

博士論文

論文題目 Numerical analysis for evolution equations
(発展方程式に対する数値解析)

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Preface

In this thesis, we shall consider numerical analysis for partial differential equations (PDEs) with time evolution. We mainly address the finite element method (FEM) for parabolic equations, as well as gradient flows of planar curves (such as the curvature flow) and the hydrostatic Stokes equations, which are linearized equations of the primitive equations. Throughout this thesis, we focus on structures of PDEs, such as the smoothing property, maximal regularity, and energy dissipation. In the context of numerical analysis for PDEs, there are two major topics on structures of equations:

- (1) Can we establish the discrete analog of a certain property?
- (2) Can we construct a numerical scheme that preserves a certain property?

We address these problems in this thesis.

As in the theory of nonlinear PDEs, structures of equations play important roles in numerical analysis, especially for nonlinear problems. For example, it is known that analyticity of (discrete) parabolic semigroups allows us to construct an efficient numerical scheme for a nonlinear parabolic equation and establish stability and error estimates. Moreover, it is also known that numerical solutions preserving energy conservation or dissipation are stable for long-time computation numerically and theoretically.

This thesis consists of four chapters. The target structures in these chapters are, respectively, the smoothing property for parabolic and hydrostatic Stokes equations, maximal regularity for parabolic problems, and energy dissipation for gradient flows of planar curves. We here present a summary of each chapter.

In chapter 1, we address maximal regularity for parabolic equations. Here we give the definition abstractly. Let X be a Banach space and A be a closed linear operator on X . Then, we say that A has maximal L^p -regularity (on a interval $J = (0, T)$) if, for every $f \in L^p(J; X)$, there exists a unique mild solution u of the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in J, \\ u(0) = 0, \end{cases} \quad (0.1)$$

with an a priori estimate

$$\|u\|_{L^p(J; X)} + \|u'\|_{L^p(J; X)} + \|Au\|_{L^p(J; X)} \leq C\|f\|_{L^p(J; X)},$$

where C is independent of f . Typical examples of such operators are the Laplace and the Stokes operator, and these properties are applied to stability analysis for quasilinear parabolic problems and the Navier-Stokes equations, respectively. Thus, it is natural to investigate whether the discrete analog of maximal regularity holds. If we can establish the discrete version of maximal regularity, it is expected that the result can be applied to numerical analysis for nonlinear PDEs.

We first introduce the temporally discretized version of maximal regularity. Let $\tau > 0$ be a time increment and $\theta \in [0, 1]$ be a fixed parameter. Assume that the operator A is bounded since we shall consider the finite dimensional problems later. Then, we discretize the problem (0.1) by the θ -method as follows:

$$\begin{cases} \frac{u^{n+1} - u^n}{\tau} = Au^{n+\theta} + f^{n+\theta}, & n = 0, 1, \dots, N_T - 1, \\ u^0 = 0, \end{cases} \quad (0.2)$$

where $N_T = \lceil T/\tau \rceil$ is the number of steps, $u = (u^n)_n \in X^{N_T+1}$ is an unknown sequence, $f = (f^n)_n \in X^{N_T+1}$ is a given sequence, and $v^{n+\theta} = \theta v^{n+1} + (1 - \theta)v^n$ for $v = (v^n)_n$. Then, we define the discrete

version of maximal L^p -regularity as follows. We say that A has maximal l^p -regularity if for every $f \in X^{N_T+1}$, there exists a unique solution $u \in X^{N_T+1}$ of (0.2) with the estimate

$$\|u_\theta\|_{l_\tau^p(N_T;X)} + \|D_\tau u\|_{l_\tau^p(N_T;X)} + \|Au_\theta\|_{l_\tau^p(N_T;X)} \leq C\|f_\theta\|_{l_\tau^p(N_T;X)},$$

uniformly with respect to τ . Here, $\|\cdot\|_{l_\tau^p(N_T;X)}$ is defined by

$$\|v\|_{l_\tau^p(N_T;X)} = \left(\sum_{n=0}^{N_T-1} \|v^n\|_X^p \tau \right)^{1/p}$$

and the sequences u_θ , $D_\tau u$, and Au_θ are given by

$$u_\theta = (u^{n+\theta})_{n=0}^{N_T-1}, \quad D_\tau u = \left(\frac{u^{n+1} - u^n}{\tau} \right)_{n=0}^{N_T-1}, \quad Au_\theta = (Au^{n+\theta})_{n=0}^{N_T-1},$$

and f_θ is as well. We here emphasize that we can address the forward Euler method ($\theta = 0$).

One of the main results of this chapter is investigation of discrete maximal regularity for the discrete Laplace operator. Let $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) be a polygonal or polyhedral domain, \mathcal{T}_h be a shape-regular triangulation of Ω with $h = \max_{K \in \mathcal{T}_h} \text{diam } K$, and $S_h \subset H_0^1(\Omega)$ be the conforming P^1 -finite element space associated with \mathcal{T}_h . We define the discrete Laplace operator $A_h: S_h \rightarrow S_h$ by

$$(A_h u_h, v_h)_h = -(\nabla u_h, \nabla v_h), \quad \forall v_h \in S_h,$$

where the bracket (\cdot, \cdot) expresses the usual L^2 -inner product over Ω and $(\cdot, \cdot)_h$ is the discrete inner product with mass-lumping associated with barycentric domains. Then, we will show that A_h has discrete maximal regularity in $X_{h,q}$ for suitable q uniformly in h , where $X_{h,q} = (S_h, \|\cdot\|_{h,q})$ is a normed space equipped with a discrete L^q -norm (Theorem 1.2.2). For $\theta < 1/2$, we impose an assumption for stability, which corresponds to the CFL condition.

As an application, we consider the linear heat equation

$$\begin{cases} \partial_t u = \Delta u + f, & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \quad (0.3)$$

where $f \in L^p(0, T; L^q(\Omega))$ and $u_0 \in L^q(\Omega)$ are given data. We discretize the problem (0.3) by FEM for the spatial variables and the θ -method for the temporal variable as follows. Find $u_h = (u_h^n)_{n=0}^{N_T} \in S_h^{N_T+1}$ that satisfies

$$\begin{cases} \left(\frac{u_h^{n+1} - u_h^n}{\tau}, v_h \right)_h + (\nabla u_h^{n+\theta}, \nabla v_h) = (f^{n+\theta}, v_h), & \forall v_h \in S_h, \quad n = 0, 1, \dots, N_T - 1, \\ (u_h^0, v_h) = (u_0, v_h), \end{cases} \quad (0.4)$$

where $f^n = f(\cdot, n\tau)$. The problem (0.4) is equivalently written as

$$\begin{cases} \frac{u_h^{n+1} - u_h^n}{\tau} = A_h u_h^{n+\theta} + Q_h f^{n+\theta}, & n = 0, 1, \dots, N_T - 1, \\ u_h^0 = P_h u_0, \end{cases}$$

where $P_h: L^1(\Omega) \rightarrow S_h$ is the L^2 -projection and $Q_h: L^1(\Omega) \rightarrow S_h$ is a projection-like operator defined by

$$(Q_h w, v_h)_h = (w, v_h), \quad \forall v_h \in S_h$$

for $w \in L^1(\Omega)$. Therefore, the discretized problem (0.4) is formulated as a Cauchy problem in $X_{h,q}$. Consequently, we can apply the discrete maximal regularity for A_h and, under appropriate regularity assumptions, we can obtain an optimal order error estimate

$$\left(\sum_{n=0}^{N_T-1} \|u_h^{n+\theta} - U^{n+\theta}\|_{L^q}^p \tau \right)^{1/p} \leq C(h^2 + \tau^{j_\theta}),$$

where $U^n = u(\cdot, n\tau)$ and $j_\theta = 2$ if $\theta = 1/2$ and $j_\theta = 1$ otherwise (Theorem 1.2.3).

Moreover, we can consider a semilinear problem

$$\begin{cases} \partial_t u = \Delta u + f(u), & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases}$$

and its discretization

$$\begin{cases} \frac{u_h^{n+1} - u_h^n}{\tau} = A_h u_h^{n+1} + Q_h f(u_h^n), & n = 0, 1, \dots, N_T - 1, \\ u_h^0 = P_h u_0. \end{cases}$$

Then, we shall establish an optimal order error estimate

$$\left(\sum_{n=1}^{N_T} \|u_h^n - U^n\|_{L^q}^p \tau \right)^{1/p} \leq C(h^2 + \tau),$$

under appropriate assumptions (Theorem 1.2.4). In its proof, we utilize not only discrete maximal regularity for A_h , but also the fractional powers of the operator $-A_h$. That is, we introduce the intermediate norms $\|(-A_h)^\alpha \cdot\|_{h,q}$ and $\|(-A_h)^\alpha \cdot\|_{l^p_T(N_T; X_{h,q})}$ for $\alpha \in (-1, 1)$, which are, roughly speaking, the discrete counterparts of the Besov norms. We will present discrete versions of the Sobolev and the Gagliardo-Nirenberg inequalities (Lemmas 1.6.4 and 1.6.5), and combine them with discrete maximal regularity. Our technique is a new method in the literature on numerical analysis for nonlinear PDEs. This chapter is based on the published paper [68].

In chapter 2, we shall consider the discretization of the smoothing property and maximal regularity for parabolic problems on smooth domains. In the previous chapter, the domain Ω is assumed to be polygonal or polyhedral. However, it is known that the regularity of the solution of PDE cannot be guaranteed when there exist corners on the boundary. Lack of regularity is troublesome in not only mathematical analysis but also numerical analysis, since it is usually assumed that the exact solution has appropriate regularity. Indeed, in our results in Chapter 1, the range of the Lebesgue exponent q is restricted due to the loss of regularity. Since it is important to choose appropriate function spaces in the theory of nonlinear PDEs, it is natural to consider numerical analysis for PDEs on smooth domains to avoid such restriction.

From this viewpoint, we consider the following parabolic problem on a smooth (and possibly non-convex) domain $\Omega \subset \mathbb{R}^N$ for general $N \in \mathbb{N}$:

$$\begin{cases} \partial_t u + (-\Delta + 1)u = f, & \text{in } \Omega \times (0, T) =: Q_T, \\ \partial_n u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \quad (0.5)$$

where ∂_n denotes the outward normal derivative on $\partial\Omega$ and $f: Q_T \rightarrow \mathbb{R}$, and $u_0: \Omega \rightarrow \mathbb{R}$ are given data. In order to construct a finite element scheme for (0.5), we introduce a triangulation of Ω . Since the boundary $\partial\Omega$ is smooth, we cannot triangulate Ω exactly. Thus, we first approximate Ω by a polygonal domain Ω_h , and then we consider the triangulation \mathcal{T}_h of Ω_h . We assume that

- for each triangle $K \in \mathcal{T}_h$, $K \cap \Omega \neq \emptyset$
- for each node P of \mathcal{T}_h , $P \in \partial\Omega_h \implies P \in \partial\Omega$

and we set $h = \max_{K \in \mathcal{T}_h} \text{diam } K$. We emphasize that $\Omega \triangle \Omega_h \neq \emptyset$ in general, where $\Omega \triangle \Omega_h$ is the symmetric difference. Then, we can introduce the conforming P^k -finite element space $V_h \subset H^1(\Omega_h)$ with respect to the triangulation \mathcal{T}_h , and construct a finite element semi-discretization scheme for (0.5) as follows. Find $u_h \in C^0(0, T; V_h)$ that satisfies

$$\begin{cases} (u_{h,t}(t), v_h)_{\Omega_h} + (\nabla u_h(t), \nabla v_h)_{\Omega_h} + (u_h(t), v_h)_{\Omega_h} = (f_h(t), v_h)_{\Omega_h}, & \forall v_h \in V_h, \\ u_h(0) = u_{h,0}, \end{cases} \quad (0.6)$$

where $f_h: (0, T) \rightarrow V_h$ and $u_{h,0} \in V_h$ are given discrete data. Throughout this chapter, the bracket $(\cdot, \cdot)_D$ means the usual L^2 -inner product over $D \subset \mathbb{R}^N$.

The main purpose of this chapter is the smoothing property of the discrete parabolic semigroup in maximum-norm and maximal regularity for the discrete Laplace operator (Theorems 2.2.1 and 2.2.2), which are the target structures of this chapter. We shall present

$$\|u_h(t)\|_{L^\infty(\Omega_h)} + \|t\partial_t u_h(t)\|_{L^\infty(\Omega_h)} \leq Ce^{-ct}\|u_{h,0}\|_{L^\infty(\Omega_h)}, \quad \forall t > 0 \quad (0.7)$$

when $f_h \equiv 0$ in (0.6), and

$$\|\partial_t u_h\|_{L^p(0,T;L^q(\Omega_h))} + \|A_h u_h\|_{L^p(0,T;L^q(\Omega_h))} \leq C\|f_h\|_{L^p(0,T;L^q(\Omega_h))} \quad (0.8)$$

for $p, q \in (1, \infty)$ when $u_{h,0} = 0$. There are several studies on these estimates for FEM of parabolic problems with the Neumann boundary condition. However, all of them assume that the domain Ω is convex and $\Omega = \Omega_h$. The latter assumption is achieved by considering pie-shaped “triangles” near the boundary, which is impossible in the three-dimensional case. In contrast to existing literature, we consider a general smooth domain.

The difficulty for the error estimate (0.7) and (0.8) is the failure of Galerkin orthogonality. When $\Omega = \Omega_h$, the solutions u and u_h satisfy the relation

$$(\partial_t(u - u_h), v_h) + (\nabla(u - u_h), \nabla v_h) + (u - u_h, v_h) = 0, \quad \forall v_h \in V_h.$$

However, in our case, there appear additional terms induced by $\Omega \triangle \Omega_h$. In order to overcome this problem, we introduce the tubular neighborhood of $\partial\Omega$, and we address these additional terms as integration over the tubular neighborhood. This strategy allows us to consider the effect of gap of domains as just a perturbation.

In Chapter 3, we consider gradient flows of planar (closed) curves described in the form

$$\mathbf{u}_t = -\text{grad } E(\mathbf{u}), \quad t > 0, \quad (0.9)$$

where $\mathbf{u} = \mathbf{u}(\zeta): (0, 1) \rightarrow \mathbb{R}^2$ is an unknown closed planar curve and $\text{grad } E$ denotes the Fréchet derivative of a given energy functional E with respect to the $L^2(ds)$ -structure. Here, $ds = |\mathbf{u}_\zeta|d\zeta$ is the line element of the curve \mathbf{u} . Typical examples of (0.9) are the curvature flow

$$\mathbf{u}_t = \kappa$$

with the corresponding energy functional $E[\mathbf{u}] = \int ds$, and the elastic flow

$$\mathbf{u}_t = -2\varepsilon^2 \left(\nabla_s^2 \kappa + \frac{1}{2} |\kappa|^2 \kappa \right) + \kappa \quad (0.10)$$

with $E[\mathbf{u}] = \varepsilon^2 \int |\kappa|^2 ds + \int ds$ ($\varepsilon > 0$), where $\kappa = \mathbf{u}_{ss}$ is the curvature vector and ∇_s is the normal component of the arc-length derivative ∂_s . Note that the latter is a fourth-order nonlinear equation. These equations have the energy dissipation property $\frac{d}{dt} E[\mathbf{u}] \leq 0$. Indeed, by the definition of $\text{grad } E$, we can observe

$$\frac{d}{dt} E[\mathbf{u}] = \int \text{grad } E(\mathbf{u}) \cdot \mathbf{u}_t ds = - \int |\mathbf{u}_t|^2 ds \leq 0. \quad (0.11)$$

In the theory of dynamical systems, this property plays an important role in the stability analysis. Therefore, a numerical solution that preserves the energy dissipation is not only physically reasonable but also expected to be stable. In fact, there are several unified approaches for gradient flows of graphs of functions, such as the Allen-Cahn and the Cahn-Hilliard equations.

This chapter is devoted to construct a numerical scheme for a general gradient flow (0.9), which preserves the energy dissipation property. In contrast to existing work, the measure of the L^2 -structure depends on the unknown function \mathbf{u} , and thus we cannot apply existing schemes directly. In order to overcome this problem, we extend the discrete gradient method, which is one of the dissipative methods for gradient flows of graphs.

Here we present the strategy of our method. The fundamental idea is discretization of the chain rule (0.11). We first replace the time derivatives $\frac{d}{dt} R[\mathbf{u}]$ and \mathbf{u}_t by the differences $E[\mathbf{u}] - E[\mathbf{v}]$ and $\mathbf{u} - \mathbf{v}$ for two closed curves \mathbf{u} and \mathbf{v} . Then, the discrete chain rule should be of the form

$$E[\mathbf{u}] - E[\mathbf{v}] = \int \mathbf{X} \cdot (\mathbf{u} - \mathbf{v}) ds,$$

for appropriate function \mathbf{X} . The function \mathbf{X} corresponds to the gradient $\text{grad} E$, and thus we write $\mathbf{X} = \text{grad}_d E(\mathbf{u}, \mathbf{v})$. Here, the line element ds may depend on the functions \mathbf{u} and \mathbf{v} . Although there are several choices for ds , we here set $ds = ds((\mathbf{u} + \mathbf{v})/2) = (1/2)|\mathbf{u}_\zeta + \mathbf{v}_\zeta|d\zeta$. Namely, the formula

$$E[\mathbf{u}] - E[\mathbf{v}] = \int_0^1 \text{grad}_d E(\mathbf{u}, \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \left| \frac{\mathbf{u}_\zeta + \mathbf{v}_\zeta}{2} \right| d\zeta$$

is our discrete chain rule, and we define the discrete gradient $\text{grad}_d E$ as a function satisfying this formula. If we can find such discrete gradient, we formulate the temporally discretized scheme by

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = -\text{grad}_d E(\mathbf{u}^{n+1}, \mathbf{u}^n), \quad n = 0, 1, \dots,$$

with the initial condition. Then, by the definition of $\text{grad}_d E$, we can obtain the discrete energy dissipation

$$\frac{E[\mathbf{u}^{n+1}] - E[\mathbf{u}^n]}{\Delta t} = \int_0^1 \text{grad}_d E(\mathbf{u}^{n+1}, \mathbf{u}^n) \cdot \partial_d \mathbf{u}^n \left| \frac{\mathbf{u}_\zeta + \mathbf{v}_\zeta}{2} \right| d\zeta = - \int_0^1 |\partial_d \mathbf{u}^n|^2 \left| \frac{\mathbf{u}_\zeta + \mathbf{v}_\zeta}{2} \right| d\zeta \leq 0$$

where $\partial_d \mathbf{u}^n = (\mathbf{u}^{n+1} - \mathbf{u}^n)/\Delta t$. We will present the derivation of $\text{grad}_d E$ by the discrete gradient method.

By multiplying a test function and integrating by parts, we can construct a weak form of the temporally semi-discrete scheme, which also preserves the energy dissipation property. Then, we use the Galerkin method for spatial discretization. If the temporally semi-discrete equation is a variational problem on the periodic H^1 -space, we can use the space of piecewise linear functions as usual. However, for fourth-order equations, such as the elastic flow (0.10), the equation becomes a variational problem on the periodic H^2 -space, and thus the space of piecewise linear functions is not available. Instead, we use the space of B-spline functions of degree p , which is a subspace of C^{p-1} -space. Owing to the smoothness of B-splines, we can naturally handle the higher order derivatives, and thus we can construct a fully discretized numerical scheme for (0.9) (Scheme 3.3), which has the discrete energy dissipation property (Lemma 3.3.1). Moreover, since B-splines require few degrees of freedom, we can reduce the computational cost for the scheme.

In the last part of this chapter, we present numerical examples of our scheme in the context of the elastic flow (0.10). We can compute numerical solutions stably until they approach the steady state. Topology-changing solutions and more complicated motion are reported. Videos illustrating our method are available on YouTube¹. This chapter is based on the published paper [66].

In Chapter 4, the hydrostatic Stokes equations are considered. In this chapter, $G \subset \mathbb{R}^2$ denotes the unit square $(0, 1)^2$ and $\Omega \subset \mathbb{R}^3$ denotes the domain $G \times (-D, 0)$ for some $D > 0$. The hydrostatic Stokes equations are given as

$$\begin{cases} u_t - \Delta u + \nabla_H p = 0, & \text{in } \Omega \times (0, T), \\ \text{div}_H \bar{v} = 0, & \text{in } \Omega \times (0, T), \\ u(0) = u_0, & \text{in } \Omega \end{cases} \quad (0.12)$$

for an unknown velocity $u: \Omega \rightarrow \mathbb{R}^2$ and pressure $p: G \rightarrow \mathbb{R}$, where $\nabla_H p = (\partial_x p, \partial_y p)^T$, $\text{div}_H v = \partial_x v_1 + \partial_y v_2$, and $\bar{v} = \int_{-D}^0 v(\cdot, z) dz$. We impose the boundary conditions

$$\begin{cases} \partial_z u = 0, & \text{on } \Gamma_u := G \times \{0\}, \\ u = 0, & \text{on } \Gamma_b := G \times \{-D\}, \\ u \text{ and } p \text{ are periodic} & \text{on } \Gamma_l := \partial G \times (-D, 0). \end{cases}$$

The system (0.12) is a linearized problem of the primitive equations

$$\begin{cases} \partial_t u + (U \cdot \nabla) u - \Delta u + \nabla_H p = 0, & \text{in } \Omega \times (0, T), \\ \partial_z p = 0, & \text{in } \Omega \times (0, T), \\ \text{div } U = 0, & \text{in } \Omega \times (0, T), \\ u(0) = u_0, & \text{in } \Omega \end{cases} \quad (0.13)$$

¹URL: https://www.youtube.com/playlist?list=PLMF3dSqWEii691oXvCtgDCI4PYijq_4L3

with boundary conditions

$$\begin{cases} \partial_z u = 0, u_3 = 0, & \text{on } \Gamma_u, \\ U = 0, & \text{on } \Gamma_b, \\ U \text{ and } p \text{ are periodic} & \text{on } \Gamma_l, \end{cases}$$

where $U = (u, u_3): \Omega \rightarrow \mathbb{R}^3$. The primitive equations are derived from the Navier-Stokes equations under the assumption that the vertical motion is relatively smaller than the horizontal one. This model is considered to describe the geophysical flows such as the ocean and the atmosphere.

In contrast to the three-dimensional Navier-Stokes equations, the primitive equations (0.13) are globally well-posed in the L^p -setting. In the proof of global well-posedness, analyticity of the hydrostatic Stokes semigroup, which is involved by the hydrostatic Stokes equations (0.12), plays an important role. Therefore, it is expected that the analytic semigroup approach is effective in numerical analysis for the primitive equations. Indeed, for the two-dimensional Navier-Stokes equations, there are several studies on the finite element approximation via the analytic semigroup approach and, therein, analysis for FEM of the non-stationary Stokes equations plays a crucial role. Therefore, it is important to study the finite element approximation for the linear problem (0.12) in the framework of analytic semigroups in view of numerical analysis for the primitive equations.

