博士論文

論文題目 Wrapping projections and decompositions of Kleinian groups
(Wrapping射影とクライン群の分解について)

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Contents

1. Introduction

A discrete subgroup of $PSL_2(\mathbb{C})$ is called a Kleinian group. $PSL_2(\mathbb{C})$ acts on the Riemann sphere $\hat{\mathbb{C}}$ as the conformal automorphisms. For a Kleinian group Γ, by considering how Γ acts on $\hat{\mathbb{C}}$, $\hat{\mathbb{C}}$ is divided into the limit set $\Lambda(\Gamma)$ and the domain of discontinuity $\Omega(\Gamma)$ (see Subsection 2.1).

$PSL_2(\mathbb{C})$ acts also on the hyperbolic 3-space $\mathbb{H}^3$ as the orientation-preserving isometries. Torsion-free Kleinian groups correspond to complete hyperbolic 3-manifolds; the quotient of $\mathbb{H}^3$ by a torsion-free Kleiniain group is a complete hyperbolic 3-manifold, and conversely a complete hyperbolic 3-manifold is given as the quotient of $\mathbb{H}^3$ by a torsion-free Kleianian group (see Subsection 2.3).

In this thesis we study Kleinian groups and hyperbolic 3-manifolds, in particular, what is related to the deformation theory of (representations into) Kleinian groups (see Subsection 2.8). Let M be a compact, orientable and hyperbolizable 3-manifold, that is, the interior $\operatorname{Int} M$ of M is homeomorphic to a complete hyperbolic 3-manifold. Let $AH(M)$ denote the space of conjugacy classes of discrete faithful representations from $\pi_1(M)$ into $PSL_2(\mathbb{C})$. We abbreviate $AH(S_{g,b} \times I)$ to $AH(S_{g,b})$, where $S_{g,b}$ is a compact orientable surface of genus g with b boundary components, and I is a closed interval in $\mathbb{R}$. The following two are known concerning the interior $\operatorname{Int} AH(M)$ of $AH(M)$ (see e.g. Chapter 7 of [18] and Section 4 of [42]):

- It was shown by works of Ahlfors, Bers ([7], [11]), Kra ([26]), Marden ([28]), Maskit ([30]), Sullivan ([44]), and Thurston that the interior $\operatorname{Int} AH(M)$ is the union of the quotients of Tiechmüller spaces (see Theorem 2.8.3).
- Ohshika ([39]), Namazi and Souto ([37]) showed that the density conjecture which asserts that the closure of $\operatorname{Int} AH(M)$ is the whole space $AH(M)$ holds (see also Theorem 2.8.4).

However, it is known that the topology of the whole deformation space is more complicated than that expected from the description of the interior $\operatorname{Int} AH(M)$ stated above and from the density conjecture. In fact, Anderson and Canary ([8]) constructed a compact, orientable and hyperbolizable 3-manifold M such that two components of $\operatorname{Int} AH(M)$ have the intersecting closures, called bumping (see Definition 2.8.5). Anderson, Canary and McCullough ([9]) gave a necessary and sufficient condition for two components of $\operatorname{Int} AH(M)$ to bump, in the case where all the boundary components of M are incompressible. Holt ([21]) refined their investigation to show that for a collection of components of $\operatorname{Int} AH(M)$ each two of which bump, there exists a point of $AH(M)$ at which those components all bump.

Let S be a closed orientable surface of genus greater than or equal to 2. By the description of $\operatorname{Int} AH(S)$ above, it has only one component. However, McMullen ([34]) observed that there exists a point at the boundary of $AH(S)$ any sufficiently small neighborhood of which has the disconnected intersection with $\operatorname{Int} AH(S)$, called self-bumping (see Definition 2.8.6). Bromberg and Holt ([14]) showed that for more general 3-manifold M, $AH(M)$ can self-bump. Ito ([22]) gave a complete description of the self-bumping points of $AH(S_{1,1})$.

In the results above [8], [9], [34], and [14], bumping and self-bumping phenomena come from the fact that the hyperbolic 3-manifold corresponding to a (self-)bumping point of $AH(M)$ has a natural projection to its associated geometric limit which wraps an annulus cusp neighborhood into a torus one.

We will express more specifically what the fact a natural projection wraps is. Let ρ be a point of $AH(M)$. We denote the image of a representation representing the conjugacy class ρ by Γ_ρ, and its quotient hyperbolic 3-manifold $\mathbb{H}^3/\Gamma_\rho$ by N_ρ. We consider the natural projection π_ρ from N_ρ to its associated geometric limit $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$. Here a geometric limit associated to ρ is a hyperbolic 3-manifold which is the geometric limit i.e. the Gromov-Hausdroff limit of a sequence of hyperbolic 3-manifolds corresponding to that of points in $AH(M)$ converging to ρ (see Subsection 2.9). We assume that N_ρ has an accidental parabolic curve γ i.e. an essential simple closed curve in an incompressible component B of the conformal boundary $\partial_c N_\rho$ of N_ρ which is homotopic in $N_\rho \cup \partial_c N_\rho$ to arbitrarily short curve (see Definition 2.7.1). We moreover assume that γ is homotoped into an annulus cusp neighborhood Q of N_ρ, that the image of Q by π_ρ is a torus cusp neighborhood $\hat{T}$ of $\hat{N}$, and that the Kleinian group $\hat{\Gamma}$ is the HNN-extension $\Gamma_0 *_\sigma$ of a subgroup Γ_0 of $\hat{\Gamma}$ extended by adding a parabolic element $\sigma \in \hat{\Gamma}$ which, together with $\rho(\gamma)$, generates the subgroup $\pi_1(\hat{T})$ of $\hat{\Gamma}$. For a non-negative integer n, we say that the projection π_ρ wraps n times if n is the smallest number such that the Kleinian group $\Gamma_0 *_{\sigma^{n+1}}$ contains Γ_ρ and the restriction of the natural projection $\bar{\pi} : N_\rho \to \mathbb{H}^3/\Gamma_0 *_{\sigma^{n+1}}$ to some compact core K for N_ρ is an embedding (see Subsection 2.10).

Meanwhile, Abikoff and Maskit ([1]) decomposed the Kleinian group Γ_ρ along the rank 1 parabolic subgroup J generated by $\rho(\gamma)$ (see Proposition 2.7.5):

$$\Gamma = \begin{cases} \Gamma_1 \underset{J}{*} \Gamma_2, & \text{if } \gamma \text{ is separating in } B, \\ \Gamma' *_\delta, & \text{if } \gamma \text{ is non-separating in } B, \end{cases}$$

where Γ_1, Γ_2 and Γ' are subgroups of Γ, and δ is an element of Γ.

In this thesis, for a point ρ of $AH(S)$ with an accidental parabolic curve γ, under some assumption of the annulus cusp neighborhood Q of N_ρ corresponding to γ, we characterize whether there exists the natural projection π_ρ from N_ρ to an associated geometric limit $\hat{N}$ with some condition of the image of Q by π_ρ which wraps n times with respect to γ. In fact, we will show the following theorem:

Theorem 3.1.1. *Suppose that $\rho \in AH(S)$ has an accidental parabolic curve γ with the corresponding annulus cusp neighborhood Q, and the cusp of Q abuts only geometrically finite ends. Let p be the fixed point of $\rho(\gamma)$, and $J = \langle \rho(\gamma) \rangle$ the rank 1 parabolic group generated by $\rho(\gamma)$. Then Γ_ρ is decomposed along J;*

$$\Gamma_\rho = \begin{cases} \Gamma_1 \underset{J}{*} \Gamma_2, & \text{if } \gamma \text{ is separating in } S, \\ \Gamma' *_\delta, & \text{if } \gamma \text{ is non-separating in } S, \end{cases}$$

where Γ_1, Γ_2 and Γ' are subgroups of Γ_ρ, and δ is an element of Γ_ρ. Moreover for a given nonnegative integer n, the following holds:

There exist an associated geometric limit $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ so that Q covers a torus cusp neighborhood $\hat{T}$ of $\hat{N}$, that the natural projection $\pi_\rho : N_\rho \to \hat{N}$ wraps n times with respect to γ, and that $\hat{N}$ has double trouble with respect to $\hat{T}$ if and only if the following holds.

- *In the case where γ is separating in S, there exist a parabolic element σ fixing the point p, (Γ_i, J)-invariant open disks C_i and D_i in $\hat{\mathbb{C}}$ for $i = 1, 2$, such that the following condition $\mathcal{DS}n$ holds:*

$$\mathcal{DS}n \begin{cases} \sigma^{n+1}(C_1) = \hat{\mathbb{C}} \setminus \overline{C_2}, \sigma^n(D_1) = \hat{\mathbb{C}} \setminus \overline{D_2}, \\ \Lambda(\Gamma_i) \textit{ separates } C_i \textit{ and } D_i, \textit{ and } \overline{C_i} \cap \Lambda(\Gamma_i) = \overline{D_i} \cap \Lambda(\Gamma_i) = p. \end{cases}$$

- *In the case where γ is non-separating in S, there exist a parabolic element σ fixing the point p, (Γ', J)-invariant open disks C and D and $(\Gamma', J^{\delta^{-1}})$-invariant open disks C' and D' in $\hat{\mathbb{C}}$ such that the following condition $\mathcal{DN}n$ holds:*

$$\mathcal{DN}n \begin{cases} \sigma^{n+1}\delta(C') = \hat{\mathbb{C}} \setminus \overline{C}, \sigma^n\delta(D') = \hat{\mathbb{C}} \setminus \overline{D}, \\ \Lambda(\Gamma') \textit{ separates } C \textit{ from } D, \textit{ and } C' \textit{ from } D', \\ \overline{C} \cap \Lambda(\Gamma') = \overline{D} \cap \Lambda(\Gamma') = p, \overline{C'} \cap \Lambda(\Gamma') = \overline{D'} \cap \Lambda(\Gamma') = \delta^{-1}(p), \textit{and} \\ (*) \textit{for any } f \in \Gamma', f(\overline{C}) \cap \overline{C'} = \emptyset, f(\overline{C}) \cap \overline{D'} = \emptyset, f(\overline{D}) \cap \overline{C'} = \emptyset, f(\overline{D}) \cap \overline{D'} = \emptyset. \end{cases}$$

Here the condition that C_i is (Γ_i, J)-invariant means that for any $f \in \Gamma_i$, $f(C_i) = C_i$ if $f \in \Gamma_i$, or $f(C_i) \cap C_i = \emptyset$ otherwise.

We note that for a point ρ of $AH(S)$ with an accidental parabolic curve γ, Evans and Holt ([20]) bounded the number n such that N_ρ has the natural projection which wraps n times with respect to γ from above, by using of hyperbolic length of curves in the conformal boundary of N_ρ which transverse to γ.

As a corollary of Theorem 3.1.1, we obtain the following:

Corollary 3.4.1. *There exists a continuous family $\rho^t \in AH(S)$, $t \geq 1$ such that ρ^t has a $\lfloor t \rfloor$-wrapping projection.*

Here $\lfloor t \rfloor$ is the largest integer less than or equal to t. In fact, we will construct such a family consisting of self-bump points of $AH(S)$ (see Subsection 3.4).

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2. Preliminaries

2.1. **Kleinian groups.** We review some definitions in Kleinian group theory. See e.g. [29] for reference.

The hyperbolic 3-space $\mathbb{H}^3$ is considered as the upper half-space

$$\{(x, y, t) \in \mathbb{R}^3 | t > 0\} \text{ equipped with the metric } ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}.$$

The Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is regarded as the boundary (at infinity) of $\mathbb{H}^3$.

Notation 2.1.1. *For a set A in $\mathbb{H}^3$, we denote its closure in $\overline{\mathbb{H}^3} := \mathbb{H}^3 \cup \hat{\mathbb{C}}$ by $\overline{A}^{\mathbb{H}^3}$, and the intersection $\overline{A}^{\mathbb{H}^3} \cap \hat{\mathbb{C}}$ by $\overline{\partial} A$.*

$PSL_2(\mathbb{C})$ acts on $\hat{\mathbb{C}}$ as the Möbius transformations. Under the above consideration, this action is naturally extended to the action on $\mathbb{H}^3$ by orientation-preserving isometries, and both the group of conformal automorphisms of $\hat{\mathbb{C}}$, denoted by $\mathrm{Aut}(\hat{\mathbb{C}})$, and the group of orientation-preserving isometries of $\mathbb{H}^3$, denoted by $\mathrm{Isom}^+ \mathbb{H}^3$, are identified with $PSL_2(\mathbb{C})$.

The hyperbolic plane $\mathbb{H}^2$ is considered as the plane in $\mathbb{H}^3$ orthogonal to $\mathbb{C}$ along $\mathbb{R}$. The subgroup $PSL_2(\mathbb{R})$ of $PSL_2(\mathbb{C})$ stabilizes this plane, and is identified the group of orientation-preserving isometries of $\mathbb{H}^2$, denoted by $\mathrm{Isom}^+ \mathbb{H}^2$. $\mathbb{H}^2$ is also regard as the upper-half plane $\{z \in \mathbb{C} | \operatorname{Im} z > 0\}$ in $\mathbb{C}$, where $\operatorname{Im} z$ is the imaginary part of z, and $PSL_2(\mathbb{R})$ is also identified with the group of conformal automorphisms of $\mathbb{H}^2$, denoted by $\mathrm{Aut}(\mathbb{H}^2)$.

Notation 2.1.2. *For an element $f \in PSL_2(\mathbb{C})$, we denoted the set of its fixed points in $\overline{\mathbb{H}^3}$ by $\mathrm{Fix}(f)$.*

An element $f \in PSL_2(\mathbb{C})$ is called either elliptic if $\emptyset \neq \mathrm{Fix}(f) \subset \mathbb{H}^3$, or parabolic if $\mathrm{Fix}(f)$ is a single point in $\hat{\mathbb{C}}$, or loxodromic if $\mathrm{Fix}(f)$ consists of two points in $\hat{\mathbb{C}}$. The following proposition is known (See e.g. Lemma 2.3.1 of [29]).

Proposition 2.1.3. *Let Γ be a subgroup of $PSL_2(\mathbb{C})$. If a loxodromic element $g \in \Gamma$ has exactly one fixed point in common with some element $h \in \Gamma$, then Γ is not discrete.*

Definition 2.1.4. *A Kleinian group is a discrete subgroup of $PSL_2(\mathbb{C})$, and a Fuchsian group is a discrete one of $PSL_2(\mathbb{R})$.*

Definition 2.1.5. *The limit set of a Kleinian group Γ, denoted by $\Lambda(\Gamma)$, is the set of accumulation points in $\overline{\mathbb{H}^3}$ of the Γ-orbit of a point $p \in \overline{\mathbb{H}^3}$. (This definition does not depend on the choice of the point p.) $\Lambda(\Gamma)$ is contained in $\hat{\mathbb{C}}$, and the domain of discontinuity of Γ, denoted by $\Omega(\Gamma)$, is the complement of $\Lambda(\Gamma)$ in $\hat{\mathbb{C}}$.*

$\Omega(\Gamma)$ is the largest open subset of $\hat{\mathbb{C}}$ upon which Γ acts properly discontinuously (see e.g. Proposition 2.10 of [32]).

Definition 2.1.6. *A Kleinian group Γ is said to be elementary if $\Lambda(\Gamma)$ consists of at most two points, and non-elementary otherwise.*

When Γ is a non-elementary Kleinian group, $\Lambda(\Gamma)$ is the closure of the set of loxodromic fixed points, and if Γ has parabolic elements, it is the closure of the set of parabolic fixed points as well.

Notation 2.1.7. *For a set B in $\overline{\mathbb{H}^3}$ and a Kleinian group Γ, we denote by $st_\Gamma(B)$ the subgroup $\{f \in \Gamma | f(B) = B\}$.*

For a component Ω of $\Omega(\Gamma)$, $st_\Gamma(\Omega)$ is called the component subgroup of Γ.

Definition 2.1.8. *For a Kleinian group Γ and its subgroup J, a set C in $\overline{\mathbb{H}^3}$ is said to be precisely invariant under J in Γ, or (Γ, J)-invariant, if $J = st_\Gamma(C)$ and $f(C) \cap C = \emptyset$ for any $f \in \Gamma \setminus J$.*

We adopt the notation "(Γ, J)-invariant" from [32] with a minor change.

Notation 2.1.9. *We denote that H is a subgroup of a group G as $H < G$, and for an element $g \in G$, we denote the conjugate group gHg^{-1} by H^g. We denote the subgroup generated by elements $g_1, \ldots, g_l$ of G by $\langle g_1, \ldots, g_l \rangle$.*

Let P be a maximal parabolic subgroup of a Kleinian group Γ. A parabolic subgroup is a subgroup consisting of parabolic elements and the identity. There exists an element $g \in G$ such that $P = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle^g$ or $\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \right\rangle^g$ for some $z \in \mathbb{C} \setminus \mathbb{R}$, where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denote the element of $PSL_2(\mathbb{C})$ represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$. For any $c > 0$, setting $H_c := \{(x, y, t) \in \mathbb{R}^3 | t \geq c\}$, P stabilizes $g(H_c)$, moreover if c is sufficiently large, $g(H_c)$ is (Γ, P)-invariant. Such $g(H_c)$ is called a horoball based at $g(\infty)$. We will always assume that parabolic subgroups are maximal.

Definition 2.1.10. *A Kleinian group Γ is said to be minimally parabolic if its parabolic subgroups are all rank 2.*

2.2. **Quasi-Fuchsian groups, generalized web groups, and degenerated groups.** We introduce several specific classes of finitely generated Kleinian groups. See e.g. [9] for reference.

Let Γ be a finitely generated Kleinian group. The Ahlfors finiteness theorem implies that for any component Ω of $\Omega(\Gamma)$, $\Lambda(st_{\Gamma(\Omega)}) = \partial\Omega$ (see Lemma 2 of [4]).

