Application of chiral anomaly to the effective theories of one dimensional interacting electron systems

一次元電子系における量子異常とその応用

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Thesis

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Contents

Chapter 1

Introduction

1.1 Motivations of this thesis

It is well accepted that the investigation of low dimensional interacting electron systems often requires the use of nonperturbative methods that can account for correlation effects which are generally unaccessible from conventional mean field theory (MFT) schemes. In one dimension (with which we will mainly be concerned with), this necessity has lead to the common use of such powerful theoretical tools as bosonization and conformal field theory [1][2]. The basic idea involved there is to start from the free (noninteracting) fermion system, and to introduce perturbing operators consistent with the symmetry requirements of the problem, which are treated by, e.g. the renormalization group.
It should be noted though, that this method is basically a weak coupling analysis. It is also worth mentioning that nonabelian bosonization and conformal field theory are founded on beautiful but rather sophisiticated mathematics - which are unique to application in one-dimensional systems, and have no obvious generalization to physics in higher dimensions - which may sometimes blur the physical context of the original condensed matter problems. For these reasons, there are cases when it is desirable to begin rather from an anticipated strong coupling fixed point, by an appropriate MFT, and then add the effects of fluctuations, which is closer in spirit to the traditional methods used in condensed matter physics. As mentioned above, though, the strong fluctuation characteristic to one dimension generally makes this approximation notoriously unreliable.
In this thesis, we will concentrate on a class of 1+1 dimensional models which may be represented by the following general Lagrangian density:

$$\mathcal{L} = \bar{\Psi}\left[\partial\!\!\!/ + mP_+U + mP_-U^\dagger\right]\Psi. \tag{1.1}$$

Since the relativistic notations involving Dirac fermions will be fixed in due course, we simply note that $P_\pm \equiv \frac{1}{2}(1 \pm \gamma^5)$ are the projection operators to the right ($\gamma^5 = 1$) and left ($\gamma^5 = -1$) chirality sectors. The external field U is generally a unitary matrix, [1] and m is a real field (in most cases taken to be a constant) with the dimension of a mass. In general the unitary matrix $U \in$SU(n) may be written as $U = e^{it^a\phi_a}$ where $\{t^a\}$ ($a = 1, \cdots n^2 - 1$) are generators of the SU(n) Lie algebra. Using this expression, it is often useful to rewrite eq.(1.1) in the following way:

$$\mathcal{L} = \bar{\psi}\left[\partial\!\!\!/ + me^{i\gamma^5 t^a\phi_a}\right]\Psi. \tag{1.2}$$

This is a Dirac fermion with a chirally rotated mass term. The chiral exponential factor in the mass term, which in general is nonabelian, will be given its precise mathematical definition in section 2.2. A Lagrangian density bearing this general form is often encountered in condensed matter applications when making a continuum approximation of a lattice model. Depending on

[1] We will later on be using the same notation U for the onsite interaction of the Hubbard and Peierls-Hubbard model. Which of the two notations it is meant to stands for should be apparent from the context of the discussion.

the mathematical property of the matrix U, it can describe various physical situations, such as the examples listed in Table 1.1. This table may be viewed as consisting of three large categories: $U \in \mathrm{U}(1)$, $U \in \mathrm{SU}(2)$, and $U \in \mathrm{SU}(2n)$. The procedures which are responsible for the appearance of the relativistic model eq.(1.1) are different for the three categories.

For the first group (abelian case) of systems, it simply appears by taking a straightforward continuum limit of the original lattice Hamiltonian, which is a noninteracting model. [2] For the second class ($U \in$SU(2)), it arises as a result of a suitable Hubbard-Stratonovich decoupling of the onsite interaction term of the Hubbard model. [3] In the last group, it arises after replicating (and decoupling the replication-generated interaction of) the original noninteracting electron system.

Table 1.1: Examples of condensed matter problems where eq.(1.1) arises

Property of U	*Applications*
$U \equiv 1$	transpolyacethylene[3]
$U \in \mathrm{U}(1)$	diatomic polymer chain [4]
$U \in \mathrm{SU}(2)$, $U = i\vec{\sigma}\cdot\vec{n}$, $\lvert\vec{n}\rvert = 1$	Hubbard chain[5] 1d (multiband) FM Kondo lattice[6] 1d double exchange[7]
$U \in \mathrm{SU}(2)$, generic	Peierls-Hubbard model[8] (discussed in present thesis)
$U \in \mathrm{SU}(2n)$, $n \to 0$	IQHE[9][10][11] impurity effect in 2-d d-wave superconductors [12] random flux localization in 2-d [13]

In this thesis, we will, starting from chapter 2, be focusing on the case with internal SU(2) symmetry. There are several good reasons for taking this particular choice:

1. The abelian case, being a noninteracting problem, has been thoroughly studied in the literature.

2. The third group of systems has additional complications and possible ambiguity coming from taking the replica limit, which should be distinguished from the issue of properly handling a system with $n \neq 0$.

3. Since the field-theoretical treatment of the nonabelian problem has *not* been extensively worked out from the scope we have described, it is natural to study the simplest case where the symmetry is SU(2). In regards to condensed-matter applications, it is important that this case corresponds to the Hubbard model and its variants, which are unarguably among the most fundamental models for studying interacting electrons in one dimension.

When U is a generic SU(2) matrix, the system descibed by eq.(1.1) or eq.(1.2) has a broken parity-symmetry. As we will show in Chapter 2, this case corresponds to the continuum model for the Peierls-Hubbard model, which is a Hubbard model with alternating hopping amplitudes. The noninvariance of this model with respect to spatial inversion about a given lattice site is reflected in this P-breaking. Since the translationally-invariant Hubbard model can also be recovered by simply putting the alternation to zero, we will be dealing with the generic (broken P) case. Beside this merit of being the generic class of models for the SU(2) case, it is also of considerable physical interest, since at half filling, its ground state is a dimerized spin chain.[4]

Our aim will be to develop, using path integral methods, a MFT+fluctuation type prescription which will yield at least qualitatively reliable results. For instance, the presence/absence of a spin gap, which is not correctly predicted within usual perturbative methods of this sort (we refer to

[2] to be described in section 2.1

[3] to be described in section 2.2

[4] Since the Peierls instability sets in even in the absence of interactions, we will distinguish the ground state of the Peierls-Hubbard model from a spin-Peierls state.

section 2.1 for details) will be properly reproduced. The central feature of the method will be the incorporation (on top of the saddle point solution (MFT)) of nonperturbative informations through *topologically nontrivial fluctuation of the order parameter*, which enters the effective theory by carefully accounting for the chiral anomalies of the Dirac fermions (the latter arises as an intermediate continuum model). The scheme should be contrasted with the usual nonabelian bosonization method: there, the topological Wess-Zumino-Witten term is present at the level of the free fermion, and its fluctuation on a manifold with a large symmetry is later regulated by addition of symmetry breaking perturbations. In the present method, in contrast, the symmetry is broken at the outset, and low energy fluctuations (both topogically trivial and nontrivial) are subsequently taken into consideration. It is reassuring, that where the two methods overlap, their results are seen to agree. Hence our method may be viewed as a simple alternative to conventional nonabelian bosonization, and we expect that a combined use of the two complementary methods should prove to be useful to obtain insight into these systems.

In addition to this, the sequence of mapping: lattice fermion model $\rightarrow$ Dirac fermion $\rightarrow$ effective nonlinear sigma type model, that is established here enables us to conveniently identify various quantum numbers associated with excitations that are induced by the topological fluctuation of the order parameter. This is a natural extension of the induced fermion numbers which were extensively studied for linearly conjugated polymers (see section 1.5), and owing to the nonabelian nature of our systems, we will find that here they come in a somewhat richer variety.

1.2 Organization of this thesis

This thesis is organized as follows. In the remaining part of this Chapter, we will provide a short review on chiral anomaly, and will introduce in some details the path-integral approach to this phenomenna, which was developed by Fujikawa. Then, taking the massless Schwinger model as an example, we will illustrate how the Fujikawa method can be used to bosonize a field theory in the path integral framework. Unfortunately, it turns out that this convenient bosonization approach (which is extensively used in the field-theoretical literature), *cannot* straightforwardly be applied to the models which we are interested in, and several nontrivial extensions and modifications must be performed. This point is also explained here, as well as in the later chapters.

In Chapter 2, we will introduce the Peierls-Hubbard model and obtain the relativistic fermionic model which arises by doing a suitable Hubbard-Stratonovich decoupling, and using the continuum approximation. We will emphasize the similarity of the resulting model with that for the diatomic polymer chain problem. This will be the starting point of our derivation of the effective bosonic theory, using chiral anomaly.

Chapter 3 constitutes the main material of this thesis. Here we perform a careful derivation of the low energy effective theory by a combined use of the standard derivative expansion and the Fujikawa approach to deriving anomalous contributions. We find some notable differences from results obtained by other authors who attempted similar calculations. We find that this discrepancy can be traced to a inappropriate evaluation of the anomaly in previous works. Furthermore, we will show that our result can be reproduced in the framework of conventional nonabelian bosonization, using a refermionization technique developed by Tsvelik. This agreement confirms the validity of our methods.

As an application and further consistency-check, we explore in Chapter 4 the possibility of various forms of induced fermion numbers which can arise in our fermionic model. These are studied by use of chiral anomaly, and we obtain the response of the charge sector at wavevector $k=\pi$ to instanton excitations in the spin sector, and study the formation of a localized spinon in the vicinity of a nonmagnetic impurity.

Chapter 5 is devoted to discussions, and concluding remarks are made in Chapter 6. There are some appendices which supplement the main text.

1.3 Brief review on chiral anomaly

Chiral anomaly refers to the impossibility, at the quantum level, to simultaneously impose vector and chiral gauge invariances on a relativistic (Dirac) fermion theory. Since these are generally a symmetry of the theory at the level of the classical Lagrangian, the anomaly is a purely quantum effect. In this section, we will review some aspects of this phenomenon which will be relevant to our subsequent discussions.

1.3.1 The Adler-Bell-Jackiw anomaly

Although foreseen in earlier works by Fukuda and Miyamoto[14], Steinberger[15], and Schwinger[16], chiral anomaly as it is known today came into full notice in the works of Adler[17], and Bell and Jackiw[18] (ABJ) in the late 1960s, in the context of the so-called PCAC (partial conservation of axial-vector current) properties of (four-dimensional) spinor electrodynamics, and its relevance to the two-photon decay of the neutral pion $\pi^0 \longrightarrow \gamma\gamma$. Without going into technical details (which will be given later when we discuss the functional integral approach developed by Fujikawa[19]), we will briefly review their results and thereby introduce the relevant concepts which are involved in the physics of chiral anomaly.

Thus consider the Lagrangian density for the four dimensional QED in Minkowski space-time, [5]

$$\mathcal{L} = \bar{\psi}[\mathrm{i}\gamma^\mu\partial_\mu - m]\psi + A^\mu j_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{1.3}$$

where ψ is a four-component spinor (particle and antiparticles with two helicities for each), $\bar{\psi} \equiv \psi^\dagger\gamma_0$, $j \equiv \mathrm{i}\bar{\psi}\gamma_\mu\psi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The gamma matrices satisfy the anticommutation relation $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, where the metric is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Let us now inspect the symmetry properties of this system. Introducing the matrix $\gamma^5 \equiv \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$, which satisfies the anticommutation relation $\{\gamma^5, \gamma_\mu\} = 0$, the projection of the fermions onto sectors with right and left helicity is made as

$$\begin{aligned} P_+\psi &\equiv \frac{1}{2}(1+\gamma^5)\psi \equiv R \\ P_-\psi &\equiv \frac{1}{2}(1-\gamma^5)\psi \equiv L. \end{aligned} \tag{1.4}$$

Chirality is defined as the eigenvalues ± 1 of γ^5. Hence for d=4, helicity and chirality may be considered as being identical concepts. [6] The 4-current j_μ can be expressed in terms of R and L as

$$\begin{aligned} j_\mu &= \mathrm{i}\psi^\dagger(P_+ + P_-)\gamma_0\gamma_\mu(P_+ + P_-)\psi \\ &= \mathrm{i}\psi^\dagger P_+\gamma_0\gamma_\mu P_+\psi + \mathrm{i}\psi^\dagger P_-\gamma_0\gamma_\mu P_-\psi \\ &\equiv \mathrm{i}\bar{R}\gamma_\mu R + \mathrm{i}\bar{L}\gamma_\mu L. \end{aligned} \tag{1.5}$$

Here we have used the relations $P_\pm\gamma_\mu = \gamma_\mu P_\mp$ and $P_\pm P_\mp = 0$. Likewise, we obtain

$$\bar{\psi}\gamma_\mu\partial_\mu\psi = \bar{R}\gamma_\mu\partial_\mu R + \bar{L}\gamma_\mu\partial_\mu L. \tag{1.6}$$

In the massless limit ($m \to 0$), it is clear from the above two equations that the Lagrangian posseses the symmetry $\mathrm{U}(1)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{L}$, i.e. the invariance with respect to seperate local U(1) gauge transformations of the right and left sectors:

$$\begin{aligned} &R \to e^{i\theta_R}R, \bar{R} \to \bar{R}e^{i\theta_R} \\ &L \to e^{i\theta_L}L, \bar{L} \to \bar{L}e^{i\theta_L}, \end{aligned} \tag{1.7}$$

[5] Our discussion will be given in Minkowki space-time only in this subsection; a more detailed list of our conventions will be presented when we start our discussions in the following subsection in the Euclidean formalism, where we actually begin to perform the manipulations.

[6] For d=2, there are no spins - in the strictly relativistic sense - so that this identification cannot be made. However, in that case an even more straightforward physical interpretation, in terms of right and left-moving dispersion branches, can be assigned to the right ($\gamma^5 = 1$) and left ($\gamma^5 = -1$) chirality sectors.

provided that the gauge field A_μ absorbs the terms arising from the phase change, by transforming as

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu(\theta_R + \theta_L). \tag{1.8}$$

Reorganizing the phases θ_R, θ_L as

$$\begin{aligned}\frac{1}{2}(\theta_R + \theta_L) &\equiv \theta_V \\ \frac{1}{2}(\theta_R - \theta_L) &\equiv \theta_A,\end{aligned} \tag{1.9}$$

where the subscripts V and A stand respectively for "vector" and "axial", the transformation of eq.(1.7) is recasted as

$$\begin{aligned}\psi &\to e^{i\theta_V + i\gamma^5\theta_A}\psi \\ \bar\psi &\to \bar\psi e^{-i\theta_V + i\gamma^5\theta_A}.\end{aligned} \tag{1.10}$$

Hereafter in this thesis, the part of the transformation involving $e^{i\theta_V} \in U(1)_V$ ($e^{i\gamma^5\theta_A} \in U(1)_A$) will be referred to as the vector (axial) transformation. According to Noether's theorem, the $\mathrm{U(1)_R} \times \mathrm{U(1)_L}$ symmetry present in the massless limit implies that in this limit, the fermion numbers for the right and left chirality sectors $R^\dagger R$ and $L^\dagger L$ are conserved seperately. From eq.(1.10), it follows that this symmetry may also be written as $\mathrm{U(1)_V} \times \mathrm{U(1)_A}$. The Noether current associated with $\mathrm{U(1)_V}$ is the current j_μ, while that for the $\mathrm{U(1)_A}$ is the *axial current* $j^5_\mu \equiv \bar\psi\gamma_\mu\gamma^5\psi$. Hence, $\mathrm{U(1)_V}$ invariance implies $\partial_\mu j_\mu = 0$ (conservation of the charge $R^\dagger R + L^\dagger L$) while $\mathrm{U(1)_A}$ invariance implies $\partial_\mu j^5_\mu = 0$ (conservation of the difference $R^\dagger R - L^\dagger L$). At the level of the classical Euler-Lagrange equations, both currents are conserved in the massless limit (of these two, the current j_μ is, of course conserved regardless of whether the mass is zero or not).

Quantum corrections to these conservation laws can be inspected by deriving the corresponding Ward-Takahashi identities (WTI), using the fermion propagator. In the diagrammtic analysis, the portion which is potentially nontrivial comes from diagrams which yield ultraviolet divergences, which must be properly regularized in a gauge-invariant manner. Since we are in four dimensions, it is obvious from power-counting that these divergences arise in loop diagrams with four or less vertices. Carrying out these procedures, ABJ found, that while $\partial_\mu j_\mu = 0$ is fully respected (as it should be) at the quantum level, the triangle diagram with two vector and one axial vertices (see Fig.1.1) modifies the conservation law for the axial current j^5_μ. They obtained

$$\begin{aligned}\partial^\mu j^5_\mu - 2im\bar\psi\gamma^5\psi &= \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda} \\ &= \frac{e^2}{2\pi^2}\vec E \cdot \vec B.\end{aligned} \tag{1.11}$$

The second contribution on the LHS of eq.(1.11) comes from the mass term, $m\bar\psi\psi$. That this piece should arise is obvious at the classical level, since the mass term is not invariant under a chiral transformation; $\psi \to e^{i\gamma^5\alpha}\psi, \bar\psi \to \bar\psi e^{i\gamma^5\alpha} \Longrightarrow m\bar\psi\psi \to m\bar\psi e^{2i\gamma^5\alpha}\psi$. In the massless limit $m = 0$, however, sectors with left and right chirality decouple, so naively the RHS was expected to vanish. The nonvanishing contribution on the RHS is the chiral anomaly. The RHS of eq.(1.11) is a total derivative, i.e. of the form $\partial^\mu G_\mu$, where the current $G_\mu \equiv \frac{e^2}{16\pi^2}\epsilon_{\mu\nu\rho\lambda}A^\nu F^{\rho\lambda}$ is the abelian analogue of what is known in topological field theory as a Chern-Simons class. Therefore it is tempting to redefine $K_\mu \equiv j_\mu - G_\mu$ as a new conserving chiral current, which gives the impression that the anomaly is resolved. However, this cannot be taken at face-value, because K_μ is gauge-noninvariant, and hence is not a physical observable. This leads to the conclusion that the vector and axial symmetries cannot be simultaneously preserved when quantizing the theory, and the anomaly is genuine. Similar types of chiral anomalies have also been found for nonabelian theories, and are generally known to arise in even space-time dimensions (d=2, d=4, *etc.*).

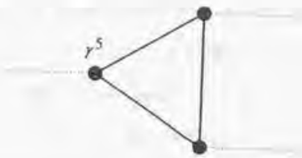

Figure 1.1: The triangle diagram of Adler-Bell-Jackiw; external lines are vector and axial gauge fields

1.3.2 Fujikawa's method

As mentioned above, the ABJ analysis was performed using perturbative methods. Fujikawa has proposed an alternative way of evaluating chiral anomalies using functional integrals[19]. The Fujikawa method has since been generalized to deal also with the nonabelian chiral anomaly, and has in fact become the standard tool for studying quantum anomalies in general, including e.g. the conformal and parity anomalies. Moreover it has proved to be an essential technique for constructing a path-integral bosonization scheme for a wide class of two-dimensional models. The evaluation of the effective theory in the main part of this thesis will also rely heavily on this powerful machinery. Here we illustrate how it works for the simple case of the massless abelian Dirac fermion, both in d=4 and d=2.

Ward-Takahashi identity via path integral
From now on we will always be using the Euclidean formalism. We start with the following Lagrangian density, [7]

$$\mathcal{L} = \bar{\psi}(\partial\!\!\!/ + \mathrm{i}e A\!\!\!/)\psi \equiv \bar{\psi} D\!\!\!\!/(A)\psi \tag{1.12}$$

where slashes denote the contraction with the gamma matrices; $a\!\!\!/ \equiv \gamma_\mu a_\mu$. The anticommutation relations for the gamma matrices are $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, and γ^5, which is unambiguously defined in even space-time dimensions, is $\gamma^5 \equiv \mathrm{i}\gamma_0\gamma_1$ for d=2, and $\gamma^5 \equiv \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$ when d=4. [8].