In this chapter, we consider the finite element approximation for (0.12). First we introduce the weak formulation. Let $V = \{v \in H^1(\Omega)^2 \mid v|_{\Gamma_b} = 0, v \text{ is periodic on } \Gamma_l\}$, $Q = L_0^2(G)$, and $H = L^2(\Omega)^2$. Then, a weak form of the hydrostatic Stokes equations (0.12) can be given as a non-stationary saddle-point problem as follows. Find $u: (0, T) \rightarrow V$ and $p: (0, T) \rightarrow Q$ satisfying

$$\begin{cases} (u_t(t), v)_H + a(u(t), v) + b(v, p(t)) = 0, & \forall v \in V, \\ b(u(t), q) = 0, & \forall q \in Q, \\ u(0) = u_0, \end{cases} \quad (0.14)$$

where

$$a(u, v) = \iiint_{\Omega} \nabla u : \nabla v \, dx dy dz, \quad b(v, q) = - \iint_G (\operatorname{div}_H \bar{v}) q \, dx dy$$

for $u, v \in V$ and $q \in Q$. Then, we consider semi-discretization scheme for (0.14) by FEM. Let \mathcal{T}_h be a tetrahedralization of Ω and $\tilde{\mathcal{T}}_h$ be the triangulation of the upper boundary Γ_u induced by \mathcal{T}_h . Then, we define $V_h \subset V$ as the conforming P^2 -finite element space or the P^1 -bubble space associated with \mathcal{T}_h with the boundary condition on Γ_b , and define $Q_h \subset Q$ as the P^1 -finite element space associated with $\tilde{\mathcal{T}}_h$. Then, the finite element approximation for (0.14) is formulated as follows. Find $u_h: (0, T) \rightarrow V_h$ and $p_h: (0, T) \rightarrow Q_h$ satisfying the variational equation

$$\begin{cases} (u_{h,t}, v_h)_H + a(u_h, v_h) + b(v_h, p_h) = 0, & \forall v_h \in V_h, \\ b(u_h, q_h) = 0, & \forall q_h \in Q_h, \\ (u_h(0), v_h)_H = (u_0, v_h)_H, & \forall v_h \in V_{h,\sigma}, \end{cases} \quad (0.15)$$

where $V_{h,\sigma} = \{v_h \in V_h \mid b(v_h, q_h) = 0, \forall q_h \in Q_h\}$.

The purpose of this chapter is the error estimate for (0.15) in the framework of analytic semigroup theory. The problem (0.14) can be viewed as an abstract evolution equation defined on a pairs of Hilbert spaces (V, Q) , and (0.15) can be viewed as an abstract Galerkin approximation. Therefore, we first establish error estimates abstractly. In fact, the error estimate for the velocity u has been already established in [87] and it is known that the estimate

$$\|u(t) - u_h(t)\|_V \leq Ch t^{-1} \|u_0\|_H, \quad \|u(t) - u_h(t)\|_H \leq Ch^2 t^{-1} \|u_0\|_H$$

holds. Although the estimate is derived for the non-stationary Stokes equations only, we can repeat the same argument in our case. Here, the singular term t^{-1} is related to the smoothing property. In the same literature, the error estimate for the pressure is given as

$$\|p(t) - p_h(t)\|_Q \leq Ch \left(t^{-1} + t^{-3/2} \right) \|u_0\|_H,$$

which is not optimal in the sense that $t^{-3/2}$ is more singular than t^{-1} .

In order to remove the singular term $t^{-3/2}$, we developed the error estimate for u_t in the topology V' , which corresponds to the space $H^{-1}(\Omega)$ for the Stokes case with the whole Dirichlet condition. Then, we shall show the error estimate

$$\|u_t(t) - u_{h,t}(t)\|_{V'} \leq Cht^{-1}\|u_0\|_H.$$

As a result, we have succeeded in deriving the error estimate for the pressure

$$\|p(t) - p_h(t)\|_Q \leq Cht^{-1}\|u_0\|_H,$$

which is strictly sharper than the previous result (Theorem 4.3.1). As in the stationary case, the inf-sup condition

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geq \beta > 0 \quad (0.16)$$

plays a crucial role, where $\beta > 0$ is independent of h .

In the last part of the chapter, we will present applications to the hydrostatic Stokes equations (0.12), as well as the non-stationary Stokes problem. We give the error estimates for the finite element approximation of the hydrostatic Stokes equations (Theorem 4.4.2). In order to guarantee the inf-sup condition (0.16), we assume that the tetrahedralization is prismatic, i.e., \mathcal{T}_h is a subdivision of some decomposition of Ω into prisms. If \mathcal{T}_h is prismatic, then we can extend the function $q_h \in Q_h$ naturally as a function defined on Ω . This extension gives the inf-sup condition for the hydrostatic Stokes problem from the usual inf-sup condition for the Stokes case, and thus we can obtain the error estimates. This chapter is based on the accepted paper [67].

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Chapter 1

Discrete maximal regularity and the finite element method for parabolic equations

Abstract

Maximal regularity is a fundamental concept in the theory of partial differential equations. In this chapter, we establish a fully discrete version of maximal regularity for parabolic equations on a polygonal or polyhedral domain Ω . We derive various stability results in the discrete $L^p(0, T; L^q(\Omega))$ norms for the finite element approximation with the mass-lumping to the linear heat equation. Our method of analysis is an operator theoretical one using pure imaginary powers of operators and might be a discrete version of the result of Dore and Venni. As an application, optimal order error estimates in that norm are proved. Furthermore, we study the finite element approximation for semilinear heat equations with locally Lipschitz continuous nonlinear terms and offer a new method for deriving optimal order error estimates. Some interesting auxiliary results including discrete Gagliardo-Nirenberg and Sobolev inequalities are also presented. This chapter is based on the published paper [68]:

- T. Kemmochi and N. Saito. Discrete maximal regularity and the finite element method for parabolic equations. *Numer. Math.*, in press. DOI: <https://doi.org/10.1007/s00211-017-0929-z>

1.1 Introduction

Let Ω be a bounded polygonal or polyhedral domain in \mathbb{R}^d , $d = 2, 3$, with the boundary $\partial\Omega$. Let $J_T = (0, T)$ be a time interval with $T \in (0, \infty]$. We consider the finite element approximation of a linear heat equation for the function $u = u(x, t)$ of $(x, t) \in \bar{\Omega} \times [0, T)$:

$$\begin{cases} \partial_t u = \Delta u + f & \text{in } \Omega \times J_T, \\ u = 0 & \text{on } \partial\Omega \times J_T, \\ u|_{t=0} = u_0 & \text{on } \Omega, \end{cases} \quad (1.1)$$

where $\partial_t u = \partial u / \partial t$, $\Delta u = \sum_{j=1}^d \partial^2 u / \partial x_j^2$, $f = f(x, t)$, and $u_0 = u_0(x)$; f and u_0 are prescribed functions. All functions and function spaces considered in this chapter are complex-valued. We will discretize the problem (1.1) by the finite element method and the finite difference time-stepping method as follows.

$$\begin{cases} \left(\frac{u_h^{n+1} - u_h^n}{\tau}, v_h \right)_h + (\nabla u_h^{n+\theta}, \nabla v_h)_{L^2} = (F^{n+\theta}, v_h)_{L^2}, & \forall v_h \in S_h, \\ (u_h^0, v_h)_{L^2} = (u_0, v_h)_{L^2}, & \forall v_h \in S_h, \end{cases} \quad (1.2)$$

for $n = 0, 1, \dots$, where h is the size of the mesh of Ω , τ is the time increment, $S_h \subset H_0^1(\Omega)$ is the conforming P_1 -finite element space with the Dirichlet boundary condition, and $F^{n+\theta} = (1-\theta)f(\cdot, n\tau) + \theta f(\cdot, (n+1)\tau)$ for a fixed parameter $\theta \in [0, 1]$. The bracket $(\cdot, \cdot)_h$ is the discrete L^2 -inner product defined by the method of mass-lumping (see (1.14)). The precise settings will be described in Section 1.2.

The purpose of this chapter is to derive various stability estimates for solution of (1.2) in the discrete version of $L^p(J_T; L^q(\Omega))$ norms defined by (1.10) with $X = L^q(\Omega)$, where $p, q \in (1, \infty)$. As applications of those estimates, we derive optimal order error estimates in those norms for the problem (1.2).

We also consider a semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + f(u) & \text{in } \Omega \times J_T \\ u = 0 & \text{on } \partial\Omega \times J_T, \\ u|_{t=0} = u_0 & \text{in } \Omega, \end{cases} \quad (1.3)$$

and its approximation

$$\begin{cases} ((D_\tau u_h)^n, v_h)_h + (\nabla u_h^{n+1}, \nabla v_h)_{L^2} = (f(u_h^n), v_h)_{L^2}, & \forall v_h \in S_h, \\ (u_h^0, v_h)_{L^2} = (u_0, v_h)_{L^2}, & \forall v_h \in S_h, \end{cases} \quad (1.4)$$

for $n = 0, 1, \dots$, where $f: \mathbb{C} \rightarrow \mathbb{C}$ is a prescribed nonlinear function. Particularly, we assume only a locally Lipschitz continuity. One of the main contribution of the present chapter is to offer a new method for optimal order error analysis for the problem (1.4).

In order to achieve these purposes, we intend to develop a discrete version of the theory of maximal regularity for evolution equations of parabolic type. To recall maximal regularity in a general context, let us consider an abstract Cauchy problem on a Banach space X :

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in J_T, \\ u(0) = 0, \end{cases} \quad (1.5)$$

where A is a densely defined closed operator on X with the domain $D(A) \subset X$, $f: J_T \rightarrow X$ is a given function, $u: J_T \rightarrow X$ is an unknown function and $u'(t) = du(t)/dt$.

Definition 1.1 (Maximal regularity, MR, CMR). Let $p \in (1, \infty)$. The operator A has *maximal L^p -regularity* on J_T , if and only if, for every $f \in L^p(J_T; X)$, there exists a unique solution $u \in W^{1,p}(J_T; X) \cap L^p(J_T; D(A))$ of (1.5) satisfying

$$\|u\|_{L^p(J_T; X)} + \|u'\|_{L^p(J_T; X)} + \|Au\|_{L^p(J_T; X)} \leq C_{\text{MR}} \|f\|_{L^p(J_T; X)}, \quad (1.6)$$

where $C_{\text{MR}} > 0$ denotes a constant that is independent of f . We say that A has *maximal regularity* (MR) if A has maximal L^p -regularity for some $p \in (1, \infty)$. To distinguish maximal (L^p -)regularity from the discrete versions introduced later, we say that A has *continuous maximal L^p -regularity* (L^p -CMR) and *continuous maximal regularity* (CMR).

It is proved that the $L^q(\Omega)$ -realization A_q of the Laplacian Δ with $D(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ has L^p -CMR for any $p, q \in (1, \infty)$ (see [72]). The problem (1.1) admits a unique solution $u \in W^{1,p}(J_T; L^q(\Omega)) \cap L^p(J_T; D(A_q))$ satisfying (1.6) with $u_0 = 0$. This result implies that $\partial_t u$ and Δu are well defined and have the same regularity as the right-hand side function f . Moreover, $\partial_t u$ and Δu cannot be in a better function space than f , since $f = \partial_t u - \Delta u$. This is not a trivial fact. For comparison, we recall the solution obtained using the analytic semigroup theory, which is a powerful method to establish the well-posedness of (1.1) and (1.3). For example, assume $f \in C^\sigma(\overline{J_T}; L^q(\Omega))$ for some $\sigma \in (0, 1)$, that is, assume

$$\sup_{t, s \in J_T, t \neq s} \frac{\|f(t) - f(s)\|_{L^q(\Omega)}}{|t - s|^\sigma} < \infty.$$

Then, by application of the analytic semigroup theory, we can prove that the problem (1.1) with $u_0 = 0$ admits a unique solution $u \in C(\overline{J_T}; L^q(\Omega)) \cap C(J_T; D(A_q)) \cap C^1(J_T; L^q(\Omega))$; see [88, Theorems 4.3.2 and 7.3.5]. However, we are able to obtain slightly less regularity $\partial_t u, \Delta u \in C(J_T; L^q(\Omega))$ than f . To obtain the same regularity $\partial_t u, \Delta u \in C^\sigma(\overline{J_T}; L^q(\Omega))$, we must further assume $f(x, 0) = 0$ for all $x \in \Omega$; see [88,

Theorem 4.3.5]. Therefore, $W^{1,p}(J_T; L^q(\Omega)) \cap L^p(J_T; D(A_q))$ is an appropriate function space to study parabolic equations such as (1.1).

Moreover, CMR is a “stronger” property than the generation of analytic semigroup in the sense that, if an operator A has CMR, then A generates the analytic (bounded) semigroup (cf. [33]). Although CMR is a concept for linear equations, it actually has many important applications to nonlinear equations, as reported in the literature (see e.g. [4, 40]). Moreover, the analytic semigroup theory and its discrete counterparts play important roles in construction and study of numerical schemes for parabolic equations (see e.g. [37, 43, 46, 91, 92, 102]). Therefore, it is natural to wonder whether a discrete version of CMR is available.

This study has another motivation. Let us consider the problem (1.3) with $f(u) = u|u|^\alpha$ for $\alpha > 0$. Without loss of generality, we can assume $0 \in \Omega$. Then, for $\lambda > 0$, the function

$$u_\lambda(x, t) = \lambda^{\frac{2}{\alpha}} u(\lambda x, \lambda^2 t)$$

also solves (1.3) where Ω and J_T are replaced, respectively, by $\Omega_\lambda = \{\lambda^{-1}x \mid x \in \Omega\}$ and J_{T/λ^2} . Moreover, if $p, q \in (1, \infty)$ satisfy

$$\frac{2}{\alpha} = \frac{d}{p} + \frac{2}{q}, \quad (1.7)$$

we have

$$\|u_\lambda\|_{L^p(J_{T/\lambda^2}; L^q(\Omega_\lambda))} = \|u\|_{L^p(J_T; L^q(\Omega))}$$

for any $\lambda > 0$. Those p, q are called the scale invariant exponents. The function space $L^p(J_T; L^q(\Omega))$ with p, q satisfying (1.7) plays a crucially important role in the study of time-local and time-global well-posedness of (1.3). Furthermore, such a scaling argument is applied to deduce a novel numerical method for solving (1.3) (see [12]). Therefore, it would be interesting to derive stability and error estimates in those norms from the dual perspectives of numerical and theoretical analysis.

Based on those motivations, we studied a time discrete version of maximal regularity for (1.5) in an earlier study [65]. Let

$$N_T = \begin{cases} \lfloor T/\tau \rfloor & (T < \infty), \\ \infty & (T = \infty). \end{cases} \quad (1.8)$$

We consider the implicit θ scheme for (1.5) given as

$$\begin{cases} \frac{u^{n+1} - u^n}{\tau} = Au^{n+\theta} + f^{n+\theta}, & n = 0, 1, \dots, N_T - 1, \\ u^0 = 0, \end{cases} \quad (1.9)$$

where $\tau > 0$ is the time increment, $\theta \in [0, 1]$, $f = (f^n)_{n=0}^{N_T}$ is a given X^{N_T+1} -valued function, and $u = (u^n)_{n=0}^{N_T}$ is an unknown X^{N_T+1} -valued function. Set

$$v^{n+\theta} = (1 - \theta)v^n + \theta v^{n+1}$$

for a sequence $v = (v^n)_n$. We moreover assume that A is bounded when $\theta \neq 1$. The function u^n might be an approximation of $u(n\tau)$ for $n = 1, \dots, N_T$.

We introduce the space $l^p(N; X)$ by setting

$$l^p(N; X) = \begin{cases} X^{N+1}, & N \in \mathbb{N}, \\ l^p(\mathbb{N}; X), & N = \infty, \end{cases}$$

and let

$$\begin{aligned} \|v\|_{l^p_\tau(N; X)} &= \left(\sum_{n=0}^{N-1} \|v^n\|_X^p \tau \right)^{1/p}, \\ D_\tau v &= \left(\frac{v^{n+1} - v^n}{\tau} \right)_{n=0}^{N-1}, \quad Av = (Av^n)_{n=0}^N, \quad v_\theta = (v^{n+\theta})_{n=0}^{N-1}, \end{aligned} \quad (1.10)$$

for $v = (v^n) \in l^p(N; X)$.

Discrete maximal regularity is then introduced as follows (see [65]).

Definition 1.2 (Discrete maximal regularity, DMR). Let $p \in (1, \infty)$. The operator A has *maximal l^p -regularity* (*l^p -DMR*) on J_T if and only if, for every $f \in l^p(N_T; X)$, there exists a unique solution $u \in X^{N_T}$ of (1.9) satisfying

$$\|u_\theta\|_{l^p_\tau(N_T; X)} + \|D_\tau u\|_{l^p_\tau(N_T; X)} + \|Au_\theta\|_{l^p_\tau(N_T; X)} \leq C_{\text{DMR}} \|f_\theta\|_{l^p_\tau(N_T; X)}, \quad (1.11)$$

uniformly with respect to τ , where $C_{\text{DMR}} > 0$ is independent of f . We say that A has *discrete maximal regularity* (DMR) if A has l^p -DMR for some $p \in (1, \infty)$.

DMR was introduced and developed in [6] and [5] for the backward Euler method ($\theta = 1$) and the Crank-Nicolson method ($\theta = 1/2$). In these literature, it was shown that CMR implies DMR (when $\theta = 0$ or $1/2$), although the constant may depend on the Banach spaces under consideration. DMR for the forward Euler method ($\theta = 0$) was studied by Blunck [13] and characterized by developing a discrete version of the operator-valued Fourier multiplier theorem. However, the dependence of τ on DMR inequalities is not clear since only the case $\tau = 1$ is studied. By contrast, we considered the general θ -method in [65] and proposed reasonable sufficient conditions for DMR, with uniform constants for τ and the Banach space. This result is summarized in Lemma 1.3.4 below.

Other temporal discretization methods are available. The discontinuous Galerkin time-stepping (DGT) method was considered by Leykekhman and Vexler [74]. They specifically examined the time-discrete version of L^p - L^q -maximal regularity for arbitrary $p, q \in [1, \infty]$ for parabolic problems. This result is valid for $p, q = 1, \infty$. However, they did not consider the R-boundedness of sets of operators, which plays an important role in the theory of maximal regularity developed by Weis [101]. The main tools in [74] were the smoothing properties of the continuous and discrete Laplace operators. Consequently, their estimate invariably contained the logarithmic term, so that the optimal error estimate is never obtained. It was established by [70] that arbitrary A-stable time-discretization preserves the time-discrete version of maximal L^p -regularity for abstract Cauchy problems and for $p \in (1, \infty)$. This result was obtained via the theory of R-boundedness. It is therefore partially the same result of our previous work [65]. An optimal error estimate was established only for the semi-discrete backward Euler scheme for a semilinear parabolic problem.

Spatial discretizations were also addressed. Let us consider semi-discretization for the spatial variables. MR for the finite element approximation of the second order elliptic operators was first obtained by Geissert [47, 48]. He considered a smooth convex domain Ω and triangulations defined on a polyhedral approximation Ω_h of Ω . (For the Neumann boundary condition case, he considered the exactly fitted triangulation.) The method is based on the discrete Green's function, which was originally introduced for L^∞ -analysis (see e.g. [94, 98]). The results were also applied to error estimates of the finite element semi-discretization for linear and semilinear parabolic problems. The estimate is local in time for the semilinear case. The result on CMR was generalized for non-smooth coefficients and non-smooth domains in [75, 77] using several modifications of estimates for the discrete Green's function. In particular, Li and Sun considered elliptic operators with Hölder continuous coefficients on polygonal or polyhedral domains in [77], which is a more general setting than ours. However, our approach is completely different, as mentioned later.

Full-discretization and its application for error estimates are the main interests of this study. Uniform DMR estimates for elliptic operators were established by Li and Sun [78] with the backward Euler scheme on smooth domains. They applied the result to error estimates for fully discretized linear parabolic problems. The convergence rate is optimal if the exact solution is smooth. Leykekhman and Vexler [74] applied their results on the DGT method to the finite element approximation. Some DMR results and error estimates were established for the finite element approximations of linear and semilinear parabolic problems. They still could not remove a logarithmic factor in their error estimates as mentioned above. Kovács et al. [70] also discussed fully discrete schemes by the finite element method for linear and semilinear problems. The error estimate is sub-optimal for the semilinear case.

In the present chapter, we take a completely different approach. We directly establish a discrete version of the method using pure imaginary powers of operators developed by [34]. To this end, we study the discrete Laplacian *with mass-lumping* A_h instead of the standard discrete Laplacian. Actually, the positivity-preserving property of the semigroup generated by A_h (see Lemma 1.3.6) plays an important role in our analysis, and the standard discrete Laplacian has no such property (see [99]). By the same reason, our attention is focused on the Laplace operator, which enables us to give a simple proof of MR for the discrete operator. Moreover, it must be borne in mind that the L^q theory for the discrete Laplacian

with *mass-lumping* is of great use in studying of nonlinear problems, such as the finite element and finite volume approximation of the Keller-Segel system modelling chemotaxis (see [91, 92, 102]).

The novel contribution of the present chapter is summarized as follows.

- After having established MR and DMR for A_h (see Theorems 1.2.1 and 1.2.2), we derive optimal order error estimates for the approximation scheme (1.2) to the linear heat equation (1.1) (see Theorem 1.2.3). We address not only unconditionally stable cases ($\theta \in [1/2, 1]$), but also *conditionally stable cases* ($\theta \in [0, 1/2)$). For the latter case, we give a useful sufficient condition for the scheme to be stable.
- We derive optimal order error estimates for fully discretized scheme (1.4) to the semilinear parabolic problem (1.3) (see Theorem 1.2.4). We propose a new method for analysis of the finite element approximation of semilinear problems, with the aid of DMR and the theory of fractional powers of operators. Since the nonlinear function f is assumed to be only locally Lipschitz continuous, the solution u might blow up in some sense. Our error estimate is valid as long as the classical solution u exists in contrast to [48]. We will also derive a sub-optimal error estimate in the $L^\infty(\Omega \times (0, T))$ norm as an intermediate result (see Theorem 1.2.5). Some auxiliary results including discrete Gagliardo-Nirenberg and Sobolev inequalities are also presented (see Lemmas 1.6.4).