Definition 2.2.1. *A finitely generated Kleinian group Γ whose domain of discontinuity has exactly two components is called quasi-Fuchsian group if Γ fixes these components and is called an extended quasi-Fuchsian group otherwise.*

Maskit showed that a quasi-Fuchsian group Γ is quasi-conformally conjugate to a Fuchsian group whose limit set is $\mathbb{R} \cup \{\infty\}$, that is, there exists a quasi-conformal homeomorphism ϕ of $\hat{\mathbb{C}}$ such that $\phi \circ \Gamma \circ \phi^{-1} = \{\phi \circ f \circ \phi^{-1} | f \in \Gamma\}$ is a Fuchsian group (see Proposition 8.7.2 of [45] and also p. 8 of [17], and see e.g. [6] for the definition of quasi-conformal map). Since $\phi(\Lambda(\Gamma)) = \Lambda(\phi \circ \Gamma \circ \phi^{-1})$ is $\mathbb{R} \cup \{\infty\}$, the limit set $\Lambda(\Gamma)$ of a quasi-Fuchsian group is a simple closed curve. The component subgroup of a extended quasi-Fuchsian group Γ' stabilizes each component of $\Omega(\Gamma')$ and thus is a qusai-Fuchsian subgroup. The limit set $\Lambda(\Gamma')$ of a extended quasi-Fuchsian group Γ' is also a simple closed curve.

Definition 2.2.2. *A web group is a finite generated Kleinian group whose domain of discontinuity has at least three components and the component subgroup of each component is quasi-Fuchsian. Quasi-Fuchsian groups, extended quasi-Fuchsian groups, and web groups are called genaralized web groups.*

For a web group Γ, each component Ω of $\Omega(\Gamma)$ is simply-connected. Indeed, since $\partial\Omega = \Lambda(st_\Gamma(\Omega))$ and $st_\Gamma(\Omega)$ is quasi-Fuchsian by definition, $\partial\Omega$ is a simple closed curve, and thus Ω is simply-connected. Hence $\Lambda(\Gamma)$ is connected, and the limit set of any generalized web group is connected.

Definition 2.2.3. *A finitely generated non-elementary Kleinan group is degenerate if both its limit set and its domain of discontinuity are non-empty and simply-connected.*

2.3. (G, X)**-structures and holonomies.** We review (G, X)-structures and holonomies. See e.g. [17] for reference.

Let X be a real analytic manifold and G a Lie group acting on X analytically and faithfully (or effectively). Let M be a manifold, possibly with boundary, having the same dimension as X.

Definition 2.3.1. *A (G, X)-structure on M is a maximal atlas, i.e. a collection of charts $\{\phi_\lambda : U_\lambda \to X\}_{\lambda\in\Lambda}$ satisfying the following conditions*:

(1) *$\{U_\lambda\}_{\lambda\in\Lambda}$ is an open covering of M*;

(2) *Each ϕ_λ is a homeomorphism onto its image. Moreover whenever $U_\lambda \cap \partial M \neq \emptyset$, $\phi_\lambda(U_\lambda \cap \partial M)$ is locally flat in X*;

(3) *For each connected component U of $U_\lambda \cap U_\mu$, there exists an element $g \in G$, called a transition function, such that $\phi_\lambda|_U = g \circ \phi_\mu|_U$.*

Here a l-submanifold L of a topological n-manifold N with $l < n$ is said to be locally flat if for any $x \in L$, there exists a neighborhood U of x in N and a homeomorphism $\tau : U \to \mathbb{R}^n$ such that $\tau(U \cap L) \subset \mathbb{R}^d \subset \mathbb{R}^n$ (see e.g. [15]).

(G, X)-structures are also called geometric structures. A hyperbolic surface is a surface with a $(PSL_2(\mathbb{R}), \mathbb{H}^2)$-structure. A hyperbolic 3-manifold is a 3-manifold with a $(PSL_2(\mathbb{C}), \mathbb{H}^3)$-structure.

Let N be a connected manifold with a (G, X)-structure, and fix a point $x_0 \in N$. A holonomy representation $\rho : \pi_1(N, x_0) \to G$ is defined as follows. Let γ be a loop with the basepoint x_0, and cover it with a finite number of charts $\{U_i\}_{i=1}^k$ so that for each $i = 1, \dots, k-1$, $U_i \cap U_{i+1}$ is non-empty and connected. For each $i = 1, \dots, k-1$, let $g_i : U_i \cap U_{i+1} \to X$ be a transition function. The holonomy of γ is defined to be the composite $g_1 \circ \cdots \circ g_{k-1}$. This definition does not depend on the choices of loops in the homotopy class of γ keeping the end points fixed, or those of charts, and thus it defines $\rho([\gamma])$.

When N is a (geodesically) complete hyperbolic 3-manifold, the image $\rho(\pi_1(N))$ is a torsion-free Kleinian group. Conversely given a torsion-free Kleinian group Γ, $\mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold (see e.g. Theorem 1.18 of [32]).

2.4. **Klein-Maskit combination theorems.** We review some terminologies and the Klein-Maskit combination theorems, see e.g. [31] for more details.

Definition 2.4.1. *A fixed point of a rank 1 parabolic subgroup J of a Kleinian group Γ is said to be doubly cusped in Γ if there exist two disjoint open circular disks C_1 and C_2 in $\hat{\mathbb{C}}$ such that $C_1 \cup C_2$ is (Γ, J)-invariant. $C_1 \cup C_2$ is called a doubly cusped region.*

For a (Γ, J)-invariant simple closed curve c in $\hat{\mathbb{C}}$ and a (Γ, J)-invariant open disk C in $\mathbb{H}^3$ with $\overline{\partial} C = c$, we say that C is a spanning disk for c if every rank 1 parabolic fixed point in c of J has a doubly cusped region in $\Omega(\Gamma)$ such that the pair of geodesic half-spaces on this region does not intersect C.

Theorem 2.4.2 (Klein-Maskit combination theorem I)**.** *Let Γ_1 and Γ_2 be Kleinian groups and set J=$\Gamma_1 \cap \Gamma_2$. Let c be a simple closed curve in $\hat{\mathbb{C}}$ and D_1 and D_2 the two* *components of $\hat{\mathbb{C}} \setminus c$. Suppose that each D_i is (Γ_i, J)-invariant, for $i = 1, 2$. Then the group generated by Γ_1 and Γ_2 is a Kleinan group, which is the amalgamated free product $\Gamma_1 \underset{J}{*} \Gamma_2$. Under further assumptions, $\mathbb{H}^3/\Gamma_1 \underset{J}{*} \Gamma_2$ can be described as follows. Suppose that J is geometrically finite, and that for every rank 1 parabolic fixed point x in c of J, either $st_{\Gamma_1 \underset{J}{*} \Gamma_2}(x)$ has rank 2, or x is doubly cusped in $\Gamma_1 \underset{J}{*} \Gamma_2$. Then there exists a $(\Gamma_1 \underset{J}{*} \Gamma_2, J)$-invariant open disk D_C in $\mathbb{H}^3$ spanning c, such that D_C divides $\mathbb{H}^3$ into two closed sets H_1 and H_2, where H_i is (Γ_i, J)-invariant, for $i = 1, 2$. Then $\mathbb{H}^3/\Gamma_1 \underset{J}{*} \Gamma_2$ can be obtained from the disjoint union of $\mathbb{H}^3/\Gamma_1$ with the image of H_1/J deleted and $\mathbb{H}^3/\Gamma_2$ with the image of H_2/J deleted, by identifying along their common boundary D_C/J.*

Theorem 2.4.3 (Klein-Maskit combination theorem II)**.** *Let Γ' be a Kleinian group. For $i = 1, 2$, let c_i be a simple closed curve in $\hat{\mathbb{C}}$, and D_i an open disk with $\partial D_i = c_i$, and set $J_i = st_{\Gamma'}(D_i)$. Suppose that D_1 and D_2 are disjoint, that each D_i is $(\Gamma', J_i) - invariant$, and that there exists an element $\delta \in PSL_2(\mathbb{C})$ such that $\delta(D_1) \cap D_2 = \emptyset, \delta(c_1) = c_2$ and $J_2 = J_1^{\delta}$. Then the group generated by Γ' and δ is a Kleinian group, which is the HNN-extension $\Gamma' *_\delta$. Under further assumptions, $\mathbb{H}^3/\Gamma' *_\delta$ can be described as follows. Suppose that J_i is geometrically finite, and that for every rank 1 parabolic fixed point x_i in c_i of J_i, either $st_{\Gamma' *_\delta}(x_i)$ has rank 2, or x_i is doubly cusped in $\Gamma' *_\delta$, for each i. There exists a (Γ', J_i)-invariant open disk D_{C_i} in $\mathbb{H}^3$ spanning c_i, such that $\delta(D_{C_1}) = D_{C_2}$. Let H_i be the closed topological half-space cut out of $\mathbb{H}^3$ by D_{C_i}, with $\overline{\partial} H_i = \overline{D_i}$. Then $\mathbb{H}^3/\Gamma' *_\delta$ can be obtained from $\mathbb{H}^3/\Gamma'$, by deleting the images of H_1/J_1 and H_2/J_2 and by identifying two resulting boundaries D_{C_1}/J and D_{C_2}/J by δ.*

2.5. **Spaces of structures on surfaces.** We define several spaces consisting of structures on fixed surfaces. See e.g. [29] and [35] for references.

Let Σ be a connected oriented surface of genus g with p punctures, and the number $3g - 3 + p$ be denoted by $\xi(\Sigma)$. We assume that $\xi(\Sigma) \geq 0$. A simple closed curve on Σ is said to be essential if it is neither homotopically trivial nor homotopic into a puncture. A subsurface of Σ is said to be essential if its boundary consists of essential simple close curves.

The Teichmüller space $\mathcal{T}(\Sigma)$ of Σ is the space of marked conformal structures on Σ. More precisely, $\mathcal{T}(\Sigma)$ is defined as follows. Fix a conformal structure, i.e. a 1-dimensional complex structure, on Σ. Such a surface is called a Riemann surface.

Definition 2.5.1. *The Teichmüller space $\mathcal{T}(\Sigma)$ is the set of equivalence classes of pairs $[(R', h')]$, where R' is a Riemann surface and $h' : \Sigma \to R'$ is a quasi-conformal homeomorphism, and two pairs (R_1, h_1) and (R_2, h_2) are equivalent if there exists a conformal homeomorphism $j : R_1 \to R_2$ such that $j \circ h_1$ is homotopic to h_2.*

Alternatively $\mathcal{T}(\Sigma)$ is the space of marked complete hyperbolic structures.

Fix a complete hyperbolic structure on Σ.

Definition 2.5.2. *A geodesic lamination is a closed subset on Σ which is a disjoint union of simple geodesics, called leaves of the lamination.*

Let $\mathcal{GL}(\Sigma)$ denote the space of geodesic laminations on Σ with the Hausdroff topology.

Definition 2.5.3. *A transverse measure on a geodesic lamination is a Borel measure on arcs transverse to the leaves which is invariant under homotopy of arcs preserving their topological positions with respect to the leaves. A geodesic lamination with a transverse measure is called a measured lamination.*

Let $\mathcal{ML}(\Sigma)$ denote the space of measured laminations with the weak-* topology on measures, and $\mathcal{UML}(\Sigma)$ the quotient space of $\mathcal{ML}(\Sigma)$ obtained by forgetting the measures.

Definition 2.5.4. *A measured lamination is said to be filling if it intersects transversely all the measured laminations except itself.*

Let $\mathcal{EL}(\Sigma)$ denote the image in $\mathcal{UML}(\Sigma)$ of the filling laminations in $\mathcal{ML}(\Sigma)$. Elements in $\mathcal{EL}(\Sigma)$ are called ending laminations.

2.6. **Structures of hyperbolic 3-manifolds.** We review some definitions in hyperbolic 3-manifold theory. See e.g. [29], [35] and [40] for references.

Let $N = \mathbb{H}^3/\Gamma$ be a (complete) hyperbolic 3-manifold.

Definition 2.6.1. *The convex hull of a closed set X in $\hat{\mathbb{C}}$ is the minimal convex subset of $\mathbb{H}^3$ which contains the union of geodesics connecting two points in X. The convex core of N, denoted by $C(N)$, is the quotient of the convex hull of $\Lambda(\Gamma)$ by Γ.*

Definition 2.6.2. *Both N and Γ are said to be geometrically finite if for any $\delta > 0$, the closed δ-neighborhood of $C(N)$ has finite volume.*

Definition 2.6.3. *$\Omega(\Gamma)/\Gamma$ is called the conformal boundary of N and denoted by $\partial_c N$.*

If $\pi_1(N) \cong \Gamma$ is finitely generated, then Ahlfors' finiteness theorem(see e.g. [3]) asserts that $\partial_c N$ is a finite union of Riemann surfaces of finite type.

Definition 2.6.4. *For a real number $\epsilon > 0$, the ϵ-thin part of N, denoted by N_ϵ, is the subset of N consisting of those points where the injectivity radius is less than or equal to ϵ.*

Recall that the injectivity radius of N at a point x is the real number

$$\inf \left\{ \frac{1}{2} d_{\mathbb{H}^3}(\tilde{x}, f(\tilde{x})) \middle| f \in \Gamma \setminus \{\mathrm{id}\} \right\},$$

where $\tilde{x}$ is a lift of x in $\mathbb{H}^3$.

The Margulis lemma says that there exists the universal constant ϵ_M, called the Margulis constant, such that if $\epsilon \leq \epsilon_M$, then each component of N_ϵ is either a regular neighborhood of a short geodesic, called a Margulis tube, or a quotient of a horoball by the rank 1 or 2 parabolic subgroup of Γ stabilizing it, called an annulus cusp neighborhood or a torus cusp neighborhood, respectively. In this thesis, we call the end of a cusp neighborhood a cusp. The complementary part of N of disjoint union of cusp neighborhoods of all cusps, denoted by N_0, is called the non-cuspidal part of N.

Definition 2.6.5. *An end of a manifold is the projective limit of a descending sequence of components of the complement of an ascending exhausting sequence of compact subsets.*

We consider that ends lie in the manifold.

Definition 2.6.6. *An end of N_0 is called a relative end of N.*

Definition 2.6.7. *A relative end e of N is said to be topologically tame if there exists a properly embedded compact subsurface F in N_0 such that F cut a submanifold U out of N_0, which contains e and is homeomorphic to $F \times (0, \infty)$.*

In this thesis, for the end e we call the component U of $N_0 \setminus F$ its neighborhood, and the subsurface F its face. A topologically tame end is said to be incompressible if its face is incompressible. An end which is not topologically tame is called a wild end.

Definition 2.6.8. *A topologically tame end of N is said to be geometrically finite if it has a neighborhood which does not intersect the convex core $C(N)$.*

The geometrically finite ends bijectively correspond to the components of $\partial_c N$ (see e.g. Lemma 4.6 of [38]).

Notation 2.6.9. *We denote the interior of a topological space X by* $\mathrm{Int}\, X$.

Definition 2.6.10. *An incompressible topologically tame end is said to be simply-degenerate if there exists a sequence of essential simple closed curves on* $\mathrm{Int}\, F$ *the sequence of whose geodesic representatives in $U \subset N$ tends to the end, where U is a neighborhood for the end and F is its face.*

Thurston showed that the above sequence of curves converges to an ending lamination in $\mathcal{UML}(\mathrm{Int}\, F)$ (see Chapter 8.10 of [45]). Bonahon showed that any incompressible tame end is either geometrically finite end, or simply-degenerate one (see Théorème 1.4 and Section 6 of [12]). The end invariant of N is defined to be the set consisting of a conformal structures for each geometrically finite end, and an ending lamination for each simply-degenerate end.

We consider the case where $\pi_1(N) \cong \Gamma$ is finitely generated.

Definition 2.6.11. *A compact core for an n-manifold is a compact n-submanifold whose inclusion is a homotopy equivalence. A 3-manifold pair is a pair of 3-manifold with boundary and a compact (possibly disconnected) subsurface of its boundary. A relative compact core of a 3-manifold pair is a 3-manifold pair each whose component is a compact core for the corresponding component.*

Recall that N_0 is the non-cuspidal part of N and let C be the boundary of the union of cusp neighborhoods in N. By the works of Scott ([43]), McCullough ([33]), and Kulkarni and Shalen([27]), the 3-manifold pair (N_0, C) has a relative compact core (M_0, P_0), and by the works of Agol ([2]), and Calegari and Gabai (see Theorem 0.4 of [16]), each relative end is tame, and has one of the component of $N_0 \setminus M_0$ as its neighborhood. We say that an annulus cusp abuts an end if the corresponding neighborhood does. Moreover, Brock, Canary and Minsky ([13]) gave the ending lamination theorem for incompressible ends, which asserts that the hyperbolic 3-manifold $N = \mathbb{H}^3/\Gamma$ whose relative ends are all incompressible, is determined up to isometry, by its homeomorphism type and its end invariant.

2.7. **Accidental parabolic curves and elements.** We review accidental parabolic curves, and accidental parabolic elements. See e.g. [1] and [32] for reference.

Let Γ be a torsion-free Kleinian group, and N the quotient hyperbolic 3-manifold of Γ with its holonomy representation $\rho : \pi_1(N) \to \Gamma(< PSL_2(\mathbb{C}))$.

Definition 2.7.1. *An accidental parabolic curve γ for N is an essential simple closed curve on a incompressible component of $\partial_c N$ which can be homotoped in $N \cup \partial_c N$ to have arbitrarily short length.*

An accidental parabolic curve for N is interpreted as an accidental parabolic element of Γ defined as follows.

Definition 2.7.2. *A parabolic element $f \in \Gamma$ is said to be accidental if there exist a simply-connected component Ω of $\Omega(\Gamma)$ and a conformal homeomorphism $\omega : \Omega \to \mathbb{H}^2$ such that $f \in st_\Gamma(\Omega)$ and $\omega \circ f \circ \omega^{-1}$ is a loxodromic element of $PSL_2(\mathbb{R})$.*

The preimage in Ω of the axis in $\mathbb{H}^2$ for $\omega \circ f \circ \omega^{-1}$ via ω is called the axis for f.

