博士論文

論文題目　Applications of Microlocal Sheaf Theory to Symplectic Geometry in Cotangent Bundles
（余接束のシンプレクティック幾何への超局所層理論の応用）

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Applications of Microlocal Sheaf Theory to Symplectic Geometry in Cotangent Bundles

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Abstract

The microlocal sheaf theory due to Kashiwara and Schapira can be regarded as Morse theory with sheaf coefficients. It has many applications to the study of partial differential equations and singularity theory. Recently it has been applied to symplectic geometry, after the pioneering work of Tamarkin.

In this thesis, following Tamarkin's sheaf-theoretic approach, we apply the microlocal sheaf theory to several problems in symplectic geometry in cotangent bundles. In particular, using sheaf-theoretic methods, we study (i) the intersection of two compact exact Lagrangian submanifolds, (ii) the displacement energy of two compact subsets.

First, in Chapter 3, we study intersections of compact exact Lagrangian submanifolds in cotangent bundles. We show that the total Betti number of the clean intersection of two compact exact Lagrangian submanifolds is bounded from below by the dimension of the Hom space of sheaf quantizations of the Lagrangians in Tamarkin's category. As a corollary, we give a purely sheaf-theoretic proof of a result of Nadler and Fukaya-Seidel-Smith, which asserts that the cardinality of the transverse intersection of two compact exact Lagrangians is at least the total Betti number of the base manifold.

Second, in Chapter 4, we study the displacement energy of compact subsets of cotangent bundles. We introduce a persistence-like pseudo-distance on Tamarkin's category and prove that the distance between an object and its Hamiltonian deformation is at most the Hofer norm of the Hamiltonian function. Using the distance, we show a quantitative version of Tamarkin's non-displaceability theorem, which gives a lower bound of the displacement energy of compact subsets of cotangent bundles. This theorem gives a sheaf-theoretic proof of a result of Polterovich, which says the positivity of the displacement energy of a compact subset whose interior is non-empty.

This thesis is based on the following papers of the author. Chapter 3 corresponds to [Ike17] and Chapter 4 corresponds to [AI17], which is a joint work with Tomohiro Asano.

[AI17] T. Asano and Y. Ike, Persistence-like distance on Tamarkin's category and symplectic displacement energy, *arXiv preprint*, `arXiv:1712.06847`, (2017), submitted.

[Ike17] Y. Ike, Compact exact Lagrangian intersections in cotangent bundles via sheaf quantization, *arXiv preprint*, `arXiv:1701.02057`, (2017), submitted.

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Contents

Chapter 1

Introduction

In this thesis, we study several problems in symplectic geometry in cotangent bundles using the microlocal sheaf theory. First, we review the microlocal sheaf theory and previous results on its applications to symplectic geometry. Then we present the problems (Problem 1.1.1 and Problem 1.1.2) which we will consider in this thesis.

1.1 Microlocal sheaf theory and symplectic geometry in cotangent bundles

The microlocal sheaf theory was introduced and systematically developed by Kashiwara and Schapira [KS90]. The theory can be regarded as Morse theory with sheaf coefficients. One of the key ingredients of the theory is the notion of microsupports of sheaves, which enable us to define "critical points of functions with respect to sheaves". In the sequel, let $\mathbf{k}$ be a field. Let moreover X be a C^∞-manifold without boundary and denote by $\mathbf{D}^{\mathrm{b}}(X)$ the bounded derived category of sheaves of $\mathbf{k}$-vector spaces. For an object $F \in \mathbf{D}^{\mathrm{b}}(X)$, its microsupport $\mathrm{SS}(F)$ is defined as the set of directions in which the cohomology of F cannot be extended isomorphically. The microsupport is a closed subset of the cotangent bundle T^*X of X and conic, that is, invariant under the action of $\mathbb{R}_{>0}$ on T^*X. As a generalization of classical Morse theory, we can prove that if the derivative $d\varphi$ of a C^∞-function $\varphi\colon X \to \mathbb{R}$ does not meet $\mathrm{SS}(F)$, then the cohomology of F on the sublevel set $\varphi^{-1}((-\infty, c))$ does not change. We also obtain the Morse inequality for sheaves, which describes how the cohomology of F on $\varphi^{-1}((-\infty, c))$ changes when the derivative $d\varphi$ goes across $\mathrm{SS}(F)$.

Cotangent bundles are typical symplectic manifolds and hence we can consider non-displaceability problems as explained below. In what follows, let M be a non-empty connected C^∞-manifold without boundary and denote by T^*M its cotangent bundle. We also denote by $(x;\xi)$ a local homogeneous coordinate system. We regard T^*M as an exact symplectic manifold equipped with the Liouville 1-form $\alpha_{T^*M} = \langle \xi, dx \rangle$. Let I be an open interval containing $[0,1]$. A compactly supported C^∞-function $H = (H_s)_{s \in I} \colon T^*M \times I \to \mathbb{R}$ defines a time-dependent Hamiltonian vector field $X_H = (X_{H_s})_s$ on T^*M. By the compactness of the support, X_H generates a Hamiltonian isotopy $\phi^H = (\phi^H_s)_s \colon T^*M \times I \to T^*M$. Compact subsets A and B of T^*M are said to be mutually non-displaceable if $A \cap \phi^H_1(B) \neq \emptyset$ for any compactly supported function H. Here ϕ^H_1 denotes the time-one map of the Hamiltonian isotopy ϕ^H. The problem of determining whether or not compact subsets, especially Lagrangians, are mutually non-displaceable is a central issue in symplectic geometry. As a quantitative generalization, to give an estimate of the cardinality $\#(A \cap \phi^H_1(B))$ is also an important problem. Nowadays, many symplectic geometers study the problems using pseudo-holomorphic curves and Lagrangian intersection Floer theory.

Tamarkin [Tam08] proposed a new approach to non-displaceability problems, which is based on the microlocal sheaf theory. For a conic Lagrangian submanifold of T^*M, a sheaf on M whose microsupport coincides with it (outside the zero-section) is called a *sheaf quantization* of the Lagrangian. For a non-conic Lagrangian of T^*M, one can consider a sheaf quantization by adding one more variable to the base manifold M and "conifying" it. Using sheaf quantizations, Tamarkin studied the non-displaceability of particular Lagrangian submanifolds. After his work, Guillermou-Kashiwara-Schapira [GKS12] and Guillermou [Gui12, Gui16a] proved the existence of sheaf quantizations of graphs of Hamiltonian isotopies and compact exact Lagrangian submanifolds in cotangent bundles, respectively. See Section 2.2 for more details. Using sheaf quantization, they studied the non-displaceability of the zero-sections of cotangent bundles and topological properties of compact exact Lagrangian submanifolds. Note that sheaf-theoretic approaches to symplectic geometry also appeared in [KO01, NZ09, Nad09].

We give more precise explanation on results of Tamarkin [Tam08], which we need to state our results. See Subsection 2.2.2 for more details. He introduced the category $\mathcal{D}(M)$ which is defined as a quotient category of $\mathbf{D}^{\mathrm{b}}(M \times \mathbb{R})$. For a compact subset A of T^*M, $\mathcal{D}_A(M)$ denotes the full subcategory of $\mathcal{D}(M)$ consisting of objects whose microsupports are contained in the cone of A in $T^*(M \times \mathbb{R})$. For an object $F \in \mathcal{D}(M)$ and $c \in \mathbb{R}_{\geq 0}$, there is a canonical morphism $\tau_{0,c}(F)\colon F \to T_{c*}F$, where $T_c\colon M \times \mathbb{R} \to M \times \mathbb{R}$ is the translation map $(x,t) \mapsto (x, t+c)$. Moreover, the category $\mathcal{D}(M)$ admits an internal Hom functor $\mathcal{H}om^\star$ such that $\mathrm{Hom}_{\mathcal{D}(M)}(F,G) \simeq H^0 R\Gamma_{M \times [0,+\infty)}(M \times \mathbb{R}; \mathcal{H}om^\star(F,G))$ for any $F, G \in \mathcal{D}(M)$. Denote by $q_{\mathbb{R}}\colon M \times \mathbb{R} \to \mathbb{R}$ the projection and let A and B be compact subsets of T^*M. Tamarkin proved the following two theorems:

(i) (Tamarkin's separation theorem) If there exist $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$ such that $Rq_{\mathbb{R}*}\mathcal{H}om^\star(F,G) \neq 0$, then $A \cap B \neq \emptyset$.

(ii) (Tamarkin's non-displaceability theorem) If there exist $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$ such that $\tau_{0,c}(Rq_{\mathbb{R}*}\mathcal{H}om^\star(F,G)) \neq 0$ for any $c \in \mathbb{R}_{\geq 0}$, then A and B are mutually non-displaceable.

The aim of this thesis is to give quantitative generalizations of Tamarkin's theorems in two different directions. More concretely, we consider the following two problems.

Problem 1.1.1. Guillermou dealt with only one Lagrangian submanifold and did not consider the intersection of two Lagrangian submanifolds. Moreover, Tamarkin's separation theorem concerns only the non-emptiness of the intersection and says nothing about its cardinality. We wish to estimate the cardinality or the total Betti number of the intersection using Guillermou's sheaf quantizations and the functor $\mathcal{H}om^\star$.

Problem 1.1.2. Tamarkin dealt with only the non-displaceability of two compact subsets and did not consider displaceable subsets. Even if two compact subsets are displaceable, we would like to estimate their displacement energy using Tamarkin's category $\mathcal{D}(M)$.

We study Problem 1.1.1 in Chapter 3 and Problem 1.1.2 in Chapter 4. We state our results for each problems in Section 1.2 and Section 1.3, respectively.

1.2 Compact exact Lagrangian intersections in cotangent bundles via sheaf quantization

In Chapter 3, we prove that the cardinality of the transverse intersection of two compact exact Lagrangian submanifolds in cotangent bundles is bounded from below by the dimension of the local cohomology of $\mathcal{H}om^\star$ applied to sheaf quantizations of the Lagrangians. More generally, provided $\mathbf{k} = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, we show that a clean version of the estimate holds with "cardinality" replaced by "total $\mathbb{F}_2$-Betti number".

1.2.1 Our results

In this part, we assume that M is compact. A submanifold L of dimension $\dim M$ in T^*M is said to be exact Lagrangian if $\alpha_{T^*M}|_L$ is exact. The main result of this part is the following. See Section 2.2 for the definitions of simple sheaf quantizations and the category $\mathcal{T}(M)$.

Theorem 1.2.1 (see Theorem 3.4.7)**.** *For $i = 1, 2$, let L_i be a compact connected exact Lagrangian submanifolds and $F_i \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R})$ be a simple sheaf quantization associated with L_i and a function $f_i \colon L_i \to \mathbb{R}$ satisfying $df_i = \alpha_{T^*M}|_{L_i}$. Assume that L_1 and L_2 intersect cleanly, that is, $L_1 \cap L_2$ is a submanifold of T^*M and $T_p(L_1 \cap L_2) = T_pL_1 \cap T_pL_2$ for any $p \in L_1 \cap L_2$. Let $L_1 \cap L_2 = \bigsqcup_{j=1}^n C_j$ be the decomposition into connected components and define $f_{21}(C_j) := f_2(p) - f_1(p)$ for some $p \in C_j$ (independent of the choice of p). Let moreover $a, b \in \mathbb{R}$ with $a < b$ or $a \in \mathbb{R}, b = +\infty$. Then, for $\mathbf{k} = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, one has*

$$\begin{aligned} &\sum_{a \le f_{21}(C_j) < b} \sum_{k \in \mathbb{Z}} \dim_{\mathbb{F}_2} H^k(C_j; \mathbb{F}_2) \\ &\qquad \geq \sum_{k \in \mathbb{Z}} \dim_{\mathbb{F}_2} H^k R\Gamma_{M \times [a,b)}((-\infty, b); \mathcal{H}om^\star(F_2, F_1)). \end{aligned} \tag{1.2.1}$$

In particular,

$$\sum_{j=1}^n \sum_{k \in \mathbb{Z}} \dim_{\mathbb{F}_2} H^k(C_j; \mathbb{F}_2) \geq \sum_{k \in \mathbb{Z}} \dim_{\mathbb{F}_2} \mathrm{Hom}_{\mathcal{T}(M)}(F_2, F_1[k]). \tag{1.2.2}$$

If L_1 and L_2 intersect transversally, the inequalities hold for any field $\mathbf{k}$, not only for $\mathbb{F}_2$.

We also have

$$\mathrm{Hom}_{\mathcal{T}(M)}(F_2, F_1[k]) \simeq H^k(M; \mathcal{L}) \quad \text{for any } k \in \mathbb{Z}, \tag{1.2.3}$$

where $\mathcal{L}$ is the locally constant sheaf of rank 1 on M associated with F_1 and F_2 (see Proposition 3.1.2 for details). Combining this with Theorem 1.2.1, we obtain a purely sheaf-theoretic proof of the following result of Nadler [Nad09] and Fukaya-Seidel-Smith [FSS08], as a corollary.

Corollary 1.2.2 ([Nad09, Theorem 1.3.1] and [FSS08, Theorem 1])**.** *Let L_1 and L_2 be compact connected exact Lagrangian submanifolds of T^*M intersecting transversally. Then*

$$\#(L_1 \cap L_2) \geq \sum_{k \in \mathbb{Z}} \dim H^k(M; \mathcal{L}) \tag{1.2.4}$$

for any rank 1 locally constant sheaf $\mathcal{L}$ on M over any field $\mathbf{k}$. In particular, $\#(L_1 \cap L_2) \geq \sum_{k \in \mathbb{Z}} \dim H^k(M; \mathbf{k})$.

The proof of Theorem 1.2.1 goes as follows. First, we apply the Morse-Bott inequality for sheaves (see Theorem 2.1.10) to the object $\mathcal{H} := \mathcal{H}om^\star(F_2, F_1)$ and the function

$M \times \mathbb{R} \to \mathbb{R}, (x,t) \mapsto t$, and obtain

$$\begin{aligned}\sum_{a \le c < b} \sum_{k \in \mathbb{Z}} \dim H^k R\Gamma \left(M \times \{c\}; R\Gamma_{M \times [c,+\infty)}(\mathcal{H})|_{M \times \{c\}}\right) \\ \ge \sum_{k \in \mathbb{Z}} \dim H^k R\Gamma_{M \times [a,b)}(M \times (-\infty, b); \mathcal{H}).\end{aligned} \tag{1.2.5}$$

In order to calculate the left hand side of (1.2.5), we use the functor $\mu hom \colon \mathbf{D}^{\mathrm{b}}(X)^{\mathrm{op}} \times \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(T^*X)$ introduced by Kashiwara-Schapira [KS90]. Using the functor, we show the isomorphism

$$R\Gamma\left(M \times \{c\}; R\Gamma_{M \times [c,+\infty)}(\mathcal{H})|_{M \times \{c\}}\right) \simeq R\Gamma(\Omega_+; \mu hom(T_{c*}F_2, F_1)|_{\Omega_+}), \tag{1.2.6}$$

where $T_c \colon M \times \mathbb{R} \to M \times \mathbb{R}, (x,t) \mapsto (x, t+c)$ and $\Omega_+ := \{\tau > 0\} \subset T^*(M \times \mathbb{R})$ with $(t;\tau)$ being the homogeneous symplectic coordinate on $T^*\mathbb{R}$. The object $\mu hom(T_{c*}F_2, F_1)|_{\Omega_+}$ is supported in $\{(x,t;\tau\xi,\tau) \mid \tau > 0, (x;\xi) \in L_1 \cap L_2, t = f_2(x;\xi) - f_1(x;\xi) = c\}$ and isomorphic to a shift of the constant sheaf of rank 1 on the support. This completes the proof.

Remark 1.2.3. Even if the intersection is degenerate, (1.2.5) and (1.2.6) still hold, but the object $\mu hom(T_{c*}F_2, F_1)|_{\Omega_+}$ is not necessarily locally constant on the support. In this sense, the family of sheaves $\{\mu hom(T_{c*}F_2, F_1)|_{\Omega_+}\}_c$ encodes the "contribution" from each possibly degenerate component of the intersection $L_1 \cap L_2$. We will also explore the contribution in degenerate cases in Section 3.A Appendix I.

1.2.2 Relation to Lagrangian intersection Floer theory

Although our approach is purely sheaf-theoretic, it seems to be closely related to Floer cohomology and Fukaya categories. We briefly remark the relation below. The category $\mathcal{T}(M)$ has the following properties:

(i) Hamiltonian invariance ([Tam08, GS14]),

(ii) the dimension of the cohomology of the clean intersection of two compact exact Lagrangian submanifolds is bounded from below by the dimension of the Hom space of simple sheaf quantizations (Theorem 1.2.1).

Moreover, as pointed out by T. Kuwagaki, the following also holds in $\mathcal{T}(M)$:

(iii) a simple sheaf quantization associated with any compact connected exact Lagrangian submanifold is isomorphic to a simple sheaf quantization associated with the zero-section of T^*M (see Proposition 3.1.4).

The Floer cohomology $HF^*(L_2, L_1)$ has similar properties to (i) and (ii), though the approach is totally different. Floer cohomology for clean Lagrangian intersections was studied by Poźniak [Poź99], Frauenfelder [Fra04], Fukaya-Oh-Ohta-Ono [FOOO09a, FOOO09b], and Schmäschke [Sch16]. Moreover, Nadler [Nad09] and Fukaya-Seidel-Smith [FSS08, FSS09] proved the following, which corresponds to (iii): in the infinitesimal Fukaya category of T^*M, any relatively spin compact connected exact Lagrangian submanifold of T^*M with vanishing Maslov class is isomorphic to a shift of the zero-section. Note that their assumptions of relatively spin and vanishing Maslov class can be removed, thanks to results of Abouzaid [Abo12], and Abouzaid and Kragh [Kra13], respectively. We also remark that Guillermou [Gui12, Gui16a] gave a sheaf-theoretic proof for the relatively spin property and the vanishing of the Maslov class.

1.3 Persistence-like distance on Tamarkin's category and symplectic displacement energy

In Chapter 4, we introduce a pseudo-distance on Tamarkin's category, inspired by the recent work by Kashiwara-Schapira [KS17] on the sheaf-theoretic interpretation of the interleaving distance for persistence modules. We also propose a new sheaf-theoretic method to estimate the displacement energy of compact subsets of cotangent bundles, which is a quantitative generalization of Tamarkin's non-displaceability theorem.

For a compact subset of a symplectic manifold, its displacement energy measures the minimal energy of Hamiltonian isotopies which displace the subset. In this part, we consider the displacement energy of subsets of cotangent bundles. Following Hofer [Hof90], we define the norm of a compactly supported Hamiltonian function $H\colon T^*M \times I \to \mathbb{R}$ by

$$\|H\| := \int_0^1 \left(\max_p H_s(p) - \min_p H_s(p) \right) ds. \tag{1.3.1}$$

For compact subsets A and B of T^*M, we define their *displacement energy* $e(A, B)$ by

$$e(A,B) := \inf \left\{ \|H\| \;\middle|\; \begin{array}{l} H\colon T^*M \times I \to \mathbb{R} \text{ with compact support,} \\ A \cap \phi_1^H(B) = \emptyset \end{array} \right\}. \tag{1.3.2}$$

Note that if $e(A, B) = +\infty$, then $A \cap \phi_1^H(B) \neq \emptyset$ for any compactly supported function H. In this part, we give a lower bound of $e(A, B)$ in terms of the microlocal sheaf theory.

1.3.1 Main results

First, using the $\mathbb{R}$-direction of $M \times \mathbb{R}$, we introduce the following pseudo-distance $d_{\mathcal{D}(M)}$ on Tamarkin's category $\mathcal{D}(M)$, which is similar to the interleaving distance for persistence modules (see [CCSG+09, CdSGO16]). Our definition is inspired by the pseudo-distances on the derived categories of sheaves on vector spaces recently introduced by Kashiwara-Schapira [KS17]. See also Remark 4.2.7 for their relation.

Definition 1.3.1.

(i) Let $F, G \in \mathcal{D}(M)$ and $a, b \in \mathbb{R}_{\geq 0}$. Then F is said to be (a, b)*-isomorphic* to G if there exist morphisms $\alpha, \delta\colon F \to T_{a*}G$ and $\beta, \gamma\colon G \to T_{b*}F$ satisfying the following conditions:

(1) $F \xrightarrow{\alpha} T_{a*}G \xrightarrow{T_{a*}\beta} T_{a+b*}F$ is equal to $\tau_{0,a+b}(F)\colon F \to T_{a+b*}F$ and $G \xrightarrow{\gamma} T_{b*}F \xrightarrow{T_{b*}\delta} T_{a+b*}G$ is equal to $\tau_{0,a+b}(G)\colon G \to T_{a+b*}G$,

(2) $\tau_{a,2a}(G) \circ \alpha = \tau_{a,2a}(G) \circ \delta$ and $\tau_{b,2b}(F) \circ \beta = \tau_{b,2b}(F) \circ \gamma$.

(ii) For objects $F, G \in \mathcal{D}(M)$, one defines

$$d_{\mathcal{D}(M)}(F, G) := \inf\{a + b \in \mathbb{R}_{\geq 0} \mid a, b \in \mathbb{R}_{\geq 0}, F \text{ is } (a,b)\text{-isomorphic to } G\}, \tag{1.3.3}$$

and calls $d_{\mathcal{D}(M)}$ the *translation distance*.

Now, let us consider the distance between an object in $\mathcal{D}(M)$ and its Hamiltonian deformation. Let $H\colon T^*M \times I \to \mathbb{R}$ be a compactly supported Hamiltonian function. Then, using the sheaf quantization associated with the Hamiltonian isotopy ϕ^H due to Guillermou-Kashiwara-Schapira [GKS12] one can define a functor $\Psi_1^H\colon \mathcal{D}(M) \to \mathcal{D}(M)$, which induces a functor $\Psi_1^H\colon \mathcal{D}_A(M) \to \mathcal{D}_{\phi_1^H(A)}(M)$ for any compact subset A of T^*M. Our first result is the following:

Theorem 1.3.2 (see Theorem 4.2.13)**.** *Let $G \in \mathcal{D}(M)$ and $H \colon T^*M \times I \to \mathbb{R}$ be a compactly supported Hamiltonian function. Then $d_{\mathcal{D}(M)}(G, \Psi_1^H(G)) \le \|H\|$.*

The outline of the proof is as follows. First we prove that the distance between two objects is controlled by the angle of a cone which contains the microsupport of a "homotopy sheaf" connecting them. Then using the sheaf quantization associated with ϕ^H, we can construct a homotopy sheaf $G' \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times I)$ such that $G'|_{M \times \mathbb{R} \times \{0\}} \simeq G, G'|_{M \times \mathbb{R} \times \{1\}} \simeq \Psi_1^H(G)$ and $\mathrm{SS}(G') \subset T^*M \times \gamma_H$, where

$$\gamma_H = \left\{ (t, s; \tau, \sigma) \;\middle|\; -\max_p H_s(p) \cdot \tau \le \sigma \le -\min_p H_s(p) \cdot \tau \right\} \subset T^*(\mathbb{R} \times I). \tag{1.3.4}$$

We thus obtain the result.

Next, we use the above result to estimate the displacement energy. Recall that one can define an internal Hom functor $\mathcal{H}om^\star$ on the category $\mathcal{D}(M)$. Let $q_{\mathbb{R}} \colon M \times \mathbb{R} \to \mathbb{R}$ denote the projection as before. Using these notions, we make the following definition.

Definition 1.3.3. For $F, G \in \mathcal{D}(M)$, one defines

$$\begin{aligned} e_{\mathcal{D}(M)}(F, G) &:= d_{\mathcal{D}(\mathrm{pt})}(Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, G), 0) \\ &= \inf\{c \in \mathbb{R}_{\ge 0} \mid \tau_{0,c}(Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, G)) = 0\}. \end{aligned} \tag{1.3.5}$$

Our main theorem is the following:

Theorem 1.3.4 (see Theorem 4.3.2)**.** *Let A and B be compact subsets of T^*M. Then, for any $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$, one has*

$$e(A, B) \ge e_{\mathcal{D}(M)}(F, G). \tag{1.3.6}$$

In particular, for any $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$,

$$e(A, B) \ge \inf\{c \in \mathbb{R}_{\ge 0} \mid \mathrm{Hom}_{\mathcal{D}(M)}(F, G) \to \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}G) \text{ is zero}\}. \tag{1.3.7}$$

This theorem implies, in particular, that $\tau_{0,c}(Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, G))$ is non-zero for any $c \in \mathbb{R}_{\ge 0}$, then A and B are mutually non-displaceable. In this sense, the theorem is a quantitative version of Tamarkin's non-displaceability theorem (see Tamarkin [Tam08, Theorem 3.1] and Guillermou-Schapira [GS14, Theorem 6.2]).

Theorem 1.3.4 is proved by Tamarkin's separation theorem and Theorem 1.3.2 as follows. Suppose that a compactly supported Hamiltonian function H satisfies $A \cap \phi_1^H(B) = \emptyset$. Then, by Tamarkin's separation theorem, $Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, \Psi_1^H(G)) \simeq 0$. Thus, by fundamental properties of $d_{\mathcal{D}(M)}$ and Theorem 1.3.2, we obtain

$$\begin{aligned} e_{\mathcal{D}(M)}(F, G) &= d_{\mathcal{D}(\mathrm{pt})}(Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, G), 0) \\ &\le d_{\mathcal{D}(M)}(\mathcal{H}om^\star(F, G), \mathcal{H}om^\star(F, \Psi_1^H(G))) \\ &\le d_{\mathcal{D}(M)}(G, \Psi_1^H(G)) \le \|H\|. \end{aligned} \tag{1.3.8}$$

As an application of Theorem 1.3.4, we prove that the displacement energy of the image of the compact exact Lagrangian immersion

$$S^m = \{(x, y) \in \mathbb{R}^m \times \mathbb{R} \mid \|x\|^2 + y^2 = 1\} \longrightarrow T^*\mathbb{R}^m \simeq \mathbb{R}^{2m}, \quad (x, y) \longmapsto (x; yx) \tag{1.3.9}$$

is greater than or equal to $2/3$ (see Example 4.4.1). Using this estimate, we give a purely sheaf-theoretic proof of the following theorem of Polterovich [Pol93], for subsets of cotangent bundles. Note that he proved the result for more general class of symplectic manifolds, using pseudo-holomorphic curves.

Proposition 1.3.5 ([Pol93, Corollary 1.6])**.** *Let A be a compact subset of T^*M whose interior is non-empty. Then the displacement energy of A is positive: $e(A, A) > 0$.*

1.3.2 Related topics

The interleaving distance for persistence modules is now widely used in topological data analysis (see, for example, [CCSG+09, CdSGO16]). Recently, Kashiwara-Schapira [KS17] interpreted the distance as that on the derived category of sheaves. In symplectic geometry, the notion of persistence modules was introduced by Polterovich-Shelukhin [PS16] (see also Polterovich-Shelukhin-Stojisavljević [PSS17]). For barcodes of chain complexes over Novikov fields such as Floer cohomology complexes, see also Usher-Zhang [UZ16]. Note also that Theorem 1.3.2 seems to be related to the results of Schwarz [Sch00] and Oh [Oh05] for continuation maps, although they did not use persistence modules.

As remarked in Tamarkin [Tam08, Section 1], for $F, G \in \mathcal{D}(M)$, one can associate a submodule $H(F,G)$ of $\prod_{c\in\mathbb{R}} \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}G)$, which is a module over a Novikov ring $\Lambda_{0,\mathrm{nov}}(\mathbf{k})$ (with a formal variable T). Using this module, we can express (1.3.7) in Theorem 1.3.4 as

$$e(A,B) \geq \inf\{c \in \mathbb{R}_{\geq 0} \mid H(F,G) \text{ is } T^c\text{-torsion}\}. \tag{1.3.10}$$

See Remark 4.3.5 for more details. This inequality seems to be closely related to the estimate of the displacement energy discussed in Fukaya-Oh-Ohta-Ono [FOOO09a, FOOO09b, Theorem J] and [FOOO13, Theorem 6.1].

1.4 Organization

This thesis is organized as follows.

Chapter 2 is devoted to an introduction of some notions and a review of previous results. In Section 2.1, we recall some definitions and results in the microlocal sheaf theory due to Kashiwara and Schapira [KS90]. In Section 2.2, we review results of [Tam08, GKS12, GS14, Gui12, Gui16a] about Tamarkin's non-displaceability theorem, and sheaf quantization of Hamiltonian isotopies and compact exact Lagrangian submanifolds in cotangent bundles.

Chapter 3 concerns compact exact Lagrangian intersections in cotangent bundles. In Section 3.1, we prove the isomorphism (1.2.3) and the non-displaceability of two compact exact Lagrangian submanifolds as a corollary. In Section 3.2, we apply the Morse-Bott inequality for sheaves to $\mathcal{H}om^\star$ and obtain (1.2.5). Then, in Section 3.3, we interpret the local cohomology in the left hand side of (1.2.5) using the μhom functor. Finally, in Section 3.4, we prove Theorem 1.2.1. In Section 3.A Appendix I, we briefly remark that our method can deal with degenerate Lagrangian intersections, using very simple examples. In Section 3.B Appendix II, we prove the "functoriality" of simple sheaf quantizations with respect to Hamiltonian isotopies. In Section 3.C Appendix III by Tomohiro Asano, we relate the shift of a simple sheaf quantization of a Lagrangian to the grading in Lagrangian intersection Floer cohomology theory.

Chapter 4 concerns the relation between the displacement energy and Tamarkin's theorem. In Section 4.1, we give a complementary result on torsion objects. In Section 4.2, we introduce the translation distance $d_{\mathcal{D}(M)}$ on Tamarkin's category and prove Theorem 1.3.2. Then, in Section 4.3, we show Theorem 1.3.4. Finally, in Section 4.4, we give some examples and applications.

Chapter 2

Preliminaries on microlocal sheaf theory and its applications to symplectic geometry

2.1 Preliminaries on microlocal sheaf theory

In this thesis, all manifolds are assumed to be real manifolds of class C^∞ without boundary. Throughout this thesis, let $\mathbf{k}$ be a field.

In this section, we recall some definitions and results from [KS90]. We mainly follow the notation in [KS90]. Until the end of this section, let X be a C^∞-manifold without boundary.

2.1.1 Geometric notions ([KS90, §4.3, §A.2])

For a locally closed subset A of X, we denote by $\overline{A}$ its closure and by $\mathrm{Int}(A)$ its interior. We also denote by Δ_X or simply Δ the diagonal of $X \times X$. We denote by $\tau_X \colon TX \to X$ the tangent bundle of X, and by $\pi_X \colon T^*X \to X$ the cotangent bundle of X. If there is no risk of confusion, we simply write τ and π instead of τ_X and π_X, respectively. For a submanifold M of X, we denote by $T_M X$ the normal bundle to M in X, and by $T^*_M X$ the conormal bundle to M in X. In particular, $T^*_X X$ denotes the zero-section of T^*X. We set $\mathring{T}^*X := T^*X \setminus T^*_X X$. For two subsets S_1 and S_2 of X, we denote by $C(S_1, S_2) \subset TX$ the normal cone of the pair (S_1, S_2).

Let $f \colon X \to Y$ be a morphism of manifolds. With f we associate the following morphisms and commutative diagram:

$$
\begin{tikzcd}
T^*X \arrow[d, "\pi_X"'] & X \times_Y T^*Y \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*Y \arrow[d, "\pi_Y"] \\
X \arrow[r, equal] & X \arrow[r, "f"'] & Y,
\end{tikzcd}
\tag{2.1.1}
$$

where f_π is the projection and f_d is induced by the transpose of the tangent map $f' \colon TX \to X \times_Y TY$.

We denote by $(x; \xi)$ a local homogeneous coordinate system on T^*X. The cotangent bundle T^*X is an exact symplectic manifold with the Liouville 1-form $\alpha_{T^*X} = \langle \xi, dx \rangle$. We denote by $a \colon T^*X \to T^*X, (x;\xi) \mapsto (x; -\xi)$ the antipodal map. For a subset A of T^*X, we denote by A^a its image under the map a. We also denote by $\mathbf{h} \colon T^*T^*X \xrightarrow{\sim} TT^*X$ the

Hamiltonian isomorphism given in local coordinates by $\mathbf{h}(dx_i) = -\partial/\partial\xi_i$ and $\mathbf{h}(d\xi_i) = \partial/\partial x_i$.

