学位論文

On superconformal field theories at large charge

(超対称共形場理論における電荷の大きな極限について)

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Abstract

We study superconformal field theories (SCFTs) with a one-complex-dimensional moduli space of vacua in three and four dimensions. We develop an effective field theory on moduli space to describe the large-R-charge sector of the theories, and by using it we calculate physical quantities associated to operators carrying large R-charge.

In the case of a $\mathcal{N} = 2$ SCFT in three dimensions, we compute the anomalous dimension of certain low-lying operators carrying large R-charge J. We find that the lowest and second-lowest scalar primary operators have vanishing anomalous dimension up to and including $O(J^{-3})$, and this result is consistent with the fact that they are in protected supermultiplets. We also show that that the anomalous dimension of the third-lowest primary operator carrying large R-charge must be nonpositive, making use of unitarity of moduli scattering and the absence of superluminal signal propagation in the effective dynamics of the complex modulus.

In the case of $\mathcal{N} = 2$ SCFTs of in four dimensions with a one-complex-dimensional Coulomb branch, we show that two-point functions of chiral primary operators have a nontrivial but universal asymptotic expansion at large R-charge. In particular, the asymptotic expansion depends on the difference between the a-coefficient of the Weyl anomaly of the underlying SCFT and that of the effective theory of the Coulomb branch. For Lagrangian SCFT, we check our predictions for the logarithm of the two-point functions against exact results from supersymmetric localization, and find reasonably good agreement. In this way, we show the large-R-charge expansion serves as a bridge from the world of unbroken superconformal symmetry and SCFT data, to the world of the low-energy dynamics of the moduli space of vacua.

Contents

1 Introduction and summary

The idea of macroscopic or classical phenomena as aggregates of a huge number of microscopic constituents goes back to the atomism of Leucippus and Democritus, which is first confirmed by Einstein's exhaustive explanation of Brownian motion [7]. In quantum mechanics, Bohr's correspondence principle [8] states that the behavior of quantum mechanical systems reproduces the classical behavior in the limit of large quantum numbers. These ideas are enlarged and sophisticated by Wilson, who introduced the notion of renormalization group [9, 10]. The renormalization group is a theoretical tool which represents a consecutive application of scaling and coarse graining operations. If one is interested in the dynamics of a physical system at a very large length scale L, details of the system at short distances, such as effects of particles much heavier than L^{-1} are generically not important, and just encoded as undetermined parameters of an effective theory describing only low energy degrees of freedom. The most distinguished low energy effective theories include Einstein's general relativity, the standard model in particle physics and hydrodynamics. The renormalization group is why these models work extremely well without knowing the dynamics at short distances such as quantum gravity, particle physics beyond the TeV scale or the detailed chemical composition of fluids.

Although renormalization group and the effective theoretic description of low energy dynamics are very powerful, there are interesting systems where these techniques are not of immediate practical use, due to the strongly coupled nature of the systems. In such systems, the effective Hamiltonian describing the low energy physics contains infinitely many terms of the same order of magnitude, and it is quite difficult to compute physical quantities by making use of it.

Nonetheless, one can perform the effective-field-theoretic analysis for some *sectors* in such strongly coupled systems. One example is the Regge trajectories in hadron physics. Experimentally observed spectra of baryons and mesons obey the simple linear formula $J \simeq \alpha' M^2$, where M and J are the mass and angular momentum of hadrons, respectively [11]. This formula is easily derived by an effective theory of rotating strings [12] at large angular momentum J, in which the loops and higher-derivative corrections[1] to the classical result are suppressed by powers of J^{-1}, and works remarkably well even at J of order one. This kind of effective field theory (EFT) at sectors where some quantum number (conserved charge) is large has been used to extract nontrivial information from various strongly coupled systems, in which the standard perturbative method loses its power. For instance, in [15,16] the large charge limit has played a key role in the correspondence between single-trace operators in planar $\mathcal{N} = 4$ super-Yang–Mills in four dimensions and fundamental strings in $\mathrm{AdS}_5 \times \mathbb{S}^5$. In this thesis we analyze the large charge sector of conformal field theories using EFT.

1.1 Conformal field theories

Conformal field theories (CFTs) generically arise at ultraviolet (UV) and infrared (IR) fixed points of renormalization group flow of Poincaré-invariant quantum field theories. At a fixed point, coupling constants do not run under renormalization group flow and therefore it is invariant under rescaling of the energy scale. In general[2] scale invariance is enhanced to conformal invariance, which is very powerful to study theories at fixed points.

Also, critical phenomena in statistical mechanics and condensed matter systems are often described in terms of CFTs.[3] When a system is approaching a second-order phase transition,

[1] The first correction to the linear spectrum predicted by the effective theory of rotating strings is later rederived by the S-matrix bootstrap [13,14].

[2] In two dimensions, it is proven [17] that unitary scale- and Poincaré-invariant fixed points of renormalization group flow are also conformal. The same statement is proven in four dimensions [18–20] in specific setups under mild assumptions.

[3] See [21,22] and references therein for condensed matter applications of CFT.

its correlation length diverges and the system becomes scaleless at longer distances than the intrinsic scales of the theory such as the lattice spacing.

An important point is that dynamical properties of a CFT do not depend on its microscopic description. For instance, let us take a three-dimensional Ising model on a cubic lattice, which describes antiferromagnetic systems such as FeF_2. At the critical temperature and in the long-distance limit, it is well known that the system exhibits the order-disorder phase transition and is described by a $\mathbb{Z}_2$-invariant CFT, which we call the Ising CFT in three dimensions. The same CFT is reached by taking the low-energy limit of the ϕ^4-theory in three dimensions in Euclidean signature, whose action is[4]

$$S = \int d^3x \left(\frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 \right). \tag{1.1}$$

In addition, the liquid-vapor critical point of water[5] has the same critical exponents as the three-dimensional Ising CFT. So, the same CFT appears in many, different physical systems, and physical observables do not depend on its microscopic description, that is, where it comes from. This property, called universality, allows CFT to describe physics in nature and thus makes it a very important class of quantum field theory.

The AdS/CFT correspondence [23] is another reason why CFT is worth being studied. Every CFT in d dimensions is believed to be equivalent to some UV-complete quantum gravitational theory in $(d+1)$-dimensional anti-de Sitter (AdS) spacetime. Especially, CFT gives a nonperturbative definition of quantum gravity in AdS, and one can study rather enigmatic gravitational physics such as the dynamics of black holes by analyzing CFT.

1.2 Conventional techniques

Despite its ubiquity and importance, CFTs are very often strongly coupled, and therefore the ordinary perturbative method loses its power in such theories. The following approaches are frequently used to tackle nonperturbative CFTs:

- The ϵ-expansion [24]. In order to study some, for instance, three-dimensional CFT such as the ϕ^4-theory above, one formally analytically continues the CFT to $d = 4 - \epsilon$ dimensions and regards ϵ as small. Physical observables are perturbatively expanded in powers of ϵ, and resumed somehow to find values in $d = 3$ dimensions. This approach is helpful only when the theory becomes free in the $\epsilon \to 0$ limit.

- The large-N expansion [25–27]. If a theory contains a large (either gauge or global) symmetry such as $\mathrm{U}(N)$ or $\mathrm{O}(N)$ with $N \gg 1$, one can perturbatively expand physical observables in N^{-1}. However, the parameter N is usually small in physically interesting systems, and therefore the large-N technique is not applicable straightforwardly.

[4] In order for the IR of this theory to be conformal, one must fine-tune the value of λ/m.

[5] There, the order parameter is the density.

- The numerical conformal bootstrap [28].[6] Using the crossing symmetry of four-point functions in CFT, one can rigorously numerically bound the spectrum of low-lying operators and their operator product expansion (OPE) coefficients. This method is quite successful in hunting up "minimal" interacting CFTs, *e.g.*, those with a small number of relevant operators and small stress-tensor two-point functions. On the other hand, it is computationally difficult to access to more generic, "nonminimal" theories.

- Monte Carlo simulations. If a CFT is realized as a system on a lattice, one may be able to use Monte Carlo methods to evaluate physical observables. However, a notorious problem, the so-called sign problem, often appears when one deal with systems involving fermions. Also, there are CFTs which cannot be put on a lattice, due to the gravitational anomaly [31, 32].

Still, there are quite many strongly coupled theories to which none of the above methods can be applied.

1.3 Large-charge expansion

Instead of dealing with a particular single theory, one can discuss general features of physical quantities in some *sector* of CFTs. CFTs with global symmetries exhibit alluring and useful simplifications in the sector of large global charge (either an internal symmetry such as spin, or an internal global symmetry such as a $\mathrm{U}(n)$ or $\mathrm{O}(n)$ symmetry). Thanks to these simplifications, one can compute physical quantities associated with operators carrying large global charge as asymptotic expansions in negative powers of the global charge. One prominent example is the large-spin expansion of the anomalous dimension and three-point functions of operators with large spin [33, 34]. This method is applicable to the most general CFTs, as it uses only the eikonal limit of the analytic conformal bootstrap and does not use any Lagrangian description as its input. The results of the large-spin expansion looks particularly intuitive in AdS holographic duals.

More recently, in [35] and related work [36–45] it was noted that CFTs with global symmetries simplify in some familiar strongly coupled systems, when one considers the operator dimensions and three-point functions of low-lying operators of large global charge J.[7] The large-global-charge expansion employs an effective-field-theoretic description of superfluids [48]. Superfluids are states with finite density and correspond to operators with large

[6] See also recent reviews [29, 30] and references therein.

[7] These large-quantum-number limits themselves appear to be special cases of an even more general situation, a "macroscopic limit" in which one takes pure or mixed states or density matrices into some extreme direction in Hilbert space. The "eigenstate thermalization hypothesis" (ETH) [46], when it holds, is perhaps the most famous example of this behavior. For a recent bootstrap derivation of some properties of the large charge behavior in [35–37] making use of Regge theory and the conceptual connection with the ETH, see [47].

global charge, *via* the operator-state correspondence reviewed in section 2.2. Vacuum correlation functions of operators with large global charge are mapped to correlators at finite charge density, and therefore can be computed in a systematic perturbation in J^{-1} by using the EFT of superfluids. As discussed in detail in section 3.1, the EFT works due to a large hierarchy between the IR cutoff $E_{\rm IR}$ and the scale defined by the charge density ρ, $E_{\rm IR} \ll |\rho|^{1/d-1}$, and the loops and the higher-derivatives are suppressed by powers of $E_{\rm IR}/\rho^{1/d-1}$. This situation is analogous to that of the chiral perturbation theory of pion physics [49], where one considers the EFT in the regime the pion mass m_π is much smaller than the scale defined by the decay constant f_π, and physical observables are controlled by a perturbative expansion in m_π/f_π.

We regard these large-charge expansions as a new, useful tool for studying strongly coupled CFTs. Although operators with large global charge have the large operator dimension Δ and therefore highly irrelevant ($\Delta \gg d$) in the sense of renormalization group flow, it is possible to learn some physical consequences from them. First of all, we can bound the behavior of physical quantities in the large charge limit as we do in section 3 and 4, so theories violating the bounds at large charge cannot be unitary CFTs. This is similar in spirit to the bounds found in the numerical bootstrap [28], but is complementary in the sense that the numerical conformal bootstrap usually bounds quantities associated with operators of low dimension whereas the large-charge expansion gives constraints for operators of large dimension. Also, the large-charge expansion may be useful for studying quantum gravity in AdS *via* the AdS/CFT correspondence. In [50] it has been pointed out that the scaling behavior of operator dimensions studied in [35] is precisely that of the extremal non-Bogomol'nyi–Prasad–Sommerfeld (BPS) AdS-Reissner–Nordström (RN) black brane, which is a electrically charged solution of the Einstein–Maxwell theory in AdS. Since the models studied in [35] do not have weakly coupled gravity duals, the result of [35] does not give immediate strong implications for notorious problems in quantum gravity such as the weak gravity conjecture [51] for now. However, given the nontrivial correspondence between the CFT operator dimensions and the energy of the extremal AdS-RN black blane at large charge, it may be possible that the same technique applied to CFTs with weakly coupled gravity description shed light on these problems. Furthermore, we note that condensed matter literature [52] has studied a very similar EFT description of a strongly interacting Fermi gas, which is realized in experiments (see, *e.g.*, [53] and references therein), where the large number density suppresses the loops and higher-derivatives.

Thus, the large-charge expansion has a potentially wide range of applications in physics. So far, most of the existing literature studying the large-global-charge expansion of CFTs has considered the universality classes including the Wilson–Fisher fixed points with an $\mathrm{O}(N)$ global symmetry [35–39], with a notable exception being the large-R-charge expansion of four-dimensional $\mathcal{N}=4$ super-Yang–Mills theory in the large-N limit [15,16]. In particular, a general structure of the EFT at large charge in the case of CFTs with a family of Lorentz-invariant vacua, *i.e.*, moduli space, is completely unknown. CFTs with moduli space very often have a fermionic spacetime symmetry called supersymmetry, and then the conformal group is enlarged to a supergroup, termed the superconformal group.

There are various motivations to study supersymmetric CFTs (SCFTs) with moduli space of vacua using the large-global-charge expansion. First of all, the behavior of SCFTs with moduli space in the large-global-charge limit is very different from that of generic non-supersymmetric CFTs because of special properties of SCFTs such as a stronger unitarity bound called the BPS bound, and the existence of a ring structure formed by certain scalar operators. In addition, it is worth pointing out that some SCFTs do not have a known UV Lagrangian description (see, *e.g.*, [54]). In these theories, it is impossible to compute nontrivial physical quantities using conventional methods such as the Monte Carlo simulations. The large-global-charge expansion can be regarded as a new tool to study such mysterious SCFTs. Furthermore, it is suggested that a three-dimensional SCFT with moduli space, which is exactly the one we study in section 3, may be realized in systems where supersymmetry emerges at quantum critical points [55]. So, the large-charge expansion may be useful to estimate the measurable critical exponents associated with operators carrying large charge density.

1.4 Summary of this thesis

In this thesis, we explore the large-charge expansion in three- and four-dimensional SCFTs with a continuous family of Lorentz-invariant vacua (moduli space), in which behavior of physical quantities at large charge is interesting and very different from that in theories with no moduli space.

This thesis is organized as follows.

In section 2 we review basic facts about CFT such as the conformal algebra, radial quantization, the unitarity bounds and OPE. These are necessary to understand the later sections.

In section 3 we consider the spectrum of operators in a $\mathcal{N} = 2$ SCFT in three dimensions with one-complex-dimensional moduli spaces, that is the IR fixed point of three free chiral superfields X, Y, Z perturbed by a relevant superpotential $W = XYZ$. We develop an EFT on each branch of moduli space, and compute the operator dimensions of certain low-lying operators with large $\mathrm{U}(1)_\mathrm{R}$-charge J. This EFT is quite different from the one describing the nonsupersymmetric O(2) Wilson–Fisher CFT in three dimensions, as it should be since there the operator dimension Δ carrying large charge J behaves like $\Delta \propto J^{3/2}$, whereas in the present case the lowest operator dimension should be linear in R-charge due to the unitarity bound. While the superconformal primary operator with the lowest dimension is a chiral primary, the scalar primary with the second-lowest dimension is in a so-called semishort representation, with dimension exactly $J + 1$, a fact that is hard to discern from the Lagrangian of the $W = XYZ$ theory. The scalar primary operator with the third-lowest dimension is in a long representation with dimension $J + 2 - bJ^{-3} + O(J^{-4})$, with b being an unknown positive constant. The coefficient b is proportional to the lowest-derivative superconformally invariant (or super-Weyl-invariant) interaction term in the effective action on moduli space, which we explicitly construct in section 3.1.3. The fact that b is positive is not a consequence of supersymmetry, but follows from unitarity of the forward $2 \to 2$ scattering amplitude and the absence of superluminal fluctuations in the low-energy dynamics of the moduli field [56]. We also present a proof of a lemma which states that scalar semishort primary operators can be naturally considered as a module over the chiral ring, by usual multiplication of local operators. This lemma, together with the existence of scalar semishort primary operators at large charge J, proves that scalar semishort primary operators exist for all values of $J \in 2\mathbb{N}/3$. The contents in section 3 are heavily based on work [43] done in collaboration with Simeon Hellerman and Masataka Watanabe.

It is very interesting to compare predictions made from the effective-field-theoretic analysis in the large-charge limit, with other methods which make maximal use of the full superconformal symmetry. Recent studies [1–4] have shown that correlation functions of a number of chiral primary operators and a single antichiral primary operator in $\mathcal{N} = 2$ Lagrangian SCFTs in four dimensions can be exactly computed by supersymmetric localization. In order to compare predictions from the EFT with these exact results, in section 4 we consider two-point functions of chiral and antichiral primary operators with large $\mathrm{U}(1)_{\mathrm{R}}$-charge, in $\mathcal{N} = 2$ SCFTs in four dimensions using the EFT at large $\mathrm{U}(1)_{\mathrm{R}}$-charge. We restrict ourselves on the case of one-complex-dimensional Coulomb branch, and make use of the effective-field-theoretic techniques developed in section 3, to evaluate the asymptotic behavior of the two-point function

$$\mathcal{Y}_n := |x-y|^{2n\Delta_{\mathcal{O}}} \left\langle \mathcal{O}^n(x)\bar{\mathcal{O}}^n(y)\right\rangle \tag{1.2}$$

in the limit where the operator insertion $\mathcal{O}^n$ has large R-charge $2n\Delta_{\mathcal{O}}$, which is equivalent to taking $n \to \infty$. The asymptotic expansion of $\mathcal{Y}_n$ at large n is shown to be nontrivial but universal, and of the form

$$\mathcal{Y}_n = (n\Delta_{\mathcal{O}})! \left|\frac{\mathbf{N}_{\mathcal{O}}}{2\pi}\right|^{2n\Delta_{\mathcal{O}}} (n\Delta_{\mathcal{O}})^{\alpha} \tilde{\mathcal{Y}}_n, \tag{1.3}$$

where $\tilde{\mathcal{Y}}_n$ converges to a constant in the large-n limit, and $\mathbf{N}_{\mathcal{O}}$ is a constant which is independent of n and characterizes the normalization of $\mathcal{O}$ relative to the effective Abelian vector multiplet scalar. We also show that the constant α is positive and proportional to the difference between the so-called a-coefficient of the Weyl anomaly of the underlying SCFT and that of the EFT of the Coulomb branch. For SCFTs with a Lagrangian description,[8] we compare our predictions from the EFT for the logarithm of the two-point function $\mathcal{B}_n = \log \mathcal{Y}_n$, up to $O(\log n)$, with exact computations obtained by supersymmetric localization [1–4]. In the case of $\mathrm{SU}(2)$ $\mathcal{N} = 4$ super-Yang–Mills theory we achieve

[8] Most of known SCFTs with one-complex-dimensional Coulomb branch do not have a known Lagrangian description. [57–60]

explicit analytic agreement, and in the case of SU(2) $\mathcal{N}=2$ supersymmetric quantum chromodynamics (SQCD) with four fundamental hypermultiplets we see fairly nice numerical agreement at $n \simeq 30$ employing the zero-instanton approximation to the sphere partition function. The contents in section 4 are heavily based on work [44] done in collaboration with Simeon Hellerman.

Section 5 is devoted to the conclusion.

2 Review on conformal field theories

We first review basic properties of CFTs in $d \geq 3$ dimensions,[9] which are necessary to understand the later sections. See, *e.g.*, [29, 30, 64–66] for more complete reviews. In this section we discuss what the conformal symmetry and unitarity can say about a general CFT. Their consequences will turn out to be very powerful, and play a very important role in later sections where we derive the large-R-charge expansion of dynamical quantities in SCFTs in three and four dimensions.

2.1 Conformal algebra

The conformal group is the subgroup of general coordinate transformations $x_\mu \mapsto x'_\mu(x)$ which leaves the spacetime metric invariant up to an overall factor,

$$g_{\mu\nu}(x) \mapsto g'_{\mu\nu}(x') = c(x) g_{\mu\nu}(x). \tag{2.1}$$

We work in Euclidean signature, where the conformal group is $\mathrm{SO}(d+1,1)$ in d dimensions. The defining property (2.1) of conformal transformations implies that the Jacobian of the coordinate transformation $x_\mu \mapsto x'_\mu(x)$ must be of the form

$$\frac{\partial}{\partial x_\nu} x'_\mu(x) = \sqrt{c(x)}\, \mathsf{M}_{\mu\nu}(x), \tag{2.2}$$

with some matrix $\mathsf{M}(x) \in \mathrm{SO}(d)$ and $c(x)$ being the factor which appears in (2.1).

Conformal transformations are generated by the $\mathrm{SO}(d)$ rotation $M_{\mu\nu}$, momentum P_μ, special conformal K_μ and dilatation D, which are given by

$$M_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu), \quad P_\mu = -i\partial_\mu, \quad K_\mu = i\big(2x_\mu x_\nu \partial_\nu - |x|^2 \partial_\mu\big), \quad D = i x_\mu \partial_\mu, \tag{2.3}$$

[9] Two-dimensional CFTs are special due to the infinite-dimensional Virasoro symmetry [61]. For details, see [62, 63] and references therein.

where the Greek indices run over $1, 2, \cdots, d$. Their commutation relations are given by[10]

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = -i(\delta_{\mu\sigma}M_{\nu\rho} + \delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\rho}M_{\nu\sigma} - \delta_{\nu\sigma}M_{\mu\rho}), \\
&[M_{\mu\nu},\ P_\rho\ ] = -i(\delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu), \\
&[M_{\mu\nu},\ K_\rho\ ] = -i(\delta_{\nu\rho}K_\mu - \delta_{\mu\rho}K_\nu), \\
&[\ D\ ,\ P_\rho\ ] = -iP_\rho, \\
&[\ D\ ,\ K_\rho\ ] = +iK_\rho, \\
&[\ P_\mu\ ,\ K_\rho\ ] = -2i(\delta_{\mu\rho}D + M_{\mu\rho}),
\end{aligned} \tag{2.4}$$

with all other commutators vanishing. In our conventions the action of the dilatation generator D on an operator $\mathcal{O}$ at the origin is[11] $[D, \mathcal{O}(0)] = -i\Delta_{\mathcal{O}}\mathcal{O}(0)$, where $\Delta_{\mathcal{O}}$ is the conformal dimension of $\mathcal{O}$ which is nonnegative real number in unitary CFTs, as shown in section 2.4. Note that P_μ (K_μ) are raising (lowering) operators with respect to D. Since D and $M_{\mu\nu}$ commute we can simultaneously diagonalize them and label physical states by the conformal dimension and the SO(d) spins.

Since special conformal transformations generated by K_μ may be less familiar than the others, let us make some comments. A finite special conformal transformation maps a point x_μ in $\mathbb{R}^d$ to

$$\frac{x_\mu + v_\mu|x|^2}{1 + 2v_\mu x_\mu + |v|^2|x|^2}, \tag{2.5}$$

where v_μ is an arbitrary real vector. From (2.5) it is easy to show that a special conformal transformation is generated by an inversion

$$I : x_\mu \mapsto x'_\mu := -\frac{x_\mu}{|x|^2}, \tag{2.6}$$

followed by a translation $x'_\mu \mapsto x'_\mu + v_\mu$ followed by a second inversion. We note that we do not claim that the inversion operation I is a symmetry of CFTs in general.[12] One can show that I belongs to $\mathrm{O}(d+1, 1)$ but not to $\mathrm{SO}(d+1, 1)$, so I is merely an outer automorphism of the conformal group. Instead, the combined action of the inversion and a parity transformation such as $x_{\mu=1} \mapsto -x_{\mu=1}$ is an element of the conformal group $\mathrm{SO}(d+1, 1)$ [66]. Therefore, CFTs invariant under parity are invariant under the inversion and vice versa. Chern–Simons-matter CFTs in three dimensions (see, *e.g.*, [77, 78] and references therein) are the typical examples of parity-violating CFTs. As we will see in section 2.3, the action of I is equivalent

[10]These commutation relations are the same as those used in, *e.g.*, [67, 68].

[11]Hereafter we assume D is diagonalizable, which is the case in reflection-positive theories, as we discuss in section 2.3. In theories where reflection positivity is lost D can admit a nontrivial Jordan block decomposition. Details can be found in [69–72] for two-dimensional theories and [73] for higher-dimensional ones.

[12]Related comments are made in [29, 30, 66, 74–76].

to the time reversal on the cylinder $\mathbb{R}\times\mathbb{S}^{d-1}$, which may or may not be a symmetry of the theory on the cylinder.

It is known [79–81] that special conformal invariance requires the stress tensor to be traceless in flat space,

$$T_{\mu\mu}=0, \tag{2.7}$$

so that the corresponding Noether current $J^K_{\mu\rho}$ can be constructed as

$$J^K_{\mu\rho}=\big(\delta_{\nu\rho}|x|^2-2x_\nu x_\rho\big)T_{\mu\nu}. \tag{2.8}$$

2.1.1 Primary states and descendants

A primary state is defined as a state annihilated by the generators of special conformal transformations K_μ,

$$K_\mu\left|\mathsf{primary}\right\rangle=0,\qquad \mu=1,2,\cdots,d. \tag{2.9}$$

Given a primary state $\left|\mathsf{primary}\right\rangle$, we can construct a series of state of higher conformal dimensions, which are dubbed as descendants, by acting on $\left|\mathsf{primary}\right\rangle$ with momentum operators. For any primary state, there is an associated infinite-dimensional representation, the Verma module, which is given by

$$\mathcal{V}_{\Delta,\{\ell\}}\coloneqq\mathrm{span}\left\{\prod_{\mu=1}^{d}P_\mu^{n_\mu}\left|\Delta,\{\ell\}\right\rangle\,\middle|\,n_\mu\in\mathbb{N}\right\}, \tag{2.10}$$

where $\left|\Delta,\{\ell\}\right\rangle$ is a primary state of conformal dimension Δ, *i.e.*,

$$D\left|\Delta,\{\ell\}\right\rangle=-i\Delta\left|\Delta,\{\ell\}\right\rangle, \tag{2.11}$$

and $\{\ell\}=\big\{\ell_1,\ell_2,\cdots,\ell_{\lfloor d/2\rfloor}\big\}$ denotes a set of SO(d) spins.

2.2 Radial quantization and the operator-state correspondence

Here we review radial quantization in CFT, which allows one to identify states in the Hilbert space with local operators. This property will be essential when we derive the anomalous dimension of near-BPS operators carrying large R-charge in section 3.

2.2.1 General remarks

In general, quantization of a quantum field theory is related to the choice of a foliation of the spacetime. That is, the d-dimensional spacetime is expressed as a series of $(d-1)$-dimensional spatial leaves which evolves in time, and each leaf is endowed with its own Hilbert space.

In CFTs, correlation functions of operators are the most basic observables. They are often defined by the path integral

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = \int [\mathcal{D}\phi]\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)e^{-S[\phi]}. \tag{2.12}$$

Once the foliation is specified, a correlation function can be interpreted as a time-ordered expectation value

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = \langle 0|\mathsf{T}\left[\widetilde{\mathcal{O}}_1(x_1)\widetilde{\mathcal{O}}_2(x_2)\cdots\widetilde{\mathcal{O}}_n(x_n)\right]|0\rangle\,, \tag{2.13}$$

where T denotes the time ordering with respect to the foliation, $|0\rangle$ is the vacuum state in the Hilbert space $\mathcal{H}$ living on a spatial leaf,[13] and $\widetilde{\mathcal{O}}_i(x_i)$ are quantum operators acting on $\mathcal{H}$ which correspond to the path integral insertions $\mathcal{O}_i(x_i)$.

A different foliation of the spacetime leads to a different Hilbert space and a time ordering T'. However, if we rearrange the right-hand side of (2.13) we get the same correlator on the left-hand side of (2.13).

There is a particularly useful quantization scheme in CFTs. In d-dimensional theories, let us take a series of $(d-1)$-spheres of various radii with center at the origin as the foliation, and the associated time direction as the radial direction. Therefore, the Hamiltonian, which generates the time translation, is identified with the dilatation generator D.

This choice of the foliation is called radial quantization. An important fact is that there is a bijection between local operators and states in the Hilbert space in radial quantization. Let us review how this correspondence works in theories with a Lagrangian description following [30, 82].

2.2.2 To states from local operators

First we would like to see how to construct states in the Hilbert space in radial quantization from operator insertions. Let $|\phi_b\rangle$ be a field eigenstate, where $\phi_b = \phi_b(n)$ is a field configuration on a sphere of, for instance, unit radius $r = 1$ and n is a vector of unit length. A set of field eigenstates spans the Hilbert space in radial quantization. Then, a general state $|\Phi\rangle$ is expressed as a linear combination of field eigenstates,

$$|\Phi\rangle = \int [\mathcal{D}\phi_b]\,|\phi_b\rangle\,\langle\phi_b|\Phi\rangle\,, \tag{2.14}$$

where the measure $[\mathcal{D}\phi_b]$ indicates the path integral is over the field on unit $\mathbb{S}^{d-1}$.

For the vacuum $|\Phi\rangle = |0\rangle$, the inner product $\langle\phi_b|0\rangle$, *i.e.*, the wave function in the "position representation", is given by the path integral over the interior of unit $\mathbb{S}^{d-1}$ with

[13] Hereafter we assume that a vacuum state exists as a unique state belonging to the kernel of all the conformal generators.

boundary condition $\phi(r=1,n)=\phi_b(n)$ and without operator insertions,

$$\langle\phi_b|0\rangle = \int\limits_{\substack{\phi(r=1,n)=\phi_b(n)\\ r\leq 1}} [\mathcal{D}\phi(r,n)]e^{-S[\phi]}. \tag{2.15}$$

A general state created by an operator $\mathcal{O}$ of definite conformal dimension can be defined in a similar manner. Let us define a state $\mathcal{O}(r,n)\,|0\rangle$ by the path integral with an operator insertion,

$$\langle\phi_b|\mathcal{O}(r,n)|0\rangle \coloneqq \int\limits_{\substack{\phi(r=1,n)=\phi_b(n)\\ r\leq 1}} [\mathcal{D}\phi(r,n)]e^{-S[\phi]}\mathcal{O}(r,n). \tag{2.16}$$

In this language, we can state that the vacuum state $|0\rangle$ is defined by inserting the identity operator $\mathcal{O}=\mathbb{1}$ in the path integral.

2.2.3 To local operators from states and back

Now we would like to show operators can be constructed from states. Let $|\Phi\rangle$ be a state in the Hilbert space in radial quantization, which is an eigenstate of the dilatation operator with dimension Δ. We would like to define from $|\Phi\rangle$ correlation functions of local operators involving an operator Φ. We cut a $(d-1)$-ball B of unit radius centered at the origin in the path integral with operator insertions outside B, and glue the state $|\Phi\rangle$ at the boundary of B. From this procedure, we have

$$\int [\mathcal{D}\phi_b]\,\langle\phi_b|\Phi\rangle \int\limits_{\phi|_{\partial B}=\phi_b} [\mathcal{D}\phi]e^{-S[\phi]}\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n), \qquad x_1,\ x_2,\ \cdots,\ x_n\notin B, \tag{2.17}$$

where the first path integral with the measure $[\mathcal{D}\phi_b]$ is carried out over the field on the boundary ∂B, and the second path integral is performed over the region outside B with the boundary condition $\phi\Big|_{\partial B}=\phi_b$ on ∂B. The quantity (2.17) should behave exactly the same way as the correlation function of local operators

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n)\Phi(0)\rangle\,, \tag{2.18}$$

under symmetry transformations. If there are insertions inside the $(d-1)$-ball B of unit radius, we can perform a scaling symmetry transformation $x\mapsto\lambda x$ with sufficiently large λ so that the insertions become sufficiently far from the origin. On the other hand, in the limit where B shrinks to a point, the path integral over the boundary of B can be regarded as defining a local operator at the origin. We emphasize that the dilatation symmetry is crucial in this construction.

The mapping from states to operators made here is by construction inverse to the one in section 2.2.2. This is called the state-operator correspondence [83], which states that there is a one-to-one mapping between local operators at the origin and states in the Hilbert space in radial quantization,

$$\mathcal{O}(0) \longleftrightarrow |\mathcal{O}\rangle = \mathcal{O}(0)\,|0\rangle\,. \tag{2.19}$$

For a primary operator $\mathcal{O}_{\Delta,\{\ell\}}$ of conformal dimension Δ and SO(d) spin $\{\ell\}$, we have the correspondence

$$\begin{aligned} \left[\ D\ \ ,\mathcal{O}_{\Delta,\{\ell\}}(0)\right] &= \quad -i\Delta\mathcal{O}_{\Delta,\{\ell\}}(0) \quad &\longleftrightarrow \quad \quad D\left|\mathcal{O}_{\Delta,\{\ell\}}\right\rangle \quad &= \quad -i\Delta\left|\mathcal{O}_{\Delta,\{\ell\}}\right\rangle, \\ \left[M_{\mu\nu},\mathcal{O}_{\Delta,\{\ell\}}(0)\right] &= -i\Sigma^{\{\ell\}}_{\mu\nu}\mathcal{O}_{\Delta,\{\ell\}}(0) \quad &\longleftrightarrow \quad M_{\mu\nu}\left|\mathcal{O}_{\Delta,\{\ell\}}\right\rangle &= -i\Sigma^{\{\ell\}}_{\mu\nu}\left|\mathcal{O}_{\Delta,\{\ell\}}\right\rangle, \end{aligned} \tag{2.20}$$

where $\Sigma^{\{\ell\}}_{\mu\nu}$ is the matrix acting on the spin indices.

In the discussion above we have assumed there is a Lagrangian description, but it is believed that the operator-state correspondence works for theories without an UV Lagrangian description.

2.2.4 Cylindrical point of view

One can perform a Weyl transformation which transforms flat space $\mathbb{R}^d$ to the cylinder $\mathbb{R}\times\mathbb{S}^{d-1}$,

$$\begin{aligned} ds^2\left[\mathbb{R}^d\right] = dr^2 + r^2 ds^2\left[\mathbb{S}^{d-1}\right] &= r^2\left((d\log r)^2 + ds^2\left[\mathbb{S}^{d-1}\right]\right) \\ = e^{2\tau}\left(d\tau^2 + ds^2\left[\mathbb{S}^{d-1}\right]\right) &= e^{2\tau} ds^2\left[\mathbb{R}\times\mathbb{S}^{d-1}\right], \end{aligned} \tag{2.21}$$

where $\tau := \log r$ plays a role of the time coordinate on the cylinder. A dilatation $r \to \lambda r$ is equivalent to a shift $\tau \to \tau + \log\lambda$. Radial quantization on flat space $\mathbb{R}^d$ is equivalent to the usual quantization on the cylinder $\mathbb{R}\times\mathbb{S}^{d-1}$. An operator at the origin of flat $\mathbb{R}^d$ is mapped to an operator at the past infinity $\tau \to -\infty$ on the cylinder, which specifies the initial boundary condition. That is, the path integral on flat $\mathbb{R}^d$ with an operator insertion $\mathcal{O}(0)$ at the origin is equivalent to the path integral on the cylinder with a fixed initial state $|\mathcal{O}\rangle$.

Under a general Weyl transformation $g_{\mu\nu} = \delta_{\mu\nu} \to [\Omega(x)]^2\delta_{\mu\nu}$, a scalar operator $\mathcal{O}$ of conformal dimension Δ transforms as[14]

$$\mathcal{O}(x) \to [\Omega(x)]^{\Delta}\mathcal{O}(x), \tag{2.22}$$

and correlation functions on flat space and on the cylinder are related to each other by[15]

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_{\delta_{\mu\nu}} = \langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_{[\Omega(x)]^2\delta_{\mu\nu}}\prod_{i=1}^{n}[\Omega(x_i)]^{\Delta_i}. \tag{2.23}$$

[14]See, *e.g.*, [29] for a general conformal symmetry action on operators with spin.

[15]Here we have implicitly normalized the correlation functions dividing by the partition functions, so that the Weyl anomaly in even dimensions [84, 85] cancels between the denominator and the numerator.

In the case of the cylinder, $\Omega(x) = e^{-\tau}$ and we define a scalar operator on the cylinder by

$$\mathcal{O}_{\text{cylinder}}(\tau, n) \coloneqq e^{\tau\Delta}\mathcal{O}_{\text{flat}}(r, n) \tag{2.24}$$

and the general relation (2.23) becomes

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_{\mathbb{R}^d} = \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_{\mathbb{R}\times\mathbb{S}^{d-1}} \prod_{i=1}^{n} e^{-\tau\Delta_i}. \tag{2.25}$$

2.3 Unitarity and reflection positivity

2.3.1 On flat space

In Lorentzian signature we are interested in unitary theories, where the energy and momentum operators are Hermitian and the evolution is unitary. When Wick rotated to Euclidean signature, unitarity becomes a property dubbed as reflection positivity.