Proposition 2.7.3. *For any accidental parabolic curve γ for N, $\rho(\gamma)$ is an accidental parabolic element of Γ. Conversely, for any accidental parabolic element $f \in \Gamma$ which is primitive, i.e. not a non-trivial power, the axis for f projects to an accidental parabolic curve for N.*

We check this proposition. Let $\pi : \mathbb{H}^3 \cup \Omega(\Gamma) \to N \cup \partial_c N$ be the projection. We note that the following holds (see e.g. Proposition 2.35 of [32]):

Proposition 2.7.4. *Let Σ be a connected surface in $N \cup \partial_c N$, and Δ a component of $\pi^{-1}(\Sigma)$. Σ is incompressible in $N \cup \partial_c N$ if and only if Δ is simply-connected.*

Proof of Proposition 2.7.3. Let γ be an accidental parabolic curve for N. By definition, $\rho(\gamma)$ is parabolic. Let B be the component of $\partial_c N$ in which γ lies, and Ω the component of $\Omega(\Gamma)$ which covers B. Since $\gamma \subset B \subset \partial_c N$, $\rho(\gamma) \in st_\Gamma(\Omega)$. Since B is incompressible, by Proposition 2.7.4, Ω is simply-connected, and by the Riemann mapping theorem (see e.g. [5]), there exists a conformal homeomorphism $\omega : \Omega \to \mathbb{H}^2$. Since Γ acts on Ω conformally, $\omega \circ \rho(\gamma) \circ \omega^{-1} \in PSL_2(\mathbb{R})$. Since $\rho(\gamma)$ is a translation along a lift of γ, $\omega \circ \rho(\gamma) \circ \omega^{-1}$ is a translation along a lift of γ via $\mathbb{H}^2 \overset{\omega^{-1}}{\to} \Omega \overset{\pi|_\Omega}{\to} \Omega / st_\Gamma(\Omega)$. Since γ is essential, $\omega \circ \rho(\gamma) \circ \omega^{-1}$ is loxodromic. Thus $\rho(\gamma)$ is an accidental parabolic element of Γ. Conversely, let f be an accidental parabolic element which is primitive in Γ, Ω and ω, a component and a conformal homeomorphism as in Definition 2.7.2. The axis for $\omega \circ f \circ \omega^{-1}$ in $\mathbb{H}^2$ projects to a homotopically non-trivial curve γ in $\Omega / st_\Gamma(\Omega) \subset \partial_c N$. Since Ω is simply-connected, by Proposition 2.7.4, $\Omega / st_\Gamma(\Omega)$ is incompressible. Since f is loxodromic, γ is not homotopic into a puncture. Since γ also represents the parabolic element f, it can be homotoped to have arbitrarily short length in $N \cup \partial_c N$. Since f is primitive in Γ, γ is simple and thus is an accidental parabolic curve. Hence Proposition 2.7.3 holds. ☐

Let γ be an accidental parabolic curve on a component B of $\partial_c N$, p the parabolic fixed point of $\rho(\gamma)$, and $\tilde{\gamma}$ the axis for $\rho(\gamma)$ in $\Omega(\Gamma)$ starting and ending at p.

Proposition 2.7.5. Γ *is decomposed along the parabolic subgroup $J := \langle \rho(\gamma) \rangle$;*

$$\Gamma = \begin{cases} \Gamma_1 \underset{J}{*} \Gamma_2, & \textit{if } \gamma \textit{ is separating in } B, \\ \Gamma' *_\delta, & \textit{if } \gamma \textit{ is non-separating in } B, \end{cases}$$

where Γ_1*,* Γ_2 *and* Γ' *are subgroups of* Γ*, and* δ *is an element of* Γ*.*

Proof of Proposition 2.7.5. This proposition follows from Lemmas 2 and 3 of [1] as follows. Since γ is simple, $\tilde{\gamma} \cup \{p\}$ is (Γ, J)-invariant. The Γ-translates of $\tilde{\gamma} \cup \{p\}$ divide up $\hat{\mathbb{C}}$ into infinitely many connected components. Let T_1 and T_2 be the connected components whose boundaries contain $\tilde{\gamma} \cup \{p\}$. Set $\Gamma_i := st_\Gamma(T_i)$, for $i = 1, 2$.

First, we suppose that γ is separating in B. We note that T_1 and T_2 are not Γ-equivalent, that is, there exists no element $f \in \Gamma$ such that $f(T_2) = T_1$. Let D_i be the open disk bounded by $\tilde{\gamma} \cup \{p\}$ where $D_i \cap T_i = \emptyset$. In the proof of Lemma 2 of [1], Abikoff and Maskit asserted that D_i is (Γ_i, J)-invariant, and that Γ is generated by Γ_1 and Γ_2. Thus the Klein-Maskit combination theorem I implies that $\Gamma = \Gamma_1 \underset{J}{*} \Gamma_2$.

Next, we suppose that γ is non-separating in B. We note that T_1 and T_2 are Γ-equivalent, that is, there exists an element $f \in \Gamma$ such that $f(T_2) = T_1$. In the proof of Lemma 3 of [1], Abikoff and Maskit asserted that there exist an open disk D_1 bounded by $\tilde{\gamma} \cup \{p\}$ which is (Γ_1, J)-invariant, and an open disk D_2 bounded by $f(\tilde{\gamma} \cup \{p\})$ which is (Γ_1, J^f)-invariant, such that for any $h \in \Gamma_1$, $h(D_1) \cap D_2 = \emptyset$, and that Γ is generated by Γ_1 and f. Set $\Gamma' := \Gamma_1$, $\delta := f$, and $T' := T_1$. By the Klein-Maskit combination theorem II, we have $\Gamma = \Gamma' *_\delta$. □

We have two remarks for Proposition 2.7.5.

Remark 2.7.6. *Abikoff and Maskit also showed that when* Γ *is finitely generated, the decomposition process described in Proposition 2.7.5 must end after a finitely many number of steps (see Section 6 of* [1]*), and the resulted groups are generalized web groups or degenerate groups without accidental parabolic elements (see Theorem 1 of* [1]*).*

Remark 2.7.7. *By definition, there exists a cusp neighborhood* Q *of* N *which* γ *is homotoped into. Moreover by the Cylinder theorem (see e.g. Section 3.7 of* [29]*), there exists a once-punctured disk* A *in* N *whose boundary (at infinity) is* γ *and whose puncture lies on the cusp of* Q*, which decomposes* N*. If* Q *is an annulus cusp neighborhood whose cusp abuts only geometrically finite ends, then p is doubly cusped in* Γ*, and by the Klein-Maskit combination theorems,* N *is decompsed along the once-punctured disk* A *into parts of* $\mathbb{H}^3/\Gamma_1$ *and* $\mathbb{H}^3/\Gamma_2$ *in the case where* A *is separating in* N*, or into a part of* $\mathbb{H}^3/\Gamma'$ *in the case where* A *is non-separating in* N*. Similarly if* Q *is a torus cusp neighborhood, then* $st_\Gamma(p)$ *has rank 2, and by the Klein-Maskit combination theorem II,* N *is decompsed along* A *into a part of* $\mathbb{H}^3/\Gamma'$*.*

For a torus cusp of N, we define the following.

Definition 2.7.8. *When* N *has a torus cusp neighborhood* T*,* N *is said to have double trouble with respect to* T *if there exist essential simple closed curves in* ∂T *and in two distinct incompressible components of* $\partial_c N$*, respectively, which are mutually homotopic in* $N \cup \partial_c N$*.*

We note that the above curves on $\partial_c N$ are accidental parabolic curves.

2.8. **Deformation spaces.** We review deformation theory of Kleinian groups. See e.g. [18] for more details.

Let M be a compact orientable hyperbolizable 3-manifold. Here "hyperbolizable" means that $\operatorname{Int} M$ admits a complete hyperbolic structure. Let $AH(M)$ denote the set of conjugacy classes of discrete faithful representations of $\pi_1(M)$ into $PSL_2(\mathbb{C})$.

We will define the topology of $AH(M)$. Let $\operatorname{Hom}_T(\pi_1(M), PSL_2(\mathbb{C}))$ denote the set of representations $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ with the property that $\rho(\gamma)$ is parabolic if γ is a non-trivial element of a rank 2 abelian subgroup of $\pi_1(M)$. Fix a generating set $\{\gamma_1, \ldots, \gamma_l\}$ of $\pi_1(M)$, and identify $\operatorname{Hom}_T(\pi_1(M), PSL_2(\mathbb{C}))$ with the affine algebraic variety, denoted by $R_T(M)$ in $PSL_2(\mathbb{C})^l$ via the map $\rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_l))$. We note that another generating set induces a canonically isomorphic affine algebraic variety. The quotient of $\operatorname{Hom}_T(\pi_1(M), PSL_2(\mathbb{C}))$ by conjugation is called the $(PSL_2(\mathbb{C}))$-character variety of $\pi_1(M)$ and denoted by $X_T(M)$. More precisely $X_T(M)$ is the Mumford quotient $\operatorname{Hom}_T(\pi_1(M), PSL_2(\mathbb{C}))//PSL_2(\mathbb{C})$ (see e.g. [24] for precise definition). Culler and Shalen showed that $X_T(M)$ is a closed algebraic set (see Corollary 1.4.5 [19]). $AH(M)$ is topologized as the subspace of $X_T(M)$. This topology is called the algebraic topology. The Mumford quotient $\operatorname{Hom}_T(\pi_1(M), PSL_2(\mathbb{C}))//PSL_2(\mathbb{C})$ is not isomorphic to the usual quotient $\operatorname{Hom}_T(\pi_1(M), PSL_2(\mathbb{C}))/PSL_2(\mathbb{C})$ by conjugation, however the Mumford and usual quotients of the subset of $\operatorname{Hom}_T(\pi_1(M), PSL_2(\mathbb{C}))$ consisting of representations whose images are not contained in parabolic groups are isomorphic (see [24]). Form now on, we suppose that $\pi_1(M)$ is non-abelian. It follows from the above that a sequence ρ_m converges to ρ in $AH(M)$ if and only if there exist representatives ξ_m and ξ of ρ_m and ρ, respectively, such that for any $\gamma \in \pi_1(M)$, $\xi_m(\gamma)$ converges to $\xi(\gamma)$ in $PSL_2(\mathbb{C})$. Convergences in $AH(M)$ are called algebraic convergences. By a work of Jørgensen, $AH(M)$ is closed in $X_T(M)$ (see Theorem 1 of [23]).

Notation 2.8.1. *From now on, given $\rho \in AH(M)$, we denote its representative also by ρ.*

The interior of $AH(M)$ can be characterized as follows. Let $MP(M)$ denote the subspace of $AH(M)$ consisting of those representations with geometrically finite and minimally parabolic images. It follows from Marden's Isomorphism theorem (Theorem 8.1 of [28]) that $MP(M)$ is open in $AH(M)$ as a subset in $X_T(M)$. Together with a work of Sullivan ([44]), this implies that $MP(M)$ is the interior of $AH(M)$. Furthermore the topology of $\operatorname{Int} AH(M)$ has been well understood.

Definition 2.8.2. *An element $\rho' \in AH(M)$ is said to be quasi-conformally conjugate to an element $\rho \in AH(M)$ if there exists a quasi-conformal homeomorphism $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that for any $\gamma \in \pi_1(M)$, $\rho'(\gamma) = \phi \circ \rho(\gamma) \circ \phi^{-1}$.*

For an element $\rho \in AH(M)$, let $QC(\rho)$ denote the subspace of $AH(M)$ consisting of those representations which are quasi-conformally conjugate to ρ. Marden observed that for any $\rho \in MP(M)$, $QC(\rho) \subset MP(M)$ (see Proposition 9.1 of [28]).

Let $\mathcal{A}(M)$ denote the set of equivalence classes of pairs $[(M', h')]$, where M' is a compact, oriented hyperbolizable 3-manifold and $h' : M \to M'$ is a homotopy equivalence, and two pairs (M_1, h_1) and (M_2, h_2) are equivalent if there exists an orientation-preserving homeomorphism $j : M_1 \to M_2$ such that $j \circ h_1$ is homotopic to h_2. For an element $[(M', h')] \in \mathcal{A}(M)$, let $Mod_0(M')$ denote the group of isotopy classes of orientation-preserving homeomorphisms of M' which are homotopic to the identity, and $\partial_{NT} M'$ the non-toroidal boundary components of M'.

The map $\Phi : AH(M) \to \mathcal{A}(M)$ is defined by $\Phi(\rho) := [(M_\rho, h_\rho)]$, where for each $\rho \in AH(M)$, M_ρ is a compact core for $\mathbb{H}^3/\rho(\pi_1(M))$, and $h_\rho : M \to M_\rho$ is a homotopy equivalence such that the conjugacy class of the induced homeomorphism $h_{\rho *} : \pi_1(M) \to \pi_1(M_\rho)$ is ρ. Such a homotopy equivalence is called a marking of ρ.

By works of Ahlfors, Bers ([7], [11]), Kra ([26]), and Maskit ([30]), for any $\rho \in MP(M)$ with $\Phi(\rho) = [(M', h')] \in \mathcal{A}(M)$, $QC(\rho) \cong \mathcal{T}(\partial_{NT} M')/Mod_0(M')$ holds.

Marden's Isomorphism theorem (Theorem 8.1 of [28]) implies that for any $\rho, \rho' \in MP(M)$, ρ' is quasi-conformally conjugate to ρ if and only if $\Phi(\rho') = \Phi(\rho)$. Thurston's geometrization theorem (see Morgan [36], or Otal [41]) implies that the restriction $\Phi|_{MP(M)}$ is surjective. Marden's stability theorem (Proposition 9.1 of [28]) implies that Φ is continuous i.e. locally constant. In conclusion, we have got,

Theorem 2.8.3.

$$\mathrm{Int}\, AH(M) = MP(M) = \bigcup_{\rho \in MP(M)} QC(\rho) \cong \bigsqcup_{[(M', h')] \in \mathcal{A}(M)} \mathcal{T}(\partial_{NT} M')/Mod_0(M').$$

In this thesis, we will consider the case where $M = S \times I$. Here S is a closed surface of genus $g \geq 2$ and I is a closed interval. We abbreviate $AH(S \times I)$ and $MP(S \times I)$ to $AH(S)$ and $MP(S)$, respectively. Let $QF(S)$ denote the subset of $AH(S)$ consisting of those points whose images are quasi-Fuchsian groups. We recall that any quasi-Fuchsian group is quasi-conformally conjugate to a Fuchsian group (see Subsection 2.2). Thus $QF(S)$ is contained in $MP(S)$. Since any two Fuchsian representations of $AH(S)$ are quasi-conformally conjugate, $MP(S) = QF(S)$ and $\Phi(MP(S))$ is a single point. Then, the results above are stated as

$$\mathrm{Int}\, AH(S) = MP(S) = QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(S)$$

.

We go back to a general hyperbolizable compact 3-manifold M. Ohshika (Theorem 1.1 of [39]), Namazi and Souto (Theorem 1.1 of [37]) showed that

Theorem 2.8.4 (Density theorem)**.**

$$AH(M) = \overline{\mathrm{Int}\, AH(M)}$$

.

However the topology of $AH(M)$ is still mysterious. Anderson and Canary discovered the first example of the bumping phenomena (see Theorem 3.1 of [8]).

Definition 2.8.5. *Two distinct components W_1 and W_2 of $\mathrm{Int}\, AH(M)$ are said to bump at a point ρ in the boundary $\partial AH(M)$ if $\rho \in \overline{W_1} \cap \overline{W_2}$, where $\overline{W_i}$ is the closure of W_i in $X_T(M)$ for $i = 1, 2$.*

Anderson, Canary and McCullough characterized exactly when two components of $AH(M)$ can bump in the case where the boundary components of M are all incompressible (Corollary 1 of [9]).

Furthermore McMullen first demonstrated that $AH(S)$ can self-bump (see Theorem A.1 of [34]).

Definition 2.8.6. *A component W of the interior of $AH(M)$ are said to self-bump at a point ρ in the boundary $\partial AH(M)$ if for any sufficiently small neighborhood V of ρ in $AH(M)$, $V \cap W$ is disconnected.*

Bromberg and Holt generalized this result by using different technique (see Theorem 4.5 of [14]). These bumponomics phenomena are related to non-trivial wrapping projections which will be stated in Subsection 2.10. In fact, the existence of bumpings in [8] and [9], and self-bumpings in [34] and [14] was shown under existence of non-trivial wrapping projections.

2.9. **Geometric limits.** We review geometric limits (see e.g. [10]), and some terminologies and a theorem given by Ohshika and Soma in [40].

Definition 2.9.1. *A sequence of hyperbolic 3-manifolds with basepoints (N_m, x_m) is said to converge geometrically to a hyperbolic manifold with a basepoint $(\hat{N}, \hat{x})$ if there exist sequences of real numbers $\{R_m > 0\}$ and $\{K_m \geq 1\}$ with $R_m \to \infty$ and $K_m \to 1$, and a sequence of biLipschitz diffeomorphisms into images $\{f_m : B(\hat{x}, R_m) \to N_m\}$ such that for each m, f_m is K_m-biLipschitz and maps $\hat{x}$ to x_m, where $B(\hat{x}, R_m)$ is the open ball of radius R_m centered at $\hat{x}$ in $\hat{N}$.*

We often omit basepoints of geometrically convergent sequences.