2.1.2 Microsupports of sheaves ([KS90, §5.1, §5.4, §6.1])

We denote by $\mathbf{k}_X$ the constant sheaf with stalk $\mathbf{k}$ and by $\mathrm{Mod}(\mathbf{k}_X)$ the abelian category of sheaves of $\mathbf{k}$-vector spaces on X. Moreover we denote by $\mathbf{D}^{\mathrm{b}}(X) = \mathbf{D}^{\mathrm{b}}(\mathrm{Mod}(\mathbf{k}_X))$ the bounded derived category of $\mathrm{Mod}(\mathbf{k}_X)$. One can define Grothendieck's six operations between derived categories of sheaves $R\mathcal{H}om, \otimes, Rf_*, f^{-1}, Rf_!, f^!$ for a morphism of manifolds $f\colon X \to Y$. Since we work over the field $\mathbf{k}$, we simply write $\otimes$ instead of $\overset{L}{\otimes}$. Moreover, for $F \in \mathbf{D}^{\mathrm{b}}(X)$ and $G \in \mathbf{D}^{\mathrm{b}}(Y)$, we define their external tensor product $F \boxtimes G \in \mathbf{D}^{\mathrm{b}}(X \times Y)$ by $F \boxtimes G := q_X^{-1}F \otimes q_Y^{-1}G$, where $q_X \colon X \times Y \to X$ and $q_Y \colon X \times Y \to Y$ are the projections. For a locally closed subset Z of X, we denote by $\mathbf{k}_Z$ the zero-extension of the constant sheaf with stalk $\mathbf{k}$ on Z to X, extended by 0 on $X \setminus Z$. Moreover, for a locally closed subset Z of X and $F \in \mathbf{D}^{\mathrm{b}}(X)$, we define $F_Z, R\Gamma_Z(F) \in \mathbf{D}^{\mathrm{b}}(X)$ by

$$F_Z := F \otimes \mathbf{k}_Z, \quad R\Gamma_Z(F) := R\mathcal{H}om(\mathbf{k}_Z, F). \tag{2.1.2}$$

One denotes by $\omega_X \in \mathbf{D}^{\mathrm{b}}(X)$ the dualizing complex on X, that is, $\omega_X := a_X^! \mathbf{k}$, where $a_X \colon X \to \mathrm{pt}$ is the natural morphism. Note that ω_X is isomorphic to $\mathrm{or}_X[\dim X]$, where or_X is the orientation sheaf on X. More generally, for a morphism of manifolds $f\colon X \to Y$, we denote by $\omega_f = \omega_{X/Y} := f^! \mathbf{k}_Y \simeq \omega_X \otimes f^{-1}\omega_Y^{\otimes -1}$ the relative dualizing complex. For $F \in \mathbf{D}^{\mathrm{b}}(X)$, we define the Verdier dual of F by $\mathbb{D}_X F := R\mathcal{H}om(F, \omega_X)$.

Let us recall the definition of the *microsupport* $\mathrm{SS}(F)$ of an object $F \in \mathbf{D}^{\mathrm{b}}(X)$.

Definition 2.1.1 ([KS90, Definition 5.1.2])**.** Let $F \in \mathbf{D}^{\mathrm{b}}(X)$ and $p \in T^*X$. One says that $p \notin \mathrm{SS}(F)$ if there is a neighborhood U of p in T^*X such that for any $x_0 \in X$ and any C^∞-function φ on X (defined on a neighborhood of x_0) satisfying $d\varphi(x_0) \in U$, one has $R\Gamma_{\{\varphi \geq \varphi(x_0)\}}(F)_{x_0} \simeq 0$.

One can check the following properties:

(i) The microsupport of an object in $\mathbf{D}^{\mathrm{b}}(X)$ is a conic (i.e., invariant under the action of $\mathbb{R}_{>0}$ on T^*X) closed subset of T^*X.

(ii) For an object $F \in \mathbf{D}^{\mathrm{b}}(X)$, one has $\mathrm{SS}(F) \cap T^*_X X = \pi(\mathrm{SS}(F)) = \mathrm{Supp}(F)$.

(iii) The microsupports satisfy the triangle inequality: if $F_1 \longrightarrow F_2 \longrightarrow F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathbf{D}^{\mathrm{b}}(X)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for $j \neq k$.

We also use the notation $\mathring{\mathrm{SS}}(F) := \mathrm{SS}(F) \cap \mathring{T}^*X = \mathrm{SS}(F) \setminus T^*_X X$.

Example 2.1.2. (i) If F is a locally constant sheaf on X, then $\mathrm{SS}(F) \subset T^*_X X$. Conversely, if $\mathrm{SS}(F) \subset T^*_X X$ then the cohomology sheaves $H^k(F)$ are locally constant for all $k \in \mathbb{Z}$.

(ii) Let M be a closed submanifold of X. Then $\mathrm{SS}(\mathbf{k}_M) = T^*_M X \subset T^*X$.

(iii) Let $\varphi\colon X \to \mathbb{R}$ be a C^∞-function and assume that $d\varphi(x) \neq 0$ for any $x \in \varphi^{-1}(0)$. Set $U := \{x \in X \mid \varphi(x) > 0\}$ and $Z := \{x \in X \mid \varphi(x) \geq 0\}$. Then

$$\begin{aligned} \mathrm{SS}(\mathbf{k}_U) &= T^*_X X|_U \cup \{(x; \lambda d\varphi(x)) \mid \varphi(x) = 0, \lambda \leq 0\}, \\ \mathrm{SS}(\mathbf{k}_Z) &= T^*_X X|_Z \cup \{(x; \lambda d\varphi(x)) \mid \varphi(x) = 0, \lambda \geq 0\}. \end{aligned} \tag{2.1.3}$$

The following proposition is called (a particular case of) the microlocal Morse lemma. See [KS90, Proposition 5.4.17 and Corollary 5.4.19] for more details. The classical theory corresponds to the case F is the constant sheaf $\mathbf{k}_X$.

Proposition 2.1.3. *Let $F \in \mathbf{D}^{\mathrm{b}}(X)$ and $\varphi \colon X \to \mathbb{R}$ be a C^∞-function. Let moreover $a, b \in \mathbb{R}$ with $a < b$ or $a \in \mathbb{R}, b = +\infty$. Assume*

(1) *φ is proper on* $\operatorname{Supp}(F)$*,*

(2) *$d\varphi(x) \notin \operatorname{SS}(F)$ for any $x \in \varphi^{-1}([a, b))$.*

Then the canonical morphism

$$R\Gamma(\varphi^{-1}((-\infty, b)); F) \longrightarrow R\Gamma(\varphi^{-1}((-\infty, a)); F) \tag{2.1.4}$$

is an isomorphism.

By using microsupports, we can microlocalize the category $\mathbf{D}^{\mathrm{b}}(X)$. Let $A \subset T^*X$ be a subset and set $\Omega = T^*X \setminus A$. We denote by $\mathbf{D}^{\mathrm{b}}_A(X)$ the subcategory of $\mathbf{D}^{\mathrm{b}}(X)$ consisting of sheaves whose microsupports are contained in A. By the triangle inequality, the subcategory $\mathbf{D}^{\mathrm{b}}_A(X)$ is a triangulated subcategory. We define $\mathbf{D}^{\mathrm{b}}(X; \Omega)$ as the localization of $\mathbf{D}^{\mathrm{b}}(X)$ by $\mathbf{D}^{\mathrm{b}}_A(X)$: $\mathbf{D}^{\mathrm{b}}(X; \Omega) := \mathbf{D}^{\mathrm{b}}(X)/\mathbf{D}^{\mathrm{b}}_A(X)$. A morphism $u \colon F \to G$ in $\mathbf{D}^{\mathrm{b}}(X)$ becomes an isomorphism in $\mathbf{D}^{\mathrm{b}}(X; \Omega)$ if u is embedded in a distinguished triangle $F \xrightarrow{u} G \longrightarrow H \xrightarrow{+1}$ with $\operatorname{SS}(H) \cap \Omega = \emptyset$. For a closed subset B of Ω, $\mathbf{D}^{\mathrm{b}}_B(X; \Omega)$ denotes the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(X; \Omega)$ consisting of F with $\operatorname{SS}(F) \cap \Omega \subset B$. In the case $\Omega = \{p\}$ with $p \in T^*X$, we simply write $\mathbf{D}^{\mathrm{b}}(X; p)$ instead of $\mathbf{D}^{\mathrm{b}}(X; \{p\})$. Note that our notation is the same as in [KS90] and slightly differs from that of [Gui12, Gui16a].

2.1.3 Functorial operations ([KS90, §5.4])

We consider bounds for the microsupports of proper direct images, non-characteristic inverse images, and $R\mathcal{H}om$.

Definition 2.1.4 ([KS90, Definition 5.4.12])**.** Let $f \colon X \to Y$ be a morphism of manifolds and A be a closed conic subset of T^*Y. The morphism f is said to be *non-characteristic* for A if

$$f_\pi^{-1}(A) \cap f_d^{-1}(T^*_X X) \subset X \times_Y T^*_Y Y. \tag{2.1.5}$$

See (2.1.1) for the notation f_π and f_d. In particular, any submersion from X to Y is non-characteristic for any closed conic subset of T^*Y. Note that submersions are called smooth morphisms in [KS90]. One can show that if $f \colon X \to Y$ is non-characteristic for a closed conic subset A of T^*Y, then $f_d f_\pi^{-1}(A)$ is a closed conic subset of T^*X.

Theorem 2.1.5 ([KS90, Proposition 5.4.4 and Proposition 5.4.13])**.** *Let $f \colon X \to Y$ be a morphism of manifolds, $F \in \mathbf{D}^{\mathrm{b}}(X)$, and $G \in \mathbf{D}^{\mathrm{b}}(Y)$.*

(i) *Assume that f is proper on* $\operatorname{Supp}(F)$*. Then* $\operatorname{SS}(Rf_*F) \subset f_\pi f_d^{-1}(\operatorname{SS}(F))$.

(ii) *Assume that f is non-characteristic for* $\operatorname{SS}(G)$*. Then the canonical morphism $f^{-1}G \otimes \omega_f \to f^!G$ is an isomorphism and* $\operatorname{SS}(f^{-1}G) \cup \operatorname{SS}(f^!G) \subset f_d f_\pi^{-1}(\operatorname{SS}(G))$.

Proposition 2.1.6 ([KS90, Proposition 5.4.2])**.** *For $i = 1, 2$, let X_i be a manifold and denote by q_i the projection $X_1 \times X_2 \to X_i$. Let moreover $F_i \in \mathbf{D}^{\mathrm{b}}(X_i)$ for $i = 1, 2$. Then*

$$\operatorname{SS}(R\mathcal{H}om(q_2^{-1}F_2, q_1^{-1}F_1)) \subset \operatorname{SS}(F_1) \times \operatorname{SS}(F_2)^a. \tag{2.1.6}$$

For closed conic subsets A and B of T^*X, let us denote by $A+B$ the fiberwise sum of A and B, that is,

$$A+B := \{(x;a+b) \mid x \in \pi(A)\cap\pi(B), a \in A\cap\pi^{-1}(x), b \in B\cap\pi^{-1}(x)\} \subset T^*X. \tag{2.1.7}$$

Proposition 2.1.7 ([KS90, Proposition 5.4.14]). *Let $F, G \in \mathbf{D}^{\mathrm{b}}(X)$.*

(i) *If $\mathrm{SS}(F)\cap\mathrm{SS}(G)^a \subset T^*_X X$, then*

$$\mathrm{SS}(F\otimes G) \subset \mathrm{SS}(F)+\mathrm{SS}(G). \tag{2.1.8}$$

(ii) *If $\mathrm{SS}(F)\cap\mathrm{SS}(G) \subset T^*_X X$, then*

$$\mathrm{SS}(R\mathcal{H}om(F,G)) \subset \mathrm{SS}(F)^a+\mathrm{SS}(G). \tag{2.1.9}$$

Moreover if F is cohomologically constructible (see [KS90, §3.4] for the definition), the natural morphism $R\mathcal{H}om(F,\mathbf{k}_X)\otimes G \to R\mathcal{H}om(F,G)$ is an isomorphism.

2.1.4 Non-proper direct images ([Tam08, GS14])

We consider estimates of the microsupports of non-proper direct images in special cases. Let V_1 and V_2 be finite-dimensional real vector spaces and consider a constant linear map $u\colon X\times V_1 \to X\times V_2$. That is, we assume that there exists a linear map $u_V\colon V_1\to V_2$ satisfying $u = \mathrm{id}_X \times u_V$. The map u induces the maps

$$\begin{tikzcd}[column sep=tiny] & T^*X\times V_1\times V_2^* \arrow[dl, "u_d"'] \arrow[dr, "u_\pi"] & \\ T^*X\times V_1\times V_1^* \arrow[dr, "v_\pi"'] & & T^*X\times V_2\times V_2^* \arrow[dl, "v_d"] \\ & T^*X\times V_2\times V_1^*. & \end{tikzcd} \tag{2.1.10}$$

Note that for a subset A of $T^*(X\times V_1)$, we have $u_\pi(u_d^{-1}(A)) = v_d^{-1}(v_\pi(A))$.

Definition 2.1.8. Let $u\colon X\times V_1\to X\times V_2$ be a constant linear map and $A\subset T^*(X\times V_1)$ be a closed subset. One sets

$$u_\sharp(A) := v_d^{-1}\left(\overline{v_\pi(A)}\right). \tag{2.1.11}$$

Proposition 2.1.9 ([Tam08, Lemma 3.3] and [GS14, Theorem 1.16]). *Let $u\colon X\times V_1 \to X\times V_2$ be a constant linear map and $F\in\mathbf{D}^{\mathrm{b}}(X\times V_1)$. Then*

$$\mathrm{SS}(Ru_*F)\cup\mathrm{SS}(Ru_!F) \subset u_\sharp(\mathrm{SS}(F)). \tag{2.1.12}$$

2.1.5 Morse-Bott inequality for sheaves ([ST92])

In this subsection, we give the Morse-Bott inequality for sheaves, which is a slight generalization of the Morse inequality for sheaves by Kashiwara-Schapira [KS90, Proposition 5.4.20] and was proved by Schapira-Tose [ST92]. For a bounded complex W of $\mathbf{k}$-vector spaces with finite-dimensional cohomology, we set

$$b_j(W) := \dim H^j(W), \quad b_l^*(W) := (-1)^l\sum_{j\le l}(-1)^j b_j(W). \tag{2.1.13}$$

Let $F \in \mathbf{D}^{\mathrm{b}}(X)$ and $\varphi\colon X \to \mathbb{R}$ be a C^∞-function. We set

$$\Gamma_{d\varphi} := \{(x; d\varphi(x)) \mid x \in X\} \subset T^*X. \tag{2.1.14}$$

We consider the following assumptions:

(1) $\mathrm{Supp}(F) \cap \varphi^{-1}((-\infty, t])$ is compact for any $t \in \mathbb{R}$,

(2) the set $\varphi(\pi(\mathrm{SS}(F) \cap \Gamma_{d\varphi}))$ is finite, say $\{c_1, \dots, c_N\}$ with $c_1 < \cdots < c_N$,

(3) the object

$$W_i := R\Gamma\left(\varphi^{-1}(c_i); R\Gamma_{\{\varphi \ge c_i\}}(F)|_{\varphi^{-1}(c_i)}\right) \tag{2.1.15}$$

has finite-dimensional cohomology for any $i = 1, \dots, N$.

Theorem 2.1.10 ([ST92, Theorem 1.1], see also [KS90, Proposition 5.4.20])**.** *Assume that* (1)–(3) *are satisfied. Then*

(i) *$R\Gamma(X; F)$ has finite-dimensional cohomology,*

(ii) *one has*

$$b_l^*(R\Gamma(X; F)) \le \sum_{i=1}^{N} b_l^*(W_i) \tag{2.1.16}$$

for any $l \in \mathbb{Z}$.

The proof is the same as [KS90, Proposition 5.4.20], since

$$R\Gamma_{[t,+\infty)}(R\varphi_* F)_t \simeq R\Gamma\left(\varphi^{-1}(t); R\Gamma_{\{\varphi \ge t\}}(F)|_{\varphi^{-1}(t)}\right). \tag{2.1.17}$$

Note also that (2.1.16) implies

$$b_k(R\Gamma(X; F)) \le \sum_{i=1}^{N} b_k(W_i) \tag{2.1.18}$$

for any $k \in \mathbb{Z}$.

2.1.6 Kernels ([KS90, §3.6])

For $i = 1, 2, 3$, let X_i be a manifold. We write $X_{ij} := X_i \times X_j$ and $X_{123} := X_1 \times X_2 \times X_3$ for short. We use the same symbol q_i for the projections $X_{ij} \to X_i$ and $X_{123} \to X_i$. We also denote by q_{ij} the projection $X_{123} \to X_{ij}$. Similarly, we denote by p_{ij} the projection $T^*X_{123} \to T^*X_{ij}$. One denotes by p_{12^a} the composite of p_{12} and the antipodal map on T^*X_2.

Let $A \subset T^*X_{12}$ and $B \subset T^*X_{23}$. We set

$$A \circ B := p_{13}(p_{12^a}^{-1} A \cap p_{23}^{-1} B) \subset T^*X_{13}. \tag{2.1.19}$$

We define the operation of composition of kernels as follows:

$$\begin{aligned} \underset{X_2}{\circ} \colon \mathbf{D}^{\mathrm{b}}(X_{12}) \times \mathbf{D}^{\mathrm{b}}(X_{23}) &\to \mathbf{D}^{\mathrm{b}}(X_{13}) \\ (K_{12}, K_{23}) &\mapsto K_{12} \underset{X_2}{\circ} K_{23} := Rq_{13!}\,(q_{12}^{-1} K_{12} \otimes q_{23}^{-1} K_{23}). \end{aligned} \tag{2.1.20}$$

If there is no risk of confusion, we simply write $\circ$ instead of $\underset{X_2}{\circ}$. By Theorem 2.1.5 and Proposition 2.1.7, we have the following:

Proposition 2.1.11. *Let $K_{ij} \in \mathbf{D}^{\mathrm{b}}(X_{ij})$ and set $\Lambda_{ij} := \mathrm{SS}(K_{ij}) \subset T^*X_{ij}$ $(ij = 12, 23)$. Assume*

(1) *q_{13} is proper on $q_{12}^{-1} \operatorname{Supp}(K_{12}) \cap q_{23}^{-1} \operatorname{Supp}(K_{23})$,*

(2) *$p_{12^a}^{-1}\Lambda_{12} \cap p_{23}^{-1}\Lambda_{23} \cap (T^*_{X_1}X_1 \times T^*X_2 \times T^*_{X_3}X_3) \subset T^*_{X_{123}}X_{123}$.*

Then

$$\mathrm{SS}(K_{12} \underset{X_2}{\circ} K_{23}) \subset \Lambda_{12} \circ \Lambda_{23}. \tag{2.1.21}$$

2.1.7 Microlocalization and μhom functors ([KS90, §4.3, §4.4])

Let M be a closed submanifold of X. The microlocalization functor along M is a functor $\mu_M \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(T^*_M X)$ (see [KS90, §4.3] for more details). Microlocalization is related to local cohomology as follows. Let $p \in \mathring{T}^*X$ and $\varphi \colon X \to \mathbb{R}$ be a C^∞-function such that $\varphi(\pi(p)) = 0$ and $d\varphi(\pi(p)) = p$. Then, for $F \in \mathbf{D}^{\mathrm{b}}(X)$, we have

$$R\Gamma_{\{\varphi \geq 0\}}(F)_{\pi(p)} \simeq \mu_{\varphi^{-1}(0)}(F)_p. \tag{2.1.22}$$

Under suitable assumptions, the functoriality of microlocalization with respect to proper direct images and non-characteristic inverse images holds as follows:

Proposition 2.1.12 ([KS90, Proposition 4.3.4 and Corollary 6.7.3])**.** *Let $f \colon X \to Y$ be a morphism of manifolds. Let moreover N be a closed submanifold of Y and assume that $M = f^{-1}(N)$ is also a closed submanifold of X. Denote by $f_{Md} \colon M \times_N T^*_N Y \to T^*_M X$ the morphism induced by f_d and by $f_{M\pi} \colon M \times_N T^*_N Y \to T^*_N Y$ the morphism induced by f_π (see (2.1.1)).*

(i) *Let $F \in \mathbf{D}^{\mathrm{b}}(X)$. Assume that f is proper on $\operatorname{Supp}(F)$ and $f_{Md} \colon M \times_N T^*_N Y \to T^*_M X$ is surjective. Then*

$$Rf_{M\pi !} f_{Md}^{-1} \mu_M(F) \xrightarrow{\sim} \mu_N(Rf_* F). \tag{2.1.23}$$

(ii) *Let $G \in \mathbf{D}^{\mathrm{b}}(Y)$. Assume that f is non-characteristic for $\mathrm{SS}(F)$ and $f|_M \colon M \to N$ is a submersion. Then*

$$\mu_M(f^! G) \xrightarrow{\sim} Rf_{Md*} f^!_{M\pi} \mu_N(G). \tag{2.1.24}$$

We also recall the functor μhom. Let $q_1, q_2 \colon X \times X \to X$ be the projections. We identify $T^*_{\Delta_X}(X \times X)$ with T^*X through the first projection $(x, x; \xi, -\xi) \mapsto (x; \xi)$.

Definition 2.1.13 ([KS90, Definition 4.4.1])**.** For $F, G \in \mathbf{D}^{\mathrm{b}}(X)$, one defines

$$\mu hom(F, G) := \mu_{\Delta_X} R\mathcal{H}om(q_2^{-1}F, q_1^! G) \in \mathbf{D}^{\mathrm{b}}(T^*X). \tag{2.1.25}$$

Proposition 2.1.14 ([KS90, Proposition 4.4.2 and Proposition 4.4.3])**.** *Let $F, G \in \mathbf{D}^{\mathrm{b}}(X)$.*

(i) *$R\pi_* \mu hom(F, G) \simeq R\mathcal{H}om(F, G)$.*

(ii) *If F is cohomologically constructible (see [KS90, §3.4] for the definition), then $R\pi_! \mu hom(F, G) \simeq R\mathcal{H}om(F, \mathbf{k}_X) \otimes G$.*

(iii) *For a closed submanifold M of X, $\mu hom(\mathbf{k}_M, F) \simeq i_* \mu_M(F)$, where $i \colon T^*_M X \to T^*M$ is the embedding.*

Proposition 2.1.15 ([KS90, Corollary 5.4.10 and Corollary 6.4.3])**.** *Let $F, G \in \mathbf{D}^{\mathrm{b}}(X)$. Then*

$$\begin{aligned} \mathrm{Supp}(\mu hom(F,G)) &\subset \mathrm{SS}(F) \cap \mathrm{SS}(G), \\ \mathrm{SS}(\mu hom(F,G)) &\subset -\mathbf{h}^{-1}(C(\mathrm{SS}(G), \mathrm{SS}(F))), \end{aligned} \tag{2.1.26}$$

where $C(S_1, S_2)$ is the normal cone and $\mathbf{h}\colon T^*T^*X \xrightarrow{\sim} TT^*X$ is the Hamiltonian isomorphism (see Subsection 2.1.1).

Proposition 2.1.16. *Let $\varphi\colon X \to \mathbb{R}$ be a C^∞-function and assume that $d\varphi(x) \neq 0$ for any $x \in \varphi^{-1}(0)$. Set $M := \varphi^{-1}(0)$ and define an open subset $T^{*+}_M X$ of $T^*_M X$ by*

$$T^{*+}_M X := \{(x; \lambda d\varphi(x)) \mid x \in M, \lambda > 0\}. \tag{2.1.27}$$

Denote moreover by $\pi_{M+}\colon T^{*+}_M X \to M$ the projection. Let $F \in \mathbf{D}^{\mathrm{b}}(X)$. Then

$$R\Gamma_{\{\varphi \geq 0\}}(F)|_M \simeq R\pi_{M+*}\mu hom(\mathbf{k}_{\{\varphi\geq 0\}}, F)|_{T^{*+}_M X} \simeq R\pi_{M+*}\mu_M(F)|_{T^{*+}_M X}. \tag{2.1.28}$$

In particular,

$$R\Gamma(M; R\Gamma_{\{\varphi\geq 0\}}(F)|_M) \simeq R\Gamma\left(T^{*+}_M X; \mu_M(F)|_{T^{*+}_M X}\right). \tag{2.1.29}$$

Proof. Consider the distinguished triangle

$$R\pi_! \mu hom(\mathbf{k}_{\{\varphi\geq 0\}}, F) \to R\pi_*\mu hom(\mathbf{k}_{\{\varphi\geq 0\}}, F) \to R\mathring{\pi}_*\mu hom(\mathbf{k}_{\{\varphi\geq 0\}}, F)|_{\mathring{T}^*X} \xrightarrow{+1}. \tag{2.1.30}$$

By Proposition 2.1.15, $\mathrm{Supp}(\mu hom(\mathbf{k}_{\{\varphi\geq 0\}}, F)|_{\mathring{T}^*X}) \subset T^{*+}_M X$. Hence we have

$$R\mathring{\pi}_*\mu hom(\mathbf{k}_{\{\varphi\geq 0\}}, F)|_{\mathring{T}^*X} \simeq \left(R\pi_{M+*}\mu hom(\mathbf{k}_{\{\varphi\geq 0\}}, F)|_{T^{*+}_M X}\right)_M. \tag{2.1.31}$$

On the other hand, since $\mathbf{k}_{\{\varphi\geq 0\}}$ is cohomologically constructible, by Proposition 2.1.14 (i) and (ii), we get

$$\begin{aligned} R\pi_!\mu hom(\mathbf{k}_{\{\varphi\geq 0\}}, F) &\simeq R\mathcal{H}om(\mathbf{k}_{\{\varphi\geq 0\}}, \mathbf{k}_X) \otimes F \simeq R\Gamma_{\{\varphi\geq 0\}}(\mathbf{k}_X) \otimes F, \\ R\pi_*\mu hom(\mathbf{k}_{\{\varphi\geq 0\}}, F) &\simeq R\mathcal{H}om(\mathbf{k}_{\{\varphi\geq 0\}}, F) \simeq R\Gamma_{\{\varphi\geq 0\}}(F). \end{aligned} \tag{2.1.32}$$

Since $R\Gamma_{\{\varphi\geq 0\}}(\mathbf{k}_X)|_M \simeq 0$, restricting the distinguished triangle (2.1.30) to M, we obtain the first isomorphism in (2.1.28). Moreover since $\mathrm{SS}(\mathbf{k}_{\{\varphi>0\}}) \cap T^{*+}_M X = \emptyset$, by Proposition 2.1.15, we have

$$\mu hom(\mathbf{k}_{\{\varphi\geq 0\}}, F)|_{T^{*+}_M X} \xrightarrow{\sim} \mu hom(\mathbf{k}_{\{\varphi=0\}}, F)|_{T^{*+}_M X}. \tag{2.1.33}$$

Thus the second isomorphism in (2.1.28) follows from Proposition 2.1.14 (iii). □

2.1.8 Simple sheaves and quantized contact transformations ([KS90, §7.5])

Let $\Lambda \subset \mathring{T}^*X$ be a locally closed conic Lagrangian submanifold and $p \in \Lambda$. Simple sheaves along Λ at p are defined in [KS90, Definition 7.5.4]. In this subsection, we recall them.

Let $\varphi\colon X \to \mathbb{R}$ be a C^∞-function such that $\varphi(\pi(p)) = 0$ and $\Gamma_{d\varphi}$ intersects Λ transversally at p. For $p \in \Gamma_{d\varphi} \cap \Lambda$, we define the following Lagrangian subspaces in T_pT^*X:

$$\lambda_\infty(p) := T_p(T^*_{\pi(p)}X), \quad \lambda_\Lambda(p) := T_p\Lambda, \quad \lambda_\varphi(p) := T_p\Gamma_{d\varphi}. \tag{2.1.34}$$

Here, our notation $\lambda_\infty(p)$ is different from that of [KS90], where the authors write $\lambda_0(p)$ for $T_p(T^*_{\pi(p)}X)$. In this thesis, we do *not* use the symbol $\lambda_0(p)$. We briefly recall the definition of the inertia index of a triple of Lagrangian subspaces (see [KS90, §A.3]). Let (E, σ) be a symplectic vector space and $\lambda_1, \lambda_2, \lambda_3$ be three Lagrangian subspaces of E. We define a quadratic form q on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ by $q(v_1, v_2, v_3) = \sigma(v_1, v_2) + \sigma(v_2, v_3) + \sigma(v_3, v_1)$. Then the *inertia index* $\tau_E(\lambda_\infty, \lambda_1, \lambda_3)$ of the triple is defined as the signature of q. Using the inertia index and the notation (2.1.34), one sets

$$\tau_\varphi = \tau_{p,\varphi} := \tau_{T_pT^*X}(\lambda_\infty(p), \lambda_\Lambda(p), \lambda_\varphi(p)). \tag{2.1.35}$$

Proposition 2.1.17 ([KS90, Proposition 7.5.3])**.** *For $i = 1, 2$, let $\varphi_i \colon X \to \mathbb{R}$ be a C^∞-function such that $\varphi_i(\pi(p)) = 0$ and $\Gamma_{d\varphi_i}$ intersects Λ transversally at p. Let $F \in \mathbf{D}^{\mathrm{b}}(X)$ and assume that $\mathrm{SS}(F) \subset \Lambda$ in a neighborhood of p. Then*

$$R\Gamma_{\{\varphi_1 \geq 0\}}(F)_{\pi(p)} \simeq R\Gamma_{\{\varphi_2 \geq 0\}}(F)_{\pi(p)} \left[\tfrac{1}{2}(\tau_{\varphi_2} - \tau_{\varphi_1})\right]. \tag{2.1.36}$$

Definition 2.1.18 ([KS90, Definition 7.5.4])**.** In the situation of Proposition 2.1.17, F is said to have microlocal type $L \in \mathbf{D}^{\mathrm{b}}(\mathrm{Mod}(\mathbf{k}))$ with shift $d \in \frac{1}{2}\mathbb{Z}$ at p if

$$R\Gamma_{\{\varphi \geq 0\}}(F)_{\pi(p)} \simeq L\left[d - \tfrac{1}{2}\dim X - \tfrac{1}{2}\tau_\varphi\right] \tag{2.1.37}$$

for some (hence for any) C^∞-function φ such that $\varphi(\pi(p)) = 0$ and $\Gamma_{d\varphi}$ intersects Λ transversally at p. If moreover $L \simeq \mathbf{k}$, F is said to be *simple* along Λ at p. If F is simple at all points of Λ, one says that F is simple along Λ.

One can prove that if $F \in \mathbf{D}^{\mathrm{b}}(X)$ is simple along Λ, then $\mu hom(F, F)|_\Lambda \simeq \mathbf{k}_\Lambda$. When Λ is a conormal bundle to a closed submanifold M of X in a neighborhood of p, that is, $\pi|_\Lambda \colon \Lambda \to X$ has constant rank, then $F \in \mathbf{D}^{\mathrm{b}}(X)$ is simple along Λ at p if $F \simeq \mathbf{k}_M[d]$ in $\mathbf{D}^{\mathrm{b}}(X; p)$ for some $d \in \mathbb{Z}$.

Example 2.1.19. Let $X = \mathbb{R}^{n+1}$ and consider the hyperplane $M = \mathbb{R}^n \times \{0\}$. Then $\mathbf{k}_M$ is simple with shift $1/2$ along $T^*_M X$.

We also recall the notion of quantized contact transformations. Let $\chi \colon T^*X \supset \Omega_1 \xrightarrow{\sim} \Omega_2 \subset T^*X$ be a contact transformation. A *quantized contact transformation* associated with χ is a kernel $K \in \mathbf{D}^{\mathrm{b}}(X \times X)$ which is simple along $(\mathrm{id}_X \times a)^{-1}\mathrm{Graph}(\chi)$ in $\Omega_2 \times \Omega_1^a$ and satisfies some properties (see [KS90, §7.2] for details). A quantized contact transformation K induces an equivalence of categories

$$K \circ (*) \colon \mathbf{D}^{\mathrm{b}}(X; \Omega_1) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(X; \Omega_2). \tag{2.1.38}$$

Proposition 2.1.20 ([KS90, Theorem 7.2.1])**.** *Let $K \in \mathbf{D}^{\mathrm{b}}(X \times X)$ be a quantized contact transformation associated with a contact transformation $\chi \colon T^*X \supset \Omega_1 \xrightarrow{\sim} \Omega_2 \subset T^*X$. Let moreover $F, G \in \mathbf{D}^{\mathrm{b}}(X; \Omega_1)$. Then*

$$\mu hom(K \circ F, K \circ G)|_{\Omega_2} \simeq \chi_*(\mu hom(F, G)|_{\Omega_1}). \tag{2.1.39}$$

The behavior of the shift of a simple sheaf under a quantized contact transformation is described by the inertia index.