In unitary field theories in Lorentzian signature, a local operator $\mathcal{O}_{\text{L}}(t, \mathbb{x})$ obeys a unitary spacetime translation,

$$\mathcal{O}_{\text{L}}(t, \mathbb{x}) = e^{iHt - i\mathbb{x}\cdot\mathbb{P}_{\text{L}}}\mathcal{O}_{\text{L}}(0,0)e^{-iHt + i\mathbb{x}\cdot\mathbb{P}_{\text{L}}}, \tag{2.26}$$

where the generators H and $\mathbb{P}_{\text{L}}$ are Hermitian in the usual flat space quantization. For the sake of simplicity suppose that $\mathcal{O}_{\text{L}}(t, \mathbb{x})$ is a Hermitian scalar operator,

$$[\mathcal{O}_{\text{L}}(t, \mathbb{x})]^\dagger = \mathcal{O}_{\text{L}}(t, \mathbb{x}), \tag{2.27}$$

and let us Wick rotate the theory to Euclidean signature. Then, (2.26) becomes

$$\mathcal{O}_{\text{E}}(t_{\text{E}}, \mathbb{x}) \coloneqq \mathcal{O}_{\text{L}}(-it_{\text{E}}, \mathbb{x}) = e^{Ht_{\text{E}} - i\mathbb{x}\cdot\mathbb{P}_{\text{L}}}\mathcal{O}_{\text{L}}(0,0)e^{-Ht_{\text{E}} + i\mathbb{x}\cdot\mathbb{P}_{\text{L}}}, \tag{2.28}$$

which implies that the Euclidean operator $\mathcal{O}_{\text{E}}(t_{\text{E}}, \mathbb{x})$ satisfies

$$[\mathcal{O}_{\text{E}}(t_{\text{E}}, \mathbb{x})]^\dagger = \mathcal{O}_{\text{E}}(-t_{\text{E}}, \mathbb{x}). \tag{2.29}$$

Let us call an Euclidean operator real if it is Hermitian in Lorentzian signature. Unlike the Lorentzian case, the Hermitian conjugation in Euclidean signature depends on which direction we choose as the time direction. Let us consider a state

$$|\psi\rangle \coloneqq \mathcal{O}_{\text{E},1}(t_{\text{E},1}, \mathbb{x}_1)\mathcal{O}_{\text{E},2}(t_{\text{E},2}, \mathbb{x}_2)\cdots\mathcal{O}_{\text{E},n}(t_{\text{E},n}, \mathbb{x}_n)\,|0\rangle \tag{2.30}$$

with $\mathcal{O}_{i,\text{E}}$ being real Euclidean operators. The conjugate of this state is

$$\langle\psi| = (|\psi\rangle)^\dagger = \langle 0|\,\mathcal{O}_{\text{E},1}(-t_{\text{E},1}, \mathbb{x}_1)\mathcal{O}_{\text{E},2}(-t_{\text{E},2}, \mathbb{x}_2)\cdots\mathcal{O}_{\text{E},n}(-t_{\text{E},n}, \mathbb{x}_n) \tag{2.31}$$

That is, $\langle\psi|$ is given by the vacuum state in the future acted on by operators in a time-reflected way. If the theory under consideration is equivalent to a unitary Lorentzian theory *via* the Wick rotation, the positivity of the norm

$$\langle\psi|\psi\rangle > 0 \tag{2.32}$$

should hold. In Euclidean signature, from (2.31) we see that any $2n$-point functions with operators inserted in a time-reflection-invariant way must be positive. This property is called reflection positivity.

2.3.2 On the cylinder

Let us consider reflection positivity for CFTs on the cylinder. We define the Hermitian conjugation operation on the conformal generators as

$$P_\mu^\ddagger = K_\mu, \qquad M_{\mu\nu}^\ddagger = -M_{\mu\nu}, \qquad D^\ddagger = D. \tag{2.33}$$

Here, we have used the symbol $\ddagger$ for the Hermitian conjugation in radial quantization, instead of $\dagger$ in the usual quantization in flat space, to emphasize that the Hermitian conjugation operation depends on the quantization scheme used. This definition of the Hermitian conjugation is an involutive anti-automorphism of the Euclidean conformal algebra (2.4), known as the Belavin–Polyakov–Zamolodchikov conjugation [61], and is geometrically equivalent to an inversion (2.6). To see the latter statement, let us first understand how the conformal generators are expressed in the cylindrical variables (τ, n). Assuming $(r, n_1, \cdots, n_{d-1})$ as independent variables and $n_d \equiv \sqrt{1-\left(n_1^2+\cdots+n_{d-1}^2\right)}$, from (2.3) we have

$$\begin{aligned}
M_{\mu\nu} &= -i\left[n_\mu(1-\delta_{\nu d})\frac{\partial}{\partial n_\nu} - n_\nu(1-\delta_{\mu d})\frac{\partial}{\partial n_\mu}\right], \\
P_\mu &= -ie^{-\tau}\left[n_\mu\frac{\partial}{\partial\tau} + (1-\delta_{\mu d})\frac{\partial}{\partial n_\mu} - \sum_{j=1}^{d-1} n_\mu n_j\frac{\partial}{\partial n_j}\right], \\
K_\mu &= +ie^{+\tau}\left[n_\mu\frac{\partial}{\partial\tau} - (1-\delta_{\mu d})\frac{\partial}{\partial n_\mu} + \sum_{j=1}^{d-1} n_\mu n_j\frac{\partial}{\partial n_j}\right], \\
D &= i\frac{\partial}{\partial\tau}.
\end{aligned} \tag{2.34}$$

From this expression, we see that an inversion $(\tau, n) \mapsto (-\tau, -n)$ and a complex conjugation transform the conformal generators transform in the same way as the conjugation (2.33). Therefore, real Euclidean operators on the cylinder satisfies the property

$$[\mathcal{O}_{\mathrm{E,cylinder}}(\tau, n)]^\ddagger = \mathcal{O}_{\mathrm{E,cylinder}}(-\tau, -n). \tag{2.35}$$

2.4 Unitarity bounds

Demanding unitarity in CFTs in Lorentzian signature, or equivalently reflection positivity in Euclidean signature, imposes several bounds on conformal dimensions of primary states [67, 86, 87]. Here, we derive the bounds using only conformal algebra. Similar bounds for $\mathcal{N}=2$ SCFTs in three dimensions are discussed in appendix A.

2.4.1 Level one

Let $|\Delta,\{\ell\}\rangle$ be a primary state of conformal dimension Δ and the $\mathrm{SO}(d)$ spin $\{\ell\}$. Assuming reflection positivity, we impose the condition that the matrix element

$$A_{\mu\nu} \coloneqq \langle\Delta,\{\ell\}|P^{\dagger}_{\mu}P_{\nu}|\Delta,\{\ell\}\rangle = \langle\Delta,\{\ell\}|K_{\mu}P_{\nu}|\Delta,\{\ell\}\rangle \tag{2.36}$$

have only nonnegative eigenvalues. Using (2.4) and (2.20), we have

$$A_{\mu\nu} = 2\,\langle\Delta,\{\ell\}|\big(\delta_{\mu\nu}\Delta-\Sigma^{\{\ell\}}_{\mu\nu}\big)|\Delta,\{\ell\}\rangle\,. \tag{2.37}$$

Now, we rewrite the spin matrix $\Sigma^{\{\ell\}}_{\mu\nu}$ as

$$\Sigma^{\{\ell\}}_{\mu\nu} = \frac{1}{2}L_{\mu\nu,\mu'\nu'}\Sigma^{\{\ell\}}_{\mu'\nu'}, \qquad L_{\mu\nu,\mu'\nu'} \coloneqq (\delta_{\mu\mu'}\delta_{\nu\nu'}-\delta_{\mu\nu'}\delta_{\nu\mu'}). \tag{2.38}$$

The matrix L defined here is the generator of $\mathrm{SO}(d)$ in the vector representation V. Writing $A=(\mu',\nu')$ for an adjoint index of $\mathrm{SO}(d)$ and interpreting L^A and Σ_A to be acting on the representation $\mathsf{V}\otimes\mathsf{R}_{\{\ell\}}$, where $\mathsf{R}_{\{\ell\}}$ is the representation of $\mathrm{SO}(d)$ to which the state $|\Delta,\{\ell\}\rangle$ belongs, we have

$$\big(L^A\Sigma_A\big)_{\mu\nu} = \frac{1}{2}\big[(V+\Sigma)^2-V^2-\Sigma^2\big]. \tag{2.39}$$

Then, the requirement of the nonnegativity of the matrix (2.37) is equivalent to

$$\Delta \geq \frac{1}{2}\Big[c_2(\mathsf{R})+c_2(\mathsf{V})-\min_{\mathsf{R}'}c_2(\mathsf{R}')\Big], \tag{2.40}$$

where R' runs over all possible irreducible representations which appear in the Clebsch–Gordan decomposition of $\mathsf{R}\otimes\mathsf{V}$, and c_2 is the quadratic Casimir invariant of $\mathrm{SO}(d)$.

Let us explicitly evalaute the lower bounds of (2.40) for the spin-ℓ symmetric traceless representation, which has

$$c_2\Big(\boxed{1}\boxed{2}\boxed{\cdots}\boxed{\ell}\Big) = \ell(\ell+d-2), \qquad \ell=0,1,2,\cdots\,. \tag{2.41}$$

Furthermore, in this case it is easy to show that

$$\min_{\mathsf{R}'}c_2(\mathsf{R}') = c_2\Big(\boxed{1}\boxed{2}\boxed{\cdots}\boxed{\ell{-}1}\Big) = (\ell-1)(\ell+d-3), \qquad \ell=1,2,3,\cdots\,, \tag{2.42}$$

where R' runs over irreducible representations which appear in the Clebsch–Gordan decomposition of $\square \otimes \boxed{1}\boxed{2}\boxed{\cdots}\boxed{\ell}$. From (2.40), (2.41) and (2.42), we get the unitarity bound for scalar primaries ($\ell = 0$)

$$\Delta_{\ell=0} \geq 0, \tag{2.43}$$

and for primaries in the spin-ℓ symmetric traceless representation with $\ell \geq 1$,

$$\Delta_\ell \geq \ell + d - 2, \qquad \ell = 1, 2, 3, \cdots . \tag{2.44}$$

The bound (2.43) for scalar primary operators is saturated by the identity operator, whereas (2.44) is saturated by conserved current operators of spin ℓ. To see this is the case, we notice that the bound (2.40) for R being the spin-ℓ symmetric traceless representation is saturated when R' is the spin-$(\ell - 1)$ symmetric traceless representation, so the scalar state $P_\nu \left| J_{\nu\mu_1\cdots\mu_{\ell-1}} \right\rangle$ is null. This implies the corresponding primary operator $J_{\nu\mu_1\cdots\mu_{\ell-1}}$ satisfies the conservation law $\partial_\nu J_{\nu\mu_1\cdots\mu_{\ell-1}}(x) = 0$.

We note that conserved currents with $\ell = 1$ and $\ell = 2$ are usual global symmetry currents and stress tensors.[16] If there are conserved currents with $\ell \geq 3$ in $d \geq 3$ dimensions, the theory is free [88, 89].

Strictly speaking, if there are global symmetries in the theory under consideration there are always the associated Noether charge operators, but it is not guaranteed that these charge operators are generated by conserved current operators. In this thesis we postulate that CFTs with global symmetries always have the associated conserved current operators.

2.4.2 Level two

Let $|\Delta\rangle$ be a primary state of conformal dimension Δ and spin 0, and we consider the matrix element

$$\langle\Delta|(P\cdot P)^\dagger(P\cdot P)|\Delta\rangle = \langle\Delta|(K\cdot K)(P\cdot P)|\Delta\rangle = 8d\Delta\left(\Delta - \frac{d-2}{2}\right)\langle\Delta|\Delta\rangle\,. \tag{2.45}$$

If $\Delta = 0$, we see from (2.37) that the state $|\Delta = 0, \ell = 0\rangle$ is annihilated by all the conformal generators. That is, $\Delta = 0$ automatically means the corresponding operator is the identity operator. On the other hand, if $\Delta > 0$, in order for this matrix element to be nonnegative, we need the condition

$$\Delta \geq \frac{d-2}{2}. \tag{2.46}$$

This bound is saturated by a free scalar. Indeed, if the bound is saturated (2.45) implies that the state $(P\cdot P)\,|\Delta\rangle$ is null. Since $P_\mu = -i\partial_\mu$, this means in the operatorial language that the corresponding operator satisfies the Klein–Gordon equation $\partial^2\mathcal{O}(x) = 0$.

[16] There is always a unique stress tensor which generates global conformal transformations. However, if, for example, the theory under consideration is a sum of several decoupled CFTs, there are multiple spin-2 conserved currents.

It can be shown that considering higher levels than two (*i.e.*, descendants with three or more P_μ's) does not yield tighter bounds than obtained above. See [86] for details in four dimensions.

2.5 Operator product expansion

Operator product expansion (OPE) is the idea that we can expand a product of operators, in the limit where they are close to each other, in terms of local operators. In general, a state created by two operators $\mathcal{O}_1$ and $\mathcal{O}_2$, $\mathcal{O}_1(x)\mathcal{O}_2(0)\left|0\right\rangle$, is a linear combination of basis states of the Hilbert space. In CFT, since any state of definite dimension corresponds to a local operator at the origin, one can write

$$\mathcal{O}_1(x)\mathcal{O}_2(0)\left|0\right\rangle = \sum_i C_{ijk}(x,P)\mathcal{O}_i(0)\left|0\right\rangle , \tag{2.47}$$

where the sum runs over all the primary operators, and $C_{ijk}(x,P)$ represents an operator which encodes the contribution of the primary operator $\mathcal{O}_i$ and all of its descendants. The OPE (2.47) is often expressed as the operatorial equation,

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_i C_{ijk}(x,-i\partial)\mathcal{O}_i(0), \tag{2.48}$$

which is understood to be inside correlation functions.

One important aspect of OPE is that (2.48) converges if all the other operators in a correlation function are outside the ball of radius $|x|$. That is, if we write a general correlation function as

$$\begin{aligned} &\left\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\cdots\mathcal{O}_n(x_n)\right\rangle \\ &= \sum_i C_{12k}(x_1-x_2,-i\partial_{x_2})\left\langle \mathcal{O}_k(x_2)\mathcal{O}_3(x_3)\cdots\mathcal{O}_n(x_n)\right\rangle , \end{aligned} \tag{2.49}$$

it is converging as long as

$$|x_1-x_2| < \min_{i=3,\cdots,n} |x_i-x_2|. \tag{2.50}$$

This fact follows from the theorem which states that the overlap of two states is convergent when one of the two is expanded in terms of an orthonormal basis of the Hilbert space [90,91].

3 Operator dimensions from moduli

As mentioned in section 1, generic CFTs do not have to be weakly interacting, in the sense of lying in discrete or continuous families with certain limits in which the theory is simplified to a weakly interacting field theory or a theory which is exactly solvable in a more general manner. However strongly interacting theories may have a set of observables with limits in which they are calculable by using a weakly coupled field theory. The simplest such limit is that of large global charge.

In [35] and following work [36–39] various familiar CFTs in three dimensions with global symmetries have been shown to simplify in this sense, when the conformal dimensions of low-lying primary operators carrying large global charge J is considered. In [35] the authors have studied the Wilson–Fisher fixed point of the O(2) model and the IR fixed point of the so-called Wess–Zumino model, *i.e.*, the $\mathcal{N} = 2$ supersymmetric field theory which consists of a chiral superfield Φ and cubic superpotential $W = \Phi^3$. In each case, the conformal dimension of the lowest primary operator with global charge J is a scalar primary operator with dimension $\Delta_J = b_{3/2} J^{3/2} + b_{1/2} J^{1/2} - 0.093 + O\left(J^{-1/2}\right)$, where the unknown coefficients $b_{3/2}$, $b_{1/2}$ may be different in different theories but the constant term, which is equal to -0.093, is universal.[17] In the case of the O(2) Wilson–Fisher CFT, this large-charge behavior is in agreement with the results from lattice Monte Carlo simulations [45]. The common form of the large-global-charge expansions for Δ_J is because of the fact that an EFT in the same universality class governs both of the two CFTs at large global charge. The EFT contains a compact real scalar field $\chi \simeq \chi + 2\pi$ transforming as $\chi \to \chi + (\text{const.})$ under global symmetry transformations. One can use this EFT to compute the conformal dimensions of low-lying operators carrying large global charge by radially quantizing the theory on the sphere of radius r, and taking the Wilsonian cutoff Λ to satisfy $\mathsf{r}^{-1} \ll \Lambda \ll \sqrt{J}\mathsf{r}^{-1}$. Then higher-derivative interaction terms in the effective action and quantum loop corrections are both suppressed by inverse powers of large global charge J, and the RG equation at the IR fixed point obligates all Λ-dependence in physical quantities to vanish at each order in J. It is important to note that conformal invariance is a symmetry of the EFT at the quantum level, which forces the Λ-independent terms in the effective action to be classically conformally invariant (indeed, Weyl invariant in curved backgrounds) and, in addition, uniquely determines the form of the Λ-dependent terms in the effective action in terms of the Λ-independent terms.

The $\Delta_J \sim J^{3/2}$ scaling in the O(2) Wilson–Fisher and $\mathcal{N} = 2$ Wess–Zumino fixed points in three dimensions follows from the fact that both theories do not have a moduli space of Lorentz-invariant vacua on flat spacetime $\mathbb{R}^3$. Accordingly, the Ricci curvature of the spatial slice is never relevant for high-energy states on $\mathbb{S}^2$ (*i.e.*, states carrying large conformal dimension), and therefore the relationship between the charge density ρ and the energy density $\mathcal{H}$ in the large-global-charge ground state can only be $\mathcal{H} \sim |\rho|^{3/2}$.

In the case of the IR fixed point of the $\mathcal{N} = 2$ Wess–Zumino model, the $\Delta_J \sim J^{3/2}$ scaling follows from the fact that supersymmetry is broken spontaneously at large global charge (R-charge in this case). Supersymmetry must be broken because the chiral ring of

[17] The value of the constant term is corrected in [92] from the one originally appearing in [35], which misused ζ-function regularization.

the $\mathcal{N} = 2$ Wess–Zumino model truncates; the only chiral primary operators in this model[18] are the identity operator $\mathbb{1}$ and $\phi := \Phi\big|_{\theta=\bar{\theta}=0}$, and there is no chiral primary operator of the form ϕ^n with $n \geq 2$. Nevertheless it is hard to understand the fact that the breaking of supersymmetry is parametrically large at large R-charge from the structure of the chiral ring, and only the application of the EFT can reveal it.

Theories with an infinite-dimensional chiral ring exhibit a quite different behavior at large global charge. $\mathcal{N} = 2$ SCFTs with an infinite-dimensional chiral ring have a continuous family of Lorentz-invariant vacua on flat space $\mathbb{R}^3$, whose holomorphic coordinate ring forms the chiral ring [93]. Also, if the theory has a m-complex-dimensional moduli space of vacua, it is described by a freely-generated chiral ring with m algebraically independent ring generators, or $m + k$ generators with k algebraic relations.

When the curvature of the spatial slice vanishes, the existence of a moduli space indicates that the energy spectrum is degenerate, and therefore the curvature cannot be irrelevant in the relationship between the energy density $\mathcal{H}$ and the charge density ρ on the sphere, unlike the nondegenerate case. For SCFTs with an infinite-dimensional chiral ring, low-lying states carrying large R-charge satisfy $\mathcal{H} \sim |\rho|\sqrt{\mathcal{R}/2}$, where $\mathcal{R}$ is the Ricci scalar of the cylinder $\mathbb{R} \times \mathbb{S}^2$. When radially quantizing the theory, this means $\Delta_J \sim |J|$ at large J. This relation is satisfied exactly by the scalar primary state of R-charge J and the lowest dimension, which is always chiral or antichiral if the theory has an infinite-dimensional chiral ring, and J obeys a certain quantization condition.

Conceivably the simplest nontrivial SCFT having a moduli space of vacua may be the so-called XYZ theory, the IR fixed point of the $\mathcal{N} = 2$ supersymmetric field theory of three free chiral superfields X, Y, Z perturbed by a relevant superpotential $W = gXYZ$ where g is a coupling constant with mass dimension $[g] = [\text{mass}]^{1/2}$. This theory may find condensed matter applications in systems where supersymmetry emerges at quantum critical points [55]. At mass scales much lower than g^2 this theory flows to a strongly coupled SCFT in the sense that the OPE coefficients and anomalous dimensions of the fields[19] X, Y, Z are all of order one. As shown by a Leigh–Strassler type argument [94] and studied recently in [95], this SCFT has one exactly marginal operator, given by the integral of the superpotential

$$\int d^2\theta \big(X^3 + Y^3 + Z^3\big) + \text{c.c.}, \tag{3.1}$$

which breaks the $\mathrm{U}(1)_X \times \mathrm{U}(1)_Y \times \mathrm{U}(1)_Z$ flavor symmetry and lowers the dimension of the moduli space, but no exactly marginal operators uncharged under the full symmetry group. The complex dimension of the chiral ring of this SCFT is one since there are three generators with two algebraic relations. The three chiral ring generators X, Y, Z

[18]That is, primary operators annihilated by the supercharges $\bar{Q}_\alpha$, S^α and $\bar{S}^\alpha$. See appendix A for a review on the superconformal algebra and its representations.

[19]We will use the same notation for the superfields X, Y, Z and their $\theta = \bar{\theta} = 0$ bosonic components.

satisfy the algebraic relation $XY = XZ = YZ = 0$ modulo descendants, and therefore the moduli space of vacua consists of three branches, coordinatized by X, Y, Z, respectively. Correspondingly, the chiral ring of this SCFT is spanned by the monomials, $\{X^q, Y^q, Z^q \,|\, q \in \mathbb{N}\}$. As mentioned above, there are three independent U(1) flavor symmetries $\mathrm{U}(1)_{X,Y,Z}$ under which X, Y, and Z have charge $+1$, respectively. The R-charge J_R, which appears in the $\mathcal{N} = 2$ superconformal algebra in three dimensions (A.7), is a linear combination of the $\mathrm{U}(1)_{X,Y,Z}$-charges $J_{X,Y,Z}$, that is,

$$J_\mathrm{R} = \frac{2}{3}(J_X + J_Y + J_Z). \tag{3.2}$$

Primary operators in the chiral ring satisfy $\Delta = J_\mathrm{R}$. If primary operators are not in the chiral ring, but obey $(\Delta - J_\mathrm{R})/J_\mathrm{R} \ll 1$, one can regard them as "near-BPS", and their conformal dimensions should be calculable by a perturbative expansion in J_R^{-1}.

In this section we show that conformal dimensions of near-BPS operators can be computed in a straightforward manner by using a perturbative expansion in J_R^{-1} in the XYZ model. Especially, we extend the method of [35] to analyze the low-lying spectrum of near-BPS scalar primary operators carrying large R-charge on the "X-branch", *i.e.*, those satisfying $J_X \gg 1$, $J_\mathrm{R} \gg 1$, and $\Delta - J_\mathrm{R}$ and $J_X - 3J_\mathrm{R}/2$ of order one. We radially quantize the theory and utilize the EFT of the X-branch of moduli space, to calculate the conformal dimension, which is equivalent to the energy eigenvalue of the state on $\mathbb{S}^2$ *via* the state-operator correspondence. In the large-R-charge limit, the EFT of the X-branch of moduli space becomes practically useful: Both quantum loop corrections and higher-derivative terms in the effective action are suppressed by inverse powers of large R-charge.

We obtain various fascinating results:

- Whereas the primary state carrying large R-charge J_R and the lowest conformal dimension is always chiral, the primary state with the same R-charge and the next-to-lowest conformal dimension is also a scalar primary and in a "semishort" representation, which means that its $\bar{Q}^2$-descendant state has a vanishing norm and its conformal dimension is exactly equal to $J_\mathrm{R} + 1$. (Basic properties of the $\mathcal{N} = 2$ superconformal algebra and superconformal multiplets are summarized in appendix A.)

- The primary state with the same R-charge and the third-lowest conformal dimension is in a long multiplet and has an anomalous dimension of order J_R^{-3}.

- The anomalous dimension of order J_R^{-3} emerges from a insertion of the lowest-derivative interaction term invariant under superconformal (or super-Weyl) transformations in the effective action on the X-branch.

- We cannot compute the coefficient of this interaction term perturbatively, but by using unitarity of the forward $2 \to 2$ scattering amplitude of the moduli field it is possible to show that its sign is positive [56]. Consequently, the anomalous dimension of order $J_{\rm R}^{-3}$ of the scalar primary state with the third-lowest dimension is shown to be negative.

The contents in this section are based on work [43] done in collaboration with Simeon Hellerman and Masataka Watanabe.

3.1 Effective theory of the X-branch

As in [35], we first argue that at large values of the fields X, Y, Z, the effective action admits an expansion in powers of $\Lambda/E_{\rm UV}$, where Λ is the Wilsonian cutoff and $E_{\rm UV}$ is the UV scale defined by the vacuum expectation values (VEVs) of the scalar fields X, Y, Z. When the VEV of one of the three fields is nonzero and those of the other two are zero, the other two become massive, with masses much heavier than the Wilsonian cutoff Λ. By integrating these massive fields out, one can obtain an EFT for only one of the three.

3.1.1 Structure of the effective action

Let the VEV of X be nonzero and large. The EFT of the X field has the following form. In flat $\mathbb{R}^3$, each term in the effective action can be expressed as an integral over the full superspace,

$$\int d^3x d^2\theta d^2\bar{\theta}\mathcal{I}, \tag{3.3}$$

where $\mathcal{I}$ is an operator with $J_X = J_{\rm R} = 0$.

Terms in the effective action can be divided into two types: classical and quantum terms. Classical terms do not depend on Λ and invariant under the Weyl transformation, under which the X field transforms as $X \to e^{2\sigma/3}X$ and the fermionic superpartner ψ of X transforms as $\psi \to e^{7\sigma/6}\psi$. The scaling dimension of $\mathcal{I}$ has to be exactly 1.

Quantum terms depend on positive powers of Λ. They are not scale invariant as terms in the effective action since the Λ-dependence breaks scale invariance. These terms are in general of the form

$$\mathcal{I} = \Lambda^q \mathcal{I}_{1-q}, \tag{3.4}$$

where $\mathcal{I}_{1-q}$ is an operator of scaling dimension $1-q$ and $q > 0$. We note that the condition $q > 0$ is not always true in general EFTs: When a shell of modes between Λ and $\Lambda - \delta\Lambda$ is integrated out, the propagators are proportional to Λ^{-2} and naïvely it would seem that negative powers of Λ can appear in the effective action. Instead, the condition $q > 0$ follows from the fact that the EFT on moduli space is free at deep IR. As a result, the one-particle irreducible (1PI) effective action for X is finite when we expand it around a nonzero VEV, since X itself is an observable in the IR theory. So we conclude that the Wilsonian effective action must be finite in the $\Lambda \to 0$ limit, because the 1PI effective action is just the $\Lambda \to 0$ limit of the Wilsonian effective action.

As discussed above, our Wilsonian EFT is perturbatively controlled by taking $E_{\rm IR} \ll \Lambda \ll E_{\rm UV}$, where $E_{\rm IR}$ is an IR scale and $E_{\rm UV} = |X|^{3/2}$. We use conformal invariance as an input, which implies that the effect of integrating out a shell of modes and changing the Wilsonian cutoff infinitesimally from Λ to $\Lambda - \delta\Lambda$ has to be compensated exactly by rescaling the momenta by a factor of $(1 - \delta\Lambda/\Lambda)^{-1}$, and redefining the effective fields in order to retain the original normalization of the kinetic term. The RG equation can be used to obtain the coefficients of the Λ-dependent quantum terms from the Λ-independent classical terms, order by order in $\Lambda/E_{\rm UV} = \Lambda|X|^{-3/2}$. That is, once we choose the regulator and use conformal invariance as an input, the Λ-dependent quantum terms have no independent information.

Specifically speaking, the RG evolution of the classical Lagrangian is of the form[20]

$$\delta^{[\rm RG]}\mathcal{L}_{\rm classical} = \sum_{q_i > 0} c_i \Lambda^{q_i} \mathcal{O}_i, \tag{3.5}$$

where $\mathcal{O}_i$ is an operator of dimension $\Delta_i = 3 - q_i$. As explained above, the RG evolution of the Λ-independent classical Lagrangian (3.5) must be cancelled by the Λ-dependent quantum terms,

$$\Lambda \frac{\delta^{[\rm RG]}}{\delta\Lambda} \mathcal{L}_{\rm quantum}(\Lambda) = -\sum_{q_i > 0} c_i \Lambda^{q_i} \mathcal{O}_i = -\delta^{[\rm RG]}\mathcal{L}_{\rm classical}, \tag{3.6}$$

so that the total Lagrangian $\mathcal{L}_{\rm classical} + \mathcal{L}_{\rm quantum}(\Lambda)$ is conformally invariant.

It is not necessary to know the actual form of the Λ-dependent quantum terms for most practical purposes. In physical observables, their only role is to compensate the Λ-dependence from quantum amplitudes at each order in $E_{\rm IR}/E_{\rm UV}$. Practically speaking, we can simply quantize the Λ-independent classical action with a (sufficiently supersymmetric) cutoff, and add local counterterms with positive powers of Λ to cancel any divergences. Since the underlying theory is conformal, there is no harm in obtaining the wrong answer by doing this.

For tree-level amplitudes we do not have to consider the Λ-dependent quantum terms at all, and for one-loop amplitudes the simplest regularization scheme may be a scale-free (and sufficiently supersymmetric) one, such as or dimensional regularization ζ-function. Since we do not compute loop amplitudes in this section, we do not have to consider the Λ-dependent quantum terms further.

[20] In principle there can be terms of the form $\Lambda^{q_i} [\log(\Lambda/E_{\rm UV})]^{s_i}$ with $q_i > 0$. These can be included into the RG equation but we omit them for the sake of simplicity.

3.1.2 Weyl invariance and super-Weyl invariance

Weyl invariance in curved backgrounds imposes constraints on terms more strongly than scale invariance, and super-Weyl invariance constrains them more strongly still. Since the original CFT is super-Weyl-invariant and can be formulated on an arbitrary curved background,[21] the same must be true for the EFT of the X-branch. We use super-Weyl invariance to determine low-derivative terms in the effective action.

First, let us see why certain scale-invariant low-derivative terms do not have a Weyl-invariant completion on curved backgrounds, and therefore cannot appear in the effective action, even without imposing constraints which come from supersymmetry.

Let us first define the field[22]

$$\phi := X^{3/4}, \tag{3.7}$$

so that ϕ has dimension $1/2$, which is that of a free scalar in three dimensions. Then we see that the term which has scaling dimension 3 and the largest X-scaling would be $|\phi|^6 = |X|^{9/2}$, which has the X-scaling $9/2$. This term is disallowed by supersymmetry, however.

The term which has the next-to-largest X-scaling and dimension 3 would be the kinetic term $(\partial_\mu \phi)(\partial^\mu \bar{\phi})$. This term is scale-invariant but not Weyl-invariant and therefore not conformally invariant either. However it has a Weyl-invariant completion obtained by adding the conformal coupling to the Ricci scalar, $\mathcal{R}|\phi|^2/8$. That is,

$$\int d^3x \sqrt{|g|}\mathcal{O}_{\text{kinetic}}, \qquad \mathcal{O}_{\text{kinetic}} := (\partial_\mu \phi)(\partial^\mu \bar{\phi}) + \frac{1}{8}\mathcal{R}|\phi|^2 \tag{3.8}$$

is absolutely invariant under Weyl transformations. This term can be made super-Weyl-invariant by adding the kinetic term for the fermionic superpartner.

3.1.3 Leading interaction term

We would like to discuss the interaction term and how it contributes to the spectrum of the X-branch. There are no bosonic operators with three derivatives and scaling dimension 3. This is an immediate consequence of Lorentz invariance and parity. At the four-derivative level, there is a unique bosonic operator of scaling dimension 3 [96–100] which is called the operator of Fradkin–Tesytlin–Paneitz–Riegert (FTPR) operator [100–103]. In flat space, it is given by

$$\mathcal{O}_{\text{FTPR}}\Big|_{\mathbb{R}^{1,2}} = \frac{1}{\bar{\phi}}\partial^2\partial^2\frac{1}{\phi}, \tag{3.9}$$

[21] At least, a smooth one of nonnegative scalar curvature.

[22] This transformation is harmless when calculating conformal dimensions, but may be subtle when calculating correlation functions of operators carrying large global charge, when the classical solution can have 0 and ∞ and the path integral may be sensitive to the singular branch points there. See section 4.1.2 for further comments on this point.

and in curved backgrounds it takes the form [100]

$$\mathcal{O}_{\text{FTPR}} = \frac{1}{\bar{\phi}}\left[\nabla^2\nabla^2 + \nabla_\mu\left(\frac{5}{4}g^{\mu\nu}\mathcal{R} - 4\mathcal{R}^{\mu\nu}\right)\nabla_\nu - \frac{1}{8}(\nabla^2\mathcal{R}) + \mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} - \frac{23}{64}\mathcal{R}^2\right]\frac{1}{\phi}, \tag{3.10}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor and $\mathcal{R}$ is the Ricci scalar constructed out of the background metric. The operator $\mathcal{O}_{\text{FTPR}}$ is a tensor of Weyl weight 3, so

$$\int d^3x\sqrt{|g|}\mathcal{O}_{\text{FTPR}} \tag{3.11}$$

is invariant under Weyl transformations.

For purposes of this thesis, we only have to find the action on the cylinder $\mathbb{R}\times\mathbb{S}^2$, which is conformally flat. So we could have obtain the additional terms appearing in (3.10) for this particular geometry by transforming the FTPR term in flat space (3.9) under the conformal transformation

$$ds^2_{\mathbb{R}^{1,2}} \to ds^2_{\mathbb{R}\times\mathbb{S}^2} = \frac{\mathsf{r}^2}{|x|^2}ds^2_{\mathbb{R}^{1,2}}, \tag{3.12}$$

where x^μ are the linear coordinates on $\mathbb{R}^{1,2}$ and r is the radius of the sphere. In principle, there may be terms in the effective action which are invariant under Weyl transformations but do not have a super-Weyl-invariant completion. However, the $\mathcal{N}=2$ super-Weyl-invariant completion of the bosonic FTPR term exists [104]. In flat space, it can be expressed as an integral over the full superspace,

$$\int d^3x\mathcal{O}_{\text{super-FTPR}} = \int d^3x d^2\theta d^2\bar{\theta}\mathcal{I}_{\text{super-FTPR}}, \qquad \mathcal{I}_{\text{super-FTPR}} = \frac{\partial_\mu\Phi\partial^\mu\bar{\Phi}}{\left(\Phi\bar{\Phi}\right)^2}, \tag{3.13}$$

where $\Phi \equiv \phi + \sqrt{2}\theta\psi + \cdots$ is a chiral superfield, whose complete expression is given in (A.4).

By coupling the supersymmetric FTPR term (3.13) to a background supergravity multiplet (see for instance [105, 106]), in principle we should be able to determine the component field expression for the couplings of the fermionic superpartner to the background fields in general curved backgrounds as well. However in practice, it is quite cumbersome to determine the explicit component form of the super-FTPR term in general curved backgrounds from the curved superspace expression. Instead, we directly Weyl-transform the flat-space expression (3.13) to obtain the expression in the cylinder $\mathbb{R}\times\mathbb{S}^2$. For a sphere of radius r and in Lorentzian signature, we get

$$\mathcal{O}_{\text{super-FTPR}} = \mathcal{L}_{\text{4b}} + \mathcal{L}_{\text{2b2f}} + \mathcal{L}_{\text{4f}}, \tag{3.14}$$

$$\mathcal{L}_{\text{4b}} = \mathcal{O}_{\text{FTPR}} = \frac{1}{\bar{\phi}}\left[\left(\nabla^2\right)^2 + \frac{3}{2\mathsf{r}^2}\nabla^2 + \frac{4}{\mathsf{r}^2}\partial_t^2 + \frac{9}{16\mathsf{r}^4}\right]\frac{1}{\phi}, \tag{3.15}$$

$$\mathcal{L}_{\text{2b2f}} = -\bar{\psi}^\alpha\left[\left(\nabla^2 - \frac{3i}{\mathsf{r}}\partial_t + \frac{2}{\mathsf{r}^2}\right)\frac{1}{\bar{\phi}^2}\right]\left[\left(\gamma^\mu_{\alpha\beta}\nabla_\mu + \frac{i}{\mathsf{r}}\gamma^t_{\alpha\beta}\right)\frac{1}{\phi^2}\right]\psi^\beta, \tag{3.16}$$

$$\mathcal{L}_{\text{4f}} = -\frac{\bar{\psi}_\beta\bar{\psi}^\beta}{\bar{\phi}^3}\left(\nabla^2 - \frac{1}{4\mathsf{r}^2}\right)\frac{\psi^\alpha\psi_\alpha}{\phi^3}, \tag{3.17}$$

where t is the Lorentzian time coordinate of $\mathbb{R} \times \mathbb{S}^2$, and by $\bar{\phi}$ and $\bar{\psi}$ we mean the Hermitian conjugation $\ddagger$ of ϕ and ψ in radial quantization discussed in section 2.3.2. Here ∇^2 is the Laplacian of the Lorentzian $\mathbb{R} \times \mathbb{S}^2$.

3.1.4 Sign constraint

The purely bosonic term with four derivatives in the flat-space classical action comes exclusively from the bosonic FTPR term (3.9). It has been shown [56] that such a term can only appear with a positive coefficient in the effective action for a massless field, if the underlying theory is unitary. A negative coefficient would causes superluminal signal propagation and violates unitarity in the $2 \to 2$ forward scattering amplitude, within the regime of validity of the EFT. When we compute the operator spectrum, we will see that the positivity of the coefficient (which we shall call κ) of the super-FTPR term shall imply immediately the negativity of the leading large-J correction to the conformal dimension of the lowest unprotected scalar primary operator carrying large R-charge J.