Definition 2.9.2. *A sequence of Kleinian groups $\{\Gamma_m\}$ is said to converge to a Kleinian group $\hat{\Gamma}$ in the Chabauty topology if the following two conditions hold*:

(1) *each $\hat{f} \in \hat{\Gamma}$ is the limit of some sequence $\{f_m \in \Gamma_m\}$ in $PSL_2(\mathbb{C})$;*

(2) *for any sequence $\{f_m \in \Gamma_m\}$, whenever its subseuence $\{f_{m(l)}\}$ converges to an element $\hat{f}$ in $PSL_2(\mathbb{C})$, $\hat{f}$ lies in $\hat{\Gamma}$.*

It is known that a sequence of Kleinian groups $\{\Gamma_m\}$ converges to a Kleinian group $\hat{\Gamma}$ in the Chabauty topology if and only if the quotient manifolds $\mathbb{H}^3/\Gamma_m$ converges geometrically to $\mathbb{H}^3/\hat{\Gamma}$.

Jørgensen and Marden showed that every sequence $\{\rho_m\}$ converging to ρ in $AH(M)$ has a subsequence, also denoted by $\{\rho_m\}$ such that $\rho_m(\pi_1(M))$ converges to a Kleinian group $\hat{\Gamma}$ in the Chabauty topology (see Propositions 3.8 and 3.10 of [25]). Since $\rho(\gamma) = \lim_{m\to\infty} \rho_m(\gamma)$ for any $\gamma \in \pi_1(M)$, by the condition (2) of the definition above, $\rho(\pi_1(M))$ is a subgroup of $\hat{\Gamma}$, and thus there exists a locally isometric covering $\mathbb{H}^3/\rho(\pi_1(M)) \to \mathbb{H}^3/\hat{\Gamma}$, called the natural projection. Both $\mathbb{H}^3/\hat{\Gamma}$ and $\hat{\Gamma}$ are called associated geometric limits of ρ.

We will consider geometric limits of Kleinian groups isomorphic to $\pi_1(S)$, where S is a closed surface of genus g with p punctures such that $\xi(S) \geq 0$. Ohshika and Soma ([40]) introduced biLipschitz models for such geometric limits, called labelled brick manifolds, and classified those geometric limits. Following [40], we will state the definition of labelled brick manifold and their theorem. For a compact essential subsurface F of S with $\xi(S) \geq 0$, and an interval J, which is either $[0,1]$, $[0,1)$, or $(0,1]$, a 3-manifold homeomorphic to $F \times J$ is called a brick. For a brick $B = F \times J$, $F \times \{0\}$ and $F \times \{1\}$ are denoted by $\partial_- B$ and $\partial_+ B$, and called the lower front and the upper front, respectively, and for each $t \in J$ and $x \in F$, $F \times \{x\}$ and $\{t\} \times J$ are called horizontal and vertical leaves, respectively. When a brick is homeomorphic to either $F \times [0,1)$ or $F \times (0,1]$, it is called a half-open brick and a front which is not contained in it is called the ideal front.

Definition 2.9.3. *A finite brick complex is a finitely many bricks $\mathcal{K} = \{B_1, \ldots, B_l\}$ realized as subsets of a 3-manifold with pairwise disjoint interiors satisfying the following two conditions*:

(1) $\displaystyle\bigcup_{i=1}^{l} B_i$ *is connected*;

(2) *for any two bricks B_i and B_j in $\mathcal{K}$ with $F_{ij} := B_i \cap B_j \neq \emptyset$, there exists a leaf-preserving embedding $\eta : B_i \cup B_j \to S \times [-1,1]$ with $\eta(B_i) \subset S \times [-1,0]$, $\eta(B_j) \subset S \times [0,1]$ such that $\eta(F_{ij})$ is an essential subsurface of $S \times \{0\}$.*

The finite union $\bigcup_{B \in \mathcal{K}} B$ is called a finite brick manifold with brick decomposition $\mathcal{K}$.

Definition 2.9.4. *For an ascending sequence $\{\mathcal{K}_m\}_{m=1}^{\infty}$ of finite brick complexes, the union $\mathcal{K} := \displaystyle\bigcup_{m=1}^{\infty} \mathcal{K}_m$ is called a brick complex, and the union $\displaystyle\bigcup_{B \in \mathcal{K}} B$ a brick manifold with brick decomposition $\mathcal{K}$.*

Definition 2.9.5. *A labelled brick manifold is a brick manifold labelled as follows. Categorize its half-open bricks into geometrically finite bricks and simply-degenerate bricks. Then for each geometrically finite brick $B = F \times J$, attach a point in $\mathcal{T}(\mathrm{Int}\, F)$ to its ideal front, and for each simply-degenerate brick $B = F \times J$, a point in $\mathcal{UML}(\mathrm{Int}\, F)$ to the end corresponding to its ideal front.*

In this thesis, we denote the union of all the (marked) conformal structures attached to geometrically finite bricks by $\partial_c M$. We note that this symbol is denoted by $\partial_\infty M$ in [40].

Ohshika and Soma showed the following theorem (see Theorem C of [40]).

Theorem 2.9.6. *Suppose that M is a labelled brick manifold satisfying the following conditions $(i) - (iv)$ and (EL) :*

(i) *each component of ∂M is either a torus or an open annulus*;

(ii) *there exists no proper embedded incompressible annulus in M whose boundary components lie on distinct boundary component of M*;

(iii) *if there exists an embedded, incompressible half-open annulus $S^1 \times [0,\infty)$ in M such that $S^1 \times \{t\}$ tends to a wild end e of M as $t \to \infty$, then its core curve is homotopic into an open annulus component of ∂M tending to e*;

(iv) *the manifold M is (not necessary properly) embedded in $S \times (0,1)$ in such a way that each brick has a form $F \times J$ with an interval J and an essential* *subsurface F of S with respect to the product structure of $S \times (0,1)$ and the ends of geometrically finite bricks lie in $S \times \{0,1\}$;*

(EL) *the ending laminations of two simply-degenerate ends of M are not homotopic to each other in M.*

Then M has a block decomposition, and if we put on M the model metric associated with the decomposition, then there exists a non-elementary geometric limit G of quasi-Fuchsian groups isomorphic to $\pi_1(S)$ such that $N_G = \mathbb{H}^3/G$ admits a K-biLipschitz homeomorphism $f : M \to (N_G)_0$ which can be extended to continuously to a conformal map $\partial_c M \to \partial_c N_G$ between the boundaries at infinity for a constant $K \geq 1$ depending only on $\chi(S)$.

In [40], a block decomposition is a brick decomposition constructed from that of M by removing repeatedly some union of solid tori specified by the information of the end invariants of M, so that each brick is homeomorphic to either $\Sigma_{0,3} \times J$, $\Sigma_{1,1} \times J$, or $\Sigma_{0,4} \times J$, where $\Sigma_{0,3}$ is a thrice-holed sphere, $\Sigma_{1,1}$ is a once-holed torus, $\Sigma_{0,4}$ is a 4-holed sphere, and J is an interval, and the model metric on its brick manifold is the piecewise Riemannian metric with a preassigned metric on each brick.

Remark 2.9.7. *We note that in the proof of Theorem C of* [40], *each torus cusp of N_G is* (*hyperbolic*) *Dehn filled, that is, attached by a solid torus with some specific gluing, in each hyperbolic 3-manifold of the converging sequence to N_G.*

2.10. **Wrapping projections.** We explain wrapping projections. See [20].

Let M be a compact orientable hyperbolizable 3-manifold at least one of whose boundary components is incompressible. Let ρ be an element of $AH(M)$ and set $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$. Recall that ρ denotes also its representative. Let $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ be an associated geometric limit of ρ with the natural projection $\pi_\rho : N_\rho \to \hat{N}$.

We assume the following:

(i) ρ has an accidental parabolic curve γ;
(ii) the cusp neighborhood Q corresponding to γ is an annulus one in N_ρ which covers a torus one $\hat{T}$ in $\hat{N}$ via π_ρ; and
(iii) there exist a parabolic element $\sigma \in \hat{\Gamma}$ and a subgroup Γ_0 of $\hat{\Gamma}$ such that σ and $\rho(\gamma)$ generate $\pi_1(\hat{T})$, and $\hat{\Gamma} = \Gamma_0 *_\sigma$.

Then, we say that the natural projection $\pi_\rho : N_\rho \to \hat{N}$ wraps n times with respect to γ if n is the smallest number such that there exists a hyperbolic 3-manifold $\bar{N} = \mathbb{H}^3/\bar{\Gamma}$ with $\Gamma_\rho < \bar{\Gamma} < \hat{\Gamma}$ satisfying the following condition $\mathcal{W}n$:

$$\mathcal{W}n \begin{cases} \pi_\rho \text{ factors through } \bar{\pi} : N_\rho \to \bar{N} \text{ and } \hat{\pi} : \bar{N} \to \hat{N}, \\ \text{there exists a compact core } K \text{ for } N_\rho \text{ so that } \bar{\pi}|_K \text{ is an embedding, and} \\ \bar{\Gamma} = \Gamma_0 *_{\sigma^{n+1}} . \end{cases}$$

We call a natural projection π_ρ with nonzero wrapping the wrapping projection.

3. Main theorem and its proof

In this section, we state our main theorem, which characterize wrapping projections by the actions of decomposed groups along the accidental parabolic subgroup, and give its proof and an example as a corollary. Let S be a closed orientable surface of genus $g \geq 2$. Recall that for $\rho \in AH(S)$, we denote its representative also by ρ. From now on, for simplicity, we denote its image $\rho(\pi_1(S))$ by Γ_ρ.

3.1. **The statement of the main theorem.** In this thesis, we show the following.

Theorem 3.1.1. *Suppose that $\rho \in AH(S)$ has an accidental parabolic curve γ with the corresponding annulus cusp neighborhood Q, and the cusp of Q abuts only geometrically finite ends. Let p be the fixed point of $\rho(\gamma)$, and $J = \langle \rho(\gamma) \rangle$ the rank 1 parabolic group generated by $\rho(\gamma)$. Then Γ_ρ is decomposed along J;*

$$\Gamma_\rho = \begin{cases} \Gamma_1 \underset{J}{*} \Gamma_2, & \text{if } \gamma \text{ is separating in } S, \\ \Gamma' *_\delta, & \text{if } \gamma \text{ is non-separating in } S, \end{cases}$$

where Γ_1, Γ_2 and Γ' are subgroups of Γ_ρ, and δ is an element of Γ_ρ. Moreover for a given nonnegative integer n, the following holds:

There exist an associated geometric limit $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ so that Q covers a torus cusp neighborhood $\hat{T}$ of $\hat{N}$, that the natural projection $\pi_\rho : N_\rho \to \hat{N}$ wraps n times with respect to γ, and that $\hat{N}$ has double trouble with respect to $\hat{T}$ if and only if the following holds.

- *In the case where γ is separating in S, there exist a parabolic element σ fixing the point p, (Γ_i, J)-invariant open disks C_i and D_i in $\hat{\mathbb{C}}$ for $i = 1, 2$, such that the following condition $\mathcal{DS}n$ holds:*

$$\mathcal{DS}n \begin{cases} \sigma^{n+1}(C_1) = \hat{\mathbb{C}} \setminus \overline{C_2}, \sigma^n(D_1) = \hat{\mathbb{C}} \setminus \overline{D_2}, \\ \Lambda(\Gamma_i) \text{ separates } C_i \text{ and } D_i, \text{ and } \overline{C_i} \cap \Lambda(\Gamma_i) = \overline{D_i} \cap \Lambda(\Gamma_i) = p. \end{cases}$$

- *In the case where γ is non-separating in S, there exist a parabolic element σ fixing the point p, (Γ', J)-invariant open disks C and D and $(\Gamma', J^{\delta^{-1}})$-invariant open disks C' and D' in $\hat{\mathbb{C}}$ such that the following condition $\mathcal{DN}n$ holds:*

$$\mathcal{DN}n \begin{cases} \sigma^{n+1}\delta(C') = \hat{\mathbb{C}} \setminus \overline{C}, \sigma^n\delta(D') = \hat{\mathbb{C}} \setminus \overline{D}, \\ \Lambda(\Gamma') \text{ separates } C \text{ from } D, \text{ and } C' \text{ from } D', \\ \overline{C} \cap \Lambda(\Gamma') = \overline{D} \cap \Lambda(\Gamma') = p, \overline{C'} \cap \Lambda(\Gamma') = \overline{D'} \cap \Lambda(\Gamma') = \delta^{-1}(p), \text{and} \\ (*)\text{for any } f \in \Gamma', f(\overline{C}) \cap \overline{C'} = \emptyset, f(\overline{C}) \cap \overline{D'} = \emptyset, f(\overline{D}) \cap \overline{C'} = \emptyset, f(\overline{D}) \cap \overline{D'} = \emptyset. \end{cases}$$

See Figures 3.1.1 and 3.1.2. (Note that all the figures in this thesis are schematic pictures.)

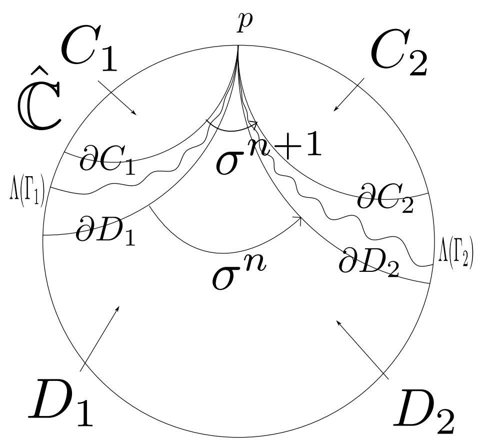

FIGURE 3.1.1. The disks satisfying the condition $\mathcal{DS}n$

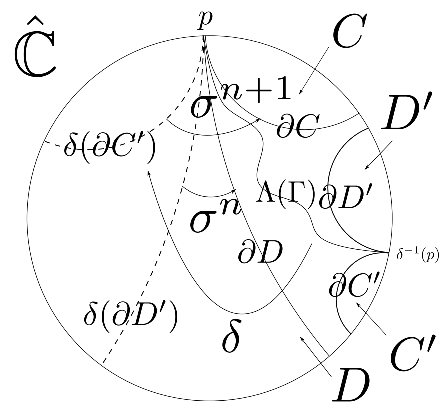

FIGURE 3.1.2. The disks satisfying the condition $\mathcal{DN}n$

Remark 3.1.2. *We note that in the separating case, since $\overline{C_i} \cap \Lambda(\Gamma_i) = \overline{D_i} \cap \Lambda(\Gamma_i) = p$, and C_i and D_i are disks, each of them is contained in some component of $\Omega(\Gamma_i)$, and that "$\Lambda(\Gamma_i)$ separates C_i and D_i" means C_i and D_i lie in different components of $\Omega(\Gamma_i)$, and similarly for the non-separating case. We labelled the last condition in $\mathcal{DN}n$, which requires that for any $f \in \Gamma'$, $f(\overline{C}) \cap \overline{C'} = \emptyset$, $f(\overline{C}) \cap \overline{D'} = \emptyset$, $f(\overline{D}) \cap \overline{C'} = \emptyset$, $f(\overline{D}) \cap \overline{D'} = \emptyset$, by $(*)$.*

Now we prove Theorem 3.1.1. Since N_ρ has the accidental parabolic curve γ, by Proposition 2.7.5, Γ_ρ is decomposed along $J = \langle \rho(\gamma) \rangle$;

$$\Gamma_\rho = \begin{cases} \Gamma_1 \underset{J}{*} \Gamma_2, & \text{if } \gamma \text{ is separating in } S, \\ \Gamma' *_\delta, & \text{if } \gamma \text{ is non-separating in } S, \end{cases}$$

where Γ_1, Γ_2 and $\Gamma' < \Gamma_\rho$, and $\delta \in \Gamma_\rho$.

3.2. **The actions of the decomposed groups imply wrapping projection.** In this subsection, given a parabolic element σ and the disks satisfying the condition $\mathcal{DS}n$ in the case where γ is separating in S, or $\mathcal{DN}n$ in the case where γ is non-separating in S, we will construct an associated geometric limit $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ with the natural projection $\pi_\rho : N_\rho \to \hat{N}$ such that Q covers a torus cusp neighborhood $\hat{T}$ of $\hat{N}$ via π_ρ, that π_ρ wraps n times with respect to γ, and that $\hat{N}$ has double trouble with respect to $\hat{T}$.

The case where γ is separating in S. By assumption, $\hat{\mathbb{C}}$ is divided into the $({\Gamma_1}^{\sigma^{n+1}}, J)$-invariant disk $\sigma^{n+1}(C_1)$ and the (Γ_2, J)-invariant disk C_2. The Klein-Maskit combination theorem I implies that ${\Gamma_1}^{\sigma^{n+1}} \underset{J}{*} \Gamma_2$ is discrete. We define Γ_0 to be this Kleinian group.

We show the next claim:

Claim 3.2.1. *$f(C_i) \cap D_i = \emptyset$ for any $f \in \Gamma_i$ and $i = 1, 2$.*

Proof of Claim 3.2.1. Suppose that $f(C_i) \cap D_i \neq \emptyset$ for some $f \in \Gamma_i$. Let Ω_i be the component of $\Omega(\Gamma_i)$ such that $D_i \subset \Omega_i$. Since $f(C_i) \cap D_i \neq \emptyset$, $f(C_i) \cap \Omega_i \neq \emptyset$, thus $f(C_i) \subset \Omega_i$.

Since the cusp of Q abuts only geometrically finite ends by assumption, $N_\rho = \mathbb{H}^3/\Gamma_\rho$ is cut along a once-punctured disk corresponding to J which connects the accidental parabolic curve γ to the cusp of Q, and divided into (parts of) $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$ as in Remark 2.7.7, and the cusps in $\partial_c\mathbb{H}^3/\Gamma_i$ corresponding to J are contained in two different components B_{i+} and B_{i-} of $\partial_c\mathbb{H}^3/\Gamma_i$. See Figure 3.2.1.

