博士論文

Quasi-integrable extensions of the discrete Toda lattice equation

(離散戸田格子方程式の準可積分拡張)

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Abstract

In this thesis, we introduce extensions to the two-dimensional Toda lattice equation. In these extensions, the resulted equations preserve the co-primeness property which is an algebraic reinterpretation of singularity confinement used in the research of the discrete Painlevé equations. We call these extensions quasi-integrable extensions. To our best knowledge, this equation is the first example of quasi-integrable discrete equation defined over three dimensional lattice.

First, we treat the quasi-integrable equations in polynomial form, which is an analog of the bilinear form of integrable equations. We prove that general iterates of the equation are irreducible Laurent polynomials of the initial data and that every pair of two iterates is co-prime, which indicates confined singularities of the equation. By reducing the equation to two- or one-dimensional lattices, we obtain quasi-integrable extensions to the one-dimensional Toda lattice equation and the Somos-4 recurrence.

Then, we investigate extensions of the nonlinear form of two-dimensional Toda lattice equation, which cannot be obtained by variable transformation from the above polynomial forms. By extending to non-autonomous polynomial forms, we then prove that their tau function analogues possess irreducibility and the Laurent property with respect to their initial variables and some extra terms related to the non-autonomous terms. Using irreducibility and this Laurent property, we prove that these equations satisfy the co-primeness conditions in some extended Laurent polynomial rings.We also introduce nonlinear forms of the extended Somos-4 recurrence obtained by reduction of the extended 2 dimensional discrete Toda lattice equations, and prove that they also possess the extended Laurent property and satisfy co-primeness conditions.

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Chapter 1

Introduction

1.1 Toda lattice equation and co-primeness property

The Toda lattice equation was derived by Toda as a model of one-dimensional chain of masses connected by springs with nonlinear interaction force [1]. The Toda lattice is one of the completely integrable systems with multi-soliton solutions. The equation of motion of the Toda lattice is

$$\frac{d^2}{dx^2}w_n = \exp(w_{n-1} - w_n) - \exp(w_n - w_{n+1}),$$

where w_n is the position of the n-th particle. The time-discretization of the Toda lattice was obtained by Hirota [2], and was given as a bilinear form:

$$\tau_{t+1,n}\tau_{t-1,n} = \tau_{t,n}^2 + \tau_{t,n-1}\tau_{t,n+1}. \tag{1.1}$$

Two-dimensional Toda lattice (2D-Toda) equation was introduced by Leznov and Saveliev [3]:

$$\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right) w_n = \exp(w_{n-1} - w_n) - \exp(w_n - w_{n+1}).$$

Its bilinear form (2D-dToda) was found by Hirota, Tsujimoto and Imai [4] as

$$\tau_{t+1,n,m+1}\tau_{t-1,n+1,m} = \tau_{t,n+1,m}\tau_{t,n,m+1} + \tau_{t,n,m}\tau_{t,n+1,m+1}, \tag{1.2}$$

where the dependent variable τ is defined on the three-dimensional lattice $(t, n, m) \in \mathbb{Z}^3$.

There are several integrability criteria of discrete equations. Particularly important two of them are 'the singularity confinement test' [5], and 'the algebraic entropy test' [6]. A singularity of a discrete equation is said to be confined, if it is eliminated after a finite number of iteration steps, and at that stage, the dependence on the initial data is recovered. The discrete equation passes the singularity confinement test (SC test), if all the singularities of the equation are confined. The SC test was extremely effective in distinguishing 'integrable' discrete systems, and also in constructing non-autonomous integrable equations. For example, discrete analogs of the Painlevé equations were discovered by searching for non-autonomous extensions to the QRT mappings that conserve their singularity patterns [7]. However, a 'counter example' to SC test has been proposed [8]. The equation they have proposed, which is now called the Hietarinta-Viallet equation, passes the SC test, although it is considered to be non-integrable in the sense that it has chaotic orbits of iterates and has no conserved quantities. They have proposed to use the algebraic entropy

to deal with this type of equations. Algebraic entropy is a quantity to measure the degree growth of the iterates of the equations. The criterion is that the equation is integrable if and only if its algebraic entropy is zero. This criterion is quite strong: the Hietarinta-Viallet equation has positive ($\log((3+\sqrt{5})/2) > 0$) algebraic entropy. Recently, with the aim of refining these integrability criteria, the 'irreducibility' and the 'co-primeness' properties have been proposed to distinguish integrable mappings [9]. Let us study a discrete mapping whose iterates are always Laurent polynomials of the initial variables. If the equation has this property, it is said to have the Laurent property [10]. The mapping has the irreducibility, if every iterate is an irreducible Laurent polynomial of the initial variables. Here we assume that the equation is well-defined under a suitable boundary condition. The equation satisfies the co-primeness condition, if every pair of two iterates is co-prime as Laurent polynomials. We can also define the co-primeness condition, even if the iterates are not necessarily Laurent polynomials. Moreover, it is also possible to relax the co-primeness condition as follows: the equation passes the co-primeness condition if every pair of two iterates 'which is separeted by a fixed finite distance' is always co-prime. The irreducibility and co-primeness are found out to be useful in formulating the integrability of discrete equations defined over the lattice of dimension more than one. For example, It was proved that these two properties can be also formulated for the discrete Toda equation (both the τ-function form and the nonlinear form) with various boundary conditions: i.e., open, the Dirichlet, and the periodic boundaries [11].

A discrete equation is called 'quasi-integrable', if it passes the SC test, and at the same time, has exponential growth of the degrees of the iterates. Note that, in one-dimensional systems, the latter statement is equivalent to saying that the algebraic entropy of the equation is positive, however, for higher-dimensional case we cannot define the entropy in its usual sense. To construct quasi-integrable extensions to known discrete equations, we introduce parameters on the powers of the terms. Some of the equation we study are already introduced by Fomin and Zelevinsky as examples of equations whose iterates are always Laurent polynomials [10]. In this thesis, in addition to the Laurent property, we prove the irreducibility and the co-primeness properties for these equations. Let us prepare a small lemma on Laurent polynomials:

Lemma 1
Let R be a ring of Laurent polynomials. For two Laurent polynomials $f, g \in R$, let us suppose that f is irreducible in R and, at the same time, f contains a non-unit variable that is not in g. Then f and g are co-prime in R.

Proof Since f is irreducible, the common factor of f and g should be either a unit or f itself. However, from the assumption, there exists a variable that is in f but not in g, thus f cannot be a factor of g. Thus f and g are co-prime. ∎

Note 1
For example, in a ring $R = \mathbb{Z}[x^{\pm}]$, $1/x$ and $1/x^2$ are units because they are invertible in R, however, $2 \in R$ is not a unit since $1/2 \notin R$. Another example is that $(x^2+1)/x^3 \in R$ is irreducible and is co-prime with $(x^2+3)/x \in R$, but is not co-prime with $(x^4-1) \in R$.

1.2 The contents of the thesis

In chapter 2, we consider a polynomial form of the quasi-integrable two dimensional discrete Toda lattice equation. Our quasi-integrable extension to the 2 dimensional discrete Toda lattice equation (2D-dToda) is given in τ-function form as follows:

$$\tau_{t+1,n,m+1}\tau_{t-1,n+1,m} = \tau_{t,n+1,m}^{k_1}\tau_{t,n,m+1}^{k_2} + \tau_{t,n,m}^{l_1}\tau_{t,n+1,m+1}^{l_2}, \tag{1.3}$$

where k_1, k_2, l_1, l_2 are positive integers, with $(k_1, k_2, l_1, l_2) \neq (1, 1, 1, 1)$. The set of initial values of equation (1.3) is $\{\tau_{0,n,m}, \tau_{1,n,m} | n, m \in \mathbb{Z}\}$: i.e., set of all the entries in the $t = 0$ and $t = 1$ planes. The evolutions of (1.3) goes upward in the t-axis to $t \geq 2$. The equation (1.3) is quasi-integrable in the sense that its degree growth is exponential (as we shall explain in Proposition 1) and that it has co-primeness property(Theorem 1).

Proposition 1
The degrees $\deg(\tau_{t,n,m})$ *of the iterates of* (1.3) *grow exponentially with respect to* t.

Proof Let us suppose that the initial values are $\tau_{0,n,m} = 1$, $\tau_{1,n,m} = a$ for every n, m, and prove that $\deg(\tau_{t,n,m}) := d_t$ grows exponentially. Since $\tau_t \equiv \tau_{t,n,m}(\forall n, m)$ becomes a polynomial in a, and τ_{t-1} is a factor of τ_t, it is readily obtained that $d_{t+1} = Md_t - d_{t-1}$, with $d_0 = 0$, $d_1 = 1$, where $M := \max(k_1 + k_2, l_1 + l_2)$. Unless $k_1 = k_2 = l_1 = l_2 = 1$, we have

$$\lim_{t\to+\infty}(d_t)^{(1/t)} = \frac{M + \sqrt{M^2 - 4}}{2} > 1.$$

We have proved an exponential growth for one particular degenerate case, which constitutes the lower bound of the degrees of the iterates. ∎

Our goal in chapter 2 is to prove the Laurent property, the irreducibility and the co-primeness property of (1.3), all the three of them are indication of integrable-like nature of the equation in terms of the singularity analysis. For the simplicity of our arguments, we will assume that the greatest common divisor of (k_1, k_2, l_1, l_2) is equal to $2^m (m \in \mathbb{Z}_{\geq 0})$, which is equivalent to the statement that the polynomial

$$x^{k_1}y^{k_2} + z^{l_1}w^{l_2} \tag{1.4}$$

is irreducible in $\mathbb{Z}[x, y, z, w]$[*1]. Our main theorem in chapter 2 is Theorem 1, which states that every iterate $\tau_{t,n,m}$ of (1.3) is an irreducible Laurent polynomial of the initial variables in $\mathbb{Z}$ coefficients, and that two iterates are always co-prime. To make our strategy for proof clear, we first prove the co-primeness property of extended Somos-4 recurrence, which is obtained from equation (1.3) by reduction.

We note that the equation (1.2) of 2D-dToda is essentially the same as the Hirota-Miwa equation [12]. Therefore, we can think of (1.3) as a quasi-integrable extension to the Hirota-Miwa equation.

In chapter 3, we consider the following discrete lattice equation

$$\frac{(U_{t+1,n,m+1} - 1)(U_{t-1,n+1,m} - 1)}{(U_{t,n+1,m} - 1)^{k_1}(U_{t,n,m+1} - 1)^{k_2}} = \frac{U_{t,n,m}^{l_1}U_{t,n+1,m+1}^{l_2}}{U_{t,n+1,m}^{k_1}U_{t,n,m+1}^{k_2}}. \tag{1.5}$$

[*1] Even if the greatest common divisor is not a nonnegative power of two, we can prove that they are co-prime as Laurent polynomials of the initial values.

where $t \in \mathbb{Z}_{\geq 0}$, $(n, m) \in \mathbb{Z}^2$ and k_1, k_2, l_1, l_2 are arbitrary positive integers. When $k_1 = k_2 = l_1 = l_2 = 1$, (1.5) is called a two-dimensional Toda lattice equation[4, 13]. In fact, if we define

$$\tilde{U}_{t,n,m} := \frac{-1 + \delta\epsilon}{\delta\epsilon} U_{t,n,m},$$

(1.5) with $k_1 = k_2 = l_1 = l_2 = 1$ turns into

$$\frac{\left(1 + \delta\epsilon(\tilde{U}_{t+1,n,m+1} - 1)\right)\left(1 + \delta\epsilon(\tilde{U}_{t-1,n+1,m} - 1)\right)}{\left(1 + \delta\epsilon(\tilde{U}_{t,n+1,m} - 1)\right)\left(1 + \delta\epsilon(\tilde{U}_{t,n,m+1} - 1)\right)} = \frac{\tilde{U}_{t,n,m}\tilde{U}_{t,n+1,m+1}}{\tilde{U}_{t,n+1,m}\tilde{U}_{t,n,m+1}}.$$

We denote $\tilde{U}_{t,n,m} := U_t(n\delta, m\epsilon)$. By fixing the values $x := n\delta$ and $y := m\epsilon$ and taking $\delta \to 0$ and $\epsilon \to 0$, we obtain

$$\frac{\partial^2}{\partial x \partial y} \log U_t(x, y) = U_{t+1}(x, y) - 2U_t(x, y) + U_{t-1}(x, y), \tag{1.6}$$

which is the canonical form of the two dimensional Toda lattice equation[14, 15].

For a given solution of (1.3), if we put

$$U_{t,n,m} := \frac{\tau_{t+1,n,m+1}\tau_{t-1,n+1,m}}{\tau_{t,n+1,m}^{k_1}\tau_{t,n,m+1}^{k_2}}, \tag{1.7}$$

$U_{t,n,m}$ satisfies (1.5). Hence one might expect that (1.5) also has the co-primeness property and this conjecture will be proved easily from the irreducibility of (1.3). However, for a solution of (1.5), $\tau_{t,n,m}$ is in (1.7) does not necessarily satisfy (1.3). The equation for $\tau_{t,n,m}$ depends on the initial values of $U_{t,n,m}$ and we cannot directly use the irreducibility of $\tau_{t,n,m}$ for the discussion of the initial value dependence of $U_{t,n,m}$ in (1.5). In fact, (1.5) has a slightly different co-primeness property from those previously discussed, which is what we wish to explain in this chapter. The main theorem in chapter 3 is Theorem 5. Before proving the main theorem, we study a simpler case: a nonlinear discrete mapping which is given by a reduction of (1.5) and that can be regarded as a nonlinear form of an extended Somos-4 recurrence relation[16].

Chapter 4 is devoted to concluding remarks. Some lemmas and propositon are shown in appendix.

Chapter 2

Polynomial form of quasi-integrable two dimensional discrete Toda lattice equation

2.1 Quasi-integrable Somos-4 sequence

Before dealing with our main target (1.3), let us study the properties of a one-dimensional recurrence relation (2.1), which is obtained by a reduction of the equation (1.3) on a line: we identify all the iterates $\tau_{t,n,m}$ such that $N = 1 + 2t + n + m$ and introduce a new variable $x_N := \tau_{t,n,m}$. The form of the reduced equation is

$$x_{n+4}x_n = x_{n+3}^l x_{n+1}^m + x_{n+2}^k, \tag{2.1}$$

where l, m, k are positive integers, which we shall call the *quasi-integrable* Somos-4 sequence. In fact, (2.1) is quasi-integrable for every set of positive integers l, m, k except for $(l, m, k) = (1, 1, 1), (1, 1, 2)$. The flow of the discussion here for the quasi-integrable Somos-4 case helps us to construct the proof of our main theorem. It is worth noting that equation (2.1) is given by Fomin and Zelevinsly as the 'generalized Somos-4 sequence' in their paper (Refer to Example 3.3 in [10]. The original Somos-4 is the case of $(l, m, k) = (1, 1, 2)$.), and that the Laurent property of the equation is proved using the ideas in cluster algebras. Our Proposition 2 studies not only the Laurent property but also the irreducibility and co-primeness.

Proposition 2
Let us assume that the greatest common divisor of (l, m, k) is 1 or a positive power of 2. Then every iterate x_n of equation (2.1) is an irreducible Laurent polynomial of the initial variables x_0, x_1, x_2, x_3. If $n \neq m$, x_n and x_m are co-prime.

We note that the condition on (l, m, k) is equivalent to the irreducibility of the polynomial $x^l y^m + z^k \in \mathbb{Z}[x, y, z]$ (Proposition 8). The following Lemma 2 is used to prove Proposition 2.

Lemma 2
For every $n \geq 4$ we have

$$x_n \in R_0 := \mathbb{Z}[x_0^{\pm}, x_1^{\pm}, x_2^{\pm}, x_3^{\pm}],$$

and that every pair from $x_n, x_{n-1}, x_{n-2}, x_{n-3}$ is co-prime.

Note 2
The assumption that '$x^l y^m + z^k \in \mathbb{Z}[x, y, z]$ is irreducible' is not used in Lemma 2. Thus the Laurent property holds for every positive integer l, m, k.

Proof Proof of Lemma 2 is done by induction. It is easy for $n \leq 7$. We just give a proof for $n = 7$, assuming Lemma 2 for $n \leq 6$. Since $x_7 = (x_6^l x_4^m + x_5^k)/x_3$, x_7 is trivially a Laurent polynomial of $x_i (i = 0, 1, 2, 3)$. We also obtain the co-primeness of x_7 and x_6 as follows: if x_7 has a common factor (which is not a unit) with x_6, that factor must also divide x_5, which leads us to a contradiction with the induction hypothesis that x_6 is co-prime with x_5.