3.1.5 Global symmetries

We present in table 1 the action of the global symmetries on the fields of the UV description and on the ϕ and ψ fields of the moduli space of the X-branch. As mentioned earlier, the XYZ model has three independent U(1) global symmetries, so two out of the five are redundant, but we present them for the sake of usefulness. Note that the $\mathrm{U}(1)_\phi$ and $\mathrm{U}(1)_\psi$ symmetries are separately conserved as exact symmetries in the moduli space EFT. These separate boson- and fermion-number conservation laws simplify the classification of operators and states in the large-R-charge EFT to a great extent.

3.2 Quantization of the effective X-branch theory

Now we would like to derive the Feynman rules for the EFT of the X-branch. We have a double hierarchy $E_{\mathrm{IR}} \ll \Lambda \ll E_{\mathrm{UV}}$, where E_{IR} is the IR scale defined by the inverse of the radius r of the sphere, Λ is the Wilsonian cutoff (of unspecified form) and E_{UV} is the UV scale defined by the "VEV" of $|\phi|^2$. We will be working in finite spatial volume, so the "VEV" does not truly mean an expectation value in the vacuum state; however we shall call it a "VEV" informally. In section 3.3.2 we will comment on the precise meaning of the "VEV" in the sense we use it here. For the moment, it is enough to refer to it by its practical meaning: We define the path integral by separating $\phi = X^{3/4}$ into a "VEV" ϕ_0 and a fluctuation F, and path integrate over F in the standard way, imposing Feynman boundary conditions on it.

	$\mathrm{U}(1)_X$	$\mathrm{U}(1)_{YZ}$	$\mathrm{U}(1)_{\mathrm{R}}$	$\mathrm{U}(1)_\phi$	$\mathrm{U}(1)_\psi$
W	0	0	2	2	-2
Q	0	0	-1	-1	1
$\bar{Q}$	0	0	1	1	-1
X	1	0	$^2/_3$	$^4/_3$	0
Y	$-^1/_2$	1	$^2/_3$	$^1/_3$	-1
Z	$-^1/_2$	-1	$^2/_3$	$^1/_3$	-1
ϕ	$^3/_4$	0	$^1/_2$	1	0
ψ	$^3/_4$	0	$-^1/_2$	0	1

Table 1: R and non-R global charges. $W = gXYZ$ is the superpotential and Q and $\bar{Q}$ are the Poincaré supercharges. The charge assignments in the EFT are $J_\phi = 2J_X/3 + J_{\mathrm{R}}$ and $J_\psi = 2J_X/3 - J_{\mathrm{R}}$. The $\mathrm{U}(1)_{YZ}$ symmetry is generated by $J_{YZ} \coloneqq 2(J_Y - J_Z)$ and acts trivially on all light states on the X-branch. The fermion-number symmetry $\mathrm{U}(1)_\psi$ is unbroken even when ϕ gets an expectation value, and organizes Feynman rules in large R-charge states.

3.2.1 VEV and fluctuations

Let us first define the superfield

$$\Phi \coloneqq X^{3/4}, \tag{3.18}$$

and divide the bosonic component $\phi = \Phi\big|_{\theta=\bar{\theta}=0}$ into

$$\phi = \phi_0 + F, \qquad \phi_0 = \exp\left(\frac{it}{2\mathsf{r}}\right)\varphi_0, \qquad F = \exp\left(\frac{it}{2\mathsf{r}}\right)f, \tag{3.19}$$

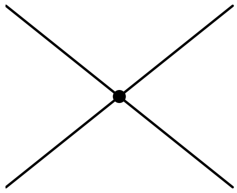
where F is presumed to satisfy the free equation of motion on $\mathbb{R} \times \mathbb{S}^2$,

$$\nabla^2 F = \frac{1}{8}\mathcal{R}F = \frac{1}{4\mathsf{r}^2}F, \tag{3.20}$$

and φ_0 is a constant of order $\sqrt{J}$, as we will explain in section 3.3.1. We then decompose the FTPR term (3.15) into the VEV and fluctuations, and keep terms of four or fewer fluctuations for the purpose of this section. Note that substituting ϕ_0 into the FTPR term will only yield zero, so that the classical correction vanishes.

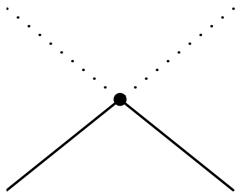
Here we would like to list a few terms which will be relevant later. By explicit computation, when we expand the bosonic FTPR term in the cylinder, there are no terms quadratic in fluctuations, modulo terms proportional to the leading-order equation of motion. There are also terms with three bosonic fluctuations, as well as two fermions and one bosonic fluctuation, but they do not contribute to the physical quantities we will compute and therefore we do not list them. Generally, the term in the effective action with n_{B} bosonic fluctuations and n_{F} fermions, scales as $|\phi_0|^{-(n_{\mathrm{B}}+n_{\mathrm{F}}+2)}$, so the cubic terms scale as $|\phi_0|^{-5} \propto J^{-5/2}$ and the quartic terms scale as $|\phi_0|^{-6} \propto J^{-3}$. Hereafter we denote the propagators of the bosonic fluctuation by solid lines and those of the fermions by dotted lines.

The four-point bosonic vertex is



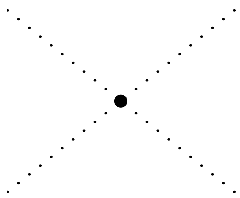
$$= \frac{2\bar{F}^2}{\bar{\phi}_0^3} \left[\left(\nabla^2\right)^2 + \frac{3}{2\mathsf{r}^2}\nabla^2 + \frac{4}{\mathsf{r}^2}\partial_t^2 + \frac{9}{16\mathsf{r}^4} \right] \frac{2F^2}{\phi_0^3} \propto J^{-3}. \tag{3.21}$$

The vertex with two fermions and two bosonic fluctuations is



$$= -4\bar{\psi}^\alpha \left[\left(\nabla^2 - \frac{3i}{\mathsf{r}}\partial_t + \frac{2}{\mathsf{r}^2} \right) \frac{\bar{F}}{\bar{\phi}_0^3} \right] \left[\left(\gamma^\mu_{\alpha\beta}\nabla_\mu + \frac{i}{\mathsf{r}}\gamma^t_{\alpha\beta} \right) \frac{F}{\phi_0^3} \right] \psi^\beta \propto J^{-3}, \tag{3.22}$$

and the vertex with four fermions and no bosonic fluctuations is



$$= \frac{\bar{\psi}_\alpha \bar{\psi}^\alpha}{\bar{\phi}_0^3} \left(\nabla^2 - \frac{1}{4\mathsf{r}^2} \right) \frac{\psi_\beta \psi^\beta}{\phi_0^3} \propto J^{-3}. \tag{3.23}$$

As we will explain later when we provide a precise meaning of the "VEV" ϕ_0, the fluctuation F field obeys Feynman boundary conditions, and therefore has the usual Feynman propagator.

3.2.2 Feynman rules and J-scaling of diagrams

Having divided the field into the "VEV" and fluctuations, we can most easily understand the scaling of corrections by writing the Feynman rules for the F and ψ fields. A diagram with m FTPR vertices with k_1, k_2, $\cdots$, k_m lines on each vertex, will scale as $|\phi_0|^{-2m-\sum_i k_i}$, and therefore as $J^{-m-\sum_i k_i/2}$.

3.3 Corrections to operator dimensions

3.3.1 Dynamics on $\mathbb{R} \times \mathbb{S}^2$

Now we would like to analyze the theory in radial quantization, in which conformal dimensions of operators are equivalent to energies of the corresponding states on the cylinder of radius r, in units of r^{-1}. We will focus on the lowest primary states carrying large R-charge J, and low-lying excited states carrying the same R-charge. We will see that the lowest state is described by a classical solution with a particular symmetry, and that the fluctuations around the classical solution are weakly coupled when the global charges are large. Therefore, the physical observables such as conformal dimensions of the low-lying states are computable in a perturbative expansion in J^{-1}.

Classical solutions with lowest energy for a given global charge. There is a certain family of classical solutions on the cylinder which saturates the lower bound on the energy E for a given R-charge J_{R}, that is, $E \geq J_{\mathrm{R}}/\mathsf{r}$, where r is the radius of the spatial sphere.

This solution exists irrespective of the form of the terms in the effective action for the moduli field X. This follows from the fact that the lowest classical solution with a given value of a conserved charge J, always has a "helical" symmetry, *i.e.*, a combined symmetry under a time translation and action of the corresponding global symmetry transformation by Poisson brackets. Moreover, the angular frequency of the global symmetry action is given by $\omega \coloneqq \mathrm{d}E/\mathrm{d}J$. In the case where the global symmetry is the R-charge, the lowest classical solution with a given R-charge J_{R} is invariant under a combined time translation and R-symmetry transformation, and the angular frequency of the R-symmetry rotation is exactly r^{-1} for any value of J_{R}. That is, the lowest classical solution on the X-branch carrying a given value of the R-charge has a helical symmetry, with the moduli field X depending on the cylindrical time as $\exp\bigl(2it(3\mathsf{r})^{-1}\bigr)$.

Since the theory has the superpotential $W = gXYZ$, there is no such solution with more than one of X, Y, Z turned on simultaneously, the classical solutions carrying a given R-charge and the lowest energy are simply $X = X_0 \exp\bigl(2it(3\mathsf{r})^{-1}\bigr)$, and then two other branches of solutions, with X replaced by Y or Z. These branches of solutions have X, Y and Z charge equal to $3/2$ times their R-charge, respectively. Due to the S_3 symmetric group permuting the three branches, there is no harm in neglecting the Y and Z branches, and we focus only on the properties of the X-branch.

We emphasize that we do not assume any relation between classical solutions of the UV free XYZ model, and solutions of the moduli space EFT. Rather, we are simply using known structure of the moduli space and the known properties of the moduli space effective action to describe the large charge states in the IR. In particular, each branch of moduli space has exactly one light complex scalar field with a particular combination of global charges.

Unlike the values of E and ω for a given J_{R}, which are universal, the absolute value $|X_0|$

of the helical solution as a function of the R-charge J_{R} depends on the unknown form of the effective action. However it is possible to estimate the dependency of $|X_0|$ on $J = J_{\mathrm{R}}$ using dimensional analysis. Each scale-invariant bosonic term in the effective action takes the form of a polynomial in derivatives of X and $\bar{X}$ in the numerator, divided by the appropriate power of $|X|$ in the denominator to make the term scale-invariant. So each additional derivative (or curvature) in the numerator requires an additional power of $|X|^{3/2}$ (or two) in the denominator. Similarly, terms with fermions are also suppressed by powers of $|X|^{-1}$. Thus the derivative (and curvature) expansion of the effective action is also an expansion in $|X|^{-1}$, because of the underlying conformal invariance of the theory, which we make use of as an input in constraining the effective action. It follows that the leading term (3.8) in the effective action for X (or ϕ), which is simply the free kinetic term with conformal coupling to the Ricci scalar, controls the leading large-J asymptotics of the magnitude of $|X_0|$ in the helical solution. Since the free charge density is proportional to $\bar{\phi}\dot{\phi} + \text{c.c.}$, we conclude that $|\phi_0|$ is proportional to $(J/\mathsf{r})^{1/2}$ and $|X_0|$ is proportional to $(J/\mathsf{r})^{3/4}$ at leading order, with a coefficient depending on the normalization of the kinetic term and corrections that are subleading at large J.

3.3.2 Meaning of the "VEV"

Free-field matrix elements with a "VEV" are coherent state matrix elements. Spontaneous symmetry breaking does not occur in finite volume. This is because the expectation value of a charged operator in a charge eigenstate, always vanishes. This statement is true only if the state is an exact eigenstate of the charge operator. However, it is possible to construct states with exactly Gaussian correlation functions for charged free fields, as coherent states. Let $a^\ddagger$ be a creation operator for an excitation of the ϕ field in the s-wave. Then the coherent state defined by

$$|[v]\rangle\!\rangle \coloneqq e^{v a^\ddagger} |0\rangle \tag{3.24}$$

is an eigenstate of the annihilation operator a,

$$a\,|[v]\rangle\!\rangle = v\,|[v]\rangle\!\rangle\,, \tag{3.25}$$

and correlation functions of the oscillators in the coherent state $|[v]\rangle\!\rangle$ are exactly Gaussian. Thus, free fields ϕ, $\bar{\phi}$ constructed from the oscillators have the property that $f \coloneqq \phi - \langle\phi\rangle$ and $\bar{f} \coloneqq \bar{\phi} - \langle\bar{\phi}\rangle$ have the same correlation functions as the vacuum correlation functions of ϕ and $\bar{\phi}$:

$$\big\langle [v]\big|\mathcal{O}\big[\phi,\bar{\phi}\big]\big|[v]\big\rangle = \big\langle 0\big|\mathcal{O}\big[f,\bar{f}\big]\big|0\big\rangle\,, \tag{3.26}$$

where we have defined

$$|[v]\rangle \coloneqq \frac{|[v]\rangle\!\rangle}{\langle\!\langle [v]|[v]\rangle\!\rangle^{1/2}} = \exp\left(-\frac{|v|^2}{2}\right)|[v]\rangle\!\rangle\,. \tag{3.27}$$

Now let us compare the expectation values in a large-J eigenstate with those in a coherent state,[23] and show that the latter approximates the former in the large-J limit, with computable corrections. Using

$$|J\rangle := \frac{\left(a^{\ddagger}\right)^{J}}{\sqrt{J!}}\,|0\rangle\,, \tag{3.28}$$

the definition (3.24) can be expressed as

$$|[v]\rangle\!\rangle = \sum_{J} \frac{v^{J}}{\sqrt{J!}}\,|J\rangle\,, \tag{3.29}$$

and by inverting this we find

$$|J\rangle = \frac{1}{2\pi i}\oint \frac{dv}{v^{J+1}}\,|[v]\rangle\!\rangle\,. \tag{3.30}$$

This state has exactly Gaussian correlators, with a connected two-point function identical to that of the (uncharged) vacuum. It follows that the relation between the vacuum and coherent-state two-point function is simply a shift of the one-point functions, by a free classical solution. We can therefore use Feynman diagrams in a "background" given by the classical solution represented by the coherent state expectation value, to calculate arbitrary free-field correlation functions in the coherent state. So the usual Feynman diagrammatic perturbation theory with a "VEV" given by a nontrivial free classical solution for the scalar field, is simply a way of doing time-ordered perturbation theory in the coherent state in finite volume. This is relevant to the large-J expansion for definite-J matrix elements, because as we will now see, large-J matrix elements in charged Fock states are approximated at leading order by matrix elements in the corresponding coherent state.

Relationship between Fock states and coherent states. A consequence of this representation is that expectation values for Fock states are approximated at leading order in J by expectation values in coherent states, up to (calculable) subleading large-J corrections. To see this concretely, we need the following facts:

- The definite-J Fock matrix element is given by a double contour integral of coherent-state matrix elements;
- The double contour integral for a neutral operator can be evaluated by saddle point;
- Fluctuation corrections to the saddle-point approximation are suppressed by powers of J; and

[23] This calculation was developed by Simeon Hellerman and Ian Swanson, and used to estimate corrections to the energies of rotating relativistic strings [12].

- The leading saddle-point approximation is simply given by the coherent-state matrix element in the coherent state where the expectation value of the charge, is J.

First we write the expectation value $A_{\mathcal{O}}[J] \coloneqq \langle J|\mathcal{O}|J\rangle$ in the state J as a double contour integral,

$$A_{\mathcal{O}}[J] = \frac{1}{(2\pi)^2} \oint \frac{dw}{w^{J+1}} \oint \frac{dv}{v^{J+1}} \langle\!\langle [w]|\mathcal{O}|[v]\rangle\!\rangle \,. \tag{3.31}$$

One combination of the two integrals simply projects onto operators $\mathcal{O}$ that commute with $\hat{J}$. Assume without loss of generality that $\mathcal{O}$ carries a definite charge $J_{\mathcal{O}}$. If $J_{\mathcal{O}} \neq 0$, then clearly its expectation value in Fock states must vanish. The first of the two contour integrals simply implements the projection that causes the Fock state expectation value to vanish. If $\mathcal{O}$ is uncharged, the remaining contour integral is nonzero, and can be evaluated by saddle point when J is large, with fluctuation corrections that can be calculated as a series in J^{-1}. Define $F_{\mathcal{O}}[J]$ as the expectation value of an uncharged operator $\mathcal{O}$ in a coherent state of classical charge equal to J:

$$F_{\mathcal{O}}[J] \coloneqq \langle [w]|\mathcal{O}|[v]\rangle \Big|_{J=wv} \,. \tag{3.32}$$

Then the Fock expectation value $A_{\mathcal{O}}[J]$ is given by

$$A_{\mathcal{O}}[J] = \sum_{m,n\geq 0} \frac{1}{2} \mathcal{R}_{mn} J^m \frac{\mathrm{d}^n}{\mathrm{d}J^n} F_{\mathcal{O}}[J], \tag{3.33}$$

where the leading coefficient $\mathcal{R}_{00}$ is 1, and all the other coefficients are given by the generating function

$$\sum_{m,n\geq 0} \mathcal{R}_{mn} x^m y^n = e^{-xy}(1+y)^x. \tag{3.34}$$

Note that $\mathcal{R}_{mn} = 0$ unless $m \leq n/2$, so there are only a finite number of nonzero terms at a given order in J. Concretely, if we expand

$$A_{\mathcal{O}}[J] = \sum_{k\geq 0} A_{\mathcal{O}}^{(k)}[J] \tag{3.35}$$

where $A_{\mathcal{O}}^{(k)}$ is the relative order J^{-k} contribution to the Fock-state expectation value, then

$$A_{\mathcal{O}}^{(k)}[J] = \sum_{n-m=k} \mathcal{R}_{mn} J^m \frac{\mathrm{d}^n}{\mathrm{d}J^n} F_{\mathcal{O}}[J], \tag{3.36}$$

and the first few contributions are

$$A_{\mathcal{O}}^{(0)} = F_{\mathcal{O}}[J], \qquad A_{\mathcal{O}}^{(1)} = -\frac{1}{2} J \frac{\mathrm{d}^2 F_{\mathcal{O}}[J]}{\mathrm{d}J^2}, \qquad A_{\mathcal{O}}^{(2)} = \frac{1}{8} J^2 \frac{\mathrm{d}^4 F_{\mathcal{O}}[J]}{\mathrm{d}J^4} + \frac{1}{3} J \frac{\mathrm{d}^3 F_{\mathcal{O}}[J]}{\mathrm{d}J^3}. \tag{3.37}$$

As expected, the leading approximation $A_{\mathcal{O}}^{(0)}[J]$ is simply equal to the coherent-state expectation value.

Conical deficit and ϕ-charge quantization. The change of variables from X to $\phi \coloneqq X^{3/4}$ is well-behaved at large values of X (compared to the IR scale) but singular at $X = 0$. The classical helical solution never comes near the origin $X = 0$ of field space, nor do fixed-energy perturbations of the helical solution in the limit of large-J. So one would expect that the singularity of the change of variables is irrelevant in large-J perturbation theory.

On the one hand, this expectation is entirely accurate, in the sense that the details of the "resolution" of the singularity are really irrelevant to all orders in the J^{-1} expansion. Any two physically well-defined resolutions of the singularity, must necessarily correspond to different Hamiltonians $H_{1,2}$ which modify the moduli space effective action in a neighborhood of ϕ-field space of size $M^{1/2}$ (equivalently, a neighborhood of X-field space of size $M^{3/4}$), where M is some UV scale. If the correction terms scale like $M^{k/2}/|\phi|^k$ at long distances in field space, then the corresponding large-J corrections are of order $(M/J)^{k/2}$. If the two resolutions of the geometry are both exactly conical outside a region of field space $|\phi| < M^{1/2}$, then the corrections to observables from the modified geometry vanish to all orders in J^{-1}. This is the precise sense in which the singularity at the origin is "irrelevant" for large-J physics: At large J, the field does not live at the origin or anywhere near it.

However the conical deficit of the moduli space is a property of the geometry which is visible asymptotically, and the EFT should know about all properties of the moduli space geometry where the VEV is large compared to the IR scale. The quantization rule for ϕ-charge is precisely the property of the quantum EFT in which the conical deficit at large VEV is encoded. In order to compute a large-J asymptotic expansion in the EFT, one may simply take X-charge to be a multiple of 3, in which case the number of ϕ-excitations is an integer, and in particular a multiple of 4.

In order to confirm that the conical deficit only imposes the quantization rule, one can simply redo any computation in the moduli space EFT using a logarithmic superfield defined as $L \coloneqq \log \Phi$. In terms of L, the only effect of the conical deficit is to change the periodicity of $\operatorname{Im} L$; otherwise the effective action is completely unaffected by the deficit. We conclude that the conical deficit does not affect the energy spectrum to all orders in J^{-1}, so long as the classical solution uniformly satisfies $|\phi|^2 \gg E_{\text{IR}}$.

3.3.3 BPS property and vanishing of the vacuum correction

The classical energy of the large-J ground state. First, as a consistency check, we would like to study the energies of the chiral primary states, at the classical and one-loop level. By general multiplet-shortening arguments [67, 107–109], these energies must be uncorrected and equal to the R-charge of the state. However even at the classical level, it is not so obvious that the super-FTPR term does not contribute to the energies of the chiral primary states. The super-FTPR term is a sum of many contributions with certain coefficients determined by super-Weyl invariance, none of which individually vanishes for the helical classical solution. Nevertheless the sum of the terms in the FTPR expression

(3.15) indeed combines to give zero when evaluated on the helical solution:

$$\mathcal{O}_{\mathrm{FTPR}}\Big|_{\phi=\phi_0} = \frac{1}{\bar{\phi}}\left[\left(\nabla^2\right)^2 + \frac{3}{2\mathsf{r}^2}\nabla^2 + \frac{4}{\mathsf{r}^2}\partial_t^2 + \frac{9}{16\mathsf{r}^4}\right]\frac{1}{\phi}\Big|_{\phi=\phi_0} = 0, \tag{3.38}$$

for any spherically homogeneous helical solution with frequency $(2\mathsf{r})^{-1}$, that is,

$$\phi = \phi_0 = \varphi_0 \exp\left(\frac{it}{2\mathsf{r}}\right). \tag{3.39}$$

This gives us some confidence in the applicability of the moduli space effective action to compute energies consistently. Next, we would like to compute the one-loop energies of the ground states, as well as semiclassical and one-loop energies of first-excited states as a consistency check, to build further confidence in our methods.

One-loop energy of the large-J ground state. We now check the one-loop energy of the large-J ground state, by expanding the action around the helical solution to quadratic order in fluctuations, and summing $\pm\omega$ over bosonic and fermionic fluctuations with frequency ω, with the sign appropriate to the statistics. At the free level, the contributions from the bosonic and fermionic fluctuations cancel with each other. The super-FTPR term does not contributes to the energies at order J^{-2} because as noted in section 3.2, the super-FTPR term, when expanded around the helical solution, does not have terms quadratic in fermions or in bosonic fluctuations, and therefore there is no energy correction at absolute order J^{-2} (which is relative order J^{-3}) even without any further nontrivial Bose-Fermi cancellation. So we see that the energy of the chiral primary ground state is therefore uncorrected up to and including order J^{-2}, as it must be to all orders in J. A nontrivial check of the large-J expansion would be to prove that the energy of the chiral primary state is uncorrected to all orders in the large-J expansion. It may be that some type of superfield formalism adapted to quantization about the helical ground state would make such cancellations more transparent.

3.3.4 Semishort property of the s-wave one-particle state

Next we would like to compute energies of the first-excited states at large J. The lowest state above the large-J ground state with the same U(1) quantum numbers, is the state with an additional ϕ excitation and $\bar{\phi}$ excitation, both in the $\ell = 0$ mode, *i.e.*, the s-wave. At the free level, each fluctuation has frequency $\omega = (2\mathsf{r})^{-1}$, and we have seen that the frequency is uncorrected by the super-FTPR term up to and including order J^{-2}. Therefore, the energy of the first-excited state is simply $J + 1$ up to and including order J^{-2}. Since this state is not a chiral primary, one might be interested in computing corrections to its energy in the large-J expansion. However the one-loop correction actually vanishes, because this state is semishort and the dimension is exactly equal to $J + 1$.

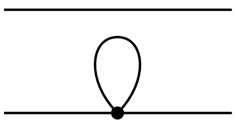
Heuristically speaking, the first-excited state can be thought of as obtained by shifting the R-charge of the ground state from $J \to J+1$, and then cancelling it by adding a single quantum of $\bar{\phi}$ in the s-wave. It is possible to show explicitly that the one-loop correction to the energy of the state coming from the super-FTPR term vanishes. The one nontrivial aspect of this cancellation is the operator ordering of the super-FTPR term. A convenient description is in terms of normal-ordered operators in the Hilbert space on the sphere: All operators appearing have no less than two $\bar{\phi}$-multiplet annihilation oscillators ordered to the right, and thus the perturbing Hamiltonian does not affect the energy of the semishort state, which has only a single $\bar{\phi}$ excitation.

The existence of semishort states with the appropriate charges is visible in the superconformal index; we have included an expression for the superconformal index in appendix D, as well as its expansion to several orders. It is interesting to note that the semishort states of the X-branch persist down to $J = 0$: The "moment map" operator is semishort on general grounds (see appendix A.3), because it is the superconformal primary whose descendant is the $\mathrm{U}(1)_X$ current [67, 110, 111]. This operator can be thought of as the conformal Kähler potential itself for the EFT of the X-branch, namely $K \propto (X\bar{X})^{3/4}$. This expression in terms of the X-field is not a well-defined, controlled operator generally, but this expression is well-defined and precise in matrix elements between large-J states.

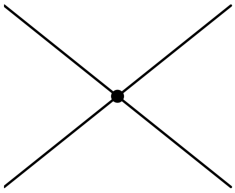
The one-particle states with nonzero spin are also in semishort representations at the free-field level. At the interacting level, it is easy to prove in many cases that the semishort property persists, because there are no other states with the appropriate angular momentum, $\mathrm{U}(1)_{\mathrm{R}}$, and $\mathrm{U}(1)_X$ charges to be combined with the semishort states to become a full long representation at weak but finite coupling. For instance, the vector states obtained by acting on the chiral primary state corresponding to the operator ϕ^{2J+1}, with the $\ell = 1$ modes of the $\bar{\phi}$ field, can be shown to be protected by such an argument. This prediction is also verified by the superconformal index in appendix D.

3.3.5 Correction to the two-particle energies

Semishort property means no disconnected diagrams. Let us compute the energy correction to the state with two $\bar{\phi}$'s on top of the chiral primary state corresponding to the operator ϕ^{2J+2}. By expanding the super-FTPR term in VEV and fluctuations, two diagrams we need to consider at order J^{-3} are as follows:

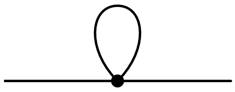
 $\propto J^{-3}$ (3.40)

and



$$\propto J^{-3}. \tag{3.41}$$

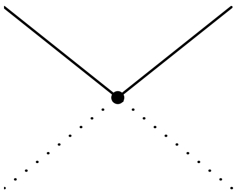
Incidentally, we know from the argument in section 3.3.4 that the following diagram should vanish:



$$= 0. \tag{3.42}$$

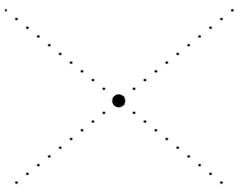
Note that these Feynman diagrams with loops in them have scheme dependence, *i.e.*, how loop integrals are regularized and renormalized – once we choose one scheme which is sufficiently supersymmetric on the cylinder, the expression makes sense, and the diagram (3.42) exactly vanishes. Hence the only contribution to the energy correction at order J^{-3} is the diagram (3.41).

The above statement is true even when some of the $\bar{\phi}$'s are replaced by $\bar{\psi}$, the $\bar{Q}$-descendant of $\bar{\phi}$. The only diagram which contributes to the energy correction to the state $\bar{\psi}\bar{\phi}\left|\phi^J\right\rangle$ is



$$\propto J^{-3}, \tag{3.43}$$

while for $\bar{\psi}\bar{\psi}\left|\phi^J\right\rangle$ this is



$$\propto J^{-3}. \tag{3.44}$$

Two-particle states energy correction. Now we would like to compute the energy correction to the state $\bar{\phi}\bar{\phi}\left|\phi^J\right\rangle$ by expanding the super-FTPR term in VEV and fluctuations. Here note that we only need to care about a spatially uniform field because of the above-mentioned argument, that is, we lose nothing by truncating the fluctuation f to s-waves.

By doing so we obtain the Lagrangian density for the spatially uniform fluctuation $f(t)$,

$$\begin{gathered}\mathcal{L} = \mathcal{L}_0 + \kappa \mathcal{L}_{\text{int}}, \\ \mathcal{L}_0 = \dot{\bar{F}}\dot{F} - \frac{1}{4}F^2 = \dot{\bar{f}}\dot{f} + \frac{i}{2}\left(\dot{\bar{f}}f - \bar{f}\dot{f}\right), \qquad \mathcal{L}_{\text{int}} = -\frac{24}{|\varphi_0|^6}\bar{f}^2\dot{f}^2.\end{gathered} \tag{3.45}$$

Here we have set $\mathsf{r} = 1$ for the sake of simplicity, as we shall continue to do in the rest of this section. Dots represent derivative with respect to t. The conjugate momenta in terms of f and $\bar{f}$ are $\Pi \coloneqq \partial\mathcal{L}_0/\partial\dot{f} = \dot{\bar{f}} - i\bar{f}/2$ and $\bar{\Pi} \coloneqq \dot{f} + if/2$, respectively, and the Hamiltonian H, which is 4π times the Hamiltonian density, is

$$\begin{gathered}H = H_0 + \kappa H_{\text{int}}, \\ H_0 = 4\pi\left(\Pi + \frac{i}{2}\bar{f}\right)\left(\bar{\Pi} + \frac{i}{2}f\right) \qquad H_{\text{int}} = \frac{96\pi}{|\varphi_0|^6}\bar{f}^2\dot{f}^2.\end{gathered} \tag{3.46}$$

We define creation and annihilation operators as

$$\begin{aligned} a^\ddagger &= \sqrt{4\pi}\left(\Pi + \frac{i}{2}\bar{f}\right), & a &= \sqrt{4\pi}\left(\bar{\Pi} - \frac{i}{2}f\right), \\ b^\ddagger &= \sqrt{4\pi}\left(\bar{\Pi} + \frac{i}{2}f\right), & b &= \sqrt{4\pi}\left(\Pi - \frac{i}{2}\bar{f}\right), \end{aligned} \tag{3.47}$$

and in terms of these the interaction Hamiltonian H_{int} becomes

$$H_{\text{int}} = -\frac{6}{\pi|\varphi_0|^6}\left(a^\ddagger - b\right)^2 a^2. \tag{3.48}$$

The energy correction to the state $\left(a^\ddagger\right)^2|0\rangle$, which corresponds to $\bar{\phi}\bar{\phi}\left|\phi^J\right\rangle$, is computed as

$$\Delta E = -\frac{6\kappa}{\pi|\varphi_0|^6}\frac{\langle 0|a^2\left[\left(a^\ddagger - b\right)^2 a^2\right]\left(a^\ddagger\right)^2|0\rangle}{\langle 0|a^2\left(a^\ddagger\right)^2|0\rangle} = -\frac{12\kappa}{\pi|\varphi_0|^6}. \tag{3.49}$$

The energy correction to $\bar{\psi}\bar{\phi}\left|\phi^J\right\rangle$ and $\bar{\psi}\bar{\psi}\left|\phi^J\right\rangle$ can be computed in a similar way and is equal to (3.49), as it should be due to supersymmetry. Detailed calculations are given in appendix C.

Note that the form of the interaction Hamiltonian (3.48) is normal-ordered rather than time-ordered. This is needed in order for supersymmetry to be preserved, and can be achieved directly from the necessity of the existence of a set of generators implementing the $\mathcal{N} = 2$ superconformal algebra on the cylinder; in fact, it can be understood just from the consistency of a smaller subalgebra generated by half the generators, namely those which annihilate the chiral primary states corresponding to the operators of the form X^J. In appendix E, we discuss how the closure of the subalgebra in the interacting theory implies the form of the operator perturbation of the Hamiltonian in a toy model, obtained by truncating the antichiral $\bar{\phi}$ multiplet down to its s-wave mode on the cylinder.

In the energy correction (3.49), there are no disconnected contributions coming from vacuum bubbles and propagator corrections. The vanishing of these contributions come from the nonrenormalization of the vacuum energy and semishort one-particle energy, respectively.

As noted in section 3.1, the sign of κ appearing in (3.49) must be positive due to unitarity of the $2 \to 2$ forward scattering amplitude of the moduli field [56], and therefore the order J^{-3} contribution to the operator dimension is negative.

3.4 Operator algebras and the semishort spectrum

In this section we return to the question of the semishort energy (non)correction. We have seen explicitly that the one-loop correction to the energy of the scalar semishort primary state vanishes, and there is an independent algebraic argument which states that the it is uncorrected to all orders in J^{-1}: The energy can be corrected only if the semishort primary state combines with other primary states to form a long multiplet. However, there are no states which have appropriate charges and energy approximately equal to $\Delta = J + 1$ at large J, to fill out a long multiplet. Therefore, there is a scalar semishort primary state at sufficiently large J. In this section we would like to show that semishort primary operators form a module over the chiral ring, and as a result associativity of the algebra relates semishort primary operators at large values of J to those at low values of J.

3.4.1 Nonsingularity of certain OPE structure functions

As shown in appendix A.3, all superconformal primary operators in any unitary $\mathcal{N} = 2$ SCFT in three dimensions satisfy the following bound,

$$\begin{aligned} &\Delta = |R|, \qquad && \ell = 0 \\ &\quad \text{or} && \\ &\Delta \geq |R| + \ell + 1, \qquad && \ell \geq 0. \end{aligned} \tag{3.50}$$

where ℓ is the spin quantum number and R is the R-charge. The first bound is saturated if and only if the operator is annihilated by Q_α or $\bar{Q}^\alpha$, the energy-raising supercharges. For instance, an element of the chiral ring, *i.e.*, a scalar superconformal primary operator $\mathcal{O}$ satisfying

$$\left[\bar{Q}_\alpha, \mathcal{O}\right] = 0, \tag{3.51}$$

has dimension equal to its R-charge

$$R = \Delta. \tag{3.52}$$

Let $\mathcal{O}_{\text{SSS}}$ be a scalar primary operator in a semishort multiplet. That is, $\mathcal{O}_{\text{SSS}}$ satisfies $\left\{\bar{Q}^\alpha, \left[\bar{Q}_\alpha, \mathcal{O}_{\text{SSS}}\right]\right\} = 0$ but $\left[\bar{Q}^\alpha, \mathcal{O}_{\text{SSS}}\right]$ does not vanish. Such an operator $\mathcal{O}_{\text{SSS}}$ saturates the inequality in (3.50) with $\ell = 0$.

It is well known that chiral parimary operators have a ring structure because their OPE is automatically nonsingular [112, 113]. The argument follows immediately from superconformal invariance and especially the formula (3.50). Let $\mathcal{O}_{1,2}$ two chiral primary operators. Their R-charges and conformal dimensions obey $\Delta_{1,2} = R_{1,2}$. In general their OPE has the following form

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_i f_i(x)\mathcal{O}_i(0), \tag{3.53}$$

where $\mathcal{O}_i$ are operators of R-charge $R_i = R_1 + R_2$, dimension Δ_i and spin ℓ_i. The function f_i has x-scaling $\gamma_i := \Delta_i - \Delta_1 - \Delta_2 = \Delta_i - R_i$.[24] So by (3.50) and the fact that all the super-Poincaré generators P_μ, Q_α and $\bar{Q}_\alpha$ satisfy $\Delta \geq R + \ell$, we have $\gamma_i \geq \ell_i \geq 0$, and therefore all the functions f_i vanish at $x = 0$ if $\ell_i > 0$. When $\mathcal{O}_i$ is a scalar the function $f_i(x)$ has a finite limit. Therefore, the OPE (3.53) is nonsingular in the $|x| \to 0$ limit, and we can define an associative multiplication by the ordinary product of two chiral primary operators at coincident points, which is the multiplicative structure of the chiral ring.