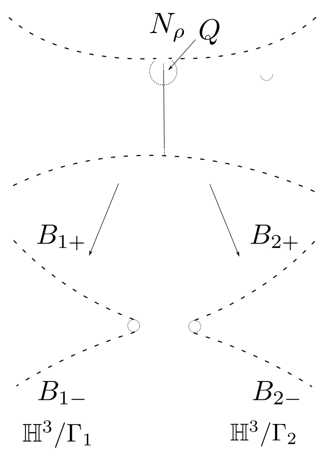

Figure 3.2.1. The decomposition of N_ρ

If $f^{-1}(\Omega_i) \neq \Omega_i$ (see Figure 3.2.2), then the cusps in $\partial_c\mathbb{H}^3/\Gamma_i$ are contained in only one component of $\partial_c\mathbb{H}^3/\Gamma_i$.

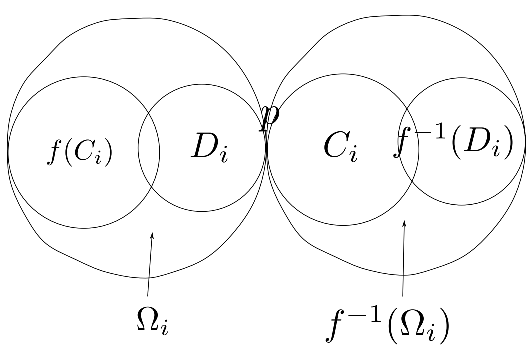

Figure 3.2.2. The components of $\Omega(\Gamma_i)$ around p if $f^{-1}(\Omega_i) \neq \Omega_i$

Thus $f^{-1}(\Omega_i) = \Omega_i$, and hence $C_i = f^{-1}(f(C_i)) \subset f^{-1}(\Omega_i) = \Omega_i$. This violates the assumption that $\Lambda(\Gamma_i)$ separates C_i and D_i. Hence, Claim3.2.1 holds. $\square$

We observe the following (see Figures 3.2.3, 3.2.4 and 3.2.5):

Observation 3.2.2. *By Claim 3.2.1 and the* $({\Gamma_1}^{\sigma^{n+1}}, J)$*-invariance of* $\sigma^{n+1}(D_1)$*, we see that for any* $f \in {\Gamma_1}^{\sigma^{n+1}} \setminus J$*,*

$$f(\sigma^{n+1}(D_1)) \subset \hat{\mathbb{C}} \setminus \overline{\sigma^{n+1}(C_1)} = C_2 \text{ and } f(\sigma^{n+1}(D_1)) \cap \sigma^{n+1}(D_1) = \emptyset.$$

Observation 3.2.3. *By the* (Γ_2, J)*-invariance of* C_2 *and Claim 3.2.1, we see that for any* $g \in \Gamma_2 \setminus J$*,*

$$g(C_2) \subset \hat{\mathbb{C}} \setminus \overline{C_2} = \sigma^{n+1}(C_1) \text{ and } g(C_2) \cap D_2 = \emptyset.$$

Observation 3.2.4. *By the* $({\Gamma_1}^{\sigma^{n+1}}, J)$*-invariance of* $\sigma^{n+1}(C_1)$ *and Claim 3.2.1, we see that for any* $f \in {\Gamma_1}^{\sigma^{n+1}} \setminus J$*,*

$$f(\sigma^{n+1}(C_1)) \subset \hat{\mathbb{C}} \setminus \overline{\sigma^{n+1}(C_1)} = C_2 \text{ and } f(\sigma^{n+1}(C_1)) \cap \sigma^{n+1}(D_1) = \emptyset.$$

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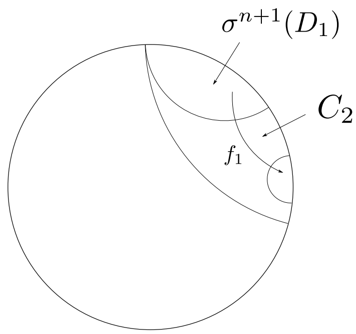

Figure 3.2.3.
Observation 3.2.2

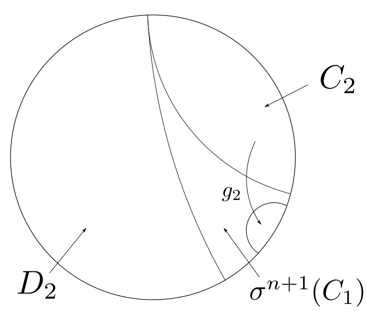

Figure 3.2.4.
Observation 3.2.3

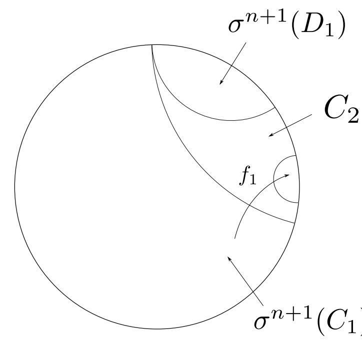

Figure 3.2.5.
Observation 3.2.4

We will show that $\sigma^{n+1}(D_1)$ is (Γ_0, J)-invariant. Since D_1 is J-invariant and σ commutes with J, $\sigma^{n+1}(D_1)$ is also J-invariant. Given any $h \in \Gamma_0 \setminus J$, since $\Gamma_0 = {\Gamma_1}^{\sigma^{n+1}} \underset{J}{*} \Gamma_2$, we can express it as a reduced word of the form $\cdots g_4 f_3 g_2 f_1$, or $\cdots g_3 f_2 g_1$, where $f_i \in {\Gamma_1}^{\sigma^{n+1}} \setminus J$, $g_j \in \Gamma_2 \setminus J$. Then we can check that $h(\sigma^{n+1}(D_1)) \cap \sigma^{n+1}(D_1) = \emptyset$ by looking at $f_1(\sigma^{n+1}(D_1))$, $g_2 f_1(\sigma^{n+1}(D_1))$, $f_3 g_2 f_1(\sigma^{n+1}(D_1))$, $g_4 f_3 g_2 f_1(\sigma^{n+1}(D_1)), \ldots$ using Observations 3.2.2, 3.2.3, 3.2.4, 3.2.3, …, or by looking at $g_1 \sigma^{n+1}(D_1)$, $f_2 g_1 \sigma^{n+1}(D_1)$, $g_3 f_2 g_1 \sigma^{n+1}(D_1), \ldots$ using Observations 3.2.3, 3.2.4, 3.2.3, ….

Similarly, D_2 is (Γ_0, J)-invariant. Noting $C_2 \cap D_2 = \emptyset$, we have $h(\sigma^{n+1}(D_1)) \cap D_2 = \emptyset$ for any $h \in \Gamma_0$. Applying the Klein-Maskit combination theorem II to D_2 and $\sigma^{n+1}(D_1)$, we see that $\Gamma_0 *_\sigma$ is discrete. We define $\hat{\Gamma}$ to be this Kleinian group.

We define $\bar{\Gamma}$ to be $\Gamma_0 *_{\sigma^{n+1}}$. By definition, $\bar{\Gamma}$ is a subgroup of $\hat{\Gamma}$, and since $\Gamma_0 = {\Gamma_1}^{\sigma^{n+1}} \underset{J}{*} \Gamma_2$ and $\Gamma_\rho = \Gamma_1 \underset{J}{*} \Gamma_2$, $\bar{\Gamma}$ contains Γ_ρ. Let π_ρ, $\bar{\pi}$ and $\hat{\pi}$ be the natural projections corresponding to $\Gamma_\rho < \bar{\Gamma} < \hat{\Gamma}$:

$$\begin{array}{ccc} N_\rho = \mathbb{H}^3/\Gamma_\rho & \xrightarrow{\bar{\pi}} & \bar{N} = \mathbb{H}^3/\bar{\Gamma} \\ & \searrow^{\pi_\rho} & \downarrow \hat{\pi} \\ & & \hat{N} = \mathbb{H}^3/\hat{\Gamma} \end{array}$$

We will construct $\mathbb{H}^3/\Gamma_0$ and $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ combinatorially from $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$. Using the Klein-Maskit combination theorem I, we construct $\mathbb{H}^3/\Gamma_0$ as follows. Since p is the unique parabolic fixed point of Γ_0 in ∂C_2, which is doubly cusped in Γ_0, we can take a (Γ_0, J)-invariant spanning disk $D_{\partial C_2}$ in $\mathbb{H}^3$ for ∂C_2. Recall that $\sigma^{n+1}(\partial C_1) = \partial C_2$ divides $\hat{\mathbb{C}}$ into $\sigma^{n+1}(C_1)$ and C_2. Let H_{C_i} be the closed topological half-space in $\mathbb{H}^3$ with $\overline{\partial} H_{C_i} = \overline{C_i}$ for each i, such that $\mathbb{H}^3$ is divided by $D_{\partial C_2}$ into $\sigma^{n+1}(H_{C_1})$ and H_{C_2}. It is clear that H_{C_i} is (Γ_i, J)-invariant. Then $\mathbb{H}^3/\Gamma_0$ can be regarded as the union of $(\mathbb{H}^3/\Gamma_2)\backslash(H_{C_2}/J)$ and $(\mathbb{H}^3/{\Gamma_1}^{\sigma^{n+1}})\backslash(\sigma^{n+1}(H_{C_1})/J)$ along $D_{\partial C_2}/J$. Here $(\mathbb{H}^3/{\Gamma_1}^{\sigma^{n+1}})\backslash(\sigma^{n+1}(H_{C_1})/J)$ is isometric to $(\mathbb{H}^3/\Gamma_1)\backslash(H_{C_1}/J)$. Using the Klein-Maskit combination theorem II, we construct $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ as follows. Since p is the unique parabolic fixed point of $\hat{\Gamma}$ in both $\partial_2 D$ and in $\sigma^{n+1}(\partial D_1)$, which is rank 2 in $\hat{\Gamma}$, we can take (Γ_0, J)-invariant spanning disks $D_{\partial D_2}$ for ∂D_2 and $D_{\sigma^{n+1}(\partial D_1)}$ for $\sigma^{n+1}(\partial D_1)$, such that $\sigma(D_{\partial D_2}) = D_{\sigma^{n+1}(\partial D_1)}$, and that $D_{\partial C_2}$ does not intersect either $D_{\partial D_2}$ or $D_{\sigma^{n+1}(\partial D_1)}$. Set $D_{\partial D_1} = \sigma^{-(n+1)}(D_{\sigma^{n+1}(\partial D_1)})$. Let H_{D_i} be the closed topological half-space cut out of $\mathbb{H}^3$ by $D_{\partial D_i}$ with $\overline{\partial} H_{D_i} = \overline{D_i}$ for each i. Then $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ can be obtained by gluing $(\mathbb{H}^3/\Gamma_0)\backslash((H_{D_2} \cup \sigma^{n+1}(H_{D_1}))/J)$ with itself along $D_{\partial D_2}/J$ and $\sigma^{n+1}(D_{\partial D_1})/J$ via the identification given by σ.

By the constructions of $\mathbb{H}^3/\Gamma_0$ and $\hat{N}$, $\hat{N}$ is homeomorphic to a manifold obtained by removing a simple closed curve homotopic to γ from the interior of $S \times [0,1]$. A tubular neighborhood of γ gives rise to the torus cusp neighborhood $\hat{T}$. Since $D_{\partial C_2}/J$ and $D_{\partial D_2}/J$ are embedded once-punctured disks in $\hat{N}$, $\hat{N}$ has double trouble with respect to $\hat{T}$.

We will construct a compact core K for N_ρ such that the restriction $\bar{\pi}|_K$ is an embedding. For each $i = 1, 2$, we take an embedded thickened surface $\hat{S}_i$ in $\hat{N}$ whose cusp lies on the one of $\hat{T}$, and the inclusion of whose lift is a homotopy equivalence to $\mathbb{H}^3/\Gamma_i$, so that $\hat{S}_1$ and $\hat{S}_2$ are disjoint. Let S_i be a lift of $\hat{S}_i$ in N_ρ, which is homeomorphic to $\hat{S}_i$. We construct a compact core K for N_ρ by removing a smaller annulus cusp neighborhood P in the cusp neighborhood Q from S_1 and S_2 and connecting them by a thickened annulus R in P. See Figure 3.2.6.

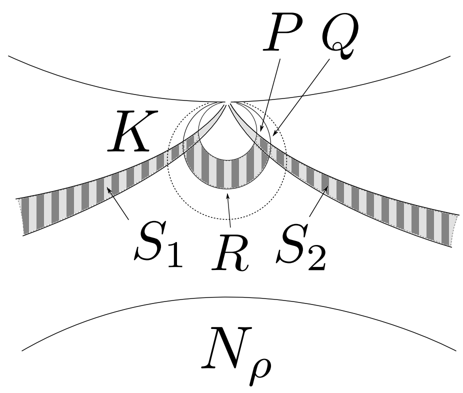

FIGURE 3.2.6. The construction of K in the separating case

We consider this surgery in $\mathbb{H}^3$ for lifts of these components. Let H be the horoball at p which covers P. By the constructions of $\mathbb{H}^3/\Gamma_0$ and $\hat{N}$ above, we can take a lift $\tilde{S}_i$ in $\mathbb{H}^3$ of both S_i in N_ρ and $\hat{S}_i$ in $\hat{N}$ such that $\tilde{S}_i$ lies between H_{C_i} and $D_{\partial D_i}$, and $\overline{\partial}\tilde{S}_i = p$. Since $D_{\partial D_1} \cup D_{\partial D_2} \subset \mathbb{H}^3 \setminus (H_{C_1} \cup H_{C_2})$, we have $\tilde{S}_1 \cup \tilde{S}_2 \subset \mathbb{H}^3 \setminus (H_{C_1} \cup H_{C_2})$. By removing H from these two lifts and connecting them by the thickened strip $\tilde{R}$ in H which covers R, we construct a lift $\tilde{K}$ of K in $\mathbb{H}^3$. See Figure 3.2.7.

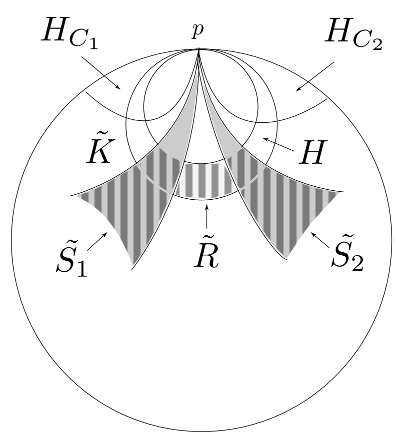

FIGURE 3.2.7. The construction of $\tilde{K}$ in the separating case

Since S_1 and S_2 are disjointly embedded in $\hat{N}$ via π_ρ, $K \setminus R$ is embedded in $\bar{N}$ via $\bar{\pi}$. On the other hand, since $\sigma^{n+1}(\mathbb{H}^3 \setminus (H_{C_1} \cup H_{C_2})) \cap (\mathbb{H}^3 \setminus (H_{C_1} \cup H_{C_2})) = \emptyset$, we have $\sigma^{n+1}(\tilde{R}) \cap \tilde{R} = \emptyset$. So taking H so that $H/\langle \rho(\gamma), \sigma^{n+1} \rangle = H/\bar{\Gamma}$, R is also embedded in $\bar{N}$ via $\bar{\pi}$. Therefore, the compact core K is embedded in $\bar{N}$ via $\bar{\pi}$.

The case where γ is non-separating in S. By assumptions, we see that $C = \sigma^{n+1}\delta(\hat{\mathbb{C}} \setminus \overline{C'})$ is (Γ', J)-invariant, C' is $(\Gamma', J^{\delta^{-1}})$-invariant, and $f(C) \cap C' = \emptyset$ for any $f \in \Gamma'$. The Klein-Maskit combination theorem II implies that $\Gamma' *_{\sigma^{n+1}\delta}$ is discrete. We define Γ_0 to be this Kleinian group.

