博士論文

Continuous Analogue of the Existence Problem of Compact Clifford–Klein Forms
(コンパクトClifford–Klein形の存在問題の連続類似)

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Acknowledgement

My heartfelt appreciation goes to my supervisor Taro Yosino. He gives me constructive comments and warm encouragement during the preparation of this thesis. This thesis could not exist without his support.

I wish to express my thanks to Toshiyuki Kobayashi for giving me insightful advice. I am greatly indebted to Takayuki Okuda for his collaboration in proving the non-existence for compact standard Clifford–Klein forms for the case $G/H = E_{6(-14)}/F_{4(-20)}$ in Chapter 1.

Discussions of Clifford–Klein forms and representation theory with Yoshinori Tanimura, Yosuke Morita, Yuichiro Tanaka, Keichi Maeta, Yuu Sunahara, Masaaki Tsuchiya, Mizuki Arata, Ryosuke Nakahama, Masatoshi Kitagawa, Bent Ørsted, Hideyuki Ishi, Hiroshi Tamaru and Atsumu Sasaki have been illuminating.

This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

Contents

Chapter 1

Classification of irreducible symmetric spaces admitting compact standard Clifford–Klein forms

Abstract

We give a complete classification of irreducible symmetric spaces which admit standard compact Clifford–Klein forms by using representation theory, embeddability of semisimple Lie algebras into simple Lie algebras and the criterion for proper action on homogeneous space of reductive type by T. Kobayashi.

1.1 Introduction

1.1.1 Background

A Clifford–Klein form is a double coset space $\Gamma\backslash G/H$ equipped with a manifold structure, where G is a Lie group, H is a closed subgroup of G and Γ is a discontinuous group for G/H (see Definition 1.2.1 for more details). For example, symmetric spaces, Klein's bottle and compact Riemannian surfaces are Clifford–Klein forms. It is important to consider the existence of compact Clifford–Klein forms in this field,

Problem A ([Ko89, Ko96b])**.** Which homogeneous space G/H admits a compact Clifford–Klein form?

A special case of Problem A includes:

Fact 1.1.1 ([Bo63])**.** Let G be a linear reductive Lie group and H a compact subgroup of G. Then G/H always admits compact Clifford–Klein forms.

Opposite extremal case occurs if H is non-compact:

Fact 1.1.2 ([CM62])**.** Let $\Gamma\backslash O(n+1,1)/O(n,1)$ $(n \geq 2)$ be a Clifford–Klein form. Then Γ is a finite group. In particular, $O(n+1,1)/O(n,1)$ never admits compact Clifford–Klein forms.

On the other hand, in the case where H is non-compact, Kulkarni found examples admitting compact Clifford–Klein forms:

Fact 1.1.3 ([Ku81])**.** Homogeneous spaces $SO(2,2n)/SO(1,2n)$ and $SO(4,4n)/SO(3,4n)$ $(n \geq 1)$ admit compact Clifford–Klein forms.

These results are for specific homogeneous spaces, but in 1980's, a systematic study of the existence problem of compact Clifford Klein forms was started by T. Kobayashi, which deals with a wide class of homogeneous spaces containing pseudo-Riemannian symmetric space. His breakthrough on the problem is to introduce "continuous analogue" of discontinuous group and to give a sufficient condition for the existence of compact Clifford–Klein forms:

Fact 1.1.4 ([Ko89, Theorem 4.7])**.** A homogeneous space G/H of reductive type has a compact Clifford–Klein form $\Gamma\backslash G/H$ if it admits a reductive subgroup L of G whose natural action on G/H is proper and cocompact.

Then, Problem A is trivial for Riemannian symmetric spaces and group manifolds.

Remark 1.1.5 (Trivial case)**.** A group manifold $(G' \times G')/\operatorname{diag}_\tau G'$ admits a compact Clifford–Klein form, where we put $\operatorname{diag}_\tau G' := \{(g, \tau(g)) : g \in G'\} \subset G' \times G'$ for an involution τ on G'.

As other examples, T. Kobayashi found 12 series of irreducible symmetric spaces which admit compact Clifford–Klein forms.

Fact 1.1.6 ([KY05, Corollary 3.3.7])**.** Symmetric spaces in the following Table 1.1 admit compact Clifford–Klein forms. Here $n = 1, 2, \cdots$.

Table 1.1: Symmetric spaces which admit compact Clifford–Klein forms.

	G/H	L
1	$SU(2,2n)/Sp(1,n)$	$U(1,2n)$
2	$SU(2,2n)/U(1,2n)$	$Sp(1,n)$
3	$SO(2,2n)/U(1,n)$	$SO(1,2n)$
4	$SO(2,2n)/SO(1,2n)$	$U(1,n)$
5	$SO(4,4n)/SO(3,4n)$	$Sp(1,n)$
6	$SO(4,4)/SO(4,1) \times SO(3)$	$Spin(4,3)$
7	$SO(4,3)/SO(4,1) \times SO(2)$	$G_{2(2)}$
8	$SO(8,8)/SO(7,8)$	$Spin(1,8)$
9	$SO(8,\mathbb{C})/SO(7,\mathbb{C})$	$Spin(1,7)$
10	$SO(8,\mathbb{C})/SO(7,1)$	$Spin(7,\mathbb{C})$
11	$SO^*(8)/U(3,1)$	$Spin(1,6)$
12	$SO^*(8)/SO^*(6) \times SO^*(2)$	$Spin(1,6)$

Remark 1.1.7. $SO(6,2)/U(3,1)$, $SO^*(8)/U(3,1)$ and $SO^*(8)/SO^*(6)\times SO^*(2)$ are infinitesimally isomorphic.

So far, Problem A has been attacked by not only T. Kobayashi but also many other mathematicians. However, Problem A is not yet solved completely even for irreducible symmetric spaces, which was classified by M. Berger [Br57].

1.1.2 Continuous analogue of Problem A and main theorem

We want to consider a continuous analogue of Problem A based on Fact 1.1.4.

Problem B. Classify homogeneous spaces G/H of reductive type which have a reductive subgroup L of G acting properly and cocompactly.

In this paper, we give a solution to this problem for irreducible symmetric spaces:

Theorem 1.1.8. Let G/H be a noncompact irreducible symmetric space where G is the group of displacements. Suppose that G/H admits a reductive subgroup L of G acting on G/H properly and cocompactly. Then G/H is locally isomorphic as a symmetric space to one of the following list:

- (trivial case) a Riemannian symmetric space,
- (trivial case) a group manifold,
- a symmetric space in Table 1.1.

Here, recall that T. Kobayashi gave the following:

Conjecture 1.1.9 ([KY05, Conjecture 3.3.10])**.** Let G/H be a homogeneous space of reductive type. If G/H admits a compact Clifford–Klein form, then G/H admits a reductive subgroup L of G acting on G/H properly and cocompactly.

No counter example to Conjecture 1.1.9 has been known as of now. Evidence of Conjecture 1.1.9 includes the nonexistence theorems of compact Clifford–Klein forms in various settings proved by K. Ono [KO90], R.J Zimmer [Z94], R. Lipsman [Li95], Y. Benoist [B96], F. Labourie, S. Mozes [LMZ95], G.A. Margulis [Ma97], H. Oh, D. Witte [OW00], T. Yoshino [KY05], Y. Morita [M15].

If this conjecture is true, then we complete the classification of irreducible symmetric spaces admitting compact Clifford–Klein forms from Theorem 1.1.8.

Note that Theorem 1.1.8 only claims the existence of L. We also want to classify such reductive subgroup L.

Problem C. Suppose G/H is locally isomorphic as a symmetric space to a symmetric space in Table 1.1. Classify a reductive subgroup L of G acting on G/H properly and cocompactly.

From the following Fact 1.1.10 , it is natural to deal with group manifolds in Problem C.

Fact 1.1.10 ([Fl86, Theorem 2(iv)])**.** Let X be a symmetric space and $G = G(X)$ its group of displacements. Then X is irreducible if and only if either X has dimension one or $\mathfrak{g}$ is simple or $\mathfrak{g}$ is the direct sum of two isomorphic simple ideals, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1$, and $\sigma(X, Y) = (Y, X)$ for all $X, Y \in \mathfrak{g}_1$.

Remark 1.1.11. The classification of reductive subgroups acting properly and cocompacly on group manifolds is more difficult than the case where G is simple.

We also give a solution to Problem C.

Theorem 1.1.12. Suppose G/H is locally isomorphic to a symmetric space in Table 1.1 except for $\mathrm{Lie}(G) \simeq \mathfrak{so}(2,2)$. Let L' be a reductive subgroup of G acting on G/H properly and cocompactly. Then L' is locally isomorphic to the corresponding L "up to compact factor".

See Definition 1.2.25 for the definition of "up to compact factor". Moreover, we will give all the embeddings of $\mathfrak{l} \subset \mathfrak{g}$ up to $\mathrm{Int}(\mathfrak{g})$ in Section 1.5.

Remark 1.1.13. If $\mathrm{Lie}(G)$ is isomorphic to $\mathfrak{so}(2,2)$, then G/H is a group manifold.

1.1.3 Observation on the results

Remark that symmetric spaces in Table 1.1 have the following good properties:

Observation 1. Suppose $(G/H, L)$ is in Table 1.1. Then the following conditions are satisfied:

- $\mathrm{rank}_{\mathbb{R}} H = 1$ or $\mathrm{rank}_{\mathbb{R}} L = 1$,
- $\mathrm{rank}_{\mathbb{R}} H + \mathrm{rank}_{\mathbb{R}} L = \mathrm{rank}_{\mathbb{R}} G$.

The author does not know a direct proof of these conditions. However, the proof of Theorem 1.1.8 could be simpler, if one has shown these conditions directly for $(G/H,\ L)$ satisfying the assumption of Theorem 1.1.8.

1.2 Preliminary, setting and strategy

1.2.1 Clifford–Klein form

In this subsection, we prepare terminology for our problem and definition of Clifford–Klein forms of reductive type.

Proposition and Definition 1.2.1 (Clifford–Klein form (See [Ko96b, §0] for more details.))**.** Let G be a Lie group and H a closed subgroup of G. Suppose a discrete subgroup Γ of G acts on G/H properly discontinuously and freely, then the quotient space $\Gamma\backslash G/H$ has the natural manifold structure such that $G/H \to \Gamma\backslash G/H$ is a C^∞-covering map. We call the manifold $\Gamma\backslash G/H$ *Clifford–Klein form* of G/H and the discrete subgroup Γ a *discontinuous group*.

In Conjecture 1.1.9, T. Kobayashi assumed that G/H is of reductive type and subgroup acting on G/H properly and cocompactly is a reductive in G. So, let us recall:

Definition 1.2.2 ([Ko89])**.** Let G be a linear reductive Lie group and H a reductive subgroup of G. We say the homogeneous space G/H is *of reductive type*.

Throughout this paper, we shall work in the following:

Setting 1. G is a linear reductive Lie group. H and L are reductive subgroups of G.

Let us recall the definition of "linear reductive" and "reductive subgroup".

Definition 1.2.3 (linear reductive Lie group, See [Ko89] for more details)**.** Let G be a Lie group. We say G is a *linear reductive* Lie group if G is contained a connected complex reductive Lie group $G_{\mathbb{C}}$ with Lie algebra isomorphism $\mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{Lie}(G_{\mathbb{C}})$ (see Definition [Ko89]). Then G has a global Cartan involution θ. We call the dimension of $\dim G/K = \dim \mathfrak{g}^{-\theta}$ *noncompact dimension* of G, which is denoted by $d(G)$.

Definition 1.2.4 (reductive subgroup [Ko89])**.** Let G be a linear reductive Lie group and H a closed subgroup of G. We say H is *reductive in G* or a *reductive subgroup* of G if there exists a Cartan involution θ on G such that $\theta(H) = H$ and H has finitely many connected components.

Definition 1.2.5 (reductive Lie subalgebra)**.** Let G be a linear reductive Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{l}$ be a subalgebra of $\mathfrak{g}$. We say $\mathfrak{l}$ is a *reductive subalgebra* of $\mathfrak{g}$ if there exists a Cartan involution $\tilde{\theta}$ on G such that $d\tilde{\theta}(\mathfrak{l}) = \mathfrak{l}$. By $d(\mathfrak{l})$ we denote its noncompact dimension $\dim \mathfrak{l}^{-d\tilde{\theta}}$ of $\mathfrak{l}$.

Remark 1.2.6. Let G be a connected linear reductive Lie group and L a connected closed subgroup of G. Then we have

$$L \text{ is reductive in } G \iff \mathfrak{l} \text{ is a reductive subalgebra of } \mathfrak{g}.$$

Definition 1.2.7 (standard Clifford–Klein form [KK16])**.** Let G/H be a homogeneous space of reductive type and Γ a discontinuous group for G/H. A Clifford–Klein form $\Gamma\backslash G/H$ is called *standard* if there exists a reductive subgroup containing Γ and acting on G/H properly.

Remark 1.2.8. Problem B is equivalent to the classification of irreducible symmetric spaces admitting standard compact Clifford–Klein forms.

Remark 1.2.9 ([KY05, Remark 3.3.11])**.** There exist non-standard compact Clifford–Klein forms.

1.2.2 Kobayashi's criterion for proper action

In this subsection, we recall the criterion for proper action and cocompactness by T. Kobayashi.

Definition 1.2.10. We call the action of L on G/H is proper if the following subset $L_S \subset L$ is compact for all compact subsets $S \subset G/H$.

$$L_S := \{\ell \in L : \ell S \cap S \neq \emptyset\}.$$

Remark 1.2.11. For discrete subgroup $\Gamma \subset G$, Γ action is propely discontinuous if and only if the action is proper.

Let us recall the definition of "proper in G" and "similar in G" to describe useful criterion for proper action.

Definition 1.2.12 ([Ko96a])**.** We say the pair $(H,\, L)$

- is proper in G, denoted by $H \pitchfork L$ in G if For any compact subset S of G, $SHS^{-1} \cap L$ is relatively compact.
- is similar in G, denoted by $H \sim L$ in G if there exists a compact subset of G such that $L \subset SHS^{-1}$ and $H \subset SLS^{-1}$.

Fact 1.2.13 ([Ko96a])**.** Let G be a Lie group and L, L' and H closed subgroup of G. Then we have

$$\begin{aligned} L\text{-action on } G/H \text{ is proper } &\iff L \pitchfork H \text{ in } G, \\ \text{If } L \sim L' \text{ in } G, \text{ then } L \pitchfork H \text{ in } G &\iff L' \pitchfork H \text{ in } G. \end{aligned}$$

Remark 1.2.14. Let L be a closed subgroup of G. Let L' be a closed subgroup of L such that $L' \subset L$. If L-action on G/H is proper, then so is L'-action.

Fact 1.2.15 ([Ko89])**.** In Setting 1, we fix a Cartan involution on G and fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Take Cartan involutions θ_1 and θ_2 on G such that $\theta_1(H) = H$ and $\theta_2(L) = L$. Take maximal abelian subspaces $\mathfrak{a}'_H \subset \mathfrak{h}^{-\theta_1}$ and $\mathfrak{a}'_L \subset \mathfrak{l}^{-\theta_2}$. Then we can and do take $\alpha_1, \alpha_2 \in \mathrm{Int}(\mathfrak{g})$ such that $\mathfrak{a}_H := \alpha(\mathfrak{a}'_H), \mathfrak{a}_L := \alpha_2(\mathfrak{a}'_L) \subset \mathfrak{a}$. Then the following conditions are equivalent:

(i) the natural action of L on G/H is proper,

(ii) $W\mathfrak{a}_H \cap \mathfrak{a}_L = \{0\}$.

Here $W = W(\mathfrak{g}, \mathfrak{a})$ is the Weyl group coming from the restricted root system of $\mathfrak{g}$ with maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$.

From Fact 1.2.15, we obtain the following:

Remark 1.2.16. In Setting 1, if L-action on G/H is proper, then the following inequality holds:

$$\mathrm{rank}_{\mathbb{R}} L + \mathrm{rank}_{\mathbb{R}} H \leq \mathrm{rank}_{\mathbb{R}} G.$$

Remark 1.2.17. From Fact 1.2.15, in Setting 1, we can consider the properness in Lie algebra level.

Fact 1.2.18 ([Ko89, Theorem 4.7])**.** In Setting 1, under the assumption that L-action on G/H is proper, the following conditions on G, H and L is equivalent:

(i) $L\backslash G/H$ is compact,

(ii) $d(G) = d(L) + d(H)$.

1.2.3 Exact formulation of Problem C

Problem C has non-essential parts. So we elliminate them, namely

(i) conjugate for L (see Definition 1.2.21),

(ii) compact factor for L (see Definition 1.2.25).

Eventually, we reach the following:

Problem C'. Suppose G/H is locally isomorphic as a symmetric space to one in Table 1.1. Classify a reductive subgroup L of G acting on G/H properly and cocompactly up to conjugate and compact factor.

■ **conjugate**

Remark 1.2.19. Let Γ be a discontinuous group for G/H. Take $g \in G$. Then $\Gamma\backslash G/H$ and $g\Gamma g^{-1}\backslash G/H$ are differmorphic as a $(G, G/H)$-manifold.

Definition 1.2.20 ((G,G/H)-structure, see [LMZ95, M15] for example)**.** A manifold M is said to be locally modelled on a homogeneous space of G/H or said to be $(G, G/H)$-structure, if it is covered by open sets that are diffeomorphic to open sets of G/H and the transition functions are locally given by transitions by elements of G satisfying the cocycle condition.

From the above remark, we introduce the following equivalent relation on reductive subgroup of G:

Definition 1.2.21. Let L_1, L_2 be reductive subgroups of G. We denote by $L_1 \sim_{conj} L_2$ in G if there exists an element $g \in G_0$ such that $g(L_1)_0 g^{-1} = (L_2)_0$.

■ **compact factor**

For the problem for the existence of a reductive subgroup L acting on G/H properly and cocompacly, "compact factor" of L is not essential. Therefore we introduce a equivalence class in reductive subgroups of G which preserves properness and cocompactness.

Fix a linear reductive Lie group G. Put $\mathfrak{g} = \mathrm{Lie}(G)$.

Definition 1.2.22. Let L_1, L_2 be reductive subgroups of G. We denote by $L_1 <_c L_2$ in G if there exists a compact subgroup K of G satisfying the following two conditions:

(i) $K \subset Z_G\left((L_1)_0\right)$,

(ii) $K(L_1)_0 = (L_2)_0$.

Remark 1.2.23. A compact subgroup of a linear reductive Lie group is reductive in G.

Remark 1.2.24. The relation "$<_c$" in reductive subgroups of G is pre-order. Namely the following conditions hold:

(i) $L_1 <_c L_1$,

(ii) $L_1 <_c L_2$, $L_2 <_c L_3 \implies L_1 <_c L_3$.

However "$L_1 <_c L_2$ and $L_2 <_c L_1 \implies L_1 = L_2$" does not hold.

Proof. (i) is clear by definition. We show (ii). There exist compact reductive subgroups K_1 and K_2 and such that

$$K_1(L_1)_0 = (L_2)_0, K_1 \subset Z_G\left((L_1)_0\right),$$
$$K_2(L_2)_0 = (L_3)_0, K_2 \subset Z_G\left((L_2)_0\right).$$

Put $K_3 = K_2K_1$. It is enough to show that K_3 satisfies $K_3 \subset Z_G((L_1)_0)$.These comes from $K_1 \subset Z_G((L_1)_0)$ and $K_2 \subset Z_G((L_2)_0) = Z_G(K_1(L_1)_0)$. $\square$

Definition 1.2.25. Let $\sim_c$ be the equivalence relation of reductive subgroups of G generated by the pre-order $<_c$. We denote by $L_1 \sim_c L_2$ in G if L_1 is equivalent to L_2 in the sense of $\sim_c$.

The equivalence relation "$\sim_c$" can be described in Lie algebra level.

Definition 1.2.26. Let $\mathfrak{l}_1$, $\mathfrak{l}_2$ be reductive subalgebras of $\mathfrak{g}$. We denote by $\mathfrak{l}_1 <_c \mathfrak{l}_2$ in $\mathfrak{g}$ if there exists a compact Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ and satisfying the following two conditions:

- $[\mathfrak{k}, \mathfrak{l}_1] = 0$,
- $\mathfrak{k} \oplus \mathfrak{l}_1 = \mathfrak{l}_2$.

This is pre-order in reductive Lie subalgebras of $\mathfrak{g}$. Let $\sim_c$ be the equivalence relation of reductive subalgebras of $\mathfrak{g}$ generated by the pre-order $<_c$. We denote by $\mathfrak{l}_1 \sim_c \mathfrak{l}_2$ in $\mathfrak{g}$ if $\mathfrak{l}_1$ is equivalent to $\mathfrak{l}_2$ in the sense of $\sim_c$.

Proposition 1.2.27. For reductive subgroups L_1, L_2 of a linear reductive Lie group G and their Lie algebras $\mathfrak{l}_1 := \mathrm{Lie}(L_1)$, $\mathfrak{l}_2 := \mathrm{Lie}(L_2)$, the following conditions are equivalent:

(i) $L_1 \sim_c L_2$ in G,

(ii) $\mathfrak{l}_1 \sim_c \mathfrak{l}_2$ in $\mathfrak{g}$.

Proof. It is enough to show that $L_1 <_c L_2$ in $G \iff \mathfrak{l}_1 <_c \mathfrak{l}_2$ in $\mathfrak{g}$. This is clear by definition. $\square$

Remark 1.2.28. For reductive subgroups L_1 and L_2 of G, $d(L_1) = d(L_2)$ and $L_1 \sim L_2$ in G holds if $L_1 \sim_c L_2$ in G.

1.2.4 Strategy of our proof

Our proof depends on the following three theories:

(i) representation theory of semisimple Lie algebras,

(ii) criterion for embeddability of semisimple Lie algebras into simple Lie algebras,

(iii) criterion for proper action on homogeneous spaces of reductive type.

Conditions of our problem can be described in terms of representation of Lie algebra. From (i), we can parameterize representations of semisimple Lie algebras in terms of highest weight and calculate the dimensions of the representations. From (ii), we can reduce the candidates of reductive subgroups L by the criterion for embedding semisimple Lie algebras into simple Lie algebras. From (iii), we can determine whether or not the corresponding subgroup L acts on G/H properly in Lie algebra level.

1.2.5 Methods and key idea

In this section, we see our strategy to prove Theorem 1.1.8 and 1.1.12 and prepare methods and some lemmas used in common by the following proofs in Section 1.3 and 1.4. Our methods depending on a representation theory of Lie algebras works for both Problem B and C', but for Problem C', we investigate Lie subalgebra more precisely after classification in the level of representations of Lie algebras.

We overview our strategy for Problem B and C' step by step.

step 1 We work on the classification of symmetric pair by M. Berger. We reduce candidates of symmetric spaces G/H by using a necessary condition to admit compact standard Clifford–Klein forms that the corresponding tangential symmetric space G_θ/H_θ admits compact Clifford–Klein forms. In this step, we reduce candidates as shown in Table 1.4.

step 2 We exclude the cases $(G,\, H)$=$(Sp(2n,\mathbb{R}),\, Sp(n,\mathbb{C}))$ $(n \geq 2)$ in Table 1.4 by Fact 1.2.35.

step 3 By step 2, candidates of symmetric pair $(G,\, H)$ are classical types or $(E_{6(-14)}, F_{4(-20)})$. We consider the classical types here (we use other methods for $(E_{6(-14)}, F_{4(-20)})$. Then we describe conditions for a existence of reductive subgroups acting properly and cocompactly in terms of a existence of representations ρ of Lie algebras $\mathfrak{l} = \mathrm{Lie}(L)$.

step 4 For each candidate G/H, we obtain upper bound of dimension of representation ρ coming from Fact 1.2.18 and Remark 1.2.16. By using Weyl's dimensionality formula, we obtain finite number of candidates of representation of "primary simple factor" (see 1.2.31 for the definition).

step 5 We reduce candidates by using criterion for the embeddability of semisimple Lie algebras into simple Lie algebras and properness.

step 6 For each pair of a Lie algebras and its representation $(\mathfrak{l}, \rho)$ which induces proper and cocompact L-action, we determine which images $\rho(\mathfrak{l})$ are conjugate by $\mathrm{Int}(\mathfrak{g})$ (see Section 1.5).

key idea in step 4 and 5

In this subsection, we see the key idea (algorithm) to determine representations of Lie algebras which induce reductive subgroups acting on G/H properly and cocompactly, and prepare lemmas used in step 4 and 5.

We introduce an equivalent relation in pairs of a Lie algebra and its representation as follows:

Definition 1.2.29. Let $(\mathfrak{l}, \rho)$ and $(\mathfrak{l}', \rho')$ be pairs of Lie algebras and their representations. We say $(\mathfrak{l}, \rho)$ is equivalent to $(\mathfrak{l}', \rho')$ if there exists a Lie algebra isomorphism $\varphi : \mathfrak{l} \to \mathfrak{l}'$ such that ρ is equivalent to $\rho'\varphi$ as a representation of $\mathfrak{l}$.

In the case where G/H with $\mathrm{rank}_{\mathbb{R}}\, G - \mathrm{rank}_{\mathbb{R}}\, H = 1$, which is easier case than the case $\mathrm{rank}_{\mathbb{R}}\, G - \mathrm{rank}_{\mathbb{R}}\, H \geq 2$, it is enough to consider simple Lie algebras with real rank one such as $\mathfrak{so}(k,1)$, $\mathfrak{su}(k,1)$, $\mathfrak{sp}(k,1)$ and $f_{4(-20)}$ by Remark 1.2.16 and the following Table 1.2.

Table 1.2: Simple Lie algebras with $\mathrm{rank}_{\mathbb{R}}\, L = 1$ and their noncompact dimensions

$\mathfrak{l}$	$d(L)$
$\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{sp}(1,\mathbb{R}) \simeq \mathfrak{su}(1,1)$	2
$\mathfrak{sp}(1,\mathbb{C}) \simeq \mathfrak{o}(3,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{o}(3,1)$	3
$\mathfrak{su}^*(4) \simeq \mathfrak{o}(1,5)$	5
$\mathfrak{so}^*(6) \simeq \mathfrak{su}(1,3)$	6
$\mathfrak{so}(k,1)$	k
$\mathfrak{su}(k,1)$	$2k$
$\mathfrak{sp}(k,1)$	$4k$
$\mathfrak{f}_{4(-20)}$	16

Next, we consider the case where G/H with $\mathrm{rank}_{\mathbb{R}}\, G - \mathrm{rank}_{\mathbb{R}}\, H \geq 2$, which is difficult case because there are a lot of possible combinations of representations of reductive Lie algebras, which are not necessarily simple. Therefore we focus on "primary simple factor", which is "the largest" simple ideal of $\mathfrak{l}$ in the sense of ratio of the noncompact dimension to the real rank.

Assume that L is a reductive subgroup of G acting on G/H properly and cocompacly. Since $\mathfrak{l} := \mathrm{Lie}(L)$ is a reductive, we have a Levi decomposition as follows:

$$\mathfrak{l} = \mathfrak{z} \oplus \mathfrak{l}^{ss},$$
$$\mathfrak{l}^{ss} = \oplus_{i=1}^{s} \mathfrak{l}_i$$

Here $\mathfrak{z}$ is the center of $\mathfrak{l}$, and $\mathfrak{l}^{ss}$ is the semisimple ideal of $\mathfrak{l}$, and $\mathfrak{l}_i$ $(i = 1, \cdots s)$ are simple ideals of $\mathfrak{l}$.

Remark 1.2.30. For the classification of L up to compact factor, we can and do assume that $\mathfrak{l}_i$ $(i = 1, \cdots, s)$ are noncompact.

Then we label simple ideals as follows:

Setting 2.

$$\frac{d(L_i)}{\operatorname{rank}_{\mathbb{R}} L_i} \geq \frac{d(L_{i+1})}{\operatorname{rank}_{\mathbb{R}} L_{i+1}} \ (i = 1, \cdots, s-1).$$

Definition 1.2.31. We call $\mathfrak{l}_1$ primary simple factor.

We surmmarize properties related to primary factor of real rank and noncompact dimension of semisimple Lie algebra.

Remark 1.2.32. Let $\mathfrak{l}^{ss} = \oplus_{i=1}^{s} \mathfrak{l}_i$ be a decomposition into simple Lie algebras. Then we have the following:

(i) $d(L^{ss}) = \sum_{i=1}^{s} d(L_i)$, $\operatorname{rank}_{\mathbb{R}} L^{ss} = \sum_{i=1}^{s} \operatorname{rank}_{\mathbb{R}} L_i$.

(ii) $\operatorname{rank}_{\mathbb{R}} L_1 \leq \operatorname{rank}_{\mathbb{R}} L^{ss}$

Moreover, we assume that $\frac{d(L_i)}{\operatorname{rank}_{\mathbb{R}} L_i} \geq \frac{d(L_{i+1})}{\operatorname{rank}_{\mathbb{R}} L_{i+1}}$ $(1 \leq i \leq s-1)$. Then we have:

(iii) $\frac{d(L^{ss})}{\operatorname{rank}_{\mathbb{R}} L^{ss}} \leq \frac{\sum_{i=1}^{\ell} d(L_i)}{\sum_{i=1}^{\ell} \operatorname{rank}_{\mathbb{R}} L_i}$ for any $1 \leq \ell \leq s$,

(iv) $\frac{d(L^{ss})}{\operatorname{rank}_{\mathbb{R}} L^{ss}} \leq \frac{d(L_1)}{\operatorname{rank}_{\mathbb{R}} L_1}$.

Proof. Properties (i) and (ii) are clear. The inequality (iv) comes from (iii). So, we prove (iii). This comes from the following inequalities about real numbers with $\frac{a_i}{b_i} \geq \frac{a_{i+1}}{b_{i+1}}$ $(i = 1, \cdots, n-1)$ for fixed $n \in \mathbb{N}$.

- $\frac{a_{i+1}}{b_{i+1}} \leq \frac{a_i + a_{i+1}}{b_i + b_{i+1}} \leq \frac{a_i}{b_i}$,
- $\frac{\sum_{i=n-\ell+1}^{n} a_i}{\sum_{i=n-\ell+1} b_i} \leq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \leq \frac{\sum_{i=1}^{\ell} a_i}{\sum_{i=1}^{\ell} b_i}$ for $1 \leq \ell \leq n$

These are easily checked. So, we omit the proof. □

■ Outline of the proof for the case $\operatorname{rank}_{\mathbb{R}} G - \operatorname{rank}_{\mathbb{R}} H \geq 2$:
From step 1 and 2, it is enough to consider the following three types of symmetric spaces for classical types.

Table 1.3: possible symmetric spaces with $\operatorname{rank}_{\mathbb{R}} G - \operatorname{rank}_{\mathbb{R}} H \geq 2$

step $\backslash G/H$	$SO(8,\mathbb{C})/SO(7,1)$	$SU(2p,2q)/Sp(p,q)$	$SO_0(2p,q+1)/SO_0(2p,1)\times SO(q)$
step a	Remark 1.4.40	Lemma 1.3.17	Lemma 1.3.49
step b	Lemma 1.4.41	Lemma 1.3.20	Remark 1.3.52
step c	Lemma 1.4.46	Lemma 1.3.30	Lemma 1.3.57
step d	Remark 1.4.48	Remark 1.4.49	Lemma 1.3.59

- In step a: we give a upper bound of the dimension of irreducible components π of the restriction of ρ to primary factor $\mathfrak{l}_1$ and reduce candidates of π.

- In step b: By using criterion for embeddability of semisimple Lie lagebras into simple Lie algebras, we reduce candidates of irreducible components π of $\rho|_{\mathfrak{l}_1}$.
- In step c: Considering possible combinations of the other factors $\mathfrak{l}_i$ $(i = 2, \cdots, s)$, we determine possible pairs $(\mathfrak{l}^{ss},\ \rho|_{\mathfrak{l}}^{ss})$ of semisimple parts and restriction of ρ to $\mathfrak{l}^{ss}$.
- In step d: By verifying that the center of $\mathfrak{l}$ has no noncompact part or possible parameter of p and q, we determine pairs $(\mathfrak{l},\ \rho)$.

1.2.6 Tangential symmetric space

In this subsection, we reduce candidates of symmetric spaces G/H admitting reductive subgroups acting on G/H properly and cocompactly by using Fact 1.2.33, 1.2.34 and 1.2.35.

Fact 1.2.33. Let G/H be a homogeneous space of reductive type. If G/H admits standard compact Clifford–Klein forms, then the tangential homogeneous space G_θ/H_θ admits compact Clifford–Klein forms.

Proof. This comes form Fact 2.2.7. □

Fact 1.2.34. Let G be a connected simple Lie group and G/H symmetric space of reductive type with H noncompact. If the tangential symmetric space G_θ/H_θ associated with G/H admits compact Clifford–Klein forms, then G/H is infinitesimally isomorphic to one of the following symmetric spaces:

Table 1.4: possible candidates of symmetric spaces whose corresponding tangential symmetric spaces admit compact Clifford–Klein forms

Symmetric space	condition	$\mathrm{rank}_{\mathbb{R}}\, G - \mathrm{rank}_{\mathbb{R}}\, H$
$Sp(2n,\mathbb{R})/Sp(n,\mathbb{C})$	$n \geq 2$	n
$SO(p,q+1)/SO(p,q)$	$1 \leq q < HR(p)$	1
$SU(2p,2q)/Sp(p,q)$	$1 \leq q \leq p$	q
$SO(p,q+1)/SO(p,1) \times SO(q)$	$2 \leq q < HR(p)$	q
$E_{6(-14)}/F_{4(-20)}$		1
$SU(2p,2)/U(2p,1)$	$p \geq 1$	1
$SO(2p,2)/U(p,1)$	$p \geq 2$	1
$SO(8,\mathbb{C})/SO(1,7)$		3
$SO(8,\mathbb{C})/SO(7,\mathbb{C})$		1
$SO^*(8)/U(3,1)$		1
$SO^*(8)/SO^*(6) \times SO^*(2)$		1

Here $HR(n)$ is the Hurwitz–Radon number (see Fact 2.7.9, Remark 2.7.8 for the definition of Hurwitz–Radon number).

Proof. This comes from classification of irreducible semisimple symmetric spaces and Theorem 2.1.10, 2.1.11, 2.1.12 and 2.1.13. □

Fact 1.2.35 ([Ko89, Example (4.11)])**.** A symmetric space $Sp(2n,\mathbb{R})/Sp(n,\mathbb{C})$ does not admit compact Clifford–Klein forms for any positive integer n.

Remark 1.2.36. To show Theorem 1.1.8, from Fact 1.1.6, 1.2.34 and 1.2.35, it is enough to consider the following symmetric spaces:

- $SO(p,q+1)/SO(p,q)$ $(1 \leq q < HR(p))$,
- $SO(p,q+1)/SO(p,1) \times SO(q)$ $(2 \leq q < HR(p))$,
- $SU(2p,2q)/Sp(p,q)$ $(1 \leq q \leq p)$,
- $E_{6(-14)}/F_{4(-20)}$.

1.2.7 Lemmas used in the following sections

In this subsection, we prepare facts and lemmas about representation of semisimple Lie algebra, embeddability of semisimple Lie algebra into simple Lie algebras and the relation between real representations and complex representations, which are used in the following proofs.
■ representation of semisimple Lie algebras

Lemma 1.2.37. Let V be a vector space over $\mathbb{C}$ and π an irreducible component of a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(V)$ of a semisimple Lie algebra $\mathfrak{l}$. Put $m(\mathfrak{l}) := \min\{\dim \pi : \pi \text{ is a nontrivial irreducible representation of } \mathfrak{l}\}$. If $\dim \pi + m(\mathfrak{l}) > \dim \rho$, then we have $\rho = \pi \oplus \oplus^{\dim \rho - \dim \pi} \mathrm{triv}$.

Proof. Let π' be a nontrivial irreducible component of ρ. Then $\dim \pi + m(\mathfrak{l}) \leq \dim \pi + \dim \pi' \leq \dim \rho$. □

Fact 1.2.38. Let $\mathfrak{l}^{ss}$ be a semisimple Lie algebra and $\rho : \mathfrak{l}^{ss} \to \mathfrak{sl}(V)$ a representation of $\mathfrak{l}^{ss}$. Then we have the decomposition $\rho = \oplus_i^t \rho_i$ into irreducible components and have the following:

(i) $\dim \rho = \sum_{i=1}^t \dim \rho_i$,

(ii) each irreducible component ρ_i can be written as an external tensor product $(\rho_i = \pi_1^i \boxtimes \cdots \boxtimes \pi_s^i,\ V_1^i \otimes \cdots \otimes V_s^i)$ where π_k^i is an irreducible representation of $\mathfrak{l}_k$.

(iii) $\dim \rho_i = \prod_{j=1}^s \dim \pi_j^i$

(iv) $\dim \rho = \sum_{i=1}^t \prod_{j=1}^s \dim \pi_j^i \geq \sum_{(i,j)\in I} \dim \pi_j^i$.

Here $I := \{(i,j) \in \{1,\cdots,t\}\times\{1,\cdots,s\} : \pi_j^i \text{ is a nontrivial irreducible representation of } \mathfrak{l}_j\}$.

Proof. (iv) comes from that dimensions of nontrivial representations of simple Lie algebras are greater than or equal to 2 and that the inequality $ab \geq a + b$ holds for $a, b \geq 2$. □

Lemma 1.2.39. Assume ρ is injective. Let π be an irreducible component of $\rho|_{\mathfrak{l}_1}$. If the inequality $\dim \pi + 2 > \dim \rho$ holds, then we have $\mathfrak{l}^{ss} = \mathfrak{l}_1$.

Proof. Since the minimum dimension of nontrivial irreducible representations of simple Lie algebras is two, $\dim \pi + 2 \leq \dim \rho$ if $s \geq 2$. □

■ Lemmas for embeddability of semisimple Lie algebras into simple Lie algebras
See Notation 1.6.15 for the definition of "$\subset_{\mathrm{Int}}$", which is used in the following lemmas.

Lemma 1.2.40. Let $\rho : \mathfrak{l}^{ss} \to \mathfrak{sl}(V)$ be a representation of semisimple Lie algebra $\mathfrak{l}^{ss} = \oplus_{i=1}^{s} \mathfrak{l}_i$. Suppose an irreducible component π of $\rho|_{\mathfrak{l}_1}$ satisfies $2 \dim \pi > \dim \rho$. Then the following conditions hold:

(i) $\pi \simeq \overline{\pi}$ and $\mathrm{index}_{\tau_1} \pi = 1$ if $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{sl}(n, \mathbb{R})$,

(ii) $\pi \simeq \overline{\pi}$ and $\mathrm{index}_{\tau_1} \pi = -1$ if $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{su}^*(2\frac{n}{2})$,

(iii) $\pi \simeq \pi^{\vee}$ and $\mathrm{index}_{\theta_1} \pi = 1$ if $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{so}(n, \mathbb{C})$,

(iv) $\pi \simeq \pi^{\vee}$ and $\mathrm{index}_{\theta_1} \pi = -1$ if $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{sp}(\frac{n}{2}, \mathbb{C})$,

(v) $\pi \simeq \pi^*$ if $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{su}(p, q)$ for some $p + q = n$ $(p, q \in \mathbb{Z}_{\geq 0})$.

Here $n = \dim V$, τ_1 is the involution on $\mathfrak{l}_1^{\mathbb{C}}$ defining $\mathfrak{l}_1$ and θ_1 is the Cartan involution on $\mathfrak{l}_1^{\mathbb{C}}$.

Proof. We show the condition (i). The similar argument works for (ii), (iii), (iv) and (v). If we put $\pi \not\simeq \overline{\pi}$, then from Lemma 1.2.42, we have $2 \dim \pi \leq \dim \rho < 2 \dim \pi$. This is contradiction. So we obtain $\pi \simeq \overline{\pi}$. The same argument induces $m = [\pi : \rho|_{\mathfrak{l}_1}] = 1$. By applying the criterion Proposition 1.6.17 to $\rho|_{\mathfrak{l}_1}$, we obtain $1 = (\mathrm{index}_{\tau_1} \pi)^m = \mathrm{index}_{\tau_1} \pi$. □

Remark 1.2.41. For an irreducible representation π of simple Lie algebras, we can easily check whether of not π is self-conjugate (or dual, adjoint) and the $\mathrm{index}_{\tau} \pi \in \{\pm 1\}$ by using "diagram". See Appendix or [Oni] for example.

Lemma 1.2.42. Let $\rho : \mathfrak{l}^{ss} \to \mathfrak{sl}(V)$ be a representation of $\mathfrak{l}^{ss}$ and π a irreducible component of $\rho|_{\mathfrak{l}_1}$, where $\mathfrak{l}_1$ is a simple ideal of $\mathfrak{l}^{ss}$. Put $n := \dim V$. Then we have

(i) $[\pi : \rho|_{\mathfrak{l}_1}] = [\overline{\pi} : \rho|_{\mathfrak{l}_1}]$ if $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{sl}(n, \mathbb{R})$ or $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{su}^*(2\frac{n}{2})$,

(ii) $[\pi : \rho|_{\mathfrak{l}_1}] = [\pi^{\vee} : \rho|_{\mathfrak{l}_1}]$ if $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{so}(n, \mathbb{C})$ or $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{sp}(\frac{n}{2})$,

(iii) $[\pi : \rho|_{\mathfrak{l}_1}] = [\pi^* : \rho|_{\mathfrak{l}_1}]$ if $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{su}(p, q)$ for some $p + q = n$.

Proof. These come from Proposition 1.6.17, 1.6.18, 1.6.20, 1.6.21 and 1.6.22. □

Lemma 1.2.43. Let $\rho : \mathfrak{l}^{ss} \to \mathfrak{sl}(V)$ be a representation of $\mathfrak{l}^{ss}$. Suppose that $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{so}(p, q)$ for some $p + q = \dim V$. ($\pi \simeq \overline{\pi}$ and $\mathrm{index}_{\tau_1} \pi = -1$) or ($\pi \simeq \pi^{\vee}$, $\mathrm{index}_{\theta_1} \pi = -1$) implies $[\pi : \rho|_{\mathfrak{l}_1}]$ is even.

Proof. This comes from $\rho|_{\mathfrak{l}_1}(\mathfrak{l}_1) \subset_{\mathrm{Int}} \mathfrak{sl}(n, \mathbb{R})$, $\rho|_{\mathfrak{l}_1}(\mathfrak{l}_1) \subset_{\mathrm{Int}} \mathfrak{so}(n, \mathbb{C})$ and Proposition 1.6.17 and 1.6.20. Here, $n = \dim V$. $\square$

■ relation between complex representation and real representation

Remark 1.2.44. Let $\mathfrak{l}$ be a Lie algebra over $\mathbb{R}$ and V be a complex vector space and $\rho : \mathfrak{l} \to \mathfrak{gl}(V)$ an representation. We consider the cofficient restricton of ρ, denoted by $\rho_{\mathbb{R}} : \mathfrak{l} \to \mathfrak{gl}(V_{\mathbb{R}})$. Then we have $(\rho_{\mathbb{R}})^{\mathbb{C}} \simeq \rho \oplus \overline{\rho}$, where $(\rho_{\mathbb{R}})^{\mathbb{C}} : \mathfrak{l} \to \mathfrak{gl}((V_{\mathbb{R}})^{\mathbb{C}})$

Proof. Let i be a complex structure of V and $\sqrt{-1}$ a complex structure of of represetatkve $(\rho_{\mathbb{R}})^{\mathbb{C}}$ We define $\mathbb{C}$-linear linear isomomorphism.

$$\begin{aligned} V_{\mathbb{R}} + \sqrt{-1}V_{\mathbb{R}} &\to V \oplus \overline{V} \\ v + \sqrt{-1}v' &\mapsto (v + iv', \overline{v - iv'}) \end{aligned}$$

The above map induces the equivalence between $(\rho_{\mathbb{R}})^{\mathbb{C}}$ and $\rho \oplus \overline{\rho}$. $\square$

Remark 1.2.45. Let $\rho : \mathfrak{l} \to \mathfrak{gl}(V)$ be a representation of real Lie algebra. Then $\rho_{\mathbb{R}} \simeq (\overline{\rho})_{\mathbb{R}}$ holds.

Lemma 1.2.46. Let $(\rho, \mathfrak{l})$ be a representation of real semisimple Lie algebra and π an irreducible representation of $\mathfrak{l}$. If $(\rho_{\mathbb{R}})^{\mathbb{C}} \simeq \pi \oplus \overline{\pi}$ then $\rho_{\mathbb{R}} \simeq \pi_{\mathbb{R}} (\simeq \overline{\pi}_{\mathbb{R}})$.

Proof. From the above Remark 1.2.44 and the assumption, we have $(\rho_{\mathbb{R}})^{\mathbb{C}} \simeq \rho \oplus \overline{\rho} \simeq \pi \oplus \overline{\pi}$. Since π is irreducible, we obtain $\pi \simeq \rho$ or $\overline{\pi} \simeq \rho$. Thus $\rho_{\mathbb{R}} \simeq \pi_{\mathbb{R}} (\simeq \overline{\pi}_{\mathbb{R}})$ holds. $\square$

1.3 Proof of non-existence part

From Remark 1.2.36 and a property of Hurwitz–Radon number, to show Theorem 1.1.8, it is enough to conisider the following symmetric spaces G/H:

- $SO(2p, q + 1)/SO(2p, q)$ $(1 \leq q < HR(2p))$,
- $SU(2p, 2q)/Sp(p, q)$ $(1 \leq q \leq p)$,
- $SO(2p, q + 1)/SO(2p, 1) \times SO(q)$ $(2 \leq q < HR(2p))$,
- $E_{6(-14)}/F_{4(-20)}$.

Therefore we give a proof for the above four types of symmetric spaces in the following subsections.

1.3.1 $(G,\ H)$=$(SO(2p,q+1),\ SO(2p,q))$ $(1 \leq q < HR(2p))$

In this subsection, we consider the case $(G,H) = (SO_0(2p,q+1), SO_0(2p,q))$ $(1 \leq q < HR(2p),\ p \geq 2)$. Our goal is the following

Proposition 1.3.1. Let $G/H = SO_0(2p,q+1)/SO_0(2p,q)$ $(1 \leq q < HR(2p),\ p \geq 2)$. There exists a closed subgroup L which is reductive in G and acts on G/H properly and cocompactly if and only if $q = 1$ or ($q = 3$ and p is even) or ($q = 7$ and $p = 4$). Moreover, L is locally isomorphic to $SU(1,p)$, $Sp(1,\frac{p}{2})$ or $Spin_0(1,8)$ respectively.

Proof. This comes from Lemma 1.3.2 and Proposition 1.3.3. $\square$

Lemma 1.3.2. Let $G/H = SO_0(2p,q+1)/SO_0(2p,q)$ $(1 \leq q < HR(2p),\ p \geq 2)$ and $n = 2p+q+1$. There exists a reductive subgroup of G acting on G/H properly and cocompactly if and only if there exist a pair of a simple Lie algebra $\mathfrak{l}$ and its faithful representation $\rho : \mathfrak{l} \to \mathfrak{sl}(n,\mathbb{C})$ of satisfying the following conditions:

(i) $\rho(\mathfrak{l}) \subset \mathfrak{so}(2p,q+1) \subset \mathfrak{sl}(n,\mathbb{C})$,

(ii) $\rho(\mathfrak{l})$ is preserved by matrix transpose,

(iii) $\mathfrak{a}_L \cap W\mathfrak{a}_H = \{0\}$,

(iv) $d(L) = d(G) - d(H) (= 2p)$.

Here the inclusion $\mathfrak{so}(2p,2) \subset \mathfrak{sl}(n,\mathbb{C})$ is realized by the standard inclusion which is preserved by matrix transpose, L is the analytic subgroup corresponding to $\rho(\mathfrak{l}) \subset \mathfrak{so}(2p,q+1)$, and $W \simeq N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is the Weyl group of G.

Proof. This comes from Fact 1.2.15 and 1.2.18. $\square$

Proposition 1.3.3. A pair of a simple Lie algebra $\mathfrak{l}$ and its representation satisfying the conditions (i) to (iv) in Lemma 1.3.2 is equivalent to one of the following:

- the representation $\rho_{\varpi_1} \oplus \overline{\rho_{\varpi_1}} : \mathfrak{l} = \mathfrak{su}(p,1) \to \mathfrak{so}(2p,2)$ $(p \geq 2)$,
- the representation $\rho_{\varpi_1} \oplus \overline{\rho_{\varpi_1}} : \mathfrak{l} = \mathfrak{sp}(p',1) \to \mathfrak{so}(4p',4)$ $(p' \geq 1)$,
- the spin representation $\rho_{\varpi_4} : \mathfrak{l} = \mathfrak{so}(1,8) \to \mathfrak{so}(8,8)$.

Here ρ_{ϖ_i} denotes irreducible representation with highest weight ϖ_i and $\overline{\rho}$ denotes complex conjugate representation of ρ.

Proof. This follows from Lemma 1.3.7, 1.3.8, 1.3.9 1.3.10, 1.3.11 and 1.3.12. $\square$

Remark 1.3.4. Suppose a pair $(\mathfrak{l},\ \rho)$ of Lie algebra and its representation satisfies the conditions in Lemma 1.3.2. Then the following inequalities hold:

$$\begin{cases} 1 \leq q < HR(2p) \leq 2p \\ d(L) = 2p \\ \dim \rho = 2p + q + 1 \end{cases}$$

In particular, we have $\dim \rho \leq d(L) + HR(d(L)) \leq 2d(L) \geq 8$.

Lemma 1.3.5. Suppose $\rho : \mathfrak{l} \to \mathfrak{sl}(2p+2, \mathbb{C})$ satisfies the conditions (i) to (iv) of Lemma 1.3.2. Let π be a nontrivial irrreducible component of ρ. Then the following conditions are satisfied:

(i) $\dim \pi \leq d(L) + HR(d(L)) \leq 2d(L) \geq 8$,

(ii) $2 \dim \pi > d(L) + HR(d(L)) \implies \pi \simeq \overline{\pi} \simeq \pi^{\vee}$ and $\mathrm{index}_{\tau} \pi = \mathrm{index}_{\theta} \pi = 1$.

(iii) $\dim \pi + m(\mathfrak{l}) > d(L) + HR(d(L)) \implies \mathrm{rank}\, \pi(X) = 2(q+1)$ for any $X \in \mathfrak{p}_L \setminus \{0\}$.

Here τ is the involution on $\mathfrak{l}_{\mathbb{C}}$ such that $\mathfrak{l}_{\mathbb{C}}^{\tau} = \mathfrak{l}$ and θ is the Cartan involution on $\mathfrak{l}_{\mathbb{C}}$.

Proof. (i) This comes from Remark 1.3.4.

(ii) Assume that $2 \dim \pi > d(L) + HR(d(L))$. Then we have $2 \dim \pi > d(L) + HR(d(L)) \geq \dim \rho$. Therefore we obtain the desired condition by Lemma 1.2.40.

(iii) From Lemma 1.2.37 and 1.2.14, it is enough to show that for $0 \neq X \in \mathfrak{p} \subset \mathrm{Sym}(2p+q+1, \mathbb{R})$, $X \in \mathrm{Int}(\mathfrak{g})\mathfrak{a}_H \iff \mathrm{rank}\, X \leq 2(q+1)$. This comes from the fact that $\mathrm{Ad}(K)\mathfrak{a} = \mathfrak{p}$ and $\mathrm{Int}(\mathfrak{g})$ action on $\mathfrak{g}$ preserve the rank as a matrix.

$\square$

Remark 1.3.6. Since $d(L) = 2p$ is even, for $\mathfrak{l} = \mathfrak{so}(k', 1)$ case, we consider the case where k' is even.

Lemma 1.3.7. Let $(\mathfrak{l},\ \pi)$ be a pair of a simple Lie algebra with $\mathrm{rank}_{\mathbb{R}} L = 1$ and its irreducible representation over $\mathbb{C}$. Suppose $(\mathfrak{l},\ \pi)$ satisfies the condition (i) of Lemma 1.3.5. Then $(\mathfrak{l},\ \pi)$ is equivalent to one of the following list:

Table 1.5: a pair of a simple Lie algebra and its irreducible representation satisfying (i) in Lemma 1.3.5

$\mathfrak{l}$	π	$\dim \pi$	not satisfy
$\mathfrak{so}(2k,1)$ $(k \geq 2)$	ϖ_1	$2k+1$	(iii)
$\mathfrak{so}(8,1)$	ϖ_4	16	
$\mathfrak{so}(6,1)$	ϖ_3	8	(ii)
$\mathfrak{su}(k,1)$ $(k \geq 2)$	ϖ_1, ϖ_k	$k+1$	
$\mathfrak{su}(4,1)$	ϖ_2, ϖ_3	10	(ii)
	$2\varpi_1$, $2\varpi_4$	15	(ii)
$\mathfrak{su}(3,1)$	ϖ_2	6	(ii)
$\mathfrak{su}(2,1)$	$2\varpi_1$, $2\varpi_2$	6	(ii)
	$\varpi_1 + \varpi_2$	8	(iii)
$\mathfrak{sp}(k,1)$ $(k \geq 1)$	ϖ_1	$2k+2$	
$\mathfrak{sp}(2,1)$	ϖ_2	14	(iii)
	ϖ_3	14	(ii)

Lemma 1.3.8. Let $(\mathfrak{l},\, \pi)$ be a pair of a simple Lie algebra and its irreducible representation which is equivalent to one of the following table. Then $(\mathfrak{l},\, \pi)$ does not satrisfy the condition (ii) of Lemma 1.3.5.

Table 1.6: pairs of a simple Lie algebra and its irreducible representation which do not satisfy (ii) in Lemma 1.3.5

$\mathfrak{l}$	π	$\dim \pi$	$d(L) + HR(d(L))$	selfconj? selfdual?
$\mathfrak{so}(6,1)$	ϖ_3	8	8	$\mathrm{index}_\tau\, \pi = -1$
$\mathfrak{su}(4,1)$	ϖ_2, ϖ_3	10	16	$\pi \not\simeq \overline{\pi}$
	$2\varpi_1$, $2\varpi_4$	15	16	$\pi \not\simeq \overline{\pi}$
$\mathfrak{su}(3,1)$	ϖ_2	6	8	$\mathrm{index}_\tau\, \pi = -1$
$\mathfrak{su}(2,1)$	$2\varpi_1$, $2\varpi_2$	6	8	$\pi \not\simeq \overline{\pi}$
$\mathfrak{sp}(2,1)$	ϖ_3	14	16	$\mathrm{index}_\tau\, \pi = -1$

Proof. This comes from the data in the above table. Here τ is the involution on $\mathfrak{l}_\mathbb{C}$ such that $\mathfrak{l}_\mathbb{C}^\tau = \mathfrak{l}$. □

Lemma 1.3.9. Let $(\mathfrak{l},\, \pi)$ be a pair of simple Lie algebra and its irreducible representation which is equivalent to one of the following Table 1.7. Then $(\mathfrak{l},\, \pi)$ does not satrisfy the condition (iii) of Lemma 1.3.5.

Table 1.7: pairs of a simple Lie algebra and its irreducible representation which do not satisfy (iii) in Lemma 1.3.5

$\mathfrak{l}$	π	$\dim \pi$	$d(L)$
$\mathfrak{so}(2k,1)$	ϖ_1	$2k+1$	$2k$
$\mathfrak{su}(2,1)$	$\varpi_1 + \varpi_2$	8	4
$\mathfrak{sp}(2,1)$	ϖ_2	14	8

Proof.
- In the case $(\mathfrak{l},\, \pi)$=$(\mathfrak{so}(2k,1),\, \varpi_1)$ $(k \geq 2)$.
  Since the inequality $\dim \pi + m(\mathfrak{so}(2k,1)) \geq (2k+1)+(2k) > 4k = 2d(L) \geq d(L) + HR(d(L))$ holds, it is enough to show that there exists $X \in \mathfrak{p}_L \backslash \{0\}$

such that $\operatorname{rank}\pi(X) < 2(q+1)$. Take $X = E_{1,2k+1} + E_{2k+1,1} \in \mathfrak{p}_L$, then $\operatorname{rank}\pi(X) = 2 < 2(q+1)$.

- In the case $(\mathfrak{l},\ \pi)$=$(\mathfrak{su}(2,1),\ \varpi_1+\varpi_2)$.
  Since the inequality $\dim\pi+m(\mathfrak{su}(2,1)) = 8+3 > 4+4 = d(L)+HR(d(L))$ holds, it is enough to show that there exists $0 \neq X \in \mathfrak{p}_L$ such that $\operatorname{rank}\pi(X) < 2(q+1)$. We have $p = 2$, $q = 3$ from $d(L) = 4 = 2p$ and $8 = \dim\pi \leq \dim\rho \leq 2p+q+1$ and $q < HR(2p)$. Therefore we have $\pi = \operatorname{ad} : \mathfrak{su}(2,1) \to \mathfrak{so}(4,4) \subset \mathfrak{sl}(\mathfrak{su}(2,1))$. Take $0 \neq X \in \mathfrak{p}_L$ then $\operatorname{rank}\pi(X) < 8 = 2(q+1)$. In fact, $\operatorname{rank}\pi(X) < 8$ means $\pi(X) : \mathfrak{su}(2,1) \to \mathfrak{su}(2,1)$ is not injective and $\pi(X)(X) = \operatorname{ad}(X)(X) = 0$.

- In the case $(\mathfrak{l},\ \pi)$=$(\mathfrak{sp}(2,1),\ \varpi_2)$.
  Since the inequality $\dim\pi+m(\mathfrak{sp}(2,1)) = 14+6 > 8+8 = d(L)+HR(d(L))$ holds, and we have $p = 4$ and $5 \leq q \leq 7$ from $8 = d(L) = 2p$, $\dim\pi \leq \dim\rho = 2p+q+1$ and $1 \leq q < HR(2p)$ coming from Remark 1.3.4. it is enough to show that there exists $X \in \mathfrak{p}_L\backslash\{0\}$ such that $\operatorname{rank}\pi(X) \leq 10(< 2(q+1))$.

  We realize $\mathfrak{sp}(2,1) = \mathfrak{sp}(3,\mathbb{C})^\tau$ as follows:

$$\begin{aligned}
&\mathfrak{sp}(3,\mathbb{C}) = \{X \in M(6,\mathbb{C}) : {}^tXJ + JX = 0\},\\
&\tau : \mathfrak{sp}(3,\mathbb{C}) \to \mathfrak{sp}(3,\mathbb{C}),\ X \mapsto -I_{2,1;2,1}X^*I_{2,1;2,1},
\end{aligned}$$

  where, $J = \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix}$ and $I_{2,1;2,1} = \operatorname{diag}(1,1,-1,1,1,-1)$. The representation of $\mathfrak{sp}(2,1)$ with highest weight ϖ_2 can be realized as follows

$$\begin{aligned}
&\mathfrak{sp}(2,1) \times \ker\varphi \to \ker\varphi\\
&\qquad (X, v\wedge w) \mapsto Xv\wedge w + v\wedge Xw
\end{aligned}$$

  Here φ is the following linear map:

$$\varphi : \mathbb{C}^6 \wedge \mathbb{C}^6 \to \mathbb{C}, v\wedge w \mapsto {}^tvJw$$

  Take $X := S_{2,6} + S_{3,5} \in \mathfrak{p}_L$. It is enough to show that $\operatorname{rank}\pi(X) \leq 10$, that is $\dim\ker\pi(X) \geq 4$. Let $v_1 := -2e_1\wedge e_4 + e_2\wedge e_5 + e_3\wedge e_6$, $v_2 := e_2\wedge e_3 + e_5\wedge e_6$, $v_3 := -e_2\wedge e_6 + e_3\wedge e_5$, $v_4 := e_2\wedge e_6 + e_3\wedge e_5$. Then v_i $(i = 1,2,3,4)$ are linearly independent and $\pi(X)v_i = 0$.

  □

Next, we consider the pairs $(\mathfrak{l},\ \pi)$= $(\mathfrak{so}(8,1),\ \varpi_4)$, $(\mathfrak{su}(k,1),\ \varpi_1)$, $(\mathfrak{sp}(k,1),\ \varpi_1)$ $(k \geq 1)$.

Lemma 1.3.10. Let ρ be a representation of a simple Lie algebra $\mathfrak{so}(8,1)$ satisfying the conditions (i) to (iv) of Lemma 1.3.2. Suppose that $\pi \simeq \rho_{\varpi_4}$ is an irreducible component of ρ. Then we have $\rho = \pi$.

Proof. This comes from $16 = \dim\pi \leq \dim\rho \leq d(L) + HR(d(L)) = 16$ and Lemma 1.2.37. □

Lemma 1.3.11. Let ρ be a representation of a simple Lie algebra $\mathfrak{su}(k,1)$ ($k \geq 2$) satisfying the conditions (i) to (iv) of Lemma 1.3.2. Suppose that $\pi = \rho_{\varpi_1}$ is an irreducible component of ρ. Then we have $k = p$, $q = 1$ and $\rho \simeq \pi \oplus \overline{\pi}$.

Proof. From $d(L) = 2k = 2p$, we have $k = p$. Then, from Lemma 1.2.42 (i), Fact 1.2.38, and $\pi \not\simeq \overline{\pi}$, we have $2[\pi : \rho](p+1) \leq \dim \rho \leq 4p$, that is $[\pi : \rho] = [\overline{\pi} : \rho] = 1$. Let π' be another irreducible component of ρ. Then π' is equivalent to one of Table 1.5 in Lemma 1.3.7. However, $\dim \pi + \dim \overline{\pi} + \dim \pi' \leq d(L) + HR(d(L))$ does not hold. Therefore, $\rho \simeq \pi \oplus \overline{\pi} \oplus \mathrm{triv}^{q-1}$ Then there exists an element $X \in \mathfrak{p}_L$ such that $\operatorname{rank} \rho(X) = 4$. From properness, we have $\operatorname{rank} \rho(X) = 4 \geq 2(q+1)$ (See proof for Lemma 1.3.5(iii)). So, we have $q = 1$ and $\rho \simeq \pi \oplus \overline{\pi}$. □

Lemma 1.3.12. Let ρ be a representation of a simple Lie algebra $\mathfrak{sp}(k,1)$ ($k \geq 1$) satisfying the conditions (i) to (iv) of Lemma 1.3.2. Suppose that $\pi = \rho_{\varpi_1}$ is an irreducible component of ρ Then we have p is even, $k = \frac{p}{2}$, $q = 3$ and $\rho \simeq \pi \oplus \overline{\pi}$.

Proof. From $d(L) = 4k = 2p$, we have p is even and $k = \frac{p}{2}$. Moreover, from $\operatorname{index}_\tau \pi = -1$, Lemma 1.2.43 and $[\pi : \rho] \dim \pi = [\pi : \rho](p+2) \leq 4p = 2d(L)$, we have $[\pi : \rho] = 2$. Next, we show that $q = 3$, which induces that $\rho = \pi \oplus \overline{\pi}$ from $2 \dim \pi = 2p + 4 = \dim \rho$. From the inequality $2 \dim \pi = 2(p+2) \leq \dim \rho = 2p + q + 1$, we obtain $q \geq 3$.

Claim. There is no irreducible component other than ρ_{ϖ_1}.

To prove this claim, it is enough to consider the case $\mathfrak{l} = \mathfrak{sp}(2,1)$ from Table 1.5 and this claim can be easily checked by the ineqality about dimension of the representation. Then for an appropriate $0 \neq X \in \mathfrak{p}_L$, $\operatorname{rank} \pi(X) = 8$ holds by the definition of ϖ_1. Therefore, from properness of L-action (see proof for Lemma 1.3.5(iii)), we have $2(q+1) \leq 8$, that is, $q \leq 3$. □

1.3.2 $(G,\ H)$=$(SU(2p,2q),\ Sp(p,q))$ $(1 \leq q \leq p)$

In this subsection, we consider the case $(G,\ H)$=$(SU(2p,2q),\ Sp(p,q))$ $(1 \leq q \leq p)$. Our goal in this subsection is the following:

Proposition 1.3.13. Let $G/H = SU(2p,2q)/Sp(p,q)$ $(p \geq q \geq 1)$. There exists a closed subgroup L which is reductive in G acting on G/H properly and cocompactly if and only if $q = 1$. Moreover, L is locally isomorphic to $SU(2p,1)$ up to compact factor.

Proof. This follows from Lemma 1.3.14 and Porposition 1.3.15. □

Lemma 1.3.14. Let $G/H = SU(2p,2q)/Sp(p,q)$ $(1 \leq q \leq p)$ and $n := 2p+2q$. There exists a reductive subgroup of G acting on G/H properly and cocompactly if and only if there exists a reductive Lie algebra $\mathfrak{l}$ and its faithful representation $\rho : \mathfrak{l} \to \mathfrak{sl}(n,\mathbb{C})$ satisfying the following conditions:

(i) $\rho(\mathfrak{l}) \subset \mathfrak{su}(2p,2q) \subset \mathfrak{sl}(n,\mathbb{C})$,

(ii) $\rho(\mathfrak{l})$ is preserved by matrix adjoint,

(iii) $\mathfrak{a}_{\mathfrak{l}} \cap W\mathfrak{a}_{\mathfrak{h}} = \{0\}$,

(iv) $d(L) = d(G) - d(H)$.

Here the inclusion $\mathfrak{su}(2p, 2q) \subset \mathfrak{sl}(n, \mathbb{C})$ is realized as the standard inclusion which is preserved matrix adjoint, L is the analytic subgroup corresponding to $\rho(\mathfrak{l}) \subset \mathfrak{su}(2p, 2q)$ and $W \simeq N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is the Weyl group of G.

Proof. This comes from Fact 1.2.15 and 1.2.18. □

Proposition 1.3.15. There exists a reductive Lie algebra $\mathfrak{l}$ and its faithful representation $\rho : \mathfrak{l} \to \mathfrak{sl}(n, \mathbb{C})$ satisfying conditions (i) to (iv) of Lemma 1.3.14 if and only if $q = 1$. Moreover, such $\mathfrak{l}$ is isomorphic to $\mathfrak{su}(2p, 1)$ up to compact factor and ρ is equivalent to $\rho_{\varpi_1} \oplus \mathrm{triv}$.

We devote this subsection below to showing Proposition 1.3.15.

First, for each symmetric pair $(\mathfrak{su}(2p, 2q), \mathfrak{sp}(p, q))$, we reduce candidates of primary simple factors and their irreducible components.

Lemma 1.3.16. Suppose a $(\mathfrak{l}, \rho)$ satisfies the conditions (i) to (iv) of Lemma 1.3.14 and $\mathfrak{l}_1$ is the primary factor of $\mathfrak{l}$ and π is a nontrivial irreducible component of $\rho|_{\mathfrak{l}_1}$. Then $(\mathfrak{l}_1, \pi)$ is equivalent to one of the following table

Table 1.8: pairs of a simple Lie algebra and its irreducible representation π satisfying the conditions (i) and (ii) of Lemma 1.3.17

$\mathfrak{l}$	π	$\dim \pi$	not satisfy 1.3.20
$\mathfrak{sl}(n, \mathbb{C})$ $(n \geq 2)$	$\varpi_1 \boxtimes \mathrm{triv}$	n	(i)
$\mathfrak{su}^*(2n)$ $n \geq 2$	ϖ_1	$2n$	(i)
$\mathfrak{su}(k, \ell)$ $(2 \leq k \geq \ell \geq 1)$	ϖ_1	$k + \ell$	(ii) if $\ell \geq 2$
$\mathfrak{su}(3, 1)$	ϖ_2	6	(iii)
$\mathfrak{so}(2n + 1, \mathbb{C})$ $(n \geq 2)$	$\varpi_1 \boxtimes \mathrm{triv}$	$2n + 1$	(i)
$\mathfrak{so}^*(4n + 2)$ $(n \geq 2)$	ϖ_1	$4n + 2$	(iii)
$\mathfrak{sp}(n, \mathbb{C})$ $(n \geq 2)$	$\varpi_1 \boxtimes \mathrm{triv}$	$2n$	(i)
$\mathfrak{sp}(k, \ell)$ $(k \geq \ell \geq 1)$	ϖ_1	$2(k + \ell)$	(ii)
$\mathfrak{g}_2^{\mathbb{C}}$	$\varpi_1 \boxtimes \mathrm{triv}$	7	(i)

Proof. This comes from Lemma 1.3.17 and Weyl's dimensionality formula. □

Lemma 1.3.17. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(V)$ satisfies the conditions (i) to (iv) of Lemma 1.3.14 and a pair $(\mathfrak{l}_1, \pi)$ of a primary factor of $\mathfrak{l}$ and its irreducible component of $\rho|_{\mathfrak{l}_1}$. Then $(\mathfrak{l}_1, \pi)$ satisfies the following condition:

(i) $\dim \pi \leq \frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1}$,

(ii) $\max(4, 2\sqrt{d(L_1)}) \leq \frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1}$.

Proof. (i) From Lemma 1.3.18, 1.3.19, by taking $A = \mathrm{rank}_{\mathbb{R}} L^{ss}$, $B = d(L^{ss})$ and $C = \dim \rho$, we have $\dim \pi \leq \dim \rho|_{\mathfrak{l}_1} = \dim \rho \leq 2 \,\mathrm{rank}_{\mathbb{R}} L^{ss} + \frac{d(L^{ss})}{2 \,\mathrm{rank}_{\mathbb{R}} L^{ss}} \leq \frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1}$. Here we used the inequality of Remark 1.2.32 (iv).

(ii) We obtain in the same way as (i) above, that is, we have $\max(4, 2\sqrt{d(L_1)}) \leq \max(4, 2\sqrt{d(L^{ss})}) \leq \dim \rho \leq \frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1}$.

$\square$

Lemma 1.3.18. Suppose a pair $(\mathfrak{l}, \rho)$ of a reductive Lie algebra and its representation satisfies the conditions (i) to (iv) of Lemma 1.3.14. Then the following inequalities hold:

$$\begin{cases} \mathrm{rank}_{\mathbb{R}} L^{ss} + t \leq q, \\ d(L^{ss}) + t = 4pq, \\ \dim \rho = 2(p+q). \end{cases}$$

Here $t = \dim \rho(\mathfrak{z})^{-\theta}$, where $\mathfrak{z}$ is the center of $\mathfrak{l}$.

Proof. The condition (ii) implies the condition (ii)' $\mathrm{rank}_{\mathbb{R}} L \leq \mathrm{rank}_{\mathbb{R}} G - \mathrm{rank}_{\mathbb{R}} H = q$. $\square$

Lemma 1.3.19. Let $1 \leq A \leq B$ and $C \geq 4$. There exist $1 \leq q \leq p$ and $t \geq 0$ such that

$$\begin{cases} A + t \leq q \\ B + t = 4pq \\ C = 2(p+q) \end{cases}$$

if and only if $\max(4, 2\sqrt{B}) \leq C \leq 2A + \frac{B}{2A}$. Moreover, then we have $2A + \frac{B}{2A} \leq \frac{B}{A}$.