Consider some continuous symmetry of the system (for instance gauge symmetry) which is respected at the classical level. Let us perform a **local** infinitesmal transformation corresponding to that symmetry, parametrized by an infinitesmal field $\alpha(x)$ (x stands for the d-dimensional Euclidean coordinate), which is a scalar function for the example of a gauge transformation, and a pseudoscalar (chiral angle) for the case of a chiral transformation: [9]

$$\begin{aligned} \psi &\rightarrow \psi + \delta\psi_{\alpha(x)} \equiv \psi' \\ \bar{\psi} &\rightarrow \bar{\psi} + \delta\bar{\psi}_{\alpha(x)} \equiv \bar{\psi}', \end{aligned} \tag{1.13}$$

where ψ' and $\bar{\psi}'$ are of the same order as α. It must be noted that here the gauge field should be regarded as an external field, and the transformation only affects the fermionic fields. Accordingly, the action will transform in the following manner:

$$S \rightarrow S' = S + \int d^d x \Lambda_{\alpha(x)} \tag{1.14}$$

[7] The neglect of a mass term here is merely for simplicity; a mass term explicitely breaks chiral symmetry, and will only give trivial contributions (which are already present classically) to the anomalous WTI.
[8] A list of the Euclidean conventions used in the text is summarised in appendix A
[9] Unless explicitly stated to the contrary, the term "gauge transformation" will mean a vector gauge transformation, whereas the term "chiral transformation" will be used in the same sense as an axial gauge transformation.

Then, since the transformation corresponds to a symmetry of the system, the function $\Lambda(x)$ must be the divergence of some Noether current.

In order to promote the classical conservation law to a WTI and thereby seek possible quantum corrections, we view eq.(1.13) as a change of variables in the generating functional

$$Z_F = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S[A,\bar{\psi},\psi]}. \tag{1.15}$$

In terms of the new variables, Z_F becomes

$$Z_F = J[\alpha(x)] \int \mathcal{D}\bar{\psi}'\mathcal{D}\psi' e^{-S[A,\bar{\psi}',\psi']-\int d^dx \Lambda_{\alpha(x)}} \tag{1.16}$$

where $J[\alpha(x)]$ is the Jacobian of the functional measure,

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \rightarrow J[\alpha(x)]\mathcal{D}\bar{\psi}'\mathcal{D}\psi'. \tag{1.17}$$

It is sometimes useful to express $J[\alpha(x)]$ as the ratio between the fermionic determinants before and after the transformation,

$$\det D(A) = J[\alpha(x)] \det D(A)' \tag{1.18}$$

where $D(A)'$ denotes the transformed fermionic determinant, i.e. $S' \equiv \int d^dx \bar{\psi}' \not{D}(A)' \psi'$. To make equation (1.18) sensible, the fermionic determinants on both sides of the equation must be evaluated using a suitable regularization which respects gauge invariance; in fact we may say that this is the key point around which the entire discussion of this thesis centers around. Now, since eq.(1.13) is merely a change in variables, Z_F should not depend on $\alpha(x)$, so that from eq.(1.16) we obtain

$$-\frac{1}{Z_F}\frac{\delta Z_F[\alpha(x')]}{\delta\alpha(x)} \equiv 0 = \left\langle \frac{\delta}{\delta\alpha(x)} \int d^dx' \Lambda_{\alpha(x')} \right\rangle - \frac{\delta \ln J[\alpha(x')]}{\delta\alpha(x)}. \tag{1.19}$$

where $< ... >$ denotes expectation with respect to the weight e^{-S}. Eq.(1.19) is the WTI.
Now let us consider the case of an abelian gauge transformation, for which

$$\begin{aligned} \delta\psi &= i\alpha(x)\psi \\ \delta\bar{\psi} &= -i\alpha(x)\bar{\psi} \end{aligned} \tag{1.20}$$

and

$$\Lambda_{\alpha(x)} = -\alpha(x)\partial_\mu j_\mu. \tag{1.21}$$

In this case, $D[A]$ transforms covariantly, so apparently eq.(1.18) implies that $J[\alpha(x)] = 1$. Plugging these into eq.(1.19), we arrive at

$$\langle \partial_\mu j_\mu \rangle = 0, \tag{1.22}$$

meaning that gauge invariance is preserved, as it must, at the quantum level.

Anomalous Ward-Takahashi identities

Now we turn to the case of chiral transformations. An infinitesmal transformation is given by

$$\begin{aligned} \delta\psi &= i\alpha(x) \\ \delta\bar{\psi} &= i\alpha(x)\bar{\psi} \end{aligned} \tag{1.23}$$

and accordingly, $\Lambda_{\alpha(x)}(x)$ is

$$\Lambda_{\alpha(x)}(x) = -i\alpha(x)\partial_\mu(\bar{\psi}\gamma_\mu\gamma^5\psi). \tag{1.24}$$

In order to calculate the Jacobian, we expand the fermion fields $\psi(x)$ and $\bar{\psi}(x)$ in terms of some orthonormalised complete set of functions,

$$\begin{cases} \psi = \sum_n a_n \varphi_n \\ \bar{\psi} = \sum_n b_n \varphi_n^\dagger \end{cases}. \tag{1.25}$$

where $\int d^dx\varphi_n^\dagger(x)\varphi_m(x) = \delta_{nm}$, $\sum_n \varphi_n(x)\varphi_m^\dagger(y) = \delta^d(x-y)$, and the $\{a_n\}$s and the $\{b_n\}$s are elements of an infinite Grassman algebra.

It is convenient to take as φ_n the eigenfunctions of the Dirac operator $\not{D} = (\not{\partial} + \mathrm{i}e\not{A})$, i.e.

$$\not{D}\varphi_n = \lambda_n\varphi_n. \tag{1.26}$$

The transformed field $\psi(x)'$ is also expanded in terms of this basis as

$$\psi(x)' = \sum_n a_n'\varphi_n(x), \tag{1.27}$$

where

$$a_n' = \int d^dx\varphi_n^\dagger(x)\psi(x)' = a_n + \sum_{m\neq n} C_{nm}a_m, \tag{1.28}$$

where

$$C_{nm} = \mathrm{i}\int d^dx\varphi_n^\dagger(x)\alpha(x)\gamma^5\varphi_m(x). \tag{1.29}$$

This will affect the measure as [10]

$$\prod_n \mathrm{d}a_n = \det C\prod_n \mathrm{d}a_n'. \tag{1.30}$$

Repeating the same steps for the b_ns, we find our Jacobian to be

$$\begin{aligned} J &= (\det C)^2 = \mathrm{e}^{2\mathrm{Tr}\ln C} \\ &= \exp\left[2\mathrm{i}\int d^dx\alpha(x)\sum_n \varphi_n^\dagger(x)\gamma^5\varphi_n(x)\right]. \end{aligned} \tag{1.31}$$

We are thus lead to consider the quantity $\sum_n \varphi_n^\dagger(x)\gamma^5\varphi_n(x)$ which as it stands is divergent. It is therefore necessary to implement some appropriate regularization scheme, a simple example being the following:

$$\sum_n \varphi_n^\dagger(x)\gamma^5\varphi_n(x) \equiv \lim_{y\to x, M\to\infty}\sum_n \varphi_n^\dagger(y)\gamma^5 f(\lambda_n^2)\varphi_n(x), \tag{1.32}$$

where $f(x)$ is a smooth function satisfying the conditions

$$f(0) = 1, \quad f(x) \to 0 \quad (|x| \to \infty). \tag{1.33}$$

It is important to note that, as the λ_ns are eigenvalues of $\not{D}$, this regularization is manifestly gauge invariant. For concreteness [11] we follow Fujikawa and choose the following form:

$$\begin{aligned} \sum_n \varphi_n^\dagger(x)\gamma^5\varphi_n(x) &= \lim_{y\to x, M\to\infty}\sum_n \varphi_n^\dagger(y)\gamma^5 \mathrm{e}^{-\lambda_n^2/M^2}\varphi_n(x) \\ &= \lim_{y\to x, M\to\infty} \mathrm{Tr}\left[\gamma^5\exp\left(-\frac{\not{D}^2}{M^2}\right)\delta^d(x-y)\right], \end{aligned} \tag{1.34}$$

where cyclicity of the trace was used in the final expression. From this expression, we see that what we are doing is a heat-kernel regularization which cuts off contributions from large $|\lambda_n|$. Using $\delta^d(x-y) = \int \frac{d^dk}{(2\pi)^d}\mathrm{e}^{\mathrm{i}k(x-y)}$ and $\not{D}^2 = D_\mu D_\mu + \frac{\mathrm{i}e}{4}[\gamma_\mu, \gamma_\nu]F_{\mu\nu}$, we have

$$\begin{aligned} \sum_n \varphi_n^\dagger(x)\gamma^5\varphi_n(x) &= \lim_{M^2\to\infty}\mathrm{Tr}\int\frac{d^dk}{(2\pi)^d}\left[\gamma^5\mathrm{e}^{-\frac{k^2}{M^2}}\exp\left(-\frac{\mathrm{i}e}{4M^2}[\gamma_\mu,\gamma_\nu]F_{\mu\nu}\right)\right] \\ &= \lim_{M^2\to\infty} M^d\mathrm{Tr}\int\frac{d^dk}{(2\pi)^d}\gamma^5\exp\left[-k^2 - \frac{\mathrm{i}e}{4M^2}[\gamma_\mu,\gamma_\nu]F_{\mu\nu}\right], \end{aligned} \tag{1.35}$$

[10] Because of the properties of Grassman numbers, the factor on the RHS is the inverse of what is obtained in the case of a bosonic theory.

[11] It is actually possible to carry out the calculations that follows by solely assuming the asymptotics of eq.(1.33).

where a shift of integration variable $k_\mu \to k_\mu - \mathrm{i}eA_\mu$ has been taken, and we have rescaled the variables ($k_\mu \to Mk_\mu$) in the last line. The next step involves expanding the exponential in inverse powers of M^2, and picking up the term which survives in the limit $M^2 \to \infty$. The result is dependent on the space-time dimensionality. Since chiral anomaly is known to occur in even space-time dimensions, we evaluate the above for the cases d=2 and d=4.

(1) d=2 case

The only nonvanishing contribution comes from the order $O(1/M^2)$ term of the expansion:

$$\sum_n \varphi_n^\dagger(x)\gamma^5\varphi_n(x) = -\lim_{M^2\to\infty} M^2 \mathrm{Tr}\gamma^5 \frac{\mathrm{i}e}{4M^2}[\gamma_\mu, \gamma_\nu]F_{\mu\nu} \int \frac{d^2k}{(2\pi)^2} e^{-k^2} \tag{1.36}$$

Using the properties of the 2-d γ-matrices, $\gamma^5[\gamma_\mu, \gamma_\nu] = 2\mathrm{i}\epsilon_{\mu\nu}\mathbf{1}$ and $\mathrm{Tr}\mathbf{1} = 2$, this further becomes

$$\begin{aligned} \sum_n \varphi_n^\dagger(x)\gamma^5\varphi_n(x) &= \frac{e}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu} \\ &= \frac{e}{2\pi}F_{01}. \end{aligned} \tag{1.37}$$

Inserting this into eq.(1.31), we finally obtain

$$\begin{aligned} J &= \exp\left[\mathrm{i}\frac{e}{\pi}\int \mathrm{d}^2x\,\alpha(x)F_{01}(x)\right] \\ &= \exp\left[\mathrm{i}\frac{e}{2\pi}\int \mathrm{d}^2x\,\alpha(x)\epsilon_{\mu\nu}F_{\mu\nu}(x)\right]. \end{aligned} \tag{1.38}$$

Substituting eqs.(1.24) and (1.38) into the general formula, eq.(1.19), we arrive at the anomalous WTI. [12]

$$\begin{aligned} \langle \partial_\mu j_\mu^5 \rangle &= -\frac{e}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu} \\ &= -\frac{e}{\pi}F_{01}. \end{aligned} \tag{1.39}$$

(2) d=4 case

For this case, the only nonvanishing contribution comes from the order $O(1/M^4)$ term of the expansion:

$$\begin{aligned} \sum_n \varphi_n^\dagger(x)\gamma^5\varphi_n(x) &= \lim_{M^2\to\infty} M^4 \mathrm{Tr}\gamma^5 \frac{1}{2!}\left(\frac{\mathrm{i}e}{4M^2}[\gamma_\mu, \gamma_\nu]F_{\mu\nu}\right)^2 \int \frac{d^4k}{(2\pi)^2} e^{-k^2} \\ &= -\frac{e^2}{32\pi^2}\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}. \end{aligned} \tag{1.40}$$

Accordingly, the Jacobian and the anomalous WTI become

$$J = \exp\left[-\int d^4x \frac{e^2}{16\pi^2}\alpha(x)F_{\mu\nu}F^*_{\mu\nu}\right] \tag{1.41}$$

and [13]

$$\langle \partial_\mu j_\mu^5 \rangle = \frac{e^2}{16\pi^2}F_{\mu\nu}F^*_{\mu\nu}. \tag{1.42}$$

[12] Actually, the general WTI which is derived by adding fermionic source fields to the action in the generating functional is of the form $\partial_\mu \mathrm{T}\left\langle j_\mu^5(x)\cdots\right\rangle = -\mathrm{T}\left\langle \frac{e}{\pi}F_{01}(x)\cdots\right\rangle$, where T denotes time-ordering, and $(\cdots)$ is some product of the fermionic fields generated by a functional derivative of the generating functional. Eq.(1.39) is a special case of this formula. General WTI of similar form can also be derived for the d=4 case.

[13] Our definition of the dual gauge field strength tensor is $F^*_{\mu\nu} \equiv \epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}$.

Eq.(1.42) coincides with the chiral limit (i.e. massless limit) of the ABJ result, eq.(1.11).

General remarks

We make two short observations on the Fujikawa method.

(1) Tracing back our steps which lead to eqs.(1.39) and (1.42), it is easy to identify what was the crucial factor in deriving a chiral anomaly: it is apparently the fact that we have used a gauge invariant regularization. Let us compare this with what would have happened if a gauge *variant* regulator was used. For instance, we could have chosen as the basis functions $\{\varphi_n\}$ the eigenfunctions of $\partial\!\!\!/$ (instead of those of $D\!\!\!\!/$), and the heat-kernel regulator $e^{-(\partial\!\!\!/)^2}$. Then we would have had, instead of eq.(1.34),

$$\begin{aligned}\sum_n \varphi_n^\dagger(x)\gamma^5\varphi_n(x) &= \lim_{M^2\to\infty} \mathrm{Tr}\gamma^5 \int \frac{d^dk}{(2\pi)^d} e^{-k^2/M^2} \\ &\equiv 0,\end{aligned} \tag{1.43}$$

which would immediately lead to the conclusion that $J[\alpha(x)] \equiv 1$, and hence to $\langle \partial_\mu j_\mu^5 \rangle = 0$. In addition, it can be shown that we would obtain for this case an anomalous WTI for the *vector* current. This observation illustrates once again that the chiral anomaly originates from the impossibility to quantize a theory in a way that maintains both the vector and axial symmetries of the classical action.

(2) Chiral anomaly is known to have deep topological meanings, which the Fujikawa approach exposes in a direct manner. While a rigorous account will require considerably sophisticated mathematics, our survey will be very incomplete with an omission of this aspect. Hence we shall give a simplified version of this argument. To this end, let us go back to eq.(1.31) and consider the case $\alpha(x) \equiv$ const.. The first observation to make, is that since $\{\gamma^5, D\!\!\!\!/\} = 0$, we have

$$D\!\!\!\!/\gamma^5\varphi_n(x) = -\lambda_n\varphi_n(x) \quad (\text{for } \lambda_n \neq 0). \tag{1.44}$$

That is, for each eigenfunction $\varphi_n(x)$ with a nonvanishing eigenvalue λ_n, there exists a corresponding eigenfunction $\gamma^5\varphi_n(x)$ with the eigenvalue $-\lambda_n$. Let us consider a global chiral transformation, in which case the evaluation of the anomaly reduces to the integration of $\sum_n \varphi_n^\dagger(x)\gamma^5\varphi_n(x)$ over space-time. Then, since eigenfunctions with different eigenvalues are orthogonal to each other, contributions from eigenfunctions with nonvanishing λ_ns drop out. In other words, we see that only the zero modes (eigenfunctions with zero eigenvalue) are contributing to the anomaly. Meanwhile, the zero modes can always be chosen to be eigenvalues of γ^5. From these considerations, we are lead to the simple expression

$$\int d^dx \sum_n \varphi_n^\dagger(x)\gamma^5\varphi_n(x) = (n_R - n_L), \tag{1.45}$$

where $d = 2$ or 4, and n_R and n_L are the numbers of zero modes with right and left chirality, respectively. On the other hand, we know that the gauge invariant regularization discussed above implies that

$$\int d^dx \sum_n \varphi_n^\dagger(x)\gamma^5\varphi_n(x) = \left\{ \begin{array}{ll} \frac{e}{4\pi}\int d^2x\epsilon_{\mu\nu}F_{\mu\nu} & (d=2) \\ -\frac{e^2}{32\pi^2}\int d^4xF_{\mu\nu}F^*_{\mu\nu} & (d=4) \end{array} \right. . \tag{1.46}$$

Since the integrand on the RHS, for both d=2 and d=4, is a total derivative, integrating it over the Euclidean space will give a topological quantity, in the sense that it will be completely determined by the global topology of the gauge field configuration. From the above two equations, we conclude that $n_R - n_L$, which is an *index*[14] which charecterises the spectral property of the Dirac fermion

[14] It is proportional to what is known in topology as the η-invariant.

(it is an invariant since nonzero-mode states always appear in chirality pairs), is to be identified with a topological invariant of the gauge field.[15] This is an example of the Atiyah-Singer index theorem.

1.4 Path-integral bosonization in 1+1d

Let us emphasize once more that the original Fujikawa derivation of the chiral anomaly was based on the calculation of the Jacobian (we will henceforth call it the Fujikawa Jacobian (FJ)) for an *infinitesmal* chiral transformation. Soon after the paper of Fujikawa appeared, the method was generalised by several authors to deal with Jacobians associated with a *finite* transformation. This opened up a new and powerful way to obtain effective field theories (and in some special cases exactly *solve* them), both abelian and nonabelian, for relativistic fermions in even space-time dimensions, most notably in 1+1 d. [16] In general, topology-related, and therefore nonperturbative information enters into the theory due to the chiral anomaly. This may be considered as a path integral approach to 1+1 d bosonization. Here we will discuss, mainly following the work of Gamboa Saraví et al. [20], how this formalism works, using the prototypical example of the Schwinger model. The main theme of this thesis is basically an application of this method to 1+1d interacting electron systems with internal SU(2) (spin-rotational) symmetry, it will involve several very nontrivial extensions and generalizations (a part of this will be mentioned below in detail in this subsection), which will later be explained.

Fermion-gauge field decoupling and some related comments

The Schwinger model is electrodynamics in 2d with massless fermions. The Lagrangian density is

$$\mathcal{L} = \bar{\psi}(\partial\!\!\!/ + \mathrm{i}e A\!\!\!/)\psi + \frac{1}{4}F_{\mu\nu}^2. \tag{1.47}$$

In the so called chiral basis, in which γ^5 is diagonal and $\psi =^t [R, L]$, the algebras of the 2-d gamma matrices can be fullfilled by choosing the following representations: $\gamma^5 = \tau_3$, $\gamma_0 = \tau_1$, $\gamma_1 = -\tau_2$, where $\{\tau_a\}$ are Pauli matrices. Hence the explicit representation of the fermionic part of $\mathcal{L}$ in this basis is

$$\mathcal{L}_F = [R^\dagger, L^\dagger] \begin{bmatrix} \partial_\tau - \mathrm{i}\partial_x + \mathrm{i}e(A_\tau - \mathrm{i}A_x) & 0 \\ 0 & \partial_\tau + \mathrm{i}\partial_x + \mathrm{i}e(A_\tau + \mathrm{i}A_x) \end{bmatrix} \begin{bmatrix} R \\ L \end{bmatrix}. \tag{1.48}$$

Here we have set the light velocity to unity, $c = 1$. We now perform a combined vector and axial gauge transformations on the fermion,

$$\begin{cases} \psi \to e^{\gamma^5\phi(x)+\mathrm{i}\eta(x)}\psi \\ \bar{\psi} \to \bar{\psi}e^{\gamma^5\phi(x)-\mathrm{i}\eta(x)} \end{cases} \tag{1.49}$$

Notice that there are no factors of "i" in front of γ^5 in the exponents, which appears to be rather peculiar. It turns out that this is in fact the correct form of a chiral transformation within our Euclidean conventions on the gauge fields. That is, if we intend to treat the vector and axial gauge fields on an equal footing, a chiral transformation will necesarrily bear this form. Though implicit in the literature, this is not often clearly stated. However, as this point will be central to the investigations in the following chapters, we will need to digress on this here. [17]

[15] When this Dirac operator is interpreted as the Hamiltonian density of a Dirac fermion in one dimension higher, the index gives the spectral asymmetry of that Hamiltonian, and the anomaly which occurred in even space-time dimensions now becomes related to the parity anomaly.