Let us consider the OPE of a chiral primary operator $\mathcal{O}_{\text{chiral}}$ of R-charge $R_{\mathcal{O}_{\text{chiral}}}$ with a scalar semishort primary operator $\mathcal{O}_{\text{SSS}}$ of R-charge $R_{\mathcal{O}_{\text{SSS}}}$. The conformal dimensions of the operators are given by $\Delta_{\mathcal{O}_{\text{chiral}}} = R_{\mathcal{O}_{\text{chiral}}}$ and $\Delta_{\mathcal{O}_{\text{SSS}}} = R_{\mathcal{O}_{\text{SSS}}} + 1$. This OPE may be singular and contain the following operator

$$\mathcal{O}_{\text{chiral}}(x)\mathcal{O}_{\text{SSS}}(0) = \cdots + \frac{c_{\text{chiral}}}{|x|}\mathcal{O}'_{\text{chiral}}(0) + \cdots, \tag{3.54}$$

where $\mathcal{O}'_{\text{chiral}}$ is a BPS scalar primary of dimension $\Delta_{\mathcal{O}'_{\text{chiral}}} = R_{\mathcal{O}'_{\text{chiral}}} = R_{\mathcal{O}_{\text{chiral}}} + R_{\mathcal{O}_{\text{SSS}}}$. There may also be the nonsingular term

$$\mathcal{O}_{\text{chiral}}(x)\mathcal{O}_{\text{SSS}}(0) = \cdots + c_{\text{SSS}}\mathcal{O}'_{\text{SSS}}(0) + \cdots, \tag{3.55}$$

which will be of principal interest in this section. Let us show that the smooth term (3.55) in the OPE defines an associative multiplication of the chiral ring elements on the scalar semishort primary operators. We cannot reach this argument directly from the OPE above, since the nonsingular terms in a generic OPE are not associative generally; only the sum of all terms in the OPE, singular and not, obeys associativity when taken together. However by taking a $\bar{Q}$-descendant, we can establish associativity of (3.55) indirectly. Let $\bar{Q}_\uparrow$ be the energy- and R-charge-raising supercharge which carries the third component of the spin $\ell_3 = 1/2$. By taking the $\bar{Q}_\uparrow$-descendant, we can define an associative multiplication of the chiral ring elements on the superpartners $\left[\bar{Q}_\uparrow, \mathcal{O}_{\text{SSS}}\right]$ of scalar semishort primary operators. By the same arguments as above, any function $\tilde{f}_i$ appearing in the OPE of $\mathcal{O}_{\text{chiral}}$ and $\bar{Q}^\alpha \cdot \mathcal{O}_{\text{SSS}} := \left[\bar{Q}^\alpha, \mathcal{O}_{\text{SSS}}\right]$,

$$\mathcal{O}_{\text{chiral}}(x)\left(\bar{Q}^\alpha \cdot \mathcal{O}_{\text{SSS}}\right)(0) = \sum_i \tilde{f}_i(x)\mathcal{O}_i(0) \tag{3.56}$$

[24] That is, $f_i(x)$ satisfies $f_i(\lambda x) = \lambda^{\gamma_i} f_i(x)$.

must scale as $|x|^{\tilde{\gamma}_i}$, with the exponent $\tilde{\gamma}_i$ defined as

$$\tilde{\gamma}_i := \Delta_{\mathcal{O}_i} - \Delta_{\mathcal{O}_{\text{chiral}}} - \Delta_{\bar{Q}^\alpha \cdot \mathcal{O}_{\text{SSS}}} = \Delta_{\mathcal{O}_i} - R_{\mathcal{O}_{\text{chiral}}} - R_{\mathcal{O}_{\text{SSS}}} - \frac{3}{2}, \tag{3.57}$$

where $\Delta_{\mathcal{O}_i}$ and $R_{\mathcal{O}_i} = R_{\mathcal{O}_{\text{chiral}}} + R_{\mathcal{O}_{\text{SSS}}} + 1$ are the dimension and R-charge of $\mathcal{O}_i$ appearing in (3.56). In the case where the $\mathcal{O}_i$ is a superpartner of a scalar semishort primary operator, then $\ell_{\mathcal{O}_i} = 1/2$, and

$$\Delta_{\mathcal{O}_i} = R_{\mathcal{O}_{\text{chiral}}} + R_{\mathcal{O}_{\text{SSS}}} + \frac{3}{2} \qquad \Leftrightarrow \qquad \tilde{\gamma}_i = 0. \tag{3.58}$$

In this case, $\tilde{f}_i$ can have two possible Lorentz tensor structures, that is,

$$\mathcal{O}_{\text{chiral}}(x)\big(\bar{Q}^\alpha \cdot \mathcal{O}_{\text{SSS}}\big)(0) = \cdots + c_{\mathsf{A}} \mathcal{O}^\alpha_{\mathsf{A}}(0) + c_{\mathsf{B}} \gamma^{\alpha\beta}_\mu \frac{x^\mu}{|x|} \mathcal{O}^\beta_{\mathsf{B}}(0) + \cdots . \tag{3.59}$$

However, in a $\mathcal{N} = 2$ SCFT in three dimensions such as the XYZ model, it can be shown that $c_{\mathsf{B}} = 0$.[25] Therefore, the OPE (3.59) becomes

$$\mathcal{O}_{\text{chiral}}(x)\big(\bar{Q}^\alpha \cdot \mathcal{O}_{\text{SSS}}\big)(0) = \cdots + c_{\mathsf{A}} \mathcal{O}^\alpha_{\mathsf{A}}(0) + \cdots , \tag{3.60}$$

where $\mathcal{O}^\alpha_{\mathsf{A}}$ is an operator of spin $1/2$ satisfying $\Delta = R + 1/2$. Any such operator has to be the $\bar{Q}^\alpha$-descendant of a scalar semishort primary operator, as shown by the following argument. By virtue of the unitarity bound (3.50), all other operators on the right-hand side of (3.60) vanish in the limit $x \to 0$. Therefore the $\bar{Q}$-descendants of scalar semishort primary operators form a module over the commutative ring of the chiral primary operators:[26]

$$\mathcal{O}_{\text{chiral}}(x)\big(\bar{Q}^\alpha \cdot \mathcal{O}_{\text{SSS}}\big)(0) = c\big(\bar{Q}^\alpha \cdot \mathcal{O}'_{\text{SSS}}\big)(0) + (\text{terms vanishing at } x = 0). \tag{3.61}$$

From this, it follows that the scalar semishort primary operators themselves form a module over the chiral ring. Naïvely this would appear to follow without any further justification, as one expects that the OPEs of descendant operators are completely determined by the OPEs of primary operators. For nonsupersymmetric conformal invariance this is indeed the case, a fact synonymous with the existence and uniqueness of the conformal blocks. For superconformally covariant OPEs, this argument does not generalize; there are multiparameter families of nontrivial, functions of three-point functions in superspace [114] which is invariant under superconformal transformations. By multiplying a superconformally covariant three-point function by such a superconformally invariant function of three copies of superspace, one can obtain another three-point function which is also covariant under superconformal transformations. However, in the case where one of the three operators is a

[25] One can show by using the formulation of [114] that superspace three-point functions of scalar superfields is uniquely determined up to an overall constant, if any one of the superfields is a chiral superfield. Since $c_{\mathsf{B}} = 0$ in the free theory of a chiral superfield, one concludes that $c_{\mathsf{B}} = 0$ in any theory.

[26] In four dimensions this module structure is discussed in [115].

chiral primary operator, there is no such problem: The identity, up to an overall constant, is the only function on three copies of superspace that is annihilated by the superderivative $\bar{D}_\alpha$ acting on any one of the three points. Therefore, the OPE coefficient c appearing in (3.61) uniquely determines the OPE coefficient c_{SSS} appearing in (3.55) and vice versa. So, we can define the module structure of scalar semishort primary operators indirectly from the nonsingular OPE (3.61).

Heuristically, we can discuss about the action of the chiral ring elements on scalar semishort primary operators directly through the following operation,

$$\mathcal{O}_{\mathrm{chiral}} \times \mathcal{O}_{\mathrm{SSS}} = \mathcal{O}'_{\mathrm{SSS}} = \left(\bar{Q}_\uparrow\right)^{-1} \cdot \mathcal{O}_{\mathrm{chiral}} \cdot \bar{Q}_\uparrow \cdot \mathcal{O}_{\mathrm{SSS}}. \tag{3.62}$$

Since the chiral ring element $\mathcal{O}_{\mathrm{chiral}}$ is annihilated by $\bar{Q}_\uparrow$, the operator $\mathcal{O}_{\mathrm{chiral}}$ indeed formally commutes with $\bar{Q}_\uparrow$, justifying the above definition. However, we need the uniqueness of superconformally covariant three-point functions of three copies of superspace with one chiral primary operator in order to make logical sense of equation (3.62).

The fact that scalar semishort primary operators form a module over the chiral ring implies the existence of scalar semishort primary operators at low values of J as well. Starting with the moment map operator, which is the primary operator of the flavor current multiplet and may be expressed as $\left(X\bar{X}\right)^{3/4} = \phi\bar{\phi}$ in the EFT, we act $2J$ times with ϕ to obtain a scalar semishort primary operator $\bar{\phi}\phi^{2J+1}$, which carries R-charge J. Algebraically, this state could in principle vanish: *a priori* the representation of the chiral ring on the module of scalar semishort primary operators does not have to be faithful, *i.e.*, the OPE coefficient c_{A} appearing in (3.60) could vanish for some intermediate value of J. However we have seen already that the scalar semishort primary operator $\bar{\phi}\phi^{2J+1}$ is nonvanishing for sufficiently large value of J, using the effective description. By associativity, we conclude that any of the intermediate products $\bar{\phi}\phi^{k+1}$ cannot vanish, for any nonnegative value of k satisfying the certain quantization condition.

The existence of scalar semishort primary operators for all values of k can in principle be checked *via* the superconformal index; and in appendix D we expand the superconformal index to several orders and verify our prediction. However we would like to emphasize that the same conclusion can be obtained in a less arduous way by using the large-J perturbation theory.

3.5 Conclusion of section 3

In this section we have considered the $\mathcal{N} = 2$ superconformal XYZ model in three dimensions, and calculated the conformal dimensions of certain low-lying operators carrying large charges to the first nontrivial order in an expansion in large R-charge J_{R} and large X-charge $J_X \sim 3J_{\mathrm{R}}/2$. In order to do so, we have radially quantized the theory and used the EFT on the X-branch of the moduli space. In this theory, both loop corrections and higher-derivative interaction terms in the effective action are suppressed by powers of $|\phi| = |X|^{3/4}$

when $|\phi|$ is large and scales as $J^{1/2}$. We have seen that the state with one $\bar{\phi}$ excitation in the s-wave is protected because it is in a scalar semishort multiplet [67, 107–109] The third-lowest scalar primary operator can be viewed as the state with two $\bar{\phi}$ quanta in the s-wave, atop a sea of $(2J_{\mathrm{R}} + 2)$ ϕ-quanta in the s-wave. This state is in a long multiplet and has a nontrivial anomalous dimension which is proportional to the coefficient of the leading interaction term in the effective action – the superconformal extension of the FTPR term. Due to unitarity and causality of the EFT [56], the coefficient of the super-FTPR term must be positive, and therfore the anomalous dimension of the third-lowest state must be negative. There is an curious similarity between the large-R-charge expansion of the anomalous dimension, and the large-spin expansion of the anomalous dimension of operators with large SO(d) spin [33, 34], although the two expansions are based on rather different logical arguments. It would be interesting to understand these two expansions within a unified framework of operator dimensions with large charges. For recent work in this direction, see [41].

One advantage of the large-R-charge expansion, in the case where the theory has a family of Lorentz-invariant vacua, is that it gives us the tools to connect properties of a SCFT which are called the CFT data, *i.e.*, conformal dimensions and OPE constants, with those which can be expressed in terms of EFT on the moduli space of vacua.

In the low-energy dynamics of moduli space, superconformal invariance is spontaneously broken and physical observables can be calculated perturbatively by using the EFT. Such perturbative calculations do not depend on any weak coupling limit of the underlying SCFT; the perturbative parameter in the context of moduli space EFT is the ratio of the IR to the UV energy scale, which in the present case is simply an inverse power of the charge eigenvalue of the state. Therefore we can make use of the large-charge EFT to calculate physical observables associated with near-BPS primary states as a perturbative expansion in J^{-1}.

As a consistency check, we have checked that the scalar primary operators in protected multiplets with large R- and X-charge exist. For chiral primary operators this is straightforward, and it is a bit more nontrivial for scalar semishort primary operators. Curiously, once the existence of scalar semishort primary operators at large values of X-charge is verified, the module structure of scalar semishort primary operators over the chiral ring indicates that scalar semishort primary operators exist at low values of X-charge as well. This prediction agrees with the operator spectrum of protected multiplets obtained by explicitly expanding the superconformal index. Thus it seems that the combination of holomorphy with the large-J expansion is quite powerful in studying SCFTs. It could be possible that this combination of points of view provides some insights into the dynamics of other interesting SCFTs as well.

4 On the large-R-charge expansion in $\mathcal{N} = 2$ superconformal field theories in four dimensions

In section 3 we have analyzed the large-R-charge expansion for operator dimensions in a three-dimensional SCFT with a one-complex-dimensional moduli space, and have quantized the EFT on moduli space in radial quantization, in order to compute operator dimensions of near-BPS primary operators of large R-charge. It is reasnoable to try to make a further connection between the large-R-charge expansion and other methods which maximally uses superconformal invariance. In order to do so, one would like to find a set of physical quantities associated with operators carrying large R-charge, which both has a nontrivial expansion in large R-charge, like the near-BPS operator dimensions calculated in section 3, and can also be computed directly by making use of exact superconformal invariance. The three-point functions of two chiral and one antichiral primary operators in SCFTs with eight or more Poincaré supercharges are prime candidates and studied in this section. These three-point functions are equivalent to the OPE coefficients of chiral ring elements and have a nontrivial dependence on R-charges of the operators, unlike the dimensions of chiral primaries, whose dependence on R-charge is determined by the superconformal algebra. For SCFTs with a UV Lagrangian description, one can in principle calculate these three-point functions exactly by supersymmetric localization and in some cases they have been worked out explicitly [1–4, 116–123] (see also an earlier work [124]).

In this section we calculate three-point functions of two chiral primary operators and one antichiral primary operator in $\mathcal{N} \geq 2$ SCFTs in four dimensions, with a one-complex-dimensional Coulomb branch. As explained in section 4.1, considering three-point functions is equivalent to considering two-point functions of chiral and antichiral primary operators,

$$\mathcal{Y}_n \coloneqq |x-y|^{2n\Delta_{\mathcal{O}}} \left\langle \mathcal{O}^n(x)\bar{\mathcal{O}}^n(y)\right\rangle_{\mathbb{R}^4}, \tag{4.1}$$

which is independent of the positions x, y in a CFT, and therefore invariant under conformal transformations. For SCFTs with a one-complex-dimensional Coulomb branch, the chiral ring is generated by a single chiral primary operator $\mathcal{O}$, whose dimension $\Delta_{\mathcal{O}}$ and R-charge $J_{\mathcal{O}}$ satisfy the following relation,

$$\Delta_{\mathcal{O}} = \frac{1}{2}|J_{\mathcal{O}}|. \tag{4.2}$$

Here, we normalize the $\mathrm{U}(1)_{\mathrm{R}}$-charge $J_{\mathcal{O}}$ so that the unitarity bound for scalar primary operators is $\Delta \geq |J|/2$, and a free vector multiplet scalar ϕ has $\Delta_\phi = |J_\phi|/2 = 1$. Note that a four-dimensional $\mathcal{N} = 2$ SCFT has an $\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{U}(1)_{\mathrm{R}}$ R-symmetry, and chiral primary operators are neutral under $\mathrm{SU}(2)_{\mathrm{R}}$. The parameter n appearing in (4.1) is related to the R-charge of the chiral primary operator $\mathcal{O}^n$ as $J_{\mathcal{O}^n} = 2n\Delta_{\mathcal{O}}$, and therefore the large-R-charge limit is the limit where the parameter n is large.

We show in section 4.2.3 that the asymptotic behavior of the logarithm of the two-point function $\mathcal{Y}_n$ in the large-n limit is universal, behaving as

$$\mathcal{B}_n \coloneqq \log \mathcal{Y}_n = \log\left[(n\Delta_{\mathcal{O}})!\right] + b_{-1} n\Delta_{\mathcal{O}} + \alpha \log\left(n\Delta_{\mathcal{O}}\right) + O\big(n^0\big). \tag{4.3}$$

The coefficient b_{-1} appearing in (4.3) is independent of n but depends on the normalization of the metric on the Coulomb branch relative to the normalization of the chiral primary operator $\mathcal{O}$ itself. The coefficient α appearing in the logarithmic term of (4.3) is shown to be determined by the the a-coefficients of the Weyl anomaly.[27] The definition of the anomaly coefficients is convention-dependent, but α is not. The value of α can be expressed in a convention-independent manner as

$$\alpha \coloneqq \frac{5\Delta a}{12 a_{\mathrm{favm}}}, \tag{4.4}$$

where $\Delta a \coloneqq a_{\mathrm{CFT}} - a_{\mathrm{EFT}}$ is the difference between the a-coefficient a_{CFT} of the underlying SCFT and the a-coefficient a_{EFT} of the low-energy EFT of massless moduli fields. The constant a_{favm} in the denominator is the a-coefficient of a free Abelian vector multiplet of $\mathcal{N} = 2$ supersymmetry in four dimensions. We have expressed the value of α in this form so that it is independent of the overall normalization of the a-coefficient of the Weyl anomaly. In one popular convention used in [6] by Anselmi, Erlich, Freedman and Johansen (AEFJ), the value of a_{favm} is $5/24$, and so α is given by

$$\alpha = 2\Big(a^{[\mathrm{AEFJ}]}_{\mathrm{CFT}} - a^{[\mathrm{AEFJ}]}_{\mathrm{EFT}}\Big), \tag{4.5}$$

in that convention. In table 2, we provide a list of values for α in all known $\mathcal{N} \geq 2$ SCFTs[28] whose Coulomb branch space at a generic point has only one Abelian vector multiplet (plus possibly massless hypermultiplets), in the convention of [6]. For instance, $\mathcal{N} = 4$ super-Yang–Mills theory with gauge algebra $\mathfrak{su}(2)$, has $\alpha = 1$.

As in section 3, an operator carrying large R-charge is equivalent to a state of large R-charge on $\mathbb{S}^3$ *via* radial quantization. Although we will not be radially quantizing the theory or making use of the cylindrical frame in this section, the underlying physics is the same as in section 3. The large-n limit of the two-point functions $\mathcal{Y}_n$ is the large-R-charge limit, in which the EFT becomes weakly interacting.

The first term on the right-hand side of (4.3) comes from the effective action evaluated on the classical solution created by the operator insertions $\mathcal{O}^n(x)\bar{\mathcal{O}}^n(y)$, the constant b_{-1} depends on the normalization of the operator $\mathcal{O}$, and the third term $\alpha\log(n\Delta_{\mathcal{O}})$ comes from the Wess–Zumino Lagrangian (also known as the dilaton effective action) [5, 125] in the EFT on the Coulomb branch. The remaining terms of order n^0 are contributed by quantum loops within the EFT, as well as superconformally invariant higher-derivative interaction

[27] Basics facts about the Weyl anomaly are summarized in appendix H.

[28] As of December 2017.

terms in the effective action of the moduli space, with coefficients nontrivially depending on the underlying theory.[29] Quantum loops within the EFT contributes only at order n^0 and smaller; even those are summed up entirely by the free-field action. That is, the only quantum loops contributing at order n^0 are determinants in the free EFT, and these can be computed by Wick contractions of the free Abelian vector multiplet scalar describing the Coulomb branch.

The contents in this section are based on work [44] done in collaboration with Simeon Hellerman.

4.1 Large-R-charge expansion of two-point functions

The strategy for the computation is analogous with that of section 3. Some of the differences are the number of spacetime dimensions (four here versus three in section 3) and the number of supercharges (eight Poincaré supercharges in the present section versus only four in section 3), but these differences are not significant when considering the large-R-charge expansion. In particular, one can regard a four-dimensional SCFT with eight Poincaré supercharges as a special case of $\mathcal{N} = 1$ SCFT in four dimensions, which becomes by dimensional reduction a theory with $\mathcal{N} = 2$ supersymmetry in three dimensions, as in the case of the theory analyzed in section 3.

Our computation applies to all four-dimensional $\mathcal{N} \geq 2$ SCFTs with a one-complex-dimensional Coulomb branch. Such theories are sometimes called rank-one theories. SCFTs with $\mathcal{N} \geq 3$ supersymmetry are regarded as special cases of $\mathcal{N} = 2$ SCFTs. When we refer to dimensions of moduli spaces, we will always be using the $\mathcal{N} = 2$ terminology, in terms of which, *e.g.*, $\mathcal{N} = 4$ $\mathrm{SU}(N)$ super-Yang–Mills theory, can be considered as an $\mathcal{N} = 2$ super-Yang–Mills theory with gauge group $\mathrm{SU}(N)$ and a single adjoint hypermultiplet, and its moduli space is described by $N-1$ vector multiplets and $N-1$ massless hypermultiplets, again in the $\mathcal{N} = 2$ terminology.

4.1.1 Basics

Two-point functions as three-point functions. For four-dimensional $\mathcal{N} = 2$ SCFTs with a one-complex-dimensional Coulomb branch parametrized by the chiral primary operator $\mathcal{O}$, we consider the three-point functions

$$\left\langle \mathcal{O}^{n_1}(x)\mathcal{O}^{n_2}(y)\bar{\mathcal{O}}^{n_1+n_2}(z)\right\rangle, \qquad n_1,\ n_2 \in \mathbb{N}. \tag{4.6}$$

[29] Some low-derivative interaction terms in effective actions have been constructed in [126–129]. However there is no complete classification of superconformally invariant interaction terms even at low orders in the derivative expansion. Furthermore, nothing is known about the coefficients of higher-derivative interaction terms even for simple $\mathcal{N} = 2$ SCFT.

Similarly to the three-dimensional case in section 3.4.1, due to the unitarity bounds the OPE of chiral primary operators is nonsingular,

$$\mathcal{O}^{n_1}(x)\mathcal{O}^{n_2}(y) = \mathcal{O}^{n_1+n_2}(y) + (\text{terms vanishing at } x = y). \tag{4.7}$$

So three-point functions of two chiral primary operators and one antichiral primary operator can be reduced to two-point functions, by taking the two chiral primary operators to lie at the same point. In order to get rid of the position dependence, we define

$$\begin{aligned} \mathcal{Y}_n &\coloneqq \lim_{y\to x} |x-z|^{2m\Delta_{\mathcal{O}}} |y-z|^{2(n-m)\Delta_{\mathcal{O}}} \left\langle \mathcal{O}^m(x)\mathcal{O}^{n-m}(y)\bar{\mathcal{O}}^n(z)\right\rangle \\ &= |x-z|^{2n\Delta_{\mathcal{O}}} \left\langle \mathcal{O}^n(x)\bar{\mathcal{O}}^n(z)\right\rangle . \end{aligned} \tag{4.8}$$

This quantity is independent of the integer m and the positions x, y, z, and therefore invariant under conformal transformations.

At first sight, one may wonder if there is a meaningful normalization for two-point functions in CFT, because one generically regards two-point functions of primary operators as simply being unit-normalized. However this normalization convention, while widely used, is not the natural one for chiral primary operators. Once one chooses a set of generators of the chiral ring, the higher chiral primary operators generated from them by algebraic multiplication, are defined by associativity, including their normalization. That is, if one unit-normalizes the generators $\mathcal{O}_1$ and $\mathcal{O}_2$ of the chiral ring, one cannot freely unit-normalize the product $\mathcal{O}_3 \coloneqq \mathcal{O}_1\mathcal{O}_2$.

In the case of a rank-one SCFT, one can freely choose the normalization of the chiral ring generator $\mathcal{O}$, but once it has been fixed, one does not have the freedom to normalize the higher chiral primary operators $\mathcal{O}^n$ with $n \geq 2$, and their two-point functions $\mathcal{Y}_n$ depend nontrivially on n. It is the dynamics of the SCFT that determines the value of $\mathcal{Y}_n$.

4.1.2 Free-field approximation

Two-point functions on $\mathbb{R}^4$. In the beginning of section 4.1 we have quoted some dissimilarities between the present case and the case of section 3, such as the number of supercharges and the dimensionality of spacetime, which do not quite change the structure of the computation. A more important difference is that we compute the two-point functions $\mathcal{Y}_n$ directly on flat space $\mathbb{R}^4$, instead of the cylinder as we have done in section 3. This is because the quantities we are going to compute are the two-point functions, which are viewed as the norms of the corresponding states in radial quantization, and therefore it is not easy to see them directly in radial quantization, as the Hilbert space formalism usually begins by taking the norms of the states as inputs. Radial quantization is more beneficial when we calculate the energies of the states than the overall normalizations of them. We could calculate these normalizations in radial quantization as three-point functions, as was done in [37] for the O(2) Wilson–Fisher CFT or other nonsupersymmetric CFTs whose dynamics is described by the conformally invariant EFT of a Goldstone hydrodynamic mode at large charge. It would be interesting to verify that these two methods give the same asymptotic formula for $\mathcal{Y}_n$ at large n, but we will not do it in this thesis.

Free effective field. For rank-one $\mathcal{N}=2$ SCFTs in four dimensions, the low-energy EFT consists of a single vector multiplet, plus possibly massless hypermultiplets. We ignore the hypermultiplets for now, since they do not take part in the classical solution which gives the leading contribution to the two-point function. When we consider the EFT of the vector multiplet, we will mostly[30] follow the conventions of [130, 131]. The holomorphic gauge coupling τ is defined as

$$\tau := \frac{4\pi i}{g^2} + \frac{\theta_{\mathrm{YM}}}{2\pi}. \tag{4.9}$$

An Abelian vector multiplet contains a U(1) gauge field, as well as fermions and a complex scalar field A both neutral under the U(1) gauge group. In terms of the complex scalar field A, the holomorphic effective coupling τ_{eff} is given by

$$\tau_{\mathrm{eff}} = \frac{\partial^2 \mathcal{F}_{\mathrm{eff}}}{\partial A^2}, \tag{4.10}$$

where $\mathcal{F}_{\mathrm{eff}}$ is the effective holomorphic prepotential for the Abelian vector multiplet.

The kinetic term for the complex scalar field A has nontrivial dynamical information and is related to the gauge coupling constant, but the only thing we know *a priori* about the kinetic term for A is that it is invariant under symmetry transformations. Conformal invariance and R-symmetry implies that the metric on the Coulomb branch has to be flat. Then the kinetic term for A must be of the form

$$S_{\mathrm{free}} = \int d^4x \mathcal{L}_{\mathrm{free}}, \qquad \mathcal{L}_{\mathrm{free}} := \frac{\operatorname{Im}\tau_{\mathrm{eff}}}{4\pi}(\partial_\mu A)\big(\partial_\mu \bar{A}\big). \tag{4.11}$$

The holomorphic effective gauge coupling τ_{eff} is related to the effective prepotential by (4.10) and has to be independent of A in SCFTs, so we conclude that the effective prepotential is of the form

$$\mathcal{F}(A) = \frac{\tau_{\mathrm{eff}} A^2}{2}. \tag{4.12}$$

[30] With one particular exception: For the microscopic holomorphic gauge coupling in $\mathcal{N}=2$ superconformal quantum chromodynamics, [130, 131] define

$$\tau = \frac{8\pi i}{g^2} + \frac{\theta_{\mathrm{YM}}}{\pi}$$

for reasons to do with duality and Dirac quantization condition. Instead, we will use the convention

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta_{\mathrm{YM}}}{2\pi}$$

for all gauge couplings, both microscopic and effective, uniformly in the representation of the hypermultiplets. This convention is more frequently used recently, especially in the literature on supersymmetric localization, *e.g.*, [132] and works making use of it.

We can define a scalar field with unit kinetic term by

$$\phi := \sqrt{\frac{\operatorname{Im}\tau_{\text{eff}}}{4\pi}}A \tag{4.13}$$

so that the kinetic term becomes

$$\mathcal{L}_{\text{free}} = (\partial_\mu\phi)\big(\partial_\mu\bar{\phi}\big) \tag{4.14}$$

Note that the change of the variable (4.13) is holomorphic in A and ϕ but not in the complexified coupling constant τ_{eff}.

Normalization of the effective scalar. In order to compute the term of order n in the logarithm of the two-point function $\mathcal{B}_n = \log\mathcal{Y}_n$, one would need to express the generator $\mathcal{O}$ of the chiral ring in terms of the complex scalar field A in the vector multiplet. Since A has dimension $\Delta_A = 1$, the only possibility is

$$\mathcal{O} = (\mathbf{M}_{\mathcal{O}}A)^{\Delta_{\mathcal{O}}}, \tag{4.15}$$

where $\Delta_{\mathcal{O}}$ is the conformal dimension of $\mathcal{O}$ and $\mathbf{M}_{\mathcal{O}}$ is a constant. Defining $\mathbf{N}_{\mathcal{O}} = \mathbf{N}_{\mathcal{O}}(\tau,\bar{\tau})$ such that

$$\mathbf{N}_{\mathcal{O}}\phi = \mathbf{M}_{\mathcal{O}}A, \tag{4.16}$$

the quantities $\mathbf{N}_{\mathcal{O}}$ and $\mathbf{M}_{\mathcal{O}}$ are related by

$$\mathbf{N}_{\mathcal{O}} = \sqrt{\frac{4\pi}{\operatorname{Im}\tau_{\text{eff}}}}\mathbf{M}_{\mathcal{O}}. \tag{4.17}$$

Since ϕ has the unit kinetic term in the effective action, we cannot absorb $\mathbf{N}_{\mathcal{O}}$ into the normalization of ϕ. We could of course absorb $\mathbf{N}_{\mathcal{O}}$ into the normalization of $\mathcal{O}$, but we might want to normalize $\mathcal{O}$ in some other way. For example, we might want to normalize $\mathcal{O}$ to have the unit two-point function, $\mathcal{Y}_1 = 1$. For general rank-one $\mathcal{N} = 2$ SCFTs, we do not know how to compute the constant $\mathbf{N}_{\mathcal{O}}$, provided some fixed normalization of $\mathcal{O}$: This is an interesting problem for future investigation of the large-R-charge limit. For now, we leave the constants $\mathbf{M}_{\mathcal{O}}$ and $\mathbf{N}_{\mathcal{O}}$ undetermined. The map between $\mathcal{O}$ and ϕ can be expressed as

$$\mathcal{O} = (\mathbf{N}_{\mathcal{O}}\phi)^{\Delta_{\mathcal{O}}}. \tag{4.18}$$

Multivaluedness of the map between $\mathcal{O}$ and ϕ. Note that the map between the chiral ring generator $\mathcal{O}$ and A or ϕ is not one to one in general. If $\Delta_{\mathcal{O}}$ is an integer, then the map from ϕ to $\mathcal{O}$ is single-valued, but it is one-to-one if and only if $\Delta_{\mathcal{O}} = 1$, which means that $\mathcal{O}$ is a free vector multiplet scalar and therefore the underlying SCFT is free. If $\Delta_{\mathcal{O}}$ is not an integer, the map from ϕ to $\mathcal{O}$ is multivalued.[31]

The coordinate ϕ or A should be considered only as a local holomorphic coordinate of the Coulomb branch here. As long as we are far away from the origin of the Coulomb branch, the singularity of the map (4.15) at the origin should not invalidate the use of the EFT.

When $\Delta_{\mathcal{O}}$ is a noninteger rational number, we may for simplicity restrict ourselves to the situations where $n\Delta_{\mathcal{O}}$ is an integer, so that our Wick-contraction of $n\Delta_{\mathcal{O}}$ free scalar fields is well-defined.[32] However a further transformation to a logarithmic field $\log \phi$, might validate the calculation even for $n\Delta_{\mathcal{O}} \notin \mathbb{Z}$; a similar point was made in section 3.3.2. For the purpose of calculating conformal dimensions as we have done in section 3, the existence of the logarithmic field is a persuasive reason to believe that there is no possibility of peculiar behavior happening due to the fractional part of $n\Delta_{\mathcal{O}}$. On the other hand, the computation of correlation functions is slightly different, as the multivaluedness of the map may become relevant at the insertion points of the operator $\mathcal{O}$ and its conjugate. We will leave this an open question, and for now we simply choose n such that $n\Delta_{\mathcal{O}}$ is an integer. In all SCFTs with a UV Lagrangian description, $\Delta_{\mathcal{O}}$ is always an integer, with the chiral ring being generated by traces of powers of the non-Abelian vector multiplet scalar, and therefore we can evade the above-mentioned issue. For a rank-one SCFT with a UV Lagrangian description, $\Delta_{\mathcal{O}}$ is always equal to 2, and the chiral ring generator $\mathcal{O}$ is in the multiplet of the exactly marginal operator cotangent to the microscopic holomorphic gauge coupling τ.

Calculation of the free-field contribution. Now we would like to write down the leading approximation to the two-point function in the large-n limit. For general rank-one SCFTs, we choose n so that $n\Delta_{\mathcal{O}}$ is an integer, and by Wick-contraction of $n\Delta_{\mathcal{O}}$ complex free scalar fields, we get

$$\mathcal{Y}_n \simeq |x-y|^{2n\Delta_{\mathcal{O}}} \left\langle \mathcal{O}^n(x)\bar{\mathcal{O}}^n(y)\right\rangle \Big|_{\text{free}} = \left|\frac{\mathbf{N}_{\mathcal{O}}}{2\pi}\right|^{2n\Delta_{\mathcal{O}}} (n\Delta_{\mathcal{O}})!, \tag{4.19}$$

where the factor of 2π in the denominator comes from the normalization of the free scalar propagator with the unit kinetic term, (F.3).

This is simply the free approximation and is not exact in n. However we show below that superconformally invariant higher-derivative interaction terms in the effective action only

[31] Although $\Delta_{\mathcal{O}}$ is integer in rank-one SCFTs with a UV Lagrangian description, there are various non-Lagrangian rank-one SCFTs (so-called Argyres–Douglas theories [133–135]) with fractional $\Delta_{\mathcal{O}}$. See, *e.g.*, table 1 of [60].

[32] It is believed that the conformal dimensions of chiral primary operators in four-dimensional $\mathcal{N} \geq 2$ SCFTs are always rational. This is true obviously in SCFTs with a UV Lagrangian description and also in all known non-Lagrangian $\mathcal{N} \geq 2$ SCFTs. Especially in the rank-one case, all known SCFTs in the general classification [57–60] have rational conformal dimensions for chiral primary operators.

have n-suppressed contributions in the logarithm of the two-point function $\mathcal{B}_n = \log \mathcal{Y}_n$, using arguments similar to those we have made in section 3.

To do this, we relate the two-point function with a classical solution of the effective action with sources which correspond to the insertions of the operators. Doing this we verify that the classical approximation to $\mathcal{B}_n$ agrees with (4.19) up to terms of order $\log n$, which come from loop corrections in the path integral over the free action with sources.

4.1.3 Classical solution with operator insertions

Large-n insertions as classical sources. The path integral of the free Euclidean theory with insertions is equivalent to a path integral with sources,

$$\left\langle \prod_i \mathcal{O}_i(x_i) \right\rangle := \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \exp\left(-S_{\text{free}} - S_{\text{source}}\right), \qquad S_{\text{source}} := -\sum_i \log \mathcal{O}_i(x_i). \tag{4.20}$$

In our case,

$$\mathcal{O}_1(x_1) = [\mathcal{O}(x_1)]^{n\Delta_{\mathcal{O}}} = [\mathbf{N}_{\mathcal{O}}\phi(x_1)]^{n\Delta_{\mathcal{O}}}, \quad \mathcal{O}_2(x_2) = \left[\bar{\mathcal{O}}(x_2)\right]^{n\Delta_{\mathcal{O}}} = \left[\mathbf{N}^*_{\mathcal{O}}\bar{\phi}(x_2)\right]^{n\Delta_{\mathcal{O}}}, \tag{4.21}$$

where we have used (4.18). Then, the total action in (4.20) becomes

$$\begin{aligned} S_{\text{free}} + S_{\text{source}} &= -2n\Delta_{\mathcal{O}} \log |\mathbf{N}_{\mathcal{O}}| + \int d^4x \mathcal{L}_{\text{dyn}}, \\ \mathcal{L}_{\text{dyn}} &:= (\partial_\mu \phi)\left(\partial_\mu \bar{\phi}\right) - \delta(x - x_1) n\Delta_{\mathcal{O}} \log \phi - \delta(x - x_2) n\Delta_{\mathcal{O}} \log \bar{\phi}, \end{aligned} \tag{4.22}$$

so the equations of motion for the scalar fields are given by

$$\partial^2 \bar{\phi}(x) = -\frac{n\Delta_{\mathcal{O}}}{\phi(x)} \delta(x - x_1), \qquad \partial^2 \phi(x) = -\frac{n\Delta_{\mathcal{O}}}{\bar{\phi}(x)} \delta(x - x_2). \tag{4.23}$$

Solution to the equation of motion. The classical solution to (4.23) has a $\mathrm{U}(1)_{\mathrm{R}}$-phase zero mode and therefore is not unique. One can regard this phase zero mode as an R-symmetry Goldstone boson of the solution,[33] which shifts by a constant under a $\mathrm{U}(1)_{\mathrm{R}}$ transformation. An EFT including this R-symmetry Goldstone boson has been analyzed in [136]. At higher orders in n^{-1} the path integral over the phase zero mode develops corrections to the effective action through the path integral measure, but these are suppressed and contribute only at order n^{-1} or smaller. As we are calculating the effective action only up to and including order $\log n$ in this section, we do not have to consider such quantum corrections.