Similarly to Claim 3.2.1, we have:

Claim 3.2.5. *$f(C) \cap D = \emptyset$ and $f(C') \cap D' = \emptyset$ for any $f \in \Gamma'$.*

We observe the following (see Figures 3.2.8, 3.2.9 and 3.2.10):

Observation 3.2.6. *By Claim 3.2.5, the (Γ', J)-invariance of D, the $(\Gamma', J^{\delta^{-1}})$-invariance of D', and the condition $(*)$ of $\mathcal{DN}n$, we see that*

$$f(D) \subset \hat{\mathbb{C}} \setminus \overline{C \cup D \cup C' \cup D'} \text{ for any } f \in \Gamma' \setminus J,\ f(D) = D \text{ for any } f \in J,$$

$$f(D') \subset \hat{\mathbb{C}} \setminus \overline{C \cup D \cup C' \cup D'} \text{ for any } f \in \Gamma' \setminus J^{\delta^{-1}},\ \text{and } f(D') = D' \text{ for any } f \in J^{\delta^{-1}}.$$

Observation 3.2.7. *We see that*

$$\sigma^{n+1}\delta(\hat{\mathbb{C}} \setminus \overline{C'}) = C, \text{ and } (\sigma^{n+1}\delta)^{-1}(\hat{\mathbb{C}} \setminus \overline{C}) = C'.$$

Observation 3.2.8. *By Claim 3.2.5, the (Γ', J)-invariance of C, the $(\Gamma', J^{\delta^{-1}})$-invariance of C', and the condition $(*)$ of $\mathcal{DN}n$, we see that*

$$f(C) \subset \hat{\mathbb{C}} \setminus \overline{C \cup D \cup C' \cup D'} \text{ for any } f \in \Gamma' \setminus J,\ f(C) = C \text{ for any } f \in J,$$

$$f(C') \subset \hat{\mathbb{C}} \setminus \overline{C \cup D \cup C' \cup D'} \text{ for any } f \in \Gamma' \setminus J^{\delta^{-1}},\ \text{and } f(C') = C' \text{ for any } f \in J^{\delta^{-1}}.$$

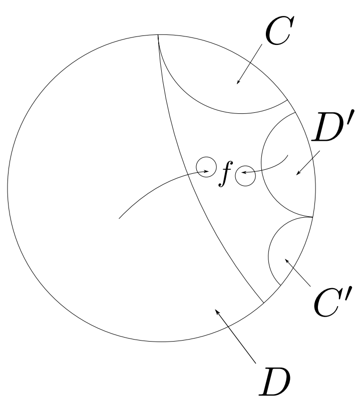

FIGURE 3.2.8. Observation 3.2.6

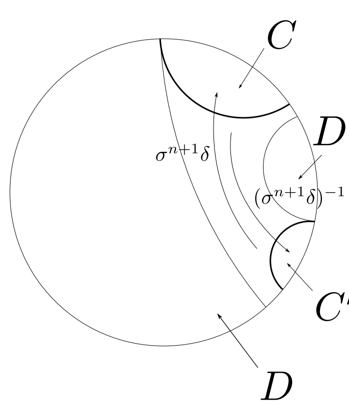

FIGURE 3.2.9. Observation 3.2.7

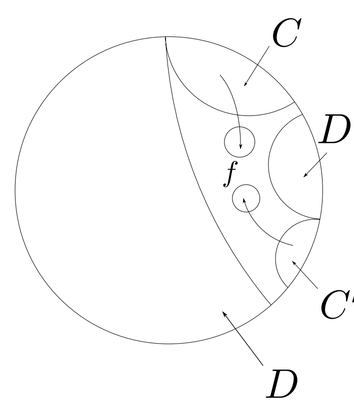

FIGURE 3.2.10. Observation 3.2.8

We will show that D is (Γ_0, J)-invariant in a similar way to the separating case. Given any $h \in \Gamma_0 \setminus J$, since $\Gamma_0 = (\Gamma') *_{\sigma^{n+1}\delta}$, we can express it as a reduced word of the form $\cdots g_4 f_3 g_2 f_1$, or $\cdots g_3 f_2 g_1$, where $f_i \in \Gamma'$ and $g_j = \sigma^{n+1}\delta$ or $(\sigma^{n+1}\delta)^{-1}$. Since $(\sigma^{n+1}\delta)^{-1} f \sigma^{n+1}\delta = f^{\delta^{-1}}$ for any $f \in J$, the above word does not have $(\sigma^{n+1}\delta)^{-1} f \sigma^{n+1}\delta$ for any $f \in J$, or $\sigma^{n+1}\delta f (\sigma^{n+1}\delta)^{-1}$ for any $f \in J^{\delta^{-1}}$, as its consecutive subsequence of letters. We can check that $h(D) \cap D = \emptyset$, by chasing

$f_1(D)$, $g_2f_1(D)$, $f_3g_2f_1(D)$, $g_4f_3g_2f_1(D), \ldots$ using Observations 3.2.6, 3.2.7, 3.2.8, 3.2.7, ..., or by chasing $g_3f_2g_1(D)$, $g_3f_2g_1(D)$, $g_3f_2g_1(D), \ldots$ using Observations 3.2.7, 3.2.8, 3.2.7, ...

Similarly, D' is $(\Gamma_0, J^{\delta^{-1}})$-invariant and thus, $\sigma^{n+1}\delta(D')$ is (Γ_0, J)-invariant. Noting that $\sigma^{n+1}\delta(D') \subset C$, $h(\sigma^{n+1}\delta(D')) \cap D = \emptyset$ for any $h \in \Gamma_0$. Applying the Klein-Maskit combination theorem II to D and $\sigma^{n+1}\delta(D')$, we see that $\Gamma_0 *_\sigma$ is discrete. We define $\hat{\Gamma}$ to be this Kleinian group.

We define $\bar{\Gamma}$ to be $\Gamma_0 *_{\sigma^{n+1}}$. By definition, $\bar{\Gamma}$ is a subgroup of $\hat{\Gamma}$, and since $\Gamma_0 = \Gamma' *_{\sigma^{n+1}\delta}$ and $\Gamma_\rho = \Gamma' *_\delta$, $\bar{\Gamma}$ contains Γ_ρ.

We will construct $\mathbb{H}^3/\Gamma_0$ and $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ combinatorially from $\mathbb{H}^3/\Gamma'$. Using the Klein-Maskit combination theorem II, we construct $\mathbb{H}^3/\Gamma_0$ as follows. Since p and $\delta^{-1}(p)$ are the unique parabolic fixed points of Γ_0 in ∂C and in $\partial C'$, respectively, which are doubly cusped in Γ_0, we can take a (Γ', J)-invariant spanning disk $D_{\partial C}$ for ∂C and $(\Gamma', J^{\delta^{-1}})$-invariant spanning one $D_{\partial C'}$ for $\partial C'$ such that $\sigma^{n+1}\delta(D_{\partial C'}) = D_{\partial C}$. Let H_C be the closed topological half-space cut out of $\mathbb{H}^3$ by $D_{\partial C}$ with $\overline{\partial} H_C = \overline{C}$ and $H_{C'}$ the closed topological half-space cut out of $\mathbb{H}^3$ by $D_{\partial C'}$ with $\overline{\partial} H_{C'} = \overline{C'}$. Then $\mathbb{H}^3/\Gamma_0$ can be obtained by gluing $(\mathbb{H}^3/\Gamma') \setminus ((H_C \cup H_{C'})/J)$ with itself along $D_{\partial C}/J$ and $D_{\partial C'}/J$ via the identification given by $\sigma^{n+1}\delta$. Using the Klein-Maskit combination theorem II, we construct $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ as follows. Since p is the unique parabolic fixed point of $\hat{\Gamma}$ in ∂D and in $\sigma^{n+1}\delta(\partial D')$, which is doubly cusped in $\hat{\Gamma}$, we can take (Γ_0, J)-invariant spanning disks $D_{\partial D}$ for ∂D and $D_{\sigma^{n+1}\delta(\partial D')}$ for $\sigma^{n+1}\delta(\partial D')$, such that $\sigma(D_{\partial D}) = D_{\sigma^{n+1}\delta(\partial D')}$ and that $D_{\partial C}$ does not intersect $D_{\partial D}$ or $\sigma^{n+1}\delta(D_{\partial D'})$. Set $D_{\partial D'} := (\sigma^{n+1}\delta)^{-1}(D_{\sigma^{n+1}\delta(\partial D')})$. Let H_D be the closed topological half-space cut out of $\mathbb{H}^3$ by $D_{\partial D}$ with $\overline{\partial} H_D = \overline{D}$, and $H_{D'}$ the closed topological half-space cut out of $\mathbb{H}^3$ by $D_{\partial D'}$ with $\overline{\partial} H_{D'} = \overline{D'}$. Then $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ can be obtained by gluing $(\mathbb{H}^3/\Gamma_0) \setminus ((H_D \cup \sigma^{n+1}\delta(H_{D'}))/J)$ with itself, along $D_{\partial D}/J$ and $\sigma^{n+1}\delta(D_{\partial D'})/J$ via the identification given by σ.

By the constructions of $\mathbb{H}^3/\Gamma_0$ and $\hat{N}$, $\hat{N}$ is homeomorphic to a manifold obtained by removing a simple closed curve homotopic to γ from the interior of $S \times [0, 1]$. A tubular neighborhood of γ gives rise to the torus cusp neighborhood $\hat{T}$. Since $D_{\partial C}/J$ and $D_{\partial D}/J$ is embedded once-punctured disks in $\hat{N}$, $\hat{N}$ has double trouble with respect to $\hat{T}$. Let π_ρ, $\bar{\pi}$ and $\hat{\pi}$ be the natural projections corresponding to $\Gamma_\rho < \bar{\Gamma} < \hat{\Gamma}$, similarly to the separating case.

We will construct a compact core K for N_ρ such that the restriction $\bar{\pi}|_K$ is an embedding in a way similar to the separating case. We take an embedded thickened surface $\hat{S}'$ in $\hat{N}$ whose cusps lie on the one of $\hat{T}$, and the inclusion of whose lift is a homotopy equivalence to $\mathbb{H}^3/\Gamma'$. Let S' be a lift of $\hat{S}'$ in N_ρ, which is homeomorphic to $\hat{S}'$. We construct a compact core K for N_ρ by removing a smaller annulus cusp neighborhood P in the cusp neighborhood Q from S' and connecting it with a thickened annulus R in P. See Figure 3.2.11.

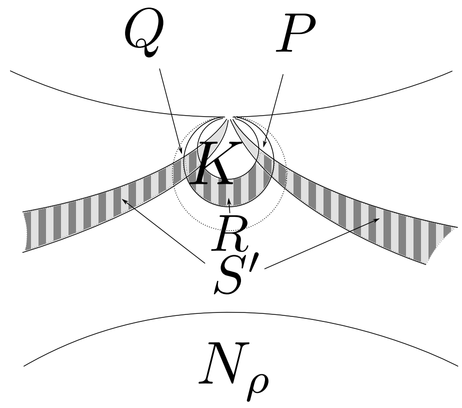

FIGURE 3.2.11. The construction of K in the non-separating case

We consider this surgery in $\mathbb{H}^3$ with lifts of these components. Let H be the horoball at p which covers P. By the constructions of $\mathbb{H}^3/\Gamma_0$ and $\hat{N}$ above, we can take a lift $\tilde{S}'$ in $\mathbb{H}^3$ of both S' in N_ρ and $\hat{S}'$ in $\hat{N}$ such that $\tilde{S}'$ lies in between H_C, $D_{\partial D}$, $H_{C'}$ and $D_{\partial D'}$, and $\overline{\partial}\tilde{S}' = \{p, \delta^{-1}(p)\}$. Since $\delta(D_{\partial D'}) \cup D_{\partial D} \subset \mathbb{H}^3 \setminus (\delta(H_{C'}) \cup H_C)$, we have $\delta(\tilde{S}') \cup \tilde{S}' \subset \mathbb{H}^3 \setminus (\delta(H_{C'}) \cup H_C)$. By removing H and $\delta^{-1}(H)$ from the lift and connecting it with the thickened strip $\tilde{R}$ in H which covers R and connects $\delta(\tilde{S}')$ and $\tilde{S}'$, we construct a lift $\tilde{K}$ of K in $\mathbb{H}^3$. See Figure 3.2.12.

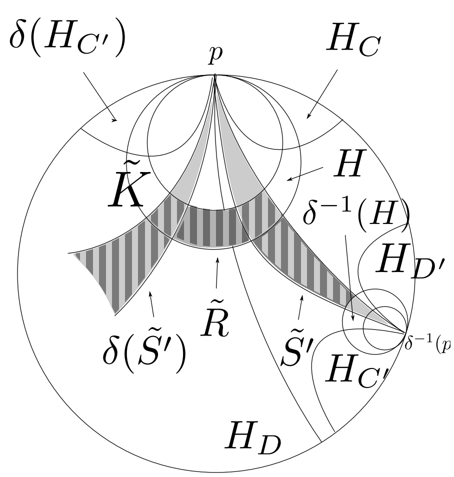

FIGURE 3.2.12. The construction of $\tilde{K}$ in the non-separating case

Since S' is embedded in $\hat{N}$ via π_ρ, $K \setminus R$ is embedded in $\bar{N}$ via $\bar{\pi}$. On the other hand, since $\sigma^{n+1}(\mathbb{H}^3 \setminus (\delta(H_{C'}) \cup H_C)) \cap (\mathbb{H}^3 \setminus (\delta(H_{C'}) \cup H_C)) = \emptyset$, we have $\sigma^{n+1}(\tilde{R}) \cap \tilde{R} = \emptyset$. So taking H so that $H/\langle \rho(\gamma), \sigma^{n+1} \rangle = H/\bar{\Gamma}$, R is also embedded in $\bar{N}$ via $\bar{\pi}$. Therefore, the compact core K is embedded in $\bar{N}$ via $\bar{\pi}$.

The hyperbolic 3-manifold $\hat{N}$. From here we consider the both cases in the same time. We have obtained the hyperbolic 3-manifolds $\hat{N}$ and $\bar{N}$ satisfying the condition $\mathcal{W}n$. If there exists a hyperbolic 3-manifold $\bar{N}' = \mathbb{H}^3/\bar{\Gamma}'$ with $\Gamma_\rho < \bar{\Gamma}' < \hat{\Gamma}$ satisfying the condition $\mathcal{W}k$, instead of $\bar{N} = \mathbb{H}^3/\bar{\Gamma}$. The projection $\bar{\pi}' : N_\rho \to \bar{N}'$ embeds some compact core K' for N_ρ and $\bar{\Gamma}' = \Gamma_0 *_{\sigma^{k+1}}$. By considering this projection around the cusp neighborhood of Q, we get $k \geq n$. So n is the smallest of such numbers.

Finally we will check that $\hat{N}$ is an associated geometric limit of ρ. Let $f : S \times [0,1] \to N_\rho$ be a marking of ρ, i.e. a homotopy equivalence corresponding to ρ, such that $f(S \times [0,1]) = K$.

We will apply the work of Ohshika and Soma [40] stated as in Subsection 2.9 to $\hat{N}$. Since $\hat{N}$ is homeomorphic to $S \times (0,1)$ from which a simple closed curve (homotopic to γ) has been deleted by the construction, we can take a labelled brick manifold M such that M is homeomorphic to the non-cuspidal part $\hat{N}_0$ of $\hat{N}$, has the same end invariant as $\hat{N}$, and satisfies the conditions (i)-(iv) and (EL) in Theorem 2.9.6. By Theorem 2.9.6, there exists a geometrically convergent sequence of quasi-Fuchsian manifolds $\{N_m\}$ such that the non-cuspidal part $(N_G)_0$ of N_G is homeomorphic to M, N_G has the same end invariant as M, where N_G is the geometric limit of $\{N_m\}$. So $(N_G)_0$ and $\hat{N}_0$ are homeomorphic and have the same end invariant. Since $\pi_1(N_G) \cong \pi_1(\hat{N})$ is finitely generated, and both N_G and $\hat{N}$ have only incompressible tame ends, by the ending lamination theorem for incompressible ends stated as in Subsection 2.6, N_G is isometric to $\hat{N}$. We may assume that the sequence of quasi-Fuchsian manifolds with basepoints (N_m, x_m) converges geometrically to $(\hat{N}, \hat{x})$ for some basepoint $\hat{x}$.

For each m, let $f_m : B(\hat{x}, R_m) \to N_m$ be a K_m-biLipschitz diffeomorphism stated in the definition of geometric convergence. Since K is compact, taking a subsequence if necessary, we can assume that $\pi_\rho(K) \subset B(\hat{x}, R_m)$ for all m. By an argument similar to that indicating that $\bar{\pi}|_K$ is an embedding, we can check that $\pi_\rho(K)$ wraps n times around the cusp of $\hat{T}$ in $\hat{N}$, that is, $\pi_\rho(K)$ is homotopic to the union of the embedded thickened surface $\hat{S}_K$ homeomorphic to a thickening of S and the n-fold thicken torus $\hat{T}_K$ around the cusp of $\hat{T}$. See Figure 3.2.13.

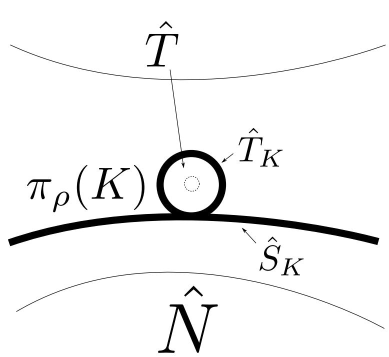

FIGURE 3.2.13. $\hat{N}$ and $\pi_\rho(K)$

Since $\{f_m\}$ is a diffeomorphism and converges to an isometry, we may assume that $f_m|_{\hat{S}_K} : \hat{S}_K \to N_m$ is a homotopy equivalence and $f_m(\hat{T}_K)$ wraps around a solid torus in N_m corresponding to the cusp of $\hat{T}$ (see Remark 2.9.7 and Figure 3.2.14), and this wrapping can be eliminated by a homotopy in N_m.

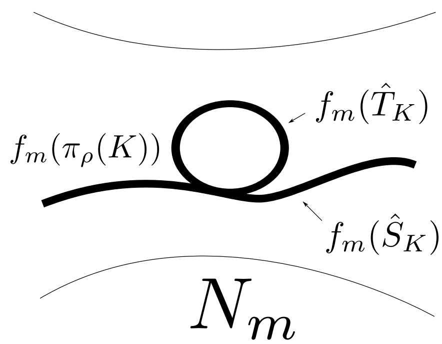

FIGURE 3.2.14. N_m and $f_m(\pi_\rho(K))$

This implies that $f_m \circ \pi_\rho \circ f : S \to N_m$ is a homotopy equivalence. For each m, let ρ_m be the holonomy of the pull-backed hyperbolic metric on $S \times [0,1]$ induced from N_m via $f_m \circ \pi_\rho \circ f$. Since $f_m \circ \pi_\rho \circ f : S \to N_m$ is a homotopy equivalence, it is a marking of ρ_m. Hence ρ_m can be regarded as the holonomy of N_m. Since N_m converges geometrically to $\hat{N}$, ρ_m converges algebraically to ρ. Thus $\hat{N}$ is an associated geometric limit of ρ. Therefore we have got the desired geometric limit associated to ρ with the wrapping projection.

3.3. **Wrapping projection implies the actions of the decomposed groups.** In this subsection, we will show the opposite direction of the proof. We assume that there exists an associated geometric limit $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ so that Q covers a torus cusp neighborhood $\hat{T}$ of $\hat{N}$, that the natural projection $\pi_\rho : N_\rho \to \hat{N}$ wraps n times with respect to γ, and that $\hat{N}$ has double trouble with respect to $\hat{T}$. We will give the disks satisfying the condition $\mathcal{DS}n$ in the case where γ is separating in S, or $\mathcal{DN}n$ in the case where γ is non-separating in S.