Next we prove the case of larger n. Let us suppose that Lemma 2 is satisfied for every integer less than $n + 1$ and prove it for $n + 1$. By continued iterations we have

$$\begin{aligned}
x_{n+1}x_{n-3} &= x_n^l x_{n-2}^m + x_{n-1}^k \\
&= \left(\frac{x_{n-1}^l x_{n-3}^m + x_{n-2}^k}{x_{n-4}}\right)^l \left(\frac{x_{n-3}^l x_{n-5}^m + x_{n-4}^k}{x_{n-6}}\right)^m + \left(\frac{x_{n-2}^l x_{n-4}^m + x_{n-3}^k}{x_{n-5}}\right)^k \\
&= \frac{x_{n-2}^{kl} x_{n-4}^{km} (x_{n-4}^l x_{n-6}^m + x_{n-5}^k) + O(x_{n-3})}{x_{n-4}^l x_{n-5}^k x_{n-6}^m} \\
&= \frac{x_{n-2}^{kl} x_{n-4}^{km} x_{n-3} x_{n-7} + O(x_{n-3})}{x_{n-4}^l x_{n-5}^k x_{n-6}^m}. \qquad (2.2)
\end{aligned}$$

From induction hypotheses, the right hand side of (2.2) $(x_n^l x_{n-2}^m + x_{n-1}^k)$ must be a Laurent polynomial of $x_i (i = 0, 1, 2, 3)$. In equation (2.2), the term x_{n-3} must be co-prime with x_{n-4}, x_{n-5} and x_{n-6}. Thus, by dividing the both sides of (2.2) by x_{n-3}, we obtain that the numerator of (2.2) must be divisible by $x_{n-4}^l x_{n-5}^k x_{n-6}^m$. Therefore we obtain that x_{n+1} is also a polynomial in $x_i^{\pm}$ $(i = 0, 1, 2, 3)$. Since $x_{n+1}x_{n-3} = x_n^l x_{n-2}^m + x_{n-1}^k$, x_{n+1} is co-prime with x_n, x_{n-1}, x_{n-2}. ∎

Next we prove the irreducibility. We prepare the following Lemma 3 to assist the proof for the $n \geq 9$ case. Let us define y_n as a value of x_n when we substitute $x_0 = x_1 = x_2 = x_3 = 1$:

$$y_n = x_n|_{\{x_0 = x_1 = x_2 = x_3 = 1\}}.$$

Lemma 3
The integer sequence $\{y_n\}$ is strictly increasing for $n \geq 3$. If $l = m = k = 1$, we have

$$y_{10} > y_8 y_4 > y_9 > y_7 y_4,$$

and if otherwise, we have

$$y_9 > y_8 y_4.$$

Proof If $k = l = m = 1$, we have

$$y_4 = 2, \quad y_5 = 3, \quad y_6 = 5, \quad y_7 = 13, \quad y_8 = 22, \quad y_9 = 41, \quad y_{10} = 111,$$

and the lemma is readily obtained. In other cases, we have

$$y_9 - y_8 y_4 = y_9 - 2y_8 = \frac{y_8^l y_6^m + y_7^k}{y_5} - 2y_8 = \frac{y_8 (y_8^{l-1} y_6^m - 2y_5) + y_7^k}{y_5}.$$

If $l \geq 2$, the right hand side is positive. In the case of $l = 1$, we have $y_6^m = (3 + 2^k)^m > 2y_5$, thus the right hand side is also positive. ∎

Proof (**Proposition 2**)
It is sufficient to prove that x_n $(n \geq 4)$ is irreducible. $x_4 = (x_3^l x_1^m + x_2^k)/x_0$ is trivially irreducible because of the assumption. We use Lemma 12 on the factorization of Laurent polynomials under a variable transformation, which has been introduced in [9]. Lemma 12 is reproduced in the appendix of this thesis. We take

$$M = 4,\ \{q_1, q_2, q_3, q_4\} = \{x_0, x_1, x_2, x_3\},\ \{p_1, p_2, p_3, p_4\} = \{x_1, x_2, x_3, x_4\},$$
$$f(x_1, x_2, x_3, x_4) = x_n\ (n \geq 5).$$

First x_4 is trivially a Laurent polynomial of $\{q_1, q_2, q_3, q_4\}$, since $x_4 \in R_0$. Also, since equation (2.1) is invertible, x_0 is a Laurent polynomial of $\{x_1, x_2, x_3, x_4\}$ and is irreducible. Thus we can factorize x_n $(n \geq 5)$ as

$$x_n = x_4^\alpha f_{irr},$$

where f_{irr} is some irreducible Laurent polynomial in R_0. Since we have already proved that $x_m \in R_0$, the parameter α must be a non-negative integer. From Lemma 2, x_4 is co-prime with x_5, x_6, x_7, thus we have $\alpha = 0$. Therefore x_5, x_6, x_7 are irreducible. For x_n $(n \geq 8)$, Lemma 2 does not tell us if x_8 is co-prime with x_4, thus we take another approach. We will prove that, if suitable initial values are taken for $(x_0, x_1, x_2, x_3) \in \mathbb{C}^4$, then we have at the same time $x_4 = 0$ and $x_8 \neq 0$. Then we can conclude that x_8 does not contain a factor x_4 when factorized, and thus $\alpha = 0$. Let us investigate the case of x_8. By a direct computation, we have

$$\begin{aligned} x_8 = \frac{1}{x_1^m x_2^k x_3^l} \Big\{ & x_3^{km} x_5^{kl} x_0 + l x_2^k x_3^{km} x_5^{k(l-1)} x_6^l x_4^{m-1} + \\ & + m x_2^{k+m} x_3^{k(m-1)} x_5^{kl} x_4^{l-1} + k x_1^m x_3^{(k-1)m+l} x_5^{(k-1)l} x_4^{k-1} + O(x_4) \Big\}. \end{aligned} \tag{2.3}$$

Since $x_4 = \dfrac{x_3^l x_1^m + x_2^k}{x_0}$, the value of x_4 is zero, when we take $x_0 = x_1 = x_3 = 1$, $x_2 = t$, $t = \boldsymbol{e}^{\sqrt{-1}\pi/k}$ as initial values. In this case,

$$x_5 = \frac{x_3^k}{x_1} = 1, \quad x_6 = t^{-1}, \quad x_7 = \frac{x_5^k}{x_3} = 1,$$

and from (2.3),

$$x_8 = (-1) \left\{ 1 + \delta_{m,1}(-1) l t^{-l} + \delta_{l,1}(-1) m t^m + k \delta_{k,1} \right\}, \tag{2.4}$$

where $\delta_{p,q}$ is the Kronecker delta. From (2.4), we have $x_8 \neq 0$ for every $(k, l, m) \in \mathbb{Z}^3_{>0}$ with the exception of $(k, l, m) = (1, 1, 2), (1, 2, 1), (3, 1, 1)$. We can study these three cases separately and can find at least one set of $(x_0, x_1, x_2, x_3) \in \mathbb{C}^4$ such that $x_4 = 0$ and $x_8 \neq 0$. (In the case of $(k, l, m) = (3, 1, 1)$, for example, if we take the initial values as $x_1 = -1, x_0 = x_2 = x_3 = 1$, we have $x_4 = 0$ and $x_8 = 3 \neq 0$.) Thus the irreducibility of x_8 is proved.

The iterate $x_n (n \geq 9)$ has the following two factorizations from Lemma 12,

$$x_n = x_4^\alpha f_{irr} = x_5^{\beta_1} x_6^{\beta_2} x_7^{\beta_3} x_8^{\beta_4} g_{irr} \quad (n \geq 9), \tag{2.5}$$

where f_{irr}, g_{irr} are both irreducible Laurent polynomials of the initial variables. To obtain (2.5), we have chosen $\{p_1, p_2, p_3, p_4\} = \{x_5, x_6, x_7, x_8\}$ for the second equality

and have applied Lemma 12. Let us suppose that $x_n (n \geq 9)$ is reducible and derive a contradiction. From (2.5), a factorization of x_n is limited to the following type:

$$x_n = x_4 x_j \times \text{unit} \qquad (n \geq 9,\ j \in \{5,6,7,8\}),$$

where 'unit' is a unit element in R_0. When we substitute $x_0 = x_1 = x_2 = x_3 = 1$ in the above equation, the 'unit' goes to 1 and we have

$$y_n = y_4 y_j \qquad (n \geq 9,\ j \in \{5,6,7,8\}),$$

which is impossible from Lemma 3. Thus $x_n (n \geq 9)$ is irreducible. We have completed the proof that x_n is irreducible for every $n \geq 1$. Since the sequence $\{y_n\}$ is strictly increasing for $n \geq 3$, two iterates x_n and $x_{n'}$ with $n \neq n'$ cannot be equal to each other. Two irreducible distinct elements must be co-prime, and the proof of Proposition 2 is finished. ∎

2.2 Co-primeness of quasi-integrable 2D-dToda

Next we move on to our main equation (1.3). For ease of notation, let us shift all the variables $\tau_{t,n,m}$ to $\tau_{t,n+t/2,m-t/2}$. These shifts produce half-integer lattice points, however, the evolution of equation (1.3) is simplified since it is now described using six vertices of an octahedron. For simplicity let us define the following symbols in (n,m)-plane: $\boldsymbol{n} = (n,m)$ and

$$\boldsymbol{e}_1 = \left(\frac{1}{2}, \frac{1}{2}\right),\ \boldsymbol{e}_2 = \left(-\frac{1}{2}, \frac{1}{2}\right),\ \boldsymbol{e}_3 = \left(-\frac{1}{2}, -\frac{1}{2}\right),\ \boldsymbol{e}_4 = \left(\frac{1}{2}, -\frac{1}{2}\right).$$

From here on, $\hat{\tau}$ denotes a up-shift in the t-axis, and $\check{\tau}$ denotes a downshift in the t-axis. Then equation (1.3) can be expressed as

$$\hat{\tau}_{\boldsymbol{n}} \check{\tau}_{\boldsymbol{n}} = \tau_{\boldsymbol{n}+\boldsymbol{e}_4}^{k_1} \tau_{\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} + \tau_{\boldsymbol{n}+\boldsymbol{e}_3}^{l_1} \tau_{\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}. \tag{2.6}$$

We have the following main theorem on the irreducibility and co-primeness of 2D-dToda:

Theorem 1
Let us assume that the greatest common divisor of (k_1, k_2, l_1, l_2) is a non-negative power of 2. Then each iterate $\tau_{t,\boldsymbol{n}}$ of equation (2.6) is an irreducible Laurent polynomial of the initial variables

$$\left\{ \tau_{t=0,\boldsymbol{n}},\ \tau_{t=1,\boldsymbol{m}} \,\middle|\, \boldsymbol{n} \in \mathbb{Z}^2, \boldsymbol{m} \in \left(\mathbb{Z} + \frac{1}{2}\right)^2 \right\}.$$

Every pair of the iterates is always co-prime.

Proof We will rewrite $x_{\boldsymbol{n}} := \tau_{t=0,\boldsymbol{n}}$, $y_{\boldsymbol{n}} := \tau_{t=1,\boldsymbol{n}}$, and so on: i.e., we use $z_{\boldsymbol{n}}, u_{\boldsymbol{n}}, v_{\boldsymbol{n}}, w_{\boldsymbol{n}}, p_{\boldsymbol{n}}, q_{\boldsymbol{n}}$ for the values of $\tau_{t=i,\boldsymbol{n}}$ at $i = 2,3,4,5,6,7$. We will prove Theorem 1 step by step from $x_{\boldsymbol{n}}$ to $p_{\boldsymbol{n}}$ and beyond.

1. The case of $t = 2$: If $\boldsymbol{n} \neq \boldsymbol{n}'$, then $z_{\boldsymbol{n}}$ and $z_{\boldsymbol{n}'}$ are irreducible in

$$R := \mathbb{Z}\left[\boldsymbol{x}_{\boldsymbol{n}}^{\pm}, \boldsymbol{y}_{\boldsymbol{n}'}^{\pm} \,\middle|\, \boldsymbol{n} \in \mathbb{Z}^2, \boldsymbol{n}' \in (\mathbb{Z} + 1/2)^2\right],$$

and are co-prime.
$\because$) Since

$$z_{\boldsymbol{n}} = \frac{1}{x_{\boldsymbol{n}}}\left(y_{\boldsymbol{n}+\boldsymbol{e}_4}^{k_1} y_{\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} + y_{\boldsymbol{n}+\boldsymbol{e}_3}^{l_1} y_{\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}\right),$$

if $\boldsymbol{n} \neq \boldsymbol{n}'$, two iterates $z_{\boldsymbol{n}}$ and $z_{\boldsymbol{n}'}$ have at most two variables in the $(t=1)$-plane (y_*) in common. Thus from Lemma 1, two iterates must be co-prime. ∎

2. The case of $t = 3$: if $\boldsymbol{n} \neq \boldsymbol{n}'$, two iterates $u_{\boldsymbol{n}}$ and $u_{\boldsymbol{n}'}$ are irreducible in $R = \mathbb{Z}[\boldsymbol{x}^{\pm}, \boldsymbol{y}^{\pm}]$ and are co-prime. Each $u_{\boldsymbol{n}}$ is co-prime with $z_{\boldsymbol{n}'}$ for all $\boldsymbol{n}' \in \mathbb{Z}^2$.
$\because$) From

$$u_{\boldsymbol{n}} = \frac{1}{y_{\boldsymbol{n}}}\left(z_{\boldsymbol{n}+\boldsymbol{e}_4}^{k_1} z_{\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} + z_{\boldsymbol{n}+\boldsymbol{e}_3}^{l_1} z_{\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}\right)$$

and from Lemma 12, we obtain the factorization of $u_{\boldsymbol{n}}$ as

$$u_{\boldsymbol{n}} = \left(\prod_{k\,\text{finite}} z_{\boldsymbol{n}_k}^{\alpha_k}\right) f_{irr} \qquad (\boldsymbol{n}_k \in \mathbb{Z}^2,\ \alpha_k \in \mathbb{Z}_{\geq 0}),$$

where f_{irr} is irreducible in R. Since $u_{\boldsymbol{n}}$ does not have the factor $z_{\boldsymbol{n}+\boldsymbol{e}_i}$ for $i = 1,2,3,4$, we have $\alpha_k = 0$ if $\boldsymbol{n}_k = \boldsymbol{n} + \boldsymbol{e}_i$ $(i = 1,2,3,4)$. (Note that $\{z_{\boldsymbol{n}}\}$ is mutually co-prime, and thus, two distinct z_*'s are not identical.) For other $\boldsymbol{n}_k$, the iterate $z_{\boldsymbol{n}_k}$ contains at least one term $y_{\boldsymbol{n}_k+\boldsymbol{e}_i}$ that does not appear in $u_{\boldsymbol{n}}$. Since $z_{\boldsymbol{n}_k}$ is binomial with respect to the variables $\{y_{\boldsymbol{n}}\}$ in $t = 1$, the term $y_{\boldsymbol{n}_k+\boldsymbol{e}_i}$ cannot be eliminated by multiplying some unit element in R. Thus from Lemma 1, two iterates $z_{\boldsymbol{n}_k}$ and $u_{\boldsymbol{n}}$ are co-prime, and we have $\alpha_k = 0$. We have proved that $u_{\boldsymbol{n}}$ is irreducible. It is readily obtained that each $u_{\boldsymbol{n}}$ is co-prime with $z_{\boldsymbol{n}'}$ for every $\boldsymbol{n}'$. The final step is to prove that $u_{\boldsymbol{n}}$ and $u_{\boldsymbol{n}'}$ are co-prime if $\boldsymbol{n} \neq \boldsymbol{n}'$. Each iterate $u_{\boldsymbol{n}}$, when expanded as a Laurent polynomial in R, contains nine terms y_* in $(t=1)$-plane, none of which is cancelled out by multiplying unit elements in R. When $\boldsymbol{n} \neq \boldsymbol{n}'$, there must be at least one term y_* that does not appear simultaneously in the iterates $u_{\boldsymbol{n}}$ and $u_{\boldsymbol{n}'}$. Therefore, using Lemma 1, we obtain the co-primeness of $u_{\boldsymbol{n}}$ and $u_{\boldsymbol{n}'}$. ∎

3. The case of $t = 4$ (Part I): $v_{\boldsymbol{n}} \in R$:
$\because$) Let us denote 9 points $a, b, c, \ldots, i$ in the lattice plane $\mathbb{Z}^2$ on which the variables $v_{\boldsymbol{n}}, z_{\boldsymbol{n}}, x_{\boldsymbol{n}}$ lie, and denote 8 points $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ in the lattice plane $(\mathbb{Z}+1/2)^2$ on which the variables $u_{\boldsymbol{n}}, y_{\boldsymbol{n}}$ lie, as in Figure 2.1. Let us take the point 'e' at the center as $\boldsymbol{n} = e$. By a direct computation, we have

$$\begin{aligned}
z_e v_e &= u_\delta^{k_1} u_\alpha^{k_2} + u_\gamma^{l_1} u_\beta^{l_2} \\
&= \left(\frac{z_i^{k_1} z_e^{k_2} + z_h^{l_1} z_f^{l_2}}{y_\delta}\right)^{k_1} \left(\frac{z_e^{k_1} z_a^{k_2} + z_d^{l_1} z_b^{l_2}}{y_\alpha}\right)^{k_2} \\
&\quad + \left(\frac{z_h^{k_1} z_d^{k_2} + z_g^{l_1} z_e^{l_2}}{y_\gamma}\right)^{l_1} \left(\frac{z_f^{k_1} z_b^{k_2} + z_e^{l_1} z_c^{l_2}}{y_\beta}\right)^{l_2} \\
&= \frac{(z_h^{l_1 k_1} z_f^{l_2 k_1} z_d^{l_1 k_2} z_b^{l_2 k_2})(y_\gamma^{l_1} y_\beta^{l_2} + y_\delta^{k_1} y_\alpha^{k_2}) + O(z_e)}{y_\delta^{k_1} y_\alpha^{k_2} y_\gamma^{l_1} y_\beta^{l_2}} \\
&= \frac{z_e x_e \cdot z_h^{l_1 k_1} z_f^{l_2 k_1} z_d^{l_1 k_2} z_b^{l_2 k_2} + O(z_e)}{y_\delta^{k_1} y_\alpha^{k_2} y_\gamma^{l_1} y_\beta^{l_2}}.
\end{aligned} \tag{2.7}$$

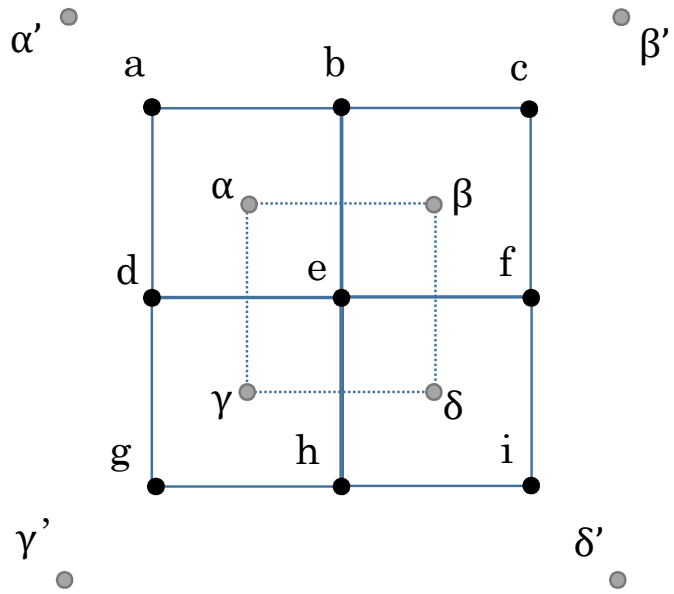

Fig. 2.1 Numbering of the lattice points

By eliminating z_e from both sides, we conclude that v_e is a Laurent polynomial. The calculation of $\tau_{t,\boldsymbol{n}}\tau_{t-2,\boldsymbol{n}}$ can be done for $t \geq 5$ in the same manner as we have done for $t = 4$ in (2.7). Therefore the Laurent property (not the irreducibility) in Theorem 1 can be partially proved first: i.e., if we suppose that $\tau_{t,\boldsymbol{n}} \in R$ and is irreducible in R for every $0 \leq t \leq t_0$, then we readily conclude that $\tau_{t,\boldsymbol{n}} \in R$ for $t = t_0 + 1$. However the irreducibility for $t = t_0 + 1$ needs more careful treatment.