[16] In 3+1 d, it has potential applicabilites to e.g. skyrmion physics in Hadron phenomenology.

[17] in the previous subsection we bypassed this problem by following the original Fujikawa argument, and implicitly changing conventions on the definition of the Euclidean gauge field.

Under the transformation eq.(1.49), $\bar{\psi}\not{\partial}\psi$ transforms as

$$\bar{\psi}\not{\partial}\psi \to \bar{\psi}\not{\partial}\psi + \bar{\psi}(\not{\partial}\phi\gamma^5)\psi + i\bar{\psi}(\not{\partial}\eta)\psi. \tag{1.50}$$

Using the crucial relation (unique to 2d)

$$\gamma_\mu\gamma^5 = i\epsilon_{\mu\nu}\gamma_\nu, \tag{1.51}$$

we have

$$\delta(\bar{\psi}\not{\partial}\psi) = \bar{\psi}\left(-i\gamma_\mu\epsilon_{\mu\nu}\partial_\nu\phi + i\gamma_\mu\partial_\mu\eta\right)\psi \tag{1.52}$$

Hence this additional term will decouple the fermions and gauge fields, if

$$\bar{\psi}ie\not{A}\psi = -\delta(\bar{\psi}\not{\partial}\psi), \tag{1.53}$$

i.e. if

$$A_\mu = \frac{1}{e}(\epsilon_{\mu\nu}\partial_\nu\phi - \partial_\mu\eta), \tag{1.54}$$

which is just the Helmholtz decomposition of a 2d vector into divergence-free and curl-free portions. This decoupling property of the fermion and gauge-field sectors will simplify the subsequent derivation of the effective theory considerably.

We note that if an "i" was present in the chiral transformation, then when we denote $\delta(\bar{\psi}\not{\partial}\psi) \equiv i\bar{\psi}\gamma_\mu B_\mu\psi$, the obtained expression for B_μ would be a mixture of real ($\propto \partial_\mu\eta$) and pure imaginary ($\propto \epsilon_{\mu\nu}\partial_\nu\phi$) terms, which is impossible to absorb by a gauge transformation. This implies that if a chiral transformation is to behave as a subgroup $U(1)_A$ of the $U(1)$ gauge symmetry (so that ϕ and η can be treated on an equal footing) it is necessary, in the Euclidean formalism, for it to be of the form $U_5 = e^{\gamma^5\phi}$. The same statement applies for the nonabelian cases.

At this point, we would like to make some important remarks on the methods used in our later investigations, in relation to the observations just made. Later on, we will be mainly concerned with models (having their origins in condensed matter physics) which, when straightforwardly expressed in relativistic notations using the definitions for the gamma matrices, have the following type of Euclidean Lagrangian density:

$$\mathcal{L} = \bar{\psi}\left(\not{\partial} + me^{i\gamma^5\phi}\right)\psi, \tag{1.55}$$

where ϕ is either a real pseudoscalar field or is of the form $\phi = \varphi_a t_a$, where the $\{\varphi_a\}$ are real and $\{t_a\}$ are generators of an $SU(n)$ algebra (e.g. $\frac{\sigma_a}{2}$ for $SU(2)$). As will be described later, there we will be performing "chiral transformations" such as

$$\begin{cases} \psi \to e^{-i\gamma^5\frac{\phi}{2}}\psi \\ \bar{\psi} \to \bar{\psi}e^{-i\gamma^5\frac{\phi}{2}} \end{cases}, \tag{1.56}$$

which will prove to be the key step towards deriving a low energy effective theory for eq.(1.55). The corresponding FJ will give rise to the important topological terms which determine the properties of the theory.

As we have just mentioned, however, eq.(1.56) is *not* the chiral transformation in the usual sense. So what we are actually doing here, implicitly or explicitly, is to start, instead of eq.(1.55), from the Lagrangian density

$$\mathcal{L} = \bar{\psi}\left(\not{\partial} + me^{\gamma^5\phi}\right)\psi, \tag{1.57}$$

eliminate the chiral factors in the mass term via the (well-defined) chiral transformation

$$\begin{cases} \psi \to e^{-\gamma^5\frac{\phi}{2}}\psi \\ \bar{\psi} \to \bar{\psi}e^{-\gamma^5\frac{\phi}{2}} \end{cases}, \tag{1.58}$$

evaluate the corresponding FJ, and finally, *complexify* (via analytic continuation) the chiral angle: $\phi \to i\phi$. In this way, what was initially the anomaly associated with, for example, the SU(2) chiral transformation, will change into one involving its complexification, SL(2,**C**). This should *not* be confused with a Wick rotation, as we are always working within the Euclidean formalism.

The crucial question is, then, whether such a procedure is legitimate. The validity of such methods has been confirmed, at least in the massless QCD_2 case by Gamboa Saraví et al[21], who found a natural connection of their results to the Wess-Zumino-Witten model. This result invites us to explore possible extensions to the massive case. We will indeed see, that for our massive model (where an additional complication -due to the impossibility of completely decoupling the fermionic and background field sectors- makes the issue even more nontrivial) we can make connections of our obtained results to the Polyakov-Wiegmann identity [22], which plays a key role in conventional nonabelian bosonization.

Effective action for the Schwinger model

We continue our derivation of the effective action for the Schwinger model. Though the transformation defined by eqs.(1.49) and (1.54) has decoupled the fermions from the gauge field, i.e.

$$\mathcal{L} \to \mathcal{L} = \frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}\partial\!\!\!/\psi, \tag{1.59}$$

, with $F_{\mu\nu} = -\frac{1}{e}\Box\phi$ (where $\Box \equiv \partial_\mu\partial_\nu$, the Schwinger mechanism does not show up at this level. To treat the problem at the quantum level, we consider the generating functional,

$$Z = \int \mathcal{D}\phi\mathcal{D}\eta\triangle_{FP}\mathcal{D}\bar{\psi}\mathcal{D}\psi J e^{-S}, \tag{1.60}$$

J is the FJ accompanying transformation (1.49), and $\triangle_{FP}$ is the Faddev-Popov Jacobian associated to transformation (1.54). It is convenient to work in the Coulomb gauge condition, which is fullfilled by insertion of $\delta(\partial_\mu\eta)$, which eliminates the η integration.

Turning our attention to J, we first note that since eq.(1.49) is a finite transformation, the methods of the previous sections must be generalized to deal with such a case. For this purpose, we introduce a one-parameter family of transformations

$$\left\{ \begin{array}{l} \psi = e^{t(\gamma^5\phi+i\eta)}\psi_t \\ \bar{\psi} = \bar{\psi}_t e^{t(\gamma^5\phi-i\eta)} \end{array} \right. , \tag{1.61}$$

where $t \in [0,1]$, and perform a successive sequence of infinitesmal transformations so that t grows from 0 to 1. Let us write the fermionic part of the Lagrangian density as

$$\mathcal{L}_F = \bar{\psi}D_t\psi, \tag{1.62}$$

where

$$\begin{aligned} D_t &\equiv e^{t(\gamma^5\phi-i\eta)}\,D\!\!\!/\,e^{t(\gamma^5\phi-i\eta)} \\ &= \partial\!\!\!/ + ie\,A\!\!\!/ + it\gamma_\mu(-\epsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\eta) \\ &= \partial\!\!\!/ + ie(1-t)A\!\!\!/. \end{aligned} \tag{1.63}$$

From here on, we will omit the part of A_μ that involves η by means of the above gauge fixing condition. Now, since the generating functional $Z_F = J(t)\det D_t$ cannot depend on the parameter t, we have

$$\frac{dZ_F}{dt} = 0 = \frac{dJ}{dt}\det D_t + J(t)\frac{d}{dt}(\det D_t), \tag{1.64}$$

which is formally integrated to give

$$J \equiv J(1) = \exp\left[-\int_0^1 dt w'(t)\right] \tag{1.65}$$

where

$$w'(t) = \frac{\mathrm{d}w}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\ln\det D_t. \tag{1.66}$$

It is easily seen that

$$D_{t+\Delta t} = D_t + \{\gamma^5\phi, D_t\}\Delta t + O(\Delta t^2). \tag{1.67}$$

Substituting this into

$$\begin{aligned} w'(t) &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\ln\det D_{t+\Delta t} - \ln\det D_t\right] \\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\mathrm{tr}\ln(D_{t+\Delta t})D_t^{-1}\right], \end{aligned} \tag{1.68}$$

we get

$$\begin{aligned} w'(t) &= \lim_{\Delta t\to 0}\mathrm{tr}\ln(1 + 2\gamma^5\phi\Delta t) \\ &= 2\mathrm{tr}\gamma^5\phi. \end{aligned} \tag{1.69}$$

To regularize this formal expression, we will use a regularization similar to that of Fujikawa:

$$w'(t) = \lim_{M^2\to\infty}\mathrm{tr}\left[2\cdot\frac{1}{(2\pi)^2}\int d^2x d^2k e^{ikx} e^{-D_t^2/M^2}\gamma^5 e^{-ikx}\right] \tag{1.70}$$

which is motivated by the anticipation that the insertion of a Heat kernel which uses the "instantaneous Dirac operator" D_t for each step of the successive transformations should be the best way to respect the full symmetry of the system at all stages. Proceeding as in the previous section,

$$\begin{aligned} w'(t) &= \lim_{M^2\to\infty}\mathrm{tr}\gamma^5\frac{1}{(2\pi)^2}\int d^2x d^2k\phi\exp\left[-(D_t + \not{k})^2/M^2\right] \\ &= \lim_{M^2\to\infty}\mathrm{tr}\gamma^5\frac{1}{(2\pi)^2}\int d^2x d^2k\phi e^{-k^2/M^2}\left(1 - D_t^2/M^2 + O(1/M^4)\right) \\ &= -\frac{1}{2\pi}\mathrm{tr}\gamma^5\gamma_\mu\gamma_\nu\int d^2x(D_t)_\mu(D_t)_\nu\phi \\ &= \frac{e}{2\pi}(1-t)\int d^2x\phi\epsilon_{\mu\nu}F_{\mu\nu}, \end{aligned} \tag{1.71}$$

where terms surviving the Dirac trace were collected in the final expression. Inserting this into eq.(1.65), we arrive at

$$\ln J = \frac{e}{4\pi}\int d^2x\epsilon_{\mu\nu}F_{\mu\nu}\phi = -\frac{e^2}{2\pi}\int d^2x A_\mu\left[\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right]A_\nu. \tag{1.72}$$

The generating functional then reads, including source terms,

$$Z[\eta,\bar\eta] = \int \mathcal{D}A_\mu\delta(\partial_\mu A_\mu)\mathcal{D}\bar\psi\mathcal{D}\psi e^{-S_{eff}}, \tag{1.73}$$

where

$$S_{eff} = \frac{1}{4}F_{\mu\nu}^2 + \frac{e^2}{2\pi}A_\mu^2 + \bar\psi\not{\partial}\psi + \bar\psi e^{\gamma^5\phi}\eta + \bar\eta e^{\gamma^5\phi}\psi. \tag{1.74}$$

We see that the photon has acquired a mass, which is the Schwinger mechanism. From this derivation it is clear that the chiral anomaly was responsible for this to occur. Since the effective action is completely decoupled into free fermions and gauge fields, each with simple propagators, it is a straightforward task to obtain via functional derivatives, any correlators for the original fermions, since they will factorize into the fermion and gauge parts.

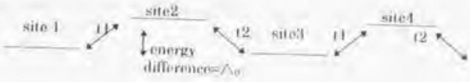

Figure 1.2: The Rice-Mele model for a diatomic polymer chain

1.5 Condensed matter applications - polymer chains

The Fermi surface is a central feature of electronic systems which arise in condensed matter physics. Since it is not a Lorentz invariant object, the relativistic field theory arguments of the previous sections cannot be immediately applied to these systems. However, in one dimension, low energy effective theories are often constucted by linearizing the spectrum with repect to the two Fermi-points $k = \pm k_F$. In such cases, there is a possibility that chiral anomaly of 1+1 d Dirac fermions will play an important role in the effective theory. This is actually known to be the case in several situations. The most obvious and well known applications involves charge density wave and solitonic excitations in quasi-linear organic compounds[1] [23][24]. Though they have been thoroughly investigated using abelian bosonization techniques[2][1], chiral anomaly, when applied carefully, provides an alternative and rather direct way of understanding these subjects. A less trivial application is the dynamics of superfluid ^{3}He[25], where the higher-dimensional fermi surface is approximated as a collection of 1+1 d Dirac fermions.

As an introduction to the next chapter, we describe in this section the soliton-induced fermion numbers in diatomic polymer chains. Induced fermion numbers are generally associated with the polarization of the fermion vacuum caused by an interaction with background fields which have nontrivial topology. In conjugated polymer chains, this occurs when bond solitons induce localised electronic states with fractional charge. A simple illustration of this phenomenon in terms of field theory was given by Goldstone and Wilczek (GW)[4] whose work we rephrase in a manner which will emphasize the role played by (abelian) chiral anomaly. Later on, we will see that our treatment of the Peierls-Hubbard model turns out to be a *direct nonabelian extension of the GW theory*. Therefore, the presentation of this section will be used to draw analogies and to establish some notations for subsequent chapters.

The Rice-Mele model[31] , which describes a diatomic polymer chain with alternating hopping amplitudes and valence energy levels, has the following Hamiltonian:

$$\begin{aligned}\mathcal{H}_{RM} &= \sum_i (t - (-1)^i \delta t) c_i^\dagger c_{i+1} + h.c. \\ &\quad + \Delta \sum_{i\sigma} (-1)^i n_{i\sigma}.\end{aligned} \tag{1.75}$$

The low-energy physics of this model can be described, following Jackiw and Semenoff[33] , in the continuum approximation by a Dirac fermion with a *chirally rotated mass term*,

$$\mathcal{L}_{GW} = \bar{\Psi}[\partial\!\!\!/ + i s\!\!\!/ + m e^{i\gamma_5 \phi}]\Psi, \tag{1.76}$$

where a source field s_μ has been inserted. [18] This model, which is a 2-d analogue of the linear sigma model, will be called the GW model. In the classification of Table 1.1, this corresponds to

[18] A detailed derivation of the continuum fermionic model, for a generalized version of eq.(1.75) will be given in the next section.

the abelian case, where the background field $U \in$U(1). Notations for eqs.(1.75) and (1.76) are as follows: δt is the alternation of the hopping amplitude, 2Δ is the site valence energy difference between adjacent sites, $m \equiv (\Delta^2+\delta t^2)^{1/2}$, and $\phi \equiv \tan^{-1}(\delta t/\Delta)$. For simplicity we have neglected the electron spin which does not play a role in eq.(1.75), so the Dirac spinor Ψ consists of two components, which are in the chiral basis the right and left movers, $\Psi = {}^t[R, L]$. In later sections we will introduce the spin degree of freedom, which will double the number of components. In terms of R and L, eq.(1.76) reads

$$\mathcal{L} = [R^\dagger, L^\dagger] \begin{bmatrix} \partial_-, \mathrm{m}e^{-i\phi} \\ \mathrm{m}e^{i\phi}, \partial_+ \end{bmatrix} \begin{bmatrix} R \\ L \end{bmatrix}, \tag{1.77}$$

where $\partial_\pm \equiv \partial_\tau \pm iv_F\partial_x$ and $v_F = 2at_0$.

Anticipating a slow variation in the bond-alternation strength, the general stategy now is to derive an effective low energy theory in the form of a gradient expansion of the pseudo-scalar field $\phi(x, \tau)$. Hence at this point we perform a chiral transformation [19]

$$\begin{aligned} \Psi &\rightarrow e^{-i\frac{\phi}{2}\gamma^5}\Psi \\ \bar{\Psi} &\rightarrow \bar{\Psi}e^{-i\frac{\phi}{2}\gamma^5}. \end{aligned} \tag{1.78}$$

This has two consequences. One is the transformation of eq.(1.76) to

$$\mathcal{L} = \bar{\Psi}[\partial\!\!\!/ + \mathrm{i}s\!\!\!/ + \frac{\mathrm{i}}{2}\partial\!\!\!/\phi + m]\Psi. \tag{1.79}$$

Since the mass term is now real, and the gradient of ϕ has explicitly appeared in the Dirac operator (we will henceforth call the operator in the action which is bilinear in the fields Ψ and $\bar{\Psi}$ the Dirac operator) this is a suitable starting point for doing a gradient expansion. The second consequence is that, as is well known, the change of variables in eq.(1.76) gives rise to a nontrivial Jacobian factor, the Fujikawa Jacobian, due to chiral anomaly. This factor will contribute the term

$$\mathcal{L}_{anomaly} = \frac{\mathrm{i}}{2\pi}\epsilon_{\mu\nu}\partial_\mu s_\nu\phi. \tag{1.80}$$

to the effective theory. It turns out that only this term contributes to the fermion number, and so the induced current due to a variation in $\phi(\mathrm{x}, \tau)$ is obtained as

$$< j_\mu >_{ind} = -\mathrm{i}\frac{\delta S_{eff}}{\delta s_\mu} = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\nu\phi. \tag{1.81}$$

When there is a kink in the alternation pattern of eq.(1.75), this implies that a localized charge of $Q = \frac{\Delta\phi}{2\pi}$ will be induced in its vicinity.

[19] As emphasized in section 1.4, it is important to notice that the transformations and its corresponding anomalies and FJ must be understood as having been obtained via a complexification of the chiral angle in the model $\mathcal{L} = \bar{\psi}(\partial\!\!\!/ + me^{-\gamma^5\phi})\psi$.

Chapter 2

Continuum form of the Peierls-Hubbard model

2.1 Statement of the problem

The path integral-based derivation of effective field theories for interacting electrons - especially of the semiclassical type, where one starts from an appropriate saddle point solution - becomes a highly nontrivial task in 1+1 dimension. Let us restrict our discussions to the repulsive Hubbard model at half-filling. We know that this is equivalent to considering the S=1/2 antiferromagnetic (AF) Heisenberg model, which, without perturbations is critially ordered. On the other hand, the natural place to start the semiclassical analysis would be in this case a spin-density-wave (SDW) mean-field theory (MFT). A convenient method of proceeding from the MFT, is a derivative expansion, in powers of space/time derivatives of the orientation of the SDW vectorial order parameter. (This is a regime where amplitude fluctuations are strongly massive, and are therefore suppressed.) The standard treatment[27] will yield an O(3) nonlinear sigma (NLσ) model. However, it is known that the NLσ model is always disordered in 1+1 dimension, and the correct behaviour cannot be obtained. It is now known, from Haldane's work[28], that this discrepancy stems from the failure to incorporate into the NLσ model its so-called topological term, which has its origin in the Berry phases of the electron spins. Since this important term is related to the global topology of the spin system in Euclidean space-time, it seems a hopeless task to try to salvage it from a perturbative expansion in terms of local AF fluctuations.