The plan of this chapter is as follows. In Section 1.2, we introduce the notion of finite element approximation and state the main results (Theorems 1.2.1–1.2.5). We summarize some preliminary results used in the proofs of Theorems in Section 1.3. Some auxiliary lemmas related to MR, DMR and A_h are described there. A useful sufficient condition for DMR to hold is also described there (Lemma 1.3.3). In Section 1.4, we prove Theorems 1.2.1 and 1.2.2 by a discrete version of the method of [34] using pure imaginary powers of operators. The proof of error estimate (Theorem 1.2.3) for the linear equation (1.1) is described in Section 1.5. The semilinear equation (1.3) is studied in Section 1.6. Therein, we also present auxiliary results including discrete Gagliardo-Nirenberg and Sobolev inequalities and provide useful results related to the fractional powers of A_h . Combining those results, we show the final error estimate, Theorems 1.2.4 and 1.2.5.

1.2 Main results

Throughout this chapter, Ω is assumed to be a bounded polygonal or polyhedral domain in \mathbb{R}^d , $d = 2, 3$, with the boundary $\partial\Omega$. We follow the notation of [1]. As an abbreviation, we write $L^q = L^q(\Omega)$, $W^{s,q} = W^{s,q}(\Omega)$ and $H^s = W^{s,2}$ for $q \in [1, \infty]$ and $s > 0$. We use $W_0^{1,q} = \{v \in W^{1,q} \mid v|_{\partial\Omega} = 0\}$ and $H_0^1 = W_0^{1,2}$. Generic positive constants which are independent of discretization parameters, h and τ , are denoted by C . Their values might be different in each appearance.

Since the boundary $\partial\Omega$ is not smooth, we make the following shape assumption on Ω .

Assumption 1.1 (Shape assumption on Ω). There exists $\mu > d$ satisfying

$$\|v\|_{W^{2,q}} \leq C \|\Delta v\|_{L^q}, \quad \forall v \in W^{2,q} \cap W_0^{1,q}, \quad (1.12)$$

for $q \in (1, \mu)$, where $C > 0$ depends only on Ω and q .

For example, if Ω is a convex polygonal domain in \mathbb{R}^2 , then one can find $\mu > 2$ satisfying Assumption 1.1 (see [50]).

Let \mathcal{T}_h be a conforming triangulation (tetrahedralization) of Ω with the granularity parameter $h = \max_{K \in \mathcal{T}_h} h_K$, where $h_K = \text{diam } K$. We assume the following.

Assumption 1.2 (Regularity of $\{\mathcal{T}_h\}_h$). There exists $\nu > 0$ such that

$$h_K \leq \nu \rho_K, \quad \forall K \in \mathcal{T}_h, \quad \forall h > 0,$$

where ρ_K denotes the radius of the inscribed circle or sphere of K .

Here we consider the P_1 -finite element method. Let V_h be the conforming P_1 -finite element space on Ω with respect to the mesh \mathcal{T}_h and let $S_h = V_h \cap H_0^1$. We denote the nodes (which may be on $\partial\Omega$) of \mathcal{T}_h and the corresponding basis functions of V_h by P_j and ϕ_j , respectively for $j = 1, 2, \dots, \bar{N}_h := \dim V_h$. Namely, $\phi_j(P_i) = \delta_{ij}$, where δ_{ij} is Kronecker's delta.

Moreover, we presume that $\{\mathcal{T}_h\}_h$ satisfies the following conditions if necessary.

(H1) (Quasi-uniformity) There exists $\gamma > 0$ such that

$$h \leq \gamma h_K, \quad \forall K \in \mathcal{T}_h, \quad \forall h > 0.$$

(H2) (Acuteness) For each $h > 0$ and for each $i, j \in \{1, 2, \dots, N_h\}$ with $i \neq j$,

$$\int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j \, dx \leq 0. \quad (1.13)$$

Remark 1.1. In the two-dimensional case, let $\sigma \subset \Omega$ be an edge of the triangulation \mathcal{T}_h and K and L be the triangles of \mathcal{T}_h meeting in σ . We denote the interior angle of K and L opposite to the edge σ by α^K and α^L , respectively. Then, the condition (1.13) is equivalent to the equation of $\alpha^K + \alpha^L \leq \pi$. See [69, Corollary 3.48] for the detail.

Remark 1.2 (Discrete maximum principle). The condition (H2) is equivalent to the discrete maximum principle, i.e., the following conditions are equivalent.

- (i) The triangulation \mathcal{T}_h fulfills the acuteness condition.
- (ii) Let $u_h \in V_h$ be the solution of the following problem for $f \in L^2$ and $g_h \in V_h$:

$$\begin{cases} (\nabla u_h, \nabla v_h)_{L^2} = (f, v_h)_{L^2}, & \forall v_h \in S_h, \\ u_h|_{\partial\Omega} = g_h. \end{cases}$$

Then, $u_h \geq 0$ in Ω provided that $f \geq 0$ in Ω and $g_h \geq 0$ on $\partial\Omega$.

See [69, Theorem 3.49] for details.

Remark 1.3. We will discuss the possibility to relax the mesh condition (H1). We impose (H1) for the stability of the L^2 - and H^1 -projections P_h and R_h (see Lemma 1.3.8) and the global inverse inequality. The weaker conditions for the stability of P_h can be found in [14, 7], and the relaxation for R_h was studied in [31]. Thus we can relax the mesh conditions for Theorems 1.2.1–1.2.3 below. However, since the global inverse inequality is essential for the proof of the discrete Gagliardo-Nirenberg inequality (Lemma 1.6.4), we cannot relax (H1) for the semilinear case (Theorem 1.2.4 and 1.2.5). Throughout this chapter, we will impose the stronger condition (H1) for simplicity. See also Remark 1.6.

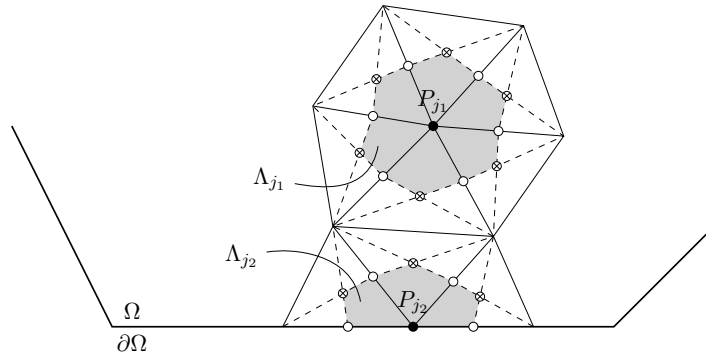


Figure 1.1: Barycentric domains in 2-D. P_{j1} : interior node, P_{j2} : boundary node. \bullet : node, \circ : midpoint of an edge, \otimes : barycenter of a triangle.

We describe the method of mass-lumping. For a node P_j , we designate the corresponding barycentric domain as $\Lambda_j = \{x \in \Omega \mid \lambda_j(x) \geq \lambda_i(x), \forall i \neq j\}$, where λ_j is the corresponding barycentric coordinate (see Figure 1.1 for illustration and see [69] for the definition). We denote the characteristic function of Λ_j by χ_j for $j = 1, \dots, N_h = \dim S_h$. Then, we set

$$\bar{S}_h = \text{span}\{\chi_j\}_{j=1}^{N_h}$$

and define the lumping operator $M_h: S_h \rightarrow \bar{S}_h$ by

$$M_h v_h = \sum_{j=1}^{N_h} v_h(P_j) \chi_j.$$

Moreover, we define $K_h = M_h^* M_h$, where M_h^* is the adjoint operator of M_h with respect to the L^2 -inner product. As one might expect, M_h is invertible and therefore K_h is as well. We define the mesh-dependent norms and inner product as

$$\|v_h\|_{h,q} = \|M_h v_h\|_{L^q}, \quad (u_h, v_h)_h = (M_h u_h, M_h v_h), \quad u_h, v_h \in S_h \quad (1.14)$$

for $q \in [1, \infty]$. In fact, $\|\cdot\|_{h,q}$ is an equivalent norm to $\|\cdot\|_{L^q}$ in S_h for each $q \in [1, \infty]$. Indeed, one can see that

$$C^{-1} \|v_h\|_{L^q} \leq \|M_h v_h\|_{L^q} \leq C \|v_h\|_{L^q}, \quad v_h \in S_h, \quad q \in [1, \infty]$$

for some $C > 0$ depending only on q and Ω . Moreover, if $\{\mathcal{T}_h\}_h$ satisfies (H1) when $q \neq 2$, then there exists $C > 0$ depending only on q and Ω such that

$$C^{-1} \|v_h\|_{L^q} \leq \|K_h v_h\|_{L^q} \leq C \|v_h\|_{L^q}, \quad v_h \in S_h, \quad q \in [1, \infty]. \quad (1.15)$$

For the proof, see [43]. We finally define the discrete L^q -space by $X_{h,q} := (S_h, \|\cdot\|_{h,q})$ for $q \in [1, \infty]$. We also equip the space $X_{h,2}$ with the structure of the Hilbert space with respect to the inner product $(\cdot, \cdot)_h$.

At this stage, we introduce the discrete Laplacian as follows. Define the operator A_h on S_h by

$$(A_h u_h, v_h)_h = -(\nabla u_h, \nabla v_h), \quad \forall v_h \in S_h, \quad (1.16)$$

for $u_h \in S_h$. We designate A_h as the discrete Laplacian *with mass-lumping*. From the Poincaré inequality, A_h is injective so that it is invertible due to $\dim S_h < \infty$. We equip the space S_h with the norm $\|A_h \cdot\|_{h,q}$ and denote it by $D(A_h)$.

We are now in a position to state the main results of this study. In the theorems below, we always presume that Assumptions 1.1 and 1.2 are satisfied, unless otherwise stated explicitly. The first one is about CMR for A_h . Recall that $J_T = (0, T) \subset \mathbb{R}$.

Theorem 1.2.1 (CMR for A_h). *Let $T \in (0, \infty]$, $p \in (1, \infty)$, and $q \in (1, \mu)$. Assume that (H1) and (H2) are satisfied when $q \neq 2$. Then, A_h has L^p -CMR on J_T in $X_{h,q}$ uniformly in $h > 0$. That is, for every $f_h \in L^p(J_T; X_{h,q})$, there exists a unique solution $u_h \in W^{1,p}(J_T; X_{h,q}) \cap L^p(J_T; D(A_h))$ of the equation*

$$\begin{cases} u_h'(t) = A_h u_h(t) + f_h(t), & t \in J_T, \\ u_h(0) = 0 \end{cases} \quad (1.17)$$

that satisfies

$$\|u_h\|_{L^p(J_T; X_{h,q})} + \|u_h'\|_{L^p(J_T; X_{h,q})} + \|A_h u_h\|_{L^p(J_T; X_{h,q})} \leq C \|f_h\|_{L^p(J_T; X_{h,q})},$$

where $C > 0$ is independent of f_h , h , and T .

Remark 1.4. Since (1.17) is a system of (inhomogeneous) linear ordinary differential equations, the unique existence of a solution follows immediately.

Next, we state results about DMR for A_h . To state them, we set

$$\theta_q = \arccos |1 - q/2|, \quad (1.18)$$

$$\kappa_h = \min_{K \in \mathcal{T}_h} \kappa_K, \quad (1.19)$$

where κ_K denotes the minimum length of perpendiculars of K . Recall that $v^{n+\theta} = (1 - \theta)v^n + \theta v^{n+1}$ and $v_\theta = (v^{n+\theta})_n$ for a sequence $v = (v^n)_n$.

Theorem 1.2.2 (DMR for A_h). *Let $T > 0$, $p \in (1, \infty)$, $q \in (1, \mu)$, and $\theta \in [0, 1]$. Assume that (H1) and (H2) are satisfied when $q \neq 2$. We choose ε and τ sufficiently small so that*

$$\frac{\tau}{\kappa_h^2} \leq \frac{2 \sin \theta_q - \varepsilon}{(1 - 2\theta)(d + 1)^2} \quad (1.20)$$

is satisfied, when $\theta \in [0, 1/2)$. Then, for every $f_h \in l^p(N_T; X_{h,q})$, there exists a unique solution $u_h \in l^p(N_T; X_{h,q})$ of the equation

$$\begin{cases} (D_\tau u_h)^n = A_h u_h^{n+\theta} + f_h^{n+\theta}, & n = 0, 1, \dots, N_T - 1, \\ u_h^0 = 0 \end{cases}$$

that satisfies

$$\|u_{h,\theta}\|_{l_\tau^p(N_T; X_{h,q})} + \|D_\tau u_h\|_{l_\tau^p(N_T; X_{h,q})} + \|A_h u_{h,\theta}\|_{l_\tau^p(N_T; X_{h,q})} \leq C \|f_{h,\theta}\|_{l_\tau^p(N_T; X_{h,q})} \quad (1.21)$$

where $C > 0$ is independent of f_h , T , h , and τ .

Moreover, for $\theta = 1$, if $u_{0,h} \in (X_{h,q}, D(A_h))_{1-1/p,p}$, then there exists a unique solution $u_h \in l^p(N_T; X_{h,q})$ of the equation

$$\begin{cases} (D_\tau u_h)^n = A_h u_h^{n+1} + f_h^{n+1}, & n = 0, 1, \dots, N_T - 1, \\ u_h^0 = u_{0,h}, \end{cases}$$

that satisfies

$$\|u_{h,1}\|_{l_\tau^p(N_T; X_{h,q})} + \|D_\tau u_h\|_{l_\tau^p(N_T; X_{h,q})} + \|A_h u_{h,1}\|_{l_\tau^p(N_T; X_{h,q})} \leq C [\|f_{h,1}\|_{l_\tau^p(N_T; X_{h,q})} + \|u_{0,h}\|_{h,1-1/p,p}],$$

with the same constant C as in (1.21).

Therein, $(X_{h,q}, D(A_h))_{1-1/p,p}$ and $\|\cdot\|_{h,1-1/p,p}$ respectively denote the real interpolation space and its norm.

Those theorems are applicable for error analysis of the fully discretized finite element approximation for heat equations. First, we consider a linear heat equation (1.1) for $T \in (0, \infty)$, $f \in L^p(J_T; L^q)$ and $u_0 \in L^q$. We further assume $f \in C^0(\bar{J}_T; L^q)$. We consider the solution $u_h = (u_h^n)_n \in l^p(N_T; S_h)$ of (1.2), where $\tau \in (0, 1)$, $\theta \in [0, 1]$, $t_n = n\tau$, and $F^n = f(\cdot, t_n)$.

Let P_h be the L^2 -projection onto S_h defined by

$$(P_h v, v_h)_{L^2} = (v, v_h)_{L^2}, \quad \forall v_h \in S_h \quad (1.22)$$

for $v \in L^1$. Then, (1.2) is equivalently written as

$$\begin{cases} (D_\tau u_h)^n = A_h u_h^{n+\theta} + K_h^{-1} P_h F^{n+\theta}, & n = 0, 1, \dots, N_T - 1, \\ u_h^0 = P_h u_0. \end{cases} \quad (1.23)$$

Since A_h is invertible, there exists a unique solution of (1.23). We introduce

$$j_\theta = \begin{cases} 2, & \theta = 1/2, \\ 1, & \text{otherwise} \end{cases} \quad (1.24)$$

and $\mu_d = \max\{\mu', d/2\}$. Since $\mu' < d' \leq 2 \leq d < \mu$, it might be apparent that

$$\mu_d = \begin{cases} \mu', & d = 2, \\ d/2 = 3/2, & d = 3. \end{cases}$$

Theorem 1.2.3 (Error estimate for a linear equation). *Let $T > 0$, $p \in (1, \infty)$, and $q \in (\mu_d, \mu)$. Let $u_h = (u_h^n)_n \in l^p(N_T; S_h)$ be the solution of (1.23) and u be that of (1.1). Assume $u \in W^{1,p}(J_T; W^{2,q}) \cap W^{2,p}(J_T; W^{1,q}) \cap W^{1+j_\theta,p}(J_T; L^q)$ and set $U^n = u(\cdot, t_n)$. Assume that (H1) and (H2) are satisfied.*

Moreover, we choose ε and τ sufficiently small so that (1.20) is satisfied, when $\theta \in [0, 1/2)$. Then, there exists a positive constant C such that

$$\left(\sum_{n=0}^{N_T-1} \|u_h^{n+\theta} - U^{n+\theta}\|_{L^q}^p \tau \right)^{1/p} \leq C(h^2 + \tau^{j_\theta}). \quad (1.25)$$

The constant C is taken as

$$C = C' \cdot (\|u\|_{W^{1,p}(J_T; W^{2,q})} + \|\partial_t u\|_{W^{1,p}(J_T; W^{1,q})} + \|u\|_{W^{1+j_\theta,p}(J_T; L^q)}),$$

where C' depends only on Ω , p , q , and θ , but is independent of h , τ , and T .

For $q \in (1, \infty)$, let A_q be the realization of the Dirichlet Laplacian:

$$D(A_q) = W^{2,q} \cap W_0^{1,q}, \quad A_q u = \Delta u. \quad (1.26)$$

We are assuming Assumption 1.1 so that $\|A_q \cdot\|_{L^q}$ is an equivalent norm on $D(A_q)$ for $q \in (1, \mu)$. We consider a semilinear heat equation (1.3) under the following basic assumptions:

$$u_0 \in (L^q, D(A_q))_{1-1/p, p}, \quad (1.27)$$

$$f: \mathbb{C} \rightarrow \mathbb{C} \text{ is locally Lipschitz continuous with } f(0) = 0. \quad (1.28)$$

Restriction $f(0) = 0$ is set for simplicity. It is noteworthy that the solution u of (1.3) might blow-up: let $T_\infty \in (0, \infty]$ be the life span of u (the maximal existence time of u). To avoid unnecessary difficulties, we restrict our consideration to a semi-implicit scheme for (1.3) given by (1.4) or, equivalently,

$$\begin{cases} (D_\tau u_h)^n = A_h u_h^{n+1} + K_h^{-1} P_h f(u_h^n), & n = 0, 1, \dots, N_T - 1, \\ u_h^0 = P_h u_0. \end{cases} \quad (1.29)$$

Since A_h is invertible, there exists a unique solution of (1.29). Our final theorem is the following error estimate for semilinear equation. Our error estimate remains valid as long as the classical solution of (1.3) exists and requires no size condition on u_0 .

Theorem 1.2.4 (Error estimate for semilinear equations). *Let $p \in (1, \infty)$, $q \in (\mu_d, \mu)$ and $p > 2q/(2q-d)$. Assume that (H1) and (H2) are satisfied. Presuming that (1.3) admits a sufficiently smooth solution u under the conditions (1.27) and (1.28), then, for every $T \in (0, T_\infty)$ and the solution $u_h = (u_h^n)_{n=0}^{N_T}$ of (1.29), we have*

$$\left(\sum_{n=1}^{N_T} \|u_h^n - U^n\|_{L^q}^p \tau \right)^{1/p} \leq C(h^2 + \tau),$$

where $U^n = u(\cdot, n\tau)$. The constant C depends on p , q , d , Ω , T , the constants of the shape-regularity and the quasi-uniformity of $\{\mathcal{T}_h\}_h$, and the norms of u , but is independent of h and τ .

In the proof of Theorem 1.2.4 (Section 1.6), the following sub-optimal error estimate, which is worth stating separately, will be used.

Theorem 1.2.5 (L^∞ -estimate for semilinear equations). *Assume the same hypotheses as in Theorem 1.2.4. Let $\alpha_{p,q,d} = 1 - 1/p - d/(2q)$. Then, for every $\alpha \in (0, \alpha_{p,q,d})$ and $T \in (0, T_\infty)$, the following error estimate holds:*

$$\max_{0 \leq n \leq N_T} \|u_h^n - U^n\|_{L^\infty} \leq C(h^{2\alpha} + \tau),$$

where $U^n = u(\cdot, n\tau)$. The constant C depends on p , q , d , α , Ω , T , the constants of the shape-regularity and the quasi-uniformity of $\{\mathcal{T}_h\}_h$, and the norms of u , but is independent of h and τ .

1.3 Preliminaries

As explained in this section, we collect some preliminary results used for this study. In the first two subsections 1.3.1 and 1.3.2 below, we present general results on CMR and DMR. Therein, we are concerned with a general linear operator A on a general Banach space X . In the last subsection 1.3.3, we focus on the discrete Laplacian, which is denoted by A_h . The continuous Laplacian on L^q is denoted by A_q (see (1.26)). We will also use the notation $J_T = (0, T)$ as in the previous sections.

1.3.1 Imaginary powers of operators and CMR

In order to show CMR for the discrete Laplacian, we utilize the theory of the imaginary powers of operators. The next lemma, which is the celebrated result of Dore and Venni [34, Theorem 3.2], is the key factor of the proof of Theorem 1.2.2 (see also [4, Section III.4]).

Lemma 1.3.1. *Let $p \in (1, \infty)$, X be a UMD space, and A be a densely defined and closed operator on X . Assume that $[0, \infty) \subset \rho(A)$ and A has the properties*

$$\|(1 + \lambda)(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq K, \quad \forall \lambda \geq 0 \quad (1.30)$$

and

$$\|(-A)^{it}\|_{\mathcal{L}(X)} \leq M e^{\vartheta|t|}, \quad \forall t \in \mathbb{R} \quad (1.31)$$

with some $K > 0$, $M \geq 1$, and $\vartheta \in [0, \pi/2)$. Then, A has L^p -CMR on J_T for any $T \in (0, \infty]$. Moreover, the constant $C_{\text{MR}} > 0$ depends only on X , K , M , and ϑ , but is independent of the individual operator A .

Herein, a UMD space is the Banach space that is characterized by the Hilbert transform (see e.g. [3, section III.4]). The imaginary power $(-A)^{it}$ is defined by H^∞ -functional calculus (see [72, chapter II]). The dependence of the constant C_{MR} on the Banach space X is derived from the boundedness of imaginary powers of the time-differential operator on $L^p(J_T; X)$. See [4, Lemma III.4.10.5]. Chasing the constants appearing in the proofs, we can obtain the following property.

Lemma 1.3.2. *Let $p \in (1, \infty)$, X be a UMD space, $X_0 \subset X$ be a closed subspace, and A be a densely defined and closed operator on X_0 . Assume that A has the same properties as in Lemma 1.3.1 with X replaced by X_0 for some $K > 0$, $M \geq 1$, and $\vartheta \in [0, \pi/2)$. Then A has L^p -CMR on J_T for any $T \in (0, \infty]$. Moreover, the constant $C_{\text{MR}} > 0$ depends only on X , K , M , and ϑ , but is independent of X_0 and the individual operator A .*

We will apply the above lemma with $X = L^q$ and $X_0 = X_{h,q}$.