Proof. This can be easily checked by a fudamental argument on inequalites. So, we omit the proof. $\square$

Lemma 1.3.20. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(V)$ satisfies the conditions (i) to (iv) of Lemma 1.3.14 and a pair $(\mathfrak{l}_1, \pi)$ of a primary factor of $\mathfrak{l}$ and its nontrivial irreducible component of $\rho|_{\mathfrak{l}_1}$. Then $(\mathfrak{l}_1, \pi)$ satisfies the following conditions:

(i) $\pi \simeq \pi^*$,

(ii) $\dim \pi + m(\mathfrak{l}_1) > \frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1} \implies \pi(X)\pi(X)^* \in \mathrm{Herm}(2p, \mathbb{C})$ has a odd dimensional eigen space for any $X \in \mathfrak{p}_L \setminus \{0\}$, where we identify $\mathfrak{p}$ with $M(2p, 2q; \mathbb{C})$,

(iii) $\dim \pi = \frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1} \implies d(L_1) \leq 4(\mathrm{rank}_{\mathbb{R}} L_1)^2$.

Here $m(\mathfrak{l}_1) := \min\{\dim \pi' : \pi' \text{ is a nontrivial irreducible representation of } \mathfrak{l}_1\}$.

Proof. (i) From Lemma 1.2.40, we have $2 \dim \pi \leq \frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1}$ or $\pi \simeq \pi^*$. From Remark 1.3.21, we obtain the desired conclusion.

(ii) From Lemma 1.2.37, we have $\rho|_{\mathfrak{l}_1} = \pi \oplus \oplus^{\dim\rho - \dim\pi}\mathrm{triv}$. We identify $\mathfrak{p}$ with $M(2p, 2q; \mathbb{C})$, where $(k_1, k_2) \in S(U(2p)\times U(2q))$ acts on $M(2p, 2q; \mathbb{C})$ as $X \mapsto k_1 X k_2^{-1}$. Then it is enough to show that for $X \in M(2p, 2q; \mathbb{C})$, $X \in \mathrm{Ad}(K)\mathfrak{p}_H$ holds if and only if the dimension of any eigenspace of $XX^* \in \mathrm{Herm}(2p, \mathbb{C})$ is even. This comes from that we can take maximal abelian subspace of $\mathfrak{p}_H$ such as $\mathfrak{a}_H = \{(a_1, \cdots, a_q, a_1, \cdots, a_q) \in \mathbb{R}^{2q} : a_i \in \mathbb{R}\ (i = 1, \cdots, q)\}$ by taking an appropriate coordinate, the Weyl group is isomorphic to $\mathfrak{S}_{2q} \ltimes (\mathbb{Z}_2)^{2q}$ and the property that the dimension of any eigenspace of $XX^* \in \mathrm{Herm}(2p, \mathbb{C})$ is even is invariant under the action of $\mathrm{Ad}(K)$.

(iii) Assume $\dim\pi = \frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1}$. Then since we have the inequality $\frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1} = \dim\pi \leq \dim\rho \leq \frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1}$, we obtain $\mathfrak{l}^{ss} = \mathfrak{l}_1$. Therefore we obtain $d(L_1) = 4(\mathrm{rank}_{\mathbb{R}} L_1)^2$ from $\frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1} = \dim\pi = \dim\rho \leq 2\,\mathrm{rank}_{\mathbb{R}} L_1 + \frac{d(L_1)}{2\,\mathrm{rank}_{\mathbb{R}} L_1} \leq \frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1}$.

□

Remark 1.3.21. Let $\mathfrak{l}$ be a noncompact simple Lie algebra and π a nontrivial irreducible representation of $\mathfrak{l}$. Then a inequality $2\dim\pi > \frac{d(L)}{\mathrm{rank}_{\mathbb{R}} L}$ holds.

Proof. This comes from the classification of simple Lie algebras and Weyl's dimensionality formula. □

In the case of $\mathfrak{g} = \mathfrak{su}(2p, 2q)$, we can determine the properness by the equivalent class of representation from the following:

Proposition 1.3.22. Suppose G is a linear reductive Lie group such that its Lie algebra $\mathfrak{g}$ is a noncompact real form of $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$. Suppose $\mathfrak{l}$ is a reductive Lie algebra and $\rho_i : \mathfrak{l} \to \mathfrak{sl}(n, \mathbb{C})$ $(i = 1, 2)$ are faithful representations of $\mathfrak{l}$ such that $\rho_i(\mathfrak{l})$ $(i = 1, 2)$ are reductive subalgebra of $\mathfrak{g}$ and G, H and L_i $(i = 1, 2)$ are analytic subgroups of $SL(n, \mathbb{C})$. If ρ_1 and ρ_2 are equivalent as a representation of $\mathfrak{l}$, L_1 action on G/H is proper if and only if L_2 action on G/H is proper.

Proof. Since we have $\rho_1 \simeq \rho_2$ as a complex representation, there exist $\alpha_0 \in \mathrm{Int}(\mathfrak{sl}(n, \mathbb{C}))$ such that $\rho_2 = \alpha_0\rho_1$. Assume that L_1-action on G/H is not proper. Then we have $\mathrm{Ad}(K)\mathfrak{p}_{\rho_1(\mathfrak{l})} \cap \mathfrak{p}_H \neq \{0\}$ from Fact 1.2.15. Take $0 \neq X \in \mathfrak{p}_{\rho_1(\mathfrak{l})}$ and $k \in K$ such that $\mathrm{Ad}(k)X \in \mathfrak{p}_H$. From Fact 1.4.16, it is enough to show the following:

Claim. $\mathrm{Int}(\mathfrak{g})\rho_2(\mathfrak{l}) \cap \mathfrak{p}_H \neq \{0\}$.

From Remark 1.3.23, $\alpha_0(X) \in \rho_2(\mathfrak{l})$ is hyperbolic in $\mathfrak{g}$. So, we can take $\alpha_1 \in \mathrm{Int}(\mathfrak{g})$ such that $\alpha_1(X) = \alpha_0(X)$. Therefore, $0 \neq \mathrm{Ad}(k)X = (\mathrm{Ad}(k)(\alpha_1)^{-1})\alpha_1(X) \in \mathfrak{p}_H \cap \mathrm{Int}(\mathfrak{g})\rho_2(\mathfrak{l})$. □

Remark 1.3.23 (See [O13] for example)**.** For $X \in \mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$, X is hyperbolic in $\mathfrak{g}$ if and only if X is hyperbolic in $\mathfrak{g}_{\mathbb{C}}$.

Fact 1.3.24 ([O13, Proposition 4.5 (i)])**.** Let $\mathfrak{g}$ be a non-compact real form of a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$. For a pair of hyperbolic elements A_1 and A_2 in $\mathfrak{g}$, the following two conditions are equivalent:

- A_1 and A_2 are $(\operatorname{Int}\mathfrak{g})$-conjugate in $\mathfrak{g}$.
- A_1 and A_2 are $(\operatorname{Int}\mathfrak{g}_{\mathbb{C}})$-conjugate in $\mathfrak{g}_{\mathbb{C}}$.

Lemma 1.3.25. Let $(\mathfrak{l}, \pi)$ be a pair of a simple Lie algebra and its irreducible representation which is equivalent to one of the following Table 1.9. Then $(\mathfrak{l}, \pi)$ does not satisfy the condition (i) of Lemma 1.3.20.

Table 1.9: pairs of a simple Lie algebra and its irreducible representation in Table 1.8 which do not satisfy (i) of Lemma 1.3.20

$\mathfrak{l}$	π
$\mathfrak{sl}(n,\mathbb{C})$ $n \geq 2$	$\varpi_1 \boxtimes \mathrm{triv}$
$\mathfrak{su}^*(2n)$ $n \geq 2$	ϖ_1, ϖ_{2n-1}
$\mathfrak{so}(2n+1,\mathbb{C})$ $n \geq 2$	$\varpi_1 \boxtimes \mathrm{triv}$
$\mathfrak{sp}(n,\mathbb{C})$ $n \geq 2$	$\varpi_1 \boxtimes \mathrm{triv}$
$\mathfrak{g}_2^{\mathbb{C}}$	$\varpi_1 \boxtimes \mathrm{triv}$

Proof. The above irreducible representations π satisfy $\pi \not\simeq \pi^*$. See Appendix or Table 5 [Oni] for example. $\square$

Lemma 1.3.26. Let $(\mathfrak{l}, \pi)$ be a pair of a simple Lie algebra and its irreducible representation which is equivalent to one of the following table. Then $(\mathfrak{l}, \pi)$ does not satisfy the condition (ii) of Lemma 1.3.20.

Table 1.10: pairs of a simple Lie algebra and its irreducible representation in Table 1.8 which do not satisfy (ii) of Lemma 1.3.20

$\mathfrak{l}$	π
$\mathfrak{su}(k,\ell)$ $(k \geq \ell \geq 2)$	ϖ_1
$\mathfrak{sp}(k,\ell)$ $(k \geq \ell \geq 1)$	ϖ_1

Proof.

- In the case $\mathfrak{l}_1 = \mathfrak{su}(k,\ell)$ $(k \geq \ell \geq 2)$:
  We have $\dim \pi = k + \ell = m(\mathfrak{su}(k,\ell))$ and $\frac{d(L_1)}{\operatorname{rank}_{\mathbb{R}} L_1} = 2k$. So, the inequality $\dim \pi + m(\mathfrak{l}_1) > \frac{d(L_1)}{\operatorname{rank}_{\mathbb{R}} L_1}$ holds. We realize $\mathfrak{su}(k,\ell) \subset \mathfrak{sl}(k+\ell,\mathbb{C})$ as follows:

$$\begin{aligned} \sigma : \mathfrak{sl}(k+\ell,\mathbb{C}) &\to \mathfrak{sl}(k+\ell,\mathbb{C}) \\ X &\mapsto -I_{k,\ell} X^* I_{k,\ell} \\ \mathfrak{su}(k,\ell) &:= \mathfrak{sl}(k+\ell,\mathbb{C})^{\sigma} \end{aligned}$$

  We take $X \in \mathfrak{p}_L$ as follows:

$$X := \begin{pmatrix} & B \\ B^* & \end{pmatrix}, B := \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \\ & & & & \end{pmatrix} \in M(k,\ell;\mathbb{C}).$$

Then we have $\rho_{\varpi_1}(X)\rho_{\varpi_1}(X)^* = BB^* = \mathrm{diag}(1,1,0,\cdots,0) \in \mathrm{Herm}(2p,\mathbb{C})$ under the identification $\mathfrak{p} \simeq M(2p,2q;\mathbb{C})$. Since dimensions of all the eigenspaces of $\mathrm{diag}(1,1,0,\cdots 0)$ are even, the condition (ii) of Lemma 1.3.20 is not satisfied.

- In the case $\mathfrak{l}_1 = \mathfrak{sp}(k,\ell)$ $(k \geq \ell \geq 1)$:
  We have $\dim\pi = 2(k+\ell) = m(\mathfrak{sp}(k,\ell))$ and $\frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1} = 4k$. So, the inequality $\dim\pi + m(\mathfrak{l}_1) > \frac{d(L_1)}{\mathrm{rank}_{\mathbb{R}} L_1}$ holds. We realize $\mathfrak{sp}(k,\ell) \subset \mathfrak{su}(2k,2\ell)$ as follows:

$$\begin{aligned}\sigma : \mathfrak{su}(2k,2\ell) &\to \mathfrak{su}(2k,2\ell),\\ X &\mapsto (J \otimes I_{k+\ell})\overline{X}(J \otimes I_{k+\ell})^{-1},\\ \mathfrak{sp}(k,\ell) &:= \mathfrak{su}(2k,2\ell)^{\sigma}.\end{aligned}$$

  We take $X \in \mathfrak{p}_L$ as follows:

$$X := \begin{pmatrix} & B \\ B^* & \end{pmatrix}, B := \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \\ & & & & \end{pmatrix} \in M(2k,2\ell;\mathbb{C}).$$

  Then we can prove that the condition (ii) of Lemma 1.3.20 is not satisfied in the same way as the above case $\mathfrak{l}_1 = \mathfrak{su}(k,\ell)$ $(k \geq \ell \geq 2)$.

□

Lemma 1.3.27. Let $(\mathfrak{l}, \pi)$ be a pair of a simple Lie algebra and its irreducible representation which is equivalent to one of the following table. Then $(\mathfrak{l}, \pi)$ does not satisfy the condition (iii) of Lemma 1.3.20.

Table 1.11: pairs of a simple Lie algebra and its irreducible representation in Table 1.8 which do not satisfy (iii) of Lemma 1.3.20

$\mathfrak{l}$	π	$\dim\pi$	$\frac{d(L)}{\mathrm{rank}_{\mathbb{R}} L}$	$d(L)$	$\mathrm{rank}_{\mathbb{R}} L_1$
$\mathfrak{su}(3,1)$	ϖ_2	6	6	6	1
$\mathfrak{so}^*(4n+2)$ $n \geq 2$	ϖ_1	$4n+2$	$4n+2$	$2n(2n+1)$	n

Proof. We can easily check from the data in table above. □

Lemma 1.3.28. Suppose a reprensetation ρ of $\mathfrak{l}$ satisfies the conditions (i) to (iv) of Lemma 1.3.14 and π is an irreducible component of $\rho|_{\mathfrak{l}_1}$. Then $(\mathfrak{l}_1, \pi)$ is equivalent to one of the following Table 1.12.

Table 1.12: pairs of simple Lie algebras $\mathfrak{l}_1$ and their irreducible representation which satisfy conditions (i) to (iii) of Lemma 1.3.20

$\mathfrak{l}_1$	π
$\mathfrak{su}(k,1)$ $(k \geq 2)$	ϖ_1

Proof. This comes from Lemma 1.3.16, 1.3.20, 1.3.25, 1.3.26 and 1.3.27. $\square$

Next, we determine the pair $(\mathfrak{l}^{ss},\ \rho|_{\mathfrak{l}}^{ss})$. It is enough to consider the case $\mathfrak{l}_1 = \mathfrak{su}(k,1)$ $(k \geq 2)$ from Lemma 1.3.28. We show the following:

Lemma 1.3.29. $(\mathfrak{l}^{ss},\ \rho|_{\mathfrak{l}^{ss}})$ is one of the following Table 1.13:

Table 1.13: candidates of pairs of semisimple Lie algebras $\mathfrak{l}^{ss}$ and their representation $\rho|_{\mathfrak{l}^{ss}}$

$\mathfrak{l}^{ss}$	$\rho\|_{\mathfrak{l}^{ss}}$
$\mathfrak{su}(k,1)$ $(k \geq 2)$	$\varpi_1 \oplus \oplus^{2(p+q)-(k+1)}\mathrm{triv}$

Proof. From Lemma 1.3.30, it is enough to show that $(\mathfrak{l}_1, \pi)$ satisfies the inequality $d(L_1) < 2\dim\pi - 1$. Since we have $d(L_1) = 2k$ and $\dim\pi = k+1$, this inequality holds. $\square$

Lemma 1.3.30. Let $G/H = SU(2p,2q)/Sp(p,q)$ $(p \geq q \geq 1)$. If a pair $(\mathfrak{l}_1,\ \pi)$ of a simple Lie algebra and its nontrivial representation satisfies the conditions $\mathrm{rank}_{\mathbb{R}} L_1 = 1$ and $d(L_1) < 2\dim\pi - 1$, then we have $\mathfrak{l}^{ss} = \mathfrak{l}_1$.

Proof. Assume that $d(L_1) < 2\dim\pi - 1$ and that there exists another simple factor $\mathfrak{l}_2$, which satisfies $d(L_1) \geq \frac{d(L_2)}{\mathrm{rank}_{\mathbb{R}} L_2} \geq \frac{d(L_i)}{\mathrm{rank}_{\mathbb{R}} L_i}$ for $2 \leq i \leq s$. Then by the injectivity of ρ, there exists an irreducible representation π' of $\mathfrak{l}_2$ such that $\dim\pi + \dim\pi' \leq \dim\rho \leq \frac{d(L_1)+d(L_2)}{1+\mathrm{rank}_{\mathbb{R}} L_2}$. From Lemma 1.3.31, we have $\dim\pi' < \frac{d(L_2)}{\mathrm{rank}_{\mathbb{R}} L_2} - 1$. From Weyl's dimensionality formula, a pair $(\mathfrak{l}_2, \pi')$ of a simple Lie algebra and its nontrivial representation is a standard representation of $\mathfrak{sp}(k',\ell')$ or $\mathfrak{su}(k',\ell')$. From Lemma 1.3.32 and 1.3.33, we have $2\dim\pi + 2 \leq d(L_1) < 2\dim\pi - 1$. This is contradiction. $\square$

The following Lemmas 1.3.31, 1.3.32 and 1.3.33 are used to prove Lemma 1.3.30.

We apply Lemma 1.3.31 by substituting $d(L_1)$, $\dim\pi$, $\mathrm{rank}_{\mathbb{R}} L_2$ and $d(L_2)$ for B, C, p and q respectively.

Lemma 1.3.31. Let $1 \leq B$, $2 \leq C$. The following conditions on B and C are equivalent:

- For any $p, q \in \mathbb{R}$, if $1 \leq p \leq q$, $\frac{q}{p} \leq B$, then $\frac{B+q}{1+p} - C < \frac{q}{p} - 1$,
- $B < 2C - 1$.

Proof. We prove that the following negative propositions are equivalent:

- There exist $p, q \in \mathbb{R}$ such that $1 \leq p \leq q$, $\frac{q}{p} \leq B$ and $\frac{B+q}{1+p} - C \geq \frac{q}{p} - 1$.
- $B \geq 2C - 1$.

$\frac{B+q}{1+p} - C \geq \frac{q}{p} - 1 \iff q \leq -(C-1)p(p - \frac{B+1-C}{C-1}) =: f(p)$. Then there exist such p, q if and only if

$$f(1) \geq 1.$$

We can easily obtain the above equivalence by considering pq plane with convex curve $q = f(p)$. This is equivalent to the following:

$$B \geq 2C - 1.$$

□

The following lemma is for the case $\mathfrak{l}_2 = \mathfrak{sp}(k', \ell')$ $(k' \geq \ell' \geq 1)$.

Lemma 1.3.32. Let $1 \leq B$, $2 \leq C$. If there exist $1 \leq \ell' \leq k'$ such that

$$\begin{cases} 4k' \leq B \\ C \leq \frac{B+4k'\ell'}{1+\ell'} - 2(k' + \ell') \end{cases},$$

then we have $2C + 4 \leq B$.

Proof. This comes from the following:

$$\begin{aligned} C &\leq \frac{B + 4k'\ell'}{1 + \ell'} - 2(k' + \ell') \\ &= \frac{B + 4k'(1 + \ell') - 4k'}{1 + \ell'} - 2(k' + \ell') \\ &= \frac{B - 4k'}{1 + \ell'} + 2k' - 2\ell' \\ &\leq \frac{B}{2} - 2. \end{aligned}$$

Here we have the last inequality by substituting 1 for ℓ' considering that $\frac{B-4k'}{1+\ell'} + 2k' - 2\ell'$ is monotone decreasing with regard to ℓ'. □

The following lemma is for the case $\mathfrak{l}_2 = \mathfrak{su}(k', \ell')$ $(k' \geq \ell' \geq 1)$.

Lemma 1.3.33. If there exist $1 \leq \ell' \leq k'$ such that

$$\begin{cases} 2k' \leq B \\ C \leq \frac{B+2k'\ell'}{1+\ell'} - (k' + \ell') \end{cases}$$

then $2C + 2 \leq B$.

Proof. We can prove this by the similar argument with Lemma 1.3.32. □

Finally, we determine pairs $(\mathfrak{l},\ \rho)$ satisfying the conditions (i) to (iv) of Lemma 1.3.14.

Lemma 1.3.34. If a pair $(\mathfrak{l}, \rho)$ of a reductive Lie algebra and its faithful representation satisfies the conditions (i) to (iv) of Lemma 1.3.14, then $(\mathfrak{l}, \rho)$ is equivalent to one of the following table:

Table 1.14: candidates of pairs of reductive Lie algebras $\mathfrak{l}$ and their representation ρ.

$\mathfrak{l}$	ρ
$\mathfrak{su}(2p,1)$	$\varpi_1 \oplus \mathrm{triv}$

Proof. We show $\mathfrak{l} = \mathfrak{l}^{ss} = \mathfrak{su}(2p,1)$ (i.e. $k = 2p$, $q = 1$), and $\rho = \varpi_1 \oplus \mathrm{triv}$. Suppose $\mathfrak{l}^{ss} = \mathfrak{su}(k,1)$ satisfies conditions (i) to (iv) of Lemma 1.3.14. Then we have the following inequalities

$$\begin{cases} 1 \leq q \leq p \\ 1+t \leq q \\ 2k+t = 4pq \\ k+1 \leq 2(p+q) \end{cases}$$

Here $t = \dim \rho(\mathfrak{z})^{-\theta}$, where $\mathfrak{z}$ is the center of $\mathfrak{l}$. It is enough to show $t = 0$ by Remark 1.4.49. Assume $t \geq 1$. Then we have

$$4pq = 2k+t \leq (1+t)k+(1+t)-1 \leq (1+t)(k+1)-1 \leq 2(p+q)q-1 \leq 4pq-1.$$

This is contradiction. So we obtain $t = 0$. Then we have $k = 2pq$ from the third equality. From the fourth inequality, we have $2pq+1 \leq 2(p+q) \iff (p-1)(q-1) \leq \frac{1}{2}$, which implies $q = 1$, $k = 2p$. As a result, we obtain $\rho = \pi \oplus \mathrm{triv}$. $\square$

1.3.3 $(G, H) = (SO(p,q+1), SO(p,1) \times SO(q))$ $(2 \leq q < HR(p))$

In this subsection, we consider the case $(G, H) = (SO(p,q+1), SO(p,1) \times SO(q))$ $(2 \leq q < HR(p))$. Our goal in this subsection is the following:

Proposition 1.3.35. Let $G = SO(p,q+1)$ and $H = SO(p,1) \times SO(q)$ $(2 \leq q < HR(p))$. If there exists a reductive subgroup L of G acting on G/H properly and cocompactly, then $(p,q) = (4,3)$ or $(4,2)$. Moreover L is locally isomorphic to $Spin(4,3)$ or $G_{2(2)}$ respectively up to compact factor.

Proof. This comes from Lemma 1.3.36 and Proposition 1.3.37. $\square$

The condition $2 \leq q < HR(p)$ implies that p is even and $p \geq 4$. So, we consider the case $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(2p,q+1), \mathfrak{so}(2p,1) \oplus \mathfrak{so}(q))$ with $2 \leq q < HR(2p)$ and $p \geq 2$.

Lemma 1.3.36. Let p, q be integers satisfying $2 \leq q < HR(2p)$ and $G/H = SO_0(2p,q+1)/SO_0(2p,1) \times SO(q)$. There exists a reductive subgroup of G acting on G/H properly and cocompactly if and only if there exists a reductive Lie algebra $\mathfrak{l}$ and its faithful representation $\rho : \mathfrak{l} \to \mathfrak{sl}(2p+q+1, \mathbb{C})$ satisfying the following conditions:

(i) $\rho(\mathfrak{l}) \subset \mathfrak{so}(2p,q+1) \subset \mathfrak{sl}(2p+q+1, \mathbb{C})$,

(ii) $\rho(\mathfrak{l})$ is preserved by matrix transpose,

(iii) $\mathfrak{a}_L \cap W\mathfrak{a}_H = \{0\}$,

(iv) $d(L) = d(G) - d(H)$.

Here W is the Weyl group of $SO_0(2p, q+1)$ and L is the analytic subgroup of G corresponding to $\rho(\mathfrak{l}) \subset \mathfrak{so}(2p, q+1)$. We consider subalgebra $\mathfrak{so}(2p, q+1) \subset \mathfrak{sl}(2p+q+1, \mathbb{C})$ as standard inclusion preserverd by matrix transpose ($X \mapsto {}^tX$),

Proof. This comes from Fact 1.2.15 and 1.2.18. □

Proposition 1.3.37. Let p, q be integers satisfying $2 \leq q < HR(2p)$ and $G/H = SO_0(2p, q+1)/SO_0(2p, 1) \times SO(q)$. There exists a reductive Lie algebra $\mathfrak{l}$ and its faithful representation $\rho : \mathfrak{l} \to \mathfrak{sl}(2p+q+1, \mathbb{C})$ satisfying the conditions (i) to (iv) of Lemma 1.3.36 if and only if $(p, q) = (2, 3)$ or $(2, 2)$. Moreover, such pair $(\mathfrak{l}, \rho)$ is equivalent to $(\mathfrak{so}(4, 3), \rho_{\varpi_3})$ or $(\mathfrak{g}_{2(2)}, \rho_{\varpi_1})$ up to compact factor.

Proof. Our proof consists of step a, b, c and d (see Outline of the proof for the case $\operatorname{rank}_{\mathbb{R}} G - \operatorname{rank}_{\mathbb{R}} H \geq 2$ in subsection 1.2.5). This comes from Lemma 1.3.7, Remark 1.3.52, Lemma 1.3.57 and 1.3.59. □

Notation 1.3.38. We put $M^{ss} = \frac{d(L^{ss})}{\operatorname{rank}_{\mathbb{R}} L^{ss}}$, $M_1 := \frac{d(L_1)}{\operatorname{rank}_{\mathbb{R}} L_1}$.

step a : reduce candidates by upper bound of the dimension of representations

Lemma 1.3.39. Let $\rho : \mathfrak{l} \to \mathfrak{sl}(n, \mathbb{C})$ be a representation satisfying the conditions (i) to (iv) of Proposition 1.3.36 and π an irreducible component of $\rho|_{\mathfrak{l}_1}$. Then the pair $(\mathfrak{l}_1, \pi)$ satisfies the following conditions:

(i) $\dim \pi \leq \min(\operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1, 2M_1)$,

(ii) $\max(7, 1 + \sqrt{4d(L_1) + 1}) \leq 2M_1$.

Proof. The condition (ii) and the inequality $\dim \pi \leq 2M_1$ in (i) of Lemma 1.3.39 comes from Lemma 1.3.40, $d(L^{ss}) \leq d(L_1)$ and $M^{ss} \leq M_1$. Next we show the inequality $\dim \pi \leq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$. From Lemma 1.3.40, we have $\dim \rho \leq \operatorname{rank}_{\mathbb{R}} L^{ss} + M^{ss} + 1 \leq \operatorname{rank}_{\mathbb{R}} L^{ss} + M_1 + 1$. So, it is enough to show that

$$\dim \pi - \operatorname{rank}_{\mathbb{R}} L_1 \leq \dim \rho - \operatorname{rank}_{\mathbb{R}} L^{ss}.$$

This comes from Lemma 1.3.44. □

Lemma 1.3.40. Suppose a reductive Lie algebra $\mathfrak{l}$ and its faithful representation $\rho : \mathfrak{l} \to \mathfrak{sl}(V)$ satisfy the conditions (i) to (iv) of Lemma 1.3.36 for some positive integers p, q with $2 \leq q < HR(2p)$. Then the following inequality holds:

$$\max(7, 1 + \sqrt{4d(L^{ss}) + 1}) \leq \dim \rho \leq \operatorname{rank}_{\mathbb{R}} L^{ss} + M^{ss} + 1 \leq 2M^{ss}.$$

Proof. This comes from the following Lemmas 1.3.41 and 1.3.42. □

Lemma 1.3.41. Fix positive integers p, q with $2 \leq q < HR(2p)$. Suppose a reductive Lie algebra $\mathfrak{l}$ and its faithful representation $\rho : \mathfrak{l} \to \mathfrak{sl}(n, \mathbb{C})$ ($n := 2p+q+1$) satisfy the conditions (i) to (iv) in Proposition 1.3.36. Then we have the following inequalities:

$$\begin{cases} \operatorname{rank}_{\mathbb{R}} L^{ss} + t \leq q \leq 2p-1, \\ d(L^{ss}) + t = 2pq, \\ 7 \leq \dim \rho = 2p+q+1. \end{cases}$$

Here $d(L^{ss})$, $\operatorname{rank}_{\mathbb{R}} L^{ss}$ are the noncompact dimension, real rank of semisimple Lie subalgebra $\mathfrak{l}^{ss}$ of $\mathfrak{l}$ respectively and $t = \dim \rho(\mathfrak{z})^{-\theta}$, where $\mathfrak{z}$ is the center of $\mathfrak{l}$.

Proof. This comes from Remark 1.2.16 and Fact 1.2.18. □

Lemma 1.3.42. Let $1 \leq A \leq B$ and $C \geq 6$. There exist real numbers $1 \leq p, q$ and $t \geq 0$ such that

$$\begin{cases} A + t \leq q \leq 2p-1 \\ B + t = 2pq \\ C = 2p+q+1 \end{cases}$$

if and only if $\max(6, 1+\sqrt{4B+1}) \leq \dim \rho \leq A + \frac{B}{A} + 1$. Moreover, then we have $A + \frac{B}{A} + 1 \leq \frac{2B}{A}$.

Proof. We can easily check this lemma by fundamental argument on inequality. So, we omit the proof. □

Remark 1.3.43. Let $\mathfrak{l}$ be a simple Lie algebra over $\mathbb{R}$ and π a nontrivial irreducible representation. Then we have

$$\dim \pi \geq \operatorname{rank}_{\mathbb{R}} L + 1.$$

This comes from the classification of simple Lie algebras.

Lemma 1.3.44. Let $\rho : \mathfrak{l}^{ss} \to \mathfrak{sl}(V)$ be a faithful representation of semisimple Lie algebra $\mathfrak{l}^{ss} = \oplus_{i=1}^{s} \mathfrak{l}_i$. Let π be a nontrivial irreducible component of $\rho|_{\mathfrak{l}_1}$. Then the following inequality holds:

$$\dim \pi - \operatorname{rank}_{\mathbb{R}} L_1 - 1 \leq \dim \rho - \operatorname{rank}_{\mathbb{R}} L^{ss} - s.$$

Proof. From Remark 1.2.38(iv), the injectivity of ρ and Remark 1.3.43,

$$\begin{aligned} \dim \rho &\geq \dim \pi + \sum_{i=2}^{s} m(\mathfrak{l}_i) \\ &\geq \dim \pi + \sum_{i=2}^{s} (\operatorname{rank}_{\mathbb{R}} L_i + 1) \\ &= \dim \pi + \operatorname{rank}_{\mathbb{R}} L^{ss} - \operatorname{rank}_{\mathbb{R}} L_1 + s - 1. \end{aligned}$$

□

step b: reduce candidates of primary factor by using criterion for embeddability of semisimple Lie algebras

Lemma 1.3.45. Let $\rho : \mathfrak{l} \to \mathfrak{sl}(n, \mathbb{C})$ be a representation satisfying the conditions (i) to (iv) of Lemma 1.3.36 and π an irreducible component of $\rho|_{\mathfrak{l}_1}$. The pair $(\mathfrak{l}_1, \pi)$ satisfies the following conditions:

(i) (a) If the inequality $\dim \pi > M_1$ holds, then the following three conditions are satisfied:

i. $\overline{\pi} \simeq \pi$ and $\operatorname{index}_{\tau_1} \pi = 1$,

ii. $\pi^\vee \simeq \pi$ and $\operatorname{index}_{\theta_1} \pi = 1$,

iii. $\pi \simeq \pi^*$.

Here, τ_1 is the involution on $\mathfrak{l}_1^{\mathbb{C}}$ detemining $\mathfrak{l}_1$ and θ_1 is the Cartan involution on $\mathfrak{l}_1^{\mathbb{C}}$.

(b) If the inequality $2 \dim \pi > M_1$ holds, then at least one of the following conditions holds:

i. $\pi \simeq \overline{\pi}$,

ii. $\pi \simeq \pi^\vee$,

iii. $\pi \simeq \pi^*$.

(c) Assume the following conditions:

i. $2 \dim \pi > M_1$,

ii. ($\pi \simeq \overline{\pi}$, $\operatorname{index}_{\tau_1} \pi = -1$) or ($\pi \simeq \pi^\vee$ and $\operatorname{index}_{\theta_1} \pi = -1$)

Then we have $\pi^* \simeq \pi$ as a representation of $\mathfrak{l}_1$ and $[\pi : \rho|_{\mathfrak{l}_1}] = 2$.

(ii) $\dim \pi > M_1$ and $\dim \pi + m(\mathfrak{l}_1) > \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1 \implies \operatorname{rank} \pi(A) \geq 4$ for any $A \in \mathfrak{p}_{L_1} \setminus \{0\}$.

Here $m(\mathfrak{l}_i) := \min\{\dim \pi' : \pi' \text{ is a nontrivial irreducible representation of } \mathfrak{l}_i\}$ $(i = 1, \cdots, s)$.

Proof. (i) (a) This comes from Lemma 1.2.40 and $\dim \rho \leq 2M_1$.

(b) This comes from Lemma 1.3.47 and $\dim \rho \leq 2M_1$.

(c) This comes from Lemma 1.3.48 and $\dim \rho \leq 2M_1$.

(ii) It is enough to show that $\rho|_{\mathfrak{l}_1} = \pi \oplus \oplus^{\dim V - \dim \pi} \mathrm{triv}$. In fact, the rank condition comes in the same way as the case (G, H)=$(SO(2p, q+1), SO(2p, q))$.

From the assumption $\dim \pi > M_1$, we have $[\pi : \rho|_{\mathfrak{l}_1}]$. Assume that $\rho|_{\mathfrak{l}_1}$ has another nontrivial irreducible component π'. From Lemma 1.2.38 and injectivity of ρ, we have

$$\dim \pi + \dim \pi' + \sum_{i=2}^{s} m(\mathfrak{l}_i) \leq \dim \rho \leq \operatorname{rank}_{\mathbb{R}} L^{ss} + M_1 + 1.$$

From Remark 1.3.43, we have $\operatorname{rank}_{\mathbb{R}} L^{ss} \leq \operatorname{rank}_{\mathbb{R}} L_1 + \sum_{i=2}^{s} m(\mathfrak{l}_i)$. From the above two inequalities and the assumption $\dim \pi + m(\mathfrak{l}_1) > \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$, we have

$$\dim \pi + \dim \pi' \leq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1 < \dim \pi + m(\mathfrak{l}_1) \leq \dim \pi + \dim \pi'.$$

This is contradiction.

□

Remark 1.3.46. The rank condition (ii) of Lemma 1.3.45 is preserved by basis transformation. So we can discuss properness up to equivalent class of representation.

Lemma 1.3.47. Let $\rho : \mathfrak{l}^{ss} = \mathfrak{l}_1 \oplus \cdots \oplus \mathfrak{l}_k \to \mathfrak{sl}(V)$ be a representation of semisimple Lie algebra $\mathfrak{l}$ such that $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{so}(p,q)$ for some $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q = \dim_{\mathbb{C}} V$. Let $\mathfrak{l}_1$ be a simple ideal of $\mathfrak{l}^{ss}$ and π an irreducible component of $\rho|_{\mathfrak{l}_1}$. If the inequality $4 \dim \pi > \dim \rho$ holds, then at least one of the following conditions holds:

(i) $\pi \simeq \overline{\pi}$,

(ii) $\pi \simeq \pi^{\vee}$,

(iii) $\pi \simeq \pi^{*}$.

Proof. We show the contraposition. Suppose $\pi \not\simeq \overline{\pi}$, $\pi \not\simeq \pi^{\vee}$ and $\pi \not\simeq \pi^{*}$. From $\rho|_{\mathfrak{l}_1}(\mathfrak{l}_1) \subset_{\mathrm{Int}} \mathfrak{so}(p,q)$ for some $p + q = \dim V$ $(p, q \geq 0)$, $\rho|_{\mathfrak{l}_1}$ has at least one π, $\overline{\pi}$, $\pi^{\vee}$ and π^{*} as an irreducible component respectively by Lemma 1.2.42. Therefore we have $4 \dim \pi \leq \dim \rho$. □

Lemma 1.3.48. Let $\rho : \mathfrak{l}_1 \oplus \cdots \oplus \mathfrak{l}_k = \mathfrak{l}^{ss} \to \mathfrak{sl}(V)$ be a representation of semisimple Lie algebra such that $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{so}(p,q)$ for some $p + q = \dim_{\mathbb{C}} V$. Let $\mathfrak{l}_1$ be a simple ideal of $\mathfrak{l}^{ss}$ and π_1 an irreducible component of $\rho|_{\mathfrak{l}_1}$ satisfying the two conditions:

(i) $\dim \rho < 4 \dim \pi_1$,

(ii) $(\pi \simeq \overline{\pi}, \operatorname{index}_{\tau_1} \pi = -1)$ or $(\pi \simeq \pi^{\vee}$ and $\operatorname{index}_{\theta_1} \pi = -1)$

Then we have $\pi^{*} \simeq \pi$ as a representation of $\mathfrak{l}_1$ and $[\pi : \rho|_{\mathfrak{l}_1}] = 2$.

Proof. Form Lemma 1.2.43 and the assumption (i), we have $[\pi : \rho|_{\mathfrak{l}_1}] = 2 = [\pi^{*} : \rho|_{\mathfrak{l}_1}]$. If we put $\pi^{*} \not\simeq \pi$, then we have $4 \dim \pi \leq \dim \rho < 4 \dim \pi$, which is contradiction. So we obtain $\pi^{*} \simeq \pi$. □

Lemma 1.3.49. Suppose a pair $(\mathfrak{l}, \rho)$ of a reductive Lie algebra and its faithful representation satisfying the conditions (i) to (iv) of Lemma 1.3.36. Let $\mathfrak{l}_1$ be a primary factor of $\mathfrak{l}^{ss}$ and π a nontrivial irreducible component of $\rho|_{\mathfrak{l}_1}$. Then $(\mathfrak{l}_1, \pi)$ is equivalent to one of the following:

Table 1.15: pairs of simple Lie algebras $\mathfrak{l}$ and their irreducible representations π which satisfy the condition (i) and (ii) of Lemma 1.3.39

$\mathfrak{l}$	π	$\dim \pi$	not satisfy the condition of Lemma 1.3.45
$\mathfrak{sl}(n,\mathbb{C})$ $(n \geq 3)$	$\varpi_1 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_1$,	n	(i)
$\mathfrak{sl}(3,\mathbb{C})$	$2\varpi_1 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes 2\varpi_1$,	6	(i)
$\mathfrak{sl}(4,\mathbb{C})$	$\varpi_2 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_2$	6	(i)
$\mathfrak{sl}(5,\mathbb{C})$	$\varpi_2 \boxtimes \mathrm{triv}$, $\varpi_3 \boxtimes \mathrm{triv}$,	10	(i)
$\mathfrak{sl}(n,\mathbb{R})$ $(n \geq 3)$	ϖ_1	n	(i)
$\mathfrak{su}^*(2n)$ $(n \geq 2)$	ϖ_1	$2n$	(i)
$\mathfrak{su}(k,\ell)$ $(k+\ell \geq 3)$	ϖ_1, $\varpi_{k+\ell-1}$	$k+\ell$	
$\mathfrak{su}(2,1)$	$2\varpi_1$, $2\varpi_2$	6	(i)
$\mathfrak{su}(3,1)$	ϖ_2	6	
$\mathfrak{su}(4,1)$	ϖ_2, ϖ_3	10	(i)
$\mathfrak{so}(2n+1,\mathbb{C})$ $(n \geq 2)$	$\varpi_1 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_1$	$2n+1$	
$\mathfrak{so}(7,\mathbb{C})$	$\varpi_3 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_3$	8	(i)
$\mathfrak{so}(2n,\mathbb{C})$ $(n \geq 4)$	$\varpi_1 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_1$	$2n$	(i)
$\mathfrak{so}(8,\mathbb{C})$	$\varpi_3 \boxtimes \mathrm{triv}$, $\varpi_4 \boxtimes \mathrm{triv}$	8	(i)
$\mathfrak{so}(k,\ell)$ $(k+\ell \geq 5)$	ϖ_1	$k+\ell$	(ii)
$\mathfrak{so}(6,1)$	ϖ_3	8	(i)
$\mathfrak{so}(5,2)$	ϖ_3	8	(i)
$\mathfrak{so}(4,3)$	ϖ_3	8	
$\mathfrak{so}(7,1)$	ϖ_3, ϖ_4	8	(i)
$\mathfrak{so}(6,2)$	ϖ_3, ϖ_4	8	(i)
$\mathfrak{so}(5,3)$	ϖ_3, ϖ_4	8	(i)
$\mathfrak{so}^*(4n)$ $(n \geq 3)$	ϖ_1	$4n$	(i)
$\mathfrak{so}^*(4n+2)$ $(n \geq 2)$	ϖ_1	$4n+2$	
$\mathfrak{sp}(2,\mathbb{C})$	$\varpi_1 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_1$	4	
$\mathfrak{sp}(n,\mathbb{C})$ $(n \geq 3)$	$\varpi_1 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_1$	$2n$	
$\mathfrak{sp}(n,\mathbb{R})$ $(n \geq 2)$	ϖ_1	$2n$	(i)
$\mathfrak{sp}(k,\ell)$ $(k+\ell \geq 2)$	ϖ_1	$2(k+\ell)$	
$\mathfrak{g}_2^{\mathbb{C}}$	$\varpi_1 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_1$	7	
$\mathfrak{g}_{2(2)}$	ϖ_1	7	

Here $k \geq \ell \geq 1$,

Proof. This follows from the classification of simple Lie algebras and Weyl's dimensionality formula. $\square$

Lemma 1.3.50. The following pairs $(\mathfrak{l}, \pi)$ of a simple Lie algebra and its irrducible representation does not satisfy the condition (i) of Lemma 1.3.45.

Table 1.16: simple Lie algebra which does not satisfy the condition (i) of Lemma 1.3.45

$\mathfrak{l}$	π	$\dim \pi$	M_1	data	(a), (b) or (c)
$\mathfrak{sl}(n, \mathbb{C})$ $(n \geq 3)$	$\varpi_1 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_1$	n	$n+1$	$\pi^\vee \not\simeq \pi \not\simeq \overline{\pi}, \pi \not\simeq \pi^*$	(b)
$\mathfrak{sl}(3, \mathbb{C})$	$2\varpi_1 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes 2\varpi_1$,	6	4	$\pi \not\simeq \overline{\pi}$	(a)
$\mathfrak{sl}(4, \mathbb{C})$	$\varpi_2 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_2$	6	5	$\pi \not\simeq \overline{\pi}$	(a)
$\mathfrak{sl}(5, \mathbb{C})$	$\varpi_2 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_2$	10	6	$\pi \not\simeq \overline{\pi}$	(a)
$\mathfrak{sl}(n, \mathbb{R})$ $(n \geq 3)$	ϖ_1	n	$\frac{n+2}{2}$	$\pi \not\simeq \pi^\vee$	(a)
$\mathfrak{su}^*(2n)$ $(n \geq 2)$	ϖ_1	$2n$	$2n+1$	$\pi^* \not\simeq \pi$	(c)
$\mathfrak{su}(2, 1)$	$2\varpi_1$	6	4	$\pi \not\simeq \overline{\pi}$	(a)
$\mathfrak{su}(4, 1)$	ϖ_2	10	8	$\pi \not\simeq \overline{\pi}$	(a)
$\mathfrak{so}(7, \mathbb{C})$	$\varpi_3 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_3$	8	7	$\pi \not\simeq \overline{\pi}$	(a)
$\mathfrak{so}(2n, \mathbb{C})$ $(n \geq 4)$	$\varpi_1 \boxtimes \mathrm{triv}$, $\mathrm{triv} \boxtimes \varpi_1$	$2n$	$2n-1$	$\pi \not\simeq \overline{\pi}$	(a)
$\mathfrak{so}(8, \mathbb{C})$	$\varpi_3 \boxtimes \mathrm{triv}$, $\varpi_4 \boxtimes \mathrm{triv}$	8	7	$\pi \not\simeq \overline{\pi}$	(a)
$\mathfrak{so}(6, 1)$	ϖ_3	8	6	$\mathrm{index}_{\tau_1} \pi = -1$	(a)
$\mathfrak{so}(5, 2)$	ϖ_3	8	5	$\mathrm{index}_{\tau_1} \pi = -1$	(a)
$\mathfrak{so}(7, 1)$	ϖ_3, ϖ_4	8	7	$\pi \not\simeq \overline{\pi}$	(a)
$\mathfrak{so}(6, 2)$	ϖ_3, ϖ_4	8	6	$\mathrm{index}_{\tau_1} \pi = -1$	(a)
$\mathfrak{so}(5, 3)$	ϖ_3, ϖ_4	8	$\frac{15}{2}$	$\pi \not\simeq \overline{\pi}$	(a)
$\mathfrak{so}^*(4n) (n \geq 3)$	ϖ_1	$4n$	$4n-2$	$\mathrm{index}_{\tau_1} \pi = -1$	(a)
$\mathfrak{sp}(n, \mathbb{R})$ $(n \geq 2)$	ϖ_1	$2n$	$n+1$	$\mathrm{index}_{\theta_1} \pi = -1$	(a)

Proof. This follows from the data in the above table. $\square$

Lemma 1.3.51. Let $\rho : \mathfrak{l} \to \mathfrak{sl}(V)$ be a faithful representation of a reductive Lie algebra $\mathfrak{l}$ and $\mathfrak{l}_1 = \mathfrak{so}(k, \ell)$ $(k + \ell \geq 5)$ the primary factor of $\mathfrak{l}$. If $\rho|_{\mathfrak{l}_1}$ has a irreducible component $\pi \simeq \rho_{\varpi_1}$, then ρ does not satisfy the conditions (i) to (iv) of Lemma 1.3.36.

Proof. We show this by using Lemma 1.3.45 (iv). We have $\dim \pi = k + \ell > k = M_1$. Since we have $m(\mathfrak{so}(k, \ell)) \geq 4$ for any $k+\ell \geq 5$, we have $\dim \pi + m(\mathfrak{l}_1) \geq k+\ell+4 > \ell+k+1 = \mathrm{rank}_{\mathbb{R}} L_1 + M_1 + 1$. However, for $S_{1,k+1} = E_{1,k+1} + E_{k+1,1} \in \mathfrak{p}_L$, we have $\mathrm{rank}\, \pi(S_{1,k+1}) = 2$. $\square$

Remark 1.3.52. Let $\rho : \mathfrak{l} \to \mathfrak{sl}(V)$ be a faithful representation of a reductive Lie algebra satisfying the conditions (i) to (iv) of Lemma 1.3.36. Then a pair $(\mathfrak{l}_1, \pi)$ of primary factor and its irreducible representation is equivalent to one of the following:

Table 1.17: pairs of a simple Lie algebra $\mathfrak{l}$ and its irreducible representation π which satisfy the condition (i), (ii), (iii) and (iv) of Lemma 1.3.45

$\mathfrak{l}$	π	$\dim \pi$	$\operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$	selfconj? $\operatorname{index}_{\tau_1} \pi$
$\mathfrak{su}(k, \ell)$ $(k + \ell \geq 3)$	ϖ_1	$k + \ell$	$2k + \ell + 1$	$\pi \not\simeq \overline{\pi}$
$\mathfrak{su}(3, 1)$	ϖ_2	6	8	$\operatorname{index}_{\tau_1} \pi = -1$
$\mathfrak{so}(2n + 1, \mathbb{C})$ $(n \geq 2)$	$\varpi_1 \boxtimes \mathrm{triv}$	$2n + 1$	$3n + 2$	$\pi \not\simeq \overline{\pi}$
$\mathfrak{so}(4, 3)$	ϖ_3	8	8	
$\mathfrak{so}^*(4n + 2)$ $(n \geq 2)$	ϖ_1	$4n + 2$	$5n + 3$	$\operatorname{index}_{\tau_1} \pi = -1$
$\mathfrak{sp}(2, \mathbb{C})$	$\varpi_1 \boxtimes \mathrm{triv}$	4	8	
$\mathfrak{sp}(n, \mathbb{C})$ $(n \geq 3)$	$\varpi_1 \boxtimes \mathrm{triv}$	$2n$	$3n + 2$	$\pi \not\simeq \overline{\pi}$
$\mathfrak{sp}(k, \ell)$ $(k + \ell \geq 1)$	ϖ_1	$2(k + \ell)$	$4k + \ell + 1$	$\operatorname{index}_{\tau_1} \pi = -1$
$\mathfrak{g}_2^{\mathbb{C}}$	$\varpi_1 \boxtimes \mathrm{triv}$	7	10	$\pi \not\simeq \overline{\pi}$
$\mathfrak{g}_{2(2)}$	ϖ_1	7	7	

This comes from Lemma 1.3.49, 1.3.50 and 1.3.51.

step c : determine the pairs $(\mathfrak{l}^{ss}, \rho|_{\mathfrak{l}}^{ss})$

Lemma 1.3.53. Let p and q be positive integers with $2 \leq q < HR(2p)$. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(2p + q + 1, \mathbb{C})$ satisfies conditions (i) to (iv) of Lemma 1.3.36. Let $\mathfrak{l}_1$ be a primary simple factor of $\mathfrak{l}$ and π a nontrivial irreducible component of $\rho|_{\mathfrak{l}_1}$. Assume at least one of the following two conditions (i) and (ii) is satisfied:

(i) $(\pi \not\simeq \overline{\pi}$ or $(\pi \simeq \overline{\pi}$ and $\operatorname{index}_{\tau_1} \pi = -1))$ and $2 \dim \pi \geq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$,

(ii) $\dim \pi \geq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$.

Then we have $\mathfrak{l}^{ss} = \mathfrak{l}_1$. Moreover, the last equality $2 \dim \pi = \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$ of (i) is attained if the condition (i) is satisfied, and the equality $\dim \pi = \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$ is attained if the condition (ii) is satisfied.

Proof. Let $\mathfrak{l}^{ss} = \oplus_{i=1}^{s} \mathfrak{l}_i$ be the decomposition into simple ideals, where $\mathfrak{l}_i$ $(i = 1, \cdots, s)$ are simple ideals.

(i)
- In the case $\pi \not\simeq \overline{\pi}$ and $2 \dim \pi \geq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$:
  From Lemma 1.3.54, we have $2 \dim \pi + (s - 1) \leq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1 \leq 2 \dim \pi$. Thus we obtain $s = 1$. Note that we have $2 \dim \pi \leq \dim \rho$.
- In the case $\pi \simeq \overline{\pi}$, $\operatorname{index}_{\tau_1} \pi = -1$ and $2 \dim \pi \geq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$:
  We show this case in the same way as above by using Lemma 1.3.54. It is enough to show that $[\pi : \rho|_{\mathfrak{l}_1}] = 2$. This comes from Lemma 1.2.43 and $4 \dim \pi \geq 2(\operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1) > 2M_1 \geq \dim \rho$.

In the both cases, from Lemma 1.3.40, we have

$$\operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1 \leq 2 \dim \pi \leq \dim \rho \leq \operatorname{rank}_{\mathbb{R}} L^{ss} + M^{ss} + 1 = \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1.$$

Thus we have $2 \dim \pi = \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$.

(ii) From Lemma 1.3.44 and 1.3.40, we have

$$\dim \pi + s - 1 + \operatorname{rank}_{\mathbb{R}} L^{ss} - \operatorname{rank}_{\mathbb{R}} L_1 \leq \dim \rho \leq \operatorname{rank}_{\mathbb{R}} L^{ss} + M_1 + 1.$$

Therefore we have $\dim \pi + s - 1 \leq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$. By the assumption $\operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1 \leq \dim \pi$, we obtain $s = 1$ and $\dim \pi = \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$.

□

Lemma 1.3.54. Let p and q be positive integers with $2 \leq q < HR(2p)$. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(2p+q+1, \mathbb{C})$ satisfies conditions (i) to (iv) of Lemma 1.3.36. Let $\mathfrak{l}_1$ be a primary simple factor of $\mathfrak{l}$ and π a nontrivial irreducible component of $\rho|_{\mathfrak{l}_1}$. Assume at least one of the following two conditions (i) and (ii) is satisfied:

(i) $\pi \not\simeq \overline{\pi}$,

(ii) $[\pi : \rho|_{\mathfrak{l}_1}] = 2$ and $\pi \simeq \overline{\pi}$ and $\operatorname{index}_{\tau_1} \pi = -1$.

Then we have

$$2 \dim \pi + (s-1) \leq \operatorname{rank}_{\mathbb{R}} L_1 + \frac{d(L_1)}{\operatorname{rank}_{\mathbb{R}} L_1} + 1.$$

Here s is the number of simple ideals of $\mathfrak{l}^{ss} = [\mathfrak{l}, \mathfrak{l}]$.

Proof. (i) In the case $\pi \not\simeq \overline{\pi}$:
From Lemma 1.2.42, 1.2.38 and injectivity of ρ, we have

$$2 \dim \pi + \sum_{i=2}^{s} m(\mathfrak{l}_i) \leq \dim \rho \leq \operatorname{rank}_{\mathbb{R}} L^{ss} + M_1 + 1.$$

From Remark 1.3.43, we have

$$\operatorname{rank} L^{ss} - \operatorname{rank}_{\mathbb{R}} L_1 + s - 1 \leq \sum_{i=2}^{s} m(\mathfrak{l}_i).$$

Therefore, we obtain the desired inequality from the above two inequalities.

(ii) In the case $\pi \simeq \overline{\pi}$ and $\operatorname{index}_{\tau_1} \pi = -1$ and $[\pi : \rho|_{\mathfrak{l}_1}] = 2$:
Let $\rho|_{\mathfrak{l}^{ss}} = \oplus_{i=1}^{t} \rho_i$ be the decomposition into irreducible components. From Lemma 1.3.55, there exist $j \neq j' \in \{1, \cdots, t\}$ such that $\rho_j \simeq \pi \boxtimes \mathrm{triv}$, $\rho_{j'} \simeq \pi \boxtimes \mathrm{triv}$. Then from Lemma 1.2.38, we have

$$2 \dim \pi + \sum_{i=2}^{s} m(\mathfrak{l}_i) \leq \dim \rho.$$

Therefore we can prove this case in the same way as above.

□

Lemma 1.3.55. Let $\rho : \mathfrak{l}^{ss} = \oplus_{i=1}^{s}\mathfrak{l}_i \to \mathfrak{sl}(V)$ be a representation of a semisimple Lie algebra without compact factor, where $\mathfrak{l}_i$ $(i = 1, \cdots, s)$ are noncompact simple ideals. Let $\rho = \oplus_{j=1}^{t}\rho_j$ be the decomposition into irreducible components. Suppose that $\rho(\mathfrak{l}^{ss}) \subset_{\mathrm{Int}} \mathfrak{sl}(n,\mathbb{R})$, $\pi \simeq \overline{\pi}$, $\mathrm{index}_{\tau_1}\pi = -1$ and $[\pi : \rho|_{\mathfrak{l}_1}] = 2$, where π is a nontrivial irreducible component of $\rho|_{\mathfrak{l}_1}$. Then there exist $j \neq j' \in \{1, \cdots, t\}$ such that $\rho_j = \pi\boxtimes\mathrm{triv}$, $\rho_{j'} = \pi\boxtimes$ triv.

Proof. If there do not exist the above j, j', then there exists j such that $\rho_j = \pi \boxtimes \pi'$ and $\dim\pi' = 2$ where π' is the irreducible representation of $\oplus_{i=2}^{s}\mathfrak{l}_i$. Nontrivial irreducible representations of noncompact simple Lie algebra with dimension two are only the standard representations of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{C})$. In these cases, since $\mathrm{index}_\tau\rho_j = -1$ for $\mathfrak{sl}(2,\mathbb{R})$, $\rho_j \not\simeq \overline{\rho_j}$ for $\mathfrak{sl}(2,\mathbb{C})$, these do not induce embeddings into $\mathfrak{sl}(n,\mathbb{R})$ by Proposition 1.6.17. $\square$

Lemma 1.3.56. Let $\rho : \mathfrak{l} \to \mathfrak{gl}(V)$ be a faithful representation of a reductive Lie algebra $\mathfrak{l}$. If the pair $(\mathfrak{l}_1, \pi)$ of the primary simple factor and its irreducible component of $\rho|_{\mathfrak{l}}$ is equivalent to one of the following table, then ρ does not satisfy the conditions (i) to (iv) of Lemma 1.3.36.

Table 1.18: pairs of a simple Lie algebra $\mathfrak{l}$ and its irreducible representation π

$\mathfrak{l}_1$	π	$\dim\pi$	$\mathrm{rank}_\mathbb{R} L_1 + M_1 + 1$	selfconj? $\mathrm{index}_{\tau_1}\pi$
$\mathfrak{su}(k,\ell)$ $(k+\ell \geq 3)$	ϖ_1	$k+\ell$	$2k+\ell+1$	$\pi \not\simeq \overline{\pi}$
$\mathfrak{su}(3,1)$	ϖ_2	6	8	$\mathrm{index}_{\tau_1}\pi = -1$
$\mathfrak{so}(2n+1,\mathbb{C})$ $(n \geq 2)$	$\varpi_1\boxtimes\mathrm{triv}$	$2n+1$	$3n+2$	$\pi \not\simeq \overline{\pi}$
$\mathfrak{so}^*(4n+2)$ $(n \geq 2)$	ϖ_1	$4n+2$	$5n+3$	$\mathrm{index}_{\tau_1}\pi = -1$
$\mathfrak{sp}(n,\mathbb{C})$ $(n \geq 3)$	$\varpi_1\boxtimes\mathrm{triv}$	$2n$	$3n+2$	$\pi \not\simeq \overline{\pi}$
$\mathfrak{sp}(k,\ell)$ $(k+\ell \geq 1)$	ϖ_1	$2(k+\ell)$	$4k+\ell+1$	$\mathrm{index}_{\tau_1}\pi = -1$
$\mathfrak{g}_2^\mathbb{C}$	$\varpi_1\boxtimes\mathrm{triv}$	7	10	$\pi \not\simeq \overline{\pi}$

Proof.

- In the case $(\mathfrak{l}_1, \pi)$=$(\mathfrak{su}(k,\ell), \rho_{\varpi_1})$:
The assumption (i) of Lemma 1.3.53 is satisfied. In fact, we have $\pi \not\simeq \overline{\pi}$ and $2\dim\pi = 2(k+\ell) \geq 2k+\ell+1$. Therefore we have $\mathfrak{l}^{ss} = \mathfrak{l}_1$ and $2(k+\ell) = k+2\ell+1$, that is, $\ell = 1$. Then from Lemma 1.3.41, we have
$$\begin{cases} 1+t \leq q \leq 2p-1, \\ 2k+t = 2pq, \\ 2(k+1) = 2p+q+1. \end{cases}$$
This implies $q = 1$, which contradicts $q \geq 2$.

- In the case $(\mathfrak{l}_1, \pi)$=$(\mathfrak{su}(3,1), \rho_{\varpi_2})$:
The assumpotion (i) of Lemma 1.3.53 is satisfied. In fact, we have $\pi \simeq \overline{\pi}$ and $\mathrm{index}_{\tau_1} = -1$ and $2\dim\pi = 12 \geq 1+6+1 = \mathrm{rank}_\mathbb{R} L_1 + M_1 + 1$. However, we have $2\dim\pi \neq \mathrm{rank}_\mathbb{R} L_1 + M_1 + 1$, which is contradiction.

- In the case $(\mathfrak{l}_1, \pi)$=$(\mathfrak{so}(2n+1,\mathbb{C}), \rho_{\varpi_1}\boxtimes\mathrm{triv})$ $(n \geq 2)$:
The assumpotion (i) of Lemma 1.3.53 is satisfied. In fact, we have $\pi \not\simeq \overline{\pi}$ and $2\dim\pi = 4n+2 \geq 3n+2 = \mathrm{rank}_\mathbb{R} L_1 + M_1 + 1$. However, we have $2\dim\pi \neq \mathrm{rank}_\mathbb{R} L_1 + M_1 + 1$, which is contradiction.

- In the case $(\mathfrak{l}_1,\, \pi)$=$(\mathfrak{so}^*(4n+2),\, \rho_{\varpi_1})$ $(n \geq 2)$:
The assumpotion (i) of Lemma 1.3.53 is satisfied. In fact, we have $\pi \simeq \overline{\pi}$, $\operatorname{index}_{\tau_1} \pi = -1$ and $2\dim \pi = 8n+4 \geq 5n+3 = \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$. However, we have $2\dim \pi \neq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$, which is contradiction.

- In the case $(\mathfrak{l}_1,\, \pi)$=$(\mathfrak{sp}(n,\mathbb{C}),\, \rho_{\varpi_1} \boxtimes \mathrm{triv})$ $(n \geq 3)$:
The assumption (i) of Lemma 1.3.53 is satisfied. In fact, we have $\pi \not\simeq \overline{\pi}$ and $2\dim \pi = 4n \geq 3n+2 = \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$. However, we have $2\dim \pi \neq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$ if $n \geq 3$, which is contradiction.

- In the case $(\mathfrak{l}_1,\, \pi)$=$(\mathfrak{sp}(k,\ell),\, \rho_{\varpi_1})$ $(k+\ell \geq 1)$:
The assumption (i) of Lemma 1.3.53 is satisfied. In fact, we have $\pi \simeq \overline{\pi}$, $\operatorname{index}_{\tau_1} \pi = -1$ and $2\dim \pi = 4(k+\ell) \geq 4k+\ell+1 = \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$. However, we have $2\dim \pi \neq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$, which is contradiction.

- In the case $(\mathfrak{l}_1,\, \pi)$=$(\mathfrak{g}_2^{\mathbb{C}},\, \rho_{\varpi_1} \boxtimes \mathrm{triv})$:
The assumption (i) of Lemma 1.3.53 is satisfied. In fact, we have $\pi \not\simeq \overline{\pi}$ and $2\dim \pi = 14 \geq 10 = \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$. However, we have $2\dim \pi \neq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$, which is contradiction.

$\square$

Lemma 1.3.57. Let $\rho : \mathfrak{l} \to \mathfrak{gl}(V)$ be a representation of a reductive Lie algebra satisfying the conditions (i) to (iv) of Lemma 1.3.36. Then the pair $(\mathfrak{l}^{ss}, \rho|_{\mathfrak{l}^{ss}})$ is equivalent to one of the following:

- $(\mathfrak{so}(4,3),\, \rho_{\varpi_3})$,
- $(\mathfrak{g}_{2(2)},\, \rho_{\varpi_1})$,
- $(\mathfrak{sp}(2,\mathbb{C}),\, (\rho_{\varpi_1} \boxtimes \mathrm{triv}) \oplus (\mathrm{triv} \boxtimes \rho_{\varpi_1}))$.

Proof. From Remark 1.3.52 and Lemma 1.3.56, it is enough to consider the pairs $(\mathfrak{l}_1,\, \pi)$ =$(\mathfrak{so}(4,3),\, \rho_{\varpi_3})$, $(\mathfrak{g}_{2(2)},\, \rho_{\varpi_1})$, $(\mathfrak{sp}(2,\mathbb{C}),\, \rho_{\varpi_1} \boxtimes \mathrm{triv})$ of primary factor and its irreducible component of $\rho|_{\mathfrak{l}_1}$.

(i) In the case $(\mathfrak{l}_1,\, \pi)$=$(\mathfrak{so}(4,3),\, \rho_{\varpi_3})$:
The assumpotion (ii) of Lemma 1.3.53 is satisfied. In fact, $\dim \pi = 8 = \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$ holds. Therefore we obtain $\mathfrak{l}^{ss} = \mathfrak{l}_1 = \mathfrak{so}(4,3)$ and $\rho|_{\mathfrak{l}^{ss}} = \pi$ from $\operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1 \leq \dim \pi = \dim \rho \leq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$.

(ii) In the case $(\mathfrak{l}_1,\, \pi)$=$(\mathfrak{g}_{2(2)},\, \rho_{\varpi_1})$:
The assumpotion (ii) of Lemma 1.3.53 is satisfied. In fact, $\dim \pi = 7 = \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$ holds. Therefore we obtain $\mathfrak{l}^{ss} = \mathfrak{l}_1 = \mathfrak{g}_{2(2)}$ and $\rho|_{\mathfrak{l}^{ss}} = \pi$ from $\operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1 \leq \dim \pi = \dim \rho \leq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$.

(iii) In the case $(\mathfrak{l}_1,\, \pi)$=$(\mathfrak{sp}(2,\mathbb{C}),\, \rho_{\varpi_1} \boxtimes \mathrm{triv})$:
The assumpotion (i) of Lemma 1.3.53 is satisfied. In fact, we have $\pi \not\simeq \overline{\pi}$ and $2\dim \pi = 8 = \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$. Therefore we obtain $\mathfrak{l}^{ss} = \mathfrak{l}_1$ and $\rho|_{\mathfrak{l}^{ss}} = \pi \oplus \overline{\pi}$ from $[\pi : \rho|_{\mathfrak{l}^{ss}}] = [\overline{\pi} : \rho|_{\mathfrak{l}^{ss}}]$ and $\operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1 \leq 2\dim \pi = \dim \rho \leq \operatorname{rank}_{\mathbb{R}} L_1 + M_1 + 1$.

$\square$

step d : determine the pairs $(\mathfrak{l}, \rho)$

Lemma 1.3.58. Let $\rho : \mathfrak{l} \to \mathfrak{sl}(V)$ be a faithful representation of a reductive Lie algebra. Suppose $\mathfrak{l}^{ss} \simeq \mathfrak{sp}(2, \mathbb{C})$ and $\rho|_{\mathfrak{l}^{ss}} \simeq (\rho_{\varpi_1} \boxtimes \mathrm{triv}) \oplus (\mathrm{triv} \boxtimes \rho_{\varpi_1})$. Then there does not exist positive integers p, q with $2 \leq q < HR(2p)$ satisfying the conditions (i) to (iv) of Lemma 1.3.36.

Proof. From Lemma 1.3.41, it is enough to show that there does not exist positive integers p, q and non-negative integer t such that

$$\begin{cases} 2 + t \leq q \leq 2p - 1, \\ 10 + t = 2pq, \\ 8 = 2p + q + 1. \end{cases}$$

This can be easily checked. $\square$

Lemma 1.3.59. Let $\rho : \mathfrak{l} \to \mathfrak{gl}(V)$ be a representation of a reductive Lie algebra satisfying the conditions (i) to (iv) of Lemma 1.3.36. Suppose ($\mathfrak{l}^{ss}$, $\rho|_{\mathfrak{l}^{ss}}$) is equivalent to ($\mathfrak{so}(4,3)$, ρ_{ϖ_3}) or ($\mathfrak{g}_{2(2)}$, ρ_{ϖ_1}). Then we have $\mathfrak{l} = \mathfrak{l}^{ss}$ and $(p, q) = \begin{cases} (2,3) & \text{if } \mathfrak{l} \simeq \mathfrak{so}(4,3), \\ (2,2) & \text{if } \mathfrak{l} \simeq \mathfrak{g}_{2(2)} \end{cases}$.

Proof.

- In the case ($\mathfrak{l}^{ss}$, $\rho|_{\mathfrak{l}^{ss}}$) is equivalent to ($\mathfrak{so}(4,3)$, ρ_{ϖ_3})
  Let p, q be positive integers and t non-negative integer. It is enough to show that the following inequalities implies $(p, q) = (2, 3)$ and $t = 0$, which can be easily checked.

$$\begin{cases} 3 + t \leq q < HR(2p) \leq 2p \\ 12 + t = 2pq \\ 8 = 2p + q + 1. \end{cases}$$

- In the case ($\mathfrak{l}^{ss}$, $\rho|_{\mathfrak{l}^{ss}}$) is equivalent to ($\mathfrak{g}_{2(2)}$, ρ_{ϖ_1})
  Let p, q be positive integers and t non-negative integer. It is enough to show that the following inequalities implies $(p, q) = (2, 2)$ and $t = 0$, which can be easily checked.

$$\begin{cases} 2 + t \leq q < HR(2p) \leq 2p \\ 8 + t = 2pq \\ 7 = 2p + q + 1. \end{cases}$$

$\square$

1.3.4 $(G,\ H)$=$(E_{6(-14)},\ F_{4(-20)})$

In this subsection, we consider the case $(G,\ H)$=$(E_{6(-14)},\ F_{4(-20)})$. Our goal in this section is the following:

Proposition 1.3.60. Let $(G,\, H)$=$(E_{6(-14)},\, F_{4(-20)})$. There does not exist a reductive subgroup L of G acting on G/H properly and cocompactly.

Proof. From Lemma 1.3.62, it is enough to consider the case $\mathfrak{l} \simeq \mathfrak{so}(16,1)$, $\mathfrak{su}(8,1)$, $\mathfrak{sp}(4,1)$ and $\mathfrak{f}_{4(-20)}$. From Lemma 1.3.63, 1.3.65 and 1.3.67, we obtain the desired conclusion. $\square$

We have the following data:

$$\begin{aligned}&\operatorname{rank}_{\mathbb{R}} G = 2, d(G) = 32,\\ &\operatorname{rank}_{\mathbb{R}} H = 1, d(H) = 16,\\ &\mathfrak{k} \simeq \mathfrak{so}(10) \oplus \mathfrak{so}(2)\end{aligned}$$

Remark 1.3.61. Since we have $\operatorname{rank}_{\mathbb{R}} G - \operatorname{rank}_{\mathbb{R}} H = 1$, it is enough to consider simple Lie groups with real rank one from Remark 1.2.16.

Lemma 1.3.62. Let L be a simple reductive subgroup of G acting on G/H properly and cocompactly. Then $\mathfrak{l} := \mathrm{Lie}(L)$ is isomorphic to one of the following Lie algebras:

- $\mathfrak{so}(16,1)$,
- $\mathfrak{su}(8,1)$,
- $\mathfrak{sp}(4,1)$,
- $\mathfrak{f}_{4(-20)}$.

Proof. From Remark 1.2.16 and Fact 1.2.18, we have $\operatorname{rank}_{\mathbb{R}} L = 1$ and $d(L) = d(G) - d(H) = 16$. This lemma comes from the classification of simple Lie algebras. $\square$

Lemma 1.3.63. Lie algebras $\mathfrak{so}(16,1)$ and $\mathfrak{su}(8,1)$ can not be realized as a reductive subalgebra of $\mathfrak{e}_{6(-14)}$.

Proof. This comes from Lemma 1.3.64, $\operatorname{rank} K = 6$, $\operatorname{rank} SO(16) = 8$ and $\operatorname{rank} U(8) = 8$. $\square$

Lemma 1.3.64. Let $\mathfrak{g}$ be a linear reductive Lie algebra and $\mathfrak{l}$ a reductive subalgebra. Then $\operatorname{rank} K_L \leq \operatorname{rank} K$ holds. Here K, K_L is the analytic subgroups of G, L corresponding to maximal compact subalgebras $\mathfrak{k}$, $\mathfrak{k}_L$ respectively.