It is here that the notion of chiral anomaly may potentially turn out to be of great use in interacting electron systems. We have seen that the chiral anomaly is intimitely related to the global topology of a background gauge field. If the directional fluctuation of the SDW order parameter can somehow be viewed as a sort of gauge field which acts back on the electron system, -which turns out to be possible- we have a situation which is strikingly similar to those of the preceeding chapter. The only difference would now be that we are concerned with nonrelativistic fermions. This, however is not a problem in 1+1d, since it is a common practice to approximate, when considering low energies, the dispersion of a tight-binding band by linearized ones near the two Fermi points. The opening of the mass gap at the fermi surface, due to the finite amplitude of the SDW order parameter would explicitly violate chiral symmetry, but we can expect that additional violation, of purely quantum mechanical origin, would occur due to the back ground (spin-)gauge fields. In this way, we may hope that a MFT approach, when supplemented with the nonperturbative contribution of the chiral anomaly, may give reliable results, adding a new entry to the list of theoretical methods available in 1+1 d.

The above idea was originally described by Nagaosa and Oshikawa (NO) [5]. They demonstrated, using a relativistic continuum limit model for the Hubbard model away from half-filling, that a pair of appropriately chosen chiral transformations, each coupling to the Maxwellian and spin gauge fields would yield the acceleration of the charge phason by the electric field, and the

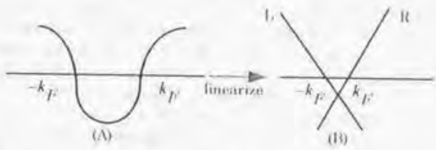

Figure 2.1: linearization of dispersion

topological term (with the proper coefficient, i.e. vacuum angle, conventionally denoted as θ) for the NLσ model respectively. The remaining parts of the action were treated by the derivative expansion, to give the kinetic part of the NLσ model, and the Tomonaga-Luttinger model for the charge phase field. Thus they have basically succeeded in providing a reliable MFT approach, and most importantly, have shown that the contributions from chiral anomalies are indispensible.

Subsequently, we have reinspected the NO method, and have discussed a natural extension to the case with bond alternation of the hopping amplitudes, i.e. the Peierls-Hubbard (PH) model[8, 11]. Through this investigation, we have found that several essential corrections to the original NO scheme must be brought about in order for it to properly account for the effects of the chiral anomaly, especially regarding the spin part. They are:

1. First of all, the "chiral anomaly" considered here in Euclidean formalism, turns out to be the complexified case, which we have mentioned in section 1.4. Hence the FJ calculation used here is in fact not the standard one, and one must be fully aware of the nontriviality of the implicitly assumed analytic continuation.

2. The chiral transformation (with the above point understood) which is relevant for the spin part, is not an abelian one, associated to the CP^1 gauge field (which arises as a representation for a U(1) subgroup of the SU(2) spin internal symmetry), but is actually a nonabelian one (SU(2) or its $SL(2,\mathbf{C})$ complexification). In order to respect the symmetry of the system, it is clear that we must treat the full nonabelian version of the anomaly. In addition, since we are dealing with a finite chiral rotation, the corresponding generalization for the FJ calculation, as illustrated in section 1.4 is required. Indeed, we have found that when the generalization to the PH model is made, the θ-value for the topological term which coincides with nonabelian bosonization is obtained only when these cautions are taken.

3. The FJ contains, in addition to the Wess-Zumino type topological term, a contribution to the kinetic part of the NLσ model. [1] Rather surprisingly, we have found that in fact that the terms contributing to the kinetic part is exhausted by that from the FJ. We will see that this agrees with a refermionization technique developed by Tsvelik.

The purpose of the following two chapters is to give a detailed account of the PH model investigation mentioned above.

Though similar in spirit to the NO work, we believe there are, as we have listed above, important changes, which we hope will contribute towards establishing theoretical frameworks which are both reliable and economical.

Motivated largely by the discovery of the inorganic spin-Peierls (SP) compound $CuGeO_3$, the SP system has in recent years revived as an important topical subject, and there already exists a huge body of theoretical studies on various aspects of these systems. Though the physical origin

[1] This is essentially due to a nonabelian analogue of the Schwinger mechanism.

may be considered as distinct from the spin Peierls system, the PH model at half-filling also has a dimerized spin chain as its ground state, so its study may have some bearings on aspects of the spin Peierls state. In this respect, it might be worthwhile to study the doped version of the problem, in view of the current interest in the disordered spin-Peierls system.

Hence our focus will be, besides the primary goal of establishing a general formulation that yields reliable effective theories, the application of the formalism to *induced fermion numbers*, which are topology-related phenomenna that are generally difficult to investigate by means of standard perturbative methods (and so perhaps illustrate best the merits of our formulation).

Before going into the details, it is perhaps appropriate to comment on why we think the PH model is interesting from the view-point of induced-fermion numbers. (From the viewpoint of constructing the formalism itself, choosing this particular model as the subject of study is almost self-explanatory, as we will later see - c.f. table 1.1.) In the prototypical case of polyacethylene physics, the "counting rule" for induced fermion numbers involves a single pseudo-scalar field (see eq.(1.81) below). The only nontrivial topological object that this background field can realize is a kink. On the other hand, as noticed in early works by Ho[29] and by Horowitz[30] , when one considers similar rules for spin-density-wave (SDW) systems, the pseudo-vector character of the background field gives rise to a richer variety of background topologies, and hence to nontrivial counting rules. In the PH model case, we will find that in addition to the SDW field, there will also arise a pseudo scalar field which represents the dimerization pattern of the system. Furthermore, when (nonmagnetic) impurities are doped, we will need to consider yet another scalar field, which will account for the charge degree of freedom. In this case the relevant background defect is a composite object of the dimerization-field kink and charge density field kink. It is the interplay between all these fields which makes the PH model an unusually interesting system for investigating this type of phenomenna. It is our hope that this approach will provide some fresh views on relevant matters such as the interrelation between spin dynamics and charge dynamics in the (pure) PH system, and the coexistence of dimer and antiferromagnetic order in the doped case[34].

2.2 Derivation of the fermionic model

2.2.1 The nonabelian GW model

We now proceed to discuss the derivation of the effective theory for the spin-Peierls system. Our starting Hamiltonian is the Peierls-Hubbard model at half filling, given by

$$H = \sum_{i\sigma} t_0(1 + \alpha(u_{i+1} - u_i))(c^\dagger_{i\sigma} c_{i+1\sigma} + h.c.)$$
$$+U \sum_i n_{i\uparrow} n_{i\downarrow} + \frac{K}{2} \sum_i u_i^2, \tag{2.1}$$

where $U > 0$, and u_i represents the deviation of the i-th lattice site from its equlibrium position. The coupling of the electron system to the lattice is included in the hopping term, and the 3rd term is the distortion energy of the lattice.

Though it is recently becoming clear that for the spin-Peierls compound $CuGeO_3$, the nonadiabatic nature of the lattice plays an important role, we shall here be content in assuming classical behaviour for the lattice, i.e., that quantum fluctuations, and retardation effects can be neglected for our purpose, owing to its 3-d nature. In one spatial dimension, nesting becomes perfect, and at low temperatures, it is well known that phonons undergo a $q = 2k_F$ condensation in favor of the energy-gain of the electron system (Peierls instability). When that happens, we may put $u_i = (-1)^i u_0$ so that

$$H \ = \ \sum_{i \in odd, \sigma} t_1(c^\dagger_{i\sigma} c_{i+1\sigma} + h.c.)$$

$$
\begin{aligned}
&+ \quad \sum_{j\in even,\sigma} t_2(c_{j\sigma}^\dagger c_{j+1\sigma} + h.c.) \\
&+ \quad U\sum_i n_{i\uparrow}n_{i\downarrow} + \frac{K}{2}\sum_i u_{oi}^2,
\end{aligned} \tag{2.2}
$$

where $t_1 = t_0 + \delta t, t_2 = t_0 - \delta t$, and $\delta t \equiv 2t_0\alpha u_0$.

In the following we assume strong coupling, i.e. $U >> t_0$. Then it is clear that eq.(2.2) effectively describes a dimerized spin chain, since second order perturbation in $\frac{t_0}{U}$ gives the AF Heisenberg model with alternating exchange interaction, and RG shows bond alternation to be a relevant perturbation at the S=1/2 uniform Heisenberg model fixed point [32]. Since this description will be sufficient for our later discussions on induced fermion numbers, we employ eq.(2.2) as the minimal model of a spin-Peierls system, and neglect other perturbations which may possibly be relevant experimentally, such as next nearest neighbor interactions. The passage from eq.(2.2) to the continuum model (eq.(2.4) below) is given in details in the following subsection. For the sake of clarity, here we shall outline the main steps that are carried out. First, in order to deal with the interaction term, we follow NO and rewrite it using the identity $Un_{i\uparrow}n_{i\downarrow} = \frac{U}{2}(n_{i\uparrow}+n_{i\downarrow}) - \frac{U}{6}(c_i^\dagger\vec{\sigma}c_i)^2$ and decouple it by introducing a vector auxiliary field $\vec{\varphi}_i$ via the Hubbard-Stratonovich transformation $\exp(\frac{U}{6}(c_i^\dagger\vec{\sigma}c_i)^2) = \int d\vec{\varphi}_i \exp[-\frac{U}{6}\vec{\varphi}_i^2 + \frac{U}{3}\vec{\varphi}_i\cdot c_i^\dagger\vec{\sigma}c_i]$. Then, writing $\vec{\varphi}_i = (-1)^i m\vec{n}_i$ ($|\vec{n}| = 1$) the decoupled Hamiltonian becomes

$$
\begin{aligned}
H &= \sum_{i\in odd} t_1(c_i^\dagger c_{i+1} + h.c.) + \sum_{j\in even} t_2(c_j^\dagger c_{j+1} + h.c.) \\
&+ \quad m\sum_n(\vec{n}_{2n}\cdot c_{2n}^\dagger\vec{\sigma}c_{2n} - \vec{n}_{2n-1}\cdot c_{2n-1}^\dagger\vec{\sigma}c_{2n-1}).
\end{aligned} \tag{2.3}
$$

At this point, it is worth noting that *if* there were AF order, we may simply put $\vec{n} \equiv \hat{z}$ and eq.(2.3) would reduce to a sum of two Hamiltonians, one for each spin component, with exactly the same form as eq.(1.75) of the previous section. (The only new feature would be the opposite signs of the "site energy difference" Δ for the up and down spins.) This suggests that we are dealing with a system that is closely related to the diatomic polymer chain. This expectation will be partially confirmed in section 4.1 where counting rules that are analogous to eq.(1.81) will be derived. At the same time, since strong AF fluctuation is present in this case, we expect that new complications should enter when taking into account the space-time variation of the vector field $\vec{n}$ below.

We now make a continuum approximation for this Hamiltonian by linearizing the spectrum around the two fermi points $k = \pm k_F$ and grouping the fermions into spinors, where the two (or four, including spin indices) entries correspond in the chiral base to right and left movers. We refer to the following subsection for details. The resulting Lagrangian density is, in relativistic notations,

$$
\mathcal{L} = \bar{\Psi}\left[\mathbf{1}\otimes\not{\partial} + \Delta_0 Q e^{i\gamma^5 Q\frac{\phi}{2}}\right]\Psi, \tag{2.4}
$$

where and the symbol $\otimes$ represents the direct product between the spin-space and chirality space (i.e. the SU(2) space spanned by the Dirac gamma matrices); $\mathbf{1}$ is the identity operator in the SU(2) space, $Q \equiv \vec{n}\cdot\vec{\sigma}$. Finally, Δ_0 and ϕ are introduced through the relations $\Delta_0\cos\frac{\phi}{2} = m, \Delta_0\sin\frac{\phi}{2} = 2\delta t$. The *nonabelian chiral factor* is defined as

$$
e^{i\gamma^5 Q\frac{\phi}{2}} \equiv \mathbf{1}\otimes\mathbf{1}\cos\frac{\phi}{2} + iQ\otimes\gamma^5\sin\frac{\phi}{2}. \tag{2.5}
$$

Written explicitly in the chiral basis, in which the representation of Ψ is $\Psi = {}^t[R_\uparrow, R_\downarrow, L_\uparrow, L_\downarrow]$, eq.(2.4) reads

$$
\mathcal{L} = [R^\dagger, L^\dagger]\begin{bmatrix} \partial_-\mathbf{1}, \Delta_0 Qe^{-iQ\frac{\phi}{2}} \\ \Delta_0 Qe^{iQ\frac{\phi}{2}}, \partial_+\mathbf{1} \end{bmatrix}\begin{bmatrix} R \\ L \end{bmatrix}, \tag{2.6}
$$

where $R \equiv {}^t[R_\uparrow, R_\downarrow]$ and $L \equiv {}^t[L_\uparrow, L_\downarrow]$. The action is rotationaly invariant in spin space; and the field $Q(\mathrm{x}, \tau)$, which may be written in the form $Q(\mathrm{x}, \tau) \equiv U(\mathrm{x}, \tau)\sigma_3 U^\dagger(\mathrm{x}, \tau)$, where $U \in \mathrm{SU}(2)$, accounts for the *directional fluctuation* (with a fixed amplitude) of the SDW order parameter, which is expected to be relevant at low energies. Note that when $\phi = 0$ (no dimerization), we recover the fermionic model studied by NO[5]. In the following we fix the value of Δ_0 at the mean field value since the amplitude mode is massive, and only consider the directional fluctuation of $\vec{n}$ (or equivalently Q). (We are for now assuming an undisturbed dimerization pattern, i.e., ϕ=const., in which case a spectral gap $2\Delta_0$ opens up. Later we will also consider the case where the bond alternation strength ϕ acquires a spatial dependence.)

One can notice a strong resemblance between our model and the GW model, eq.(2.4), for a diatomic polymer chain. Indeed, equation (2.4) can be regarded as the direct nonabelian extension of of the GW model, and many similar physical properties can be found.

2.2.2 Details of the derivation

The derivation given here is a slight extension of Jackiw and Semenoff's mapping of a diatomic polymer chain into a continuum relativistic model [33]. First we write the Peierls-Hubbard Hamiltonian as

$$\begin{aligned} H &= \sum_j t_{j+1,j}(a_j^\dagger b_{j+1} + b_{j+1}^\dagger a_j) \\ &\quad + \sum_l t_{l+1,l}(b_l^\dagger a_{l+1} + a_{l+1}^\dagger b_l) \\ &\quad + U(\sum_j n_{j\uparrow}n_{j\downarrow} + \sum_l n_{l\uparrow}n_{l\downarrow}) + H_{phonon}, \end{aligned} \tag{2.7}$$

where $j \in$ odd sites and $l \in$ even sites, and $t_{n+1,n} = t_0 - \gamma(y_{n+1} - y_n)$. For the moment we concentrate on the hopping part. Introducing the variables

$$\begin{aligned} a_j &= (\sqrt{-1})^j \sqrt{2s}U(js) \\ b_j &= (\sqrt{-1})^j \sqrt{2s}V(js) \end{aligned} \tag{2.8}$$

and $y_n = (-1)^n \alpha(ns)$ where s is the lattice constant and further putting

$$\begin{aligned} &U(m+s) - U(m-s) \equiv 2sU'(m) \\ &V(m+s) - V(m-s) \equiv 2sV'(m) \\ &U(2ns-s) + U(2ns+s) \equiv 2U(2ns) \\ &V(2ns) + V(2ns+2s) \equiv 2V(2ns+s), \end{aligned} \tag{2.9}$$

it is straightforward to rewrite the hopping part as

$$\begin{aligned} H_{hop} = -4it_0 s^2 &\sum_n (U^\dagger(2ns+s)V'(2ns+s) \\ &\quad + V^\dagger(2ns)U'(2ns)) \\ +4is\gamma &\sum_n \alpha(ns)\left(U^\dagger(ns)V(ns) - V^\dagger(ns)U(ns)\right). \end{aligned} \tag{2.10}$$

Using the relations

$$\begin{aligned} &\sum_n U^\dagger(2ns+s)V'(2ns+s) \\ = \ &\frac{1}{4}\sum_n (U^\dagger(2ns) + U^\dagger(2ns+2)) \end{aligned}$$

$$\begin{aligned}
& \times (V'(2ns) + V'(2ns+2s)) \\
= & \ \frac{1}{4}\sum_n U^\dagger(2ns) \\
& \times (V'(2ns-2s) + 2V'(2ns) + V'(2ns+2s)) \\
= & \ \sum_n U^\dagger(2ns)V'(2ns) + O(s^2)
\end{aligned} \tag{2.11}$$

(where boundary terms were discarded in the last line) and,

$$\sum_n V^\dagger(2ns)U'(2ns) = \sum_n V^\dagger(2ns+s)U(2ns+s), \tag{2.12}$$

this is reexpressed as

$$\begin{aligned}
H_{hop} &= -2\mathrm{i}t_0 s^2 \sum_n \left(U^\dagger(ns)V'(ns) + V^\dagger(ns)U'(ns)\right) \\
&+ 4\mathrm{i}s\gamma \sum_n \alpha(ns)\left(U^\dagger(ns)V'(ns) - V^\dagger(ns)U(ns)\right).
\end{aligned} \tag{2.13}$$

Finally taking the continuum limit $ns \to \mathrm{x}$ and $s\sum_n \to \int d\mathrm{x}$, we obtain $H_{hop} = \int d\mathrm{x}\mathcal{H}_{hop}$, where

$$\mathcal{H}_{hop} = [U^\dagger, V^\dagger]\left[\begin{array}{c} 0, -\mathrm{i}2st_0\frac{d}{d\mathrm{x}} + 4\mathrm{i}\gamma\alpha(\mathrm{x}) \\ -\mathrm{i}2st_0\frac{d}{d\mathrm{x}} - 4\mathrm{i}\gamma\alpha(\mathrm{x}), 0 \end{array}\right]\left[\begin{array}{c} U \\ V \end{array}\right]. \tag{2.14}$$

Next we turn to the onsite repulsion term. Using the Hubbard-Stratonovich transformation

$$e^{\frac{U}{6}(c^\dagger\vec{\sigma}c)^2} = \int d\vec{\varphi} e^{-\frac{U}{6}\vec{\varphi}^2 + \frac{U}{3}\vec{\varphi}\cdot c^\dagger\vec{\sigma}c}, \tag{2.15}$$

this term becomes

$$H_{int} = \frac{U}{6}\sum_n \vec{\varphi}^2(ns) - \frac{U}{3}\left(\sum_j \vec{\varphi}_j a_j^\dagger\vec{\sigma}a_j + \sum_l \vec{\varphi}_l b_l^\dagger\vec{\sigma}b_l\right). \tag{2.16}$$

Putting $\vec{\varphi}(ns) = (-1)^n m\vec{n}(ns)$ ($|\vec{n}| = 1$), and assuming $\vec{n}(ns)$ to be the slowly-varying variable, we can use the same procedures as in eqs.(2.11) and (2.12) and get

$$\begin{aligned}
H_{int} &= \frac{U}{6}\sum_n 1 + \frac{U}{3}m\sum_n sU^\dagger(ns)(\vec{n}(ns)\cdot\vec{\sigma})U(ns) \\
&\quad - \frac{U}{3}m\sum_n sV^\dagger(ns)(\vec{n}(ns)\cdot\vec{\sigma})V(ns) \\
&\to \int d\mathrm{x}\,[U^\dagger, V^\dagger]\left[\begin{array}{c} \frac{U}{3}mQ(\mathrm{x}), 0 \\ 0, -\frac{U}{3}mQ(\mathrm{x}) \end{array}\right]\left[\begin{array}{c} U \\ V \end{array}\right] \\
&\quad + \text{const},
\end{aligned} \tag{2.17}$$

where $Q(\mathrm{x}) \equiv \vec{n}(\mathrm{x})\cdot\vec{\sigma}$. We now switch to the chiral basis by the transformation

$$\left[\begin{array}{c} R \\ L \end{array}\right] = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} -1 & 1 \\ 1 & 1 \end{array}\right]\left[\begin{array}{c} U \\ V \end{array}\right]. \tag{2.18}$$

Summing up eqs.(2.14) and (2.17), the total Hamiltonian density becomes, aside from a constant

$$\mathcal{H} = [R^\dagger, L^\dagger]\left[\begin{array}{c} -\mathrm{i}v_F\frac{d}{d\mathrm{x}}, \frac{U}{3}mQ - 4\mathrm{i}\gamma\alpha(\mathrm{x}) \\ \frac{U}{3}mQ + 4\mathrm{i}\gamma\alpha(\mathrm{x}), \mathrm{i}v_F\frac{d}{d\mathrm{x}} \end{array}\right]\left[\begin{array}{c} R \\ L \end{array}\right], \tag{2.19}$$

where $v_F = 2st_0$. Using the parametrization

$$\begin{aligned} \frac{U}{3} m &\equiv \Delta_0 \cos\frac{\phi}{2} \\ 4\gamma\alpha &\equiv \Delta_0 \sin\frac{\phi}{2} \end{aligned} \tag{2.20}$$

we arrive at the final form,

$$\mathcal{H} = [R^\dagger, L^\dagger] \left[\begin{array}{c} -iv_F\frac{d}{dx}, \Delta_0 Q e^{-iQ\frac{\phi}{2}} \\ \Delta_0 Q e^{iQ\frac{\phi}{2}}, iv_F\frac{d}{dx} \end{array} \right] \left[\begin{array}{c} R \\ L \end{array} \right] \tag{2.21}$$

2.3 Physical meaning of the fermionic model

Before proceeding to the derivation of a low energy effective action, it is perhaps worthwhile at this point to consider the physical meaning of our fermionic model.