4. The case of $t = 4$ (Part II): Iterates $v_{\boldsymbol{n}}$ ($\boldsymbol{n} \in \mathbb{Z}^2$) are irreducible Laurent polynomials and are mutually co-prime. They are also co-prime with every $u_{\boldsymbol{n}'}$ ($\boldsymbol{n}' \in (\mathbb{Z} + 1/2)^2$) and $z_{\boldsymbol{n}'}$ ($\boldsymbol{n}' \in \mathbb{Z}^2$).
$\because$) Let us take $\boldsymbol{n} = e$ and use the numbering of the lattice points in Figure 2.1. We will prove that v_e is an irreducible Laurent polynomial. By using Lemma 12, we obtain the following factorization of v_e:

$$v_e = \left(\prod_{\boldsymbol{n}'} z_{\boldsymbol{n}'}^{\alpha_{\boldsymbol{n}'}} \right) f_{irr} \quad (\alpha_{\boldsymbol{n}'} \in \mathbb{Z}_{\geq 0}),$$

where f_{irr} is irreducible in R. For $\boldsymbol{n}'$ with $\boldsymbol{n}' \notin \{a, b, ..., i\}$, there exists a term y_* of $t = 2$, that is contained in the iterate $z_{\boldsymbol{n}'}$, and at the same time, is not contained in v_e. This term y_* cannot be cancelled out by multiplying some unit element, and thus from Lemma 1 we have $\alpha_{\boldsymbol{n}'} = 0$.
The proof is completed if we prove that $\alpha_{\boldsymbol{n}'} = 0$ for $\boldsymbol{n}' \in \{a, b, ..., i\}$. Let us define two ideals of R as $I_1 := z_a \cdot R$ and $I_2 := z_e^2 \cdot R$. From the symmetry of the evolution equation and the configuration of the variables, it is sufficient to prove that $\alpha_a = \alpha_b = \alpha_e = 0$.

■(Proof of $\alpha_a = 0$) Since z_e and z_a are co-prime, we only have to prove that $z_e v_e \notin I_1$. Let us suppose that $z_e v_e \in I_1$ and lead us to a contradiction. Note that z_a does not have a term $y_{\delta'}$ when written with the initial variables. Thus it is necessary that the term of the highest order of $z_e v_e$ be divisible by z_a, when the terms of $z_e v_e$ is re-arranged with respect to $y_{\delta'}$ (as a Laurent polynomial

of $y_{\delta'}$). We have

$$z_e v_e = u_\delta^{k_1} u_\alpha^{k_2} + u_\gamma^{l_1} u_\beta^{l_2} = \left(\frac{z_i^{k_1} z_e^{k_2} + z_h^{l_1} z_f^{l_2}}{y_\delta} \right)^{k_1} u_\alpha^{k_2} + u_\gamma^{l_1} u_\beta^{l_2}. \tag{2.8}$$

Among the eight terms z_i, z_e, z_h, z_f, y_δ, u_α, u_γ, u_β that appear in (2.8), only z_i contains the term $y_{\delta'}$ in its expansion. Therefore when we re-arrange the terms of $z_e v_e$ as a Laurent polynomial of the variable $y_{\delta'}$, its degree is k_1^3. The coefficient of the term $y_{\delta'}^{k_1^3}$ is equal to

$$x_i^{-k_1^2} y_\delta^{k_1^2 k_2 - k_1} z_e^{k_1 k_2} u_\alpha^{k_2},$$

and it should be divisible by z_a. This leads us to a contradiction because we have already proved that u_α and z_e are both co-prime with z_a. We have $z_e v_e \notin I_1$ and thus $\alpha_a = 0$.
Proof of $\alpha_b = 0$ can be done in a similar manner to the previous step and is omitted in this thesis.

■(Proof of $\alpha_e = 0$) Let us suppose that $z_e v_e \in I_2$. The four variables $y_{\alpha'}, y_{\beta'}, y_{\gamma'}, y_{\delta'}$ are not used to construct z_e. Thus, when an element of I_2 is considered as a Laurent polynomial in $y_{\alpha'}$, the coefficient of its highest term should be divisible by z_e^2. By further expanding the iterates $u_\alpha, u_\beta, u_\gamma$ in the right hand side of equation (2.8), we have

$$\begin{aligned}
I_2 &= z_e v_e + I_2 \\
&= y_\delta^{-k_1} y_\alpha^{-k_2} (k_1 z_i^{k_1} z_e^{k_2} z_h^{l_1(k_1-1)} z_f^{l_2(k_1-1)} + z_h^{l_1 k_1} z_f^{l_2 k_1}) \\
&\quad \times (k_2 z_e^{k_1} z_a^{k_2} z_d^{l_1(k_2-1)} z_b^{l_2(k_2-1)} + z_d^{l_1 k_2} z_b^{l_2 k_2}) \\
&\quad + y_\gamma^{-l_1} y_\beta^{-l_2} (l_1 z_g^{l_1} z_e^{l_2} z_h^{k_1(l_1-1)} z_d^{k_2(l_1-1)} + z_h^{k_1 l_1} z_d^{k_2 l_1}) \\
&\quad \times (l_2 z_e^{l_1} z_c^{l_2} z_f^{k_1(l_2-1)} z_b^{k_2(l_2-1)} + z_f^{k_1 l_2} z_b^{k_2 l_2}) + I_2 \\
&= y_\delta^{-k_1} y_\alpha^{-k_2} (k_1 z_i^{k_1} z_e^{k_2} z_h^{l_1(k_1-1)} z_f^{l_2(k_1-1)} z_d^{l_1 k_2} z_b^{l_2 k_2} \\
&\quad + k_2 z_e^{k_1} z_a^{k_2} z_d^{l_1(k_2-1)} z_b^{l_2(k_2-1)} z_h^{l_1 k_1} z_f^{l_2 k_1}) \\
&\quad + y_\gamma^{-l_1} y_\beta^{-l_2} (l_1 z_g^{l_1} z_e^{l_2} z_h^{k_1(l_1-1)} z_d^{k_2(l_1-1)} z_f^{k_1 l_2} z_b^{k_2 l_2} \\
&\quad + l_2 z_e^{l_1} z_c^{l_2} z_f^{k_1(l_2-1)} z_b^{k_2(l_2-1)} z_h^{k_1 l_1} z_d^{k_2 l_1}) \\
&\quad + (y_\delta^{-k_1} y_\alpha^{-k_2} + y_\gamma^{-l_1} y_\beta^{-l_2}) z_h^{l_1 k_1} z_f^{l_2 k_1} z_d^{l_1 k_2} z_b^{l_2 k_2} + I_2.
\end{aligned} \tag{2.9}$$

Here the term with z_e^2 is absorbed in the ideal I_2. Among all the iterates on $t = 2$ plane (i.e., $z_{\boldsymbol{n}}$), the only iterate that contain $y_{\alpha'}$ in its expansion is z_a. Let us re-arrange the right hand side of (2.9) as a Laurent polynomial of $y_{\alpha'}$. Then the coefficient of the highest order (k_2^2-th order) is

$$k_2 x_a^{-k_2} y_\delta^{-k_1} y_\alpha^{k_1 k_2 - k_2} z_e^{k_1} z_d^{l_1(k_2-1)} z_b^{l_2(k_2-1)} z_h^{l_1 k_1} z_f^{l_2 k_1},$$

which should be divisible by z_e^2. Since every pair of two terms of $z_{\boldsymbol{n}}$ is co-prime, we conclude that $k_1 \geq 2$. The same arguments also show that $l_1 \geq 2$, $l_2 \geq 2$ and $k_2 \geq 2$. (We consider (2.9) as a Laurent polynomial of $y_{\beta'}$, $y_{\gamma'}$, and $y_{\delta'}$ each.) Therefore, in (2.9), the first two terms are divisible by z_e^2 and belong

to the ideal I_2. Thus the last term $(y_\delta^{-k_1}y_\alpha^{-k_2}+y_\gamma^{-l_1}y_\beta^{-l_2})z_h^{l_1k_1}z_f^{l_2k_1}z_d^{l_1k_2}z_b^{l_2k_2}$ of (2.9) is also in I_2. On the other hand, from the evolution equation, we have

$$\begin{aligned}&(y_\delta^{-k_1}y_\alpha^{-k_2}+y_\gamma^{-l_1}y_\beta^{-l_2})z_h^{l_1k_1}z_f^{l_2k_1}z_d^{l_1k_2}z_b^{l_2k_2}\\ =&y_\delta^{-k_1}y_\alpha^{-k_2}y_\gamma^{-l_1}y_\beta^{-l_2}z_h^{l_1k_1}z_f^{l_2k_1}z_d^{l_1k_2}z_b^{l_2k_2}x_e\cdot z_e,\end{aligned}\tag{2.10}$$

which indicates that (2.10) is divisible by z_e only once. This leads us to a contradiction. Thus $z_ev_e\notin I_2$, and therefore $\alpha_e=0$ is proved.
Now we have finished the proof of the irreducibility of v_e.
Next let us prove that each $v_{\boldsymbol{n}}$ is co-prime with every iterates below $t=4$. Let us substitute $x_{\boldsymbol{n}}\to 1$, $y_{\boldsymbol{n}}\to 1$. Then $\tau_{\boldsymbol{n}}$ is a constant independent of a choice of $\boldsymbol{n}$ for a fixed t: we define $\tilde{\tau}:=\tau_{\boldsymbol{n}}|_{x_{\boldsymbol{n}}\to 1,y_{\boldsymbol{n}}\to 1}$ and use symbols such as $\tilde{z}$ for $\tilde{\tau}(t=2)$ and so on. Then we have $\tilde{z}=2$, $\tilde{u}=2^{k_1+k_2}+2^{l_1+l_2}$, $\tilde{v}=(\tilde{u}^{k_1+k_2}+\tilde{u}^{l_1+l_2})/2$, which indicates that $\tilde{v}>\tilde{u},\tilde{z}$. Therefore $v_{\boldsymbol{n}}$ cannot have a common factor with $\{u_{\boldsymbol{n}'}\}$ or $\{z_{\boldsymbol{n}'}\}$. Finally we note that $v_{\boldsymbol{n}}$ and $v_{\boldsymbol{n}'}$ are co-prime if $\boldsymbol{n}\neq\boldsymbol{n}'$. This is readily proved using Lemma 1, since $v_{\boldsymbol{n}}$ and $v_{\boldsymbol{n}'}$ are Laurent polynomials of the same degree, and they have distinct terms y_* in the $t=1$ plane. ∎

5. Proof of the case $t=5$: Let us prove that $w_{\boldsymbol{n}}$ is an irreducible Laurent polynomial in R and every pair is co-prime. Also we prove that $w_{\boldsymbol{n}}$ is co-prime with $v_{\boldsymbol{n}'}$, $u_{\boldsymbol{n}'}$, $z_{\boldsymbol{n}'}$.
∵) Let us use Lemma 12 to consider the possible factorizations of $w_{\boldsymbol{n}}$ as we have done for the quasi-Somos-4 sequence ($n\geq 9$) in Proposition 2. Let us suppose that $w_{\boldsymbol{n}}$ is reducible. Then we have only two types of factorizations as follows:
$$w_{\boldsymbol{n}}=\text{unit}\times z_{\boldsymbol{n}'}u_{\boldsymbol{n}''}\text{ or }w_{\boldsymbol{n}}=\text{unit}\times z_{\boldsymbol{n}'}v_{\boldsymbol{n}''}.$$
However, since $\tilde{w}>\tilde{z}\tilde{v}>\tilde{z}\tilde{u}$, we have a contradiction. Thus $w_{\boldsymbol{n}}$ is irreducible. By a discussion similar to that in the previous part, we conclude that $w_{\boldsymbol{n}}$ and $w_{\boldsymbol{n}'}$ are co-prime if $\boldsymbol{n}\neq\boldsymbol{n}'$. ∎
6. The proof of the case $t=6$: All the iterates $p_{\boldsymbol{n}}$ are irreducible Laurent polynomials in R and are pairwise co-prime. Moreover they are co-prime with $w_{\boldsymbol{n}'}$, $v_{\boldsymbol{n}'}$, $u_{\boldsymbol{n}'}$, $z_{\boldsymbol{n}'}$.
∵) The discussion proceeds in the same way as in the previous part. ∎
7. The proof of the case $t\geq 7$: For $t\geq 7$, each term $\tau_{t,\boldsymbol{n}}$ is an irreducible Laurent polynomial in R and is co-prime with every iterate $\tau_{s,\boldsymbol{n}'}$ ($s\leq t$).
∵) From the discussion in the case of $t=4$ (Part I), the iterate $q_{\boldsymbol{n}}\in R$ for every $t\geq 7$. By using Lemma 12 for $q_{\boldsymbol{n}}$, we have three types of factorizations:
$$\begin{aligned}q_{\boldsymbol{n}}&=\left(\prod_{\boldsymbol{n}'}z_{\boldsymbol{n}'}^{\alpha_{\boldsymbol{n}'}}\right)f_{irr}\\&=\left(\prod_{\boldsymbol{n}'}u_{\boldsymbol{n}'}^{\beta_{\boldsymbol{n}'}}\right)\left(\prod_{\boldsymbol{n}'}v_{\boldsymbol{n}'}^{\beta'_{\boldsymbol{n}'}}\right)g_{irr}\\&=\left(\prod_{\boldsymbol{n}'}w_{\boldsymbol{n}'}^{\gamma_{\boldsymbol{n}'}}\right)\left(\prod_{\boldsymbol{n}'}p_{\boldsymbol{n}'}^{\gamma'_{\boldsymbol{n}'}}\right)h_{irr},\end{aligned}$$
where f_{irr}, g_{irr}, h_{irr} are irreducible in R. These factorizations cannot be compatible unless $q_{\boldsymbol{n}}$ is irreducible in R (in that case, $\alpha_{\boldsymbol{n}'}=\cdots=\gamma'_{\boldsymbol{n}'}=0$ and $f_{irr}=g_{irr}=h_{irr}$). By the same argument to the previous step, each pair of $q_{\boldsymbol{n}}$ and $q_{\boldsymbol{n}'}$ is co-prime if $\boldsymbol{n}\neq\boldsymbol{n}'$.

8. The proof for $t \geq 8$ is done inductively. ∎

2.3 Quasi-integrable 1D discrete Toda equation

The following equation (2.11) is obtained from a reduction of equation (1.3) to a two-dimensional lattice. Let us make a transformation $n + m \to N$ and identify all $\tau_{t,n,m}$ with $N = n + m$. Then $\tau_{t,N} := \tau_{t,n,m}$ satisfy

$$\tau_{t+1,N}\tau_{t-1,N} = \tau_{t,N}^{k} + \tau_{t,N-1}^{l_1}\tau_{t,N+1}^{l_2} \qquad (k, l_1, l_2 \in \mathbb{Z}_+). \tag{2.11}$$

For an arbitrary $(k, l_1, l_2) \in \mathbb{Z}_+$, equation (2.11) passes the singularity confinement test and has irreducibility and co-primeness properties. If $(k, l_1, l_2) \neq (1, 1, 1), (2, 1, 1)$, the equation (2.11) has exponential growth of the degrees of its iterates. The equation (2.11) is the discrete Toda equation (1.1) if $k = 2$, $l_1 = 1$, $l_2 = 1$. In the case of $(k, l_1, l_2) = (1, 1, 1)$, we can prove that the degree of its iterates grows according to a polynomial of degree one, by applying a discussion in [17]. We note that equation (2.11) is already mentioned in [10] as 'Number walls' and its Laurent property is proved. We shall call (2.11) a 'quasi-integrable 1D discrete Toda equation' and include it in the category of quasi-integrable systems.

Proposition 3
Let us define the evolution of the equation (2.11) *from the initial variables* $\tau_{0,n}$, $\tau_{1,n}$ $(n \in \mathbb{Z})$, *upward on the t-axis. Then every iterate* $\tau_{t,n}$ *for* $t \geq 3$ *is an irreducible Laurent polynomial in*

$$\mathbb{Z}\left[\tau_{0,n}^{\pm}, \tau_{1,n}^{\pm} \,|\, n \in \mathbb{Z}\right],$$

and every pair of the iterates is co-prime.

Note that the proof of Proposition 3 is not directly transferred from that of Theorem 1, since the irreducibility and the co-primeness are not necessarily conserved under the reduction. Proof of this proposition is omitted in this paper, since it can be done inductively with respect to t, with the help of Lemma 12.