Proof. This is clear by the definition of rank. $\square$

Lemma 1.3.65. $\mathfrak{sp}(4,1)$ can not be realized as a reductive subalgebra of $\mathfrak{e}_{6(-14)}$.

Proof. Assume that $\mathfrak{l} := \mathfrak{sp}(4,1)$ is a reductive subalgebra of $\mathfrak{g} := \mathfrak{e}_{6(-14)}$. Then $\mathfrak{k}_L \simeq \mathfrak{sp}(4) \oplus \mathfrak{sp}(1)$ is a reductive subalgebra of $\mathfrak{k} \simeq \mathfrak{so}(10) \oplus \mathfrak{so}(2)$.

Claim. $\mathfrak{k}_L \simeq \mathfrak{sp}(4) \oplus \mathfrak{sp}(1)$ is containd in $\mathfrak{so}(10) \subset \mathfrak{k}$.

proof of Claim. Let $\iota : \mathfrak{k}_L \to \mathfrak{k} \simeq \mathfrak{so}(10) \oplus \mathfrak{so}(2)$ be the inclusion map and $pr_2 : \mathfrak{so}(10) \oplus \mathfrak{so}(2) \to \mathfrak{so}(2)$ the projection map to the second component. Assume that $\mathfrak{k} \not\subset \mathfrak{so}(10)$. Then the kernel $\ker \mathrm{pr}_2 \iota$ is a codimension one ideal of $\mathfrak{k}_L$. However there are no such ideals in $\mathfrak{k}_L \simeq \mathfrak{sp}(4) \oplus \mathfrak{sp}(1)$. $\square$

By the above Claim and $\operatorname{rank} K_L = \operatorname{rank} SO(10)$, $\mathfrak{sp}(4) \oplus \mathfrak{sp}(1)$ is a regular subalgebra of $\mathfrak{so}(10)$. Here $\mathfrak{so}(10)$ is simply laced but $\mathfrak{sp}(4) \oplus \mathfrak{sp}(1)$ is not simply laced. This is contradiction by Lemma 1.3.66. $\square$

Lemma 1.3.66. Let $\mathfrak{g}_{\mathbb{C}}$ be a simple Lie algebra over $\mathbb{C}$ and $\mathfrak{g}'_{\mathbb{C}}$ a semisimple subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with $\operatorname{rank} \mathfrak{g}_{\mathbb{C}} = \operatorname{rank} \mathfrak{g}'_{\mathbb{C}}$. If $\mathfrak{g}_{\mathbb{C}}$ is simply laced, then each simple ideal of $\mathfrak{g}'_{\mathbb{C}}$ is simply laced.

Proof. Since $\mathfrak{g}'_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ are semisimple and $\operatorname{rank} \mathfrak{g}_{\mathbb{C}} = \operatorname{rank} \mathfrak{g}'_{\mathbb{C}}$, we can take common Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}'_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$. Therefore we have a natural inclusion $\Delta(\mathfrak{g}'_{\mathbb{C}}, \mathfrak{h}) \subset \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$, which implies each simple ideal of $\mathfrak{g}'_{\mathbb{C}}$ is simply laced. $\square$

Lemma 1.3.67. There does not exist a simple Lie subgroup L of G such that $\mathrm{Lie}(L) \simeq \mathfrak{f}_{4(-20)}$ acting on G/H properly and cocompactly.

Proof. Let L be a simple subgroup of G such that $\mathfrak{l} := \mathrm{Lie}(L) \simeq \mathfrak{f}_{4(-20)}$. We show that there exists a hyperbolic orbit in $\mathfrak{g}$ which meets both $\mathfrak{l}$ and $\mathfrak{h}$. From Fact 1.3.68, there exists an inner automorphism $\alpha \in \mathrm{Int}(\mathfrak{g}^{\mathbb{C}})$ such that $\alpha(\mathfrak{l}^{\mathbb{C}}) = \mathfrak{h}^{\mathbb{C}} \simeq \mathfrak{f}_4^{\mathbb{C}}$. Since $\alpha(\mathfrak{l})$ and $\mathfrak{h}$ are isomorphic as a real form of $\mathfrak{h}^{\mathbb{C}} \simeq \mathfrak{f}_4^{\mathbb{C}}$, we can take $\alpha \in \mathrm{Int}(\mathfrak{g}^{\mathbb{C}})$ such that $\alpha(\mathfrak{l}) = \mathfrak{h}$ (Remark 1.3.69). Take a hyperbolic element $0 \neq X \in \mathfrak{l}$ in $\mathfrak{g}$. Then $\alpha(X) \in \mathfrak{h}$ is also a hyperbolic elment in $\mathfrak{g}_{\mathbb{C}}$. From Fact 1.3.24, we can take $\alpha' \in \mathrm{Int}(\mathfrak{g})$ such that $\alpha'(X) \in \mathfrak{h}$. Thus the hyperbolic orbit $\mathrm{Int}(\mathfrak{g})X$ meets both $\mathfrak{l}$ and $\mathfrak{h}$. $\square$

Fact 1.3.68 ([Dy52, Table 25, 39])**.** Let $\mathfrak{l}$ and $\mathfrak{l}'$ be a subalgebra over $\mathbb{C}$ of $\mathfrak{e}_6^{\mathbb{C}}$ which are isomorphic to $\mathfrak{f}_4^{\mathbb{C}}$. Then there exists $\alpha \in \mathrm{Int}(\mathfrak{e}_6^{\mathbb{C}})$ such that $\mathfrak{l}' = \alpha(\mathfrak{l})$.

Remark 1.3.69. Let $\mathfrak{l}$, $\mathfrak{l}'$ be real forms with real rank one of $\mathfrak{f}_4^{\mathbb{C}}$. Then there exists $\alpha \in \mathrm{Int}(\mathfrak{f}_4^{\mathbb{C}})$ such that $\alpha(\mathfrak{l}) = \mathfrak{l}'$.

1.4 Classification of reductive subgroups in the representation level

In this section, we deal with Problem C' in the representation level. We classify pairs of a reductive subalgebra $\mathfrak{l}$ and its faithful representation inducing proper and cocompact action on each G/H in Table 1.1 up to compact factor. We shall classify the embedding of $\mathfrak{l} \subset \mathfrak{g}$ up to $\mathrm{Int}(\mathfrak{g})$ in the following Chapter 1.5.

1.4.1 $(G,\ H)$=$(SU(2p,2),\ U(2p,1))$ $(p \geq 1)$

Our goal in this subsection is the following

Proposition 1.4.1. Let $G/H = SU(2p,2)/U(2p,1)$ $(p \geq 1)$. There exists a closed subgroup L which is reductive in G and acts on G/H properly and cocompactly. Moreover, $L \subset G$ is locally isomorphic to $Sp(p,1)$ up to compact factor.

Proof. It is enough to show "moreover" part. From Lemma 1.4.2 and Proposition 1.4.3, it is enough to consider $\mathfrak{l} = \mathfrak{sp}(p,1)$ and ρ which has standard representation ϖ_1 as an irreducible component. Since the equality $\dim \pi = 2p+2 = \dim \rho$ holds, we obtain $\rho = \pi$ by Lemma 1.2.37. □

Lemma 1.4.2. Let $G/H = SU(2p,2)/U(2p,1)$ $(p \geq 1)$ and $n = 2p+2$. There exists a closed subgroup L which is reductive in G and acts on G/H properly and cocompactly if and only if there exist a faithful representation $\rho : \mathfrak{l} \to \mathfrak{sl}(n, \mathbb{C})$ of a simple Lie algebra $\mathfrak{l}$ satisfying the following conditions:

(i) $\rho(\mathfrak{l}) \subset \mathfrak{su}(2p,2) \subset \mathfrak{sl}(n,\mathbb{C})$,

(ii) $\rho(\mathfrak{l})$ is preserved by matrix adjoint,

(iii) $\mathfrak{a}_L \cap W\mathfrak{a}_H = \{0\}$,

(iv) $d(L) = d(G) - d(H) (= 4p)$.

Here, the above inclusion $\mathfrak{su}(2p,2) \subset \mathfrak{sl}(n,\mathbb{C})$ is realized by the standard inclusion which is preserved by matrix adjoint and L is the analytic subgroup of G corresponding to $\rho(\mathfrak{l}) \subset \mathfrak{g}$.

Proof. This comes from Fact 1.2.15 and 1.2.18. □

Proposition 1.4.3. If a pair $(\mathfrak{l}, \rho)$ of a simple Lie algebra and its representation satisfies the conditions (i) to (iv) of Lemma 1.4.2, then $(\mathfrak{l}, \rho)$ is equivalent to $(\mathfrak{sp}(p,1), \rho_{\varpi_1}(\text{standard representation}))$.

Proof. Let π be an irreducible component of ρ. From Lemma 1.4.6, it is enough to consider the pair $(\mathfrak{l},\ \pi) \simeq (\mathfrak{su}(k,1),\ \varpi_1)$ and $(\mathfrak{sp}(k,1),\ \varpi_1)$. A pair $(\mathfrak{l},\ \pi) \simeq (\mathfrak{su}(k,1),\ \varpi_1)$ does not satisfy the condition (ii) of Lemma 1.4.4. In the case $(\mathfrak{l},\ \pi) \simeq (\mathfrak{sp}(k,1),\ \varpi_1)$, we have $k = p$ from $d(L) = 4k = 4p = d(G) - d(H)$. Since $\dim \pi = 2p+2 = \dim \rho$ holds, we obtain $\pi \simeq \rho$ from Lemma 1.2.37. □

We reduce candidates of pairs of simple Lie algebras and their irreducible components of ρ by the following:

Lemma 1.4.4. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(2p+2, \mathbb{C})$ of a simple Lie algebra $\mathfrak{l}$ satisfies the conditions (i) to (iv) of Lemma 1.4.2. Let π be a nontrivial irreducible component of ρ. Then π satisfies the following conditions:

(i) $\dim \pi \leq \frac{1}{2}d(L) + 2$ and $\operatorname{rank}_{\mathbb{R}} L = 1$,

(ii) $\dim \pi + m(\mathfrak{l}) > \frac{1}{2}d(L) + 2 \implies \operatorname{rank} \pi(X) \geq 4$ for any $X \in \mathfrak{p}_L \setminus \{0\}$.

Proof. (i) This comes from Lemma 1.4.5.

(ii) From Lemma 1.2.37 and 1.2.14, it is enough to show that for $X \in \operatorname{Ad}(K)\mathfrak{p}_H \iff \operatorname{rank} \pi(X) \leq 2$. We realize $\mathfrak{g} = \mathfrak{su}(2p, 2)$ and $\mathfrak{h} = \mathfrak{u}(2p, 1)$ as follows.

$$\begin{aligned} \mathfrak{su}(2p, 2) &:= \{X \in \mathfrak{sl}(2p+2, \mathbb{C}) : X^* I_{2p,2} + I_{2p,2} X = 0\} \\ \tau : \mathfrak{su}(2p, 2) &\to \mathfrak{su}(2p, 2) \\ X &\mapsto I_{2p+1,1} X I_{2p+1,1}^{-1} \\ \mathfrak{u}(2p, 1) &:= \mathfrak{su}(2p, 2)^\tau \end{aligned}$$

Then we can identify $\mathfrak{p}$ with $M(2p, 2; \mathbb{C})$ with K-action $(k_1, k_2), X \mapsto k_1 X k_2^{-1}$, where $k_1 \in U(2p), k_2 \in U(2)$. This action preserves matrix rank. So, (ii) follows from the descrption of $\mathfrak{p}_H$.

□

Lemma 1.4.5. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(n, \mathbb{C})$ of a simple Lie algebra satisfies the conditions (i) to (iv) in Lemma 1.4.2. Then the following equalities hold:

$$\begin{cases} d(L) = d(G) - d(H) = 4p \\ \dim \rho = 2p + 2 \end{cases}$$

In particular, we have $\dim \rho = \frac{1}{2}d(L) + 2$.

This is clear from Lemma 1.4.2. So we omit the proof.

Lemma 1.4.6. Suppose a pair $(\mathfrak{l}, \pi)$ of a simple Lie algebra and its irreducible representation satisfies the condition (i) of Lemma 1.4.4. Then $(\mathfrak{l}, \pi)$ is equivalent to one of the following:

$\mathfrak{l}$	π (highest weight)	$\dim \pi$	not satisfy
$\mathfrak{su}(k, 1)$ $k \geq 1$	ϖ_1	$k+1$	(ii)
$\mathfrak{sp}(k, 1)$ $k \geq 1$	ϖ_1	$2k+2$	

Proof. This comes from Weyl's dimensionality formula (see Appendix 1.6.2). □

1.4.2 (G,H)=$(SO_0(2p, 2),\ U(p, 1))$ $(p \geq 2)$

Our goal of this subsection is the following:

Proposition 1.4.7. Let $G/H = SO_0(2p, 2)/U(p, 1)$ $(p \geq 2)$. There exists a closed subgroup L which is reductive in G and acts on G/H properly and cocompactly. Moreover, $L \subset G$ is locally isomorphic to $SO_0(2p, 1)$ up to compact factor.

Proof. This comes from Lemma 1.4.8 and Proposition 1.4.9. □

Lemma 1.4.8. Let $G/H = SO_0(2p,2)/U(p,1)$ $(p \geq 2)$ and $n = 2p+2$. There exists a closed subgroup L which is reductive in G and acts on G/H properly and cocompactly if and only if there exists a faithful representation $\rho : \mathfrak{l} \to \mathfrak{sl}(n,\mathbb{C})$ of a simple Lie algebra $\mathfrak{l}$ satisfying the following conditions:

(i) $\rho(\mathfrak{l}) \subset \mathfrak{so}(2p,2) \subset \mathfrak{sl}(n,\mathbb{C})$,

(ii) $\rho(\mathfrak{l})$ is preserved by matrix transpose,

(iii) $\mathrm{Int}(\mathfrak{g})\mathfrak{p}_{\rho(\mathfrak{l})} \cap \mathfrak{h} = \{0\}$

(iv) $d(L) = d(G) - d(H) (= 2p)$.

Here the inclusion $\mathfrak{so}(2p,2) \subset \mathfrak{sl}(n,\mathbb{C})$ is realized by the standard inclusion which is preserved by matrix transpose and $\mathfrak{p}_{\rho(\mathfrak{l})} = \rho(\mathfrak{l})^{-\theta}$ for a Cartan involution on $\mathfrak{g}$ such that $\theta(\rho(\mathfrak{l})) = \rho(\mathfrak{l})$.

Proof. This comes from Fact 1.4.16 and 1.2.18. □

Proposition 1.4.9. Suppose a pair ($\mathfrak{l}$, ρ) of a simple Lie algebra and its representation satisfying the conditions (i) to (iv) of Lemma 1.4.8. Then ($\mathfrak{l}$, ρ) is equivalent to ($\mathfrak{so}(2p,1)$, $\rho_{\varpi_1} \oplus \mathrm{triv}$).

Proof. From Lemma 1.4.12 and 1.4.13, it is enough to consider the cases $(\mathfrak{l},\pi)$=($\mathfrak{so}(1,2k)$ $(k \geq 2)$, ϖ_1), ($\mathfrak{su}(1,k)$ $(k \geq 2)$, ϖ_1). Moreover, from Lemma 1.4.14, it is enough to consider the case $(\mathfrak{l},\pi)$=($\mathfrak{so}(1,2k)$ $(k \geq 2)$, ϖ_1). We have $k = p$, $\dim \pi = 2p+1$ by $d(L) = 2k = 2p$. $(\mathfrak{l},\rho)$ is equivalent to $(\mathfrak{so}(1,2p), \pi \oplus \mathrm{triv})$ from Lemma 1.2.37. □

Lemma 1.4.10. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(2p+2,\mathbb{C})$ of a simple Lie algebra satisfies the conditions (i) to (iv) of Lemma 1.4.8. Let π be a nontrivial irreducible component of ρ. Then π satisfies the following conditions:

(i) $\dim \pi \leq d(L) + 2 \geq 6$,

(ii) $2\dim \pi > d(L) + 2 \implies \pi \simeq \overline{\pi} \simeq \pi^\vee$ and $\mathrm{index}_\tau\, \pi = \mathrm{index}_\theta\, \pi = 1$.

Here τ is the real structure on $\mathfrak{l}_\mathbb{C}$ such that $\mathfrak{l}_\mathbb{C}^\tau = \mathfrak{l}$ and θ is a Cartan involution on $\mathfrak{l}_\mathbb{C}$.

Proof. (i) This comes from Lemma 1.4.11 and $\dim \pi \leq \dim \rho$.

(ii) This comes from Lemma 1.2.40 and $\rho(\mathfrak{l}) \subset \mathfrak{so}(2p,2) \subset_{\mathrm{Int}} \mathfrak{so}(2p+2,\mathbb{C})$, $\mathfrak{sl}(2p+2,\mathbb{R})$.

□

Lemma 1.4.11. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(n,\mathbb{C})$ of a simple Lie algebra satisfies the conditions (i) to (iv) in Lemma 1.4.8. Then the following inequalities hold:

$$\begin{cases} d(L) = 2p \\ \dim \rho = 2p + 2 \geq 6. \end{cases}$$

In particular, we have $\dim \rho = d(L) + 2 \geq 6$.

We can easily checked the above lemma. So we omit the proof.

Lemma 1.4.12. Suppose a pair $(\mathfrak{l},\ \pi)$ of a simple Lie algebra and its irreducible representation satisfies the condition (i) of Lemma 1.4.10. Then $(\mathfrak{l},\ \pi)$ is equivalent to one of the following Table 1.19:

Table 1.19: pairs of simple Lie algebras $\mathfrak{l}$ and their irreducible representations π which satisfy the conditions (i) in Lemma 1.4.10.

$\mathfrak{l}$	π	$\dim \pi$	does not satisfy
$\mathfrak{so}(1, 2k)\ (k \geq 2)$	ϖ_1	$2k+1$	
$\mathfrak{so}(1, 6)$	ϖ_3	8	(ii)
$\mathfrak{so}(1, 4)$	ϖ_2	4	(ii)
$\mathfrak{su}(1, k)\ (k \geq 2)$	ϖ_1	$k+1$	
$\mathfrak{su}(1, 4)$	$\varpi_2,\ \varpi_3$	10	(ii)
$\mathfrak{su}(1, 3)$	ϖ_2	6	(ii)
$\mathfrak{su}(1, 2)$	$2\varpi_1,\ 2\varpi_2$	6	(ii)
$\mathfrak{sp}(1, k)\ (k \geq 1)$	ϖ_1	$2k+2$	(ii)

Proof. This comes from Weyl's dimensionality formula (see Appendix 1.6.2). $\square$

Lemma 1.4.13. Let $(\mathfrak{l},\ \pi)$ be a pair of a simple Lie algebra and its irreducible representation which is equivalent to one of the following Table 1.20. Then $(\mathfrak{l},\ \pi)$ does not satisfy the condition (ii) of Lemma 1.4.10.

Table 1.20: pairs of simple Lie algebras $\mathfrak{l}$ and their irreducible representations π which do not satisfy the conditions (ii) in Lemma 1.4.10.

$\mathfrak{l}$	π	$\dim \pi$	$d(L)+2$	property
$\mathfrak{so}(1, 6)$	ϖ_3	8	8	$\mathrm{index}_\tau\, \pi = -1$
$\mathfrak{so}(1, 4)$	ϖ_2	4	6	$\mathrm{index}_\tau\, \pi = -1$
$\mathfrak{su}(1, 4)$	$\varpi_2,\ \varpi_3$	10	10	$\overline{\pi} \not\simeq \pi$
$\mathfrak{su}(1, 3)$	ϖ_2	6	8	$\mathrm{index}_\tau\, \pi = -1$
$\mathfrak{su}(1, 2)$	$2\varpi_1,\ 2\varpi_2$	6	6	$\overline{\pi} \not\simeq \pi$
$\mathfrak{sp}(1, k)$	ϖ_1	$2k+2$	$4k+2$	$\mathrm{index}_\tau\, \pi = -1$

Proof. This is clear from the data in Table 1.20 (see Appendix to check the property). $\square$

Lemma 1.4.14. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(2p+2, \mathbb{C})$ of a simple Lie algebra $\mathfrak{l}$ satisfies the conditions (i) to (iv) in Lemma 1.4.8 and π is a nontrivial irreducible component of ρ. Then $(\mathfrak{l}, \pi)$ is not equivalent to $(\mathfrak{su}(k, 1), \varpi_1)$ $(k \geq 2)$.

Proof. Assume that $\rho : \mathfrak{l} \to \mathfrak{sl}(2p+2, \mathbb{C})$ satisfies the conditions (i), (ii) and (iv) in Lemma 1.4.8 and π is an irreducible component of ρ such that $(\mathfrak{l}, \pi)$ is equivalent to $(\mathfrak{su}(k, 1), \rho_{\varpi_1})$. We show that ρ does not satisfy the condition (iii) of Lemma 1.4.8. From (iv), we have $k = p$.

Claim. $\rho \simeq \rho_{\varpi_1} \oplus \overline{\rho_{\varpi_1}}$.

Here this claim comes from $[\rho_{\varpi_1} : \rho] = [\overline{\rho_{\varpi_1}} : \rho]$, which comes from Lemma 1.2.42, the properties $\rho_{\varpi_1} \not\simeq \overline{\rho_{\varpi_1}}$, $\dim \rho_{\varpi_1} = p+1$ and $\dim \rho = 2p+2$. We can regard the representation ρ as a real representation $\rho_0 : \mathfrak{l} \to \mathfrak{sl}(2p+2, \mathbb{R})$ with $(\rho_0)^{\mathbb{C}} = \rho \simeq \rho_{\varpi_1} \oplus \overline{\rho_{\varpi_1}}$. From Lemma 1.2.46, we have $\rho_0 \simeq (\rho_{\varpi_1})_{\mathbb{R}}$ as a real representation. From Remark 1.4.15, it is enough to check that action on G/H induced by one of the representatives does not satisfy the condition (iii). There exist a representative $r_0 \in \mathrm{Hom}_0(\mathfrak{l}, \mathfrak{g})$ and an element $\alpha \in \mathrm{Int}(\mathfrak{so}(2p,2))$ such that $\alpha(r_0(\mathfrak{l})) \subset \mathfrak{h}$. Take $0 \neq X \in \mathfrak{a}(\mathfrak{l}) \subset r_0(\mathfrak{l})$, then we have $\alpha(X) \in \mathfrak{h}$, which implies that the condition (iii) is not satisfied. $\square$

Remark 1.4.15. To discuss properness of the action induced from the above representation $[\rho_{\varpi_1}] : \mathfrak{l} \to \mathfrak{so}(2p,2) \subset \mathfrak{sl}(2p+2, \mathbb{R})$, we can choose any representative:
Let $\mathfrak{g} = \mathfrak{so}(2p,2)$ $(p \geq 2)$ and $\mathfrak{l} = \mathfrak{su}(p,1)$. Suppose Lie algebra embedding $\varphi : \mathfrak{l} \to \mathfrak{g}$ satisfies $\iota\varphi \simeq (\rho_{\varpi_1})_{\mathbb{R}} : \mathfrak{l} \to \mathfrak{sl}(2p+2, \mathbb{R})$ as a real representation of $\mathfrak{l}$ where $\iota : \mathfrak{g} \to \mathfrak{sl}(2p+2, \mathbb{R})$ is a natural embedding. Then we have $[\varphi] \in \{[r], [\mathrm{Ad}(I_{2p-1,1,2})r], [\mathrm{Ad}(I_{2p,1,1})r], [\mathrm{Ad}(I_{2p-1,1,1,1})r]\} \subset \mathrm{Int}(\mathfrak{g})\backslash \mathrm{Hom}_0(\mathfrak{l}, \mathfrak{g}) / \mathrm{Aut}(\mathfrak{l})$ (see Section 1.5.5 for more details). $\mathrm{Int}(\mathfrak{g})\mathfrak{a}(\mathfrak{l})$ coincide for noncompact part $\mathfrak{a}(\mathfrak{l})$ coming from the above representatives. In particular, Therefore, from the criterion Fact 1.4.16, the properness does not depend on the choice of representatives.

Here we prepare criterion Fact 1.4.16 for properness in terms of hyperbolic orbit for the proof above. Let G be a linear reductive Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ a semisimple Lie algebra over $\mathbb{R}$, and H, L reductive subgroups of G. Take Cartan involutions θ on G and θ_1, θ_2 which preserve H, L respectively and maximal abelian subspaces $\mathfrak{a}(\mathfrak{h})$, $\mathfrak{a}(\mathfrak{l})$ of $\mathfrak{g}^{-\theta_1}$, $\mathfrak{g}^{-\theta_2}$. We can and do take $\alpha_1, \alpha_2 \in \mathrm{Int}(\mathfrak{g})$ such that $\mathfrak{a}_{\mathfrak{h}} := \alpha_1(\mathfrak{a}(\mathfrak{h})), \mathfrak{a}_{\mathfrak{l}} := \alpha_2(\mathfrak{a}(\mathfrak{l})) \subset \mathfrak{a}$. Then we have the following:

Fact 1.4.16 ([Ko89])**.** In Setting 1, the following conditions on G, H, L are equivalent:

(i) the natural L action on G/H is proper

(ii) $\mathfrak{a}_{\mathfrak{l}} \cap W(\mathfrak{g}, \mathfrak{a})\mathfrak{a}_{\mathfrak{h}} = \{0\}$

(iii) $\mathrm{Int}(\mathfrak{g})\mathfrak{a}(\mathfrak{l}) \cap \mathfrak{a}(\mathfrak{h}) = \{0\}$

Moreover in the case $\mathfrak{h} = \mathfrak{g}^{\sigma}$ for some involution σ on $\mathfrak{g}$, (iii) is equivalent to the following condition:

$$\mathrm{Int}(\mathfrak{g})\mathfrak{a}(\mathfrak{l}) \cap \mathfrak{g}^{\sigma} = \{0\}.$$

In the same setting, the criterion of proper action can be described in terms of hyperbolic orbit as follows:

Fact 1.4.17 ([O13], Theorem 4.1)**.** The following conditions on G, H, L is equivalent:

(i) The natural L-action on G/H is proper,

(ii) No hyperbolic orbit meets both $\mathfrak{l}$ and $\mathfrak{h}$ other than zero-orbit.

1.4.3 $(G,\ H)$=$(SO^*(8),\ U(3,1))$

Our goal of this subsection is the following:

Proposition 1.4.18. Let $G/H = SO^*(8)/U(3,1)$. There exists a reductive subgroup L of G acting on G/H properly and cocompactly. Moreover, L is locally isomorphic to $Spin(1,6)$ up to compact factor.

Proof. This comes from Lemma 1.4.19 and Proposition 1.4.20. □

Lemma 1.4.19. Let $G/H = SO^*(8)/U(3,1)$. There exists a reductive subgroup L of G acting on G/H properly and cocompactly if and only if there exists a faithful representation $\rho : \mathfrak{l} \to \mathfrak{sl}(8,\mathbb{C})$ of a simple Lie algebra $\mathfrak{l}$ satisfying the following conditions:

(i) $\rho(\mathfrak{l}) \subset \mathfrak{so}^*(8) \subset \mathfrak{sl}(8,\mathbb{C})$,

(ii) $\rho(\mathfrak{l})$ is preserved by matrix adjoint,

(iii) $\mathrm{Int}(\mathfrak{g})\mathfrak{p}_{\rho(\mathfrak{l})} \cap \mathfrak{h} = \{0\}$,

(iv) $d(L) = d(G) - d(H) = 6$.

Here the inclusion $\mathfrak{so}^*(8) \subset \mathfrak{sl}(8,\mathbb{C})$ is realized by the standard inclusion which is preserved by matrix adjoint.

Proof. This comes from Fact 1.4.16 and 1.2.18. □

Proposition 1.4.20. Suppose a pair $(\mathfrak{l},\ \rho)$ of a simple Lie algebra and its representation satisfying the conditions (i) to (iv) of Lemma 1.4.19. Then $(\mathfrak{l},\ \rho)$ is equivalent to $(\mathfrak{so}(6,1),\ \rho_{\varpi_3})$.

Proof. From Lemma 1.4.23, 1.4.24, 1.4.25 and 1.4.26, it is enough to show $\rho \simeq \pi$ for the case $(\mathfrak{l},\ \pi)$=$(\mathfrak{so}(6,1),\ \rho_{\varpi_3})$. This comes from $\dim \pi = \dim \rho = 8$. □

Lemma 1.4.21. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(8,\mathbb{C})$ of a simple Lie algebra $\mathfrak{l}$ satisfies the conditions (i) to (iv) of Lemma 1.4.19. Let π be a nontrivial irreducible component of ρ. Then π satisfies the following conditions:

(i) $\dim \pi \le 8$,

(ii) $\mathrm{rank}_{\mathbb{R}} L = 1$, $d(L) = 6$,

(iii) $\dim \pi \ge 5 \Rightarrow \pi \simeq \pi^\vee \simeq \overline{\pi}$ and $\mathrm{index}_\theta\, \pi = 1$ and $\mathrm{index}_\tau\, \pi = -1$,

(iv) $\dim \pi + m(\mathfrak{l}) > 8 \Rightarrow \mathrm{rank}\, \pi(X) = 8$ for any $X \in \mathfrak{p}_L \setminus \{0\}$.

Here τ is the real structure on $\mathfrak{l}_{\mathbb{C}}$ such that $\mathfrak{l}_{\mathbb{C}}^\tau = \mathfrak{l}$ and θ is a Cartan involution on $\mathfrak{l}_{\mathbb{C}}$.

Proof. (i) This is clear.

(ii) This is clear from Remark 1.2.16, Fact 1.2.18.

(iii) This comes from Lemma 1.2.40.

(iv) From Lemma 1.2.37 and Remark 1.2.14, it is enough to show that for $X \in \mathfrak{p}$, $X \in \mathrm{Ad}(K)\mathfrak{p}_H \iff \mathrm{rank}\,\pi(X) \leq 7$. We realize $\mathfrak{h} = \mathfrak{u}(3,1) \subset \mathfrak{so}^*(8)$ as follows:

$$\begin{aligned} \mathfrak{so}^*(8) &= \{X \in \mathfrak{sl}(8,\mathbb{C}) : J\overline{X} = XJ, {}^tX + X = 0\}, \\ \sigma : \mathfrak{so}^*(8) &\to \mathfrak{so}^*(8), X \mapsto I_{3,1;3,1}\overline{X}I_{3,1;3,1}, \\ \mathfrak{u}(3,1) &:= \mathfrak{so}^*(8)^\sigma. \end{aligned}$$

Then we have

$$\begin{aligned} \mathfrak{p}_H = \{i\begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A = \begin{pmatrix} & & & a_1 \\ & & & a_2 \\ & & & a_3 \\ -a_1 & -a_2 & -a_3 & \end{pmatrix}, B = \begin{pmatrix} & & & b_1 \\ & & & b_2 \\ & & & b_3 \\ -b_1 & -b_2 & -b_3 & \end{pmatrix}, a_i, b_i \in \mathbb{R}\} \\ \subset \mathfrak{p} = \{i\begin{pmatrix} A & B \\ B & -A \end{pmatrix} A, B \in \mathrm{Alt}(4,\mathbb{R})\}. \end{aligned}$$

Adjoint action of $K \simeq U(4)$ on $\mathfrak{p} \simeq \mathrm{Alt}(4,\mathbb{C})$ is equivalent to the action of $U(4)$ on $\mathrm{Alt}(4,\mathbb{C})$, $(k, A) \mapsto kA{}^tk$ where $k \in U(4)$ and $A \in \mathrm{Alt}(4,\mathbb{C})$. This action preserves rank. By the description of $\mathfrak{p}$ and $\mathfrak{p}_H$, the rank of $X \in \mathfrak{p}$ is divided by four and $X \in \mathrm{Ad}(K)\mathfrak{p}_H$ holds if and only if $\mathrm{rank}\,X \leq 4$. Therefore we obtain $X \in \mathrm{Ad}(K)\mathfrak{p}_H \iff \mathrm{rank}\,X \leq 7$ for $X \in \mathfrak{p}$.

□

Remark 1.4.22. The rank condition (iii) of Lemma 1.4.21 is preserved by basis transformation. So we can discuss properness up to equivalent class of representations.

Lemma 1.4.23. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(8,\mathbb{C})$ of a simple Lie algebra satisfies the conditions (i) to (iv) of Lemma 1.4.19 and π is an irreducible component of ρ. Then $(\mathfrak{l}, \pi)$ is equivalent to one of the following table.

Table 1.21: a pairs $(\mathfrak{l}, \pi)$ of a simple Lie algebra and its irreducible representation satisfying (i) and (ii)

$\mathfrak{l}$	π	$\dim\pi$	not satisfy
$\mathfrak{so}(6,1)$	ϖ_1	7	(iii)
	ϖ_3	8	
$\mathfrak{su}(3,1)$	ϖ_1	4	
	ϖ_2	6	(iv)

Proof. This comes from the conditions (i) and (ii) of Lemma 1.4.21 and Weyl's dimensionality formula. □

Lemma 1.4.24. Let $\rho : \mathfrak{so}(6,1) \to \mathfrak{sl}(8,\mathbb{C})$ be a representation with the irreducible component $\pi \simeq \rho_{\varpi_1}$. Then ρ does not satisfy the conditions (i) to (iv) of Lemma 1.4.19.

Proof. We use Lemma 1.4.21(iii). The assumption $\dim \pi = 7 \geq 5$ is satisfied. However, we have $\mathrm{index}_\tau \pi = 1$. $\square$

Lemma 1.4.25. Let $\rho : \mathfrak{su}(3,1) \to \mathfrak{sl}(8,\mathbb{C})$ be a representation with an irreducible component $\pi \simeq \rho_{\varpi_2}$. Then ρ does not satisfy the conditions (i) to (iv) of Lemma1.4.19.

Proof. We use Lemma 1.4.21(iv). The assumption $\dim \pi + m(\mathfrak{l}) = 6 + 4 > 8$ is satisfied. Take $X := S_{14} \in \mathfrak{p}_L \setminus \{0\}$. Then it is obvious that $\mathrm{rank}\, \pi(X) \leq 6$ holds from the dimension of π. $\square$

Lemma 1.4.26. Let $\rho : \mathfrak{su}(3,1) \to \mathfrak{sl}(8,\mathbb{C})$ be a representation of $\mathfrak{su}(3,1)$ with an irreducible component $\pi \simeq \rho_{\varpi_1}$. Then ρ does not satisfy the conditions (i) to (iv) of Lemma1.4.19 .

Proof. From Lemma 1.2.42, we have $[\pi : \rho] = [\overline{\pi} : \rho]$. Since we have $\rho_{\varpi_1} \not\simeq \overline{\rho_{\varpi_1}}$ and $2 \dim \pi = 8$, we obtain $\rho \simeq \pi \oplus \overline{\pi}$. Take $0 \neq S_{13} = E_{1,3} + E_{3,1} \in \mathfrak{p}_L$. Then we have $\mathrm{rank}\, \rho(S_{13}) = 4$. Therefore the L-action on G/H is not proper. $\square$

1.4.4 $(G,\ H)$=$(SO^*(8),\ SO^*(6) \times SO^*(2))$

Our goal of this subsection is the following:

Proposition 1.4.27. Let $G/H = SO^*(8)/SO^*(6) \times SO^*(2)$. There exists a reductive subgroup L of G acting on G/H properly and cocompactly. Moreover, L is locally isomorphic to $Spin(1,6)$ up to compact factor.

Proof. This comes from Lemma 1.4.28 and Proposition 1.4.29. $\square$

Lemma 1.4.28. Let $G/H = SO^*(8)/SO^*(6) \times SO^*(2)$. There exists a reductive subgroup of G acting on G/H properly and cocompactly if and only if there exists a faithful representation $\rho : \mathfrak{l} \to \mathfrak{sl}(8,\mathbb{C})$ of a simple Lie algebra $\mathfrak{l}$ satisfying the following conditions:

(i) $\rho(\mathfrak{l}) \subset \mathfrak{so}^*(8) \subset \mathfrak{sl}(8,\mathbb{C})$,

(ii) $\rho(\mathfrak{l})$ is preserved by matrix adjoint,

(iii) $\mathrm{Int}(\mathfrak{g})\mathfrak{p}_{\rho(\mathfrak{l})} \cap \mathfrak{h} = \{0\}$,

(iv) $d(L) = d(G) - d(H) = 6$.

Here the inclusion $\mathfrak{so}^*(8) \subset \mathfrak{sl}(8,\mathbb{C})$ is realized by the standard inclusion which is preserved by matrix adjoint and L is the analytic subgroup of G corresponding to $\rho(\mathfrak{l}) \subset \mathfrak{g}$.

Proof. This comes from Fact 1.4.16 and 1.2.18. $\square$

Proposition 1.4.29. Suppose a pair $(\mathfrak{l},\ \rho)$ of a simple Lie algebra and its representation satisfying the conditions (i) to (iv) of Lemma 1.4.28. Then $(\mathfrak{l},\ \rho)$ is equivalent to $(\mathfrak{so}(6,1),\ \rho_{\varpi_3})$.

Proof. We can prove this proposition in the same way as Proposition 1.4.20 because conditions key Lemma 1.4.30 and properness conditions are same as the case $G/H = SO^*(8)/U(3,1)$. $\square$

Lemma 1.4.30. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(8,\mathbb{C})$ satisfies the conditions (i) to (iv) of Lemma 1.4.28. Let π be a nontrivial irreducible component of ρ. Then π satisfies the following conditions:

(i) $\dim \pi \leq 8$,

(ii) $\operatorname{rank}_{\mathbb{R}} L = 1$, $d(L) = 6$,

(iii) $\dim \pi \geq 5 \Rightarrow \pi \simeq \pi^\vee \simeq \overline{\pi}$ and $\operatorname{index}_{\theta_1} \pi = 1$ and $\operatorname{index}_{\tau_1} \pi = -1$,

(iv) $\dim \pi + m(\mathfrak{l}) > 8 \Rightarrow \operatorname{rank} \pi(X) = 8$ for any $X \in \mathfrak{p}_L \setminus \{0\}$.

Proof. We can prove this in the same way as Lemma 1.4.21 for the case $G/H = SO^*(8)/U(3,1)$ in the previous subsection. $\square$

1.4.5 (G,H)=$(SO(8,\mathbb{C}),\ SO(7,\mathbb{C}))$

Our goal of this subsection is the following:

Proposition 1.4.31. Let $G/H = SO(8,\mathbb{C})/SO(7,\mathbb{C})$. Then there exists a closed subgroup L which is reductive in G and acts on G/H properly and cocompactly. Moreover, L is isomorphic to $Spin(1,7)$ up to compact factor.

Proof. This comes from Lemma 1.4.32 and Proposition 1.4.33. $\square$

Lemma 1.4.32. Let $G/H = SO(8,\mathbb{C})/SO(7,\mathbb{C})$. If there exists a closed subgroup which is reductive in G and acts on G/H properly and cocompactly if and only if there exists a simple Lie algebra $\mathfrak{l}$ and its representation $\rho : \mathfrak{l} \to \mathfrak{sl}(8,\mathbb{C})$ satisfying the following conditions:

(i) $\rho(\mathfrak{l}) \subset \mathfrak{so}(8,\mathbb{C}) \subset \mathfrak{sl}(8,\mathbb{C})$,

(ii) $\rho(\mathfrak{l})$ is preserved by matrix adjoint,

(iii) $\mathfrak{a}_L \cap W\mathfrak{a}_H = \{0\}$,

(iv) $d(L) = d(G) - d(H) = 7$.

Here the inclusion $\mathfrak{so}(8,\mathbb{C}) \subset \mathfrak{sl}(8,\mathbb{C})$ is realized by the standard inclusion which is preserved by matrix adjoint and L is the analytic subgroup of G corresponding to $\rho(\mathfrak{l}) \subset \mathfrak{g}$.

Proof. This comes from Fact 1.2.15 and 1.2.18. $\square$

Proposition 1.4.33. Suppose a pair $(\mathfrak{l},\ \rho)$ of a simple Lie algebra and its representation satisfies the conditions (i) to (iv) of Lemma 1.4.32. Then $(\mathfrak{l},\ \rho)$ is equivalent to $(\mathfrak{so}(1,7),\ \rho_{\varpi_3})$.

Proof. From Remark 1.4.35 and Lemma 1.4.36, it is enough to consider the case ($\mathfrak{l}$, π)=($\mathfrak{so}(1,7)$, ϖ_3). From $\dim \pi = 8 = \dim \rho$ and Lemma 1.2.37, we obtain $\pi \simeq \rho$. $\square$

Lemma 1.4.34. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(8, \mathbb{C})$ of a simple Lie algebra $\mathfrak{l}$ satisfies the conditions (i) to (iv) of Lemma 1.4.32. Let π be a nontrivial irreducible component of ρ. Then the following conditions are satisfied:

(i) $\dim \pi \leq 8$,

(ii) $\operatorname{rank}_{\mathbb{R}} L = 1$, $d(L) = 7$,

(iii) $\dim \pi + m(\mathfrak{l}) > 8 \implies \operatorname{rank} \pi(X) = 8$ for any $X \in \mathfrak{p}_L \setminus \{0\}$.

Proof. (i) This is clear from $\dim \pi \leq \dim \rho = 8$.

(ii) This is clear from Remark 1.2.16 and Fact 1.2.18.

(iii) From Lemma 1.2.37 and 1.2.14, it is enough to show that for $X \in \mathfrak{p} = i\mathfrak{o}(8)$, $X \in \operatorname{Ad}(K)\mathfrak{p}_H \iff \operatorname{rank} \pi(X) \leq 7$. This comes from that the adjoint action of K on $\mathfrak{p} = i\mathfrak{o}(8)$ preserves rank.

$\square$

Remark 1.4.35. The conditions (i) and (ii) of Lemma 1.4.34 imply that $(\mathfrak{l}, \pi)$ is equivalent to one of the following:

$\mathfrak{l}$	π	not satisfy
$\mathfrak{so}(1,7)$	ϖ_1	(iii)
	ϖ_3	

Lemma 1.4.36. A pair of a simple Lie algebra and an irreducible component of ρ satisfies the conditions (i) to (iii) of Lemma 1.4.34, then ($\mathfrak{l}$, π) is equivalent to ($\mathfrak{so}(7,1)$, ρ_{ϖ_3}).

Proof. From Remark 1.4.35, it is enough to show that the pair ($\mathfrak{so}(7,1)$, ρ_{ϖ_1}) does not satisfy the condition (iii) of Lemma 1.4.34. Put $X = i(E_{1,8} - E_{8,1}) \in \mathfrak{p}_{\mathfrak{so}(7,1)}$ Then we have $\operatorname{rank} \pi(X) = 2$. $\square$

1.4.6 $(G,\ H)$=$(SO(8, \mathbb{C}),\ SO_0(7,1))$

Our goal of this subsection is the following:

Proposition 1.4.37. Let $G/H = SO(8, \mathbb{C})/SO_0(7,1)$. Then there exists a closed subgroup L of G which is reductive in G and acts on G/H properly and cocompactly. Moreover L is locally isomorphic to $Spin(7, \mathbb{C})$ up to compact factor.

Proof. This comes Lemma 1.4.38 and Proposition 1.4.39. $\square$

Lemma 1.4.38. Let $G/H = SO(8, \mathbb{C})/SO_0(7,1)$. There exists a reductive subgroup of G acting on G/H properly and cocompactly if and only if there exists a reductive Lie algebra $\mathfrak{l}$ and its faithful representation $\rho : \mathfrak{l} \to \mathfrak{sl}(8, \mathbb{C})$ satisfying the following conditions:

(i) $\rho(\mathfrak{l}) \subset \mathfrak{so}(8, \mathbb{C}) \subset \mathfrak{sl}(8, \mathbb{C})$,

(ii) $\rho(\mathfrak{l})$ is preserved by matrix adjoint,

(iii) $\mathfrak{a}_{\mathfrak{l}} \cap W \mathfrak{a}_{\mathfrak{h}} = \{0\}$,

(iv) $d(L) = d(G) - d(H)$.

Here the inclusion $\mathfrak{so}(8, \mathbb{C}) \subset \mathfrak{sl}(8, \mathbb{C})$ is realized by the standard inclusion which is preserved matrix adjoint and L is the analytic subgroup of G corresponding to $\rho(\mathfrak{l}) \subset \mathfrak{g}$.

Proof. This comes from Fact 1.4.16 and 1.2.18. □

Proposition 1.4.39. Let $G/H = SO(8, \mathbb{C})/SO_0(7, 1)$. If a pair $(\mathfrak{l}, \rho)$ of reductive Lie algebra and its representation satisfies the conditions (i) to (iv) of Lemma 1.4.32, then $(\mathfrak{l}, \rho)$ is equivalent to $(\mathfrak{so}(7, \mathbb{C}), \rho_{\varpi_3} \boxtimes \mathrm{triv})$ up to compact factor.

Proof. This comes from Remark 1.4.40, Lemma 1.4.44, 1.4.45 and 1.4.47 (see Outline of the proof for the case $\operatorname{rank}_{\mathbb{R}} G - \operatorname{rank}_{\mathbb{R}} H \geq 2$ in Section 1.2.5). □

First we reduce candidates by upper bound of the dimension of representations by the following:

Remark 1.4.40. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(8, \mathbb{C})$ of a reductive Lie algebra $\mathfrak{l}$ satisfies conditions (i) to (iv) of Lemma 1.4.38 and π is an irreducible component of $\rho|_{\mathfrak{l}_1}$. Then π satisfies $\dim \pi \leq 8$.

From Weyl's dimensionality formula, we have the following list of pairs of a simple Lie algebra and its irreducible representation π satisfying $\dim \pi \leq 8$:

Table 1.22: pairs of simple Lie algebras $\mathfrak{l}$ and their irreducible representations π which satisfy $\dim \pi \leq 8$.

$\mathfrak{l}$	π	not satisfy
$\mathfrak{sl}(n,\mathbb{C})$ $(5 \leq n \leq 8)$	$\varpi_1 \boxtimes$ triv	(i)
$\mathfrak{sl}(n,\mathbb{C})$ $(2 \leq n \leq 4)$	$\varpi_1 \boxtimes$ triv	(ii)
$\mathfrak{sl}(4,\mathbb{C})$	$\varpi_2 \boxtimes$ triv	(ii)
$\mathfrak{sl}(3,\mathbb{C})$	$(\varpi_1+\varpi_2) \boxtimes$ triv	(ii)
$\mathfrak{sl}(2,\mathbb{C})$	$k\varpi_1 \boxtimes$ triv	(ii)
$\mathfrak{su}(4,4)$	ϖ_1	(i)
$\mathfrak{su}(4,\ell)$ $(1 \leq \ell \leq 3)$	ϖ_1	(iv)
$\mathfrak{su}(k,\ell)$ $(1 \leq \ell \leq k \leq 3)$	ϖ_1	(ii)
$\mathfrak{sl}(n,\mathbb{R})$ $(5 \leq n \leq 8)$	ϖ_1	(i)
$\mathfrak{sl}(n,\mathbb{R})$ $(2 \leq n \leq 4)$	ϖ_1	(ii)
$\mathfrak{su}^*(2n)$ $(3 \leq n \leq 4)$	ϖ_1	(iv)
$\mathfrak{su}^*(4)$	ϖ_1, ϖ_2	(ii)
$\mathfrak{su}(k,\ell)$ $(k+\ell=4)$	ϖ_2	(ii)
$\mathfrak{sl}(4,\mathbb{R})$	ϖ_2	(ii)
$\mathfrak{sl}(3,\mathbb{R})$	$\varpi_1+\varpi_2$	(ii)
$\mathfrak{su}(2,1)$	$\varpi_1+\varpi_2$	(ii)
$\mathfrak{sl}(2,\mathbb{R})$	$k\varpi_1$ $(1 \leq k \leq 7)$	(ii)
$\mathfrak{so}(7,\mathbb{C})$	$\varpi_1 \boxtimes$triv	(v)
$\mathfrak{so}(5,\mathbb{C})$	$\varpi_1 \boxtimes$ triv	(ii)
$\mathfrak{so}(7,\mathbb{C})$	$\varpi_3 \boxtimes$ triv	
$\mathfrak{so}(5,\mathbb{C})$	$\varpi_3 \boxtimes$ triv	(ii)
$\mathfrak{so}(k,\ell)$ $(k+\ell=7)$	ϖ_1, ϖ_3	(ii)
$\mathfrak{so}(k,\ell)$ $(k+\ell=5)$	ϖ_1, ϖ_2	(ii)
$\mathfrak{sp}(4,\mathbb{C})$	$\varpi_1 \boxtimes$triv	(i)
$\mathfrak{sp}(3,\mathbb{C})$	$\varpi_1 \boxtimes$triv	(iv)
$\mathfrak{sp}(4,\mathbb{R})$	ϖ_1	(i)
$\mathfrak{sp}(k,\ell)$ $(k+\ell=4)$	ϖ_1	(iii)
$\mathfrak{sp}(3,\mathbb{R})$	ϖ_1	(ii)
$\mathfrak{sp}(2,1)$	ϖ_1	(iv)
$\mathfrak{so}(8,\mathbb{C})$	$\varpi_1 \boxtimes$triv	(i)
$\mathfrak{so}(4,4)$	ϖ_1, ϖ_3, ϖ_4	(i)
$\mathfrak{so}(5,3)$	ϖ_1, ϖ_3, ϖ_4	(ii)
$\mathfrak{so}(6,2)$	ϖ_1, ϖ_3, ϖ_4	(ii)
$\mathfrak{so}(7,1)$	ϖ_1, ϖ_3, ϖ_4	(iii)
$\mathfrak{so}^*(8)$	ϖ_1, ϖ_3, ϖ_4	(ii)
$\mathfrak{g}_2^{\mathbb{C}}$	$\varpi_1 \boxtimes$triv	(iii)
$\mathfrak{g}_{2(2)}$	ϖ_1	(ii)

Next, we reduce candidates of pairs of primary simple factor $\mathfrak{l}_1$ and its irreducible representation π by using the following:

Lemma 1.4.41. If a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(8,\mathbb{C})$ of a reductive Lie algebra $\mathfrak{l}$ satisfies conditions (i) to (iv) of Lemma 1.4.38, then irreducible components π

of $\rho|_{\mathfrak{l}_1}$ satisfy the following conditions, where $\mathfrak{l}_1$ is the primary simple factor of $\mathfrak{l}$:

(i) $\operatorname{rank}_{\mathbb{R}} L_1 \leq 3$

(ii) $7 \leq \frac{d(L_1)}{\operatorname{rank}_{\mathbb{R}} L_1}$,

(iii) $\dim \pi_1 \geq 7 \implies d(L_1) - \operatorname{rank}_{\mathbb{R}} L_1 \geq 18$,

(iv) $\dim \pi \geq 5 \implies \pi^\vee \simeq \pi$ and $\operatorname{index}_{\theta_1} \pi = 1$,

(v) $\dim \pi + m(\mathfrak{l}_1) \geq 9 \implies \operatorname{rank} \pi(X) \geq 3$ for any $X \in \mathfrak{p}_L \setminus \{0\}$.

Here $m(\mathfrak{l}_1) := \min\{\dim \pi' : \pi'$ is a nontrivial irreducible representation of $\mathfrak{l}_1\}$ and θ_1 is a Cartan involution on $(\mathfrak{l}_1)_{\mathbb{C}}$.

Proof. (i) This is clear from Remark 1.2.14 and 1.2.16.

(ii) From Lemma 1.4.43, we have $\dim \pi \leq \dim \rho = 8 \leq \frac{d(L^{ss})}{\operatorname{rank}_{\mathbb{R}} L^{ss}} + 1 \leq \frac{d(L_1)}{\operatorname{rank}_{\mathbb{R}} L_1} + 1$.

(iii) If the number of simple factors in $\mathfrak{l}^{ss}$ is greater than or equal to two, then we have $\dim \pi + 2 \leq \dim \rho = 8$. So, we have $\mathfrak{l}^{ss} = \mathfrak{l}^1$ if $\dim \pi_1 \geq 7$. In this case, the equality $21 - d(L_1) \leq 3 - \operatorname{rank}_{\mathbb{R}} L_1$ holds from Lemma 1.4.43.

(iv) This comes form Lemma 1.2.40.

(v) It is enough to show $\rho|_{\mathfrak{l}_1} = \pi \oplus \mathrm{triv}$ if the inequality $\dim \pi + m(\mathfrak{l}_1) \geq 9$ holds. We show the contraposition. If there exists another irreducible component π' in $\rho|_{\mathfrak{l}_1}$, we have $\dim \pi + m(\mathfrak{l}_1) \leq \dim \pi + \dim \pi' \leq \dim \rho|_{\mathfrak{l}_1} = \dim \rho = 8$. $\square$

The following lemma comes from Remark 1.2.16 immediately.

Lemma 1.4.42. Suppose a representation (ρ, V) of $\mathfrak{l}$ satisfies conditions (i) to (iv) in Lemma 1.4.38, then the following inequalities hold:

$$\begin{cases} \operatorname{rank}_{\mathbb{R}} L^{ss} + t \leq 3 (= \operatorname{rank}_{\mathbb{R}} G - \operatorname{rank}_{\mathbb{R}} H), \\ d(L^{ss}) + t = 21 (= d(G) - d(H)). \end{cases}$$

Here $t := \dim \rho(\mathfrak{z})^{-\theta}$ and $\mathfrak{z}$ is the center of $\mathfrak{l}$ and θ is a Cartan involution on $\mathfrak{g}$ such that $\theta(\rho(\mathfrak{l})) = \rho(\mathfrak{l})$.

Lemma 1.4.43. There exists $t \geq 0$ such that

$$\begin{cases} \operatorname{rank}_{\mathbb{R}} L^{ss} + t \leq 3 \\ d(L^{ss}) + t = 21 \end{cases}$$

if and only if $0 \leq 21 - d(L^{ss}) \leq 3 - \operatorname{rank}_{\mathbb{R}} L^{ss}$, which implies $\dim \rho = 8 \leq \frac{d(L^{ss})}{\operatorname{rank}_{\mathbb{R}} L^{ss}} + 1$.

This can be easily checked by a fundamental argument on inequalities. So, we omit the proof.

By using Lemma 1.4.41, we obtain the candidates of pairs of a primary simple factor $\mathfrak{l}_1$ and its irreducible component of $\rho|_{\mathfrak{l}_1}$:

Lemma 1.4.44. Suppose a representation $\rho : \mathfrak{l} \to \mathfrak{sl}(8, \mathbb{C})$ of a reductive Lie algebra $\mathfrak{l}$ satisfies conditions (i) to (iv) of Lemma 1.4.38 and π is an irreducible component of $\rho|_{\mathfrak{l}_1}$, where $\mathfrak{l}_1$ is the primary simple factor of $\mathfrak{l}$. Then $(\mathfrak{l}_1, \pi)$ is equivalent to one of the following Table 1.23:

Table 1.23: pairs of simple Lie algebras $\mathfrak{l}$ and their irreducible representations π which satisfy the conditions (i) to (v) Lemma 1.4.41 with $\dim \pi \leq 8$.

$\mathfrak{l}_1$	π
$\mathfrak{so}(7, \mathbb{C})$	$\varpi_3 \boxtimes \mathrm{triv}$

Proof. This comes from Remark 1.4.40 and Lemma 1.4.41 and Tables 1.22, 24, 25, 26, 27, 28. □

Table 1.24: pairs of simple Lie algebras $\mathfrak{l}$ and their irreducible representations π which do not satisfy the condition (i) of Lemma 1.4.41.

$\mathfrak{l}$	π	$\mathrm{rank}_{\mathbb{R}} L$
$\mathfrak{sl}(n, \mathbb{C})$ $(5 \leq n \leq 8)$	$\varpi_1 \boxtimes$ triv	n
$\mathfrak{su}(4, 4)$	ϖ_1	4
$\mathfrak{sl}(n, \mathbb{R})$ $(5 \leq n \leq 8)$	ϖ_1	n
$\mathfrak{sp}(4, \mathbb{C})$	$\varpi_1 \boxtimes \mathrm{triv}$	4
$\mathfrak{sp}(4, \mathbb{R})$	ϖ_1	4
$\mathfrak{so}(8, \mathbb{C})$	$\varpi_1 \boxtimes \mathrm{triv}$	4
$\mathfrak{so}(4, 4)$	ϖ_1, ϖ_3, ϖ_4	4

Table 1.25: pairs of simple Lie algebras $\mathfrak{l}$ and their irreducible representations π which do not satisfy the condition (ii) of Lemma 1.4.41.

$\mathfrak{l}$	π	M_1
$\mathfrak{sl}(n,\mathbb{C})$ $(2 \leq n \leq 4)$	$\varpi_1 \boxtimes$ triv	$n+1$
$\mathfrak{sl}(4,\mathbb{C})$	$\varpi_2 \boxtimes$ triv	5
$\mathfrak{sl}(3,\mathbb{C})$	$(\varpi_1+\varpi_2)\boxtimes$ triv	4
$\mathfrak{sl}(2,\mathbb{C})$	$k\varpi_1 \boxtimes$ triv	3
$\mathfrak{su}(k,\ell)$ $(1 \leq \ell \leq k \leq 3)$	ϖ_1	$2k$
$\mathfrak{sl}(n,\mathbb{R})$ $(2 \leq n \leq 4)$	ϖ_1	$\frac{n+2}{2}$
$\mathfrak{su}^*(4)$	ϖ_1	5
$\mathfrak{su}(k,\ell)$ $(k+\ell=4)$	ϖ_2	$2k$
$\mathfrak{sl}(4,\mathbb{R})$	ϖ_2	3
$\mathfrak{su}^*(4)$	ϖ_2	5
$\mathfrak{sl}(3,\mathbb{R})$	$\varpi_1+\varpi_2$	$\frac{5}{2}$
$\mathfrak{su}(2,1)$	$\varpi_1+\varpi_2$	4
$\mathfrak{sl}(2,\mathbb{R})$	$k\varpi_1$ $(1 \leq k \leq 7)$	2
$\mathfrak{so}(5,\mathbb{C})$	$\varpi_1 \boxtimes$ triv	5
$\mathfrak{so}(5,\mathbb{C})$	$\varpi_3 \boxtimes$ triv	5
$\mathfrak{so}(k,\ell)$ $(k+\ell=7)$	ϖ_1, ϖ_3	k
$\mathfrak{so}(k,\ell)$ $(k+\ell=5)$	ϖ_1, ϖ_2	k
$\mathfrak{sp}(3,\mathbb{R})$	ϖ_1	4
$\mathfrak{so}(5,3)$	ϖ_1, ϖ_3, ϖ_4	5
$\mathfrak{so}(6,2)$	ϖ_1, ϖ_3, ϖ_4	6
$\mathfrak{so}^*(8)$	ϖ_1, ϖ_3, ϖ_4	6
$\mathfrak{g}_{2(2)}$	ϖ_1	4

Table 1.26: pairs of simple Lie algebras $\mathfrak{l}$ and their irreducible representations π which do not satisfy the condition (iii) of Lemma 1.4.41.

$\mathfrak{l}$	π	$\dim\pi$	$d(L)-\operatorname{rank}_{\mathbb{R}} L$
$\mathfrak{sp}(k,\ell)$ $(k+\ell=4)$	ϖ_1	8	$4k-\ell$
$\mathfrak{so}(7,1)$	ϖ_1, ϖ_3, ϖ_4	8	6
$\mathfrak{g}_2^{\mathbb{C}}$	$\varpi_1\boxtimes$triv	7	12

Table 1.27: pairs of simple Lie algebras $\mathfrak{l}$ and their irreducible representations π which do not satisfy the condition (iv) of Lemma 1.4.41.

$\mathfrak{l}$	π	$\dim\pi$	selfdual?
$\mathfrak{su}(4,\ell)$ $(1 \leq \ell \leq 3)$	ϖ_1	$4+\ell$	$\pi \not\simeq \pi^\vee$
$\mathfrak{su}^*(2n)$ $(3 \leq n \leq 4)$	ϖ_1	$2n$	$\pi \not\simeq \pi^\vee$
$\mathfrak{sp}(3,\mathbb{C})$	$\varpi_1\boxtimes$triv	6	$\operatorname{index}_\theta(\varpi_1\boxtimes\text{triv})=-1$
$\mathfrak{sp}(2,1)$	ϖ_1	6	$\operatorname{index}_\theta \varpi_1 = -1$

Table 1.28: pairs of simple Lie algebras $\mathfrak{l}$ and their irreducible representations π which do not satisfy the condition (v) of Lemma 1.4.41.

$\mathfrak{l}_1$	π	not satisfy
$\mathfrak{so}(7,\mathbb{C})$	$\varpi_1 \boxtimes \mathrm{triv}$	(v)

Next, we determine the pair $(\mathfrak{l}^{ss},\ \rho|_{\mathfrak{l}^{ss}})$ as follows:

Lemma 1.4.45. $(\mathfrak{l}^{ss}, \rho|_{\mathfrak{l}^{ss}})$ is eqivalent to one of the following Table 1.29

Table 1.29: pairs of semisimple Lie algebras $\mathfrak{l}^{ss}$ and their representations $\rho|_{\mathfrak{l}^{ss}}$.

$\mathfrak{l}^{ss}$	$\rho\|_{\mathfrak{l}^{ss}}$
$\mathfrak{so}(7,\mathbb{C})$	$\varpi_3 \boxtimes \mathrm{triv}$

Proof. This comes from Lemma 1.4.44, 1.4.46 and $\dim \pi = 8$. □

Lemma 1.4.46. If irreducible component π of $\rho|_{\mathfrak{l}_1}$ satisfies $\dim \pi = \dim \rho = 8$, then we have $\mathfrak{l}^{ss} = \mathfrak{l}_1$ and $\rho|_{\mathfrak{l}^{ss}} = \pi$.

Proof. $\mathfrak{l}^{ss} = \mathfrak{l}_1$ comes from Lemma 1.2.39 and $\rho|_{ss} = \pi$ comes from Lemma 1.2.37. □

Finally we determine the pair $(\mathfrak{l},\ \rho)$ as follows:

Lemma 1.4.47. $(\mathfrak{l}, \rho)$ is equivalent to one of the following Table 1.30:

Table 1.30: candidates of pairs of reductive Lie algebras $\mathfrak{l}$ and their representations ρ.

$\mathfrak{l}$	ρ
$\mathfrak{so}(7,\mathbb{C})$	$\varpi_3 \boxtimes \mathrm{triv}$

Proof. This comes from Lemma 1.4.45 and Remark 1.4.48. □

Remark 1.4.48. If $\mathrm{rank}_{\mathbb{R}} L^{ss} = \mathrm{rank}_{\mathbb{R}} G - \mathrm{rank}_{\mathbb{R}} H$, then $L = L^{ss}$ up to compact factor.

Remark 1.4.49. To show $\mathfrak{l}^{ss} = \mathfrak{l}$ up to compact factor, it is enough to prove $\dim \rho(\mathfrak{z})^{-\theta} = 0$. Here $\mathfrak{z}$ is the center of $\mathfrak{l}$ and θ is a Cartan involution on $\mathfrak{g}$ such that $\theta(\rho(\mathfrak{l})) \subset \rho(\mathfrak{l})$.

1.5 Classification of embeddings of $\mathfrak{l}$ up to $\mathrm{Int}(\mathfrak{g})$

To solve Problem C', we classify reductive subalgebras $\mathfrak{l} \subset \mathfrak{g}$ which induces proper and cocompact action on G/H up to conjugate by $\mathrm{Int}(\mathfrak{g})$. In Section 1.3 and 1.4, we classified reductive subalgebras in the representation level, namely up to conjugate by $\mathrm{Ad}(SL(n,\mathbb{C}))$ or $\mathrm{Ad}(GL(n,\mathbb{R}))$, where $\mathfrak{g} \subset \mathfrak{sl}(n,\mathbb{C})$ or $\mathfrak{gl}(n,\mathbb{R})$ is a natural realization. In the representation level, we identify $\mathfrak{l}$ and $\mathfrak{l}'$ which are not conjugate by $\mathrm{Int}(\mathfrak{g})$ but are conjugate by $\mathrm{Ad}(SL(n,\mathbb{C}))$ (or

$\mathrm{Ad}(GL(n,\mathbb{R}))$). To distinguish them, it is enough to investigate the inverse image of the following maps $\Phi_{\mathbb{C}}$ and $\Phi_{\mathbb{R}}$, where $\iota : \mathfrak{g} \to \mathfrak{sl}(n,\mathbb{C})$ or $\mathfrak{gl}(n,\mathbb{R})$ is a fixed realization.

$$\Phi_{\mathbb{C}} : \mathfrak{D}(\mathfrak{l},\mathfrak{g}) := \mathrm{Int}(\mathfrak{g})\backslash \mathrm{Hom}_0(\mathfrak{l},\mathfrak{g})/\mathrm{Aut}(\mathfrak{l}) \to \mathrm{Ad}(SL(n,\mathbb{C}))\backslash \mathrm{Hom}_0(\mathfrak{l},\mathfrak{sl}(n,\mathbb{C}))/\mathrm{Aut}(\mathfrak{l}) =: \mathfrak{D}(\mathfrak{l},\mathfrak{sl}(n,\mathbb{C}))$$
$$[\varpi] \mapsto [\iota \circ \varpi]$$
$$\Phi_{\mathbb{R}} : \mathfrak{D}(\mathfrak{l},\mathfrak{g}) := \mathrm{Int}(\mathfrak{g})\backslash \mathrm{Hom}_0(\mathfrak{l},\mathfrak{g})/\mathrm{Aut}(\mathfrak{l}) \to \mathrm{Ad}(GL(n,\mathbb{R}))\backslash \mathrm{Hom}_0(\mathfrak{l},\mathfrak{sl}(n,\mathbb{R}))/\mathrm{Aut}(\mathfrak{l}) =: \mathfrak{D}(\mathfrak{l},\mathfrak{sl}(n,\mathbb{R}))$$
$$[\varpi] \mapsto [\iota \circ \varpi]$$

Here $\mathrm{Hom}_0(\mathfrak{l},\mathfrak{g}) := \{f : \mathfrak{l} \to \mathfrak{g} : f \text{ is an injective homomorphism}\}$, and $\mathrm{Aut}(\mathfrak{l})$ acts on $\mathrm{Hom}_0(\mathfrak{l},\mathfrak{g})$ as follows:

$$\mathrm{Hom}_0(\mathfrak{l},\mathfrak{g}) \times \mathrm{Aut}(\mathfrak{l}) \to \mathrm{Hom}_0(\mathfrak{l},\mathfrak{g})$$
$$(f,\alpha) \mapsto f \circ \alpha$$

1.5.1 general method

We want to determine the set $\Phi_{\mathbb{K}}^{-1}([\iota \circ \rho_0])$ for $(\mathfrak{g},\, \mathfrak{l},\, \rho_0)$ in Table 1.31, where ρ_0 is the element of $\mathrm{Hom}_0(\mathfrak{l},\mathfrak{g})$ which induces proper and cocompact L-action on G/H. Here we consider a realization $G \subset SL(n,\mathbb{C})$ or $G \subset GL(n,\mathbb{R})$. For our purpose, we use some methods (Lemma 1.5.1, 1.5.2, 1.5.3 and 1.5.5).

Lemma 1.5.1. Let $\tilde{\tau}$ be an involution on $SL(n,\mathbb{C})$. Put $G := SL(n,\mathbb{C})^{\tilde{\tau}}$, $\mathfrak{g} := \mathrm{Lie}(G)$, $\tau := d\tilde{\tau} : \mathfrak{sl}(n,\mathbb{C}) \to \mathfrak{sl}(n,\mathbb{C})$ and

$$M = \{g \in SL(n,\mathbb{C}) \mid g^{-1}\tilde{\tau}(g) \in \mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{C}^n)\}.$$

Let F be a set of generators of M as a G_0-set. Then we have

$$\Phi_{\mathbb{C}}^{-1}([\iota \circ \rho_0]) = \{[\mathrm{Ad}(f)\rho_0] \mid f \in F\}.$$

Proof. First, we prove $\Phi_{\mathbb{C}}^{-1}([\iota \circ \rho_0]) \supset \{[\mathrm{Ad}(f)\rho_0] \mid f \in F\}$. It is enough to show that $\Phi_{\mathbb{C}}([\mathrm{Ad}(f)\rho_0]) = [\iota \circ \rho_0]$ for any $f \in F$, namely, there exists $g \in SL(n,\mathbb{C})$ such that $\mathrm{Ad}(g)\,\mathrm{Ad}(f)\rho_0(\mathfrak{l}) = \rho_0(\mathfrak{l})$. By taking $g := f^{-1} \in SL(n,\mathbb{C})$, we get the desired equality.

Next, we prove $\Phi_{\mathbb{C}}^{-1}([\iota \circ \rho_0]) \subset \{[\mathrm{Ad}(f)\rho_0] \mid f \in F\}$. Let $[\varphi] \in \Phi_{\mathbb{C}}^{-1}([\iota \circ \rho_0])$. Then there exists $g \in SL(n,\mathbb{C})$ such that $\mathrm{Ad}(g)\rho_0(\mathfrak{l}) = \varphi(\mathfrak{l}) \subset \mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})^{\tau}$, that is, $\mathrm{Ad}(g^{-1}\tilde{\tau}(g))X = X$ for any $X \in \rho_0(\mathfrak{l})$. Therefore we have $g \in M$. From the definition of F, there exist $g_0 \in G_0$ and $f \in F$ such that $g = g_0 f$. Thus we have $\mathrm{Ad}(g_0)\,\mathrm{Ad}(f)\rho_0(\mathfrak{l}) = \varphi(\mathfrak{l})$, namely, $[\varphi] = [\mathrm{Ad}(f)\rho_0]$. □

Lemma 1.5.2. Let $\tilde{\tau}$ be an involution on $GL(n,\mathbb{R})$ such that $\tilde{\tau}(SL(n,\mathbb{R})) = SL(n,\mathbb{R})$. Put $G := SL(n,\mathbb{R})^{\tilde{\tau}}$, $\mathfrak{g} := \mathrm{Lie}(G)$, $\tau := d\tilde{\tau} : \mathfrak{gl}(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$ and

$$M = \{g \in GL(n,\mathbb{R}) \mid g^{-1}\tilde{\tau}(g) \in \mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{R}^n)\}.$$

Let F be a set of generators of M as a G_0-set. Then we have

$$\Phi_{\mathbb{R}}^{-1}([\iota \circ \rho_0]) = \{[\mathrm{Ad}(f)\rho_0] \mid f \in F\}.$$

This can be proved in the same way as Lemma 1.5.1. So, we omit the proof.

