_4\left(\dot{\mathcal{U}}_\kappa + L\dot{\mathit{\Pi}}_\kappa\right). \tag{E.62}$$

Finally, we reached the following functional flow equation for the effective potential,

$$\begin{aligned}\dot{\mathcal{U}}_\kappa + L\dot{\mathit{\Pi}}_\kappa &= \beta\left(\partial_\chi^2\mathcal{U}_\kappa, \partial_\chi^2\mathit{\Pi}_\kappa;\kappa\right) \\ &= H^D\frac{\pi^{(D-1)/2}}{16(D-1)\Gamma\left(\frac{D-1}{2}\right)(2\pi)^{D-2}}\frac{\kappa^{D+1}}{\kappa^2+\partial_\chi^2\left(\mathcal{U}_\kappa+L\mathit{\Pi}_\kappa\right)}B_D(\nu_\kappa,\kappa),\end{aligned} \tag{E.63}$$

where B_D is given by

$$\begin{aligned}B_D(\nu_\kappa,\kappa) = \mathrm{e}^{-\pi\Im(\nu_\kappa)}\Big\{&((D-1)^2-2\nu_\kappa^2+2\kappa^2)|H_{\nu_\kappa}^{(1)}(\kappa)|^2 \\ &+2\kappa^2|H_{\nu_\kappa}^{(1)\prime}(\kappa)|^2-2(D-1)\kappa\Re[H_{\nu_\kappa}^{(1)*}(\kappa)H_{\nu_\kappa}^{(1)\prime}(\kappa)]\Big\}.\end{aligned} \tag{E.64}$$

Note that this functional flow equation is an extension of previous result (Eq.(2.20) in [80]), and we recover the same result when we take the zero electric field limit $L \to 0$ in our expression (E.63).

It seems appealing to numerically investigate the physical consequences of the flow equation with the electric field (E.63) after the fashion of previous works [80]. Once the unknown functions $\mathcal{U}_\kappa(\chi^2), \mathit{\Pi}_\kappa(\chi^2)$ are expanded as serieses, for instance,

$$\mathcal{U}_\kappa(\chi^2) = \sum_{n=0}^{N} c_{2n,\kappa}\chi^{2n}, \quad \mathit{\Pi}_\kappa(\chi^2) = \sum_{n=0}^{N} d_{2n,\kappa}\chi^{2n}, \tag{E.65}$$

the flow equation merely yields the coupled differential equations for the coefficients $c_{2n,\kappa}$ and $d_{2n,\kappa}$. Therefore we can, in principle, solve the flow equation (E.63) numerically.

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