1.3.2 Discrete maximal regularity

We investigated a sufficient condition for DMR on J_∞ in the UMD case in [65]. More precisely, we proved the following result [65, Corollary 3.3]. Herein, for $\omega \in (0, \pi)$, the set $\Sigma_\omega = \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \omega\}$ denotes the sector domain, the set $S(A) \subset \mathbb{C}$ is the numerical range of A (cf. [88]), and $r(A) = \max_{z \in S(A)} |z|$. Note that $r(A)$ is not the spectral radius of A .

Lemma 1.3.3. *Let $p \in (1, \infty)$, $\theta \in [0, 1]$, X be a UMD space, $X_0 \subset X$ be a closed subspace, and A be a bounded operator on X_0 . Assume that A has L^p -CMR on J_∞ with the constant C_{MR} . Furthermore, we suppose that the following conditions (condition $(\text{NR})_{\delta, \varepsilon}$) are satisfied when $\theta \in [0, 1/2)$:*

(NR1) *There exists $\delta \in (0, \pi/2)$ such that $S(A) \subset \mathbb{C} \setminus \Sigma_{\delta+\pi/2}$.*

(NR2) *There exists $\varepsilon > 0$ such that $(1 - 2\theta)\tau r(A) + \varepsilon \leq 2 \sin \delta$.*

Then, A has l^p -DMR on J_∞ . Moreover, the constant C_{DMR} depends only on p , θ , δ , ε , X , and C_{MR} , but is independent of X_0 .

The following lemma states that DMR on finite intervals is obtained from the infinite-interval case. Although the inequality (1.32) below is slightly different from (1.11), it does not affect error analysis. The proof is easy and thus we will omit it.

Lemma 1.3.4. *Let $p \in (1, \infty)$, $\theta \in [0, 1]$, X be a Banach space, and A be a bounded operator on X . Assume that A has l^p -DMR on J_∞ with $C_{\text{DMR}} = C_0$. Then, for every $T > 0$ and for every $f \in l^p(N_T - 1; X)$, there exists a unique solution $u \in l^p(N_T; X)$ of the equation*

$$\begin{cases} \frac{u^{n+1} - u^n}{\tau} = Au^{n+\theta} + f^n, & n = 0, 1, \dots, N_T - 1, \\ u^0 = 0, \end{cases}$$

and it satisfies

$$\|u_\theta\|_{l_\tau^p(N_T; X)} + \|D_\tau u\|_{l_\tau^p(N_T; X)} + \|Au_\theta\|_{l_\tau^p(N_T; X)} \leq C_0 \|f\|_{l_\tau^p(N_T; X)}. \quad (1.32)$$

An a priori estimate with non-zero initial value is obtained only in the case where $\theta = 1$. See [6, Theorem 2.3.1] for $T < \infty$ and [65, Theorem 4.2] for $T = \infty$. Recall that $v_1 = (v^{n+1})_{n \geq 0}$ for a sequence $v = (v^n)_{n \geq 0}$.

Lemma 1.3.5. *Let $p \in (1, \infty)$, $T \in (0, \infty]$, X be a UMD space, $X_0 \subset X$ be a closed subspace, and A be a bounded operator on X_0 . Assume that A has l^p -DMR on J_T . Then, for each $f \in l^p(N_T; X_0)$ and for each $u_0 \in (X_0, D(A))_{1-1/p, p}$, there exists a unique solution $u \in l^p(N_T; X_0)$ of the equation*

$$\begin{cases} \frac{u^{n+1} - u^n}{\tau} = Au^{n+1} + f^{n+1}, & n = 0, 1, \dots, N_T - 1, \\ u^0 = u_0, \end{cases}$$

which satisfies

$$\|u_1\|_{l^p_\tau(N_T; X_0)} + \|D_\tau u\|_{l^p_\tau(N_T; X_0)} + \|Au_1\|_{l^p_\tau(N_T; X_0)} \leq C_{\text{DMR}} (\|f_1\|_{l^p_\tau(N_T; X_0)} + \|u_0\|_{1-1/p, p}),$$

where $C_{\text{DMR}} > 0$ is independent of f , u_0 , and X_0 .

1.3.3 Operator-theoretical properties of the discrete Laplacian

In this subsection, we present several properties concerning on the discrete Laplace operator A_h defined by (1.16). We begin with the properties on the discrete heat semigroup generated by A_h . A semigroup $T(t)$ on a Lebesgue space $X = L^q(\Omega, \mu)$ ($q \in [1, \infty]$) is said to be positivity-preserving if

$$u \geq 0 \text{ } \mu\text{-a.e. in } \Omega \implies T(t)u \geq 0 \text{ } \mu\text{-a.e. in } \Omega$$

for every $t > 0$ and $u \in X$. In the proofs of the following two lemmas, the discrete maximum principle (Remark 1.2) plays a crucially important role. The proofs can be found in [97, Theorem 15.5] and [26, Theorem 4.1].

Lemma 1.3.6. *Let $q \in [1, \infty]$. Assume that the family of triangulations $\{\mathcal{T}_h\}$ satisfies the acuteness condition (H2). Then, the semigroup e^{tA_h} generated by A_h is positivity-preserving in $X_{h,q}$.*

Lemma 1.3.7. *Let $q \in [1, \infty]$. Assume that the family of triangulations $\{\mathcal{T}_h\}$ satisfies the acuteness condition (H2). Then, A_h generates an analytic and contraction semigroup on $X_{h,q}$. Moreover, if $q \in (1, \infty)$, then A_h satisfies the condition (NR1) with the angle θ_q defined by (1.18).*

Remark 1.5. In [97] and [26], the authors considered the two-dimensional case only. However, the essential factor is the component-wise non-negativity of the inverse of the stiffness matrix, which is derived from the condition (H2). Thus we can obtain the same results for the three-dimensional case.

We introduce several mesh-depending operators on S_h . The L^2 projection P_h is defined by (1.22). Let R_h be the Ritz projection onto S_h defined by

$$(\nabla R_h u, \nabla v_h)_{L^2} = (\nabla u, \nabla v_h)_{L^2}, \quad \forall v_h \in S_h$$

for $u \in W^{1,1}$. These operators have the following well-known properties. See [16] for the proofs.

Lemma 1.3.8. *Assume that $\{\mathcal{T}_h\}_h$ satisfies (H1). Then, there exists $C > 0$ depending only on Ω and q such that*

$$\begin{aligned} \|P_h v\|_{L^q} &\leq C \|v\|_{L^q}, & \forall v \in L^q, & \quad \forall q \in [1, \infty], \\ \|P_h v\|_{W^{1,q}} &\leq C \|v\|_{W^{1,q}}, & \forall v \in W^{1,q}, & \quad \forall q \in [1, \infty], \\ \|R_h v\|_{W^{1,q}} &\leq C \|v\|_{W^{1,q}}, & \forall v \in W^{1,q}, & \quad \forall q \in (1, \infty], \\ \|v - P_h v\|_{L^q} &\leq Ch^2 \|v\|_{W^{2,q}}, & \forall v \in W^{2,q}, & \quad \forall q \in (d/2, \infty], \\ \|v - R_h v\|_{L^q} &\leq Ch^2 \|v\|_{W^{2,q}}, & \forall v \in W^{2,q} \cap W_0^{1,q}, & \quad \forall q \in (\mu', \infty), \end{aligned} \tag{1.33}$$

where μ' is the Hölder conjugate of μ . When $q = 2$, (H1) is not required for all inequalities above except for (1.33).

We use the standard discrete Laplacian L_h defined by

$$(L_h u_h, v_h) = -(\nabla u_h, \nabla v_h), \quad \forall v_h \in S_h,$$

for $u_h \in S_h$. We designate L_h the discrete Laplacian *without* mass-lumping. From the Poincaré inequality, L_h is injective. Consequently, it is invertible due to $\dim S_h < \infty$. Then, by the definitions given above, it is apparent that

$$L_h = K_h A_h, \quad R_h = L_h^{-1} P_h A_q, \quad (1.34)$$

where M_h and K_h are the operators for the method of mass-lumping introduced in Section 1.2 and the operator A_q is the L^q -realization of the Laplacian defined by (1.26). From these relations, the following estimate is obtained.

Lemma 1.3.9. *Assume that $\{\mathcal{T}_h\}_h$ satisfies (H1) when $q \neq 2$. Then, for $q \in (1, \mu)$, there exists $C > 0$ satisfying*

$$\|v_h\|_{h,q} \leq C \|A_h v_h\|_{h,q}, \quad \forall v_h \in S_h,$$

where C depends only on Ω and q .

Proof. By (1.15) and (1.34), it suffices to show

$$\|v_h\|_{L^q} \leq C \|L_h v_h\|_{L^q}$$

for all $v_h \in S_h$. Fix $v_h \in S_h$ arbitrarily and set $f_h = L_h v_h$ and $v = A_q^{-1} f_h \in D(A_q)$. Then, noting that $P_h f_h = f_h$ and from (1.34), one obtains

$$v_h = L_h^{-1} P_h f_h = L_h^{-1} P_h A_q v = R_h v.$$

Therefore, we have

$$\|v_h\|_{L^q} \leq \|R_h v\|_{W^{1,q}} \leq C \|v\|_{W^{1,q}} \leq C \|v\|_{W^{2,q}} \leq C \|A_q v\|_{L^q} = C \|L_h v_h\|_{L^q}$$

by Lemma 1.3.8 and (1.12). \square

1.4 Proofs of CMR and DMR for the discrete Laplacian

The aim of this section is to establish CMR and DMR for A_h . We first consider the time-continuous case via the method of imaginary powers of operators. Then, we obtain DMR for A_h by our previous result (Lemma 1.3.3). We also present a useful criterion to check the condition $(\text{NR})_{\delta,\varepsilon}$.

Proof of Theorem 1.2.1. In view of Lemmas 1.3.1 and 1.3.2, it suffices to show that A_h satisfies (1.30) and (1.31) for some K , M , and ϑ independent of h . The first condition (1.30) is a simple consequence of Lemmas 1.3.7 and 1.3.9. It is clear that the constant K in (1.30) is independent of h since the contractivity of the semigroup e^{tA_h} implies uniform resolvent estimates $\|(\lambda I - A_h)^{-1}\|_{\mathcal{L}(X_{h,q})} \leq 1/\lambda$ for any $\lambda > 0$.

In order to check the condition (1.31), we refer to the Duong's result [35, Theorem 2] (see also [24]), which states that an operator admits a bounded H^∞ -functional calculus (cf. [32]) on Σ_θ for any $\theta \in (\pi/2, \pi)$, if it satisfies (1.30) and generates a contractive and positivity-preserving semigroup. Here, $\Sigma_\theta = \{z \in \mathbb{C} \mid |\arg z| < \theta\}$. Owing to Lemmas 1.3.6 and 1.3.7, we can apply this result and we have

$$\|m(-A_h)\|_{\mathcal{L}(X_{h,q})} \leq M \|m\|_{L^\infty(\Sigma_\theta)} \quad (1.35)$$

for each bounded and holomorphic functions m on Σ_θ and for $\theta \in (\pi/2, \pi)$. Moreover, chasing the constants in the proof of [35, Theorem 2], we can see that M is independent of h . In particular, taking $m(z) = z^{it}$ for $t \in \mathbb{R}$, we have

$$\|(-A_h)^{it}\|_{\mathcal{L}(X_{h,q})} \leq M e^{\theta|t|}, \quad \forall t \in \mathbb{R}.$$

for any $\theta \in (\pi/2, \pi)$.

Now, we are ready to show (1.31). We first assume that $q = 2$. In this case, $X_{h,2}$ is a Hilbert space and $-A_h$ is self-adjoint and positive definite without conditions on the triangulation by Poincaré

inequality. Moreover, in the Hilbert case, the imaginary powers $(-A_h)^{it}$ can be defined by the spectral decomposition, and it coincides with the definition by the H^∞ -functional calculus (see e.g. [3, Theorem III.4.6.7]). Thus we have

$$\|(-A_h)^{it}\|_{\mathcal{L}(X_{2,h})} \leq \int_0^\infty dE_{-A_h}(\lambda) = 1$$

for all $t \in \mathbb{R}$, where E_{-A_h} is the spectral resolution of $-A_h$.

Next we suppose $q \neq 2$. Set

$$\vartheta_{q,r} = \frac{q^{-1} - 2^{-1}}{r^{-1} - 2^{-1}}$$

for $r \neq 2$. Since $q \neq 2$, we can choose $r \in (1, \infty)$ satisfying $\vartheta_{q,r} \in (0, 1)$. Then, by the Riesz-Thorin theorem, we can obtain

$$\|(-A_h)^{it}\|_{\mathcal{L}(X_{h,q})} \leq \|(-A_h)^{it}\|_{\mathcal{L}(X_{h,2})}^{1-\vartheta_{q,r}} \|(-A_h)^{it}\|_{\mathcal{L}(X_{h,r})}^{\vartheta_{q,r}} \leq M^{\vartheta_{q,r}} e^{\theta \vartheta_{q,r} |t|}$$

for any $t \in \mathbb{R}$ and $\theta \in (\pi/2, \pi)$, where $M > 0$ is the constant in (1.35). Since $\vartheta_{q,r} \in (0, 1)$, we can take θ as

$$\frac{\pi}{2} < \theta < \frac{\pi}{2\vartheta_{q,r}},$$

which implies (1.31) with $M = M^{\vartheta_{q,r}}$ and $\vartheta = \theta \vartheta_{q,r} < \pi/2$. This completes the proof of Theorem 1.2.1. \square

Owing to Lemma 1.3.3 and Theorem 1.2.1, we are able to obtain DMR for A_h . To apply Lemma 1.3.3, it is necessary to verify that the condition $(\text{NR})_{\delta,\varepsilon}$ is satisfied when $\theta < 1/2$. From Lemma 1.3.7, the condition (NR1) is always satisfied. Therefore, what is left is to check the condition (NR2). We begin with the following lemma, which is a generalization of [41, Lemma 2]. No condition on the triangulation is required.

Lemma 1.4.1. *Let $r \in [1, \infty)$. Then, we have*

$$\|\nabla v_h\|_{L^r} \leq \frac{d+1}{\kappa_h} \|v_h\|_{h,r}, \quad \forall v_h \in S_h,$$

where κ_h is defined by (1.19).

Proof. Fix $K \in \mathcal{T}_h$ arbitrarily. Then it suffices to show that

$$\|\nabla v_h\|_{L^r(K)} \leq \frac{d+1}{\kappa_h} \|v_h\|_{h,r,K}, \quad \forall v_h \in S_h,$$

where $\|v_h\|_{h,r,K} = \|M_h v_h\|_{L^r(K)}$. Let Q_j ($j = 0, \dots, d$) be the vertex of K , λ_j be the corresponding barycentric coordinate in K , and κ_j be the length of the perpendicular from Q_j in K . Then it is well-known that $|\nabla \lambda_j| = 1/\kappa_j$. Take $v_h \in S_h$ arbitrarily and set $v_j = v_h(Q_j)$. Since $v_h|_K = \sum_{j=0}^d v_j \lambda_j$, we have

$$\|\nabla v_h\|_{L^r(K)} \leq \sum_{j=0}^d \frac{|v_j|}{\kappa_j} |K|^{1/r} \leq \frac{(d+1)^{1/r'}}{\kappa_h} \left(|K| \sum_{j=0}^d |v_j|^r \right)^{1/r} \quad (1.36)$$

by Hölder's inequality, where r' is the Hölder conjugate of r . Moreover, it is readily apparent that

$$\|v_h\|_{h,r,K} = \left(\frac{1}{d+1} |K| \sum_{j=0}^d |v_j|^r \right)^{1/r}.$$

This, together with (1.36), implies that

$$\|\nabla v_h\|_{L^r(K)} \leq \frac{(d+1)^{1/r'+1/r}}{\kappa_h} \|v_h\|_{h,r,K} = \frac{d+1}{\kappa_h} \|v_h\|_{h,r,K}.$$

Thereby we complete the proof. \square

Now, we describe a sufficient condition for (NR2) to hold.

Lemma 1.4.2 (A sufficient condition for (NR2)). *Assume $\theta \in [0, 1/2)$ and $q \in (1, \infty)$. Let $\theta_q = \arccos|1 - 2/q|$. If we choose ε and τ sufficiently small so that (1.20) is satisfied for every h , then the condition (NR2) with $\delta = \theta_q$ is fulfilled.*

Proof. With $v_h \in S_h$, we associate $v_h^* \in S_h$ defined by $v_h^*(P) = |v_h(P)|^{q-2}v_h(P)$ for every node P of \mathcal{T}_h . Recall that the numerical range of A_h is expressed as

$$S(A_h) = \{(A_h v_h, v_h^*)_h \mid v_h \in S_h, \|v_h\|_{h,q} = 1\}.$$

Then, by Lemma 1.4.1, we have

$$|(A_h v_h, v_h^*)_h| \leq \|\nabla v_h\|_{L^q} \|\nabla v_h^*\|_{h,q'} \leq \frac{(d+1)^2}{\kappa_h^2} \|v_h\|_{h,q}^q$$

for $v_h \in S_h$. Hence we can deduce (NR2) from the assumption (1.20). \square

Proof of Theorem 1.2.2. The first assertion is a consequence of Theorem 1.2.1 and Lemmas 1.3.3 and 1.4.2. The second one can be obtained by the first assertion and Lemma 1.3.5. \square

1.5 Application to error estimates: linear case

This section is devoted to error analysis of the solution $u_h = (u_h^n) \in l^p(N_T; S_h)$ of (1.23). We begin by presenting some lemmas. Recall that $J_T = (0, T)$

Lemma 1.5.1. *Let X be a Banach space, $T > 0$, $p \in (1, \infty)$ and $\tau \in (0, 1)$. Set $t_n = n\tau$ for $n = 0, 1, \dots, N_T$. Then, there exists $C_S > 0$ satisfying*

$$\left(\sum_{n=0}^{N_T-1} \|v(t_n)\|_X^p \tau \right)^{1/p} + \left(\sum_{n=1}^{N_T} \|v(t_n)\|_X^p \tau \right)^{1/p} \leq C_S \|v\|_{W^{1,p}(J_T; X)} \quad (1.37)$$

for all $v \in W^{1,p}(J_T; X)$, where C_S depends only on p , but is independent of T , τ , and X .

Proof. By the Sobolev embedding $W^{1,p}(0, 1; X) \hookrightarrow L^\infty(0, 1; X)$, there exists $C_1 > 0$ such that

$$\|v\|_{L^\infty(0,1;X)} \leq C_1 \|v\|_{W^{1,p}(0,1;X)}$$

for $v \in W^{1,p}(0, 1; X)$. One can check that C_1 is independent of X . See the proof of [17, Theorem 8.8]. Then, by rescaling the variable, we have

$$\|v(t_n)\|_X \leq \|v\|_{L^\infty(t_n, t_{n+1}; X)} \leq C_1 (1 + \tau) \tau^{-1/p} \|v\|_{W^{1,p}(t_n, t_{n+1}; X)}$$

for each $n \in \mathbb{N}$. Therefore, we have (1.37) with $C_S = 2C_1$. \square

The next lemma is shown readily by Taylor's theorem. Therefore, we skip the proof.

Lemma 1.5.2. *Let X be a Banach space, $T > 0$, $p \in (1, \infty)$, $\theta \in [0, 1]$ and $\tau \in (0, 1)$. Set $t_n = n\tau$ for $n = 0, 1, \dots, N_T$ and*

$$r^n = \frac{v(t_{n+1}) - v(t_n)}{\tau} - \left[(1 - \theta) \frac{dv}{dt}(t_n) + \theta \frac{dv}{dt}(t_{n+1}) \right]$$

for $v \in W^{j_\theta+1,p}(J_T; X)$, where j_θ is defined by (1.24). Then, there exists $C > 0$ such that

$$\left(\sum_{n=0}^{N_T-1} \|r^n\|_X^p \tau \right)^{1/p} \leq C \tau^{j_\theta} \|v\|_{W^{j_\theta+1,p}(J_T; X)},$$

where C is independent of τ and X .

Now we can state the following proof.