First we point out that the special case $\phi \equiv 0$ corresponds to the continuum approximation for the Hubbard model at half-filling, investigated e.g. by NO. This limit should give a useful check to the subsequent calculations.

The essence of eq.(2.4)

We note that in the above derivation, we had first decoupled the interaction by a SDW MFT, and *then* added the bond alternation. This is because we are considering the case with a finite onsite repulsion, in which case the ground state at half-filing is expected to be in the universality class of a Heisenberg model with staggered exchange interaction. In appendix B, we show that the leading operator from a point splitting of the staggered interaction for the WZW model has the same form as the continuum limit of the bond alternating hopping term in the PH model, which supports this expectation.

It may also be convenient to explain the physical meaning of eq.(2.4) in terms of the effective potential that it yields. Considering for a moment the extreme case in which $\vec{n} \equiv z$, the mass term of this model becomes

$$\begin{aligned} \mathcal{L}_{mass} &= \bar{\Psi}\Delta_0\sigma_3 e^{i\gamma^5\sigma_3\frac{\phi}{2}}\Psi \\ &= L^\dagger\Delta_0\sigma_3 e^{i\sigma_3\frac{\phi}{2}} R + R^\dagger\Delta_0\sigma_3 e^{-i\sigma_3\frac{\phi}{2}} L. \end{aligned} \tag{2.22}$$

Putting back on the factors $e^{\pm ik_F x}$ where appropriate, this gives the following potential energy term:

$$V(\mathrm{x}) = \begin{cases} 2\Delta_0 \sin(2k_F \mathrm{x} + (\frac{\pi}{2} + \frac{\phi}{2})) & (\sigma = \uparrow) \\ 2\Delta_0 \sin(2k_F \mathrm{x} - (\frac{\pi}{2} + \frac{\phi}{2})) & (\sigma = \downarrow) \end{cases} \tag{2.23}$$

whose minima are located at $\mathrm{x}_{min} = ia$ (a: lattice constant), where

$$i = \begin{cases} (2n-1) - \frac{\phi}{2\pi}, n \in \mathbf{Z} & (\sigma = \uparrow). \\ 2n + \frac{\phi}{2\pi}, n \in \mathbf{Z} & (\sigma = \downarrow). \end{cases} \tag{2.24}$$

This suggests the following picture. When $\phi = 0$, odd sites (even sites) are occupied by up-spins (down-spins). This is just the commensurate SDW mean-field theory. Turning on the electron-lattice coupling shifts the position of these up-spins (down-spins) to the left (the right). This will result in a regular array of strong bonds (even-odd bonds) on which a pair of opposing spins will move close to each other, forming a singlet, and weak bonds (odd-even bonds). This is in conformity with the intuitive description of a spin-Peierls state, and also has certain similarities with the abelian bosonization (phase Hamiltonian) approach.

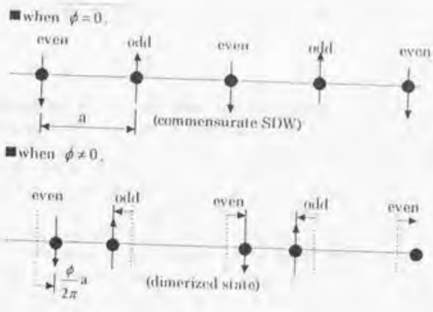

Figure 2.2: Inferred physical picture for the fermionic model

If the directional fluctuation of $\vec{n}$ takes place on a longer scale than a, this picture is still expected to be valid, with the up-spins (down-spins) replaced by spins aligned with (antialigned to) the local direction of $\vec{n}$. Technically, this is done by performing a unitary transformation in spin-space so that the local spin-quantization axis always coincides with the z-axis (cf. Nagaosa-Oshikawa[5]); $\Psi \to U\Psi$, $\bar{\Psi} \to \bar{\Psi}U^\dagger$, $U \in SU(2)$ s.t. $U\sigma_3 U^\dagger = \vec{n}\cdot\vec{\sigma}$. Eq.(2.4) now becomes

$$\mathcal{L} = \bar{\Psi}\left[\not\partial + U^\dagger \not\partial U + \Delta_0 \sigma_3 e^{i\gamma^5 \sigma_3 \frac{\phi}{2}}\right]\Psi. \tag{2.25}$$

If the directional fluctuation is sufficiently slow the second term on the right hand side of eq.(2.25) can be treated as a perturbation, and the physical situation is as stated above. This problem can also be handled systematically by the nonabelian bosonization technique.

Symmetries

Based on these physical pictures, the following symmetry is expected to be respected in the effective theory:

Invariance under simultaneous operations of $\phi \to -\phi$, $\vec{n} \to -\vec{n}$ and $\phi \to \phi + 2\pi$. (The last operation corresponds to a translation by one lattice constant.)

2.4 SU(2n) extension

Besides its application to magnetic systems, the NLσ sigma model also frequently appears in the literature in the context of disordered electron systems, especially in connection with localization properties in two spatial-dimensions [35]. In closing this section, we mention a direct generalization of our continuum model eq.(2.4), which has an important application to this latter class of problems. The generalized model can also be investigated by the bosonization scheme which is developed in this thesis, suggesting another relevant area in which our method can be useful.

In general, the "effective field theory" in this case refers to the exponent of the generating functional for the disorder-averaged advanced and retarded Green functions.

$$W = \int \mathcal{D}V \mathrm{P}[V] \int \mathcal{D}\psi^\dagger \mathcal{D}\psi e^{-\int d^2 r \psi^\dagger (H - E \pm i s_\pm)\psi} \equiv e^{-S_{eff}}. \tag{2.26}$$

(To distinguish from usual effective actions, these are also called transport actions.) In the above equation, we are considering noninteracting electrons in 2 dimensions under a magnetic field, which are subject to a local random potential $V(x, y)$ with a gaussian probability distribution $\mathrm{P}[V]$, and $s_\pm$ are source terms for the advanced and retarded sectors. To make the disorder averaging feasible, one often introduces replicas of the electrons, of n-colors each for both the advanced and retarded sectors, where the limit $\mathrm{n} \to 0$ is to be taken at the end. This replication will generate a four-fermion term, which in turn is decoupled by an SU(2n)/SU(n)×SU(n)-valued auxiliary field Q. (The unitary class appears because of the presence of the magnetic field.) Interpreting one of the spatial coordinates as the imaginary time, one formally obtains a 1+1d fermionic system. Lee[9] and Wang[10], starting from a Chalker-Coddington network model for the IQHE with alternating tunneling amplitudes between the edge states (see fig.2.3), used a continuum approximation to obtain the following fermionic model:

$$\mathcal{L} = \bar{\Psi} \left[\mathbf{1}_{2n} \otimes \not{\partial} + mQ e^{i\gamma^5 \frac{\phi}{2} Q} \right] \Psi, \tag{2.27}$$

where m is the mass generated by the saddle point approximation, and the field Q describes the directional fluctuation within the SU(2n)/SU(n)×SU(n) Grassmanian manifold, which may be written as $Q = U \Lambda U^\dagger$, in which $\Lambda \equiv \mathrm{diag}(+1, +1, \cdots, -1, -1)$ (with n-entries for both +1 and -1) and $U \in$SU(2n). Eq.(2.27) is precisely the generalization of our eq.(2.4) to the case with an SU(2n) internal symmetry, and corresponds to the third category of problems listed in table 1.1. (Here too, the phase ϕ in the chiral exponential factor parametrizes the strength of the alternation between the tunneling amplitudes.) Integrating out the fermions, one expects to find a NLσ model-type effective theory. Fortunately, it is almost a trivial task to extend the derivation of the effective theory for the SU(2) case described in the next chapter to the general SU(2n) model. In ref.[10], Wang attempted to evaluate the effective action for this system by a simple adoption of the procedure of ref.[5]. However, we have pointed out in ref.[11] that this is insufficient, and the correct result (especially the topological contribution) can only be extracted by taking full account of the nonabelian nature and the finiteness of the chiral rotation involved. Needless to say, applications are not limited to this specific model, and suitable modifications enables us to investigate a variety of different physical situations, such as layered structures[10], random flux localization[13], and (to list a topic with a more remote appearance) magnetic impurity effects in 2d d-wave superconductors [12], which is relevant to the issue of quasiparticle-localization in the cuprates.

Though the persistance of the obtained result down to the replica limit n→0 is a delicate problem (which makes it important to compare, when possible, with alternative methods, such as the supersymmetric sigma model approach) this suggests an interesting direction to seak further applications.

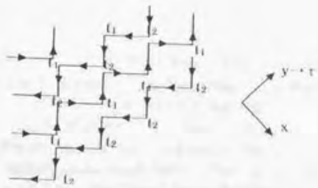

Figure 2.3: Network of edge states with alternating tunneling amplitudes.

Chapter 3

Effective theory

This chapter explains how one can start from eq.(2.4) to obtain a low energy effective theory in terms of the directional fluctuation $\vec{n}$ and $\tilde{\phi}$, which - as we have remarked several times, can be viewed as a new and convenient nonabelian bosonization scheme. Following section 3.1 where we outline the sequence of procedures employed and list the results obtained at each step, we present in section 3.2 the calculational details involved in the central step, namely the calculation of the FJ for the chiral transformation used here. The charge sector, which has so far been left out of the analysis (since we are considering the half-filling case), is briefly discussed in section 3.3. This completes the derivation of our bosonic effective theory. However, as we have emphasized in sections 1.3 and 2.1, a very nontrivial issue regarding the analytic continuation of the chiral angle is implicit within our scheme. In order to validate our results and our methods, we make a detailed comparison with a perturbed WZW approach in the final section. This will enable us to gain a deeper understanding of the role played by chiral anomaly in the present approach.

3.1 General scheme and summary of results

General scheme

When deriving the effective bosonic theory of massless Dirac fermions, the role played by chiral transformations and their corresponding anomalies is completely clear; they are used to decouple the fermion sector from their background fields. This was demonstrated in details in Chapter 1 for the Schwinger model, and a similar procedure will yield the effective theory for massless QCD_2.

However, we are now trying to develop a similar procedure for a massive model. The presence of the mass terms renders it impossible to perform a complete decoupling, which makes it less obvious to decide how to use the chiral anomaly to extract all nonperturbative informations included within the fermionic determinant. So far, no systematic schemes analogous to the massless case have been established in the field theory literature.

Hence we will proceed by using the abelian analogue, the polymer problem of section 2.1 as a guide. As in that case, we will perform a suitable sequence of chiral transformation, which will eliminate the chiral factors from the mass term, and make the mass real. These transformations will generate additional terms in the action, which are derivatives of Q and ϕ. The latter will be treated as a perturbation, using the propagator of the Dirac fermion with a real mass. Meanwhile, the FJ of the chiral transformations are expected to give the topological terms of the effective action. Combining contributions from the FJ and the derivative expansion will complete the derivation of the desired effective theory. We should note at this point that we have no guarantee that the method will successfully yield the correct action, especially the topological term; there always remains the possibility that portions of the topological term are still left unextracted from the fermionic determinant, due to an inappropriate choice of chiral transformations. However, we will find results that strongly suggest the validity of our scheme, which also agree exactly with a perturbed WZW model approach which relies on the conventional nonabelian bosonization theory. In addition, the nontrivial fermion numbers correctly obtained in the next section by our method gives further evidence that we have found the correct general framework. We will therefore be able to safely conclude, that a natural extension of the path integral bosonization method commonly used for massless fermions does exist, for the case with a nonabelian chirally rotated mass term.

The chiral transformations

In order to perform a derivative expansion, we first rotate away the chiral factor from the mass term of eq.(2.4). This is achieved in two steps;

1. First we apply a nonabelian chiral transformation $\Psi \to U_5\Psi$, $\bar\Psi \to \bar\Psi U_5$, where $U_5 \equiv e^{i\gamma^5 Q(\frac{\pi}{4}-\frac{\phi}{4})}$ yielding
$$\mathcal{L} \to \mathcal{L} = \bar\Psi\left[\mathbf{1} \otimes \partial\!\!\!/ + U_5 \partial\!\!\!/ U_5 + i\Delta_0\gamma^5\right]\Psi. \tag{3.1}$$
2. This is followed by a second transformation, $\Psi \to \tilde{U}_5\Psi$, $\bar\Psi \to \bar\Psi\tilde{U}_5$, $\tilde{U}_5 \equiv e^{i\gamma^5\frac{\pi}{4}}$, after which
$$\mathcal{L} \to \mathcal{L} = \bar\Psi\left[\mathbf{1} \otimes \partial\!\!\!/ + U_5 \partial\!\!\!/ U_5 + \Delta_0\right]\Psi. \tag{3.2}$$

Fujikawa Jacobian

The Jacobian associated with the first of these transformations, U_5, is calculated in the next section. This gives rise, beside others, to the so-called topological term for the nonlinear sigma model;

$$\begin{aligned}\mathcal{L}_{WZ} &= \frac{\theta}{16\pi}\mathrm{Tr}[\epsilon_{\mu\nu}Q\partial_\mu Q\partial_\nu Q] \\ &= \mathrm{i}\frac{\theta}{4\pi}\vec{n}\cdot\partial_\tau\vec{n}\times\partial_x\vec{n}\end{aligned} \tag{3.3}$$

where the "vacuum angle" θ is given as

$$\theta = \pi - \phi(\mathrm{x}) - \sin\phi(\mathrm{x}). \tag{3.4}$$

In distinction with conventional spin-chain mappings, the vacuum angle has acquired a spatial (and possibly, temporal) dependence, which will become relevant for disordered systems. It is worth mentioning that this spatial dependence, together with the gradient terms for the ϕ-fields derived below, also has considerable similarities with the effective action for the random-flux localization problem proposed by Zhang and Arovas [13]. Meanwhile, the transformation $\tilde{U}_5$, being an abelian global chiral transformation, does not couple with the $U_5\partial\!\!\!/ U_5$ term, and gives no topological contributions. The Jacobian contains in addition kinetic contributions also. These are (when written in the form of the Euclidean action),

$$S_{kinetic} = \int \mathrm{d}\tau\mathrm{d}x\left[\frac{1}{8\pi}(\partial_\mu\phi)^2 + \frac{1}{8\pi}(1+\cos\phi)(\partial_\mu\vec{n})^2\right]. \tag{3.5}$$

Relation with intuitive picture

A qualitative argument (which unfortunately does not reproduce the precise value of the above θ), based on eqs.(2.23) and (2.24), which illustrates why the vacuum angle θ should shift from $\theta = \pi$ can be given as follows. When $\phi \equiv 0$, it is well known [28] that the WZ term is a summation of Berry phases from the individual spin $\frac{1}{2}$s residing on each site, each proportional to the solid angle ω subtended by the spin during its evolution in imaginary time:

$$S_{WZ} = \mathrm{i}\sum_j(-1)^j\frac{1}{2}\omega(\vec{n}_j)$$

$$\begin{aligned}
&\simeq \frac{i}{4}\int dx\partial_x\omega(\vec{n}(x)) \\
&= \frac{i}{4}\int_0^\beta d\tau\int dx\vec{n}\cdot\partial_\tau\vec{n}\times\partial_x\vec{n}.
\end{aligned} \tag{3.6}$$

This corresponds to $\theta=\pi$. For nonzero ϕ, the semiclassical picture as suggested by eq.(2.24) is that each of the $\omega(\vec{n}_j)$ s get modified in the following way:

$$\begin{aligned}
&\omega(\vec{n}_j) \longrightarrow \omega(\vec{n}(x_j+\delta x_j)) \\
&\simeq \omega(\vec{n}(x_j)) + (-1)^j\frac{\phi}{2\pi}a\partial_x\omega(\vec{n}(x_j))
\end{aligned} \tag{3.7}$$

Due to this modification, using $\delta\omega = \vec{n}\cdot\delta\vec{n}\times\partial_\tau\vec{n}$, we have in the continuum approximation

$$S_{WZ} = \int_0^\beta d\tau\int dx\frac{i}{4\pi}(\pi-\phi)\vec{n}\cdot\partial_\tau\vec{n}\times\partial_x\vec{n}. \tag{3.8}$$

Derivative expansion

We now proceed with the derivative expansion of the fermion determinant of eq.(3.2). This is an expansion in powers of $U_5\partial\!\!\!/ U_5$, which represents (1) the directional fluctuation of $\vec{n}$, and (2) possible slow variations of ϕ:

$$\begin{aligned}
&\mathrm{Tr}\ln[\partial\!\!\!/ + U_5\partial\!\!\!/ U_5 + \Delta_0] = \mathrm{Tr}\ln[\partial\!\!\!/ + \Delta_0] + \mathrm{Tr}\ln\left[1+\frac{1}{\partial\!\!\!/ + \Delta_0}U_5\partial\!\!\!/ U_5\right] \\
&= \mathrm{Tr}\ln[\partial\!\!\!/ + \Delta_0] + \mathrm{Tr}\left[\frac{1}{\partial\!\!\!/ + \Delta_0}U_5\partial\!\!\!/ U_5\right] - \frac{1}{2}\mathrm{Tr}\left[\left(\frac{1}{\partial\!\!\!/ + \Delta_0}U_5\partial\!\!\!/ U_5\right)^2\right] \\
&\quad + \cdots,
\end{aligned} \tag{3.9}$$

In the momentum representation, $\partial\!\!\!/ \to ip\!\!\!/$, where $p\equiv(\omega, v_F k)$. Since $(ip\!\!\!/ + \Delta_0)(-ip\!\!\!/ + \Delta_0) = p^2+\Delta_0^2$, the first order term in the expansion is

$$S_1 = -\mathrm{Tr}\left[\frac{-ip\!\!\!/ + \Delta_0}{p^2+\Delta_0^2}U_5\partial\!\!\!/ U_5\right] = 0. \tag{3.10}$$

(The term $\propto -ip\!\!\!/$ vanishes because it is an odd function of p. The term $\propto \Delta_0$ vanishes because the γ_μ matrices are uncontracted and yields zero when taking the Dirac trace.) Hence the leading contribution starts at a higher order.

In principle, we can evaluate each term of the series eq.(3.9), by shifting p-dependent terms to the left, and x-dependent terms to the right, in the course of which the noncommutativity between p and $U_5\partial\!\!\!/ U_5$ must be properly accounted for. A general recipe for carrying this task out for arbitrary orders of the expansion was given by Fraser and Aitchison [36].

However, the 2nd order terms are potentially dangerous, since there arise diagrams that contain both vector and axial vertices, which, if nonvanishing, may give rise to anomalous contributions. Therefore, in order to avoid obtaining possible spurious terms, which can easily occur from mishandling regularization ambiguities, we resort here to a more neat (though, of course, equivalent) scheme which will give us a definitive answer.