Lemma 1.5.3. Let $\tilde{\tau}_i : SL(n,\mathbb{C}) \to SL(n,\mathbb{C})$ be involutions ($i = 1, 2$) such that $\tilde{\tau}_1\tilde{\tau}_2 = \tilde{\tau}_2\tilde{\tau}_1$. Put $G := (SL(n,\mathbb{C})^{\tilde{\tau}_1})^{\tilde{\tau}_2}$, $\mathfrak{g} := \mathrm{Lie}(G)$, $\tau_i := d\tilde{\tau}_i : \mathfrak{sl}(n,\mathbb{C}) \to \mathfrak{sl}(n,\mathbb{C})$ and

$$M := \{g \in SL(n,\mathbb{C}) \mid g^{-1}\tilde{\tau}_i(g) \in \mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{C}^n)\ (i = 1, 2)\},$$

where M admits G_0-left action. Let F be a set of generators of M as a G_0-set. Then we have

$$\Phi_{\mathbb{C}}^{-1}([\iota \circ \rho_0]) = \{[\mathrm{Ad}(f)\rho_0] \mid f \in F\}.$$

This can be proved in the same way as Lemma 1.5.1. So, we omit the proof.

For cases where ρ_0 is irredusible, the following fact is useful:

Fact 1.5.4 ([Ta96, Theorem 8.7])**.** Let $\mathfrak{l}$ be a real Lie algebra, $\rho : \mathfrak{l} \to \mathfrak{gl}(V)$ its complex irreducible representation and $r : \mathfrak{l} \to \mathfrak{gl}(E)$ the corresponding irreducible real representation by Cartan's fundamental theorem. Then we have

$$\begin{aligned}\mathrm{End}_{r(\mathfrak{l})}(E) &:= \{f \in \mathrm{End}(E) : fr(X) = r(X)f \text{ for all } X \in \mathfrak{l}\}\\ &\simeq \begin{cases}\mathbb{R} \text{ if } \rho \simeq \overline{\rho} \text{ and } \mathrm{index}_\tau\, \rho = 1,\\ \mathbb{H} \text{ if } \rho \simeq \overline{\rho} \text{ and } \mathrm{index}_\tau\, \rho = -1,\\ \mathbb{C} \text{ if } \rho \not\simeq \overline{\rho}.\end{cases}\end{aligned}$$

Here τ is the involution on $\mathfrak{l}_{\mathbb{C}}$ such that $\mathfrak{l}_{\mathbb{C}}^{\tau} = \mathfrak{l}$.

Lemma 1.5.5. Let G be a connected linear reductive Lie group and $\mathfrak{g}$ its Lie algebra. Fix a Cartan involution on $\mathfrak{g}$. Let $\mathfrak{l}_i$ ($i = 1, 2$) be semisimple subalgebras of $\mathfrak{g}$ such that $\theta(\mathfrak{l}_i) = \mathfrak{l}_i$. If there exists an element $\alpha \in \mathrm{Int}(\mathfrak{g})$ such that $\alpha(\mathfrak{l}_1) = \mathfrak{l}_2$, then there exists an element $k \in K = G^\theta$ such that $\mathrm{Ad}(k)\mathfrak{l}_1 = \mathfrak{l}_2$.

Proof. Take $e^X k \in G$ such that $\alpha = \mathrm{Ad}(e^X k)$. Then we have $\mathrm{Ad}(e^X)(\mathrm{Ad}(k)\mathfrak{p}_1) = \mathfrak{p}_2 \subset \mathfrak{p}$ and $\mathrm{Ad}(k)\mathfrak{p}_1 \subset \mathfrak{p}$. From Fact 1.5.6, for any $H \in \mathrm{Ad}(k)\mathfrak{p}_1$, $\mathrm{Ad}(e^X)H = H$ holds. Therefore we have $\mathrm{Ad}(k)\mathfrak{p}_1 = \mathfrak{p}_2$. Since we have $\mathfrak{l}_i = [\mathfrak{p}_i, \mathfrak{p}_i] + \mathfrak{p}_i$ from semisimplicity of $\mathfrak{l}_i$ ($i = 1, 2$), we obtain $\mathrm{Ad}(k)\mathfrak{l}_1 = \mathfrak{l}_2$. $\square$

The following fact, which is used in the proof of the above lemma, was proved by Takayuki Okuda in his master thesis.

Fact 1.5.6. Fix $H \in \mathfrak{p}$. Take $H_0 \in \mathfrak{p}$. If $\exp(\mathrm{ad}(H_0))H \in \mathfrak{p}$ holds, then $\exp(\mathrm{ad}(H_0))H = H$ holds.

Proof. Let θ be the Cartan involution on $\mathfrak{g}$. Since $\exp(\mathrm{ad}(H_0))H \in \mathfrak{p}$ holds, we have $\theta(\exp(\mathrm{ad}(H_0))H) = -\exp(\mathrm{ad}(H_0))H$. On the other hand, we have

$$\begin{aligned}\theta(\exp(\mathrm{ad}(H_0))H) &= \exp(\mathrm{ad}(\theta(H_0)))\theta(H)\\ &= \exp(\mathrm{ad}(-H_0))(-H)\\ &= -\exp(\mathrm{ad}(-H_0))H.\end{aligned}$$

Therefore, we have $H \in \ker(\exp(\mathrm{ad}(H_0)) - \exp(\mathrm{ad}(-H_0)))$. Since all the eigenvalues of $\mathrm{ad}(H_0)$ are in $\mathbb{R}$ and $\mathrm{ad}(H_0)$ is diagonalizable, $\exp(\mathrm{ad}(H_0)) - \exp(\mathrm{ad}(-H_0))$ acts on V_λ as a scalar $e^\lambda - e^{-\lambda}$, where V_λ is λ-eigenspace of $\mathrm{ad}(H_0)$. So, we have $H \in \ker(\mathrm{ad}(H_0))$. Thus we obtain $\exp(\mathrm{ad}(H_0))H = H$. $\square$

We consider the cases which appear in the classification of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ in the level of representation. Let ρ_0 be an embedding of $\mathfrak{l}$ into $\mathfrak{g}$ induced by the fixed representation which induces proper and cocompact action of L on G/H. In the following table, the number of $\Phi_{\mathbb{K}}^{-1}([\iota\rho_0])$ means the number of embeddings which is distinguished by $\mathrm{Int}(\mathfrak{g})$. We see what kinds of embeddings appear in the following subsections.

Table 1.31: The cardinarity of the inverse image $\Phi_{\mathbb{K}}^{-1}([\iota\rho_0])$ for reductive subalgebras $\mathfrak{l}$ which induce proper and cocompact action on G/H

$\mathfrak{g}$	$\mathfrak{l}$	$\Phi_{\mathbb{K}}^{-1}([\iota\rho_0])$	$\mathfrak{h}$
$\mathfrak{su}(2p,2)$	$\mathfrak{sp}(p,1)$ $(p \geq 1)$	one point	$\mathfrak{u}(2p,1)$
$\mathfrak{su}(2p,2)$	$\mathfrak{su}(2p,1)$ $(p \geq 1)$	$\begin{cases} \text{two points if } p=1 \\ \text{one point if } p \geq 2 \end{cases}$	$\mathfrak{sp}(p,1)$
$\mathfrak{so}(2p,2)$	$\mathfrak{so}(2p,1)$ $(p \geq 2)$	one point	$\mathfrak{u}(p,1)$
$\mathfrak{so}(2p,2)$	$\mathfrak{su}(p,1)$ $(p \geq 2)$	two points	$\mathfrak{so}(2p,1)$
$\mathfrak{so}(4p,4)$	$\mathfrak{sp}(p,1)$ $(p \geq 1)$	four points	$\mathfrak{so}(4p,3)$
$\mathfrak{so}(8,8)$	$\mathfrak{spin}(1,8)$	four points	$\mathfrak{so}(7,8)$
$\mathfrak{so}(4,4)$	$\mathfrak{spin}(3,4)$	four points	$\mathfrak{so}(4,1) \oplus \mathfrak{so}(3)$
$\mathfrak{so}(4,3)$	$\mathfrak{g}_{2(2)}$	two points	$\mathfrak{so}(4,1) \oplus \mathfrak{so}(2)$
$\mathfrak{so}(8,\mathbb{C})$	$\mathfrak{spin}(1,7)$	two points	$\mathfrak{so}(7,\mathbb{C})$
$\mathfrak{so}(8,\mathbb{C})$	$\mathfrak{spin}(7,\mathbb{C})$	two points	$\mathfrak{so}(7,1)$
$\mathfrak{so}^*(8)$	$\mathfrak{spin}(1,6)$	one point	$\mathfrak{u}(3,1)$
			$\mathfrak{so}^*(6) \oplus \mathfrak{so}^*(2)$

We describe all the points in $\Phi_{\mathbb{K}}^{-1}([\iota\rho_0])$ in the following subsection for each pair $(\mathfrak{g}, \rho_0(\mathfrak{l}))$.

1.5.2 $(\mathfrak{g}, \mathfrak{l}) = (\mathfrak{su}(2p,2), \mathfrak{sp}(p,1))$ $(p \geq 1)$

In this subsection, we consider the case $(\mathfrak{g},\ \mathfrak{l})$=$(\mathfrak{su}(2p,2),\ \mathfrak{sp}(p,1))$ for the symmetric pair $(\mathfrak{g},\ \mathfrak{h})$=$(\mathfrak{su}(2p,2),\ \mathfrak{u}(2p,1))$. From Proposition 1.4.3, it is enough to consider the standard representation $\rho_0 := \rho_{\varpi_1} : \mathfrak{sp}(p,1) \to \mathfrak{su}(2p,2) \subset \mathfrak{sl}(2p+2,\mathbb{C})$. Our goal of this subsection is the following:

Proposition 1.5.7. $\Phi_{\mathbb{C}}^{-1}([\rho_0]) \subset \mathfrak{D}(\mathfrak{l},\mathfrak{g})$ consists of one point, namely, $\Phi_{\mathbb{C}}^{-1}([\rho_0]) = \{[\rho_0]\}$.

We realize $G = SU(2p,2) := SL(2p+2)^{\tilde{\tau}}$, $\mathfrak{g} = \mathfrak{su}(2p,2) := \mathfrak{sl}(2p+2,\mathbb{C})^\tau$ by the following involutions $\tilde{\tau}$, τ. Let $\rho_0 : \mathfrak{l} \to \mathfrak{g}$ be the standard embedding, which image $\rho_0(\mathfrak{l}) = \mathfrak{sp}(p,1) \subset \mathfrak{su}(2p,2)$ is described as follows.

$$\begin{aligned}
\tilde{\tau} : SL(2p+2,\mathbb{C}) &\to SL(2p+2,\mathbb{C}), \\
g &\mapsto I_{2p,2} g^{*-1} I_{2p,2}^{-1} \\
\tau : \mathfrak{sl}(2p+2,\mathbb{C}) &\to \mathfrak{sl}(2p+2,\mathbb{C}), \\
X &\mapsto -I_{2p,2} X^* I_{2p,2}^{-1}, \\
\sigma : \mathfrak{su}(2p,2) &\to \mathfrak{su}(2p,2) \\
X &\mapsto -\begin{pmatrix} J_p & \\ & J_1 \end{pmatrix} X^* \begin{pmatrix} J_p & \\ & J_1 \end{pmatrix}^{-1} \\
\mathfrak{sp}(p,1) &= \mathfrak{su}(2p,2)^{\sigma}
\end{aligned}$$

Proof of Proposition 1.5.7. We use Lemma 1.5.1.

Claim.

$$M = \begin{cases} SU(2,2) \cdot \{I_4, I_2 \otimes J\} \text{ if } p = 1 \\ SU(2p,2) \text{ if } p \geq 2 \end{cases}$$

proof of Claim. The inclusion $\supset$ is clear by definition. We show the inclusion $\subset$. Let $g \in M$. Since we have $\mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{C}^{2p+2}) = \mathbb{C} I_{2p+2}$ from Lemma 1.5.8, we have $\tilde{\tau}(g) = ag$, that is, $ag^* I_{2p,2} g = I_{2p,2}$ for some $a \in \mathbb{C}^*$. By taking determinant and adjoint, we have $a \in \{\pm 1\}$. In the case $p \geq 2$, from Sylvester's law of inertia, we obtain that $a = 1$, that is, $g \in SU(2p,2)$. In the case $p = 1$ and $a = -1$, we have $gJ_2^{-1} \in SU(2,2)$, namely, $g \in SU(2,2)(I_2 \otimes J)$. ☐

From the above Claim, we can take F of Lemma 1.5.1 as follows:

$$F = \begin{cases} \{I_4, I_2 \otimes J\} \text{ if } p = 1 \\ \{I_{2p+2}\} \text{ if } p \geq 2 \end{cases}$$

Since $\mathrm{Ad}(J_2)$ preserves $\mathfrak{sp}(1,1)$, we have the desired conclusion from Lemma 1.5.1. ☐

Lemma 1.5.8. We have

$$\mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{C}^{2p+2}) = \{aI_{2p+2} : a \in \mathbb{C}\} \simeq \mathbb{C}.$$

Proof. This comes from Schur's lemma over $\mathbb{C}$ and that the representation $\iota\rho_0 : \mathfrak{sp}(p,1) \to \mathfrak{su}(2p,2) \subset \mathfrak{sl}(2p+2,\mathbb{C})$ is irreducible. ☐

1.5.3 $(\mathfrak{g},\mathfrak{l}) = (\mathfrak{su}(2p,2), \mathfrak{su}(2p,1))$ $(p \geq 1)$

In this subsection, we consider the case $(\mathfrak{g},\mathfrak{l}) = (\mathfrak{su}(2p,2), \mathfrak{su}(2p,1))$ $(p \geq 1)$ for a symmetric pair $(\mathfrak{g}, \mathfrak{h})$=$(\mathfrak{su}(2p,2), \mathfrak{sp}(p,1))$. From Proposition 1.3.15, it is enough to consider the standard representation $\rho_0 = \rho_{\varpi_1} \oplus$triv: $\mathfrak{su}(2p,1) \to \mathfrak{su}(2p,2) \subset \mathfrak{sl}(2p+2,\mathbb{C})$. Our goal in this subsection is the following:

Proposition 1.5.9. We have

$$\Phi_{\mathbb{C}}^{-1}([\iota \circ \rho_0]) = \begin{cases} \{[\rho_0]\} \quad (p \geq 2), \\ \{[\rho_0], [\mathrm{Ad}(J_2)\rho_0]\} \quad (p = 1). \end{cases}$$

Here $J_2 := \begin{pmatrix} & -I_2 \\ I_2 & \end{pmatrix}$.

We realize $G = SU(2p, 2) = SL(2p + 2, \mathbb{C})^{\tilde{\tau}}$, $\mathfrak{g} = \mathfrak{su}(2p, 2) = \mathfrak{sl}(2p + 2, \mathbb{C})^{\tau}$ by the following involutions $\tilde{\tau}$, τ:

$$\tilde{\tau} : SL(2p + 2, \mathbb{C}) \to SL(2p + 2, \mathbb{C}), g \mapsto I_{2p,2} g^{*-1} I_{2p,2}^{-1},$$
$$\tau : \mathfrak{sl}(2p + 2, \mathbb{C}) \to \mathfrak{sl}(2p + 2, \mathbb{C}), X \mapsto -I_{2p,2} X^* I_{2p,2}^{-1}.$$

We fix $\rho_0 : \mathfrak{su}(2p, 1) \to \mathfrak{su}(2p, 2)$ as follows:

$$\rho_0 : \mathfrak{su}(2p, 1) \to \mathfrak{su}(2p, 2),$$
$$X \mapsto \begin{pmatrix} X & \\ & 0 \end{pmatrix}.$$

Lemma 1.5.10. We have

$$\mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{C}^{2p+2}) = \{\begin{pmatrix} aI_{2p+1} & \\ & b \end{pmatrix} \in M(2p + 2, \mathbb{C}) : a, b \in \mathbb{C}\}.$$

Proof. This immediately comes from Schur's lemma over $\mathbb{C}$. □

Proof of Proposition 1.5.9. We use Lemma 1.5.1.

Claim.

$$M = \begin{cases} SU(2, 2) \cdot \{I_4, I_2 \otimes J\} \cdot \{\mathrm{diag}(aI_3, a^{-3}) : a > 0\} \text{ if } p = 1 \\ SU(2p, 2) \cdot \{\mathrm{diag}(aI_{2p+1}, a^{-2p-1}) : a > 0\} \text{ if } p \geq 2 \end{cases}$$

proof of Claim. The inclusion $\supset$ is clear by definition. We show the inclusion $\subset$. Let $g \in M$. From Lemma 1.5.10, there exist $a, b \in \mathbb{C}$ such that $g^{-1}\tilde{\tau}(g) = \begin{pmatrix} aI_{2p+1} & \\ & b \end{pmatrix}$. By taking matrix adjoint and determinant, we obtain $a, b \in \mathbb{R}^{\times}$, $b = a^{-(2p+1)}$.

- in the case $p \geq 2$: By Sylvester's law of inertia, we have $a > 0$. Then we have $I_{2p,2} = g^* I_{2p,2} g \,\mathrm{diag}(a, \cdots, a, a^{-2p-1}) \iff g \,\mathrm{diag}(\sqrt{a}, \cdots, \sqrt{a}, \sqrt{a^{-2p-1}}) \in SU(2p, 2)$. So, there exists an element $g_0 \in SU(2p, 2)$ such that $g = g_0 \,\mathrm{diag}(\sqrt{a}, \cdots, \sqrt{a}, \sqrt{a^{-2p-1}})^{-1}$.
- in the case $p = 1$: If $a > 0$, we get $g \in SU(2, 2) \cdot \{\mathrm{diag}(aI_3, a^{-3}) : a > 0\}$ in the same way above. If $a < 0$, we have

$$g^* I_{2,2} g \,\mathrm{diag}(a, a, a, a^{-3}) = I_{2,2}$$
$$\iff (g \,\mathrm{diag}(\sqrt{-a}, \sqrt{-a}, \sqrt{-a}, \sqrt{-a^{-3}}) J_2^{-1})^* I_{2,2} (g \,\mathrm{diag}(\sqrt{-a}, \sqrt{-a}, \sqrt{-a}, \sqrt{-a^{-3}}) J_2^{-1}) = I_{2,2}.$$

Therefore, we get $g\operatorname{diag}(\sqrt{-a},\sqrt{-a},\sqrt{-a},\sqrt{-a^{-3}})J_2^{-1}\in SU(2,2)$, namely, $g\in SU(2,2)\cdot\{I_2\otimes J\}\cdot\{\operatorname{diag}(aI_3,a^{-3}):a>0\}$.

□

From the above Claim, we can take F of Lemma 1.5.1 as follows:

$$F=\begin{cases}\{I_4,I_2\otimes J\}\cdot\{\operatorname{diag}(aI_3,a^{-3}):a>0\} \text{ if } p=1\\ \{\operatorname{diag}(aI_{2p+1},a^{-2p-1}):a>0\} \text{ if } p\geq 2\end{cases}$$

Since $\operatorname{Ad}(\operatorname{diag}(aI_{2p+1},a^{-3}))$ preserve the image $\rho_0(\mathfrak{l})$ for $p\geq 1$, it is enough to show the following:

Claim. $[\rho_0]\neq[\operatorname{Ad}(J_2)\rho_0]\in\mathfrak{D}(\mathfrak{l},\mathfrak{g})$, namely, $\rho_0(\mathfrak{l})=\mathfrak{su}(2,1)$ is not $\operatorname{Int}(\mathfrak{su}(2,2))$-conjugate to $\operatorname{Ad}(J_2)\rho_0(\mathfrak{l})$.

proof of Claim. Assume there exists an element $g\in SU(2,2)$ such that $g^{-1}\mathfrak{su}(2,1)g=J_2\mathfrak{su}(2,1)J_2^{-1}$. Put $X=i\operatorname{diag}(0,0,1,-1)\in J_2\mathfrak{su}(2,1)J_2^{-1}$. Then we have $gXg^{-1}\in\mathfrak{su}(2,1)$. We describe $g=\begin{pmatrix}A&B\\C&D\end{pmatrix}$. Then by direct calculation, we have

$$gXg^{-1}=i\begin{pmatrix}B\operatorname{diag}(-1,1)B^* & B\operatorname{diag}(1,-1)D^*\\ D\operatorname{diag}(-1,1)B^* & D\operatorname{diag}(1,-1)D^*\end{pmatrix}\in\mathfrak{su}(2,1)$$

So there exists $d'\in\mathbb{R}$ such that $D\operatorname{diag}(1,-1)D^*=\operatorname{diag}(id',0)$. But Since D is in $GL(2,\mathbb{C})$ by Remark 1.5.11, this is contradiction.

□

□

Remark 1.5.11. For $g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in SL(4,\mathbb{C})$, $A,B,C,D\in M(2,\mathbb{C})$, we have

$$g\in SU(2,2)\iff g^*I_{2,2}g=I_{2,2}\iff\begin{pmatrix}A^*A-C^*C & A^*B-C^*D\\ B^*A-D^*C & B^*B-D^*D\end{pmatrix}=I_{2,2}$$

So, we have $D^*D=I_2+B^*B\in\operatorname{Herm}_{>0}(2,\mathbb{C})=\{H\in\operatorname{Herm}(2,\mathbb{C}):\det H>0\}$.

1.5.4 $(\mathfrak{g},\mathfrak{l})=(\mathfrak{so}(2p,2),\mathfrak{so}(2p,1))$ $(p\geq 2)$

In this subsection, we consider the case $(\mathfrak{g},\mathfrak{l})$=$(\mathfrak{so}(2p,2)$, $\mathfrak{so}(2p,1))$ $(p\geq 2)$ for the symmetric pair $(\mathfrak{g},\mathfrak{h})=(\mathfrak{so}(2p,2),\mathfrak{u}(p,1))$. It is enough to consider the representation $\rho\simeq\rho_{\varpi_1}\oplus$triv: $\mathfrak{l}\to\mathfrak{sl}(2p+2,\mathbb{C})$ from Proposition 1.4.9. The representation $\rho_{\varpi_1}\oplus$trivial factors $\mathfrak{so}(2p,2)$, so we denote the embedding into $\mathfrak{so}(2p,2)\subset\mathfrak{sl}(2p+2,\mathbb{R})$ by ρ_0. Moreover, such ρ_0 is unique up to equivalence class as a real representation of $\mathfrak{so}(2p,1)$ by Cartan's fundamental theorem (see Appendix 1.6.3). Our goal in this subsection is the following:

Proposition 1.5.12. $\Phi_{\mathbb{R}}^{-1}([\iota \circ \rho_0])$ is a one point set, namely, $\Phi_{\mathbb{R}}^{-1}([\iota \circ \rho_0]) = \{[\rho_0]\}$.

We realize $G = SO(2p, 2)$, $\mathfrak{g} = \mathfrak{so}(2p, 2)$ as follows:

$$\begin{aligned}\tilde{\tau} : GL(2p+2, \mathbb{R}) &\to GL(2p+2, \mathbb{R}),\\ g &\mapsto I_{2p,2}{}^t g^{-1} I_{2p,2}^{-1},\\ G &:= SL(2p+2, \mathbb{R})^{\tilde{\tau}},\\ \tau : \mathfrak{sl}(2p+2, \mathbb{R}) &\to \mathfrak{sl}(2p+2, \mathbb{R}),\\ \tau(X) &:= -I_{2p,2}{}^t X I_{2p,2}^{-1}\\ \mathfrak{so}(2p, 2) &:= \mathfrak{sl}(2p+2, \mathbb{R})^{\tau}.\end{aligned}$$

Moreover, we realize $\rho_{\varpi_1} \oplus \mathrm{triv}$ as follows:

$$\begin{aligned}\mathfrak{so}(2p, 1) \to \mathfrak{so}(2p, 2) &\subset \mathfrak{sl}(2p+2, \mathbb{R})\\ X &\mapsto \begin{pmatrix} X & \\ & 0 \end{pmatrix}\end{aligned}$$

Lemma 1.5.13. we have

$$\mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{R}^{2p+2}) = \{\begin{pmatrix} aI_{2p+1} & \\ & b \end{pmatrix} \in M(2p+2, \mathbb{R}) : a, b \in \mathbb{R}\}.$$

Proof. This immediately comes from Schur's lemma over $\mathbb{R}$ (Fact 1.5.4). □

Proof of Proposition 1.5.12. We use Lemma 1.5.2.

Claim. $M = SO_0(2p, 2) \cdot \{I_{2p+2}, \mathrm{diag}(I_{2p-1}, -1, 1, 1), \mathrm{diag}(I_{2p+1}, -1), \mathrm{diag}(I_{2p-1}, -1, 1, -1)\} \cdot \{\mathrm{diag}(aI_{2p+1}, b) : a, b > 0\}$.

proof of Claim. The inclusion $\supset$ is clear by definition. We show the inclusion $\subset$. Let $g \in M$. From Lemma 1.5.13, there exists $a, b \in \mathbb{R}$ such that $g^{-1}\tilde{\tau}(g) = \begin{pmatrix} aI_{2p+1} & \\ & b \end{pmatrix}$. By Sylvester's law of inertia, we have $a > 0$ and $b > 0$. Therefore we have $g\,\mathrm{diag}(\sqrt{a}I_{2p+1}, \sqrt{b}) \in O(2p, 2)$, which implies the desired conclusion. □

From the above Claim, we can take F of Lemma 1.5.2 as follows:

$$F = \{I_{2p+2}, \mathrm{diag}(I_{2p-1}, -1, 1, 1), \mathrm{diag}(I_{2p+1}, -1), \mathrm{diag}(I_{2p-1}, -1, 1, -1)\} \cdot \{\mathrm{diag}(aI_{2p+1}, b) : a, b > 0\}.$$

Here, $\mathrm{Ad}(x)$ $(x \in \{I, \mathrm{diag}(I_{2p-1}, -1, 1, 1), \mathrm{diag}(I_{2p+1}, -1), \mathrm{diag}(I_{2p-1}, -1, 1, -1)\})$ and $\mathrm{Ad}(\mathrm{diag}(\sqrt{a}I_{2p+1}, \sqrt{b})$ $(a, b > 0)$ preserve $\rho_0(\mathfrak{l})$. Therefore, we obtain $\Phi_{\mathbb{R}}^{-1}([\iota \circ \rho_0]) = \{[\rho_0]\}$ from Lemma 1.5.2. □

1.5.5 $(\mathfrak{g}, \mathfrak{l}) = (\mathfrak{so}(2p,2), \mathfrak{su}(p,1))$ $(p \geq 2)$

In this subsection, we consider the case $(\mathfrak{g}, \mathfrak{l}) = (\mathfrak{so}(2p,2), \mathfrak{su}(p,1))$ $(p \geq 2)$ for the symmetric pair $(\mathfrak{g}, \mathfrak{h})$=$(\mathfrak{so}(2p,2), \mathfrak{so}(2p,1))$. It is enough to consider the irreducible representation $\rho_0 = (\rho_{\varpi_1})_{\mathbb{R}}$ over $\mathbb{R}$ from Proposition 1.3.3 and Remark 1.2.44 and Lemma 1.2.46. Our goal in this subsection is the following:

Proposition 1.5.14. $\Phi_{\mathbb{R}}^{-1}([\rho_0])$ consists of two points. Moreover, the two points are given as follows:

$$\{[\rho_0], [\mathrm{Ad}(\mathrm{diag}(I_{2p-1}, -1, I_2))\rho_0]\}.$$

We realize $G = SO(2p,2)$, $\mathfrak{g} = \mathfrak{so}(2p,2)$ in the same way as Subsection 1.5.4 and realize $\mathfrak{su}(p,1) \subset \mathfrak{so}(2p,2)$ as follows:

$$\begin{aligned}
\sigma : \mathfrak{so}(2p,2) &\to \mathfrak{so}(2p,2), \\
X &\mapsto (J \otimes I_{p+1})X(J \otimes I_{p+1})^{-1}, \\
\mathfrak{su}(p,1) &:= \{X = (x_{i,j}) \in \mathfrak{so}(2p,2)^{\sigma} : \sum_{i=1}^{p+1} x_{2i,2i-1} = 0\}.
\end{aligned}$$

From Fact 1.5.4 and the realization of $\mathfrak{su}(p,1)$, we have

Lemma 1.5.15.

$$\mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{R}^{2p+2}) = \{aI_{2p+2} + bJ_1 \otimes I_{p+1} : a, b \in \mathbb{R}\} \simeq \mathbb{C}.$$

Proof of Proposition 1.5.14. We use Lemma 1.5.2.

Claim. $M = SO_0(2p,2) \cdot \{I_{2p+2}, \mathrm{diag}(I_{2p-1}, -1, 1, 1), \mathrm{diag}(I_{2p+1}, -1), \mathrm{diag}(I_{2p-1}, -1, 1, -1)\} \cdot \{aI_{2p+2} : a > 0\}$.

proof of Claim. The inclusion $\supset$ is clear by definition. We show the inclusion $\subset$. Let $g \in M$. From Lemma 1.5.15, there exist $a, b \in \mathbb{R}$ such that $g^{-1} I_{2p,2} {}^t g^{-1} I_{2p,2} = aI_{2p+2} + bJ_1 \otimes I_{p+1}$. By taking transpose and Sylvester's law of inertia, we have $b = 0$ and $a > 0$. Therefore we obtain $g \in O(2p,2)\sqrt{a}^{-1} I_{2p+2}$, which implies the desired conclusion. □

From the above Claim, we can take F of Lemma 1.5.2 as follows:

$$F = \{I_{2p+2}, \mathrm{diag}(I_{2p-1}, -1, 1, 1), \mathrm{diag}(I_{2p+1}, -1), \mathrm{diag}(I_{2p-1}, -1, 1, -1)\} \cdot \{\mathrm{diag}(aI_{2p+2}) : a > 0\}.$$

Here $\mathrm{Ad}(aI_{2p+2})$ preserves $\rho_0(\mathfrak{l})$ for $a > 0$. From Lemma 1.5.16, $\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(\mathrm{diag}(I_{2p-1}, -1, I_2))\rho_0(\mathfrak{l})$ is not $\mathrm{Int}(\mathfrak{g})$-conjugate. Moreover,

- In the case p is even: $\mathrm{Ad}(\mathrm{diag}(I_{2p}, 1, -1))\rho_0(\mathfrak{l})$ is conjugate to $\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(\mathrm{diag}(I_{2p-1}, -1, 1, -1))\rho_0(\mathfrak{l})$ is conjugate to $\mathrm{Ad}(\mathrm{diag}(I_{2p-1}, -1, I_2))\rho_0(\mathfrak{l})$.
- In the case p is odd: $\mathrm{Ad}(\mathrm{diag}(I_{2p}, 1, -1))\rho_0(\mathfrak{l})$ is conjugate to $\mathrm{Ad}(\mathrm{diag}(I_{2p-1}, -1, I_2)$ and $\mathrm{Ad}(\mathrm{diag}(I_{2p-1}, -1, 1, -1))\rho_0(\mathfrak{l})$ is conjugate to $\rho_0(\mathfrak{l})$.

Thus, we obtain the desired conclusion.

$\square$

Lemma 1.5.16. $\mathrm{Ad}(\mathrm{diag}(I_{2p-1}, -1, I_2))\rho_0(\mathfrak{l})$ is not $\mathrm{Int}(\mathfrak{g})$-conjugate to $\rho_0(\mathfrak{l})$.

Proof. Let θ be a Cartan involution θ on $\mathfrak{g}$ as follows:

$$\theta : \mathfrak{g} \to \mathfrak{g}, X \to -{}^tX.$$

Put $\mathfrak{p}_{L'} := (\mathrm{Ad}(\mathrm{diag}(I_{2p-1}, -1, I_2))\rho_0(\mathfrak{l}))^{-\theta}$. Assume $\mathrm{Ad}(I_{2p-1}, -1, I_2))\rho_0(\mathfrak{l})$ is $\mathrm{Int}(\mathfrak{g})$-conjugate. Then, there exists $k \in K = SO(2p) \times SO(2)$ such that $\mathrm{Ad}(k)\mathfrak{p}_L = \mathfrak{p}_{L'} = \mathrm{Ad}(\mathrm{diag}(I_{2p-1}, -1, I_2))\mathfrak{p}_L$, that is $\mathrm{Ad}(k^{-1}\,\mathrm{diag}(I_{2p-1}, -1, I_2))\mathfrak{p}_L = \mathfrak{p}_L$. This is contradiction by Lemma 1.5.18. $\square$

Lemma 1.5.17. $\mathrm{Ad}(\mathrm{diag}(I_{2p}, 1, -1))\rho_0(\mathfrak{l})$ is $\mathrm{Int}(\mathfrak{g})$-conjugate to $\rho_0(\mathfrak{l})$ if and only if p is even. Moreover, $\mathrm{Ad}(\mathrm{diag}(I_{2p-1}, -1, 1, -1))\rho_0(\mathfrak{l})$ is $\mathrm{Int}(\mathfrak{g})$-conjugate to $\rho_0(\mathfrak{l})$ if and only if p is odd.

Proof. We can prove "Moreover part" in the same way as tha former part. So, we only prove the former part. Let θ be a Cartan involution θ on $\mathfrak{g}$ as follows:

$$\theta : \mathfrak{g} \to \mathfrak{g}, X \mapsto -{}^tX.$$

Then $\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(\mathrm{diag}(I_{2p}, 1, -1))\rho_0(\mathfrak{l})$ are θ-stable and their noncompact part $\mathfrak{p}_L := \rho_0(\mathfrak{l})^{-\theta}$ and $\mathfrak{p}_{L'} := (\mathrm{Ad}(\mathrm{diag}(I_{2p}, 1, -1))\rho_0(\mathfrak{l}))^{-\theta}$ are described as follows:

$$\begin{aligned}
\mathfrak{p}_L &= \{\begin{pmatrix} & \big(v, (J \otimes I_p)v\big) \\ {}^t\big(v, (J \otimes I_p)v\big) & \end{pmatrix} : v \in \mathbb{R}^{2p}\}, \\
\mathfrak{p}_{L'} &= \mathrm{Ad}(\mathrm{diag}(S, \cdots, S, I_2))\mathfrak{p}_L.
\end{aligned}$$

Here $S := \mathrm{diag}(1, -1)$, S apears p times in $\mathrm{diag}(S, \cdots, S, S, I_2)$. From Lemma 1.5.5, we have

$$\begin{aligned}
&\mathrm{Ad}(\mathrm{diag}(I_{2p}, 1, -1))\rho_0(\mathfrak{l}) \text{ is } \mathrm{Int}(\mathfrak{g})\text{-conjugate} \\
\iff &\text{there exist } k \in SO(2p) \times SO(2) \text{ such that } \mathrm{Ad}(k)\mathfrak{p}_{L'} = \mathfrak{p}_L \\
\iff &\text{there exist } k \in SO(2p) \times SO(2) \text{ such that } \mathrm{Ad}(k\,\mathrm{diag}(S, \cdots, S, I_2))\mathfrak{p}_L = \mathfrak{p}_L \\
\iff &p \text{ is even.}
\end{aligned}$$

Here, in the last implication, we used Lemma 1.5.18.

$\square$

Lemma 1.5.18. Put $\mathfrak{p}_L := \rho_0(\mathfrak{l})^{-\theta}$ and $A = \{g = (g_1, k_2) \in GL(2p, \mathbb{R}) \times SO(2) : \mathrm{Ad}(g)\mathfrak{p}_L = \mathfrak{p}_L\}$. Then we have

$$A = \{(g_1, k_2) \in O(2p) \times SO(2) : g_1(J \otimes I_p) = (J \otimes I_p)g_1\}.$$

In particular, we have $\det g_1 > 0$ if $g = (g_1, k_2) \in A$.

Proof. "In particular" part is clear from the description of A. So, we show the former part. We can easily check that $\mathrm{Ad}(I_p, k_2)\mathfrak{p}_L = \mathfrak{p}_L$. Therefore for $(g_1, k_2) \in GL(2p, \mathbb{R}) \times SO(2)$, we have

$$\begin{aligned}
&\mathrm{Ad}(g_1, k_2)\mathfrak{p}_L = \mathfrak{p}_L, \\
\iff &\mathrm{Ad}(g_1, I_2)\mathfrak{p}_L = \mathfrak{p}_L, \\
\iff &\begin{pmatrix} & g_1(v, (J \otimes I_p)v) \\ {}^t(v, (J \otimes I_p)v)g_1^{-1} & \end{pmatrix} \in \mathfrak{p}_L \text{ for all } v \in \mathbb{R}^{2p}, \\
\iff &\begin{cases} g_1(J \otimes I_p)v = (J \otimes I_p)g_1 v, \\ {}^t(g_1 v) = {}^t v g_1^{-1} \end{cases} \qquad \text{for all } v \in \mathbb{R}^{2p}, \\
\iff &\begin{cases} g_1(J \otimes I_p) = (J \otimes I_p)g_1, \\ {}^t g_1 g_1 = I_{2p}. \end{cases}
\end{aligned}$$

□

1.5.6 $(\mathfrak{g}, \mathfrak{l}) = (\mathfrak{so}(4p, 4), \mathfrak{sp}(p, 1))$ $(p \geq 1)$

In this subsection, we consider the case $(\mathfrak{g}, \mathfrak{l}) = (\mathfrak{so}(4p, 4), \mathfrak{sp}(p, 1))$ for the symmetric pair $(\mathfrak{g}, \mathfrak{h})$=$(\mathfrak{so}(4p, 4), \mathfrak{so}(4p, 3))$. It is enough to consider the irreducible representation $\rho_0 = \rho_{\varpi_1} : \mathfrak{sp}(p, 1) \to \mathfrak{so}(4p, 4)$ over $\mathbb{R}$ from Proposition 1.3.3, Remark 1.2.44 and Lemma 1.2.46. Our goal in this subsection is the following:

Proposition 1.5.19. $\Phi_{\mathbb{R}}^{-1}([\rho_0])$ consists of four points. Moreover, the four points are given as follows:

$$\{[\rho_0], [\mathrm{Ad}(\mathrm{diag}(I_{4p-1}, -1, I_4))\rho_0], [\mathrm{Ad}(\mathrm{diag}(I_{4p}, I_3, -1))\rho_0], [\mathrm{Ad}(\mathrm{diag}(I_{4p-1}, -1, I_3, -1))\rho_0]\}.$$

We realize $G = SO(4p, 4) = SL(4p + 4, \mathbb{R})^{\tilde{\tau}}$, $\mathfrak{sp}(p, 1) \subset \mathfrak{so}(4p, 4) = \mathfrak{sl}(4p + 4, \mathbb{R})^{\tau}$ as follows:

$$\begin{aligned}
\tau &: \mathfrak{sl}(4p + 4, \mathbb{R}) \to \mathfrak{sl}(4p + 4, \mathbb{R}),\ X \mapsto -I_{4p,4}{}^t X I_{4p,4}^{-1}, \\
\tilde{\tau} &: GL(4p + 4, \mathbb{R}) \to GL(4p + 4, \mathbb{R}),\ g \mapsto I_{4p,4}{}^t g^{-1} I_{4p,4}^{-1}, \\
\sigma_1 &: \mathfrak{so}(4p, 4) \to \mathfrak{so}(4p, 4),\ X \mapsto (J \otimes I_2 \otimes I_{p+1})X(J \otimes I_2 \otimes I_{p+1})^{-1}, \\
\sigma_2 &: \mathfrak{so}(4p, 4) \to \mathfrak{so}(4p, 4),\ X \mapsto (S \otimes J \otimes I_{p+1})X(S \otimes J \otimes I_{p+1})^{-1}, \\
&\mathfrak{sp}(p, 1) := (\mathfrak{so}(4p, 4)^{\sigma_1})^{\sigma_2}, \\
&\mathfrak{p}_L = \left\{ \begin{pmatrix} & & & B_1 \\ & & & \vdots \\ & & & B_p \\ {}^t B_1 & \dots & {}^t B_p & \end{pmatrix} : B_i \in \mathbb{R}\text{-span}\{I_4, J \otimes S, I \otimes J, J \otimes T\} \right\}.
\end{aligned}$$

Here $S = \mathrm{diag}(1, -1), T = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \in GL(2, \mathbb{R})$. By Fact 1.5.4 and the realization of $\mathfrak{sp}(p, 1)$, we have the following:

Lemma 1.5.20.

$\mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{R}^{4p+4}) = \mathbb{R}\text{-span}\{I_{4p+4}, J\otimes I\otimes I_{p+1}, S\otimes J\otimes I_{p+1}, T\otimes J\otimes I_{p+1} \in M(4p+4,\mathbb{R})\}.$

Proof of Proposition 1.5.19. We use Lemma 1.5.2.

$$\textbf{Claim. } M = \begin{cases} SO_0(4p,4)\cdot\{I_{4p+4}, \mathrm{diag}(I_{4p-1},-1,I_4), \mathrm{diag}(I_{4p},I_3,-1), \mathrm{diag}(I_{4p-1},-1,I_3,-1)\} \\ \cdot\{aI_{4p+4} : a>0\} \text{ if } p\geq 2, \\ \\ SO_0(4,4)\cdot\{I_8, \mathrm{diag}(I_3,-1,I_4), \mathrm{diag}(I_4,I_3,-1), \mathrm{diag}(I_3,-1,I_3,-1)\} \\ \cdot\{aI_8 : a>0\}\cdot\{I_8, J_4\} \text{ if } p=1. \end{cases}$$

proof of Claim. The inclusion $\supset$ is clear by definition. We show the inclusion $\subset$. Let $g \in M$. From Lemma 1.5.20, there exist $a,b,c,d \in \mathbb{R}$ such that $g^{-1}I_{4p,4}{}^tg^{-1}I_{4p,4} = aI_{4p+4} + bJ\otimes I\otimes I_{p+1} + cS\otimes J\otimes I_{p+1} + dT\otimes J\otimes I_{p+1}$. By taking matrix transpose, we have $b=c=d=0$.

- In the case $p \geq 2$: By Sylvester's law of inertia, we obtain $a>0$. therefore we have $g \in O(4p,4)\sqrt{a}^{-1}I_{4p+4}$, which implies the desired conclusion.
- In the case $p=1$: we have ${}^t(\sqrt{|a|}g)I_{4,4}(\sqrt{|a|}g) = I_{4,4}$ or $-I_{4,4}$, namely, $g \in O(4,4)\sqrt{|a|}^{-1}\cdot\{I_8, J_4\}$, which implies the desired conclusion.

□

From the above Claim, we can take F of Lemma 1.5.2 as follows:

$$F = \begin{cases} \{I_{4p+4}, \mathrm{diag}(I_{4p-1},-1,I_4), \mathrm{diag}(I_{4p},I_3,-1), \mathrm{diag}(I_{4p-1},-1,I_3,-1)\} \\ \cdot\{\mathrm{diag}(aI_{4p+4}) : a>0\} \text{ if } p\geq 2, \\ \\ \{I_8, \mathrm{diag}(I_3,-1,I_4), \mathrm{diag}(I_4,I_3,-1), \mathrm{diag}(I_3,-1,I_3,-1)\} \\ \cdot\{\mathrm{diag}(aI_8) : a>0\}\cdot\{I_8, J_4\} \text{ if } p=1. \end{cases}$$

Since $\mathrm{Ad}(aI_{4p+4})$ $(a>0)$ preserves $\rho_0(\mathfrak{l})$ for $p\geq 1$ and $\mathrm{Ad}(J_4)$ preserves $\rho_0(\mathfrak{l})$ for $p=1$, it is enough to consider $\mathrm{Ad}(f)\rho_0$ for $f \in \{I_{4p+4}$, $\mathrm{diag}(I_{4p-1},-1,I_4)$, $\mathrm{diag}(I_{4p},I_3,-1)$, $\mathrm{diag}(I_{4p-1},-1,I_3,-1)\}$. From Lemma 1.5.21, we obtain the desired conclusion.

□

Lemma 1.5.21. $\rho_0(\mathfrak{l})$, $\mathrm{Ad}(\mathrm{diag}(I_{4p-1},-1,I_4))\rho_0(\mathfrak{l})$, $\mathrm{Ad}(\mathrm{diag}(I_{4p},I_3,-1))\rho_0(\mathfrak{l})$, and $\mathrm{Ad}(\mathrm{diag}(I_{4p-1},-1,I_3,-1))\rho_0(\mathfrak{l})$ are not $\mathrm{Int}(\mathfrak{g})$-conjugate each other.

Proof. Assume two subalgebras $\mathfrak{m}$, $\mathfrak{m}'$ of them such that $\mathfrak{m} \neq \mathfrak{m}'$ are conjugate by $\mathrm{Int}(\mathfrak{g})$. From Lemma 1.5.5, there exist $k \in K = SO(4p)\times SO(4)$ such that $\mathrm{Ad}(k)\mathfrak{m} = \mathfrak{m}'$. Then we have $\mathrm{Ad}(\mathrm{diag}(k_1,k_2))\mathfrak{p}_L = \mathfrak{p}_L$ where $k_1 \in O(4p)$, $k_2 \in O(4)$. Here we have $\det k_1 = -1$ or $\det k_2 = -1$. This is contradiction by Lemma 1.5.22. □

Lemma 1.5.22. Suppose that $(k_1, k_2) \in O(4p) \times O(4)$ satisfies $\mathrm{Ad}(k_1, k_2)\mathfrak{p}_L = \mathfrak{p}_L$. Then we have $\det k_1, \det k_2 > 0$.

Proof. We identify $\mathfrak{p} \simeq M(4p, 4; \mathbb{R})$ with the action $X \mapsto k_1' X {}^t k_2'$ for $(k_1', k_2') \in O(4p) \times O(4)$. By taking ${}^t(0, \cdots, I_4, \cdots 0) \in \mathfrak{p}_L$, we can describe k_1 as follows for some $v_i \in \mathfrak{p}_L \subset M(4p, 4; \mathbb{R})$ $(i = 1, \cdots, p)$.

$$k_1 = (v_1, \cdots v_p) \operatorname{diag}(k_2, \cdots, k_2).$$

Here $(v_1, \cdots, v_p) \in O(4p)$.

$$\begin{aligned} & k_1 X {}^t k_2 \in \mathfrak{p}_L \text{ for all } X \in \mathfrak{p}_L \\ \iff & (v_1, \cdots v_p) \operatorname{diag}(k_2, \cdots, k_2) X {}^t k_2 \in \mathfrak{p}_L \text{ for all } X \in \mathfrak{p}_L \\ \iff & \operatorname{diag}(k_2, \cdots, k_2) X {}^t k_2 \in \mathfrak{p}_L \text{ for all } X \in \mathfrak{p}_L \quad (\because \text{Remark 1.5.23}) \\ \Rightarrow & \det k_2 > 0. \end{aligned}$$

Here we used Remark 1.5.25 and 1.5.24 for the last implication. The condition $\det k_1 > 0$ comes from $k_1 = (v_1, \cdots, v_p) \operatorname{diag}(k_2, \cdots, k_2)$ and $(v_1, \cdots, v_p) \in SO(4p)$. $\square$

Remark 1.5.23. $\mathbb{R}$-span$\{I_4, J \otimes S, I \otimes J, J \otimes T\} \subset M(4, \mathbb{R})$ is closed by matrix transpose and matrix multiplication.

Remark 1.5.24. For $B \in \mathrm{Alt}(2n, \mathbb{R})$, $\mathrm{Pfaff}(gB{}^t g) = \det g \, \mathrm{Pfaff}(B)$, where $\mathrm{Pfaff}(B)$ means pfaffian of B.

Remark 1.5.25. For $0 \neq X \in ImH(4, \mathbb{R}) := \mathbb{R}\text{-span}\{J \otimes S, I \otimes J, J \otimes T\}$, $\mathrm{Pfaff}(X) < 0$ holds.

1.5.7 $(\mathfrak{g}, \mathfrak{l})$=$(\mathfrak{so}(3,4), \mathfrak{g}_{2(2)})$

In this subsection, we consider $(\mathfrak{g}, \mathfrak{l})$=$(\mathfrak{so}(3,4), \mathfrak{g}_{2(2)})$ for the symmetric pair $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(3,4), \mathfrak{so}(2) \oplus \mathfrak{so}(1,4))$. From Proposition 1.3.37 and Cartan's fundamental theorem (see Fact 1.6.5), it is enough to consider the irreducible representation $\rho_0 := \rho_{\varpi_1} : \mathfrak{g}_{2(2)} \to \mathfrak{so}(3,4)$ over $\mathbb{R}$. Our goal in this subsection is the following:

Proposition 1.5.26. Let $(\mathfrak{g}, \mathfrak{l}) = (\mathfrak{so}(3,4), \mathfrak{g}_{2(2)})$. The inverse image $\Phi_{\mathbb{R}}^{-1}([\iota \circ \rho_0])$ consists of two points. Moreover, the two points are given as follows:

$$\{[\rho_0], [\mathrm{Ad}(\operatorname{diag}(I_3, I_{3,1}))\rho_0]\}.$$

Here, ρ_0 is the standard embedding of $\mathfrak{g}_{2(2)}$ by the standard representation ρ_{ϖ_1} with highest weight ϖ_1.

Remark 1.5.27. The irreducible representation ρ_{ϖ_1} of $\mathfrak{g}_{2(2)}$ factors $\mathfrak{so}(3,4) \subset \mathfrak{sl}(7, \mathbb{R})$ and it is unique as a real representation up to equivalence by Cartan's fundamental theorem.

From Lemma 1.5.4, we have

Lemma 1.5.28.

$$\mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{R}^7) = \mathbb{R}I_7.$$

We realize $G = SO(3,4) = SL(7,\mathbb{R})^{\tilde{\tau}}$ by a involution $\tilde{\tau} : GL(7,\mathbb{R}) \to GL(7,\mathbb{R}), g \mapsto I_{3,4}{}^t g^{-1} I_{3,4}^{-1}$.

Proof of Proposition 1.5.26. We use Lemma 1.5.2.

Claim. $M = SO_0(3,4) \cdot \{I_7, \mathrm{diag}(I_3, I_{3,1}), \mathrm{diag}(I_{2,1}, I_4), \mathrm{diag}(I_{2,1}, I_{3,1})\} \cdot \{aI_7 : a > 0\}$

proof of Claim. The inclusion $\supset$ is clear by definition. We show the inclusion $\subset$. Let $g \in M$. From Lemma 1.5.28, there exists $a \in \mathbb{R}^\times$ such that $g^{-1}\tilde{\tau}(g) = aI_7$. By Sylvester's law of inertia, we have $a > 0$. Therefore we have the desired conclusion. □

From the above Claim, we can take F of Lemma 1.5.2 as follows

$$F = \{I_7, \mathrm{diag}(I_3, I_{3,1}), \mathrm{diag}(I_{2,1}, I_4), \mathrm{diag}(I_{2,1}, I_{3,1})\} \cdot \{aI_7 : a > 0\}.$$

Since $\mathrm{Ad}(aI_7)$ preserves $\rho_0(\mathfrak{l})$ for $a > 0$, it is enough to show that $\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(\mathrm{diag}(I_3, I_{3,1}))\rho_0(\mathfrak{l})$ are not $\mathrm{Int}(\mathfrak{g})$-conjugate, and $\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(\mathrm{diag}(I_{2,1}, I_4))\rho_0(\mathfrak{l})$ are $\mathrm{Int}(\mathfrak{g})$-conjugate. These comes from Lemma 1.5.29.

□

Lemma 1.5.29. $\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(\mathrm{diag}(I_3, I_{3,1}))\rho_0(\mathfrak{l})$ are not $\mathrm{Int}(\mathfrak{g})$-conjugate. $\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(\mathrm{diag}(I_{2,1}, I_4))\rho_0(\mathfrak{l})$ are $\mathrm{Int}(\mathfrak{g})$-conjugate.

To prove the above Lemma 1.5.29, we realize noncompact part $\mathfrak{p}_L$ of $\rho_0(\mathfrak{l})$, which can be considered as a subspace of $M(4,3;\mathbb{R}) \simeq \mathfrak{p}$ and its orthogonal complement subspace $\mathfrak{p}_L^\perp$ of $\mathfrak{p}_L$ in $M(4,3;\mathbb{R})$ with regard to the following $\mathrm{Ad}(K)$ invariant inner product:

$$\begin{aligned} M(4,3;\mathbb{R}) \times M(4,3;\mathbb{R}) &\to \mathbb{R}, \\ (X,Y) &\mapsto \mathrm{trace}({}^tXY). \end{aligned}$$

We use the elements H_1, H_2, X_i and $Y_i \in \mathfrak{g}_{2(2)}$ $(i = 1, \cdots 6)$ with relation given in Table 22.1 of the book [FH] and weight vectors v_4, v_3, v_1, u, w_1, w_3 and w_4 of standard representation ρ_{ϖ_1} in Lecture 22 of the book [FH]. We realize $\mathfrak{so}(3,4)$ as follows:

$$\begin{aligned} \tau : \mathfrak{sl}(7,\mathbb{R}) &\to \mathfrak{sl}(7,\mathbb{R}), \\ X &\mapsto -I_{3,4}{}^tXI_{3,4} \\ \mathfrak{so}(3,4) &:= \mathfrak{sl}(7,\mathbb{R})^\tau \end{aligned}$$

First, we describe $\mathfrak{p}_L$ as a subspace $M(4,3;\mathbb{R}) \simeq \mathfrak{p}$. Take a basis $\{v_4^+, v_3^+, v_1^+, u, v_1^-, v_3^-, v_4^-\}$ on the representation space of the standard representation of $\mathfrak{g}_{2(2)}$ as follows:

$$\begin{aligned}
v_4^+ &:= v_4 + w_4, \\
v_3^+ &:= v_3 + w_3, \\
v_1^+ &:= v_1 + w_1, \\
v_1^- &:= v_1 - w_1 \\
v_3^- &:= v_3 - w_3 \\
v_4^- &:= v_4 - w_4.
\end{aligned}$$

Then we have the following matrix representation of a basis $\{H_1, H_2, X_1 + Y_1, X_2 + Y_2, X_3 + Y_3, X_4 + Y_4, X_5 + Y_5, X_6 + Y_6\}$ of $\mathfrak{p}_{\mathfrak{g}_{2(2)}}$ with regard to the above basis:

$$\rho(H_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho(H_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho(X_1 + Y_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho(X_2 + Y_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho(X_3 + Y_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho(X_4 + Y_4) = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho(X_5 + Y_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho(X_6 + Y_6) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore we have the following description of $\mathfrak{p}_L \subset M(4,3;\mathbb{R}) \simeq \mathfrak{p}$:

$$\mathfrak{p}_L = \{ \begin{pmatrix} 2f & 2e & 2c \\ -e-g & d+f & 2a-b \\ c-h & -a+b & d-f \\ a & c+h & -e+g \end{pmatrix} : a,b,c,d,e,f,g,h \in \mathbb{R}\}$$

Therefore its orthogonal complement subspace $\mathfrak{p}_L^\perp$ of $\mathfrak{p}_L$ is given as follows:

$$\begin{aligned} \mathfrak{p}_L^\perp &= \{ \begin{pmatrix} a & b & -c \\ b & -a & -d \\ c & -d & a \\ d & c & b \end{pmatrix} : a,b,c,d \in \mathbb{R}\} \\ &= \{(v, (-J \otimes S)v, (I \otimes J)v : v \in \mathbb{R}^4\}. \end{aligned}$$

Lemma 1.5.30. Put $\mathfrak{A} := \{(k_1,k_2) \in O(4) \times O(3) : k_1 \mathfrak{p}_L^\perp k_2^{-1} \subset \mathfrak{p}_L^\perp\}$. Then we have $\det k_1 > 0$ if $(k_1,k_2) \in \mathfrak{A}$.

Proof. Let $(k_1,k_2) \in \mathfrak{A}$. and $pr_2 : \mathfrak{so}(4) \oplus \mathfrak{so}(3) \to \mathfrak{so}(3)$ the second projection. Since the composition map of $pr_2 \circ \rho_0|_{\mathfrak{k}_{\mathfrak{g}_{2(2)}}} : \mathfrak{k}_{\mathfrak{g}_{2(2)}} \simeq \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \to \mathfrak{so}(4) \oplus \mathfrak{so}(3) \to \mathfrak{so}(3)$ is surjective, there exist $k_L \in K_L$ such that $(k_1,k_2) = (k', \operatorname{diag}(I_2,\varepsilon))k_L$ for some $k' \in O(4)$, $\varepsilon \in \{\pm 1\}$, where K_L is the analytic subgroup of $\mathfrak{k}_L$. It is enough to show that $\det k' > 0$.

$$\begin{aligned} & k_1 \mathfrak{p}_L^\perp k_2^{-1} \subset \mathfrak{p}_L^\perp \\ \iff & k' \mathfrak{p}_L^\perp \operatorname{diag}(I_2,\varepsilon) \subset \mathfrak{p}_L^\perp \\ \iff & (k'v, k'(-J\otimes S)v, \varepsilon k'(I \otimes J)v) \in \mathfrak{p}_L^\perp \text{ for all } v \in \mathbb{R}^4 \\ \iff & k'(-J \otimes S)k'^{-1} = -J \otimes S \text{ and } \varepsilon k'(I \otimes J)k'^{-1} = I \otimes J \end{aligned}$$

From the second condition, k' has the following form:

$$k' = \operatorname{diag}(I_2, \varepsilon I_2) \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

Here $A, B \in M(2,\mathbb{R})$. Therefore, we have $\det k' > 0$. $\square$

Proof of Lemma 1.5.29. First we show that $\rho_0(\mathfrak{l})$ and $\operatorname{Ad}(I_3, I_{3,1})\rho_0(\mathfrak{l})$ are not $\operatorname{Int}(\mathfrak{g})$-conjugate. Assume $\rho_0(\mathfrak{l})$ and $\operatorname{Ad}(I_3, I_{3,1})\rho_0(\mathfrak{l})$ are $\operatorname{Int}(\mathfrak{g})$-conjugate. Then from Lemma 1.5.5, there exist $(k_1,k_2) \in SO(4) \times SO(3)$ such that $k_1 I_{3,1} \mathfrak{p}_L k_2^{-1} \subset \mathfrak{p}_L$. Since we have $k_1 I_{3,1} \mathfrak{p}_L k_2^{-1} \subset \mathfrak{p}_L \iff k_1 I_{3,1} \mathfrak{p}_L^\perp k_2 \subset \mathfrak{p}_L^\perp$, we have $(k_1 I_{3,1}, k_2) \in \mathfrak{A}$. This is contradict $\det(k_1 I_{3,1}) < 0$.

Next, we show that $\rho_0(\mathfrak{l})$ and $\operatorname{Ad}(I_{2,1}, I_4)\rho_0(\mathfrak{l})$ are $\operatorname{Int}(\mathfrak{g})$-conjugate. It is enough to show that there exists $(k_1,k_2) \in SO(4) \otimes SO(3)$ such that $k_1 \mathfrak{p}_L^\perp k_2^{-1} = \mathfrak{p}_L^\perp I_{2,1}$. Take $k_1 := I \otimes S$, $k_2 = I_3$. Then we have $k_1 \mathfrak{p}_L^\perp k_2^{-1} = \mathfrak{p}_L^\perp I_{2,1}$. $\square$

1.5.8 $(\mathfrak{g},\ \mathfrak{l})$=$(\mathfrak{so}(4,4),\ \mathfrak{spin}(3,4))$

In this subsection, we consider $(\mathfrak{g},\ \mathfrak{l})$=$(\mathfrak{so}(4,4),\ \mathfrak{spin}(3,4))$ for the symmetric pair $(\mathfrak{g},\mathfrak{h})$=$(\mathfrak{so}(4,4),\ \mathfrak{so}(4,1)\oplus\mathfrak{so}(3))$. From Proposition 1.3.37 and Cartan's fundamental theorem (see Fact 1.6.5), it is enough to consider the irreducible representation $\rho_0 := \rho_{\varpi_3} : \mathfrak{so}(3,4) \to \mathfrak{so}(4,4)$ over $\mathbb{R}$. Our goal of this subsection is the following:

Proposition 1.5.31. $\Phi_{\mathbb{R}}^{-1}([\rho_0])$ consists of four points. Moreover, the four points are given as follows:

$$\{[\rho_0], [\mathrm{Ad}(\mathrm{diag}(I_{3,1}, I_4))\rho_0], [\mathrm{Ad}(\mathrm{diag}(I_4, I_{3,1}))\rho_0], [\mathrm{Ad}(\mathrm{diag}(I_{3,1}, I_{3,1}))\rho_0]\}$$

We realize $SO(4,4) = SL(8,\mathbb{R})^{\tilde{\tau}}$, $\mathfrak{so}(4,4) = \mathfrak{sl}(8,\mathbb{R})^{\tau}$ by the following involutions τ, $\tilde{\tau}$:

$$\begin{aligned}\tau : \mathfrak{sl}(8,\mathbb{R}) &\to \mathfrak{sl}(8,\mathbb{R}),\\ X &\mapsto -I_{4,4}{}^tX I_{4,4}^{-1},\\ \tilde{\tau} : GL(8,\mathbb{R}) &\to GL(8,\mathbb{R}),\\ g &\mapsto I_{4,4}{}^tg^{-1} I_{4,4}^{-1}.\end{aligned}$$

From Lemma 1.5.4, we have

Lemma 1.5.32.

$$\mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{R}^8) = \mathbb{R}I_8.$$

Proof of Proposition 1.5.31. We use Lemma 1.5.2.

Claim.

$$M = SO_0(4,4)\cdot\{I_8, \mathrm{diag}(I_{3,1}, I_4), \mathrm{diag}(I_4, I_{3,1}), \mathrm{diag}(I_{3,1}, I_{3,1})\}\cdot\{I_8, I\otimes I\otimes J\}\cdot\{aI_8 : a > 0\}.$$

proof of Claim. The inclusion $\supset$ is clear by definition. We show the inclusion $\subset$. Let $g \in M$. From Lemma 1.5.32, there exists $a \in \mathbb{R}$ such that $g^{-1}\tilde{\tau}(g) = aI_8$, that is, $a{}^tgI_{4,4}g = I_{4,4}$, which implies the desired conclusion. □

From the above Claim, we can take F of Lemma 1.5.2 as follows:

$$F = \{I_8, \mathrm{diag}(I_{3,1}, I_4), \mathrm{diag}(I_4, I_{3,1}), \mathrm{diag}(I_{3,1}, I_{3,1})\}\cdot\{I_8, I\otimes I\otimes J\}\cdot\{aI_8 : a > 0\}.$$

Since $\mathrm{Ad}(aI_8)$ $(a > 0)$ and $\mathrm{Ad}(I \otimes I \otimes J)$ preserves the image of ρ_0 from the description of spin representation below, it is enough to show that $\rho_0(\mathfrak{l})$, $\mathrm{Ad}(\mathrm{diag}(I_{3,1}, I_4))\rho_0(\mathfrak{l})$, $\mathrm{Ad}(\mathrm{diag}(I_4, I_{3,1}))\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(\mathrm{diag}(I_{3,1}, I_{3,1}))\rho_0(\mathfrak{l})$ are not $\mathrm{Int}(\mathfrak{g})$-conjugate each other. Assume that two subalgebras $\mathrm{Ad}(k_1)\rho_0(\mathfrak{l})$, $\mathrm{Ad}(k_2)\rho_0(\mathfrak{l})$ of them are $\mathrm{Int}(\mathfrak{g})$-conjugate, where $k_1, k_2 \in \{I_8, \mathrm{diag}(I_{3,1}, I_4), \mathrm{diag}(I_4, I_{3,1}), \mathrm{diag}(I_{3,1}, I_{3,1})\}$ such that $k_1 \neq k_2$. Then from Lemma 1.5.5, there exists $k \in SO(4)\times SO(4)$ such that $\mathrm{Ad}(k)\,\mathrm{Ad}(k_1)\mathfrak{p}_L = \mathrm{Ad}(k_2)\mathfrak{p}_L$, which is equivalent to $\mathrm{Ad}(k_2^{-1}kk_1)\mathfrak{p}_L^{\perp} = \mathfrak{p}_L^{\perp}$. From Lemma 1.5.34, this is contradiction. □

We consider the following realization $\rho_0 := \varphi_3\varphi_2\varphi_1$ of spin representation (see section 4 in [KY05] for more details): Put $A_{ij} = -E_{ij}+E_{ji}$, $S_{ij} = E_{ij}+E_{ji}$.

$$\begin{aligned}
\mathfrak{so}(3,4) &:= \{X \in \mathfrak{sl}(7,\mathbb{R}) : {}^tXI_{3,4} + I_{3,4}X = 0\} \\
\varphi_1 : \mathfrak{so}(3,4) &\to C_{even}(3,4) \\
A_{i,j} &\mapsto -\frac{1}{2}v_i^+v_j^+ \ (1 \le i < j \le 3) \\
A_{i+3,j+3} &\mapsto \frac{1}{2}v_i^-v_j^- \ (1 \le i < j \le 4) \\
S_{i,j+3} &\mapsto \frac{1}{2}v_i^+v_j^- \ (1 \le i \le 3, 1 \le j \le 4) \\
\varphi_2 : C_{even}(3,4) &\to C(3,3) \to C(1,1) \otimes C(1,1) \otimes C(1,1) \simeq M(8,\mathbb{R})
\end{aligned}$$

Here we use the maps in the following Fact 1.5.33 (i), (ii) and (iv):

Fact 1.5.33 ([KY05]). (i) Put $\mathfrak{so}(p,q) := \{X \in \mathfrak{sl}(p+q,\mathbb{R}) : {}^tXI_{p,q}+I_{p,q}X = 0\}$. Then the following map gives an Lie algebra injective map:

$$\begin{aligned}
\mathfrak{so}(p,q) &\to C_{even}(p,q) \\
A_{i,j} &\mapsto -\frac{1}{2}v_i^+v_j^+ \quad (1 \le i < j \le p) \\
A_{i+p,j+p} &\mapsto \frac{1}{2}v_i^-v_j^- \quad (1 \le i < j \le q) \\
S_{i,j+p} &\mapsto \frac{1}{2}v_i^+v_j^- \quad (1 \le i \le p, 1 \le j \le q)
\end{aligned}$$

Here $A_{i,j} := -E_{i,j} + E_{j,i}$ and $S_{i,j} := E_{i,j} + E_{j,i}$.

(ii) For $p \ge 0$, $q \ge 1$, the following map gives an algebra isomorphism:

$$\begin{aligned}
C(p,q-1) &\to C_{even}(p,q) \\
v_i^+ &\mapsto v_i^+v_q^- \quad (1 \le i \le p) \\
v_j^- &\mapsto v_j^-v_q^- \quad (1 \le j \le q-1)
\end{aligned}$$

(iii) Let $K = (k^+,k^-)$ and $L = (p,q) \in \mathbb{Z}^2_{\ge 0}$, the following map gives an algebra isomorphism if $k^+ - k^- \equiv 1 \pmod 4$:

$$\begin{aligned}
C(K+L) &\to C(K+L^\vee) \\
v_i^+ &\mapsto v_i^+ \quad (1 \le i \le k^+) \\
v_j^- &\mapsto v_j^- \quad (1 \le j \le k^-) \\
v_{k^++i}^+ &\mapsto V_Kv_{k^-+i}^- \quad (1 \le i \le p) \\
v_{k^-+j}^- &\mapsto V_Kv_{k^++j}^+ \quad (1 \le j \le q)
\end{aligned}$$

Here $V_K := v_1^+ \cdots v_{k^+}^+ v_1^- \cdots v_{k^-}^-$.

(iv) For non-negative integers p and q, the following map gives an algebra isomorphism:

$$\begin{aligned} C(p+1,q+1) &\to C(1,1)\otimes C(p,q) \\ v_1^+ &\mapsto v_1^+ \otimes 1 \\ v_1^- &\mapsto v_1^- \otimes 1 \\ v_{i+1}^+ &\mapsto v_1^+ v_1^- \otimes v_i^+ \quad (1 \le i \le p) \\ v_{j+1}^- &\mapsto v_1^+ v_1^- \otimes v_j^- \quad (1 \le j \le q) \end{aligned}$$

We use the following identification:

$$\begin{aligned} C(1,1) &\to M(2,\mathbb{R}) \\ v_1^+ &\mapsto T := \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \\ v_1^- &\mapsto J := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \\ v_1^+ v_1^- &\mapsto S := \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \end{aligned}$$

Let $(,)$ be a standard inner product on $\mathbb{R}^8$. Then $\varphi_2\varphi_1(\mathfrak{so}(3,4))$-invariant scalar product B on $\mathbb{R}^8$ is given as follows:

$$B(v,w) := {}^t v(J\otimes T\otimes J)w.$$

Put $g_0 := \frac{1}{\sqrt{2}}(I\otimes I\otimes I - J\otimes T\otimes T)$ and compose the basis transformation φ_3 to $\varphi_2\varphi_1$

$$\begin{aligned} \varphi_3 : M(8,\mathbb{R}) &\to M(8,\mathbb{R}), \\ X &\mapsto g_0 X g_0^{-1} \\ \rho_0 &:= \varphi_3\varphi_2\varphi_1 \end{aligned}$$

Then $\rho_0(\mathfrak{so}(3,4))$-invariant scalar product B_0 on $\mathbb{R}^8$ is given as follows:

$$B_0(v,w) := {}^t v I\otimes I\otimes S w.$$

Moreover the image $\mathfrak{p}_L$ of $\mathfrak{p}_{\mathfrak{so}(3,4)}$ by ρ_0 is given as follows:

$$
\begin{aligned}
&\mathfrak{so}(4,4) := \{X \in M(8,\mathbb{R}) : {}^tX(I\otimes I\otimes S) + (I\otimes I\otimes S)X = 0\}\\
&\mathfrak{so}(3,4) \to \mathfrak{so}(4,4)\\
&\quad 2S_{14} \mapsto -T\otimes T\otimes T\\
&\quad 2S_{15} \mapsto I\otimes S\otimes T\\
&\quad 2S_{16} \mapsto J\otimes S\otimes J\\
&\quad 2S_{17} \mapsto S\otimes T\otimes T\\
&\quad 2S_{24} \mapsto S\otimes I\otimes T\\
&\quad 2S_{25} \mapsto -J\otimes J\otimes T\\
&\quad 2S_{26} \mapsto I\otimes J\otimes J\\
&\quad 2S_{27} \mapsto T\otimes I\otimes T\\
&\quad 2S_{34} \mapsto -T\otimes S\otimes T\\
&\quad 2S_{35} \mapsto I\otimes T\otimes T\\
&\quad 2S_{36} \mapsto -J\otimes T\otimes J\\
&\quad 2S_{37} \mapsto S\otimes S\otimes T
\end{aligned}
$$

We definie $\mathrm{Ad}(K)$-invariant inner product on $M(4,\mathbb{R}) \simeq \mathfrak{p}$ by

$$
\begin{aligned}
M(4,\mathbb{R}) \times M(4,\mathbb{R}) &\to \mathbb{R}\\
(X,Y) &\mapsto \operatorname{trace}({}^tXY)
\end{aligned}
$$

Take the orthogonal complement subspace $\mathfrak{p}_L^\perp$ of $\mathfrak{p}_L$ in $\mathfrak{p} \simeq M(4,\mathbb{R})$ with respect to the above inner product. Then we have:

$$
\begin{aligned}
\mathfrak{p}_L^\perp &= \left\{ \begin{pmatrix} a & -b & c & -d\\ b & a & -d & -c\\ -c & d & a & -b\\ d & c & b & a \end{pmatrix} : a,b,c,d \in \mathbb{R} \right\}\\
&= \mathbb{R}(I\otimes I) + \mathbb{R}(J\otimes I) + \mathbb{R}(S\otimes J) + \mathbb{R}(T\otimes J) \simeq \mathbb{H}.
\end{aligned}
$$

Lemma 1.5.34. If $(k_1,k_2) \in O(4)\times O(4)$ satisfies $k_1\mathfrak{p}_L^\perp k_2^{-1} \subset \mathfrak{p}_L^\perp$, then $\det k_1 = \det k_2 = 1$.