Proof of Theorem 1.2.3. We set $e_h^n = u_h^n - P_h U^n$ so that

$$u_h^n - U^n = e_h^n + (P_h - I)U^n.$$

Then, by Lemmas 1.3.8 and 1.5.1, we have

$$\sum_{n=0}^{N_T-1} \|(P_h - I)U^{n+\theta}\|_{L^q}^p \tau \leq Ch^{2p} \|u\|_{W^{1,p}(J_T; W^{2,q})}^p. \quad (1.38)$$

It remains to derive an estimation for e_h^n . Set $V^n = \partial_t u(\cdot, t_n)$ and

$$r_h^{n,\theta} = (K_h^{-1} P_h A_q - A_h P_h) U^{n+\theta} + P_h \left(\frac{u(t_{n+1}) - u(t_n)}{\tau} \right) - K_h^{-1} P_h V^{n+\theta}.$$

Then, by a simple computation, we have

$$\begin{cases} (D_\tau(A_h^{-1} e_h))^n = A_h(A_h^{-1} e_h^{n+\theta}) + A_h^{-1} r_h^{n,\theta}, & n = 0, 1, \dots, N_T - 1, \\ A_h^{-1} e_h^0 = 0. \end{cases}$$

Consequently, according to Theorem 1.2.2, we can obtain

$$\sum_{n=0}^{N_T-1} \|e_h^{n+\theta}\|_{L^q}^p \tau = \sum_{n=0}^{N_T-1} \|A_h(A_h^{-1} e_h^{n+\theta})\|_{L^q}^p \tau \leq C \sum_{n=0}^{N_T-1} \|A_h^{-1} r_h^{n,\theta}\|_{L^q}^p \tau. \quad (1.39)$$

We divide $r_h^{n,\theta}$ into two parts as

$$r_h^{n,\theta} = r_{1,h}^{n,\theta} + r_{2,h}^{n,\theta},$$

where

$$\begin{aligned} r_{1,h}^{n,\theta} &= (K_h^{-1} P_h A_q - A_h P_h) U^{n+\theta}, \\ r_{2,h}^{n,\theta} &= P_h \left(\frac{u(t_{n+1}) - u(t_n)}{\tau} \right) - K_h^{-1} P_h V^{n+\theta}. \end{aligned}$$

We first address $r_{1,h}^{n,\theta}$. Noting the relation (1.34), we have

$$A_h^{-1} r_{1,h}^{n,\theta} = [(K_h A_h)^{-1} P_h A_q - P_h] U^{n+\theta} = (R_h - P_h) U^{n+\theta},$$

so that

$$\left(\sum_{n=0}^{N_T-1} \|A_h^{-1} r_{1,h}^{n,\theta}\|_{L^q}^p \tau \right)^{1/p} \leq Ch^2 \|u\|_{W^{1,p}(J_T; W^{2,q})} \quad (1.40)$$

by Lemmas 1.3.8 and 1.5.1. Also, $A_h^{-1} r_{2,h}^{n,\theta}$ is expressed as

$$\begin{aligned} A_h^{-1} r_{2,h}^{n,\theta} &= A_h^{-1} P_h \left[\frac{u(t_{n+1}) - u(t_n)}{\tau} - V^{n+\theta} \right] + A_h^{-1} (I - K_h^{-1}) P_h V^{n+\theta} \\ &=: \rho_{1,h}^n + \rho_{2,h}^n. \end{aligned}$$

According to Lemmas 1.3.8, 1.3.9, and 1.5.2, we have

$$\left(\sum_{n=0}^{N_T-1} \|\rho_{1,h}^n\|_{L^q}^p \tau \right)^{1/p} \leq C \tau^{j_\theta} \|u\|_{W^{j_\theta+1,p}(J_T; L^q)}.$$

It is known that A_h and K_h satisfy the inequality

$$\|A_h^{-1} (I - K_h^{-1}) v_h\|_{L^q} \leq Ch^2 \|\nabla v_h\|_{L^q} \quad (1.41)$$

for all $v_h \in S_h$ (see [91, Lemma 4.6]). Thus we can address $\rho_{2,h}^n$ and obtain

$$\begin{aligned} \left(\sum_{n=0}^{N_T-1} \|\rho_{2,h}^n\|_{L^q}^p \tau \right)^{1/p} &\leq Ch^2 \left(\sum_{n=0}^{N_T-1} \|\nabla P_h V^{n+\theta}\|_{L^q}^p \tau \right)^{1/p} \\ &\leq Ch^2 \|\partial_t u\|_{W^{1,p}(J_T; W^{1,q})}, \end{aligned}$$

thanks to Lemmas 1.3.8 and 1.5.1. Therefore, we can establish

$$\left(\sum_{n=0}^{N_T-1} \|A_h^{-1} r_{2,h}^{n,\theta}\|_{L^q}^p \tau \right)^{1/p} \leq C \tau^{j_\theta} \|u\|_{W^{j_\theta+1,p}(J_T; L^q)} + Ch^2 \|\partial_t u\|_{W^{1,p}(J_T; W^{1,q})}. \quad (1.42)$$

Combining (1.38), (1.39), (1.40), and (1.42), we can obtain the error estimate (1.25). \square

1.6 Application to error estimates: semilinear case

This section is devoted to analysis of semilinear problems (1.3) and (1.29). We first prove several auxiliary lemmas.

1.6.1 Embedding and trace theorems

For $N \in \mathbb{N} \cup \{\infty\}$ and $v_h = (v_h^n)_{n=0}^N \in S_h^{N+1}$, we set

$$\|v_h\|_{Y_{h,\tau,N}^{p,q}} = \|v_{h,1}\|_{l_\tau^p(N; X_{h,q})} + \|A_h v_{h,1}\|_{l_\tau^p(N; X_{h,q})} + \|D_\tau v_h\|_{l_\tau^p(N; X_{h,q})} \quad (1.43)$$

and $Y_{h,\tau,N}^{p,q} = \left(S_h^{N+1}, \|\cdot\|_{Y_{h,\tau,N}^{p,q}} \right)$, where $v_{h,1} = (v_h^{n+1})_{n=0}^{N-1} \in S_h^N$. For abbreviation, we write $Y_{h,\tau}^{p,q} = Y_{h,\tau,\infty}^{p,q}$ and

$$\|v_h\|_{Y_T} = \|v_h\|_{Y_{h,\tau,N_T}^{p,q}} \quad (1.44)$$

for $T \in (0, \infty]$, where N_T is defined by (1.8).

The aim of this subsection is to show the following trace result for Y_T . We recall that $\|\cdot\|_{h,1-1/p,p}$ is the norm of the space $(X_{h,q}, D(A_h))_{1-1/p,p}$, where $D(A_h) = (S_h, \|A_h \cdot\|_{h,q})$.

Lemma 1.6.1. *Let $N \in \mathbb{N}$, $q \in (1, \mu)$ and $p \in (1, \infty)$. Assume that the family $\{\mathcal{T}_h\}_h$ satisfies (H1) and (H2) when $q \neq 2$. Then, there exists $C > 0$ independent of N , h , and τ such that*

$$\max_{0 \leq n \leq N} \|v_h^n\|_{h,1-1/p,p} \leq C \left(\|v_h\|_{Y_{h,\tau,N}^{p,q}} + \|v_h^0\|_{h,1-1/p,p} \right)$$

for every $v_h \in Y_{h,\tau,N}^{p,q}$.

It is known that the corresponding inequality holds for infinite sequences (see [6, Theorem 2.3.1]). The result is the discrete counterpart of the characterization of the real interpolation space via the analytic semigroup (see e.g. [84, Lemma 6.2]).

In order to prove Lemma 1.6.1, we need to extend each element of $Y_{h,\tau,N}^{p,q}$ to that of $Y_{h,\tau,\infty}^{p,q}$. For this purpose, we show the following extension lemma, which corresponds to [4, Lemma 7.2].

Lemma 1.6.2. *Let X be a Banach space and L be a linear operator which has discrete maximal regularity and which satisfies $0 \in \rho(L)$. Let $N \in \mathbb{N} \cup \{\infty\}$ and set*

$$\|v\|_{p,N} = \|v_1\|_{l_\tau^p(N; X)} + \|Lv_1\|_{l_\tau^p(N; X)} + \|D_\tau v\|_{l_\tau^p(N; X)}$$

for $v \in X^{N+1}$ and $Y_N^p = \{v \in X^{N+1} \mid v^0 \in (X, D(L))_{1-1/p,p}, \|v\|_{p,N} < \infty\}$. Then, for $M \in \mathbb{N}$ with $M < N$, there exists a map $\text{ext}_M: Y_M^p \rightarrow Y_N^p$ satisfying

$$(\text{ext}_M v)^n = v^n, \quad n = 0, \dots, M,$$

and

$$\|\text{ext}_M v\|_{p,N} \leq C (\|v\|_{p,M} + \|v^0\|_{1-1/p,p}),$$

where C depends only on the constant of the DMR property of L , but is independent of τ and M .

Proof. For $v \in Y_M^p$, we define $g \in Y_N^p$ by

$$g^n = \begin{cases} (D_\tau v)^n - Lv^{n+1}, & n = 0, \dots, M-1, \\ 0, & \text{otherwise.} \end{cases}$$

Let V be the solution of

$$\begin{cases} (D_\tau V)^n = LV^{n+1} + g^n, & n \in \mathbb{N}, \\ V^0 = v^0, \end{cases}$$

which is uniquely solvable by discrete maximal regularity of L . Then, if we set $\text{ext}_M v = V$, it satisfies the desired properties. Indeed, since $w^n = v^n - V^n$ satisfies

$$\begin{cases} (D_\tau w)^n = Lw^n, & n = 0, \dots, M-1, \\ w^0 = 0, \end{cases}$$

we can obtain $w^n = (I - \tau L)^{-n} w^0 = 0$ for $n = 0, \dots, M$. Moreover, by discrete maximal regularity, we have

$$\|V\|_{p,N} \leq C (\|g\|_{l_\tau^p(N;X)} + \|V^0\|_{1-1/p,p}) \leq C (\|v\|_{p,M} + \|v^0\|_{1-1/p,p}),$$

which is the desired estimate. \square

We are now ready to show Lemma 1.6.1.

Proof of Lemma 1.6.1. We first assume $N = \infty$. In this case, owing to [6, Theorem 2.3.1], we can find a constant $C > 0$ satisfying

$$\sup_{n \geq 1} \|v_h^n\|_{1-1/p,p} \leq C \|v_h\|_{Y_{h,\tau}^{p,q}} \quad (1.45)$$

for every $v_h \in Y_{h,\tau}^{p,q}$. Chasing the constants in the proof, we can check that C depends only on p .

Now, let $N \in \mathbb{N}$ and $v_h \in Y_{h,\tau,N}^{p,q}$. Then, by (1.45) and Lemma 1.6.2, we have

$$\begin{aligned} \max_{0 \leq n \leq N} \|v_h^n\|_{1-1/p,p} &\leq \sup_{n \in \mathbb{N}} \|(\text{ext}_N v_h)^n\|_{1-1/p,p} \\ &\leq C \left(\|\text{ext}_N v_h\|_{Y_{h,\tau}^{p,q}} + \|v_h^0\|_{1-1/p,p} \right) \\ &\leq C \left(\|v_h\|_{Y_{h,\tau,N}^{p,q}} + \|v_h^0\|_{1-1/p,p} \right), \end{aligned}$$

which is the desired estimate. \square

1.6.2 Fractional powers of the discrete Laplacian

We will use the fractional power $(-A_h)^z$ for $z \in (0, 1)$ and $z \in (-1, 0)$; see [88]. The negative powers are defined by

$$(-A_h)^{-z} v_h = \frac{\sin(\pi z)}{\pi} \int_0^\infty \mu^{-z} (\mu I - A_h)^{-1} v_h d\mu \quad (1.46)$$

for $z \in (0, 1)$. Since A_h satisfies (1.30) uniformly in h , i.e.,

$$\|(\mu I - A_h)^{-1}\|_{\mathcal{L}(X_{h,q})} \leq \frac{C}{1 + \mu}, \quad \forall \mu \geq 0, \quad (1.47)$$

the operator $(-A_h)^{-z}$ is well-defined for each $z \in (0, 1)$. One can check that $(-A_h)^{-z}$ is invertible. Consequently, the positive power $(-A_h)^z$ is defined by the inverse operator of $(-A_h)^{-z}$ for $z \in (0, 1)$. Fractional powers satisfy the following interpolation properties:

$$\|(-A_h)^z v_h\|_{h,q} \leq C \|v_h\|_{h,q}^{1-z} \|A_h v_h\|_{h,q}^z, \quad (1.48)$$

$$\|(-A_h)^{-z} v_h\|_{h,q} \leq C \|v_h\|_{h,q}^{1-z} \|A_h^{-1} v_h\|_{h,q}^z, \quad (1.49)$$

for each $z \in (0, 1)$ and $v_h \in S_h$, uniformly in h . Consequently, we have

$$\|(-A_h)^{-z} v_h\|_{h,q} \leq C \|v_h\|_{h,q}, \quad \forall v_h \in S_h \quad (1.50)$$

uniformly in h , because of Lemma 1.3.9. Below we set $(-A_h)^0 = I$ and $(-A_h)^1 = -A_h$. The main purpose of this subsection is to show the following result.

Lemma 1.6.3. *Let $p \in (1, \infty)$, $q \in (\mu_d, \mu)$, $p > 2q/(2q - d)$ and $\alpha_{p,q,d} = 1 - 1/p - d/(2q)$. Assume that the family $\{\mathcal{T}_h\}_h$ satisfies (H1) and (H2). Then, for every $\alpha \in (0, \alpha_{p,q,d})$, there exists $C > 0$ independent of h and τ that satisfies*

$$\max_{0 \leq n \leq N} \|v_h\|_{L^\infty} \leq C \left(\|(-A_h)^{-\alpha} v_h\|_{Y_{h,\tau,N}^{p,q}} + \|(-A_h)^{-\alpha} v_h^0\|_{1-\frac{1}{p},p} \right),$$

for all $N \in \mathbb{N}$ and $v_h \in S_h^{N+1}$.

We begin with the discrete Gagliardo-Nirenberg type inequality for the proof of the above lemma. The following result is a generalization of [58, Lemma 3.3], which states the L^2 -case. The proof is almost the same and thus we will omit it.

Lemma 1.6.4 (Discrete Gagliardo-Nirenberg type inequality). *Let $q \in (\mu_d, \mu)$. Assume that the family $\{\mathcal{T}_h\}_h$ satisfies (H1) and (H2). Then, we have*

$$\|v_h\|_{L^\infty} \leq C \|A_h v_h\|_{h,q}^{\frac{d}{2q}} \|v_h\|_{h,q}^{1-\frac{d}{2q}}, \quad \forall v_h \in S_h, \quad (1.51)$$

where C is independent of h .

Remark 1.6. In the proof of (1.51), the global inverse inequality

$$\|v_h\|_{L^\infty} \leq C h^{-d/q} \|v_h\|_{L^q}, \quad \forall v_h \in S_h$$

for $q \in [1, \infty]$ is essential. Therefore, we cannot relax the condition (H1) in the following arguments.

Thanks to the above inequality, we can obtain the following embedding lemma, which is itself an interesting result.

Lemma 1.6.5 (Discrete Sobolev inequality). *Assume that the family $\{\mathcal{T}_h\}_h$ satisfies (H1) and (H2). For every $q > d/2$ and $\gamma \in (d/(2q), 1)$, there exists $C > 0$ independent of h , which fulfills the inequality*

$$\|v_h\|_{L^\infty} \leq C \|(-A_h)^\gamma v_h\|_{L^q},$$

for all $v_h \in S_h$.

Proof. It suffices to show that

$$\|(-A_h)^{-\gamma} f_h\|_{L^\infty} \leq C \|f_h\|_{h,q}, \quad \forall f_h \in S_h. \quad (1.52)$$

By the definition (1.46), it is necessary to address $\|(\mu I - A_h)^{-1} f_h\|_{L^\infty}$. Lemma 1.6.4 and the resolvent estimates (1.47) imply

$$\|(\mu I - A_h)^{-1} f_h\|_{L^\infty} \leq C(1 + \mu)^{-1+\frac{d}{2q}} \|f_h\|_{h,q}$$

for all $\mu \geq 0$. Consequently,

$$\|(-A_h)^{-\gamma} f_h\|_{L^\infty} \leq C \int_0^\infty \mu^{-\gamma} (1 + \mu)^{-1+\frac{d}{2q}} d\mu \|f_h\|_{h,q}. \quad (1.53)$$

Since $\gamma \in (d/(2q), 1)$, the integral in the right-hand-side of (1.53) is finite. Therefore, we can obtain the estimate (1.52). \square

Now, we are in a position to show Lemma 1.6.3.

Proof of Lemma 1.6.3. We first note that

$$\|(-A_h)^\beta v_h\|_{h,q} \leq C \|v_h\|_{1-\frac{1}{p},p}, \quad \forall v_h \in S_h \quad (1.54)$$

for any $\beta \in (0, 1 - 1/p)$, with a constant C independent of h . Indeed, by the general embedding theorem for positive operators [84, Proposition 4.7], we have

$$(X_{h,q}, D(A_h))_{\beta,1} \hookrightarrow D((-A_h)^\beta),$$

where we define $D((-A_h)^\beta) = (S_h, \|(-A_h)^\beta \cdot\|_{h,q})$. Moreover, $\beta < 1 - 1/p$ implies

$$(X_{h,q}, D(A_h))_{1-\frac{1}{p},p} \hookrightarrow (X_{h,q}, D(A_h))_{\beta,1},$$

which is also a basic result (see e.g. [84]). Chasing the constants in these proofs, we can show that both embedding properties are uniform in h . Thus we can check (1.54).

Now, we show the desired estimate. Since $\alpha + d/(2q) < 1 - 1/p$, we can find $\beta \in (0, 1 - 1/p)$ that satisfies $\beta - \alpha \in (d/(2q), 1)$. Thus, owing to Lemmas 1.6.5 and 1.6.1 and the embedding result (1.54), we have

$$\begin{aligned} \|v_h^n\|_{L^\infty} &\leq C \|(-A_h)^{\beta-\alpha} v_h^n\|_{h,q} \leq C \|(-A_h)^{-\alpha} v_h^n\|_{1-\frac{1}{p},p} \\ &\leq C \left(\|(-A_h)^{-\alpha} v_h\|_{Y_{h,\tau,N}^{p,q}} + \|(-A_h)^{-\alpha} v_h^0\|_{1-\frac{1}{p},p} \right), \end{aligned}$$

for $v_h = (v_h^n)_n \in S_h^{N+1}$ and $N \in \mathbb{N}$. This completes the proof. \square

1.6.3 Completion of the proofs of Theorems 1.2.4 and 1.2.5

Let u and $u_h = (u_h^n)_{n=0}^{N_T}$ be solutions of (1.3) and (1.29), respectively. Set $U^n = u(n\tau)$. We consider the error $e_h = (e_h^n)_{n=0}^{N_T} \in S_h^{N_T+1}$ defined by

$$e_h^n = u_h^n - P_h U^n \quad (n = 0, 1, \dots, N_T).$$

We first state the sub-optimal error estimate for a *globally* Lipschitz nonlinear function f . If f is globally Lipschitz continuous, then (1.3) admits a unique time-global solution. Thus the life span is infinite, i.e., $T_\infty = \infty$.

Lemma 1.6.6. *In addition to the hypotheses of Theorem 1.2.4, we assume that f is a globally Lipschitz continuous function. Then, for every $\alpha \in [0, 1]$ and $T \in (0, \infty)$, we have*

$$\|(-A_h)^{-\alpha} e_h\|_{Y_T} \leq C(h^{2\alpha} + \tau), \quad (1.55)$$

where the norm $\|\cdot\|_{Y_T}$ is defined by (1.43) and (1.44). The constant C depends on $p, q, d, \alpha, \Omega, T$, the constants of the shape-regularity and the quasi-uniformity of $\{\mathcal{T}_h\}_h$, and the norms of u , but is independent of h and τ .

Proof. The proof is divided into two steps.

Step 1. We prove that there exists $T_1 = T_1(u_0, T) \in (0, T)$ satisfying

$$\|(-A_h)^{-\alpha} e_h\|_{Y_{T_1}} \leq C(h^{2\alpha} + \tau). \quad (1.56)$$

The error e_h satisfies

$$\begin{cases} (D_\tau e_h)^n = A_h e_h^{n+1} + r_h^n, & n = 0, 1, \dots, \\ e_h^0 = 0, \end{cases}$$

where $r_h^n = F_h(u_h^n) - P_h(D_\tau U)^n + A_h P_h U^{n+1}$ and $F_h = K_h^{-1} \circ P_h \circ f$. We decompose r_h^n into two parts:

$$r_h^n = r_{1,h}^n + r_{2,h}^n, \quad r_{1,h}^n = F(u_h^n) - F(P_h U^n), \quad r_{2,h}^n = r_h^n - r_{1,h}^n.$$

We will use the notation $r_{1,h} = (r_{1,h}^n)_n$ and so on.

We perform an estimation for $r_{2,h}^n$. Let $V^n = \partial_t u(\cdot, n\tau)$. Noting that $V^{n+1} = A_q U^{n+1} + f(U^{n+1})$, the residual term $r_{2,h}^n$ can be decomposed as

$$\begin{aligned} r_{2,h}^n &= R_{1,h}^n + R_{2,h}^n + R_{3,h}^n, \\ R_{1,h}^n &= A_h(P_h - R_h)U^{n+1}, \\ R_{2,h}^n &= (K_h^{-1} - I)P_h V^{n+1} + P_h(V^{n+1} - (D_\tau U)^n), \\ R_{3,h}^n &= [F_h(P_h U^n) - F_h(U^n)] + [F_h(U^n) - F_h(U^{n+1})]. \end{aligned}$$

From the interpolation property (1.48) and the inverse inequality, we have

$$\|(-A_h)^\gamma v_h\|_{h,q} \leq Ch^{-2\gamma} \|v_h\|_{h,q},$$

for $\gamma \in (0, 1)$. Therefore, the first term $R_{1,h}^n$ is addressed as

$$\|(-A_h)^{-\alpha} R_{1,h}^n\|_{h,q} = \|(-A_h)^{1-\alpha} (P_h - R_h) U^{n+1}\|_{h,q} \leq Ch^{2\alpha} \|U^{n+1}\|_{W^{2,q}}$$

for every $n \in \mathbb{N}$, which implies

$$\|(-A_h)^{-\alpha} R_{1,h}\|_{l_\tau^p(N_S; X_{h,q})} \leq Ch^{2\alpha} \quad (1.57)$$

due to Lemma 1.5.1. Furthermore, from (1.49), (1.41), and Lemma 1.3.8, we have

$$\|(-A_h)^{-\alpha} (K_h^{-1} - I) P_h V^{n+1}\|_{h,q} \leq Ch^{2\alpha} \|V^{n+1}\|_{W^{1,q}}.$$

Combining this inequality with Lemmas 1.5.1 and 1.5.2, we have

$$\|(-A_h)^{-\alpha} R_{2,h}\|_{l_\tau^p(N_S; X_{h,q})} \leq C(h^{2\alpha} + \tau) \quad (1.58)$$

for any $S \leq T$. Since f is globally Lipschitz continuous, we have

$$\begin{aligned} \|(-A_h)^{-\alpha} R_{3,h}\|_{l_\tau^p(N_S; X_{h,q})} &\leq CL \left[\left(\sum_{n=0}^{N-1} \|(P_h - I) U^n\|_{h,q}^p \tau \right)^{1/p} + \left(\sum_{n=0}^{N-1} \|U^{n+1} - U^n\|_{h,q}^p \tau \right)^{1/p} \right] \\ &\leq CL(h^2 + \tau), \end{aligned} \quad (1.59)$$

for $S \leq T$ by (1.50) and Lemma 1.3.8, where L is the Lipschitz constant of f . The equations (1.57), (1.58), and (1.59) yield

$$\|(-A_h)^{-\alpha} r_{2,h}\|_{l_\tau^p(N_S; X_{h,q})} \leq C(h^{2\alpha} + \tau) \quad (1.60)$$

for $S \leq T$.

Now, we are ready to show (1.56). We designate some constants appearing in this proof. Since A_h has discrete maximal regularity on the interval $(0, \infty)$ in $X_{h,q}$ uniformly with respect to h , there exists $C_{\text{DMR}} > 0$ depending only on p, q, Ω satisfying

$$\|v_h\|_{Y_S} \leq C_{\text{DMR}} (\|\varphi_{h,1}\|_{l_\tau^p(N_S; X_{h,q})} + \|x_h\|_{h,1-1/p,p}), \quad (1.61)$$

for every $\varphi_h = (\varphi_h^n)_n \in l^p(N_S; X_{h,q})$, $x_h \in S_h$, and $v_h = (v_h^n)_n$ with the relation

$$\begin{cases} (D_\tau v_h)^n = A_h v_h^{n+1} + \varphi_h^{n+1}, & n = 0, \dots, N_S - 1, \\ v_h^0 = x_h. \end{cases}$$

In view of (1.50) and the Lipschitz continuity of f , we have

$$C_{\text{Lip}} = \sup \left\{ \frac{\|(-A_h)^{-\alpha} (F_h(v_h) - F_h(w_h))\|_{h,q}}{\|v_h - w_h\|_{h,q}} \mid \begin{array}{l} h > 0, v_h, w_h \in S_h, \\ v_h \neq w_h \end{array} \right\} < \infty,$$

which is the Lipschitz constant of $(-A_h)^{-\alpha} \circ F_h$. We denote the constant in Lemma 1.6.3 by C_{tr} . Finally, we set

$$C_0 = C_{\text{DMR}} C_{\text{Lip}} C_{\text{tr}} |\Omega|^{1/q},$$

where $|\Omega|$ denotes the d -dimensional Lebesgue measure.