Proposition 2.1.21 ([KS90, Proposition 7.5.6 and Theorem 7.5.11])**.** *Let $F \in \mathbf{D}^{\mathrm{b}}(X)$ and assume that F is simple with shift d along Λ at p. Let $\chi \colon T^*X \supset \Omega_1 \xrightarrow{\sim} \Omega_2 \subset T^*X$ be a contact transformation defined in a neighborhood of p and $K \in \mathbf{D}^{\mathrm{b}}(X \times X)$ be a quantized contact transformation associated with χ. Assume that K is simple with shift d' along $(\mathrm{id}_X \times a)^{-1}\mathrm{Graph}(\chi)$ at $(\chi(p), p^a)$. Then $K \circ F$ is simple with shift $d + d' - \delta$ along $\chi(\Lambda)$ at $\chi(p)$, where*

$$\delta := \frac{1}{2}\dim X + \frac{1}{2}\tau(\lambda_\infty(p), \lambda_\Lambda(p), \chi^{-1}(\lambda_\infty(\chi(p)))). \tag{2.1.40}$$

2.2 Sheaf quantization and Tamarkin's non-displaceability theorem

In what follows, until the end of this thesis, let M be a non-empty connected manifold without boundary.

In this section, we review Tamarkin's approach to non-displaceability problems in symplectic geometry based on the microlocal sheaf theory. We also review sheaf quantization of Hamiltonian isotopies and compact exact Lagrangian submanifolds in cotangent bundles.

2.2.1 Sheaf quantization of Hamiltonian isotopies ([GKS12])

Guillermou-Kashiwara-Schapira [GKS12] constructed sheaf quantizations of Hamiltonian isotopies. Since the microsupports of sheaves are conic subsets of cotangent bundles, the microlocal sheaf theory is related to the exact (homogeneous) symplectic structures rather than the symplectic structures of cotangent bundles. For the sheaf-theoretic study of non-homogeneous Hamiltonian isotopies and non-conic Lagrangian submanifolds of cotangent bundles, an important trick is to add one more variable to the base manifolds and "conify" the Lagrangians, which is an idea of Tamarkin.

Denote by $(x;\xi)$ a local homogeneous symplectic coordinate system on T^*M and by $(t;\tau)$ the homogeneous symplectic coordinate system on $T^*\mathbb{R}$. We set $\Omega_+ := \{\tau > 0\} = \{(x,t;\xi,\tau) \mid \tau > 0\} \subset T^*(M \times \mathbb{R})$ and define the map

$$\begin{array}{rcl} \rho\colon \Omega_+ & \longrightarrow & T^*M \\ \cup\!\!\!\!\;\text{\rotatebox{90}{}} & & \\ (x,t;\xi,\tau) & \longmapsto & (x;\xi/\tau). \end{array} \tag{2.2.1}$$

Let I be an open interval in $\mathbb{R}$ containing 0. Let moreover $H\colon T^*M \times I \to \mathbb{R}$ be a compactly supported Hamiltonian function and denote by $\phi^H = (\phi^H_s)_{s\in I}\colon T^*M \times I \to T^*M$ the Hamiltonian isotopy generated by H. Note that the Hamiltonian vector field is defined by $d\alpha_{T^*M}(X_{H_s},*) = -dH_s$ and ϕ^H is the identity for $s=0$. One can conify ϕ^H and construct a homogeneous lift $\widehat{\phi}$ of ϕ^H as follows. Define $\widehat{H}\colon T^*M \times \mathring{T}^*\mathbb{R} \times I \to \mathbb{R}$ by $\widehat{H}_s(x,t;\xi,\tau) := \tau \cdot H_s(x;\xi/\tau)$. Note that $\widehat{H}$ is homogeneous of degree 1, that is, $\widehat{H}_s(x,t;c\xi,c\tau) = c \cdot \widehat{H}_s(x,t;\xi,\tau)$ for any $c \in \mathbb{R}_{>0}$. The Hamiltonian isotopy $\widehat{\phi}\colon T^*M \times \mathring{T}^*\mathbb{R} \times I \to T^*M \times \mathring{T}^*\mathbb{R}$ generated by $\widehat{H}$ makes the following diagram commute:

$$\begin{tikzcd} \Omega_+ \times I \arrow[r, "\widehat{\phi}"] \arrow[d, "\rho\times\mathrm{id}"'] & \Omega_+ \arrow[d, "\rho"] \\ T^*M \times I \arrow[r, "\phi^H"'] & T^*M. \end{tikzcd} \tag{2.2.2}$$

Moreover there exists C^∞-function $u\colon T^*M \times I \to \mathbb{R}$ such that

$$\widehat{\phi}_s(x,t;\xi,\tau) = (x', t + u_s(x;\xi/\tau); \xi', \tau), \tag{2.2.3}$$

where $(x';\xi'/\tau) = \phi^H_s(x;\xi/\tau)$. By construction, $\widehat{\phi}$ is a homogeneous Hamiltonian isotopy: $\widehat{\phi}_s(x,t;c\xi,c\tau) = c \cdot \widehat{\phi}_s(x,t;\xi,\tau)$ for any $c \in \mathbb{R}_{>0}$. See [GKS12, Subsection A.3] for more details. We define a conic Lagrangian submanifold $\Lambda_{\widehat{\phi}} \subset T^*M \times \mathring{T}^*\mathbb{R} \times T^*M \times \mathring{T}^*\mathbb{R} \times T^*I$ by

$$\Lambda_{\widehat{\phi}} := \left\{ \left(\widehat{\phi}_s(x,t;\xi,\tau), (x,t;-\xi,-\tau), (s; -\widehat{H}_s \circ \widehat{\phi}_s(x,t;\xi,\tau))\right) \;\middle|\; \begin{array}{c} (x;\xi) \in T^*M, \\ (t;\tau) \in \mathring{T}^*\mathbb{R}, \\ s \in I \end{array} \right\}. \tag{2.2.4}$$

By construction, we have

$$\widehat{H_s} \circ \widehat{\phi}_s(x,t;\xi,\tau) = \tau \cdot (H_s \circ \phi_s^H(x;\xi/\tau)). \tag{2.2.5}$$

Note that

$$\begin{aligned}\Lambda_{\widehat{\phi}} \circ T_s^* I = \left\{ \left(\widehat{\phi}_s(x,t;\xi,\tau), (x,t;-\xi,-\tau) \right) \,\middle|\, (x,t;\xi,\tau) \in T^*M \times \mathring{T}^*\mathbb{R} \right\} \\ \subset T^*M \times \mathring{T}^*\mathbb{R} \times T^*M \times \mathring{T}^*\mathbb{R}\end{aligned} \tag{2.2.6}$$

for any $s \in I$ (see (2.1.19) for the definition of $A \circ B$).

Theorem 2.2.1 ([GKS12, Theorem 4.3])**.** *In the preceding situation, there exists a unique object $K \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times M \times \mathbb{R} \times I)$ satisfying the following conditions:*

(1) $\mathring{\mathrm{SS}}(K) \subset \Lambda_{\widehat{\phi}}$,

(2) $K|_{M\times\mathbb{R}\times M\times\mathbb{R}\times\{0\}} \simeq \mathbf{k}_{\Delta_{M\times\mathbb{R}}}$, *where $\Delta_{M\times\mathbb{R}}$ is the diagonal of $M \times \mathbb{R} \times M \times \mathbb{R}$.*

Moreover K is simple along $\Lambda_{\widehat{\phi}}$ and both projections $\mathrm{Supp}(K) \to M \times \mathbb{R} \times I$ are proper.

Remark 2.2.2. In [GKS12, Theorem 4.3], it was proved that $K|_{M\times\mathbb{R}\times M\times\mathbb{R}\times J}$ is a bounded object for any relatively compact interval J of I. Since we assume that H has compact support, we find that $K \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times M \times \mathbb{R} \times I)$.

The object K is called the *sheaf quantization* of $\widehat{\phi}$ or associated with ϕ^H. Set $K_s := K|_{M\times\mathbb{R}\times M\times\mathbb{R}\times\{s\}} \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times M \times \mathbb{R})$. Note that $\mathring{\mathrm{SS}}(K_s) \subset \Lambda_{\widehat{\phi}} \circ T_s^* I$ and K_s is a quantized contact transformation associated with $\widehat{\phi}_s \colon \Omega_+ \xrightarrow{\sim} \Omega_+$.

2.2.2 Tamarkin's separation and non-displaceability theorems ([Tam08, GS14])

Compact subsets A and B of T^*M are said to be *mutually non-displaceable* if $A \cap \phi_1^H(B) \neq \emptyset$ for any Hamiltonian isotopy $\phi^H = (\phi_s^H)_s \colon T^*M \times [0,1] \to T^*M$ generated by a compactly supported Hamiltonian function H. For simplicity, hereafter in this thesis, such an isotopy is called a Hamiltonian isotopy with compact support. Tamarkin [Tam08] (see also Guillermou-Schapira [GS14]) considered some categories consisting of sheaves on $M \times \mathbb{R}$ and deduced a new sheaf-theoretic criterion for non-displaceability using them.

We denote by $(x;\xi)$ a local homogeneous coordinate system on T^*M and by $(t;\tau)$ the homogeneous coordinate system on $T^*\mathbb{R}$ as before. We define the maps

$$\begin{aligned}\tilde{q}_1, \tilde{q}_2, s_{\mathbb{R}} \colon M \times \mathbb{R} \times \mathbb{R} \longrightarrow M \times \mathbb{R}, \\ \tilde{q}_1(x,t_1,t_2) = (x,t_1),\ \tilde{q}_2(x,t_1,t_2) = (x,t_2),\ s_{\mathbb{R}}(x,t_1,t_2) = (x,t_1+t_2).\end{aligned} \tag{2.2.7}$$

If there is no risk of confusion, we simply write s for $s_{\mathbb{R}}$. We also set

$$i \colon M \times \mathbb{R} \to M \times \mathbb{R},\ (x,t) \longmapsto (x,-t). \tag{2.2.8}$$

Definition 2.2.3 ([Tam08] and [GS14])**.** For $F, G \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R})$, one sets

$$F \star G := Rs_!(\tilde{q}_1^{-1} F \otimes \tilde{q}_2^{-1} G), \tag{2.2.9}$$

$$\mathcal{H}om^\star(F,G) := R\tilde{q}_{1*}\, R\mathcal{H}om(\tilde{q}_2^{-1} F, s^! G) \tag{2.2.10}$$

$$\simeq Rs_*\, R\mathcal{H}om(\tilde{q}_2^{-1} i^{-1} F, \tilde{q}_1^! G). \tag{2.2.11}$$

Note that the functor $\star$ is a left adjoint to $\mathcal{H}om^\star$.

The functor

$$\mathbf{k}_{M\times[0,+\infty)} \star (*) \colon \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R}) \longrightarrow \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R}) \tag{2.2.12}$$

defines a projector on the left orthogonal ${}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})$, where $\{\tau\le 0\}$ denotes the closed subset $\{(x,t;\xi,\tau) \mid \tau\le 0\}$ of $T^*(M\times\mathbb{R})$. Similarly, the functor

$$\mathcal{H}om^\star(\mathbf{k}_{M\times[0,+\infty)}, *) \colon \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R}) \longrightarrow \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R}) \tag{2.2.13}$$

defines a projector on the right orthogonal $\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})^\perp$. By using these projectors, Tamarkin proved that the localized category $\mathbf{D}^{\mathrm{b}}(M\times\mathbb{R};\{\tau>0\})$ is equivalent to both the left orthogonal ${}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})$ and the right orthogonal $\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})^\perp$:

$$\begin{aligned} P_l &:= \mathbf{k}_{M\times[0,+\infty)} \star (*) \colon \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R};\{\tau>0\}) \xrightarrow{\sim} {}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R}), \\ P_r &:= \mathcal{H}om^\star(\mathbf{k}_{M\times[0,+\infty)}, *) \colon \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R};\{\tau>0\}) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})^\perp. \end{aligned} \tag{2.2.14}$$

Note also the inclusion ${}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R}), \mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})^\perp \subset \mathbf{D}^{\mathrm{b}}_{\{\tau\ge 0\}}(M\times\mathbb{R})$. We set $\Omega_+ = \{\tau>0\} \subset T^*(M\times\mathbb{R})$ and $\rho\colon \Omega_+ \to T^*M, (x,t;\xi,\tau)\mapsto (x;\xi/\tau)$ as before.

Definition 2.2.4 ([Tam08])**.** One defines

$$\mathcal{D}(M) := \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R};\Omega_+) \simeq {}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R}) \simeq \mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})^\perp. \tag{2.2.15}$$

For a compact subset A of T^*M, one also defines a full subcategory $\mathcal{D}_A(M)$ of $\mathcal{D}(M)$ by

$$\mathcal{D}_A(M) := \mathbf{D}^{\mathrm{b}}_{\rho^{-1}(A)}(M\times\mathbb{R};\Omega_+). \tag{2.2.16}$$

For $F\in\mathcal{D}(M)$, we take the canonical representative $P_l(F)\in{}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})$ unless otherwise specified. For a compact subset A of T^*M and $F\in\mathcal{D}_A(M)$, the canonical representative $P_l(F)\in{}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})$ satisfies $\mathrm{SS}(P_l(F))\subset\overline{\rho^{-1}(A)}$. Note also that if $F\in{}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})$ then $\mathcal{H}om^\star(F,G)\in\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})^\perp$. Thus $\mathcal{H}om^\star$ induces an internal Hom functor $\mathcal{H}om^\star\colon\mathcal{D}(M)^{\mathrm{op}}\times\mathcal{D}(M)\to\mathcal{D}(M)$.

Remark 2.2.5. Let $f\colon M\to N$ be a morphism of manifolds and set $\widetilde{f} := f\times\mathrm{id}_{\mathbb{R}}\colon M\times\mathbb{R}\to N\times\mathbb{R}$. Then, for $F\in{}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})$, we have $R\widetilde{f}_!F\in{}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(N\times\mathbb{R})$. Similarly, for $G\in\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M\times\mathbb{R})^\perp$, we have $R\widetilde{f}_*G\in\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(N\times\mathbb{R})^\perp$. In other words, the morphism f induces functors $\mathcal{D}(M)\to\mathcal{D}(N)$.

Proposition 2.2.6 ([GS14, Lemma 3.18])**.** *Let $F,G\in\mathcal{D}(M)$. Then*

$$\mathrm{Hom}_{\mathcal{D}(M)}(F,G) \simeq H^0 R\Gamma_{M\times[0,+\infty)}(M\times\mathbb{R};\mathcal{H}om^\star(F,G)). \tag{2.2.17}$$

The following separation theorem was proved by Tamarkin [Tam08]. Using the theorem, we can prove the non-emptiness of the intersection of two compact subsets .

Theorem 2.2.7 ([Tam08, Theorem 3.2] and [GS14, Theorem 3.28])**.** *Let A and B be compact subsets of T^*M and assume that $A\cap B=\emptyset$. Denote by $q_{\mathbb{R}}\colon M\times\mathbb{R}\to\mathbb{R}$ the second projection. Then, for any $F\in\mathcal{D}_A(M)$ and $G\in\mathcal{D}_B(M)$, one has $Rq_{\mathbb{R}*}\mathcal{H}om^\star(F,G)\simeq 0$. In particular, for any $F\in\mathcal{D}_A(M)$ and $G\in\mathcal{D}_B(M)$, one has* $\mathrm{Hom}_{\mathcal{D}(M)}(F,G)\simeq 0$.

Using sheaf quantization of Hamiltonian isotopies, we can define Hamiltonian deformations in Tamarkin's category $\mathcal{D}(M)$ as follows. Let $\phi^H = (\phi_s^H)_s \colon T^*M \times I \to T^*M$ be a Hamiltonian isotopy with compact support, where I is an open interval containing the closed interval $[0,1]$. Let $\widehat{\phi} \colon \mathring{T}^*(M \times \mathbb{R}) \times I \to \mathring{T}^*(M \times \mathbb{R})$ be the associated homogeneous Hamiltonian isotopy and $K \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times M \times \mathbb{R} \times I)$ the sheaf quantization of $\widehat{\phi}$. Then, for any $s \in I$, the composition with $K_s := K|_{M \times \mathbb{R} \times M \times \mathbb{R} \times \{s\}} \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times M \times \mathbb{R})$ defines a functor

$$\Psi_s^H := K_s \circ (*) \colon \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R}) \longrightarrow \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R}), \tag{2.2.18}$$

which induces a functor $\Psi_s^H \colon \mathcal{D}(M) \to \mathcal{D}(M)$ (see [GS14, Proposition 3.29]). Moreover, for a compact subset A of T^*M and $F \in \mathcal{D}_A(M)$, Proposition 2.1.11 and the commutative diagram (2.2.2) imply

$$\mathrm{SS}(K_s \circ F) \cap \Omega_+ \subset (\Lambda_{\widehat{\phi}} \circ T_s^* I) \circ \rho^{-1}(A) = \widehat{\phi}_s(\rho^{-1}(A)) \subset \rho^{-1}(\phi_s^H(A)). \tag{2.2.19}$$

In other words, $\Psi_s^H = K_s \circ (*)$ induces a functor $\mathcal{D}_A(M) \to \mathcal{D}_{\phi_s^H(A)}(M)$ for any compact subset A on T^*M.

Tamarkin [Tam08] proved the non-displaceability theorem by using the category $\mathcal{D}(M)$ and torsion objects, which we will explain below. Moreover, Guillermou-Schapira [GS14] proved that torsion objects form a triangulated subcategory and introduced the quotient category $\mathcal{T}(M)$, which is invariant under Hamiltonian deformations. For $c \in \mathbb{R}$, we define the translation map

$$T_c \colon M \times \mathbb{R} \to M \times \mathbb{R}, (x,t) \mapsto (x, t+c). \tag{2.2.20}$$

For $F \in \mathbf{D}^{\mathrm{b}}_{\{\tau \geq 0\}}(M \times \mathbb{R})$ and $c \leq d$, there exists a canonical morphism $\tau_{c,d}(F) \colon T_{c*}F \to T_{d*}F$. In Section 4.1 below, we will recall the construction of the morphism and detailed results on torsion objects due to Guillermou-Schapira [GS14]. Recall that $\mathcal{D}(M)$ is regarded as a full subcategory of $\mathbf{D}^{\mathrm{b}}_{\{\tau \geq 0\}}(M \times \mathbb{R})$ via the projector P_l or P_r.

Definition 2.2.8 ([Tam08])**.** An object $F \in \mathbf{D}^{\mathrm{b}}_{\{\tau \geq 0\}}(M \times \mathbb{R})$ is said to be a *torsion object* if $\tau_{0,c}(F) = 0$ for some $c \in \mathbb{R}_{\geq 0}$. Denote by $\mathcal{N}_{\mathrm{tor}}$ the subcategory of torsion objects in $\mathcal{D}(M)$.

Let $F \in \mathbf{D}^{\mathrm{b}}_{\{\tau \geq 0\}}(M \times \mathbb{R})$ and assume that $\mathrm{Supp}(F) \subset M \times C$ for some compact subset C of $\mathbb{R}$. Then F is a torsion object.

Proposition 2.2.9 ([GS14, Theorem 5.4])**.** *The subcategory $\mathcal{N}_{\mathrm{tor}}$ is a full triangulated subcategory of $\mathcal{D}(M)$.*

Definition 2.2.10 ([GS14, Definition 5.6])**.** The triangulated category $\mathcal{T}(M)$ is defined as the quotient category of $\mathcal{D}(M)$ by $\mathcal{N}_{\mathrm{tor}}$: $\mathcal{T}(M) := \mathcal{D}(M)/\mathcal{N}_{\mathrm{tor}}$.

Hom spaces in $\mathcal{T}(M)$ are described as inductive limits of those in $\mathcal{D}(M)$.

Proposition 2.2.11 ([GS14, Proposition 5.7])**.** *Let $F, G \in \mathcal{D}(M)$. Then*

$$\varinjlim_{c \to +\infty} \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}G) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}(M)}(F, G). \tag{2.2.21}$$

The following is the Hamiltonian invariance theorem due to Tamarkin [Tam08].

Theorem 2.2.12 ([Tam08, Theorem 3.9] and [GS14, Theorem 6.1])**.** *Let $\phi^H : T^*M \times I \to T^*$ be a Hamiltonian isotopy with compact support and $s \in I$. Let moreover $\Psi_s^H : \mathcal{D}(M) \to \mathcal{D}(M)$ be the functor defined above. Then, for any $F \in \mathcal{D}(M)$, one has*

$$F \simeq \Psi_s^H(F) \quad \textit{in } \mathcal{T}(M). \tag{2.2.22}$$

The following is Tamarkin's non-displaceability theorem. The second assertion follows from the first one, Proposition 2.2.6, and Proposition 2.2.11. Note that the second assertion also follows from Theorem 2.2.7, Proposition 2.2.11, and Theorem 2.2.12.

Theorem 2.2.13 ([Tam08, Theorem 3.1] and [GS14, Theorem 6.2])**.** *Let A and B be compact subsets of T^*M. Assume that there exist $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$ such that $Rq_{\mathbb{R}*} \mathcal{H}om^\star(F,G)$ is not torsion, that is, $\tau_{0,c}(Rq_{\mathbb{R}*} \mathcal{H}om^\star(F,G)) \neq 0$ for any $c \in \mathbb{R}_{\geq 0}$. Then A and B are mutually non-displaceable. In particular, if there exist $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$ such that $\mathrm{Hom}_{\mathcal{T}(M)}(F,G) \neq 0$, then A and B are mutually non-displaceable.*

In this thesis, we give quantitative generalizations of Theorem 2.2.13 in two different directions. First, in Chapter 3, we prove that $\mathrm{Hom}_{\mathcal{T}(M)}(F,G)$ gives a lower bound of the cardinality of the intersection when A and B are compact exact Lagrangian submanifolds, and F and G are associated simple sheaf quantizations (see Subsection 2.2.3 below). Second, in Chapter 4, we show that the infimum of $\{c \in \mathbb{R}_{\geq 0} \mid \tau_{0,c}(Rq_{\mathbb{R}*} \mathcal{H}om^\star(F,G)) = 0\}$ gives a lower bound of the displacement energy of the pair (A,B).

2.2.3 Guillermou's sheaf quantization of compact exact Lagrangian submanifolds ([Gui12, Gui16a])

In this subsection, we assume that M is compact. Recall that a Lagrangian submanifold L of T^*M is said to be *exact* if the restriction of the Liouville 1-form $\alpha_{T^*M}|_L$ is exact. Guillermou [Gui12, Gui16a] proved the existence of sheaf quantizations of compact exact Lagrangian submanifolds of T^*M.

Let L be a compact connected exact Lagrangian submanifold of T^*M and choose a primitive of the Liouville 1-form $f : L \to \mathbb{R}$ satisfying $df = \alpha_{T^*M}|_L$. We define the *conification* $\widehat{L}_f \subset \Omega_+$ of L with respect to f by

$$\widehat{L}_f := \{(x,t;\tau\xi,\tau) \mid \tau > 0, (x;\xi) \in L, t = -f(x;\xi)\}. \tag{2.2.23}$$

If there is no risk of confusion, we simply write $\widehat{L}$ instead of $\widehat{L}_f$.

Let us consider the category $\mathbf{D}^{\mathrm{b}}_{\widehat{L} \cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R})}(M \times \mathbb{R})$ consisting of sheaves whose microsupports are contained in $\widehat{L} \cup T^*_{M\times\mathbb{R}}(M \times \mathbb{R})$. By the compactness of L, there is $A \in \mathbb{R}_{>0}$ such that $\widehat{L} \subset T^*(M \times (-A,A))$. Hence for any $F \in \mathbf{D}^{\mathrm{b}}_{\widehat{L} \cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R})}(M \times \mathbb{R})$, the restrictions $F|_{M\times(-\infty,-A)}$ and $F|_{M\times(A,+\infty)}$ are locally constant.

Definition 2.2.14 ([Gui12, Definition 20.1] and [Gui16a, Definition 13.1])**.** Let $A \in \mathbb{R}_{>0}$ satisfying $\widehat{L} \subset T^*(M \times (-A,A))$. For an object $F \in \mathbf{D}^{\mathrm{b}}_{\widehat{L} \cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R})}(M \times \mathbb{R})$, one defines $F_-, F_+ \in \mathbf{D}^{\mathrm{b}}(M)$ by

$$F_- := F|_{M\times\{-t\}}, \quad F_+ := F|_{M\times\{t\}} \tag{2.2.24}$$

for any $t > A$ (independent of t). One also defines $\mathbf{D}^{\mathrm{b}}_{\widehat{L} \cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R}),+}(M \times \mathbb{R})$ as the full subcategory of $\mathbf{D}^{\mathrm{b}}_{\widehat{L} \cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R})}(M \times \mathbb{R})$ consisting of F such that $F_- \simeq 0$.

Guillermou [Gui12, Gui16a] proved the following existence and uniqueness of sheaf quantizations of compact exact Lagrangian submanifolds.

Theorem 2.2.15 ([Gui12, Theorem 26.1] and [Gui16a, Theorem 18.1])**.** *Let L, f, and $\widehat{L} = \widehat{L}_f$ be as above.*

(i) *For any rank 1 locally constant sheaf $\mathcal{L} \in \mathrm{Mod}(\mathbf{k}_M)$, there exists an object $F \in \mathbf{D}^{\mathrm{b}}_{\widehat{L} \cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R}),+}(M \times \mathbb{R})$ satisfying $F_+ \simeq \mathcal{L}$.*

(ii) *Moreover F in* (i) *is unique up to a unique isomorphism and simple along $\widehat{L}$.*

We call the object $F \in \mathbf{D}^{\mathrm{b}}_{\widehat{L} \cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R}),+}(M \times \mathbb{R})$ in (i) the *simple sheaf quantization* of $\widehat{L}$ with respect to the rank 1 locally constant sheaf $\mathcal{L}$. Moreover, if $\mathcal{L}$ is the constant sheaf $\mathbf{k}_M$, that is, $F_+ \simeq \mathbf{k}_M$, then F is said to be the *canonical sheaf quantization* of $\widehat{L}$. Note that the simple sheaf quantization of $\widehat{L}$ with respect to $\mathcal{L}$ is of the form $F \otimes q_M^{-1}\mathcal{L}$, where F is the canonical sheaf quantization and $q_M \colon M \times \mathbb{R} \to M$ is the projection. We sometimes write a sheaf quantization associated with L (and f) instead of $\widehat{L}$ for simplicity.

Chapter 3

Compact exact Lagrangian intersections in cotangent bundles via sheaf quantization

In this chapter, we study intersections of compact exact Lagrangian submanifolds in cotangent bundles, using Tamarkin's category and Guillermou's sheaf quantizations. In particular, we prove Theorem 1.2.1, a Morse-Bott-type inequality for clean Lagrangian intersections. Throughout this chapter, we assume that M is compact. Moreover, for $i = 1, 2$, let L_i be a compact connected exact Lagrangian submanifold and $f_i \colon L_i \to \mathbb{R}$ be a primitive of the Liouville 1-form satisfying $df_i = \alpha_{T^*M}|_{L_i}$. We denote by $\Lambda_i := \widehat{L_i}$ the conification of L_i with respect to f_i. Let furthermore $F_i \in \mathbf{D}^{\mathrm{b}}_{\Lambda_i \cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R}),+}(M \times \mathbb{R})$ be a simple sheaf quantization of Λ_i. Until the end of Section 3.3, we do *not* assume that L_1 and L_2 intersect cleanly.

3.1 Non-displaceability of compact exact Lagrangian submanifolds

In this section, we prove that the Hom space in $\mathcal{T}(M)$ between the canonical sheaf quantizations associated with compact exact Lagrangian submanifolds is isomorphic to the cohomology of the base manifold M. Combined with Theorem 2.2.13, this implies the non-displaceability.

First, we give a preliminary result useful to calculate Hom spaces in $\mathcal{D}(M)$.

Lemma 3.1.1. *Let L be a compact connected exact Lagrangian submanifold of T^*M and $\Lambda = \widehat{L}$ be the conification of L with respect to some primitive. Then*

$$\mathbf{D}^{\mathrm{b}}_{\Lambda \cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R}),+}(M \times \mathbb{R}) \subset {}^{\perp}\mathbf{D}^{\mathrm{b}}_{\{\tau \le 0\}}(M \times \mathbb{R}). \tag{3.1.1}$$

Proof. By compactness, there exists a constant $B \in \mathbb{R}$ such that $\Lambda \subset T^*(M \times (B, +\infty))$. Let $F \in \mathbf{D}^{\mathrm{b}}_{\Lambda \cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R}),+}(M \times \mathbb{R})$ and $G \in \mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M \times \mathbb{R})$. Since $\Lambda \subset \{\tau > 0\}$, by Proposition 2.1.7, we have $\mathrm{SS}(R\mathcal{H}om(F, G)) \subset \{\tau \le 0\}$. Applying the microlocal Morse lemma (Proposition 2.1.3) to $R\mathcal{H}om(F, G)$ and the function $t \colon M \times \mathbb{R} \to \mathbb{R}, (x, t) \mapsto t$, we get $R\mathrm{Hom}(F, G) \simeq 0$ by the inclusion $\mathrm{Supp}(R\mathcal{H}om(F, G)) \subset M \times [B, +\infty)$. □

Proposition 3.1.2. *Let $\mathcal{L}_i := (F_i)_+ \in \mathrm{Mod}(\mathbf{k}_M)$ be the locally constant sheaf of rank 1 associated with the simple sheaf quantization F_i for $i = 1, 2$. Then there exists $c_0 \in \mathbb{R}_{\ge 0}$*

such that $\mathrm{Hom}_{\mathcal{D}(M)}(F_2, T_{c*}F_1[k])$ *is isomorphic to* $H^k(M; \mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes -1})$ *for any* $c \geq c_0$ *and* $k \in \mathbb{Z}$. *In particular,*

$$\mathrm{Hom}_{\mathcal{T}(M)}(F_2, F_1[k]) \simeq H^k(M; \mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes -1}) \quad \textit{for any } k \in \mathbb{Z}. \tag{3.1.2}$$

Proof. The proof is very similar to those of [Gui12, Theorem 20.3] and [Gui16a, Theorem 13.3]. By Lemma 3.1.1, for any $k \in \mathbb{Z}$, we have

$$\mathrm{Hom}_{\mathcal{D}(M)}(F_2, T_{c*}F_1[k]) = \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(M\times\mathbb{R})}(F_2, T_{c*}F_1[k]). \tag{3.1.3}$$

By the compactness of L_1 and L_2, there exists $A \in \mathbb{R}_{>0}$ satisfying $\Lambda_1, \Lambda_2 \subset T^*(M \times (-A, A))$. Take a sufficiently large $c_0 \in \mathbb{R}_{\geq 0}$ such that $c_0 > 2A$. Then, by the isomorphism $F_2|_{M\times(A,+\infty)} \simeq \mathcal{L}_2 \boxtimes \mathbf{k}_{(A,+\infty)}$ and the inclusion $\mathrm{Supp}(T_{c*}F_1) \subset M \times (c - A, +\infty)$, we get

$$\begin{aligned} R\mathrm{Hom}(F_2, T_{c*}F_1) &\simeq R\mathrm{Hom}(\mathcal{L}_2 \boxtimes \mathbf{k}_{\mathbb{R}}, T_{c*}F_1) \\ &\simeq R\Gamma(M \times \mathbb{R}; F_1 \otimes (\mathcal{L}_2^{\otimes -1} \boxtimes \mathbf{k}_{\mathbb{R}})) \end{aligned} \tag{3.1.4}$$

for any $c \geq c_0$. Since $\mathrm{SS}(F_1 \otimes (\mathcal{L}_2^{\otimes -1} \boxtimes \mathbf{k}_{\mathbb{R}})) \subset \{\tau \geq 0\}$, we can apply the microlocal Morse lemma (Proposition 2.1.3) and obtain

$$\begin{aligned} R\Gamma(M \times \mathbb{R}; F_1 \otimes (\mathcal{L}_2^{\otimes -1} \boxtimes \mathbf{k}_{\mathbb{R}})) &\simeq R\Gamma(M \times (A, +\infty); F_1 \otimes (\mathcal{L}_2^{\otimes -1} \boxtimes \mathbf{k}_{\mathbb{R}})) \\ &\simeq R\Gamma(M \times (A, +\infty); (\mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes -1}) \boxtimes \mathbf{k}_{\mathbb{R}}) \\ &\simeq R\Gamma(M; \mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes -1}). \end{aligned} \tag{3.1.5}$$

The second assertion follows from Proposition 2.2.11. ☐

Remark 3.1.3. In the special case where both L_1 and L_2 are the zero-section $T^*_M M$ of T^*M, (3.1.2) was already obtained by Guillermou-Schapira [GS14]. The outline of the proof is as follows. The simple sheaf quantization associated with the zero-section $T^*_M M$ and a rank 1 locally constant sheaf $\mathcal{L} \in \mathrm{Mod}(\mathbf{k}_M)$ is isomorphic to $\mathcal{L} \boxtimes \mathbf{k}_{[0,+\infty)}$. In [GS14], Guillermou and Schapira proved that the functor

$$\mathbf{D}^{\mathrm{b}}(M) \longrightarrow \mathcal{T}(M), \; F \longmapsto F \boxtimes \mathbf{k}_{[0,+\infty)} \tag{3.1.6}$$

is fully faithful (see [GS14, Corollary 5.8]). We thus obtain

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}(M)}(\mathcal{L}_2 \boxtimes \mathbf{k}_{[0,+\infty)}, \mathcal{L}_1 \boxtimes \mathbf{k}_{[0,+\infty)}[k]) &\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(M)}(\mathcal{L}_2, \mathcal{L}_1[k]) \\ &\simeq H^k(M; \mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes -1}) \end{aligned} \tag{3.1.7}$$

for rank 1 locally constant sheaves $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Mod}(\mathbf{k}_M)$.