The classical solution to (4.23) is of the form

$$\phi(x) = \frac{c_\phi}{|x - x_2|^2}, \qquad \bar{\phi}(x) = \frac{c_{\bar{\phi}}}{|x - x_1|^2}, \tag{4.24}$$

[33] Note that scaling invariance is explicitly broken by the sources, and therefore there is no scaling zero mode in the solution.

with c_ϕ and $c_{\bar{\phi}}$ satisfying

$$\frac{(2\pi)^2 c_\phi c_{\bar{\phi}}}{|x_1 - x_2|^2} = n\Delta_{\mathcal{O}}. \tag{4.25}$$

The absolute value of ϕ is evaluated as

$$|\phi(x)| = \frac{\sqrt{n\Delta_{\mathcal{O}}}}{2\pi} \frac{|x_1 - x_2|}{|x - x_1||x - x_2|}. \tag{4.26}$$

The value of the action (4.22) at the saddle point (4.24) is given by

$$\int d^4x \mathcal{L}_{\mathrm{dyn}} = n\Delta_{\mathcal{O}}[-\log(n\Delta_{\mathcal{O}}) + 1 + 2\log|x_1 - x_2| + 2\log(2\pi)]. \tag{4.27}$$

Classical approximation to the free two-point function. The classical action is of order $n \log n$. Thus the R-charge $2n\Delta_{\mathcal{O}}$ of the operator plays a role of an inverse Planck constant $\hbar^{-1}$ in a perturbative expansion, and suppresses quantum fluctuations of any product $\prod_i \mathcal{O}'_i$ of operators $\mathcal{O}'_i$ carrying R-charge of order one, inserted into the two-point function. Especially, one can divide the scalar field ϕ into its classical solution plus a fluctuation, $\phi = \phi_{\mathrm{cl}} + \phi_{\mathrm{fluc}}$, and n plays a role of a parameter which suppresses quantum corrections relative to the classical partition function $Z_{\mathrm{cl}} \coloneqq \exp(-S_{\mathrm{free}} - S_{\mathrm{source}})\big|_{\phi=\phi_{\mathrm{cl}}}$. In other words, we expect

$$\begin{aligned} \log Z_{\text{free+sources}} \coloneqq\ & \log\left[\int \mathcal{D}\phi\mathcal{D}\bar{\phi} \exp\left(-S_{\mathrm{free}} - S_{\mathrm{source}}\right)\right] \\ \simeq\ & \log Z_{\mathrm{cl}} = -(S_{\mathrm{free}} + S_{\mathrm{source}})\Big|_{\phi=\phi_{\mathrm{cl}}}, \end{aligned} \tag{4.28}$$

with errors of relative order n^{-1}.

We would like to verify this prediction explicitly. Since the classical partition function $Z_{\text{free+sources}}$ is exactly determined by the Wick contraction result (4.19), we only need to calculate the value of the classical free action with sources at the saddle point (4.24), and check if it agrees with the asymptotic formula of the logarithm of the Wick contraction result (4.19). The right-hand side of (4.28) is

$$-(S_{\mathrm{free}} + S_{\mathrm{source}})\Big|_{\phi=\phi_{\mathrm{cl}}} = \log\left[\left(\frac{n\Delta_{\mathcal{O}}|\mathbf{N}_{\mathcal{O}}|}{2\pi|x_1 - x_2|}\right)^{2n\Delta_{\mathcal{O}}} e^{-n\Delta_{\mathcal{O}}}\right], \tag{4.29}$$

so from (4.8) we have

$$\mathcal{Y}_n \simeq \left(\frac{n\Delta_{\mathcal{O}}|\mathbf{N}_{\mathcal{O}}|}{2\pi}\right)^{2n\Delta_{\mathcal{O}}} e^{-n\Delta_{\mathcal{O}}}. \tag{4.30}$$

This approximation can be viewed as Stirling's approximation to the Wick contraction result $(2\pi)^{-2n\Delta_{\mathcal{O}}}(n\Delta_{\mathcal{O}})!$ of the two-point function, multiplied by the normalization factor $|\mathbf{N}_{\mathcal{O}}|^{2n\Delta_{\mathcal{O}}}$.

So we have verified the prediction (4.28), and that the R-charge $2n\Delta_{\mathcal{O}}$ indeed plays a role of a parameter which suppresses corrections to the classical approximation. If we were satisfied with the approximation (4.30), which is accurate only at order $n \log n$ and n, it would be a rather unsophisticated way to approximate a free two-point function; if the effective action were exactly free, then we would just use the exact formula (4.19). However the approximation of the large-n Wick contraction by a classical saddle point, allows us to go beyond free-field ones, and include corrections coming from interaction terms in the effective action. In section 4.2 we compute contributions to $\mathcal{B}_n = \log \mathcal{Y}_n$ of order $\log n$, which come from interaction terms in the effective action.

4.2 Contributions of interaction terms

In (4.30) an approximation to the two-point function $\mathcal{Y}_n$ is given, with the symbol "$\simeq$" implying that interaction terms beyond the free kinetic term in the effective action are discarded. We would like to understand how accurate the approximation (4.30) is. To do this, we need to know how interaction terms contribute in the large-n limit. Since n controls the magnitude of the classical value (4.26) of $|\phi|$, which is of order $\sqrt{n}$, we can determine the order of contribution of each interaction term in the effective action at large n from its $|\phi|$-scaling.

4.2.1 n-scalings and the dressing rule

The $|\phi|$-scaling of an interaction term in the effective action means the number of ϕ's and $\bar{\phi}$'s appearing in the numerator of the term, minus that appearing in the denominator. In the denominator, the fields can only appear without derivatives. This rule, long (correctly) regarded as obvious for study of moduli space EFTs, is based on the nontrivial fact that moduli spaces exist, and so the leading term in the denominator in a moduli space effective action, must be an undifferentiated field.

More generally, in any EFT where scale invariance is spontaneously broken, the field appearing in the denominator of an interaction term needs to be a dimensionful field that has an expectation value in the spontaneously broken vacuum. The spontaneously broken scale invariance and R-symmetry severely constrain interaction terms which can appear in the effective action, even without making full use of superconformal invariance. The most important point is that, since ϕ and $\bar{\phi}$ themselves are the lowest-dimension fields with VEVs, they are the unique dressing fields to make a term invariant under scaling and R-symmetry transformations. For instance, let $(\mathsf{term})_{\text{undressed}}$ be some monomial in $\partial^k \phi$, $\partial^k \bar{\phi}$, the curvature of the Abelian gauge field and its derivatives, and the fermions in the vector multiplet and their derivatives. Then it has a unique scale-invariant and $\mathrm{U}(1)_\mathrm{R}$-symmetry-invariant dressing by ϕ and $\bar{\phi}$:

$$
\begin{gathered}
(\mathsf{term})_{\text{dressed}} = \phi^{-\ell}\bar{\phi}^{-\tilde{\ell}}(\mathsf{term})_{\text{undressed}}, \\
\ell = \frac{1}{2}\Delta_{\text{undressed}} + \frac{1}{4}J_{\text{undressed}}, \qquad \tilde{\ell} = \frac{1}{2}\Delta_{\text{undressed}} - \frac{1}{4}J_{\text{undressed}},
\end{gathered}
\tag{4.31}
$$

where $\Delta_{\text{undressed}}$ and $J_{\text{undressed}}$ represent the scaling dimension and R-charge of $(\mathsf{term})_{\text{undressed}}$, respectively.

In the $\mathcal{N}=2$ superspace formalism [137, 138], we can express each term in the effective action as an integral over all the Grassmann variables ($\mathcal{N}=2$ D-terms) or a subset ($\mathcal{N}=2$ F-terms and θ^6-integrals), and therefore make it manifestly supersymmetric. The dressing rule can then be enforced at the level of the superspace integrands themselves, taking into consideration the contributions of the integration measure to the scaling dimension and R-charge of the term.

Let Φ be a superfield consisting of the effective vector multiplet. The dressing rules implies that one need to dress each (super-)derivative ∂_μ or D^i_α, $\bar{D}^i_{\dot\alpha}$ to make it invariant under scale and $\mathrm{U}(1)_\mathrm{R}$-transformations, using undifferentiated Φ's and $\bar{\Phi}$'s themselves. Each ∂_μ has to be dressed by $(\Phi\bar{\Phi})^{-1/2}$ and each D^i_α or $\bar{D}^i_{\dot\alpha}$ has to be dressed by $\Phi^{-1/2}$ or $\bar{\Phi}^{-1/2}$, respectively.

Before coming to the classification of terms, several comments are in order:

- Dressing with the appropriate amount of Φ and $\bar{\Phi}$ in the denominator, is necessary but clearly not a sufficient condition for a term which can appear in the effective action: Besides rigid scale invariance and supersymmetry, there are many other conditions such as invariance under special conformal and $\mathrm{SU}(2)_\mathrm{R}$ transformations, as well as super-Weyl covariance on curved backgrounds. These conditions give additional severe constraints on terms in the effective action, but as we shall see supersymmetry, $\mathrm{U}(1)_\mathrm{R}$-symmetry, and scale invariance alone deny any contribution from superconformally invariant interaction terms up to order n^0.

- We classify terms in flat space; one should be aware of the fact that this classification can underestimate the n-scaling of a term in a general curved background. There can be a Weyl-invariant term involving curvature tensors, where the n-scaling of the curvature-dependent piece of the term in a curved background is larger than the n-scaling of the term in flat space. For instance, the large-charge EFT of the non-supersymmetric O(2) Wilson–Fisher CFT [35] contains the following Weyl-invariant term,

$$
|\partial\chi|\mathcal{R} + 2\frac{(\partial|\partial\chi|)^2}{|\partial\chi|},
\tag{4.32}
$$

where the first piece scaling as $n^{1/2}$, and the second piece scaling only as $n^{-1/2}$ for a helical solution in the cylinder.[34]

- A systematic classification of superconformally invariant terms including their super-Weyl-invariant curvature completions in general curved backgrounds requires to develop a superconformal version of the formalism of [37], which is based on the general Callan–Coleman–Wess–Zumino (CCWZ) methodology [139, 140]. One can make use of this formalism in large-charge EFTs such as [35–38, 40–43] in order for the classification of terms in effective actions when some combinations of internal symmetry and conformal symmetry are preserved with the rest spontaneously broken.[35] In the case of $\mathcal{N} = 2$ SCFTs in three dimensions such as the model discussed in section 3, a conformal supergravity formalism such as [104] would give a superconformal extension of [37], and actually the leading superconformally invariant interaction term, the super-FTPR term, is constructed in a most general curved background by utilizing the formalism of [104].

- However in the present EFT one can easily see that the only possible scale- and $\mathrm{U}(1)_{\mathrm{R}}$-invariant curvature-dependent term scaling as greater than n^0, is given by $\mathcal{R}|\phi|^2$. This term is just the usual conformal coupling which is always needed to make the flat-space kinetic term $|\partial\phi|^2$ Weyl-invariant, with coefficient[36] $1/6$. One more Ricci scalar would necessitate an additional factor of $|\phi|^{-2}$. Tensorial curvature terms are not dangerous either: The Ricci tensor $\mathcal{R}^{\mu\nu}$, for instance, has weight 4 under a rigid Weyl transformation, and therefore would have to be compensated with $\left(\partial_\mu\phi\partial_\nu\bar{\phi}\right)|\phi|^{-2}$ in order to have weight 4 so that it can appear in the effective Lagrangian. In this case the n-scaling of the resulting term is zero. Terms involving curvature tensors with more free indices, have even higher Weyl weight, and therefore require even more undifferentiated $|\phi|$'s in the denominator in order to make a term of total weight 4 after contracting tensor indices with derivatives of the fields. So in the present case we never underestimate the n-scaling of superconformally invariant terms contributing larger than n^0, by classifying them simply in flat space.

- Our classification is valid only for superconformally invariant terms in the effective action. Since the underlying SCFT is invariant under superconformal transformations modulo the anomaly, so must the effective action be invariant, modulo the anomaly.

[34] The field χ is the O(2) Goldstone boson, *i.e.*, a compact scalar transforming as $\chi \to \chi + (\text{const.})$ under an O(2) transformation. The gradient $\partial\chi$ scales as $n^{1/2}$ whereas $\partial^p\chi$ with $p \geq 2$ scales as n^0.

[35] While enlarging the previous scope of the CCWZ formalism [139,140], the formalism of [37] still requires a ground state to be invariant under spatial translations. It would be interesting to understand how the formalism of [37] can be generalized to cases where the ground state is not spatially homogeneous at large charge [36,39].

[36] When the kinetic term is unit-normalized, the coefficient of $\mathcal{R}|\phi|^2$ must be $(d-2)/(4d-4)$ in $d \geq 2$ dimensions.

That is, the full effective action must nontrivially transform under superconformal transformations in order to correctly reproduce the anomaly of the underlying SCFT. So, the effective action should contain terms which are not invariant under superconformal transformations. Such terms are known as the Wess–Zumino terms or the dilaton effective action [5, 125], and compensate the difference of the Weyl anomaly between the underlying SCFT and the EFT of moduli space. The coefficients of these terms are c-numbers, independent of the state, and therefore cannot depend on n. The Wess–Zumino terms scale as n^0 in flat space, but it is enhanced to $\log n$ in a curved background, as we see explicitly in section 4.3.

4.2.2 Dressing and n-scaling of superconformally invariant interaction terms

Here we demonstrate that no superconformally invariant interaction term can contribute at order n^0 or larger to $\mathcal{B}_n$.

Higher-derivative terms for vector multiplets. The analysis here is very similar to that in section 3: each superderivative D^i_α or $\bar{D}^i_{\dot\alpha}$ in a superspace integrand needs to be accompanied by no less than one factor of $\Phi^{-1/2}$ or $\bar\Phi^{-1/2}$ respectively, and each derivative ∂_μ has to be accompanied by no less than one factor of $|\phi|^{-1}$, in order for scale invariance to be maintained. The measure $d^8\theta$ for full superspace integrals, D-terms, have scaling dimension 4, and therefore their integrands have to satisfy

$$D\text{-term dressing:} \qquad N_{\Phi+\bar\Phi} = -N_\partial - \frac{1}{2}N_{D+\bar{D}}, \tag{4.33}$$

where $N_{\Phi+\bar\Phi}$ is the $|\Phi|$-scaling of an integrand, and N_∂ and $N_{D+\bar{D}}$ are the numbers of derivatives and superderivatives in the numerator, respectively. Therefore, even the n-scaling of a D-term with no derivatives or superderivatives would be n^0. However, integrands without derivatives or superderivatives are just constants, which vanish when integrated over superspace. Any nontrivial D-term must have definitely negative n-scaling.

Integrands of half-superspace integrals (F-terms) have scaling dimension 2, and therefore must satisfy

$$F\text{-term dressing:} \qquad N_{\Phi+\bar\Phi} = 2 - N_\partial - \frac{1}{2}N_{D+\bar{D}}, \tag{4.34}$$

The F-term integrand with zero derivatives is merely the kinetic term proportional to Φ^2 (see, *e.g.*, section 7.3 of [138]). It is known [126–129] that the term with the second largest value of $N_{\Phi+\bar\Phi}$ has $N_\partial = 2$ as a half-superspace integrand, given in equation (5.13) of [126]. The integrand is of the form $\mathcal{G}[\Phi](\partial^\mu\Phi)(\partial_\mu\Phi)$ for some holomorphic function $\mathcal{G}[\Phi]$. Since the half-superspace measure $d^4\theta$ has scaling dimension 2, this half-superspace integrand needs to have scaling dimension 2 in order for scale invariance to be preserved. Therefore, the only possible form for the function $\mathcal{G}[\Phi]$ is $\mathcal{G}[\Phi] = c_\mathcal{G}\Phi^{-2}$, where $c_\mathcal{G}$ is a constant. However, in order for R-symmetry to be preserved, the R-charge of any half-superspace integrand must be $J_F = 4$. Since the R-charge of $\Phi^{-2}(\partial^\mu \Phi)(\partial_\mu \Phi)$ is obviously zero, this term is disallowed. So we conclude that any half-superspace integral in the effective action must have strictly negative n-scaling, except for the kinetic term.

Inclusion of massless hypermultiplets. So far we have considered superconformally invariant terms constructed only out of vector multiplets. If massless hypermultiplets are present in the low-energy dynamics of the underlying SCFT, the classification of terms becomes a little subtler, because there can be superconformally invariant higher-derivative terms which involve both vector and hypermultiplets. In the $\mathcal{N} = 2$ superspace formalism, such terms can be expressed as integrals over six out of the eight Grassmann variables. Such a $^3/_4$-superspace integrand must satisfy

$$\text{mixed term dressing:} \qquad N_{\Phi+\bar{\Phi}} = 1 - N_\partial - \frac{1}{2} N_{D+\bar{D}} - N_h, \tag{4.35}$$

where N_h is the number of powers of hypermultiplets appearing in the integrand.

In the classical solution, only the complex scalar field in the vector multiplet has a nonvanishing value, because there are sources only for the vector multiplet scalar, and the metric of the moduli space factorizes into hypermultiplet and vector multiplet factors. All fields in the hypermultiplets are vanishing in the classical solution in order for $\mathrm{SU}(2)_\mathrm{R}$ to be preserved, and therefore the terms which contribute with the maximal n-scaling are those consisting of degrees of freedom of the vector multiplets. That is, the only terms which are nonvanishing when evaluated at the saddle point, are those with $N_h = 0$, and these n-scaling cannot be larger than $n^{1/2}$. However any superconformally invariant term which contains only vector multiplets must be equal to an F-term, and therefore the n-scaling of such a term is strictly less than n^0 as shown above. Also, in [126] it is shown that any $^3/_4$-superspace integrand involving both vector and hypermultiplets satisfies $N_\partial + N_{D+\bar{D}}/2 > 1$. So we conclude that the effective action can contain mixed $^3/_4$-superspace terms, but their classical value at the saddle point is always zero and they contribute only through their quantum fluctuations. Each loop is suppressed by powers of n^{-1} relative to the maximum n-scaling determined by (4.35). Therefore, any contribution coming from mixed $^3/_4$-superspace terms which really involve vector and hypermultiplets must be strictly smaller than n^0.

Order $\log n$ contribution from the Wess–Zumino term. We have shown above that contributions coming from superconformally invariant interaction terms are of order strictly smaller than n^0. Now we turn to consider the Wess–Zumino anomaly term and its supersymmetric completion [5,125,136]. It is not superconformally invariant and compensates the difference of the Weyl and $\mathrm{U}(1)_\mathrm{R}$ anomalies between the underlying SCFT and the moduli space EFT. Contributions coming from this term are of order $\log n$ and n^0, and therefore provide a power-law factor n^α in the two-point function $\mathcal{Y}_n$, where its coefficient α is determined by the a-coefficient of the Weyl anomaly. Since we are only calculating $\mathcal{B}_n = \log \mathcal{Y}_n$

up to and including order $\log n$, the Wess–Zumino anomaly term is the only interaction term in the effective action we have to consider. Furthermore, we only need to compute its classical contribution, as its quantum contributions to $\mathcal{B}_n$ are suppressed by inverse powers of n^{-1}.

The only contribution of order $\log n$ comes from the coupling of the dynamical effective dilaton τ to the Euler density of the background metric; all other contributions coming from the Wess–Zumino term are of order n^0 and smaller. The Euler density is vainishing in flat space $\mathbb{R}^4$, and we need to transform to the $\mathbb{S}^4$ frame in order to see the contribution of order $\log n$ clearly. We shall elaborate why it is necessary to do so in section 4.2.4.

4.2.3 Structure of the large-n expansion

Before computing the $O(\log n)$ contribution from the Wess–Zumino term, let us clarify the structure of the large-n expansion of the logarithm of the two-point function, $\mathcal{B}_n = \log \mathcal{Y}_n$.

We have all the data which are necessary to determine the large-n expansion of $\mathcal{B}_n$ up to and including order $\log n$. The two-point function $\mathcal{Y}_n$ can be regarded as a partition function with sources. That is, it can be expressed as

$$\mathcal{Y}_n = e^{\mathcal{B}_n} = |x_1 - x_2|^{2n\Delta_{\mathcal{O}}} Z_n, \tag{4.36}$$

where Z_n is defined by the path integral with the integrand $\exp(-S_n)$, where

$$S_n \coloneqq S_{\text{CFT}} - n \log [\mathcal{O}(x_1)] - n \log \left[\bar{\mathcal{O}}(x_2)\right]. \tag{4.37}$$

In the large-n limit, the path integral is approximated by the saddle point corresponding to the classical solution (4.24), in which the value of ϕ is large. In this regime the action S_{CFT} is approximated by by its moduli space effective action and we identify the chiral primary operator $\mathcal{O}$ with $(\mathbf{N}_{\mathcal{O}}\phi)^{\Delta_{\mathcal{O}}}$. The quantity $\mathcal{B}_n$ has a well-behaved perturbative expansion in n, which can be computed as a sum of connected diagrams. In the large-n expansion, the terms nonanalytic in n, if any, cannot come from singular low-energy dynamics of the EFT, since the EFT is free at deep IR. The logarithmic terms arise since there are explicitly nonanalytic terms in the effective action, as a function of the scalar field ϕ: Since ϕ scales as $\sqrt{n}$ and the action (4.37) explicitly has an $n\Delta_{\mathcal{O}} \log \phi + \text{c.c.}$ term, the quantity $\mathcal{B}_n$ has contributions of order $n \log n$ as well as of order $\log n$. The latter, $O(\log n)$ contribution can be understood as a one-loop contribution in quantizing the theory around the saddle point, or more simply as the next-to-next-to-leading term in the expansion of $\log [(n\Delta_{\mathcal{O}})!]$ by Stirling's formula at large n.

So far we have shown that superconformally invariant interaction terms only provide contributions of order smaller than n^0 in the effective action. The only term in the effective action which gives a contribution larger than n^0 is the Wess–Zumino term, whose contribution is of order $\log n$ with a coefficient determined by the difference of the a-coefficients, $\Delta a \coloneqq a_{\text{CFT}} - a_{\text{EFT}}$, as we will elaborate in section 4.3.

To sum up, we have the following large-n expansion of $\mathcal{B}_n$ up to contributions of order n^0,

$$\mathcal{B}_n = \log\left[(n\Delta_{\mathcal{O}})!\right] + b_{-1} n\Delta_{\mathcal{O}} + \alpha \log\left(n\Delta_{\mathcal{O}}\right) + O\big(n^0\big), \qquad b_{-1} = 2\log\left|\frac{\mathbf{N}_{\mathcal{O}}}{2\pi}\right|, \tag{4.38}$$

where α is determined by the Weyl anomaly, and $\mathbf{N}_{\mathcal{O}}$ is an n-independent constant which describes the normalization of the chiral ring generator relative to the effective vector multiplet scalar. The expansion (4.38) gives the large-n expansion of the two-point function $\mathcal{Y}_n$,

$$\mathcal{Y}_n = (n\Delta_{\mathcal{O}})! \left|\frac{\mathbf{N}_{\mathcal{O}}}{2\pi}\right|^{2n\Delta_{\mathcal{O}}} (n\Delta_{\mathcal{O}})^{\alpha} \tilde{\mathcal{Y}}_n, \tag{4.39}$$

where $\tilde{\mathcal{Y}}_n$ converges to a constant in the $n \to \infty$ limit. These large-n asymptotic expansions are our main formulæ.

Sum and product rules. Although $\mathcal{N} = 2$ superconformal symmetry simplifies correlation functions of chiral and antichiral primary operators, it is still nontrivial to compute them. For some rank-one SCFTs, one might be able to get only approximate or numerical data, with which one might want to compare our predictions (4.38) and (4.39) in the large-n limit. In such theories, it is convenient to express the properties of the large-n expansion as product/quotient rules for the two-point function $\mathcal{Y}_n$, or equivalently as sum rules for its logarithm $\mathcal{B}_n$, which extract particular terms in the expansion. The simplest rules are simply limits for $\mathcal{B}_n$,

$$\frac{\mathcal{B}_n}{\log\left[(n\Delta_{\mathcal{O}})!\right]} = 1 + O\left(\frac{1}{\log n}\right). \tag{4.40}$$

The error on the right-hand side is due to the $O(n)$ term in (4.38), which depends on the normalization of the chiral ring generator. The inverse of $\log n$ falls off quite slowly, and therefore (4.40) is not so useful. If we already knew the value of the constant $\mathbf{N}_{\mathcal{O}}$, we could have the more accurate limit,

$$\frac{\mathcal{B}_n - b_{-1} n\Delta_{\mathcal{O}}}{\log\left[(n\Delta_{\mathcal{O}})!\right]} = 1 + O\big(n^{-1}\big). \tag{4.41}$$

However, it is burdensome to determine the value of $b_{-1} = 2\log|\mathbf{N}_{\mathcal{O}}/(2\pi)|$, since it depends on the normalization of the chiral ring generator itself and we have to deal with information from a lot of choices of conventions which may be troublesome to compare among definitions of the chiral ring generator. We would like to obtain sum rules which do not involve the factor $\mathbf{N}_{\mathcal{O}}$, so that we need not to bother to deal with the normalization issue. The simplest such sum rule may be

$$n\mathcal{B}_{n+1} - (n+1)\mathcal{B}_n = n\Delta_{\mathcal{O}} - \left(\alpha + \frac{1}{2}\right)\log\left(n\Delta_{\mathcal{O}}\right) + O\big(n^0\big). \tag{4.42}$$

This sum rule looks useful, because each term on the left-hand side is of order $n^2 \log n$ whereas the error on the left-hand side is of order n^0, and furthermore the normalization constant $\mathbf{N}_\mathcal{O}$ is absent.

It may be burdensome analytically or costly numerically to extract the logarithm on the right-hand side of (4.42), so in some cases it may be better to use product or quotient rules for $\mathcal{Y}_n$ rather than sum rules for $\mathcal{B}_n$. The exponential of the sum rule (4.42) is given by

$$\frac{(\mathcal{Y}_{n+1})^n}{(\mathcal{Y}_n)^{n+1}} = e^{n\Delta_\mathcal{O}}(n\Delta_\mathcal{O})^{-\alpha-1/2}\big[O(n^0) + O(n^{-1}) + \cdots\big]. \tag{4.43}$$

The difficulty in using this quotient rule is that each of the numerator and denominator of the left-hand side is of order larger than $(n\Delta_\mathcal{O})!$ to the n^{th} power, and we need to take ratios of them. That is, both the numerator and denominator have digits of order $n^2 \log n$, which cancel with a precision of digits of order n, so quite a lot of significant figures of precision are squandered.

The use of three adjacent values let a product or quotient rule evade this difficulty and still be independent of the normalization factor $\mathbf{N}_\mathcal{O}$, while both the numerator and denominator of the left-hand side have only $O(n \log n)$ digits each. We have:

$$\frac{\mathcal{Y}_{n+2}\mathcal{Y}_n}{(\mathcal{Y}_{n+1})^2} = \big[1 + O(n^{-3})\big] \exp\left(n^{-1}\Delta_\mathcal{O} - n^{-2}\left[\Delta_\mathcal{O} + \frac{1}{2} + \alpha\right]\right). \tag{4.44}$$

This product rule is equivalent to the following sum rule for $\mathcal{B}_n$,

$$\mathcal{B}_{n+2} - 2\mathcal{B}_{n+1} + \mathcal{B}_n = n^{-1}\Delta_\mathcal{O} - n^{-2}\left(\Delta_\mathcal{O} + \frac{1}{2} + \alpha\right) + O(n^{-3}). \tag{4.45}$$

This rule may be the most useful one, because we can make three independent consistency checks from it, at orders $n^{0,-1,-2}$, respectively, without encountering neither terrible computational difficulty nor the need to know the value of the normalization constant $\mathbf{N}_\mathcal{O}$. The three independent checks for $\mathcal{B}_n$ can be expressed as:

$$\lim_{n\to\infty} (\mathcal{B}_{n+2} - 2\mathcal{B}_{n+1} + \mathcal{B}_n) = 0, \tag{4.46}$$

$$\lim_{n\to\infty} n(\mathcal{B}_{n+2} - 2\mathcal{B}_{n+1} + \mathcal{B}_n) = \Delta_\mathcal{O}, \tag{4.47}$$

$$\lim_{n\to\infty} \big[n^2(\mathcal{B}_{n+2} - 2\mathcal{B}_{n+1} + \mathcal{B}_n) - n\Delta_\mathcal{O}\big] = -\left(\Delta_\mathcal{O} + \frac{1}{2} + \alpha\right). \tag{4.48}$$

The multiplicative version of the rule is

$$\frac{\mathcal{Y}_{n+2}\mathcal{Y}_n}{(\mathcal{Y}_{n+1})^2} = 1 + n^{-1}\Delta_\mathcal{O} + n^{-2}\left(\frac{\Delta_\mathcal{O}^2}{2} - \Delta_\mathcal{O} - \frac{1}{2} - \alpha\right) + O(n^{-3}), \tag{4.49}$$

from which we can make the following three independent checks,

$$\lim_{n\to\infty} \frac{\mathcal{Y}_{n+2}\mathcal{Y}_n}{(\mathcal{Y}_{n+1})^2} = 1, \tag{4.50}$$

$$\lim_{n\to\infty} n\left(\frac{\mathcal{Y}_{n+2}\mathcal{Y}_n}{(\mathcal{Y}_{n+1})^2} - 1\right) = \Delta_{\mathcal{O}}, \tag{4.51}$$

$$\lim_{n\to\infty} n^2\left(\frac{\mathcal{Y}_{n+2}\mathcal{Y}_n}{(\mathcal{Y}_{n+1})^2} - 1 - n^{-1}\Delta_{\mathcal{O}}\right) = \frac{\Delta_{\mathcal{O}}^2}{2} - \Delta_{\mathcal{O}} - \frac{1}{2} - \alpha. \tag{4.52}$$

In section 4.4 we shall check (4.45) in two Lagrangian SCFTs with $\Delta_{\mathcal{O}} = 2$ using exact results from supersymmetric localization, with the value of α determined by the a-coefficients of the Weyl anomaly computed in appendix H.2.

4.2.4 Correlation functions on $\mathbb{R}^4$ versus $\mathbb{S}^4$

The determination of the contribution of order $\log n$ to the two-point function, will be the major nontrivial portion of the EFT computation. As the next step, we Weyl-transform the system to the round four-sphere $\mathbb{S}^4$, because the term of order $\log n$ cannot be seen easily in flat space.

Why do we have to consider the four-sphere at all? First we would like to explain why we have to use the spherical conformal frame. Eventually, our basic strategy is to quantize the EFT around the classical solution, and in principle we should be able to do this either on $\mathbb{R}^4$ or $\mathbb{S}^4$. That is, if the term of order $\log n$ is hidden on $\mathbb{R}^4$, then where is it and why cannot we see it?

In order to comprehend why the computation cannot be performed on $\mathbb{R}^4$, we would like to recall the basic scheme for understanding corrections to large-n quantities in EFT, as done in [35, 37, 43]. As in [35, 37, 43] the EFT can be regularized and renormalized at an energy scale Λ much smaller than the UV scale E_{UV}, while keeping Λ much larger than the IR scale E_{IR}. The EFT then has a perturbative expansion in n^{-1} if

$$E_{\mathrm{IR}} \ll E_{\mathrm{UV}}. \tag{4.53}$$

This criterion is satisfied if

$$E_{\mathrm{UV}} = n^p E_{\mathrm{IR}}, \qquad p > 0. \tag{4.54}$$

In the present case, the UV scale is given by the expectation value of $|\phi|$,

$$E_{\mathrm{UV}} = \langle |\phi| \rangle\,. \tag{4.55}$$

On $\mathbb{R}^4$, the only IR scale is

$$E_{\mathrm{IR}} = |x_1 - x_2|^{-1}, \tag{4.56}$$

so the criterion (4.53) becomes

$$|x_1 - x_2|^{-1} \ll \langle|\phi|\rangle\,. \tag{4.57}$$

In the large-n limit, one can make $\langle|\phi|\rangle$ as large as we want in the region in $\mathbb{R}^4$ containing the points x_1 and x_2, as one can see explicitly from the classical solution (4.24). However we must be careful: The VEV of $|\phi|$ is not a global but a local quantity, and in the classical solution (4.24), the value of $|\phi|$ falls off to zero sufficiently far away from the insertion points. So we cannot use the EFT straightforwardly, because $\mathbb{R}^4$ is not completely in the regime of validity of the EFT.

However this is not disastrous: The criterion (4.57) is not a necessary but only a sufficient one. In a CFT, there is clearly a less tighter criterion which is still sufficient to make the two-point function under control in the large-n limit. Large-n corrections are controlled if the criterion (4.57) is satisfied in any one of conformal frames at all, and it does not have to be the $\mathbb{R}^4$ conformal frame. Especially, if we conformally transform to $\mathbb{S}^4$ of radius $r \gtrsim |x_1 - x_2|$, in which the insertion points have an angular separation of order one, then the criterion (4.57) is satisfied in the conformally transformed spherical frame, which includes a conformal transformation of the scalar field ϕ:

$$\phi_{\mathbb{S}^4}(x') = \left[\det\left(\frac{\partial x'}{\partial x}\right)\right]^{-1}\phi_{\mathbb{R}^4}(x), \tag{4.58}$$

where x' denotes the coordinates on $\mathbb{S}^4$ and x denotes the coordinates on $\mathbb{R}^4$.

If we were to compute $\mathcal{B}_n$ up to and including corrections of order n^0, we would have to perform this transformation explicitly, in order to evaluate the fluctuation determinant around the classical solution and the spacetime integral of the Wess–Zumino term. Since we only want to calculate up to order $\log n$, the situation is much simpler. In the $\mathbb{R}^4$ frame, the integral of the Wess–Zumino term is divergent at infinity; in the $\mathbb{S}^4$ frame, the contributions of order $\log n$ coming from the Wess–Zumino term and fluctuation determinant are finite. It is now clear that the term of order $\log n$ in flat space is hiding in the fluctuation determinant, in the region where the EFT is broken down and it is not easy to compute it directly in flat space. However by conformally transforming to the $\mathbb{S}^4$ frame we can recover it.

Note that the EFT we will be using preserves the full $\mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{R}$ R-symmetry group and the entire $\mathrm{SO}(5,1)$ conformal group modulo the anomaly, rather than the smaller group preserved by the D-term deformation used to calculate the $\mathbb{S}^4$ partition function by supersymmetric localization in [132] and used to calculate correlation functions of chiral and antichiral primary operators in [1–4, 116–120]. While the results from these exact methods are used in section 4.4 to check our predictions in the large-n limit, our EFT is never deformed by the supergravity background from the maximally superconformal one. So our chiral ring does not have any ambiguity caused by curvature-dependent contact terms; our four-sphere background is equivalent to flat space just by a conformal transformation, not by a D-term deformation.

Instead of computing the classical solution on $\mathbb{S}^4$ explicitly, we can obtain it indirectly by conformally transforming the classical solution on $\mathbb{R}^4$ by using (4.58). We would have to know the explicit form of the solution on $\mathbb{S}^4$ in order to calculate the terms of order n^0 in $\mathcal{B}_n$, but since we are only calculating the terms of order $\log n$ in the present section, we only have to understand some qualitative properties of the classical solution, and the expression (4.58) suffices to do so.

4.3 Anomaly terms

In section 4.2.2, we have shown that no superconformally invariant term in the effective action can make contributions of order larger than n^0. However the Wess–Zumino term is not superconformally invariant, and they evade the analysis in section 4.2.2.

In order to calculate the contribution of order $\log n$ correctly, we have to express the Wess–Zumino term taking care of their normalization. We will concentrate on the normalization of the coefficient of the $O(\log n)$ contribution to the effective action. We will see that the form of the term of order $\log n$ generated from the Wess–Zumino term is simple and comes only from the coupling term of the effective dilaton with the Euler density.

Let us begin with the Wess–Zumino term which captures the Weyl and $\mathrm{U}(1)_{\mathrm{R}}$ anomalies in general $\mathcal{N}=1$ SCFTs [125, 136]. Its explicit form in Lorentzian signature is given by

$$S_{\mathrm{WZ}} = \int d^4x \sqrt{|g|}\mathcal{L}_{\mathrm{WZ}}, \tag{4.59}$$

with

$$\begin{aligned} \mathcal{L}_{\mathrm{WZ}} \coloneqq\ & \tau\big(\Delta c^{[\mathrm{KS}]} W^2(g) - \Delta a^{[\mathrm{KS}]} E_4(g) - 6\Delta c^{[\mathrm{KS}]} F^2\big) \\ & + \beta\Big[2\big(5\Delta a^{[\mathrm{KS}]} - 3\Delta c^{[\mathrm{KS}]}\big) F\tilde{F} + \big(\Delta c^{[\mathrm{KS}]} - \Delta a^{[\mathrm{KS}]}\big) R\tilde{R}\Big] \\ & - \Delta a^{[\mathrm{KS}]}\bigg[4\bigg(R^{\mu\nu}(g) - \frac{1}{2}R(g)g^{\mu\nu}\bigg)\partial_\mu\tau\partial_\nu\tau - 2(\partial\tau)^2\big(2\Box\tau - (\partial\tau)^2\big)\bigg], \end{aligned} \tag{4.60}$$

where in our context $\Delta c \coloneqq c_{\mathrm{CFT}} - c_{\mathrm{EFT}}$ and $\Delta a \coloneqq a_{\mathrm{CFT}} - a_{\mathrm{EFT}}$. The fields τ and β are Goldstone fields corresponding to the conformal and $\mathrm{U}(1)_{\mathrm{R}}$-symmetries, respectively. The Komargodski–Schwimmer a-theorem [5] requires Δa to be positive. Note that the normalization of the coefficients a and c used in [5, 18, 125, 136] and here, which we denote by the superscript [KS], is different from the one used in [6] and in (4.5). The two differ by a factor of $16\pi^2$ (see appendix H.2), that is,

$$(a,c)^{[\mathrm{KS}]} = \frac{1}{16\pi^2}(a,c)^{[\mathrm{AEFJ}]}. \tag{4.61}$$

In (4.60), we have only expressed the scalar components of the Wess–Zumino term, rather than a fully supersymmetric completion. It has been argued [125, 141] that a supersymmetric version of the Wess–Zumino term can be expressed as manifestly supersymmetrically invariant but conformal- and R-symmetry-breaking terms in superspace. The fermionic as well as gauge contributions are suppressed by inverse powers of $M_{\text{UV}} \propto |\phi| \propto \sqrt{n}$, and therefore of order smaller than n^0.