Let $e_{j,h} = (e_{j,h}^n)_n$ ($j = 1, 2$) be the solution of

$$\begin{cases} (D_\tau e_{j,h})^n = A_h e_{j,h}^{n+1} + r_{j,h}^n, & n = 0, \dots, N_T - 1, \\ e_{j,h}^0 = 0. \end{cases} \quad (1.62)$$

It is apparent that $e_h = e_{1,h} + e_{2,h}$. Moreover, for every $S \in (0, \infty)$ and $\alpha \in [0, 1]$, one can obtain

$$\|(-A_h)^{-\alpha} e_{2,h}\|_{Y_S} \leq C(h^{2\alpha} + \tau) \quad (1.63)$$

by (1.60) and

$$\|e_{2,h}\|_{l^p(N_S; X_{h,q})} \leq C(h^2 + \tau) \quad (1.64)$$

by the same technique as in (1.39).

Next, it is necessary to derive an estimation for $e_{1,h}$. Take $S < T$ arbitrarily. Since $e_{1,h}$ is the solution of (1.62), we can deduce

$$\begin{aligned} \|(-A_h)^{-\alpha} e_{1,h}\|_{Y_S} &\leq C_{\text{DMR}} C_{\text{Lip}} \|e_h\|_{l^p_\tau(N_S; X_{h,q})} \\ &\leq C_{\text{DMR}} C_{\text{Lip}} |\Omega|^{1/q} S^{1/p} \max_{0 \leq n \leq N_S-1} \|e_{1,h}^n\|_{L^\infty} + C(h^2 + \tau) \\ &\leq C_0 S^{1/p} \|(-A_h)^{-\alpha} e_{1,h}\|_{Y_S} + C(h^2 + \tau) \end{aligned}$$

from (1.61), (1.64), and Lemma 1.6.3. Consequently, taking $S \leq (2C_0)^{-p}$, we obtain

$$\|(-A_h)^{-\alpha} e_{1,h}\|_{Y_S} \leq C(h^2 + \tau). \quad (1.65)$$

This, together with (1.63), implies (1.56) with $T_1 = (2C_0)^{-p}$.

Step 2. We prove (1.55) for any $T \in (0, \infty)$. Then we shall show that

$$\|(-A_h)^{-\alpha} e_{1,h}^{+N_S}\|_{Y_\sigma} \leq C(\|(-A_h)^{-\alpha} e_{1,h}\|_{Y_S} + h^2 + \tau) \quad (1.66)$$

for all $S < T$ and $\sigma \leq \min\{T_1, T - S\}$. Take $S < T$ and $\sigma \leq \min\{T_1, T - S\}$ arbitrarily, and set $w_{j,h}^n = e_{j,h}^{n+N_S}$ ($j = 1, 2$). Then, $w_{1,h}$ satisfies

$$\begin{cases} (D_\tau w_{1,h})^n = A_h w_{1,h}^{n+1} + F_h(u_h^{n+N_S}) - F_h(P_h U^{n+N_S}), & n = 0, \dots, N_{T-S}, \\ w_{1,h}^0 = e_{1,h}^{N_S}. \end{cases}$$

Therefore, thanks to (1.61), (1.64), and (1.65) and Lemmas 1.6.1 and 1.6.3, we can obtain

$$\begin{aligned} &\|(-A_h)^{-\alpha} w_{1,h}\|_{Y_\sigma} \\ &\leq C_{\text{DMR}} C_{\text{Lip}} \|w_{1,h} + w_{2,h}\|_{l^p_\tau(N_\sigma; X_{h,q})} + C_{\text{DMR}} \|(-A_h)^{-\alpha} e_{1,h}^{N_S}\|_{h,1-1/p,p} \\ &\leq C_0 \sigma^{1/p} \left(\|(-A_h)^{-\alpha} w_{1,h}\|_{Y_\sigma} + \|(-A_h)^{-\alpha} e_{1,h}^{N_S}\|_{h,1-1/p,p} \right) + C(h^2 + \tau) + C\|(-A_h)^{-\alpha} e_{1,h}\|_{Y_S} \\ &\leq \frac{1}{2} \|(-A_h)^{-\alpha} w_{1,h}\|_{Y_\sigma} + C\|(-A_h)^{-\alpha} e_{1,h}\|_{Y_S} + C(h^2 + \tau), \end{aligned}$$

since $\sigma \leq T_1 = (2C_0)^{-p}$, which yields (1.66).

Noting that $N_{S+\sigma} \leq N_S + N_\sigma$, one obtains

$$\|v_h\|_{Y_{S+\sigma}} \leq \|v_h\|_{Y_S} + \|v_h^{+N_S}\|_{Y_\sigma}$$

for $v_h \in l^p(N_S + N_\sigma; S_h)$ and $S, \sigma > 0$. Therefore, we can inductively establish the desired estimate (1.55) from (1.63), (1.65), and (1.66). \square

Finally, we state the following proof.

Proof of Theorems 1.2.4 and 1.2.5. Observe that

$$\|u_h^n - U^n\|_{L^q} \leq \|e_h^n\|_{L^q} + \|P_h U^n - U^n\|_{L^q} \leq \|e_h^n\|_{L^q} + Ch^2 \|U^n\|_{W^{2,q}}$$

by Lemma 1.3.8. Therefore, it suffices to prove

$$\left(\sum_{n=1}^{N_T} \|e_h^n\|_{L^q}^p \tau \right)^{1/p} \leq C(h^2 + \tau) \quad \text{and} \quad \max_{0 \leq n \leq N_T} \|e_h^n\|_{L^\infty} \leq C(h^{2\alpha} + \tau)$$

for $\alpha \in (0, \alpha_{p,q,d})$. To this end, let

$$M = \|u\|_{L^\infty(\Omega \times (0,T))} + \sup_{h>0} \|P_h u\|_{L^\infty(\Omega \times (0,T))}$$

for the solution u of (1.3) and $T \in (0, T_\infty)$. It is apparent that M is finite since the L^2 -projection P_h is stable in the L^∞ -norm (Lemma 1.3.8). We introduce

$$\tilde{f}(z) = \tilde{f}_M(z) = \begin{cases} f(z), & |z| \leq M, \\ f\left(M \frac{z}{|z|}\right), & |z| > M. \end{cases}$$

Then, \tilde{f} is a globally Lipschitz continuous function. We consider the problems (1.3) and (1.29) with replacement of f by \tilde{f} , and denote the corresponding solutions by \tilde{u} and \tilde{u}_h , respectively. Moreover, we consider the error $\tilde{e}_h = (\tilde{e}_h^n)_{n=0}^{N_T} \in S_h^{N_T+1}$, where $\tilde{e}_h^n = \tilde{u}_h^n - P_h \tilde{u}(n\tau)$.

In view of Lemma 1.6.6, the following error estimate holds:

$$\|(-A_h)^{-\alpha} \tilde{e}_h\|_{Y_T} \leq C(h^{2\alpha} + \tau)$$

for any $\alpha \in [0, 1]$. By setting $\alpha = 1$, we obtain

$$\left(\sum_{n=1}^{N_T} \|\tilde{e}_h^n\|_{L^q}^p \tau \right)^{1/p} \leq C(h^2 + \tau). \quad (1.67)$$

Applying Lemma 1.6.3, we can deduce

$$\max_{0 \leq n \leq N_T} \|\tilde{e}_h^n\|_{L^\infty} \leq C(h^{2\alpha} + \tau), \quad (1.68)$$

for $\alpha \in (0, \alpha_{p,q,d})$.

At this stage, we have $\tilde{u} = u$ by the unique solvability of (1.3). Indeed, $\|u\|_{L^\infty(\Omega \times (0,T))} \leq M$ implies $\tilde{f}(u(x, t)) = f(u(x, t))$ for every $(x, t) \in \Omega \times (0, T)$. Moreover, according to (1.68), we deduce

$$\max_{0 \leq n \leq N_T} \|\tilde{u}_h^n\|_{L^\infty} \leq C(h^{2\alpha} + \tau) + \sup_{h>0} \|P_h u\|_{L^\infty(\Omega \times (0,T))}$$

for $\alpha \in (0, \alpha_{p,q,d})$. Therefore, there exist $h_0 > 0$ and $\tau_0 > 0$ such that

$$\max_{0 \leq n \leq N_T} \|\tilde{u}_h^n\|_{L^\infty} \leq M, \quad \forall h \leq h_0, \quad \forall \tau \leq \tau_0,$$

which implies that $\tilde{f}(\tilde{u}_h^n) = f(\tilde{u}_h^n)$. Again, the unique solvability of (1.29) yields $\tilde{u}_h = u_h$ for $h \leq h_0$ and $\tau \leq \tau_0$. Hence we can replace \tilde{e}_h^n by e_h^n in (1.67) and (1.68), which completes the proof of Theorems 1.2.4 and 1.2.5. \square

Chapter 2

Stability and analyticity in maximum-norm and maximal regularity for FEM of parabolic equations on smooth domains

Abstract

In this chapter, we consider the finite element semi-discretization of smoothing property in maximum-norm and maximal regularity for a parabolic problem on a smooth domain $\Omega \subset \mathbb{R}^N$ with the Neumann boundary condition. We emphasize that the domain can be non-convex in general. We implement the finite element method for this problem by constructing a family of polygonal or polyhedral domains $\{\Omega_h\}_h$ that approximate the original domain Ω . The main result of this study is the smoothing property for the discrete semigroup and the maximal regularity results for the discrete elliptic operators. The difficulty of this study is the effect of gap of domains, since the symmetric difference $\Omega \triangle \Omega_h$ is not empty in general. In order to address the effect of the symmetric difference, we introduce the tubular neighborhood of the original boundary $\partial\Omega$. Moreover, we will propose a slightly new technique to establish the L^1 -type estimates for the regularized Green's function without a strong super-approximation property.

2.1 Introduction

In this chapter, we consider the finite element method (FEM) for a parabolic problem on a bounded domain $\Omega \subset \mathbb{R}^N$ with general $N \in \mathbb{N}$, which can be non-convex. We assume that the boundary $\partial\Omega$ is sufficiently smooth. The target problem of the present chapter is the following parabolic equation on Ω :

$$\begin{cases} \partial_t u + Au = f, & \text{in } \Omega \times (0, T) =: Q_T, \\ \partial_n u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \quad (2.1)$$

where $A = -\Delta + 1$, $f: \Omega \times (0, T) \rightarrow \mathbb{R}$, $u_0: \Omega \rightarrow \mathbb{R}$, and ∂_n denotes the outward normal derivative on $\partial\Omega$. Although we can consider general (strongly) elliptic operators of second order, we here address the operator $-\Delta + 1$ for simplicity. As is well-known, when $f \equiv 0$, the problem (2.1) has the stability and smoothing property in $L^p(\Omega)$ -norm

$$\|u(t)\|_{L^p(\Omega)} + \|t\partial_t u(t)\|_{L^p(\Omega)} \leq C\|u_0\|_{L^p(\Omega)}, \quad \forall t > 0 \quad (2.2)$$

for $u_0 \in L^p(\Omega)$ with $p \in [1, \infty)$ and $u_0 \in C^0(\overline{\Omega})$ with $p = \infty$. It also has the maximal regularity

$$\|\partial_t u\|_{L^p(0, T; L^q(\Omega))} + \|Au\|_{L^p(0, T; L^q(\Omega))} \leq C\|f\|_{L^p(0, T; L^q(\Omega))} \quad (2.3)$$

for $p, q \in (1, \infty)$ when $u_0 = 0$. These estimates play crucial roles in the analysis for nonlinear partial differential equations, such as well-posedness and stability.

The purpose of this chapter is to investigate spatial discretization of the stability and smoothing property in maximum-norm, and maximal regularity. Let us consider the finite element approximation for (2.1). If we can establish the discrete counterpart of the estimates (2.2) and (2.3), we can expect that these results can be applied to numerical analysis for nonlinear partial differential equations. Indeed, the discrete counterpart of the smoothing property is applied to construct and study numerical schemes for linear and nonlinear parabolic problems (e.g. [37, 43, 46, 97, 92, 102]), and the discrete version of maximal regularity is as well (e.g. [47, 48, 76, 71, 68]). We remark that all of these studies assume that Ω is a polygonal or smooth convex domain.

In the context of FEM, the domain Ω is usually assumed to be a polygonal or polyhedral domain so that triangulations can be exactly implemented. However, it is known that the regularity of the solution cannot be guaranteed if there exist corners in the boundary of the domain (see e.g. [50]). Lack of regularity of solutions is troublesome in numerical analysis for partial differential equations. For example, in [92, 102], finite element and finite volume schemes for the Keller-Segel system on a polygonal domain are considered. In their error estimates ([92, Theorem 2.4] and [102, Theorem 3.1]), the convergence rate in $L^\infty(0, T; L^p(\Omega))$ -norm is $O(h^{1-N/p})$, in contrast to the expected rate $O(h)$, where h is the mesh size. This shortcoming is caused by the corner singularity of the boundary. Indeed, it is shown that the convergence rate is $O(h)$ if the boundary is smooth [92, Section 5.1]. Moreover, in [68], the authors consider discretization of maximal regularity on polygonal or polyhedral domains. However, the Lebesgue exponent q of the spatial variables are restricted, whereas (2.3) holds for all $q \in (1, \infty)$, which also results from the loss of regularity.

In view of the theory of nonlinear partial differential equations, appropriate regularity, such as smoothing property and maximal regularity, is essential for analysis of equations. Therefore, it is natural to assume the boundary is smooth, and consequently, it is important to consider FEM for such problems.

There are several studies on FEM for parabolic problems on smooth domains. For example, in [15, 94, 98, 75, 73], the stability, analyticity, and maximum-norm error estimates for FEM of parabolic equations are studied. In particular, [94] gives a general method for these problems using the regularized Green's function. However, all of them assume that the domain is convex. For homogeneous Dirichlet problems (e.g., [15, 98]), they consider a family of polynomial (or polyhedral) domains $\{\Omega_h\}_h$ whose vertices lie in $\partial\Omega$, and introduced a space of piecewise polynomials associated with a triangulation of Ω_h that vanishes on $\partial\Omega_h$. Then, they extend each functions in such a space by zero in $\Omega \setminus \Omega_h$. Therefore, discrete functions can be viewed as elements in $H_0^1(\Omega)$, yet this procedure is available for convex domains and for homogeneous Dirichlet problems. For Neumann problems (e.g., [94, 75]), it is assumed that the domain is exactly triangulated. That is, piecewise polynomial functions are extended by considering pie-shaped element near the boundary. However, this extension is unavailable for the three-dimensional case, even if the domain is convex, as pointed out in [94, page 1356]. In [94], it is also mentioned that their argument can be applied if one considers an extended domain which has the Hausdorff distance of order $O(h^2)$ and a polyhedral approximation of the extended domain that includes the original one. However, this procedure is not practical. The same assumptions are imposed in the context of (spatial) discrete maximal regularity on smooth domains [47, 48, 75].

In contrast to these works, we never assume that Ω is convex and thus $\Omega \triangle \Omega_h \neq \emptyset$ in general, where $\Omega \triangle \Omega_h$ is the symmetric difference. We here describe the implementation of FEM on a smooth non-convex domain Ω . We first approximate Ω by a family of polygonal domains. Let $\Omega_h \subset \mathbb{R}^N$ be a polygonal (or polyhedral) domain whose vertices lie on $\partial\Omega$. We make a conforming triangulation \mathcal{T}_h of Ω_h . Here, we assume that

- for each triangle $K \in \mathcal{T}_h$, $K \cap \Omega \neq \emptyset$
- for each node P of \mathcal{T}_h , $P \in \partial\Omega_h \implies P \in \partial\Omega$

and we set $h = \max_{K \in \mathcal{T}_h} \text{diam } K$. Then, we define $V_h \in H^1(\Omega_h)$ as the conforming P^k -finite element space associated with \mathcal{T}_h for $k \geq 1$. Now, the finite element approximation for (2.1) can be formulated as follows. Find $u_h \in C^0([0, T]; V_h)$ that satisfies

$$\begin{cases} (u_{h,t}(t), v_h)_{\Omega_h} + a_{\Omega_h}(u_h(t), v_h) = (f_h(t), v_h)_{\Omega_h}, & \forall v_h \in V_h, \\ u_h(0) = u_{h,0}, \end{cases} \quad (2.4)$$

for each $t \in (0, T)$, where $f_h: (0, T) \rightarrow V_h$ and $u_{h,0} \in V_h$ are given discrete data and the bracket $(\cdot, \cdot)_D$ denotes the usual L^2 -inner product over $D \subset \mathbb{R}^N$. This procedure is adopted in basic softwares on FEM such as FreeFEM++ [60] and FEniCS [82], and thus it is important to investigate theoretical properties of the approximation scheme (2.4).

Our main results are the smoothing property in maximum-norm and maximal regularity for the discrete Laplace operator A_h . Here, we define A_h by

$$(A_h u_h, v_h)_{\Omega_h} = (\nabla u_h, \nabla v_h)_{\Omega_h} + (u_h, v_h)_{\Omega_h}, \quad \forall u_h, v_h \in V_h,$$

which is a discrete analog of Green's formula. We shall show that the estimate

$$\|u_h(t)\|_{L^\infty(\Omega_h)} + \|t \partial_t u_h(t)\|_{L^\infty(\Omega_h)} \leq C e^{-ct} \|u_{h,0}\|_{L^\infty(\Omega_h)}, \quad \forall t > 0,$$

holds $f_h \equiv 0$ (Theorem 2.2.1), and

$$\|\partial_t u_h\|_{L^p(0,T;L^q(\Omega_h))} + \|A_h u_h\|_{L^p(0,T;L^q(\Omega_h))} \leq C \|f_h\|_{L^p(0,T;L^q(\Omega_h))},$$

holds for $p, q \in (1, \infty)$ when $u_{h,0} \equiv 0$ and $f_h \in L^p(0, T; V_h)$ (Theorem 2.2.2). The smoothing effect holds also in $L^p(\Omega_h)$ for $p \in [1, \infty)$.

Since Ω is not convex, we should take great care of the effect of gap of domains $\Omega \triangle \Omega_h$. In order to address the integration over $\Omega \triangle \Omega_h$, we introduce the tubular neighborhood of $\partial\Omega$. As in the analysis of FEM for elliptic equations, the Galerkin orthogonality is essential in the analysis for FEM of parabolic problems (cf. [94]). However, since $\Omega \triangle \Omega_h \neq \emptyset$, it does not hold and there appear additional terms (see Lemma 2.4.1). We shall address these terms using the tubular neighborhood as in the elliptic case discussed in our forthcoming paper [62]. This procedure allows us to consider the effect of the gap of domains as just a perturbation, and thus we can obtain the same estimates as in [94, Theorem 2.1] and [47, Theorem 3.2].

The main strategy of the proof of the above estimates is similar to [94, 47]. That is, we introduce the regularized delta function, regularized Green's function Γ , and its finite element approximation Γ_h . Then, we reduce the stability estimates to the L^1 -type estimates for $F = \Gamma_h - \tilde{\Gamma}$ (Lemma 2.4.2), where $\tilde{\Gamma}$ denotes an extension of Γ to Ω_h in the sense of Sobolev spaces. We will introduce a space-time dyadic decomposition $Q_{h,j}$ by parabolic annuli (see (2.47)) and we address the integration of F over each $Q_{h,j}$. In the proof of the estimates for F , we shall take a slightly new approach. In [94], they prove local energy error estimates with kick-back argument. For this purpose, they show the strong super-approximation property for the discrete space V_h [94, Section 5]. The argument of [98] is similar and they consider a delicate estimate with a special cut-off function [98, pages 387–388]. Finally, local estimates are integrated with respect to the dyadic decomposition, and the L^1 -estimates are obtained. In contrast to these arguments, we will use the kick-back argument after summation. Our strategy does not require the strong super-approximation property and special cut-off functions. Therefore, our argument gives an alternative proof for stability, analyticity, and spatially discrete maximal regularity for FEM.

The rest of this chapter is organized as follows. In Section 2.2, we present our notation and state the main results. In Section 2.3, we summarize preliminary results on FEM, tubular neighborhood, and the regularized Green's function Γ . The estimates on the gap of domains stated in subsection 2.3.2 will be used repeatedly in this chapter. Section 2.4 is devoted to the proof of the stability and the smoothing property. We will postpone the proof of L^1 -type estimates for F , which is given in Section 2.5. The estimates for Γ over the tubular neighborhood are also given here (Lemma 2.5.2). Finally, we will present the proof of maximal regularity for the discrete elliptic operator A_h in Section 2.6.

2.2 Notation and main result

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with general $N \in \mathbb{N}$. We assume that $\partial\Omega$ is sufficiently smooth. The target problem of the present chapter is the parabolic equation on Ω :

$$\begin{cases} \partial_t u + Au = f, & \text{in } \Omega \times (0, T) =: Q_T, \\ \partial_n u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \quad (2.5)$$

where $A = -\Delta + 1$, $f: \Omega \times (0, T) \rightarrow \mathbb{R}$, $u_0: \Omega \rightarrow \mathbb{R}$, and ∂_n denotes the outward normal derivative on $\partial\Omega$. In this chapter, we address the operator $A = -\Delta + 1$ for simplicity, although our method is available for general parabolic equations with strongly coercive elliptic operators (if the coefficients are sufficiently smooth). As is well-known, the parabolic problem (2.5) has the following two properties.

(i) Stability and analyticity. Let $f \equiv 0$. Then,

$$\|u(t)\|_{C^0(\bar{\Omega})} + \|tu_t(t)\|_{C^0(\bar{\Omega})} \leq Ce^{-ct}\|u_0\|_{C^0(\bar{\Omega})}, \quad \forall t > 0, \quad (2.6)$$

provided that $u_0 \in C^0(\bar{\Omega})$, and

$$\|u(t)\|_{L^p(\Omega)} + \|tu_t(t)\|_{L^p(\Omega)} \leq Ce^{-ct}\|u_0\|_{L^p(\Omega)}, \quad \forall t > 0, \quad (2.7)$$

provided that $u_0 \in L^p(\Omega)$ and $1 \leq p < \infty$, where $C > 0$ and $c > 0$ are independent of u_0 and t .