Hence let us go back once more to eq.(2.4), and perform the global chiral transform $\tilde{U}_5$ in the fermions, which gives us the following:

$$\mathcal{L} = \bar{\Psi}\left[\partial\!\!\!/ + \Delta_0 e^{-i2\alpha Q\gamma^5}\right]\Psi, \tag{3.11}$$

where $\alpha\equiv\frac{\pi}{4}-\frac{\phi}{4}$. As before, we next perform the transformation $U_5\equiv e^{i\varphi\gamma^5 Q}$, but this time, we are going to take advantage of the fact that (1)the fermion determinant is invariant with respect to a vector unitary transformation of the fermions, and that (2)the combined operation of U_5 will still eliminate the chiral factor of the mass term in eq.(3.11). Hence we will choose the transformation

$$\begin{cases} \Psi \to e^{-i\alpha Q \mathbf{1}} U_5 \Psi \\ \bar{\Psi} \to \bar{\Psi} U_5 e^{+i\alpha Q \mathbf{1}} \end{cases} \tag{3.12}$$

where the $\mathbf{1}$ was inserted to emphasize that it is a vector transformation. By this transformation, our new Lagrangian becomes

$$\mathcal{L} = \bar{\Psi}\left[\partial\!\!\!/ + \Delta_0\right]\Psi + L^\dagger U^\dagger(\partial_+ U)L, \tag{3.13}$$

where $\partial_+ \equiv \partial_\tau + i\partial_x$, and $U \equiv e^{-i2\alpha Q}$. It is useful to rewrite the last term on the RHS using

$$L^\dagger(U^\dagger \partial_+ U)L = \bar{\Psi}\gamma_\mu P_-(U^\dagger \partial_\mu U)P_-\Psi, \tag{3.14}$$

which can be checked easily. (The P_- is the projection to left chirality, $P_- = \frac{1}{2}(1-\gamma^5)$.) This term will now be the perturbation part. It is easy to see that the first order term vanishes again. Turning to the second order expansion, which is

$$\mathrm{Tr}\left[\frac{-ip\!\!\!/ + \Delta_0}{p^2+\Delta_0^2}\gamma_\mu P_-(U^\dagger\partial_\mu U)P_-\frac{-ip\!\!\!/ + \Delta_0}{p^2+\Delta_0^2}\gamma_\nu P_-(U^\dagger\partial_\nu U)P_-\right], \tag{3.15}$$

we first consider the diagram where the numerator of both of the fermionic propagators contains an m. The Dirac trace of this piece reads

$$\begin{aligned} \mathrm{tr}\gamma_\mu P_- P_- \gamma_\nu P_- P_- &= \mathrm{tr}\gamma_\mu P_- \gamma_\nu P_- \\ &= \mathrm{tr}\gamma_\mu\gamma_\nu P_+ P_- \\ &= \mathrm{tr}0. \end{aligned} \tag{3.16}$$

Hence there are no contributions from here, which is a remarkable result. This has two consequences:

1. Since this is the only place where an anomalous term proportional to $\mathrm{Tr}[\epsilon_{\mu\nu}Q\partial_\mu Q\partial_\nu Q]$ could have arised in the perturbation, we now know that there are no corrections from the expansion to the topological term in the FJ. This leads us to expect that the topological contributions contained in the fermion determinant was completely extracted from the determinant into the FJ part.
2. In addition, this imples that also no kinetic terms of the NLσ model arises from the perturbation either.

The leading contribution of the derivative expansion, coming from the $p\!\!\!/ p\!\!\!/$ part of eq.(3.15), is $\mathrm{Tr}\frac{\pi}{\Delta_0^2}[\partial_-(U(\partial_+ U^\dagger))]^2$, and is a higher derivative term. The above result relied only on the properties of projection operators and are free of ambiguities. In this way we are lead to conclude the following:

The effective theory for eq.(2.4), up to second order in derivatives of Q, comes only from the FJ accompanying the chiral transformation U_5.

We conclude by writing down the final form of our effective theory,

$$S_{kinetic} = \int \mathrm{d}\tau\mathrm{d}x\left[\frac{1}{8\pi}(\partial_\mu\phi)^2 + \frac{1}{8\pi}(1+\cos\phi)(\partial_\mu\vec{n})^2 + \frac{1}{4\pi}(\pi-\phi-\sin\phi)\vec{n}\cdot\partial_\tau\vec{n}\times\partial_x\vec{n}\right] \tag{3.17}$$

[1]

[1] The time derivative part of the kinetic action for ϕ may not be valid in the Peierls-Hubbard context since we are using the adiabatic approximation.

3.2 Details of FJ evaluation -SU(2n) formulation-

It turns out that the calculation of the FJ can be done in a unified manner for general SU(2n). The spin chain result is obtained by the substitution n=1 at the end of the calculation, while, as we have mentioned in section 2.4, the replica limit (n→0) of the SU(2n) case, eq.(2.27), is known to be relevant to 2d disordered electrons in a magnetic field, e.g. the integer quantum Hall effect. As we have said, for the latter case, there have been confusion regarding the value of the vacuum angle of the topological term in the literature[10] which has its origin in an incorrect treatment of the anomaly. This suggests how nontrivial the issue is. It is therefore worthwhile to present the details of the evaluation of the topological contributions to the FJ in the general SU(2n) formulation. This will be the subject of this subsection.

We also note in passing that the NO paper [5] proposed to obtain the Jacobian in the rotating frame, which we mentioned in the previous section. In fact, it can be shown that the FJ, for eliminating the chiral factor of the mass term in *any* frame reached by a vector transformation is the same. That will be proved in Appendix C.

Consider the Lagrangian

$$\mathcal{L} = \bar{\Psi}(\mathbf{I} \otimes \not{\partial} + mU\Lambda U^{\dagger} e^{iU\Lambda U^{\dagger}\frac{\phi}{2}\gamma^5})\Psi \tag{3.18}$$

where $U \in SU(2n)$. Λ is given by

$$\Lambda = \begin{pmatrix} \mathbf{I}_n & 0 \\ 0 & -\mathbf{I}_n \end{pmatrix}. \tag{3.19}$$

Note the property $(U\Lambda U^{\dagger})^2 = U\Lambda U^{\dagger}U\Lambda U^{\dagger} = \mathbf{I}_{2n}$. Since $U\Lambda U^{\dagger} = i\gamma^5 e^{-i\gamma^5 U\Lambda U^{\dagger}\frac{\pi}{2}}$, the chiral transformation that makes the mass term look like $\sim m\bar{\Psi}\Psi$ is:

$$\begin{aligned} \Psi &\to e^{+iU\Lambda U^{\dagger}(\frac{\pi}{4}-\frac{\phi}{2})\gamma^5}\Psi \\ \bar{\Psi} &\to \bar{\Psi}e^{+iU\Lambda U^{\dagger}(\frac{\pi}{4}-\frac{\phi}{2})\gamma^5}. \end{aligned} \tag{3.20}$$

(As explained in the preceding section, actually one ends up with a purely imaginary mass, and making this real requires performing an additional abelian global chiral transformation. Since the latter will not affect the calculations below, it will not be done explicitly.) The transformation in eq.(3.20) is an example of a ***nonabelian*** chiral transformation, and is defined more precisely as

$$\begin{aligned} &e^{+iU\Lambda U^{\dagger}(\frac{\pi}{4}-\frac{\phi}{2})\gamma^5} \\ &\equiv \cos(\frac{\pi}{4}-\frac{\phi}{2})\mathbf{I}\otimes\mathbf{I} + iU\Lambda U^{\dagger}\otimes\gamma^5\sin(\frac{\pi}{4}-\frac{\phi}{2}), \end{aligned} \tag{3.21}$$

where the $\otimes$-symbols (which will usually be omitted below) are inserted to distinguish explicitly between operators in the SU(2n)-internal spin space and the SU(2) iso-spin (chirality) space. Following methods developed originally by the La Plata group [37],we introduce

$$U_5(t) \equiv e^{+itU\Lambda U^{\dagger}(\frac{\pi}{4}-\frac{\phi}{2})\gamma^5}, \tag{3.22}$$

in terms of which the Fujikawa Jacobian J for the transformation of eq.(3.20) can be expressed as

$$\ln J = -\frac{1}{2\pi}\int_0^1 dt \mathrm{Tr}\left[iU\Lambda U^{\dagger}(\frac{\pi}{4}-\frac{\phi}{2})\gamma^5 D_t^2\right], \tag{3.23}$$

where

$$D_t \equiv U_5(t)(\not{\partial}U_5(t)) = \not{\partial} + U_5(t)\not{\partial}U_5(t). \tag{3.24}$$

The second equality in eq.(3.24) was obtained by the use of the relation $U_5(t)\gamma^{\mu} = \gamma^{\mu}U_5(-t)$. Now we proceed to obtain the topological contributions (i.e. terms proportional to the topological charge density) that are included in eq.(3.23). First we note that assuming ϕ = const, (as far as the topological term is concerned, derivatives of ϕ do not play a role) $\partial_\mu U_5(t) = i\partial_\mu Q\gamma^5 \sin t(\frac{\pi}{4} - \frac{\phi}{4})$ where $Q \equiv U\Lambda U^\dagger$. Hence

$$\begin{aligned}
&U_5(t)\partial\!\!\!/ U_5(t) = \gamma^\mu U_5(-t)\partial_\mu U_5(t) \\
&= \gamma^\mu \left(\cos t(\frac{\pi}{4} - \frac{\phi}{4}) - iQ\gamma^5 \sin t(\frac{\pi}{4} - \frac{\phi}{4})\right) \\
&\quad \cdot i\partial_\mu Q\gamma^5 \sin t(\frac{\pi}{4} - \frac{\phi}{4}) \\
&= i\partial_\mu Q\gamma^\mu\gamma^5 \frac{1}{2} \sin t(\frac{\pi}{2} - \frac{\phi}{2}) \\
&\quad + \gamma^\mu Q\partial_\mu Q\frac{1}{2}(1 - \cos t(\frac{\pi}{2} - \frac{\phi}{2})).
\end{aligned} \tag{3.25}$$

With the use of eq.(3.25), we find that the following two terms from $(U_5(t)\partial\!\!\!/ U_5(t))^2$ survive the trace in eq.(3.24):

(A) $i\gamma^\mu\gamma^5 i\gamma^\nu\gamma^5\partial_\mu Q\partial_\nu Q\frac{1}{4}\sin^2 t(\frac{\pi}{2} - \frac{\phi}{2})$
$\rightarrow -i\epsilon_{\mu\nu}\gamma^5\frac{1}{4}\sin^2 t(\frac{\pi}{2} - \frac{\phi}{2})\partial_\mu Q\partial_\nu Q$

(B) $\gamma^\mu\gamma^\nu(Q\partial_\mu Q)(Q\partial_\nu Q)\frac{1}{4}(1 - \cos t(\frac{\pi}{2} - \frac{\phi}{2}))^2$.
$\rightarrow +i\epsilon_{\mu\nu}\gamma^5\frac{1}{4}(1 - \cos t(\frac{\pi}{2} - \frac{\phi}{2}))^2\partial_\mu Q\partial_\nu Q$.

Likewise, the term from $\gamma^\mu\partial_\mu(U_5(t)\partial\!\!\!/ U_5(t))$ that contributes is

(C) $\gamma^\mu\gamma^\nu\partial_\mu Q\partial_\nu Q\frac{1}{2}(1 - \cos t(\frac{\pi}{2} - \frac{\phi}{2}))$.
$\rightarrow -i\epsilon_{\mu\nu}\gamma^5\frac{1}{2}(1 - \cos t(\frac{\pi}{2} - \frac{\phi}{2}))\partial_\mu Q\partial_\nu Q$.

Adding these three terms, we find

$$\begin{aligned}
&\mathrm{Tr}\left[Q\gamma^5((\mathrm{A}) + (\mathrm{B}) + (\mathrm{C}))\right] \\
&= \frac{1}{4}\cdot 2\cdot(1 - \cos t(\pi - \phi))\mathrm{Tr}[(-i\epsilon_{\mu\nu})Q\partial_\mu Q\partial_\nu Q],
\end{aligned} \tag{3.26}$$

where the trace over the γ-matrices, i.e. $\mathrm{Tr}(\gamma^5)^2 = 2$ was taken in the final expression. Substituting this into eq.(3.23), we finally obtain

$$\begin{aligned}
\ln J &= -\frac{1}{2\pi}(-i)\frac{1}{2}(i)\int_0^1 dt\frac{1}{4}(\pi - \phi)(1 - \cos t(\pi - \phi)) \\
&\quad \times \mathrm{Tr}[\epsilon_{\mu\nu}Q\partial_\mu Q\partial_\nu Q] \\
&= -\frac{1}{16\pi}(\pi - \phi - \sin\phi)\mathrm{Tr}[\epsilon_{\mu\nu}Q\partial_\mu Q\partial_\nu Q].
\end{aligned} \tag{3.27}$$

We now examine eq.(3.27) for the special case of n=1, i.e. SU(2). In this case $\Lambda = \sigma_3$, and so the matrix Q can be written in terms of the unit vector $\vec{n}$ as $Q = \vec{n}\cdot\vec{\sigma}$. Then using the identity $\mathrm{Tr}\left[(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})(\vec{c}\cdot\vec{\sigma})\right] = 2i\vec{a}\cdot\vec{b}\times\vec{c}$, we immediately obtain

$$\begin{aligned}
\ln J &= -\frac{2i}{16\pi}(\pi - \phi - \sin\phi)\int d^2x\epsilon_{\mu\nu}\vec{n}\cdot\partial_\mu\vec{n}\times\partial_\nu\vec{n} \\
&= -i(\pi - \phi - \sin\phi)Q_{x\tau},
\end{aligned} \tag{3.28}$$

where $Q_{x\tau} \equiv \frac{1}{4\pi}\int d^2x\vec{n}\cdot\partial_\tau\vec{n}\times\partial_x\vec{n}$ is the skyrmion number.
This completes the evaluation of the topological term. The kinetic part can be obtained in exactly the same way, which yields eq.(3.5).

3.3 Description of the charge sector

Up to now, we have not mentioned the charge sector. This is because we are concerned with the half-filling case, for which the Hubbard gap freezes the charge excitations. It is however, important for the consistency of the formulation, to be able to describe the seperation of the charge and spin sectors. Of course, this will also become necessary when we want to extend our formulation to cases away from half-filling. Here we shall briefly discuss how this may be done.

Following ref.[5], we can introduce an additional U(1) chiral phase factor $e^{i\gamma^5\Phi}$ to the mass term of model (2.4):

$$\mathcal{L} = [R^\dagger, L^\dagger] \begin{bmatrix} \partial_- \mathbf{1}, & \Delta_0 Q e^{-iQ\frac{\pi}{2}} e^{-i\Phi} \\ \Delta_0 Q e^{iQ\frac{\pi}{2}} e^{i\Phi}, & \partial_+ \mathbf{1} \end{bmatrix} \begin{bmatrix} R \\ L \end{bmatrix}, \tag{3.29}$$

The pseudoscalar field Φ represents the charge sector, whose origin can be understood as follows. We begin by noting that the introduction of a hole at some site will (1) flip the spin, which is descibed by inverting sign of the spin-field Q to its right, and (2) invert the sign of the dimerization field ϕ. These effects can be incorporated by making (at the level of the decoupled lattice Hamiltonian) the following adjustment:

$$\Delta_0 \to \Delta_0 \cos[\pi \int_{-\infty}^{x} dy n(y)] \tag{3.30}$$

where $n(y)$ is the number of holes residing in the interval $(-\infty, y)$. Accordingly, the continuum form of this piece will become

$$(-1)^j \Delta_0 \to \Delta_0 \cos[2k_F x + \Phi(x)] \tag{3.31}$$

where $\Phi(x) \equiv -\pi \int_{-\infty}^{x} dy n(y)$, and $k_F \equiv \pi(1 - n_h)/2$, with n_h the number of holes. Expressing the fermion field operator in terms of left and right go-ers, we arrive at eq.(3.29).

The effective action for Φ can be obtained in analogy with what we have done for the spin field Q: we perform a U(1) (i.e. spin-independent) chiral transformation $R_\sigma \to e^{-i\frac{\Phi}{2}} R_\sigma$, $L_\sigma \to e^{+i\frac{\Phi}{2}} L_\sigma$, which eliminates this chiral factor (and produces a new term $\bar\psi \frac{1}{2} \partial\!\!\!/ \Phi \gamma^5 \psi$, to be treated as a perturbation), obtain the corresponding FJ and the perturbative contributions from the derivative expansion. The result is a free scalar field action,

$$\mathcal{L}_\Phi = \frac{1}{4\pi v_F} (\partial_\mu \Phi)^2. \tag{3.32}$$

(No cross-terms between Q and Φ arises.) It is worth noting that precisely as in the spin sector, this term comes solely from the chiral anomaly.

Interactions should renormalize the charge velocity and stifness. It is customary in conventional bosonization to incorporate this effect by adding the forward scattering term,

$$H_{forward} = U(\bar\Psi \gamma^0 \Psi)^2, \tag{3.33}$$

which, using the relation $R^\dagger R = \frac{1}{4\pi v_F} \partial_+ \Phi$ and $L^\dagger L = \frac{1}{4\pi v_F} \partial_- \Phi$, is added to eq.(3.32) to give

$$\mathcal{L}_\Phi = \frac{1}{4\pi v_\rho K_\rho} [(\partial_\tau \Phi)^2 + v_\rho^2 (\partial_x \Phi)^2], \tag{3.34}$$

where $K_\rho = (1 + U/2\pi v_F)^{-1/2}$ and $v_\rho = v_F(1 + U/2\pi v_F)^{-1/2}$. To understand how the forward scattering enters into our formulation, we need to go back to decoupling of the density-density product of the Hubbard interaction term,

$$n_\uparrow n_\downarrow = \frac{\rho^2}{4} - (\vec{S} \cdot \vec{n})^2. \tag{3.35}$$

Up to now, the first term in the right hand side had been ignored. A correct description of the charge sector requires us to take it into account. For this purpose, an additional auxiliary field Δ_c has to be introduced,

$$e^{-Un_\uparrow n_\downarrow} = \int d\Delta_c dm d\vec{n} \exp[-(\frac{\Delta_c^2}{U} + i\Delta_c\rho + \frac{m^2}{U} - 2m\vec{n}\cdot\vec{S})]. \tag{3.36}$$

When the abelian chiral transformation mentioned above is done, the second term in the exponent of the right hand side of eq.(3.36) can be grouped together with the induced term containing $\partial_x\Phi$. (Both $\partial_x\Phi$ and $i\Delta_c$ represent the charge density.) Thus the modified action concerning Φ and Δ_c is

$$\mathcal{L} = \frac{1}{4\pi v_F}\left[(\partial_\tau\Phi)^2 + v_F^2(\partial_x\Phi + \frac{2i}{v_F}\Delta_c)^2\right] + \frac{1}{U}\Delta_c^2. \tag{3.37}$$

Integrating out the massive field Δ_c yields eq.(3.34).

In the case of half-filling, this must further be supplemented by the unklapp term (which we write here in the lattice form)

$$\mathcal{H}_{Umklapp} = U(R_\uparrow^\dagger(n)L_\uparrow(n)R_\downarrow^\dagger(n)L_\downarrow(n)e^{i4k_Fn} + h.c.), \tag{3.38}$$

since the $4k_F$ oscillating factors are unity in this case. This term was originally dropped in the passage to the continuum model.

When the aforementioned chiral transformation is performed, the two terms on the RHS of eq.(3.38) acquire the phase factors $e^{\pm 2i\Phi}$, so that accordingly the action for Φ becomes

$$\mathcal{L}_\Phi = \frac{1}{4\pi v_\rho K_\rho}[(\partial_\tau\Phi)^2 + v_\rho^2(\partial_x\Phi)^2] + A\cos 2\Phi, \tag{3.39}$$

where A is a constant. In the present formalism, this cosine potential energy term can also be picked up from the square of $\cos(2k_FR_j + \Phi_j)$, in which $k_F = \frac{\pi}{2}$.

We thus see that at half-filling, the umklapp process fixes the charge phase field Φ, as it should.