Proof. Take $I_4 \in \mathfrak{p}_L^\perp$. There exists $v \in \mathfrak{p}_L^\perp$ such that $k_1k_2^{-1} = v$. Take $J\otimes I \in \mathfrak{p}_L^\perp$. Since $\mathfrak{p}_L^\perp$ is closed by matrix transpose and multiplication, Then we have $k_1(J\otimes I)k_2^{-1} \in \mathfrak{p}_L^\perp$, that is, $k_2(J\otimes I){}^tk_2 \in \mathfrak{p}_L^\perp$. Here, we have

$$\det k_2 \operatorname{Pfaff}(J\otimes I) = \operatorname{Pfaff}(k_2(J\otimes I){}^tk_2) > 0.$$

Therefore we have $\det k_2 = \det k_1 = 1$. $\square$

1.5.9 $(\mathfrak{g},\ \mathfrak{l})$=$(\mathfrak{so}(8,8),\ \mathfrak{spin}(1,8))$

In this subsection, we consider $(\mathfrak{g},\ \mathfrak{l})$=$(\mathfrak{so}(8,8),\ \mathfrak{spin}(8,1))$ for the symmetric pair $(\mathfrak{g},\ \mathfrak{h})$=$(\mathfrak{so}(8,8),\ \mathfrak{so}(8,7))$. From Proposition 1.3.3 and Cartan's fundamental theorem (see Fact 1.6.5), it is enough to consider the irreducible representation $\rho_0 := \rho_{\varpi_4} : \mathfrak{so}(8,1) \to \mathfrak{so}(8,8)$. Our goal of this subsection is the following:

Proposition 1.5.35. The inverse image $\Phi_{\mathbb{R}}^{-1}([\iota \circ \rho_0])$ consists of four points. Moreover, the four points are given as follows:

$$\{[\rho_0], [\mathrm{Ad}(\mathrm{diag}(I_{7,1}, I_8))\rho_0], [\mathrm{Ad}(\mathrm{diag}(I_8, I_{7,1}))\rho_0], [\mathrm{Ad}(\mathrm{diag}(I_{7,1}, I_{7,1}))\rho_0]\}.$$

We realize $G = SO(8,8) = SL(16,\mathbb{R})^{\tilde{\tau}}$, $\mathfrak{so}(8,8) = \mathfrak{sl}(8,\mathbb{R})^{\tau}$ by the following involutions $\tilde{\tau}$, τ:

$$\begin{aligned}
\tilde{\tau} : GL(16,\mathbb{R}) &\to GL(16,\mathbb{R}), \\
g &\mapsto I_{8,8}{}^t g^{-1} I_{8,8}^{-1}, \\
\tau : \mathfrak{sl}(8,\mathbb{R}) &\to \mathfrak{sl}(8,\mathbb{R}), \\
X &\mapsto -I_{8,8}{}^t X I_{8,8}^{-1}.
\end{aligned}$$

From Lemma 1.5.4, we have

Lemma 1.5.36.
$$\mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{R}^{16}) = \{aI_{16} : a > 0\}.$$

Proof of Proposition 1.5.35. We use Lemma 1.5.2.

Claim.

$$M = SO_0(8,8) \cdot \{I_{16}, \mathrm{diag}(I_{7,1}, I_8), \mathrm{diag}(I_8, I_{7,1}), \mathrm{diag}(I_{7,1}, I_{7,1})\} \cdot \{I_{16}, J_8\} \cdot \{aI_{16} : a > 0\}.$$

This claim can be proved in the same way as the case $(\mathfrak{g}, \mathfrak{l}) = \mathfrak{so}(4,4), \mathfrak{spin}(3,4))$ in Subsection 1.5.8. So, we omit the proof.

From the above Claim, we can take F of Lemma 1.5.2 as follows:

$$F = \{I_{16}, \mathrm{diag}(I_{7,1}, I_8), \mathrm{diag}(I_8, I_{7,1}), \mathrm{diag}(I_{7,1}, I_{7,1})\} \cdot \{I_{16}, J_8\} \cdot \{aI_{16} : a > 0\}.$$

Here $\mathrm{Ad}(aI_{16})$ $(a > 0)$ and $\mathrm{Ad}(J_8)$ preserve the image $\rho_0(\mathfrak{l})$, which can be proved by the description of spin representation of $\mathfrak{so}(8,1)$.

So, it is enough to show that the following four subalgebras $\rho_0(\mathfrak{l})$, $\mathrm{Ad}(\mathrm{diag}(I_{7,1}, I_8))\rho_0(\mathfrak{l})$, $\mathrm{Ad}(\mathrm{diag}(I_8, I_{7,1}))\rho_0(\mathfrak{l})$, $\mathrm{Ad}(\mathrm{diag}(I_{7,1}, I_{7,1}))\rho_0(\mathfrak{l})$ are not $\mathrm{Int}(\mathfrak{g})$-conjugate each other.

Since the other cases can be proved in the same way, we show only $\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(\mathrm{diag}(I_{7,1}, I_8))\rho_0(\mathfrak{l})$ are not $\mathrm{Int}(\mathfrak{g})$-conjugate. Assume $\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(\mathrm{diag}(I_{7,1}, I_8))\rho_0(\mathfrak{l})$ are $\mathrm{Int}(\mathfrak{g})$-conjugate. Then, from Lemma 1.5.5, there exists $k \in SO(8) \times SO(8)$ such that $\mathrm{Ad}(k)\rho_0(\mathfrak{p}_L) = \mathrm{Ad}(\mathrm{diag}(I_{7,1}, I_8))\rho_0(\mathfrak{p}_L)$, that is, $\mathrm{Ad}(k^{-1}\,\mathrm{diag}(I_{7,1}, I_8))\mathfrak{p}_L = \mathfrak{p}_L$. Therefore we have $k^{-1}\,\mathrm{diag}(I_{7,1}, I_8) \in \mathfrak{A}$ of Lemma 1.5.37. This contradict $k^{-1}\,\mathrm{diag}(I_{7,1}, I_8) \notin SO(8) \times SO(8)$. $\square$

Lemma 1.5.37. Put $\mathfrak{A} := \{k = (k_1, k_2) \in O(8) \times O(8) : \mathrm{Ad}(k)\mathfrak{p}_L \subset \mathfrak{p}_L\}$. Then we have $\mathfrak{A} = \{(I_8, I_8), (-I_8, I_8)\} \cdot K_L \subset SO(8) \times SO(8)$ where K_L is the analytic subgroup of $\mathfrak{k}_L$.

Proof. $\{(I_8, I_8), (-I_8, I_8)\} \cdot K_L \subset \mathfrak{A}$ is clear by the definition of $\mathfrak{A}$. We show that $\mathfrak{A} \in \{(I_8, I_8), (-I_8, I_8)\} \cdot K_L$. First we consider the description of $\mathfrak{p}_L$. $\mathfrak{spin}(8,1) \subset \mathfrak{so}(8,8) = \mathfrak{g}$ is given as a image of spin representation of $\mathfrak{so}(8,1)$. In this proof, we realize $\mathfrak{spin}(8,1) \subset \mathfrak{so}(8,8) := \{X \in M(16,\mathbb{R}) : {}^tX(I \otimes I \otimes S \otimes I) + (I \otimes I \otimes S \otimes I)X\}$ by the image of the composition of the following two maps ι and φ.

$$\begin{aligned}\iota : \mathfrak{so}(8,1) &\to C_{even}(8,1) \to C(8,0) \to C(5,3)\\ &\to C(1,1) \otimes C(1,1) \otimes C(1,1) \otimes C(2,0)\\ &\to C(1,1) \otimes C(1,1) \otimes C(1,1) \otimes C(1,1)\\ &\to M(2,\mathbb{R}) \otimes M(2,\mathbb{R}) \otimes M(2,\mathbb{R}) \otimes M(2,\mathbb{R})\end{aligned}$$

Here we used maps in Fact 1.5.33 (i) to (iv) and the following isomorphisim

$$\begin{aligned}C(2,0) &\to C(1,1),\\ v_1^+ &\mapsto v_1^+,\\ v_2^+ &\mapsto v_1^+ v_1^-.\end{aligned}$$

Put $g_0 := \frac{1}{2}(I \otimes I \otimes I \otimes I + J \otimes T \otimes T \otimes I) \in SO(16)$. Then we have $\varphi\iota(\mathfrak{so}(8,1)) \subset \mathfrak{so}(8,8)$. We consider the following basis transformation φ:

$$\begin{aligned}\varphi : M(2,\mathbb{R}) \otimes M(2,\mathbb{R}) \otimes M(2,\mathbb{R}) \otimes M(2,\mathbb{R}) &\to M(2,\mathbb{R}) \otimes M(2,\mathbb{R}) \otimes M(2,\mathbb{R}) \otimes M(2,\mathbb{R})\\ X &\mapsto g_0^{-1} X g_0\end{aligned}$$

By identification $\mathfrak{p}$ with $M(8,\mathbb{R})$ which compatible with the $\mathrm{Ad}(K)$ action, we have the following description of $\mathfrak{p}_L \subset M(8,\mathbb{R})$:

$$\begin{aligned}\mathfrak{p}_L = \mathbb{R}\text{-span}\{&S \otimes T \otimes I, T \otimes I \otimes I, S \otimes S \otimes I, T \otimes J \otimes T,\\ &T \otimes J \otimes S, S \otimes J \otimes J, J \otimes I \otimes J, I \otimes I \otimes J\}\\ = \{&(v, A_1 v, A_2 v, A_3 v, A_4 v, A_5 v, A_6 v, A_7 v) \in M(8,\mathbb{R}) : v \in \mathbb{R}^8\}.\end{aligned}$$

Here $A_i \in GL(8,\mathbb{R})$ $(i = 1, \cdots, 7)$ are given as follows

$$A_1 := \begin{pmatrix} -J & & & \\ & -J & & \\ & & J & \\ & & & -J \end{pmatrix}, \qquad A_2 := \begin{pmatrix} & S & & \\ -S & & & \\ & & & -I \\ & & I & \end{pmatrix},$$

$$A_3 := \begin{pmatrix} & -T & & \\ T & & & \\ & & & J \\ & & J & \end{pmatrix}, \qquad A_4 := \begin{pmatrix} & & -I & \\ & & & -S \\ I & & & \\ & S & & \end{pmatrix},$$

$$A_5 := \begin{pmatrix} & & -J & \\ & & & T \\ -J & & & \\ & -T & & \end{pmatrix}, \qquad A_6 := \begin{pmatrix} & & & S \\ & & -I & \\ & I & & \\ -S & & & \end{pmatrix},$$

$$A_7 := \begin{pmatrix} & & & -T \\ & & -J & \\ & -J & & \\ T & & & \end{pmatrix}.$$

Let $(k_1, k_2) \in \mathfrak{A}$. From the simplicity of $\mathfrak{so}(8) \subset \mathfrak{so}(8,1)$ and the description of spin representation, the maps $\mathrm{pr}_i \circ \varphi \circ \iota|_{\mathfrak{so}(8)} \to \mathfrak{so}(8)$ $(i = 1, 2)$ are surjective. Therefore, we can take $k_L \in K_L$ such that $(k_1, k_2) = (k, \mathrm{diag}(I_7, \varepsilon))k_L$ for some $k \in O(8)$, $\mathrm{diag}(I_7, \varepsilon) \in O(8)$ $(\varepsilon \in \{\pm 1\})$. Thus it is enough to show that $k = \pm I_8$, $\varepsilon = 1$.

$$\begin{aligned} & \mathrm{Ad}(k_1, k_2)\mathfrak{p}_L \subset \mathfrak{p}_L \\ \iff & \mathrm{Ad}(k, \mathrm{diag}(I_7, \varepsilon))\mathfrak{p}_L \subset \mathfrak{p}_L \\ \iff & (kv, kA_1v, kA_2v, kA_3v, kA_4v, kA_5v, \varepsilon kA_6v) \in \mathfrak{p}_L \ \text{ for any } v \in \mathbb{R}^8 \\ \iff & kA_ik^{-1} = A_i \ (i = 1, 2, 3, 4, 5, 6) \text{ and } \varepsilon kA_7k^{-1} = A_7 \end{aligned}$$

By direct calculation, we obtain $k = \pm I_8$ and $\varepsilon = 1$. □

1.5.10 $(\mathfrak{g}, \mathfrak{l}) = (\mathfrak{so}^*(8), \mathfrak{spin}(1,6))$

In this subsection, we consider $(\mathfrak{g}, \mathfrak{l}) = (\mathfrak{so}^*(8), \mathfrak{spin}(1,6))$ for the symmetric pairs $(\mathfrak{g}, \mathfrak{h})$=$(\mathfrak{so}^*(8), \mathfrak{u}(3,1))$ and $(\mathfrak{so}^*(8), \mathfrak{so}^*(6) \oplus \mathfrak{so}^*(2))$. From Proposition 1.4.20 and 1.4.29, it is enough to consider the irreducible representation $\rho_0 := \rho_{\varpi_3} : \mathfrak{so}(1,6) \to \mathfrak{so}^*(8)$. Our goal of this subsection is the following:

Proposition 1.5.38. The inverse image $\Phi_{\mathbb{C}}^{-1}([\iota \circ \rho_0])$ consists of one point, namely $\Phi_{\mathbb{C}}^{-1}([\iota \circ \rho_0]) = \{[\rho_0]\}$.

We realize $\mathfrak{so}^*(8)$ and $SO^*(8)$ as follows:

$$\begin{aligned}
\tau_1, \tau_2 : \mathfrak{sl}(8,\mathbb{C}) &\to \mathfrak{sl}(8,\mathbb{C}),\\
\tau_1(X) &:= -{}^tX,\\
\tau_2(X) &:= J\overline{X}J^{-1},\\
\mathfrak{so}^*(8) &:= \{X \in \mathfrak{sl}(8,\mathbb{C}) : \tau_1(X) = X = \tau_2(X)\},\\
\tilde{\tau_1}, \tilde{\tau_2} : SL(8,\mathbb{C}) &\to SL(8,\mathbb{C})\\
\tilde{\tau_1}(g) &:= {}^tg^{-1},\\
\tilde{\tau_2}(g) &:= J_4\overline{g}J_4^{-1},\\
SO^*(8) &:= \{g \in SL(8,\mathbb{C}) : \tilde{\tau_1}(g) = g = \tilde{\tau_2}(g)\}.
\end{aligned}$$

Remark 1.5.39. For $g \in GL(2n,\mathbb{C})$, the conditions ${}^tgg = I_{2n}$ and $J_n\overline{g} = gJ_n$ implies $\det g = 1$. Therefore, we have $SO^*(8) = \{g \in GL(n,\mathbb{C}) : \tilde{\tau_1}(g) = g = \tilde{\tau_2}(g)\}$, where $\tilde{\tau_1}$ and $\tilde{\tau_2}$ are natural extension of the above maps to $GL(2n,\mathbb{C}) \to GL(2n,\mathbb{C})$.

The image $\rho_0(\mathfrak{l}) \subset \mathfrak{so}^*(8)$ is given by the composition of the following injective maps ι_1 and A_{g_0}.

$$\begin{aligned}
\iota_1 : \mathfrak{so}(1,6) &\to C_{even}(1,6) \to C(1,5)\\
&\to C(1,1) \otimes C(0,4)\\
&\to C(1,1) \otimes C(1,3)\\
&\to C(1,1) \otimes C(1,1) \otimes C(0,2)\\
&\to M(2,\mathbb{R}) \otimes M(2,\mathbb{R}) \otimes M(2,\mathbb{C})
\end{aligned}$$

Here we use Fact 1.5.33 and the following injective map:

$$\begin{aligned}
C(0,2) &\to M(2,\mathbb{C}),\\
v_1^- &\mapsto J,\\
v_2^- &\mapsto iS,\\
v_1^-v_2^- &\mapsto iT.
\end{aligned}$$

Put $g_0 := \frac{1}{\sqrt{2}}(I \otimes I \otimes I + J \otimes T \otimes Ji)$. We define $A_{g_0} : \mathfrak{sl}(8,\mathbb{C}) \to \mathfrak{sl}(8,\mathbb{C})$ by

$$A_{g_0}X := g_0^{-1}Xg_0.$$

From Schur's lemma, we have

Lemma 1.5.40.

$$\mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{C}^8) = \{aI_8 : a \in \mathbb{C}\}.$$

Proof of Proposition 1.5.38. We use Lemme 1.5.3.

Claim.

$$M = SO^*(8) \cdot \{I_8, I_2 \otimes I_2 \otimes S\} \cdot \{aI_8 : a \in \mathbb{C}, a^8 = 1\}.$$

proof of Claim. The inclusion $\supset$ is clear by definition. We show the inclusion $\subset$. Let $g \in M$. From Lemma 1.5.40, there exists $\alpha \in \mathbb{C}^\times$ such that $g^{-1}\tilde{\tau}_1(g) = \alpha I_8$. Therefore there exist $h \in O(8, \mathbb{C})$ and $\beta \in \mathbb{C}^\times$ such that $g = \beta h$. Since we have $g^{-1}\tilde{\tau}_2(g) = \beta^{-1}\overline{\beta}h^{-1}\tilde{\tau}_2(h) \in \mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{C}^8) = \{aI_8 : a \in \mathbb{C}\}$, there exists $\gamma \in \mathbb{C}^\times$ such that ${}^t h J_4 \overline{h} = \gamma J_4$. By taking matrix adjoint and determinant, we obtain $\gamma \in \{\pm 1\}$, which implies $h \in SO^*(8) \cdot \{I_8, I_2 \otimes I_2 \otimes S\}$. Since $g = \beta h \in SL(8, \mathbb{C})$ holds, we have $\beta^8 = 1$ by taking determinant. Thus we obtain $M = SO^*(8) \cdot \{I_8, I_2 \otimes I_2 \otimes S\} \cdot \{aI_8 : a \in \mathbb{C}, a^8 = 1\}$. □

From the above Claim, we can take F of Lemma 1.5.3 as follows.

$$F = \{I_8, I_2 \otimes I_2 \otimes S\} \cdot \{aI_8 : a \in \mathbb{C}, a^8 = 1\}.$$

Here, $\mathrm{Ad}(aI_8)$ ($a \in \mathbb{C}^\times$) and $\mathrm{Ad}(I \otimes I \otimes S)$ preserves $\rho_0(\mathfrak{l})$, which can be checked by the above realization.Thus we obtain the desired conclusion.

□

1.5.11 $(\mathfrak{g},\ \mathfrak{l})$=$(\mathfrak{so}(8, \mathbb{C}),\ \mathfrak{spin}(1, 7))$

In this subsection, we consider $(\mathfrak{g}, \mathfrak{l})$=$(\mathfrak{so}(8, \mathbb{C}), \mathfrak{spin}(1, 7))$ for the symmetric pair $(\mathfrak{g},\ \mathfrak{h})$=$(\mathfrak{so}(8, \mathbb{C}),\ \mathfrak{so}(7, \mathbb{C}))$. From Proposition 1.4.33, it is enough to consider the irreducible representation $\rho_0 := \rho_{\varpi_3}$. Note that the representation ρ_{ϖ_4} of $\mathfrak{so}(1, 7)$ is equivalent to ρ_{ϖ_3} in the sense of Definition 1.2.29. Our goal of this subsection is the following:

Proposition 1.5.41. $\Phi^{-1}([\rho_0])$ consists of two points. Moreover the two points given as follows:

$$\{[\rho_0], [\mathrm{Ad}(I_{7,1})\rho_0]\}.$$

We realize $G = SO(8, \mathbb{C}) := SL(8, \mathbb{C})^{\tilde{\tau}}$, $\mathfrak{g} = \mathfrak{so}(8, \mathbb{C}) := \mathfrak{sl}(8, \mathbb{C})^{\tau}$, where $\tilde{\tau} : SL(8, \mathbb{C}) \to SL(8, \mathbb{C})$, $g \mapsto {}^t g^{-1}$, $\tau : \mathfrak{sl}(8, \mathbb{C}) \to \mathfrak{sl}(8, \mathbb{C})$, $X \mapsto -{}^t X$. By using maps in Fact 1.5.33, we obtain a realization of $\mathfrak{spin}(1, 7) \subset \mathfrak{so}(8, \mathbb{C})$ by the composition of the following maps ι_1 and A_{g_0}:

$$\begin{aligned} \iota_1 : \mathfrak{so}(1, 7) &\to C_{even}(1, 7) \to C(1, 6) \to C(3, 4) \\ &\to C(1, 1) \otimes C(1, 1) \otimes C(1, 1) \otimes C(0, 1) \\ &\simeq M(2, \mathbb{R}) \otimes M(2, \mathbb{R}) \otimes M(2, \mathbb{R}) \otimes \mathbb{C} \end{aligned}$$

Put $g_0 := \frac{1}{\sqrt{2}}(I \otimes I \otimes I + J \otimes T \otimes Ji) \in GL(8, \mathbb{C})$. We define $A_{g_0} : M(8, \mathbb{C}) \to M(8, \mathbb{C})$ by

$$A_{g_0}X := g_0^{-1}Xg_0.$$

Then we have $A_{g_0}\iota_1(\mathfrak{so}(1, 7)) = \mathfrak{spin}(1, 7) \subset \mathfrak{so}(8, \mathbb{C})$.

From Schur's lemma, we have

Lemma 1.5.42.

$$\mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{C}^8) = \mathbb{C}I_8.$$

Proof of Proposition 1.5.41. We use Lemma 1.5.1.

Claim.

$$M = SO(8,\mathbb{C}) \cdot \{I_8, I_{7,1}\} \cdot \{aI_8 : a^8 = 1, a \in \mathbb{C}\}.$$

proof of Claim. The inclusion $\supset$ is clear by definition. We show the inclusion $\subset$. Let $g \in M$. From Lemma 1.5.42, there exists $a \in \mathbb{C}$ such that $a^t gg = I_8$. Therefore we have $g \in SO(8,\mathbb{C}) \cdot \{I_8, I_{7,1}\} bI_8$ for some $b \in \mathbb{C}$. Since we have $g \in SL(8,\mathbb{C})$, we get $b^8 = 1$. □

From the above Claim, we can take F of Lemma 1.5.1 as follows.

$$F = \cdot\{I_8, I_{7,1}\} \cdot \{aI_8 : a^8 = 1, a \in \mathbb{C}\}.$$

Since $\mathrm{Ad}(aI_8)$ ($a \in \mathbb{C}^\times$) preserves $\rho_0(\mathfrak{l})$, it is enough to show that $\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(I_{7,1})\rho_0(\mathfrak{l})$ are not $\mathrm{Int}(\mathfrak{g})$-conjugate. This comes from Fact 1.5.43. Take a Cartan involution $\theta : X \mapsto -X^*$ on $\mathfrak{so}(8,\mathbb{C})$ and a maximal abelian subspace $\mathfrak{a} := \mathbb{R}\text{-span}\{iA_{12}, iA_{3,4}, iA_{5,6}, iA_{7,8}\}$. Then we can take $\mathfrak{a}_L = \mathbb{R}J \otimes S \otimes Si = \mathbb{R}A_{g_0}\iota_1(S_{1,5}) \subset \mathfrak{spin}(1,7)$. Since we have $\mathrm{Ad}(I_{7,1})J \otimes S \otimes Si \notin W\mathfrak{a}_L$, where $W \simeq \mathfrak{S}_4 \ltimes (\mathbb{Z}_2)^3$ is the Weyl group of $\mathfrak{so}(8,\mathbb{C})$, the images of the Cartan projection of $\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(I_{7,1})\rho_0(\mathfrak{l})$ do not coincide. □

Fact 1.5.43 (See [Ko96b] for example)**.** Let G be a linear reductive Lie group and $\mathfrak{g}$ its Lie algebra and $\mathfrak{l}$ and $\mathfrak{l}'$ reductive subalgebras of $\mathfrak{g}$. If there exists $\alpha \in \mathrm{Int}(\mathfrak{g})$ such that $\mathfrak{l}' = \alpha(\mathfrak{l})$, then the images of Cartan projection of L and L' coincide, where L and L' are analytic subgroups of $\mathfrak{l}$ and $\mathfrak{l}'$.

1.5.12 $(\mathfrak{g}, \mathfrak{l})$=$(\mathfrak{so}(8,\mathbb{C}), \mathfrak{spin}(7,\mathbb{C}))$

In this subsection, we consider $(\mathfrak{g}, \mathfrak{l})$=$(\mathfrak{so}(8,\mathbb{C}), \mathfrak{spin}(7,\mathbb{C}))$ for the symmetric pair $(\mathfrak{g}, \mathfrak{h})$=$(\mathfrak{so}(8,\mathbb{C}), \mathfrak{so}(7,1))$. From Proposition 1.4.39, it is enough to consider the irreducible representation $\rho_0 := \rho_{\varpi_3} \boxtimes \mathrm{triv}$ of $\mathfrak{so}(7,\mathbb{C})$. Our goal of this subsection is the following:

Proposition 1.5.44. $\Phi_{\mathbb{C}}^{-1}([\rho_0])$ consists of two points. Moreover the two points given as follows:

$$\{[\rho_0], [\mathrm{Ad}(I_{7,1})\rho_0]\}.$$

We can prove in the same way as the case $(\mathfrak{so}(8,\mathbb{C}), \mathfrak{spin}(1,7))$ by using Fact 1.5.43.

We realize $\mathfrak{so}(8,\mathbb{C}) = \mathfrak{sl}(8,\mathbb{C})^\tau$ and $SO(8,\mathbb{C}) = SL(8,\mathbb{C})^{\tilde{\tau}}$ in the same way as subsection 1.5.11. By using maps in Fact 1.5.33, we obtain realization of $\mathfrak{spin}(7,\mathbb{C}) \subset \mathfrak{so}(8,\mathbb{C})$ by the complexification of the following map ι_1.

$$\begin{aligned}\iota_1 : \mathfrak{so}(7) &\to C_{even}(0,7) \to C(0,6) \to C(3,3)\\ &\to C(1,1) \otimes C(1,1) \otimes C(1,1)\\ &\simeq M(2,\mathbb{R}) \otimes M(2,\mathbb{R}) \otimes M(2,\mathbb{R}).\end{aligned}$$

Proof of Proposition 1.5.44. We use Lemma 1.5.1. Since $\mathrm{End}_{\rho_0(\mathfrak{l})}(\mathbb{C}^8) = \mathbb{C}I_8$, we obtain the same M and can take F as in Subsection 1.5.11. Thus it is enough to show that $\rho_0(\mathfrak{l})$ and $\mathrm{Ad}(I_{7,1})\rho_0(\mathfrak{l})$ are not $\mathrm{Int}(\mathfrak{g})$-conjugate. This comes from Fact 1.5.43. Take a Cartan involution $\theta : \mathfrak{so}(8,\mathbb{C}) \to \mathfrak{so}(8,\mathbb{C})$, $X \mapsto -X^*$ and a maximal abelian subspace $\mathfrak{a} := \mathbb{R}\text{-span}\{iA_{12}, iA_{3,4}, iA_{5,6}, iA_{7,8}\}$ of $\mathfrak{p} := \mathfrak{so}(8,\mathbb{C})^{-\theta}$. Then we can take

$$\begin{aligned}
\mathfrak{a}_L &= \mathbb{R}\text{-span}\{i\rho_0(A_{1,7}), i\rho_0(A_{2,6}), i\rho_0(A_{3,5})\} \\
&= \mathbb{R}\text{-span}\{iJ \otimes I \otimes I, iJ \otimes I \otimes S, iJ \otimes S \otimes I\} \\
&\simeq \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 : a_1 + a_4 = a_2 + a_3\},
\end{aligned}$$

where we used the coordinate by $\{iA_{1,2}, iA_{3,4}, iA_{5,6}, iA_{7,8}\}$. Therefore we obtain

$$\mathrm{Ad}(I_{7,1})\mathfrak{a}_L \not\subset W\mathfrak{a}_L = \mathfrak{a}(L).$$

Here $W \simeq \mathfrak{S}_4 \ltimes (\mathbb{Z}_2)^3$ is the Weyl group of $\mathfrak{so}(8,\mathbb{C})$. $\square$

1.6 Appendix

1.6.1 dimension of irreducible representation of simple Lie algebra

We prepare the Weyl's dimensionality formula: We consider simple Lie algebras over $\mathbb{C}$. Let $\{\alpha_i\}_{i=1}^n$ be a simple system and Δ^+ the set of positive roots.

Fact 1.6.1 (see [Kn] for example)**.** Let $\lambda = \sum_{i=1}^n k_i \varpi_i$ be a highest weight, where ϖ_i $(i = 1, \cdots, n)$ are the fundamental weights for the corresponding types. Then the dimension of the irreducible representation with highest weights λ is given as follows:

$$\dim \rho_\lambda = \prod_{\sum m_i \alpha_i \in \Delta^+} \frac{\sum (k_i + 1) m_i (\alpha_i, \alpha_i)}{\sum m_i (\alpha_i, \alpha_i)}.$$

Here m_i $(i = 1, \cdots n)$ are non-negative integers.

We summarize the each factor coming from positive roots for classical case below:

- Type A_n $(n \geq 1)$,
  Then the dimension of the irreducible representation with highest weight λ is given by the multiplication of the following factors:

$$\frac{k_i + \cdots k_j + j - i + 1}{j - i + 1} \quad (1 \leq i \leq j \leq n)$$

- Type B_n $(n \geq 2)$,
  the dimension of the irreducible representation with highest weight λ is given by the multiplication of the following factors:

$$\frac{k_i + \cdots k_j + j - i + 1}{j - i + 1} \quad (1 \leq i \leq j \leq n-1)$$
$$\frac{2(k_i + \cdots + k_{n-1}) + k_n + 2n - 2i + 1}{2n - 2i + 1} \quad (1 \leq i \leq n-1)$$
$$k_n + 1$$
$$\frac{(k_i + \cdots + k_{j-1}) + 2(k_j + \cdots + k_{n-1}) + k_n + 2n - j - i - 1}{2n - j - i + 1} \quad (1 \leq i < j \leq n-1)$$
$$\frac{(k_i + \cdots + k_{n-1}) + k_n + n - i + 1}{n - i + 1} \quad (1 \leq i \leq n-1)$$

- Type C_n $(n \geq 2)$,
  the dimension of the irreducible representation with highest weight λ is given by the multiplication of the following factors:

$$\frac{k_i + \cdots k_j + j - i + 1}{j - i + 1} \quad (1 \leq i \leq j \leq n-1)$$
$$\frac{k_i + \cdots k_n + n - i + 1}{n - i + 1} \quad (1 \leq i \leq n)$$
$$\frac{(k_i + \cdots + k_{j-1}) + 2(k_j + \cdots + k_n) + 2n - j - i + 2}{2n - j - i + 2} \quad (1 \leq i < j \leq n)$$

- Type D_n $(n \geq 2)$,
  Then the dimension of the irreducible representation with highest weight λ is given by the multiplication of the following factors:

$$\frac{k_i + \cdots k_j + j - i + 1}{j - i + 1} \quad (1 \leq i \leq j \leq n-1)$$
$$\frac{k_i + \cdots k_{j-1} + 2(k_j + \cdots + k_{n-2}) + k_{n-1} + k_n + 2n - j - i}{2n - j - i} \quad (1 \leq i < j \leq n-2)$$
$$\frac{k_i + \cdots + k_{n-1} + k_n + n - i + 1}{n - i + 1} \quad (1 \leq i \leq n-2)$$
$$\frac{k_i + \cdots k_{n-2} + k_n + n - i}{n - i} \quad (1 \leq i \leq n-2)$$
$$k_n + 1$$

We use the following data in our proof.

Fact 1.6.2 (minmum dimension)**.** Minimum dimension of non trivial irreducible representations of exeptional Lie algebra $\mathfrak{g}$ is given as follows

- $\mathfrak{g} = \mathfrak{g}_2^{\mathbb{C}}$: 7

- $\mathfrak{g} = \mathfrak{f}_4^{\mathbb{C}}$: 26
- $\mathfrak{g} = \mathfrak{e}_6^{\mathbb{C}}$: 27
- $\mathfrak{g} = \mathfrak{e}_7^{\mathbb{C}}$: 56
- $\mathfrak{g} = \mathfrak{e}_8^{\mathbb{C}}$: 248

The second smallest dimension of non trivial irreducible representations of $\mathfrak{g}_2^{\mathbb{C}}$ is 14.

1.6.2 dimension of irreducible representation of real rank one Lie algebras

We list irreducible representations π of real rank one Lie algebras $\mathfrak{l} = \mathfrak{f}_{4(-20)}$, $\mathfrak{so}(1,2p)$, $\mathfrak{su}(1,p)$ and $\mathfrak{sp}(1,p)$ to use in Section 1.3.1, 1.4.1, 1.4.2 satisfying the following:

$$\dim \pi \leq \begin{cases} 2d(L) \\ \frac{1}{2}d(L)+2 \\ d(L)+2 \end{cases}$$

respectively.

Remark 1.6.3. We have $d(\mathfrak{f}_{4(-20)}) = 16$. Only 26-dimensional representation satisfies $\dim \pi \leq 2d(L)$ among nontrivial irreducible representations of $\mathfrak{l} = \mathfrak{f}_{4(-20)}$.

■ **Irreducible representation of** $\mathfrak{l} = \mathfrak{so}(1,2p)(p \geq 2)$ **satisfying** $\dim \pi \leq 4p = 2d(L), 2p+2 = d(L)+2, p+2 = \frac{1}{2}d(L)+2$

The following tables are lists of irreducible representations π of $\mathfrak{so}(1,2p)$ satisfying $\dim \pi \leq 4p$, $2p+2$ and $p+2$ respectively:

p	π	$\dim \pi$
≥ 2	ϖ_1	$2p+1$
4	ϖ_4	16
3	ϖ_3	8
2	ϖ_2	4

p	π	$\dim \pi$
≥ 2	ϖ_1	$2p+1$
3	ϖ_3	8
2	ϖ_2	4

p	π	$\dim \pi$
2	ϖ_2	4

Here ϖ_i are fundamental weights of $\mathfrak{so}(2p+1,\mathbb{C})$.

■ **irreducible representation** π **of** $\mathfrak{l} = \mathfrak{su}(1,p)(p \geq 1)$ **satisfying** $\dim \pi \leq 4p = 2d(L)$**,** $2p+2 = d(L)+2$**,** $p+2 = \frac{1}{2}d(L)+2$ **respectively**

The following tables are lists of irreducible representations π of $\mathfrak{su}(1,p)$ satisfying $\dim \pi \leq 4p$, $2p+2$ and $p+2$ respectively:

p	π	$\dim\pi$
≥ 1	$\varpi_1,\ \varpi_p$	$p+1$
7	$\varpi_2,\ \varpi_6$	28
6	$\varpi_2,\ \varpi_5$	21
5	$\varpi_2,\ \varpi_4$	15
	ϖ_3	20
4	$\varpi_2,\ \varpi_3$	10
	$2\varpi_1,\ 2\varpi_4$	15
3	ϖ_2	6
	$2\varpi_1,\ 2\varpi_3$	10
2	$2\varpi_1,\ 2\varpi_2$	6
	$\varpi_1+\varpi_2$	8
1	$2\varpi_1$	3
	$3\varpi_1$	4

p	π	$\dim\pi$
≥ 1	$\varpi_1,\ \varpi_p$	$p+1$
4	$\varpi_2,\ \varpi_3$	10
3	ϖ_2	6
2	$2\varpi_1,\ 2\varpi_2$	6
1	$2\varpi_1$	3
	$3\varpi_1$	4

p	π	$\dim\pi$
≥ 1	$\varpi_1,\ \varpi_p$	$p+1$
1	$2\varpi_1$	3

■ **irreducible representations of** $\mathfrak{l}=\mathfrak{sp}(1,p)$ **(**$p\geq 2$**) satisfying** $\dim\pi\leq 8p=2d(L)$**,** $4p+2=d(L)+2$ **and** $2p+2=\frac{1}{2}d(L)+2$ **respectively**

The following tables are lists of irreducible representations π of $\mathfrak{sp}(1,p)$ satisfying $\dim\pi\leq 8p$, $4p+2$ and $2p+2$ respectively:

p	π	$\dim\pi$
≥ 1	ϖ_1	$2p+2$
2	ϖ_2	14
	ϖ_3	14

p	π	$\dim\pi$
≥ 1	ϖ_1	$2p+2$

p	π	$\dim\pi$
≥ 1	ϖ_1	$2p+2$

■ **nontrivial irreducible representation π of simple Lie algebras with** $\dim \pi \leq 8$

Table 1.32: pairs $(\mathfrak{l}, \pi)$ of a noncompact simple Lie algebra and its nontrivial irreducible representation with $\dim \pi \leq 8$

$\mathfrak{l}$	π	$\dim \pi$
$\mathfrak{sl}(n, \mathbb{C})(2 \leq n \leq 8)$	$\varpi_1 \boxtimes \mathrm{triv}$	n
$\mathfrak{sl}(4, \mathbb{C})$	$\varpi_2 \boxtimes \mathrm{triv}$	6
$\mathfrak{sl}(3, \mathbb{C})$	$2\varpi_1 \boxtimes \mathrm{triv}$	6
$\mathfrak{sl}(3, \mathbb{C})$	$(\varpi_1 + \varpi_2) \boxtimes$ triv	8
$\mathfrak{sl}(2, \mathbb{C})$	$k\varpi_1 \boxtimes \mathrm{triv}$ $(1 \leq k \leq 7)$	$k+1$
$\mathfrak{sl}(n, \mathbb{C})^\tau$ $(2 \leq n \leq 8)$	ϖ_1	n
$\mathfrak{sl}(4, \mathbb{C})^\tau$	ϖ_2	6
$\mathfrak{sl}(3, \mathbb{C})^\tau$	$2\varpi_1$	6
$\mathfrak{sl}(3, \mathbb{C})^\tau$	$\varpi_1 + \varpi_2$	8
$\mathfrak{sl}(2, \mathbb{C})^\tau$	$k\varpi_1$ $(1 \leq k \leq 7)$	$k+1$
$\mathfrak{so}(7, \mathbb{C})$	$\varpi_1 \boxtimes \mathrm{triv}$	7
$\mathfrak{so}(5, \mathbb{C})$	$\varpi_1 \boxtimes \mathrm{triv}$	5
$\mathfrak{so}(7, \mathbb{C})$	$\varpi_3 \boxtimes \mathrm{triv}$	8
$\mathfrak{so}(5, \mathbb{C})$	$\varpi_2 \boxtimes \mathrm{triv}$	4
$\mathfrak{so}(7, \mathbb{C})^\tau$	ϖ_1	7
$\mathfrak{so}(5, \mathbb{C})^\tau$	ϖ_1	5
$\mathfrak{so}(7, \mathbb{C})^\tau$	ϖ_3	8
$\mathfrak{so}(5, \mathbb{C})^\tau$	ϖ_2	4
$\mathfrak{sp}(4, \mathbb{C})$	$\varpi_1 \boxtimes \mathrm{triv}$	8
$\mathfrak{sp}(3, \mathbb{C})$	$\varpi_1 \boxtimes \mathrm{triv}$	6
$\mathfrak{sp}(4, \mathbb{C})^\tau$	ϖ_1	8
$\mathfrak{sp}(3, \mathbb{C})^\tau$	ϖ_1	6
$\mathfrak{so}(8, \mathbb{C})$	$\varpi_1 \boxtimes \mathrm{triv}$	8
$\mathfrak{so}(8, \mathbb{C})^\tau$	ϖ_1, ϖ_3, ϖ_4	8
$\mathfrak{g}_2^{\mathbb{C}}$	$\varpi_1 \boxtimes \mathrm{triv}$	7
$(\mathfrak{g}_2^{\mathbb{C}})^\tau$	ϖ_1	7

Here τ means a real structure.

1.6.3 Cartan's fundamental theorem and Iwahori's criterion

In this subsection, we quickly review Cartan's fundamental theorem (Fact 1.6.5) and Iwahori's criterion (Fact 1.6.8). See [Iw59] for more details.

Setting 3. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\tau : \mathfrak{g} \to \mathfrak{g}$ an anti-holomorphic involution.

To state Cartan's fundamental theorem and Iwahori's criterion, we introduce the following:

Notation 1.6.4. We use the following notation;

$$\begin{aligned}
C(\mathfrak{g}^\tau) &:= \{ \text{ an irreducible complex representation of } \mathfrak{g}^\tau.\}/\simeq, \\
R(\mathfrak{g}^\tau) &:= \{ \text{ an irreducible real representation of } \mathfrak{g}^\tau.\}/\simeq, \\
C^I(\mathfrak{g}^\tau) &:= \{\rho \in C(\mathfrak{g}^\tau) |\ \rho_{\mathbb{R}} \text{ is not irreducible.}\}, \\
C^{II}(\mathfrak{g}^\tau) &:= \{\rho \in C(\mathfrak{g}^\tau) |\ \rho_{\mathbb{R}} \text{ is irreducible.}\}, \\
R^I(\mathfrak{g}^\tau) &:= \{\rho \in R(\mathfrak{g}^\tau) |\ \rho_{\mathbb{C}} \text{ is irreducible as a complex representation.}\}, \\
R^{II}(\mathfrak{g}^\tau) &:= \{\rho \in R(\mathfrak{g}^\tau) |\ \rho_{\mathbb{C}} \text{ is not irreducible as a complex representation.}\}.
\end{aligned}$$

Here, for complex representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, we write $\rho_{\mathbb{R}}$ for the corresponding real representation $\rho_{\mathbb{R}} : \mathfrak{g} \to \mathfrak{gl}(V_{\mathbb{R}})$. For real representation $\rho : \mathfrak{g} \to \mathfrak{gl}(E)$, we write $\rho_{\mathbb{C}}$ for the corresponding complex representation $\rho_{\mathbb{C}} : \mathfrak{g} \to \mathfrak{gl}(E \otimes \mathbb{C})$. V, E are vector spaces over $\mathbb{C}$, $\mathbb{R}$ respectively. We write $V_{\mathbb{R}}$ instead of V when we regard it as a real vector space.

$\mathbb{Z}_2$ acts on $C(\mathfrak{g}^\tau)$ by taking the complex conjugate representation. We put

$$\hat{C}(\mathfrak{g}^\tau) := C(\mathfrak{g}^\tau)/\mathbb{Z}_2.$$

Since $\rho_{\mathbb{R}} \simeq (\overline{\rho})_{\mathbb{R}}$ holds, the $\mathbb{Z}_2$-action on $C(\mathfrak{g}^\tau)$ preserves the subsets $C^I(\mathfrak{g}^\tau)$ and $C^{II}(\mathfrak{g}^\tau)$. So, we put

$$\hat{C}^{II}(\mathfrak{g}^\tau) := C^{II}(\mathfrak{g}^\tau)/\mathbb{Z}_2.$$

Fact 1.6.5 ([Iw59, Theorem 1] Cartan's fundamental theorem)**.** The following maps are bijective;

$$\begin{aligned}
R^I_n(\mathfrak{g}^\tau) \to C^I_n(\mathfrak{g}^\tau), &\qquad (\rho : \mathfrak{g}^\tau \to \mathfrak{gl}(E)) \mapsto (\rho_{\mathbb{C}} : \mathfrak{g}^\tau \to \mathfrak{gl}(E \otimes \mathbb{C})), \\
\hat{C}^{II}_n(\mathfrak{g}^\tau) \to R^{II}_{2n}(\mathfrak{g}^\tau), &\qquad (\rho : \mathfrak{g}^\tau \to \mathfrak{gl}(V)) \mapsto (\rho_{\mathbb{R}} : \mathfrak{g}^\tau \to \mathfrak{gl}(V_{\mathbb{R}})).
\end{aligned}$$

Here, the subscripts n and $2n$ means the dimensions of representations.

By Cartan's fundamental theorem, to study real representations, the problem is to determine which classes given complex representation belongs to. Iwahori's criterion gives the solution. To state Iwahori's criterion, we prepare the terms "self conjugate" and "$\operatorname{index}_\tau \rho \in \{\pm 1\}$".

Definition 1.6.6 ([Iw59, §9] Definition of index)**.**

- We call representation (ρ, V) of $\mathfrak{g}^\tau$ *self conjugate* if $\rho \simeq \overline{\rho}$ as a representation of $\mathfrak{g}^\tau$.
- Let (ρ, V) be a self conjugate irreducible representation of $\mathfrak{g}^\tau$. Then we can take an anti holomorphic isomorphism J such that $J^2 = c\,\mathrm{id}_V$ $(c \in \mathbb{R}^\times)$ given by Remark 1.6.7, so we put

  $$\operatorname{index}_\tau \rho := \begin{cases} 1 & (c > 0), \\ -1 & (c < 0). \end{cases}$$

  This is independent on the choice of J.

Remark 1.6.7. (i) A representation (ρ, V) of $\mathfrak{g}^\tau$ is self conjugate if and only if there exists an anti holomorphic isomorphism $J : V \to V$ satisfying $J\rho(X) = \rho(X)J$ for all $X \in \mathfrak{g}^\tau$. Moreover if ρ is irreducible, then there exists $c \in \mathbb{C}^\times$ such that $J^2 = c\,\mathrm{id}_V$ from Schur's Lemma.

(ii) If $J : V \to V$ is a anti holomorphic isomorphism satisfying $J^2 = c\,\mathrm{id}_V$ for some $c \in \mathbb{C}^\times$, then $c \in \mathbb{R}^\times$ holds.

(iii) The signature of $c \in \mathbb{R}^\times$ given by (i), (ii) above is independent of the choice of J.

We can now state the following:

Fact 1.6.8 ([Iw59, Lemma 4])**.** Let (ρ, V) be a complex irreducible representation of Lie algebra $\mathfrak{g}^\tau$. Then $\rho : \mathfrak{g}^\tau \to \mathfrak{gl}(V)$ is of class $C^I(\mathfrak{g}^\tau)$ if and only if the following conditions (i) and (ii) are satisfied:

(i) ρ is self conjugate (i.e. $\rho \simeq \overline{\rho}$ as a representation of $\mathfrak{g}^\tau$),

(ii) $\mathrm{index}_\tau\, \rho = 1 \in \{\pm 1\}$.

In Setting 4, we can check the condition (i) and (ii) of Fact 1.6.8 easily by using "diagram" (see Fact 1.6.12).

Notation 1.6.9. Take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, a corresponding root system $\Delta \subset \mathfrak{t}_\mathbb{R}^\vee$ and a simple system $\Pi := \{\alpha_1, \cdots, \alpha_n\} \subset \Delta$. Let θ be a Cartan involution on $\mathfrak{g}$, σ the split real structure on $\mathfrak{g}$ associated with Π, which satisfy $\theta\sigma = \sigma\theta$. For an anti holomorphic involution τ on $\mathfrak{g}$, there exists $\alpha \in \mathrm{Int}(\mathfrak{g})$ such that $\alpha\tau\alpha^{-1}\theta = \theta\alpha\tau\alpha^{-1}$ and $\alpha\tau\alpha^{-1}\sigma = \sigma\alpha\tau\alpha^{-1}$ (see [Oni] §4 Theorem 2). Put $\tau' = \alpha\tau\alpha^{-1}$. Let pr_1 be the first projection $\mathrm{Aut}(\Pi) \ltimes \mathrm{Int}(\mathfrak{g}) \to \mathrm{Aut}(\Pi)$. Then we put

$$\begin{aligned} s_- &:= pr_1\varphi(\tau'\sigma) \in \mathrm{Aut}(\Pi) \\ s_* &:= pr_1\varphi(\tau'\theta) \\ s_\vee &:= pr_1\varphi(\sigma\theta), \end{aligned}$$

where φ is the map of Remark 1.6.10 below. The elements s_-, s_* and $s_\vee$ are uniquely well-defined for a real form $\mathfrak{g}^\tau$. An element of $\mathrm{Aut}(\Pi)$ induces the action on $\mathfrak{t}_\mathbb{R}^\vee$. So, for $\lambda \in \mathfrak{t}_\mathbb{R}^\vee$, the notation $s_-(\lambda)$, $s_*(\lambda)$ and $s_\vee(\lambda)$ make sense. Moreover, s_-, s_* and $s_\vee \in \mathrm{Aut}(\Pi)$ induce automorphisms of Dynkin diagram of $\mathfrak{g}$.

Remark 1.6.10 (see [Oni] §4 Theorem 1)**.** We have a natural bijection as follows:

$$\varphi : \mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\Pi) \ltimes \mathrm{Int}\,\mathfrak{g}$$

Here $\mathrm{Aut}(\Pi) := \{\varphi \in O(\mathfrak{t}_\mathbb{R}^\vee) : \varphi\Pi = \Pi\}$.

Remark 1.6.11. For a highest weight $\lambda \in \mathfrak{t}_{\mathbb{R}}^{\vee}$, we have the following:

$$\begin{aligned}\overline{\rho_\lambda} &\simeq \rho_{s_-(\lambda)} \text{ as a representation of } \mathfrak{g}^\tau,\\ \rho_\lambda^* &\simeq \rho_{s_*(\lambda)} \text{ as a representation of } \mathfrak{g}^\tau,\\ \rho_\lambda^\vee &\simeq \rho_{s_\vee(\lambda)} \text{ as a representation of } \mathfrak{g}.\end{aligned}$$

These comes from $\rho \simeq \overline{\rho}\sigma \simeq \rho^*\theta \simeq \rho^\vee\sigma\theta$ (as a representation of $\mathfrak{g}$).

Fact 1.6.12 ([Iw59, Theorem 2])**.** Let $\rho : \mathfrak{g}^\tau \to \mathfrak{sl}(V)$ be an irreducible representation with a highest weight $\lambda = \sum_{i=1}^n m_i\varpi_i$, where ϖ_i $(i = 1, \cdots n)$ are the fundamental weights. Then the following conditions are equivalent:

(i) $\rho \simeq \overline{\rho}$ as a representation of $\mathfrak{g}^\tau$ (self conjugate),

(ii) $m_i = m_{p(i)}$.

Here $p \in \mathfrak{S}_n$ is the permutation induced by $s_- \in \mathrm{Aut}(\Pi)$, namely $s_-(\alpha_i) = \alpha_{p(i)}$. Moreover, for the self conjugate irreducible representation, we have

$$\mathrm{index}_\tau\,\rho = \prod_{i\in\{1,\cdots,n\}^p} (\mathrm{index}_\tau\,\rho_{\varpi_i})^{m_i}.$$

Here, $i \in \{1, \cdots, n\}^p \iff p(i) = i$ and ρ_{ϖ_i} is the fundamental representation with fundamental weight ϖ_i.

We can check the above conditions by seeing each simple factor. Suppose $\mathfrak{g}^\tau = \oplus_i^s \mathfrak{g}_i^{\tau_i}$ is the decomposition into simple ideals $\mathfrak{g}_i^{\tau_i}$ $(i = 1, \cdots, s)$ and $\rho : \mathfrak{g}^\tau \to \mathfrak{sl}(V)$ is an irreducible representation of $\mathfrak{g}^\tau$. Then ρ has the description $\rho_1 \boxtimes \cdots \boxtimes \rho_s$, where ρ_i is an irreducible representation of $\mathfrak{g}_i^{\tau_i}$. Then we have the following:

Fact 1.6.13 ([Iw59, Lemma 6])**.** $\rho \simeq \overline{\rho}$ holds as a representation of $\mathfrak{g}^\tau$ if and only if $\rho_i \simeq \overline{\rho_i}$ holds as a representation of $\mathfrak{g}_i^{\tau_i}$ for any $i \in \{1, \cdots, s\}$. In this case, we have

$$\mathrm{index}_\tau\,\rho = \prod_{i=1}^s \mathrm{index}_{\tau_i}\,\rho_i.$$

Notation 1.6.14 (Iwahori diagram, see [Oni] Table 5 also)**.** From Fact 1.6.12, for given irreducible representation ρ of $\mathfrak{g}^\tau$, we can determine whether or not ρ is self conjugate and calculate $\mathrm{index}_\tau\,\rho$ by the permutation $p \in \mathfrak{S}_n$ induced by $s_- \in \mathrm{Aut}(\Pi)$ and $\mathrm{index}_\tau\,\rho_{\varpi_i}$ for $i \in \{1, \cdots, n\}^p$. We describe the information on Dynkin diagram. We connect the nodes α_i and $\alpha_{p(i)}$ by arrows if $p(i) \neq i$ $(i = 1, \cdots, n)$ and use black nodes $\bullet$ for corresponding simple roots α_i if $i \in \{1, \cdots, n\}^p$ and $\mathrm{index}_\tau\,\rho_{\varpi_i} = -1$ and use white nodes $\circ$ otherwise. We call the diagram *Iwahori diagram* of $\mathfrak{g}^\tau$. For simple Lie algebras, Iwahori diagram was drawn as in Table 1.33.

1.6.4 embeddability of semisimple Lie algebras into $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{su}^*(2n)$, $\mathfrak{so}(n,\mathbb{C})$ and $\mathfrak{sp}(n,\mathbb{C})$

We consider embeddability by representation of semisimple Lie algebras into simple Lie algebras such as $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{su}^*(2n)$, $\mathfrak{so}(n,\mathbb{C})$ and $\mathfrak{sp}(n,\mathbb{C})$. Our goals in this subsection are Proposition 1.6.17, 1.6.18, 1.6.20 and 1.6.21.

Notation 1.6.15. Let $\mathfrak{g}$ be a Lie algabra and $\mathfrak{l}$ and $\mathfrak{h}$ Lie subalgebra of $\mathfrak{g}$. $\mathfrak{l} \subset_{\mathrm{Int}} \mathfrak{h}$ denotes if there exist $\alpha \in \mathrm{Int}(\mathfrak{g})$ such that $\alpha(\mathfrak{l}) \subset \mathfrak{h}$.

Setting 4. In this subsection, $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{C}$. τ is an anti holomorphic involution on $\mathfrak{g}$ and θ is a Cartan involution on $\mathfrak{g}$. $\rho : \mathfrak{g} \to \mathfrak{sl}(V)$ is a representation over $\mathbb{C}$. Put $n := \dim_{\mathbb{C}} V$.

Remark 1.6.16. We have a natural bijection as follows:

$$\begin{aligned}\{\text{a representation } \rho : \mathfrak{g} \to \mathfrak{gl}(V)\} &\to \{\text{a representation } \rho : \mathfrak{g}^\tau \to \mathfrak{gl}(V)\} \\ \rho &\mapsto \rho|_{\mathfrak{g}^\tau}\end{aligned}$$

We sometimes identify $\rho|_{\mathfrak{g}^\tau}$ with ρ. To clearify the domain if necessaly, we say "as a representation of $\mathfrak{g}^\tau$".

Proposition 1.6.17. In Setting 4, the following conditions are equivalent:

(i) $\rho(\mathfrak{g}^\tau) \subset_{\mathrm{Int}} \mathfrak{sl}(n,\mathbb{R})$,

(ii) $\rho \simeq \overline{\rho}$ as a representation of $\mathfrak{g}^\tau$ and $(\mathrm{index}_\tau \pi)^{m_\pi} = 1$ for any $\pi \in SCIR(\mathfrak{g}^\tau)$.

Here $m_\pi = [\pi : \rho]$ is the multiplicity of π and

$$\begin{aligned}SCIR(\mathfrak{g}^\tau) &= \{\pi \in C(\mathfrak{g}^\tau) : \pi \simeq \overline{\pi}\} \\ &= \{\text{self conjugate irreducible representation of } \mathfrak{g}^\tau\}/\sim .\end{aligned}$$

Proposition 1.6.18. In Setting 4, the following conditions are equivalent:

(i) $\rho(\mathfrak{g}^\tau) \subset_{\mathrm{Int}} \mathfrak{su}^*(2\frac{n}{2})$

(ii) $\rho \simeq \overline{\rho}$ as a representation of $\mathfrak{g}^\tau$ and $(-\,\mathrm{index}_\tau \pi)^{m_\pi} = 1$ for any $\pi \in SCIR(\mathfrak{g}^\tau)$,

where $m_\pi = [\pi : \rho]$.

Remark 1.6.19. In the above Proposition 1.6.17 and 1.6.18, $\mathfrak{sl}(n,\mathbb{R}), \mathfrak{su}^*(2\frac{n}{2}) \subset \mathfrak{sl}(V)$ are subalgebras coming from involtutions on $\mathfrak{sl}(V)$. They are unique up to $\mathrm{Int}(\mathfrak{sl}(V))$.

Proposition 1.6.20. In Setting 4, the following conditions are equivalent:

(i) $\rho(\mathfrak{g}^\tau) \subset_{\mathrm{Int}} \mathfrak{so}(n,\mathbb{C})$,

(ii) $\rho \simeq \rho^\vee$ as a representation of $\mathfrak{g}$ and $(\operatorname{index}_\theta \pi)^{m_\pi} = 1$ for any $\pi \in SCIR(\mathfrak{g}^\theta)$,

where $m_\pi = [\pi : \rho]$.

Proposition 1.6.21. In Setting 4, the following conditions are equivalent:

(i) $\rho(\mathfrak{g}^\tau) \subset_{\mathrm{Int}} \mathfrak{sp}(\frac{n}{2}, \mathbb{C})$,

(ii) $\rho \simeq \rho^\vee$ as a representation of $\mathfrak{g}$ and $(-\operatorname{index}_\theta \pi)^{m_\pi} = 1$ for any $\pi \in SCIR(\mathfrak{g}^\theta)$,

where $m_\pi = [\pi : \rho]$.

Proposition 1.6.22 (see [Oni, Theorem 3] for an irreducible representation case)**.** In Setting 4, the following conditions are equivalent:

(i) $\rho(\mathfrak{g}^\tau) \subset_{\mathrm{Int}} \mathfrak{su}(p, q)$ for some $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q = n$,

(ii) $\rho \simeq \rho^*$ as a representation of $\mathfrak{g}^\tau$.

Proof. (i) holds if and only if there exists a $\rho(\mathfrak{g}^\tau)$-invariant Hermitian form h on V. This is equivalent to (ii), which comes from $h^\sharp : V \to V^*$, $v \mapsto h(v, \cdot)$ induces intertwining operator between ρ and ρ^*. $\square$

Remark 1.6.23. In the above Proposition 1.6.20 and 1.6.21, $\mathfrak{so}(n, \mathbb{C}), \mathfrak{sp}(\frac{n}{2}, \mathbb{C}) \subset \mathfrak{sl}(V)$ are subalgebras coming from involutions on $\mathfrak{sl}(V)$. They are unique up to $\mathrm{Int}(\mathfrak{sl}(V))$.

Proof of Proposition 1.6.17 and 1.6.18. This comes from Lemma 1.6.24 and 1.6.25. $\square$

Lemma 1.6.24. Let $\mathfrak{g}_0$ a real Lie algebra and $\rho : \mathfrak{g}_0 \to \mathfrak{sl}(V)$ be a representation of $\mathfrak{g}_0$ over $\mathbb{C}$. Put $n = \dim_{\mathbb{C}} V$. $\rho(\mathfrak{g}_0) \subset_{\mathrm{Int}} \mathfrak{sl}(n, \mathbb{R})$ (resp. $\mathfrak{su}^*(2\frac{n}{2})$) if and only if there exists anti-holomorphic map $J : V \to V$ such that $\rho(X)J = J\rho(X)$ for all $X \in \mathfrak{g}_0$ and $J^2 = \mathrm{id}$ (resp. $-\mathrm{id}$).

Proof. only if part: This comes from classification of anti-holomorphic involution on $\mathfrak{sl}(V)$.
if part: Define a anti-holomorphic involution $\tilde{J}$ on $\mathfrak{sl}(V)$ by $f \mapsto JfJ^{-1}$. Then, we have $\mathfrak{sl}(V)^{\tilde{J}} \simeq \begin{cases} \mathfrak{sl}(n, \mathbb{R}) \ (J^2 = \mathrm{id}) \\ \mathfrak{su}^*(2\frac{n}{2}) \ (J^2 = -\mathrm{id}) \end{cases}$ and $\rho(\mathfrak{g}_0) \subset \mathfrak{sl}(V)^{\tilde{J}}$. $\square$

Lemma 1.6.25. In Setting 4, the following conditions are equivalent

(i) There exists an anti-holomorphic linear isomorphism $J : V \to V$ such that $J\rho(X) = \rho(X)J$ for all $X \in \mathfrak{g}^\tau$,

(ii) $\rho \simeq \overline{\rho}$ as a representation of $\mathfrak{g}^\tau$, Moreover, for $\varepsilon \in \{\pm 1\}$, there exists an anti-holomorphic linear isomorphism $J : V \to V$ such that $J^2 = \varepsilon\,\mathrm{id}_V$ and $J\rho(X) = \rho(X)J$ for all $X \in \mathfrak{g}^\tau$ if and only if $(\varepsilon\,\mathrm{index}_\tau\,\pi)^{m_\pi} = 1$ for all $\pi \in SCIR(\mathfrak{g}^\tau)$, where $m_\pi := [\pi : \rho]$.

Proof. Former part is clear by the definition of complex conjugate representation. So we prove "Moreover" part.
only if part: Let $\pi \in SCIR(\mathfrak{g}^\tau)$. Let V_π be a representation space of π. Then there exists an $\mathbb{C}$-linear isomorphism $\varphi : V_\pi \to \overline{V_\pi}$ such that $\pi(X)\varphi = \varphi\pi(X)$ for all $X \in \mathfrak{g}^\tau$. Take $\sigma_1 : V_\pi \to \overline{V_\pi}$, $v \mapsto \overline{v}$. We can describe $J|_{V_\pi \otimes \mathbb{C}^{m_\pi}} : V_\pi \otimes \mathbb{C}^{m_\pi} \to V_\pi \otimes \mathbb{C}^{m_\pi}$ as $J|_{V_\pi \otimes \mathbb{C}^{m_\pi}} = \sigma^{-1}(\varphi \otimes A)$ for some $A \in GL(m_\pi, \mathbb{C})$ by Schur's Lemma. Here $\sigma = \sigma_1 \otimes \sigma_2$ and $\sigma_2 : \mathbb{C}^{m_\pi} \to \overline{\mathbb{C}^{m_\pi}}$, $w \mapsto \overline{w}$. Therefore we have

$$\varepsilon\,\mathrm{id}\,|_{V_\pi \otimes \mathbb{C}^{m_\pi}} = J|^2_{V_\pi \otimes \mathbb{C}^{m_\pi}} = \sigma_1^{-1}\varphi\sigma_1^{-1}\varphi \otimes \sigma_2^{-1}A\sigma_2^{-1}A = a(\mathrm{index}_\tau\,\pi)\,\mathrm{id}_{V_\pi} \otimes \sigma_2^{-1}A\sigma_2^{-1}A,$$

for some positive number $a \in \mathbb{R}$, which implies $\varepsilon\,\mathrm{id}_{\mathbb{C}^{m_\pi}} = a(\mathrm{index}_\tau\,\pi)\sigma_2^{-1}A\sigma_2^{-1}A$. By taking determinant of both sides, we obtain $(\varepsilon\,\mathrm{index}_\tau\,\pi)^{m_\pi} a^{m_\pi} |\det A|^2 = 1$, which implies $(\varepsilon\,\mathrm{index}_\tau\,\pi)^{m_\pi} = 1$.
if part: Assume $(\varepsilon\,\mathrm{index}_\tau\,\pi)^{m_\pi} = 1$ for all $\pi \in SCIR(\mathfrak{g}^\tau)$. In the case $m_\pi := [\pi, \rho] \geq 1$, it is enough to show that there exists an anti-holomorphic map J with $J^2 = \varepsilon\,\mathrm{id}_V$ commuting with $\rho(X)$ for all $X \in \mathfrak{g}^\tau$ on

(i) $V_\pi \oplus \overline{V_\pi}$ for $\pi \notin SCIR(\mathfrak{g}^\tau)$,

(ii) $\oplus^{m_\pi} V_\pi$ for $\pi \in SCIR(\mathfrak{g}^\tau)$.

(i): Define an anti-holomorphic map $J : V_\pi \oplus \overline{V_\pi} \to V_\pi \oplus \overline{V_\pi}$ by $J(v, \overline{w}) = (\varepsilon w, \overline{v})$. Then the following two conditions are satisfied

- $J^2 = \varepsilon\,\mathrm{id}$,
- $(\pi \oplus \overline{\pi})(X)J = J(\pi \oplus \overline{\pi})(X)$ for all $X \in \mathfrak{g}^\tau$.

(ii): It is enough to show the following:

Claim. Let $\pi \in SCIR(\mathfrak{g}^\tau)$. For $\mu \in \{\pm 1\}$, there exists an anti-holomorphic map $J : V_\lambda \oplus V_\lambda \to V_\lambda \oplus V_\lambda$ such that $J^2 = \mu\,\mathrm{id}$ and $J(\pi \oplus \pi)(X) = (\pi \oplus \pi)(X)J$ for all $X \in \mathfrak{g}^\tau$.

proof of Claim. Take an anti-holomorphic map $J_\pi : V_\pi \to V_\pi$ such that $J_\pi^2 = \mathrm{index}_\tau\,\pi\,\mathrm{id}$ and $\pi(X)J_\pi = J_\pi\pi(X)$ for all $X \in \mathfrak{g}^\tau$.

- in the case when $\mathrm{index}_\tau\,\pi = \mu$:
  Put $J := J_\pi \oplus J_\pi : V_\pi \oplus V_\pi \to V_\pi \oplus V_\pi$, $(v, w) \mapsto (J_\pi v, J_\pi w)$. Then J is the desired map.
- in the case when $\mathrm{index}\,\pi = -\mu$:
  Put $J : V_\pi \oplus V_\pi \to V_\pi \oplus V_\pi$, $(v, w) \mapsto (J_\pi w, -J_\pi v)$. Then J is the desired map.

□

□

Proof of Proposition 1.6.20 and 1.6.21. This comes from Proposition 1.6.17, 1.6.18, Example 1.6.30 and that $\rho(\mathfrak{g}^\tau)_{cu} = \rho(\mathfrak{g})_u =_{\mathrm{Int}} \rho(\mathfrak{g}^\theta)$. □

1.6.5 associated duality for semisimple Lie algebra

Our goal of this subsection is the following:

Proposition 1.6.26. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and $\mathfrak{l}$ and $\mathfrak{h}$ real semisimple Lie subalgerbras of $\mathfrak{g}$. Then we have $\mathfrak{l}_{cu} \subset_{\mathrm{Int}} \mathfrak{h} \iff \mathfrak{l} \subset_{\mathrm{Int}} \mathfrak{h}_{uc}$.

Definition 1.6.27. Fix a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Let $\mathfrak{h}$ be a semisimple Lie subalgebra over $\mathbb{R}$. $\mathfrak{h}_c$ denotes inner complexification of $\mathfrak{h}$ in $\mathfrak{g}$. $\mathfrak{h}_u$ denotes maximal compact subalgebra of $\mathfrak{h}$, which is well-defined up to $\mathrm{Int}(\mathfrak{h})$.

Lemma 1.6.28. The operation c and u have the following properties.

(i) $c^2 = c$ (i.e. $(\mathfrak{h}_c)_c =_{\mathrm{Int}} \mathfrak{h}_c$),

(ii) $u^2 = u$ (i.e. $(\mathfrak{h}_u)_u =_{\mathrm{Int}} \mathfrak{h}_u$),

(iii) $cuc = c$ (i.e. $((\mathfrak{h}_c)_u)_c =_{\mathrm{Int}} \mathfrak{h}_c$),

(iv) $ucu = u$ (i.e. $((\mathfrak{h}_u)_c)_u =_{\mathrm{Int}} \mathfrak{h}_u$),

(v) For a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we have $\mathfrak{h}_u \subset \mathfrak{h} \subset \mathfrak{h}_c$.

(vi) $\mathfrak{l} \subset_{\mathrm{Int}} \mathfrak{h} \implies \mathfrak{l}_{\mathbb{C}} \subset_{\mathrm{Int}} \mathfrak{h}_{\mathbb{C}}, \mathfrak{l}_u \subset_{\mathrm{Int}} \mathfrak{h}_u$.

Here "$\mathfrak{h} =_{\mathrm{Int}} \mathfrak{l}$" means there exists $\alpha \in \mathrm{Int}(\mathfrak{g})$ such that $\alpha(\mathfrak{h}) = \mathfrak{l}$.

Proof. This can be easily checked. So we omit the proof. □

Proposition 1.6.29. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ and $\mathfrak{l}$ real semisimple Lie subalgebras of $\mathfrak{g}$. Then the following conditions are equivalent:

(i) $\mathfrak{l}_{cu} \subset_{\mathrm{Int}} \mathfrak{h}$,

(ii) $\mathfrak{l}_{cu} \subset_{\mathrm{Int}} \mathfrak{h}_u$,

(iii) $\mathfrak{l}_c \subset_{\mathrm{Int}} \mathfrak{h}_{uc}$,

(iv) $\mathfrak{l} \subset_{\mathrm{Int}} \mathfrak{h}_{uc}$.