(ii) Maximal regularity. Let $u_0 \equiv 0$. Then,

$$\|u_t\|_{L^p(0,T;L^q(\Omega))} + \|Au\|_{L^p(0,T;L^q(\Omega))} \leq C\|f\|_{L^p(0,T;L^q(\Omega))}, \quad (2.8)$$

provided that $f \in L^p(0,T;L^q(\Omega))$ and $p, q \in (1, \infty)$, where $C > 0$ is independent of f .

In the theory of nonlinear partial differential equations, these two properties play crucial roles for investigation of well-posedness and stability.

The purpose of this chapter is (spatial) discretization of the above estimates. If we can establish the discrete counterparts of (2.6), (2.7), and (2.8) when the problem (2.5) is discretized for numerical computation, then it is expected that such estimates are available for numerical analysis of nonlinear partial differential equations. From this viewpoint, we consider the finite element approximation of (2.5). In contrast to existing work (e.g., [94, 98, 47]), the domain Ω is not convex in general. Therefore, we first approximate the domain Ω by polygonal (or polyhedral) domains. Let $\Omega_h \subset \mathbb{R}^N$ be a polygonal domain and \mathcal{T}_h be a triangulation (decomposition into simplexes in general) of Ω_h with $h = \max_{K \in \mathcal{T}_h} \text{diam } K$. Throughout this chapter, we assume that Ω_h and \mathcal{T}_h enjoy the following conditions.

- For each simplex $K \in \mathcal{T}_h$, $K \cap \Omega \neq \emptyset$.
- For each node P of \mathcal{T}_h , $P \in \partial\Omega_h \implies P \in \partial\Omega$.

Moreover, we suppose that \mathcal{T}_h is shape-regular and quasi-uniform. Note that $\Omega_h \triangle \Omega \neq \emptyset$ in general and the identity

$$\int_{\Omega} \varphi dx - \int_{\Omega_h} \varphi dx = \int_{\Omega \setminus \Omega_h} \varphi dx - \int_{\Omega_h \setminus \Omega} \varphi dx \quad (2.9)$$

holds for $\varphi \in L^1(\Omega \cup \Omega_h)$. We also set $Q_{h,T} := \Omega_h \times (0, T)$.

Let $V_h \subset H^1(\Omega_h)$ be the conforming P^k -finite element space with respect to \mathcal{T}_h ($k \geq 1$). Then, we can formulate the finite element approximation of (2.5) as follows. Find $u_h \in C^0([0, T]; V_h)$ that satisfies

$$\begin{cases} (u_{h,t}(t), v_h)_{\Omega_h} + a_{\Omega_h}(u_h(t), v_h) = (f_h(t), v_h)_{\Omega_h}, & \forall v_h \in V_h, \\ u_h(0) = u_{h,0}, \end{cases} \quad (2.10)$$

for each $t \in (0, T)$, where $f_h: (0, T) \rightarrow V_h$ and $u_{h,0} \in V_h$ are given data. Here, and hereafter, the bracket $(\cdot, \cdot)_D$ denotes the L^2 -inner product over a domain $D \subset \mathbb{R}^N$ and

$$a_D(u, v) = (\nabla u, \nabla v)_D + (u, v)_D, \quad u, v \in H^1(D)$$

for $D \subset \mathbb{R}^N$. Moreover, we define the discrete Laplace operator $A_h: V_h \rightarrow V_h$ by

$$(A_h v_h, w_h)_{\Omega_h} = a_{\Omega_h}(v_h, w_h), \quad \forall w_h \in V_h$$

for $v_h \in V_h$.

The main results of this chapter are the stability and analyticity of the discrete semigroup e^{-tA_h} in $L^p(\Omega_h)$ and (spatially) discrete maximal regularity for A_h .

Theorem 2.2.1 (Stability and analyticity of the discrete semigroup). *Let $p \in [1, \infty]$ and \mathcal{T}_h be a shape-regular and quasi-uniform triangulation of Ω . Let u_h be the solution of (2.10) for $f_h \equiv 0$. Then, we have*

$$\|u_h(t)\|_{L^p(\Omega_h)} + t\|\partial_t u_h(t)\|_{L^p(\Omega_h)} \leq Ce^{-ct}\|u_{h,0}\|_{L^p(\Omega_h)}, \quad \forall t > 0, \quad (2.11)$$

where $C > 0$ and $c > 0$ are independent of h , $u_{h,0}$, and t .

Theorem 2.2.2 (Discrete maximal regularity). *Let $p, q \in (1, \infty)$ and \mathcal{T}_h be a shape-regular and quasi-uniform triangulation of Ω . Let u_h be the solution of (2.10) for $u_{h,0} = 0$ and $f_h \in L^p(0, T; V_h)$. Then, we have*

$$\|A_h u_h\|_{L^p(0, T; L^q(\Omega_h))} + \|\partial_t u_h\|_{L^p(0, T; L^q(\Omega_h))} \leq C\|f_h\|_{L^p(0, T; L^q(\Omega_h))}, \quad (2.12)$$

where $C > 0$ is independent of h , f_h , and T .

2.3 Preliminaries

2.3.1 Projection and interpolation

We introduce the projection and interpolation operators with respect to V_h . We denote the $L^2(\Omega_h)$ -projection by P_h . That is,

$$(P_h v, w_h)_{\Omega_h} = (v, w_h)_{\Omega_h}, \quad \forall v \in L^1(\Omega_h), \forall w_h \in V_h.$$

We denote the node-wise interpolation operator by I_h . Furthermore, we construct a “quasi-interpolation” operator \tilde{I}_h acting on the Sobolev space $W^{1,1}(\Omega_h)$, whereas I_h acts on the space of continuous functions. For construction, see [47, Section 5] (especially, definition of \tilde{I}_h^N). For these operators, the following stability and error estimates hold. The proofs can be found in [16] and [47] (see also [98, Lemma 2.1]).

Lemma 2.3.1. *Assume that \mathcal{T}_h is shape-regular and quasi-uniform.*

(i) *For each $p \in [1, \infty]$, we have*

$$\begin{aligned} \|P_h v\|_{L^p(\Omega_h)} &\leq C\|v\|_{L^p(\Omega_h)}, \quad \forall v \in L^p(\Omega_h), \\ \|P_h v\|_{W^{1,p}(\Omega_h)} &\leq C\|v\|_{W^{1,p}(\Omega_h)}, \quad \forall v \in W^{1,p}(\Omega_h). \end{aligned}$$

(ii) *Let $0 \leq l \leq m \leq k+1$ be integers. Then, for each $K \in \mathcal{T}_h$, we have*

$$\|D^l(v - I_h v)\|_{L^\infty(K)} \leq Ch^{m-l}\|D^m v\|_{L^\infty(K)}, \quad \forall v \in C^m(\overline{K}),$$

where C is independent of h , K , and v .

(iii) *Let $K \in \mathcal{T}_h$ and $M_K := \bigcup\{\overline{T} \in \mathcal{T}_h \mid \overline{T} \cap \overline{K} \neq \emptyset\}$. Then, for each $p \in [1, \infty]$, we have*

$$\begin{aligned} \|v - \tilde{I}_h v\|_{L^p(K)} &\leq Ch\|\nabla v\|_{L^p(M_K)}, \quad \forall v \in W^{1,p}(M_K), \\ \|v - \tilde{I}_h v\|_{L^p(K)} &\leq Ch^2\|D^2 v\|_{L^p(M_K)}, \quad \forall v \in W^{2,p}(M_K), \\ \|\nabla(v - \tilde{I}_h v)\|_{L^p(K)} &\leq Ch\|D^2 v\|_{L^p(M_K)}, \quad \forall v \in W^{2,p}(M_K), \end{aligned}$$

where each C is independent of h , K , and v .

2.3.2 Tubular neighborhood

In order to address the integrals over the symmetric difference $\Omega \triangle \Omega_h$, we introduce the tubular neighborhood of $\Omega \triangle \Omega_h$. If h is sufficiently small, we can construct a homeomorphism $\pi: \partial\Omega_h \rightarrow \partial\Omega$ based on the signed distance function with respect to $\partial\Omega$. Then, the inverse map $\pi^*: \partial\Omega \rightarrow \partial\Omega_h$ is of the form $\pi^*(x) = x + t^*(x)n(x)$ ($x \in \partial\Omega$), where $n(x)$ is the outward unit normal vector of $\partial\Omega$ at x and $t^* \in C^0(\partial\Omega; \mathbb{R})$. We refer the reader to [49, Section 14.6] for construction and properties of π . It is known that $\|t^*\|_{L^\infty(\partial\Omega)} \leq c_0 h^2$ for some $c_0 > 0$ depending only on Ω . In what follows, we set $\varepsilon := c_0 h^2$ for such c_0 . We introduce the tubular neighborhood of $\partial\Omega$ by

$$T(\varepsilon) := \{x \in \mathbb{R}^N \mid \text{dist}(x, \partial\Omega) < \varepsilon\},$$

where $\text{dist}(x, D) = \inf_{y \in D} |x - y|$ for $x \in \mathbb{R}^N$ and $D \subset \mathbb{R}^N$. Then, from the above observation, we have $\Omega \triangle \Omega_h \subset T(\varepsilon)$.

Here, we collect some estimates related to $T(\varepsilon)$. For the proofs of the following inequalities, we refer to [63, Appendix]. Note that the local estimates (2.16) and (2.17) can be obtained by the same argument given there, since the proofs are based on local coordinates.

Lemma 2.3.2. (i) For $f \in L^1(T(\varepsilon))$, we have

$$\left| \int_{\partial\Omega} f ds - \int_{\partial\Omega_h} f \circ \pi ds \right| \leq C\varepsilon \|f\|_{L^1(\partial\Omega)}.$$

(ii) For $f \in W^{1,p}(T(\varepsilon))$ and $p \in [1, \infty]$, we have

$$\|f - f \circ \pi\|_{L^p(\partial\Omega_h)} \leq C\varepsilon^{1-\frac{1}{p}} \|\nabla f\|_{L^p(T(\varepsilon))}, \quad (2.13)$$

$$\|f\|_{L^p(T(\varepsilon))} \leq C\varepsilon^{1/p} \|f\|_{L^p(\partial\Omega)} + C\varepsilon \|\nabla f\|_{L^p(T(\varepsilon))}, \quad (2.14)$$

$$\|f\|_{L^p(T(\varepsilon))} \leq C\varepsilon^{1/p} \|f\|_{L^p(\partial\Omega_h)} + C\varepsilon \|\nabla f\|_{L^p(T(\varepsilon))}, \quad (2.15)$$

and the local estimate

$$\|f - f \circ \pi\|_{L^p(\partial\Omega_h \cap D)} \leq C\varepsilon^{1-\frac{1}{p}} \|\nabla f\|_{L^p(T(\varepsilon) \cap D_\varepsilon)}, \quad (2.16)$$

$$\|f\|_{L^p(T(\varepsilon) \cap D)} \leq C\varepsilon^{1/p} \|f\|_{L^p(\partial\Omega_h \cap D_\varepsilon)} + C\varepsilon \|\nabla f\|_{L^p(T(\varepsilon) \cap D_\varepsilon)}, \quad (2.17)$$

for $D \subset \mathbb{R}^N$ and $D_d := \{x \in \mathbb{R}^N \mid \text{dist}(x, D) < d\}$ for $d > 0$.

(iii) Letting n_h be the outward unit normal vector of $\partial\Omega_h$, we have

$$\|n_h - n \circ \pi\|_{L^\infty(\partial\Omega_h)} \leq Ch. \quad (2.18)$$

Here, each C is independent of h and f .

Let $\tilde{\Omega} := \Omega \cup T(\varepsilon) = \Omega_h \cup T(\varepsilon)$. As mentioned above $\Omega \triangle \Omega_h \neq \emptyset$ in general. Therefore, we sometimes need to extend a function on Ω to Ω_h in the sense of Sobolev spaces. We denote such extension by \tilde{w} for a given function w defined over Ω . Although the extension map may be different up to the regularity of the function, we use the same notation. Such extension can be constructed by reflection and is well-defined as a function over $\tilde{\Omega}$. We can check the following global and local stability of the extension operators (see [1]):

$$\|\tilde{w}\|_{W^{s,p}(\tilde{\Omega})} \leq C \|w\|_{W^{s,p}(\Omega)}, \quad (2.19)$$

$$\|\tilde{w}\|_{W^{s,p}(T(\varepsilon))} \leq C \|w\|_{W^{s,p}(\Omega \cap T(\varepsilon))},$$

$$\|\tilde{w}\|_{W^{s,p}(D \cap T(\varepsilon))} \leq C \|w\|_{W^{s,p}(\Omega \cap D_{2\varepsilon} \cap T(\varepsilon))}, \quad D \subset \mathbb{R}^N, \quad (2.20)$$

for $w \in W^{s,p}(\Omega)$, where C depends only on s , p , and Ω .

2.3.3 Regularized Green's function

As in the previous work on maximum-norm stability estimates for FEM, we introduce the regularized delta and Green's functions. In what follows, we fix $x_0 \in \Omega_h$ arbitrarily and let $K_0 \in \mathcal{T}_h$ be a triangle such that $x_0 \in \bar{K}_0$. Moreover, the symbol C denotes a positive constant that is independent of h and x_0 . Although its value may be different in each appearance, we use the same symbol.

Then, we can construct a smooth function $\bar{\delta} = \bar{\delta}_{x_0} \in C_0^\infty(K_0)$ that fulfills

$$P(x_0) = (P, \bar{\delta})_{K_0}, \quad \forall P \in \mathcal{P}^k(K_0),$$

where $\mathcal{P}^k(K_0)$ is the set of all polynomials of degree $\leq k$ over K_0 . Moreover, $\bar{\delta}$ satisfies $\text{supp } \bar{\delta} \subset \Omega \cap \Omega_h$ (i.e., $\text{supp } \bar{\delta} \cap T(\varepsilon) = \emptyset$) and

$$\|\bar{\delta}\|_{W^{s,p}(K_0)} \leq C_{s,p} h^{-s - (1-\frac{1}{p})N}, \quad \forall s \geq 0, \quad \forall p \in [1, \infty], \quad (2.21)$$

where $C_{s,p}$ is independent of h and x_0 . For construction, see [93, Appendix]. Further, we have

$$|(P_h \bar{\delta})(x)| \leq Ch^{-N} e^{-c|x_0-x|/h}, \quad \forall x \in \Omega_h, \quad (2.22)$$

where C and c are independent of h , x_0 , and x . The proofs can be found in [100, Lemma 7.2].

We then define the regularized Green's function Γ as the solution of the homogeneous problem

$$\begin{cases} \partial_t \Gamma + A\Gamma = 0, & \text{in } Q_T, \\ \partial_n \Gamma = 0, & \text{on } \partial\Omega \times (0, T), \\ \Gamma(0) = \bar{\delta}, & \text{in } \Omega. \end{cases} \quad (2.23)$$

Note that $\Gamma \in C^\infty(\overline{Q_T})$ since $\bar{\delta}$ and $\partial\Omega$ are sufficiently smooth. Furthermore, we define Γ_h as the finite element approximation of Γ as follows.

$$\begin{cases} (v_h, \Gamma_{h,t}(t))_{\Omega_h} + a_{\Omega_h}(v_h, \Gamma_h(t)) = 0, & \forall v_h \in V_h, \\ \Gamma_h(0) = P_h \bar{\delta}. \end{cases} \quad (2.24)$$

We finally set $F := \Gamma_h - \tilde{\Gamma}$, which is a function defined on Ω_h .

We recall the pointwise estimates for the usual Green's function (fundamental solution). Let $G = G(x, y; t)$ be the solution of

$$\begin{cases} \partial_t G + AG = 0, & \text{in } Q_T, \\ \partial_n G = 0, & \text{on } \partial\Omega \times (0, T), \\ G(0) = \delta_y, & \text{in } \Omega. \end{cases}$$

where $y \in \Omega$ and δ_y is the Dirac δ -function with respect to y . Then, the following pointwise estimates are known.

$$|\partial_t^k \partial_x^\alpha G(x, y; t)| \leq C \left(\sqrt{t} + |x - y| \right)^{-N-2k-|\alpha|} e^{-c|x-y|^2/t}, \quad \forall x, y \in \Omega, \quad \forall t > 0, \quad (2.25)$$

for any non-negative integer k and multi-index α , where C and c are independent of x , y , and t . See [30] for the proof.

2.4 Proof of stability and analyticity

According to [94], we reduce the stability estimate (2.11) to the L^1 -error estimates for Γ and Γ_h for $T \leq 1$. In the argument of [94], the Galerkin orthogonality

$$(v_h, (\Gamma - \Gamma_h)_t)_\Omega + a_\Omega(v_h, \Gamma - \Gamma_h) = 0, \quad \forall v_h \in V_h$$

holds since $\Omega = \Omega_h$, and this identity is used repeatedly. However, in our case, there appears additional terms induced by the gap of domains. Thus we begin this section by the *asymptotic* Galerkin orthogonality in a general setting. In what follows, ∂_{n_h} denotes the outward normal derivative on $\partial\Omega_h$.

Lemma 2.4.1 (Asymptotic Galerkin orthogonality). *Assume z solves*

$$\begin{cases} z_t + Az = \varphi, & \text{in } Q_T, \\ \partial_n z = \psi, & \text{on } \partial\Omega \times (0, T), \end{cases}$$

and z_h solves

$$(z_{h,t}, v_h)_{\Omega_h} + a_{\Omega_h}(z_h, v_h) = (\tilde{\varphi}, v_h)_{\Omega_h} + (\tilde{\psi}, v_h)_{\partial\Omega_h}, \quad \forall v_h \in V_h$$

for given $\varphi \in C(\overline{Q_T})$ and $\psi \in C(\partial\Omega \times (0, T))$. Then, we have

$$((z_h - \tilde{z})_t, v_h)_{\Omega_h} + a_{\Omega_h}(z_h - \tilde{z}, v_h) = -(\tilde{z}_t + A\tilde{z} - \tilde{\varphi}, v_h)_{\Omega_h \setminus \Omega} - (\partial_{n_h} \tilde{z} - \tilde{\psi}, v_h)_{\partial\Omega_h} \quad (2.26)$$

Proof. We observe that the formula

$$(\nabla v, \nabla w)_{\Omega \setminus \Omega_h} - (\nabla v, \nabla w)_{\Omega_h \setminus \Omega} = (\partial_n v, w)_{\partial \Omega} - (\partial_{n_h} v, w)_{\partial \Omega_h} - (\Delta v, w)_{\Omega \setminus \Omega_h} + (\Delta v, w)_{\Omega_h \setminus \Omega} \quad (2.27)$$

holds for $v \in H^2(\tilde{\Omega})$ and $w \in H^1(\tilde{\Omega})$ by integration by parts. Now, from the identity (2.9), we have

$$(\tilde{z}_t, v_h)_{\Omega_h} + a_{\Omega_h}(\tilde{z}, v_h) = I_1 + I_2,$$

where

$$I_1 = (z_t, \tilde{v}_h)_{\Omega} + a_{\Omega}(z, \tilde{v}_h) = (\varphi, \tilde{v}_h)_{\Omega} + (\psi, \tilde{v}_h)_{\partial \Omega}$$

and

$$I_2 = -(z_t, \tilde{v}_h)_{\Omega \setminus \Omega_h} - a_{\Omega \setminus \Omega_h}(z, \tilde{v}_h) + (\tilde{z}_t, v_h)_{\Omega_h \setminus \Omega} + a_{\Omega_h \setminus \Omega}(\tilde{z}, v_h).$$

Again, the equality (2.9) yields

$$I_1 = (\tilde{\varphi}, v_h)_{\Omega_h} + (\varphi, \tilde{v}_h)_{\Omega \setminus \Omega_h} - (\tilde{\varphi}, v_h)_{\Omega_h \setminus \Omega} + (\psi, \tilde{v}_h)_{\partial \Omega}.$$

Moreover, due to the formula (2.27), we have

$$\begin{aligned} I_2 &= -(z_t + Az, \tilde{v}_h)_{\Omega \setminus \Omega_h} + (\tilde{z}_t + A\tilde{z}, v_h)_{\Omega_h \setminus \Omega} - (\partial_n z, \tilde{v}_h)_{\partial \Omega} + (\partial_n \tilde{z}, v_h)_{\partial \Omega_h} \\ &= -(\varphi, \tilde{v}_h)_{\Omega \setminus \Omega_h} + (\tilde{z}_t + A\tilde{z}, v_h)_{\Omega_h \setminus \Omega} - (\psi, \tilde{v}_h)_{\partial \Omega} + (\partial_n \tilde{z}, v_h)_{\partial \Omega_h} \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} (\tilde{z}_t, v_h)_{\Omega_h} + a_{\Omega_h}(\tilde{z}, v_h) &= (\tilde{\varphi}, v_h)_{\Omega_h} + (\tilde{z}_t + A\tilde{z} - \tilde{\varphi}, v_h)_{\Omega_h \setminus \Omega} + (\partial_n \tilde{z}, v_h)_{\partial \Omega_h} \\ &= (\tilde{\varphi}, v_h)_{\Omega_h} + (\tilde{\psi}, v_h)_{\partial \Omega_h} + (\tilde{z}_t + A\tilde{z} - \tilde{\varphi}, v_h)_{\Omega_h \setminus \Omega} + (\partial_n \tilde{z} - \tilde{\psi}, v_h)_{\partial \Omega_h}, \end{aligned}$$

which implies the desired equality owing to the definition of z_h . \square

Now, we state the following L^1 -type estimate (cf. [94, Proposition 3.2]). Recall that $F = \Gamma_h - \tilde{\Gamma}$.

Lemma 2.4.2. *Assume $T \leq 1$. Then, we have*

$$\|F_t\|_{L^1(Q_{h,T})} + \|tF_{tt}\|_{L^1(Q_{h,T})} \leq C,$$

where C is independent of h and x_0 .

We postpone the proof of Lemma 2.4.2 in Section 2.5 and here present the proof of Theorem 2.2.1 for $T \leq 1$. The argument below is almost the same as in [94]; nevertheless, we give an outline for the reader's convenience.

Proof of Theorem 2.2.1 for $T \leq 1$. It suffices to show (2.11) for $p = \infty$. Indeed, (2.11) for $p = 2$ is obtained by substituting $v_h = u_h(t)$ as a test function, and thus the Riesz-Thorin theorem yields (2.11) for $p \in [2, \infty]$, provided that the case for $p = \infty$ is accomplished. Moreover, since A_h is symmetric, (2.11) for $p \in [1, 2]$ can be obtained by duality.