Since $\hat{N}$ has double trouble with respect to $\hat{T}$, as in Subsection 2.6, there exist accidental parabolic curves γ_+ and γ_- in some components B_+ and B_- of $\partial_c\hat{N}$, respectively, and an essential simple closed curve $\hat{\gamma}$ in $\partial\hat{T}$, which are mutually homotopic in $\hat{N} \cup \partial_c\hat{N}$. As in Remark 2.7.7, there exist disjoint embedded once-punctured disks $\hat{A}_+$ and $\hat{A}_-$ such that for each $j = +, -$, the boundary (at infinity) of $\hat{A}_j$ is γ_j, and the puncture of $\hat{A}_j$ lies on the cusp of $\hat{T}$. Since γ_+ and γ_- are accidental parabolic curves in $\hat{N}$ and $\hat{\gamma}$ lies in $\partial\hat{T}$, they also represent $\rho(\gamma)$. We can consider one of these two curves, say γ_-, as the image of the accidental parabolic curve γ by $\pi_\rho : N_\rho \cup \partial_c N_\rho \to \hat{N} \cup \partial_c\hat{N}$.

The union of the closures of all the connected components of preimage of $\hat{A}_-$ in $\overline{\mathbb{H}^3}$ which contain p divides $\overline{\mathbb{H}^3}$ into infinitely many regions with boundary being a pair of closures of components of preimage of $\hat{A}_-$. As in Remark 2.7.7 and in the proof of Proposition 2.7.5, the decomposition of $\hat{N}$ along the once-punctured disk $\hat{A}_-$ gives a decomposition $\hat{\Gamma} = \Gamma_0 *_\sigma$ such that $\Lambda(\Gamma_0)$ is contained in the closure $\overline{\Delta_0}^{\mathbb{H}^3}$ of some region Δ_0 stated above.

We will construct a compact core K for N_ρ as follows. As in Remark 2.7.6, Abikoff and Maskit showed that Γ_ρ is decomposed along all accidental parabolic subgroups into a finite number of subgroups $\{\Gamma'_1, \dots, \Gamma'_l\}$ such that for each $j = 1, \dots, l$, Γ'_j is either a generalized web group, or a degenerate group without accidental parabolic elements (see Theorem 1 of [1] and also Lemma 3.2 of [9]). Moreover in pp. 721–723 of [9] (see also Lemma 9 of [20]), Anderson, Canary and McCullough showed that there exists a pairwise disjoint collection of relative compact carries for $\{\Gamma'_1, \dots, \Gamma'_l\}$ in $\hat{N}$. Here $\hat{K}'_j$ is called a compact carry for Γ'_j in $\hat{N}$ if there exists a relative compact core K'_j for the non-cuspidal part $(\mathbb{H}^3/\Gamma'_j)_0$ of $\mathbb{H}^3/\Gamma'_j$ such that the natural projection $\mathbb{H}^3/\Gamma'_j \to \hat{N}$ is injective on K'_j, and that $\hat{K}'_j$ is the image of K'_j in $\hat{N}$. We take each $\hat{K}'_j$ so that $\hat{K}'_j \cap (\hat{A}_+ \cup \hat{A}_-) = \emptyset$. We construct a compact core K for N_ρ by connecting the lifts of $\hat{K}'_1, \dots, \hat{K}'_l$ in N_ρ via π_ρ by a thickened annulus in an annulus cusp neighborhood corresponding to each accidental parabolic curve of N_ρ which is homotoped into its cusp.

The case where γ is separating in S. For each $i = 1, 2$, let K_i be the component of $K \setminus Q$ such that $\rho(\pi_1(K_i)) = \Gamma_i$. We will show that for each $i = 1, 2$, $\pi_\rho|_{K_i}$ is an embedding. Suppose that $\pi_\rho|_{K_i}$ is not an embedding. By the construction of K, $K_i \cap (N_\rho)_0$ is embedded in $\hat{N}_0$, where $(N_\rho)_0$ and $\hat{N}_0$ are the non-cuspidal part of N_ρ and that of $\hat{N}$, respectively. In p. 724 of [9], Anderson, Canary and McCullough showed that π_ρ maps different cusp neighborhoods of N_ρ to different cusp neighborhoods of $\hat{N}$. Thus there exists an annulus cusp neighborhood Q_α of N_ρ whose cusp does not coincide with that of Q such that $\pi_\rho|_{K \cap Q_\alpha}$ is not an embedding. Let H_α be a horoball which covers Q_α, and $\tilde{K}_\alpha$ a connected component of preimage of K via the projection $\mathbb{H}^3 \to N_\rho$ which meets H_α. Since $\pi_\rho|_{K \cap Q_\alpha}$ is not an embedding, $\sigma_\alpha(\tilde{K}_\alpha)$ intersects $\tilde{K}_\alpha$ transversely, for some element $\sigma_\alpha \in st_{\hat{\Gamma}}(H_\alpha)$.

Since π_ρ wraps n times with respect to γ, the condition $\mathcal{W}n$ holds, that is, Γ_ρ is a subgroup of $\Gamma_0 *_{\sigma^{n+1}}$ and there exists a compact core K' for N_ρ such that $\bar{\pi}'|_{K'}$ is an embedding, where $\bar{\pi}' : N_\rho \to \mathbb{H}^3/\Gamma_0 *_{\sigma^{n+1}}$ is the natural projection. Since K and K' are homotopic to each other in N_ρ, $\tilde{K}_\alpha$ is homotopic to a connected component $\tilde{K}'_\alpha$ of preimage of K' without moving the boundary (at infinity). Since $\sigma_\alpha(\tilde{K}_\alpha)$ intersects $\tilde{K}_\alpha$ transversely, $\sigma_\alpha(\tilde{K}'_\alpha)$ intersects $\tilde{K}'_\alpha$ transversely.

By the existence of σ_α, Q_α covers a torus cusp neighborhood $\hat{T}_\alpha$ whose cusp does not coincide with that of $\hat{T}$, and we have $\sigma_\alpha \in \pi_1(\hat{T}_\alpha)$. Since $\hat{N} = \mathbb{H}^3/\Gamma_0 *_\sigma$ is obtained by gluing a part of $\mathbb{H}^3/\Gamma_0$ to itself as in the Klein-Maskit combination theorem II, we can consider that $\hat{T}_\alpha \subset \mathbb{H}^3/\Gamma_0$ and $\sigma_\alpha \in \Gamma_0$. Since $\sigma_\alpha(\tilde{K}'_\alpha)$ intersects $\tilde{K}'_\alpha$ transversely, $\bar{\pi}'|_{K'}$ is not an embedding, which contradicts that $\bar{\pi}'|_{K'}$ is an embedding. Hence $\pi_\rho|_{K_i}$ is an embedding, for $i = 1, 2$.

We will show that $\Gamma_\rho < \Gamma_0 *_{\sigma^{k+1}}$ for some $k \in \mathbb{Z}_{\geq -1}$. By the construction of K, K_i lifts into $\mathbb{H}^3/\Gamma_i$ isometrically via the natural projection $\mathbb{H}^3/\Gamma_i \to N_\rho$, for $i = 1, 2$. Since $\hat{N} = \mathbb{H}^3/\Gamma_0 *_\sigma$ is constructed by gluing a part of $\mathbb{H}^3/\Gamma_0$ to itself as in the Klein-Maskit combination theorem II and the resulting joint in $\hat{N}$ is the once-punctured disk $\hat{A}_-$, which is disjoint from $\pi_\rho(K_i)$, $\pi_\rho(K_i)$ lifts into $\mathbb{H}^3/\Gamma_0$ isometrically via the natural projection $\mathbb{H}^3/\Gamma_0 \to \hat{N}$, for $i = 1, 2$. Since the restriction $\pi_\rho|_{K_i}$ is an isometry, by comparing these two lifts we see that there exists an element $f_i \in \Gamma_0 *_\sigma$ such that $\Gamma_i^{f_i} < \Gamma_0$, for $i = 1, 2$. Moreover, since the lift of $\overline{K_i} \cap \partial Q$ in $\mathbb{H}^3/\Gamma_i$ is an annulus which represents $\rho(\gamma) \in \Gamma_i$ and the lift of $\pi_\rho(\overline{K_i} \cap \partial Q)$ in $\mathbb{H}^3/\Gamma_0$ also represents $\rho(\gamma) \in \Gamma_0$, we see that $J^{f_i} = J$, for $i = 1, 2$. Thus for each $i = 1, 2$, we may assume that $f_i = \sigma^{a_i}$ for some $a_i \in \mathbb{Z}$. By replacing ${\Gamma_0}^{\sigma^{-a_2}}$ by Γ_0, we have $\Gamma_2 < \Gamma_0$ and ${\Gamma_1}^{\sigma^{(a_1-a_2)}} < \Gamma_0$. We have $a_1 - a_2 \geq 0$ by replacing σ^{-1} by σ if necessary, and set $k := a_1 - a_2 - 1$. Since ${\Gamma_1}^{\sigma^{k+1}} < \Gamma_0$ and $\Gamma_2 < \Gamma_0$, we have $\Gamma_\rho (= \Gamma_1 \underset{J}{*} \Gamma_2) < \Gamma_0 *_{\sigma^{k+1}}$. Let $\bar{\pi} : N_\rho \to \mathbb{H}^3/\Gamma_0 *_{\sigma^{k+1}}$ be the natural projection.

Let H be the horoball based at p which covers Q. Let $\tilde{K}_{0i} (i = 1, 2)$ and $\tilde{A}_+$ be the connected component of preimage of $\pi_\rho(K_i)$ and that of $\hat{A}_+$ via the natural projection $\mathbb{H}^3 \to \hat{N}$, respectively, which is contained in Δ_0 and which meets H. Since $\hat{A}_+$ lies between $\pi_\rho(K_1)$ and $\pi_\rho(K_2)$ around $\hat{T}$ (see Figure 3.3.1), one of components Δ_{01} of $\Delta_0 \setminus \tilde{A}_+$ contains $\tilde{K}_{01}$ and the other component Δ_{02} contains $\tilde{K}_{02}$. Since $\pi_1(\pi_\rho(K_1)) = {\Gamma_1}^{\sigma^{k+1}} < \Gamma_0$ and $\pi_1(\pi_\rho(K_2)) = \Gamma_2 < \Gamma_0$ acts on $\tilde{K}_{01}$ and on $\tilde{K}_{02}$ as translations, respectively, by the definition of limit set, we have $\Lambda({\Gamma_1}^{\sigma^{k+1}}) \subset \overline{\Delta_{01}}^{\mathbb{H}^3}$ and $\Lambda(\Gamma_2) \subset \overline{\Delta_{02}}^{\mathbb{H}^3}$.

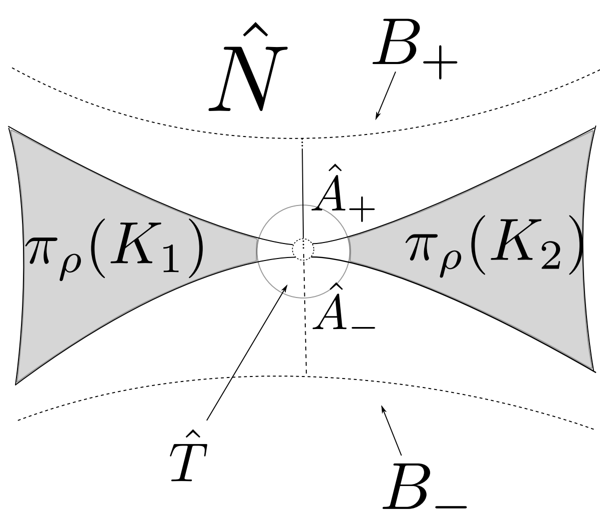

FIGURE 3.3.1. $\hat{N}$, $\pi_\rho(K_1)$ and $\pi_\rho(K_2)$ around $\hat{T}$

Set $\Delta_1 := \sigma^{-(k+1)}(\Delta_{01})$ and $\Delta_2 := \Delta_{02}$. We have $\Lambda(\Gamma_1) \subset \overline{\Delta_1}^{\mathbb{H}^3}$ and $\Lambda(\Gamma_2) \subset \overline{\Delta_2}^{\mathbb{H}^3}$. For each $i = 1, 2$, let C_i be the component of $\hat{\mathbb{C}} \setminus \overline{\Delta_i}^{\mathbb{H}^3}$ which is disjoint from Δ_{3-i}, and D_i the other component. We see that $\overline{C_i} \cap \Lambda(\Gamma_i) = \overline{D_i} \cap \Lambda(\Gamma_i) = p$, for $i = 1, 2$. See Figure 3.3.2.

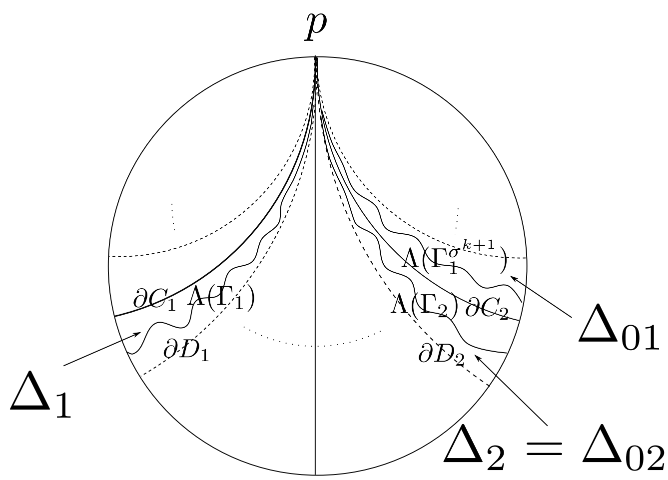

FIGURE 3.3.2. Δ_1, $\Delta_2 = \Delta_{02}$, and Δ_{01}

We will show that for each $i = 1, 2$, $\Lambda(\Gamma_i)$ separates C_i and D_i. We reorder the subgroups $\{\Gamma'_1, \ldots, \Gamma'_l\}$ of Γ_ρ stated above so that $J < \Gamma'_i < \Gamma_i$ for $i = 1, 2$. Since Γ'_i is either a generalized web group or a degenerate group without accidental parabolic elements, $\Lambda(\Gamma'_i)$ is connected. Considering the action of J on $\Lambda(\Gamma'_i)$, by the connectedness of $\Lambda(\Gamma'_i)$, we see that $\Lambda(\Gamma'_i)$, thus $\Lambda(\Gamma_i)$, separates C_i and D_i.

We will check that $k \geq 0$ and that $\sigma^{k+1}(C_1) = \hat{\mathbb{C}} \setminus \overline{C_2}$ and $\sigma^k(D_1) = \hat{\mathbb{C}} \setminus \overline{D_2}$. By the choice of $\hat{A}_-$, the preimage of $\hat{A}_-$ in N_ρ via π_ρ is a once-punctured disk A_- between K_1 and K_2. We assume that Δ_1 and Δ_2 are in the order of the σ-direction. K_1, A_- and K_2 are in this order, since π_ρ preserves this order, $\pi_\rho(K_1)$, $\hat{A}_- = \pi_\rho(A_-)$ and $\pi_\rho(K_2)$ are in this order. Thus $\pi_\rho(K_1)$, $\hat{A}_-$, $\pi_\rho(K_2)$ and $\hat{A}_+$ are cyclically in this order around $\hat{T}$, and $\tilde{K}_{02}$, $\tilde{A}_+$ and $\tilde{K}_{01}$ are in this order. In particular, Δ_{02} and Δ_{01} are in this order. Thus we see that $k \geq 0$ and that $\sigma^{k+1}(C_1) = \hat{\mathbb{C}} \setminus \overline{C_2}$ and $\sigma^k(D_1) = \hat{\mathbb{C}} \setminus \overline{D_2}$.

We will show that C_i is (Γ_i, J)-invariant. By construction, ∂C_i is $(\hat{\Gamma}, J)$-invariant, in particular (Γ_i, J)-invariant. Suppose that $f(C_i) \cap C_i \neq \emptyset$ for some $f \in \Gamma_i$. Since ∂C_i is (Γ_i, J)-invariant, we get $f(C_i) \subset C_i$ or $C_i \subset f(C_i)$. In the both cases, f has one fixed point on $\overline{C_i}$. Since $\overline{C_i} \cap \Lambda(\Gamma_i) = p$, this fixed point is p. Since $\hat{\Gamma}$ is discrete, by Proposition 2.1.3, f is the parabolic element fixing p. Since J is the maximal parabolic subgroup of Γ_ρ, we have $f \in J$. By the J-invariance of ∂C_i, C_i is (Γ_i, J)-invariant. Similarly, D_i is also (Γ_i, J)-invariant. Hence the open disks C_1, D_1, C_2 and D_2 satisfy the condition $\mathcal{DS}k$.