Proof. $\underline{\text{(ii)} \implies \text{(i)}}$: This comes from $\mathfrak{h}_u \subset \mathfrak{h}$
$\underline{\text{(i)} \implies \text{(ii)}}$: Take u to the both sides.
$\underline{\text{(ii)} \implies \text{(iii)}}$: Take c to the both sides.
$\underline{\text{(iii)} \implies \text{(ii)}}$: Take u to the both sides.
$\underline{\text{(iv)} \implies \text{(iii)}}$: Take c to the both sides.
$\underline{\text{(iii)} \implies \text{(iv)}}$: This comes from $\mathfrak{l} \subset \mathfrak{l}_{\mathbb{C}}$. □

Example 1.6.30. Put $\mathfrak{h} = \mathfrak{sl}(n, \mathbb{R}) \subset \mathfrak{sl}(n, \mathbb{C}) =: \mathfrak{g}$. Then for a semisimple Lie subalgebra $\mathfrak{l} \subset \mathfrak{g}$, The following conditions are equivalent:

(i) $\mathfrak{l}_{cu} \subset_{\mathrm{Int}} \mathfrak{sl}(n, \mathbb{R})$,

(ii) $\mathfrak{l}_{cu} \subset_{\mathrm{Int}} \mathfrak{so}(n)$,

(iii) $\mathfrak{l}_{c} \subset_{\mathrm{Int}} \mathfrak{so}(n, \mathbb{C})$,

(iv) $\mathfrak{l} \subset_{\mathrm{Int}} \mathfrak{so}(n, \mathbb{C})$.

Put $\mathfrak{h} = \mathfrak{su}^*(2n) \subset \mathfrak{sl}(2n, \mathbb{C}) =: \mathfrak{g}$. Then for a semisimple Lie subalgebra $\mathfrak{l} \subset \mathfrak{g}$, The following conditions are equivalent:

(i) $\mathfrak{l}_{cu} \subset_{\mathrm{Int}} \mathfrak{su}^*(2n)$,

(ii) $\mathfrak{l}_{cu} \subset_{\mathrm{Int}} \mathfrak{sp}(n)$,

(iii) $\mathfrak{l}_{c} \subset_{\mathrm{Int}} \mathfrak{sp}(n, \mathbb{C})$,

(iv) $\mathfrak{l} \subset_{\mathrm{Int}} \mathfrak{sp}(n, \mathbb{C})$.

1.6.6 embeddability for an irreducible representation

Our goal of this section is the following Propositions 1.6.31, 1.6.32, 1.6.33 and 1.6.34:

Setting 5. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, τ a real structure on $\mathfrak{g}$ and (ρ, V) an irreducible representation of $\mathfrak{g}^\tau$. Put $n := \dim_{\mathbb{C}} V$.

Proposition 1.6.31. In Setting 5, The following conditions are equivalent:

(i) $\rho(\mathfrak{g}^\tau) \subset_{\mathrm{Int}} \mathfrak{so}(p, q)$ for some $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q = n$,

(ii) $\overline{\rho} \simeq \rho \simeq \rho^\vee$ and $\mathrm{index}_\tau\, \rho = 1 = \mathrm{index}_\theta\, \rho$.

Proposition 1.6.32. In Setting 5, The following conditions are equivalent:

(i) $\rho(\mathfrak{g}^\tau) \subset_{\mathrm{Int}} \mathfrak{sp}(\frac{n}{2}, \mathbb{R})$,

(ii) $\overline{\rho} \simeq \rho \simeq \rho^\vee$ and $\mathrm{index}_\tau\, \rho = 1 = -\mathrm{index}_\theta\, \rho$.

Proposition 1.6.33. In Setting 5, The following conditions are equivalent:

(i) $\rho(\mathfrak{g}^\tau) \subset_{\mathrm{Int}} \mathfrak{so}^*(2\frac{n}{2})$,

(ii) $\overline{\rho} \simeq \rho \simeq \rho^\vee$ and $-\mathrm{index}_\tau\, \rho = 1 = \mathrm{index}_\theta\, \rho$.

Proposition 1.6.34. In Setting 5, The following conditions are equivalent:

(i) $\rho(\mathfrak{g}^\tau) \subset_{\mathrm{Int}} \mathfrak{sp}(p, q)$ for some $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q = \frac{n}{2}$,

(ii) $\overline{\rho} \simeq \rho \simeq \rho^\vee$ and $\mathrm{index}_\tau\, \rho = -1 = \mathrm{index}_\theta\, \rho$.

Remark 1.6.35. In Setting 5, we have the following table, which means $\rho(\mathfrak{g}^\tau)$ is contained in the simple Lie algebra by $\mathrm{Int}(\mathfrak{g})$ if the conditions of left side and up side are satisfied.

conditions	$\rho \simeq \rho^\vee$, $\mathrm{index}_\theta\, \rho = 1$	$\rho \simeq \rho^\vee$, $\mathrm{index}_\theta\, \rho = 1$	$\rho \not\simeq \rho^\vee$
$\rho \simeq \overline{\rho}$, $\mathrm{index}_\tau\, \rho = 1$	$\mathfrak{so}(p,q)$ $(p+q=n)$	$\mathfrak{sp}(\frac{n}{2},\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{R})$
$\rho \simeq \overline{\rho}$, $\mathrm{index}_\tau\, \rho = -1$	$\mathfrak{so}^*(2\frac{n}{2})$	$\mathfrak{sp}(p,q)$ $(p+q=\frac{n}{2})$	$\mathfrak{su}^*(\frac{n}{2})$
$\rho \not\simeq \overline{\rho}$	$\mathfrak{so}(n,\mathbb{C})$	$\mathfrak{sp}(\frac{n}{2},\mathbb{C})$	$\mathfrak{su}(p,q)$ $(p+q=n)$ if $\rho \simeq \rho^*$

In the following part of this subsection, we prove Propositions 1.6.31, 1.6.32, 1.6.33 and 1.6.34. The implication (i) to (ii) is clear from Propositions 1.6.17, 1.6.18, 1.6.20 and 1.6.21. A key argument to show the implication (ii) to (i) is the commutativity of two involutions (see Lemma 1.6.42).

Notation 1.6.36 ([Oni] §6)**.** Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and τ and $\tilde{\tau}$ Lie algebra endomorphisms of $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Suppose $f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. We denote by $\tau \uparrow_f \tilde{\tau}$ if $f\tau = \tilde{\tau} f$ holds.

First we prove (ii) $\Longrightarrow$ (i) of Propositions 1.6.31, 1.6.32, 1.6.33 and 1.6.34 by using Lemma 1.6.42.

Proof of (ii) $\Longrightarrow$ (i) of Propositions 1.6.31, 1.6.32, 1.6.33 and 1.6.34. From the assumption (ii), there exists an anti holomorphic involution $\tilde{\tau}$ on $\mathfrak{sl}(V)$ and holomorphic involution $\tilde{\omega}$ on $\mathfrak{sl}(V)$ such that $\tau \uparrow_\rho \tilde{\tau}$, $\mathrm{id}_{\mathfrak{g}} \uparrow_\rho \tilde{\omega}$ and

$$\mathfrak{sl}(V)^{\tilde{\tau}} \simeq \begin{cases} \mathfrak{sl}(n,\mathbb{R}) \text{ if } \mathrm{index}_\tau\, \rho = 1, \\ \mathfrak{su}^*(2\frac{n}{2}) \text{ if } \mathrm{index}_\theta\, \rho = -1, \end{cases} \quad \text{and } \mathfrak{sl}(V)^{\tilde{\omega}} \simeq \begin{cases} \mathfrak{so}(n,\mathbb{C}) \text{ if } \mathrm{index}_\theta\, \rho = 1, \\ \mathfrak{sp}(\frac{n}{2},\mathbb{C}) \text{ if } \mathrm{index}_\theta\, \rho = -1. \end{cases}$$

It is enough to show that $\tilde{\tau}\tilde{\omega} = \tilde{\omega}\tilde{\tau}$ from Fact 1.6.37 and the classification of simple Lie algebras. This comes from Lemma 1.6.42. $\square$

Fact 1.6.37 ([Oni, Proposition 1 in §6])**.** Let $\mathfrak{g}$ and $\mathfrak{h}$ be complex Lie algebras and τ and $\tilde{\tau}$ anti-holomorphic involutions on $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Let $f : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of complex Lie algebra. Then $f(\mathfrak{g}^\tau) \subset \mathfrak{h}^{\tilde{\tau}}$ if and only if $\tau \uparrow_f \tilde{\tau}$ holds.

We devote the remaining part of this subsection to showing Lemmma 1.6.42. We use the concept of "S-homomorphism" and Fact 1.6.39.

Definition 1.6.38 (see [Oni] for more details)**.** Let $f : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of complex Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. f is said to be a S-homomorphism if $\mathrm{id}_{\mathfrak{g}} \uparrow_f \varphi$ implies $\varphi = \mathrm{id}_{\mathfrak{h}}$ for any $\varphi \in \mathrm{Int}(\mathfrak{h})$.

Fact 1.6.39 ([Oni, §6, Lemma 1])**.** Let $\mathfrak{g}$ and $\mathfrak{h}$ be complex Lie algebras. Let $f : \mathfrak{g} \to \mathfrak{h} \subset \mathfrak{gl}(V)$ be an irreducible complex representation of $\mathfrak{g}$. Then f is a S-homomorphism. Conversely, if a representation $f : \mathfrak{g} \to \mathfrak{sl}(V)$ of a semisimple complex Lie algebra $\mathfrak{g}$ is an S-homomorphism, then f is irreducible.

A key lemma to prove Lemma 1.6.42 is the following:

Lemma 1.6.40. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and ρ a irreducible representation of $\mathfrak{g}$. Suppose $\tau : \mathfrak{g} \to \mathfrak{g}$ and $\tilde{\tau} : \mathfrak{sl}(V) \to \mathfrak{sl}(V)$ are holomorphic or anti-holomorphic and $\tau \uparrow_\rho \tilde{\tau}$. If τ is involutive, then so is $\tilde{\tau}$.

Proof of Lemma 1.6.40. If τ, $\tilde{\tau}$ are holomorphic, it is clear from $\mathrm{id}_{\mathfrak{g}} \uparrow_\rho \tilde{\tau}^2 \in \mathrm{Int}(\mathfrak{sl}(V))$ and Fact 1.6.39. So, we consider the case where τ and $\tilde{\tau}$ are anti holomorphic. Take Cartan involutions $\theta : \mathfrak{g} \to \mathfrak{g}$, $\tilde{\theta} : \mathfrak{sl}(V) \to \mathfrak{sl}(V)$ such that $\theta \uparrow_\rho \tilde{\theta}$ and $\tau\theta = \theta\tau$. It is enough to show that $\tilde{\tau}\tilde{\theta}\tilde{\tau}\tilde{\theta} = \mathrm{id}$ and $\tilde{\theta}\tilde{\tau} = \tilde{\tau}\tilde{\theta}$. These imply that $\tilde{\tau}^2 = \mathrm{id}$. In fact, $\mathrm{id} = \tilde{\tau}\tilde{\theta}\tilde{\tau}\tilde{\theta} = \tilde{\tau}\tilde{\theta}\tilde{\theta}\tilde{\tau} = \tilde{\tau}^2$.

- Let us show $\tilde{\tau}\tilde{\theta}\tilde{\tau}\tilde{\theta} = \mathrm{id}$. This follows from $\mathrm{id} = \tau\theta\tau\theta \uparrow_\rho \tilde{\tau}\tilde{\theta}\tilde{\tau}\tilde{\theta}$ and $(\tilde{\tau}\tilde{\theta})^2 \in \mathrm{Int}(\mathfrak{sl}(V))$.
- Let us show $\tilde{\tau}\tilde{\theta}\tilde{\tau}^{-1}\tilde{\theta} = \mathrm{id}$, which is equivalent to $\tilde{\theta}\tilde{\tau} = \tilde{\tau}\tilde{\theta}$.
  $\mathrm{id} = \tau\theta\tau^{-1}\theta \uparrow_\rho \tilde{\tau}\tilde{\theta}\tilde{\tau}^{-1}\tilde{\theta}$. So, it is enough to show $\tilde{\tau}\tilde{\theta}\tilde{\tau}^{-1}\tilde{\theta} = (\tilde{\tau}\tilde{\theta})(\tilde{\theta}\tilde{\tau})^{-1} \in \mathrm{Int}(\mathfrak{sl}(V))$. This follows from Remark 1.6.41 and the structure of $\mathrm{Aut}(\mathfrak{sl}(V))/\,\mathrm{Int}(\mathfrak{sl}(V))$.

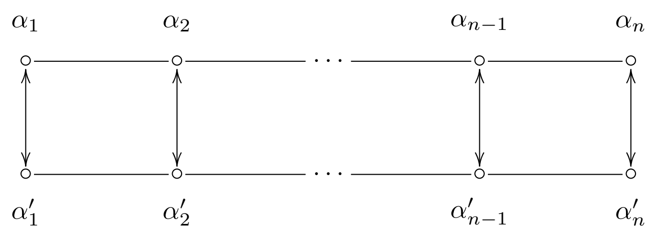
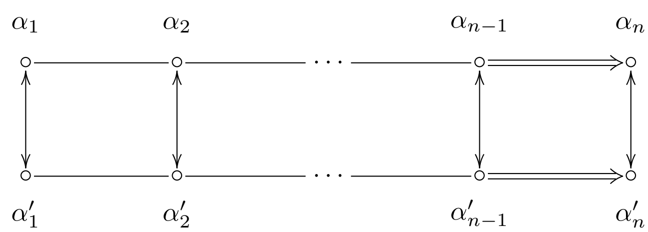
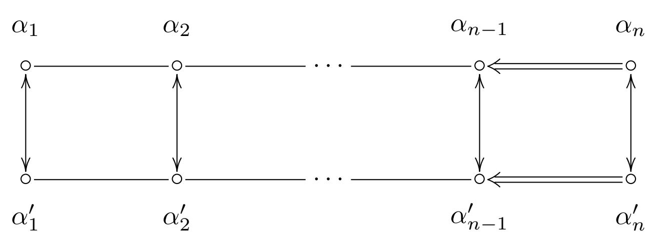
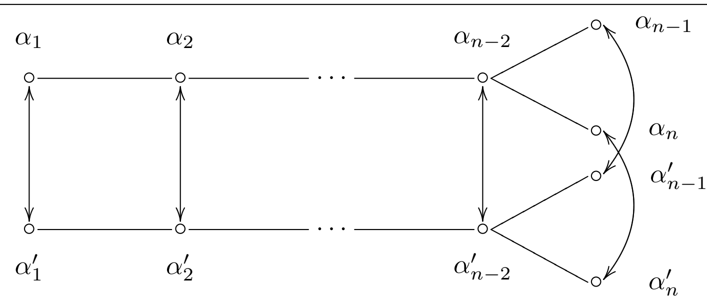
□

Remark 1.6.41. Let V be a complex vector space. Suppose α and $\beta \in \mathrm{Aut}_{\mathbb{R}}(\mathfrak{sl}(V))$ are anti-holomorphic. Then $\alpha\beta \in \mathrm{Int}(\mathfrak{sl}(V))$ implies $\beta\alpha \in \mathrm{Int}(\mathfrak{sl}(V))$,

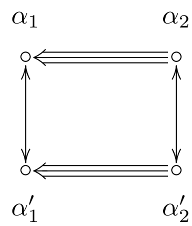
Lemma 1.6.42. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and ρ an irreducible representation of $\mathfrak{g}$. Suppose $\tau_i : \mathfrak{g} \to \mathfrak{g}$ and $\tilde{\tau}_i : \mathfrak{sl}(V) \to \mathfrak{sl}(V)$ are holomorphic or anti-holomorphic homomorphisms such that $\tau_i \uparrow \tilde{\tau}_i$ $(i = 1, 2)$. If τ_1 and τ_2 are commutative involutions on $\mathfrak{g}$, then $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are commutative involutions on $\mathfrak{sl}(V)$.

Proof. From Lemma 1.6.40, we get ${\tilde{\tau}_1}^2 = \mathrm{id}$, ${\tilde{\tau}_2}^2 = \mathrm{id}$. By applying Lemma 1.6.40 to an involution $\tau_1\tau_2 = \tau_2\tau_1$, we obtain $(\tilde{\tau}_1\tilde{\tau}_2)^2 = \mathrm{id}$. Since $\tilde{\tau}_1$, $\tilde{\tau}_2$ are involutive, $\tilde{\tau}_1\tilde{\tau}_2 = \tilde{\tau}_2\tilde{\tau}_1$ holds. □

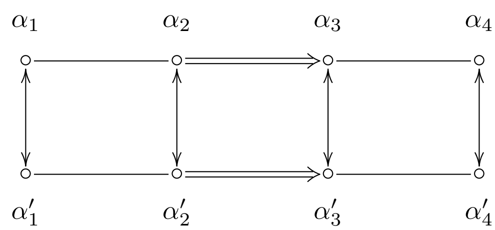
Table 1.33: Iwahori diagrams of simple Lie algebras

$\mathfrak{sl}(n + 1, \mathbb{C})$ $(n \geq 1)$	
$\mathfrak{so}(2n + 1, \mathbb{C})$ $(n \geq 1)$	
$\mathfrak{sp}(n, \mathbb{C})$ $(n \geq 2)$	
$\mathfrak{so}(2n, \mathbb{C})$ $(n \geq 4)$	

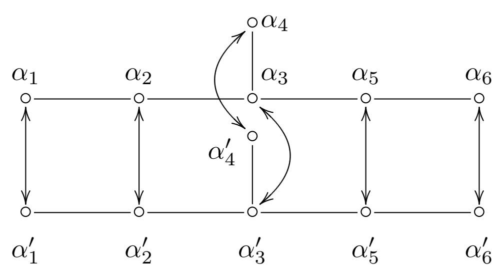
$\mathfrak{g}_2^{\mathbb{C}}$



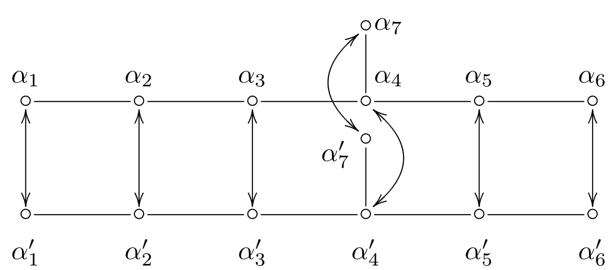
$\mathfrak{f}_4^{\mathbb{C}}$



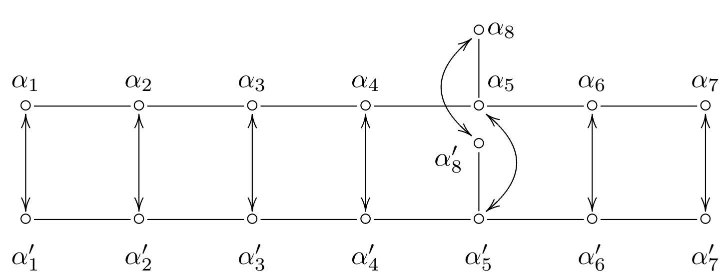
$\mathfrak{e}_6^{\mathbb{C}}$



$\mathfrak{e}_7^{\mathbb{C}}$

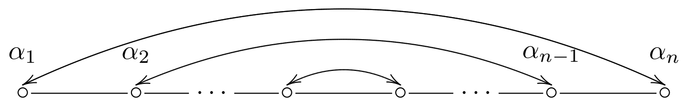


$\mathfrak{e}_8^{\mathbb{C}}$

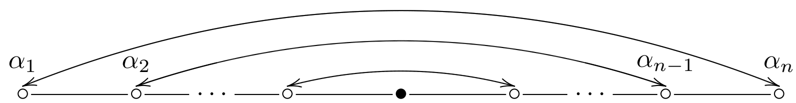


compact Lie algebra

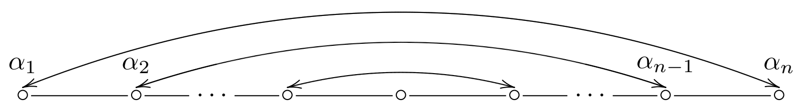
$\mathfrak{su}(n+1)$
n is even



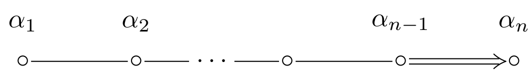
$\mathfrak{su}(n+1)$
$n \equiv 1 \pmod 4$



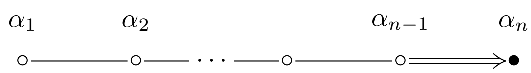
$\mathfrak{su}(n+1)$
$n \equiv 3 \pmod 4$



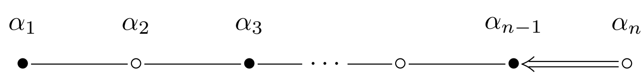
$\mathfrak{so}(2n+1)$
$n \equiv 0$ or $3 \pmod 4$



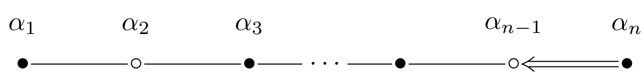
$\mathfrak{so}(2n+1)$
$n \equiv 0$ or $3 \pmod 4$



$\mathfrak{sp}(n)$
n is even



$\mathfrak{sp}(n)$
n is odd

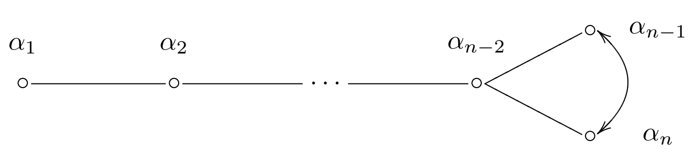


$\circ$ and $\bullet$ appear alternately

$\mathfrak{so}(2n)$ $(n \geq 4)$
$n \equiv 0 \pmod 4$

$\mathfrak{so}(2n)$
$n \equiv 2 \pmod 4$

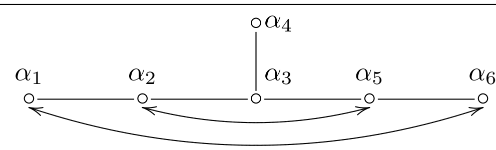
$\mathfrak{so}(2n)$
$n \equiv 1$ or $3 \pmod 4$



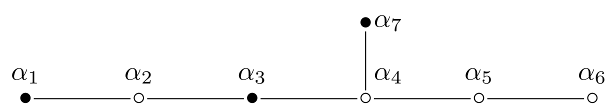
$\mathfrak{g}_2$

$\mathfrak{f}_4$

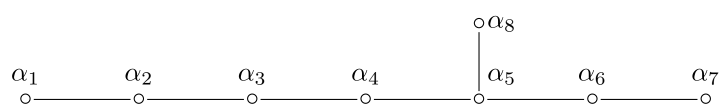
$\mathfrak{e}_6$



$\mathfrak{e}_7$



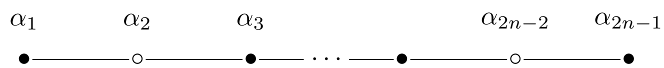
$\mathfrak{e}_8$



non-compact simple Lie algebra

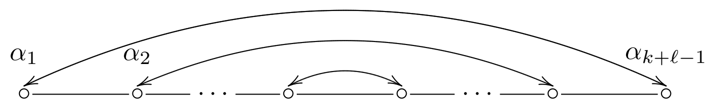
$\mathfrak{sl}(n+1,\mathbb{R})$
$(n \geq 1)$

$\mathfrak{su}^*(2n)$
n is even

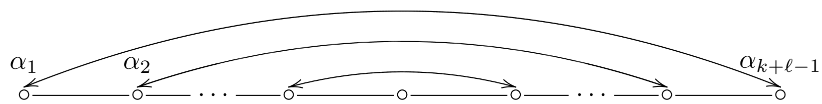


○ and ● appear alternately

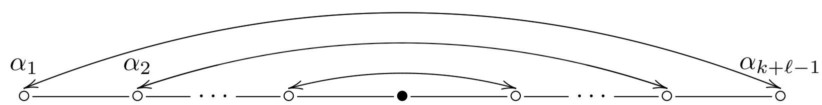
$\mathfrak{su}(k,\ell)$ $(1 \leq \ell \leq k)$
$k+\ell$ is odd



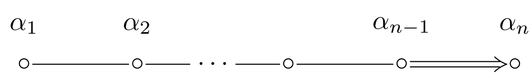
$\mathfrak{su}(k,\ell)$ $(1 \leq \ell \leq k)$
$k+\ell$ is even,
$k-\ell \equiv 0 \pmod 4$



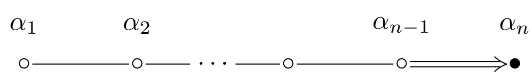
$\mathfrak{su}(k,\ell)$ $(1 \leq \ell \leq k)$
$k+\ell$ is even,
$k-\ell \equiv 2 \pmod 4$



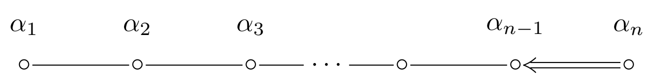
$\mathfrak{so}(k,\ell)$ $(1 \leq \ell \leq k)$
$k-\ell \equiv 1$ or $7 \pmod 8$
$5 \leq k+\ell = 2n+1$



$\mathfrak{so}(k,\ell)$ $(1 \leq \ell \leq k)$
$k-\ell \equiv 3$ or $5 \pmod 8$

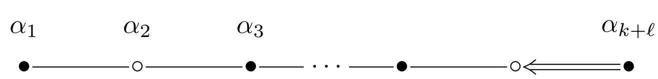


$\mathfrak{sp}(n,\mathbb{R})$



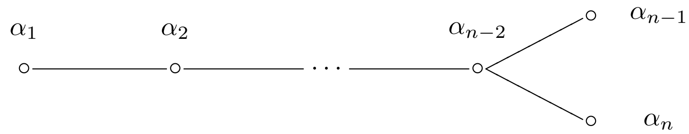
$\mathfrak{sp}(k,\ell)$ $(1 \leq \ell \leq k)$
$k+\ell$ is even

$\mathfrak{sp}(k,\ell)$ $(1 \leq \ell \leq k)$
$k+\ell$ is odd

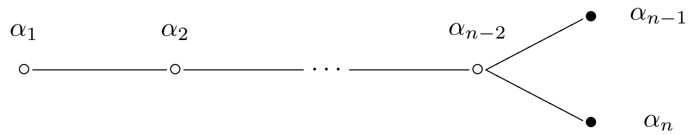


○ and ● appear alternately

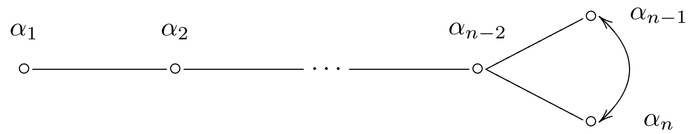
$\mathfrak{so}(k,\ell)$ $(1 \leq \ell \leq k)$
$8 \leq k+\ell = 2n$ is even,
$k-\ell \equiv 0 \pmod 8)$



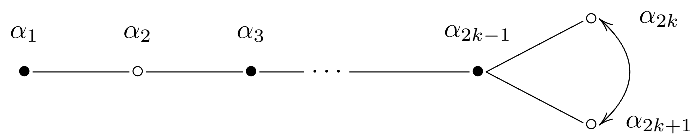
$\mathfrak{so}(k,\ell)$ $(1 \leq \ell \leq k)$
$8 \leq k+\ell = 2n$ is even,
$k-\ell \equiv 4 \pmod 8)$



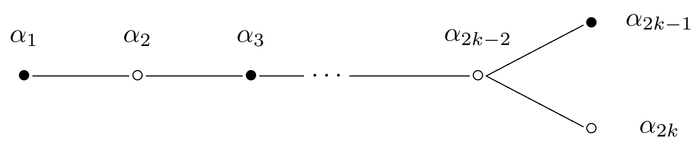
$\mathfrak{so}(k,\ell)$ $(1 \leq \ell \leq k)$
$8 \leq k+\ell = 2n$ is even,
$k-\ell \equiv 2$ or $6 \pmod 8)$



$\mathfrak{so}^*(2(2k+1))$ $(2 \leq k)$,



$\mathfrak{so}^*(2(2k))$ $(2 \leq k)$,

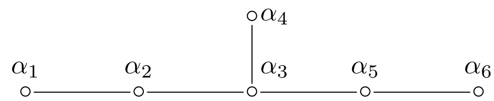


$\mathfrak{g}_{2(2)}$

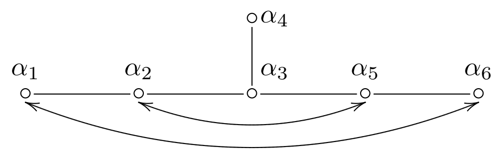
$\mathfrak{f}_{4(4)}$

$\mathfrak{f}_{4(-20)}$

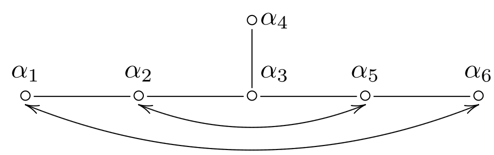
$\mathfrak{e}_{6(6)}$



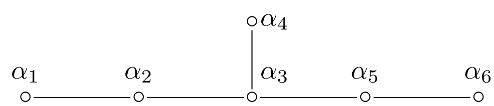
$\mathfrak{e}_{6(2)}$



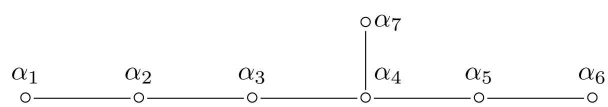
$\mathfrak{e}_{6(-14)}$



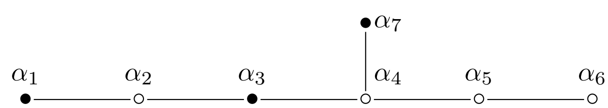
$\mathfrak{e}_{6(-26)}$



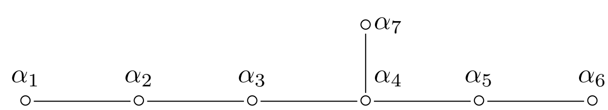
$\mathfrak{e}_{7(7)}$



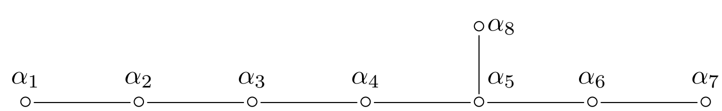
$\mathfrak{e}_{7(-5)}$



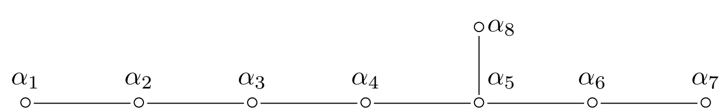
$\mathfrak{e}_{7(-25)}$



$\mathfrak{e}_{8(8)}$



$\mathfrak{e}_{8(-24)}$



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Chapter 2

Obstruction for the existence of tangential symmetric spaces

Abstract

For a homogeneous space G/H of reductive type, we consider the tangential homogeneous space G_θ/H_θ. In this paper, we give obstructions for the existence of compact Clifford–Klein forms for such tangential symmetric spaces and obtain new tangential symmetric spaces which do not admit compact Clifford–Klein forms. As a result, in the class of irreducible semisimple symmetric spaces, we have only three types of symmetric spaces which are not proved not to admit compact Clifford–Klein forms.

The existence problem of compact Clifford–Klein forms for homogeneous spaces of reductive type, which was initiated by T. Kobayashi in 1980's, has been studied by various methods but is not completely solved yet. On the other hand, one for tangential homogeneous spaces has been studied since 2000's and a criterion was already obtained by T. Yoshino. Our obstructions for the existence of compact Clifford–Klein forms for tangential symmetric spaces depend on the criterion and are related to various fields of Mathematics such as associated pair of symmetric space, Calabi-Markus phenomenon, trivializability of vector bundle (parallelizability, Pontrjagin class), Hurwitz–Radon number and Pfister's theorem (the existence problem of common zero points of polynomials of odd degree).

2.1 Introduction and Main results

In this paper, we give some obstructions for the existence of compact Clifford–Klein forms of tangential symmetric spaces and obtain new examples which do not admit compact Clifford–Klein forms.

Let G be a Lie group and H a closed subgroup of G. Geometry of Clifford–Klein forms has been enriched by the following:

Open Problem 2.1.1 ([Ko96b, Problem 1.7(2)])**.** When does G/H admit compact Clifford–Klein forms?

This is still open even if we restrict the problem to semisimple irreducible symmetric spaces, although M. Berger [Br57] classified them. A systematical study was initiated and Open Problem 2.1.1 was raised by T. Kobayashi in 1980's. These results are summarized in the papers [Ko96b, 16, KY05].

In this paper, we consider Problem 2.1.1 for tangential symmetric spaces G_θ/H_θ (See Definition 2.2.3) corresponding to semisimple symmetric pairs (G, H).

Problem 2.1.2 (tangential case)**.** Classify semisimple irreducible symmetric spaces G/H with regard to whether or not the corresponding tangential symmetric spaces G_θ/H_θ admit compact Clifford–Klein forms.

For Problem 2.1.2, the following Fact 2.1.3, 2.1.4 and 2.1.8 are known as partial solutions:

Fact 2.1.3 ([19, Theorem 3])**.** Let G/H be an irreducible symmetric space, and G a complex reductive Lie group. G_θ/H_θ has a compact Clifford–Klein form if and only if G/H is locally isomorphic to one of the following list:

- a Riemannian symmetric spaces G/K,
- Group manifolds $(G \times G)/(\mathrm{diag}_\tau G)$, where we put $\mathrm{diag}_\tau G := \{(g, \tau(g)) : g \in G\} \subset G \times G$ for each involution τ on G,
- $SO(8, \mathbb{C})/SO(7, \mathbb{C})$,
- $SO(8, \mathbb{C})/SO_0(7, 1)$.

The unpublished paper [19] will be published.

Fact 2.1.4 ([KY05, Proposition 5.5.1])**.** The following conditions on the pair (p, q) of positive integers are equivalent:

(i) The tangential symmetric space of $SO_0(p, q+1)/SO_0(p, q)$ admits a compact Clifford–Klein form.

(ii) $q < \rho(p, \mathbb{R})$.

Here, $\rho(p, \mathbb{R})$ is Hurwitz-Randon number (see Definition 2.7.7).

The main parts of the above two facts are non-existence results for tangential symmetric space. On the other hand, existence results are also known. To state it, we introduce the notion of standard Clifford–Klein form:

Definition 2.1.5 ([11, Definition 1.4])**.** Let G be a linear reductive Lie group. A Clifford–Klein form $\Gamma\backslash G/H$ of G/H is standard if Γ is contained in some reductive subgroup L of G acting properly on G/H.

Remark 2.1.6. If there exists a standard compact Clifford–Klein form of G/H of reductive type, then its tangential homogeneous space G_θ/H_θ admits a compact Clifford–Klein form. That is, non-existence of compact Clifford–Klein forms of G_θ/H_θ implies non-existence of standard compact Clifford–Klein forms of homogeneous space G/H of reductive type. See Section 2.2.2 for more details.

Remark 2.1.7. Any tangential symmetric spaces associated with Riemannian symmetric spaces G/K and group manifolds $(G \times G)/(\mathrm{diag}_\tau G)$ admit standard compact Clifford–Klein forms. Therefore, for Problem 2.1.2, we focus on the case where G is simple and H is not compact.

Fact 2.1.8 ([KY05, Corollary 3.3.7])**.** Let (G, H) be a symmetric pair which is locally isomorphic to one in Table 2.1 and suppose that G is connected. Then the tangential symmetric space G_θ/H_θ corresponding to symmetric space G/H in the following table admits compact Clifford–Klein forms.

Table 2.1: Symmetric pairs (G, H) which admit compact standard Clifford–Klein forms.

G	H	L	G	H	L
$SO_0(2,2n)$	$SO_0(1,2n)$	$U(1,n)$	$SU(2,2n)$	$U(1,2n)$	$Sp(1,n)$
$SO_0(4,4n)$	$SO_0(3,4n)$	$Sp(1,n)$	$SU(2,2n)$	$Sp(1,n)$	$U(1,2n)$
$SO_0(4,4)$	$SO_0(4,1) \times SO(3)$	$Spin(4,3)$	$SO(8,\mathbb{C})$	$SO(7,\mathbb{C})$	$Spin(1,7)$
$SO(4,3)_0$	$SO_0(4,1) \times SO(2)$	$G_{2(2)}$	$SO(8,\mathbb{C})$	$SO(7,1)$	$Spin(7,\mathbb{C})$
$SO_0(8,8)$	$SO_0(7,8)$	$Spin(1,8)$	$SO^*(8)$	$SO^*(6) \times SO^*(2)$	$Spin(1,6)$
$SO_0(2,2n)$	$U(1,n)$	$SO_0(1,2n)$	$SO^*(8)$	$U(3,1)$	$Spin(1,6)$

Here L is a reductive subgroup of G acting on G/H properly and cocompactly.

Remark 2.1.9. For a symmetric pair (G, H), both of implications between the conditions "Existence of compact Clifford–Klein forms for G/H" and the condition "Existence of compact Clifford–Klein forms for G_θ/H_θ" have not been proved in the existence literatures.

In this paper, we give new examples which do not admit compact compact Clifford–Klein forms in the class of irreducible semisimple tangential symmetric spaces. To show the non-existence of compact Clifford–Klein forms of tangential symmetric spaces, it is enough to consider symmetric spaces up to associated pairs. This is one of the reasons why Our Problem 2.1.2 is easier to deal with than the case when G/H is of reductive type. We see it in Proposition 2.2.10 in the following section.

We use the following five methods to give necessary conditions for the existence of compact Clifford–Klein forms of tangential symmetric spaces (Theorem 2.3.1, 2.4.1, 2.5.1, 2.6.1):

(i) Calabi-Markus phenomenon,

(ii) Applications of Pfister's theorem.

(iii) Maximality of non-compactness,

(iv) Non-triviality of symmetric spaces as vector bundles,

(v) Applications of Adams's theorem,

The following tangential symmetric spaces G_θ/H_θ are typical examples which are proved not to admit compact Clifford–Klein forms by each method:

(i) $SL(n,\mathbb{C})_\theta/SL(n,\mathbb{R})_\theta$ $(n \geq 2)$,

(ii) $SL(2n,\mathbb{R})_\theta/Sp(n,\mathbb{R})_\theta$ $(n \geq 2)$.

(iii) $SO_0(p_1+p_2,q_1+q_2)_\theta/(SO_0(p_1,q_1)\times SO_0(p_2,q_2))_\theta$ $(0 < p_1 \leq p_2, q_1, q_2)$,

(iv) $SO_0(2p,2q)_\theta/U(p,q)_\theta$ $(2 \leq p \leq q)$,

(v) $SU(p,2)_\theta/U(p,1)_\theta$ (p is odd),

Lists of non-existence results obtained by each method shall be given in the corresponding section.

Theorem 2.1.10. Let (G, H) be a symmetric pair which is locally isomorphic to one in Table 2.2 and suppose that G is connected. Then the tangential symmetric space G_θ/H_θ does not admit compact Clifford–Klein forms:

Table 2.2: Symmetric pairs whose tangential symmetric spaces G_θ/H_θ do not admit compact Clifford–Klein forms.

G	H	G	H
$SL(p+q,\mathbb{C})$ $(p,q\geq 1)$	$S(GL(p,\mathbb{C})\times GL(q,\mathbb{C}))$ $SU(p,q)$	$Sp(p+q,\mathbb{C})$ $(p,q\geq 1)$	$Sp(p,\mathbb{C})\times Sp(q,\mathbb{C})$ $Sp(p,q)$
$SL(n,\mathbb{C})$ $(n\geq 2)$	$SL(n,\mathbb{R})$ $SO(n,\mathbb{C})$	$Sp(n,\mathbb{C})$ $(n\geq 1)$	$Sp(n,\mathbb{R})$ $GL(n,\mathbb{C})$
$SL(p+q,\mathbb{R})$ $(p,q\geq 1)$	$S(GL(p,\mathbb{R})\times GL(q,\mathbb{R}))$ $SO_0(p,q)$	$Sp(p+q,\mathbb{R})$ $(p,q\geq 1)$	$Sp(p,\mathbb{R})\times Sp(q,\mathbb{R})$ $U(p,q)$
$SU(p,q)$ $(p,q\geq 1)$	$SO_0(p,q)$	$Sp(p,q)$ $(p,q\geq 1)$	$U(p,q)$
$SU(n,n)$ $(n\geq 1)$	$GL^{\pm}(n,\mathbb{C})$	$SU(n,n)$ $(n\geq 2)$	$Sp(n,\mathbb{R})$ $SO^*(2n)$
$SU^*(2n)$ $(n\geq 2)$	$S'L(n,\mathbb{C})$ $SO^*(2n)$	$SO(2n,\mathbb{C})$ $(n\geq 2)$	$GL(n,\mathbb{C})$ $SO^*(2n)$
$SU^*(2(p+q))$ $(p,q\geq 1)$	$S(U^*(2p)\times U^*(2q))$ $Sp(p,q)$	$Sp(n,\mathbb{R})$ $(n\geq 1)$	$GL(n,\mathbb{R})$
$SO_0(n,n)$ $(n\geq 1)$	$GL(n,\mathbb{R})$ $SO(n,\mathbb{C})$	$Sp(n,n)$ $(n\geq 1)$	$U^*(2n)$ $Sp(n,\mathbb{C})$
$SO^*(2n)$ $(n\geq 2)$	$SO(n,\mathbb{C})$	$SO^*(4n)$ $(n\geq 1)$	$U^*(2n)$

$GL^{\pm}(n,\mathbb{C})$ is the subgroup of $SU(n,n)$, which have the following realization:

$$SU(n,n) = \{g \in GL(2n,\mathbb{C}) : g^* \begin{pmatrix} & I_n \\ I_n & \end{pmatrix} g = \begin{pmatrix} & I_n \\ I_n & \end{pmatrix}\},$$
$$GL^{\pm}(n,\mathbb{C}) = \{g \in SU(n,n) : gI_{n,n} = I_{n,n}g\}.$$

$SL'(n,\mathbb{C})$ is the subgroup of $SU^*(2n)$, which have the following realization:

$$SU^*(2n) = \{g \in SL(2n,\mathbb{C}) : \overline{g}J_n = J_n g\},$$
$$S'L(n,\mathbb{C}) = \{g \in SU^*(2n) : \overline{g} = g\}.$$

See Proposition 2.3.4 for the proof.

Theorem 2.1.11. Let (G, H) be a symmetric pair, where G is a connected linear reducitive Lie group. Suppose corresponding symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is one of the Table 2.5' or Table 2.5". Then G_θ/H_θ does not admit compact Clifford–Klein forms.

See Proposition 2.3.5 for the proof.

Theorem 2.1.12. Let $(G,\ H)$ be a symmetric pair, where G is a connected linear reductive Lie group. Suppose corresponding symmetric pair $(\mathfrak{g},\ \mathfrak{h})$ is one of the following table. Then G_θ/H_θ does not admit compact Clifford–Klein forms.

Table 2.3: symmetric pairs whose corresponding tangential symmetric space G_θ/H_θ does not admit compact Clifford–Klein forms

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{e}_6^{\mathbb{C}}$	$\mathfrak{f}_4^{\mathbb{C}}$
	$\mathfrak{e}_{6(-26)}$
$\mathfrak{e}_{6(2)}$	$\mathfrak{so}^*(10) \oplus \mathfrak{u}(1)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{su}^*(6) \oplus \mathfrak{su}(2)$
	$\mathfrak{f}_{4(4)}$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$
$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(2) \oplus \mathfrak{so}^*(12)$
	$\mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2)$
$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2)$

See Section 2.6 for the proof.

Theorem 2.1.13. Let $(G,\, H)$ be a symmetric pair which is locally isomorphic to one in Table 2.4 and suppose that G is connected. Then the tangential symmetric space G_θ/H_θ does not admit compact Clifford–Klein froms.

Table 2.4: symmetric pairs (G,H) whose tangential symmetric spaces G_θ/H_θ do not admit compact Clifford–Klein forms.

G	H	condition
$SO^*(2(p+q))$	$SO^*(2p)\times SO^*(2q)$ $U(p,q)$ $(1\le p\le q)$	$p\ge 2$ or ($q\ne 1$ and $q\ne 3$)
$SO(p+q,\mathbb{C})$	$SO(p,\mathbb{C})\times SO(q,\mathbb{C})$ $SO_0(p,q)$ $(1\le p\le q)$	$(p,q)\ne(1,1),(1,3),(1,7)$
$SO_0(p,q)$	$SO_0(p_1,q_1)\times SO_0(p_2,q_2)$ $(0\le p_1\le p_2, q_1,q_2\ge 1)$	$p_1\ge 1$ or ($q_1\ge 2$ and $q_2\ge 2$)
$SU(p,q)$	$S(U(p_1,q_1)\times U(p_2,q_2))$ $(0\le p_1\le p_2, q_1,q_2\ge 1)$	$p_1\ge 1$ or $q_1\ge 2$ or $q_2\ge 2$ or p_2 is odd.
$Sp(p,q)$	$Sp(p_1,q_1)\times Sp(p_1,p_2)$ $(0\le p_1\le p_2, q_1,q_2\ge 1)$	$0\le p_1\le p_2, q_1,q_2\ge 1$
$SL(2n,\mathbb{C})$	$Sp(n,\mathbb{C})$ $(n\ge 2)$ $SU^*(2n)$	$n\ge 2$
$SL(2n,\mathbb{R})$	$Sp(n,\mathbb{R})$ $(n\ge 2)$ $S'L(n,\mathbb{C})$	$n\ge 2$
$SO_0(2p,2q)$	$U(p,q)$ $(1\le p\le q)$	$p\ge 2$

Here, $S'L(n,\mathbb{C})$ is a subgroup of $SL(2n,\mathbb{R})$ realized as follows:

$$S'L(n,\mathbb{C}) := \{g\in SL(2n,\mathbb{R}) : gJ_n = J_n g\},\ J_n := \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}.$$

Proof. Theorem 2.1.13 follows from Proposition 2.3.3, 2.4.3, 2.5.4, 2.6.5 and 2.7.1. □

Remark 2.1.14. From the above theorems, we reached the complete classification of tangential symmetric spaces associated with irreducible semisimple symmetric spaces which admit compact Clifford–Klein forms except for three types. In the class of irreducible semisimple symmetric spaces, for the following semisimple symmetric pairs (G,H), corresponding tangential symmetric spaces G_θ/H_θ was not proved not to admit compact Clifford–Klein forms.

- $(Sp(2n,\mathbb{R}), Sp(n,\mathbb{C}))$ $(n\ge 2)$,
- $(SU(2p,2q), Sp(p,q))$ $(2\le p,q)$,
- $(E_{6(-14)}, F_{4(-20)})$.

Remark 2.1.15. $Sp(2,\mathbb{R})_\theta/Sp(1,\mathbb{C})_\theta$ does not admit compact Clifford–Klein forms. This comes from that symmetric pairs $(\mathfrak{sp}(2,\mathbb{R}),\mathfrak{sp}(1,\mathbb{C}))$ and $(\mathfrak{so}(3,2),\mathfrak{so}(3,1))$ are isomorphic to each other and that $SO_0(3,2)_\theta/SO_0(3,1)_\theta$ do not admit compact Clifford–Klein forms (see Fact 2.1.4).

2.2 Preliminary

In this chapter, we consider tangential symmetric spaces G_θ/H_θ associated with irreducible semisimple symmetric space G/H. We prepare the precise setting and notions in Subsection 2.2.1. In the next subsection, we review a criterion for the existence of compact Clifford–Klein forms for tangential homogeneous spaces given by [KY05]. In Subsection 2.2.3, we see that it is enough to consider symmetric spaces up to associated pair for our problem.

2.2.1 Setting and Notation

Throughout this paper, unless otherwise noted, we assume that G is a linear reductive and connected semisimple Lie group and that H is an open subgroup of $G^\sigma := \{g \in G : \sigma g = g\}$, where σ is the involution determining symmetric pair. Then, the symmetric space G/H is of reductive type ([Ko96b, Example 2.6.3]).

Remark 2.2.1. The existence problem of compact Clifford–Klein forms for tangential homogeneous spaces associated with homogeneous spaces of reductive type depends only on the set of orbits $\mathrm{Ad}(K)\mathfrak{p}_H$ of the adjoint action of the maximal compact subgroup K of G on $\mathfrak{p}$ (see Fact 2.2.5). So, we can assume that H is the identity component of G^σ for our purpose.

Now, we recall the definition of a tangential homogeneous space G_θ/H_θ for a homogeneous space G/H of reductive type.

Definition 2.2.2 (Cartan motion group, See [KY05, Subsection 5.1])**.** Let θ be a Cartan involution of G. The Cartan motion group G_θ of G is defined by

$$G_\theta := K \ltimes_{\mathrm{Ad}} \mathfrak{p}.$$

Here $K = G^\theta$ is a maximal compact Lie subgroup of G and $\mathfrak{p} = \mathfrak{g}^{-\theta}$.

Let G/H be a homogeneous space of reductive type, then we can take a Cartan involution θ of G such that $\theta|_H$ is also a Cartan involution of H. Then we get a closed subgroup $H_\theta := K_H \ltimes \mathfrak{p}_H$ of G_θ where $K_H = K \cap H$ and $\mathfrak{p}_H = \mathfrak{p} \cap \mathfrak{h}$.

Definition 2.2.3 ([KY05, Definition 5.1.2])**.** We call $(G/H)_\theta := G_\theta/H_\theta$ the *tangential homogeneous space* of G/H.

Remark 2.2.4. If (G, H) is a symmetric pair, then so is (G_θ, H_θ).

2.2.2 Tangential analogue of Kobayashi's criterion

By the following fact, the existence problem of compact Clifford–Klein form for a tangential homogeneous space G_θ/H_θ reduces to how large subspace of $\mathfrak{p}$ satisfying condition Fact 2.2.5 (ii) we can take.

Fact 2.2.5 ([KY05, Theorem 5.3.2])**.** Let G_θ/H_θ be a tangential homogeneous space of a homogeneous space G/H of reductive type. Then, the following two conditions are equivalent:

(i) The homogeneous space G_θ/H_θ admits compact Clifford–Klein forms.

(ii) There exists a subspace W in $\mathfrak{p}$ satisfying the following two conditions (a) and (b).

  (a) $\mathfrak{a}(W) \cap \mathfrak{a}(H) = \{0\}$,

  (b) $\dim W + d(H) = d(G)$.

  Here, $\mathfrak{a}$ is a fixed maximally abelian subspace of $\mathfrak{p}$ and $d(G) = \dim \mathfrak{p}$, $d(H) = \dim \mathfrak{p}_H$ are non-compact dimension of G, H respectively ([Ko89]). For a subset L in the Cartan motion group $G = K \ltimes \mathfrak{p}$, we put $\mathfrak{a}(L) := KLK \cap \mathfrak{a}$.

Remark 2.2.6. In Fact 2.2.5, the condition (ii)(a) is equivalent to the following condition (ii)(a'):

$$\text{(ii)(a')}\ W \cap \mathrm{Ad}(K)\mathfrak{p}_H = \{0\}.$$

Proof of Remark 2.2.6. We have $\mathfrak{a}(H) = \mathfrak{a} \cap \mathrm{Ad}(K)\mathfrak{p}_H$ and $\mathfrak{a}(W) = \mathfrak{a} \cap \mathrm{Ad}(K)W$. Therefore, we have

$$\begin{aligned}\mathfrak{a}(W) \cap \mathfrak{a}(H) &= \mathfrak{a} \cap \mathrm{Ad}(K)W \cap \mathrm{Ad}(K)\mathfrak{p}_H \\ &= \mathfrak{a} \cap \mathrm{Ad}(K)(W \cap \mathrm{Ad}(K)\mathfrak{p}_H) \\ &= \mathfrak{a}(W \cap \mathrm{Ad}(K)\mathfrak{p}_H).\end{aligned}$$

Thus, Remark 2.2.6 follows from the observation that for a subset X of $\mathfrak{p}$ containing 0, $X = \{0\}$ holds if and only if $\mathfrak{a}(X) = \{0\}$ holds. □

Let us see Remark 2.1.6 in detail. The implication in Remark 2.1.6 comes from the following Fact 2.2.7, 2.2.8 by taking $\mathfrak{p}_L$ as W in Fact 2.2.5.

Fact 2.2.7 ([Ko89, Theorem 4.1])**.** Let H, L be reductive subgroups of a real reductive linear group G. Then the following conditions on H, L are equivalent:

(i) The L-action on G/H is proper,

(ii) $\mathrm{Ad}(K)\mathfrak{p}_H \cap \mathfrak{p}_L = \{0\}$.

Fact 2.2.8 ([Ko89, Theorem 4.7])**.** Let H, L be reductive subgroups of a real reductive linear group G. Under the conditions in Fact 2.2.7, the following conditions are equivalent:

(i) The double coset space $L\backslash G/H$ is compact,

(ii) $d(G) = d(H) + d(L)$.

2.2.3 Associated pair

In this subsection, we show that the existence problem of compact Clifford–Klein forms for the tangential symmetric space corresponding to a symmetric pair (G, H) is equivalent to that for the tangential symmetric pair corresponding to (G, H^a), where (G, H^a) is the associated pair of (G, H) defined as follows:

Let σ be an involution of G which define the symmetric pair (G, H). We take a Cartan involution θ of G satisfying $\theta \circ \sigma = \sigma \circ \theta$. Then $\theta|_H$ is also a Cartan involution of H and $\theta \circ \sigma$ is also an involution of G.

Definition 2.2.9 ([20])**.** We call the symmetric pair (G, H^a) defined by $\theta \circ \sigma$ the *associated pair* of (G, H).

By the definition of θ and σ, one can easily see that the associated pair of (G, H^a) is (G, H).

Proposition 2.2.10 ([19, Theorem 20])**.** Let (G, H) be a semisimple symmetric pair and (G, H^a) the associated pair of (G, H). Then G_θ/H_θ admits compact Clifford–Klein forms if and only if G_θ/H^a_θ admits compact Clifford–Klein forms.

Proof of Proposition 2.2.10. It is enough to show "only if" part. Let B be the restriction on $\mathfrak{p}$ of the Killing form on $\mathfrak{g}$. Take a subspace V in $\mathfrak{p}$ such that $\dim V = d(G) - d(H) = d(H^a)$ and $\mathrm{Ad}(K)\mathfrak{p}_H \cap V = \{0\}$. By taking the orthogonal complement of V, we obtain the subspace $V^\perp$ in $\mathfrak{p}$ satisfying the two conditions, $\dim V^\perp = d(G) - d(H^a)$ and $\mathrm{Ad}(K)\mathfrak{p}_{H^a} \cap V^\perp = \{0\}$, which follow from the fact that the representation Ad is unitary and orthogonal complement of $\mathfrak{p}_H$ is $\mathfrak{p}_{H^a}$ with regard to B. $\square$

Remark 2.2.11. Proposition 2.2.10 holds for a symmetric pair of reductive type without semisimplicity. We can prove it in the same way by taking a $\mathrm{Ad}(G)$-invariant and $d\sigma$-invariant inner product on $\mathfrak{p}$.

2.3 Calabi-Markus phenomenon

In this section, we see that a necessary condition of the existence of compact Clifford–Klein forms of homogeneous spaces G/H of reductive type is also one of tangential homogeneous spaces G_θ/H_θ.

Theorem 2.3.1. If a homogeneous space G/H of reductive type satisfies that $\mathrm{rank}_\mathbb{R} G = \mathrm{rank}_\mathbb{R} H$ and G/H is non-compact, then its tangential homogeneous space G_θ/H_θ does not admit compact Clifford–Klein forms.

Proof. This comes from Fact 2.2.5 and the fact that the condition $\mathrm{rank}_\mathbb{R} G = \mathrm{rank}_\mathbb{R} H$ implies $\mathrm{Ad}(K)\mathfrak{p}_H = \mathfrak{p}$. $\square$

Remark 2.3.2. The condition $\mathrm{rank}_\mathbb{R} G = \mathrm{rank}_\mathbb{R} H$ is a criterion of the Calabi-Markus phenomenon [Ko89, Corollary (4.4)] for a homogeneous space G/H of reductive type. By a similar argument to the reductive case, we can see that only a finite subgroup of G_θ can acts properly on G_θ/H_θ if the condition $\mathrm{rank}_\mathbb{R} G = \mathrm{rank}_\mathbb{R} H$ is satisfied.

Proposition 2.3.3. Let $(G,\ H)$ and $(G,\ H^a)$ be symmetric pairs which are locally isomorphic to one of the following list and suppose that G is connected. Then neither G_θ/H_θ nor G_θ/H_θ^a admit compact Clifford–Klein forms.

- $(G, H, H^a) = (SO^*(2(p+q)), SO^*(2p) \times SO^*(2q), U(p,q))$
- $(G, H, H^a) = (SO(p+q, \mathbb{C}), SO(p, \mathbb{C}) \times SO(q, \mathbb{C}), SO_0(p,q))$

Here, p and q are positive integers and p or q is even.

Proof. This follows form the Fact 2.3.1 and $\operatorname{rank}_{\mathbb{R}} G - \operatorname{rank}_{\mathbb{R}} H = \lfloor \frac{p+q}{2} \rfloor - (\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor)$. □

For a semisimple irreducible symmetric pair (G, H), we consider the following two conditions A and B.

A : $\operatorname{rank}_{\mathbb{R}} G = \operatorname{rank}_{\mathbb{R}} H$,

B : the associated pair satisfies the condition A.

Proposition 2.3.4. Let (G, H) be a symmetric pair which is locally isomorphic to one in Table 2.5 and suppose that G is connected. Then G_θ/H_θ does not admit compact Clifford–Klein forms.

Table 2.5: Symmetric pairs (G, H) satisfying A or B.

G	H	$\mathrm{rank}_{\mathbb{R}}\, G$	$\mathrm{rank}_{\mathbb{R}}\, H$	A or B
$SL(p+q,\mathbb{C})$	$S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$	$p+q-1$	$p+q-1$	A
$p,q \geq 1$	$SU(p,q)$		$\min(p,q)$	B
$SL(n,\mathbb{C})$	$SL(n,\mathbb{R})$	$n-1$	$n-1$	A
$n \geq 2$	$SO(n,\mathbb{C})$		$\lfloor \frac{n}{2} \rfloor$	B
$SL(p+q,\mathbb{R})$	$S(GL(p,\mathbb{R}) \times GL(q,\mathbb{R}))$	$p+q-1$	$p+q-1$	A
$p,q \geq 1$	$SO_0(p,q)$		$\min(p,q)$	B
$SO(2n,\mathbb{C})$	$GL(n,\mathbb{C})$	n	n	A
$n \geq 2$	$SO^*(2n)$		$\lfloor \frac{n}{2} \rfloor$	B
$SO_0(n,n)$	$GL(n,\mathbb{R})$	n	n	A
$n \geq 1$	$SO(n,\mathbb{C})$		$\lfloor \frac{n}{2} \rfloor$	B
$SU(n,n)$ $n \geq 1$	$GL^{\pm}(n,\mathbb{C})$	n	n	A
$SU(n,n)$	$Sp(n,\mathbb{R})$	n	n	A
$n \geq 1$	$SO^*(2n)$		$\lfloor \frac{n}{2} \rfloor$	B
$SU(p,q)$ $p,q \geq 1$	$SO_0(p,q)$	$\min(p,q)$	$\min(p,q)$	A
$Sp(n,n)$	$U^*(2n)$	n	n	A
$n \geq 1$	$Sp(n,\mathbb{C})$		n	A
$Sp(p,q)$ $p,q \geq 1$	$U(p,q)$	$\min(p,q)$	$\min(p,q)$	A
$Sp(p+q,\mathbb{C})$	$Sp(p,\mathbb{C}) \times Sp(q,\mathbb{C})$	n	n	A
$p,q \geq 1$	$Sp(p,q)$		$\min(p,q)$	B
$Sp(n,\mathbb{C})$	$Sp(n,\mathbb{R})$	n	n	A
$n \geq 1$	$GL(n,\mathbb{C})$		n	A
$Sp(n,\mathbb{R})$ $n \geq 1$	$GL(n,\mathbb{R})$	n	n	A
$Sp(p+q,\mathbb{R})$ $p,q \geq 1$	$Sp(p,\mathbb{R}) \times Sp(q,\mathbb{R})$	n	n	A
	$U(p,q)$		$\min(p,q)$	B
$SU^*(2n)$	$S'L(n,\mathbb{C})$	$n-1$	$n-1$	A
$n \geq 2$	$SO^*(2n)$		$\lfloor \frac{n}{2} \rfloor$	B
$SU^*(2(p+q))$	$S(U^*(2p) \times U^*(2q))$	$n-1$	$n-1$	A
$p,q \geq 1$	$Sp(p,q)$		$\min(p,q)$	B
$SO^*(2n)$	$SO(n,\mathbb{C})$	$\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	A
$SO^*(4n)$ $n \geq 1$	$U^*(2n)$	n	n	A

Here H^a coming from the associated pair of (G, H) is written in the same cell with H.

Proposition 2.3.5. Let (G, H) be a symmetric pair with G connected. Suppose corresponding symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is one of the following table. Then G_θ/H_θ does not admit compact Clifford–Klein froms.

Table 2.5': symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ which satisfy $\mathrm{rank}_{\mathbb{R}}\, G - \mathrm{rank}_{\mathbb{R}}\, H = 0$ or $\mathrm{rank}_{\mathbb{R}}\, G - \mathrm{rank}_{\mathbb{R}}\, H^a = 0$.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathrm{rank}_{\mathbb{R}}\, G$	$\mathrm{rank}_{\mathbb{R}}\, H$	A or B
$\mathfrak{g}_2^{\mathbb{C}}$	$\mathfrak{g}_{2(2)}$	2	2	A
	$\mathfrak{sp}(1,\mathbb{C}) \oplus \mathfrak{sp}(1,\mathbb{C})$	2	2	A
$\mathfrak{g}_{2(2)}$	$\mathfrak{sp}(1,\mathbb{R}) \oplus \mathfrak{sp}(1,\mathbb{R})$	2	2	A
$\mathfrak{f}_4^{\mathbb{C}}$	$\mathfrak{f}_{4(-20)}$	4	1	B
	$\mathfrak{so}(9,\mathbb{C})$	4	4	A
$\mathfrak{f}_4^{\mathbb{C}}$	$\mathfrak{sp}(1,\mathbb{C}) \oplus \mathfrak{sp}(3,\mathbb{C})$	4	4	A
	$\mathfrak{f}_{4(4)}$	4	4	A
$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(2,1) \oplus \mathfrak{su}(2)$	4	1	B
	$\mathfrak{so}(4,5)$	4	4	A
$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(1,\mathbb{R}) \oplus \mathfrak{sp}(3,\mathbb{R})$	4	4	A
$\mathfrak{f}_{4(-20)}$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1,2)$	1	1	A
$\mathfrak{f}_{4(-20)}$	$\mathfrak{sp}(1,8)$	1	1	A
$\mathfrak{e}_6^{\mathbb{C}}$	$\mathfrak{e}_{6(2)}$	6	4	B
	$\mathfrak{sp}(1,\mathbb{C}) \oplus \mathfrak{sl}(6,\mathbb{C})$	6	6	A
	$\mathfrak{e}_{6(-14)}$	6	2	B
	$\mathfrak{so}(2,\mathbb{C}) \oplus \mathfrak{so}(10,\mathbb{C})$	6	6	A
	$\mathfrak{sp}(4,\mathbb{C})$	6	4	B
	$\mathfrak{e}_{6(6)}$	6	6	A
$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(2,2)$	6	2	B
	$\mathbb{R} \oplus \mathfrak{so}(5,5)$	6	6	A
	$\mathfrak{sp}(1,\mathbb{R}) \oplus \mathfrak{sl}(6,\mathbb{R})$	6	6	A
	$\mathfrak{sp}(4,\mathbb{R})$	6	4	B
$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(4,2) \oplus \mathfrak{su}(2)$	4	2	B
	$\mathfrak{u}(1) \oplus \mathfrak{so}(6,4)$	4	4	A
	$\mathfrak{sp}(3,1)$	4	1	B
	$\mathfrak{f}_{4(4)}$	4	4	A
	$\mathfrak{sp}(1,\mathbb{R}) \oplus \mathfrak{su}(3,3)$	4	4	A
	$\mathfrak{sp}(4,\mathbb{R})$	4	4	A
$\mathfrak{e}_{6(-14)}$	$\mathfrak{sp}(1) \oplus \mathfrak{su}(2,4)$	2	2	A
	$\mathfrak{sp}(1,\mathbb{R}) \oplus \mathfrak{su}(5,1)$	2	2	A
	$\mathfrak{u}(1) \oplus \mathfrak{so}^*(10)$	2	2	A
	$\mathfrak{u}(1) \oplus \mathfrak{so}(8,2)$	2	2	A
	$\mathfrak{sp}(2,2)$	2	2	A

Table 2.5": symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ which satisfy $\mathrm{rank}_{\mathbb{R}} G - \mathrm{rank}_{\mathbb{R}} H = 0$ or $\mathrm{rank}_{\mathbb{R}} G - \mathrm{rank}_{\mathbb{R}} H^a = 0$.

$\mathfrak{g}$	$\mathfrak{h}, \mathfrak{h}^a$	$\mathrm{rank}_{\mathbb{R}} G$	$\mathrm{rank}_{\mathbb{R}} H$	A or B
$\mathfrak{e}_{6(-26)}$	$\mathfrak{sp}(1) \oplus \mathfrak{su}^*(6))$	2	2	A
	$\mathfrak{sp}(1,3)$	2	1	B
	$\mathbb{R} \oplus \mathfrak{so}(1,9)$	2	2	A
	$\mathfrak{f}_{4(-20)}$	2	1	B
$\mathfrak{e}_7^{\mathbb{C}}$	$\mathfrak{e}_{7(-5)}$	7	4	B
	$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(12,\mathbb{C})$	7	7	A
	$\mathfrak{e}_{7(-25)}$	7	3	B
	$\mathbb{C} \oplus \mathfrak{e}_6^{\mathbb{C}}$	7	7	A
	$\mathfrak{sl}(8,\mathbb{C})$	7	7	A
	$\mathfrak{e}_{7(7)}$	7	7	A
$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2)$	7	4	3
	$\mathfrak{su}(2) \oplus \mathfrak{so} * (12)$	7	3	4
	$\mathfrak{su}(4,4)$	7	4	B
	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(6,6)$	7	6	A
	$\mathfrak{su}^*(8)$	7	3	B
	$\mathbb{R} \oplus \mathfrak{e}_{6(6)}$	7	7	A
	$\mathfrak{sl}(8,\mathbb{R})$	7	7	A
$\mathfrak{e}_{7(-5)}$	$\mathfrak{su}(6,2)$	4	2	B
	$\mathfrak{u}(1) \oplus \mathfrak{e}_{6(2)}$	4	4	A
	$\mathfrak{su}(4,4)$	4	4	A
	$\mathfrak{su}(2) \oplus \mathfrak{so}(8.4)$	4	4	A
	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}^*(12)$	4	4	A
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$	3	2	B
	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(2,10)$	3	3	A
	$\mathfrak{su}(6,2)$	3	2	B
	$\mathfrak{su}(2) \oplus \mathfrak{so}^*(12)$	3	3	A
	$\mathbb{R} \oplus \mathfrak{e}_{6(-26)}$	3	3	A
	$\mathfrak{su}^*(8)$	3	3	A
$\mathfrak{e}_8^{\mathbb{C}}$	$\mathfrak{e}_{8(24)}$	8	4	B
	$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{e}_7^{\mathbb{C}}$	8	8	A
	$\mathfrak{so}(16,\mathbb{C})$	8	8	A
	$\mathfrak{e}_{8(8)}$	8	8	A
$\mathfrak{e}_{8(8)}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{e}_{7(7)}$	8	8	A
	$\mathfrak{so}^*(16)$	8	4	B
	$\mathfrak{so}(8,8)$	8	8	A
$\mathfrak{e}_{8(-24)}$	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{e}_{7(-25)}$	4	4	A
	$\mathfrak{su}(2) \oplus \mathfrak{e}_{7(-5)}$	4	4	A
	$\mathfrak{so}(4,12)$	4	4	A
	$\mathfrak{so}^*(16)$	4	4	A

Table 2.6: classical irreducible symmetric pairs we consider in the following sections.

G	H	methods
$SO^*(2(2p+2q+2))$ $p,q \geq 0,\ (p,q) \neq (0,0)$	$SO^*(2(2p+1)) \times SO^*(2(2q+1))$ $U(2p+1, 2q+1)$	(iii) Maximality, (v) Applications of Adams's theorem
$SO(2p+2q+2,\mathbb{C})$ $p,q \geq 0,\ (p,q) \neq (0,0)$	$SO(2p+1,\mathbb{C}) \times SO(2q+1,\mathbb{C}))$ $SO_0(2p+1, 2q+1)$	(v) Applications of Adams's theorem
$SO_0(p_1+p_2, q_1+q_2)$ $0 \leq p_1 \leq p_2, q_1, q_2 \geq 1$	$SO_0(p_1,q_1) \times SO_0(p_2,q_2)$	(iii) Maximality, (iv) Non-triviality, (v) Applications of Adams's theorem
$SU(p_1+p_2, q_1+q_2)$ $0 \leq p_1 \leq p_2, q_1, q_2 \geq 1$	$S(U(p_1,q_1) \times U(p_2,q_2))$	(iii) Maximality, (iv) Non-triviality, (v) Applications of Adams's theorem
$Sp(p_1+p_2, q_1+q_2)$ $0 \leq p_1 \leq p_2, q_1, q_2 \geq 1$	$Sp(p_1,q_1) \times Sp(p_2,q_2)$	(iii) Maximality, (iv) Non-triviality
$SL(2n,\mathbb{C})$ $n \geq 2$	$Sp(n,\mathbb{C})$ $SU^*(2n)$	(ii) Pfister's Theorem
$SL(2n,\mathbb{R})$ $n \geq 2$	$Sp(n,\mathbb{R})$ $S'L(n,\mathbb{C})$	(ii) Pfister's Theorem
$SO_0(2p,2q)$ $1 \leq p \leq q$	$U(p,q)$	(iv) Non-triviality

2.4 Applications of Pfister's theorem

In this section, we give a necessary condition for the existence of compact Clifford–Klein forms for tangential symmetric spaces (Theorem 2.4.1) and apply it to two types of symmetric pairs (Proposition 2.4.3). We use Pfister's theorem (see Fact 2.4.2) to prove Theorem 2.4.1.

Theorem 2.4.1. Let G/H be a semisimple symmetric space and $\mathfrak{g} \subset \mathfrak{sl}(2n,\mathbb{K})$ a subalgebra, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If the following two conditions are satisfied, then G_θ/H_θ does not admit compact Clifford–Klein forms.

(i) $d(G) - d(H) \geq n$,

(ii) For $X \in \mathfrak{p} \subset M(2n,\mathbb{K})$, if the characteristic polynomial of X over $\mathbb{K}$ is even, then X is in $\mathrm{Ad}(K)\mathfrak{p}_H$.

Fact 2.4.2 ([21]. See also [8, Example 13.1(c)])**.** Let V be a real vector space with dimension $n+1$. Suppose $f_i : V \to \mathbb{R}$ $(i = 1, \cdots, n)$ are homogeneous polynomial functions on V of odd degree. Then $\{f_i\}_{i=1}^n$ has common zero points in $V \setminus \{0\}$.