Moreover, we can prove (3.1.2) for general compact exact Lagrangians L_1 and L_2 using (3.1.7) and Proposition 3.1.4 below. The following was pointed out to the author by T. Kuwagaki.

Proposition 3.1.4. *Let* L *be a compact connected exact Lagrangian submanifold of* T^*M. *Let* $\mathcal{L} \in \mathrm{Mod}(\mathbf{k}_M)$ *be a locally constant sheaf of rank 1 and* $F \in \mathbf{D}^{\mathrm{b}}_{\widehat{L} \cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R}),+}(M \times \mathbb{R})$ *be the simple sheaf quantization associated with* L *satisfying* $F_+ \simeq \mathcal{L}$. *Then*

$$F \simeq \mathcal{L} \boxtimes \mathbf{k}_{[0,+\infty)} \quad \textit{in } \mathcal{T}(M). \tag{3.1.8}$$

Proof. By the compactness of L, we can take a sufficiently large $A \in \mathbb{R}_{>0}$ such that $\widehat{L} \subset T^*(M \times (-A, A))$. Since $F|_{M\times(A,+\infty)} \simeq \mathcal{L} \boxtimes \mathbf{k}_{(A,+\infty)}$, there exists a canonical morphism

$$F \longrightarrow \mathcal{L} \boxtimes \mathbf{k}_{[A+1,+\infty)}. \tag{3.1.9}$$

The cone of this morphism is supported in $M \times [-A, A+1]$ and hence a torsion object. Therefore the morphism (3.1.9) is an isomorphism in $\mathcal{T}(M)$. A similar argument shows that the morphism $\mathcal{L} \boxtimes \mathbf{k}_{[0,+\infty)} \to \mathcal{L} \boxtimes \mathbf{k}_{[A+1,+\infty)}$ is an isomorphism in $\mathcal{T}(M)$. □

By Theorem 2.2.13 and Proposition 3.1.2, we obtain the following:

Corollary 3.1.5. *In the same notation as in Proposition 3.1.2, assume that F_i is the canonical sheaf quantization of $\widehat{L}_i$, that is, $\mathcal{L}_i \simeq \mathbf{k}_M$ for $i = 1, 2$. Then*

$$\mathrm{Hom}_{\mathcal{T}(M)}(F_2, F_1[k]) \simeq H^k(M; \mathbf{k}) \quad \textit{for any } k \in \mathbb{Z}. \tag{3.1.10}$$

In particular, L_1 and L_2 are mutually non-displaceable.

3.2 Morse-Bott inequality for $\mathcal{H}om^\star$

In this section, we shall apply the Morse-Bott inequality for sheaves to $\mathcal{H}om^\star(F_2, F_1)$. For this purpose, we estimate $\mathrm{SS}(\mathcal{H}om^\star(F_2, F_1))$. Recall the isomorphism

$$\mathcal{H}om^\star(F_2, F_1) \simeq Rs_* \, R\mathcal{H}om(\tilde{q}_2^{-1} i^{-1} F_2, \tilde{q}_1^! F_1), \tag{3.2.1}$$

where $\tilde{q}_1, \tilde{q}_2 \colon M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R}$ are the projections, $s \colon M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R}$ is the addition map, and $i \colon M \times \mathbb{R} \to M \times \mathbb{R}$ is the involution $(x, t) \mapsto (x, -t)$. Since $\tilde{q}_2$ and $\tilde{q}_1$ are submersions, by Theorem 2.1.5 (ii), we have inclusions

$$\begin{aligned} \mathring{\mathrm{SS}}(\tilde{q}_2^{-1} i^{-1} F_2) &\subset \tilde{q}_{2d} \tilde{q}_{2\pi}^{-1} \mathring{\mathrm{SS}}(i^{-1} F_2) \\ &= \left\{ (x, t_1, t_2; \tau_2 \xi_2, 0, -\tau_2) \;\middle|\; \begin{array}{l} \tau_2 > 0, (x; \xi_2) \in L_2, \\ t_1 \in \mathbb{R}, t_2 = f_2(x; \xi_2) \end{array} \right\} \end{aligned} \tag{3.2.2}$$

and

$$\begin{aligned} \mathring{\mathrm{SS}}(\tilde{q}_1^! F_1) &\subset \tilde{q}_{1d} \tilde{q}_{1\pi}^{-1} \mathring{\mathrm{SS}}(F_1) \\ &= \left\{ (x, t_1, t_2; \tau_1 \xi_1, \tau_1, 0) \;\middle|\; \begin{array}{l} \tau_1 > 0, (x; \xi_1) \in L_1, \\ t_1 = -f_1(x; \xi_1), t_2 \in \mathbb{R} \end{array} \right\}. \end{aligned} \tag{3.2.3}$$

Hence $\mathring{\mathrm{SS}}(\tilde{q}_2^{-1} i^{-1} F_2) \cap \mathring{\mathrm{SS}}(\tilde{q}_1^! F_1) = \emptyset$, and by Proposition 2.1.7, we obtain

$$\begin{aligned} &\mathring{\mathrm{SS}}(R\mathcal{H}om(\tilde{q}_2^{-1} i^{-1} F_2, \tilde{q}_1^! F_1)) \subset \mathring{\mathrm{SS}}(\tilde{q}_2^{-1} i^{-1} F_2)^a + \mathring{\mathrm{SS}}(\tilde{q}_1^! F_1) \\ &= \left\{ (x, t_1, t_2; \tau_1 \xi_1 - \tau_2 \xi_2, \tau_1, \tau_2) \;\middle|\; \begin{array}{l} \tau_1, \tau_2 > 0, \\ (x; \xi_1) \in L_1, (x; \xi_2) \in L_2, \\ t_1 = -f_1(x; \xi_1), t_2 = f_2(x; \xi_2) \end{array} \right\} \\ &=: \Lambda_{M\times\mathbb{R}\times\mathbb{R}}. \end{aligned} \tag{3.2.4}$$

Lemma 3.2.1. *One has*

$$\begin{aligned} &v_d^{-1} \left(\overline{v_\pi(\Lambda_{M\times\mathbb{R}\times\mathbb{R}} \cup T^*_{M\times\mathbb{R}\times\mathbb{R}}(M \times \mathbb{R} \times \mathbb{R}))} \right) \\ &= v_d^{-1} v_\pi(\Lambda_{M\times\mathbb{R}\times\mathbb{R}} \cup T^*_{M\times\mathbb{R}\times\mathbb{R}}(M \times \mathbb{R} \times \mathbb{R})) \\ &= s_\pi s_d^{-1}(\Lambda_{M\times\mathbb{R}\times\mathbb{R}} \cup T^*_{M\times\mathbb{R}\times\mathbb{R}}(M \times \mathbb{R} \times \mathbb{R})). \end{aligned} \tag{3.2.5}$$

In other words,

$$\begin{aligned} & s_\sharp(\Lambda_{M\times\mathbb{R}\times\mathbb{R}} \cup T^*_{M\times\mathbb{R}\times\mathbb{R}}(M\times\mathbb{R}\times\mathbb{R})) \\ & = s_\pi s_d^{-1}(\Lambda_{M\times\mathbb{R}\times\mathbb{R}} \cup T^*_{M\times\mathbb{R}\times\mathbb{R}}(M\times\mathbb{R}\times\mathbb{R})). \end{aligned} \tag{3.2.6}$$

See Subsection 2.1.4 for the notation v_π, v_d, and $s_\sharp$ associated with the constant linear map $s\colon M\times\mathbb{R}\times\mathbb{R} \to M\times\mathbb{R}$.

Proof. Define $\Lambda' \subset T^*M\times\mathbb{R}\times(\mathbb{R}\times\mathbb{R})$ by

$$\Lambda' := \left\{ ((x;\tau_1\xi_1-\tau_2\xi_2),(t;\tau_1,\tau_2)) \;\middle|\; \begin{array}{l} \tau_1,\tau_2>0,(x;\xi_1)\in L_1,(x;\xi_2)\in L_2, \\ t=f_2(x;\xi_2)-f_1(x;\xi_1) \end{array} \right\}. \tag{3.2.7}$$

Then the set $v_\pi(\Lambda_{M\times\mathbb{R}\times\mathbb{R}} \cup T^*_{M\times\mathbb{R}\times\mathbb{R}}(M\times\mathbb{R}\times\mathbb{R}))$ is equal to $\Lambda'\cup(T^*_MM\times\mathbb{R}\times\{(0,0)\}) \subset T^*M\times\mathbb{R}\times(\mathbb{R}\times\mathbb{R})$. It suffices to check that $\Lambda'\cup(T^*_MM\times\mathbb{R}\times\{(0,0)\})$ is equal to its closure. By the compactness of L_1 and L_2, there exists $C\in\mathbb{R}_{>0}$ such that $|\xi|\le C(|\tau_1|+|\tau_2|)$ for any $((x;\xi),(t;\tau_1,\tau_2))\in\Lambda'$. Therefore the same inequality holds on the closure $\overline{\Lambda'}$ of Λ'. Hence if $((x;\xi),(t;\tau_1,\tau_2))\in\overline{\Lambda'}$ and $\tau_1=\tau_2=0$ then $\xi=0$, which proves the equality. □

By Proposition 2.1.9, Lemma 3.2.1, and (3.2.4), $\mathring{\mathrm{SS}}(\mathcal{H}om^\star(F_2,F_1))$ is estimated as

$$\begin{aligned} \mathring{\mathrm{SS}}(\mathcal{H}om^\star(F_2,F_1)) & \subset s_\sharp(\Lambda_{M\times\mathbb{R}\times\mathbb{R}} \cup T^*_{M\times\mathbb{R}\times\mathbb{R}}(M\times\mathbb{R}\times\mathbb{R})) \cap \mathring{T}^*(M\times\mathbb{R}) \\ & = s_\pi s_d^{-1}(\Lambda_{M\times\mathbb{R}\times\mathbb{R}} \cup T^*_{M\times\mathbb{R}\times\mathbb{R}}(M\times\mathbb{R}\times\mathbb{R})) \cap \mathring{T}^*(M\times\mathbb{R}) \\ & \subset \left\{ (x,t;\tau(\xi_1-\xi_2),\tau) \;\middle|\; \begin{array}{l} \tau>0, \\ (x;\xi_1)\in L_1,(x;\xi_2)\in L_2, \\ t=f_2(x;\xi_2)-f_1(x;\xi_1) \end{array} \right\} \\ & =: \Lambda_{M\times\mathbb{R}}. \end{aligned} \tag{3.2.8}$$

Let $t\colon M\to\mathbb{R}$ be the function $(x,t)\mapsto t$. Then, by (3.2.8), we obtain

$$\Gamma_{dt}\cap\mathrm{SS}(\mathcal{H}om^\star(F_2,F_1)) \subset \left\{ (x,t;0,1) \;\middle|\; \begin{array}{l} \exists\,(x;\xi)\in L_1\cap L_2, \\ t=f_2(x;\xi)-f_1(x;\xi) \end{array} \right\}. \tag{3.2.9}$$

By this inclusion, we find that $R\Gamma_{M\times[c,+\infty)}(\mathcal{H}om^\star(F_2,F_1))|_{M\times\{c\}} \simeq 0$ if $c\notin\{f_2(p)-f_1(p) \mid p\in L_1\cap L_2\}$.

Proposition 3.2.2. *Let $a,b\in\mathbb{R}$ with $a<b$ or $a\in\mathbb{R}, b=+\infty$. Assume*

1. *the point $a\in\mathbb{R}$ is not an accumulation point of $\{f_2(p)-f_1(p) \mid p\in L_1\cap L_2\}\subset\mathbb{R}$,*
2. *the set $\{f_2(p)-f_1(p) \mid p\in L_1\cap L_2\}\cap[a,b)$ is finite,*
3. *the object $R\Gamma(M\times\{c\};R\Gamma_{M\times[c,+\infty)}(\mathcal{H}om^\star(F_2,F_1))|_{M\times\{c\}})$ has finite-dimensional cohomology for any $a\le c<b$.*

Then

$$\begin{aligned} & \sum_{a\le c<b} \dim H^kR\Gamma\left(M\times\{c\};R\Gamma_{M\times[c,+\infty)}(\mathcal{H}om^\star(F_2,F_1))|_{M\times\{c\}}\right) \\ & \qquad \ge \dim H^kR\Gamma_{M\times[a,b)}(M\times(-\infty,b);\mathcal{H}om^\star(F_2,F_1)) \end{aligned} \tag{3.2.10}$$

for any $k\in\mathbb{Z}$.

Proof. We set $\mathcal{H} := \mathcal{H}om^\star(F_2, F_1)$. By the assumption (1), we can take $a' < a$ such that

$$f_1(p) - f_2(p) \notin [a', a) \quad \text{for any } p \in L_1 \cap L_2. \tag{3.2.11}$$

By (3.2.8) and (3.2.11), we have $\mathring{\mathrm{SS}}(\mathcal{H}) \cap \mathring{\mathrm{SS}}(\mathbf{k}_{M\times[a',+\infty)}) = \emptyset$. Hence, by Proposition 2.1.7, we obtain

$$\begin{aligned}\mathring{\mathrm{SS}}(R\Gamma_{M\times[a',+\infty)}(\mathcal{H})) &= \mathring{\mathrm{SS}}(R\mathcal{H}om(\mathbf{k}_{M\times[a',+\infty)}, \mathcal{H})) \\ &\subset \Lambda_{M\times\mathbb{R}} \cap \pi^{-1}(\{t > a'\}) + \{(x, a'; 0, -\tau') \mid \tau' > 0\}.\end{aligned} \tag{3.2.12}$$

Set $\mathcal{H}' := R\Gamma_{M\times[a',+\infty)}(\mathcal{H})|_{M\times(-\infty,b)} \in \mathbf{D}^{\mathrm{b}}(M \times (-\infty, b))$ and let $t \colon M \times (-\infty, b) \to \mathbb{R}$ be the function $(x, t) \mapsto t$. We shall apply the Morse-Bott inequality for sheaves (Theorem 2.1.10) to H' and $t \colon M \times (-\infty, b) \to \mathbb{R}$. Combining (3.2.8) with (3.2.12), we get

$$\Gamma_{dt} \cap \mathrm{SS}(\mathcal{H}') \subset \{(x, t; 0, 1) \mid \exists p \in L_1 \cap L_2, x = \pi(p), a' < t = f_2(p) - f_1(p) < b\}. \tag{3.2.13}$$

Hence, the conditions in Theorem 2.1.10 are satisfied by (3.2.11), and the assumptions (2) and (3). Hence we have the inequality

$$\begin{aligned}\sum_{a'<c<b} &\dim H^k R\Gamma\left(M \times \{c\}; R\Gamma_{M\times[c,+\infty)}(\mathcal{H})|_{M\times\{c\}}\right) \\ &\geq \dim H^k R\Gamma_{M\times[a',b)}(M \times (-\infty, b); \mathcal{H})\end{aligned} \tag{3.2.14}$$

for any $k \in \mathbb{Z}$. Moreover, by (3.2.8), (3.2.11), and (3.2.12), we get $\Gamma_{dt} \cap \mathrm{SS}(\mathcal{H}') \cap \pi^{-1}(M \times [a', a)) = \emptyset$. Applying the microlocal Morse lemma (Proposition 2.1.3), we have

$$\begin{aligned}R\Gamma_{M\times[a',a)}(M \times (-\infty, a); \mathcal{H}) &\simeq R\Gamma(M \times (-\infty, a); \mathcal{H}') \\ &\simeq R\Gamma((-\infty, a'); \mathcal{H}') \simeq 0.\end{aligned} \tag{3.2.15}$$

Thus we get $R\Gamma_{M\times[a,b)}(M \times (-\infty, b); \mathcal{H}) \simeq R\Gamma_{M\times[a',b)}(M \times (-\infty, b); \mathcal{H})$. On the other hand, by (3.2.11), $R\Gamma_{M\times[c,+\infty)}(\mathcal{H})|_{M\times\{c\}} \simeq 0$ for $c \in [a', a)$ and the left hand side of (3.2.14) is equal to that of (3.2.10). This completes the proof. □

Remark 3.2.3. C. Viterbo announced that he found some relation between the section of $\mathcal{H}om^\star(F_2, F_1)$ on $M \times (-\infty, \lambda)$ and the Floer cohomology complex $CF_{<\lambda}(L_2, L_1)$ filtered by $\{p \in L_1 \cap L_2 \mid f_2(p) - f_1(p) < \lambda\}$. Inspired by his work, in Proposition 3.2.2, we consider not only the section on $M \times \mathbb{R}$ but also that on $M \times (-\infty, b)$.

3.3 Microlocalization of $\mathcal{H}om^\star$

In this section, we describe $R\Gamma(M \times \{c\}; R\Gamma_{M\times[c,+\infty)}(\mathcal{H}om^\star(F_2, F_1))|_{M\times\{c\}})$ in terms of the functor μhom. Applying T_{c*} to F_2, we may assume $c = 0$. The following lemma follows from Proposition 2.1.16.

Lemma 3.3.1. *Set* $V_+ := \{(x, 0; 0, \tau) \mid \tau > 0\} \subset T^*_{M\times\{0\}}(M \times \mathbb{R})$. *Then*

$$\begin{aligned}&R\Gamma(M \times \{0\}; R\Gamma_{M\times[0,+\infty)}(\mathcal{H}om^\star(F_2, F_1))|_{M\times\{0\}}) \\ &\quad\simeq R\Gamma(V_+; \mu_{M\times\{0\}}(\mathcal{H}om^\star(F_2, F_1))|_{V_+}).\end{aligned} \tag{3.3.1}$$

Recall the isomorphism

$$\mathcal{H}om^\star(F_2, F_1) \simeq Rs_*\delta^! \, R\mathcal{H}om(q_2^{-1}i^{-1}F_2, q_1^! F_1), \tag{3.3.2}$$

where $s\colon M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R}$ is the addition map, $\delta\colon M \times \mathbb{R} \times \mathbb{R} \to M \times M \times \mathbb{R} \times \mathbb{R}$ is the diagonal embedding, and $q_i\colon M \times \mathbb{R} \times M \times \mathbb{R} \to M \times \mathbb{R}$ is the i-th projection. The morphism s induces the following commutative diagram, where we omit T^*M (resp. $T^*_M M$) in the first (resp. second) row and use the same symbol s for the addition map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$\begin{tikzcd}
T^*(\mathbb{R} \times \mathbb{R}) & (\mathbb{R} \times \mathbb{R}) \times_{\mathbb{R}} T^*\mathbb{R} \arrow[l, "s_d"'] \arrow[r, "s_\pi"] & T^*\mathbb{R} \\
T^*_{s^{-1}(0)}(\mathbb{R} \times \mathbb{R}) \arrow[u, hook] & s^{-1}(0) \times_{\{0\}} T^*_0\mathbb{R} \arrow[l, "\sim"'] \arrow[r, "s_\pi"'] \arrow[u, hook] & T^*_0\mathbb{R}. \arrow[u, hook]
\end{tikzcd} \tag{3.3.3}$$

We denote by $\pi_s\colon T^*_M M \times T^*_{s^{-1}(0)}(\mathbb{R} \times \mathbb{R}) \to T^*_M M \times T^*_0\mathbb{R} \simeq T^*_{M\times\{0\}}(M \times \mathbb{R})$ the induced morphism in the second row in the above diagram. On the other hand, the morphism δ induces the following commutative diagram, where we omit $T^*_{s^{-1}(0)}(\mathbb{R} \times \mathbb{R})$:

$$\begin{tikzcd}
T^*M & M \times_{M\times M} T^*(M \times M) \arrow[l, "\delta_d"'] \arrow[r, "\delta_\pi"] & T^*(M \times M) \\
T^*_M M \arrow[u, hook] \arrow[d, equal] & M \times_{\Delta_M} T^*_{\Delta_M}(M \times M) \arrow[l] \arrow[r, "\sim"] \arrow[u, hook] \arrow[d, equal] & T^*_{\Delta_M}(M \times M) \arrow[u, hook] \arrow[d, equal] \\
M & T^*M \arrow[l, "\pi_M"] \arrow[r, equal] & T^*M.
\end{tikzcd} \tag{3.3.4}$$

Let moreover $\iota\colon T^*\mathbb{R} \simeq T^*_{\Delta_{\mathbb{R}}}(\mathbb{R} \times \mathbb{R}) \xrightarrow{\sim} T^*_{s^{-1}(0)}(\mathbb{R} \times \mathbb{R})$ be the isomorphism of line bundles defined by $(t_1, t_2, \tau, -\tau) \mapsto (t_1, -t_2, \tau, \tau)$. We also use the same symbol ι for the induced isomorphism $T^*(M \times \mathbb{R}) \simeq T^*M \times T^*_{\Delta_{\mathbb{R}}}(\mathbb{R} \times \mathbb{R}) \xrightarrow{\sim} T^*M \times T^*_{s^{-1}(0)}(\mathbb{R} \times \mathbb{R})$.

Proposition 3.3.2. *Set $V_+ := \{(x, 0; 0, \tau) \mid \tau > 0\} \subset T^*_{M\times\{0\}}(M \times \mathbb{R})$ as in Lemma 3.3.1 and keep the notation defined above:*

$$\begin{aligned}
&\pi_s\colon T^*_M M \times T^*_{s^{-1}(0)}(\mathbb{R} \times \mathbb{R}) \to T^*_M M \times T^*_0\mathbb{R} \simeq T^*_{M\times\{0\}}(M \times \mathbb{R}),\\
&\pi_M\colon T^*M \times T^*_{s^{-1}(0)}(\mathbb{R} \times \mathbb{R}) \to T^*_M M \times T^*_{s^{-1}(0)}(\mathbb{R} \times \mathbb{R}),\\
&\iota\colon T^*(M \times \mathbb{R}) \simeq T^*M \times T^*_{\Delta_{\mathbb{R}}}(\mathbb{R} \times \mathbb{R}) \xrightarrow{\sim} T^*M \times T^*_{s^{-1}(0)}(\mathbb{R} \times \mathbb{R}).
\end{aligned}$$

Then

$$\mu_{M\times\{0\}}(\mathcal{H}om^\star(F_2, F_1))|_{V_+} \simeq (R\pi_{s*}R\pi_{M*}\iota_*\mu hom(F_2, F_1))|_{V_+}. \tag{3.3.5}$$

Proof. (a) Set $\mathcal{H} := \mathcal{H}om^\star(F_2, F_1)$. First, we note that $\mu_{M\times\{0\}}(\mathcal{H}) \simeq \mu_{M\times\{0\}}(\mathcal{H}|_{M\times(-1,1)})$. Set $U := M \times (-1, 1) \subset M \times \mathbb{R}$. There exists a sufficiently large $A \in \mathbb{R}_{>0}$ such that F_1 and F_2 are constant on $M \times (A - 2, +\infty)$. Then $\tilde{q}_1^! F_1 \simeq \tilde{q}_1^{-1}F_1[1]$ is constant on $s^{-1}(U) \cap (M \times \mathbb{R} \times (-\infty, -A + 1))$, which implies isomorphisms

$$\begin{aligned}
&R\mathcal{H}om(\tilde{q}_2^{-1}i^{-1}\mathbf{k}_{M\times[A,+\infty)}, \tilde{q}_1^! F_1)|_{s^{-1}(U)}\\
&\simeq R\mathcal{H}om(\mathbf{k}_{M\times\mathbb{R}\times(-\infty,-A]}, \mathbf{k}_{M\times\mathbb{R}\times\mathbb{R}}[1])|_{s^{-1}(U)}\\
&\simeq R\varGamma_{s^{-1}(U)\cap(M\times\mathbb{R}\times(-\infty,-A])}(\mathbf{k}_{s^{-1}(U)})[1].
\end{aligned} \tag{3.3.6}$$

Therefore we obtain

$$(Rs_* R\mathcal{H}om(\tilde{q}_2^{-1}i^{-1}\mathbf{k}_{M\times[A,+\infty)}, \tilde{q}_1^! F_1))|_U \simeq 0. \tag{3.3.7}$$

By the distinguished triangle

$$F_2' \longrightarrow F_2 \longrightarrow \mathbf{k}_{M\times[A,+\infty)} \xrightarrow{+1} \tag{3.3.8}$$

with F_2' supported in some compact subset, we find that

$$(Rs_* R\mathcal{H}om(\tilde{q}_2^{-1}i^{-1}F_2, \tilde{q}_1^! F_1))|_U \simeq (Rs_* R\mathcal{H}om(\tilde{q}_2^{-1}i^{-1}F_2', \tilde{q}_1^! F_1))|_U \tag{3.3.9}$$

and s is proper on $\mathrm{Supp}(R\mathcal{H}om(\tilde{q}_2^{-1}i^{-1}F_2', \tilde{q}_1^! F_1))$.
(b) Since s is proper on the support, by Proposition 2.1.12 (i), we have

$$\mu_{M\times\{0\}}(Rs_* R\mathcal{H}om(\tilde{q}_2^{-1}i^{-1}F_2', \tilde{q}_1^! F_1)) \simeq R\pi_{s*}\mu_{M\times s^{-1}(0)}(R\mathcal{H}om(\tilde{q}_2^{-1}i^{-1}F_2', \tilde{q}_1^! F_1)). \tag{3.3.10}$$

Moreover since δ is non-characteristic for $\mathrm{SS}(R\mathcal{H}om(\tilde{q}_2^{-1}i^{-1}F_2', \tilde{q}_1^! F_1))$ and $\delta|_{M\times s^{-1}(0)} \colon M\times s^{-1}(0) \to \Delta_M \times s^{-1}(0)$ is a submersion, by Proposition 2.1.12 (ii), we obtain

$$\begin{aligned}\mu_{M\times s^{-1}(0)}(R\mathcal{H}om(\tilde{q}_2^{-1}i^{-1}F_2', \tilde{q}_1^! F_1)) &\simeq \mu_{M\times s^{-1}(0)}(\delta^! R\mathcal{H}om(q_2^{-1}i^{-1}F_2', q_1^! F_1)) \\ &\simeq R\pi_{M*}\mu_{\Delta_M\times s^{-1}(0)} R\mathcal{H}om(q_2^{-1}i^{-1}F_2', q_1^! F_1).\end{aligned} \tag{3.3.11}$$

Let $i_2 \colon M\times\mathbb{R}\times\mathbb{R} \to M\times\mathbb{R}\times\mathbb{R}$ be the involution $(x,t_1,t_2)\mapsto(x,t_1,-t_2)$. Note that the associated automorphism of $T^*M\times T^*(\mathbb{R}\times\mathbb{R})$ induces $\iota\colon T^*M\times T^*_{\Delta_\mathbb{R}}(\mathbb{R}\times\mathbb{R}) \xrightarrow{\sim} T^*M\times T^*_{s^{-1}(0)}(\mathbb{R}\times\mathbb{R})$. Then, by Proposition 2.1.12 (i) again, we have

$$\begin{aligned}\mu_{\Delta_M\times s^{-1}(0)} R\mathcal{H}om(i_2^{-1}q_2^{-1}F_2', q_1^! F_1) &\simeq \mu_{\Delta_M\times s^{-1}(0)} i_{2*} R\mathcal{H}om(q_2^{-1}F_2', q_1^! F_1) \\ &\simeq \iota_*\mu_{\Delta_{M\times\mathbb{R}}} R\mathcal{H}om(q_2^{-1}F_2', q_1^! F_1) \\ &\simeq \iota_*\mu hom(F_2', F_1).\end{aligned} \tag{3.3.12}$$

(c) By Proposition 2.1.15, we have

$$\mathrm{Supp}(\mu hom(\mathbf{k}_{M\times[A,+\infty)}, F_1)) \subset T^*_{M\times\mathbb{R}}(M\times\mathbb{R}). \tag{3.3.13}$$

Thus, by the distinguished triangle (3.3.8), we get

$$\mu hom(F_2', F_1)|_{\{\tau>0\}} \xrightarrow{\sim} \mu hom(F_2, F_1)|_{\{\tau>0\}}, \tag{3.3.14}$$

which completes the proof. □

We define an open subset Ω_+ of $T^*(M\times\mathbb{R}) \simeq T^*M\times T^*\mathbb{R}$ by $\Omega_+ := \{\tau > 0\} \subset T^*(M\times\mathbb{R})$. Combining Proposition 3.2.2 with Lemma 3.3.1 and Proposition 3.3.2, we obtain the following:

Proposition 3.3.3. *Let $a, b \in \mathbb{R}$ with $a < b$ or $a\in\mathbb{R}, b = +\infty$. Assume*

(1) *the point $a \in \mathbb{R}$ is not an accumulation point of $\{f_2(p) - f_1(p) \mid p \in L_1\cap L_2\} \subset \mathbb{R}$,*

(2) *the set $\{f_2(p) - f_1(p) \mid p \in L_1\cap L_2\} \cap [a, b) \subset \mathbb{R}$ is finite,*

(3) *the object* $R\Gamma\left(\Omega_+; \mu hom(T_{c*}F_2, F_1)|_{\Omega_+}\right)$ *has finite-dimensional cohomology for any* $a \le c < b$.

Then

$$\begin{aligned}\sum_{a \le c < b} &\dim H^k R\Gamma\left(\Omega_+; \mu hom(T_{c*}F_2, F_1)|_{\Omega_+}\right) \\ &\ge \dim H^k R\Gamma_{M\times[a,b)}(M \times (-\infty, b); \mathcal{H}om^\star(F_2, F_1))\end{aligned} \tag{3.3.15}$$

for any $k \in \mathbb{Z}$.

3.4 Clean intersections of compact exact Lagrangian submanifolds

Throughout this section, we assume the following:

Assumption 3.4.1. *The Lagrangian submanifolds* L_1 *and* L_2 *intersect cleanly, that is,* $L_1 \cap L_2$ *is a submanifold of* T^*M *and* $T_p(L_1 \cap L_2) = T_pL_1 \cap T_pL_2$ *for any* $p \in L_1 \cap L_2$.

Under the assumption, the intersection $L_1 \cap L_2$ has finitely many connected components, which are compact submanifolds of T^*M, and the value $f_2(p) - f_1(p)$ is constant on each component. In particular, the set $\{f_2(p) - f_1(p) \mid p \in L_1 \cap L_2\} \subset \mathbb{R}$ is finite. For a component C of $L_1 \cap L_2$, we define $f_{21}(C) := f_2(p) - f_1(p)$, taking some $p \in C$.