Chapter 3

Nonlinear forms of the quasi-integrable two-dimensional Toda lattice equation

3.1 Nonlinear recurrence related to Somos-4

Let us consider the following nonlinear mapping:

$$\frac{(u_{n+4}-1)(u_n-1)}{(u_{n+2}-1)^k} = \frac{u_{n+3}^m u_{n+1}^l}{u_{n+2}^k} \quad (k,l,m \in \mathbb{Z}_{>0}). \tag{3.1}$$

Equation (3.1) is obtained from (1.5) by the reduction:

$$u_{2t+n+m} := U_{t,n,m},$$

with $k_1 + k_2 \to k$, $l_1 \to l$ and $l_2 \to m$. Putting

$$u_{n+2} := \frac{x_{n+4}x_n}{x_{n+2}^k}, \tag{3.2}$$

and substituting (3.2) in (3.1), we have

$$\begin{aligned}(\hat{S}^2 - k + \hat{S}^{-2})\log(u_{n+2}-1) &= (m\hat{S} - k + l\hat{S}^{-1})\log u_{n+2} \\ &= (m\hat{S} - k + l\hat{S}^{-1})(\hat{S}^2 - k + \hat{S}^{-2})\log x_{n+2},\end{aligned}$$

where $\hat{S}$ is an up-shift operator with respect to n. Thus we obatin

$$\begin{aligned}&(\hat{S}^2 - k + \hat{S}^{-2})\Big(\log(u_{n+2}-1) - (m\hat{S} - k + l\hat{S}^{-1})\log x_{n+2}\Big) \\ &= (\hat{S}^2 - k + \hat{S}^{-2})\log\left(\frac{x_{n+4}x_n - x_{n+2}^k}{x_{n+3}^m x_{n+1}^l}\right) = 0.\end{aligned} \tag{3.3}$$

Let us introduce a new variable F_n by

$$F_n := \frac{x_{n+4}x_n - x_{n+2}^k}{x_{n+3}^m x_{n+1}^l}. \tag{3.4}$$

Then we can rewrite (3.3) as

$$\frac{F_{n+4}F_n}{F_{n+2}^k} = 1. \tag{3.5}$$

Equation (3.5) is a nine-term recurrence relation for $\{x_n\}$, whose initial variables are $x_0, x_1, ..., x_7$. An iterate x_n is a rational function of these initial variables.

The recurrence (3.5) is, in fact, explicitly solvable. Let us introduce

$$\begin{aligned} f_0 := F_0 &= \frac{x_4 x_0 - x_2^k}{x_3^m x_1^l}, \quad f_1 := F_2 = \frac{x_6 x_2 - x_4^k}{x_5^m x_3^l}, \\ g_0 := F_1 &= \frac{x_5 x_1 - x_3^k}{x_4^m x_2^l}, \quad g_1 := F_3 = \frac{x_7 x_3 - x_5^k}{x_6^m x_4^l}, \end{aligned} \tag{3.6}$$

and the sequence $\{a_n\}$ defined by

$$a_{i+1} - k a_i + a_{i-1} = 0 \ (i = 0, 1, 2, ...), \quad a_{-1} = -1, a_0 = 0, \tag{3.7}$$

($a_1 = 1$, $a_2 = k$, $a_3 = k^2 - 1$, $a_4 = k^3 - 2k, ...$).

Then it is easy to prove by induction that

$$F_{2i} = \frac{f_1^{a_i}}{f_0^{a_{i-1}}}, \quad F_{2i+1} = \frac{g_1^{a_i}}{g_0^{a_{i-1}}} \quad (i \geq 0). \tag{3.8}$$

Therefore the following pair of recurrences is equivalent to (3.5):

$$x_{2i+4} x_{2i} = x_{2i+2}^k + \left(\frac{f_1^{a_i}}{f_0^{a_{i-1}}} \right) x_{2i+3}^m x_{2i+1}^l \tag{3.9a}$$

$$x_{2i+5} x_{2i+1} = x_{2i+3}^k + \left(\frac{g_1^{a_i}}{g_0^{a_{i-1}}} \right) x_{2i+4}^m x_{2i+2}^l. \tag{3.9b}$$

Note that (3.9a) and (3.9b) are trivially satisfied for $i = 0, 1$. In case $^\forall n$, $F_n = 1$, $k = 2$ and $l = m = 1$, we have

$$x_{n+4} x_n = x_{n+3} x_{n+1} + x_{n+2}^2$$

which is the Somos-4 recurrence[16]. Hence, we may regard (3.1) as a nonlinear form of an extended Somos-4 recurrence.

3.1.1 Extended Laurent property

Our main results are the "extended Laurent property" and "irreducibility" of (3.9a) and (3.9b), and the "co-primeness" of (3.1). Let us first introduce the former result: Proposition 4 which states the Laurent property of (3.9a) and (3.9b).

Proposition 4
x_n is an extended Laurent polynomial of the initial data: i.e.,

$$x_n \in \mathcal{R} := \mathbb{Z}\left[x_4^{\pm}, x_5^{\pm}, x_6^{\pm}, x_7^{\pm}, f_0^{\pm}, f_1^{\pm}, g_0^{\pm}, g_1^{\pm}\right].$$

Unlike the Laurent property in the usual sense, x_n is a Laurent polynomial of f_0, f_1, g_0, g_1, which are not initial variables themselves. Let us prepare a lemma to facilitate the proof:

Lemma 4
Let n be an integer greater than 7. If $x_j \in \mathcal{R}$ for all j with $7 \leq j \leq n$, then the four iterates x_n, x_{n-1}, x_{n-2}, x_{n-3} are mutually co-prime.

Proof (**Lemma 4**)
It is trivial that x_7, x_6, x_5, x_4 are mutually co-prime in $\mathcal{R}$. When $n = 8$, we have

$$x_8 x_4 = x_6^k + \left(\frac{f_1^{a_2}}{f_0^{a_1}}\right) x_7^m x_5^l.$$

If we suppose that x_8 has a common non-monomial factor with x_6, then this factor should also divide $x_7^m x_5^l$, which contradicts the induction hypothesis that x_7, x_6, x_5, x_4 are mutually co-prime in $\mathcal{R}$. The same argument proves that x_8 is co-prime with x_7 and x_5. Thus x_8 is co-prime with x_7, x_6, x_5. We can then prove the statement of the lemma 4 by induction. ∎

Proof (**Proposition 4**)
Let us prove by an induction. The statement is trivial if $n \leq 9$.

Let us suppose that $x_n \in \mathcal{R}$ for all n with $n \leq 2i+1$, and prove the case of $n = 2i+2$. From equation (3.9a) we have

$$x_{2i+2} x_{2i-2} = x_{2i}^k + \left(\frac{f_1^{a_{i-1}}}{f_0^{a_{i-2}}}\right) x_{2i+1}^m x_{2i-1}^l.$$

Let us focus on the factor x_{2i-2} in (3.9a) with $i \to i-2$ and write

$$x_{2i}^k = \frac{1}{x_{2i-4}^k}\left[\left(\frac{f_1^{a_{i-2}}}{f_0^{a_{i-3}}}\right)^k x_{2i-1}^{km} x_{2i-3}^{kl} + x_{2i-2} \times p_{2i}\right], \tag{3.10}$$

where p_j is a polynomial in x_i $(i \leq j-1)$. Precisely, we have

$$p_{2i} = k\left(\frac{f_1^{a_{i-2}}}{f_0^{a_{i-3}}}\right)^{k-1} x_{2i-1}^{m(k-1)} x_{2i-3}^{l(k-1)} x_{2i-2}^{k-1} + O(x_{2i-2}^{2k-1}) \tag{3.11a}$$

$$p_{2i+1} = m\left(\frac{g_1^{a_{i-2}}}{g_0^{a_{i-3}}}\right) x_{2i}^m x_{2i-1}^{k(m-1)} x_{2i-3}^{kl} x_{2i-2}^{l-1} + l\left(\frac{g_1^{a_{i-3}}}{g_0^{a_{i-4}}}\right) x_{2i-1}^{km} x_{2i-3}^{k(l-1)} x_{2i-4}^l x_{2i-2}^{m-1}$$
$$+ O(x_{2i-2}^{2l-1}) + O(x_{2i-2}^{2m-1}). \tag{3.11b}$$

From equation (3.9b) with $i \to i-2$ and $i \to i-3$,

$$x_{2i+1}^m x_{2i-1}^l = \frac{1}{x_{2i-3}^m x_{2i-5}^l}\left[x_{2i-1}^{km} x_{2i-3}^{kl} + x_{2i-2} \times p_{2i+1}\right], \tag{3.12}$$

Since $ka_{i-2} = a_{i-1} + a_{i-3}$, we have

$$\begin{aligned}
&\frac{1}{x_{2i-4}^k}\left(\frac{f_1^{a_{i-2}}}{f_0^{a_{i-3}}}\right)^k x_{2i-1}^{km} x_{2i-3}^{kl} + \frac{1}{x_{2i-3}^m x_{2i-5}^l}\left(\frac{f_1^{a_{i-1}}}{f_0^{a_{i-2}}}\right) x_{2i-1}^{km} x_{2i-3}^{kl} \\
&= \frac{x_{2i-1}^{km} x_{2i-3}^{kl}}{x_{2i-3}^m x_{2i-4}^k x_{2i-5}^l}\left(\frac{f_1^{a_{i-1}}}{f_0^{a_{i-2}}}\right)\left\{x_{2i-4}^k + \left(\frac{f_1^{a_{i-3}}}{f_0^{a_{i-4}}}\right) x_{2i-3}^m x_{2i-5}^l\right\} \\
&= \left\{\frac{x_{2i-1}^{km} x_{2i-3}^{kl}}{x_{2i-3}^m x_{2i-4}^k x_{2i-5}^l}\left(\frac{f_1^{a_{i-1}}}{f_0^{a_{i-2}}}\right) x_{2i-6}\right\} x_{2i-2}.
\end{aligned}$$

Thus,

$$x_{2i+2} x_{2i-2} = \frac{x_{2i-2}}{x_{2i-3}^m x_{2i-4}^k x_{2i-5}^l} P_{2i+2},$$

where

$$P_{2i+2} = \left(\frac{f_1^{a_{i-1}}}{f_0^{a_{i-2}}}\right) x_{2i-1}^{km} x_{2i-3}^{kl} x_{2i-6} + x_{2i-3}^{m} x_{2i-5}^{l} p_{2i} + \left(\frac{f_1^{a_{i-1}}}{f_0^{a_{i-2}}}\right) x_{2i-4}^{k} p_{2i+1}, \quad (3.13)$$

which is a polynomial of $x_j (0 \leq j \leq 2i)$. The right hand side is in $\mathcal{R}$, and at the same time, the four iterates x_{2i-2}, x_{2i-3}, x_{2i-4}, x_{2i-5} must be mutually co-prime from lemma 4. Therefore we have $x_{2i-3}^{m} x_{2i-4}^{k} x_{2i-5}^{l} \mid P_{2i+2}$. Thus $x_{2i+2} \in \mathcal{R}$ is proved. By using exactly the same argument we obtain $x_{2i+3} \in \mathcal{R}$. Thus $x_n \in \mathcal{R}$ for $n \geq 4$. ∎

3.1.2 Irreducibility and Co-primeness

Let us now introduce the latter half of our main result on the nonlinear recurrence (3.1). We first prove the irreducibility and co-primeness of x_n in $\mathcal{R}$ in Theorem 2, and then state our main theorem on the variable u_n of (3.1) in Theorem 3.

Theorem 2
The $x_n \in \mathcal{R}$ are irreducible and pairwise co-prime.

Proof Let us define $f_n := F_{2n}$, $g_n := F_{2n+1}$. Note that f_n is expressed as a monic monomial of f_0, f_1, and g_n as a monic monomial of g_0, g_1. We also define the ring $\mathcal{R}_n$ by

$$\mathcal{R}_n := \mathbb{Z}\left[x_{2n+4}^{\pm}, x_{2n+5}^{\pm}, x_{2n+6}^{\pm}, x_{2n+7}^{\pm}, f_n^{\pm}, f_{n+1}^{\pm}, g_n^{\pm}, g_{n+1}^{\pm}\right] \qquad (n = 0, 1, 2, ...),$$
$$(\mathcal{R}_0 := \mathcal{R})$$

The proof is done by induction.

- The case of $n = 8$: Note that

$$x_8 = \frac{x_6^k + \left(\frac{f_1^k}{f_0}\right) x_7^m x_5^l}{x_4},$$

  is a first order polynomial in f_0^{-1}, whose constant term $x_6^k x_4^{-1}$ is co-prime with the coefficient of f_0^{-1}. Thus x_8 is irreducible, and not a unit element.
- The case of $n = 9$:

$$x_9 = \frac{x_7^k + \left(\frac{g_1^k}{g_0}\right) x_8^m x_6^l}{x_5},$$

  is a first order polynomial of g_0^{-1}. Exactly the same argument as in the case of $n = 8$ shows that x_9 is irreducible and is not a unit.
- In the case of $n = 10$: Let us take $\boldsymbol{q} = (x_4, x_5, x_6, x_7, f_0, f_1, g_0, g_1)$ and $\boldsymbol{p} = (x_6, x_7, x_8, x_9, f_1, f_2, g_1, g_2)$, and let us use lemma 12. Since $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfy the conditions in the lemma and since the iterate x_{10} is irreducible in $\mathcal{R}_1$, we have a factorization

$$x_{10} = x_8^{i_{10}} x_9^{j_{10}} x'_{10}(\boldsymbol{q}) \qquad (i_{10},\, j_{10} \in \mathbb{Z}_{\geq 0}).$$

  (f_2, g_2 are units in $\mathcal{R}[\boldsymbol{q}^{\pm}]$, and f_0, f_1 are units in $\mathcal{R}[\boldsymbol{p}^{\pm}]$. Since x_8, x_9 are irreducible polynomials and not units, then i_{10}, $j_{10} \geq 0$.)

$$x_{10} = \frac{x_8^k + \left(\frac{f_1^{a_3}}{f_0^{a_2}}\right) x_9^m x_7^l}{x_6}.$$

If x_{10} has x_8 as a factor, x_9^m must have x_8 as a factor. However, x_8 and x_9 are co-prime with each other, which leads to a contradiction. In the same manner, we conclude that x_9 must not be a factor of x_{10}, since if it were, x_8^k must have a factor x_9, which is again a contradiction. Therefore x_{10} is irreducible. (x_{10} is trivially not a unit.) Using lemma 4, we have that x_{10} is co-prime with x_8 and x_9.

- In the case of $n = 11$: An argument similar to that in the case of $n = 10$ shows that we have a factorization

$$x_{11} = x_8^{i_{11}} x_9^{j_{11}} x'_{11}(\boldsymbol{q}) \qquad (i_{11},\, j_{11} \in \mathbb{Z}_{\geq 0}).$$

An investigation parallel to that for $n = 10$ shows that x_{11} is irreducible and that it is co-prime with x_j ($j \leq 10$).

- In the case of $n = 12$: We have a factorization of x_{12} as

$$x_{12} = x_8^{i_{12}} x_9^{j_{12}} x'_{12}(\boldsymbol{q}) \qquad (i_{12},\, j_{12} \in \mathbb{Z}_{\geq 0}).$$

Let us take $f_1 = t$ and take all the other initial variables as 1. Then we have

$$\begin{aligned} x_8 &= 1 + t^{a_2} = 1 + t^k, \\ x_9 &= 1 + x_8^m, \\ x_{10} &= x_8^k + t^{k^2-1} x_9^m, \\ x_{11} &= x_9^k + x_{10}^m x_8^l. \end{aligned}$$

If we substitute $t = \mathrm{e}^{\sqrt{-1}\pi/k}$ then

$$x_8 = 0, \quad x_9 = 1, \quad x_{10} = t^{k^2-1}, \quad x_{11} = 1.$$

Since we have

$$p_{10} = \delta_{k,1}, \qquad p_{11} = m t^{m(k^2-1)} \delta_{l,1} + l \delta_{m,1},$$

we can obtain P_{12} as

$$\begin{aligned} P_{12} &= t^{a_4} + \delta_{k,1} + t^{a_4} \left(m t^{m(k^2-1)} \delta_{l,1} + l \delta_{m,1} \right) \\ &= \delta_{k,1} + (-1)^k \left\{ 1 + m t^{m(k^2-1)} \delta_{l,1} + l \delta_{m,1} \right\}. \end{aligned}$$

Here we have used the fact that $t^{a_4} = (-1)^{k^2} = (-1)^k$ since $a_4 = k(k^2 - 2)$. Therefore we have $P_{12} \neq 0$ if $k \neq 1$. We conclude that $P_{12} = 0$ if and only if $k = 1$ and $m \geq 2$, $l \geq 2$. Thus, except for the cases of $k = 1$, $l \geq 2$, $m \geq 2$, the iterate x_{12} cannot have x_8 as a factor. Let us study the case of $k = 1$, $l \geq 2$, $m \geq 2$. We have

$$x_9 \equiv \frac{x_7}{x_5}, \quad x_{11} \equiv \frac{x_9}{x_7} \equiv \frac{1}{x_5}, \quad p_{10} \equiv 1, \quad p_{11} \equiv 0,$$

where $A \equiv B$ is considered as modulo the factor x_8. We also have $a_3 = 0$, $a_4 = -1$. Thus

$$\begin{aligned} P_{12} &\equiv \left(\frac{f_1^{a_4}}{f_0^{a_3}} \right) x_9^m x_7^l x_4 + x_7^m x_5^l \\ &\equiv \frac{x_7^{m+l} x_4}{x_5^m f_1} + x_7^m x_5^l \not\equiv 0 \end{aligned}$$

Therefore P_{12} cannot have x_8 as a factor, and thus $i_{12} = 0$. We can prove that $j_{12} = 0$ in a similar manner.

- In the case of $n = 13$, we can prove the irreducibility of x_{13} in the same manner as in the case of $n = 12$.
- We shall prove that x_n $(n \leq 13)$ are mutually co-prime. Let us define c_n as the value of x_n when we substitute 1 to all the initial data: i.e.,

$$c_n = x_n\Big|_{\substack{x_4=x_5=x_6=x_7=1\\ f_0=f_1=g_0=g_1=1}}.$$

As we have already proved the irreducibility of x_n for $n \leq 13$, it is sufficient to prove that $c_n \neq c_m$ for every $n \neq m$ $(n, m \leq 13)$. We have

$$c_8 = 2, \quad c_9 = 1 + 2^m, \quad c_{10} = 2^k + (1 + 2^m)^m, \quad c_{11} = c_9^k + c_{10}^m 2^l,$$
$$c_{12} = \frac{c_{10}^k + c_{11}^m c_9^l}{2}, \quad c_{13} = \frac{c_{11}^k + c_{12}^m c_{10}^l}{c_9}.$$

It is trivial that $2 = c_8 < c_9 < c_{10} < c_{11}$. We show inductively that $c_n < c_{n+1}$:

$$c_{n+1} = \frac{c_{n-1}^k + c_n^m c_{n-2}^l}{c_{n-3}} > \frac{c_{n-2}^l}{c_{n-3}} c_n^m > c_n^m \geq c_n,$$

for $n \geq 8$. Therefore x_n are mutually co-prime.