4.3.1 Evaluating the Wess–Zumino term on $\mathbb{S}^4$

Now we would like to compute the contribution of the Wess–Zumino term of order $\log n$ on $\mathbb{S}^4$. As we have explained above, this, rather than the computation on flat space, is in the regime of validity of the EFT.

Euler coupling of the modulus on $\mathbb{S}^4$. It is clear that on $\mathbb{S}^4$ there is only one contribution of order $\log n$ from the Wess–Zumino term, and it is topological, being proportional to the Euler density E_4. In order to determine the normalization of the contribution of the Euler term, we need some facts about the geometry of $\mathbb{S}^4$, which we have summarized in appendix G.

The natural normalization of the Euler density would be the "integer normalization" $E_4^{\mathbb{Z}}$, in which the integral of the Euler density is just the Euler number χ of the spacetime:

$$\int d^4x \sqrt{|g|} E_4^{\mathbb{Z}} = \chi \in \mathbb{Z}, \tag{4.62}$$

which is equal to 2 for $\mathbb{S}^4$, so the value of $E_4^{\mathbb{Z}}$ for $\mathbb{S}^4$ of radius r is

$$E_4^{\mathbb{Z}} = \frac{2}{\text{Area}\,(\mathbb{S}^4)} = \frac{3}{4\pi^2 \mathsf{r}^4}, \tag{4.63}$$

where we have used the fact that the area of $\mathbb{S}^4$ of radius r is given by Area $(\mathbb{S}^4) = 8\pi^2\mathsf{r}^4/3$. However, the integer-normalization convention (4.63) for E_4 is not so popular. In [5], the normalization of E_4 is defined as

$$E_4^{[\text{KS}]} := \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2, \tag{4.64}$$

in terms of which the value for $\mathbb{S}^4$ is given by

$$E_4^{[\text{KS}]} = \frac{24}{\mathsf{r}^4} = 32\pi^2 E_4^{\mathbb{Z}}. \tag{4.65}$$

Rewriting the Euler term[37] of [5] in terms of the momre intuitive integer-normalized one $E_4^{\mathbb{Z}}$, the Euler term in Euclidean signature is given by

$$\mathcal{L}_{\text{WZ}}^{\text{Euler term}} = \Delta a^{[\text{KS}]} E_4^{[\text{KS}]} \tau = 32\pi^2 \Delta a^{[\text{KS}]} E_4^{\mathbb{Z}} \tau = 2\Delta a^{[\text{AEFJ}]} E_4^{\mathbb{Z}} \tau. \tag{4.66}$$

[37] We have also changed the sign of the Euler term, which in [5] was the Lorentzian one, as suitable to compute the dilaton scattering amplitude studied in that paper. In order to compute a path integral on four-sphere, the relevant sign is the Euclidean one, in which the action is the negative of the one in Lorentzian signature after Wick rotation.

The dilaton τ is defined in [5] such that $\exp(-\tau)$ is a scalar field of scaling dimension 1; in the present case, then, the field $\exp(-\tau)$ is identified with $|\phi|$ which breaks the scale invariance spontaneously, and therefore we identify

$$\tau = -\log\frac{|\phi|}{\mu}, \tag{4.67}$$

where μ is an arbitrary mass scale, such as r^{-1}.

Boundedness of the $O(n^0)$ contribution. In order to calculate the full contribution coming from Wess–Zumino term including the $O(n^0)$ contribution, we would need to calculate the explicit profile of the dilaton τ, determined by substituting the classical solution (4.24) and the integral of the Wess–Zumino term. Since we are not going to compute the order n^0 term, we do not have to do so at all: The classical solution (4.24) for $|\phi|$ is proportional to $\sqrt{n\Delta_{\mathcal{O}}}$ in the $n \to \infty$ limit, and therefore we can decompose its logarithm as the sum of an position-independent piece of order $\sqrt{n}$, and a position-dependent piece of order n^0. That is,

$$|\phi| = \sqrt{n\Delta_{\mathcal{O}}}\left|\hat{\phi}\right|, \tag{4.68}$$

where $\left|\hat{\phi}\right|$ is of order n^0 in the $\mathbb{S}^4$ frame, with the singularities at the insertion points x_1 and x_2.

One may wonder if these singularities cause the large-n expansion to break down. In general, UV singularities are not problematic, as our theory is regularized and renormalized at a energy scale $\Lambda \ll |\phi|$. In the present situation, regularization and renormalization are not even necessary to evaluate the classical contributon of the Wess–Zumino term: Due to cancellations of the most naïvely singular contributions and the fact that the classical solution is complex rather than real, the integral of the Wess–Zumino term over $\mathbb{S}^4$ is finite. Therefore, using (4.68) we can express the Wess–Zumino Lagrangian density as

$$\mathcal{L}_{\mathrm{WZ}} = \mathcal{L}_{\mathrm{WZ}}^{\text{Euler term}} + O\big(n^0\big), \tag{4.69}$$

where the $O(n^0)$ term is finite, and we discard it as our desired order of precision is $O(\log n)$. Then, (4.67) and (4.68) give

$$\tau = -\frac{1}{2}\log\left(n\Delta_{\mathcal{O}}\right) + O\big(n^0\big), \tag{4.70}$$

and from (4.66) and the fact that the Euler number of $\mathbb{S}^4$ is $\chi_{\mathbb{S}^4} = 2$, we get

$$S_{\mathrm{WZ}} = -\alpha\log\left(n\Delta_{\mathcal{O}}\right) + O\big(n^0\big), \tag{4.71}$$

where

$$\alpha = 2\Big(a_{\mathrm{CFT}}^{[\mathrm{AEFJ}]} - a_{\mathrm{EFT}}^{[\mathrm{AEFJ}]}\Big). \tag{4.72}$$

Then Z_n, defined by the Euclidean path integral with the action (4.37), gets a multiplicative contribution,

$$\begin{aligned} Z_n &= Z_{\text{free+sources}} e^{-S_{\text{WZ}}} \left[O\big(n^0\big) + O\big(n^{-1}\big) + \cdots \right] \\ &= \left(\frac{|\mathbf{N}_{\mathcal{O}}|}{2\pi |x_1 - x_2|} \right)^{2n\Delta_{\mathcal{O}}} (n\Delta_{\mathcal{O}})! (n\Delta_{\mathcal{O}})^{\alpha} \left[O\big(n^0\big) + O\big(n^{-1}\big) + \cdots \right]. \end{aligned} \tag{4.73}$$

4.4 Localization in rank-one theories with marginal couplings

Following [1–4] we briefly review how to calculate by supersymmetric localization two-point functions of various $\mathcal{N} \geq 2$ rank-one SCFTs in four dimensions with exactly marginal couplings in $\mathbb{R}^4$. We will then apply the results of [1–4] to Lagrangian rank-one SCFTs with gauge group SU(2) (or SO(3)), and compare with our large-n expansion of $\mathcal{Y}_n$. The two interacting SCFTs with exactly marginal couplings are $\mathcal{N} = 4$ super-Yang–Mills theory with gauge group SU(2) (or SO(3)), and $\mathcal{N} = 2$ SQCD with four hypermultiplets in the fundamental representation of gauge group SU(2).

4.4.1 Relation of conventions

For rank-one SCFTs with $\Delta_{\mathcal{O}} = 2$, our two-point function $\mathcal{Y}_n$ is a function of the exactly marginal couplings τ, $\bar{\tau}$ and is identified with the two-point function $G_{2n}(\tau, \bar{\tau})$ of [3], up to powers of a normalization factor we denote by $\mathbf{K}$ such that

$$\mathcal{O}^{\text{here}} = \mathbf{K}^{\Delta_{\mathcal{O}}} \mathcal{O}_2^{\text{refs [1–4]}}. \tag{4.74}$$

The conformal dimension of the chiral ring generator $\mathcal{O}$ is $\Delta_{\mathcal{O}} = 2$ for the two SCFTs under consideration in this section. With the relative normalizations defined this way, the relationship of the two-point functions is

$$\mathcal{Y}_n(\tau, \bar{\tau}) = |\mathbf{K}|^{4n} G_{2n}(\tau, \bar{\tau}). \tag{4.75}$$

With this identification we will review the computation of two-point functions in [1–4] and then compare with our own predictions in the large-n limit.

4.4.2 Method of [1–4]

To calculate two-point functions on $\mathbb{R}^4$ by supersymmetric localization, one first needs the $\mathbb{S}^4$ partition function $Z_{\mathbb{S}^4}(\tau, \bar{\tau})$ associated with the $\mathcal{N} \geq 2$ SCFT action S_{SCFT} deformed by the chiral ring generators $\mathcal{O}_i$,

$$S_{\text{SCFT}} \to S_{\text{SCFT}} - \frac{1}{32\pi^2} \left(\int d^4x d^4\theta \mathcal{E} \sum_i \tau_i \mathcal{O}_i + \text{c.c.} \right), \tag{4.76}$$

where $\mathcal{E}$ is the chiral density of $\mathcal{N}=2$ supergravity [142, 143] and τ_i are holomorphic coupling constants. Since this deformed theory (4.76) preserves $\mathfrak{osp}(2|4)$, the massive $\mathcal{N}=2$ supersymmetry algebra on $\mathbb{S}^4$, the associated partition function can be evaluated exactly by supersymmetric localization.

In the case of rank-one SCFTs,[38] the two-point functions of the chiral ring operators defined by

$$G_{2n}(\tau,\bar{\tau}) \coloneqq \left\langle \mathcal{O}^n(0)\bar{\mathcal{O}}^n(\infty)\right\rangle, \qquad \bar{\mathcal{O}}^n(\infty) \coloneqq \lim_{|x|\to\infty} |x|^{2n\Delta_{\mathcal{O}}}\bar{\mathcal{O}}^n(x), \tag{4.77}$$

can be calculated systematically in the following way. First, one evaluates derivatives of $Z_{\mathbb{S}^4}(\tau,\bar{\tau})$ and constructs a matrix M whose (m,n)-entry $(m,n=0,1,2,\cdots)$ is given by

$$M_{m,n} \coloneqq \frac{1}{Z_{\mathbb{S}^4}(\tau,\bar{\tau})}\partial_\tau^m \partial_{\bar{\tau}}^n Z_{\mathbb{S}^4}(\tau,\bar{\tau}). \tag{4.78}$$

Then, the two-point function (4.77) can be computed by[39]

$$G_{2n}(\tau,\bar{\tau}) = 16^n \frac{\det M_{(n)}}{\det M_{(n-1)}}, \qquad n=1,2,3,\cdots, \tag{4.79}$$

where $M_{(n)}$ is the upper-left $(n+1)\times(n+1)$ submatrix of M.

4.4.3 The case of SU(2) $\mathcal{N}=4$ super-Yang–Mills

The sphere partition function of $\mathcal{N}=4$ super-Yang–Mills with gauge group SU(2) is very simple [132],

$$Z_{\mathbb{S}^4}^{\mathcal{N}=4}(\tau,\bar{\tau}) = \frac{1}{4\pi(\operatorname{Im}\tau)^{3/2}}, \tag{4.80}$$

where in this case τ is the complexified Yang–Mills coupling[40]

$$\tau = \frac{\theta_{\rm YM}}{2\pi} + \frac{4\pi i}{g^2}. \tag{4.81}$$

From (4.80) we get

$$G_{2n}(\tau,\bar{\tau}) = \frac{(2n+1)!}{(\operatorname{Im}\tau)^{2n}}, \qquad \mathcal{Y}_n(\tau,\bar{\tau}) = |\mathbf{K}|^{4n} G_{2n}(\tau,\bar{\tau}) = \left(\frac{|\mathbf{K}|^2}{\operatorname{Im}\tau}\right)^{2n}(2n+1)!. \tag{4.82}$$

[38]For Lagrangian SCFTs with multi-dimensional Coulomb branch, two-point functions can also be exactly computed, but one needs to disentangle operator mixings. For details, see [3, 4].

[39]The two-point function (4.79) satisfies the so-called $tt^\star$ equation [1, 124].

[40]Again, note that we use this convention regardless of matter content, which is different from the conventions of [130, 131].

In the large-n limit, the logarithm of $G_{2n}(\tau,\bar{\tau})$ becomes

$$\log\left[G_{2n}(\tau,\bar{\tau})\right] = 2n\log n + n(2\log 2 - 2 - 2\log(\operatorname{Im}\tau)) + \frac{3}{2}\log n + O\big(n^0\big). \tag{4.83}$$

This matches our prediction (4.39) up to and including order $\log n$, as we explicitly see in section 4.4.5.

4.4.4 Numerical analysis of SU(2) $\mathcal{N}=2$ SQCD with four fundamental hypers

One would like to find as many other rank-one $\mathcal{N}=2$ SCFTs as possible for which we could compare our predictions with two-point functions computed by supersymmetric localization. Unfortunately, there are not so many examples in the literature which have been worked out already. In [1–3] the example of SU(2) $\mathcal{N}=2$ SQCD with four fundamental hypermultiplets is studied. Even for that relatively simple SCFT the sphere partition function and therefore the two-point functions are determined nonperturbatively by a complicated integral which has a nontrivial τ-dependence. It is possible however, to compute the two-point function $\mathcal{Y}_n$ numerically for any value of τ, to good enough accuracy to obtain the coefficients of the asymptotic expansion of $\mathcal{B}_n = \log\mathcal{Y}_n$ with some precision. Especially, we can extract the coefficient α numerically and compare it with the prediction from the EFT.

The sphere partition function of SU(2) $\mathcal{N}=2$ SQCD with four fundamental hypermultiplets is given by [132, 144]

$$Z_{\mathbb{S}^4}^{\mathcal{N}=2}(\tau,\bar{\tau}) = \int_{-\infty}^{\infty} da\, a^2 e^{-4a^2\operatorname{Im}\tau}\frac{|G(1+2ia)|^4}{|G(1+ia)|^{16}}|Z_{\text{inst}}(ia,\tau)|^2, \tag{4.84}$$

where the function $G(x)$ is the so-called Barnes G-function [145], and $Z_{\text{inst}}(ia,\tau)$ is the instanton partition function [146], which is expanded at small τ as[41]

$$Z_{\text{inst}}(ia,\tau) = 1 + \frac{1}{2}\big(a^2-3\big)e^{2\pi i\tau} + O\big(e^{4\pi i\tau}\big). \tag{4.85}$$

For simplicity we focus on the region $\operatorname{Im}\tau \geq 1$ and ignore all the instanton corrections. The zero-instanton sector of the sphere partition function is independent of $\operatorname{Re}\tau$. Using (4.78) and (4.79), we compute numerically the two-point function G_{2n} up to $n \simeq 30$ for various values of $\operatorname{Im}\tau$ in section 4.4.5.

4.4.5 Comparison of exact results with the large-R-charge expansion

Now we would like to compare our predictions from the EFT with the results from supersymmetric localization, using the value of the α-coefficient determined in appendix H.2.

[41] See *e.g.*, [147] for higher order terms in this expansion.

The case of SU(2) $\mathcal{N}=4$ super-Yang–Mills. In (H.13) we have computed the α-coefficient for $\mathcal{N}=4$ super-Yang–Mills with gauge group SU(2), and it is given by

$$\alpha_{\mathrm{SU}(2)\ \mathcal{N}=4} = 1. \tag{4.86}$$

So we expect

$$\mathcal{Y}_n(\tau,\bar\tau) = (2n)!n\left|\frac{\mathbf{N}_{\mathcal{O}}}{2\pi}\right|^{4n}\tilde{\mathcal{Y}}_n, \qquad \tilde{\mathcal{Y}}_n = O\big(n^0\big). \tag{4.87}$$

The exact formula (4.82) can be written as

$$\mathcal{Y}_n(\tau,\bar\tau) = |\mathbf{K}|^{4n}G_{2n}(\tau,\bar\tau) = (2n)!n\left(\frac{|\mathbf{K}|^2}{\operatorname{Im}\tau}\right)^{2n}\left(2+\frac{1}{n}\right), \tag{4.88}$$

which agrees with the form of our asymptotic expansion, with

$$\tilde{\mathcal{Y}}_n = 2+\frac{1}{n}, \qquad \left|\frac{\mathbf{N}_{\mathcal{O}}}{2\pi}\right| = \frac{|\mathbf{K}|}{(\operatorname{Im}\tau)^{1/2}}. \tag{4.89}$$

The case of SU(2) $\mathcal{N}=2$ SQCD with four fundamental hypers. For SU(2) $\mathcal{N}=2$ SQCD with four fundamental hypermultiplets, we have $\Delta_{\mathcal{O}}=2$ and in (H.18) we have computed

$$\alpha_{\mathrm{SU}(2)\ \mathcal{N}=2\ \mathrm{SQCD}} = \frac{3}{2}. \tag{4.90}$$

In this case, the results from supersymmetric localization is only numerical. Therefore it is easier to verify the accuracy of the sum and/or product rules of section 4.2.3 than to fit the data to a curve. Our prediction for the two-point functions is expressed as the sum rule (4.45) with $\Delta_{\mathcal{O}}=2$ and $\alpha = 3/2$,

$$\mathcal{B}_{n+2} - 2\mathcal{B}_{n+1} + \mathcal{B}_n = 2n^{-1} - 4n^{-2} + O\big(n^{-3}\big), \tag{4.91}$$

which implies the following three independent checks,

$$\lim_{n\to\infty}\left(\mathcal{B}_{n+2} - 2\mathcal{B}_{n+1} + \mathcal{B}_n\right) = 0, \tag{4.92}$$

$$\lim_{n\to\infty} n\left(\mathcal{B}_{n+2} - 2\mathcal{B}_{n+1} + \mathcal{B}_n\right) = 2, \tag{4.93}$$

$$\lim_{n\to\infty}\Big[n^2\left(\mathcal{B}_{n+2} - 2\mathcal{B}_{n+1} + \mathcal{B}_n\right) - 2n\Big] = -4. \tag{4.94}$$

In figures 1, 2, 3 we plot the left-hand side of equations (4.92), (4.93), (4.94), up to $n=30$, for several (purely imaginary) values of τ. These values have been computed by the method of [1–4], approximating the sphere partition function by its zero-instanton piece alone. Even in this approximation, the large-n prediction (4.94) is close to -4 for n of order 30. Note that the agreement is best at $\operatorname{Im}\tau = 1$, which is expected to have the lowest threshold for the applicability of the large-n approximation, as the gap above the massless sector is largest there. We do not know whether the zero-instanton approximation affects the true asymptotic value of the left-hand side of the sum rule (4.94).

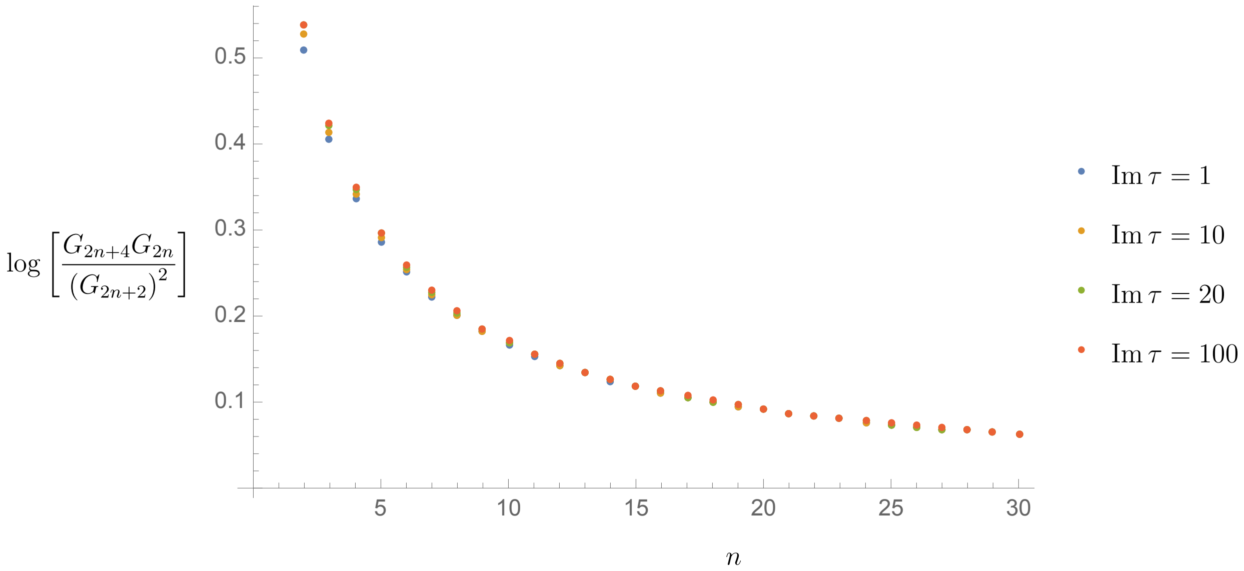


Figure 1: Approximate values of the left-hand side of sum rule (4.92) in SU(2) $\mathcal{N} = 2$ SQCD with four fundamental hypermultiplets, calculated *via* the method developed in [1–4], with instanton corrections omitted.

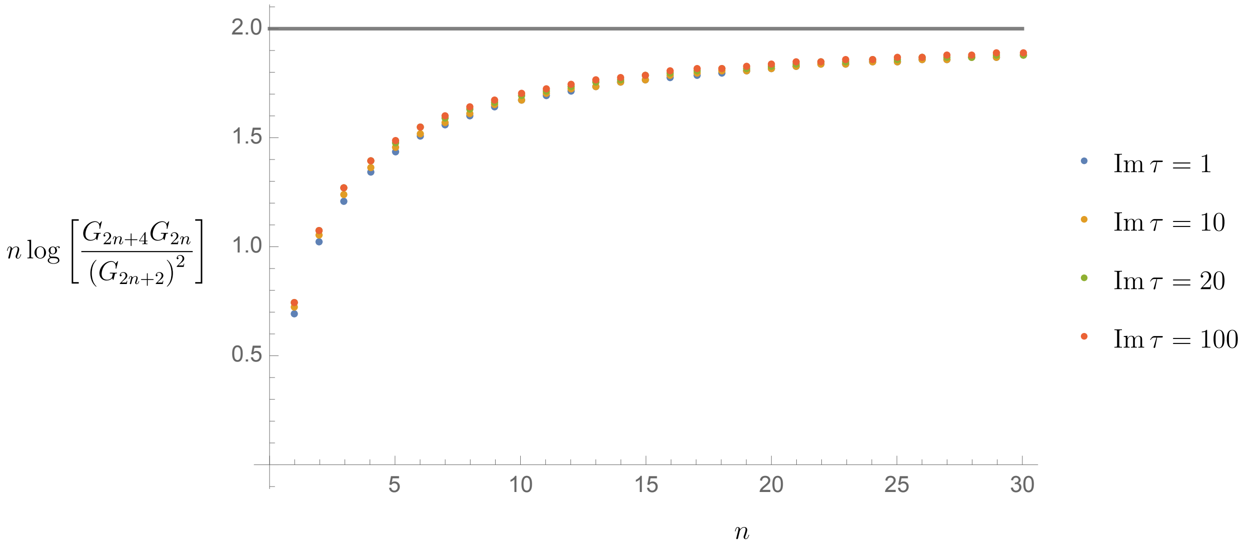


Figure 2: Approximate values of the left-hand side of the sum rule (4.93) in SU(2) $\mathcal{N} = 2$ SQCD with four fundamental hypermultiplets, computed *via* the method developed in [1–4], with instanton corrections omitted.

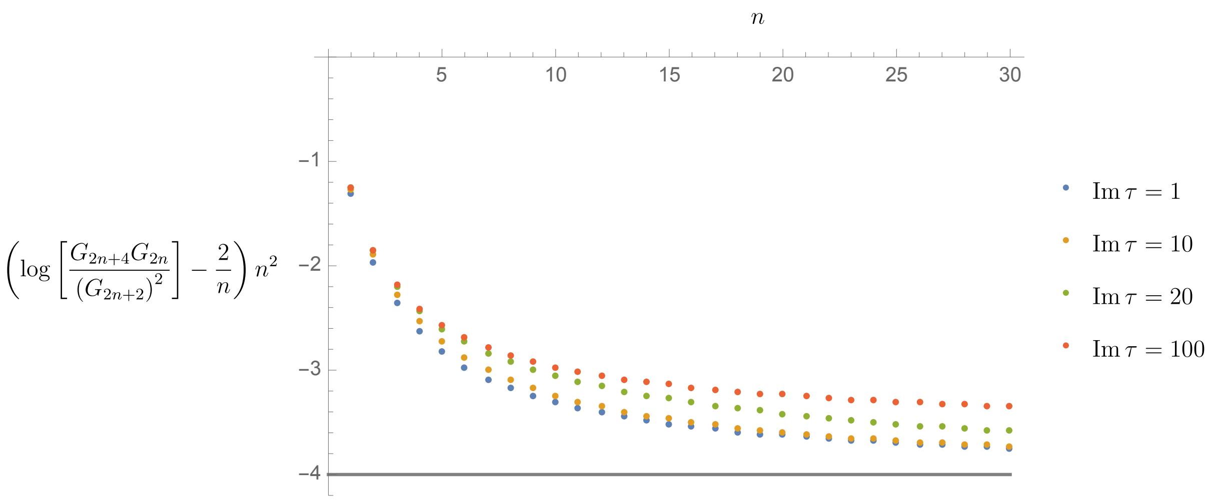


Figure 3: Approximate values of the left-hand side of sum rule (4.94) in SU(2) $\mathcal{N} = 2$ SQCD with four fundamental hypermultiplets, computed *via* the method developed in [1–4], with instanton corrections omitted. For the exact $\mathbb{S}^4$ partition function, with all instanton corrections, our analysis predicts the left-hand side of (4.94) should approach -4 for any τ, in the $n \to \infty$ limit. It would appear unlikely that the asymptotic value of of the sum rule is truly -4 for the zero-instanton approximation to the $\mathbb{S}^4$ partition function, but at present we have no theory of the error.

4.5 Discussion and conclusion of section 4

4.5.1 Other SCFTs with one-dimensional Coulomb branch

There are many other rank-one SCFTs (or more generally with a single vector multiplet and massless hypermultiplets) with or without exactly marginal couplings. Since many of them do not have UV Lagrangian description, they are harder to do explicit calculations with by supersymmetric localization and we do not have results in the literature with which we can easily compare. In order to predict correlation functions of $\mathcal{O}^n$ in the large-n limit, we need to know the dimension of the chiral ring generator $\mathcal{O}$, the a-coefficient of the Weyl anomaly of the underlying SCFT, and the massless field content of the EFT on moduli space.

Rank-one SCFTs have been the subject of intensive recent study by [57–60], in which SCFTs with one-complex-dimensional Coulomb branch have been classified under broad conditions. We make use of the nice results in [57–60] on the classification of rank-one SCFTs. In fact we will do more than just "make use of" them: We copy directly[42] a table

[42]Table 2 is created in part by copying the LaTeX code of table 1 of [60]. We are doing so with the intention of communicating our results for the a-coefficients and their relation to [60], in a context that is most easily understood by the reader. We do not claim as original work the creation of the content or appearance of our table 2 insofar as it overlaps with table 1 of [60]. According to our best understanding, this is a legitimate use of the work [60] under the arXiv non-exclusive license to distribute, https://arxiv.org/licenses/nonexclusive-distrib/1.0/license.html.

from [60], but making our own additional columns highlighted in blue, giving data on the Wess–Zumino term and the value of the α-coefficient of the theory in table 2.

	Coulomb br.: singul. type	Coulomb br.: $\Delta_{\mathcal{O}}$	Higgs br. dim.	massless hypers	central charges: $24a_{\rm CFT}$	central charges: $12c_{\rm CFT}$	Wess–Zumino term: $24a_{\rm EFT}$	Wess–Zumino term: $24\Delta a$	α-coeff.: $\alpha = 2\Delta a$
I_1 series	II^*	6	29	0	95	62	5	90	$15/2$
	III^*	4	17	0	59	38	5	54	$9/2$
	IV^*	3	11	0	41	26	5	36	3
	I_0^*	2	5	0	23	14	5	18	$3/2$
	IV	$3/2$	2	0	14	8	5	9	$3/4$
	III	$4/3$	1	0	11	6	5	6	$1/2$
	II	$6/5$	0	0	$43/5$	$22/5$	5	$18/5$	$3/10$
	I_1	1	0	0	6	3	5	1	$1/12$
I_4 series	II^*	6	16	5	82	49	10	72	6
	III^*	4	8	3	50	29	8	42	$7/2$
	IV^*	3	4	2	34	19	7	27	$9/4$
	I_0^*	2	0	1	18	9	6	12	1
	I_4	1	0	0	6	3	5	1	$1/12$
I_1^* series	II^*	6	9	4	75	42	9	66	$11/2$
	III^*	4	?	2	45	24	7	38	$19/6$
	IV^*	3	0	1	30	15	6	24	2
	I_1^*	2	0	0	17	8	5	12	1
$IV^*_{Q=1}$ ser.	II^*	6	?	3	71	38	8	63	$21/4$
	III^*	4	0	1	42	21	6	36	3
	$IV^*_{Q=1}$	3	0	0	$55/2$	$25/2$	5	$45/2$	$9/4$
I_2 ser.	I_0^*	2	0	1	18	9	6	12	1
	I_2	1	0	0	6	3	5	1	$1/12$

Table 2: Argyres, Lotito, Lü and Martone's partial list of rank-one $\mathcal{N} = 2$ SCFTs. This table has been copied directly (at the level of the LaTeX code) from [60], to clarify the identification of SCFTs, which are labelled exactly as in [60]. The convention of [6] for the a- and c-coefficients of the Weyl anomaly is used. The three columns on the right highlighted in blue are not in the original table of [60] and are created by Simeon Hellerman and the author of this thesis. The column "massless hypers" denotes the number of hypermultiplets massless at a generic point on the Coulomb branch, a situation dubbed in [60] as an "enhanced Coulomb branch" (ECB) if the number of massless hypermultiplets is nonzero.

4.5.2 Conclusion of section 4

In this section we have considered the large-R-charge expansion of two-point functions of chiral primary operators $\mathcal{O}^n$ and their conjugates, where $\mathcal{O}$ is the holomorphic generator of a chiral ring in a rank-one SCFT. To do this, we have followed earlier works and used the EFT describing the large-R-charge sector of the Hilbert space. As in section 3 on the superconformal large-n expansion, the relevant EFT is the low-energy dynamics of the supersymmetric moduli space, which is governed by spontaneously broken superconformal invariance. We have used the EFT to obtain the asymptotic expansion of the two-point function,

$$\mathcal{Y}_n = |x_1 - x_2|^{2n\Delta_{\mathcal{O}}} \left\langle \mathcal{O}^n(x_1)\bar{\mathcal{O}}^n(x_2)\right\rangle , \tag{4.95}$$

at large R-charge, *i.e.*, in the $n \to \infty$ limit. The EFT predicts that $\mathcal{Y}_n$ has an asymptotic expansion at large n, behaving as

$$\mathcal{Y}_n = (n\Delta_{\mathcal{O}})! \left|\frac{\mathbf{N}_{\mathcal{O}}}{2\pi}\right|^{2n\Delta_{\mathcal{O}}} (n\Delta_{\mathcal{O}})^{\alpha} \tilde{\mathcal{Y}}_n, \tag{4.96}$$

where $\tilde{\mathcal{Y}}_n$ approaches a constant in the $n \to \infty$ limit, and $\mathbf{N}_{\mathcal{O}}$ is an n-independent constant which depends on the normalization of the chiral ring generator relative to the effective vector multiplet scalar. We have shown that the exponent α can be computed by the coupling between the Euler density of $\mathbb{S}^4$ and the logarithm of the scalar field $|\phi|$. The coefficient of this coupling is fixed by anomaly matching to be proportional to the difference between the a-coefficient of the Weyl anomaly of the underlying SCFT and that of the EFT of massless moduli fields. In the conventions of [6], α is given by

$$\alpha = 2\Big(a_{\text{CFT}}^{\text{[AEFJ]}} - a_{\text{EFT}}^{\text{[AEFJ]}}\Big). \tag{4.97}$$

In SCFTs with a exactly marginal coupling, we have used results from supersymmetric localization [1–4] to check our predictions. In the case of $\mathcal{N} = 4$ super-Yang–Mills with gauge group SU(2) (or gauge algebra $\mathfrak{su}(2)$ in general), the exact result can be expressed in a simple closed form, and our large-R-charge expansion for the logarithm of the two-point function agrees precisely with the exact result to the precision to which we have computed, that is, up to and including the term of order $\log n$ in $\mathcal{B}_n = \log \mathcal{Y}_n$. In the case of SU(2) $\mathcal{N} = 2$ SQCD with four fundamental hypermultiplets, we compare our large-n expansion with the numerical computation of the two-point functions up to $n \simeq 30$, with the $n = 0$ expression approximated by the zero-instanton part of the $\mathbb{S}^4$ partition function. We obtain precise numerical agreement for the two leading-order behaviors, and good agreement for the next-to-next-to-leading order behavior, determined by the α-coefficient $\alpha = 3/2$, which predicts a value -4 for the left-hand side of the sum rule (4.94) in the large-n limit. Although it is not clear we should expect the sum rule to approach the value -4 exactly for the zero-instanton approximation to the initial condition $Z_{\mathbb{S}^4}$, the sum rule for $\tau = i$ appears to asymptote to a value at most -3.8, within our numerical precision. It would be great to develop a robust theory of the error in the large-n limit, provided an approximate partition function.[43]

In summary, we have shown that it is practical to use the large-R-charge expansion as a bridge from the world of unbroken superconformal symmetry, OPE data, and bootstraps, to the world of the low-energy dynamics of the moduli space of vacua.

5 Conclusion and future prospects

CFTs are important in many ways. However, since they are generically strongly interacting, we cannot study them using the conventional perturbative methods. The Wilsonian EFT is very powerful so that by using it we can investigate sectors of large global charge in strongly coupled CFTs. In this thesis we studied the large-R-charge limit of SCFTs with a moduli space of vacua.

After reviewing basic facts about CFT in section 2, we established the EFT of the large-R-charge sector of $\mathcal{N} = 2$ SCFTs with a one-dimensional moduli space in section 3. The Wilsonian effective action is given by the free action of the moduli field corrected by higher-derivative operators which are suppressed by the R-charge. We showed that the lowest and second-lowest scalar primary operators carrying large R-charge J have vanishing anomalous dimension, up to and including $O(J^{-3})$. This result is consistent with the fact that these operators are in short and semishort multiplets and therefore their dimensions are protected. The first correction for unprotected operators comes from by the supersymmetric extension (3.14) of the FTPR operator, whose coefficient has to be nonnegative by virtue of unitarity and causality. Correspondingly, the anomalous dimension of the third-lowest state with large R-charge has to be nonpositive and of order J^{-3}.

In section 4, the analysis done in 3 is generalized to four-dimensional $\mathcal{N} = 2$ SCFTs with one-dimensional Coulomb branch. We argued that the first correction to the free-theory result comes from the Wess–Zumino term, and gave an explicit formula (4.39) for the large-R-charge asymptotics of two-point functions of chiral and antichiral primary operators. This formula was checked against the exact results of supersymmetric localization. In the case of $\mathcal{N} = 4$ super-Yang–Mills we find perfect analytic agreement, and in the case of $\mathcal{N} = 2$ SQCD we find reasonably good numerical agreement using the no-instanton approximation to the $\mathbb{S}^4$ partition function.

These analyses have shown that the large-R-charge expansion of SCFTs with moduli space is described in terms of an EFT on moduli space. As a result, information about geometry of moduli space and the Weyl anomaly of the underlying theory is encoded in operator dimensions and correlation functions of operators carrying large R-charge. For instance, the fact that the logarithm of the two-point functions of chiral ring generators at

[43] We thank Z. Komargodski for correspondence on this point.

large R-charge is approximated at leading order by that of the free theory, is a consequence of the flatness of the moduli space. It would be interesting to study SCFTs with a curved moduli space by generalizing the EFT analysis developed in this thesis. In particular, in three-dimensional $\mathcal{N} \geq 4$ SCFTs with a curved Coulomb and/or Higgs branch, one may be able to make use of supersymmetric localization [121–123] to check the validity of the EFT on a curved moduli space, similarly to the analysis done in section 4.

Since moduli spaces of $\mathcal{N} \geq 4$ SCFTs in three dimensions are hyperkähler cones, one would inevitably have to extend the EFT to the case where multiple massless scalar fields are involved to study such theories using the large-charge EFT. Such an extension would also allow one to compare predictions from the EFT with localization results of other four-dimensional $\mathcal{N} = 2$ Lagrangian SCFTs, such as SU(N) SQCD ($N \geq 3$) with $2N$ fundamental hypermultiplets [4].