Under Assumption 3.4.1, we shall compute $\mu hom(T_{c*}F_2, F_1)|_{\Omega_+}$. Again, we may assume $c = 0$. Recall that we have set $\Lambda_i := \widehat{L_i}$ for simplicity of notation. The following lemma is obtained in [Gui12, Lemma 6.14].

Lemma 3.4.2. *Under Assumption 3.4.1,* $\mu hom(F_2, F_1)|_{\Omega_+}$ *is supported in* $\Lambda_1 \cap \Lambda_2$ *and has locally constant cohomology sheaves.*

Proof. For completeness, we also give a proof here. By Proposition 2.1.15, we have

$$\begin{aligned}&\mathrm{Supp}(\mu hom(F_2, F_1)|_{\Omega_+}) \subset \Lambda_1 \cap \Lambda_2, \\ &\mathrm{SS}(\mu hom(F_2, F_1)|_{\Omega_+}) \subset -\mathbf{h}^{-1}(C(\Lambda_1, \Lambda_2)) \cap T^*\Omega_+.\end{aligned} \tag{3.4.1}$$

Set $\Lambda_{12} := \Lambda_1 \cap \Lambda_2$. Since Λ_1 and Λ_2 intersect cleanly, we have

$$C(\Lambda_1, \Lambda_2) = T\Lambda_1|_{\Lambda_{12}} + T\Lambda_2|_{\Lambda_{12}}. \tag{3.4.2}$$

Since Λ_i is Lagrangian, we get $-\mathbf{h}^{-1}(T\Lambda_i) \subset T^*_{\Lambda_i}T^*(M \times \mathbb{R})$ for $i = 1, 2$. In particular, $-\mathbf{h}^{-1}(T\Lambda_i|_{\Lambda_{12}}) \subset T^*_{\Lambda_{12}}T^*(M \times \mathbb{R})$. Hence we obtain

$$-\mathbf{h}^{-1}(C(\Lambda_1, \Lambda_2)) \cap T^*\Omega_+ \subset T^*_{\Lambda_{12}}T^*(M \times \mathbb{R}). \tag{3.4.3}$$

Hence, by (3.4.1), $\mathrm{SS}(\mu hom(F_2, F_1)|_{\Omega_+}) \subset T^*_{\Lambda_{12}}T^*(M \times \mathbb{R})$, which proves the result. □

Let $C_1, \dots, C_{n_0}$ be the connected components of $L_1 \cap L_2$ with $f_{21}(C_j) = 0$ ($j = 1, \dots, n_0$). For a component C_j, we define a closed subset $\widehat{C_j}$ of $\Omega_+ \subset T^*(M \times \mathbb{R})$ by

$$\widehat{C_j} := \{(x, t; \xi, \tau) \mid \tau > 0, (x; \xi/\tau) \in C_j, t = -f_1(x; \xi/\tau)\,(= -f_2(x; \xi/\tau))\}. \tag{3.4.4}$$

Note that $\widehat{C_j}/\mathbb{R}_{>0} \simeq C_j$. We also denote by $d_i \colon \Lambda_i \to \frac{1}{2}\mathbb{Z}$ the function which assigns the shift of F_i. Since the function d_i is invariant under the $\mathbb{R}_{>0}$-action, we use the same symbol d_i for the function $L_i = \Lambda_i/\mathbb{R}_{>0} \to \frac{1}{2}\mathbb{Z}$ (see also Section 3.C Appendix III).

Theorem 3.4.3. *Under Assumption 3.4.1 and in the notation above, assume moreover* $\mathbf{k} = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$*. Then*

$$\mu hom(F_2, F_1)|_{\Omega_+} \simeq \bigoplus_{j=1}^{n_0} \mathbf{k}_{\widehat{C_j}}[-s(C_j)], \tag{3.4.5}$$

where $s(C_j) \in \mathbb{Z}$ *is given by*

$$s(C_j) := d_2(p) - d_1(p) + \frac{1}{2}(\dim M - \dim C_j) - \frac{1}{2}\tau(T_pL_2, T_pL_1, T_p(T^*_{\pi(p)}M)) \tag{3.4.6}$$

with $p \in C_j$*. In particular,*

$$R\Gamma(\Omega_+; \mu hom(F_2, F_1)|_{\Omega_+}) \simeq \bigoplus_{j=1}^{n_0} R\Gamma(C_j; \mathbf{k}_{C_j})[-s(C_j)]. \tag{3.4.7}$$

Proof. (a) By Lemma 3.4.2, $\mu hom(F_2, F_1)|_{\Lambda_1 \cap \Lambda_2}$ has locally constant cohomology sheaves. Fix $p \in C_j$ and let us compute the stalk at $p' := (p, 0; 1) \in \widehat{C_j}$. There exists a Hamiltonian isotopy with compact support $\phi^H = (\phi^H_s)_s \colon T^*M \times I \to T^*M$, where I is an open interval containing $[0,1]$, such that $\phi^H_1(L_i)$ is the graph $\Gamma_{d\varphi_i}$ of the derivative of some C^∞-function $\varphi_i \colon M \to \mathbb{R}$ in a neighborhood of $\phi^H_1(p)$ for $i = 1, 2$. Let $\widehat{\phi} \colon \mathring{T}^*(M \times \mathbb{R}) \to \mathring{T}^*(M \times \mathbb{R})$ be the homogeneous Hamiltonian isotopy associated with ϕ^H and $K \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times M \times \mathbb{R} \times I)$ be the sheaf quantization of $\widehat{\phi}$. For simplicity of notation, we set $\chi = \widehat{\phi}_1$. Set moreover $K_1 := K|_{M \times \mathbb{R} \times M \times \mathbb{R} \times \{1\}} \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times M \times \mathbb{R})$. By Proposition 2.1.20, in a neighborhood of $\chi(p')$, we have the isomorphism

$$\mu hom(K_1 \circ F_2, K_1 \circ F_1) \simeq \chi_* \mu hom(F_2, F_1). \tag{3.4.8}$$

Moreover, by Proposition 2.1.21, $K_1 \circ F_i$ is simple with shift $d_i(p) + d' - \delta_i$ along $\chi(\Lambda_i)$ at $\chi(p')$, where d' is the shift of K_1 at $(\chi(p'), p'^a)$ and

$$\delta_i := \frac{1}{2}(\dim M + 1) + \frac{1}{2}\tau\left(\lambda_\infty(p'), \lambda_{\Lambda_i}(p'), \chi^{-1}(\lambda_\infty(\chi(p')))\right). \tag{3.4.9}$$

Here, we use the symbols $\lambda_\Lambda(p)$ and $\lambda_\infty(p)$ defined in (2.1.34). Hence we obtain the isomorphism $K_1 \circ F_i \simeq \mathbf{k}_{N_i}\left[d_i(p) + d' - \delta_i - \frac{1}{2}\right]$ in $\mathbf{D}^{\mathrm{b}}(M \times \mathbb{R}; \chi(p'))$, where $N_i := \{(x,t) \in M \times \mathbb{R} \mid \varphi_i(x) + t = 0\}$ (see also Example 2.1.19). Thus we get

$$\begin{aligned} \mu hom(F_2, F_1)_{p'} &\simeq \mu hom(\mathbf{k}_{N_2}, \mathbf{k}_{N_1})_{\chi(p')}[d_1(p) - d_2(p) - \delta_1 + \delta_2] \\ &\simeq \mu_{N_2}(\mathbf{k}_{N_1})_{\chi(p')}[d_1(p) - d_2(p) - \delta_1 + \delta_2], \end{aligned} \tag{3.4.10}$$

where we used Proposition 2.1.14 (iii) for the second isomorphism. We introduce a new local coordinate system (x, t') on $M \times \mathbb{R}$ by $t' := t + \varphi_2(x)$. Then $N_2 = \{t' = 0\}$ and $N_1 = \{t' = \varphi_2(x) - \varphi_1(x)\}$. Assumption 3.4.1 implies that $\varphi := \varphi_2 - \varphi_1$ is a Morse-Bott function. Therefore, after changing the local coordinate system x on M, we may assume that $\pi(\chi(p')) = (0,0)$ in the coordinates (x,t') and $\varphi(x) = -x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_l^2$, where $l := \dim M - \dim C_j$. Note that in the coordinate system on $T^*(M \times \mathbb{R})$ associated with (x,t'), we have $\chi(p') = (0,0;0,1)$. Hence, by (2.1.22), we obtain

$$\begin{aligned} \mu_{N_2}(\mathbf{k}_{N_1})_{\chi(p')} &\simeq \mu_{\mathbb{R}^{\dim M} \times \{0\}}(\mathbf{k}_{\{t' = \varphi(x)\}})_{(0,0;0,1)} \\ &\simeq R\Gamma_{\{t' \geq 0\}}(\mathbf{k}_{\{t' = \varphi(x)\}})_0 \\ &\simeq \mathbf{k}[-\lambda]. \end{aligned} \tag{3.4.11}$$

Thus $\mu hom(F_2, F_1)|_{\widehat{C_j}}$ is concentrated in some degree and locally constant of rank 1. Since $\mathbf{k} = \mathbb{F}_2$, a locally constant sheaf of rank 1 is constant, which implies the isomorphism $\mu hom(F_2, F_1)|_{\widehat{C_j}} \simeq \mathbf{k}_{\widehat{C_j}}[d_1(p) - d_2(p) - \delta_1 + \delta_2 - \lambda]$.
(b) We shall prove

$$\lambda + \delta_1 - \delta_2 = \frac{1}{2}(\dim M - \dim C_j) - \frac{1}{2}\tau(\lambda_{L_2}(p), \lambda_{L_1}(p), \lambda_\infty(p)). \tag{3.4.12}$$

For the above coordinates x on M, we set $x' = (x_1, \dots, x_l), x'' = (x_{l+1}, \dots, x_m)$ with $m = \dim M$ and denote by $(x; \xi) = (x', x''; \xi', \xi'')$ the associated coordinates on T^*M. We also denote by $\partial^2_{x,x}\varphi(0) = (\partial^2_{x_j x_k}\varphi(0))_{j,k}$ the Hessian of φ. Then, by a similar argument to that of the proof of [KS90, Proposition 7.5.3], we get

$$\begin{aligned}\tau(\lambda_\infty(0), T_0(T^*_{\mathbb{R}^m}\mathbb{R}^m), T_0\Gamma_{d\varphi}) &= \tau(\{x = 0\}, \{\xi = 0\}, \{\xi = \partial^2_{x,x}\varphi(0) \cdot x\}) \\ &= \tau(\{x' = 0\}, \{\xi' = 0\}, \{\xi' = \partial^2_{x',x'}\varphi(0) \cdot x'\}) \\ &= -\operatorname{sgn}(\partial^2_{x',x'}\varphi(0)) = 2\lambda - l.\end{aligned} \tag{3.4.13}$$

Moreover, we have

$$\begin{aligned}\tau(\chi(\lambda_{\Lambda_2}(p')), \chi(\lambda_{\Lambda_1}(p')), \lambda_\infty(\chi(p'))) &= \tau(\lambda_{\widehat{T^*_M M}}(\chi(p')), \lambda_{\widehat{\Gamma_{-d\varphi}}}(\chi(p')), \lambda_\infty(\chi(p'))) \\ &= \tau(T_0(T^*_{\mathbb{R}^m}\mathbb{R}^m), T_0\Gamma_{-d\varphi}, \lambda_\infty(0)) \\ &= -\tau(\lambda_\infty(0), T_0(T^*_{\mathbb{R}^m}\mathbb{R}^m), T_0\Gamma_{d\varphi}).\end{aligned} \tag{3.4.14}$$

Here, we used the homogeneous symplectic coordinate system associated with (x, t') for the first equality, Lemma 3.C.2 for the second one, and Proposition 3.C.1 (i) for the last one. Combining the above two equalities, we finally obtain

$$\begin{aligned}-2\lambda + l - 2\delta_1 + 2\delta_2 &= \tau(\chi(\lambda_{\Lambda_2}(p')), \chi(\lambda_{\Lambda_1}(p')), \lambda_\infty(\chi(p'))) - 2\delta_1 + 2\delta_2 \\ &= \tau(\lambda_{\Lambda_2}(p'), \lambda_{\Lambda_1}(p'), \chi^{-1}(\lambda_\infty(\chi(p')))) \\ &\quad + \tau(\lambda_{\Lambda_1}(p'), \lambda_\infty(p'), \chi^{-1}(\lambda_\infty(\chi(p')))) \\ &\quad + \tau(\lambda_\infty(p'), \lambda_{\Lambda_2}(p'), \chi^{-1}(\lambda_\infty(\chi(p')))) \\ &= \tau(\lambda_{\Lambda_2}(p'), \lambda_{\Lambda_1}(p'), \lambda_\infty(p')) \\ &= \tau(\lambda_{L_2}(p), \lambda_{L_1}(p), \lambda_\infty(p)).\end{aligned} \tag{3.4.15}$$

Here, the second equality follows from the invariance under symplectic isomorphisms, the third one follows from the "cocycle condition" of the inertia index (Proposition 3.C.1 (ii)), and the last one follows from Lemma 3.C.2 again. Since $l = \dim M - \dim C_j$, this completes the proof. □

For a general filed $\mathbf{k}$, if L_1 and L_2 are the graphs of exact 1-forms and intersect cleanly, the locally constant object $\mu hom(F_2, F_1)|_{\Omega_+}$ is described as follows:

Proposition 3.4.4. *Let $\mathbf{k}$ be any field. Under Assumption 3.4.1, assume moreover that there exists a C^∞-function $\varphi_i \colon M \to \mathbb{R}$ such that $L_i = \Gamma_{d\varphi_i}$ and $f_i = \varphi_i \circ \pi|_{L_i}$ for $i = 1, 2$. Define a Morse-Bott function φ on M by $\varphi := \varphi_2 - \varphi_1$ and let $C_1, \dots, C_{n_0}$ be the critical components of φ with $\varphi(C_j) = 0$ $(j = 1, \dots, n_0)$. For such a critical component C_j, define $T^-_{C_j}M$ as the maximal subbundle of $T_{C_j}M$ where the restriction of the Hessian $\mathrm{Hess}(\varphi)|_{T^-_{C_j}M}$ is negative definite, and define a closed subset $\widehat{C_j}$ of Ω_+ by*

$$\widehat{C_j} := \{(x, -\varphi_1(x); \tau d\varphi_1(x), \tau) \mid \tau > 0, x \in C_j\}. \tag{3.4.16}$$

Let moreover $\mathcal{L}_i := (F_i)_+ \in \mathrm{Mod}(\mathbf{k}_M)$ be the locally constant sheaf of rank 1 associated with the simple sheaf quantization F_i for $i = 1, 2$. Then

$$\begin{aligned}\mu hom(F_2, F_1)|_{\Omega_+} &\simeq \bigoplus_{j=1}^{n_0} \pi_j^{-1}\left(\omega_{C_j/T^-_{C_j}M} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes -1}\right) \\ &\simeq \bigoplus_{j=1}^{n_0} \pi_j^{-1}\left(\mathrm{or}_{C_j/T^-_{C_j}M} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes -1}\right)[-s(C_j)],\end{aligned} \tag{3.4.17}$$

where $\pi_j \colon \widehat{C_j} \to C_j$ is the projection, $s(C_j) \in \mathbb{Z}$ is the fiber dimension of $T^-_{C_j}M$, which is equal to $s(C_j)$ given by (3.4.6) in the statement of Theorem 3.4.3, and the right hand sides denote their zero-extensions to Ω_+ by abuse of notation.

Proof. We may assume that $\varphi_1 \equiv 0, \varphi_2 \equiv \varphi$ and $\mathcal{L}_i \simeq \mathbf{k}_M$ for $i = 1, 2$. Then $F_1 \simeq \mathbf{k}_{M\times[0,+\infty)}$ and $F_2 \simeq \mathbf{k}_{\{(x,t)|\varphi(x)+t\geq 0\}}$. Take a critical component C_j of φ satisfying $\varphi(C_j) = 0$. Then, by Proposition 2.1.16, we have

$$\begin{aligned}\mu hom(\mathbf{k}_{\{(x,t)|\varphi(x)+t\geq 0\}}, \mathbf{k}_{M\times[0,+\infty)})|_{\widehat{C_j}} &\simeq \pi_j^{-1} R\Gamma_{\{(x,t)|\varphi(x)+t\geq 0\}}(\mathbf{k}_{M\times[0,+\infty)})|_{C_j\times\{0\}} \\ &\simeq \pi_j^{-1} R\Gamma_{\{(x,t)|\varphi(x)+t\geq 0\}}(\mathbf{k}_{M\times\{0\}})|_{C_j\times\{0\}} \\ &\simeq \pi_j^{-1} R\Gamma_{\{\varphi\geq 0\}}(\mathbf{k}_M)|_{C_j}.\end{aligned} \tag{3.4.18}$$

Moreover, we obtain (cf. [ST92, Corollary 1.3])

$$R\Gamma_{\{\varphi\geq 0\}}(\mathbf{k}_M)|_{C_j} \simeq R\Gamma_{C_j}(\mathbf{k}_{T^-_{C_j}M})|_{C_j} \simeq \omega_{C_j/T^-_{C_j}M}, \tag{3.4.19}$$

which completes the proof. $\square$

In the case L_1 and L_2 intersect transversally, we also obtain the following:

Proposition 3.4.5. *Let $\mathbf{k}$ be any field and assume that L_1 and L_2 intersect transversally. For $p \in L_1 \cap L_2$ with $f_2(p) - f_1(p) = 0$, define $\widehat{p} := \{(\tau p, -f_1(p); \tau) \in T^*M \times T^*\mathbb{R} \mid \tau > 0\} \subset \Omega_+$ as a special case of (3.4.4). Then*

$$\mu hom(F_2, F_1)|_{\Omega_+} \simeq \bigoplus_{\substack{p\in L_1\cap L_2,\\ f_2(p)-f_1(p)=0}} \mathbf{k}_{\widehat{p}}[-s(p)], \tag{3.4.20}$$

where $s(p) \in \mathbb{Z}$ is given by (3.4.6) in the statement of Theorem 3.4.3.

Proof. In this case, the support of $\mu hom(F_2, F_1)|_{\Omega_+}$ is contained in $\bigsqcup_p \widehat{p}$ and each $\widehat{p}$ is contractible. Hence $\mu hom(F_2, F_1)|_{\Omega_+}$ has constant cohomology sheaves on $\bigsqcup_p \widehat{p}$. The rest is exactly the same as the proof of Theorem 3.4.3. $\square$

The relation between the degree $s(C)$ and the Maslov index will be explored in Section 3.C Appendix III.

For a general field $\mathbf{k}$ and the clean intersection of two compact exact Lagrangian submanifolds, we conjecture the following:

Conjecture 3.4.6. *Let $\mathbf{k}$ be any field. Under Assumption 3.4.1, keep the notation in Theorem 3.4.3. Let U be a tubular neighborhood of L_1 in T^*M and $q \colon U \to L_1$ be the natural projection. Define a function f_{21} on $V := U \cap L_2$ by $f_{21}(p) := f_2(p) - f_1 \circ q(p)$.*

(i) *The function f_{21} is a Morse-Bott function on V whose critical set is $L_1 \cap L_2$. In particular, C_j is a critical component of f_{21} satisfying $f_{21}(C_j) = 0$ for $j = 1, \dots, n_0$.*

(ii) *For $j = 1, \dots, n_0$, define $T^-_{C_j} V$ as the maximal subbundle of $T_{C_j} V$ where the restriction of the Hessian $\mathrm{Hess}(f_{21})|_{T^-_{C_j} V}$ is negative definite. Let moreover $\mathcal{L}_i := (F_i)_+ \in \mathrm{Mod}(\mathbf{k}_M)$ be the locally constant sheaf of rank 1 associated with the simple sheaf quantization F_i for $i = 1, 2$. Then*

$$\begin{aligned} \mu hom(F_2, F_1)|_{\Omega_+} &\simeq \bigoplus_{j=1}^{n_0} \pi_j^{-1} \left(\omega_{C_j / T^-_{C_j} V} \otimes \pi_M^{-1}(\mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes -1}) \right) \\ &\simeq \bigoplus_{j=1}^{n_0} \pi_j^{-1} \left(\mathrm{or}_{C_j / T^-_{C_j} V} \otimes \pi_M^{-1}(\mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes -1}) \right) [-s(C_j)], \end{aligned} \tag{3.4.21}$$

where $\pi_j \colon \widehat{C_j} \to C_j$ is the projection and the right hand sides denote their zero-extensions to Ω_+ by abuse of notation.

Note that Proposition 3.4.4 is a special case of the conjecture. The conjecture seems to be related to the local system given in Fukaya-Oh-Ohta-Ono [FOOO09a, FOOO09b].

Theorem 3.4.7. *Under Assumption 3.4.1, let $L_1 \cap L_2 = \bigsqcup_{j=1}^{n} C_j$ be the decomposition into connected components. Recall that for a component C of $L_1 \cap L_2$, one defines $f_{21}(C) := f_2(p) - f_1(p)$, taking some $p \in C$. Let moreover $a, b \in \mathbb{R}$ with $a < b$ or $a \in \mathbb{R}, b = +\infty$. Then*

$$\sum_{a \le f_{21}(C_j) < b} \dim_{\mathbb{F}_2} H^{k - s(C_j)}(C_j; \mathbb{F}_2) \ge \dim_{\mathbb{F}_2} H^k R\Gamma_{M \times [a,b)}((-\infty, b); \mathcal{H}om^\star(F_2, F_1)) \tag{3.4.22}$$

for any $k \in \mathbb{Z}$, where $s(C_j)$ is given by (3.4.6) in the statement of Theorem 3.4.3. In particular,

$$\sum_{j=1}^{n} \dim_{\mathbb{F}_2} H^{k - s(C_j)}(C_j; \mathbb{F}_2) \ge \dim_{\mathbb{F}_2} \mathrm{Hom}_{\mathcal{T}(M)}(F_2, F_1[k]) \tag{3.4.23}$$

for any $k \in \mathbb{Z}$. If L_1 and L_2 intersect transversally, the inequalities hold for any field $\mathbf{k}$, not only for $\mathbb{F}_2$.

Proof. Since the set $\{f_2(p) - f_1(p) \mid p \in L_1 \cap L_2\} \subset \mathbb{R}$ is finite, the conditions (1) and (2) in Proposition 3.3.3 are satisfied. Moreover, by Theorem 3.4.3, the condition (3) is also satisfied. Hence, the first assertion follows from Proposition 3.3.3 and Theorem 3.4.3. For the second assertion, by Proposition 2.2.11, it is enough to show that

$$\sum_{j=1}^{n} \dim_{\mathbb{F}_2} H^{k - s(C_j)}(C_j; \mathbb{F}_2) \ge \dim_{\mathbb{F}_2} \mathrm{Hom}_{\mathcal{D}(M)}(F_2, T_{c*} F_1[k]) \tag{3.4.24}$$

for any $c \in \mathbb{R}$ and any $k \in \mathbb{Z}$. This follows from Proposition 2.2.6 and the first assertion for the case $a = 0, b = +\infty$. The last assertion follows from Proposition 3.4.5. $\square$

Corollary 3.4.8 ([Nad09, Theorem 1.3.1] and [FSS08, Theorem 1])**.** *Under Assumption 3.4.1 and in the same notation as in Theorem 3.4.7, one has*

$$\sum_{j=1}^{n} \sum_{k \in \mathbb{Z}} \dim_{\mathbb{F}_2} H^k(C_j; \mathbb{F}_2) \ge \sum_{k \in \mathbb{Z}} \dim_{\mathbb{F}_2} H^k(M; \mathbb{F}_2). \tag{3.4.25}$$

If L_1 and L_2 intersect transversally, then

$$\#(L_1 \cap L_2) \geq \sum_{k \in \mathbb{Z}} \dim H^k(M; \mathcal{L}) \tag{3.4.26}$$

for any rank 1 locally constant sheaf $\mathcal{L} \in \mathrm{Mod}(\mathbf{k}_M)$ over any field $\mathbf{k}$.

Proof. It follows from Proposition 3.1.2 and Theorem 3.4.7. □

Remark 3.4.9. Assume $L_1 = L_2 = L$ and $f_1 = f_2$, and set $\mathcal{L}_i := (F_i)_+$ for $i = 1, 2$. Then $\{\mu hom(T_{c*}F_2, F_1)|_{\Omega_+}\}_c$ is concentrated at $c = 0$ and $\mu hom(F_2, F_1)|_{\Omega_+} \simeq \pi_{\widehat{L}}^{-1}(\mathcal{L}_2 \otimes \mathcal{L}_1^{\otimes -1})$, where $\pi_{\widehat{L}} \colon \widehat{L} \to M$ is the projection, over any field $\mathbf{k}$. Let $a, b \in \mathbb{R}$ with $a < b$ or $a \in \mathbb{R}, b = +\infty$. In this case, we obtain a more precise description of the complex $R\Gamma_{M \times [a,b)}(M \times (-\infty, b); \mathcal{H}om^\star(F_2, F_1))$, not only the Morse-Bott-type inequality. Namely, if $a \leq 0 < b$, using the concentration, Lemma 3.3.1, and Proposition 3.3.2, we have

$$\begin{aligned} & R\Gamma_{M \times [a,b)}(M \times (-\infty, b); \mathcal{H}om^\star(F_2, F_1)) \\ \simeq\ & R\Gamma(M \times \{0\}; R\Gamma_{M \times [0,+\infty)}(\mathcal{H}om^\star(F_2, F_1))|_{M \times \{0\}}) \\ \simeq\ & R\Gamma(\Omega_+; \mu hom(F_2, F_1)|_{\Omega_+}) \\ \simeq\ & R\Gamma\left(\widehat{L}; \pi_{\widehat{L}}^{-1}(\mathcal{L}_2 \otimes \mathcal{L}_1^{\otimes -1})\right). \end{aligned} \tag{3.4.27}$$

This is essentially one of the results of Guillermou [Gui12, Theorem 20.4].

Appendices to Chapter 3

3.A Appendix I: Degenerate Lagrangian intersections

In this section, using very simple examples, we briefly remark that our method can also deal with degenerate Lagrangian intersections. Until the end of this section, we set $\mathbf{k} = \mathbb{Q}$. We shall consider T^*S^1 and the intersection of the zero-section S^1 and the graph of an exact 1-form $L = \Gamma_{df}$. Let $F := \mathbf{k}_{S^1 \times [0,+\infty)}$ be the canonical sheaf quantization associated with the zero-section S^1 and $G := \mathbf{k}_{\{(x,t) \in S^1 \times \mathbb{R} | f(x)+t \geq 0\}}$ be that associated with L. Assume that the intersection of S^1 and L has only one possibly degenerate component C and it is transversal outside C. Then, by Proposition 3.3.3 and similar argument to the proof of Theorem 3.4.7, we obtain

$$\begin{aligned} &\#\{p \in S^1 \cap L \mid p \text{ is a transverse intersection point}\} \\ &+ \sum_{k \in \mathbb{Z}} \dim H^k R\Gamma\left(\Omega_+ \cap \pi^{-1}(C); \mu hom(F,G)|_{\Omega_+ \cap \pi^{-1}(C)}\right) \\ &\qquad \geq \sum_k \dim \operatorname{Hom}_{\mathcal{T}(S^1)}(F, G[k]) = \sum_k \dim H^k(S^1; \mathbf{k}_{S^1}) = 2. \end{aligned} \tag{3.A.1}$$

We calculate the "contribution" $R\Gamma\left(\Omega_+ \cap \pi^{-1}(C); \mu hom(F,G)|_{\Omega_+ \cap \pi^{-1}(C)}\right)$ from C in the following two typical examples.

First, we consider the case the intersection is as in Figure 3.A.1 in a neighborhood of C. In this case, G is isomorphic to the constant sheaf supported in the shaded closed subset in Figure 3.A.2 in a neighborhood of C.

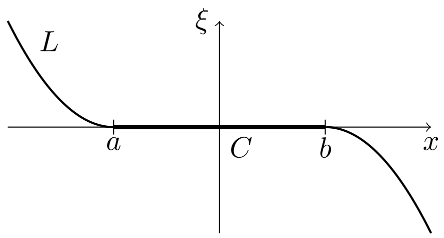

Figure 3.A.1: L in the first example

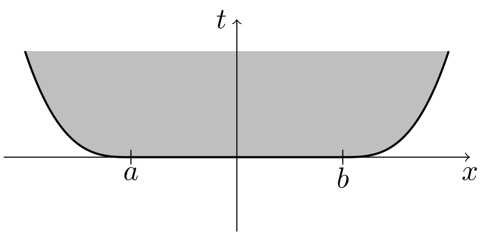

Figure 3.A.2: G in the first example

Hence, we find that $\mu hom(F,G)|_{\Omega_+ \cap \pi^{-1}(C)} \simeq \mathbf{k}_{[a,b] \times (0,+\infty)}$ and

$$R\Gamma(\Omega_+ \cap \pi^{-1}(C); \mu hom(F,G)|_{\Omega_+ \cap \pi^{-1}(C)}) \simeq R\Gamma([a,b]; \mathbf{k}_{[a,b]}) \simeq \mathbf{k}. \tag{3.A.2}$$

Thus, in this case, the contribution from C is 1 in (3.A.1), and the cardinality of the transverse intersection points is at least 1 as expected.

Next, we consider the case the intersection is as in Figure 3.A.3 in a neighborhood of C. The canonical sheaf quantization G associated with L is isomorphic to the constant sheaf supported in the shaded closed subset in Figure 3.A.4 in a neighborhood of C.

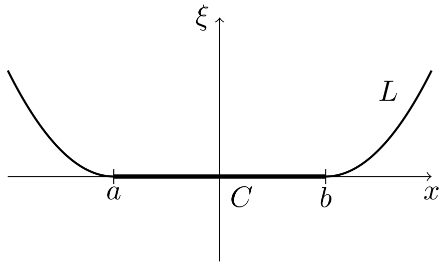

Figure 3.A.3: L in the second example

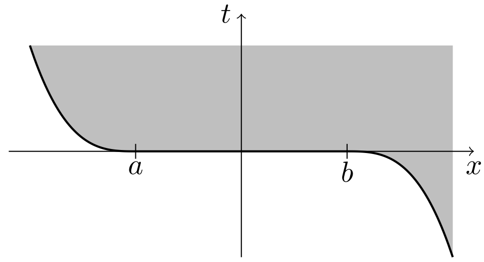

Figure 3.A.4: G in the second example

Therefore, in this case, we get $\mu hom(F,G)|_{\Omega_+ \cap \pi^{-1}(C)} \simeq \mathbf{k}_{[a,b)\times(0,+\infty)}$ and

$$R\Gamma(\Omega_+ \cap \pi^{-1}(C); \mu hom(F,G)|_{\Omega_+ \cap \pi^{-1}(C)}) \simeq R\Gamma_c([a,b); \mathbf{k}_{[a,b)}) \simeq 0. \tag{3.A.3}$$

Hence, the contribution from C is 0 in (3.A.1) and the cardinality of the transverse intersection points is at least 2 in the second case.

Remark 3.A.1. For $i = 1, 2$, let L_i be a compact connected exact Lagrangian submanifold and $f_i \colon L_i \to \mathbb{R}$ be a function satisfying $df_i = \alpha_{T^*M}|_{L_i}$. Let moreover F_i be a simple sheaf quantization associated with L_i and f_i. Proposition 3.3.3 says that the contribution from components on which $f_2(p) - f_1(p) = c$ is encoded in the sheaf $\mu hom(T_{c*}F_2, F_1)|_{\Omega_+}$ (even for possibly degenerate Lagrangian intersections). If the intersection is clean along a component C, then $\mu hom(T_{c*}F_2, F_1)|_{\Omega_+}$ is locally constant of rank 1 on the cone of C as in Lemma 3.4.2. However, as seen in the above examples, if the intersection is degenerate, then $\mu hom(T_{c*}F_2, F_1)|_{\Omega_+}$ is not necessarily locally constant.

3.B Appendix II: Functoriality of sheaf quantizations

In this section, we prove the "functoriality" of Guillermou's simple sheaf quantizations with respect to Hamiltonian isotopies. We remark that results in this section are independent of the results in Chapter 3 and not used for the proofs of them.