- In the case of $n = 14$: Using the lemma 12, we have the following two factorizations

$$x_{14} = x_8^{r_1} x_9^{r_2} x'_{14} = x_{10}^{r_3} x_{11}^{r_4} x_{12}^{r_5} x_{13}^{r_6} x''_{14},$$

where x'_{14}, x''_{14} are irreducible in $\mathcal{R}$. Let us suppose that x_{14} is **not** irreducible. Then the only possible factorization is

$$x_{14} = \alpha x_i x_j \qquad (i \in \{8, 9\},\ j \in \{10, 11, 12, 13\}),$$

where α is a unit in $\mathcal{R}$. Therefore $c_{14} = c_i c_j \leq c_9 c_{13}$. On the other hand, since we have

$$c_{14} = \frac{c_{12}^k + c_{13}^m c_{11}^l}{c_{10}},$$
$$c_{14} > c_{13} c_{11} > c_{13} c_9,$$

in the case of $m \geq 2$ or $l \geq 2$, which contradicts $c_{14} \leq c_9 c_{13}$. In the case of $m = l = 1$, we have

$$c_8 = 2, \quad c_9 = 3, \quad c_{10} = 2^k + 3, \quad c_{11} = 2^{k+1} + 3^k + 6,$$

and thus, when $k \geq 3$,

$$c_{14} > \frac{c_{13} c_{11}}{c_{10}} = c_{13} \frac{2^{k+1} + 3^k + 6}{2^k + 3} > 3c_{13} = c_{13} c_9,$$

which also leads to a contradiction. The remaining cases are $(k, l, m) = (2, 1, 1) and (1, 1, 1)$. If $(k, l, m) = (2, 1, 1)$ we can directly confirm that $c_{14} = 1529 > 3c_{13} = 942$. If $(k, l, m) = (1, 1, 1)$, we have $c_{14} = 111 = 3 \times 37$, which cannot be expressed as $c_i c_j$, $(8 \leq i \leq 9, 10 \leq j \leq 13)$. We have proved that x_{14} is irreducible.

- In the case of $n \geq 15$: If we suppose that x_{15} is not irreducible, we must have $c_n \leq c_{13} c_9 = 3c_{13}$, which is impossible when $(k, l, m) \neq (1, 1, 1)$ since we have already shown that $c_{14} > 3c_{13}$. The case of $(k, l, m) = (1, 1, 1)$ is also shown to derive a contradiction, since $c_n \geq c_{15} = 191 > 3c_{13} = 123$ for $n \geq 15$. We have proved that $x_n \in \mathcal{R}$ is irreducible, and is mutually co-prime with each other. ∎

3.1.3 Extended co-primeness property

Let us introduce one of the main theorems in this chapter, which states that the recurrence (3.1) has an "extended" type of co-primeness property, which uses extra data apart from the initial variables.

Definition 1
Let $\mathcal{A}$ be an integral domain and Quat($\mathcal{A}$) be its quotient field. Two elements f, $g \in$ Quat($\mathcal{A}$) are said to be co-prime with each other, if and only if we have the decomposition $f = f_1/f_2$, $g = g_1/g_2$ where any common factor of arbitrary two elements in $\{f_1, f_2, g_1, g_2\} \subset \mathcal{A}$ is a unit in $\mathcal{A}$.

Theorem 3
The solution u_n of equation (3.1) satisfies the following "co-primeness" property: if we suppose that $n \not\equiv n'$ (mod 2) or $|n - n'| > 4$ is satisfied, then two iterates u_n and $u_{n'}$ are co-prime in the following ring $\mathcal{R}_u$:

$$\mathcal{R}_u := \mathbb{Z}\left[\{u_j^{\pm}, (u_j - 1)^{\pm}\}_{j=2}^{5}\right]. \tag{3.14}$$

Let us prepare several lemmas. We use the following notations: $\boldsymbol{x} := \{x_4, x_5, x_6, x_7; f_0, f_1, g_0, g_1\}$, $\boldsymbol{u} := \{x_0, x_1, x_2, x_3; u_2, u_3, u_4, u_5\}$, $\xi_n(\boldsymbol{u}) := x_n(\boldsymbol{x}(\boldsymbol{u}))$.

Lemma 5
We have a birational mapping between the two sets of variables $\boldsymbol{x}$ and $\boldsymbol{u}$.

Proof We construct the rational mapping and show that it is invertible. From the definition of u_i (3.2),

$$x_4 = \frac{x_2^k}{x_0}u_2, \quad x_5 = \frac{x_3^k}{x_1}u_3, \quad x_6 = \frac{x_4^k}{x_2}u_4 = \frac{x_2^{k^2-1}}{x_0^k}u_2^k u_4, \; x_7 = \frac{x_5^k}{x_3}u_5 = \frac{x_3^{k^2-1}}{x_1^k}u_3^k u_5. \tag{3.15}$$

From (3.6) we have

$$f_0 = \frac{x_4 x_0 - x_2^k}{x_3^m x_1^l} = \frac{x_2^k(u_2 - 1)}{x_3^m x_1^l} \tag{3.16a}$$

$$f_1 = \frac{x_6 x_2 - x_4^k}{x_5^m x_3^l} = \frac{x_4^k(u_4 - 1)}{x_5^m x_3^l} = \frac{x_2^{k^2} x_1^m u_2^k (u_4 - 1)}{u_3^m x_3^{km+l} x_0^k} \tag{3.16b}$$

$$g_0 = \frac{x_5 x_1 - x_3^k}{x_4^m x_2^l} = \frac{x_3^k x_0^m (u_3 - 1)}{x_2^{mk+l} u_2^m} \tag{3.16c}$$

$$g_1 = \frac{x_7 x_3 - x_5^k}{x_6^m x_4^l} = \frac{x_5^k x_2^m (u_5 - 1)}{x_4^{mk+l} u_4^m} = \frac{x_3^{k^2} x_2^m x_0^{mk+l} u_3^k (u_5 - 1)}{x_1^k x_2^{k(mk+l)} u_2^{mk+l} u_4^m}. \tag{3.16d}$$

The inverse mapping is

$$x_3 = \frac{x_4^l x_6^m g_1 + x_5^k}{x_7}, \to x_2 = \frac{x_5^m x_3^l f_1 + x_4^k}{x_6}, \to x_1 = \frac{x_2^l x_4^m g_0 + x_3^k}{x_5}, \to x_0 = \frac{x_3^m x_1^l f_0 + x_2^k}{x_4},$$

and

$$u_j = \frac{x_{j+2} x_{j-2}}{x_j^k}, \; (j = 2, 3, 4, 5).$$

Note that in both directions, the variables are expressed as irreducible Laurent polynomials of the other variables. ∎

Lemma 6
Let us define

$$\xi_{2i+2}(\boldsymbol{u}) =: \frac{x_2^{a_{i+1}}}{x_0^{a_i}}\tilde{\xi}_{2i+2}(\boldsymbol{u}), \quad \xi_{2i+3}(\boldsymbol{u}) =: \frac{x_3^{a_{i+1}}}{x_1^{a_i}}\tilde{\xi}_{2i+3}(\boldsymbol{u}).$$

Then we have

$$\tilde{\xi}_n(\boldsymbol{u}) \in \mathbb{Z}\left[\{u_j^{\pm}, (u_j-1)^{\pm}\}_{j=2}^5\right].$$

Proof Since $x_n \in \mathcal{R}$,

$$\xi_n(\boldsymbol{u}) = x_n \in \mathbb{Z}\left[\{u_j^{\pm}, (u_j-1)^{\pm}\}_{j=2}^5, \{x_i^{\pm}\}_{i=0}^3\right].$$

Therefore we need to show that $\tilde{\xi}_n(\boldsymbol{u})$ is independent of x_0, x_1, x_2, x_3.

We inductively obtain that

$$\begin{aligned} x_{2i+2} &= \frac{x_{2i}^k}{x_{2i-2}}u_{2i} \\ &= \frac{x_2^{a_{i+1}}}{x_0^{a_i}}u_{2i}^{a_1}u_{2i-2}^{a_2}\cdots u_2^{a_i}, \qquad (3.17) \\ x_{2i+3} &= \frac{x_3^{a_{i+1}}}{x_1^{a_i}}u_{2i+1}^{a_1}u_{2i-1}^{a_2}\cdots u_3^{a_i}, \qquad (3.18) \end{aligned}$$

where a_i has been defined in (3.7). The u_i $(i \geq 6)$ can be expressed as rational functions of $u_2, ..., u_5$ from (3.1). Thus $\tilde{\xi}_n(\boldsymbol{u})$ can be expressed using only $u_2, ..., u_5$ and this expression is unique. ∎

Proposition 5
$\tilde{\xi}_n$ is irreducible in $\mathbb{Z}\left[\{u_j^{\pm}, (u_j-1)^{\pm}\}_{j=2}^5\right]$. If $n \neq r$, the two terms $\tilde{\xi}_n$ and $\tilde{\xi}_r$ are co-prime.

Proof Recall that $\xi_n(\boldsymbol{u}) \in \mathcal{R}' := \mathbb{Z}\left[\{u_j^{\pm}, (u_j-1)^{\pm}\}_{j=2}^5, \{x_i^{\pm}\}_{i=0}^3\right]$. Let us suppose a factorization $\xi_n(\boldsymbol{u}) = h_1(\boldsymbol{u})h_2(\boldsymbol{u})$, $(h_1,\, h_2 \in \mathcal{R}')$. From equations (3.16a) through (3.16d), we have

$$h_i \in \tilde{\mathcal{R}} := \mathbb{Z}\left[x_0^{\pm}, x_1^{\pm}, ..., x_7^{\pm}, f_0^{\pm}, f_1^{\pm}, g_0^{\pm}, g_1^{\pm}\right] \qquad (i = 1, 2).$$

Since $x_n(\boldsymbol{x})$ is irreducible in $\mathcal{R}$, it is also irreducible in $\tilde{\mathcal{R}}$. Therefore either h_1 or h_2 is a unit in $\tilde{\mathcal{R}}$. We can assume that h_1 is a unit and can factorize it as

$$h_1(\in \tilde{\mathcal{R}}) = \prod_{i=0}^{7} x_i^{\alpha_i} \prod_{i=0}^{1} f_i^{\beta_i} g_i^{\gamma_i} \qquad (\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}).$$

By taking the inverse transformation from $\boldsymbol{x}$ to $\boldsymbol{u}$, we have that $h_1 \in \mathcal{R}'$ is a unit. Therefore $\xi_n(\boldsymbol{u})$ is irreducible in $\mathcal{R}'$, and thus $\tilde{\xi}_n$ is irreducible in $\mathbb{Z}\left[\{u_j^{\pm}, (u_j-1)^{\pm}\}_{j=2}^5\right]$. Next we prove the co-primeness of two arbitrary iterates. Let us suppose that $\xi_n(\boldsymbol{u})$ and $\xi_r(\boldsymbol{u})$ $(n \neq r)$ have a common factor G other than

monomial ones. G is a common factor of the iterates in $\tilde{\mathcal{R}}$, and is not a unit. Therefore x_n and x_r are not co-prime in $\tilde{\mathcal{R}}$. However, from Theorem 2, they must be co-prime in $\mathcal{R}$ and thus co-prime in $\tilde{\mathcal{R}}$, which is a contradiction. ∎

Proof (**Theorem 3**)
Theorem 3 is readily obtained from

$$u_n = \frac{x_{n+2}x_{n-2}}{x_n^k} = \frac{\tilde{x}_{n+2}\tilde{x}_{n-2}}{\tilde{x}_n^k},$$

and from Proposition 5. ∎

3.2 Nonlinear extended two-dimensional discrete Toda equation

Based on the results on the nonlinear recurrence equation (3.1) in the previous section, we shall study the irreducibility and co-primeness properties of the nonlinear form of the extended two-dimensional discrete Toda equation (1.5). Let us redefine the independent variables as $n' := n + \frac{t}{2}$, $m' := m - \frac{t}{2}$, and use the notations $\boldsymbol{n} := (n', m')$, $\boldsymbol{n} \in \mathbb{Z}^2$ $(t \in 2\mathbb{Z})$, $\boldsymbol{n} \in (\mathbb{Z} + 1/2)^2$ $(t \in 2\mathbb{Z} + 1)$,

$$\boldsymbol{e}_1 = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \boldsymbol{e}_2 = \left(-\frac{1}{2}, \frac{1}{2}\right).$$

To ease notation, let us abbreviate the prime $'$ in (n', m') from here on. Then equations (1.7) and (1.5) are equivalent to the following equations:

$$U_{t,\boldsymbol{n}} = \frac{\tau_{t+1,\boldsymbol{n}}\tau_{t-1,\boldsymbol{n}}}{\tau_{t,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}, \tag{3.19}$$

$$\frac{(U_{t+1,\boldsymbol{n}} - 1)(U_{t-1,\boldsymbol{n}} - 1)}{(U_{t,\boldsymbol{n}-\boldsymbol{e}_2} - 1)^{k_1}(U_{t,\boldsymbol{n}+\boldsymbol{e}_2} - 1)^{k_2}} = \frac{U_{t,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}U_{t,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}}{U_{t,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}U_{t,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}. \tag{3.20}$$

Let us define the shift operators $\hat{S}_t$, $\hat{S}_1$, $\hat{S}_2$, which defines an up-shift in the directions of $t, \boldsymbol{e}_1$, and $\boldsymbol{e}_2$ respectively. Then the equation (3.20) can be written as:

$$\begin{aligned}
&\left(\hat{S}_t + \hat{S}_t^{-1} - k_1\hat{S}_2^{-1} - k_2\hat{S}_2\right)\log\left(U_{t,\boldsymbol{n}} - 1\right) \\
&= \left(l_1\hat{S}_1^{-1} + l_2\hat{S}_1 - k_1\hat{S}_2^{-1} - k_2\hat{S}_2\right)\log U_{t,\boldsymbol{n}} \\
&= \left(l_1\hat{S}_1^{-1} + l_2\hat{S}_1 - k_1\hat{S}_2^{-1} - k_2\hat{S}_2\right)\left(\hat{S}_t + \hat{S}_t^{-1} - k_1\hat{S}_2^{-1} - k_2\hat{S}_2\right)\log\tau_{t,\boldsymbol{n}},
\end{aligned}$$

which is equivalent to

$$\left(\hat{S}_t + \hat{S}_t^{-1} - k_1\hat{S}_2^{-1} - k_2\hat{S}_2\right)\log\left[\frac{\tau_{t+1,\boldsymbol{n}}\tau_{t-1,\boldsymbol{n}} - \tau_{t,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}{\tau_{t,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}\tau_{t,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}}\right] = 0.$$

Thus we have obtained a recurrence relation of $\tau_{\boldsymbol{n},t}$ as

$$\frac{F_{t+1,\boldsymbol{n}}F_{t-1,\boldsymbol{n}}}{F_{t,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}F_{t,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}} = 1, \tag{3.21}$$

where we have defined

$$F_{t,\boldsymbol{n}} := \frac{\tau_{t+2,\boldsymbol{n}}\tau_{t,\boldsymbol{n}} - \tau_{t+1,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t+1,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}{\tau_{t+1,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}\tau_{t+1,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}}. \tag{3.22}$$

Equation (3.21) is nonlinear and is a five-term relation with respect to t. Evolution of (3.21) can be defined by assigning the values of $\tau_{t,\boldsymbol{n}}$ at $t = 0, 1, 2, 3$, and by computing the iterations for $t \geq 4$. Let us define the sequences $a_{t,\boldsymbol{n}}$, $b_{t,\boldsymbol{n}} \in \mathbb{Z}$ by the following linear recurrence relation

$$y_{t+1,\boldsymbol{n}} - k_1 y_{t,\boldsymbol{n}-\boldsymbol{e}_2} - k_2 y_{t,\boldsymbol{n}+\boldsymbol{e}_2} + y_{t-1,\boldsymbol{n}} = 0, \tag{3.23}$$

and the initial data

$$a_{0,\boldsymbol{0}} = 1, \quad a_{0,\boldsymbol{n}\neq\boldsymbol{0}} = 0, \quad a_{1,\boldsymbol{n}} = 0, \tag{3.24}$$

$$b_{1,-\boldsymbol{e}_2} = 1, \quad b_{0,\boldsymbol{n}} = 0, \quad b_{1,\boldsymbol{n}\neq-\boldsymbol{e}_2} = 0. \tag{3.25}$$

Then we can explicitly solve $\{F_{t,\boldsymbol{n}}\}$ as

$$F_{t,\boldsymbol{n}} = \prod_{\boldsymbol{r}_0,\boldsymbol{r}_1} F_{0,\boldsymbol{r}_0}^{a_{t,\boldsymbol{n}-\boldsymbol{r}_0}} F_{1,\boldsymbol{r}_1-\boldsymbol{e}_2}^{b_{t,\boldsymbol{n}-\boldsymbol{r}_1}}, \tag{3.26}$$

where the products are taken over all integers $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$. Therefore the variable $\tau_{t,\boldsymbol{n}}$ satisfies the following equation:

$$\tau_{t+1,\boldsymbol{n}}\tau_{t-1,\boldsymbol{n}} = \tau_{t,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} + F_{t-1,\boldsymbol{n}}\tau_{t,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}\tau_{t,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}. \tag{3.27}$$

From (3.26) it follows that $F_{t-1,\boldsymbol{n}}$ is a monomial of $F_{0,\boldsymbol{n}}$, $F_{1,\boldsymbol{n}}$. Note that (3.27) is trivial for $t = 1, 2$ for which it coincides with (3.22). Let us define a ring of Laurent polynomials corresponding to $\mathcal{R}$ as

$$\mathcal{S} := \mathbb{Z}\left[\{\tau_{2,\boldsymbol{n}}^{\pm}, \tau_{3,\boldsymbol{n}}^{\pm}\}, \{F_{0,\boldsymbol{n}}^{\pm}, F_{1,\boldsymbol{n}}^{\pm}\}\right]. \tag{3.28}$$

Lemma 7
Let us suppose that $\tau_{t,\boldsymbol{n}} \in \mathcal{S}$ for every $t \geq t_0$. Then two iterates $\tau_{t_0,\boldsymbol{n}}$ and $\tau_{t_0,\boldsymbol{r}}$ are co-prime if $\boldsymbol{n} \neq \boldsymbol{r}$.