Another future direction is to consider higher order corrections to observables. From the effective-field-theoretic perspective, it amounts to classify all the possible higher-derivative interaction terms compatible with spacetime and internal symmetries order by order. For this purpose, the CCWZ method described in [37] may be useful in general situations. In addition, we would like to mention that in the case of $\mathcal{N} = 2$ SCFTs in four dimensions studied in 4, if the theory has an exactly marginal coupling, then one can make use of the so-called $tt^\star$-equation [124] satisfied by the two-point functions of chiral ring operators, in order to compute the large-R-charge expansion of them. Especially in the rank-one case, the $tt^\star$-equation is reduced to the two-dimensional semi-infinite Toda chain equation, which is integrable [148]. By making use of the $tt^\star$-equation and other physical constraints coming from, *e.g.*, the S-duality on the conformal manifold, one may be able to obtain higher order corrections to all orders in the large-R-charge expansion at least in the rank-one case.[44]

As an important application, one may hope to use the large-charge methods to estimate critical exponents associated with operators carrying large global charge in condensed matter systems. A surprising observation made in [45] is that the large-charge expansion can sometimes predict operator dimensions carrying global charge of order one, which is beyond the regime of validity of the EFT. One optimistic hope is that we may be allowed to use of the EFT to compute experimentally measurable quantities due to a hidden theoretical reasoning which makes the EFT applicable beyond its expected regime of validity, analogous to the one made in [149] in the case of the large-Lorentz-spin expansion.

In summary, in this thesis we have shown that it is practical to use the large-R-charge expansion as a bridge from the world of unbroken superconformal symmetry and SCFT data, to the world of the low-energy dynamics of the moduli space of vacua.

[44] We thank S. Hellerman, D. Orlando and M. Watanabe for discussions on this point.

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A $\mathcal{N}=2$ superconformal field theories in three dimensions

A.1 Notation for $\mathcal{N}=2$ superspace in three dimensions

Here we summarize the notation used in section 3. In Lorentzian signature, the metric on flat $\mathbb{R}^{1,2}$ is $\eta_{\mu\nu} = \mathrm{diag}\,(-,+,+)$ with $\mu = 0,1,2$. The Dirac matrices $(\gamma^\mu)_\alpha{}^\beta$ satisfy the Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\}_\alpha{}^\beta = (\gamma^\mu)_\alpha{}^\delta (\gamma^\nu)_\delta{}^\beta + (\gamma^\nu)_\alpha{}^\delta (\gamma^\mu)_\delta{}^\beta = 2\eta^{\mu\nu}\delta^\beta_\alpha. \tag{A.1}$$

Then, $\gamma^\mu_{\alpha\beta} := (\gamma^\mu)_\alpha{}^\delta \epsilon_{\delta\beta}$ is symmetric in $\alpha \leftrightarrow \beta$. One may choose $(\gamma^\mu)_\alpha{}^\beta = (i\sigma_2, \sigma_1, \sigma_3)$, so that $(\gamma^\mu)^* = \gamma^\mu$. We define the complex conjugation on products of Grassmann variables as $(\psi_1\psi_2)^* = \bar\psi_1\bar\psi_2$. A chiral superfield $\Phi\big(x,\theta,\bar\theta\big)$ is defined by $\bar{\mathcal{D}}_\alpha\Phi = 0$, where $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_\alpha$ are the superderivatives,

$$\mathcal{D}_\alpha := \frac{\partial}{\partial\theta^\alpha} - (\gamma^\mu)_\alpha{}^\beta \bar\theta_\beta \frac{\partial}{\partial x^\mu}, \qquad \bar{\mathcal{D}}_\alpha := \frac{\partial}{\partial\bar\theta^\alpha} - (\gamma^\mu)_\alpha{}^\beta \theta_\beta \frac{\partial}{\partial x^\mu}. \tag{A.2}$$

They satisfy the anticommutation relation,

$$\big\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\big\} = \big\{\mathcal{D}_\beta, \bar{\mathcal{D}}_\alpha\big\} = 2(\gamma^\mu)_{\alpha\beta}\partial_\mu. \tag{A.3}$$

$\Phi\big(x,\theta,\bar\theta\big)$ is expanded as

$$\begin{aligned}\Phi\big(x,\theta,\bar\theta\big) &= \phi(x) + \sqrt{2}\theta\phi(x) + \theta^2 F(x) \\ &\quad - \big(\theta\gamma^\mu\bar\theta\big)\partial_\mu\phi(x) - \frac{1}{\sqrt{2}}\theta^2\big(\bar\theta\gamma^\mu\partial_\mu\psi(x)\big) + \frac{1}{4}\theta^2\bar\theta^2\partial_\mu\partial^\mu\phi(x).\end{aligned} \tag{A.4}$$

The normalization for the Berezinian integral is

$$\int \theta^2\bar\theta^2 d^2\theta d^2\bar\theta = 1, \tag{A.5}$$

and it is convenient to note that, up to total derivatives,

$$\int \mathcal{I} d^2\theta d^2\bar{\theta} = \frac{1}{16}\mathcal{D}^2\bar{\mathcal{D}}^2\mathcal{I}\Big|_{\theta=\bar{\theta}=0} . \tag{A.6}$$

We can obtain the corresponding equations in Euclidean signature easily by the Wick rotation $t = -it_{\mathrm{E}}$.

A.2 Algebra

Here we present the $\mathcal{N} = 2$ supersymmetric extension of the conformal algebra in three dimensions, as we need it to understand section 3. We mostly follow the convention of [68].

In the $\mathcal{N} = 2$ superconformal algebra in three dimensions there are fermionic generators, Q_α, $\bar{Q}_\alpha$, S^α and $\bar{S}^\alpha$, in addition to the bosonic conformal generators $M_{\mu\nu}$, P_μ, K_μ and D, and the $\mathrm{U}(1)_{\mathrm{R}}$ generator R. The spinor indices $\alpha \in \{1, 2\}$ are raised and lowered by the antisymmetric tensor $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$, respectively, so that, *e.g.*, $Q_\alpha Q^\alpha = \epsilon^{\alpha\beta} Q_\alpha Q_\beta$ transforms as a scalar under SO(3) transformations. The commutation relations involving the fermionic generators are given by

$$\begin{aligned}
[\ R\ , Q_\alpha] &= -Q_\alpha, & [\ R\ , S^\alpha] &= -S^\alpha, \\
[\ R\ , \bar{Q}_\alpha] &= \bar{Q}_\alpha, & [\ R\ , \bar{S}^\alpha] &= \bar{S}^\alpha, \\
[M_{\mu\nu}, Q_\alpha] &= \frac{1}{2}\epsilon_{\mu\nu\rho}(\sigma_\rho)^\beta{}_\alpha Q_\beta, & [M_{\mu\nu}, S^\alpha] &= \frac{1}{2}\epsilon_{\mu\nu\rho}(\sigma_\rho)_\beta{}^\alpha S^\beta, \\
[M_{\mu\nu}, \bar{Q}_\alpha] &= \frac{1}{2}\epsilon_{\mu\nu\rho}(\sigma_\rho)^\beta{}_\alpha \bar{Q}_\beta, & [M_{\mu\nu}, \bar{S}^\alpha] &= \frac{1}{2}\epsilon_{\mu\nu\rho}(\sigma_\rho)_\beta{}^\alpha \bar{S}^\beta, \\
[\ D\ , Q_\alpha] &= -\frac{i}{2}Q_\alpha, & [\ D\ , S^\alpha] &= \frac{i}{2}S^\alpha, \\
[\ D\ , \bar{Q}_\alpha] &= -\frac{i}{2}\bar{Q}_\alpha, & [\ D\ , \bar{S}^\alpha] &= \frac{i}{2}\bar{S}^\alpha, \\
[\ K_\mu\ , Q_\alpha] &= (\sigma_\mu)_{\beta\alpha} S^\beta, & [\ P_\mu\ , S^\alpha] &= -(\sigma_\mu)^{\beta\alpha} Q_\beta, \\
[\ K_\mu\ , \bar{Q}_\alpha] &= (\sigma_\mu)_{\beta\alpha} \bar{S}^\beta, & [\ P_\mu\ , \bar{S}^\alpha] &= -(\sigma_\mu)^{\beta\alpha} \bar{Q}_\beta,
\end{aligned} \tag{A.7}$$

with all other commutators vanishing. The anticommutation relations between the fermionic generators are given by

$$\begin{aligned}
\{\bar{Q}_\alpha, Q_\beta\} &= P_\mu(\sigma_\mu)_{\alpha\beta}, & \{S^\alpha, \bar{Q}_\beta\} &= (iD - R)\delta^\alpha{}_\beta + \frac{1}{2}\epsilon_{\mu\nu\rho}M_{\mu\nu}(\sigma_\rho)^\alpha{}_\beta, \\
\{\bar{S}^\alpha, S^\beta\} &= K_\mu(\sigma_\mu)^{\alpha\beta}, & \{\bar{S}^\alpha, Q_\beta\} &= (iD + R)\delta^\alpha{}_\beta + \frac{1}{2}\epsilon_{\mu\nu\rho}M_{\mu\nu}(\sigma_\rho)^\alpha{}_\beta.
\end{aligned} \tag{A.8}$$

Here the Pauli matrices $(\sigma_\mu)^\alpha{}_\beta$ ($\mu \in [1, 2, 3]$) are defined by

$$(\sigma_1)^\alpha{}_\beta \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (\sigma_2)^\alpha{}_\beta \coloneqq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad (\sigma_3)^\alpha{}_\beta \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.9}$$

and we have defined

$$(\sigma_\mu)_{\alpha\beta} \coloneqq \epsilon_{\alpha\gamma}(\sigma_\mu)^\gamma{}_\beta, \qquad (\sigma_\mu)^{\alpha\beta} \coloneqq (\sigma_\mu)^\alpha{}_\gamma \epsilon^{\gamma\beta}, \qquad (\sigma_\mu)_\alpha{}^\beta \coloneqq \epsilon_{\alpha\gamma}(\sigma_\mu)^\gamma{}_\delta \epsilon^{\delta\beta}. \tag{A.10}$$

The algebra (2.4), (A.7) and (A.8) is called $\mathfrak{osp}(2|4)$ in the literature.[45] We also define the Hermitian conjugation in radial quantization acts on the fermionic generators as

$$(Q_\alpha)^\ddagger = \bar{S}^\alpha, \qquad (\bar{Q}_\alpha)^\ddagger = S^\alpha, \tag{A.11}$$

which is consistent with the algebra (A.7) and (A.8).

A.3 Superconformal primaries and unitarity bounds

There is a supersymmetric analog to conformal primary states. A superconformal primary state is defined as a state annihilated by S^α and $\bar{S}^\alpha$,

$$S^\alpha \left|\textsf{superconformal primary}\right\rangle = S^\alpha \left|\textsf{superconformal primary}\right\rangle = 0. \tag{A.12}$$

From (A.8) it is obvious that a superconformal primary state is a conformal primary. It is natural to label superconformal primary states by conformal dimension Δ, spin ℓ and R-charge R. The infinite-dimensional representation, the Verma module associated with a superconformal primary state $|\Delta, \ell, R\rangle$ is given by

$$\mathcal{V}_{\Delta,\ell,R} \coloneqq \mathrm{span}\left\{ \left(\prod_{\mu=1}^{3} P_\mu^{n_\mu}\right)\left(\prod_{\alpha,\beta=1}^{2} Q_\alpha^{n_\alpha}\bar{Q}_\beta^{n_\beta}\right)|\Delta,\ell,R\rangle \,\middle|\, \begin{array}{c} n_\mu \in \mathbb{N} \\ n_\alpha, n_\beta \in \{0,1\} \end{array} \right\}. \tag{A.13}$$

A.3.1 Unitarity bound at level one.

We proceed to derive the unitarity bounds for superconformal primary states [67,68,107]. At level one, we consider the matrix element of a $\bar{Q}_\alpha$-descendant of a superconformal primary,

$$\begin{aligned} \langle\Delta,\ell',R|(\bar{Q}_\alpha)^\ddagger\bar{Q}_\beta|\Delta,\ell,R\rangle &= \langle\Delta,\ell',R|S^\alpha\bar{Q}_\beta|\Delta,\ell,R\rangle \\ &= \langle\Delta,\ell',R|\left[(\Delta-R)\delta^\alpha{}_\beta - \frac{i}{2}\epsilon_{\mu\nu\rho}\Sigma^\ell_{\mu\nu}(\sigma_\rho)^\alpha{}_\beta\right]|\Delta,\ell,R\rangle\,. \end{aligned} \tag{A.14}$$

The condition for the matrix element (A.14) to have only nonnegative values can be derived in a similar way as was done in section 2.4.1, and is given by

$$\Delta - R \geq \frac{1}{2}\left[c_2(\ell) + c_2\left(\frac{1}{2}\right) - c_2\left(\ell-\frac{1}{2}\right)\right] = \begin{cases} 0, & (\ell=0) \\ \ell+1, & (\ell>0) \end{cases} \tag{A.15}$$

where $c_2(\ell) = \ell(\ell+1)$ is the quadratic Casimir invariant of the spin-ℓ representation of SO(3). Considering the matrix element of a Q_α-descendant of a superconformal primary,

[45] Detalis on Lie superalgebras are found in [150, 151].

one gets the bound (A.15) with R replaced by $-R$. As a result, the unitarity bound at level one is expressed as

$$\Delta \geq \begin{cases} |R|, & (\ell = 0) \\ |R| + \ell + 1. & (\ell > 0) \end{cases} \tag{A.16}$$

A.3.2 Unitarity bound at level two.

At level two, one considers the state

$$\epsilon^{\alpha\beta} \bar{Q}_\alpha \bar{Q}_\beta \left| \Delta, R \right\rangle , \tag{A.17}$$

where $|\Delta, R\rangle$ is a scalar superconformal primary of dimension Δ and R-charge R. This state has a norm

$$\langle \Delta, R| \left(\epsilon^{\alpha'\beta'} \bar{Q}_{\alpha'} \bar{Q}_{\beta'} \right)^\ddagger \epsilon^{\alpha\beta} \bar{Q}_\alpha \bar{Q}_\beta |\Delta, R\rangle = 2(\Delta - R)(\Delta - R - 1), \tag{A.18}$$

which is nonnegative if and only if $\Delta \leq R$ or $\Delta \geq R + 1$. An analogous result with R replaced by $-R$ holds for the state $\epsilon^{\alpha\beta} Q_\alpha Q_\beta \left| \Delta, R \right\rangle$. Since $\Delta < |R|$ violates the bound (A.16), at level two the unitarity bound for scalar superconformal primaries is given by

$$\Delta = |R| \qquad \text{or} \qquad \Delta \geq |R| + 1. \tag{A.19}$$

The bounds (A.16) and (A.19) can be collectively expressed as

$$\begin{aligned} &\Delta = |R|, && \ell = 0 \\ &\qquad \text{or} && \\ &\Delta \geq |R| + \ell + 1, && \ell \geq 0. \end{aligned} \tag{A.20}$$

Scalar superconformal primaries ($\ell = 0$) with $\Delta = R, -R$ are called chiral and antichiral primaries, respectively. Superconformal primaries with $\Delta = \pm R + \ell + 1$ are called semishort and anti-semishort primaries. These special superconformal primaries have a smaller number of superpartners than generic (long) primary operators with $\Delta > |R| + \ell + 1$, and their dimensions are supersymmetrically protected from quantum corrections.

The bound (A.20) is the most stringent one for general $\mathcal{N} = 2$ SCFTs in three dimensions [67, 110, 111]. That is, the matrix elements of states at higher levels do not give a stronger bound than (A.20). The primary operator of the multiplet including the stress tensor is the R-charge current, which has dimension $\Delta = 2$ and spin $\ell = 1$ and thus saturates the inequality in (A.20). A global symmetry current is a descendant of a neutral scalar primary operator of dimension $\Delta = 1$, called the moment map operator, which also saturates the inequality in (A.20) with $\ell = 0$. This fact will be used to justify the existence of scalar semishort primaries with arbitrary $R \in 2\mathbb{N}/3$ in the XYZ model in 3.4.

B Uniqueness of the super-FTPR operator on flat space

We would like to show here that on flat space there is no supersymmetric dimension-3 operator constructed with four superderivatives, except for the super-FTPR operator (3.13). We work in Lorentzian signature here. First of all, we do not have to consider operators containing any odd number of superderivatives acting on a single Φ or $\bar{\Phi}$, because such operators are always equal to ones containing only even number of superderivatives acting on a single Φ or $\bar{\Phi}$, modulo the leading-order superspace equations of motion, $\mathcal{D}^2\Phi \simeq 0$ and $\bar{\mathcal{D}}^2\bar{\Phi} \simeq 0$.

From (A.3), we have

$$\bar{\mathcal{D}}_\alpha \mathcal{D}_\beta \Phi = \bar{\mathcal{D}}_\beta \mathcal{D}_\alpha \Phi, \tag{B.1}$$

and especially $\bar{\mathcal{D}}^\alpha \mathcal{D}_\alpha \Phi = 0$. These identities are useful in decreasing the number of index structures. For instance, one can show that any candidate containing four superderivatives acting on a single Φ in any order vanishes modulo the leading-order superspace equations of motion.

By the above consideration, we conclude that only the following operators possibly survive,

$$\begin{aligned} \mathcal{O}_1^{(4)} &:= \int d^2\theta d^2\bar{\theta} \left(\frac{\bar{\mathcal{D}}_\alpha \mathcal{D}_\beta \Phi \bar{\mathcal{D}}^\alpha \mathcal{D}^\beta \Phi}{\Phi^3 \bar{\Phi}} + \text{c.c.} \right), \\ \mathcal{O}_2^{(4)} &:= \int d^2\theta d^2\bar{\theta} \frac{\bar{\mathcal{D}}_\alpha \mathcal{D}_\beta \Phi \mathcal{D}^\alpha \bar{\mathcal{D}}^\beta \bar{\Phi}}{\left(\Phi\bar{\Phi}\right)^2}. \end{aligned} \tag{B.2}$$

The operator $\mathcal{O}_2^{(4)}$ is equivalent to the super-FTPR operator (3.13), since $\left\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\right\} = 2\gamma^\mu_{\alpha\beta}\partial_\mu$. The operator $\mathcal{O}_1^{(4)}$ is also equivalent to the super-FTPR operator, because

$$\frac{\bar{\mathcal{D}}_\alpha \mathcal{D}_\beta \Phi \bar{\mathcal{D}}^\alpha \mathcal{D}^\beta \Phi}{\Phi^3 \bar{\Phi}} \sim \frac{\partial_\mu \Phi \partial^\mu \Phi}{\Phi^3 \bar{\Phi}} \sim \frac{1}{\Phi^2} \partial_\mu \left(\frac{\partial^\mu \Phi}{\bar{\Phi}} \right) \sim \frac{\partial_\mu \Phi \partial^\mu \bar{\Phi}}{\left(\Phi\bar{\Phi}\right)^2}. \tag{B.3}$$

Here, by "$\sim$" we mean modulo total superderivatives, the leading-order equations of motion, and numerical coefficients. So, there is only one supersymmetric dimension-3 operator with four superderivatives on flat space modulo total superderivatives, and it is nothing but the unique super-Weyl completion of the FTPR operator.

C Energy correction to one-boson one-fermion and two-fermion excitations

We set the radius of $\mathbb{S}^2$ to be unity here.

C.1 One-boson one-fermion excitation

Quantization of the fermionic state with lowest spin is done as follows:

$$
\begin{gathered}
\psi^{\alpha}(x)=\sum_{s=\pm}\beta_s u_s^{\alpha}(x)+\gamma_s^{\ddagger}v_s^{\alpha}(x),\\
\not{\nabla}_{\mathbb{S}^2}u_s=iu_s,\qquad \not{\nabla}_{\mathbb{S}^2}v_s=-iv_s,\\
\bar{u}^{(-)}=v^{(-)},\qquad \bar{u}^{(+)}=-v^{(+)},\qquad \gamma^0u^{(-)}=v^{(+)},\qquad \gamma^0v^{(+)}=-u^{(-)}
\end{gathered}
\tag{C.1}
$$

We also use the equation of motion for ψ:

$$
(\gamma^t)^{\alpha}{}_{\beta}\partial_t\psi^{\beta}+(\gamma^i)^{\alpha}{}_{\beta}\nabla_{\mathbb{S}^2,i}\psi^{\beta}=0. \tag{C.2}
$$

Furthermore we set

$$
\bar{u}^s_{\alpha}u^{\alpha}_r=\frac{1}{4\pi}\delta^{sr},\quad \bar{u}^s_{\alpha}v^{\alpha}_r=0 \tag{C.3}
$$

as a normalization condition. With this quantization convention we obtain the free Dirac Hamiltonian,

$$
H_0^{\text{Dirac}}=\sum_{s=\pm}\left(\beta_s^{\ddagger}\beta_s+\gamma_s^{\ddagger}\gamma_s\right), \tag{C.4}
$$

with the anticommutation relation

$$
\left\{\beta,\beta^{\ddagger}\right\}=\left\{\gamma,\gamma^{\ddagger}\right\}=1. \tag{C.5}
$$

The interaction Hamiltonian of order J^{-3} which includes the two-fermion two-boson interaction is given by

$$
\kappa H_{\text{int}}^{(2,2)}=-4\pi\kappa\times 4\bar{\psi}^{\alpha}\left[\left(-\partial_t^2+\nabla^2_{\mathbb{S}^2}-3i\partial_t+2\right)\frac{\bar{F}}{\bar{\phi}_0^3}\right]\left[\left(\gamma^{\mu}_{\alpha\beta}\nabla_{\mu}+i\gamma^t_{\alpha\beta}\right)\frac{F}{\phi_0^3}\right]\psi^{\beta}, \tag{C.6}
$$

where κ is a proportionality constant as in (3.45). Using the equation of motion and the fact that $\phi_0=e^{it/2}\varphi_0$ and then taking only the spin-1/2 and spin-0 contribution for the fermion and the scalar, respectively, we get

$$
H_{\text{int}}^{(2,2)}=-\frac{4\pi}{|\varphi_0|^6}\times 24\bar{\psi}\gamma^0\psi\times(\bar{f}-i\dot{\bar{f}})f \tag{C.7}
$$

Using the quantization of the scalar field given in section 3.3.5 and that of the fermionic field given above, we obtain

$$
H_{\text{int}}^{(2,2)}=-\frac{6}{\pi|\varphi_0|^6}(2a^{\ddagger}-b)a\times\sum_{s=\pm}\left(\beta_s^{\ddagger}\beta_s+\gamma_s^{\ddagger}\gamma_s\right), \tag{C.8}
$$

from which we compute the energy correction to the state $a^{\ddagger}\beta_+^{\ddagger}|0\rangle$ as

$$
\Delta E=-\frac{12\kappa}{\pi|\varphi_0|^6}. \tag{C.9}
$$

This agrees with the energy correction to the two-boson state, as it should be from supersymmetry.

C.2 Two-fermion excitation

The interaction Hamiltonian of order J^{-3} which includes four-fermion interaction is given by

$$\kappa H_{\text{int}}^{(0,4)} = 4\pi\kappa \times \frac{\bar{\psi}_\beta \bar{\psi}^\beta}{\bar{\phi}^3}\left(-\partial_t^2 + \nabla_{\mathbb{S}^2}^2 - \frac{1}{4}\right)\frac{\psi^\alpha \psi_\alpha}{\phi^3}, \tag{C.10}$$

where κ is a constant as in (3.45). Using the fact that $\phi_0 = e^{it/2}\varphi_0$ and taking only the spin-$1/2$ contribution for the fermionic field, we obtain

$$H_{\text{int}}^{(0,4)} = \frac{4\pi}{|\varphi_0|^6}\bar{L}\left(2L + 3i\dot{L} - \ddot{L}\right), \tag{C.11}$$

where $L = \psi\psi$. Then by using (C.1) and the normalization condition, we have

$$\begin{aligned}
\bar{L}L &= -\frac{1}{8\pi^2}\left(\gamma_-\gamma_+ + \beta_-^\dagger\beta_+^\dagger\right)\left(\gamma_+^\dagger\gamma_-^\dagger + \beta_+\beta_-\right),\\
\bar{L}\dot{L} &= \frac{i}{4\pi^2}\left(\gamma_-\gamma_+ + \beta_-^\dagger\beta_+^\dagger\right)\left(\gamma_+^\dagger\gamma_-^\dagger + \beta_+\beta_-\right),\\
\bar{L}\ddot{L} &= \frac{1}{2\pi^2}\left(\gamma_-\gamma_+ + \beta_-^\dagger\beta_+^\dagger\right)\left(\gamma_+^\dagger\gamma_-^\dagger + \beta_+\beta_-\right),
\end{aligned} \tag{C.12}$$

and the interaction Hamiltonian $H_{\text{int}}^{(0,4)}$ becomes

$$H_{\text{int}}^{(0,4)} = -\frac{12}{\pi|\varphi_0|^6}\left(\gamma_-\gamma_+ + \beta_-^\dagger\beta_+^\dagger\right)\left(\gamma_+^\dagger\gamma_-^\dagger + \beta_+\beta_-\right), \tag{C.13}$$

and the resulting energy correction to the two-fermion state $\beta_+^\dagger\beta_-^\dagger|0\rangle$ is computed as

$$\Delta E = -\frac{12\kappa}{\pi|\varphi_0|^6}, \tag{C.14}$$

which agrees with the energy correction to the two-boson state and the one-boson one-fermion state, as it should be from supersymmetry.

D Superconformal index and scalar semishort multiplets

In this appendix we check the superconformal index for the XYZ model to confirm that scalar semishort multiplets really exist in the theory. The superconformal index for the XYZ model is given by the following plethystic exponential [152, 153],

$$\begin{aligned}
I_{XYZ}(x, t_X, t_{YZ}) &\coloneqq \operatorname{Tr}\left((-1)^F z^{\Delta-R-\ell_3} x^{\Delta+\ell_3} t_X^{J_X} t_{YZ}^{J_{YZ}}\right)\\
&= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}F(x^n, t_X^n, t_{YZ}^n)\right).
\end{aligned} \tag{D.1}$$

Here, ℓ_3 is the third component of the spin on $\mathbb{S}^2$, and t_X and t_{YZ} are the fugacities for $\mathrm{U}(1)_X$ and $\mathrm{U}(1)_{YZ}$, respectively. The function $F(x, t_X, t_{YZ})$ is the so-called letter index, defined as

$$\begin{aligned} F(x, t_X, t_{YZ}) &\coloneqq f(x, t_X) + f\left(x, t_X^{-1/2} t_{YZ}\right) + f\left(x, t_X^{-1/2} t_{YZ}^{-1}\right), \\ f(x, t) &\coloneqq \frac{t x^{2/3} - t^{-1} x^{4/3}}{1 - x^2}. \end{aligned} \tag{D.2}$$

Because of the superconformal algebra, only protected multiplets can contribute to the superconformal index and therefore it is independent of the variable z in (D.1). Let $\partial_{\uparrow\uparrow}$ be the derivative which raises the third component of the spin by one, and $\bar{Q}_{\uparrow}$ be the energy- and R-charge-raising supercharge which has the third component of the spin $\ell_3 = 1/2$. In a chiral multiplet, the chiral primary operator and its $\partial_{\uparrow\uparrow}$-derivatives contribute to the superconformal index, whereas contributions from the other states in the chiral multiplet cancel between themselves. In a given scalar semishort multiplet, it is the $\bar{Q}_{\uparrow}$-descendant of the scalar semishort primary operator and its $\partial_{\uparrow\uparrow}$-derivatives that contribute to the superconformal index. When the superconformal index is expanded with respect to x, contributions from chiral multiplets have positive coefficients, whereas those from scalar semishort multiplets have negative coefficients.

In principle, the superconformal index contains all information about operators in short or semishort representations. However, the terms in a full and explicit expansion of the superconformal index do not necessarily have to correspond one-to-one with operators in short and semishort representations, because cancellations can happen. In practice, cancellations happen frequently in many familiar SCFTs, including the XYZ model in three dimensions. These cancellations can be removed by organizing the superconformal index into characters of the particular short and semishort representations which appear.

We do not do this, since the organization of the superconformal index into characters is burdensome and we are just computing the some particular terms in the superconformal index to verify its agreement with the spectrum of semishort representations as calculated by using the large-R-charge EFT. Instead, we list both the positive and negative contributions to the superconformal index separately, noting the cancellations as they happen.

To see the existence of the scalar semishort multiplets, we expand the superconformal index (D.1) with respect to x up to and including $O\left(x^{10/3}\right)$. However, some contributions from scalar semishort multiplets are canceled by those from chiral multiplets, and therefore it is impossible to see all the contributions from scalar semishort multiplets just by expanding the superconformal index. So, we separate these two kinds of contributions order by order in x, by brute force. We also identify all the positive contributions up to and including $O\left(x^{10/3}\right)$ with (descendants of) chiral primary operators. The superconformal index (D.1) is expanded with respect to x as follows:

$$
\begin{aligned}
&I_{XYZ}(x,t_X,t_{YZ}) \\
&= \underset{\mathbb{1}}{1} + x^{2/3}\left(\underset{X}{t_X} + \underset{Y}{\frac{t_{YZ}}{t_X^{1/2}}} + \underset{Z}{\frac{1}{t_X^{1/2}t_{YZ}}}\right) + x^{4/3}\left(\underset{X^2}{t_X^2} + \underset{Y^2}{\frac{t_{YZ}^2}{t_X}} + \underset{Z^2}{\frac{1}{t_X t_{YZ}^2}}\right) \\
&+ x^2\left(\underset{X^3}{t_X^3} + \underset{Y^3}{\frac{t_{YZ}^3}{t_X^{3/2}}} + \underset{Z^3}{\frac{1}{t_X^{3/2}t_{YZ}^3}}\right) \underset{\substack{\bar{Q}_\uparrow D_X \\ \bar{Q}_\uparrow D_{YZ}}}{-2x^2} \\
&+ x^{8/3}\left(\underset{X^4}{t_X^4} + \underset{Y^4}{\frac{t_{YZ}^4}{t_X^2}} + \underset{Z^4}{\frac{1}{t_X^2 t_{YZ}^4}} + \underset{\partial_{\uparrow\uparrow}X}{t_X} + \underset{\partial_{\uparrow\uparrow}Y}{\frac{t_{YZ}}{t_X^{1/2}}} + \underset{\partial_{\uparrow\uparrow}Z}{\frac{1}{t_X^{1/2}t_{YZ}}}\right) - x^{8/3}\left(t_X + \frac{t_{YZ}}{t_X^{1/2}} + \frac{1}{t_X^{1/2}t_{YZ}}\right) \\
&+ x^{10/3}\left(\underset{X\partial_{\uparrow\uparrow}Y}{t_X^{1/2}t_{YZ}} + \underset{Y\partial_{\uparrow\uparrow}Z}{\frac{1}{t_X}} + \underset{Z\partial_{\uparrow\uparrow}X}{\frac{t_X^{1/2}}{t_{YZ}}} + \underset{X^5}{t_X^5} + \underset{Y^5}{\frac{t_{YZ}^5}{t_X^{5/2}}} + \underset{Z^5}{\frac{1}{t_X^{5/2}t_{YZ}^5}} + \underset{\partial_{\uparrow\uparrow}X^2}{t_X^2} + \underset{\partial_{\uparrow\uparrow}Y^2}{\frac{t_{YZ}^2}{t_X}} + \underset{\partial_{\uparrow\uparrow}Z^2}{\frac{1}{t_X t_{YZ}^2}}\right) \\
&- x^{10/3}\left(t_X^2 + \frac{t_{YZ}^2}{t_X} + \frac{1}{t_X t_{YZ}^2}\right) + O\left(x^4\right).
\end{aligned}
\tag{D.3}
$$

The negative contribution at order x^2 is due to the $\bar{Q}_\uparrow$-descendants of the moment map operators D_X and D_{YZ}, which obviously exist since the theory has the $\mathrm{U}(1)_X$ and $\mathrm{U}(1)_{YZ}$ flavor symmetries. The negative contributions at $O\left(x^{8/3}\right)$ and at $O\left(x^{10/3}\right)$ are nontrivial, however. These cannot be descendants of the moment map operators on dimensional grounds. For instance, the $-x^{8/3}t_X$ and $-x^{10/3}t_X^2$ terms correspond to the $\bar{Q}_\uparrow$-descendants of semishort operators of spin 0 and dimension $5/3$ and $7/3$, respectively. In terms of the almost-free ϕ variables, these semishort primary states can be represented as $\phi_0\bar{\phi}_0\,|X\rangle$ and $\phi_0\bar{\phi}_0\,|X^2\rangle$, where as explained in section 3.3.2, the state $\left|X^J\right\rangle$ can be thought of as $\phi_0^{4J/3}\,|0\rangle$.

Heuristically, these semishort primary operators can be thought of as $D_X\cdot X$ and $D_X\cdot X^2$, respectively, where D_X is the scalar semishort "moment map" primary operator, whose descendant is the spin-1 $\mathrm{U}(1)_X$ current. However this description is not really accurate, because the leading term in the OPE of D_X with X^J is not the semishort operator $D_X\cdot X^J$ but the chiral primary operator X^J, and the coefficient function is singular, $|x|^{-1}$.

We emphasize that, for the purpose of understanding the spectrum directly at sufficiently large R-charge J, the power of supersymmetric representation theory is useful mainly as a convenience: The explicit computations of the large-J EFT simply agree with those of the superconformal index, with the spectrum computation becoming more reliable at large J.

The most important thing we learn directly is that there is a nonzero scalar semishort primary operator in the OPE of D_X with X^J, for sufficiently large J:

$$
D_X(x)X^J(0) = \cdots + c^{D_X X^J}{}_{X^J}|x|^{-1}X^J(0) + \cdots\,, \tag{D.4}
$$

where the OPE coefficient $c^{D_X X^J}{}_{X^J}$ can be computed semiclassically as an expectation value of D_X in the state $|X^J\rangle$, and is nonzero. This gives information about the index only asymptotically.

Combined with the power of associativity, the existence of semishort primary operators at large J has more consequences: Since the product "$\cdot$" defines an associative multiplication, and since $D_X \cdot X^J \neq 0$ at large J, then all the lower-dimensional products $D_X \cdot X$, $D_X \cdot X^2$, $\cdots$ have to automatically be nonzero as well. This is in agreement with the superconformal index as expanded above. Therefore, we see that large-J methods combined with associativity, provide information about scalar semishort primary operators at low J as well.

E Semishort superalgebra

Here we would like to set up a formalism of truncating the superconformal algebra to a finite number of degrees of freedom, which are creation and annihilation operators. This section is useful in understanding the vanishing of the one-loop energy correction to the chiral and scalar semishort primary state with given R-charge J.

E.1 Commutation relations

We radially quantize the system. We have the dilatation operator Δ, whose eigenvalues are the operator dimensions, as well as the Hermitian conjugation $\ddagger$ on the cylinder. Then it is inevitable to dismiss either P or K at least, because of the fact that the full (nonsupersymmetric) conformal algebra can only be unitary represented in the infinite-dimensional Hilbert space. Nevertheless, we are trying to find a truncation of the superconformal algebra which contains as many supercharges as possible, under such a constraint.

To specify such a subalgebra of the full superconformal algebra which has the above property, let us fix some conventions. We denote by $Q_\alpha^{(\sigma_\Delta,\sigma_{\mathrm{R}})}$ the fermionic generator which changes the conformal dimension by $\sigma_\Delta/2$ units, and the R-charge by σ_{R} units. For example, $Q_\uparrow^{(+,+)}$ is the generator which raises both dimension Δ and R-charge J_{R}, and also the third component ℓ_3 of the angular momentum ℓ_μ.

We would like to concentrate on fermionic generators which are preserved by the chiral primary states. In other words, we are only going to consider $Q_\alpha^{(+,+)}$ and $Q_\alpha^{(-,-)}$. These fermionic generators have to be related by Hermitian conjugation. Since we choose the spinorial index convention as

$$\left[s^\mu, Q_\alpha^{(\pm,\pm)}\right] = \frac{1}{2}(\sigma^\mu)^\beta{}_\alpha Q_\beta^{(\pm,\pm)}, \qquad s^\mu := \frac{1}{2}\epsilon^{\mu\nu\rho}M_{\nu\rho}, \tag{E.1}$$

these two are related by

$$\left(Q_\alpha^{(\pm,\pm)}\right)^\ddagger = \pm\epsilon^{\alpha\beta}Q_\beta^{(\mp,\mp)}. \tag{E.2}$$

Hereafter, by using these conventions, we simplify our notation the following way,

$$\mathbf{Q}_\alpha \coloneqq Q_\alpha^{(+,+)}, \qquad \mathbf{Q}^{\ddagger\alpha} = \left(Q_\alpha^{(+,+)}\right)^\ddagger = \epsilon^{\alpha\beta} Q_\beta^{(-,-)}. \tag{E.3}$$

From to the $\mathcal{N} = 2$ superconformal algebra, we choose the following (anti)commutation relations for these generators to be preserved:

$$\begin{aligned}
\left\{\mathbf{Q}_\alpha, \mathbf{Q}_\beta^\ddagger\right\} = \delta_{\alpha\beta}(\Delta - R) + \sigma^\mu_{\beta\alpha} s_\mu, \qquad \{\mathbf{Q}_\alpha, \mathbf{Q}_\beta\} = \left\{\mathbf{Q}_\alpha^\ddagger, \mathbf{Q}_\beta^\ddagger\right\} = 0,\\
[s_\mu, s_\nu] = i\epsilon_{\mu\nu\rho} s^\rho, \qquad [R, \Delta] = [R, s^\mu] = [\Delta, s^\mu] = 0,\\
[R, \mathbf{Q}_\alpha] = \mathbf{Q}_\alpha \qquad \left[R, \mathbf{Q}_\alpha^\ddagger\right] = -\mathbf{Q}_\alpha^\ddagger\\
[\Delta, \mathbf{Q}_\alpha] = \frac{1}{2}\mathbf{Q}_\alpha \qquad \left[\Delta, \mathbf{Q}_\alpha^\ddagger\right] = -\frac{1}{2}\mathbf{Q}_\alpha^\ddagger\\
[s^\mu, \mathbf{Q}_\alpha] = \frac{1}{2}(\sigma^\mu)^\beta{}_\alpha \mathbf{Q}_\beta \quad \left[s^\mu, \mathbf{Q}_\alpha^\ddagger\right] = -\frac{1}{2}(\sigma^\mu)^\beta{}_\alpha \mathbf{Q}_\beta^\ddagger.
\end{aligned} \tag{E.4}$$

Note that neither P nor K appears in this expression.