Let L be a compact connected exact Lagrangian submanifold of T^*M and f be a primitive of the Liouville form α_{T^*M}. We define the conification $\widehat{L}_f$ of L with respect to f as in (2.2.23). Let $\phi^H = (\phi^H_s)_s \colon T^*M \times I \to T^*M$ be the Hamiltonian isotopy generated by a compactly supported Hamiltonian function $H = (H_s)_s \colon T^*M \times I \to \mathbb{R}$, where I is an open interval containing $[0,1]$. We denote by X_s the associated Hamiltonian vector field on T^*M. The homogeneous lift $\widehat{\phi}$ of ϕ^H is described as follows (see [GKS12, Proposition A.6]):

$$\widehat{\phi}_1(x,t;\xi,\tau) = (x', t + u_1(x;\xi/\tau); \xi', \tau), \tag{3.B.1}$$

where $(x';\xi'/\tau) = \phi^H_1(x;\xi/\tau) = \phi^H_1(x;\xi/\tau)$ and $u_1 \colon T^*M \to \mathbb{R}$ is defined by

$$u_1(p) = \int_0^1 (H_s - \alpha_{T^*M}(X_s))(\phi^H_s(p))\, ds. \tag{3.B.2}$$

Hence we get

$$\begin{aligned}
\widehat{\phi}_1(\widehat{L}_f) &= \left\{ (x', t + u_1(x;\xi/\tau); \xi', \tau) \;\middle|\; \begin{array}{l} \tau > 0, \exists\, (x;\xi) \ \text{ s.t. } (x;\xi/\tau) \in L, \\ (x';\xi'/\tau) = \phi^H_1(x;\xi/\tau), t = -f(x;\xi/\tau) \end{array} \right\} \\
&= \left\{ (x', t'; \xi', \tau) \;\middle|\; \begin{array}{l} \tau > 0, (x';\xi'/\tau) \in \phi^H_1(L), \\ t' = -f \circ (\phi^H_1)^{-1}(x';\xi'/\tau) + u_1 \circ (\phi^H_1)^{-1}(x';\xi'/\tau) \end{array} \right\}.
\end{aligned}$$

On the other hand, we have equalities

$$\begin{aligned}(\phi_1^H)^*\alpha_{T^*M} - \alpha_{T^*M} &= \int_0^1 \left(\frac{d}{ds}(\phi_s^H)^*\alpha_{T^*M}\right) ds \\ &= \int_0^1 (\phi_s^H)^*(L_{X_s}\alpha_{T^*M})\, ds \\ &= \int_0^1 (\phi_s^H)^*(d\iota_{X_s}\alpha_{T^*M} + \iota_{X_s}d\alpha_{T^*M})\, ds \\ &= d\int_0^1 (\phi_s^H)^*(\alpha_{T^*M}(X_s) - H_s)\, ds = -du_1.\end{aligned} \tag{3.B.3}$$

Here, for a vector field X, L_X denotes the Lie derivative with respect to X, and the third equality follows from Cartan's formula. Moreover, the fourth equality follows from the definition of the Hamiltonian vector field: $d\alpha_{T^*M}(X_s, *) = -dH_s$. Hence, setting $\widetilde{f} := (f - u_1) \circ (\phi_1^H)^{-1} \colon \phi_1^H(L) \to \mathbb{R}$, we get

$$\begin{aligned}\alpha_{T^*M}|_{\phi_1^H(L)} &= ((\phi_1^H)^{-1})^*(\alpha_{T^*M}|_L - du_1|_L) \\ &= ((\phi_1^H)^{-1})^*(df - du_1|_L) = d\widetilde{f}.\end{aligned} \tag{3.B.4}$$

Thus we find that $\widetilde{f}$ is a primitive of α_{T^*M} on $\phi_1^H(L)$ and obtain the following:

Lemma 3.B.1. *One has*

$$\widehat{\phi}_1(\widehat{L}_f) = \widehat{\phi_1^H(L)}_{\widetilde{f}} \subset T^*(M \times \mathbb{R}). \tag{3.B.5}$$

Proposition 3.B.2. *Let $\mathcal{L} \in \mathrm{Mod}(\mathbf{k}_M)$ be a locally constant sheaf of rank 1 and F_L be the simple sheaf quantization of $\widehat{L}_f$ satisfying $F_{L+} \simeq \mathcal{L}$. Let $\phi^H \colon T^*M \times I \to T^*M$ be the Hamiltonian isotopy generated by a compactly supported Hamiltonian function H. Let moreover $\Psi_1^H \colon \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R}) \to \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R})$ be the functor associated with the time-one map ϕ_1^H (see (2.2.18)). Define $\widetilde{f} := (f - u_1) \circ (\phi_1^H)^{-1} \colon \phi_1^H(L) \to \mathbb{R}$ as above and denote by $\widehat{\phi_1^H(L)}_{\widetilde{f}}$ the conification of $\phi_1^H(L)$ with respect to $\widetilde{f}$. Let furthermore $F_{\phi_1^H(L)}$ be the simple sheaf quantization of $\widehat{\phi_1^H(L)}_{\widetilde{f}}$ satisfying $\left(F_{\phi_1^H(L)}\right)_+ \simeq \mathcal{L}$. Then*

$$\Psi_1^H(F_L) \simeq F_{\phi_1^H(L)}. \tag{3.B.6}$$

Proof. By Lemma 3.B.1, we have

$$\Psi_1^H(F_L) \in \mathbf{D}^{\mathrm{b}}_{\widehat{\phi_1^H(L)}_{\widetilde{f}} \cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R})}(M \times \mathbb{R}). \tag{3.B.7}$$

By the uniqueness of simple sheaf quantizations (Theorem 2.2.15), it remains to show that

$$\Psi_1^H(F_L)_- \simeq 0, \quad \Psi_1^H(F_L)_+ \simeq \mathcal{L}. \tag{3.B.8}$$

Let $\widehat{\phi} \colon \mathring{T}^*(M \times \mathbb{R}) \times I \to \mathring{T}^*(M \times \mathbb{R})$ be the associated homogeneous Hamiltonian isotopy and $K \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times M \times \mathbb{R} \times I)$ be its sheaf quantization. Consider the composite $K \circ F_L \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times I)$. By the compactness of L and the support of H, there exists $A \in \mathbb{R}_{>0}$ satisfying

$$\bigcup_{s\in I} \widehat{\phi}_s(\widehat{L}_f) \subset T^*(M \times (-A, A)). \tag{3.B.9}$$

Set $G := (K \circ F_L)|_{M \times (A,+\infty) \times I} \in \mathbf{D}^{\mathrm{b}}(M \times (A, +\infty) \times I)$. We shall show that

$$\mathrm{SS}(G) \subset T^*_{M \times (A,+\infty) \times I}(M \times (A, +\infty) \times I). \tag{3.B.10}$$

First, by Proposition 2.1.11, we have

$$\mathrm{SS}(K \circ F_L) \subset (\Lambda_{\widehat{\phi}} \circ \widehat{L}_f) \cup T^*_{M \times \mathbb{R} \times I}(M \times \mathbb{R} \times I). \tag{3.B.11}$$

By the definition of $\Lambda_{\widehat{\phi}}$ (see (2.2.4)), we obtain

$$(\Lambda_{\widehat{\phi}} \circ \widehat{L}) \cap (T^*_{M \times \mathbb{R}}(M \times \mathbb{R}) \times T^*I) \subset T^*_{M \times \mathbb{R} \times I}(M \times \mathbb{R} \times I). \tag{3.B.12}$$

Denote by $i_s \colon M \times \mathbb{R} \times \{s\} \hookrightarrow M \times \mathbb{R} \times I$ the closed embedding for any $s \in I$. Then, by the definition of $\Lambda_{\widehat{\phi}}$, we also have

$$(i_s)_d (i_s)_\pi^{-1} (\Lambda_{\widehat{\phi}} \circ \widehat{L}_f) = \widehat{\phi}_s(\widehat{L}_f). \tag{3.B.13}$$

Moreover, by (3.B.9), we get

$$\widehat{\phi}_s(\widehat{L}_f) \cap T^*(M \times (A, +\infty)) = \emptyset \tag{3.B.14}$$

for any $s \in I$. Hence the inclusion (3.B.10) follows from the above estimates (3.B.12), (3.B.13), and (3.B.14). Since I is contractible, we have $G \simeq q^{-1}(G|_{M \times (A,+\infty) \times \{0\}})$, where $q \colon M \times (A, +\infty) \times I \to M \times (A, +\infty)$ is the projection. In particular, we get

$$\begin{aligned}
\Psi^H_1(F_L)|_{M \times (A,+\infty)} &= G|_{M \times (A,+\infty) \times \{1\}} \\
&\simeq G|_{M \times (A,+\infty) \times \{0\}} \\
&\simeq (F_L)|_{M \times (A,+\infty)} \simeq \mathcal{L} \boxtimes \mathbf{k}_{(A,+\infty)}
\end{aligned} \tag{3.B.15}$$

and $\Psi^H_1(F_L)_+ \simeq \mathcal{L}$. A similar argument shows that $\Psi^H_1(F_L)_- \simeq 0$. □

3.C Appendix III: Relation to grading in Lagrangian Floer cohomology theory, by Tomohiro Asano

In this section, we relate the absolute grading of $\mathcal{H}om^\star$ to that of Lagrangian Floer cohomology.

3.C.1 Inertia index and Maslov index

In this subsection, we recall some properties of the inertia index and the Maslov index. First we list some properties of the inertia index.

Proposition 3.C.1 ([KS90, Theorem A.3.2]). *Let E be a symplectic vector space and denote by $\mathcal{L}(E)$ the Lagrangian Grassmannian of E. The inertia index $\tau \colon \mathcal{L}(E)^3 \to \mathbb{Z}$ satisfies the following properties.*

(i) *For any $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{L}(E)$, $\tau(\lambda_1, \lambda_2, \lambda_3) = -\tau(\lambda_2, \lambda_1, \lambda_3) = -\tau(\lambda_1, \lambda_3, \lambda_2)$.*

(ii) *The inertia index satisfies the "cocycle condition": for any quadruple $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathcal{L}(E)$,*

$$\tau(\lambda_1, \lambda_2, \lambda_3) = \tau(\lambda_1, \lambda_2, \lambda_4) + \tau(\lambda_2, \lambda_3, \lambda_4) + \tau(\lambda_3, \lambda_1, \lambda_4). \tag{3.C.1}$$

(iii) *If $\lambda_1, \lambda_2, \lambda_3$ move continuously in the Lagrangian Grassmannian $\mathcal{L}(E)$ so that $\dim(\lambda_1 \cap \lambda_2), \dim(\lambda_2 \cap \lambda_3), \dim(\lambda_3 \cap \lambda_1)$ remain constant, then $\tau(\lambda_1, \lambda_2, \lambda_3)$ remains constant.*

(iv) *Let E' be another symplectic vector space, and let $\lambda_1, \lambda_2, \lambda_3$ (resp. $\lambda_1', \lambda_2', \lambda_3'$) be a triple of Lagrangian subspaces of E (resp. E'). Then*

$$\tau_{E\oplus E'}(\lambda_1 \oplus \lambda_1', \lambda_2 \oplus \lambda_2', \lambda_3 \oplus \lambda_3') = \tau_E(\lambda_1, \lambda_2, \lambda_3) + \tau_{E'}(\lambda_1', \lambda_2', \lambda_3'). \tag{3.C.2}$$

Let M be a compact connected manifold without boundary and T^*M be its cotangent bundle. Let moreover $\mathcal{L}_{T^*M}$ be the fiber bundle over T^*M whose fiber is the Lagrangian Grassmannian, that is, $\mathcal{L}_{T^*M,p} = \mathcal{L}(T_pT^*M)$. Denote by $\lambda_\infty \colon T^*M \to \mathcal{L}_{T^*M}, p \mapsto T_pT^*_{\pi(p)}M$ be the section which assigns the fiber to p. A Lagrangian submanifold L of T^*M defines a section $\lambda_L \colon L \to \mathcal{L}_{T^*M}, p \mapsto T_pL$ over L.

Lemma 3.C.2. *For $i = 1, 2$, let L_i be a compact connected exact Lagrangian submanifold and $f_i \colon L_i \to \mathbb{R}$ be a function such that $df_i = \alpha_{T^*M}|_{L_i}$ and set $\Lambda_i := \widehat{L_i}_{f_i}$, the conification of L_i with respect to f_i. Let $p \in L_1 \cap L_2$ and assume $f_1(p) = f_2(p)$. Set $p' := (p, -f_1(p); 1) \in \Lambda_1 \cap \Lambda_2 \subset T^*(M \times \mathbb{R})$. Then*

$$\tau_{T_{p'}T^*(M\times\mathbb{R})}(\lambda_{\Lambda_2}(p'), \lambda_{\Lambda_1}(p'), \lambda_\infty(p')) = \tau_{T_pT^*M}(\lambda_{L_2}(p), \lambda_{L_1}(p), \lambda_\infty(p)). \tag{3.C.3}$$

Proof. Take a local homogeneous symplectic coordinate system $(x, t; \xi, \tau)$ on $T^*(M \times \mathbb{R})$. Using the coordinate system, we identify $T_{p'}T^*(M \times \mathbb{R})$ with $\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$. In this coordinate system, we get $\lambda_\infty(p') = 0 \times 0 \times \mathbb{R}^m \times \mathbb{R}$. Write $p = (x; \xi)$ by the coordinate. Then $\lambda_{\Lambda_i}(p')$ is spanned by

$$(0, 0; \xi, 1), (v, -Tf_i(v_i); \zeta_i, 0) \ \ ((v_i, \zeta_i) \in T_pL_i)\,. \tag{3.C.4}$$

For $r \in [0, 1]$, let $\lambda_{\Lambda_i}(p'; r)$ be the Lagrangian linear subspace spanned by

$$(0, 0; r\xi, 1), (v_i, -r \cdot Tf_i(v_i); \zeta_i, 0) \ \ ((v_i, \zeta_i) \in T_pL_i)\,. \tag{3.C.5}$$

Then, by Proposition 3.C.1 (iii), we have

$$\tau_{T_{p'}T^*(M\times\mathbb{R})}(\lambda_{\Lambda_2}(p'), \lambda_{\Lambda_1}(p'), \lambda_\infty(p')) = \tau_{T_{p'}T^*(M\times\mathbb{R})}(\lambda_{\Lambda_2}(p'; r), \lambda_{\Lambda_1}(p'; r), \lambda_\infty(p')) \tag{3.C.6}$$

for any $r \in [0, 1]$. Since $\lambda_{\Lambda_i}(p'; 0) = \lambda_{L_i}(p) \oplus \mathbb{R}\langle(0; 1)\rangle$, by Proposition 3.C.1 (iv), we obtain

$$\begin{aligned} &\tau_{T_{p'}T^*(M\times\mathbb{R})}(\lambda_{\Lambda_2}(p'), \lambda_{\Lambda_1}(p'), \lambda_\infty(p')) \\ &= \tau_{T_{p'}T^*(M\times\mathbb{R})}(\lambda_{\Lambda_2}(p'; 0), \lambda_{\Lambda_1}(p'; 0), \lambda_\infty(p')) \\ &= \tau_{T_pT^*M}(\lambda_{L_2}(p), \lambda_{L_1}(p), \lambda_\infty(p)). \end{aligned}$$ □

Next, we recall some properties of the Maslov index (see, for example, Leray [Ler81], Robbin-Salamon [RS93], and de Gosson [dG09]).

Proposition 3.C.3. *Let E be a symplectic vector space and denote by $\widetilde{\mathcal{L}}(E)$ the universal covering of the Lagrangian Grassmannian $\mathcal{L}(E)$ of E. For $\widetilde{\lambda_i} \in \widetilde{\mathcal{L}}(E) (i \in \mathbb{N})$, denote its projection to $\mathcal{L}(E)$ by λ_i. The Maslov index $\mu \colon \widetilde{\mathcal{L}}(E)^2 \to \frac{1}{2}\mathbb{Z}$ satisfies the following properties.*

(i) *For any $\widetilde{\lambda_1}, \widetilde{\lambda_2} \in \widetilde{\mathcal{L}}(E)$, $\mu(\widetilde{\lambda_1}, \widetilde{\lambda_2}) = -\mu(\widetilde{\lambda_2}, \widetilde{\lambda_1})$*

(ii) *The coboundary of μ is given by τ :* $\mu(\widetilde{\lambda_1}, \widetilde{\lambda_2}) + \mu(\widetilde{\lambda_2}, \widetilde{\lambda_3}) + \mu(\widetilde{\lambda_3}, \widetilde{\lambda_1}) = \frac{1}{2}\tau(\lambda_1, \lambda_2, \lambda_3)$

(iii) *If $\widetilde{\lambda_1}$ and $\widetilde{\lambda_2}$ move continuously in $\widetilde{\mathcal{L}}(E)$ so that* $\dim(\lambda_1 \cap \lambda_2)$ *remains constant, then* $\mu(\widetilde{\lambda_1}, \widetilde{\lambda_2})$ *remains constant.*

(iv) *For any $\widetilde{\lambda_1}, \widetilde{\lambda_2} \in \widetilde{\mathcal{L}}(E)$,* $\mu(\widetilde{\lambda_1}, \widetilde{\lambda_2}) \equiv \frac{1}{2}(\dim(\lambda_1 \cap \lambda_2) + \frac{1}{2}\dim E) \mod \mathbb{Z}$.

Remark 3.C.4. Notation for the Maslov index differs by authors. Our μ is equal to half of μ in [dG09]. Note that (ii) and (iii) of the above proposition determine the function $\mu\colon \widetilde{\mathcal{L}}(E)^2 \to \frac{1}{2}\mathbb{Z}$ uniquely.

3.C.2 Graded Lagrangian submanifolds and Maslov index

Next, we recall the notion of graded Lagrangian submanifolds due to Seidel [Sei00]. Denote by $\widetilde{\mathcal{L}}_{T^*M}$ the fiberwise universal cover of $\mathcal{L}_{T^*M}$ whose fiber over p is identified with the space of the homotopy classes of paths in $\mathcal{L}_{T^*M,p}$ from λ_∞. We also denote by $\mu\colon \widetilde{\mathcal{L}}_{T^*M} \times_{T^*M} \widetilde{\mathcal{L}}_{T^*M} \to \frac{1}{2}\mathbb{Z}$ the Maslov index on T^*M. For a Lagrangian submanifold L of T^*M, a *grading* of L is a lift $\widetilde{\lambda}\colon L \to \widetilde{\mathcal{L}}_{T^*M}$ of λ_L. A *graded Lagrangian submanifold* is a pair $(L, \widetilde{\lambda})$ consisting of a Lagrangian submanifold L and a grading $\widetilde{\lambda}$ of L.

$$\begin{tikzcd}
& \widetilde{\mathcal{L}}_{T^*M} \arrow[d, "\mathbb{Z}"] \\
& \mathcal{L}_{T^*M} \arrow[d, shift right] \\
L \arrow[uur, dashed, "\widetilde{\lambda}"] \arrow[ur, "\lambda_L"'] \arrow[r, hook] & T^*M \arrow[u, shift right, "\lambda_\infty"']
\end{tikzcd} \tag{3.C.7}$$

Now, let $(L_1, \widetilde{\lambda_1})$ and $(L_2, \widetilde{\lambda_2})$ be graded Lagrangian submanifolds of T^*M intersecting cleanly. For a connected component C of $L_1 \cap L_2$, we define the absolute grading $\mathrm{gr}(L_2, L_1; C)$ of C by taking $p \in C$ and

$$\mathrm{gr}(L_2, L_1; C) = \frac{1}{2}(\dim M - \dim C) - \mu(\widetilde{\lambda_2}(p), \widetilde{\lambda_1}(p)), \tag{3.C.8}$$

which induces the absolute grading of Lagrangian Floer cohomology. Note that by Proposition 3.C.3 (i) and (ii), the grading $\mathrm{gr}(L_2, L_1; C)$ is written as

$$\begin{aligned}
&\mathrm{gr}(L_2, L_1; C) \\
&= \frac{1}{2}(\dim M - \dim C) + \mu(\widetilde{\lambda_1}(p), \lambda_\infty(p)) + \mu(\lambda_\infty(p), \widetilde{\lambda_2}(p)) - \frac{1}{2}\tau(\lambda_2(p), \lambda_1(p), \lambda_\infty(p)) \\
&= \frac{1}{2}(\dim M - \dim C) + \mu(\lambda_\infty(p), \widetilde{\lambda_2}(p)) - \mu(\lambda_\infty(p), \widetilde{\lambda_1}(p)) - \frac{1}{2}\tau(T_pL_2, T_pL_1, \lambda_\infty(p)),
\end{aligned} \tag{3.C.9}$$

where the point $\lambda_\infty(p)$ is regarded as (the homotopy class of) the constant path.

3.C.3 Shifts of simple sheaf quantizations

Let L be a compact exact Lagrangian submanifold of T^*M and $f\colon L \to \mathbb{R}$ be a primitive of the Liouville 1-form. Denote by $\widehat{L} \subset T^*(M \times \mathbb{R})$ the conification of L with respect to f and let $F \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R})$ be a simple sheaf quantization of $\widehat{L}$. By Theorem 2.2.15, the object F is simple along $\widehat{L}$ and the shift of F at a point of $\widehat{L}$ defines a function $d\colon \widehat{L} \to \frac{1}{2}\mathbb{Z}$. Since $d(c \cdot p') = d(p')$ for any $p' \in \widehat{L}$ and $c \in \mathbb{R}_{>0}$, and $\widehat{L}/\mathbb{R}_{>0} = L$, we also regard d as a function $L \to \frac{1}{2}\mathbb{Z}$.

Proposition 3.C.5. *There is a grading* $\widetilde{\lambda}\colon L \to \widetilde{\mathcal{L}}_{T^*M}$ *such that*

$$\mu(\lambda_\infty(p), \widetilde{\lambda}(p)) + \frac{1}{2}(\dim M + 1) = d(p), \tag{3.C.10}$$

where λ_∞ *denotes the constant path.*

Proof. Let $U_L \subset \mathcal{L}_{T^*M}|_L$ be the open subset of Lagrangian Grassmannian restricted over L consisting of Lagrangian subspaces transversal to λ_∞ and λ_L. Let moreover $U \subset U_L$ be a connected open subset of U_L which has a local section $\gamma\colon \pi(U) \to U$. Note that the set of such $\pi(U)$ covers L. For $p \in L$, we set $p' := (p, -f(p); 1) \in \widehat{L}$. Take a local section $\gamma'\colon \rho^{-1}(\pi(U)) \to \mathcal{L}_{T^*(M\times\mathbb{R})}|_{\widehat{L}}$ so that $\gamma'(p') = \gamma(p) \oplus \mathbb{R}\langle(1;0)\rangle \subset T_pT^*M \oplus T_{(-f(p);1)}T^*\mathbb{R}$ holds for every $p \in \pi(U)$. By Proposition 3.C.3 and the same homotopy $\lambda_{\widehat{L}}(p';r)$ as in the proof of Lemma 3.C.2, we get

$$\frac{1}{2}\tau(\lambda_\infty(p'), \lambda_{\widehat{L}}(p'), \gamma'(p')) = \mu(\lambda_\infty(p), \widetilde{\lambda}(p)) + \mu(\widetilde{\lambda}(p), \widetilde{\gamma}(p)) + \mu(\widetilde{\gamma}(p), \lambda_\infty(p)), \tag{3.C.11}$$

where $\widetilde{\gamma}$ and $\widetilde{\lambda}$ are locally defined lifts of γ and λ_L. Since the image of γ is contained in a connected component of U_L, both $\mu(\widetilde{\lambda}(p), \widetilde{\gamma}(p))$ and $\mu(\widetilde{\gamma}(p), \lambda_\infty(p))$ are constant on $\pi(U)$. The difference of the shifts can be calculated as

$$\begin{aligned} d(p) - d(q) &= \frac{1}{2}\left(\tau(\lambda_\infty(p'), \lambda_{\widehat{L}}(p'), \gamma(p')) - \tau(\lambda_\infty(q'), \lambda_{\widehat{L}}(q'), \gamma(q'))\right) \\ &= \mu(\lambda_\infty(p), \widetilde{\lambda}(p)) - \mu(\lambda_\infty(q), \widetilde{\lambda}(q)) \end{aligned} \tag{3.C.12}$$

(see [Gui12, Section 8]). Hence the function $d(p) - \mu(\lambda_\infty(p), \widetilde{\lambda}(p))$ is constant on $\pi(U)$ with value in $\frac{1}{2}\mathbb{Z}$. Hence $\widetilde{\lambda}$ can be extended to the whole of L and L has a grading. Moreover, since $\mu(\widetilde{\lambda}(p), \gamma(p)) \equiv \mu(\gamma(p), \lambda_\infty(p)) \equiv \frac{1}{2}\dim M \mod \mathbb{Z}$, we have

$$d(p) - \mu(\lambda_\infty(p), \widetilde{\lambda}(p)) \equiv \frac{1}{2}\dim(M \times \mathbb{R}) = \frac{1}{2}(\dim M + 1) \mod \mathbb{Z}, \tag{3.C.13}$$

which completes the proof. □

Next, we consider the degree of $\mathcal{H}om^\star(F_2, F_1)$. Let L_1 and L_2 be compact exact Lagrangian submanifolds of T^*M intersecting cleanly. For $i = 1, 2$, take a primitive $f_i\colon L_i \to \mathbb{R}$ of the Liouville 1-form and denote by $\widehat{L}_i$ the conification of L_i with respect to f_i. Let $F_i \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R})$ be a simple sheaf quantization of $\widehat{L}_i$. We also denote by $d_i\colon L_i \to \frac{1}{2}\mathbb{Z}$ the function which assigns the shift of F_i. Then, by Theorem 3.4.3, the degree associated with a component C of $L_1 \cap L_2$ in $\mathcal{H}om^\star(F_2, F_1)$ is given by

$$d_2(p) - d_1(p) + \frac{1}{2}(\dim M - \dim C) - \frac{1}{2}\tau(T_pL_2, T_pL_1, \lambda_\infty(p)) \tag{3.C.14}$$

for any $p \in C$. Thus, combining Proposition 3.C.5 with (3.C.9) and (3.C.14), we obtain the following theorem.

Theorem 3.C.6. *For* $i = 1, 2$, *let* $\widetilde{\lambda}_i\colon L_i \to \widetilde{\mathcal{L}}_{T^*M}$ *be the grading of* L_i *given in Proposition 3.C.5. Then the degree associated with a component* C *of* $L_1 \cap L_2$ *in* $\mathcal{H}om^\star(F_2, F_1)$ *is equal to* $\mathrm{gr}(L_2, L_1; C)$.

Chapter 4

Persistence-like distance on Tamarkin's category and symplectic displacement energy

In this chapter, we introduce a pseudo-distance on Tamarkin's category $\mathcal{D}(M)$. We prove that the distance between an object and its Hamiltonian deformation via sheaf quantization is less than or equal to the Hofer norm of the Hamiltonian function. Using the result, we also show a quantitative version of Tamarkin's non-displaceability theorem, which gives a lower bound of the displacement energy. In this chapter, we do *not* assume that M is compact.

4.1 Complements on torsion objects

Torsion objects were introduced by Tamarkin [Tam08] and the category of torsion objects was systematically studied by Guillermou-Schapira [GS14]. In this section, we introduce the notion of c-torsion for $c \in \mathbb{R}_{\geq 0}$, which we will use to estimate the displacement energy. Note that the results in this section are essentially due to Guillermou-Schapira [GS14].

First, we recall the microlocal cut-off lemma in a general setting. Let V be a finite-dimensional real vector space and γ be a closed convex cone with $0 \in \gamma$ in V. Define the maps

$$\begin{gathered}\tilde{q}_1, \tilde{q}_2, s_V \colon M \times V \times V \longrightarrow M \times V, \\ \tilde{q}_1(x, v_1, v_2) = (x, v_1),\ \tilde{q}_2(x, v_1, v_2) = (x, v_2),\ s_V(x, v_1, v_2) = (x, v_1 + v_2).\end{gathered} \tag{4.1.1}$$

For $F \in \mathbf{D}^{\mathrm{b}}(M \times V)$, the canonical morphism $\mathbf{k}_{M\times\gamma} \to \mathbf{k}_{M\times\{0\}}$ induces the morphism

$$Rs_{V*}(\tilde{q}_1^{-1}\mathbf{k}_{M\times\gamma} \otimes \tilde{q}_2^{-1}F) \longrightarrow Rs_{V*}(\tilde{q}_1^{-1}\mathbf{k}_{M\times\{0\}} \otimes \tilde{q}_2^{-1}F) \simeq F. \tag{4.1.2}$$

The following is called the microlocal cut-off lemma due to Kashiwara-Schapira [KS90, Proposition 5.2.3], which is reformulated by Guillermou-Schapira [GS14, Proposition 3.9]. For a cone γ with $0 \in \gamma$ in V, we define its polar cone $\gamma^\circ \subset V^*$ by

$$\gamma^\circ := \{w \in V^* \mid \langle w, v\rangle \geq 0 \text{ for any } v \in \gamma\}. \tag{4.1.3}$$

We also identify T^*V with $V \times V^*$.

Proposition 4.1.1. *Let V be a finite-dimensional real vector space and γ be a closed convex cone with $0 \in \gamma$ in V. Then, for $F \in \mathbf{D}^{\mathrm{b}}(M \times V)$, $\mathrm{SS}(F) \subset T^*M \times V \times \gamma^\circ$ if and only if the morphism $Rs_{V*}(\tilde{q}_1^{-1}\mathbf{k}_{M\times\gamma} \otimes \tilde{q}_2^{-1}F) \to F$ is an isomorphism.*

If $\mathrm{Int}(\gamma) \neq \emptyset$, then $\tilde{q}_1^{-1}\mathbf{k}_{M\times\gamma} \simeq R\mathcal{H}om(\mathbf{k}_{M\times\mathrm{Int}(\gamma)\times V}, \mathbf{k}_{M\times V\times V})$. Hence, by Proposition 2.1.7 (ii), we have

$$Rs_{V*}(\tilde{q}_1^{-1}\mathbf{k}_{M\times\gamma} \otimes \tilde{q}_2^{-1}F) \simeq Rs_{V*}R\Gamma_{M\times\mathrm{Int}(\gamma)\times V}(\tilde{q}_2^{-1}F). \tag{4.1.4}$$

Now we return to the case $V = \mathbb{R}$ and $\gamma = [0, +\infty)$. Let $F \in \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R})$. Then, by Proposition 4.1.1, $F \in \mathbf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(M\times\mathbb{R})$ if and only if

$$Rs_*(\tilde{q}_1^{-1}\mathbf{k}_{M\times[0,+\infty)} \otimes \tilde{q}_2^{-1}F) \xrightarrow{\sim} F. \tag{4.1.5}$$

Recall that $T_c\colon M\times\mathbb{R} \to M\times\mathbb{R}$ denotes the translation map $(x,t) \mapsto (x, t+c)$ for $c \in \mathbb{R}$. For $F \in \mathbf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(M\times\mathbb{R})$, by (4.1.5), we have

$$Rs_*(\tilde{q}_1^{-1}\mathbf{k}_{M\times[c,+\infty)} \otimes \tilde{q}_2^{-1}F) \xrightarrow{\sim} T_{c*}F \tag{4.1.6}$$

for any $c \in \mathbb{R}$. Hence, for $c \leq d$, the canonical morphism $\mathbf{k}_{M\times[c,+\infty)} \to \mathbf{k}_{M\times[d,+\infty)}$ induces a morphism of functors from $\mathbf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(M\times\mathbb{R})$ to $\mathbf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(M\times\mathbb{R})$:

$$\tau_{c,d}\colon T_{c*} \longrightarrow T_{d*}. \tag{4.1.7}$$

Definition 4.1.2 (cf. [Tam08])**.** Let $c \in \mathbb{R}_{\geq 0}$. An object $F \in \mathbf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(M\times\mathbb{R})$ is said to be *c-torsion* if the morphism $\tau_{0,c}(F)\colon F \to T_{c*}F$ is zero.