Proof of Theorem 2.4.1. From Fact 2.2.5, it is enough to prove that for any $\mathbb{R}$-subspace V with dimension n of $\mathfrak{p}$, $\mathrm{Ad}(K)\mathfrak{p}_H \cap V \neq \{0\}$ holds. By the assumption, it is enough to show that there exists a non-zero element $X \in V$ such that $f_X(x)$ is even, where f_X denotes the characteristic polynomial of $X \in V \subset M_{2n}(\mathbb{K})$. Let V be a subspace of $\mathfrak{p}$ such that $\dim_{\mathbb{R}} V = n$. We define maps $\tau_i : V \to \mathbb{R}$ $(i = 0, 1, \cdots, 2n)$ by

$$f_X(x) = \det(xI - X) = \sum_{i=0}^{2n} \tau_i(X) x^i \text{ for } X \in V.$$

Then $\tau_{2n} = 1$, $\tau_{2n-1}(X) = \mathrm{trace}(X) = 0$ for all $X \in V$ by definition. Since τ_{2i-1} $(i = 1, 2, \cdots, n-1)$ are homogeneous polynomials on V of odd degree, by using the Fact 2.4.2, we can take a non-zero element $X \in V$ such that $f_X(x)$ is even. $\square$

Proposition 2.4.3. Let $(G,\ H)$ and $(G,\ H^a)$ be symmetric pairs which are locally isomorphic to one of the following list and suppose that G is connected. Then neither G_θ/H_θ nor G_θ/H^a_θ admit compact Clifford–Klein forms.

- $(G, H, H^a) = (SL(2n, \mathbb{R}), Sp(n, \mathbb{R}), S'L(n, \mathbb{C}))$ $(n \geq 2)$,
- $(G, H, H^a) = (SL(2n, \mathbb{C}), Sp(n, \mathbb{C}), SU^*(2n))$ $(n \geq 2)$.

Proof. Let (G, H) be a symmetric pair $(SL(2n, \mathbb{K}), Sp(n, \mathbb{K}))$ where $K = \mathbb{R}$ or $\mathbb{C}$. This comes from Theorem 2.4.1, Lemma 2.4.4 and $n \leq d(G) - d(H) = \begin{cases} n^2 - 1 \ (\mathbb{K} = \mathbb{R}), \\ 2n^2 - n - 1 \ (\mathbb{K} = \mathbb{C}) \end{cases}$ for $n \geq 2$. $\square$

We consider the case when $(G,\ H)$=$(SL(2n, \mathbb{K}),\ Sp(n, \mathbb{K}))$ $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We realize a symmetric pair $(SL(2n, \mathbb{K}), Sp(n, \mathbb{K}))$ as follows.

$$\begin{aligned} SL(2n, \mathbb{K}) &= \{g \in GL(2n, \mathbb{K}) : \det g = 1\}, \\ Sp(n, \mathbb{K}) &= \{g \in SL(2n, \mathbb{K}) : {}^t g J_n g = J_n\}, \end{aligned}$$

where $J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Then, by taking a Cartan involution $\theta : g \mapsto {}^t\overline{g}^{-1}$, we have

$$\begin{aligned} K &= \begin{cases} SO(2n) \ (\mathbb{K} = \mathbb{R}), \\ SU(2n) \ (\mathbb{K} = \mathbb{C}), \end{cases} \\ \mathfrak{p} &= \mathrm{Herm}_0(2n, \mathbb{K}) = \{X \in M(2n, \mathbb{K}) : {}^t\overline{X} = X, \mathrm{trace}\, X = 0\}, \\ \mathfrak{p}_H &= \left\{ \begin{pmatrix} A & B \\ \overline{B} & -\overline{A} \end{pmatrix} : A \in \mathrm{Herm}_0(n, \mathbb{K}), B \in \mathrm{Sym}(n, \mathbb{K}) \right\}. \end{aligned}$$

We take a maximal split abelian subspace $\mathfrak{a}_H$ of $\mathfrak{p}_H$ as follows.

$$\mathfrak{a}_H = \{\mathrm{diag}(a_1, \cdots, a_n, -a_1, \cdots, -a_n) : a_i \in \mathbb{R} \ (i = 1, \cdots, n)\}.$$

Lemma 2.4.4. For $X \in \mathfrak{p} = \mathrm{Sym}_0(2n, \mathbb{R})$, the following conditions are equivalent:

(i) $X \in \mathrm{Ad}(K)\mathfrak{p}_H$,

(ii) the characteristic polynomial $f_X(x) = \det(xI - X)$ is even.

Proof. The implication (i) $\Longrightarrow$ (ii) comes from the property that $\mathrm{Ad}(K)$-action on $\mathfrak{p}$ preserve the eigenvalues. Next, we prove (ii) $\Longrightarrow$ (i). Suppose that $X \in \mathfrak{p}$ and that the characteristic polynomial $f_X(x)$ is even. It means that there exists $k \in K$ such that $\mathrm{Ad}(k)X = \mathrm{diag}(a_1, \cdots, a_n, -a_1, \cdots, -a_n) \in \mathfrak{a}_H \subset \mathfrak{p}_H$. $\square$

2.5 Maximality of non-compactness

In this section, we give a necessary condition for the existence of compact Clifford–Klein forms for tangential symmetric spaces (Theorem 2.5.1) and apply it to five types of symmetric pairs (Proposition 2.5.4).

Theorem 2.5.1. If a symmetric space G/H of reductive type satisfies the assumption of the following Fact 2.5.2, then the corresponding tangential symmetric space G_θ/H_θ does not admit compact Clifford–Klein forms.

Proof. This comes from the fact that $\mathfrak{a}(G') \subset W_G \cdot \mathfrak{a}(H)$ is equivalent to the condition $\mathfrak{p}_{G'} \subset \mathrm{Ad}(K)\mathfrak{p}_H$. □

Fact 2.5.2 ([14, Theorem 1.5])**.** Let G/H be a homogeneous space of reductive type. If there exist a closed subgroup G' reductive in G satisfying the following two conditions, then G/H does not admit compact Clifford–Klein forms.

(i) $\mathfrak{a}(G') \subset W_G \cdot \mathfrak{a}(H)$,

(ii) $d(G') > d(H)$.

Here, $W_G := N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ is the Weyl group.

Remark 2.5.3. Since the assumptions are same in Theorem 2.5.1 and Fact 2.5.2, Non-existence results of compact Clifford–Klein forms for symmetric spaces G/H of reductive type obtained by Fact 2.5.2 imply one for corresponding tangential symmetric spaces G_θ/H_θ.

Proposition 2.5.4. Let (G, H) be a symmetric pair which is locally isomorphic to one of the following list and suppose that G is connected. Then the tangential symmetric space G_θ/H_θ does not admit compact Clifford–Klein forms.

- $(G, H) = (SO^*(2(p+q)), SO^*(2p) \times SO^*(2q))$ $(2 \leq p, q)$,
- $(G, H) = (SO^*(2(p+q)), U(p, q))$ $(2 \leq p, q)$
- $(G, H) = (SO_0(p, q), SO_0(p_1, q_1) \times SO(p_2, q_2))$ $(0 < p_1, p_2, q_1, q_2)$,
- $(G, H) = (SU(p, q), S(U(p_1, q_1) \times U(p_2, q_2)))$ $(0 < p_1, p_2, q_1, q_2)$,
- $(G, H) = (Sp(p, q), Sp(p_1, q_1) \times Sp(p_2, q_2))$ $(0 < p_1, p_2, q_1, q_2)$.

Proof. This comes from Theorem 2.5.1. See [14, Example 1.7], [KY05, Remark 3.5.8, Corollary 3.5.9]. □

Theorem 2.5.1 can be generalized as follows:

Fact 2.5.5 ([KY05, Corollary to Kobayashi-Yoshino, Theorem 5.3.2])**.** If there exists a linear subspace W of $\mathfrak{p}$ such that $W \subset \mathrm{Ad}(K)\mathfrak{p}_H$ and $\dim_{\mathbb{R}} W > \dim \mathfrak{p}_H$, then G_θ/H_θ does not admit compact Clifford–Klein forms.

2.6 Non-triviality of symmetric spaces as vector bundles

2.6.1 general method

In this subsection, we show the following necessary condition for the existence of compact Clifford–Klein forms of tangential homogeneous spaces (Theorem 2.6.1) and list up examples which are proved not to admit compact Clifford Klein forms by using the obstruction (Proposition 2.6.5 and Theorem 2.1.12).

Theorem 2.6.1. Let G/H be a semisimple symmetric space as in Section 2.2.1. If the associated vector bundle $K \times_{(K_H, \mathrm{Ad}_{\mathfrak{p}/\mathfrak{p}_H})} (\mathfrak{p}/\mathfrak{p}_H) \to K/K_H$ is not trivial bundle, then G_θ/H_θ does not admit compact Clifford–Klein forms.

Proof. This follows from the Fact 2.2.5, Remark 2.2.6 and Lemma 2.6.3 by taking $(\sigma, V) = (\mathrm{Ad}, \mathfrak{p})$ and $W_1 = \mathfrak{p}_H$. □

Remark 2.6.2 ([Ko89, Lemma 2.7])**.** Let G/H be a homogeneous space of reductive type. Then, there is a diffeomorphism $G/H \simeq K \times_{K_H} \mathfrak{p}/\mathfrak{p}_H$ as a manifold. Here, this is the associated bundle with regard to the representation $\mathrm{Ad}_{\mathfrak{p}/\mathfrak{p}_H} : K_H \to GL(\mathfrak{p}/\mathfrak{p}_H)$ which is induced by the adjoint representation $\mathrm{Ad} : K_H \to GL(\mathfrak{p})$ such that $\mathrm{Ad}(K_H)\mathfrak{p}_H \subset \mathfrak{p}_H$.

The following Lemma 2.6.3 is used to show Theorem 2.6.1.

Lemma 2.6.3. Let K be a Lie group, K_H a closed Lie subgroup of K and (σ, V) be a finite dimensional representation of K. Let W_1 be a $\sigma(K_H)$-invariant subspace of V and W_2 a subspace of V satisfying that $\sigma(K)W_1 \cap W_2 = \{0\}$. Then there exists a injective bundle map $K/K_H \times W_2 \hookrightarrow K \times_{K_H} V/W_1$ over K/K_H.

Remark 2.6.4. In the above Lemma 2.6.3, the coefficient field of vector spaces V, W can be considered as both $\mathbb{R}$ and $\mathbb{C}$. Moreover, we can replace the assumption that K is Lie group and K_H is a closed subgroup of K by an assumption that K is a topological group and K_H is a closed subgroup of K.

Proof of Lemma 2.6.3. We define a map τ by

$$\tau : K \times W_2 \to K \times V/W_1, (k, w_2) \mapsto (k, \sigma(k^{-1})w_2 + W_1).$$

Then τ is a injective K_H-equivariant bundle map over K. Here, right K_H-actions are as follows:

$$\begin{aligned} (K \times W_2) \times K_H \to K \times W_2, \quad ((k, w_2), k_H) &\mapsto (kk_H, w_2), \\ (K \times V/W_1) \times K_H \to K \times V/W_1, ((k, v + W_1), k_H) &\mapsto (kk_H, \sigma(k_H^{-1})v + W_1). \end{aligned}$$

Therefore, we get the induced bundle map $\tilde{\tau} : K/K_H \times W_2 \to K \times_{K_H} V/W_1$ over K/K_H, which is the desired injective bundle map. □

In this section, we show the tangential symmetric spaces associated with the following symmetric pair do not admit compact Clifford–Klein forms.

Proposition 2.6.5. Let p, q_1, q_2 be positive integers and (G, H) a symmetric pair which is locally isomorphic to one of the following list and suppose that G is connected. Then G_θ/H_θ does not admit compact Clifford–Klein forms.

- $(G, H) = (SO_0(p, q_1 + q_2), SO(q_1) \times SO_0(p, q_2))$ ($q_1 \geq 2$ and $q_2 \geq 2$),
- $(G, H) = (SU(p, q_1 + q_2), S(U(q_1) \times U(p, q_2)))$ ($q_1 \geq 2$ or $q_2 \geq 2$),
- $(G, H) = (Sp(p, q_1 + q_2), Sp(q_1) \times Sp(p, q_2))$ ($p \geq 1$, $q_1, q_2 \geq 1$),
- $(G, H) = (SO_0(2p, 2q), U(p, q))$ ($2 \leq p, q$).

Proof. This comes from Theorem 2.6.1, Fact 2.6.9 and Proposition 2.6.11, 2.6.12. □

Proposition 2.6.6. Let $(G,\ H,\ H^a)$=$(E_6^{\mathbb{C}},\ F_4^{\mathbb{C}},\ E_{6(-26)})$. Then neither G_θ/H_θ nor G_θ/H_θ^a admit compact compact Clifford–Klein forms.

Proof. For $G/H = E_6^{\mathbb{C}}/F_4^{\mathbb{C}}$, $K \times_{K_H} \mathfrak{p}/\mathfrak{p}_H$ is equivalent to the tangent bundle over E_6/F_4 as a vector bundle. From the following Facts 2.6.7, $K \times_{K_H} \mathfrak{p}/\mathfrak{p}_H$ is not trivial. Therefore, from Theorem 2.6.1 and 2.2.10, We obtain the desired conclusion. □

Fact 2.6.7 ([24, Theorem 2])**.** E_6/F_4 is not stably parallelizable.

Remark 2.6.8. If a tangent bundle over K/K_H is trivial, then K/K_H is stably parallelizable.

To show the non-triviality of real vector bundle, we use Pontrjagin class. The naturality of characteristic classes implies the following:

Fact 2.6.9 (See [7] for example)**.** Let $E \to M$ be a real vector bundle. If the i-th Pontrjagin class $p_i(E \to M) \in H^{4i}_{DR}(M, \mathbb{R})$ does not vanish for some $i \geq 1$, then the bundle $E \to M$ is not trivial.

By using the following fact, we can easily calculate the Pontrjagin class of associated bundles. This statement is not new, but for the sake of completeness, we give a proof in Section 2.8.

Fact 2.6.10 (See [7] for example.)**.** Let G be a connected compact Lie group, $\varpi : P \to M$ a principal G-bundle, $\rho : G \to SO(V)$ a representation of G and $E := P \times_G V$ the associated bundle. Then for any $f \in S^k(\mathfrak{so}(V)^*)^{SO(V)}$, the following equality holds.

$$[f(R)] = \omega \circ d\rho^*(f) \in H^{2k}_{DR}(M, \mathbb{R}),$$

where R is a curvature on E and $\omega : S(\mathfrak{g}^*)^G \to H^*_{DR}(M, \mathbb{R})$ is the Chern-Weil map.

By calculations of the Pontrjagin classes in the following subsections, we obtain the following Proposition 2.6.11, 2.6.12:

Proposition 2.6.11. Let p, q_1, q_2 be positive integers.

(a) Let $(G,\ H)$=$(SO_0(p, q_1+q_2),\ SO(q_1)\times SO_0(p,q_2))$. If $q_1 \geq 2$ and $q_2 \geq 2$, then the first Pontrjagin class of the vector bundle $K\times_{K_H}\mathfrak{p}/\mathfrak{p}_H \to K/K_H$ does not vanish.

(b) Let $(G,\ H)$=$(SU(p, q_1+q_2),\ S(U(q_1)\times U(p,q_2)))$. If $q_1 \geq 2$ or $q_2 \geq 2$, then the first Pontrjagin class of the vector bundle $K\times_{K_H}\mathfrak{p}/\mathfrak{p}_H \to K/K_H$ does not vanish.

(c) Let $(G,H) = (Sp(p,q_1+q_2), Sp(q_1)\times Sp(p,q_2))$. The first Pontrjagin class of the vector bundle $K\times_{K_H}\mathfrak{p}/\mathfrak{p}_H \to K/K_H$ does not vanish.

Proposition 2.6.12. Let $(G,\ H) = (SO_0(2p,2q), U(p,q))$ $(1 \leq p \leq q)$. If $p \geq 2$, then the first Pontrjagin class of the vector bundle $K\times_{K_H}\mathfrak{p}/\mathfrak{p}_H \to K/K_H$ does not vanish.

2.6.2 Calculation of first Pontrjagin class for Grassmaniann manifolds

In this subsection, we show Proposition 2.6.11 by calculating the first Pontrjagin class of corresponding vector bundles $K\times_{K_H}(\mathfrak{p}/\mathfrak{p}_H)$. Here, we use Fact 2.6.10. More precisely,

(i) Let $\tilde{p_1} \in S^2(\mathfrak{so}(\mathfrak{p}/\mathfrak{p}_H)^*)^{SO(\mathfrak{p}/\mathfrak{p}_H)}$ be all the sum of principal minors of degree two.

(ii) To determine whether $[\tilde{p_1}(R)] = \omega\circ \mathrm{ad}^*(\tilde{p_1}) \in H^4(K/K_H,\mathbb{R})$ vanishes or not, we check whether $\mathrm{ad}^*(\tilde{p_1}) \in \ker\omega$ holds or not.

More precisely, we calculate the first Pontrjagin class as follows. Here we identify $S(\mathfrak{k}^*)^K$ and $S(\mathfrak{k}_H^*)^{K_H}$ with $S(\mathfrak{t})^W$ and $S(\mathfrak{t}_H)^{W_H}$ respectively by the restriction, where $\mathfrak{t}$ and $\mathfrak{t}_H$ are maximal tori of $\mathfrak{k}$ and $\mathfrak{k}_H$ respectively.

step(0) We realize the above symmetric pairs $(G,\ H)$ and $(K,\ K_H)$ as matrix groups.

step(1) We rewrite $K\times_{K_H}\mathfrak{p}/\mathfrak{p}_H$ to an easier form to calculate Pontrjagin class.

step(2) Fix a coordinates of $\mathfrak{t} = \mathrm{Lie}(T)$ and $\mathfrak{t}_H = \mathrm{Lie}(T_H)$, where T and T_H are maximal tori of K and K_H respectively.

step(3) We write $S(\mathfrak{t}^*)^W$ and $S(\mathfrak{t}_H^*)^{W_H}$ with regard to the above coordinates.

step(4) We write $\ker\omega$ by using Fact 2.6.20.

step(5) We write $\mathrm{ad}^*(\tilde{p_1}) \in S(\mathfrak{t}_H^*)^{W_H}$ with regard to the above coordinates.

step(6) We check whether $\mathrm{ad}^*(\tilde{p_1}) \in \ker \omega$ holds or not.

■ **step(0)**: Realization of $(G,\, H)$ and $(K,\, K_H)$ as matrix groups.
We realize $G = SO_0(p,q)$ or $SU(p,q)$ $(q = q_1 + q_2)$ as the identity component of the following matrix group.

$$\{g \in GL(p+q, \mathbb{K}) : g^* I_{p,q} g = I_{p,q}, \det g = 1\}.$$

Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We realize $G = Sp(p,q)$ by the following matrix group.

$$\{g \in GL(p+q, \mathbb{H}) : g^* I_{p,q} g = I_{p,q}\}.$$

We define a involution σ of G by

$$\sigma : G \to G,\ g \mapsto I_{p,q_1,q_2} g I_{p,q_1,q_2},$$

where $I_{p,q_1,q_2} = \begin{pmatrix} I_p & & \\ & -I_{q_1} & \\ & & I_{q_2} \end{pmatrix}$. We define subgroup H of G as the identity component of G^σ. Then, we can identify $\mathfrak{p}/\mathfrak{p}_H$ with the following $\mathrm{Ad}(K_H)$-invariant subspace of $M(p+q, \mathbb{K})$.

$$\left\{ \begin{pmatrix} 0 & B & 0 \\ B^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M(p+q, \mathbb{K}) : B \in M(p, q_1; \mathbb{K}) \right\}.$$

By taking a Cartan involution θ: $g \mapsto (g^*)^{-1}$, which is commuting with σ, we obtain the realization of $(K,\, K_H)$.
■ **step(1)**: Rewrite $K \times_{K_H} \mathfrak{p}/\mathfrak{p}_H$ to easier form.
In this step, we prove the following:

Lemma 2.6.13. There exists $SO(q)$, $U(q)$ and $Sp(q)$ equivariant vector bundle isomorphisms respectively as follows:

$$K \times_{K_H} \mathfrak{p}/\mathfrak{p}_H \simeq \begin{cases} SO(q) \times_{SO(q_1)\times SO(q_2)} \mathfrak{p}/\mathfrak{p}_H \\ (when\ (a) : (G,H) = (SO_0(p, q_1+q_2), SO(q_1) \times SO_0(p, q_2))), \\ \\ U(q) \times_{U(q_1)\times U(q_2)} \mathfrak{p}/\mathfrak{p}_H \\ (when\ (b) : (G,H) = (SU(p,q), S(U(q_1) \times U(p, q_2))), \\ \\ Sp(q) \times_{Sp(q_1)\times Sp(q_2)} \mathfrak{p}/\mathfrak{p}_H \\ (when\ (c) : (G,H) = (Sp(p,q), Sp(q_1) \times Sp(p, q_2))). \end{cases}$$

Here in the right hand side, actions of $SO(q_1) \times SO(q_2)$, $U(q_1) \times U(q_2)$ and $Sp(q_1) \times Sp(q_2)$ on $\mathfrak{p}/\mathfrak{p}_H$ are given by the restriction of the action of K_H

Remark 2.6.14. From the above Lemma 2.6.13, to calculate the Pontrjagin class of $K \times_{K_H} \mathfrak{p}/\mathfrak{p}_H$, we can and do calculate it of the vector bundles in the right hand side. Thus, in the following steps, we use the notation, $(K, K_H) = (SO(q), SO(q_1) \times SO(q_2))$, $(U(q), U(q_1) \times U(q_2))$ and $(Sp(q), Sp(q_1) \times Sp(q_2))$ respectively.

Lemma 2.6.13 follows from the following Fact 2.6.15 and Remark 2.6.16.

Fact 2.6.15 ([17, Theorem 10.32])**.** Let G be a Lie group, E, M manifolds and $\pi : E \to M$ a G-equivariant vector bundle. Assume G acts on M transitively and fix $m \in M$. Then we get the following isomorphism $(\tilde{f}, f)$ from $E \to M$ to an associated bundle $G \times_H V \to G/H$ as a G-equivariant vector bundle.

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & G \times_H V \arrow[d] \\ M \arrow[r, "f"] & G/H \end{tikzcd}$$

Here $H = G_m$ is the stabilizer subgroup of G at m and $V = \pi^{-1}(m)$.

Remark 2.6.16. Let G be a Lie group and H, L Lie subgroups of G. Then the following two conditions are equivalent.

(i) L acts on G/H transitively,

(ii) $G = L \cdot H$.

Remark 2.6.17. We consider the following realization of $U(q)$ in $S(U(p) \times U(q))$.

$$U(q) = \left\{ \begin{pmatrix} \det g^{-1} & & \\ & I_{p_2 - 1} & \\ & & g \end{pmatrix} \in S(U(p) \times U(q)) : g \in U(q) \right\}.$$

■ **step(2)**: Fix coordinates of maximal tori $\mathfrak{t}$ and $\mathfrak{t}_H$ of $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{k}_H = \mathrm{Lie}(K_H)$ respectively in the sense of Remark 2.6.14.

We use the following notation.

$$A_{i,j} = E_{ij} - E_{ji} \in M(p+q, \mathbb{K}),$$

where E_{ij} is a matrix unit.

We fix maximal tori $\mathfrak{t}$, $\mathfrak{t}_H$ of $\mathfrak{k}$, $\mathfrak{k}_H$ respectively as follows.

(a) $(G, H) = (SO_0(p, q_1 + q_2), SO(q_1) \times SO_0(p, q_2))$, $q' = \lfloor \frac{q_1+q_2}{2} \rfloor$, $q_1' = \lfloor \frac{q_1}{2} \rfloor$, $q_2' = \lfloor \frac{q_2}{2} \rfloor$.

- In the case when ether q_1 or q_2 is even.

$$\begin{aligned}\mathfrak{t} =& \mathfrak{t}_H = \{\sum_{i=1}^{q_1'} t_i A_{p+2i-1,p+2i} + \sum_{i=1}^{q_2'} t_{q_1'+i} A_{p+q_1+2i-1,p+q_1+2i} \\ &\qquad \in M(p+q,\mathbb{R}) : t_i \in \mathbb{R}\ (i=1,\cdots,q'(=q_1'+q_2'))\} \\ =&\{\begin{pmatrix} 0 &&&&&&&&&&&&& \\ & \ddots &&&&&&&&&&&& \\ && 0 &&&&&&&&&&& \\ &&& 0 & t_1 &&&&&&&&& \\ &&& -t_1 & 0 &&&&&&&&& \\ &&&&& \ddots &&&&&&&& \\ &&&&&& 0 & t_{q_1'} &&&&&& \\ &&&&&& -t_{q_1'} & 0 &&&&&& \\ &&&&&&&& 0 & t_{q_1'+1} &&&& \\ &&&&&&&& -t_{q_1'+1} & 0 &&&& \\ &&&&&&&&&& \ddots &&& \\ &&&&&&&&&&& 0 & t_{q'} & \\ &&&&&&&&&&& -t_{q'} & 0 & \end{pmatrix} \\ &\quad : t_i \in \mathbb{R}\ (i=1,\cdots,q')\}.\end{aligned}$$

- In the case when both q_1 and q_2 are odd.

$$\begin{aligned}
\mathfrak{t} =&\{\sum_{i=1}^{q_1'} t_i A_{p+2i-1,p+2i} + t_{q'} A_{p+q_1,p+q_1+1} + \sum_{i=1}^{q_2'} t_{q_1'+i} A_{p+q_1+2i,p+q_1+2i+1} \\
&\in M(p+q,\mathbb{R}) : t_i \in \mathbb{R}\ (i=1,\cdots,q'(=q_1'+q_2'+1))\} \\
&\{\begin{pmatrix}
0 &&&&&&&&&&&&& \\
& \ddots &&&&&&&&&&&& \\
&& 0 &&&&&&&&&&& \\
&&& 0 & t_1 &&&&&&&&& \\
&&& -t_1 & 0 &&&&&&&&& \\
&&&&& \ddots &&&&&&&& \\
&&&&&& 0 & t_{q_1'} &&&&&& \\
&&&&&& -t_{q_1'} & 0 &&&&&& \\
&&&&&&&& 0 & t_{q'} &&&& \\
&&&&&&&& -t_{q'} & 0 &&&& \\
&&&&&&&&&& 0 & t_{q_1'+1} && \\
&&&&&&&&&& -t_{q_1'+1} & 0 && \\
&&&&&&&&&&&& \ddots & \\
&&&&&&&&&&&&& 0 \quad t_{q_1'+q_2'} \\
&&&&&&&&&&&&& -t_{q_1'+q_2'} \quad 0
\end{pmatrix} \\
&: t_i \in \mathbb{R}\ (i=1,\cdots,q')\} \\
\mathfrak{t}_H =&\{\sum_{i=1}^{q_1'} t_i A_{p+2i-1,p+2i} + \sum_{i=1}^{q_2'} t_{q'+i} A_{p+q_1+2i,p+q_1+2i+1} \\
&\in M(p+q,\mathbb{R}) : t_i \in \mathbb{R}\ (i=1,\cdots,q_1'+q_2')\} \\
&= \{\begin{pmatrix}
0 &&&&&&&&&&&&& \\
& \ddots &&&&&&&&&&&& \\
&& 0 &&&&&&&&&&& \\
&&& 0 & t_1 &&&&&&&&& \\
&&& -t_1 & 0 &&&&&&&&& \\
&&&&& \ddots &&&&&&&& \\
&&&&&& 0 & t_{q_1'} &&&&&& \\
&&&&&& -t_{q_1'} & 0 &&&&&& \\
&&&&&&&& 0 & 0 &&&& \\
&&&&&&&& 0 & 0 &&&& \\
&&&&&&&&&& 0 & t_{q_1'+1} && \\
&&&&&&&&&& -t_{q_1'+1} & 0 && \\
&&&&&&&&&&&& \ddots & \\
&&&&&&&&&&&&& 0 \quad t_{q_1'+q_2'} \\
&&&&&&&&&&&&& -t_{q_1'+q_2'} \quad 0
\end{pmatrix} \\
&: t_i \in \mathbb{R}\ (i=1,\cdots,q')\}.
\end{aligned}$$

Take coordinates $\{t_i\}_{i=1}^{q'}$, $\{t_i\}_{i=1}^{q'_1+q'_2}$ of $\mathfrak{t}$, $\mathfrak{t}_H$ by the above basis respectively.

(b) $(G,H) = (SU(p,q), S(U(q_1)\times U(p,q_2)))$

$$\mathfrak{t} = \mathfrak{t}_H = \left\{ \sqrt{-1}\sum_{i=1}^{q} t_i(-E_{11} + E_{p+i\,p+i}) \in M(p+q,\mathbb{C}) : t_i \in \mathbb{R}\ (i=1,\cdots,q) \right\}$$

$$= \left\{ \sqrt{-1} \begin{pmatrix} -\sum_{i=1}^{q} t_i & & & & & & \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & & t_1 & & \\ & & & & & \ddots & \\ & & & & & & t_q \end{pmatrix} : t_i \in \mathbb{R}\ (i=1,\cdots,q) \right\}.$$

Take a coordinate $\{t_i\}_{i=1}^{q}$ of $\mathfrak{t}$ and $\mathfrak{t}_H$ by the above basis.

(c) $(G,H) = (Sp(p,q), Sp(q_1)\times Sp(p,q_2))$

$$\mathfrak{t} = \mathfrak{t}_H = \left\{ i\sum_{\ell=1}^{q} t_\ell E_{p+\ell\,p+\ell} \in M(p+q,\mathbb{H}) : t_\ell \in \mathbb{R}\ (\ell=1,\cdots,q) \right\}$$

$$= \left\{ i \begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & & t_1 & & \\ & & & & & \ddots & \\ & & & & & & t_q \end{pmatrix} : t_\ell \in \mathbb{R}\ (\ell=1,\cdots,q) \right\}.$$

Here we consider $\mathbb{H}$ as a $\mathbb{R}$ algebra spanned by $1, i, j, k$ satisfying that $i^2 = j^2 = k^2 = -1, ijk = -1$. Take a coordinate $\{t_\ell\}_{\ell=1}^{q}$ of $\mathfrak{t}$ and $\mathfrak{t}_H$ by the above basis.

■ **step(3)**: Description of $S(\mathfrak{t}^*)^W$ and $S(\mathfrak{t}_H^*)^{W_H}$.
We use the following notation to describe $S(\mathfrak{t}^*)^W$ and $S(\mathfrak{t}_H^*)^{W_H}$.

Notation 2.6.18. For $1 \le p \le q \le n$, we denote fundamental symmetric polynomials of variables $\{t_p^2,\cdots,t_q^2\}$, $\{t_p,\cdots,t_q\}$ by $a_{(p,q)}^k \in S^{2k}((\mathbb{R}^n)^*)$, $b_{(p,q)}^k \in S^k((\mathbb{R}^n)^*)$:

$$a_{(p,q)}^k : \mathbb{R}^n \to \mathbb{R},\ t = (t_1,\cdots,t_n) \mapsto \sum_{p\le i_1<\cdots<i_k\le q} t_{i_1}^2\cdots t_{i_k}^2,$$
$$b_{(p,q)}^k : \mathbb{R}^n \to \mathbb{R},\ t = (t_1,\cdots,t_n) \mapsto \sum_{p\le i_1<\cdots<i_k\le q} t_{i_1}\cdots t_{i_k}.$$

Remark 2.6.19. For $(G,H) = (SO_0(p,q_1+q_2), SO(q_1)\times SO_0(p,q_2))$, $(SU(p,q), S(U(q_1)\times U(p,q_2)))$ and $(Sp(p,q),\ Sp(q_1)\times Sp(p,q_2))$, $S(\mathfrak{t}^*)^W$ and $S(\mathfrak{t}_H^*)^{W_H}$ can be written as follows by using the above coordinates. Here W and W_H are the Weyl groups of K and K_H respectively.

(a) $(G,H) = (SO_0(p,q_1+q_2), SO(q_1)\times SO_0(p,q_2))$

$$S(\mathfrak{t}^*)^W = \begin{cases} \mathbb{R}\big[a^1_{(1,q')},\cdots,a^{q'}_{(1,q')},b^{q'}_{(1,q')}\big] \text{ if } q=q_1+q_2 \text{ is even,} \\ \mathbb{R}\big[a^1_{(1,q')},\cdots,a^{q'}_{(1,q')}\big] \text{ otherwise.} \end{cases}$$

$$S(\mathfrak{t}_H^*)^{W_H} = \begin{cases} \mathbb{R}\big[a^1_{(1,q_1')},\cdots,a^{q_1'}_{(1,q_1')},a^1_{(q_1'+1,q_1'+q_2')},\cdots,a^{q_2'}_{(q_1'+1,q_1'+q_2')},b^{q_1'}_{(1,q_1')},b^{q_2'}_{(q_1'+1,q_1'+q_2')}\big] \\ \text{ if } q_1 \text{ and } q_2 \text{ are even,} \\ \\ \mathbb{R}\big[a^1_{(1,q_1')},\cdots,a^{q_1'}_{(1,q_1')},a^1_{(q_1'+1,q_1'+q_2')},\cdots,a^{q_2'}_{(q_1'+1,q_1'+q_2')},b^{q_1'}_{(1,q_1')}\big] \\ \text{ if } q_1 \text{ is even and } q_2 \text{ is odd,} \\ \\ \mathbb{R}\big[a^1_{(1,q_1')},\cdots,a^{q_1'}_{(1,q_1')},a^1_{(q_1'+1,q_1'+q_2')},\cdots,a^{q_2'}_{(q_1'+1,q_1'+q_2')},b^{q_2'}_{(q_1'+1,q_1'+q_2')}\big] \\ \text{ if } q_1 \text{ is odd and } q_2 \text{ is even,} \\ \\ \mathbb{R}\big[a^1_{(1,q_1')},\cdots,a^{q_1'}_{(1,q_1')},a^1_{(q_1'+1,q_1'+q_2')},\cdots,a^{q_2'}_{(q_1'+1,q_1'+q_2')}\big] \\ \text{ if } q_1 \text{ and } q_2 \text{ are odd.} \end{cases}$$

(b) $(G,H) = (SU(p,q), S(U(q_1)\times U(p,q_2)))$

$$\begin{aligned} S(\mathfrak{t}^*)^W &= \mathbb{R}[b^1_{(1,q)},\cdots,b^q_{(1,q)}], \\ S(\mathfrak{t}_H^*)^{W_H} &= \mathbb{R}[b^1_{(1,q_1)},\cdots,b^{q_1}_{(1,q_1)},b^1_{(q_1+1,q_1+q_2)},\cdots,b^{q_2}_{(q_1+1,q_1+q_2)}]. \end{aligned}$$

(c) $(G,H) = (Sp(p,q), Sp(q_1)\times Sp(p,q_2))$

$$\begin{aligned} S(\mathfrak{t}^*)^W &= \mathbb{R}[a^1_{(1,q)},\cdots,a^q_{(1,q)}], \\ S(\mathfrak{t}_H^*)^{W_H} &= \mathbb{R}[a^1_{(1,q_1)},\cdots,a^{q_1}_{(1,q_1)},a^1_{(q_1+1,q_1+q_2)},\cdots,a^{q_2}_{(q_1+1,q_1+q_2)}]. \end{aligned}$$

■ **step(4)**: Description of $\ker\omega$.
Under the identification $S(\mathfrak{k}^*)^K \simeq S(\mathfrak{t}^*)^W$, $S(\mathfrak{k}_H^*)^{K_H} \simeq S(\mathfrak{t}_H^*)^{W_H}$, we use the following:

Fact 2.6.20 ([6], See also [9])**.** Let K be a connected compact Lie group and K_H its closed connected subgroup of K. Let $\omega : S(\mathfrak{t}_H^*)^{W_H} \to H^*_{DR}(K/K_H;\mathbb{R})$ be the Chern-Weil map. Then $\ker\omega$ can be written as follows.

$$\ker\omega = (\text{ideal generated by } \bigoplus_{k=1}^{\infty} \mathrm{Im}(\mathrm{rest} : S^k(\mathfrak{t}^*)^W \to S^k(\mathfrak{t}_H^*)^{W_H}) \text{ in } S(\mathfrak{t}_H^*)^{W_H}),$$

where rest is the restriction map.

Lemma 2.6.21. For $(G, H) = (SO_0(p, q_1+q_2), SO(q_1)\times SO_0(p, q_2))$, $(SU(p, q)$, $S(U(q_1) \times U(p, q_2)))$ and $(Sp(p, q)$, $Sp(q_1) \times Sp(p, q_2))$, $\ker\omega$ can be written as follows respectively by using the above coordinates.

(a) $(G, H) = (SO_0(p, q_1 + q_2), SO(q_1) \times SO_0(p, q_2))$

$$\ker\omega = \text{ideal generated by}$$
$$\begin{cases} a^1_{(1,q')}, \cdots, a^{q'}_{(1,q')}, b^{q'}_{(1,q')} & \text{(if both } q_1 \text{ and } q_2 \text{ are even)}, \\ a^1_{(1,q'_1+q'_2)}, \cdots, a^{q'_1+q'_2}_{(1,q'_1+q'_2)} & \text{(otherwise)} \end{cases}$$
$$\text{on } S(\mathfrak{t}^*_H)^{W_H}.$$

(b) $(G, H) = (SU(p, q), S(U(q_1) \times U(p, q_2)))$

$$\ker\omega = \text{ideal generated by}$$
$$b^1_{(1,q)}, \cdots, b^q_{(1,q)}$$
$$\text{on } S(\mathfrak{t}^*_H)^{W_H}.$$

(c) $(G, H) = (Sp(p, q), Sp(q_1) \times Sp(p, q_2))$

$$\ker\omega = \text{ideal generated by}$$
$$a^1_{(1,q)}, \cdots, a^q_{(1,q)}$$
$$\text{on } S(\mathfrak{t}^*_H)^{W_H}.$$

■ **step(5)**: Description of $\mathrm{ad}^*(\tilde{p_1}) \in S(\mathfrak{t}^*_H)^{W_H}$ for the $SO(\mathfrak{p}/\mathfrak{p}_H)$-invariant polynomial $\tilde{p_1}$ on $\mathfrak{so}(\mathfrak{p}/\mathfrak{p}_H)$ of degree two.
By direct computation, we get:

Lemma 2.6.22. For $(G,H)=(SO_0(p, q_1 + q_2)$, $SO(q_1) \times SO_0(p, q_2))$, $(SU(p, q)$, $S(U(q_1) \times U(p, q_2)))$, $(Sp(p, q)$, $Sp(q_1) \times Sp(p, q_2))$, $\mathrm{ad}^*(\tilde{p_1})$ is written as follows.

(a) $(G, H) = (SO_0(p, q_1 + q_2), SO(q_1) \times SO_0(p, q_2))$

$$\mathrm{ad}^*(\tilde{p_1}) = pa^1_{(1,q'_1)},$$

(b) $(G, H) = (SU(p, q), S(U(q_1) \times U(p, q_2)))$

$$\mathrm{ad}^*(\tilde{p_1}) = pa^1_{(1,q_1)} + (q_1 b^1_{(1,q)} + 2b^1_{(1,q_1)})b^1_{(1,q)},$$

(c) $(G, H) = (Sp(p, q), Sp(q_1) \times Sp(p, q_2))$

$$\mathrm{ad}^*(\tilde{p_1}) = 2pa^1_{(1,q_1)},$$

where $\tilde{p_1}$ is all the sum of principal minors of degree two, which is a $SO(\mathfrak{p}/\mathfrak{p}_H)$-invariant polynomial of degree two on $\mathfrak{so}(\mathfrak{p}/\mathfrak{p}_H)$.

■ **step(6)**: Check whether $\mathrm{ad}^*(\tilde{p_1}) \in \ker\omega$ or not.

Proof of Proposition 2.6.11(a). It is enough to show the following:

$$q_1', q_2' \geq 1 \Rightarrow \mathrm{ad}^*(\tilde{p_1}) \notin \ker\omega.$$

Suppose $q_1', q_2' \geq 1$. From Lemma 2.6.22(a), we get $\mathrm{ad}^*(\tilde{p_1}) = p_2 a^1_{(1,q_1')}$. Since there is no generator of degree one of $\ker\omega$, if $a^1_{(1,q_1')} \in \ker\omega$, then we can write $a^1_{(1,q_1')}$ as a $\mathbb{R}$-linear combination of generators of degree 2 of $\ker\omega$. However, it is impossible to do so by seeing $\ker\omega$. □

Remark 2.6.23. We proved that if $q_1, q_2 \geq 2$, then the first Pontrjagin class of $K \times_{K_H} \mathfrak{p}/\mathfrak{p}_H$ does not vanish above. In fact, the converse is also true. That is, if $q_1 = 1$ or $q_2 = 1$, then the first Pontrjagin class of $K \times_{K_H} \mathfrak{p}/\mathfrak{p}_H$ vanishes.

Proof of Proposition 2.6.11(b). Suppose $q_1 \geq 2$ or $q_2 \geq 2$. From Lemma 2.6.22(b), we get $\mathrm{ad}^*(\tilde{p_1}) = pa^1_{(1,q_1)} + (q_1 b^1_{(1,q)} + 2b^1_{(1,q_1)})b^1_{(1,q)}$. Since $b^1_{(1,q)} \in \ker\omega$, it is enough to show $a^1_{(1,q_1)} \notin \ker\omega$. That is, we show that there is no $(c_1, c_2, c_3) \in \mathbb{R}^3$ such that

$$a^1_{(1,q_1)} = (c_1 b^1_{(1,q_1)} + c_2 b^1_{(q_1+1,q_1+q_2)})b^1_{(1,q)} + c_3 b^2_{(1,q)}.$$

By direct computation, we can find that this is true under the assumption that $q_1 \geq 2$ or $q_2 \geq 2$. □

Remark 2.6.24. We proved that if $q_1 \geq 2$ or $q_2 \geq 2$, then the first Pontrjagin class $p_1(K \times_{K_H} \mathfrak{p}/\mathfrak{p}_H)$ does not vanish. In fact, the converse is also true. That is, if $q_1 = q_2 = 1$ then $p_1(K \times_{K_H} \mathfrak{p}/\mathfrak{p}_H)$ vanish. This follows from the following equality:

$$a^1_{(1,q_1)} = b^1_{(1,q_1)} b^1_{(1,q_1+q_2)} - b^2_{(1,q_1+q_2)}.$$

Proof of Proposition 2.6.11(c). We show $\mathrm{ad}^*(\tilde{p_1}) \notin \ker\omega$. This comes from Lemma 2.6.22(c), $\mathrm{ad}^*(\tilde{p_1}) = 2pa^1_{(1,q_1)}$ and Lemma 2.6.21(c). We can find this by seeing the degree of generators of $\ker\omega$. □

2.6.3 Calculation of first Pontrjagin class of $SO_0(2p, 2q)/SU(p, q)$

We prove Proposition 2.6.12 by calculating the first Pontrjagin class of the corresponding vector bundle $K \times_{K_H} (\mathfrak{p}/\mathfrak{p}_H)$. We take the same steps with the previous subsection except for step(1).
■ step(0): Realization of (G, H) and (K, K_H) as a matrix groups.
We realize a symmetric pair $(G, H) = (SO_0(2p, 2q), U(p, q))$ as follows.

$$G = \{g \in GL(2(p+q), \mathbb{R}) : {}^t g I_{2p,2q} g = I_{2p,2q},\ \det g = 1\}_0,$$
$$H = \left\{ g \in G : g \begin{pmatrix} J_p & \\ & J_q \end{pmatrix} = \begin{pmatrix} J_p & \\ & J_q \end{pmatrix} g \right\}.$$

Then, by taking a Cartan involution $\theta : g \mapsto {}^t g^{-1}$, we have:

$$
\begin{aligned}
K = SO(2p) \times SO(2q) &= \left\{ \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix} : k_1 \in SO(2p), k_2 \in SO(2q) \right\}, \\
\mathfrak{p} &= \left\{ \begin{pmatrix} 0 & B \\ {}^t B & 0 \end{pmatrix} \in M(2(p+q), \mathbb{R}) : B \in M(2p, 2q; \mathbb{R}) \right\}, \\
\mathfrak{p}_H &= \left\{ \begin{pmatrix} 0 & B \\ {}^t B & 0 \end{pmatrix} \in M(2(p+q), \mathbb{R}) : B = \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix}, A_1, A_2 \in M(p, q; \mathbb{R}) \right\}.
\end{aligned}
$$

We can identify $\mathfrak{p}/\mathfrak{p}_H$ with the following $\mathrm{Ad}(K_H)$-invariant subspace of $M(2(p+q), \mathbb{R})$.

$$
\mathfrak{p}/\mathfrak{p}_H \simeq \left\{ \begin{pmatrix} 0 & B \\ {}^t B & 0 \end{pmatrix} \in M(2(p+q), \mathbb{R}) : B = \begin{pmatrix} B_1 & B_2 \\ B_2 & -B_1 \end{pmatrix}, B_1, B_2 \in M(p, q : \mathbb{R}) \right\}.
$$

■ step(2): Fix a coordinates of maximal tori $\mathfrak{t}$ and $\mathfrak{t}_H$ of $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{k}_H = \mathrm{Lie}(K_H)$ respectively.
Fix maximal tori $\mathfrak{t}, \mathfrak{t}_H$ of $\mathfrak{k}, \mathfrak{k}_H$ respectively as follows.

$$
\begin{aligned}
\mathfrak{t} = \mathfrak{t}_H &= \left\{ \sum_{i=1}^{p} t_i A_{i,p+i} + \sum_{j=1}^{q} t_{p+j} A_{2p+j, 2p+q+j} : t_i \in \mathbb{R} \ (i = 1, \cdots, p+q) \right\} \\
&= \left\{ \begin{pmatrix}
 & & & t_1 & & & & & & & & \\
 & & & & \ddots & & & & & & & \\
 & & & & & t_p & & & & & & \\
 -t_1 & & & & & & & & & & & \\
 & \ddots & & & & & & & & & & \\
 & & -t_p & & & & & & & & & \\
 & & & & & & & & & t_{p+1} & & \\
 & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & t_{p+q} \\
 & & & & & & -t_{p+1} & & & & & \\
 & & & & & & & \ddots & & & & \\
 & & & & & & & & -t_{p+q} & & &
\end{pmatrix} : t_i \in \mathbb{R} \ (i = 1, \cdots, p+q) \right\}.
\end{aligned}
$$

We take a coordinate $\{t_i\}_{i=1}^{p+q}$ of $\mathfrak{t} = \mathfrak{t}_H$ by the above basis.
■ step(3): Description of $S(\mathfrak{t}^*)^W$ and $S(\mathfrak{t}_H^*)^{W_H}$.
Then we get

$$
\begin{aligned}
S(\mathfrak{t}^*)^W &= \mathbb{R}[a^1_{(1,p)}, \cdots, a^p_{(1,p)}, b^p_{(1,p)}, a^1_{(p+1,p+q)}, \cdots, a^q_{(p+1,p+q)}, b^q_{(p+1,p+q)}], \\
S(\mathfrak{t}_H^*)^{W_H} &= \mathbb{R}[b^1_{(1,p)}, \cdots, b^p_{(1,p)}, b^1_{(p+1,p+q)}, \cdots, b^q_{(p+1,p+q)}].
\end{aligned}
$$

■ step(4): Description of $\ker \omega$.
Since $\mathfrak{t} = \mathfrak{t}_H$, we get:

Lemma 2.6.25.

$$\begin{aligned}&\ker\omega = \text{ideal generated by}\\ &a^1_{(1,p)}, \cdots, a^p_{(1,p)}, b^p_{(1,p)}, a^1_{(p+1,p+q)}, \cdots, a^q_{(p+1,p+q)}, b^q_{(p+1,p+q)}\\ &\text{on } S(\mathfrak{t}_H^*)^{W_H}.\end{aligned}$$

■ step(5): Description of $\mathrm{ad}^*(\tilde{p_1}) \in S(\mathfrak{t}_H^*)^{W_H}$ for the $SO(\mathfrak{p}/\mathfrak{p}_H)$-invariant polynomial $\tilde{p_1}$ on $\mathfrak{so}(\mathfrak{p}/\mathfrak{p}_H)$ of degree two.
By direct computation, we get:

Lemma 2.6.26.

$$\mathrm{ad}^*(\tilde{p_1}) = qa^1_{(1,p)} + pa^1_{(p+1,p+q)} + 2b^1_{(1,p)}b^1_{(p+1,p+q)}.$$

■ step(6): Check whether $\mathrm{ad}^*(\tilde{p_1}) \in \ker\omega$ or not.
We prove $\mathrm{ad}^*(\tilde{p_1}) \notin \ker\omega$ if $2 \leq p(\leq q)$.

Proof of Proposition 2.6.12. Suppose $p \geq 2$. From Lemma 2.6.26, we get $\mathrm{ad}^*(\tilde{p_1}) = qa^1_{(1,p)} + pa^1_{(p+1,p+q)} + 2b^1_{(1,p)}b^1_{(p+1,p+q)}$. Since $a^1_{(1,p)}$ and $a^1_{(p+1,p+q)}$ are in $\ker\omega$ from Lemma 2.6.25, it is enough to show that $b^1_{(1,p)}b^1_{(p+1,p+q)} \notin \ker\omega$. Since there is no generator of degree one of $\ker\omega$, if $b^1_{(1,p)}b^1_{(p+1,p+q)} \in \ker\omega$, it can be written as a $\mathbb{R}$-linear combination of generators of degree two of $\ker\omega$. However it is impossible. Thus, $b^1_{(1,p)}b^1_{(p+1,p+q)} \notin \ker\omega$. □

2.6.4 Calculation of the first Pontrjagin class of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{h}^a) = (\mathfrak{e}_{6(6)}, \mathfrak{su}^*(6) \oplus \mathfrak{su}(2), f_{4(4)})$

In this section, we consider the symmetric pair (G, H, H^a) where G is a connected linear reductive Lie group and corresponding Lie algebras are $(\mathfrak{e}_{6(6)}, \mathfrak{su}^*(6) \oplus \mathfrak{su}(2), \mathfrak{f}_{4(4)})$. Here $d(G) = \dim\mathfrak{p} = 42$, $d(H) = \dim\mathfrak{p}_H = 14$, $d(H^a) = 28$.

Our goal in this subsection is the following:

Proposition 2.6.27. Neithere G_θ/H_θ nor G_θ/H^a_θ admit compact Clifford–Klein forms.

Proof. This comes from Lemma 2.6.29. □

Lemma 2.6.28. Let $\mathfrak{g}$ be a semisimple Lie algebra without compact simple ideal. Then the isotropy representation $\mathrm{ad}_{\mathfrak{k}} : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$ is faithful. Moreover, $\mathrm{ad}_{\mathfrak{k}}$ is irreducible if and only if $\mathfrak{g}$ is simple.

Proof. $\ker\mathrm{ad}_{\mathfrak{k}} \subset \mathfrak{k} \subset \mathfrak{g}$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$. Since $\mathfrak{g}$ has no compact simple ideal, we obtain $\ker\mathrm{ad}_{\mathfrak{k}} = \{0\}$. □

Lemma 2.6.29. The associated bundle $K \times_{(K_H, \mathrm{Ad})} \mathfrak{p}/\mathfrak{p}_H$ over K/K_H is not trivial.

Proof. This comes from the following lemma. □

Lemma 2.6.30. The first Pontrjagin class of the associated bundle $K \times_{(K_H, \mathrm{Ad})} \mathfrak{p}/\mathfrak{p}_H$ over K/K_H does not vanish.

Proof. This comes from Fact 2.6.10 and Lemma 2.6.37. $\square$

We give a realization of the symmetric pair $\mathfrak{k} = \mathfrak{sp}(4) \supset \mathfrak{sp}(3) \oplus \mathfrak{sp}(1) = \mathfrak{k}_H$ as follows:

$$\begin{aligned}
\mathfrak{sp}(4) &:= \{X \in \mathfrak{sl}(8, \mathbb{C}) : {}^t X J + JX = 0, X^* + X = 0\} \\
&= \left\{ \begin{pmatrix} A & -B \\ \overline{B} & \overline{A} \end{pmatrix} : A \in \mathfrak{su}(4), B \in \mathrm{Sym}(4, \mathbb{C}) \right\}, \\
&\tau : \mathfrak{sp}(4) \to \mathfrak{sp}(4), X \mapsto I_{3,1:3,1} X I_{3,1:3,1}^{-1}
\end{aligned}$$

$$\mathfrak{sp}(3) := \mathfrak{sp}(4)^\tau = \{X \in \mathfrak{sp}(4) : \begin{pmatrix} A_1 & & -B_1 & \\ & \alpha & & -\beta \\ \overline{B_1} & & \overline{A_1} & \\ & \overline{\beta} & & \overline{\alpha} \end{pmatrix} : A_1 \in \mathfrak{su}(3), B_1 \in \mathrm{Sym}(3, \mathbb{C}), \alpha, \beta \in \mathbb{C}\}$$

We take maximal tori of $\mathfrak{k}$, $\mathfrak{k}_H$ as follows:

$$\mathfrak{t} = \mathfrak{t}_H = \{i \operatorname{diag}(t_1, t_2, t_3, t_4, -t_1, -t_2, -t_3, -t_4) : t_k \in \mathbb{R}\ (k = 1, 2, 3, 4)\}.$$

Then we have

$$\begin{aligned}
S(\mathfrak{t}^*)^W &= \mathbb{R}\left[t_1^2 + t_2^2 + t_3^2 + t_4^2, t_1^2 t_2^2 + t_1^2 t_3^2 + t_1^2 t_4^2 + t_2^2 t_3^2 + t_2^2 t_4^2 + t_3^2 t_4^2, t_1^2 t_2^2 t_3^3 + t_1^2 t_2^2 t_4^2 + t_1^2 t_3^2 t_4^2 + t_2^2 t_3^2 t_4^2, t_1^2 t_2^2 t_3^2 t_4^2\right] \\
S(\mathfrak{t}_H^*)^{W_H} &= \mathbb{R}\left[t_1^2 + t_2^2 + t_3^2, t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2, t_1^2 t_2^2 t_3^2, t_4^2\right]
\end{aligned}$$

From Fact 2.6.20, we have

$$\begin{aligned}
\ker \omega = \text{ideal generated by} \\
& t_1^2 + t_2^2 + t_3^2 + t_4^2, \\
& t_1^2 t_2^2 + t_1^2 t_3^2 + t_1^2 t_4^2 + t_2^2 t_3^2 + t_2^2 t_4^2 + t_3^2 t_4^2, \\
& t_1^2 t_2^2 t_3^3 + t_1^2 t_2^2 t_4^2 + t_1^2 t_3^2 t_4^2 + t_2^2 t_3^2 t_4^2, \\
& t_1^2 t_2^2 t_3^2 t_4^2 \\
& \text{on } S(\mathfrak{t}_H^*)^{W_H}
\end{aligned}$$

We consider isotropy representation of H^a, which is equivalent to the action of $\mathfrak{k}_H$ on $\mathfrak{p}/\mathfrak{p}_H$ from Remark 2.6.42.

Claim. $\mathrm{ad}_{\mathfrak{k}_{H^a}} : \mathfrak{k}_{H^a} = \mathfrak{sp}(3) \oplus \mathfrak{sp}(1) \to \mathfrak{sl}(\mathfrak{p}_{H^a})$ is the irreducible representation corresponding to one with the highest weight $\varpi_3 \boxtimes \varpi_1$ by Cartan's fundamental theorem.

Proof. Since $\mathrm{ad}_{\mathfrak{k}}$ is irreducible, $\mathrm{ad}_{\mathfrak{k}}$ can be $\pi \boxtimes \pi'$ where π, π' are irreducible representations of $\mathfrak{sp}(3)$, $\mathfrak{sp}(1)$ respectively. The dimensions of irreducible representations of $\mathfrak{sp}(3)$ are 1, 6, 14(twice), 21, 64, 70, $\cdots$. Since the dimension divide 28, the possible representations of $\mathfrak{sp}(3)$ are trivial, ϖ_2 or ϖ_3. From

Lemma 2.6.28, π is ϖ_2 or ϖ_3. On the other hand, from Lemma 2.6.28 and dimension of $\mathrm{ad}_{\mathfrak{k}}$, we have $\pi' = \varpi_1$, which is two dimensional representation. Since $\mathrm{index}_{\theta_2} \varpi_1 = -1$, $\mathrm{index}_{\theta_1} \pi = -1$ holds, where θ_i $(i = 1, 2)$ are Cartan involutions such that $\mathfrak{sp}(3, \mathbb{C})^{\theta_1} = \mathfrak{sp}(3)$ and $\mathfrak{sp}(1, \mathbb{C})^{\theta_2} = \mathfrak{sp}(1)$. So we obtain $\pi = \varpi_3$. $\square$

Next, we consider a realization of isotropy representation of G and H^a. we realize $\mathrm{ad}_{\mathfrak{k}_{H^a}} : \mathfrak{sp}(3) \oplus \mathfrak{sp}(1) \to \mathfrak{sl}(\mathbb{R}^{28})$ on the subspace of $((\bigwedge^3 \mathbb{C}^6) \otimes \mathbb{C}^2)^{\sigma'}$. Here

$$\sigma' : (\bigwedge^3 \mathbb{C}^6) \otimes \mathbb{C}^2 \to (\bigwedge^3 \mathbb{C}^6) \otimes \mathbb{C}^2, \quad (v_1 \wedge v_2 \wedge v_3) \otimes w \mapsto (J_3\overline{v_1} \wedge J_3\overline{v_2} \wedge J_3\overline{v_3}) \otimes J_1\overline{w}$$

Remark 2.6.31. The adjoint representation $\mathfrak{k} \curvearrowright \mathfrak{p} \simeq \mathbb{R}^{42}$ is equivalent to one corresponding to $(\mathfrak{sp}(4), \varpi_4)$ by Cartan's fundamental theorem (see Appendix 1.6.3), which is in class $C^I(\mathfrak{sp}(4))$. The representation space can be described as follows:

Define a anti holomorphic involution σ_k on $\bigwedge^k \mathbb{C}^8$ by

$$\sigma_k(v_1 \wedge \cdots \wedge v_k) = J_4\overline{v_1} \wedge \cdots \wedge J_4\overline{v_k}.$$

Define $\mathbb{C}$-linear map $\varphi_k : \bigwedge^k \mathbb{C}^8 \to \bigwedge^{k-2} \mathbb{C}^8$ by

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{1 \leq i < j \leq k} Q(v_i, v_j)(-1)^{i+j-1} v_1 \wedge \cdots \wedge \hat{v_i} \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_k.$$

Here $\mathbb{C}$-bilinear form $Q : \mathbb{C}^8 \times \mathbb{C}^8 \to \mathbb{C}$ is defined by $Q(v, w) := {}^t v J_4 w$.

Remark 2.6.32. We have $\varphi_k \sigma_k = \sigma_{k-2} \varphi_k$ and $\sigma_k \circ (\bigwedge^k \rho^{(8)}_{\varpi_1}(X)) = (\bigwedge^k \rho^{(8)}_{\varpi_1}(X)) \circ \sigma_k$ for all $X \in \mathfrak{sp}(4)$. Here $\bigwedge^k \rho^{(8)}_{\varpi_1}$ is the representation induced by ρ_{ϖ_1} on $\bigwedge^k \mathbb{C}^8$.

Fact 2.6.33 (see [FH] for example)**.** The fundamental representation with highest weight ϖ_k is realized by $\ker \varphi_k$.

For any $X \in \mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$, the following diagram commutes:

$$\begin{tikzcd}
(\bigwedge^3 \mathbb{C}^6) \otimes \mathbb{C}^2 \arrow[r, "\iota"] \arrow[d, "(\bigwedge^3 \rho^{(6)}_{\varpi_1} \boxtimes \rho^{(2)}_{\varpi_1})(X)"'] & \bigwedge^4 \mathbb{C}^8 \arrow[d, "\bigwedge^4 \rho^{(8)}_{\varpi_1}(X)"] \\
(\bigwedge^3 \mathbb{C}^6) \otimes \mathbb{C}^2 \arrow[r, "\iota"] \arrow[d, "\sigma'"'] & \bigwedge^4 \mathbb{C}^8 \arrow[d, "\sigma"] \\
(\bigwedge^3 \mathbb{C}^6) \otimes \mathbb{C}^2 \arrow[r, "\iota"'] & \bigwedge^4 \mathbb{C}^8
\end{tikzcd}$$

Here

$$\iota : (\bigwedge^3 \mathbb{C}^6) \otimes \mathbb{C}^2 \to \bigwedge^4 \mathbb{C}^8, \qquad (v_1 \wedge v_2 \wedge v_3) \otimes w \mapsto \iota_1(v_1) \wedge \iota_1(v_2) \wedge \iota_1(v_3) \wedge \iota_2(w)$$
$$\iota_1 : \mathbb{C}^6 \to \mathbb{C}^8, \qquad {}^t(z_1, z_2, z_3, z_4, z_5, z_6) \mapsto {}^t(z_1, z_2, z_3, 0, z_4, z_5, z_6, 0)$$
$$\iota_2 : \mathbb{C}^2 \to \mathbb{C}^8, \qquad {}^t(w_1, w_2) \mapsto {}^t(0, 0, 0, w_1, 0, 0, 0, w_2)$$

We define Hermitian form on $\bigwedge^4 \mathbb{C}^8$ by

$$H(v, w) := \frac{1}{2} \det((v_i, w_j))$$

for $v = v_1 \wedge v_2 \wedge v_3 \wedge v_4$, $w = w_1 \wedge w_2 \wedge w_3 \wedge w_4$. Here $(\cdot, \cdot) : \mathbb{C}^8 \times \mathbb{C}^8 \to \mathbb{C}$ is the standard Hermitian form on $\mathbb{C}^8$. H is $\bigwedge^4 \rho(\mathfrak{sp}(4))$-invariant by definition.

Remark 2.6.34. $H(v, w) = \overline{H(v, w)}$ if $v, w \in (\bigwedge^4 \mathbb{C}^8)^\sigma$. So we have the symmetric nondegenerate bilinear form $H_0 : (\bigwedge^4 \mathbb{C}^8)^\sigma \times (\bigwedge^4 \mathbb{C}^8)^\sigma \to \mathbb{R}$.

We construct an orthonormal basis of the representation space V of the irreducible representation $\rho_{\varpi_3} \boxtimes \rho_{\varpi_1}$ of $\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$. To calculate matrix representation easily, we make use of weight vectors.

Fact 2.6.35. Let $\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1} : \mathfrak{sp}(3) \oplus \mathfrak{sp}(1) \to \mathfrak{sl}(V)$ be an irreducible representation with highest weight $\varpi_3 \oplus \varpi_1$. The weights $W(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})$ are given as follows:

$$W(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1}) = \{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4, \pm\varepsilon_1 \pm \varepsilon_4, \pm\varepsilon_2 \pm \varepsilon_4, \pm\varepsilon_3 \pm \varepsilon_4, \}$$

Remark 2.6.36. For $\lambda \in W(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})$, the relation $\sigma(V_\lambda) = V_{-\lambda}$ holds.

Put $I_0 = \{1, 2, 3, 4, 5, 6, 7, 8\}$. For $I = (i_1, i_2, i_3, i_4) \in I_0^4$, we describe the element $e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}$ by e_I.

weight	weight vector	weight	weight vector
$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$	$e_{(1,2,3,4)}$	$-(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$	$e_{(5,6,7,8)} = \sigma(e_{(1,2,3,4)})$
$\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4$	$e_{(1,2,7,4)}$	$-(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$	$-e_{(5,6,2,8)} = \sigma(e_{(1,2,7,4)})$
$\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4$	$e_{(1,6,3,4)}$	$-(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$	$-e_{(5,2,7,8)} = \sigma(e_{(1,6,3,4)})$
$\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4$	$e_{(1,6,7,4)}$	$-(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)$	$e_{(5,2,3,8)} = \sigma(e_{(1,6,7,4)})$
$-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$	$e_{(5,2,3,4)}$	$-(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$	$-e_{(1,6,7,8)} = \sigma(e_{(5,2,3,4)})$
$-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4$	$e_{(5,2,7,4)}$	$-(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4)$	$e_{(1,6,3,8)} = \sigma(e_{(5,2,7,4)})$
$-\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4$	$e_{(5,6,3,4)}$	$-(-\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$	$e_{(1,2,7,8)} = \sigma(e_{(5,6,3,4)})$
$-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4$	$e_{(5,6,7,4)}$	$-(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)$	$-e_{(1,2,3,8)} = \sigma(e_{(5,6,7,4)})$
$\varepsilon_1 + \varepsilon_4$	$e_{(1,2,6,4)} - e_{(1,3,7,4)}$	$-(\varepsilon_1 + \varepsilon_4)$	$e_{(5,2,6,8)} - e_{(5,3,7,8)}$
$-\varepsilon_1 + \varepsilon_4$	$e_{(5,2,6,4)} - e_{(5,3,7,4)}$	$-(-\varepsilon_1 + \varepsilon_4)$	$-(e_{(1,2,6,8)} - e_{(1,3,7,8)})$
$\varepsilon_2 + \varepsilon_4$	$e_{(2,1,5,4)} - e_{(2,3,7,4)}$	$-(\varepsilon_2 + \varepsilon_4)$	$e_{(6,1,5,4)} - e_{(6,3,7,8)}$
$-\varepsilon_2 + \varepsilon_4$	$e_{(6,1,5,4)} - e_{(6,3,7,4)}$	$-(-\varepsilon_2 + \varepsilon_4)$	$-(e_{(2,1,5,8)} - e_{(2,3,7,8)})$
$\varepsilon_3 + \varepsilon_4$	$e_{(3,1,5,4)} - e_{(3,2,6,4)}$	$-(\varepsilon_3 + \varepsilon_4)$	$e_{(7,1,5,8)} - e_{(7,2,6,8)}$
$-\varepsilon_3 + \varepsilon_4$	$e_{(7,1,5,4)} - e_{(7,2,6,4)}$	$-(-\varepsilon_1 + \varepsilon_4)$	$-(e_{(3,1,5,8)} - e_{(3,2,6,8)})$

We take an orthonormal basis on the representation space of $\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1}$ as follows:

$$
\begin{aligned}
& e_I + \sigma(e_I), \sqrt{-1}(e_I - \sigma(e_I)), I \in \{1,5\} \times \{2,6\} \times \{3,7\} \times \{4\} \\
& w_1^+ := \sqrt{\tfrac{1}{2}}(e_{(1,2,6,4)} - e_{(1,3,7,4)} + \sigma(e_{(1,2,6,4)} - e_{(1,3,7,4)})), \\
& w_1^- := \sqrt{\tfrac{-1}{2}}(e_{(1,2,6,4)} - e_{(1,3,7,4)} - \sigma(e_{(1,2,6,4)} - e_{(1,3,7,4)})) \\
& w_2^+ := \sqrt{\tfrac{1}{2}}(e_{(5,2,6,4)} - e_{(5,3,7,4)} + \sigma(e_{(5,2,6,4)} - e_{(5,3,7,4)})) \\
& w_2^- := \sqrt{\tfrac{-1}{2}}(e_{(5,2,6,4)} - e_{(5,3,7,4)} - \sigma(e_{(5,2,6,4)} - e_{(5,3,7,4)})) \\
& w_3^+ := \sqrt{\tfrac{1}{2}}(e_{(2,1,5,4)} - e_{(2,3,7,4)} + \sigma(e_{(2,1,5,4)} - e_{(2,3,7,4)})) \\
& w_3^- := \sqrt{\tfrac{-1}{2}}(e_{(2,1,5,4)} - e_{(2,3,7,4)} - \sigma(e_{(2,1,5,4)} - e_{(2,3,7,4)})) \\
& w_4^+ := \sqrt{\tfrac{1}{2}}(e_{(6,1,5,4)} - e_{(6,3,7,4)} + \sigma(e_{(6,1,5,4)} - e_{(6,3,7,4)})) \\
& w_4^- := \sqrt{\tfrac{-1}{2}}(e_{(6,1,5,4)} - e_{(6,3,7,4)} - \sigma(e_{(6,1,5,4)} - e_{(6,3,7,4)})) \\
& w_5^+ := \sqrt{\tfrac{1}{2}}(e_{(3,1,5,4)} - e_{(3,2,6,4)} + \sigma(e_{(3,1,5,4)} - e_{(3,2,6,4)})) \\
& w_5^- := \sqrt{\tfrac{-1}{2}}(e_{(3,1,5,4)} - e_{(3,2,6,4)} - \sigma(e_{(3,1,5,4)} - e_{(3,2,6,4)})) \\
& w_6^+ := \sqrt{\tfrac{1}{2}}(e_{(7,1,5,4)} - e_{(7,2,6,4)} + \sigma(e_{(7,1,5,4)} - e_{(7,2,6,4)})) \\
& w_6^- := \sqrt{\tfrac{-1}{2}}(e_{(7,1,5,4)} - e_{(7,2,6,4)} - \sigma(e_{(7,1,5,4)} - e_{(7,2,6,4)}))
\end{aligned}
$$

Let $t = \sqrt{-1}\,\mathrm{diag}(t_1, t_2, t_3, t_4, -t_1, -t_2, -t_3, -t_4) \in \mathfrak{t}$. Then we have for $I \in$

$\{1,5\} \times \{2,6\} \times \{3,7\} \times \{4\}$

$$(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t)(e_I + \sigma(e_I)) = ((-1)^{\lfloor i_1/4 \rfloor} t_1 + (-1)^{\lfloor i_2/4 \rfloor} t_2 + (-1)^{\lfloor i_3/4 \rfloor} t_3 + t_4))(e_I - \sigma(e_I)),$$
$$(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t)\sqrt{-1}(e_I - \sigma(e_I)) = -((-1)^{\lfloor i_1/4 \rfloor} t_1 + (-1)^{\lfloor i_2/4 \rfloor} t_2 + (-1)^{\lfloor i_3/4 \rfloor} t_3 + t_4))(e_I + \sigma(e_I)),$$
$$\begin{aligned}
(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t) w_1^+ &= (t_1 + t_4) w_1^- \\
(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t) w_1^- &= -(t_1 + t_4) w_1^+ \\
(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t) w_2^+ &= (-t_1 + t_4) w_2^- \\
(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t) w_2^- &= -(-t_1 + t_4) w_2^+ \\
(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t) w_3^+ &= (t_2 + t_4) w_3^- \\
(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t) w_3^- &= -(t_2 + t_4) w_3^+ \\
(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t) w_4^+ &= (-t_2 + t_4) w_4^- \\
(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t) w_4^- &= -(t_2 + t_4) w_4^+ \\
(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t) w_5^+ &= (t_3 + t_4) w_5^- \\
(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t) w_5^- &= -(t_3 + t_4) w_5^+ \\
(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t) w_6^+ &= (-t_3 + t_4) w_6^- \\
(\rho^{(3)}_{\varpi_3} \boxtimes \rho^{(1)}_{\varpi_1})(t) w_6^- &= -(-t_3 + t_4) w_6^+
\end{aligned}$$

Lemma 2.6.37. (i) $\mathrm{ad}^*(\tilde{p_1}) = 10(t_1^2 + t_2^2 + t_3^2 + t_4^2) + 4t_4^2$,

(ii) $\mathrm{ad}^*(\tilde{p_1}) \notin \ker \omega$.