E.2 Oscillator realization

Let us consider the case where there is a single free multiplet with the transformation law of the s-wave mode of a free antichiral superfield $\bar{\Phi}$ on the cylinder. We shall call the bosonic oscillators a, $a^\ddagger$ and the fermionic oscillators $b_\alpha^\ddagger$. Note that here we will use the convention which assigns $b_\alpha^\ddagger$ the same transformation property under spatial rotational group SO(3) as that of $\mathbf{Q}_\alpha$. The oscillator realization of the superalgebra (E.4) is given by

$$\begin{aligned}
J = \frac{1}{2}\left(b^{\ddagger\alpha} b_\alpha - a^\ddagger a\right), \qquad s^\mu = \frac{1}{2}(\sigma^\mu)^\beta{}_\alpha b^{\ddagger\alpha} b_\beta,\\
\Delta = \frac{1}{2} a^\ddagger a + b^{\ddagger\alpha} b_\alpha, \qquad \mathbf{Q}_\alpha = b_\alpha^\ddagger a, \qquad \mathbf{Q}_\alpha^\ddagger = a^\ddagger b_\alpha,
\end{aligned} \tag{E.5}$$

where the oscillators satisfy the canonical commutation and anticommutation relations

$$\left[a, a^\ddagger\right] = 1, \qquad \left\{b_\alpha, b_\beta^\ddagger\right\} = \delta_{\alpha\beta}. \tag{E.6}$$

The generators (E.5) satisfy the algebra (E.4). On the oscillators they act as

$$\left[s^\mu, b_\alpha^\ddagger\right] = \frac{1}{2}(\sigma^\mu)^\beta{}_\alpha b_\beta^\ddagger, \qquad [s^\mu, b_\alpha] = -\frac{1}{2}(\sigma^\mu)^\beta{}_\alpha b_\beta, \tag{E.7}$$

$$\left[J, a^\ddagger\right] = -\frac{1}{2} a^\ddagger, \qquad [J, a] = \frac{1}{2} a \tag{E.8}$$

$$\left[J, b_\alpha^\ddagger\right] = \frac{1}{2} b_\alpha^\ddagger, \qquad [J, b_\alpha] = -\frac{1}{2} b_\alpha. \tag{E.9}$$

E.3 Perturbation theory

We would like to set our perturbation theory. Let us make a small change to the generators, at order ϵ, and demand that the structure of the superalgebra still be unchanged at order ϵ. Let $\mathbf{Q}_\alpha(\epsilon)$, $\mathbf{Q}^\ddagger_\alpha(\epsilon)$, $\Delta(\epsilon)$ be

$$\begin{aligned} \mathbf{Q}_\alpha(\epsilon) &= \mathbf{Q}_\alpha(0) + \epsilon\mathbf{Q}'_\alpha(0) + O\big(\epsilon^2\big), \\ \mathbf{Q}^\ddagger_\alpha(\epsilon) &= \mathbf{Q}^\ddagger_\alpha(0) + \epsilon\mathbf{Q}^{\ddagger\prime}_\alpha(0) + O\big(\epsilon^2\big), \\ \Delta(\epsilon) &= \Delta(0) + \epsilon\Delta(0) + O\big(\epsilon^2\big). \end{aligned} \tag{E.10}$$

Many first-order perturbations of the superalgebra are not physically interesting, and correspond merely to first-order redefinitions of the variables induced by transformations on Hilbert space; we fix much of this ambiguity by requiring that s^μ and J do not depend on ϵ:

$$\frac{\mathrm{d}s^\mu}{\mathrm{d}\epsilon} = \frac{\mathrm{d}J}{\mathrm{d}\epsilon} = 0. \tag{E.11}$$

Then we use the notation

$$\begin{aligned} \mathbf{Q}_\alpha &\coloneqq \mathbf{Q}_\alpha(0), \qquad & \mathbf{Q}^{\ddagger\prime}_\alpha &\coloneqq \mathbf{Q}^\ddagger_\alpha(0), \\ \mathbf{q}_\alpha &\coloneqq \mathbf{Q}'_\alpha(0), \qquad & \mathbf{q}^\ddagger_\alpha &\coloneqq \mathbf{Q}^{\ddagger\prime}_\alpha(0), \\ & & \Delta' &\coloneqq \Delta'(0), \end{aligned} \tag{E.12}$$

and the ϵ-derivative of the superalgebra (E.4), and evaluate at $\epsilon = 0$. This procedure gives a set of "easy" perturbation equations, which involve commutators with the ϵ-independent generators J and s^μ,

$$\begin{aligned} [J, \mathbf{q}_\alpha] &= \mathbf{q}_\alpha, \qquad & \big[J, \mathbf{q}^\ddagger_\alpha\big] &= -\mathbf{q}^\ddagger_\alpha, \\ [s^\mu, \mathbf{q}_\alpha] &= \frac{1}{2}(\sigma^\mu)^\beta{}_\alpha\mathbf{q}_\beta, \qquad & \big[s^\mu, \mathbf{q}^\ddagger_\alpha\big] &= -\frac{1}{2}(\sigma^\mu)^\beta{}_\alpha\mathbf{q}^\ddagger_\beta, \\ & & [s^\mu, \Delta'] &= 0, \end{aligned} \tag{E.13}$$

and "hard" perturbation equations:

$$\{\mathbf{Q}_\alpha, \mathbf{q}_\beta\} + \{\mathbf{Q}_\beta, \mathbf{q}_\alpha\} = \Big\{\mathbf{Q}^\ddagger_\alpha, \mathbf{q}^\ddagger_\beta\Big\} + \Big\{\mathbf{Q}^\ddagger_\beta, \mathbf{q}^\ddagger_\alpha\Big\} = 0, \tag{E.14}$$

$$\Big\{\mathbf{Q}_\alpha, \mathbf{q}^\ddagger_\beta\Big\} + \Big\{\mathbf{Q}^\ddagger_\beta, \mathbf{q}_\alpha\Big\} = \delta_{\alpha\beta}\Delta', \tag{E.15}$$

$$[\Delta, \mathbf{q}_\alpha] - [\mathbf{Q}_\alpha, \Delta'] = \frac{1}{2}\mathbf{q}_\alpha, \tag{E.16}$$

$$\big[\Delta, \mathbf{q}^\ddagger_\alpha\big] - \big[\mathbf{Q}^\ddagger_\alpha, \Delta'\big] = \frac{1}{2}\mathbf{q}^\ddagger_\alpha. \tag{E.17}$$

The "easy" perturbation equations (E.13) just imply that the transformation laws of the perturbed generators under the ϵ-independent generators J, s^μ are the same as those of the corresponding unperturbed generators.

We would like to solve the hard perturbation equations. Let us start with (E.14). Solving this in full generality may be cumbersome, but can be done at least in a sufficient condition way, that is, this equation is solved by the ansatz

$$\mathbf{q}_\alpha = \left[\mathbf{Q}_\alpha, \mathcal{O}_{[2]}\right], \qquad \mathbf{q}^\ddagger_\alpha = -\left[\mathbf{Q}^\ddagger_\alpha, \mathcal{O}^\ddagger_{[2]}\right]. \tag{E.18}$$

Let us introduce the notation "$\cdot$" for acting by commutation or anticommutation. We denote this by $\widetilde{\mathbf{Q}}$, and rewrite (E.18) as

$$\mathbf{q}_\alpha = \widetilde{\mathbf{Q}}_\alpha \cdot \mathcal{O}_{[2]}, \qquad \mathbf{q}^\ddagger_\alpha = -\widetilde{\mathbf{Q}}^\ddagger_\alpha \cdot \mathcal{O}^\ddagger_{[2]}. \tag{E.19}$$

The meaning of the subscript [2] will become clear shortly. The equation (E.19) is just an ansatz, but this ansatz does automatically solve (E.14). The "easy" equations (E.13) just constrain $\mathcal{O}_{[2]}$ to be a scalar operator with vanishing R-charge.

Next, we would like to solve (E.15). Contracting it with $\delta_{\alpha\beta}/2$, this equation implies

$$\Delta' = \frac{1}{2}\widetilde{\mathbf{Q}}^{\ddagger\alpha} \cdot \widetilde{\mathbf{Q}}_\alpha \cdot \mathcal{O}_{[2]} - \frac{1}{2}\widetilde{\mathbf{Q}}_\alpha \cdot \widetilde{\mathbf{Q}}^{\ddagger\alpha} \cdot \mathcal{O}^\ddagger_{[2]}. \tag{E.20}$$

An imaginary part of $\mathcal{O}_{[2]}$ contributes as a total derivative to Δ'. For real $\mathcal{A}$, when $\mathcal{O}_{[2]}$ is shifted by $i\mathcal{A}$, then Δ' is shifted by $\dot{\mathcal{A}}$. That is,

$$\mathcal{O}_{[2]} \to \mathcal{O}_{[2]} + i\mathcal{A}, \qquad \Delta' \to \Delta' + \dot{\mathcal{A}}, \tag{E.21}$$

so an imaginary part of $\mathcal{O}_{[2]}$ just corresponds to changing the Hamiltonian by conjugation by an infinitesimal unitary transformation parametrized by $\mathcal{A}$ which is scalar and R-neutral. Since ultimately we only care about the system up to change of basis, we can fix that ambiguity by simply taking the convention

$$\mathcal{O}^\ddagger_{[2]} = \mathcal{O}_{[2]}. \tag{E.22}$$

With this convention we have

$$\Delta' = \frac{1}{2}\left[\widetilde{\mathbf{Q}}^{\ddagger\alpha}, \widetilde{\mathbf{Q}}_\alpha\right] \cdot \mathcal{O}_{[2]}. \tag{E.23}$$

So now equation (E.15) reads:

$$\left[\widetilde{\mathbf{Q}}^\ddagger_\beta, \widetilde{\mathbf{Q}}_\alpha\right] \cdot \mathcal{O}_{[2]} = \delta_{\alpha\beta}\Delta'. \tag{E.24}$$

Since we can take (E.23) to define Δ', the only remaining content of (E.24) is equivalent to the condition

$$(\sigma^\mu)^\alpha{}_\beta\left[\widetilde{\mathbf{Q}}^{\ddagger\beta}, \widetilde{\mathbf{Q}}_\alpha\right] \cdot \mathcal{O}_{[2]} = 0. \tag{E.25}$$

The ansatz we are going to make to solve (E.25) is

$$\mathcal{O}_{[2]} = \frac{1}{2}\left[\widetilde{\mathbf{Q}}^{\ddagger\alpha}, \widetilde{\mathbf{Q}}_\alpha\right] \cdot \mathcal{O}_{[0]} - \mathcal{O}_{[0]}. \tag{E.26}$$

The first term would be present in flat-space supersymmetry. Indeed, the formula for Δ' in terms of four supercharges acting on $\mathcal{O}_{[0]}$, is just an operator realization of superspace perturbation theory, with $\mathcal{O}_{[0]}$ playing the role of the superspace integrand of D-term type. The second term on the right-hand side of (E.26) is not present in flat-space supersymmetry, and corresponds to a nontrivial background curvature of superspace in the sense of [105,106].

E.4 A last bit of closure of the algebra

There is one last nontrivial condition that must be satisfied. It comes from (E.16) (and its conjugate (E.17)), which reads

$$[\Delta, \mathbf{q}_\alpha] - [\mathbf{Q}_\alpha, \Delta'] = \frac{1}{2}\mathbf{q}_\alpha. \tag{E.27}$$

This equation does not impose any further independent conditions on the perturbation of the generators. In principle it is automatically satisfied and can be verified directly on the generators constructed from $\mathcal{O}_{[0]}$. To see this it is simplest to note that this equation is the commutator $[\Delta, \mathbf{Q}_\alpha] = \mathbf{Q}_\alpha/2$ at first order in ϵ, and this must hold order by order in ϵ. Thus (E.27) follows automatically from the other first-order closure equations (E.14) and (E.15) without imposing further conditions on the perturbation.

E.5 Examples

Now we would like to apply our formula to some examples of $\mathcal{O}_{[0]}$, which correspond to the superspace integrand of superspace perturbation theory, to see concretely how interaction terms made from the semishort zero mode correspond to perturbations Δ' of the Hamiltonian. In particular, we will see that all such perturbations come out automatically normal-ordered, with no less than one semishort zero mode annihilation operator on the right.

E.5.1 Example: Perturbation corresponding to quadratic deformations

The simplest sort of deformation to add would of course be $\mathcal{O}_{[0]} := \mathcal{E} a^\ddagger a$. Then using (E.5), we find that

$$\mathcal{O}_{[2]} = \widetilde{\mathbf{Q}}^{\ddagger\alpha} \cdot \widetilde{\mathbf{Q}}_\alpha \cdot \mathcal{O}_{[0]} - \widetilde{\Delta}\mathcal{O}_{[0]} - \mathcal{O}_{[0]} = \mathcal{E}\left(a^\ddagger a + b^{\ddagger\alpha} b_\alpha\right) \tag{E.28}$$

and the perturbation of the supercharges and dilatation operator simply vanish:

$$\mathbf{q}_\alpha = \mathbf{Q}^\ddagger_\alpha = \Delta' = 0. \tag{E.29}$$

E.5.2 Quartic perturbation

Now let us work out the formulae for the quartic perturbation. We define

$$\mathcal{O}_{[0]} = \frac{\mathcal{E}}{4}\left(a^{\ddagger}\right)^2 a^2. \tag{E.30}$$

Then we have

$$\mathcal{O}_{[2]} = \mathcal{E}\left(\frac{3}{4}\left(a^{\ddagger}\right)^2 a^2 + b^{\ddagger\alpha} b_\alpha a^{\ddagger} a\right), \tag{E.31}$$

and

$$\begin{aligned} \mathbf{q}_\alpha &= \frac{\mathcal{E}}{2} b^{\ddagger}_\alpha a^{\ddagger} a^2 - \mathcal{E} b^\beta b^{\ddagger}_\alpha b_\beta a, \\ \mathbf{q}^{\ddagger}_\alpha &= \frac{\mathcal{E}}{2}\left(a^{\ddagger}\right)^2 b_\alpha a - \mathcal{E} b^{\ddagger\beta} a^{\ddagger} b_\alpha b_\beta. \end{aligned} \tag{E.32}$$

The first-order modification Δ' of the Hamiltonian is

$$\Delta' = 2\mathcal{E} a^{\ddagger} b^{\ddagger\alpha} b_\alpha a + \mathcal{E}\left(a^{\ddagger}\right)^2 a^2 + \mathcal{E} b^{\ddagger\beta} b^{\ddagger\alpha} b_\alpha b_\beta. \tag{E.33}$$

E.5.3 More general perturbations with a single semishort multiplet

The most general perturbation $\mathcal{O}_{[0]}$ you can write down made from the bosonic oscillator, preserving the R-symmetry, is

$$\mathcal{O}_{[0]} \coloneqq \frac{\mathcal{E}}{p^2}\left(a^{\ddagger}\right)^p a^p. \tag{E.34}$$

So then

$$\begin{aligned} \mathcal{E}^{-1}\mathcal{O}_{[2]} &= \left(\frac{2}{p} - \frac{1}{p^2}\right)\left(a^{\ddagger}\right)^p a^p + b^{\ddagger\beta}\left(a^{\ddagger}\right)^{p-1} a^{p-1} b_\beta, \\ \mathcal{E}^{-1} q_\alpha &= \left(1 - \frac{1}{p}\right)\left(a^{\ddagger}\right)^{p-1} b^{\ddagger}_\alpha a^p - (p-1) b^{\ddagger\beta} b^{\ddagger}_\alpha \left(a^{\ddagger}\right)^{p-2} a^{p-1} b_\beta, \\ \mathcal{E}^{-1} q^{\ddagger}_\alpha &= \left(1 - \frac{1}{p}\right)\left(a^{\ddagger}\right)^p b_\alpha a^{p-1} - (p-1) b^{\ddagger\beta}\left(a^{\ddagger}\right)^{p-1} a^{p-2} b_\alpha b_\beta \\ \mathcal{E}^{-1}\Delta' &= 2(p-1) b^{\ddagger\alpha}\left(a^{\ddagger}\right)^{p-1} a^{p-1} b_\alpha + \left(2 - \frac{2}{p}\right)\left(a^{\ddagger}\right)^p a^p \\ &\quad + (p-1)^2 b^{\ddagger\beta} b^{\ddagger\alpha}\left(a^{\ddagger}\right)^{p-2} a^{p-2} b_\alpha b_\beta. \end{aligned} \tag{E.35}$$

For $p = 1$, note that q_α, $q^{\ddagger}_\alpha$ and Δ' vanish, as we found earlier.

E.5.4 The BPS zero mode multiplet

Now we introduce the BPS zero mode. Let us call it z, but we shall think of it as corresponding to ϕ_0, up to a constant. From the point of view of our small superalgebra, this operator z is actually a rather funny object. It has $J = 1/2$ and at the free level, it has $\Delta = 1/2$ too. This means it commutes with $\Delta - J$. Since it is a BPS primary field, it also commutes with the $\mathbf{Q}$ and $\mathbf{Q}^\ddagger$ as well. So, from the point of view of the small superalgebra, z is really just a c-number. However, on the other hand, z only commutes with $\Delta - J$, and not with Δ and J individually. So, since we have not yet specified the normalization of z, let us define it so that

$$\left[z^\ddagger, z\right] = 1. \tag{E.36}$$

Note that this is only possible in a unitary theory because z is the energy-raising, rather than the energy-lowering part of ϕ.

So, z has the same R-charge as a, but the same frequency as $a^\ddagger$. The composite object $\hat{A}^\ddagger := z a^\ddagger$ has frequency r^{-1} and vanishing R-charge. We can therefore make new interesting perturbations out of this operator.

Since $z, z^\ddagger$ commute with the whole superalgebra, $\hat{A}$ and $\hat{A}^\ddagger$ have the same supersymmetry representations as $a, a^\ddagger$ respectively. Defining $\hat{B}^\ddagger_\alpha := z b^\ddagger_\alpha$, we have

$$\begin{aligned}
\mathbf{Q}_\alpha \cdot \hat{A} &= 0, & \mathbf{Q}^\ddagger_\alpha \cdot \hat{A} &= -\hat{B}_\alpha, \\
\mathbf{Q}_\alpha \cdot \hat{A}^\ddagger &= \hat{B}_\alpha, & \mathbf{Q}^\ddagger_\alpha \cdot \hat{A}^\ddagger &= 0, \\
\mathbf{Q}_\alpha \cdot \hat{B}_\beta &= \delta_{\alpha\beta}\hat{A}, & \mathbf{Q}^\ddagger_\alpha \cdot \hat{B}_\beta &= 0, \\
\mathbf{Q}_\alpha \cdot \hat{B}^\ddagger_\beta &= 0, & \mathbf{Q}^\ddagger_\alpha \cdot \hat{B}^\ddagger_\beta &= \delta_{\alpha\beta}\hat{A}^\ddagger.
\end{aligned} \tag{E.37}$$

E.5.5 General bosonic perturbations involving a semishort and a BPS multiplet

So now we can make all sorts of fascinating R-symmetric perturbations such as

$$\mathcal{O}_{[0]} = \frac{1}{pq}\left(\hat{A}^\ddagger\right)^q \hat{A}^p. \tag{E.38}$$

This is non-Hermitean, but we can always add the Hermitean conjugate to make it Hermitean. So we have

$$\widetilde{\mathbf{Q}}_\alpha \cdot \mathcal{O}_{[0]} = \frac{1}{p}\hat{B}^\ddagger_\alpha \left(\hat{A}^\ddagger\right)^{q-1} \hat{A}^p, \tag{E.39}$$

and

$$\widetilde{\mathbf{Q}}^{\ddagger\alpha} \cdot \widetilde{\mathbf{Q}}_\alpha \cdot \mathcal{O}_{[0]} = \frac{2}{p}\left(\hat{A}^\ddagger\right)^q \hat{A}^p + \hat{B}^{\ddagger\alpha}\left(\hat{A}^\ddagger\right)^{q-1} \hat{A}^{p-1}\hat{B}_\alpha. \tag{E.40}$$

One major difference, now that z and $z^\ddagger$ have been introduced, is that R-neutral operators no longer necessarily commute with Δ. In particular we have

$$\begin{aligned} \left[\Delta, \hat{A}^\ddagger\right] &= \hat{A}^\ddagger, & \left[\Delta, \hat{A}\right] &= -\hat{A}, \\ \left[\Delta, \hat{B}^\ddagger_\alpha\right] &= \frac{3}{2}\hat{B}^\ddagger_\alpha, & \left[\Delta, \hat{B}_\alpha\right] &= -\frac{3}{2}\hat{B}_\alpha, \end{aligned} \tag{E.41}$$

and so

$$\left[\Delta, \left(\hat{A}^\ddagger\right)^q \hat{A}^p\right] = (q-p)\left(\hat{A}^\ddagger\right)^q \hat{A}^p. \tag{E.42}$$

The expressions for $\mathcal{O}_{[2]}$ and the perturbed generators are:

$$\begin{aligned} \mathcal{O}_{[2]} &= \frac{p+q-1}{pq}\left(\hat{A}^\ddagger\right)^q \hat{A}^p + \hat{B}^{\ddagger\alpha}\left(\hat{A}^\ddagger\right)^{q-1}\hat{A}^{p-1}\hat{B}_\alpha, \\ \mathbf{q}_\alpha &= \frac{q-1}{p}\hat{B}^{\ddagger\alpha}\left(\hat{A}^\ddagger\right)^{q-1}\hat{A}^p - (q-1)\hat{B}^\beta\hat{B}_\alpha\left(\hat{A}^\ddagger\right)^{q-2}\hat{A}^{p-1}\hat{B}_\beta, \\ \mathbf{q}^\ddagger_\alpha &= \frac{p-1}{q}\hat{B}_\alpha\left(\hat{A}^\ddagger\right)^q\hat{A}^{p-1} - (p-1)\hat{B}^{\ddagger\beta}\left(\hat{A}^\ddagger\right)^{q-1}\hat{A}^{p-2}\hat{B}_\alpha\hat{B}_\beta, \\ \Delta' &= (p+q-2)\hat{B}^{\ddagger\alpha}\left(\hat{A}^\ddagger\right)^{q-1}\hat{A}^{p-1}\hat{B}_\alpha + \left(\frac{q}{p} + \frac{p}{q} - \frac{p+q}{pq}\right)\left(\hat{A}^\ddagger\right)^q\hat{A}^p \\ &\quad + (p-1)(q-1)\hat{B}^{\ddagger\beta}\hat{B}^{\ddagger\alpha}\left(\hat{A}^\ddagger\right)^{q-2}\hat{A}^{p-2}\hat{B}_\alpha\hat{B}_\beta. \end{aligned} \tag{E.43}$$

Here, we assume that neither p nor q vanishes; otherwise (E.38) is not well-defined. If we had defined

$$\mathcal{O}_{[0]} := \left(a^\ddagger\right)^q a^p, \tag{E.44}$$

then we would have had

$$\begin{aligned} \Delta' &= pq(p+q-2)\hat{B}^{\ddagger\alpha}\left(\hat{A}\right)^{q-1}\hat{A}^{p-1}\hat{B}_\alpha + \left(q^2+p^2-p-q\right)\left(\hat{A}\right)^q\hat{A}^p \\ &\quad + (p-1)(q-1)\hat{B}^{\ddagger\beta}\hat{B}^{\ddagger\alpha}\left(\hat{A}^\ddagger\right)^{q-2}\hat{A}^{p-2}\hat{B}_\alpha\hat{B}_\beta. \end{aligned} \tag{E.45}$$

E.5.6 Protection of the semishort state

Now we see, regardless of the form of the perturbation, the Hamiltonian perturbation not only has vanishing expectation value in the semishort state, it simply annihilates the entire semishort state. This much stronger condition would seem to guarantee the protection of the semishort multiplet not just to first order, but to all orders in perturbation theory.

The nonzero modes of the free antichiral superfield $\bar{\phi}$ and the free chiral superfield ϕ should also be included; from the point of view of quantum mechanics, these are higher-spin multiplets, obtained by Kaluza–Klein reduction of the $2+1$-dimensional superfields on the spatial sphere.

F Massless scalar propagator

Define the free massless complex scalar field ϕ to have kinetic term

$$\mathcal{L} = (\partial_\mu \phi)(\partial_\mu \phi), \tag{F.1}$$

in Euclidean signature. The massless euclidean scalar propagator on $\mathbb{R}^4$ is defined as

$$\Delta_{\text{unit}}(x, y) \coloneqq \left\langle \phi(x)\bar{\phi}(y)\right\rangle_{\mathbb{R}^4}. \tag{F.2}$$

Given (F.1), the Ward identity yields the equation of motion for the propagator

$$\frac{\partial^2}{\partial x_\mu \partial x_\mu}\Delta_{\text{unit}}(x, y) = \frac{\partial^2}{\partial y_\mu \partial y_\mu}\Delta_{\text{unit}}(x, y) = -\delta^{(4)}(x - y), \tag{F.3}$$

so the propagator for ϕ has normalization

$$\Delta_{\text{unit}}(x, y) = (2\pi)^{-2}|x - y|^{-2}, \tag{F.4}$$

where we have used the identity

$$\frac{\partial^2}{\partial x_\mu \partial x_\mu}|x - y|^{-2} = -(2\pi)^2 \delta^{(4)}(x - y) \tag{F.5}$$

on $\mathbb{R}^4$. More generally, for a massless complex scalar field normalized as

$$\mathcal{L}_{\mathbf{M}} \coloneqq \mathbf{M}^2(\partial_\mu \phi)(\partial_\mu \phi) \tag{F.6}$$

the scalar propagator is

$$\left\langle \phi(x)\bar{\phi}(y)\right\rangle_{\mathbb{R}^4} = (2\pi \mathbf{M})^{-2}|x - y|^{-2}, \tag{F.7}$$

for any positive real $\mathbf{M}$. In particular, for the A-field of the effective Abelian vector multiplet, whose kinetic term is (4.11), the two-point function is

$$\left\langle A(x)\bar{A}(y)\right\rangle_{\mathbb{R}^4} = \frac{g_{\text{eff}}^2}{(2\pi)^2}|x - y|^{-2} = (\pi\, \mathrm{Im}\, \tau)^{-1}|x - y|^{-2}. \tag{F.8}$$

G Geometry of the four-sphere

The d-sphere of radius r is a symmetric space, so its Riemann tensor satisfies

$$\mathcal{R}_{\mu\nu\rho\sigma} = \frac{1}{\mathsf{r}^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{G.1}$$

So we have

$$\mathcal{R}_{\mu\rho} = g^{\nu\sigma}\mathcal{R}_{\mu\nu\rho\sigma} = \frac{d-1}{\mathsf{r}^2}g_{\mu\rho}, \qquad \mathcal{R} = g^{\mu\rho}\mathcal{R}_{\mu\rho} = \frac{d(d-1)}{\mathsf{r}^2}. \tag{G.2}$$

Now let us calculate the Euler density, according to Komargodski–Schwimmer's normalization convention (4.64). The square of the Riemann tensor, Ricci tensor and Ricci scalar are

$$\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} = \frac{2d(d-1)}{\mathsf{r}^4}, \qquad \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} = \frac{d(d-1)^2}{\mathsf{r}^4}, \qquad \mathcal{R}^2 = \frac{d^2(d-1)^2}{\mathsf{r}^4}. \tag{G.3}$$

Komargodski–Schwimmer's normalization of the Euler density is given in (4.64). For the four-sphere of radius r,

$$E_4^{[\mathrm{KS}]} = \frac{24}{\mathsf{r}^4}. \tag{G.4}$$

H Weyl anomaly

H.1 Basics

The stress tensor must be symmetric and traceless in CFTs in flat space. However, it is well known that in curved space in even dimensions the tracelessness of the stress tensor can be broken by the so-called Weyl anomaly [84, 85] (see also [76, 154–157] for reviews). It is known that in odd dimensions there is no Weyl anomaly.

In general, the one-point function of the stress tensor $\langle T_{\mu\nu}\rangle$ must vanish in flat space due to the Poincaré invariance, but it does not have to in a curved space with a nontrivial metric. One can derive the most general form of the Weyl anomaly $\langle T_\mu{}^\mu\rangle$ [158] from the Wess–Zumino consistency condition [159]. In four dimension, it is of the form

$$\langle T_\mu{}^\mu\rangle \propto aE_4 - cW_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}, \tag{H.1}$$

where E_4 is the Euler tensor of the curved background,

$$E_4 \coloneqq \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2, \tag{H.2}$$

and the square of the Weyl tensor $W_{\mu\nu\rho\sigma}$ is

$$W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 2\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \frac{1}{3}\mathcal{R}^2. \tag{H.3}$$

The a- and c-coefficients appearing in (H.1) differ in different theories. In flat space, the values of a- and c-coefficients can be obtained from the two- and three-point functions of the stress tensor [160]. Especially in SCFTs, there are techniques to exactly evaluate the values of a and c [161–163]. Also, it is shown in [5] that under a renormalization group flow from a unitary UV CFT to a unitary IR CFT, the a-coefficient is always monotonically decreasing, *i.e.*,

$$\Delta a \coloneqq a_{\mathrm{UV}} - a_{\mathrm{IR}} \geq 0. \tag{H.4}$$

This property is known as the a-theorem.

H.2 Conventions and values for the a-anomaly coefficient

Here we compare two conventions for the normalization of the a-coefficient of the Weyl anomaly (and also the c-coefficient), and we determine values of the coefficients in various $\mathcal{N} = 2$ SCFTs of interest. We also give a definition of the α-coefficient which does not depend on the normalization of the a-coefficient.

H.2.1 Translation between normalization conventions in [5] vs. [6]

The a- and c-coefficients are normalized differently in different parts of the literature. We can match by comparing anomalies for a given physical system across conventions. The simplest case is a free scalar field. In [5] the anomalies are normalized so that the contributions of a single real massless scalar field, are

$$a^{[\text{KS}]}_{\text{real massless scalar}} = \frac{1}{90(8\pi)^2}, \qquad c^{[\text{KS}]}_{\text{real massless scalar}} = \frac{1}{30(8\pi)^2}. \tag{H.5}$$

This normalization is given below (A.6) of [5]. In [6], the authors give the anomalies of a single real massless scalar field, as

$$a^{[\text{AEFJ}]}_{\text{real massless scalar}} = \frac{1}{360}, \qquad c^{[\text{AEFJ}]}_{\text{real massless scalar}} = \frac{1}{120}. \tag{H.6}$$

The relation between the two normalizations is therefore

$$(a,c)^{[\text{KS}]} = \frac{1}{16\pi^2}(a,c)^{[\text{AEFJ}]} \tag{H.7}$$

In the body of the thesis we indicate our conventions to avoid ambiguity, but we shall use the convention of [6], since it is normalized such that the anomalies of free fields, and of all $\mathcal{N} = 2$ SCFT, are rational numbers.

H.2.2 Values of the anomaly coefficient in various $\mathcal{N} = 2$ SCFTs in four dimensions

We have defined the exponent α in (4.4), which appears in the factor n^α in the asymptotic formula for the two-point function, in terms of the a-coefficient in the Weyl anomaly. The Weyl anomaly does not have a universally used normalization in the literature. So in order to find actual values for α, we need to use some particular conventions.

The a-coefficients for many Lagrangian and non-Lagrangian theories, have been given in *e.g.*, [6, 162], and we collect the relevant results here. Those authors normalize the a-coefficient according to the widely-used convention in [6], in which we have

$$a^{[\text{AEFJ}]}_{\text{fvm}_G} = \frac{5}{24}\dim G, \qquad a^{[\text{AEFJ}]}_{\text{fhm}_{\mathsf{R}}} = \frac{1}{24}\dim_{\mathbb{C}} \mathsf{R}, \tag{H.8}$$

where a_{vm_G} is the a-coefficient for the $\mathcal{N} = 2$ free vector multiplet of gauge group G, and $a_{\text{fhm}_{\mathsf{R}}}$ is the a-coefficient for the $\mathcal{N} = 2$ free hypermultiplet in some representation R of a symmetry group.

Value of the a-coefficient for $\mathcal{N}=4$ super-Yang–Mills. Organizing into $\mathcal{N}=4$ vector multiplets of gauge group G, we have

$$a^{[\text{AEFJ}]}_{\text{CFT},\,\mathcal{N}=4} = \frac{1}{4}\dim G, \tag{H.9}$$

and its Coulomb branch effective theory has

$$a^{[\text{AEFJ}]}_{\text{EFT},\,\mathcal{N}=4} = \frac{1}{4}\operatorname{rank} G, \tag{H.10}$$

so

$$\Delta a^{[\text{AEFJ}]}_{\mathcal{N}=4} = \frac{1}{4}(\dim G - \operatorname{rank} G). \tag{H.11}$$

For $\mathrm{SU}(N)$ gauge group, we have

$$\Delta a^{[\text{AEFJ}]}_{\mathrm{SU}(N)\,\mathcal{N}=4} = \frac{1}{4}\big(N^2 - N\big). \tag{H.12}$$

In particular, for $N=2$ we have

$$\Delta a^{[\text{AEFJ}]}_{\mathrm{SU}(2)\,\mathcal{N}=4} = \frac{1}{2}, \qquad \alpha_{\mathrm{SU}(2)\,\mathcal{N}=4} = 1. \tag{H.13}$$

Value of the a-coefficient for conformal $\mathcal{N}=2$ SQCD. For $\mathcal{N}=2$ SQCD with gauge group $\mathrm{SU}(N_c)$ and N_f fundamental hypermultiplets at weak coupling, we have

$$a^{[\text{AEFJ}]}_{\text{UV, SQCD}} = \frac{5}{24}\big(N_c^2 - 1\big) + \frac{1}{24}N_f N_c. \tag{H.14}$$

In the superconformal case, $N_f = 2N_c$ and we have

$$a^{[\text{AEFJ}]}_{\text{CFT, SCQCD}} = \frac{7}{24}N_c^2 - \frac{5}{24}. \tag{H.15}$$

The moduli space effective theory consists of (N_c-1) free Abelian vector multiplets and no hypers, so we have

$$a^{[\text{AEFJ}]}_{\text{EFT, SCQCD}} = \frac{5}{24}(N_c - 1), \tag{H.16}$$

so the difference in central charge is

$$\Delta a^{[\text{AEFJ}]}_{\text{SCQCD}} = a^{[\text{AEFJ}]}_{\text{CFT, SCQCD}} - a^{[\text{AEFJ}]}_{\text{EFT, SCQCD}} = \frac{1}{24}\big(7N_c^2 - 5N_c\big). \tag{H.17}$$

In particular for $N_c = 2$, we have

$$\Delta a^{[\text{AEFJ}]}_{\mathrm{SU}(2)\text{ SCQCD}} = \frac{3}{4}, \qquad \alpha_{\mathrm{SU}(2)\text{ SCQCD}} = \frac{3}{2}. \tag{H.18}$$

H.2.3 Convention-independent formula for the α-coefficient

We would like to define the α-coefficient in a convention-independent way, as a ratio of the a-anomalies. Our convention-independent formula is:

$$\alpha = \frac{5(a_{\mathrm{CFT}} - a_{\mathrm{EFT}})}{12a_{\mathrm{favm}}}, \tag{H.19}$$

where a_{favm} is the unit of a-anomaly contribution carried by a free $\mathcal{N}=2$ vector multiplet for a U(1) gauge group. In order to actually compute the value of α for some theories of interest, we must pick an actual normalization convention. The value of α in the convention of [6] is

$$\alpha = 2\left(a_{\mathrm{CFT}}^{[\mathrm{AEFJ}]} - a_{\mathrm{EFT}}^{[\mathrm{AEFJ}]}\right), \tag{H.20}$$

and in the convention of [5] it is

$$\alpha = \frac{1}{8\pi^2}\left(a_{\mathrm{CFT}}^{[\mathrm{KS}]} - a_{\mathrm{EFT}}^{[\mathrm{KS}]}\right). \tag{H.21}$$

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