3.4 Comparision with perturbed WZW model

When the global chiral transformation $\Psi \to \tilde{U}_5\Psi$, $\bar\Psi \to \bar\Psi\tilde{U}_5$, $\tilde{U}_5 \equiv e^{i\gamma^5\frac{\pi}{4}}$ which was introduced in section 3.1 is applied to eq.(2.4), the resulting fermion determinant D can be expressed in two different ways, viz

$$D = \mathrm{Tr}\ln\left[\partial\!\!\!/ + \Delta_0 e^{-2i\gamma^5 Q\alpha}\right], \tag{3.40}$$

where $\alpha = \frac{\pi}{4} - \frac{\phi}{4}$, and

$$D = \mathrm{Tr}\ln\left[\partial\!\!\!/ + \frac{1}{2}(1+\gamma^5)\Delta_0 U + \frac{1}{2}(1-\gamma^5)\Delta_0 U^\dagger\right], \tag{3.41}$$

where the SU(2) matrix U is defined as

$$U \equiv e^{-i2\alpha Q}. \tag{3.42}$$

The first expression was used in the previous sections, while the second one is of the form of eq.(1.1), introduced in section 1.1. Tsvelik has used conventional nonabelian bosonization methods, to deal with the latter type of model [39]. Since this will provide us with an important check, which will record and apply his method here. First, using the nonabelian bosonization rule: free fermion $\leftrightarrow$ level k=1 SU(2) WZW model+free scalar field, and $R_\alpha^\dagger L_\beta \leftrightarrow \frac{1}{2\pi a}e^{i\sqrt{2\pi}\Phi}g_{\alpha\beta}$ where $g \in$SU(2) and Φ is a scalar field, one can write

$$e^{D[U]} = \int Dg D\Phi \exp\left[-\int d^2x\left(W(g) + \frac{1}{2}(\partial_\mu\Phi)^2 + \frac{\Delta_0}{2\pi a}\mathrm{Tr}(Ug^\dagger e^{-i\sqrt{2\pi}\Phi} + gU^\dagger e^{i\sqrt{2\pi}\Phi})\right)\right]. \tag{3.43}$$

where $W(g)$ is the WZW model,

$$W[g] = \frac{1}{16\pi}\int d^2x \mathrm{Tr}(\partial_\mu g^{-1}\partial_\mu g) + \frac{1}{24\pi i}\int d^3 \epsilon_{\mu\nu\lambda}\mathrm{Tr} L_\mu L_\nu L_\lambda \tag{3.44}$$

with $L_\mu \equiv g^{-1}\partial_\mu g$, and the second term is defined by embedding the compactified space-time S_2 into a three-ball. (The field g takes its physical value at the surface of the solid ball, and becomes $g \equiv 1$ at the center.) Making the shift of variables:

$$g \to UG \tag{3.45}$$

which leaves the measure invariant, and using the Polyakov-Wiegmann identity,

$$W[UV] = W[U] + W[V] + \frac{1}{2\pi}\int d^2x \mathrm{Tr}(U^{-1}\partial U V \bar{\partial} V^{-1}), \tag{3.46}$$

where $\bar{\partial} = \frac{1}{2}\partial_-$ and $\partial = \frac{1}{2}\partial_+$, we have

$$e^{D[U]} = \int \mathcal{D}G\mathcal{D}\Psi \exp[-S[G,\Psi] - S_{int}[U,G] - W[U]] \tag{3.47}$$

$$S[G,\Psi] = \int d^2x [\frac{\Delta_0}{2\pi a}\mathrm{Tr}(e^{i\sqrt{2\pi}\Phi} G + G^{-1}e^{-i\sqrt{2\pi}\Phi}] + W[G] + \frac{1}{2}\int d^2x(\partial_\mu \Psi)^2 \tag{3.48}$$

$$S_{int} = \frac{1}{2\pi}\int d^2x \mathrm{Tr}(U^{-1}\partial U G \bar{\partial} G^{-1}). \tag{3.49}$$

Using the bosonization rule $L^\dagger_\alpha \tau^a_{\alpha\beta} L_\beta \leftrightarrow -\frac{1}{2\pi}\mathrm{Tr}(\tau^a g\bar{\partial}g^{-1})$ to refermiónize S_{int}, the total action besides the free scalar field part becomes

$$S = W[U] + \int d^2x \left[\bar{\Psi}(\not{\partial} + \Delta_0)\Psi + L^\dagger U^{-1}\partial U L\right]. \tag{3.50}$$

It is easy to see that the fermionic part is, (up to a constant factor of the 3rd term) equivalent to eq.(3.13). Hence from the results of section 3.1, we know that the 3rd term does not contribute to the effective action to quadratic order. Hence we are left with

$$S[U] = W[U] + O(\Delta_0^{-2}) \tag{3.51}$$

Substituting eq.(3.42) into eq.(3.51), we obtain, after some straightforward algebra, an expression which agrees completely with our eq.(3.17).

This consistency assures us that the method we have used is valid. It is worth noting that Tsvelik's strategy was also to simplify matters by isolating the real mass m from additional factors. We also mention in passing, that it is possible to make an identification between the parameter t we had introduced in order to performing infinitesmal chiral transformations successively, and the third coordinate introduced to define the three-form term in the WZW model.

Chapter 4

Nontrivial fermion numbers

Induced fermion numbers occur when the vacuum of the Dirac fermion is polarized by the topological excitation of some background field. Since this is a topology-related quantum phenomenenna with connections to chiral anomaly, we shall, as a simple application, consider in this chapter this aspect of our model.

4.1 Pure systems

As a first example of induced fermion numbers in the present model, we discuss excitations in the nondoped system, regarding the topological charge density $q_{x\tau}$ as the nontrivial background which polarizes the fermionic vacuum. We will derive a relation connecting excitations in the charge and spin sectors, which may appear counterintuitive in view of the well-accepted notion of spin-charge seperation.

Let us begin by adding the following source term to the action:

$$\mathcal{L}_s = \mathrm{i}s\bar{\Psi}\gamma^5\Psi. \tag{4.1}$$

The source field $s(\mathrm{x},\tau)$ couples to $\mathrm{i}\bar{\Psi}\gamma^5\Psi = \mathrm{i}(L^\dagger R - R^\dagger L)$ which is just the backscattering current, i.e., the $k = \pi$ component of the U(1) vector current. The total action is now

$$\begin{aligned}\mathcal{L}_{tot} &= \mathcal{L} + \mathcal{L}_s \\ &\equiv \bar{\Psi}\left[\partial\!\!\!/ + M(s)Qe^{\mathrm{i}Q\Theta(s)\gamma^5}\right]\Psi,\end{aligned} \tag{4.2}$$

where $M(s)\cos\Theta(s) \equiv m\cos\frac{\phi}{2}$ and $M(s)\sin\Theta(s) \equiv m\sin\frac{\phi}{2} + s$. Hence the effect of adding the source term can simply be viewed as a shift in the value of both the modulus and the chiral phase of the original mass term. We can therefore treat $\mathcal{L}_{tot}$ in the same way as in the previous section, by rotating away the chiral angle $\Theta(s)$. Again the Fujikawa Jacobian gives rise to the topological term

$$\mathcal{L}_{wz} = \mathrm{i}(\pi - 2\Theta(s) - \sin 2\Theta(s))q_{x\tau}. \tag{4.3}$$

We see that the source field has found its way into the vacuum angle. This leads us to expect that the background instanton configuration, characterised by the topolological charge density $q_{x\tau}$, will contribute to the expectation value of the quantity $\mathrm{i}\bar{\Psi}\gamma^5\Psi$. It is easy to confirm that this is indeed the case, viz.

$$\begin{aligned}&< \mathrm{i}\bar{\Psi}\gamma^5\Psi >_{ind} = \lim_{s\to 0}\frac{\delta S_{wz}}{\delta s} \\ &= -\mathrm{i}q_{x\tau}\lim_{s\to 0}(1+\cos 2\Theta(s))\cdot 2\frac{d\Theta(s)}{ds} \\ &= -\mathrm{i}\frac{2}{m}\cos\phi(1+\cos 2\phi)q_{x\tau}\end{aligned} \tag{4.4}$$

Figure 4.1: Illustration of a spinon induced by a 2π-shift of the phase ϕ.

The significance of the relation (4.4) lies in the fact that while the left hand, as already mentioned, describes the backscattering channel of the charge sector, the right hand side is related (through $q_{x\tau}$) to the fluctuation in the spin sector. Since the two sectors are seperated by the charge gap, the interrelation between them was rather unexpected.

The $\phi \equiv 0$ case of eq.(4.4) was previously derived by Tsvelik in the context of the 1d Kondo insulator at half filling, as a Ward-identity of the fermion system $\mathcal{L}$ [38]. (In this problem the effective theory for the spin sector is a O(3) NL σ model whose vacuum angle is either $\theta = 0$ or $\theta = \pi$, depending on microscopic details.) There the Polyakov-Wiegmann identity[22], satisfied by the WZNW model was the key to the derivation. Here we have provided an alternative route, which is a simple application of the Fujikawa-Jacobian technique used in the previous section. (Reflecting the symmetry of our problem, our eq.(4.4) is a generalization of the Kondo-lattice case to arbitrary θ.) Tsvelik further observed that since the two-point correlation function of the backscattering current is related to the $k = \pi$ component of the optical conductivity $\sigma(\omega)$, the latter will reflect informations on the two-point correlation of the topological charge density $q_{x\tau}$. Then using the asymptotics of $q_{x\tau}$'s 2-point correlator for the O(3) NLσ model with vacuum angles $\theta = 0$ and $\theta = \pi$, he was able to predict qualitatively different behaviour of $\sigma(\omega, k = \pi)$ for these two cases.

Though we can in principle follow these ideas here, there are some difficulties. First the asymptotic behaviour of the $q_{x\tau}$-$q_{x\tau}$ correlators are not available for arbritary values of the θ angle, so it is hard to make reliable predictions on the correlators of the backscattering current. Another point that should be considered is the factor of $1/m$ appearing in the right hand side of eq.(4.4). Since $m \sim U$, our relation is a finite-U effect which vanishes in the limit of infinite-U. Therefore we expect that for our case the right hand side will be of a very small order, making experimental detections difficult. For these reasons, we shall not further pursue this problem in detail. In spite of this, however, we do believe that it is theoretically interesting that certain charge excitations reflect, through topological effects, informations on the spin sector excitations. Furthermore, from a symmetry-based viewpoint, the spin-Peierls system, with its folding symmetry, is perhaps the right system to make optical measurements of this sort since the $k = \pi$ components will naturally mix into the observed optical conductivity.

4.2 Disordered systems

One way of understanding the (bulk) spin-Peierls state is as a phase in which *spinons* - domain walls of the alternating array of strong and weak bonds - are permanently confined into pairs. Let us begin by considering how to describe these excitations within the present framework.

From the simple picture provided by the effective potential energy of eq.(2.23), it is natural to expect that a domain wall in the alternation pattern can be expressed as a shift of the argument of the cosines by one lattice constant, which can in turn be absorbed into *a kink of the $\phi(x)$-field*, with a height of 2π. Since a spinon gives rise to one site with which there is no pair to form a spin-singlet bond, its spin quantum number should be $S = \frac{1}{2}$. This situation is depicted in Figure 4.1.

To check if the 2π-kink of the ϕ-field actually carries such a quantum number, we use the rotating frame of eq.(2.25). Since the transformation $\Psi \to U\Psi_{rot}$, $\bar{\Psi} \to \bar{\Psi}_{rot}U^{\dagger}$ is a vector gauge

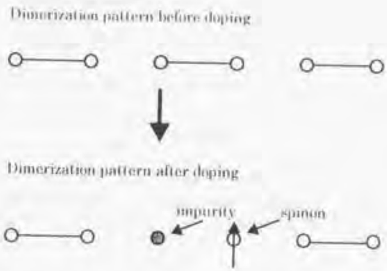

Figure 4.2: a spinon confined to a dopant site

transformation, it will not give rise to anomalous terms, and so we have the same WZ-term as in the laboratory frame (see appendix B.2 for the proof of invariance of the Fujikawa-Jacobian under a vector transformation.) Introducing the SU(2) connections a_μ^α (α=1, 2, 3) via[8] $U^\dagger \partial\!\!\!/ U \equiv i a\!\!\!/_\mu^\alpha \sigma_\alpha$, we first note that

$$\frac{\delta S_{eff}}{\delta a_\tau^3} = \mathrm{i} < \Psi_{rot}^\dagger \frac{\sigma_3}{2} \Psi_{rot} >, \tag{4.5}$$

Then using the easily-derived identities

$$(a_\mu^1)^2 + (a_\mu^2)^2 = \frac{1}{4}(\partial_\mu \vec{n})^2, \tag{4.6}$$

and

$$q_{\mathrm{x}\tau} = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\mu a_\nu^3, \tag{4.7}$$

it is apparent that the only contribution to eq.(4.5) comes from the WZ term S_{WZ}, so that

$$\begin{aligned} < \Psi_{rot}^\dagger \frac{\sigma_3}{2} \Psi_{rot} >_{induced} &= \mathrm{i}\frac{\delta S_{WZ}}{\delta a_\tau^3} \\ &= -\frac{\mathrm{i}}{2\pi}\partial_{\mathrm{x}}(\phi + \sin\phi) \end{aligned} \tag{4.8}$$

This is the counting rule for the spin quantum number induced by a ϕ-field kink. Integrating eq.(4.8) over the extent of the kink for which $\int d\mathrm{x}\partial_{\mathrm{x}}\phi = \pm 2\pi$, we obtain S$=\pm\frac{1}{2}$. Since $\phi(\mathrm{x})$ does not couple to a U(1) gauge-field (see below), it is also clear that the charge quantum number Q is $Q = 0$. From these considerations, we conclude that this kink corresponds to a spinon.

We now move onto the case in which a nonmagnetic impurity is doped into one of the sites of a chain. This will necessarily cause a depletion of charge, by one unit, at the dopant site. Then an unpaired spinon excitation, of the kind we have considered above would be released due to the unpaired singlet bond. In an isolated chain, this spinon would, in absence of frustrations such as next-nearest neighbor interactions, be mobile and can proliferate through the chain. However, in actual systems, there are interchain interactions, which disfavors a misaligned segment (caused by a mobile spinon) on an adjacent chain. This gives an effective confinement mechanism—the spinons are confined to the vicinity of the dopant site, so that the misaligend segment can be as short as possible. We therefore expect that a spinon will reside adjacent to the dopant site. As depicted in figure 4.2, this will cause only a minimal local distortion, and the global dimer pattern is restored away from the dopant-spinon composite object. How do we express this doping effect by our methods? As for the spinon part, we have already seen that the dimer modulation, accounted for by the 2π-kink of the ϕ-field gives rise to a spinon. In order to account for the charge depletion also, we need to introduce an abelian chiral phase which we shall denote here as φ_c. Hence upon doping the mass term undergoes the following change,

$$\mathcal{L}_{mass} = \bar{\Psi}\Delta Q e^{i\gamma^5 Q\frac{\phi}{2}}\Psi \rightarrow \bar{\Psi}\Delta Q e^{i\gamma^5 Q\frac{\phi+\delta\phi(x)}{2}} e^{i\gamma^5\varphi_c(x)}\Psi, \tag{4.9}$$

where the $\delta\phi(x)$ is the kink part of the SU(2) chiral angle giving rise to the spinon,

$$S = \frac{1}{4\pi}\int dx\partial_x(\delta\varphi) = \frac{1}{2}. \tag{4.10}$$

Meanwhile, the counting rule for the charge induced by the spatial modulation of the abelian chiral angle was already obtained when we reviewed the diatomic polymer problem, so we know that a unit-charge depletion corresponds to a π-kink of the $\varphi_c(x)$ field,

$$\mathrm{Q} = \frac{1}{\pi}\int dx\partial_x\varphi_c = 1. \tag{4.11}$$

Notice that the combined effect of $\phi(x) \rightarrow \pm 2\pi$ and $\varphi_c(x) \rightarrow \varphi_c(x) \pm \pi$ leaves $\bar{\Psi}\Delta Q e^{i\gamma^5 Q\frac{\phi}{2}} e^{i\gamma^5\varphi_c(x)}\Psi$ invariant. Hence we conclude that the liberation of a spinon by nonmagnetic impurity doping is consistently described by a composite object of SU(2) and U(1) kinks, where the nonabelian and abelian anomaly each induce the spinon and the charge depletion, resulting in a restoration of the global dimer pattern. Of course this is how it must turn out, and probably does not infer any new physics. (A thorough classification of soliton-like excitations and their quantum numbers may be found in ref.[2].) However, this result gives an important confirmation on the consistency of our framework, and assures us that it can be put to further applications in more realistic situations.

Chapter 5

Discussions and conclusions

5.1 Discussions

We discuss some points we have left out in the previous chapters.

First we comment on the physical reason behind the appearance of the chiral anomaly in the spin sector. When we recall the abelian case, it was the conservation of total charge which lead to the nonconservation of chiral charge in the presense of an electric field. This is an obvious fact when one thinks in terms of the original nonrelativistic band dispersion, and is reproduced in the relativistic approximation as the chiral anomaly. In the PH model, there is, besides the global U(1) symmetry, a global SU(2) symmetry which becomes manifest when rewriting the repulsion term using

$$n_\uparrow n_\downarrow = -\frac{2}{3}\vec{S}^2 + \text{const}, \tag{5.1}$$

which, of course implies that the total spin is conserved. Hence, when there is a nonzero curvature of the spin gauge field (the analogue of the electric field) which in this case is the instanton number density, conservation of chiral spin density is broken quantum mechanically (in addition to the trivial breaking by the mass), which is the origin of the chiral anomaly. From this discussion we think it is clear that the anomaly discussed here must be derived taking into account the full nonabelian nature of the problem, as we have emphasized in chapters 2 and 3.

Next we make some remarks on the difference of our scheme and the one carried out in ref[5], since we do not wish to give the impression that the two schemes are different ways of expressing an equally acceptable treatment. For simplicity we consider the nondimerized ($\phi = 0$) case,

$$\mathcal{L} = [R^\dagger, L^\dagger]\begin{bmatrix} \partial_-, & mQ \\ mQ, & \partial_+ \end{bmatrix}\begin{bmatrix} R \\ L \end{bmatrix}. \tag{5.2}$$

Following ref[5], we transfer to the rotating frame, via the SU(2) vector transformation $\Psi \to U\Psi$, $\bar{\Psi} \to \bar{\Psi}U^\dagger$, by which the above equation becomes

$$\begin{aligned} \mathcal{L} &= [R^\dagger, L^\dagger]\begin{bmatrix} \partial_- + ia^3_-\sigma_3, & m\sigma_3 \\ m\sigma_3, & \partial_+ + ia^3_+\sigma_3 \end{bmatrix}\begin{bmatrix} R \\ L \end{bmatrix} \\ &+ [R^\dagger, L^\dagger]\begin{bmatrix} ia^1_-\sigma_1 + ia^2_-\sigma_2, & 0 \\ 0, & ia^1_+\sigma_1 + ia^2_+\sigma_2 \end{bmatrix}\begin{bmatrix} R \\ L \end{bmatrix}, \end{aligned} \tag{5.3}$$

where $U^\dagger\partial_\mu U = ia^a_\mu\sigma_a$. (As we have mentioned before, the FJ calculation is the same for this and the original reference frame. We are using the rotated frame here to make a direct comparision with ref.[5].)

The field a^3_μ is the CP^1 gauge field which is often used in formally recasting the NLσ model into a gauge field theory. Beacuse of the σ_3 in the mass term, up-spin fermions and down-spin fermions have masses with opposite signs. To get rid of this sign difference, the authors of ref[5] use the transformation ${}^t[R_\uparrow, L_\uparrow] \to {}^t[R_\uparrow, L_\uparrow]$, ${}^t[R_\downarrow, L_\downarrow] \to {}^t[e^{i\frac{\pi}{2}}R_\downarrow, e^{-i\frac{\pi}{2}}L_\downarrow]$. Since this is a chiral rotation by an angle of $\pi/2$, it will give rise to anomalous terms. In evaluating the latter, one must simultaneously perform the same transformation for the second term on the right hand side as well, and contributions from this term should also be considered when evaluating the FJ. In reference [5], the FJ was calculated using only the first term. This is equivalent to treating the up and down-spin sectors as being independent of each other, which of course is not true, since the matrices σ_1 and σ_2 in the second term mix the two. In other words, the gauge field strength in this problem is $\epsilon_{\mu\nu}(\partial_\mu a_\nu^3 + ia_\mu^1 a_\nu^2)$, the second half of which comes from the second term of eq.(5.3), whose contribution was not considered in ref.[5]. In the special case of $\phi = 0$, it turns out that contributions from this term cancels, (which cannot be foreseen without explicit evaluation) which is why these authors obtained the correct result, $\theta = \pi$. However, when $\phi \neq 0$ the additional contribution does not vanish, and, as we have repeatedly emphasized, only when these are included is the result consistent with nonabelian bosonization.