Proof From the spatial symmetry of the equation, if we shift the subscripts in $\tau_{t_0,\boldsymbol{r}}$ in the direction of $\boldsymbol{n} - \boldsymbol{r}$, we obtain $\tau_{t_0,\boldsymbol{n}}$. Thus if $\tau_{t_0,\boldsymbol{r}}$ is a unit, then we have that $\tau_{t_0,\boldsymbol{n}}$ is also a unit and is co-prime with $\tau_{t_0,\boldsymbol{r}}$. Otherwise, there exists a non-unit factor $\tau'_{t_0,\boldsymbol{r}}$ such that $\tau_{t_0,\boldsymbol{r}} = \text{unit} \times \tau'_{t_0,\boldsymbol{r}}$. Since $\tau'_{t_0,\boldsymbol{r}}$ has only a finite number of variables, $\tau'_{t_0,\boldsymbol{n}}$ has at least one variable that is not in $\tau'_{t_0,\boldsymbol{r}}$. This variable is not a unit element, and thus $\tau_{t_0,\boldsymbol{r}}$ and $\tau_{t_0,\boldsymbol{n}}$ are co-prime.

Lemma 8
Let us suppose that $\tau_{t,\boldsymbol{n}}, \tau_{t-1,\boldsymbol{n}'} \in \mathcal{S}$ for all $\boldsymbol{n}, \boldsymbol{n}'$, and suppose that $\tau_{t,\boldsymbol{n}}$ is co-prime with four iterates $\tau_{t-1,\boldsymbol{n}\pm\boldsymbol{e}_1}, \tau_{t-1,\boldsymbol{n}\pm\boldsymbol{e}_2}$. Then if $\tau_{t+1,\boldsymbol{r}} \in \mathcal{S}$, the iterate $\tau_{t+1,\boldsymbol{r}}$ is also co-prime with $\tau_{t,\boldsymbol{r}\pm\boldsymbol{e}_1}, \tau_{t,\boldsymbol{r}\pm\boldsymbol{e}_2}$.

Proof Proof is immediate from equation (3.27) and lemma 7.

Proposition 6
$\tau_{t,\boldsymbol{n}} \in \mathcal{S}$.

Proof The proposition is trivial when $t \leq 5$. Let us suppose that the Proposition 6 is true for t (≥ 5). Then

$$\tau_{t+1,\boldsymbol{n}}\tau_{t-1,\boldsymbol{n}} = \tau_{t,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} + F_{t-1,\boldsymbol{n}}\tau_{t,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}\tau_{t,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2} \in \mathcal{S}.$$

By a direct calculation we have

$$\begin{aligned}
\tau_{t+1,\boldsymbol{n}}\tau_{t-1,\boldsymbol{n}} = {} & \frac{1}{\tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t-2+\boldsymbol{e}_2}^{k_2}\tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}\tau_{t-2,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}} \\
& \times \Big[\tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}\tau_{t-2,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}F_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}^{l_1k_1} \\
& \quad \cdot \tau_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}^{l_2k_1}F_{t-2,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}\tau_{t-1,\boldsymbol{n}-\boldsymbol{e}_1+\boldsymbol{e}_2}^{l_1k_2}\tau_{t-1,\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2}^{l_2k_2} \\
& + \tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t-2,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}F_{t-1,\boldsymbol{n}}\tau_{t-1,\boldsymbol{n}-\boldsymbol{e}_2-\boldsymbol{e}_1}^{k_1l_1}\tau_{t-1,\boldsymbol{n}+\boldsymbol{e}_2-\boldsymbol{e}_1}^{k_2l_1}\tau_{t-1,\boldsymbol{n}-\boldsymbol{e}_2+\boldsymbol{e}_1}^{k_1l_2}\tau_{t-1,\boldsymbol{n}+\boldsymbol{e}_2+\boldsymbol{e}_1}^{k_2l_2} \\
& \quad + \tau_{t-1,\boldsymbol{n}} \times (\text{polynomials of } \tau_{t-1,\boldsymbol{n}-2\boldsymbol{e}_2})\Big].
\end{aligned}$$

We further compute the first term in the square brackets above and obtain

$$\begin{aligned}
& \tau_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}^{l_1k_1}\tau_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}^{l_2k_1}\tau_{t-1,\boldsymbol{n}-\boldsymbol{e}_1+\boldsymbol{e}_2}^{l_1k_2}\tau_{t-1,\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2}^{l_2k_2} \\
& \quad \times \left(\tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}\tau_{t-2,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}F_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}F_{t-2,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} + \tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t-2,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}F_{t-1,\boldsymbol{n}}\right) \\
& = \tau_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}^{l_1k_1}\tau_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}^{l_2k_1}\tau_{t-1,\boldsymbol{n}-\boldsymbol{e}_1+\boldsymbol{e}_2}^{l_1k_2}\tau_{t-1,\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2}^{l_2k_2}F_{t-1,\boldsymbol{n}} \\
& \quad \times \left(\tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t-2,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} + F_{t-3,\boldsymbol{n}}\tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}\tau_{t-2,\boldsymbol{n}+\boldsymbol{e}_2}^{l_2}\right) \\
& = \tau_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}^{l_1k_1}\tau_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}^{l_2k_1}\tau_{t-1,\boldsymbol{n}-\boldsymbol{e}_1+\boldsymbol{e}_2}^{l_1k_2}\tau_{t-1,\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2}^{l_2k_2}F_{t-1,\boldsymbol{n}}\tau_{t-1,\boldsymbol{n}}\tau_{t-3,\boldsymbol{n}}.
\end{aligned}$$

Therefore there exists a polynomial $P_{t+1,\boldsymbol{n}} \in \mathcal{S}$ in $\tau_{t,\boldsymbol{n}-\boldsymbol{e}_2}$ and other iterates such that

$$\tau_{t+1,\boldsymbol{n}}\tau_{t-1,\boldsymbol{n}} = \frac{\tau_{t-1,\boldsymbol{n}}P_{t+1,\boldsymbol{n}}}{\tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t-2+\boldsymbol{e}_2}^{k_2}\tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}\tau_{t-2,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}}.$$

From lemma 8, we have that $\tau_{t-1,\boldsymbol{n}}$ is co-prime with $\tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}$, $\tau_{t-2+\boldsymbol{e}_2}$, $\tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_1}$, $\tau_{t-2,\boldsymbol{n}+\boldsymbol{e}_1}$, and satisfies $\tau_{t+1,\boldsymbol{n}}\tau_{t-1,\boldsymbol{n}} \in \mathcal{S}$. Therefore $\tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t-2+\boldsymbol{e}_2}^{k_2}\tau_{t-2,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}\tau_{t-2,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}$ divides $P_{t+1,\boldsymbol{n}}$. Thus $\tau_{t+1,\boldsymbol{n}} \in \mathcal{S}$. We have thus proved by induction that $\tau_{t,\boldsymbol{n}} \in \mathcal{S}$ for all t.

Note 3
Let us calculate $P_{t+1,\boldsymbol{n}}$:

$$P_{t+1,\boldsymbol{n}} = \frac{\tau_{t,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{t,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} + F_{t-1,\boldsymbol{n}}\tau_{t,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}\tau_{t,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}}{\tau_{t-1,\boldsymbol{n}}},$$

$$
\begin{aligned}
\tau^{k_1}_{t,\boldsymbol{n}-\boldsymbol{e}_2} &= \frac{1}{\tau^{k_1}_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}}\Big(F^{k_1}_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}\tau^{k_1l_1}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{k_1l_2}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}\\
&\quad +k_1F^{k_1-1}_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}\tau^{(k_1-1)l_1}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{(k_1-1)l_2}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{k_1}_{t-1,\boldsymbol{n}-2\boldsymbol{e}_2}\tau^{k_2}_{t-1,\boldsymbol{n}}+O(\tau^{2k_2}_{t-1,\boldsymbol{n}})\Big),\\
\tau^{k_2}_{t,\boldsymbol{n}+\boldsymbol{e}_2} &= \frac{1}{\tau^{k_2}_{t-2,\boldsymbol{n}+\boldsymbol{e}_2}}\Big(F^{k_2}_{t-2,\boldsymbol{n}+\boldsymbol{e}_2}\tau^{k_2l_1}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1+\boldsymbol{e}_2}\tau^{k_2l_2}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2}\\
&\quad +k_2F^{k_2-1}_{t-2,\boldsymbol{n}+\boldsymbol{e}_2}\tau^{(k_2-1)l_1}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1+\boldsymbol{e}_2}\tau^{(k_2-1)l_2}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2}\tau^{k_1}_{t-1,\boldsymbol{n}}\tau^{k_2}_{t-1,\boldsymbol{n}+2\boldsymbol{e}_2}+O(\tau^{2k_1}_{t-1,\boldsymbol{n}})\Big),\\
\tau^{l_1}_{t,\boldsymbol{n}-\boldsymbol{e}_1} &= \frac{1}{\tau^{l_1}_{t-2,\boldsymbol{n}-\boldsymbol{e}_1}}\Big(\tau^{l_1k_1}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{l_1k_2}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1+\boldsymbol{e}_2}\\
&\quad +l_1\tau^{k_1(l_1-1)}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{k_2(l_1-1)}_{t-1,\boldsymbol{n}+\boldsymbol{e}_2-\boldsymbol{e}_1}F_{t-2,\boldsymbol{n}-\boldsymbol{e}_1}\tau^{l_1}_{t-1,\boldsymbol{n}-2\boldsymbol{e}_1}\tau^{l_2}_{t-1,\boldsymbol{n}}+O(\tau^{2l_2}_{t-1,\boldsymbol{n}})\Big),\\
\tau^{l_2}_{t,\boldsymbol{n}+\boldsymbol{e}_1} &= \frac{1}{\tau^{l_2}_{t-2,\boldsymbol{n}+\boldsymbol{e}_1}}\Big(\tau^{l_2k_1}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{l_2k_2}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2}\\
&\quad +l_2\tau^{k_1(l_2-1)}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{k_2(l_2-1)}_{t-1,\boldsymbol{n}+\boldsymbol{e}_2+\boldsymbol{e}_1}F_{t-2,\boldsymbol{n}+\boldsymbol{e}_1}\tau^{l_1}_{t-1,\boldsymbol{n}}\tau^{l_2}_{t-1,\boldsymbol{n}+2\boldsymbol{e}_1}+O(\tau^{2l_1}_{t-1,\boldsymbol{n}})\Big).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
P_{t+1,\boldsymbol{n}} &= F_{t-1,\boldsymbol{n}}\tau^{l_1k_1}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{l_2k_1}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{l_1k_2}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1+\boldsymbol{e}_2}\tau^{l_2k_2}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2}\tau_{t-3,\boldsymbol{n}}\\
&+\tau^{l_1}_{t-2,\boldsymbol{n}-\boldsymbol{e}_1}\tau^{l_2}_{t-2,\boldsymbol{n}+\boldsymbol{e}_1}\Big(k_2F^{k_1}_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}F^{k_2-1}_{t-2,\boldsymbol{n}+\boldsymbol{e}_2}\\
&\quad\times\tau^{k_1l_1}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{k_1l_2}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{(k_2-1)l_1}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1+\boldsymbol{e}_2}\tau^{(k_2-1)l_2}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2}\tau^{k_1-1}_{t-1,\boldsymbol{n}}\tau^{k_2}_{t-1,\boldsymbol{n}+2\boldsymbol{e}_2}\\
&+k_1F^{k_2}_{t-2,\boldsymbol{n}+\boldsymbol{e}_2}F^{k_1-1}_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}\tau^{k_2l_1}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1+\boldsymbol{e}_2}\tau^{k_2l_2}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2}\tau^{(k_1-1)l_1}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{(k_1-1)l_2}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{k_1}_{t-1,\boldsymbol{n}-2\boldsymbol{e}_2}\tau^{k_2-1}_{t-1,\boldsymbol{n}}\Big)\\
&+\tau^{k_1}_{t-2,\boldsymbol{n}-\boldsymbol{e}_2}\tau^{k_2}_{t-2,\boldsymbol{n}+\boldsymbol{e}_2}F_{t-1,\boldsymbol{n}}\Big(l_2F_{t-2,\boldsymbol{n}+\boldsymbol{e}_1}\tau^{l_1k_1}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}\\
&\quad\times\tau^{l_1k_2}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1+\boldsymbol{e}_2}\tau^{k_1(l_2-1)}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{k_2(l_2-1)}_{t-1,\boldsymbol{n}+\boldsymbol{e}_2+\boldsymbol{e}_1}\tau^{l_1-1}_{t-1,\boldsymbol{n}}\tau^{l_2}_{t-1,\boldsymbol{n}+2\boldsymbol{e}_1}\\
&+l_1F_{t-2,\boldsymbol{n}-\boldsymbol{e}_1}\tau^{l_2k_1}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{l_2k_2}_{t-1,\boldsymbol{n}+\boldsymbol{e}_1+\boldsymbol{e}_2}\tau^{k_1(l_1-1)}_{t-1,\boldsymbol{n}-\boldsymbol{e}_1-\boldsymbol{e}_2}\tau^{k_2(l_1-1)}_{t-1,\boldsymbol{n}+\boldsymbol{e}_2-\boldsymbol{e}_1}\tau^{l_1}_{t-1,\boldsymbol{n}-2\boldsymbol{e}_1}\tau^{l_2-1}_{t-1,\boldsymbol{n}}\Big)+O(\tau_{t-1,\boldsymbol{n}})
\end{aligned}
\tag{3.29}
$$

Theorem 4
Every iterate $\tau_{t,\boldsymbol{n}}$ $(t\geq 2)$ is irreducible and any two iterates are co-prime.

Proof It is sufficient to prove the irreducibility for only $\tau_{t,\mathbf{0}}$ $(t\in 2\mathbb{Z})$, $\tau_{t,-\boldsymbol{e}_2}$ $(t\in 2\mathbb{Z}+1)$, because of the translational symmetries of the equation.

- In the case of $t=2,3$, the statement is trivial since $\tau_{t,\boldsymbol{n}}$ is a unit.
- In the case of $t=4$:

$$\tau_{4,\mathbf{0}}=\frac{\tau^{k_1}_{3,-\boldsymbol{e}_2}\tau^{k_2}_{3,\boldsymbol{e}_2}+F_{2,\mathbf{0}}\tau^{l_1}_{3,-\boldsymbol{e}_1}\tau^{l_2}_{3,\boldsymbol{e}_1}}{\tau_{2,\mathbf{0}}}.$$

  Since $F_{2,\mathbf{0}}=\dfrac{F^{k_1}_{1,-\boldsymbol{e}_2}F^{k_2}_{1,\boldsymbol{e}_2}}{F_{0,\mathbf{0}}}$, $\tau_{4,\mathbf{0}}$ is a first order polynomial of $F^{-1}_{0,\mathbf{0}}$, whose coefficient is co-prime with its constant term. Thus $\tau_{4,\mathbf{0}}$ is irreducible and not a unit.
- In the case of $t=5$, from lemma 12, we have

$$\tau_{5,-\boldsymbol{e}_2}=\left(\prod_{\boldsymbol{n}}\tau^{\alpha_{\boldsymbol{n}}}_{4,\boldsymbol{n}}\right)\times\tau'_{5,-\boldsymbol{e}_2}\qquad(\alpha_{\boldsymbol{n}}\in\mathbb{Z}_{\geq 0}),$$

where $\tau'_{5,-\boldsymbol{e}_2}$ is irreducible. We also have

$$\tau_{5,-\boldsymbol{e}_2} = \frac{\tau_{4,-2\boldsymbol{e}_2}^{k_1}\tau_{4,\boldsymbol{0}}^{k_2} + F_{3,-\boldsymbol{e}_2}\tau_{4,-\boldsymbol{e}_1-\boldsymbol{e}_2}^{l_1}\tau_{4,\boldsymbol{e}_1-\boldsymbol{e}_2}^{l_2}}{\tau_{3,-\boldsymbol{e}_2}}.$$

Since $\tau_{4,\boldsymbol{n}}$ and $\tau_{4,\boldsymbol{r}}$ are mutually co-prime when $\boldsymbol{n} \neq \boldsymbol{r}$ from lemma 7, we have $\alpha_{\boldsymbol{n}} = 0$ ($\boldsymbol{n} = \boldsymbol{0}$, $-2\boldsymbol{e}_2$, $\pm\boldsymbol{e}_1 - \boldsymbol{e}_2$). The term $\tau_{4,\boldsymbol{n}}$ is independent of $F_{0,\boldsymbol{r}}$ ($\boldsymbol{r} \neq \boldsymbol{n}$), and is a first order polynomial of $F_{0,\boldsymbol{n}}^{-1}$ whose constant term is non-zero. We also have that $F_{3,-\boldsymbol{e}_2}$ is a monomial of $F_{0,\boldsymbol{0}}$ and $F_{0,-2\boldsymbol{e}_2}$. Therefore $\tau_{5,-\boldsymbol{e}_2}$ is independent of all the iterates $F_{0,\boldsymbol{n}}$ ($\boldsymbol{n} \neq \boldsymbol{0}$, $-2\boldsymbol{e}_2$, $\pm\boldsymbol{e}_1 - \boldsymbol{e}_2$). Thus $\alpha_{\boldsymbol{n}} = 0$ for all $\boldsymbol{n} \neq \boldsymbol{0}$, $-2\boldsymbol{e}_2$, $\pm\boldsymbol{e}_1 - \boldsymbol{e}_2$. We have proved that $\tau_{5,-\boldsymbol{e}_2}$ is irreducible.