Here $\tilde{p_1} \in S^2(\mathfrak{p}^*_{H^a})^{SO(\mathfrak{p}_{H^a})}$ be all the sum of principal minors of degree two.

Proof. (i) By using the above basis,

$$\begin{aligned}
\mathrm{ad}^*(\tilde{p_1}) &= (t_1 + t_2 + t_3 + t_4)^2 + (t_1 + t_2 - t_3 + t_4)^2 \\
&+ (t_1 - t_2 + t_3 + t_4)^2 + (t_1 - t_2 - t_3 + t_4)^2 \\
&+ (-t_1 + t_2 + t_3 + t_4)^2 + (-t_1 + t_2 - t_3 + t_4)^2 \\
&+ (-t_1 - t_2 + t_3 + t_4)^2 + (-t_1 - t_2 - t_3 + t_4)^2 \\
&+ (t_1 + t_4)^2 + (-t_1 + t_4)^2 \\
&+ (t_2 + t_4)^2 + (-t_2 + t_4)^2 \\
&+ (t_3 + t_4)^2 + (-t_3 + t_4)^2 \\
&= 10(t_1^2 + t_2^2 + t_3^2 + t_4^2) + 4t_4^2
\end{aligned}$$

(ii) This comes from (i) and the description of $\ker \omega$ given above.

□

2.6.5 Calculation of the first Pontrjagin class of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{h}^a) = (\mathfrak{e}_{7(7)}, \mathfrak{su}(2) \oplus \mathfrak{so}^*(12), \mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2))$

In this section, we consider the symmetric pair (G, H, H^a) where G is a connected linear reductive Lie group and corresponding Lie algebras are $(\mathfrak{e}_{7(7)}, \mathfrak{su}(2) \oplus$

$\mathfrak{so}^*(12), \mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2))$. Here $\dim \mathfrak{p} = 70$, $\dim \mathfrak{p}_H = 30$ and $\dim \mathfrak{p}_{H^a} = 40$.

Our goal in this subsection is the following:

Proposition 2.6.38. Neither G_θ/H_θ nor G_θ/H_θ^a admit compact Clifford–Klein forms.

Proof. This comes from Lemma 2.6.39. □

Lemma 2.6.39. The associated bundle $K \times_{(K_H, \mathrm{Ad})} \mathfrak{p}/\mathfrak{p}_H$ is not trivial.

Proof. This comes from the following Lemma 2.6.40. □

Lemma 2.6.40. The first Pontrjagin class of the associated bundle $K \times_{(K_H,\mathrm{Ad})} \mathfrak{p}/\mathfrak{p}_H$ over K/K_H does not vanish.

Lemma 2.6.41. $\mathrm{ad} : \mathfrak{k} = \mathfrak{su}(8) \to \mathfrak{gl}(\mathfrak{p})$ comes from fundamental representation ρ_{ϖ_4} through Cartan's fundamental theorem.

Proof. Since $\mathfrak{e}_{7(7)}$ is simple, $\mathrm{ad} : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$ is an irreducible representation. So, this lemma comes from Cartan's fundamental theorem and Weyl's dimensionality formula. □

Remark 2.6.42. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation equipped with invariant bilinear form B on V and W a invariant subspace of V. Then the following representations are equivalent:

- $\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V/W)$, $\rho_1(X)(v + W) := \rho(X)v + W$,
- $\rho_2 : \mathfrak{g} \to \mathfrak{gl}(W^\perp)$, $\rho_2(X)w^\perp := \rho(X)w^\perp$.

Here $W^\perp := \{v \in V : B(v, w) = 0 \text{ for all } X \in V\}$.

Lemma 2.6.43. The representation $\mathrm{ad}|_{\mathfrak{k}_H} : \mathfrak{k}_H \to \mathfrak{gl}(\mathfrak{p}/\mathfrak{p}_H)$ is equivalent to $\mathrm{ad}|_{\mathfrak{k}_{H^a}} : \mathfrak{k}_{H^a} \to \mathfrak{gl}(\mathfrak{p}_{H^a})$. Moreover, the restriction of $\mathrm{ad}|_{\mathfrak{k}_{H^a}}$ to $\mathfrak{su}(6) \oplus \mathfrak{su}(2)$ comes from $\rho_{\varpi_3} \boxtimes \rho_{\varpi_1}$ through Cartan's fundamental theorem.

Proof. This comes from Lemma 2.6.42. For the latter statement, it is equivalent to consider the isotropy representation of $\mathfrak{e}_{6(2)}$. Since $\mathfrak{e}_{6(2)}$ is simple, the isotropy representation ρ is irreducible. From Cartan's fundamental theorem and Weyl's dimensionality formula, we obtain the "Moreover part". □

We consider a realization of $\mathrm{ad} : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$ to realize $\mathrm{ad}_{\mathfrak{k}_{H^a}} : \mathfrak{k}_{H^a} \to \mathfrak{gl}(\mathfrak{p}_{H^a})$. The representation space $\mathfrak{p}$ is given by fixed points of the star operator on $\bigwedge^4 \mathbb{C}^8$.

We recall the definition on star operator. Set an inner product H on $\bigwedge \mathbb{C}^n$ by

$$H(v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p) := \det((v_i, w_j)).$$

Here $(,)$ is the standard inner product on $\mathbb{C}^n$ and $H(v, w) = 0$ if $v \in \bigwedge^p \mathbb{C}^n$, $w \in \bigwedge^q \mathbb{C}^n$ and $p \neq q$. Fix an orthonormal basis $e_1, \cdots, e_n$.

Definition 2.6.44. We define an anti holomorphic linear isomorphism $*_n : \bigwedge \mathbb{C}^n \to \bigwedge \mathbb{C}^n$ as follows:

(i) $*_n(e_1 \wedge \cdots \wedge e_n) = 1$,

(ii) For $v \in \bigwedge^p \mathbb{C}^n$, $*_n v$ is determined by $\overline{*_n(v \wedge w)} = H(w, *_n v)$ for all $w \in \bigwedge^{n-p} \mathbb{C}^n$.

Remark 2.6.45. The star operator on $(\bigwedge \mathbb{C}^n, H)$ is determined up to signature depending on the choice of orthonormal basis.

Fact 2.6.46. For $v \in \wedge^p \mathbb{C}^n$, $*_n^2 v = (-1)^{p(n-p)} v$ holds.

Corollary 2.6.47. The restriction of $*_{2n}$ on $\bigwedge^n \mathbb{C}^{2n}$ satisfies

$$*_{2n}^2 = \begin{cases} 1 \text{ if } n \text{ is even,} \\ -1 \text{ if } n \text{ is odd.} \end{cases}$$

We realize $\mathfrak{k} = \mathfrak{su}(8)$ and $\mathfrak{k}_{H^a} = \mathfrak{k}_H \simeq \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ as follows:

$$\begin{aligned} \mathfrak{su}(8) &:= \{X \in \mathfrak{sl}(8, \mathbb{C}) : X^* + X = 0\} \\ \sigma &: \mathfrak{su}(8) \to \mathfrak{su}(8), X \mapsto I_{3,1;3,1} X I_{3,1;3,1}^{-1} \\ \mathfrak{k}_{H^a} &= \mathfrak{k}_H = \mathfrak{k}^\sigma \end{aligned}$$

We take a maximal tori $\mathfrak{t} = \mathfrak{t}_H$ as follows:

$$\mathfrak{t} = \mathfrak{t}_H = \{i \operatorname{diag}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) : t_i \in \mathbb{R}, \sum_{i=1}^{8} t_i = 0\}.$$

Remark 2.6.48.

$$\begin{aligned} S(\mathfrak{t}^*)^W &\simeq \mathbb{R}[s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8]/(s_1)_{\mathfrak{t}} \\ S(\mathfrak{t}_H^*)^{W_H} &\simeq \mathbb{R}[s'_1, s'_2, s'_3, s'_4, s'_5, s'_6, s''_1, s''_2]/(s_1)_{\mathfrak{t}_H} \end{aligned}$$

Here s_i, s'_j and s''_k are the fundamental symmetric polynomial of degree i, j and k with respect to $\{t_1, \cdots, t_8\}$, $\{t_1, t_2, t_3, t_5, t_6, t_7\}$ and $\{t_4, t_8\}$ respectively. $(s_1)_{\mathfrak{t}}$ and $(s_1)_{\mathfrak{t}_H}$ are ideals generated by s_1 over $\mathbb{R}[s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8]$ and $\mathbb{R}[s'_1, s'_2, s'_3, s'_4, s'_5, s'_6, s''_1, s''_2]$ respectively.

We take an orthonormal basis on $(\iota(\bigwedge^3 \mathbb{C}^6 \otimes \mathbb{C}^2))^* \subset \bigwedge^4 \mathbb{C}^8$ as follows, where ι is given in the previous subsection:

$$\begin{gathered} e_I \wedge e_4 + *(e_I \wedge e_4) \\ ie_I \wedge e_4 + *(ie_I \wedge e_4) \end{gathered}$$

Here $I \in \tilde{I} := \{I \subset \{1, 2, 3, 5, 6, 7\} : \#I = 3\}$. Then we have:

Lemma 2.6.49. Let $t = i \operatorname{diag}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in \mathfrak{t}$. Then we have

$$\begin{aligned} \rho_{\varpi_4}(t)(e_I \wedge e_4 + *(e_I \wedge e_4)) &= \left(\sum_{i \in I} t_i + t_4\right)(ie_I \wedge e_4 + *(ie_I \wedge e_4)), \\ \rho_{\varpi_4}(t)(ie_I \wedge e_4 + *(ie_I \wedge e_4)) &= -\left(\sum_{i \in I} t_i + t_4\right)(e_I \wedge e_4 + *(e_I \wedge e_4)). \end{aligned}$$

This claim can be easily checked by direct calculation. So we omit the proof.

Lemma 2.6.50. $\ker\omega \subset S(t_H^*)^{W_H}$ is described as follows:

$$\ker\omega \simeq \text{ideal generated by } s_2, s_3, s_4, s_5, s_6, s_7, s_8 \text{ on } S(\mathfrak{t}_H^*)^{W_H}.$$

Proof. This comes from Fact 2.6.20. □

Remark 2.6.51. For any $X \in \mathfrak{k}_{H^a} = \mathfrak{k}_H$, the following diagram commutes:

$$\begin{tikzcd}
(\bigwedge^3 \mathbb{C}^6) \otimes \mathbb{C}^2 \arrow[r, "\iota"] \arrow[d, "(\rho_{\varpi_3} \boxtimes \rho_{\varpi_1})(X)"'] & \bigwedge^4 \mathbb{C}^8 \arrow[d, "\rho_{\varpi_4}(X)"] \\
(\bigwedge^3 \mathbb{C}^6) \otimes \mathbb{C}^2 \arrow[r, "\iota"] \arrow[d, "*_6 \otimes *_2"'] & \bigwedge^4 \mathbb{C}^8 \arrow[d, "*_8"] \\
(\bigwedge^3 \mathbb{C}^6) \otimes \mathbb{C}^2 \arrow[r, "\iota"] & \bigwedge^4 \mathbb{C}^8
\end{tikzcd}$$

Here the center of $\mathfrak{k}_H = \mathfrak{k}_{H^a}$ acts on $(\bigwedge^3 \mathbb{C}^6) \otimes \mathbb{C}^2$ trivially.

Lemma 2.6.52. Let $\mathfrak{g}$ be a Lie algebra satisfying $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{C}^n)$ an unitary representation with respect to standard inner product on $\mathbb{C}^n$. Then $\bigwedge \rho : \mathfrak{g} \to \mathfrak{gl}(\bigwedge \mathbb{C}^n)$ is also unitary representation with respect to the above standard inner product H. Moreover, $(\bigwedge \rho(X))* = *(\bigwedge \rho(X)) : \bigwedge^p \mathbb{C}^n \to \bigwedge^{n-p} \mathbb{C}^n$ for all $X \in \mathfrak{g}$ $(0 \leq p \leq n)$. Here $\bigwedge \rho : \bigwedge \mathbb{C}^n \to \bigwedge \mathbb{C}^n$ is defined as follows:

$$\bigwedge \rho(X) : \bigwedge^p \mathbb{C}^n \to \bigwedge^p \mathbb{C}^n, \bigwedge \rho(X)(v_1 \wedge \cdots \wedge v_p) := \sum_{i=1}^{p} v_1 \wedge \cdots \wedge \rho(X) v_i \wedge \cdots \wedge v_p$$

Proof. It is enough to show $H(w, (\bigwedge \rho(X)) * (v_1 \wedge \cdots \wedge v_p)) = H(w, *(\bigwedge \rho(X)(v_1 \wedge \cdots \wedge v_p)))$ for all $w \in \bigwedge^{n-p} \mathbb{C}^n$ for any $X \in \mathfrak{g}$. This comes from that $\bigwedge \rho$ is unitary and one dimensional representation is trivial:

$$\begin{aligned}
&H(w, (\bigwedge \rho(X)) * (v_1 \wedge \cdots \wedge v_p)) \\
=& - H(\bigwedge \rho(X) w, *(v_1 \wedge \cdots \wedge v_p)) \\
=& - \overline{*(v_1 \wedge \cdots v_p \wedge (\bigwedge \rho(X) w))} \\
=& - \overline{*(\bigwedge \rho(X)(v_1 \wedge \cdots \wedge v_p \wedge w))} + \overline{*((\bigwedge \rho(X)(v_1 \wedge \cdots \wedge v_p)) \wedge w)} \\
=& \overline{*((\bigwedge \rho(X)(v_1 \wedge \cdots \wedge v_p)) \wedge w)} \\
=& H(w, *(\wedge \rho(X))(v_1 \wedge \cdots \wedge v_p)).
\end{aligned}$$

□

Remark 2.6.53. For a Lie algebra $\mathfrak{g}$, the following conditions are equivalent:

(i) $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$,

(ii) one dimensional representation of $\mathfrak{g}$ is trivial.

Lemma 2.6.54.

$$\mathrm{ad}^*(\tilde{p_1}) = -\sum_{I \in \tilde{I}} t_{I \cup \{4\}} t_{(I \cup \{4\})^c} + (s_1)$$

Here for $J \subset \{1, \cdots, 8\}$, put $t_J := \sum_{j \in J} t_j$, and J^c means complement subset of J in $\{1, \cdots, 8\}$.

Proof. This comes from Lemma 2.6.49. □

Lemma 2.6.55. $\mathrm{ad}^*(p_1) \notin \ker \omega$.

Proof. we have the following:

Claim. Let $f(t) + (s_1) \in S(\mathfrak{t}_H^*)^{W_H}$, where $f(t) \in \mathbb{R}[s_1', s_2', s_3', s_4', s_5', s_6', s_1'', s_2''] \subset \mathbb{R}[t_1, \cdots, t_8]$ is homogeneous polynomial of degree two. Then the following conditions are equivalent:

(i) $f(t) + (s_1) \in \ker \omega$,

(ii) There exist real numbers $a, b, c, d \in \mathbb{R}$ such that $f(t) = a \cdot s_2 + (b \cdot s_1' + c \cdot s_1'' + d)s_1$.

This claim can be easily checked by the description of $\ker \omega$. Assume that there exist real numbers $a, b, c, d \in \mathbb{R}$ satifying

$$\sum_{I \in \tilde{I}} t_{I \cup \{4\}} t_{(I \cup \{4\})^c} = as_2 + (bs_1' + cs_1'' + d)s_1.$$

Put $t_1 = -t_5$, $t_2 = -t_6$, $t_3 = -t_7$, $t_4 = -t_8$. Then the left hand side is $12(t_1^2 + t_2^t + t_3^2) + 20t_4^2$ and the right hand side is $a(t_1^2 + t_2^2 + t_3^2 + t_4^2)$. This is contradiction. □

2.6.6 Calculation of the first Pontrjagin class of $(\mathfrak{g}, \mathfrak{h} \simeq \mathfrak{h}^a) = (\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10) \oplus \mathfrak{u}(1))$

In this section, we consider the symmetric pair (G, H) where G is a connected linear reductive Lie group and corresponding Lie algebras are $(\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10) \oplus \mathfrak{u}(1))$. Here $\dim \mathfrak{p} = 40$, $\dim \mathfrak{p}_H = 20$, $\mathfrak{k} \simeq \mathfrak{su}(6) \oplus \mathfrak{su}(2)$ and $\mathfrak{k}_H \simeq \mathfrak{u}(5) \oplus \mathfrak{u}(1)$.

Our goal of this subsection is the following:

Proposition 2.6.56. G_θ / H_θ does not admit compact Clifford–Klein forms.

This comes from the following:

Lemma 2.6.57. The associated bundle $K \times_{\mathrm{Ad},K_H} \mathfrak{p}/\mathfrak{p}_H$ is not trivial.

This comes from the following:

Lemma 2.6.58. The first Pontrjagin class of the associated bundle $K \times_{\mathrm{Ad},K_H} \mathfrak{p}/\mathfrak{p}_H$ does not vanish.

Proof. This comes from Fact 2.6.10 and Lemma 2.6.66. $\square$

Fix a realization of $\mathfrak{k} = \mathfrak{su}(6) \oplus \mathfrak{su}(2)$ and $\mathfrak{k}_H = \mathfrak{u}(5) \oplus \mathfrak{u}(1)$ as follows in the same way as the case $(\mathfrak{g}, \mathfrak{h}, \mathfrak{h}^a) = (\mathfrak{e}_{7(7)}, \mathfrak{su}(2) \oplus \mathfrak{so}^*(12), \mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2))$:

$$\mathfrak{su}(6) \oplus \mathfrak{su}(2) = \{ \begin{pmatrix} A_1 & & -B & \\ & \alpha & & -\beta \\ B^* & & A_2 & \\ & \overline{\beta} & & \overline{\alpha} \end{pmatrix} : \begin{pmatrix} A_1 & -B \\ B^* & A_2 \end{pmatrix} \in \mathfrak{su}(6), \begin{pmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathfrak{su}(2), A_1, A_2 \in \mathfrak{su}(3)\}$$

$\sigma : \mathfrak{su}(6) \oplus \mathfrak{su}(2) \to \mathfrak{su}(6) \oplus \mathfrak{su}(2), X \mapsto I_{6,2} X I_{6,2}^{-1}$

$\mathfrak{u}(5) \oplus \mathfrak{u}(1) = (\mathfrak{su}(6) \oplus \mathfrak{su}(2))^\sigma$

We take maximal tori $\mathfrak{t} = \mathfrak{t}_{H^a}$ as follows:

$\mathfrak{t} = \mathfrak{t}_{H^a} = \{i \operatorname{diag}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) : t_i \in \mathbb{R}, t_1 + t_2 + t_3 + t_5 + t_6 + t_7 = 0, t_4 + t_8 = 0\}.$

Remark 2.6.59.

$$S(t^*)^W \simeq \mathbb{R}[s_1, s_2, s_3, s_4, s_5, s_6, \sigma_1, \sigma_2]/(s_1, \sigma_1)$$
$$S(t^*_{H^a})^{W_{H^a}} \simeq \mathbb{R}[s'_1, s'_2, s'_3, s'_4, s'_5, t_7, t_4, t_8]/(s_1, \sigma_1)$$

Here, s_i, s'_i and σ_i are the fundamental symmetric polynomial of degree i with respect to $\{t_1, t_2, t_3, t_5, t_6, t_7\}$, $\{t_1, t_2, t_3, t_5, t_6\}$ and $\{t_4, t_8\}$ respectively.

Remark 2.6.60. $\mathrm{ad} : \mathfrak{k} = \mathfrak{su}(6) \oplus \mathfrak{su}(2) \to \mathfrak{gl}(\mathfrak{p})$ is equivalent to the representation $(\rho_{\varpi_3} \boxtimes \rho_{\varpi_1}, (\bigwedge^3 \mathbb{C}^6 \otimes \mathbb{C}^2)^{*_6 \otimes *_2})$.

Proof. We already checked this remark in the previous subsection. $\square$

Lemma 2.6.61. $\mathrm{ad}|_{\mathfrak{k}^a_H} : \mathfrak{k}_{H^a} \to \mathfrak{gl}(\mathfrak{p}_{H^a})$, which is equivalent to $\mathrm{ad}|_{\mathfrak{k}_H} : \mathfrak{k}_H \to \mathfrak{gl}(\mathfrak{p}/\mathfrak{p}_H)$, is equivalent to the coefficient restriction to $\mathbb{R}$ of $\rho_{\varpi_2} \boxtimes 2i \boxtimes \mathrm{triv}$: $\mathfrak{su}(5) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \to \mathfrak{gl}(\bigwedge^2 \mathbb{C}^5 \otimes_{\mathbb{C}} \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C})$:

$$\mathfrak{su}(5) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \times (\bigwedge^2 \mathbb{C}^5) \otimes \mathbb{C} \otimes \mathbb{C} \to (\bigwedge^2 \mathbb{C}^5) \otimes \mathbb{C} \otimes \mathbb{C} \simeq \bigwedge^2 \mathbb{C}^5$$
$$((X, it_1, it_2), v \otimes z_1 \otimes z_2) \mapsto \rho_{\varpi_2}(X) v \otimes z_1 \otimes z_2 + v \otimes (2it_1) z_1 \otimes z_2$$

where $t_1, t_2 \in \mathbb{R}$.

Proof. Since we have $\mathfrak{h} \simeq \mathfrak{h}^a \simeq \mathfrak{so}^*(10) \oplus \mathfrak{u}(1)$, $\mathfrak{k}_{H^a} = \mathfrak{su}(5) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$, the isotropy representation on $\mathfrak{p}_{H^a}$ consists of trivial representaion of $\mathfrak{u}(1)$ and isotropy representation of $\mathfrak{so}^*(10)$. We can easily check that isotropy representation of $\mathfrak{so}^*(10)$ is equivalent to coefficient restriction to $\mathbb{R}$ of $\rho_{\varpi_2} \boxtimes 2i$ as a real representation. $\square$

To calculate $\mathrm{ad}^*(\tilde{p_1})$ by using a reasonable orthonormal basis, we consider the following emmbedding into $((\bigwedge^2 \mathbb{C}^6) \otimes \mathbb{C}^2)^{*_6 \otimes *_2}$, which is used in the previous section.

We define $\mathbb{R}$-linear injective map $\mu : \bigwedge^2 \mathbb{C}^5 \to ((\bigwedge^3 \mathbb{C}^6) \otimes \mathbb{C}^2)^{*_6 \otimes *_2}$ as follows

$$\mu(v) := \iota'(*_5(v)) \otimes e_1 + (*_6 \otimes *_2)(\iota'(*_5(v)) \otimes e_1)$$

where $\iota' : \bigwedge^3 \mathbb{C}^5 \to \bigwedge^3 \mathbb{C}^6$ is the natural inclusion induced by $\mathbb{C}^5 \to \mathbb{C}^6$, ${}^t(x_1, \cdots, x_5) \mapsto {}^t(x_1, \cdots, x_5, 0)$.

Remark 2.6.62. The $\mathbb{R}$-linear map μ is compatible with the representation of $\mathfrak{su}(5) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(6) \oplus \mathfrak{su}(2) = \mathfrak{k}$ from Lemma 2.6.52. Here the embedding $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$ into $\mathfrak{su}(6) \oplus \mathfrak{su}(2) \subset \mathfrak{su}(8)$ is given as follows:

$$\begin{aligned}
&\mathfrak{u}(1) \oplus \mathfrak{u}(1) \to \mathfrak{su}(6) \oplus \mathfrak{su}(2) \\
&\quad (it_1, it_2) \mapsto i\,\mathrm{diag}(t_1, t_1, t_1, -t_1, t_1, t_1, -5t_1, t_1) + i\,\mathrm{diag}(t_2, t_2, t_2, -3t_2, t_2, t_2, -5t_2, 3t_2)
\end{aligned}$$

We use the following orthonormal basis on $(\bigwedge^3 \mathbb{C}^6 \otimes \mathbb{C}^2)^{*_6 \otimes *_2}$:

$$\begin{aligned}
e_I \otimes e_1 + (*_6 \otimes *_2)(e_I \otimes e_1) \\
ie_I \otimes e_1 + (*_6 \otimes *_2)(ie_I \otimes e_1)
\end{aligned}$$

Here $I \in \tilde{I} := \{I \subset \{1, 2, 3, 4, 5\} : \#I = 3\}$. To make our calculation easier, we take an orthonormal basis on $(\bigwedge^4 \mathbb{C}^8)^{*_8} \supset (\bigwedge^3 \mathbb{C}^6 \otimes \mathbb{C}^2)^{*_6 \otimes *_2}$ (see the previous subsection for the embedding) corresponding to the above orthonormal basis as follows:

$$\begin{aligned}
e_J \wedge e_4 + *_8(e_J \wedge e_4) \\
ie_J \wedge e_4 + *_8(ie_J \wedge e_4)
\end{aligned}$$

Here $J \subset \tilde{J} := \{J \subset \{1, 2, 3, 5, 6\} : \#J = 3\}$

Lemma 2.6.63. Let $t := i\,\mathrm{diag}(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) \in \mathfrak{t}_H$.

$$\begin{aligned}
\mathrm{ad}(t)(e_J \wedge e_4 + *_8(e_J \wedge e_4)) &= \left(\sum_{j \in J} t_j + t_4\right)(ie_J \wedge e_4 + *_8(ie_J \wedge e_4)) \\
\mathrm{ad}(t)(ie_J \wedge e_4 + *_8(ie_J \wedge e_4)) &= -\left(\sum_{j \in J} t_j + t_4\right)(e_J \wedge e_4 + *_8(e_J \wedge e_4))
\end{aligned}$$

Lemma 2.6.64.

$$\mathrm{ad}^*(p_1) = -\sum_{J \in \tilde{J}} t_{J \cup \{4\}} t_{(J \cup \{4\})^c}.$$

Here for $J \subset \{1, \cdots, 8\}$, put $t_J := \sum_{j \in J} t_j$, and J^c means complement subset of J in $\{1, \cdots, 8\}$.

Proof. This comes from Lemma 2.6.63. ☐

Lemma 2.6.65. We have

$$\ker\omega = \text{ ideal generated by } s_2, s_3, s_4, s_5, s_6, \sigma_2 \text{ on } S(\mathfrak{t}_H^*)^{W_H}$$

Proof. This comes from Fact 2.6.20. ☐

Lemma 2.6.66. $\mathrm{ad}^*(p_1) \notin \ker\omega$.

Proof.

Claim. Let $f(t)+(s_1,\sigma_1) \in S(\mathfrak{t}_H^*)^{W_H}$, where $f(t) \in \mathbb{R}[s_1', s_2', s_3', s_4', s_5', t_7, t_4, t_8] \subset \mathbb{R}[t_1, \cdots, t_8]$ is homogeneous polynomial of degree two. Then the following conditions are equivalent:

(i) $f(t) + (s_1, \sigma_1) \in \ker\omega$,

(ii) There exist real numbers $a, b, c, d, e, a', b', c', d', e' \in \mathbb{R}$ such that $f(t) = as_2 + (bs_1' + ct_7 + dt_4 + et_8)s_1 + a'\sigma_2 + (bs_1' + ct_7 + dt_4 + et_8)\sigma_1$.

This claim can be easily checked by the description of $\ker\omega$. Assume that there exist real numbers $a, b, c, d, e, a', b', c', d', e' \in \mathbb{R}$ such that $\sum_{J\in\tilde{J}} t_{J\cup\{4\}} t^c_{J\cup\{4\}} = as_2 + (bs_1' + ct_7 + dt_4 + et_8)s_1 + a'\sigma_2 + (bs_1' + ct_7 + dt_4 + et_8)\sigma_1$. Put $t_5 = -t_1$, $t_6 = -t_2$, $t_7 = -t_3$ and $t_8 = -t_4$. Then the left hand side is $-6(t_1^2 + t_2^2 + t_3^2) - 10t^4 - 12t_3t_4$ and the right hand side is $-a(t_1^2 + t_2^2 + t_3^2) - a't_4^2$. This is contradiction. ☐

2.6.7 Calculation of the first Pontrjagin class of $(\mathfrak{g}, \mathfrak{h} \simeq \mathfrak{h}^a) = (\mathfrak{e}_{8(8)}, \mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2))$

In this section, we consider the symmetric pair (G, H) where G is a connected linear reductive Lie group and corresponding Lie algebras are $(\mathfrak{e}_{8(8)}, \mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2))$. Here $\dim\mathfrak{p} = 128$, $\dim\mathfrak{p}_H = 64$, $\mathfrak{k} \simeq \mathfrak{so}(16)$ and $\mathfrak{k}_H \simeq \mathfrak{su}(2) \oplus \mathfrak{so}(12) \oplus \mathfrak{su}(2) \simeq \mathfrak{so}(12) \oplus \mathfrak{so}(4)$. Our goal of this subsection is the following:

Proposition 2.6.67. G_θ/H_θ does not admit compact Clifford–Klein forms.

Proof. This comes from the following lemma. ☐

Lemma 2.6.68. The associated bundle $K \times_{(\mathrm{Ad}, K_H)} \mathfrak{p}/\mathfrak{p}_H$ over K/K_H is not trivial.

Proof. This comes form the following lemma. ☐

Lemma 2.6.69. The first Pontrjagin class of the associated bundle $K \times_{(\mathrm{Ad}, K_H)} \mathfrak{p}/\mathfrak{p}_H$ over K/K_H does not vanish.

Lemma 2.6.70. $\mathrm{ad} : \mathfrak{k}_{H^a} \simeq \mathfrak{so}(12)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2) \to \mathfrak{gl}(\mathfrak{p}_{H^a})$ is equivalent to the representation corresponding to $\rho_{\varpi_i}\boxtimes\rho_{\varpi_1}\boxtimes$trivial$\in C^I(\mathfrak{so}(12)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2))$ by the Cartan's fundamental theorem, where $i = 5$ or 6 (half spin representation).

Proof. Since $\mathfrak{h} \simeq \mathfrak{h}^a = \mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2)$, it is enough to check that the isotropy representation of $\mathfrak{e}_{7(-5)}$ is equivalent to the representation corresponding to $\rho_{\varpi_i} \boxtimes \rho_{\varpi_1}$ for some $i = 5$ or 6 by Cartan's fundamental theorem. $\square$

Proof of Lemma 2.6.69. (i) We put $\mathfrak{k} = \mathfrak{so}(16) = \{X \in M(16, \mathbb{R}) : {}^tX + X = 0\}$, and realize $\mathfrak{k}_{H^a} = \mathfrak{k}^\sigma$ by the involution $\sigma(X) = I_{12,4} X I_{12,4}^{-1}$, maximal tori $\mathfrak{t}$ and $\mathfrak{t}_H$ as follows:

$$\mathfrak{t} = \mathfrak{t}_H = \left\{ \sum_{i=1}^{8} t_i A_{2i-1,2i} : t_i \in \mathbb{R}\ (i = 1, \cdots, 8) \right\}.$$

(ii) We have the following description of $S(\mathfrak{t}^*)^W$ and $S(\mathfrak{t}_H^*)^{W_H}$:

$$S(\mathfrak{t}^*)^W = \mathbb{R}\Big[\sum_{i=1}^{8} t_i^2, \sum_{1 \le i < j \le 8} t_i^2 t_j^2, \sum_{1 \le i < j < k \le 8} t_i^2 t_j^2 t_k^2, \sum_{1 \le i < j < k < \ell \le 8} t_i^2 t_j^2 t_k^2 t_\ell^2, \sum_{1 \le i < j < k < \ell < m \le 8} t_i^2 t_j^2 t_k^2 t_\ell^2 t_m^2,$$
$$\sum_{1 \le i < j < k < \ell < m < n \le 8} t_i^2 t_j^2 t_k^2 t_\ell^2 t_m^2 t_n^2, \sum_{1 \le i < j < k < \ell < m < n < o \le 8} t_i^2 t_j^2 t_k^2 t_\ell^2 t_m^2 t_n^2 t_o^2, t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 \Big]$$
$$S(\mathfrak{t}_H^*)^{W_H} = \mathbb{R}\Big[\sum_{i=1}^{6} t_i^2, \sum_{1 \le i < j \le 6} t_i^2 t_j^2, \sum_{1 \le i < j < k \le 6} t_i^2 t_j^2 t_k^2, \sum_{1 \le i < j < k < \ell \le 6} t_i^2 t_j^2 t_k^2 t_\ell^2, \sum_{1 \le i < j < k < \ell < m \le 6} t_i^2 t_j^2 t_k^2 t_\ell^2 t_m^2,$$
$$t_1 t_2 t_3 t_4 t_5 t_6, t_7^2 + t_8^2, t_7 t_8 \Big]$$

(iii) From Fact 2.6.20, we have the following description of $\ker \omega$:

$\ker \omega =$ ideal generated on $S(\mathfrak{t}_H^*)^{W_H}$ by generators

$$\sum_{i=1}^{8} t_i^2, \sum_{1 \le i < j < k \le 8} t_i^2 t_j^2 t_k^2, \sum_{1 \le i < j < k < \ell \le 8} t_i^2 t_j^2 t_k^2 t_\ell^2, \sum_{1 \le i < j < k < \ell < m \le 8} t_i^2 t_j^2 t_k^2 t_\ell^2 t_m^2,$$
$$\sum_{1 \le i < j < k < \ell < m < n \le 8} t_i^2 t_j^2 t_k^2 t_\ell^2 t_m^2 t_n^2, \sum_{1 \le i < j < k < \ell < m < n < o \le 8} t_i^2 t_j^2 t_k^2 t_\ell^2 t_m^2 t_n^2 t_o^2, t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8.$$

(iv) We describe $\mathrm{ad}^*(\mathfrak{t}_H)$ and check whether it is in $\ker \omega$:

Claim. $\mathrm{ad}^*(\mathfrak{t}_H) \notin \ker \omega$.

Proof. Let $\rho_1 : \mathfrak{so}(12) \to \mathfrak{sl}(V_1)$ be a half spin representation and $\rho_2 : \mathfrak{su}(2) \to \mathfrak{sl}(V_2)$ a standard representation. From Lemma 2.6.70, there exists an anti-holomorphic involution J on $V_1 \otimes V_2$ such that $\rho_1 \boxtimes \rho_2 : \mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \to \mathfrak{sl}((V_1 \otimes V_2)^J)$ is equivalent to $\mathrm{ad} : \mathfrak{k}_{H^a} \to \mathfrak{gl}(\mathfrak{p}_{H^a})$. It is well-known that $W(\rho_1) = \{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6)\}$ where all the weight have an odd (or even) number of minus signs and $W(\rho_2) = \{\pm\varepsilon'\}$, so we have $W(\rho_1 \boxtimes \rho_2) = \{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6) \pm \varepsilon'\}$ with odd (even) number of minus signs for ε_i. We can take an orthonormal vectors on $(V_1 \boxtimes V_2)^J$ for some $\rho_1 \boxtimes \rho_2$-invariant Hermitian form on $V_1 \boxtimes V_2$ as follows:

$$\begin{aligned}&v_\lambda + Jv_\lambda,\\ &iv_\lambda + J(iv_\lambda),\end{aligned}$$

where $v_\lambda \in W(\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6) + \varepsilon')$ Then for the element t in the maximal torus $\{(\sum_{i=1}^{6} t_i A_{2i-1,2i}, \mathrm{diag}(it', -it') : t_i, t' \in \mathbb{R}\ (i = 1, \cdots, 6)\} \subset \mathfrak{so}(12) \oplus \mathfrak{su}(2)$, we have

$$\begin{aligned}(\rho_1 \boxtimes \rho_2)(t)(v_\lambda + Jv_\lambda) &= (\frac{1}{2}(\pm t_1 \pm t_2 \pm t_3 \pm t_4 \pm t_5 \pm t_6) + t')(iv_\lambda + J(iv_\lambda))\\ (\rho_1 \boxtimes \rho_2)(t)(iv_\lambda + J(iv_\lambda)) &= (-\frac{1}{2}(\pm t_1 \pm t_2 \pm t_3 \pm t_4 \pm t_5 \pm t_6) - t')(v_\lambda + J(v_\lambda)),\end{aligned}$$

So we have

$$\begin{aligned}\mathrm{ad}^*(p_1) &= \sum_{\text{with odd (even) number of minus signs}} (\frac{1}{2}(\pm t_1 \pm t_2 \pm t_3 \pm t_4 \pm t_5 \pm t_6) + t')^2\\ &= 8(t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_5^2 + t_6^2) + 32t'^2.\end{aligned}$$

Here we can write $t' = at_7 + bt_8$ for some $(a,b) \in \mathbb{R}^2 \setminus \{0\}$ from the isomorphism $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Then we have $\mathrm{ad}^*(p_1) = 8\sum_{i=1}^{6} t_i^2 + 32(a^2t_7^2 + 2abt_7t_8 + b^2t_8^2)$. Thus, we obtain $\mathrm{ad}^*(p_1) \notin \ker\omega$ from the description $\ker\omega$ and $S(\mathfrak{t}_H^*)^{W_H}$. $\square$

$\square$

2.6.8 Calculation of the first Pontrjagin class of $(\mathfrak{g}, \mathfrak{h} \simeq \mathfrak{h}^a) = (\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2))$

In this section, we consider the symmetric pair (G, H) where G is a connected linear reductive Lie group and corresponding Lie algebras are $(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2))$. Here $\dim\mathfrak{p} = 64$, $\dim\mathfrak{p}_H = 32$, $\mathfrak{k} \simeq \mathfrak{so}(12) \oplus \mathfrak{su}(2)$ and $\mathfrak{k}_H \simeq \mathfrak{so}(2) \oplus \mathfrak{so}(10) \oplus \mathfrak{u}(1)$. Our goal of this subsection is the following:

Proposition 2.6.71. The symmetric space G_θ/H_θ does not admit compact Clifford–Klein forms.

Proof. This comes from the following Lemma 2.6.72. $\square$

Lemma 2.6.72. The associated vector bundle $K \times_{(\mathrm{Ad},K_H)} \mathfrak{p}/\mathfrak{p}_H$ over K/K_H is not trivial.

Proof. This comes from the following Lemma 2.6.73. $\square$

Lemma 2.6.73. The first Pontrjagin class of the associated vector bundle $K \times_{(\mathrm{Ad},K_H)} \mathfrak{p}/\mathfrak{p}_H$ over K/K_H does not vanish.

To prove the above lemma, we prepare some lemmas below.

Lemma 2.6.74. The restriction of the isotropy representation $\mathrm{ad}\,|_{\mathfrak{so}(10)} : \mathfrak{so}(10) \to \mathfrak{gl}(\mathfrak{p}_{H^a})$ is an irreducible representation and it is the coefficient restriction to $\mathbb{R}$ of the half spin representation (ρ_{ϖ_i}, V) ($i = 4$ or 5) of $\mathfrak{so}(10)$. Moreover, the center of $\mathfrak{k}_{H^a}$ acts on V as a scalar multiplication and the scalar is not zero for the action of $\mathfrak{u}(1) \subset \mathfrak{su}(2)$.

Remark 2.6.75. The $\mathfrak{u}(1) \subset \mathfrak{su}(2)$ acts on $\mathfrak{p}_{H^a}$ nontrivially, namely the scalar is not zero. This comes from that $\mathfrak{u}(1)$-action is the restriction of standard representation of $\mathfrak{su}(2)$.

Proof of Lemma 2.6.74. The isotropy representation of $\mathfrak{h}$ is irreducible since the representation space $\mathfrak{p}_{H^a}$ comes from the simple Lie algebra $\mathfrak{e}_{6(-14)}$. So, the former part is clear. "Moreover part" is also clear from Cartan's fundamental theorem and Schur's lemma. □

We fix a realization of $\mathfrak{k} = \mathfrak{so}(12) \oplus \mathfrak{su}(2)$ and $\mathfrak{k}_H = \mathfrak{k}_{H^a} = \mathfrak{so}(10) \oplus \mathfrak{so}(2) \oplus \mathfrak{u}(1)$ as follows:

$$\begin{aligned}
&\mathfrak{k} = \mathfrak{so}(12) \oplus \mathfrak{su}(2) = \{(X, Y) \in M(12, \mathbb{R}) \oplus M(2, \mathbb{C}) : {}^tX + X = 0, Y^* + Y = 0\},\\
&\sigma : \mathfrak{k} \to \mathfrak{k}, (X, Y) \mapsto (I_{10,2} X I_{10,2}^{-1}, I_{1,1} Y I_{1,1}^{-1}),\\
&\mathfrak{k}_H = \mathfrak{k}_{H^a} = \mathfrak{k}^\sigma.
\end{aligned}$$

We realize spin representation on $M(32, \mathbb{R})$ as follows:

$$\begin{aligned}
\mathfrak{so}(10) \to C_{even}(0, 10) &\simeq C(0, 9) \simeq C(6, 3)\\
&\simeq C(1, 1) \otimes C(1, 1) \otimes C(1, 1) \otimes C(3, 0)\\
&\simeq C(1, 1) \otimes C(1, 1) \otimes C(1, 1) \otimes C(1, 2)\\
&\simeq C(1, 1) \otimes C(1, 1) \otimes C(1, 1) \otimes C(1, 1) \otimes C(0, 1)\\
&\to M(2, \mathbb{R}) \otimes M(2, \mathbb{R}) \otimes M(2, \mathbb{R}) \otimes M(2, \mathbb{R}) \otimes M(2, \mathbb{R})
\end{aligned}$$

Here we use maps of Fact 1.5.33, the following isomorphism and inclusion:

$$\begin{aligned}
C(1, 1) &\simeq M(2, \mathbb{R}), & C(0, 1) &\to M(2, \mathbb{R})\\
v_1^+ &\mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, & v_1^- &\mapsto \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},\\
v_1^- &\mapsto \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.
\end{aligned}$$

Then we obtain the following matrix representation of tori $\mathfrak{t}_{H^a}$ by the above realization of spin representation and Lemma 2.6.74:

$$\begin{aligned}
\mathfrak{so}(12) &\to M(2,\mathbb{R})\otimes M(2,\mathbb{R})\otimes M(2,\mathbb{R})\otimes M(2,\mathbb{R})\otimes M(2,\mathbb{R}),\\
(A_{1,2},0) &\mapsto \frac{1}{2}(T\otimes J\otimes 1\otimes 1\otimes 1),\\
(A_{3,4},0) &\mapsto -\frac{1}{2}(1\otimes J\otimes 1\otimes 1\otimes 1),\\
(A_{5,6},0) &\mapsto -\frac{1}{2}(1\otimes J\otimes T\otimes 1\otimes 1),\\
(A_{7,8},0) &\mapsto -\frac{1}{2}(1\otimes 1\otimes 1\otimes J\otimes 1),\\
(A_{9,10},0) &\mapsto \frac{1}{2}(T\otimes J\otimes T\otimes J\otimes J),\\
(A_{11,12},0) &\mapsto a(1\otimes 1\otimes 1\otimes 1\otimes J),\\
(0,i\begin{pmatrix}1 & \\ & -1\end{pmatrix}) &\mapsto b(1\otimes 1\otimes 1\otimes 1\otimes J),
\end{aligned}$$

for some $(a,b)\in\mathbb{R}^2\setminus\{0\}$. Moreover $b\neq 0$ holds from Remark 2.6.75. Thus we obtain

$$\mathrm{ad}^*(\tilde{p_1}) = 4(t_1^2+t_2^2+t_3^2+t_4^2+t_5^2)+16(at_6+bt_7)^2.$$

Proof of Lemma 2.6.73. It is enough to show that $p_1(K\times_{K_H}\mathfrak{p}/\mathfrak{p}_H)\neq 0\in H^4_{DR}(K/K_H,\mathbb{R})$, namely, $\mathrm{ad}^*(\tilde{p_1})\notin\ker\omega$

(i) Put maximal tori $\mathfrak{t}$ and $\mathfrak{t}_{H^a}$ of $\mathfrak{k}$ and $\mathfrak{k}_H$ respectively as follows:

$$\mathfrak{t}=\mathfrak{t}_H=\left\{(\sum_{i=1}^{6}t_iA_{2i-1,2i},\begin{pmatrix}it_7 & \\ & -it_7\end{pmatrix})\in\mathfrak{so}(12)\oplus\mathfrak{su}(2): t_i\in\mathbb{R}\ (i=1,\cdots,7)\right\}$$

(ii) We have the following description of $S(\mathfrak{t}^*)^W$ and $S(\mathfrak{t}_H^*)^{W_H}$:

$$\begin{aligned}
S(\mathfrak{t}^*)^W=\mathbb{R}\big[&\sum_{i=1}^{6}t_i^2,\sum_{1\leq i<j\leq 6}t_i^2t_j^2,\sum_{1\leq i<j<k\leq 6}t_i^2t_j^2t_k^2,\sum_{1\leq i<j<k<\ell\leq 6}t_i^2t_j^2t_k^2t_\ell^2,\\
&\sum_{1\leq i<j<k<\ell<m\leq 6}t_i^2t_j^2t_k^2t_\ell^2t_m^2,t_1t_2t_3t_4t_5t_6,t_7^2\big]\\
S(\mathfrak{t}_H^*)^{W_H}=\mathbb{R}\big[&\sum_{i=1}^{5}t_i^2,\sum_{1\leq i<j\leq 5}t_i^2t_j^2,\sum_{1\leq i<j<k\leq 5}t_i^2t_j^2t_k^2,\sum_{1\leq i<j<k<\ell\leq 5}t_i^2t_j^2t_k^2t_\ell^2,\\
&\sum_{1\leq i<j<k<\ell<m\leq 5}t_i^2t_j^2t_k^2t_\ell^2t_m^2,t_1t_2t_3t_4t_5,t_6,t_7\big]
\end{aligned}$$

(iii) description of $\ker\omega$:

$$\ker\omega = \text{ideal generated by } \sum_{i=1}^{6} t_i^2, \sum_{1\le i<j\le 6} t_i^2t_j^2, \sum_{1\le i<j<k\le 6} t_i^2t_j^2t_k^2, \sum_{1\le i<j<k<\ell\le 6} t_i^2t_j^2t_k^2t_\ell^2,$$
$$\sum_{1\le i<j<k<\ell<m\le 6} t_i^2t_j^2t_k^2t_\ell^2t_m^2, t_1t_2t_3t_4t_5t_6, t_7^2 \text{ on } S(\mathfrak{t}_H^*)^{W_H}$$

Therefore we obtain $\mathrm{ad}^*(\tilde{p_1}) = 4(t_1^2+t_2^2+t_3^2+t_4^2+t_5^2)+16(at_6+bt_7)^2 \notin \ker\omega$ from $b \neq 0$ and the above description of $\ker\omega$.

□

2.7 Applications of Adams's theorem

The goal of this section is to prove the following:

Proposition 2.7.1. Let $(G,\ H)$ and $(G,\ H^a)$ be symmetric pairs which are locally isomorphic to one of the following list and suppose that G is connected. Then neither G_θ/H_θ nor G_θ/H_θ^a admit compact Clifford–Klein forms.

- $(G,H,H^a) = (SO_0(p,q+1), SO_0(p,q), SO_0(p,1)\times SO(q))$ $(q \geq \rho(p,\mathbb{R}))$,
- $(G,H=H^a) = (SU(p,2), S(U(p,1)\times U(1)))$ (p is odd).
- $(G,H,H^a) = (SO^*(2(2p)), SO^*(2(2p-1))\times SO^*(2), U(2p-1,1))$ $(p \geq 3)$,
- $(G,H,H^a) = (SO(2(p+q)-2,\mathbb{C}), SO(2p-1,\mathbb{C})\times SO(2q-1,\mathbb{C}), SO_0(2p-1,2q-1))$ $(1 \leq p \leq q$ and $(p,q) \neq (1,1),(1,2),(1,4))$.

Remark 2.7.2. A part of Proposition 2.7.1 was obtained in the non peer-reviewed paper [26].

We apply Adams's theorem to show the above Proposition 2.7.1. To state Adams's theorem, we introduce the following Definition 2.7.3 and recall Definition 2.7.4, 2.7.7.

Definition 2.7.3. For a $\mathbb{R}$-subspace $V \subset M(p,q;\mathbb{K})$, we define $\underline{\mathrm{rank}}V$ as follows.

$$\underline{\mathrm{rank}}V := \min\{\mathrm{rank}\, v : v \in V \setminus \{0\}\}.$$

Here, $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Definition 2.7.4 ([23]). We use the following notation.

$$\ell(p,q,r;\mathbb{K}) := \max\{\dim V : V \subset M(p,q;\mathbb{K}) \text{ is a } \mathbb{R}\text{-subspace such that } \underline{\mathrm{rank}}V \geq r\},$$
$$a(n,r;\mathbb{K}) := \max\{\dim V : V \subset \mathrm{Alt}(n,\mathbb{K}) \text{ is a } \mathbb{R}\text{-subspace such that } \underline{\mathrm{rank}}V \geq r\}.$$

where $\mathrm{Alt}(n,\mathbb{K}) := \{X \in M(n,\mathbb{K}) : {}^tX + X = 0\}$ and $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$,

Remark 2.7.5. The above notation $\ell(p, q, r; \mathbb{K})$ corresponds to the notation $L_{\mathbb{K}}(p, q; r)$ on page 380 of the book [23].

Remark 2.7.6. Since the rank of alternative matrix over $\mathbb{R}$ or $\mathbb{C}$ is even, the following equality holds.

$$a(n, 2r; \mathbb{K}) = a(n, 2r - 1; \mathbb{K}),$$

where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.7.7 ([1, 2, 25])**.**

$$\begin{aligned} \rho(n, \mathbb{R}) &:= \ell(n, n, n; \mathbb{R}), \\ \rho_A(n, \mathbb{R}) &:= a(n, n; \mathbb{R}), \\ \rho(n, \mathbb{C}) &:= \ell(n, n, n; \mathbb{C}). \end{aligned}$$

Remark 2.7.8. Here, $\rho(n, \mathbb{R})$ is called Hurwitz–Radon number ([10, 22]) and the numbers $\ell(m, n, r; \mathbb{K})$ and $a(n, r; \mathbb{K})$ are its generalization.

Fact 2.7.9 ([1, 2, 25])**.** For a positive integer n, when we write $n = 2^k(2\ell + 1)$, $k = 4\alpha + \beta$ ($k, \ell, \alpha, \beta \in \mathbb{Z}_{\geq 0}$, $0 \leq \beta \leq 3$) uniquely, the following equalities holds:

$$\begin{aligned} \rho(n, \mathbb{R}) &= 8\alpha + 2^\beta, \\ \rho_A(n, \mathbb{R}) &= \rho(n, \mathbb{R}) - 1, \\ \rho(n, \mathbb{C}) &= 2k + 2. \end{aligned}$$

Remark 2.7.10. In the light of Fact 2.7.9, the following inequalities hold.

(i) $\rho(n, \mathbb{R}) \leq n$,

(ii) $\rho_A(n, \mathbb{R}) \leq n - 1$.

Here, the equalities are attained if and only if $n = 1$, 2, 4 or 8.

We introduce the number $s(G, H)$ for a homogeneous space G/H of reductive type as in Section 2.2.1, which describes how large subspace of $\mathfrak{p}$ satisfying Fact 2.2.5 (ii) we can take.

Definition 2.7.11. For a homogeneous space G/H of reductive type, we set

$$s(G, H) := \max\{\dim V : V \subset \mathfrak{p} \text{ is a } \mathbb{R}\text{-subspace such that } V \cap \mathrm{Ad}(K)\mathfrak{p}_H = \{0\}\}.$$

Remark 2.7.12. The value $s(G, H)$ is well-defined since it is independent of the choice of a Cartan involution.

Remark 2.7.13. By using $s(G, H)$, Fact 2.2.5 is reformulated as follows. For a homogeneous space G/H of reductive type, the following two conditions are equivalent:

(i) G_θ/H_θ admits compact Clifford–Klein forms,

(ii) $s(G,H) = d(G) - d(H) (= \dim \mathfrak{p} - \dim \mathfrak{p}_H)$.

To apply Adams's theorem to the above symmetric pairs, we describe $s(G,H)$ as a linear algebraic condition by using the notation in Definition 2.7.4.

Proposition 2.7.14. • For $(G,H) = (SO_0(p_1+p_2, q_1+q_2), SO_0(p_1,q_1) \times SO_0(p_2,q_2))$, $(SU(p_1+p_2, q_1+q_2), S(U(p_1,q_1) \times U(p_2,q_2)))$, $(Sp(p_1+p_2, q_1+q_2), Sp(p_1,q_1) \times Sp(p_2,q_2))$, we have

$$s(G,H) = \ell(p_1+p_2, q_1+q_2, \min(p_1,q_1) + \min(p_2,q_2) + 1; \mathbb{K})$$

where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ respectively.

- For $(G,H,H^a) = (SO^*(2(p+q)), SO^*(2p) \times SO^*(2q), U(p,q))$, we have

$$s(G,H) = a(p+q, 2(\lfloor \tfrac{p}{2} \rfloor + \lfloor \tfrac{q}{2} \rfloor) + 1; \mathbb{C}),$$
$$s(G,H^a) = a(p+q, 2\min(p,q) + 1; \mathbb{C}).$$

- For $(G,H,H^a) = (SO(p+q,\mathbb{C}), SO(p,\mathbb{C}) \times SO(q,\mathbb{C}), SO_0(p,q))$, we have

$$s(G,H) = a(p+q, 2(\lfloor \tfrac{p}{2} \rfloor + \lfloor \tfrac{q}{2} \rfloor) + 1; \mathbb{R}),$$
$$s(G,H^a) = a(p+q, 2\min(p,q) + 1; \mathbb{R}).$$

We prove only the case when $(G,\ H)$=$(SO_0(p,q),\ SO_0(p_1,q_1) \times SO_0(p_2,q_2))$ in the above symmetric pairs. The other cases are proved similarly. We realize a symmetric pair $(G,\ H)$=$(SO_0(p,q),\ SO_0(p_1,q_1) \times SO_0(p_2,q_2))$ as follows:

$$G := \{g \in GL(p+q,\mathbb{R}) : {}^t g I_{p,q} g = I_{p,q}\}_0,$$
$$H := \{g \in G : g I_{p_1,p_2,q_1,q_2} = I_{p_1,p_2,q_1,q_2} g\}_0,$$

where $I_{p,q} = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$, $I_{p_1,p_2,q_1,q_2} = \begin{pmatrix} I_{p_1,p_2} & \\ & I_{q_1,q_2} \end{pmatrix}$ and "0" means taking the identity component. Then, by taking a Cartan involution $\theta : g \mapsto {}^t g^{-1}$, we have

$$K = SO(p) \times SO(q) = \left\{ \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix} : k_1 \in SO(p), k_2 \in SO(q) \right\},$$
$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ {}^t B & 0 \end{pmatrix} \in M(p+q,\mathbb{R}) : B \in M(p,q;\mathbb{R}) \right\},$$
$$\mathfrak{p}_H = \left\{ \begin{pmatrix} 0 & B \\ {}^t B & 0 \end{pmatrix} \in M(p+q,\mathbb{R}) : B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, B_1 \in M(p_1,q_1;\mathbb{R}), B_2 \in M(p_2,q_2;\mathbb{R}) \right\}.$$

To prove the above proposition for $(G,\ H)$=$(SO_0(p,q),\ SO_0(p_1,q_1) \times SO_0(p_2,q_2))$, it is enough to show the following:

Lemma 2.7.15. Let $(G,\ H) = (SO_0(p,q),\ SO_0(p_1,q_1) \times SO_0(p_2,q_2))$. We identify $\mathfrak{p}$ with $M(p,q;\mathbb{R})$. For $X \in M(p,q;\mathbb{R})$ the following conditions are equivalent.

(i) $X \in K \cdot \mathfrak{p}_H$,

(ii) $\operatorname{rank} X \leq \min(p_1, q_1) + \min(p_2, q_2)$.

Remark 2.7.16. The adjoint representation of $K = SO(p) \times SO(q)$ on $\mathfrak{p}$ is equivalent to the following representation σ on $M(p, q; \mathbb{R})$.

$$\sigma \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} : M(p, q; \mathbb{R}) \to M(p, q; \mathbb{R}), \ X \mapsto k_1 X k_2^{-1} \text{ for } \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K.$$

We identify the adjoint representation with the above representation. Then we regard $\mathfrak{p}_H$ under the above identification as the following subspace of $M(p, q; \mathbb{R})$:

$$\mathfrak{p}_H \simeq \left\{ \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} : B_1 \in M(p_1, q_1; \mathbb{R}), B_2 \in M(p_2, q_2; \mathbb{R}) \right\}.$$

Proof of Lemma 2.7.15. Put $r := \min(p_1, q_1) + \min(p_2, q_2)$. The implication (i)$\Rightarrow$(ii) follows from that $\operatorname{rank} X \leq r$ for any $X \in \mathfrak{p}_H$ and that the K-action on $M(p, q; \mathbb{R})$ preserves the rank. On the other hand, we prove (ii)$\Rightarrow$(i). Take $X \in M(p, q, \mathbb{R})$ such that $\operatorname{rank} X \leq r$. Now, we take a maximal split abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ as follows.

$$\mathfrak{a} \simeq \{\operatorname{diag}(a_1, \cdots, a_{\min(p,q)}) \in M(p, q; \mathbb{R}) : a_i \in \mathbb{R} \ (i = 1, \cdots, \min(p, q))\}.$$

From $\operatorname{Ad}(K)\mathfrak{a} = \mathfrak{p}$, we can take $k \in K$ such that $k \cdot X = \operatorname{diag}(a_1, \cdots, a_r, 0, \cdots, 0)$. Thus, by taking appropriate $k' \in K$, we get $k'k \cdot X \in \mathfrak{p}_H$. $\square$

Proof of Proposition 2.7.1.

- For the case where $(G, H, H^a) = (SO_0(p, q+1), SO_0(p, q), SO_0(p, 1) \times SO(q))$: This comes from Fact 2.7.18 and the fact that $\operatorname{Ad}(O(p) \times O(q+1)) \cdot \mathfrak{p}_H = \operatorname{Ad}(SO(p) \times SO(q+1)) \cdot \mathfrak{p}_H$ in $\mathfrak{p} \simeq M(p, q+1; \mathbb{R})$.

- For the case where $(G, H) = (SU(p, 2), S(U(p, 1) \times U(1)))$: This comes from Proposition 2.7.14, Lemma 2.7.19(b) and $d(G) - d(H) = 2p$.

- For the case where $(G, H, H^a) = (SO^*(2(2p)), SO^*(2(2p-1)) \times SO^*(2), U(2p-1, 1))$: Since $s(G, H) = a(2p, 2p-1, \mathbb{C}) = a(2p, 2p, \mathbb{C}) \leq \rho(2p, \mathbb{C})$ and $d(G) - d(H) = 2(2p-1)$ hold, this comes from the following Lemma 2.7.17.

- For the case where $(G, H, H^a) = (SO(2(p+q)-2, \mathbb{C}), SO(2p-1, \mathbb{C}) \times SO(2q-1, \mathbb{C})), SO_0(2p-1, 2q-1))$ $(1 \leq p \leq q)$: Since $s(G, H) = a(2(p+q)-2, 2(p+q)-3; \mathbb{R}) = a(2(p+q)-2, 2(p+q)-2; \mathbb{R}) = \rho_A(2(p+q)-2, \mathbb{R})$ and $d(G) - d(H) = (2p-1)(2q-1)$ hold, this comes from the Lemma 2.7.22 and Remark 2.7.10.

$\square$

Lemma 2.7.17. Let $n \in \mathbb{Z}_{>0}$ be even. If $n \geq 6$, then $\rho(n, \mathbb{C}) < 2(n-1)$.

Proof. Let $n = 2^k(2\ell - 1)$ $(k, \ell \in \mathbb{Z}_{>0})$. Then, $n \geq 6$ if and only if $k \geq 3$ or $\ell \geq 2$. Thus,

$$\begin{aligned} 2(n-1) - \rho_{\mathbb{C}}(n) &= 2^{k+1}(2\ell - 1) - 2k - 4 \\ &\geq \begin{cases} 16(2\ell - 1) - 10 > 0 & (k \geq 3) \\ 2^{k+1}3 - 2k - 4 \geq 2^2 3 - 8 > 0 & (\ell \geq 2). \end{cases} \end{aligned}$$

Here, we use Fact 2.7.9 and the fact that $2^k - k$ is monotone increasing for $k \in \mathbb{Z}_{>0}$. □

Fact 2.7.18 ([KY05, Proposition 5.5.1])**.** The following conditions on the pair (p, q) of positive integers are equivalent:

(i) The tangential symmetric space of $O(p, q+1)/O(p, q)$ admits a compact Clifford–Klein form.

(ii) $q < \rho(p, \mathbb{R})$.

Lemma 2.7.19. (a) Let m, n be positive integers. Then

$$\ell(n, m, m; \mathbb{R}) \geq n \Leftrightarrow \rho(n, \mathbb{R}) \geq m.$$

(b) Let n be a positive integer. Then

$$\ell(n, 2, 2; \mathbb{C}) \geq 2n \Leftrightarrow n \text{ is even}.$$

Proof. (a): This comes from the following (see Definition 2.7.20 for non-singularity):

$$\begin{aligned} &\ell(n, m, m; \mathbb{R}) \geq n, \\ \Leftrightarrow\ &\text{There exists a linear injective map } \phi : \mathbb{R}^n \to M(n, m; \mathbb{R}) \text{ such that} \\ &\quad \operatorname{rank} \phi(v) \geq m \text{ for all } v \in \mathbb{R}^n \setminus \{0\}, \\ \Leftrightarrow\ &\text{There exists a non-singular bilinear map } \phi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n, \\ \Leftrightarrow\ &\text{There exists a linear injective map } \phi : \mathbb{R}^m \to M(n, \mathbb{R}) \text{ such that} \\ &\quad \operatorname{rank} \phi(v) \geq n \text{ for all } v \in \mathbb{R}^n \setminus \{0\} \\ \Leftrightarrow\ &\ell(n, n, n; \mathbb{R}) \geq m \\ \Leftrightarrow\ &\rho(n, \mathbb{R}) \geq m. \end{aligned}$$

(b): $(\Rightarrow)$: Suppose $\ell(n, 2, 2; \mathbb{C}) \geq 2n$. Then, the inequality $\ell(2n, 4, 4; \mathbb{R}) \geq 2n$ follows from Lemma 2.7.21. Therefore,

$$\begin{aligned} \ell(2n, 4, 4; \mathbb{R}) \geq 2n &\Leftrightarrow \rho(2n, \mathbb{R}) \geq 4 \\ &\Leftrightarrow n \text{ is even}. \end{aligned}$$

$(\Leftarrow)$: Assume n is even. It is enough to construct a $\mathbb{R}$-subspace V of $M(n, 2; \mathbb{C})$ such that $\dim V = 2n$ and $\underline{\operatorname{rank}} V = 2$. Such a V is given by:

$$V := \left\{ X = {}^t\begin{pmatrix} \alpha_1 & -\beta_1 & \cdots & \alpha_{\frac{n}{2}} & -\beta_{\frac{n}{2}} \\ \overline{\beta_1} & \overline{\alpha_1} & \cdots & \overline{\beta_{\frac{n}{2}}} & \overline{\alpha_{\frac{n}{2}}} \end{pmatrix} : \alpha_i, \beta_i \in \mathbb{C}\ (i = 1, \cdots, \tfrac{n}{2}) \right\}.$$

In fact, we can easily check the above conditions by $X^*X = \sum_{i=1}^{\frac{n}{2}}(|\alpha_i|^2 + |\beta_i|^2)I_2$. $\square$

Definition 2.7.20. Let U, V and W be vector spaces. A bilinear map $f : U \times V \to W$ is said to be non-singular if f satisfies that

$$f(u, v) = 0 \text{ only if } u = 0 \text{ or } v = 0.$$

Lemma 2.7.21. The following inequality holds.

$$\ell(m, n, r; \mathbb{C}) \leq \ell(2m, 2n, 2r; \mathbb{R}).$$

Proof. The following linear map ϕ is injective and has the property that $\operatorname{rank}\phi(X) = 2\operatorname{rank} X$ for $X = A + Bi \in M(m, n; \mathbb{C})$ $(A, B \in M(m, n; \mathbb{R}))$.

$$\phi : M(m, n; \mathbb{C}) \to M(2m, 2n; \mathbb{R}),\ A + Bi \to \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

$\square$

Lemma 2.7.22. For positive integers p, q, the following inequality holds:

$$p + q - 1 \leq pq.$$

Here, the equality is attained if and only if $p = 1$ or $q = 1$.

2.8 Appendix

We give a proof of Fact 2.6.10 for the sake of completeness.

Let G be a connected compact Lie group, $\varpi : P \to M$ a principal G-bundle, $\rho : G \to SO(V)$ a representation of G and $E := P \times_G V$ the associated bundle. We construct a curvature R^{∇_θ} on the vector bundle E from a curvature form θ on the principal G-bundle $\varpi : P \to M$ by using $d\rho : \mathfrak{g} \to \mathfrak{so}(V)$ and see that the curvature R^{∇_θ} is compatible with the Chern-Weil map $\omega : S(\mathfrak{g}^*)^G \to H^*(M, \mathbb{R})$.

We denote V-valued differential forms of degree q on P by $\mathfrak{A}^q(P, V)$.

Definition 2.8.1 ([12, p146])**.** We introduce the following subset of $\mathfrak{A}^q(P, V)$.

$$\mathfrak{A}^q_B(P, V) := \{\omega \in \mathfrak{A}^q(P, V) \text{ satisfying the following conditions (i),(ii)}\},$$

(i) $i(X^\sharp)\omega = 0$ for any $X \in \mathfrak{g}$,

(ii) $R_g^*\omega = \rho(g)^{-1}\omega$ for any $g \in G$,

where $X^\sharp$ is the fundamental vector field on P associated with $X \in \mathfrak{g}$.

We can define a map $\varpi_q^* : \Gamma(E \otimes \bigwedge^q T^*M) =: \mathfrak{A}^q(E) \to \mathfrak{A}^q_B(P, V)$ for any $q \in \mathbb{Z}_{\geq 0}$ by

$$(\varpi_q^* s)_p(X_1, \cdots X_q) = p^{-1}s_{\varpi(p)}(\varpi_* X_1, \cdots, \varpi_* X_q),$$

where $s \in \mathfrak{A}^q(E)$, $p \in P, X_1, \cdots, X_q \in T_pP$ and p^{-1} is the inverse map of the linear isomorphism $p : V \to E_{\varpi(p)}, v \mapsto [p, v]$. Then, we get the following:

Fact 2.8.2 ([12, Proposition 6.2.3])**.** The map $\varpi_q^* : \mathfrak{A}^q(E) \to \mathfrak{A}_B^q(P,V)$ is $\mathbb{R}$-linear isomorphism.

Fact 2.8.3 ([12, Proposition 6.3.3])**.** Let $\theta \in \mathfrak{A}^1(P,\mathfrak{g})$ be a connection form on P. Then $\nabla_\theta := \varpi_1^{*-1} \circ (d + d\rho(\theta)) \circ \varpi_0^*$ is a connection on E.

Let d^{∇_θ} be a exterior covariant differentiation defined by the connection ∇_θ and $\Omega \in \mathfrak{A}^2(P,\mathfrak{g})$ the curvature form defined by θ. Then following fact holds.

Fact 2.8.4 ([12, Proposition 6.3.3 and 6.3.10])**.** (i) d^{∇_θ} is commutative with $d + d\rho(\theta)$ through ϖ_q^*, i.e. $\varpi_{q+1}^* \circ d^{\nabla_\theta} = (d + d\rho(\theta)) \circ \varpi_q^*$ for any $q \in \mathbb{Z}_{\geq 0}$.

(ii) $(d+d\rho(\theta))\circ(d+d\rho(\theta)) = d\rho(\Omega) : \mathfrak{A}_B^q(P,V) \to \mathfrak{A}_B^{q+2}(P,V)$ for any $q \in \mathbb{Z}_{\geq 0}$.

From the above Fact 2.8.4(i), for each $q \in Z_{\geq 0}$, we get the following commutative diagram.

$$\begin{tikzcd}
\mathfrak{A}^q(E) \arrow[r, "d^{\nabla_\theta}"] \arrow[d, "\varpi_q^*"] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{A}^{q+1}(E) \arrow[r, "d^{\nabla_\theta}"] \arrow[d, "\varpi_{q+1}^*"] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{A}^{q+2}(E) \arrow[d, "\varpi_{q+2}^*"] \\
\mathfrak{A}^q(P,V) \arrow[r, "d+d\rho(\theta)"] & \mathfrak{A}^{q+1}(P,V) \arrow[r, "d+d\rho(\theta)"] & \mathfrak{A}^{q+2}(P,V)
\end{tikzcd}$$

In particular, considering the case when $q = 0$ and Fact 2.8.4(ii), we can describe the curvature R^{∇_θ} defined by ∇_θ on E as follows:

$$R^{\nabla_\theta} = \varpi_2^{*-1} \circ d\rho(\Omega) \circ \varpi_0^*.$$

For basic differential form $\alpha \in \mathfrak{A}^q(P)$ (i.e. $\alpha \in \varpi^*(\mathfrak{A}^q(M))$), we denote the corresponding differential form on M by $\overline{\alpha} \in \mathfrak{A}^q(M)$.

proof of Fact 2.6.10. We can easily check that $\varpi^*(\overline{f(\varpi_2^{*-1} \circ d\rho(\Omega) \circ \varpi_0^*)}) = f(d\rho(\Omega))$. Therefore, we get $f(\varpi_2^{*-1} \circ d\rho(\Omega) \circ \varpi_0^*) = \overline{f(d\rho(\Omega))}$. Thus,

$$\begin{aligned}
[f(R)] &= [f(R^{\nabla_\theta})] \\
&= [f(\varpi_2^{*-1} \circ d\rho(\Omega) \circ \varpi_0^*)] \\
&= [\overline{f(d\rho(\Omega))}] \\
&= \omega \circ d\rho^*(f).
\end{aligned}$$

□

2.9 Acknowledgement

I would like to express my gratitude to my supervisor Taro Yoshino for constructive comments and warm encouragement. I would also like to thank Prof. Toshiyuki Kobayashi for giving me insightful advice. Moreover, I would also like to thank Yosuke Morita and Takayuki Okuda for valuable comments and discussions. This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

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