Note that a c-torsion object is c'-torsion for any $c' \geq c$. Recall also that the category $\mathcal{D}(M) = \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R}; \{\tau > 0\})$ is regarded as a full subcategory of $\mathbf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(M\times\mathbb{R})$ via the projector $P_l\colon \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R}; \{\tau > 0\}) \to {}^{\perp}\mathbf{D}^{\mathrm{b}}_{\{\tau\leq 0\}}(M\times\mathbb{R})$ or $P_r\colon \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R}; \{\tau > 0\}) \to \mathbf{D}^{\mathrm{b}}_{\{\tau\leq 0\}}(M\times\mathbb{R})^{\perp}$. Hence we can define c-torsion objects in $\mathcal{D}(M)$.

Let I be an open interval of $\mathbb{R}$ containing the closed interval $[0,1]$. We recall a result on sheaves over $M\times\mathbb{R}\times I$ due to Guillermou-Schapira [GS14]. We denote by $(t;\tau)$ the homogeneous symplectic coordinate system on $T^*\mathbb{R}$ and by $(s;\sigma)$ that on T^*I. For $a, b \in \mathbb{R}_{>0}$, we set

$$\gamma_{a,b} := \{(\tau,\sigma) \in \mathbb{R}^2 \mid -a\tau \leq \sigma \leq b\tau\} \subset \mathbb{R}^2. \tag{4.1.8}$$

Let $q\colon M\times\mathbb{R}\times I \to M\times\mathbb{R}$ be the projection. We identify $T^*(\mathbb{R}\times I)$ with $(\mathbb{R}\times I)\times\mathbb{R}^2$.

Proposition 4.1.3 (cf. [GS14, Proposition 5.9])**.** *Let $\mathcal{H} \in \mathbf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(M\times\mathbb{R}\times I)$ and $s_1 < s_2$ be in I. Assume that there exist $a, b, r \in \mathbb{R}_{>0}$ satisfying*

$$\mathrm{SS}(\mathcal{H}) \cap \pi^{-1}(M\times\mathbb{R}\times(s_1 - r, s_2 + r)) \subset T^*M \times (\mathbb{R}\times I) \times \gamma_{a,b}. \tag{4.1.9}$$

Then $Rq_*(\mathcal{H}_{M\times\mathbb{R}\times[s_1,s_2)})$ is $(a(s_2-s_1)+\varepsilon)$-torsion and $Rq_*(\mathcal{H}_{M\times\mathbb{R}\times(s_1,s_2]})$ is $(b(s_2-s_1)+\varepsilon)$-torsion for any $\varepsilon \in \mathbb{R}_{>0}$.

Proof. The proof is essentially the same as that of [GS14, Proposition 5.9]. For the convenience of the reader, we give a detailed proof again. We only consider $Rq_*(\mathcal{H}_{M\times\mathbb{R}\times[s_1,s_2)})$. and omit the proof for the other case.
(a) Choose a diffeomorphism $\varphi\colon (s_1 - r, s_2 + r) \xrightarrow{\sim} \mathbb{R}$ satisfying $\varphi|_{[s_1,s_2]} = \mathrm{id}_{[s_1,s_2]}$ and $d\varphi(s) \geq 1$ for any $s \in (s_1 - r, s_2 + r)$. Set $\Phi := \mathrm{id}_M \times \mathrm{id}_{\mathbb{R}} \times \varphi\colon M\times\mathbb{R}\times(s_1 - r, s_2 + r) \xrightarrow{\sim} M\times\mathbb{R}\times\mathbb{R}$ and $\mathcal{H}' := \Phi_*\mathcal{H}|_{M\times\mathbb{R}\times(s_1-r,s_2+r)} \in \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R}\times\mathbb{R})$. Then, by the assumption on φ, we have

$$\mathrm{SS}(\mathcal{H}') \subset T^*M \times (\mathbb{R}\times\mathbb{R}) \times \gamma_{a,b} \tag{4.1.10}$$

and $Rq_*(\mathcal{H}_{M\times\mathbb{R}\times[s_1,s_2)}) \simeq Rq_*(\mathcal{H}'_{M\times\mathbb{R}\times[s_1,s_2)})$. Here q in the right hand side denotes the projection $M\times\mathbb{R}\times\mathbb{R} \to M\times\mathbb{R}, (x,t,s) \mapsto (x,t)$ by abuse of notation. Therefore, replacing $\mathcal{H}$ with $\mathcal{H}'$, we may assume $I=\mathbb{R}$ and (4.1.10).
(b) Set $V=\mathbb{R}^2$ and denote by $s_V\colon M\times V\times V \to M\times V$ the addition map. By Proposition 4.1.1, we have

$$Rs_{V*}R\varGamma_{M\times \mathrm{Int}(\gamma^{\circ}_{a,b})\times V}(\tilde{q}_2^{-1}\mathcal{H}) \simeq \mathcal{H}. \tag{4.1.11}$$

Note that $\mathrm{Int}(\gamma^{\circ}_{a,b}) = \{(t,s)\in\mathbb{R}^2 \mid -b^{-1}t < s < a^{-1}t\}$. Since $\mathrm{SS}(\mathbf{k}_{M\times\mathbb{R}\times[s_1,s_2)}) \subset T^*_M M\times T^*_{\mathbb{R}}\mathbb{R}\times T^*\mathbb{R}$, Proposition 2.1.7 (ii) gives $\mathcal{H}\otimes\mathbf{k}_{M\times\mathbb{R}\times[s_1,s_2)} \simeq R\varGamma_{M\times\mathbb{R}\times(s_1,s_2]}(\mathcal{H})$. Combining with (4.1.11), we obtain

$$Rq_*(\mathcal{H}_{M\times\mathbb{R}\times[s_1,s_2)}) \simeq Rq_*Rs_{V*}R\varGamma_{M\times D}(\tilde{q}_2^{-1}\mathcal{H}), \tag{4.1.12}$$

where $D=\mathrm{Int}(\gamma^{\circ}_{a,b})\times V \cap \{(t,s,t',s') \mid s_1 < s+s' \leq s_2\}$. Consider the commutative diagram

$$\begin{tikzcd}
& M\times V\times V \arrow[r, "s_V"] \arrow[dl, "\tilde{q}_2"'] \arrow[d, "\mathrm{id}_M\times\tilde{q}"] & M\times V \arrow[d, "q"] \\
M\times V & M\times\mathbb{R}\times V \arrow[l, "q_2"] \arrow[r, "\tilde{s}"'] & M\times\mathbb{R},
\end{tikzcd} \tag{4.1.13}$$

where $\tilde{q}(t,s,t',s') = (t,t',s'), q_2(x,t,t',s') = (x,t',s')$, and $\tilde{s}(x,t,t',s') = (x,t+t')$. By the adjunction of $(\mathrm{id}_M\times\tilde{q})_!$ and $(\mathrm{id}_M\times\tilde{q})^!$, we get

$$\begin{aligned} Rq_*(\mathcal{H}_{\mathbb{R}\times[s_1,s_2)}) &\simeq R\tilde{s}_*(\mathrm{id}_M\times\tilde{q})_*\,R\mathcal{H}om(\mathbf{k}_{M\times D}, (\mathrm{id}_M\times\tilde{q})^! q_2^{-1}\mathcal{H})[-1] \\ &\simeq R\tilde{s}_*\,R\mathcal{H}om(\mathbf{k}_M\boxtimes R\tilde{q}_!\mathbf{k}_D, q_2^{-1}\mathcal{H})[-1]. \end{aligned} \tag{4.1.14}$$

Here, we used $\tilde{q}^! \simeq \tilde{q}^{-1}[1]$ for the first isomorphism.
(c) Thorough the isomorphism (4.1.11), $\tau_{0,c}(\mathcal{H})$ is induced by the canonical morphism $\mathbf{k}_{\widetilde{T}_c(\mathrm{Int}(\gamma^{\circ}_{a,b})\times V)} \to \mathbf{k}_{\mathrm{Int}(\gamma^{\circ}_{a,b})\times V}$, where $\widetilde{T}_c(t,s,t',s') = (t+c,s,t',s')$. Moreover through (4.1.14), we find that $\tau_{0,c}(Rq_*(\mathcal{H}_{\mathbb{R}\times[s_1,s_2)}))$ is induced by the morphism $\mathbf{k}_{\widetilde{T}_c(D)} \to \mathbf{k}_D$. In order to prove that $R\tilde{q}_!\mathbf{k}_{\widetilde{T}_c(D)} \to R\tilde{q}_!\mathbf{k}_D$ is zero morphism for $c > a(s_2-s_1)$, we will show that $R\tilde{q}_!\mathbf{k}_D$ and $R\tilde{q}_!\mathbf{k}_{\widetilde{T}_c(D)}$ have disjoint supports.
(d) For a point $(t,t',s')\in\mathbb{R}\times V$, $\tilde{q}^{-1}(t,t',s')\cap D = \emptyset$ if $t\leq 0$ and

$$\tilde{q}^{-1}(t,t',s')\cap D = (s_1-s', s_2-s'] \cap (-b^{-1}t, a^{-1}t) \tag{4.1.15}$$

if $t>0$. This set is an empty set or a half closed interval if $t\notin (a(s_1-s'), a(s_2-s')]$. Thus $\mathrm{Supp}(R\tilde{q}_!\mathbf{k}_D)$ is contained in $\{(t,t',s') \mid t\in[a(s_1-s'),a(s_2-s')]\}$. Similarly, $\mathrm{Supp}(R\tilde{q}_!\mathbf{k}_{\widetilde{T}_c(D)})$ is contained in $\{(t,t',s') \mid t\in[a(s_1-s')+c, a(s_2-s')+c]\}$. Hence $\mathrm{Supp}(R\tilde{q}_!\mathbf{k}_D)$ and $\mathrm{Supp}(R\tilde{q}_!\mathbf{k}_{\widetilde{T}_c(D)})$ are disjoint for $c > a(s_2-s_1)$. □

4.2 Pseudo-distance on Tamarkin's category

In this section, we introduce a pseudo-distance on Tamarkin's category $\mathcal{D}(M)$. This enables us to discuss the relation between possibly non-torsion objects in $\mathcal{D}(M)$. Recall again that $\mathcal{D}(M)$ is regarded as a full subcategory of $\mathbf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(M\times\mathbb{R})$ via the projector P_l or P_r.

Definition 4.2.1. Let $F, G \in \mathbf{D}^{\mathrm{b}}_{\{\tau \geq 0\}}(M \times \mathbb{R})$ and $a, b \in \mathbb{R}_{\geq 0}$. Then F is said to be (a,b)-*isomorphic* to G if there exist morphisms $\alpha, \delta \colon F \to T_{a*}G$ and $\beta, \gamma \colon G \to T_{b*}F$ satisfying the following conditions:

(1) $F \xrightarrow{\alpha} T_{a*}G \xrightarrow{T_{a*}\beta} T_{a+b*}F$ is equal to $\tau_{0,a+b}(F) \colon F \to T_{a+b*}F$ and $G \xrightarrow{\gamma} T_{b*}F \xrightarrow{T_{b*}\delta} T_{a+b*}G$ is equal to $\tau_{0,a+b}(G) \colon G \to T_{a+b*}G$,

(2) $\tau_{a,2a}(G) \circ \alpha = \tau_{a,2a}(G) \circ \delta$ and $\tau_{b,2b}(F) \circ \beta = \tau_{b,2b}(F) \circ \gamma$.

Remark 4.2.2. Let $F, G \in \mathbf{D}^{\mathrm{b}}_{\{\tau \geq 0\}}(M \times \mathbb{R})$ and $a, b \in \mathbb{R}_{\geq 0}$.

(i) F is (a,b)-isomorphic to G if and only if G is (b,a)-isomorphic to F.

(ii) If F is (a,b)-isomorphic to G, then F is (a',b')-isomorphic to G for any $a' \geq a, b' \geq b$.

(iii) F is $(0,0)$-isomorphic to G if and only if $F \simeq G$.

(iv) F is (a,b)-isomorphic to 0 if and only if F is $(a+b)$-torsion.

Remark 4.2.3. Let $F, G \in \mathcal{D}(M)$. By Proposition 2.2.11, if F is (a,b)-isomorphic to G for some $a, b \in \mathbb{R}_{\geq 0}$, then $F \simeq G$ in $\mathcal{T}(M)$.

For the relation to the notion of "a-isomorphic" recently introduced by Kashiwara-Schapira [KS17] and the interleaving distance for persistence modules, see Remark 4.2.7

Lemma 4.2.4. *If F_0 is (a_0,b_0)-isomorphic to F_1 and F_1 is (a_1,b_1)-isomorphic to F_2, then F_0 is (a_0+a_1, b_0+b_1)-isomorphic to F_2.*

Proof. By assumption, for $i = 0,1$, there exist morphisms

$$\alpha_i, \delta_i \colon F_i \to T_{a_i *}F_{i+1}, \quad \beta_i, \gamma_i \colon F_{i+1} \to T_{b_i *}F_i \tag{4.2.1}$$

satisfying

$$\begin{aligned} T_{a_i *}\beta_i \circ \alpha_i &= \tau_{0,a_i+b_i}(F_i), & T_{b_i *}\delta_i \circ \gamma_i &= \tau_{0,a_i+b_i}(F_{i+1}), \\ \tau_{a_i,2a_i}(F_{i+1}) \circ \alpha_i &= \tau_{a_i,2a_i}(F_{i+1}) \circ \delta_i, & \tau_{b_i,2b_i}(F_i) \circ \beta_i &= \tau_{b_i,2b_i}(F_i) \circ \gamma_i. \end{aligned} \tag{4.2.2}$$

We set

$$\begin{aligned} \alpha &:= T_{a_0 *}\alpha_1 \circ \alpha_0 \colon F_0 \to T_{a_0+a_1 *}F_2, & \beta &:= T_{b_1 *}\beta_0 \circ \beta_1 \colon F_2 \to T_{b_0+b_1 *}F_1, \\ \gamma &:= T_{b_1 *}\gamma_0 \circ \gamma_1 \colon F_2 \to T_{b_0+b_1 *}F_1, & \delta &:= T_{a_0 *}\delta_1 \circ \delta_0 \colon F_0 \to T_{a_0+a_1 *}F_2. \end{aligned} \tag{4.2.3}$$

Let us consider the following commutative diagram:

$$\begin{tikzcd}
 & & & & F_0 \arrow[dll, "\alpha_0"'] \arrow[dd, "\tau_{0,a_0+b_0}(F_0)"] \\
 & & T_{a_0 *}F_1 \arrow[dll, "T_{a_0 *}\alpha_1"'] \arrow[drr, "T_{a_0 *}\beta_0"] \arrow[dd, "\tau_{a_0,a_0+a_1+b_1}(F_1)"] & & \\
T_{a_0+a_1 *}F_2 \arrow[drr, "T_{a_0+a_1 *}\beta_1"'] & & & & T_{a_0+b_0 *}F_0 \arrow[dd, "\tau_{a_0+b_0,a_0+a_1+b_0+b_1}(F_0)"] \\
 & & T_{a_0+a_1+b_1 *}F_1 \arrow[drr, "T_{a_0+a_1+b_1 *}\beta_0"'] & & \\
 & & & & T_{a_0+a_1+b_1+b_2 *}F_0.
\end{tikzcd}$$

The two triangles in the diagram commute by (4.2.2). Since we obtain the square by applying $\tau_{a_0,a_0+a_1+b_1}$ to β_0, it also commutes. Hence we have $T_{a_0+a_1*}\beta\circ\alpha = \tau_{0,a_0+a_1+b_0+b_1}(F_0)$. Similarly, we get $T_{b_0+b_1*}\delta\circ\gamma = \tau_{0,a_0+a_1+b_0+b_1}(F_2)$. Moreover, by (4.2.2) again, we obtain

$$\begin{aligned}
&\tau_{a_0+a_1,2a_0+2a_1}(F_2)\circ\alpha \\
&= \tau_{2a_0+a_1,2a_0+2a_1}(F_2)\circ\tau_{a_0+a_1,2a_0+a_1}(F_2)\circ T_{a_0*}\alpha_1\circ\alpha_0 \\
&= T_{2a_0*}\tau_{a_1,2a_1}(F_2)\circ T_{2a_0*}\alpha_1\circ\tau_{a_0,2a_0}(F_1)\circ\alpha_0 \\
&= T_{2a_0*}\tau_{a_1,2a_1}(F_2)\circ T_{2a_0*}\delta_1\circ\tau_{a_0,2a_0}(F_1)\circ\delta_0 \\
&= \tau_{a_0+a_1,2a_0+2a_1}(F_2)\circ\delta.
\end{aligned} \tag{4.2.4}$$

Similarly, we get $\tau_{b_0+b_1,2b_0+2b_1}(F_0)\circ\beta = \tau_{b_0+b_1,2b_0+2b_1}(F_0)\circ\gamma$. This completes the proof. $\square$

A similar argument to the proof of Lemma 4.2.4 shows the following lemma.

Lemma 4.2.5. *Let $F_0, F_1, G_0, G_1 \in \mathbf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(M\times\mathbb{R})$ and assume that F_0 is (a_F, b_F)-isomorphic to F_1 and G_0 is (a_G, b_G)-isomorphic to G_1. Then $\mathcal{H}om^\star(F_0, G_0)$ is $(b_F + a_G, a_F + b_G)$-isomorphic to $\mathcal{H}om^\star(F_1, G_1)$.*

Now we define a pseudo-distance on Tamarkin's category $\mathcal{D}(M)$.

Definition 4.2.6. For object $F, G \in \mathcal{D}(M)$, one defines

$$d_{\mathcal{D}(M)}(F,G) := \inf\{a+b\in\mathbb{R}_{\geq 0} \mid a,b\in\mathbb{R}_{\geq 0}, F \text{ is } (a,b)\text{-isomorphic to } G\}, \tag{4.2.5}$$

and calls $d_{\mathcal{D}(M)}$ the *translation distance*.

Remark 4.2.7. (i) Definition 4.2.1 and Definition 4.2.6 are inspired by the notion of "a-isomorphic" and the convolution distance on the derived categories of sheaves on vector spaces recently introduced by Kashiwara-Schapira [KS17]. In fact, if $M = \mathrm{pt}$ and F is (a,b)-isomorphic to G, then F and G are $2\max\{a,b\}$-isomorphic in the sense of Kashiwara-Schapira [KS17].

(ii) The translation distance $d_{\mathcal{D}(M)}$ is similar to the interleaving distance for persistence modules introduced by [CCSG+09] (see also [CdSGO16]). Their definition of "a-interleaved" corresponds to Definition 4.2.1 with $a = b$ and the condition (2) replaced by $\alpha = \delta, \beta = \gamma$. However, as remarked by Usher-Zhang [UZ16, Remark 8.5], removing the restriction $a = b$ gives a better estimate of the displacement energy. In fact, if we restrict ourselves to $a = b$ and use the associated pseudo-distance, then we can only prove $d(G_0, G_1) \leq 2\int_0^1 \|H_s\|_\infty\, ds$ in Theorem 4.2.13 below.

We summarize some properties of $d_{\mathcal{D}(M)}$.

Proposition 4.2.8. *Let $F, G, H, F_0, F_1, G_0, G_1 \in \mathcal{D}(M)$.*

(i) $d_{\mathcal{D}(M)}(F,G) = d_{\mathcal{D}(M)}(G,F)$,

(ii) $d_{\mathcal{D}(M)}(F,G) \leq d_{\mathcal{D}(M)}(F,H) + d_{\mathcal{D}(M)}(H,G)$,

(iii) $d_{\mathcal{D}(M)}(\mathcal{H}om^\star(F_0,G_0), \mathcal{H}om^\star(F_1,G_1)) \leq d_{\mathcal{D}(M)}(F_0,F_1) + d_{\mathcal{D}(M)}(G_0,G_1)$.

Let moreover $f\colon M \to N$ be a morphism of manifolds and set $\widetilde{f} := f\times\mathrm{id}_{\mathbb{R}}\colon M\times\mathbb{R}\to N\times\mathbb{R}$. Regarding F and G as objects in the right orthogonal $\mathbf{D}^{\mathrm{b}}_{\{\tau\leq 0\}}(M\times\mathbb{R})^\perp$, one has

(iv) $d_{\mathcal{D}(N)}(R\widetilde{f}_*F, R\widetilde{f}_*G) \leq d_{\mathcal{D}(M)}(F,G)$ *(see also Remark 2.2.5).*

Proof. (i) and (iv) follow from the definition of $d_{\mathcal{D}(M)}$. (ii) follows from Lemma 4.2.4 and (iii) follows from Lemma 4.2.5. □

Example 4.2.9. Assume that M is compact and $\varphi\colon M \to \mathbb{R}$ be a C^∞-function. Define

$$\begin{aligned} &Z := \{(x,t) \in M \times \mathbb{R} \mid \varphi(x) + t \geq 0\}, \\ &F := \mathbf{k}_{M\times[0,+\infty)},\ G := \mathbf{k}_Z \in {}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\leq 0\}}(M \times \mathbb{R}) \simeq \mathcal{D}(M). \end{aligned} \tag{4.2.6}$$

Set $a := \max\{\max\varphi, 0\}, b := -\min\{\min\varphi, 0\}$. Then there exist morphisms $\alpha\colon F \to T_{a*}G$ and $\beta\colon G \to T_{b*}F$ such that $T_{a*}\beta \circ \alpha = \tau_{0,a+b}(F)$ and $T_{b*}\alpha \circ \beta = \tau_{0,a+b}(G)$. This implies that F is (a,b)-isomorphic to G and

$$d_{\mathcal{D}(M)}(F,G) \leq a+b = \max\{\max\varphi, 0\} - \min\{\min\varphi, 0\}. \tag{4.2.7}$$

Since $\mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}G) \simeq H^0R\varGamma_{M\times[-c,+\infty)}(M\times\mathbb{R}; \mathcal{H}om^\star(F,G)) \simeq 0$ for any $c < \max\varphi$ and $\mathrm{Hom}_{\mathcal{D}(M)}(G, T_{c*}F) \simeq 0$ for any $c < -\min\varphi$, the equation $d_{\mathcal{D}(M)}(F,G) = a+b$ holds.

Example 4.2.10. Assume that M is compact. For $i = 1,2$, let L_i be a compact connected exact Lagrangian submanifold of T^*M and $f_i\colon L_i \to \mathbb{R}$ be a primitive of the Liouville 1-form α_{T^*M}. Then, by Corollary 3.1.5, $L_1 \cap L_2 \neq \emptyset$. For simplicity, we assume

$$\min_{p\in L_1\cap L_2}(f_2 - f_1) \leq 0 \leq \max_{p\in L_1\cap L_2}(f_2 - f_1). \tag{4.2.8}$$

Let moreover $F_i \in \mathbf{D}^{\mathrm{b}}(M\times\mathbb{R})$ be the canonical sheaf quantization associated with L_i and f_i for $i = 1,2$ (see Theorem 2.2.15). Set $a := \max_{p\in L_1\cap L_2}(f_2 - f_1)$. Then, by Proposition 3.1.2, an estimate of $\mathrm{SS}(\mathcal{H}om^\star(F_1,F_2))$ in Section 3.2, and the microlocal Morse lemma (Proposition 2.1.3), one can show that

$$\mathrm{Hom}_{\mathcal{D}(M)}(F_1, T_{a*}F_2[k]) \simeq H^k(M;\mathbf{k}_M) \tag{4.2.9}$$

for any $k \in \mathbb{Z}$. Thus there exists a morphism $\alpha\colon F_1 \to T_{a*}F_2$ corresponding to $1 \in \mathbf{k} \simeq H^0(M;\mathbf{k})$. Set $b := \max_{p\in L_1\cap L_2}(f_1 - f_2)$. Then, similarly to the above, we obtain $\mathrm{Hom}_{\mathcal{D}(M)}(F_2, T_{b*}F_1) \simeq H^0(M;\mathbf{k})$ and get a morphism $\beta\colon F_2 \to T_{b*}F_1$ corresponding to $1 \in \mathbf{k}$. By construction, we find that $T_{b*}\beta\circ\alpha = \tau_{0,a+b}(F_1)$ and $T_{a*}\alpha\circ\beta = \tau_{0,a+b}(F_2)$. Thus, F_1 is (a,b)-isomorphic to F_2 and

$$\begin{aligned} d_{\mathcal{D}(M)}(F_1,F_2) &\leq \max_{p\in L_1\cap L_2}(f_2 - f_1) + \max_{p\in L_1\cap L_2}(f_1 - f_2) \\ &= \max_{p\in L_1\cap L_2}(f_2 - f_1) - \min_{p\in L_1\cap L_2}(f_2 - f_1). \end{aligned} \tag{4.2.10}$$

Next, we prove that a "homotopy sheaf" gives an (a,b)-isomorphic pair.

Lemma 4.2.11. *Let $F \xrightarrow{u} G \xrightarrow{v} H \xrightarrow{w} F[1]$ be a distinguished triangle in $\mathbf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(M\times\mathbb{R})$ and assume that F is c-torsion. Then G is $(0,c)$-isomorphic to H.*

Proof. By assumption, we have $T_{c*}w\circ\tau_{0,c}(H) = \tau_{0,c}(F[1])\circ w = 0$. Hence, we get a morphism $\alpha\colon H \to T_{c*}G$ satisfying $\tau_{0,c}(H) = T_{c*}v\circ\alpha$.

$$\begin{tikzcd} F \arrow[r, "u"] \arrow[d] & G \arrow[r, "v"] \arrow[d] & H \arrow[r, "w"] \arrow[d] \arrow[dl, dashed, "\alpha"'] \arrow[d, phantom, "\circlearrowleft", shift right=4ex] & F[1] \arrow[d, "0"] \\ T_{c*}F \arrow[r, "T_{c*}u"'] & T_{c*}G \arrow[r, "T_{c*}v"'] & T_{c*}H \arrow[r, "T_{c*}w"'] & T_{c*}F[1] \end{tikzcd} \tag{4.2.11}$$

On the other hand, since $\tau_{0,c}(G) \circ u = T_{c*}u \circ \tau_{0,c}(F) = 0$, there exists a morphism $\delta\colon H \to T_{c*}G$ satisfying $\tau_{0,c}(G) = \delta \circ v$.

$$\begin{tikzcd} F \arrow[r, "u"] \arrow[d, "0"] & G \arrow[r, "v"] \arrow[d] \arrow[r, phantom, "\circlearrowleft", shift right=3ex] & H \arrow[r, "w"] \arrow[d] \arrow[dl, dashed, "\delta"] & F[1] \arrow[d] \\ T_{c*}F \arrow[r, "T_{c*}u"'] & T_{c*}G \arrow[r, "T_{c*}v"'] & T_{c*}H \arrow[r, "T_{c*}w"'] & T_{c*}F[1] \end{tikzcd} \tag{4.2.12}$$

Moreover, we obtain

$$\begin{aligned} \tau_{c,2c}(G) \circ \alpha &= T_{c*}\tau_{0,c}(G) \circ \alpha \\ &= T_{c*}\delta \circ T_{c*}v \circ \alpha \\ &= T_{c*}\delta \circ \tau_{0,c}(H) \\ &= T_{c*}\tau_{0,c}(G) \circ \delta = \tau_{c,2c}(G) \circ \delta. \end{aligned} \tag{4.2.13}$$

This completes the proof. □

Proposition 4.2.12. *Let $\mathcal{H} \in \mathbf{D}^{\mathrm{b}}_{\{\tau \geq 0\}}(M \times \mathbb{R} \times I)$. Assume that there exist continuous functions $f, g\colon I \to \mathbb{R}_{\geq 0}$ satisfying*

$$\mathrm{SS}(\mathcal{H}) \subset T^*M \times \{(t, s; \tau, \sigma) \mid -f(s) \cdot \tau \leq \sigma \leq g(s) \cdot \tau\}. \tag{4.2.14}$$

Then $\mathcal{H}|_{M\times\mathbb{R}\times\{0\}}$ is $\left(\int_0^1 g(s)ds + \varepsilon, \int_0^1 f(s)ds + \varepsilon\right)$-isomorphic to $\mathcal{H}|_{M\times\mathbb{R}\times\{1\}}$ for any $\varepsilon \in \mathbb{R}_{>0}$.

Proof. Set $\Lambda' := \{(t, s; \tau, \sigma) \mid -f(s)\cdot\tau \leq \sigma \leq g(s)\cdot\tau\}$. Let $s_1 < s_2$ be in $[0, 1]$ and $\varepsilon' \in \mathbb{R}_{>0}$ be an arbitrary positive number. Then there is $r \in \mathbb{R}_{>0}$ such that

$$f(s) \leq \max_{s\in[s_1,s_2]} f(s) + \frac{\varepsilon'}{2} \quad \text{and} \quad g(s) \leq \max_{s\in[s_1,s_2]} g(s) + \frac{\varepsilon'}{2} \tag{4.2.15}$$

for any $s \in (s_1 - r, s_2 + r)$, which implies

$$\Lambda' \cap \pi^{-1}(M \times \mathbb{R} \times (s_1 - r, s_2 + r)) \subset T^*M \times (\mathbb{R} \times I) \times \gamma_{a+\frac{\varepsilon'}{2}, b+\frac{\varepsilon'}{2}} \tag{4.2.16}$$

with $a = \max_{s\in[s_1,s_2]} f(s)$ and $b = \max_{s\in[s_1,s_2]} g(s)$. Let $q\colon M \times \mathbb{R} \times I \to M \times \mathbb{R}$ be the projection. By Proposition 4.1.3, $Rq_*(\mathcal{H}_{M\times\mathbb{R}\times[s_1,s_2)})$ is $(a(s_2 - s_1) + \varepsilon')$-torsion and $Rq_*(\mathcal{H}_{M\times\mathbb{R}\times(s_1,s_2]})$ is $(b(s_2 - s_1) + \varepsilon')$-torsion. Hence, by Lemma 4.2.4, Lemma 4.2.11, and the distinguished triangles

$$\begin{aligned} &Rq_*(\mathcal{H}_{M\times\mathbb{R}\times(s_1,s_2]}) \longrightarrow Rq_*(\mathcal{H}_{M\times\mathbb{R}\times[s_1,s_2]}) \longrightarrow \mathcal{H}|_{M\times\mathbb{R}\times\{s_1\}} \xrightarrow{+1}, \\ &Rq_*(\mathcal{H}_{M\times\mathbb{R}\times[s_1,s_2)}) \longrightarrow Rq_*(\mathcal{H}_{M\times\mathbb{R}\times[s_1,s_2]}) \longrightarrow \mathcal{H}|_{M\times\mathbb{R}\times\{s_2\}} \xrightarrow{+1}, \end{aligned} \tag{4.2.17}$$

we find that $\mathcal{H}|_{M\times\mathbb{R}\times\{s_1\}}$ is $(b(s_2 - s_1) + \varepsilon', a(s_2 - s_1) + \varepsilon')$-isomorphic to $\mathcal{H}|_{M\times\mathbb{R}\times\{s_2\}}$. Thus, by Lemma 4.2.4 again, $\mathcal{H}|_{M\times\mathbb{R}\times\{0\}}$ is $(b_n + \varepsilon/2, a_n + \varepsilon/2)$-isomorphic to $\mathcal{H}|_{M\times\mathbb{R}\times\{1\}}$ for any $n \in \mathbb{Z}_{>0}$, where a_n and b_n are the Riemann sums

$$a_n = \sum_{k=0}^{n-1} \frac{1}{n} \cdot \max_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]} f(s) \quad \text{and} \quad b_n = \sum_{k=0}^{n-1} \frac{1}{n} \cdot \max_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]} g(s). \tag{4.2.18}$$

Since f and g are continuous on I, there is a sufficiently large $n \in \mathbb{Z}_{>0}$ such that

$$a_n \leq \int_0^1 f(s)ds + \frac{\varepsilon}{2} \quad \text{and} \quad b_n \leq \int_0^1 g(s)ds + \frac{\varepsilon}{2}, \tag{4.2.19}$$

which completes the proof. □

Now, let us consider the distance between Hamiltonian isotopic objects in $\mathcal{D}(M)$. Using sheaf quantization of Hamiltonian isotopies (Theorem 2.2.1), we can define Hamiltonian deformations in $\mathcal{D}(M)$. From now on, assume moreover that the dimension of M is grater than 0. For a compactly supported Hamiltonian function $H = (H_s)_s \colon T^*M \times I \to \mathbb{R}$, following Hofer [Hof90], we define

$$\begin{aligned} E_+(H) &:= \int_0^1 \max_p H_s(p) ds, \qquad E_-(H) := -\int_0^1 \min_p H_s(p) ds, \\ \|H\| &:= E_+(H) + E_-(H) = \int_0^1 \left(\max_p H_s(p) - \min_p H_s(p) \right) ds. \end{aligned} \tag{4.2.20}$$

Theorem 4.2.13. *Let $H = (H_s)_s \colon T^*M \times I \to \mathbb{R}$ be a compactly supported Hamiltonian function and denote by ϕ^H the Hamiltonian isotopy generated by H. Let $K \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times M \times \mathbb{R} \times I)$ be the sheaf quantization associated with ϕ^H. Let moreover $G \in \mathcal{D}(M)$, and set $G' := K \circ G \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times I)$ and $G_s := G'|_{M \times \mathbb{R} \times \{s\}} \in \mathcal{D}(M)$ for $s \in I$. Then $G_0 = G$ is $(E_-(H) + \varepsilon, E_+(H) + \varepsilon)$-isomorphic to G_1 for any $\varepsilon \in \mathbb{R}_{>0}$. In particular, $d_{\mathcal{D}(M)}(G_0, G_1) \le \|H\|$.*

Proof. By Proposition 2.1.11 and (2.2.4), we get

$$\mathrm{SS}(G') \subset T^*M \times \left\{ (t, s; \tau, \sigma) \;\middle|\; -\max_p H_s(p) \cdot \tau \le \sigma \le -\min_p H_s(p) \cdot \tau \right\}. \tag{4.2.21}$$

Thus the result follows from Proposition 4.2.12. □

4.3 Displacement energy

In this section, we prove a quantitative version of Tamarkin's non-displaceability theorem, which gives a lower bound of the displacement energy.