- In the case of $t = 6$, from lemma 12 we have the following factorization:

$$\tau_{6,\boldsymbol{0}} = \left(\prod_{\boldsymbol{n}} \tau_{4,\boldsymbol{n}}^{\alpha_{\boldsymbol{n}}}\right) \times \tau'_{6,\boldsymbol{0}} \qquad (\alpha_{\boldsymbol{n}} \in \mathbb{Z}_{\geq 0}),$$

where $\tau'_{6,\boldsymbol{0}}$ is irreducible. Let us take the initial data as ${}^{\forall}\boldsymbol{n}$, $F_{0,\boldsymbol{n}} = t_{\boldsymbol{n}}^{-1}$, and take all the other initial data as 1. Then we have

$$\begin{aligned} F_{2,\boldsymbol{n}} &= t_{\boldsymbol{n}}, \\ F_{3,\boldsymbol{n}} &= t_{\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} t_{\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}, \\ F_{4,\boldsymbol{n}} &= t_{\boldsymbol{n}-2\boldsymbol{e}_2}^{k_1^2} t_{\boldsymbol{n}}^{2k_1k_2-1} t_{\boldsymbol{n}+2\boldsymbol{e}_2}^{k_2^2}, \end{aligned}$$

and

$$\begin{aligned} \tau_{4,\boldsymbol{n}} &= 1 + t_{\boldsymbol{n}}, \\ \tau_{5,\boldsymbol{n}} &= (1 + t_{\boldsymbol{n}-\boldsymbol{e}_2})^{k_1}(1 + t_{\boldsymbol{n}+\boldsymbol{e}_2})^{k_2} + t_{\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} t_{\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}(1 + t_{\boldsymbol{n}-\boldsymbol{e}_1})^{l_1}(1 + t_{\boldsymbol{n}+\boldsymbol{e}_1})^{l_2}, \\ \tau_{6,\boldsymbol{n}} &= \frac{\tau_{5,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{5,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} + t_{\boldsymbol{n}-2\boldsymbol{e}_2}^{k_1^2} t_{\boldsymbol{n}}^{2k_1k_2-1} t_{\boldsymbol{n}+2\boldsymbol{e}_2}^{k_2^2}\tau_{5,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1}\tau_{5,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}}{1 + t_{\boldsymbol{n}}}. \end{aligned}$$

Therefore the iterate $\tau_{6,\boldsymbol{0}}$ depends only on $t_{\boldsymbol{n}}$ ($\boldsymbol{n} \in \mathcal{I}_6$) where

$$\mathcal{I}_6 := \left\{\boldsymbol{n} = j_1\boldsymbol{e}_1 + j_2\boldsymbol{e}_2 \,\middle|\, |j_1| + |j_2| = 0,\, 2,\, (j_1, j_2) \in \mathbb{Z}^2\right\}.$$

Thus $\boldsymbol{n} \notin \mathcal{I}_6 \to \alpha_{\boldsymbol{n}} = 0$. Let us prove that $\alpha_{\boldsymbol{n}} = 0$ for $\boldsymbol{n} \in \mathcal{I}_6$, one by one.

(i) In the case of $t_{2\boldsymbol{e}_1} = -1$, with $t_{\boldsymbol{n}} = 1$ for all $\boldsymbol{n} \neq 2\boldsymbol{e}_1$:

$$\tau_{5,\pm\boldsymbol{e}_2} = 2^{k_1+k_2} + 2^{l_1+l_2} =: c_5, \quad \tau_{5,\boldsymbol{e}_1} = 2^{k_1+k_2}, \quad \tau_{5,-\boldsymbol{e}_1} = c_5.$$

Therefore we have $\tau_{6,\boldsymbol{0}} > 0$, and thus $\alpha_{2\boldsymbol{e}_1} = 0$. From the symmetry of the equation we also have $\alpha_{-2\boldsymbol{e}_1} = 0$.

(ii) In the case of $t_{2\boldsymbol{e}_2} = -1$, with $t_{\boldsymbol{n}} = 1$ for all $\boldsymbol{n} \neq 2\boldsymbol{e}_2$:

$$\tau_{5,\boldsymbol{e}_2} = (-1)^{k_2}2^{l_1+l_2}, \quad \tau_{5,-\boldsymbol{e}_2} = c_5, \quad \tau_{5,\pm\boldsymbol{e}_1} = c_5.$$

Let us note that $k_2^2 \equiv k_2 \pmod 2$ and we obtain

$$\tau_{6,\boldsymbol{0}} = (-1)^{k_2}\frac{2^{l_1+l_2}c_5^{k_1} + c_5^{l_1+l_2}}{2} \neq 0.$$

Therefore $\alpha_{2\boldsymbol{e}_2} = 0$. We also have $\alpha_{-2\boldsymbol{e}_2} = 0$ in the same way.

(iii) In the case of $t_{\boldsymbol{e}_1+\boldsymbol{e}_2} = -1$, with $t_{\boldsymbol{n}} = 0$ for all $\boldsymbol{n} \neq \boldsymbol{e}_1 + \boldsymbol{e}_2$:

$$\tau_{5,\boldsymbol{e}_2} = 1, \quad \tau_{5,-\boldsymbol{e}_2} = 1, \quad \tau_{5,\boldsymbol{e}_1} = 0, \quad \tau_{5,-\boldsymbol{e}_1} = 1.$$

Therefore

$$\tau_{6,\boldsymbol{0}} \to 1.$$

Thus $\alpha_{\boldsymbol{e}_1+\boldsymbol{e}_2} = 0$. We also have $\alpha_{-\boldsymbol{e}_1-\boldsymbol{e}_2} = 0$, $\alpha_{\boldsymbol{e}_1-\boldsymbol{e}_2} = 0$, and, $\alpha_{-\boldsymbol{e}_1+\boldsymbol{e}_2} = 0$.

(v) In the case of $\tau_{\boldsymbol{0}} = -1$, with $t_{\boldsymbol{n}} = 1$ for all $\boldsymbol{n} \neq \boldsymbol{0}$.

$$F_{3,\boldsymbol{e}_2} = (-1)^{k_1}, F_{3,-\boldsymbol{e}_2} = (-1)^{k_2}, F_{3,\boldsymbol{n}} = 1 \ (\boldsymbol{n} \neq \pm\boldsymbol{e}_2), F_{4,\boldsymbol{0}} = (-1).$$

From equation (3.29),

$$\begin{aligned}
P_{6,\boldsymbol{0}} \to\ & (-1)^{(k_1+k_2)(l_1+l_2)} + k_2\delta_{k_1,1}(-1)^{k_1k_2}(-1)^{k_1(k_2-1)}2^{(k_1+k_2-1)(l_1+l_2)+k_2} \\
& + k_1\delta_{k_2,1}(-1)^{k_1k_2}(-1)^{k_2(k_1-1)}2^{(k_1+k_2-1)(l_1+l_2)+k_1} \\
& + (-l_2)\delta_{l_1,1}2^{(k_1+k_2)(l_1+l_2-1)+l_2} + (-l_1)\delta_{l_2,1}2^{(k_1+k_2)(l_1+l_2-1)+l_1} \\
& = (-1)^{(k_1+k_2)(l_1+l_2)} + (-k_2)\delta_{k_1,1}2^{k_2(l_1+l_2+1)} + (-k_1)\delta_{k_2,1}2^{k_1(l_1+l_2+1)} \\
& + (-l_2)\delta_{l_1,1}2^{l_2(k_1+k_2+1)} + (-l_1)\delta_{l_2,1}2^{l_1(k_1+k_2+1)} \neq 0.
\end{aligned}$$

Therefore we have $\alpha_{\boldsymbol{0}} = 0$.

From these observations we conclude that $\tau_{6,\boldsymbol{0}}$ is irreducible.

- Preparation for $t \geq 7$: Let us define c_t as the value of $\tau_{t,\boldsymbol{n}}$ when we take all the initial data as 1. Note that c_t does not depend on $\boldsymbol{n}$. If we substitute 1 for all the initial data in $F_{t,\boldsymbol{n}}$, we have $F_{t,\boldsymbol{n}} \to 1$, and

$$c_3 = 1, \quad c_4 = 2, \quad c_{j+1} = \frac{c_j^{k_1+k_2} + c_j^{l_1+l_2}}{c_{j-1}} \ (j \geq 4).$$

It is easy to see that the c_j are strictly increasing. Therefore we have shown the following fact: if $\tau_{t,\boldsymbol{n}}$ and $\tau_{s,\boldsymbol{r}}$ $(s \neq t)$ are irreducible, then $\tau_{t,\boldsymbol{n}}$ and $\tau_{s,\boldsymbol{r}}$ are co-prime. Also, from lemma 12, $\tau_{t,\boldsymbol{n}}$ and $\tau_{t,\boldsymbol{r}}$ $(\boldsymbol{n} \neq \boldsymbol{r})$ are co-prime if they are both irreducible. Thus the irreducibility immediately implies the co-primeness.

- In the case of $t = 7$: From lemma 12,

$$\begin{aligned}
&\tau_{7,-\boldsymbol{e}_2} = \prod_{\boldsymbol{n}} \tau_{4,\boldsymbol{n}}^{\alpha_{\boldsymbol{n}}} \tau'_{7,-\boldsymbol{e}_2} = \prod_{\boldsymbol{n},\boldsymbol{r}} \tau_{5,\boldsymbol{n}}^{\beta_{\boldsymbol{n}}} \tau_{6,\boldsymbol{r}}^{\gamma_{\boldsymbol{r}}} \tau''_{7,-\boldsymbol{e}_2}, \\
&(\alpha_{\boldsymbol{n}}, \beta_{\boldsymbol{n}}, \gamma_{\boldsymbol{r}} \in \mathbb{Z}_{\geq 0}),
\end{aligned}$$

where $\tau'_{7,-\boldsymbol{e}_2}$, $\tau''_{7,-\boldsymbol{e}_2}$ are irreducible. If we suppose that $\tau_{7,-\boldsymbol{e}_2}$ is not irreducible, then there exist $\boldsymbol{n}$, $\boldsymbol{r}$, j such that

$$\tau_{7,-\boldsymbol{e}_2} = \text{unit} \times \tau_{4,\boldsymbol{n}}\tau_{j,\boldsymbol{r}} \qquad (j \in \{5, 6\}).$$

Therefore

$$c_7 \leq c_4c_6 = 2c_6.$$

On the other hand,

$$c_7 = \frac{c_6^{k_1+k_2} + c_6^{l_1+l_2}}{c_5} > c_6^{k_1} + c_6^{l_1} \geq 2c_6,$$

which is a contradiction.

- In the case of $t \geq 8$, the discussion goes in exactly the same manner.

The proof is complete. Our final aim is to prove the following theorem:

Theorem 5
Two iterates $U(t,\boldsymbol{n})$ and $U(s,\boldsymbol{r})$ of the equation (3.20) satisfy the following property: if $|t-s|>2$ or $\boldsymbol{r}\neq\boldsymbol{n}$, $\boldsymbol{n}\pm 2\boldsymbol{e}_2$, they are co-prime in

$$\mathcal{S}_U := \mathbb{Z}\left[\left\{U_{t,\boldsymbol{n}}^{\pm}, (U_{t,\boldsymbol{n}}-1)^{\pm}\right\}_{t=0,1}\right]. \tag{3.30}$$

Let us use the following notations:

$$\boldsymbol{\tau}_t := \{\tau_{t,\boldsymbol{n}}\}, \quad \boldsymbol{F}_t := \{F_{t,\boldsymbol{n}}\}, \quad \boldsymbol{U}_t = \{U_{t,\boldsymbol{n}}\},$$
$$\boldsymbol{T} := \{\boldsymbol{F}_0, \boldsymbol{F}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3\}, \quad \boldsymbol{W} := \{\boldsymbol{\tau}_0, \boldsymbol{\tau}_1, \boldsymbol{U}_1, \boldsymbol{U}_2\}$$

Then we have $\mathcal{S} = \mathbb{Z}\left[\boldsymbol{F}_0^{\pm}, \boldsymbol{F}_1^{\pm}, \boldsymbol{\tau}_2^{\pm}, \boldsymbol{\tau}_3^{\pm}\right] = \mathbb{Z}\left[\boldsymbol{T}\right]$. Note that from our previous results, all the iterates $\tau_{t,\boldsymbol{n}} \in \mathcal{S}$ are irreducible and mutually co-prime. As before, we have the following lemma.

Lemma 9
We have a birational mapping between $\boldsymbol{T}$ and $\boldsymbol{W}$.

Proof $\boldsymbol{W} \to \boldsymbol{T}$

$$\tau_{2,\boldsymbol{n}} = \frac{1}{\tau_{0,\boldsymbol{n}}} U_{1,\boldsymbol{n}} \tau_{1,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} \tau_{1,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}, \tag{3.31a}$$

$$\tau_{3,\boldsymbol{n}} = \frac{1}{\tau_{1,\boldsymbol{n}}} U_{2,\boldsymbol{n}} \tau_{2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} \tau_{2,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} = \frac{U_{2,\boldsymbol{n}} U_{1,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} U_{1,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} \tau_{1,\boldsymbol{n}-2\boldsymbol{e}_2}^{k_1^2} \tau_{1,\boldsymbol{n}}^{2k_1k_2-1} \tau_{1,\boldsymbol{n}+2\boldsymbol{e}_2}^{k_2^2}}{\tau_{0,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} \tau_{0,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}. \tag{3.31b}$$

Using these results,

$$F_{0,\boldsymbol{n}} = \frac{\tau_{2,\boldsymbol{n}}\tau_{0,\boldsymbol{n}} - \tau_{1,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} \tau_{1,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}{\tau_{1,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1} \tau_{1,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}} = \frac{\tau_{1,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} \tau_{1,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}{\tau_{1,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1} \tau_{1,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}} (U_{1,\boldsymbol{n}} - 1), \tag{3.31c}$$

$$\begin{aligned} F_{1,\boldsymbol{n}} &= \frac{\tau_{3,\boldsymbol{n}}\tau_{1,\boldsymbol{n}} - \tau_{2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} \tau_{2,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}{\tau_{2,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1} \tau_{2,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}} = \frac{\tau_{2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} \tau_{2,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}{\tau_{2,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1} \tau_{2,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}} (U_{2,\boldsymbol{n}} - 1) \\ &= \frac{\tau_{0,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1} \tau_{0,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}}{\tau_{0,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} \tau_{0,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}} \frac{\tau_{1,\boldsymbol{n}-2\boldsymbol{e}_2}^{k_1^2} \tau_{1,\boldsymbol{n}}^{2k_1k_2-1} \tau_{1,\boldsymbol{n}+2\boldsymbol{e}_2}^{k_2^2}}{\tau_{1,\boldsymbol{n}-2\boldsymbol{e}_1}^{1^2} \tau_{1,\boldsymbol{n}}^{2l_1l_2-1} \tau_{1,\boldsymbol{n}+2\boldsymbol{e}_1}^{l_2^2}} \frac{U_{1,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} U_{1,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}{U_{1,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1} U_{1,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2}} (U_{2,\boldsymbol{n}} - 1). \end{aligned} \tag{3.31d}$$

The inverse mapping ($\boldsymbol{T} \to \boldsymbol{W}$) is constructed as

$$\tau_{1,\boldsymbol{n}} = \frac{1}{\tau_{3,\boldsymbol{n}}} \left(F_{1,\boldsymbol{n}} \tau_{2,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1} \tau_{2,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2} + \tau_{2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} \tau_{2,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} \right), \tag{3.32a}$$

and

$$\tau_{0,\boldsymbol{n}} = \frac{1}{\tau_{2,\boldsymbol{n}}} \left(F_{0,\boldsymbol{n}} \tau_{1,\boldsymbol{n}-\boldsymbol{e}_1}^{l_1} \tau_{1,\boldsymbol{n}+\boldsymbol{e}_1}^{l_2} + \tau_{1,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} \tau_{1,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2} \right), \tag{3.32b}$$

$$U_{2,\boldsymbol{n}} = \frac{\tau_{3,\boldsymbol{n}}\tau_{1,\boldsymbol{n}}}{\tau_{2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} \tau_{2,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}, \to U_{1,\boldsymbol{n}} = \frac{\tau_{2,\boldsymbol{n}}\tau_{0,\boldsymbol{n}}}{\tau_{1,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1} \tau_{1,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}. \tag{3.32c}$$

It is immediately shown that these transformations are made up irreducible Laurent polynomials.