A second difference is that the finiteness of the chiral angle must be properly accounted for, which was also not done in ref.[5]. This procedure corresponds, in conventional nonabelian bosonization, to extending space-time to a solid ball with an extra dimensionality to construct the WZW 3-form, and neglect of this point will again yield an incorrect result for $\phi \neq 0$. Hence we believe that the neglect of these two procedures in ref.[5] is not justifiable, which lead to the modification we have discussed. We end the comparision by noting that the absence of perturbative contributions to the effective action is also a result of the proper treatment of the finite chiral rotation, and differs from the result of ref.[5], and several papers which follow their methods[10].

Lastly we comment on some drawbacks of our scheme. Our treatment relied on a linearized dispersion, which is necessary for the anomaly to arise. It is therefore possible that some subtle lattice effects associated with a finite cutoff of the dispersion may drop out from our calculations. This was pointed out in the CDW application of chiral anomaly in ref. [40]. In previous sections, part of the lattice effects, such as renormalization of the charge velocity and stiffness were shown to be accessible within our scheme. We would like to point out that the problems associated with the linearization approximation are implicit in the conventional bosonization methods as well, and there too, are remedied by adding terms which had dropped out but are permissible by the symmetry of the original problem. Mass renormalization effects which are important when electron-phonon interactions are present [1], will have to be treated by a more delicate treatment of the lattice-electron interaction, which we have not considered here. Finally, the fermion numbers of the previous section were not obtained in a self-consistent manner, but were evaluated instead using an externally imposed background field. It is difficult to give fully self-consistent analytical results because we have incorporated the directional fluctuation of the SDW field. This, we consider is the price that must be paid for respecting the symmetry of the spin sector. On the other hand, we could of course fix the direction of the SDW OP, and readily obtain a self-consistent soliton-like solution, which is essentialy equivalent to what is done in the abelian bozonization method[2]. We can *then* add the effects of directional fluctuation starting from this solution. This was done for the nondimerized case in reference[41], though again with an inadequate treatment of the anomaly.

5.2 Conclusions

We had set our goal at the construction of a mean-field theory approach which gives qualitatively reliable results for one dimensional interacting systems. Concentrating on the Hubbard and Peierls-Hubbard models, we have succeeded in obtaining effective theories that agree precisely with nonabelian bozonization. In chapter 3 we had expressed our relativistic continuum model as Dirac fermions chirally coupled to an external generic SU(2) field $U(x,\tau)$, which is a situation that was previously not considered, due to its parity breaking nature. This enabled us to see that various exotic fermion numbers can be induced by topologically nontrivial backgrounds provided by the configuarion of the U-field. It is especially noteworthy that the topological term of the NLσ model, whose physical role is usually rather abstract, directly plays a role in these fermion-number induction (e.g. spinon formation discussed in chapter 4). This is one of the major merits of using the relativistic fermion model as an intermediate step of deriving the effective model.

In conclusion, our method, though posessing some drawbacks as discussed in the previous chapter, has the merit of being relatively simple, while taking full account of the symmetry and topological fluctuations of the interacting problem, and so can be considered as a complementary approach to conventional bosonization.

Appendix A

Euclidean conventions for 1+1d-Dirac fermions

γ-matrices

(1)Unless stated otherwise, we use the Euclidean formalism. This implies that the γ matrices satisfy the anticommutation relation $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

(2)The Greek suffix runs from $\mu = 0$ (meaning $\mu = \tau$) to $\mu = 1$ ($\mu = x$).

(3)The definition of γ^5 is $\gamma^5 \equiv \mathrm{i}\gamma_0\gamma_1$. Accordingly the important duality relation $\gamma^5\gamma_\mu = -\mathrm{i}\epsilon_{\mu\nu}$ which is unique to the d=2 case is satisfied.

(4)When the explicit matrix form of the γ- matrix is needed, we employ the "chiral basis", in which $\psi =^t [R, L]$, and γ^5 is diagonal, and make the following assignment (τ_a are the Pauli matrices): $\gamma^5 = \tau_3, \gamma_0 = \tau_1, \gamma_1 = -\tau_2$.

Lagrangian

(5)Though we will be using several variants of Dirac fermions, both with and without internal symmetries, the basic form of the Lagrangian density is the same. We use for our Euclidean theory the following: $\mathcal{L} = \bar{\psi}\gamma_\mu(\partial_\mu + \mathrm{i}e A_\mu)\psi \equiv \bar{\psi}(\not{\partial} + \mathrm{i}e \not{A})\psi$.

Currents

(6) Our choice of the definitions for the vector and axial (chiral) currents are made according to the following prescription: we employ the Euclidean expression, which gives, when continued to real time ($\tau = \mathrm{i}t$), the following standard expression in terms of the right and left movers:

$j_0 \rightarrow$ in real time: $R^\dagger R + L^\dagger L$

$j_1 \rightarrow$ in real time: $R^\dagger R - L^\dagger L$

so that the current conservation in real time becomes

$\partial_\mu j_\mu = 0 \rightarrow$ in real time: $\partial_t(R^\dagger R + L^\dagger L) + \partial_x(R^\dagger R - L^\dagger L)$. It is easy to check that this is realized by using the definitions $j_\mu \equiv \mathrm{i}\bar{\psi}\gamma_\mu\psi$ and $j^5_\mu \equiv \bar{\psi}\gamma_\mu\gamma^5\psi$. The currents defined in this way satisfies the duality relation, $j^5_\mu = \epsilon_{\mu\nu}j_\nu$, which is another important feature of the two-dimensional theory.

Appendix B

OPE for $\Sigma_n (-1)^n \vec{S}_n \cdot \vec{S}_{n+1}$

In this appendix we consider the s=1/2 Heisenberg model (or more accurately, the k=1 SU(2) WZW model), and identify the most relevant operator in the OPE which arises when adding a staggered exchange interaction $\gamma \sum_n (-1)^n \vec{S}_n \cdot \vec{S}_{n+1}$. We write the spin operator in the terms of ths SU(2) currents and mass operators in the following way,

$$S_n^a \propto J^a + \bar{J}^a + (-1)^n G^a, \tag{B.1}$$

where

$$J^a = L^\dagger \sigma^a L, J^a = R^\dagger \sigma^a R \tag{B.2}$$

and

$$G^a = L^\dagger \sigma^a R + R^\dagger \sigma^a L. \tag{B.3}$$

The staggered exchange $(-1)^n \vec{S}_n \cdot \vec{S}_{n+1}$ contains the two terms $G_n^a(\bar{J}^a(n+1) + J^a(n+1))$ and $-(\bar{J}^a(n) + J^a(n))G^a(n+1)$. Taking the continuum limit, we need to point-split the operator products, and evaluate the finite contributions. Since it is easy to see from Fermi statistics that the two terms give the same contribution, we will only consider the piece $(\bar{J}^a + J^a)G^a$. Beside the completely normal ordered 4-fermi terms, there are the following contributions:

$$\begin{aligned}
(L^\dagger \sigma^a L)(L^\dagger \sigma^a R) &= L_\alpha^\dagger(x-\epsilon)L_\beta(x-\epsilon)L_\gamma^\dagger(x+\epsilon)R_\delta(x+\epsilon)\sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} \\
\rightarrow & \quad : L_\alpha^\dagger(x-\epsilon)R_\delta(x+\epsilon) :< 0|(-1)L_\gamma^\dagger(x+\epsilon)L_\beta(x-\epsilon)|0 > \sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} \\
&= +\frac{i}{4\pi\epsilon} : L_\alpha^\dagger(x-\epsilon)R_\delta(x+\epsilon) : \sigma^a_{\alpha\beta}\sigma^a_{\beta\delta}
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
(L^\dagger \sigma^a L)(R^\dagger \sigma^a L) &= L_\alpha^\dagger(x-\epsilon)L_\beta(x-\epsilon)R_\gamma^\dagger(x+\epsilon)L_\delta(x+\epsilon)\sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} \\
\rightarrow & \quad (-1) : R_\gamma^\dagger(x+\epsilon)L_\beta(x-\epsilon) :< 0|(-1)L_\alpha^a(x-\epsilon)L_\delta(x+\epsilon)|0 > \sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} \\
&= -\frac{i}{4\pi\epsilon} : R_\gamma^\dagger(x+\epsilon)L_\beta(x-\epsilon) : \sigma^a_{\alpha\beta}\sigma^a_{\gamma\alpha}
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
(R^\dagger \sigma^a R)(L^\dagger \sigma^a R) &= R_\alpha^\dagger(x-\epsilon)R_\beta(x-\epsilon)L_\gamma^\dagger(x+\epsilon)R_\delta(x+\epsilon)\sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} \\
\rightarrow & \quad (-1) : L_\gamma^\dagger(x+\epsilon)R_\beta(x-\epsilon) :< 0|(-1)R_\alpha^\dagger(x-\epsilon)R_\delta(x+\epsilon)|0 > \sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} \\
&= +\frac{i}{4\pi\epsilon} : L_\gamma^\dagger(x+\epsilon)R_\beta(x-\epsilon) : \sigma^a_{\alpha\beta}\sigma^a_{\gamma\alpha}
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
(R^\dagger \sigma^a R)(R^\dagger \sigma^a L) &= R_\alpha^\dagger(x-\epsilon)R_\beta(x-\epsilon)R_\gamma^\dagger(x+\epsilon)L_\delta(x+\epsilon)\sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} \\
\rightarrow & \quad (-1) : R_\alpha^\dagger(x-\epsilon)L_\delta(x+\epsilon) :< 0|R_\gamma^\dagger(x+\epsilon)R_\beta(x-\epsilon)|0 > \sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} \\
&= -\frac{i}{4\pi\epsilon} : R_\alpha^\dagger(x-\epsilon)L_\delta(x+\epsilon) : \sigma^a_{\alpha\beta}\sigma^a_{\beta\delta}.
\end{aligned} \tag{B.7}$$

Here the use of the well-known formulas

$$\begin{aligned} <0|L^\dagger(x-\epsilon)L(x+\epsilon)|0> &= +\frac{i}{4\pi\epsilon} \\ <0|R^\dagger(x-\epsilon)R(x+\epsilon)|0> &= -\frac{i}{4\pi\epsilon} \end{aligned} \tag{B.8}$$

were made. The ϵ in the denominator cancells with the multiplicative factor of a (the lattice constant) which appears when converting the summation to an integral, and we find that the relevant contribution from the staggered exchange interaction has the form of $i\gamma(L^\dagger R - R^\dagger L)$, which is of the same form as the continuum limit of the alternating hopping amplitude for the PH model which we consider in the main text. Hence, for a finite onsite repulsion, the PH model at half-filling is expected to be in the same universality as a Heisenber model with bond alternating exchange. The above term has scaling dimension 1/2, which implies the mass gap to open as $m \propto \gamma^{\frac{2}{3}}$.

Appendix C

Proof of invariance of Jacobian under vector-SU(2n) transformation

In order to distinguish the various transformations and reference frames which will appear bellow, we begin by establishing some notations and terminology. As before the "laboratory frame" (to be abbreviated as lab-frame) will mean the original frame of reference in which the continuum model was derived. The one-paramter family of axial-SU(2n) transformations (i.e. SU(2n) chiral transformation), used to remove the chiral factor from the lab-frame action, is denoted in this subsection as

$$U_5^{lab}(t) \equiv e^{itQ\gamma^5(\frac{\pi}{4}-\frac{\phi}{2})}. \tag{C.1}$$

As in the previous subsection we only consider the portion of the Fujikawa jacobian which involves the topological term. In the lab-frame, the part of the Dirac operator which gives rise to the topological term is simply $\partial\!\!\!/$, and transforms by $U_5^{lab}(t)$ into

$$D_t^{lab} \equiv \partial\!\!\!/ + U_5^{lab}(t)\partial\!\!\!/ U_5^{lab}(t), \tag{C.2}$$

and the corresponding Fujkawa jacobian is

$$J_{lab} = -\frac{1}{2\pi}\int_0^1 dt \mathrm{Tr}[(\frac{\pi}{4}-\frac{\phi}{4})Q\gamma^5(D_t^{lab})^2]. \tag{C.3}$$

The term "rotating frame" (rot-frame) will be used in the sense of Nagaosa-Oshikawa, i.e. as the reference frame in which the staggered magnetization always points in the $\hat{z}$-direction. The unitary (to be precise, the vector-SU(2n)) transformation from the lab-frame to the rot-frame is written as $\Psi \to U\Psi$, $\bar{\Psi} \to \bar{\Psi}U^\dagger$, where $U \in SU(2n)$ is defined by $Q = U\Lambda U^\dagger$. Since the mass term in the rot-frame is $m\bar{\Psi}\Lambda e^{i\Lambda\gamma^5\frac{\phi}{2}}\Psi$, the series of chiral transformation which removes the chiral factor here is

$$U_5^{rot}(t) \equiv e^{it\Lambda\gamma^5(\frac{\pi}{4}-\frac{\phi}{2})}. \tag{C.4}$$

By the combined transformation $\Psi \to U_5^{rot}(t)U\Psi$, $\bar{\Psi} \to \bar{\Psi}U^\dagger U_5^{rot}(t)$, the operator $\partial\!\!\!/$ becomes

$$D_t^{rot} \equiv \partial\!\!\!/ + iA\!\!\!/^{rot}(t), \tag{C.5}$$

where $iA\!\!\!/^{rot}(t) \equiv U_5^{rot}(t)iA\!\!\!/U_5^{rot}(t)$, $iA\!\!\!/ \equiv U^\dagger\partial\!\!\!/U$. Here we are again considering the case where $\phi = $ const., and therefore have left out the term $U_5^{rot}(t)\partial\!\!\!/U_5^{rot}(t)$. The Fujikawa Jacobian for this case is

$$J_{rot} = -\frac{1}{2\pi}\int_0^1 dt \mathrm{Tr}[i(\frac{\pi}{2}-\frac{\phi}{2})\Lambda\gamma^5(D_t^{rot})^2] \tag{C.6}$$

In the following we will start from the lab-frame, switch to a new reference frame by an arbitrary vector-SU(2n) transformation $\Psi \to V\Psi$, $\bar{\Psi} \to \bar{\Psi}V^\dagger$, and eliminate the chiral factor of the mass term in that frame. For the special cases $V \equiv \mathbf{I}$ and $V \equiv U$, our new frames are the lab-frame and the rot-frame, respectively. Otherwise the new frame is in between these two special frames, and therefore will be called the intermediate frame (int-frame). In analogy with the lab-frame and rot-frame cases, it is obvious that the axial transformation used in the int-frame is

$$U_5^{int}(t) \equiv e^{itV^\dagger QV\gamma^5(\frac{\pi}{2}-\frac{\phi}{2})} \tag{C.7}$$

and the corresponding Fujikawa jacobian is given by

$$J_{int} = -\frac{1}{2\pi}\int_0^1 dt \mathrm{Tr}[iV^\dagger QV\gamma^5(\frac{\pi}{2}-\frac{\phi}{2})(D_t^{int})^2], \tag{C.8}$$

where

$$D_t^{int} \equiv \partial\!\!\!/ + i A\!\!\!/^{int}(t), \tag{C.9}$$

and

$$iA\!\!\!/^{int}(t) \equiv U_5^{int}(t)\partial\!\!\!/ U_5^{int}(t) + U_5^{int}(t)V^\dagger \partial\!\!\!/ V U_5^{int}(t). \tag{C.10}$$

We will now show that $J_{int} = J_{lab}$. When this is established for arbitrary V, putting $V \equiv U$ will give $J_{rot} = J_{lab}$.

We consider the trace $\mathrm{Tr}[V^\dagger QV\gamma^5(\partial\!\!\!/ + iA\!\!\!/^{int}(t))^2]$ and try to express it in terms of quantities in the lab-frame. Using the relation

$$\begin{aligned}
&QV\gamma^5 U_5^{int}(t)(\partial\!\!\!/ U_5^{int}(t))V^\dagger \\
&= Q\gamma^5(VU_5^{int}(t)V^\dagger)V(\partial\!\!\!/ U_5^{int}(t))V^\dagger \\
&= Q\gamma^5 U_5^{lab}(t)\partial\!\!\!/ U_5^{lab}(t) - Q\gamma^5 U_5^{lab}(t)(\partial\!\!\!/ V)U_5^{int}(t)V^\dagger \\
&\qquad - Q\gamma^5(V\partial\!\!\!/ V^\dagger),
\end{aligned} \tag{C.11}$$

along with eq.(C.10), we notice the following identity

$$QV\gamma^5(\partial\!\!\!/ + iA\!\!\!/^{int}(t))V^\dagger = Q\gamma^5(\partial\!\!\!/ + U_5^{lab}(t)\partial\!\!\!/ U_5^{lab}(t)). \tag{C.12}$$

Hence, doing a cyclic permutation within the trace, we have

$$\begin{aligned}
&Tr[V^\dagger QV\gamma^5(\partial\!\!\!/ + iA\!\!\!/_{int}(t))^2] \\
&= Tr[Q\gamma^5(\partial\!\!\!/ + U_5^{lab}(t)\partial\!\!\!/ U_5^{lab}(t))^2].
\end{aligned} \tag{C.13}$$

Applying this to eq.(C.8), we conclude that

$$J_{int} = J_{lab}. \tag{C.14}$$

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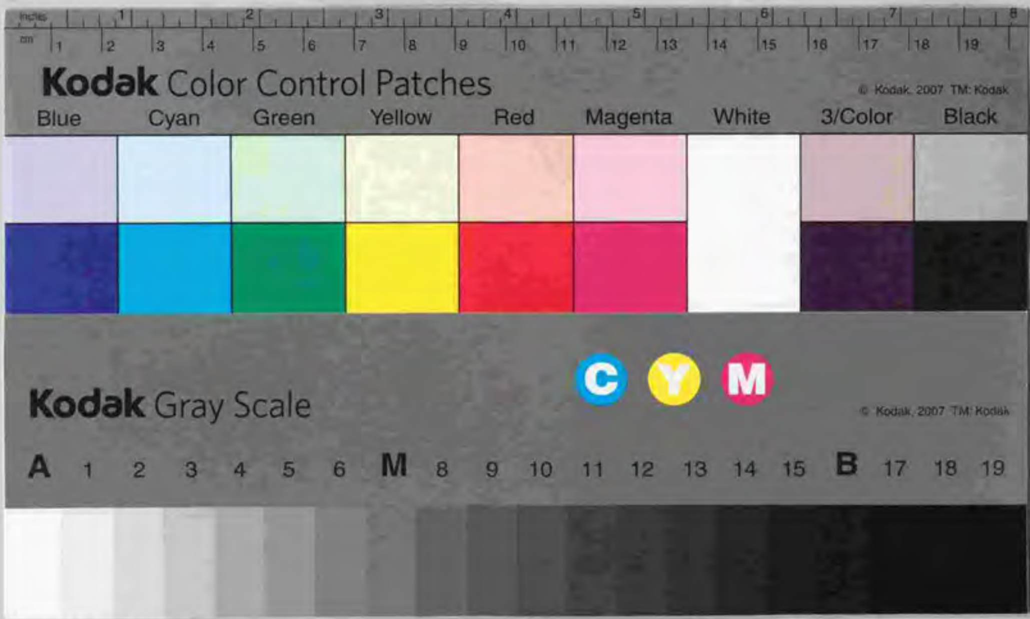

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