For compact subsets A and B of T^*M, their *displacement energy* $e(A, B)$ is defined by

$$e(A, B) := \inf \left\{ \|H\| \;\middle|\; \begin{array}{l} H \colon T^*M \times I \to \mathbb{R} \text{ with compact support,} \\ A \cap \phi_1^H(B) = \emptyset \end{array} \right\}. \tag{4.3.1}$$

For a compact subset A of T^*M, set $e(A) = e(A, A)$.

We give a sheaf-theoretic lower bound of $e(A, B)$. For that purpose, we make the following definition.

Definition 4.3.1. For $F, G \in \mathcal{D}(M)$, one defines

$$\begin{aligned} e_{\mathcal{D}(M)}(F, G) &:= d_{\mathcal{D}(\mathrm{pt})}(Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, G), 0) \\ &= \inf\{ c \in \mathbb{R}_{\ge 0} \mid Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, G) \text{ is } c\text{-torsion} \}. \end{aligned} \tag{4.3.2}$$

Theorem 4.3.2. *Let A and B be compact subsets of T^*M. Then, for any $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$, one has*

$$e(A, B) \ge e_{\mathcal{D}(M)}(F, G). \tag{4.3.3}$$

In particular, for any $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$,

$$e(A, B) \ge \inf\{ c \in \mathbb{R}_{\ge 0} \mid \mathrm{Hom}_{\mathcal{D}(M)}(F, G) \to \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}G) \text{ is zero} \}. \tag{4.3.4}$$

Proof. Suppose that a compactly supported Hamiltonian function $H\colon T^*M \times I \to \mathbb{R}$ satisfies $A \cap \phi_1^H(B) = \emptyset$. Let $K \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times M \times \mathbb{R} \times I)$ be the sheaf quantization associated with ϕ^H and define $G' := K \circ G \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times I)$ and $G_s := G'|_{M \times \mathbb{R} \times \{s\}} \in \mathcal{D}(M)$ for $s \in I$ as in Theorem 4.2.13. Since $G_1 \in \mathcal{D}_{\phi_1^H(B)}(M)$, Tamarkin's separation theorem (Theorem 2.2.7) implies $Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, G_1) \simeq 0$. On the other hand, by Theorem 4.2.13, we have $d_{\mathcal{D}(M)}(G_0, G_1) \le \|H\|$. Hence, by Proposition 4.2.8, we obtain

$$
\begin{aligned}
e_{\mathcal{D}(M)}(F,G) &= d_{\mathcal{D}(\mathrm{pt})}(Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, G_0), 0) \\
&\le d_{\mathcal{D}(M)}(\mathcal{H}om^\star(F, G_0), \mathcal{H}om^\star(F, G_1)) \\
&\le d_{\mathcal{D}(M)}(G_0, G_1) \le \|H\|,
\end{aligned} \tag{4.3.5}
$$

which proves the theorem. □

We list some properties of $e_{\mathcal{D}(M)}$.

Proposition 4.3.3. *Let $F, G \in \mathcal{D}(M)$.*

(i) *$e_{\mathcal{D}(M)}(G, F) \le e_{\mathcal{D}(M)}(F, F)$ and $e_{\mathcal{D}(M)}(F, G) \le e_{\mathcal{D}(M)}(F, F)$.*

(ii) *Assume that F and G are cohomologically constructible as objects in ${}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau \le 0\}}(M \times \mathbb{R}) \subset \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R})$. Then $e_{\mathcal{D}(M)}(F, G) = e_{\mathcal{D}(M)}(i_*\mathbb{D}_{M\times\mathbb{R}} G, i_*\mathbb{D}_{M\times\mathbb{R}} F)$.*

(iii) *Assume that there exist compact subsets A and B of T^*M such that $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$. Let $\phi^H\colon T^*M \times I \to T^*M$ be a Hamiltonian isotopy with compact support and $K \in \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R} \times M \times \mathbb{R} \times I)$ be the sheaf quantization associated with ϕ^H. Set $F' := K \circ F, G' := K \circ G$ and $F_s := F'|_{M\times\mathbb{R}\times\{s\}}, G_s := G'|_{M\times\mathbb{R}\times\{s\}}$ for $s \in I$. Then $e_{\mathcal{D}(M)}(F, G) = e_{\mathcal{D}(M)}(F_s, G_s)$ for any $s \in I$.*

Proof. (i) Assume that the morphism

$$
\begin{aligned}
\tau_{0,c}(Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, F))\colon Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, F) &\longrightarrow T_{c*} Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, F) \\
&\simeq Rq_{\mathbb{R}*} \mathcal{H}om^\star(F, T_{c*}F)
\end{aligned} \tag{4.3.6}
$$

is zero. Then the induced morphism $\mathrm{Hom}_{\mathcal{D}(M)}(F, F) \to \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}F)$ is also zero by Proposition 2.2.6, which implies $\tau_{0,c}(F) = 0$. Since the canonical morphism

$$
\begin{aligned}
\tau_{0,c}(Rq_{\mathbb{R}*} \mathcal{H}om^\star(G, F))\colon Rq_{\mathbb{R}*} \mathcal{H}om^\star(G, F) &\longrightarrow T_{c*} Rq_{\mathbb{R}*} \mathcal{H}om^\star(G, F) \\
&\simeq Rq_{\mathbb{R}*} \mathcal{H}om^\star(G, T_{c*}F)
\end{aligned} \tag{4.3.7}
$$

is induced by $\tau_{0,c}(F)$, it is also zero. This proves the first inequality. The proof for the second one is similar.

(ii) First, we show that $i_*\mathbb{D}_{M\times\mathbb{R}}\colon \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R}) \to \mathbf{D}^{\mathrm{b}}(M \times \mathbb{R})$ induces a functor $\mathcal{D}(M) \simeq {}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau \le 0\}}(M \times \mathbb{R}) \to \mathbf{D}^{\mathrm{b}}_{\{\tau \le 0\}}(M \times \mathbb{R})^\perp \simeq \mathcal{D}(M)$. Let $F \in {}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau \le 0\}}(M \times \mathbb{R})$ and $S \in \mathbf{D}^{\mathrm{b}}_{\{\tau \le 0\}}(M \times \mathbb{R})$. Then we have

$$
\begin{aligned}
\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(M\times\mathbb{R})}(S, i_*\mathbb{D}_{M\times\mathbb{R}} F) &\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(M\times\mathbb{R})}(i_*S, R\mathcal{H}om(F, \omega_{M\times\mathbb{R}})) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(M\times\mathbb{R})}(i_*S \otimes F, \omega_{M\times\mathbb{R}}) \\
&\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(M\times\mathbb{R})}(F, R\mathcal{H}om(i_*S, \omega_{M\times\mathbb{R}})).
\end{aligned} \tag{4.3.8}
$$

By Theorem 2.1.5 and Proposition 2.1.7, $R\mathcal{H}om(i_*S, \omega_{M\times\mathbb{R}}) \in \mathbf{D}^{\mathrm{b}}_{\{\tau \le 0\}}(M \times \mathbb{R})$. Hence $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(M\times\mathbb{R})}(S, i_*\mathbb{D}_{M\times\mathbb{R}} F) \simeq 0$, which implies $i_*\mathbb{D}_{M\times\mathbb{R}} F \in \mathbf{D}^{\mathrm{b}}_{\{\tau \le 0\}}(M \times \mathbb{R})^\perp$.

Now, assume that $F, G \in {}^{\perp}\mathbf{D}^{\mathrm{b}}_{\{\tau \le 0\}}(M \times \mathbb{R})$ are cohomologically constructible. Then we have

$$\begin{aligned} \mathcal{H}om^{\star}(F,G) &\simeq Rs_* \, R\mathcal{H}om(\tilde{q}_2^{-1} i^{-1} F, \tilde{q}_1^! G) \\ &\simeq Rs_* \, R\mathcal{H}om(\mathbb{D}_{M\times\mathbb{R}} \tilde{q}_1^! G, \mathbb{D}_{M\times\mathbb{R}} \tilde{q}_2^{-1} i^{-1} F) \\ &\simeq Rs_* \, R\mathcal{H}om(\tilde{q}_1^{-1} \mathbb{D}_{M\times\mathbb{R}} G, \tilde{q}_2^! i^{-1} \mathbb{D}_{M\times\mathbb{R}} F) \\ &\simeq \mathcal{H}om^{\star}(i_* \mathbb{D}_{M\times\mathbb{R}} G, i_* \mathbb{D}_{M\times\mathbb{R}} F), \end{aligned} \tag{4.3.9}$$

which proves the equality.
(iii) It is enough to show that $Rq_{\mathbb{R}*} \mathcal{H}om^{\star}(F,G) \simeq Rq_{\mathbb{R}*} \mathcal{H}om^{\star}(F_s, G_s)$ for any $s \in I$. For a compact subset C of T^*M, define $\mathrm{Cone}_H(C) \subset T^*(M \times I) \times \mathbb{R}$ by

$$\begin{aligned} &\mathrm{Cone}_H(C) \\ &:= \{(x', s; \xi', -\tau \cdot H_s(x'; \xi'/\tau), \tau) \mid \tau > 0, (x; \xi/\tau) \in C, (x'; \xi'/\tau) = \phi_s^H(x; \xi/\tau)\}. \end{aligned} \tag{4.3.10}$$

Denote by $\hat{\pi} \colon T^*(M \times I \times \mathbb{R}) \simeq T^*(M \times I) \times T^*\mathbb{R} \to T^*(M \times I) \times \mathbb{R}$ the projection. Then, by Proposition 2.1.11 and (2.2.4), we have

$$\mathrm{SS}(F') \subset \hat{\pi}^{-1}(\mathrm{Cone}_H(A)), \quad \mathrm{SS}(G') \subset \hat{\pi}^{-1}(\mathrm{Cone}_H(B)). \tag{4.3.11}$$

Let moreover $q_{I\times\mathbb{R}} \colon M \times I \times \mathbb{R} \to I \times \mathbb{R}$ be the projection. Note that $q_{I\times\mathbb{R}}$ is proper on $\mathrm{Supp}(\mathcal{H}om^{\star}(F', G'))$, where $\mathcal{H}om^{\star}$ denotes the internal Hom functor on $\mathcal{D}(M \times I)$. Then, by [GS14, Proposition 3.13 and Lemma 3.7] and Theorem 2.1.5, we obtain

$$\mathrm{SS}(Rq_{I\times\mathbb{R}*} \mathcal{H}om^{\star}(F', G')) \subset \{(s, t; 0, \tau) \mid \tau \ge 0\} \subset T^*(I \times \mathbb{R}). \tag{4.3.12}$$

Since I is contractible, there exists $S \in \mathbf{D}^{\mathrm{b}}(\mathbb{R})$ such that $Rq_{I\times\mathbb{R}*} \mathcal{H}om^{\star}(F', G') \simeq q'^{-1} S$, where $q' \colon I \times \mathbb{R} \to \mathbb{R}$ is the projection. Finally, by [GS14, Corollary 3.15], for any $s \in I$, we have

$$Rq_{I\times\mathbb{R}*} \mathcal{H}om^{\star}(F', G')|_{\{s\}\times\mathbb{R}} \simeq Rq_{\mathbb{R}*} \mathcal{H}om^{\star}(F_s, G_s), \tag{4.3.13}$$

which completes the proof. ☐

Remark 4.3.4. Assume that $F, G \in \mathcal{D}(M) \simeq {}^{\perp}\mathbf{D}^{\mathrm{b}}_{\{\tau \le 0\}}(M \times \mathbb{R})$ are constructible and have compact support. Then $Rq_{\mathbb{R}*} \mathcal{H}om^{\star}(F,G)$ is also constructible object with compact support and $\mathrm{SS}(Rq_{\mathbb{R}*} \mathcal{H}om^{\star}(F,G)) \subset \{\tau \ge 0\}$. By the decomposition result for constructible sheaves on $\mathbb{R}$ due to Guillermou [Gui16b, Corollary 7.3] (see also [KS17, Subsection 1.4]), there exist a finite family of half-closed intervals $\{[b_i, d_i)\}_{i\in I}$ and $n_i \in \mathbb{Z}$ $(i \in I)$ such that

$$Rq_{\mathbb{R}*} \mathcal{H}om^{\star}(F,G) \simeq \bigoplus_{i\in I} \mathbf{k}_{[b_i, d_i)}[n_i]. \tag{4.3.14}$$

Using this decomposition, we find that $e_{\mathcal{D}(M)}(F,G) = \max_{i\in I}(d_i - b_i)$ is the length of the longest barcodes of $Rq_{\mathbb{R}*} \mathcal{H}om^{\star}(F,G)$ in the sense of Kashiwara-Schapira [KS17].

Remark 4.3.5. Let $F, G \in \mathcal{D}(M)$. As remarked by Tamarkin [Tam08, Section 1], we can associate a module $H(F,G)$ over a Novikov ring $\Lambda_{0,\mathrm{nov}}(\mathbf{k})$ as follows. We define

$$\Lambda_{0,\mathrm{nov}}(\mathbf{k}) := \left\{ \sum_{i=1}^{\infty} c_i T^{\lambda_i} \;\middle|\; c_i \in \mathbf{k}, \lambda_i \in \mathbb{R}_{\ge 0}, \lambda_1 < \lambda_2 < \cdots, \lim_{i\to\infty} \lambda_i = +\infty \right\}. \tag{4.3.15}$$

We also define a submodule $H(F,G)$ of $\prod_{c\in\mathbb{R}} \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}G)$ by

$$\left\{ (h_c)_c \in \prod_{c\in\mathbb{R}} \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}G) \;\middle|\; \begin{array}{l} \exists\, (c_i)_{i=1}^{\infty} \subset \mathbb{R}, c_1 < c_2 < \cdots, \lim_{i\to\infty} c_i = +\infty \\ \text{such that } h_c = 0 \text{ for any } c \notin \bigcup_{i=1}^{\infty} \{c_i\} \end{array} \right\}. \quad (4.3.16)$$

For $c \in \mathbb{R}$ and $\lambda \in \mathbb{R}_{\geq 0}$, there is the canonical morphism $\tau_{c,c+\lambda}\colon \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c*}G) \to \mathrm{Hom}_{\mathcal{D}(M)}(F, T_{c+\lambda *}G)$ induced by $\tau_{c,c+\lambda}(G)\colon T_{c*}G \to T_{c+\lambda *}G$. Using this morphism, we can equip $H(F,G)$ with an action of T^λ by $T^\lambda \cdot (h_c)_c := (\tau_{c,c+\lambda}(h_c))_c$. We thus find that the Novikov ring $\Lambda_{0,\mathrm{nov}}(\mathbf{k})$ acts on $H(F,G)$.
(i) Using the $\Lambda_{0,\mathrm{nov}}(\mathbf{k})$-module $H(F,G)$, we can express (4.3.4) in Theorem 4.3.2 as

$$e(A,B) \geq \inf\{c \in \mathbb{R}_{\geq 0} \mid H(F,G) \text{ is } T^c\text{-torsion}\} \quad (4.3.17)$$

for any $F \in \mathcal{D}_A(M)$ and $G \in \mathcal{D}_B(M)$. This inequality seems to be related to the estimate of the displacement energy by Fukaya-Oh-Ohta-Ono [FOOO09a, FOOO09b, Theorem J] and [FOOO13, Theorem 6.1].
(ii) We denote by $\Lambda_{\mathrm{nov}}(\mathbf{k})$ the fraction field of $\Lambda_{0,\mathrm{nov}}(\mathbf{k})$. Then, for any $F, G \in \mathcal{D}(M)$, we have

$$H(F,G) \otimes_{\Lambda_{0,\mathrm{nov}}(\mathbf{k})} \Lambda_{\mathrm{nov}}(\mathbf{k}) \simeq \mathrm{Hom}_{\mathcal{T}(M)}(F,G) \otimes_{\mathbf{k}} \Lambda_{\mathrm{nov}}(\mathbf{k}) \quad (4.3.18)$$

Note that $\mathcal{T}(M)$ is invariant under Hamiltonian deformations (see Theorem 2.2.12). The invariance follows from Theorem 4.2.13 and Remark 4.2.3. Note also that our approach gives a more precise description of Hamiltonian deformations in the category $\mathcal{D}(M)$.

4.4 Examples and applications

In this section, we give some examples to which Theorem 4.3.2 is applicable.

The first two examples, Example 4.4.1 and Example 4.4.3, treat exact Lagrangian immersions.

Example 4.4.1. Consider $T^*\mathbb{R}^m \simeq \mathbb{R}^{2m}$ and denote by $(x;\xi)$ the homogeneous symplectic coordinate system. Let $L = S^m = \{(x,y) \in \mathbb{R}^m \times \mathbb{R} \mid \|x\|^2 + y^2 = 1\}$ and consider the exact Lagrangian immersion

$$\iota\colon L \longrightarrow T^*\mathbb{R}^m, \quad (x,y) \longmapsto (x; yx). \quad (4.4.1)$$

Setting $f\colon L \to \mathbb{R}, f(x,y) := -\frac{1}{3}y^3$, we have $df = \iota^*\alpha_{T^*\mathbb{R}^m}$. We define a locally closed subset Z of $\mathbb{R}^m \times \mathbb{R}$ by

$$Z := \left\{ (x,t) \in \mathbb{R}^m \times \mathbb{R} \;\middle|\; \|x\| \leq 1, -\frac{1}{3}(1-\|x\|^2)^{\frac{3}{2}} \leq t < \frac{1}{3}(1-\|x\|^2)^{\frac{3}{2}} \right\} \quad (4.4.2)$$

and $F := \mathbf{k}_Z \in \mathbf{D}^{\mathrm{b}}(\mathbb{R}^m \times \mathbb{R})$.

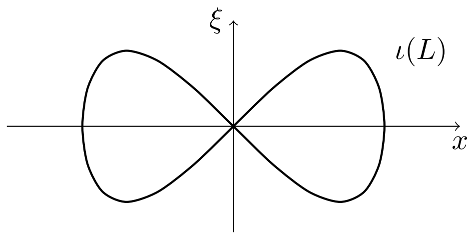

Figure 4.4.1: $\iota(L)$ in the case $m = 1$

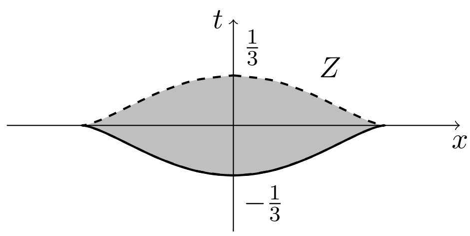

Figure 4.4.2: Z in the case $m = 1$

The object F is in ${}^{\perp}\mathbf{D}^{\mathrm{b}}_{\{\tau\leq 0\}}(\mathbb{R}^m\times\mathbb{R})$ and can be regarded as an object in $\mathcal{D}_{\iota(L)}(\mathbb{R}^m)$. For this object F, we find that

$$\mathrm{Hom}_{\mathcal{D}(\mathbb{R}^m)}(F,T_{c*}F)\simeq \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathbb{R}^m\times\mathbb{R})}(F,T_{c*}F)\simeq \begin{cases}\mathbf{k} & (0\leq c<\frac{2}{3})\\ 0 & (c\geq \frac{2}{3})\end{cases} \tag{4.4.3}$$

and the induced morphism $\mathrm{Hom}_{\mathcal{D}(\mathbb{R}^m)}(F,F)\to \mathrm{Hom}_{\mathcal{D}(\mathbb{R}^m)}(F,T_{c*}F)$ is the identity for any $0\leq c<2/3$. Hence, we obtain $e(\iota(L))\geq e_{\mathcal{D}(\mathbb{R}^m)}(F,F)\geq 2/3$ by Theorem 4.3.2. This is the same estimate as that of Akaho [Aka15]. If $m=1$, it is known that $e(\iota(L))=4/3$ by the use of Hofer-Zehnder capacity.

Using the example above, we can recover the following result of Polterovich [Pol93], for subsets of cotangent bundles.

Proposition 4.4.2 ([Pol93, Corollary 1.6, see also the first remark in p. 360])**.** *Let A be a compact subset of T^*M whose interior is non-empty. Then its displacement energy is positive: $e(A)>0$.*

Proof. Take a symplectic diffeomorphism $\psi\colon T^*M\to T^*M$ such that $T^*_MM\cap\mathrm{Int}(\psi(A))\neq\emptyset$. Since $e(\psi(A))=e(A)$, we may assume $T^*_MM\cap\mathrm{Int}(A)\neq\emptyset$ from the beginning. Take a point $x_0\in T^*_MM\cap\mathrm{Int}(A)$ and a local coordinate system $x=(x_1,\dots,x_m)$ on M around x_0. Denote by $(x;\xi)$ the associated local homogeneous symplectic coordinate system on T^*M. Using the coordinates, for $\varepsilon\in\mathbb{R}_{>0}$ we define $\iota_\varepsilon\colon S^m\to T^*M$ by $(x,y)\mapsto(\varepsilon x,\varepsilon yx)$ as in Example 4.4.1. Then, there is a sufficiently small $\varepsilon\in\mathbb{R}_{>0}$ such that the image $\iota_\varepsilon(S^m)$ is contained in $\mathrm{Int}(A)$. As in Example 4.4.1, we define $F:=\mathbf{k}_{Z_\varepsilon}\in\mathcal{D}_{\iota_\varepsilon(S^m)}(\mathbb{R}^m)$, where

$$Z_\varepsilon:=\left\{(z,t)\in\mathbb{R}^m\times\mathbb{R}\;\middle|\;\|z\|\leq\varepsilon,-\frac{1}{3\varepsilon}(\varepsilon^2-\|z\|^2)^{\frac{3}{2}}\leq t<\frac{1}{3\varepsilon}(\varepsilon^2-\|z\|^2)^{\frac{3}{2}}\right\}. \tag{4.4.4}$$

Moreover we define $G\in\mathcal{D}_{\iota_\varepsilon(S^m)}(M)$ as the zero extension of F to $M\times\mathbb{R}$. By monotonicity of the displacement energy and a similar argument to Example 4.4.1, we have

$$e(A)\geq e(\iota_\varepsilon(S^m))\geq e_{\mathcal{D}(M)}(G,G)\geq\frac{2}{3}\varepsilon^2>0. \tag{4.4.5}$$

□

For the next explicit example, our estimate is better than Akaho's estimate [Aka15].

Example 4.4.3. Let $\varphi\colon[0,1]\to(0,1]$ be a C^∞-function satisfying the following two conditions: (1) $\varphi\equiv 1$ near 0, (2) $\varphi(r)=r$ on $[1/2,1]$. Set $S^m=\{(x,y)\in\mathbb{R}^m\times\mathbb{R}\mid \|x\|^2+y^2=1\}$ and consider the exact Lagrangian immersion

$$\iota\colon S^m\longrightarrow T^*\mathbb{R}^m,\quad (x,y)\longmapsto\left(x,\left(\varphi(\|x\|)y-\frac{\varphi'(\|x\|)}{3\|x\|}y^3\right)\cdot x\right). \tag{4.4.6}$$

Setting $f\colon S^m\to\mathbb{R}, f(x,y):=-\frac{1}{3}\varphi(\|x\|)y^3$, we have $df=\iota^*\alpha_{T^*\mathbb{R}^m}$. We define a locally closed subset Z of $\mathbb{R}^m\times\mathbb{R}$ by

$$Z:=\left\{(x,t)\in\mathbb{R}^m\times\mathbb{R}\;\middle|\;\|x\|\leq 1,-\frac{1}{3}\varphi(\|x\|)(1-\|x\|^2)^{\frac{3}{2}}\leq t<\frac{1}{3}\varphi(\|x\|)(1-\|x\|^2)^{\frac{3}{2}}\right\} \tag{4.4.7}$$

and $F:=\mathbf{k}_Z\in\mathbf{D}^{\mathrm{b}}(\mathbb{R}^m\times\mathbb{R})$. Using the object F, one can show $e(\iota(S^m))\geq e_{\mathcal{D}(\mathbb{R}^m)}(F,F)\geq 2/3$ as in Example 4.4.1. On the other hand, the estimate by Akaho [Aka15] only gives $e(\iota(S^m))\geq\min_{r\in[0,\frac{1}{2}]}\{\frac{2}{3}(1-r^2)^{\frac{3}{2}}\cdot\varphi(r)\}$, which is less than $\sqrt{3}/8$.

Our theorem is also applicable to non-exact Lagrangian submanifolds. We focus on graphs of closed 1-forms here.

Example 4.4.4. Let M be a compact manifold and $\eta_i \colon M \to T^*M$ a closed 1-form for $i = 1, 2$. Set $L_i := \Gamma_{\eta_i} \subset T^*M$ the graph of η_i for $i = 1, 2$, and assume that L_1 and L_2 intersect transversally. We consider the displacement energy $e(L_1, L_2)$. The symplectic diffeomorphism ψ on T^*M defined by $\psi(x; \xi) := (x; \xi - \eta_1(x))$ sends L_1 to the zero-section M and L_2 to $\Gamma_{\eta_2 - \eta_1}$. Thus we assume $L_1 = M$ and $L_2 = \Gamma_\eta$, where η is a closed Morse 1-form from the beginning. Let $p \colon \widetilde{M} \to M$ be the abelian covering of M corresponding to the kernel of the pairing with η. Then there exists a function $f \colon \widetilde{M} \to \mathbb{R}$ such that $p^*\eta = df$. By assumption, f is a Morse function on $\widetilde{M}$. Define a closed subset Z of $\widetilde{M} \times \mathbb{R}$ by

$$Z := \{(x, t) \in \widetilde{M} \times \mathbb{R} \mid f(x) + t \geq 0\}. \tag{4.4.8}$$

Then we have $F := R(p \times \mathrm{id}_{\mathbb{R}})_* \mathbf{k}_Z \in \mathcal{D}_L(M)$ and $e(L_1, L_2) \geq e_{\mathcal{D}(M)}(\mathbf{k}_{M \times [0,+\infty)}, F)$ by Theorem 4.3.2.

Let us consider the estimate for $e_{\mathcal{D}(M)}(\mathbf{k}_{M \times [0,+\infty)}, F)$. First, we have

$$\begin{aligned} R\mathrm{Hom}(\mathbf{k}_{M \times [0,+\infty)}, T_{c*}F) &\simeq R\mathrm{Hom}(\mathbf{k}_{\widetilde{M} \times [-c,+\infty)}, \mathbf{k}_Z) \\ &\simeq R\Gamma_{\widetilde{M} \times [-c,+\infty)}\left(\widetilde{M} \times \mathbb{R}; \mathbf{k}_Z\right). \end{aligned} \tag{4.4.9}$$

Define $U_c := \{x \in \widetilde{M} \mid f(x) > c\}$ for $c \in \mathbb{R}$. Then the cohomology of the last complex $R\Gamma_{\widetilde{M} \times [-c,+\infty)}\left(\widetilde{M} \times \mathbb{R}; \mathbf{k}_Z\right)$ is isomorphic to $H^*(\widetilde{M}, U_c)$ and for $c \leq d$, $\tau_{c,d}$ is the canonical morphism induced by the map $(\widetilde{M}, U_d) \to (\widetilde{M}, U_c)$ of the pairs. Hence this persistence module is isomorphic to $(H^*(\widetilde{M}, U_c))_{c \in \mathbb{R}}$ and it is the dual of the persistence module $(H_*(\widetilde{M}, U_c))_{c \in \mathbb{R}}$. The persistence module $(H_*(\widetilde{M}, U_c))_{c \in \mathbb{R}}$ can be studied by Morse homology theory of $-f$ or Morse-Novikov theory of $-\eta$. Let v be a vector field on M which is a $(-\eta)$-gradient and satisfies the transversality condition in the sense of Pajitnov [Paj06, Chapter 3 and Chapter 4]. The existence and denseness of such vector fields hold (see Pajitnov [Paj06, Chapter 4]). Moreover let $\widetilde{v}$ be the lift of v to $\widetilde{M}$. The Morse-Novikov complex $C := C(-\eta, v)$ with respect to $\widetilde{v}$ has the filtration $(C_{\leq c})_{c \in \mathbb{R}}$ defined by the values of $-f$. Here we regard C as a finitely generated free module over the Novikov field

$$\left\{ \sum_{i=1}^{\infty} c_i T^{\lambda_i} \;\middle|\; \begin{array}{l} c_i \in \mathbf{k}, \lambda_i = \displaystyle\int_\gamma \eta \text{ for some } \gamma \in H_1(M; \mathbb{Z}), \\ \lambda_1 < \lambda_2 < \cdots, \displaystyle\lim_{i \to \infty} \lambda_i = +\infty \end{array} \right\}. \tag{4.4.10}$$

The persistence module $(H_*(C/C_{\leq c}))_{c \in \mathbb{R}}$ is isomorphic to $(H_*(\widetilde{M}, U_c))_{c \in \mathbb{R}}$ by usual Morse theoretic arguments. Each critical point generates or kills rank 1 subspace of the persistent homology. Hence one can prove that our estimate is greater than or equal to

$$\max_p \min_q \left\{ |f(p) - f(q)| \;\middle|\; \begin{array}{l} p, q \in \mathrm{Crit}(-f), |\mathrm{ind}(p) - \mathrm{ind}(q)| = 1, \\ \text{there is a flow of } \widetilde{v} \text{ connecting } p \text{ and } q \end{array} \right\}, \tag{4.4.11}$$

where $\mathrm{Crit}(-f)$ is the set of the critical points of $-f$ and $\mathrm{ind}(p)$ is the Morse index of $p \in \mathrm{Crit}(-f)$.

The persistence module $(H_*(C/C_{\leq c}))_{c \in \mathbb{R}}$ is not finitely generated in the usual sense of persistent homology theory. However we can apply the theory of Usher-Zhang [UZ16] to

C. Their result describes the "barcodes" of the persistence module $(H_*(C_{\leq c}))_c$ and one can check that our estimate in this case coincides with the length of the longest concise barcodes for $C(-\eta, v)$ defined in [UZ16].

In the last example below, our estimate determines the displacement energy.

Example 4.4.5 (Special case of Example 4.4.4)**.** Let $L = \Gamma_\eta \subset T^*S^1$ be the graph of a non-exact 1-form $\eta\colon S^1 \to T^*S^1$. Assume that L and the zero-section S^1 intersect transversally at only two points. We estimate the displacement energy $e(S^1, L)$. Let $p\colon \mathbb{R} \to S^1$ be the universal covering and take a function f on $\mathbb{R}$ such that $df = p^*\eta$. Define $F := R(p \times \mathrm{id}_\mathbb{R})_* \mathbf{k}_{\{(x,t) \in \mathbb{R} \times \mathbb{R} | f(x)+t \geq 0\}} \in \mathcal{D}_L(S^1)$. Then a similar argument to Example 4.4.4 shows that $e_{\mathcal{D}(S^1)}(\mathbf{k}_{S^1 \times [0,+\infty)}, F)$ is equal to the smaller area enclosed by S^1 and L. One can check that $e(S^1, L)$ is equal to the area.

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