Lemma 10
Let us define the new variable $\sigma_{t,\boldsymbol{n}}$ as $\sigma_{t,\boldsymbol{n}}(\boldsymbol{W}) := \tau_{t,\boldsymbol{n}}(\boldsymbol{T})$. Then we have the following factorization: $\sigma_{t,\boldsymbol{n}} = u_{t,\boldsymbol{n}}\tilde{\sigma}_{t,\boldsymbol{n}}$, where $u_{t,\boldsymbol{n}}$ is a monic Laurent monomial of $\tau_{0,\boldsymbol{r}}$, $\tau_{1,\boldsymbol{r}}$, and we have

$$\tilde{\sigma}_{t,\boldsymbol{n}} \in \mathbb{Z}\left[\{U_{s,\boldsymbol{r}}^{\pm},\ (U_{s,\boldsymbol{r}}-1)^{\pm}\}_{s=0,1}\right].$$

Proof By a direct computation, we have

$$\begin{aligned}
\tau_{2,\boldsymbol{n}} &= \frac{U_{1,\boldsymbol{n}}}{\tau_{0,\boldsymbol{n}}}\left(\tau_{1,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{1,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}\right) = \frac{\tau_{1,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{1,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}{\tau_{0,\boldsymbol{n}}}\cdot U_{1,\boldsymbol{n}},\\
\tau_{3,\boldsymbol{n}} &= \frac{U_{2,\boldsymbol{n}}}{\tau_{1,\boldsymbol{n}}}\left(\tau_{2,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{2,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}\right)\\
&= \frac{U_{2,\boldsymbol{n}}}{\tau_{1,\boldsymbol{n}}}\left(\frac{U_{1,\boldsymbol{n}-\boldsymbol{e}_2}}{\tau_{0,\boldsymbol{n}-\boldsymbol{e}_2}}\right)^{k_1}\left(\frac{U_{1,\boldsymbol{n}+\boldsymbol{e}_2}}{\tau_{0,\boldsymbol{n}+\boldsymbol{e}_2}}\right)^{k_2}\left(\tau_{1,\boldsymbol{n}-2\boldsymbol{e}_2}^{k_1^2}\tau_{1,\boldsymbol{n}}^{2k_1k_2}\tau_{1,\boldsymbol{n}+2\boldsymbol{e}_2}^{k_2^2}\right)\\
&= \frac{\tau_{1,\boldsymbol{n}-2\boldsymbol{e}_2}^{k_1^2}\tau_{1,\boldsymbol{n}}^{2k_1k_2-1}\tau_{1,\boldsymbol{n}+2\boldsymbol{e}_2}^{k_2^2}}{\tau_{0,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tau_{0,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}\cdot U_{2,\boldsymbol{n}}U_{1,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}U_{1,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2},\\
&\ \ \vdots\\
\tau_{t,\boldsymbol{n}} &= (\text{Monic Laurent monomial of } \tau_{0,\boldsymbol{r}}, \tau_{1,\boldsymbol{r}})\\
&\qquad \times (\text{Monic monomial of } U_{1,\boldsymbol{r}}, ..., U_{t-1,\boldsymbol{n}}).
\end{aligned}$$

From equation (3.20), $U_{s,\boldsymbol{n}}$ $(s = 3, 4, ...)$ can be expressed using $U_{1,\boldsymbol{r}}$, $U_{2,\boldsymbol{r}}$. Therefore $u_{t,\boldsymbol{n}}$ is a monic Laurent monomial of $\tau_{0,\boldsymbol{r}}$, $\tau_{1,\boldsymbol{r}}$, and has a factor $\tilde{\sigma}_{t,\boldsymbol{n}} \in \mathbb{Q}(\boldsymbol{U}_0, \boldsymbol{U}_1)$. On the other hand, from $\tau_{t,\boldsymbol{n}} \in \mathcal{S}$ and the transformations (3.31a) – (3.31d) we have

$$\sigma_{t,\boldsymbol{n}} \in \mathbb{Z}\left[\tau_0^{\pm}, \tau_1^{\pm}, \{U_{s,\boldsymbol{r}}^{\pm},\ (U_{s,\boldsymbol{r}}-1)^{\pm}\}_{s=0,1}\right].$$

Therefore, from the uniqueness of the factorization, we conclude that

$$\tilde{\sigma}_{t,\boldsymbol{n}} \in \mathbb{Z}\left[\{U_{s,\boldsymbol{r}}^{\pm},\ (U_{s,\boldsymbol{r}}-1)^{\pm}\}_{s=0,1}\right].$$

Proposition 7
$\tilde{\sigma}_{t,\boldsymbol{n}}$ is irreducibile in $\mathbb{Z}\left[\{U_{s,\boldsymbol{r}}^{\pm},\ (U_{s,\boldsymbol{r}}-1)^{\pm}\}_{s=0,1}\right]$. For $(t,\boldsymbol{n}) \neq (s,\boldsymbol{r})$, $\tilde{\sigma}_{t,\boldsymbol{n}}$ and $\tilde{\sigma}_{s,\boldsymbol{r}}$ are co-prime.

Proof We use an argument similar to that in the proof of Proposition 5. We have

$$\sigma_{t,\boldsymbol{n}} \in \tilde{\mathcal{S}} := \mathbb{Z}\left[\boldsymbol{\tau}_0^{\pm}, \boldsymbol{\tau}_1^{\pm}, \{U_{s,\boldsymbol{r}}^{\pm},\ (U_{s,\boldsymbol{r}}-1)^{\pm}\}_{s=0,1}\right].$$

Let us suppose that we can factor $\sigma_{t,\boldsymbol{n}}$ as $\sigma_{t,\boldsymbol{n}} = h_1(\boldsymbol{W})h_2(\boldsymbol{W})$ $(h_1,\ h_2 \in \tilde{\mathcal{S}})$. From equations (3.31a)–(3.31d) we can reformulate $(U_{s,\boldsymbol{r}} - 1)$, to obtain

$$h_1h_2 \in \mathcal{S}' := \mathbb{Z}\left[\{\boldsymbol{\tau}_i^{\pm}\}_{i=0}^{3},\ \boldsymbol{F}_t^{\pm}\right].$$

On the other hand, $\tau_{t,\boldsymbol{n}} \in \mathcal{S}$ is irreducible in $\mathcal{S}'$ and we can assume that h_1 is a unit in $\mathcal{S}'$. Therefore h_1 can be expresssed as

$$h_1 = \prod_{i=0}^{1}\prod_{\boldsymbol{n}} F_{i,\boldsymbol{n}}^{\alpha_{i,\boldsymbol{n}}}\prod_{j=0}^{4}\prod_{\boldsymbol{r}}\tau_{j,\boldsymbol{r}}^{\beta_{j,\boldsymbol{r}}} \qquad (\alpha_{i,\boldsymbol{n}}, \beta_{j,\boldsymbol{r}} \in \mathbb{Z}),$$

which implies that h_1 is a unit also in $\tilde{\mathcal{S}}$. Thus $\sigma_{t,\boldsymbol{n}}$ is irreducible in $\tilde{\mathcal{S}}$. From the uniqueness of the factorization, the iterate $\tilde{\sigma}_{t,\boldsymbol{n}}$ is irreducible in $\mathbb{Z}\left[\{U_{s,\boldsymbol{r}}^{\pm}, (U_{s,\boldsymbol{r}}-1)^{\pm}\}_{s=0,1}\right]$. Finally we prove co-primeness. Let us suppose that $\sigma_{t,\boldsymbol{n}}$ and $\sigma_{s,\boldsymbol{r}}$ are not co-prime. Then they must have a (non-unit) common factor H in $\tilde{\mathcal{S}}$. Then H is not a unit in $\mathcal{S}'$, and $\tau_{t,\boldsymbol{n}}$ and $\tau_{s,\boldsymbol{r}}$ are not co-prime in $\mathcal{S}'$. This conclusion contradicts the outcome of Theorem 4 that $\tau_{t,\boldsymbol{n}}$ and $\tau_{s,\boldsymbol{r}}$ are co-prime in $\mathcal{S}$.

Proof of Theorem 5 From equation (3.20), we have that $U_{t,\boldsymbol{n}} \in \mathbb{Q}(\boldsymbol{U})$. Thus

$$U_{t,\boldsymbol{n}} = \frac{\sigma_{t+1,\boldsymbol{n}}\sigma_{t-1,\boldsymbol{n}}}{\sigma_{t,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\sigma_{t,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}} = \frac{\tilde{\sigma}_{t+1,\boldsymbol{n}}\tilde{\sigma}_{t-1,\boldsymbol{n}}}{\tilde{\sigma}_{t,\boldsymbol{n}-\boldsymbol{e}_2}^{k_1}\tilde{\sigma}_{t,\boldsymbol{n}+\boldsymbol{e}_2}^{k_2}}$$

Therefore we obtain Theorem 5 from Proposition 7.

Chapter 4

Conclusion

In this thesis we have constructed a quasi-integrable extension to the two-dimensional discrete Toda equation (quasi 2D-dToda), and proved that it has Laurent property, the irreducibility and the co-primeness property. The quasi 2D-dToda is considered to be the first example of quasi-integrable equations defined on a three-dimensional lattice $\mathbb{Z}^3$. We have also presented a quasi-integrable Somos-4 recurrence, a quasi-integrable 1D discrete Toda equation (quasi 1D-dToda), through a reduction from quasi 2D-dToda, and have proved that they also have the Laurent, the irreducibility and the co-primeness properties (although some parts of the proofs have been omitted). Quasi-integrable Somos-4 and quasi 1D-dToda are already known to have the Laurent property [10], and in this thesis, we have added the proof for the irreducibility and the co-primeness.

Properties of the irreducibility and co-primeness are considered as strong indications of the discrete integrability (including quasi-integrability), and also are algebraic reinterpretations of singularity confinement. It is expected that further exploration into the topics related to the co-primeness, will lead us to constructing refined integrability criteria for discrete dynamical systems. For example, in some discrete equations, even when the general iterates are not irreducible, we can still prove that every pair of two iterates are co-prime. This phenomenon might lie in the boundary of integrable systems and non-integrable ones. For example, the quasi-integrable Somos-4 equation satisfies co-primeness property when we remove the condition that 'the polynomial $x^l y^m + z^k$ is irreducible in $\mathbb{Z}$'.

Another interesting topic to study is the nonlinear forms of the equations that we have investigated here. The original 2-dimensional discrete Toda equation has the τ-function bilinear form (1.2) as we have presented in this thesis, and also has a nonlinear expression related to it. We proved that the nonlinear extended discrete 2D Toda lattice equations (1.5) all have the co-primeness property in $\mathbb{Z}\left[\{U_{s,\boldsymbol{r}}^{\pm},\ (U_{s,\boldsymbol{r}}-1)^{\pm}\}_{s=0,1}\right]$ for general k_1, k_2, l_1 and l_2. The nonlinear recurrence corresponding to the extended Somos-4 (3.1), which is obtained as a reduction of (1.5), also shows the co-primeness property in the ring $\mathbb{Z}\left[\{u_j^{\pm}, (u_j-1)^{\pm}\}_{j=2}^{5}\right]$. Since the co-primeness property is an algebraic analogue of singularity confinement[5], the reason why the ring $\mathbb{Z}\left[\{u_j^{\pm}, (u_j-1)^{\pm}\}_{j=2}^{5}\right]$ appears in the property is that the "singularities"of (3.1) are not only at $u_j = 0$ but also at $u_j = 1$ $(j = 2, 3, 4, 5)$.

So far we have constructed several higher dimensional non-linear equations which have the co-primeness property[18]. A natural question is whether there is any systematic way of constructing such equations. For discrete Painlevé equations, which were first introduced as second order rational mappings with the singularity confinement property, Sakai has given a geometric construction of the so called space of initial conditions and completed the classification of the discrete Painlevé equations[7] [19].

We expect some geometric interpretation of the co-primeness property with which a systematic construction and classification of co-primeness preserving nonlinear equations becomes possible. This is one of the problems we wish to address in future works. An investigation of the continuous limits of these discrete equations is also a future problem.

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Appendix

A Conditions on the indices of the equation (1.3)

We prove the following proposition:

Proposition 8
Let r be a positive integer, and let a_1, a_2, a_3, a_4 be non-negative integers with $GCD(a_1, a_2, a_3, a_4) = 1$. Then the polynomial

$$X_1^{a_1 r} X_2^{a_2 r} + X_3^{a_3 r} X_4^{a_4 r}$$

is irreducible in $\mathbb{Z}[X_1, X_2, X_3, X_4]$ if and only if $r = 2^l$ $(l \geq 0)$.

Sketch of Proof If r has a odd prime factor, it is trivial that $X_1^{a_1 r} X_2^{a_2 r} + X_3^{a_3 r} X_4^{a_4 r}$ is reducible. We prove that, if r is a power of 2, $X_1^{a_1 r} X_2^{a_2 r} + X_3^{a_3 r} X_4^{a_4 r}$ is irreducible. It is sufficient to prove the irreducibility of

$$F := (X_1^{a_1} X_2^{a_2} X_3^{-a_3} X_4^{-a_4})^r + 1$$

as a Laurent polynomial. From Lemma 11, there exists a matrix $B = (b_{ij}) \in \mathrm{GL}_{\mathbb{Z}}(4)$ such that

$$b_{11} = a_1, \quad b_{21} = a_2, \quad b_{31} = -a_3, \quad b_{41} = -a_4.$$

Then the following ring homomorphism

$$\psi \colon \mathbb{Z}[Y_1^{\pm}, Y_2^{\pm}, Y_3^{\pm}, Y_4^{\pm}] \to \mathbb{Z}[X_1^{\pm}, X_2^{\pm}, X_3^{\pm}, X_4^{\pm}],$$

defined by

$$Y_i \mapsto X_1^{b_{i1}} X_2^{b_{i2}} X_3^{b_{i3}} X_4^{b_{i4}},$$

is in fact an isomorphism, since we can define its inverse by

$$\psi^{-1} \colon \mathbb{Z}[X_1^{\pm}, X_2^{\pm}, X_3^{\pm}, X_4^{\pm}] \to \mathbb{Z}[Y_1^{\pm}, Y_2^{\pm}, Y_3^{\pm}, Y_4^{\pm}], \quad X_i \mapsto Y_1^{c_{i1}} Y_2^{c_{i2}} Y_3^{c_{i3}} Y_4^{c_{i4}},$$

where $C = (c_{ij}) = B^{-1}$. From the definition of the map ψ, we have

$$\psi(Y_1^r + 1) = F.$$

Since r is a non-negative power of 2, $Y_1^r + 1$ is the $2r$-th cyclotomic polynomial and is irreducible. The irreducibility is preserved under the isomorphism ψ, thus F is irreducible.

Lemma 11
Suppose that $a_1, \ldots, a_N \in \mathbb{Z}$ are co-prime integers. Then there exists a matrix $B = (b_{ij}) \in \mathrm{GL}_{\mathbb{Z}}(N)$ such that

$$b_{i1} = a_i,$$

for every i.

Proof can be done using a knowledge of elementary algebra.

B Lemma on the factorization of Laurent polynomials in [9]

Let us reproduce a lemma on how the Laurent polynomial is factorized when we make a transformation to the variables, where the two sets of variables (before and after the transformation) satisfy some good conditions. In usual settings, the conditions are satisfied thanks to the Laurent property and the invertibility of the equation.

Lemma 12 ([9])
Let M be a positive integer and let $\{p_1, p_2, \cdots, p_M\}$ and $\{q_1, q_2, \cdots, q_M\}$ be two sets of independent variables with the following properties:

$$p_j \in \mathbb{Z}\left[q_1^{\pm}, q_2^{\pm}, \cdots, q_M^{\pm}\right], q_j \in \mathbb{Z}\left[p_1^{\pm}, p_2^{\pm}, \cdots, p_M^{\pm}\right],$$
$$q_j \text{ is irreducible as an element of } \mathbb{Z}\left[p_1^{\pm}, p_2^{\pm}, \cdots, p_M^{\pm}\right],$$

for $j = 1, 2, \cdots, M$. Let us take an irreducible Laurent polynomial

$$f(p_1, \cdots, p_M) \in \mathbb{Z}\left[p_1^{\pm}, p_2^{\pm}, \cdots, p_M^{\pm}\right],$$

and another (not necessarily irreducible) Laurent polynomial

$$g(q_1, \cdots, q_M) \in \mathbb{Z}\left[q_1^{\pm}, q_2^{\pm}, \cdots, q_M^{\pm}\right],$$

which satisfies $f(p_1, \cdots, p_M) = g(q_1 \cdots, q_M)$. In these settings, the function g is decomposed as

$$g(q_1, \cdots, q_M) = p_1^{r_1} p_2^{r_2} \cdots p_M^{r_M} \cdot \tilde{g}(q_1, \cdots, q_M),$$

where $r_1, r_2, \cdots, r_M \in \mathbb{Z}$ and $\tilde{g}(q_1, \cdots, q_M)$ is irreducible in $\mathbb{Z}\left[q_1^{\pm}, q_2^{\pm}, \cdots, q_M^{\pm}\right]$.

The underlying idea is the fact in algebra that the localization of a unique factorization domain preserves the irreducibility of its elements. Proof is found in reference [9].

Bibliography

[1] M. Toda: Vibration of a chain with nonlinear interaction, *J. Phys. Soc. Jpn.* **22** (1967), 431–6.

[2] R. Hirota: Nonlinear partial difference equations. II. Discrete-time Toda equation, *J. Phys. Soc. Jpn.* **43** (1977), 2074–2078.

[3] A. N. Leznov, M. V. Saveliev: Two-dimensional exactly and completely integrable dynamical systems, *Comm. Math. Phys.* **89** (1983), 59–75.

[4] R. Hirota, S. Tsujimoto, and T. Imai, Difference scheme of soliton equations, *RIMS Kokyuroku.* **822** (1977), 144-152.

[5] B. Grammaticos and A. Ramani and V. Papageorgiou: Do integrable mappings have the Painlevé property?, *Phys. Rev. Lett.* **67** (1977), 1825–1828.

[6] M. P. Bellon and C. M. Viallet: Algebraic entropy, *Comm. Math. Phys.* **204** (1999), 425–437.

[7] A. Ramani and B. Grammaticos and J. Hietarinta: Discrete versions of the Painlevé equations, *Phys. Rev. Lett.* **67** (1991), 1829–1832.

[8] J. Hietarinta and C. Viallet: Singularity confinement and chaos in discrete systems, *Phys. Rev. Lett.* **81** (1998), 325–328.

[9] M. Kanki, J. Mada and T. Mase and T. Tokihiro: Irreducibility and co-primeness as an integrability criterion for discrete equations, *J. Phys. A: Math. Theor.* **47** (2014), 465204 (15pp).

[10] S. Fomin, A. Zelevinsky: The Laurent Phenomenon, *Adv. Applied Math.* **28** (2002), 119–144.

[11] M. Kanki and J. Mada and T. Tokihiro: Irreducibility and co-primeness as an integrability criterion for discrete equations, *J. Math. Phys.* **56** (2015), 022706 (22pp).

[12] R. Hirota: Discrete Analogue of a Generalized Toda Equation, *J. Phys. Soc. Jpn.* **50** (1981), 3785–3791.

[13] Satoshi Tsujimoto: Studies on Discrete Nonlinear Integrable Systems, *Doctor Thesis, Waseda University* (1997).

[14] A.V. Mikhailov: Integrability of a two-dimensional generalization of the Toda chain, *JETP Lett.* **30** (1979), 414–418.

[15] A.P.Fordy and J.Gibbons: Integrable Nonlinear Klein-Gordon Equation and Toda Lattices, *Comm. Math. Phys.* **77** (1980), 21–30.

[16] D. Gale,: The Strange and Surprising Saga of the Somos Sequences, *Math. Intel.* **13** (1991), 40–42.

[17] T. Mase: Investigation into the role of the Laurent property in integrability, *J. Math. Phys.* **57** (2016), 022703.

[18] M. Kanki, T. Mase and T. Tokihiro: Singularity confinement and chaos in two-dimensional discrete systems, *J. Phys. A.* **49** (2015), 23LT01 .

[19] H. Sakai: Rational surfaces associated with affine root systems and geometry of the Painlevé equations, *Commun. Math. Phys.* **220** (2001), 165229.