We will check that $k = n$. Set $\tilde{K}_1 := \sigma^{-(k+1)}(\tilde{K}_{01})$ and $\tilde{K}_2 := \tilde{K}_{02}$. We take a lift $\tilde{K}$ of K in $\mathbb{H}^3$ by connecting that of K_1 in $\tilde{K}_1$ and that of K_2 in $\tilde{K}_2$ by that of $K \cap Q$ in H, which lies between C_1 and C_2. We have $\sigma^{k+1}(\tilde{K} \cap H) \cap (\tilde{K} \cap H) = \emptyset$, and $K \cap Q$ is embedded in $\bar{\pi}(Q)$ via the natural projection $\bar{\pi} : N_\rho \to \mathbb{H}^3/\Gamma_0 *_{\sigma^{k+1}}$. Since $\pi_\rho(K_1)$ and $\pi_\rho(K_2)$ are disjointly embedded in $\hat{N} \setminus \mathrm{Int}\,\hat{T}$, K_1 and K_2 are disjointly embedded in $(\mathbb{H}^3/\Gamma_0 *_{\sigma^{k+1}}) \setminus \mathrm{Int}\,\bar{\pi}(Q)$ via $\bar{\pi}$. Thus K is embedded in $\mathbb{H}^3/\Gamma_0 *_{\sigma^{k+1}}$ via $\bar{\pi}$, and the condition $\mathcal{W}k$ holds. Since $\pi_\rho : N_\rho \to \hat{N}$ wraps n times with respect to γ, n is the smallest among such numbers by definition, and we get $k \geq n$. On the other hand, recall that N_ρ has a compact core K' such that the restriction $\bar{\pi}'|_{K'}$ is an embedding, where $\bar{\pi}' : N_\rho \to \mathbb{H}^3/\Gamma_0 *_{\sigma^{n+1}}$ is the natural projection. Let $\tilde{K}$ be a connected component of preimage of K in $\mathbb{H}^3$ which meets H. Since the two compact cores K and K' for N_ρ are homotopic to each other, $\tilde{K}$ is homotopic to a conected component $\tilde{K}'$ of preimage of K' in $\mathbb{H}^3$ without moving the boundary (at infinity). If $k > n$, then $\sigma^{n+1}(\tilde{K})$ intersects $\tilde{K}$ transversely, and then by the homotopy above $\sigma^{n+1}(\tilde{K}')$ intersects $\tilde{K}'$ transversely, which contradicts that $\bar{\pi}'|_{K'}$ is an embedding. Thus we get $k \leq n$. Together $k \geq n$, we have $k = n$. Therefore the open disks C_1, D_1, C_2 and D_2 satisfy the condition $\mathcal{DS}n$.

The case where γ is non-separating in S. Set $K' := K \setminus Q$. Similarly to the separating case, $\pi_\rho|_{K'}$ is an embedding.

We will show that $\Gamma_\rho < \Gamma_0 *_{\sigma^{k+1}\delta}$ for some $k \in \mathbb{Z}_{\geq -1}$. By the construction of K, K' lifts into $\mathbb{H}^3/\Gamma'$ isometrically via the natural projection $\mathbb{H}^3/\Gamma' \to N_\rho$, and the lift of $\overline{K'} \cap \partial Q$ in $\mathbb{H}^3/\Gamma'$ is a pair of two annuli one of whose components represents $\rho(\gamma) \in \Gamma'$ and the other component represents $\rho(\gamma)^{\delta^{-1}} \in \Gamma'$. Similarly to the separating case, $\pi_\rho(K')$ lifts into $\mathbb{H}^3/\Gamma_0$ isometrically via the natural projection $\mathbb{H}^3/\Gamma_0 \to \mathbb{H}^3/\Gamma_0 *_\sigma = \hat{N}$, and the both components of the lift of $\pi_\rho(\overline{K'} \cap \partial Q)$ in $\mathbb{H}^3/\Gamma_0$ represent $\rho(\gamma) \in \Gamma_0$. Thus there exists an element $f' \in \Gamma_0 *_\sigma$ such that $\Gamma'^{f'} < \Gamma_0$ and $J^{f'} = J$. We may assume that f' is a power of σ, and have $\Gamma' < \Gamma_0$ by replacing ${\Gamma_0}^{f'^{-1}}$ by Γ_0. Moreover, there exists an element $f \in \Gamma_0$ such that $\Gamma'^f < \Gamma_0$ and $(J^{\delta^{-1}})^f = J$. We may assume that $f = \sigma^a \delta$ for some $a \in \mathbb{Z}$. We have $a \geq 0$ by replacing σ^{-1} by σ if necessary, and set $k := a - 1$. Since $\Gamma' < \Gamma_0$ and $\sigma^{k+1}\delta \in \Gamma_0$, we have $\Gamma_\rho (= \Gamma' *_\delta) < \Gamma_0 *_{\sigma^{k+1}}$.

Similarly to the separating case, let H be the horoball based at p which covers Q and $\tilde{A}_+$ the connected component of preimage of $\hat{A}_+$ via the natural projection $\mathbb{H}^3 \to \hat{N}$ which is contained in Δ_0 and which meets H. Let $\tilde{K}'$ be the connected component of preimage of $\pi_\rho(K')$ in $\mathbb{H}^3$ such that $\tilde{K}'$ is contained in Δ_0 and the boundary (at infinity) of $\tilde{K}'$ contains p and $\delta^{-1}(p)$. Since $\sigma^{k+1}\delta \in \Gamma_0$, $\sigma^{k+1}\delta(\tilde{K}')$ is also a component of preimage of $\pi_\rho(K')$ which is contained in Δ_0 and whose boundary contains p. Since $\hat{A}_+$ lies between two parts of $\pi_\rho(K')$ around $\hat{T}$ (see Figure 3.3.3), one of components Δ of $\Delta_0 \setminus \tilde{A}_+$ contains $\tilde{K}'$ and the other component Δ_0' contains $\sigma^{k+1}\delta(\tilde{K}')$. Since $\pi_1(\pi_\rho(K')) = \Gamma' < \Gamma_0$ and $\pi_1(\pi_\rho(K')) = \Gamma'^{\sigma^{k+1}\delta} < \Gamma_0$ acts on $\tilde{K}'$ and on $\sigma^{k+1}\delta(\tilde{K}')$ as translations, respectively, by the definition of limit set, we have $\Lambda(\Gamma') \subset \overline{\Delta}^{\mathbb{H}^3}$ and $\Lambda(\Gamma'^{\sigma^{k+1}\delta}) \subset \overline{\Delta_0'}^{\mathbb{H}^3}$.

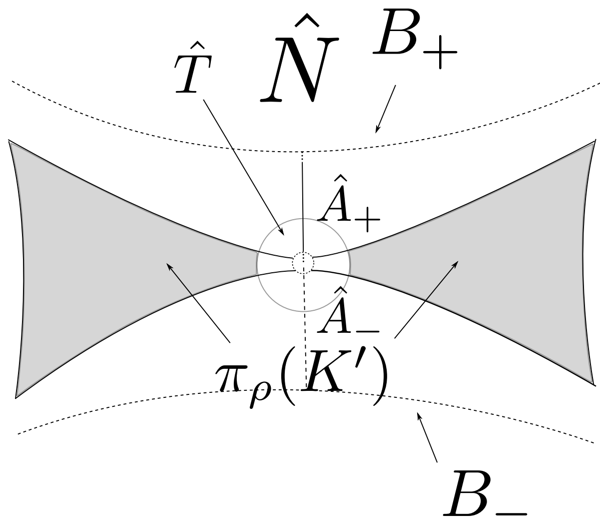

FIGURE 3.3.3. $\hat{N}$ and $\pi_\rho(K)$ around $\hat{T}$

Setting $\Delta' := \sigma^{-(k+1)}(\Delta'_0)$, we have $\Lambda(\Gamma'^{\delta}) \subset \overline{\Delta'}^{\mathbb{H}^3}$. Let C be the component of $\hat{\mathbb{C}} \setminus \overline{\Delta}^{\mathbb{H}^3}$ which is disjoint from Δ', and D the other component. Similarly, let C'^{δ} be the component of $\hat{\mathbb{C}} \setminus \overline{\Delta'}^{\mathbb{H}^3}$ which is disjoint from Δ, and D'^{δ} the other component, and set $C' := \delta^{-1}(C'^{\delta})$ and $D' := \delta^{-1}(D'^{\delta})$. See Figure 3.3.4.

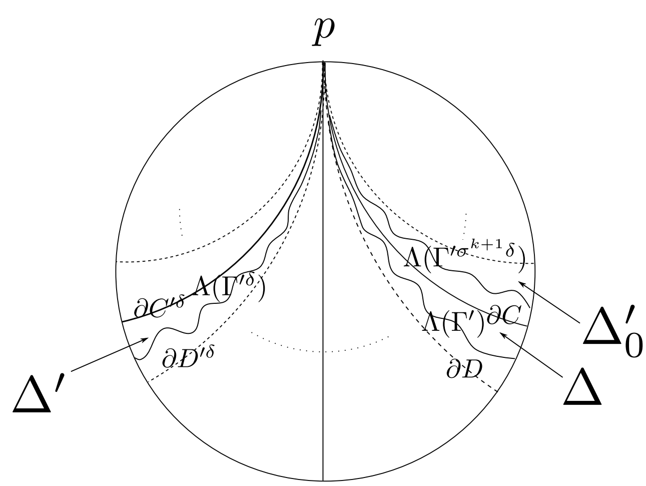

FIGURE 3.3.4. Δ', Δ and Δ'_0

By the same argument as that in the separating case, one can check that C and D are (Γ', J)-invariant open disks and C' and D' are $(\Gamma', J^{\delta^{-1}})$-invariant open disks and that $\sigma^{k+1}\delta(C') = \hat{\mathbb{C}} \setminus \overline{C}$, $\sigma^k\delta(D') = \hat{\mathbb{C}} \setminus \overline{D}$, $\Lambda(\Gamma')$ separates C from D, and C' from D', $\overline{C} \cap \Lambda(\Gamma') = \overline{D} \cap \Lambda(\Gamma') = p$, and $\overline{C'} \cap \Lambda(\Gamma') = \overline{D'} \cap \Lambda(\Gamma') = \delta^{-1}(p)$.

We will show the remaining conditions for the non-separating case, i.e. the last condition $(*)$ of $\mathcal{DN}n$. Suppose that $f(\overline{C}) \cap \overline{C'} \neq \emptyset$ for some $f \in \Gamma'$. Since $f(\partial C) \cap \partial C' = \emptyset$ by construction, we get $f(C) \subset C'$ or $C' \subset f(C)$. Since $\overline{f(C)} \cap \Lambda(\Gamma') = f(p)$, $\overline{C'} \cap \Lambda(\Gamma') = \delta^{-1}(p)$, $f(C) \cap \Lambda(\Gamma') = \emptyset$, and $C' \cap \Lambda(\Gamma') = \emptyset$, we have $f(\partial C) \ni f(p) = \delta^{-1}(p) \in \partial C'$, which contradicts $f(\partial C) \cap \partial C' = \emptyset$. Hence together with $f(\partial C) \cap \partial C' = \emptyset$, we get $f(\overline{C}) \cap \overline{C'} = \emptyset$ for any $f \in \Gamma'$. Similarly, we get $f(\overline{C}) \cap \overline{D'} = \emptyset$, $f(\overline{D}) \cap \overline{C'} = \emptyset$, and $f(\overline{D}) \cap \overline{D'} = \emptyset$, for any $f \in \Gamma'$. Hence the open disks C, D, C' and D' satisfy the condition $\mathcal{DN}k$.

By the same argument as that in the separating case, one can check that $k = n$. Therefore the open disks C, D, C' and D' satisfy the condition $\mathcal{DN}n$.

Hence in the both cases, we have got the desired open disks, and we complete the proof of Theorem 3.1.1. ☐

3.4. **An example.** In this subsection, we give an example. We suppose that $\rho \in AH(S)$ is geometrically finite, thus ρ has only geometrically finite ends, and that ρ has only one annulus cusp neighborhood, say Q. Let γ be the accidental parabolic curve corresponding to Q.

We suppose that γ is separating in S, and γ divides S into two subsurfaces S_1 and S_2. Taking the base point at γ in S, we consider $\pi_1(S) = \pi_1(S_1) \underset{\langle\gamma\rangle}{*} \pi_1(S_2)$. We choose a representative representation ρ so that $\rho(\gamma) = \begin{pmatrix} 1 & \sqrt{-1} \\ 0 & 1 \end{pmatrix}$. For $i = 1, 2$, set $\Gamma_i := \rho(\pi_1(S_i))$ and $\rho_i := \rho|_{\pi_1(S_i)}$, and set $J := \langle\rho(\gamma)\rangle$.

Recall that $\Gamma_\rho = \rho(\pi_1(S))$ and that $N_\rho = \mathbb{H}^3/\Gamma_\rho$. Let Ω_i be the component of $\Omega(\Gamma_\rho)$ which covers the component of $\partial_c N_\rho$ corresponding to S_i and $\infty \in \overline{\Omega_i}$. Replacing the numbers $i = 1, 2$ if necessary, we set $C_1 := \{z \in \mathbb{C} | \operatorname{Re} z < c_1\} \subset \Omega_1$ and $C_2 := \{z \in \mathbb{C} | \operatorname{Re} z > c_2\} \subset \Omega_2$ for some $c_1, c_2 \in \mathbb{R}$. See Figure 3.4.1. We choose c_i so that C_i is (Γ_i, J)-invariant.

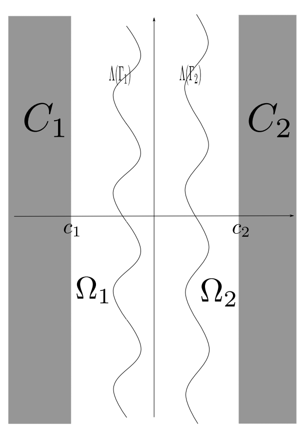

FIGURE 3.4.1. Parts of C_1 and C_2 with the real and imaginary axes

Let $\tilde{\gamma}$ be an axis in $\Omega(\Gamma_\rho)$ for the accidental parabolic element $\rho(\gamma)$. Let D_i be the component of $\hat{\mathbb{C}} \setminus (\tilde{\gamma} \cup \{\infty\})$ containing Ω_{3-i}. See Figure 3.4.2. As in Subsection 3.3, we can check that D_i is (Γ_i, J)-invariant.

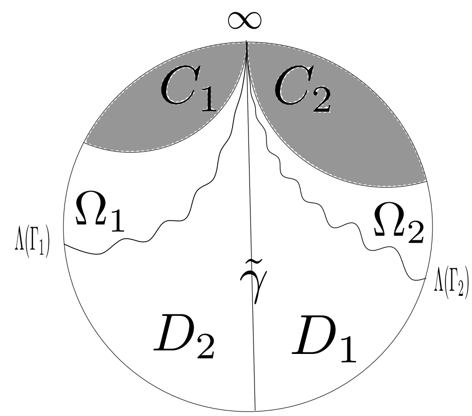

FIGURE 3.4.2. The division of $\hat{\mathbb{C}}$ along $\tilde{\gamma} \cup \{\infty\}$ with D_1 and D_2

Fix any $t \geq 0$. We define the parabolic element σ_t by $\sigma_t = \begin{pmatrix} 1 & (c_2 - c_1)t \\ 0 & 1 \end{pmatrix}$. Since $\sigma_t(D_2)$ is $({\Gamma_2}^{\sigma_t}, J)$-invariant and $\hat{\mathbb{C}} \setminus \overline{\sigma_t(D_2)} (\subset D_1)$ is (Γ_1, J)-invariant, the Klein-Maskit combination theorem I implies that $\Gamma_1 \underset{J}{*} {\Gamma_2}^{\sigma_t}$ is discrete.

Since $\rho_1(\gamma) = \rho(\gamma) = \rho_2(\gamma)^{\sigma_t}$, we can construct an isomorphism

$$\rho^t : \pi_1(S) = \pi_1(S_1) \underset{\langle\gamma\rangle}{*} \pi_1(S_2) \to \Gamma_1 \underset{J}{*} {\Gamma_2}^{\sigma_t}$$

from the isomorphisms $\rho_1 : \pi_1(S_1) \to \Gamma_1$ and ${\rho_2}^{\sigma_t} : \pi_1(S_2) \to {\Gamma_2}^{\sigma_t}$. Since $\rho^t(\pi_1(S)) = \Gamma_1 \underset{J}{*} {\Gamma_2}^{\sigma_t}$ is discrete, $\rho^t \in AH(S)$.

Let $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$ be the floor function, that is, $\lfloor x \rfloor$ is the largest integer less than or equal to x. Set

$$C_2^t := \begin{cases} \sigma_{t(1+\frac{1}{\lfloor t \rfloor})-1}(C_2), & \text{if } t \geq 1, \\ \sigma_t(C_2), & \text{if } 1 > t \geq 0. \end{cases}$$

Since $C_2^t \subset \sigma_t(C_2)$, C_2^t is $({\Gamma_2}^{\sigma_t}, J)$-invariant. Set

$$D_2^t := \begin{cases} \sigma_t(D_2), & \text{if } t \geq 1, \\ D_2, & \text{if } 1 > t \geq 0. \end{cases}$$

Since $D_2^t \subset \sigma_t(D_2)$, D_2^t is $({\Gamma_2}^{\sigma_t}, J)$-invariant. Let

$$\sigma^t := \begin{cases} \sigma_{\frac{\lfloor t \rfloor}{t}}, & \text{if } t \geq 1, \\ \sigma_{t+1}, & \text{if } 1 > t \geq 0. \end{cases}$$

We can check that our theorem applies to ρ^t, σ^t, C_1, D_1, C_2^t and D_2^t. Thus for any $t \geq 1$, ρ^t has a $\lfloor t \rfloor$-wrapping projection.

Since convergence in $AH(S)$ are defined by the algebraic convergence, $\mathbb{R}_{\geq 0} \ni t \mapsto \rho^t \in AH(S)$ is continuous.

Therefore we have got the following corollary.

Corollary 3.4.1. *There exists a continuous family $\rho^t \in AH(S)$, $t \geq 1$ such that ρ^t has a $\lfloor t \rfloor$-wrapping projection.*

Moreover, since ρ^t is geometrically finite by construction, in the same way as that in [14], one can check that for any $t \geq 1$, $AH(S)$ self-bumps at ρ^t. Thus we get a curve in $AH(S)$ consisting of self-bumping points.

4. Remarks

Finally, we give two remarks for Theorem 3.1.1.

In this thesis, we proved Theorem 3.1.1 in only the case where $M = S \times I$, S is a closed surface and I is an interval, which is the only case that there exists detailed research for geometric limits in [40], and for some kind of geometric limits.

Even in that case, it is known from [40], that some geometric limits can have very complicated topological structures. It is interesting to consider natural projections to such geometric limits.

In more general case, for example, the case where M is a compact orientable hyperbolizable 3-manifold whose boundaries components are all incompressible, we can expect that the similar results hold, if the analogy of Ohshika and Soma's works in [40] is established.

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