Now, we assume $p = \infty$. Let e^{-tA_h} be the semigroup generated by $-A_h$. Then, the solution of (2.10) for $f_h \equiv 0$ has the expression

$$u_h(t) = e^{-tA_h} u_{h,0} = (F(t), u_{h,0})_{\Omega_h} + (\tilde{\Gamma}(t), u_{h,0})_{\Omega_h}.$$

Then, since Γ is represented by

$$\Gamma(x, t) = \int_{\text{supp } \bar{\delta}} G(x, y; t) \bar{\delta}(y) dy,$$

we can obtain $\|\tilde{\Gamma}(t)\|_{L^1(\Omega_h)} \leq C$ from (2.19), (2.21), and (2.25). Moreover, if Lemma 2.4.2 is true, then

$$\|F(t)\|_{L^1(\Omega_h)} \leq \|F(0)\|_{L^1(\Omega_h)} + \int_0^t \|F_s(s)\|_{L^1(\Omega_h)} ds \leq C$$

for $t \leq 1$. Thus, we have

$$\|u_h(t)\|_{L^\infty(\Omega_h)} \leq C\|u_{h,0}\|_{L^\infty(\Omega_h)}.$$

The proof of the smoothing effect is similar. Indeed, since

$$t\partial_t u_h(t) = (tF_t(t), u_{h,0})_{\Omega_h} + (t\tilde{\Gamma}_t(t), u_{h,0})_{\Omega_h}$$

and

$$tF_t(t) = \int_0^t (sF_s)_s ds = \int_0^t (F_s + sF_{ss}) ds,$$

Lemma 2.4.2 implies the desired estimate for $T \leq 1$. \square

In the rest of this section, we show that Theorems 2.2.1 and 2.2.2 for $T \geq 1$ are derived from these results for $T \leq 1$. We first show the exponentially decaying property for the discrete semigroup e^{-tA_h} , which corresponds to [94, Lemma 3.3] in the case $\Omega = \Omega_h$.

Lemma 2.4.3. *Assume \mathcal{T}_h is shape-regular and quasi-uniform. Let $s \geq 0$ and $m > N/2$. Then, we can find $\gamma > 0$ independently of h which satisfies*

$$\|A_h^s e^{-tA_h} v_h\|_{L^\infty(\Omega_h)} \leq Ct^{-s-m} e^{-\gamma t} \|v_h\|_{L^\infty(\Omega_h)}, \quad \forall v_h \in V_h, \forall t > 0, \quad (2.28)$$

where C is independent of h .

Proof. We show that

$$\|A_h^{-1} f_h\|_{L^q(\Omega_h)} \leq C\|f_h\|_{L^p(\Omega_h)}, \quad \forall f_h \in V_h, \quad (2.29)$$

for any $1 < p < q \leq \infty$ with $1/p - 1/q < 1/N$, where C is independent of h and f_h . Once we obtain (2.29), the proof of (2.28) is similar to that of [94, Lemma 3.3].

Fix $f_h \in V_h$ arbitrarily and let \tilde{f}_h be the extension of f_h which vanishes outside of Ω_h . We consider the elliptic equation

$$\begin{cases} Au = \tilde{f}_h, & \text{in } \Omega, \\ \partial_n u = 0, & \text{on } \partial\Omega \end{cases}$$

and its discretized problem

$$a_{\Omega_h}(u_h, v_h) = (f_h, v_h)_{\Omega_h}, \quad \forall v_h \in V_h,$$

so that $u_h = A_h^{-1} f_h$. Note that $u \in W^{2,r}(\Omega)$ for arbitrary $r \in (1, \infty)$. Then, since f_h can be viewed as an extension of \tilde{f}_h , we have

$$\|u_h - P_h \tilde{u}\|_{W^{1,r}(\Omega_h)} \leq Ch\|u\|_{W^{2,r}(\Omega)} \quad (2.30)$$

for $r \in [2, \infty)$, which is proved in [62]. Now, let $1 < p < q \leq \infty$ satisfy $1/p - 1/q < 1/N$. Then, from the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow W^{2,q}(\Omega)$, the inverse inequality, the error estimate (2.30), the elliptic regularity $\|u\|_{W^{2,p}(\Omega)} \leq C\|Au\|_{L^p(\Omega)}$, and Lemma 2.3.1, we have

$$\begin{aligned} \|u_h\|_{L^q(\Omega_h)} &\leq \|u_h - P_h \tilde{u}\|_{W^{1,q}(\Omega_h)} + \|P_h \tilde{u}\|_{W^{1,q}(\Omega_h)} \\ &\leq Ch^{-N(\frac{1}{p}-\frac{1}{q})} \|u_h - P_h \tilde{u}\|_{W^{1,p}(\Omega_h)} + C\|u\|_{W^{2,p}(\Omega)} \\ &\leq C \left(h^{1-N(\frac{1}{p}-\frac{1}{q})} + 1 \right) \|u\|_{W^{2,p}(\Omega)} \\ &\leq C\|Au\|_{L^p(\Omega)} \leq C\|f_h\|_{L^p(\Omega_h)}, \end{aligned}$$

which yields (2.29). Hence we can complete the proof. \square

Proofs of Theorems 2.2.1 and 2.2.2 for $T \geq 1$. Assume that Theorem 2.2.1 holds for $T \leq 1$. Then, together with (2.28), we can extend Theorem 2.2.1 to the case $T > 1$. Consequently, e^{-tA_h} is exponentially decaying analytic semigroup on $L^q(\Omega_h)$ for any $q \in (1, \infty)$. Therefore, if Theorem 2.2.2 holds for $T \leq 1$, we can show that it holds for any $T > 0$ from the general theory of maximal regularity (cf. [33, Theorem 2.4]). \square

2.5 Local energy error estimates

In this section, we show Lemma 2.4.2. As in [94], we shall derive the desired results via the local energy error estimates addressed below. To state them, we introduce the space-time norms of L^2 -type. For $Q \subset \mathbb{R}^{N+1}$ and $l \in \mathbb{N}$, we define

$$\|v\|_Q := \|v\|_{L^2(Q)}, \quad \|v\|_{l,Q} := \sum_{|\alpha| \leq l} \|D^\alpha v\|_{L^2(Q)},$$

and we also write

$$\|v\|_D = \|v\|_{L^2(D)}, \quad \|v\|_{l,D} = \|v\|_{H^l(D)}$$

for $D \subset \mathbb{R}^N$. Now, we state the following lemma.

Lemma 2.5.1 (Local energy error estimate). *Assume $T \leq 1$ and \mathcal{T}_h is shape-regular and quasi-uniform. Let $D \subset \Omega_h$, $I = [t_0, t_1] \subset [0, T]$, $Q = D \times I$, $D_d = \{x \in \Omega_h \mid \text{dist}(x, D) < d\}$, $I_d = [t_0 - d^2, t_1] \cap [0, T]$, and $Q_d = D_d \times I_d$ for $d \in (0, \text{diam } \Omega)$. Assume that $z \in C^0([0, T]; W^{k+1, \infty}(\Omega))$ and $z_h \in C^0([0, T]; V_h)$ satisfy*

$$z_t + Az = 0, \text{ in } Q_T, \quad \partial_n z = 0, \text{ on } \partial\Omega \times (0, T),$$

and

$$(z_{h,t}, \chi)_{\Omega_h} + a_{\Omega_h}(z_h, \chi) = 0, \quad \forall \chi \in V_h,$$

respectively, and let $e = z_h - \tilde{z}$ and $\zeta = \tilde{z} - I_h \tilde{z}$. Then, for arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exist $C > 0$ and $c > 0$ such that $d \geq ch$ implies

$$\begin{aligned} \varepsilon_1 \|e_t\|_Q + d^{-1} \|e\|_{1,Q} &\leq C \kappa_d (\|e(0)\|_{1,D_d} + d^{-1} \|e(0)\|_{D_d}) \\ &\quad + \varepsilon_1 \varepsilon_2 \|e_t\|_{Q_d} + \varepsilon_1 C_{\varepsilon_2} d^{-1} \|\nabla e\|_{Q_d} \\ &\quad + C (d \|\zeta_t\|_{1,Q_d} + \|\zeta_t\|_{Q_d} + d^{-1} \|\zeta\|_{1,Q_d} + d^{-2} \|\zeta\|_{Q_d}) \\ &\quad + C (hd^{-1})^{1/2} (\|e_t\|_{Q_d} + d^{-1} \|e\|_{1,Q_d}) + Cd^{-2} \|e\|_{Q_d} \\ &\quad + C [\mathcal{G}(z, Q_d) (\|\zeta_t\|_{Q_d} + d^{-2} \|\zeta\|_{Q_d} + \|e_t\|_{Q_d} + d^{-2} \|e\|_{Q_d})]^{1/2}. \end{aligned} \quad (2.31)$$

with $C_{\varepsilon_2} > 0$ depending only on ε_2 , where $\kappa_d = 1$ if $t_0 \leq d^2$, $\kappa_d = 0$ if $t_0 > d^2$, and

$$\mathcal{G}(z, Q_0) = h^{1/2} (\|\nabla \tilde{z}\|_{(\partial\Omega_h \times (0,T)) \cap Q_0} + \|D^2 \tilde{z}\|_{(T(\varepsilon) \times (0,T)) \cap Q_0}) + \|\tilde{z}_t - A\tilde{z}\|_{Q_0 \setminus Q_T} \quad (2.32)$$

for $Q_0 \subset Q_{h,T}$.

The proof is based on that of [94, Lemma 6.1] and [98, Lemma 4.1]. In these proofs, the strong super-approximation property is introduced and proved for Lagrangian finite element spaces. However, we have succeeded in avoiding these arguments. Thus we give an alternative proof.

Proof of Lemma 2.5.1. We first introduce a cut-off function ω according to [94]. Let $\omega_1 \in C^\infty(\Omega_h)$ satisfy

$$0 \leq \omega_1 \leq 1, \quad \omega_1|_D \equiv 1, \quad \omega_1|_{\Omega_h \setminus D_d} \equiv 0, \quad |D^l \omega_1| \leq Cd^{-l}$$

for $l \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We can find such ω_1 if $d \geq 2h$ since \mathcal{T}_h is quasi-uniform. We also choose $\omega_2 \in C^1[0, t_1]$ that satisfies

$$0 \leq \omega_2 \leq 1, \quad \omega_2|_I \equiv 1, \quad \omega_2|_{[0,T] \setminus I_d} \equiv 0, \quad |\omega_2'| \leq Cd^{-2}.$$

We finally set $\omega(x, t) = \omega_1(x)\omega_2(t)$ for $(x, t) \in Q_{h,T}$.

Let $\zeta_h = z_h - I_h z = e + \zeta$. Then,

$$\frac{1}{2} \frac{d}{dt} \|\omega e\|_{\Omega_h}^2 + \|\omega \nabla e\|_{\Omega_h}^2 + \|\omega e\|_{\Omega_h}^2 = J_1 + J_2,$$

where

$$J_1 = (e_t, \omega^2 \zeta_h)_{\Omega_h} + a_{\Omega_h}(e, \omega^2 \zeta_h),$$

$$J_2 = -(e_t, \omega^2 \zeta)_{\Omega_h} + (e, \omega \omega_t e)_{\Omega_h} - a_{\Omega_h}(e, \omega^2 \zeta_h) + \|\omega \nabla e\|_{\Omega_h}^2 + \|\omega e\|_{\Omega_h}^2.$$

We can calculate J_2 as

$$J_2 = -(e_t, \omega^2 \zeta)_{\Omega_h} + (e, \omega \omega_t e)_{\Omega_h} - 2(\nabla e, \omega(\nabla \omega) \zeta_h)_{\Omega_h} - (\nabla e, \omega^2 \nabla \zeta)_{\Omega_h} - (e, \omega^2 \zeta)_{\Omega_h},$$

and thus we have

$$|J_2| \leq \varepsilon_1^2 d^2 \|\omega e_t\|_{\Omega_h}^2 + \frac{1}{2} (\|\omega \nabla e\|_{\Omega_h}^2 + \|\omega e\|_{\Omega_h}^2) + C (\|\nabla \zeta\|_{D_d}^2 + d^{-2} \|\zeta\|_{D_d}^2) + C d^{-2} \|e\|_{D_d}^2 \quad (2.33)$$

for arbitrary $\varepsilon_1 > 0$, since $\zeta_h = e + \zeta$. To address J_1 , we recall the asymptotic Galerkin orthogonality (2.26) and we have

$$\begin{aligned} J_1 &= (e_t, \omega^2 \zeta_h - \chi)_{\Omega_h} + a_{\Omega_h}(e, \omega^2 \zeta_h - \chi) - (\tilde{z}_t + A\tilde{z}, \chi)_{\Omega_h \setminus \Omega} - (\partial_{n_h} \tilde{z}, \chi)_{\partial \Omega_h} \\ &=: \sum_{i=1}^4 J_{1,i} \end{aligned}$$

for arbitrary $\chi \in V_h$. We choose $\chi = I_h(\omega^2 \zeta_h)$ so that $\text{supp } \chi \subset D_{2d}$ if $d \geq h$. We remark that the super-approximation type estimates

$$\|\omega^2 \zeta_h - \chi\|_{D_{2d}} \leq C h d^{-1} \|\zeta_h\|_{D_{2d}}, \quad (2.34)$$

$$\|\nabla(\omega^2 \zeta_h - \chi)\|_{D_{2d}} \leq C (h d^{-2} \|\zeta_h\|_{D_{2d}} + h d^{-1} \|\nabla \zeta_h\|_{D_{2d}}) \quad (2.35)$$

hold (see [98, page 386]). Thus, $J_{1,1}$ and $J_{1,2}$ can be addressed as in [94, 98] and we have

$$|J_{1,1}| + |J_{1,2}| \leq C (\|\zeta\|_{1,D_{2d}}^2 + d^{-2} \|\zeta\|_{D_{2d}}^2) + C (h^2 \|e_t\|_{D_{2d}}^2 + h d^{-1} \|e\|_{1,D_{2d}}^2) + C d^{-2} \|e\|_{D_{2d}}^2. \quad (2.36)$$

From the inverse inequality, we have $\|\chi\|_K \leq C \|\zeta_h\|_K$ for any $K \in \mathcal{T}_h$. Therefore, we have

$$|J_{1,3}| \leq C \|\tilde{z}_t + A\tilde{z}\|_{D_{2d} \setminus \Omega} (\|\zeta\|_{D_{2d}} + \|e\|_{D_{2d}}). \quad (2.37)$$

For the estimate of $J_{1,4}$, we recall that $\nabla z \cdot n = 0$ on $\partial \Omega$, and we have

$$\begin{aligned} \|\partial_{n_h} \tilde{z}\|_{\partial \Omega_h \cap D_{2d}} &\leq \|\nabla \tilde{z} \cdot (n_h - n \circ \pi)\|_{\partial \Omega_h \cap D_{2d}} + \|[\nabla \tilde{z} - (\nabla z \circ \pi)] \cdot n \circ \pi\|_{\partial \Omega_h \cap D_{2d}} \\ &\leq C h (\|\nabla \tilde{z}\|_{\partial \Omega_h \cap D_{2d}} + \|D^2 \tilde{z}\|_{T(\varepsilon) \cap D_{3d}}) \end{aligned}$$

from (2.18) and (2.16). Together with the trace inequality $\|\chi\|_{\partial \Omega_h} \leq C h^{-1/2} \|\chi\|_{\Omega_h}$, we have

$$|J_{1,4}| \leq C h^{1/2} (\|\nabla \tilde{z}\|_{\partial \Omega_h \cap D_{2d}} + \|D^2 \tilde{z}\|_{T(\varepsilon) \cap D_{3d}}) (\|\zeta\|_{D_{2d}} + \|e\|_{D_{2d}}). \quad (2.38)$$

Therefore, from equations (2.33), (2.36), (2.37), and (2.38), we can derive

$$\begin{aligned} \frac{d}{dt} \|\omega e\|_{\Omega_h}^2 + \|\omega \nabla e\|_{\Omega_h}^2 + \|\omega e\|_{\Omega_h}^2 &\leq \varepsilon_1^2 d^2 \|\omega e_t\|_{\Omega_h}^2 + C (\|\zeta\|_{1,D_{3d}}^2 + d^{-2} \|\zeta\|_{D_{3d}}^2) \\ &\quad + C (h^2 \|e_t\|_{D_{3d}}^2 + h d^{-1} \|e\|_{1,D_{3d}}^2) + C d^{-2} \|e\|_{D_{3d}}^2 \\ &\quad + C G (\|\zeta\|_{D_{2d}} + \|e\|_{D_{2d}}), \end{aligned} \quad (2.39)$$

where

$$G = \|\tilde{z}_t + A\tilde{z}\|_{D_{3d} \setminus \Omega} + h^{1/2} (\|\nabla \tilde{z}\|_{\partial \Omega_h \cap D_{2d}} + \|D^2 \tilde{z}\|_{T(\varepsilon) \cap D_{3d}}). \quad (2.40)$$

Integrating (2.39) over I_d , multiplying d^{-2} , and taking square roots, we have

$$\begin{aligned} d^{-1} (\|\omega \nabla e\|_{Q_{h,T}} + \|\omega e\|_{Q_{h,T}}) &\leq \kappa_d d^{-1} \|e(0)\|_{D_d} + \varepsilon_1 \|\omega e_t\|_{Q_{h,T}} + C (d^{-1} \|\zeta\|_{1,Q_{3d}} + d^{-2} \|\zeta\|_{Q_{3d}}) \\ &\quad + C h^{1/2} d^{-1/2} (\|e_t\|_{Q_{3d}} + d^{-1} \|e\|_{1,Q_{3d}}) + C d^{-2} \|e\|_{Q_{3d}} \\ &\quad + C d^{-1} [\mathcal{G}(z, Q_{3d}) (\|\zeta\|_{Q_{3d}} + \|e\|_{Q_{3d}})]^{1/2}, \end{aligned} \quad (2.41)$$

for arbitrary $\varepsilon_1 > 0$, where \mathcal{G} is defined by (2.32).

We next address

$$\|\omega e_t\|_{\Omega_h}^2 + \frac{1}{2} \frac{d}{dt} (\|\omega \nabla e\|_{\Omega_h}^2 + \|\omega e\|_{\Omega_h}^2) = K_1 + K_2,$$

where

$$\begin{aligned} K_1 &= (e_t, \omega^2 \zeta_{h,t})_{\Omega_h} + a_{\Omega_h}(e, \omega^2 \zeta_{h,t}), \\ K_2 &= -(e_t, \omega^2 \zeta_t)_{\Omega_h} + (\nabla e, \omega \omega_t \nabla e)_{\Omega_h} + (\nabla e, \omega^2 \nabla e_t)_{\Omega_h} + (e, \omega \omega_t e)_{\Omega_h} + (e, \omega^2 e_t)_{\Omega_h} - a_{\Omega_h}(e, \omega^2 \zeta_{h,t}). \end{aligned}$$

As in the estimate of J_2 , we have

$$\begin{aligned} K_2 &= -(e_t, \omega \zeta_t)_{\Omega_h} + (\nabla e, \omega \omega_t \nabla e)_{\Omega_h} + (e, \omega \omega_t e)_{\Omega_h} - 2(\nabla e, \nabla \omega \zeta_{h,t})_{\Omega_h} - (\nabla e, \omega^2 \nabla \zeta_t)_{\Omega_h} - (e, \omega^2 \zeta_t)_{\Omega_h} \\ &\leq \frac{1}{2} \|\omega e_t\|_{\Omega_h}^2 + C (\|d^2 \nabla \zeta_t\|_{D_d}^2 + \|\zeta_t\|_{D_d}^2) + C d^{-2} \|\nabla e\|_{D_d}^2 + C d^{-4} \|e\|_{D_d}^2. \end{aligned}$$

As in the case of J_1 , we have

$$\begin{aligned} K_1 &= (e_t, \omega^2 \zeta_{h,t} - \chi)_{\Omega_h} + a_{\Omega_h}(e, \omega^2 \zeta_{h,t} - \chi) - (\tilde{z}_t + A\tilde{z}, \chi)_{\Omega_h \setminus \Omega} - (\partial_{n_h} \tilde{z}, \chi)_{\partial \Omega_h} \\ &=: \sum_{i=1}^4 K_{1,i} \end{aligned}$$

for arbitrary $\chi \in V_h$, due to the asymptotic Galerkin orthogonality (2.26). Choose $\chi = I_h(\omega^2 \zeta_{h,t})$. Then, as observed above, we have

$$|K_{1,3}| + |K_{1,4}| \leq CG (\|\zeta_t\|_{D_{2d}} + \|e_t\|_{D_{2d}}) \quad (2.42)$$

with the same G as in (2.40). Recalling the super-approximation type estimate (2.34), we have

$$|K_{1,1}| \leq Chd^{-1} \|e_t\|_{D_{2d}}^2 + C \|\zeta_t\|_{D_{2d}}^2. \quad (2.43)$$

To address $K_{1,2}$, we take an approach slightly different from the literature. In [94], a strong super-approximation property is considered for general error estimates. In [98], they consider a special cut-off function and derive energy estimates with weights of the form $\|\omega^{2k} e\|$. In the present chapter, we just apply the super-approximation estimate (2.35) and the inverse inequality. Then, we have

$$\|\nabla(\omega^2 \zeta_{h,t} - \chi)\|_{D_d} \leq Cd^{-1} \|\zeta_{h,t}\|_{D_{2d}},$$

and thus,

$$|K_{1,2}| \leq \varepsilon_2^2 \|e_t\|_{D_{2d}}^2 + Chd^{-1} \|e_t\|_{D_{2d}}^2 + C_{\varepsilon_2} d^{-2} \|\nabla e\|_{D_{2d}}^2 + Cd^{-4} \|e\|_{D_{2d}}^2 + C \|\zeta_t\|_{D_{2d}}^2 \quad (2.44)$$

for arbitrary $\varepsilon_2 > 0$, where $C_{\varepsilon_2} > 0$ depends only on ε_2 . Summarizing (2.42), (2.43), and (2.44), we have

$$\begin{aligned} \|\omega e_t\|_{\Omega_h}^2 + \frac{d}{dt} (\|\omega \nabla e\|_{\Omega_h}^2 + \|\omega e\|_{\Omega_h}^2) &\leq \varepsilon_2^2 \|e_t\|_{D_{2d}}^2 + Chd^{-1} \|e_t\|_{D_{2d}}^2 + C_{\varepsilon_2} d^{-2} \|\nabla e\|_{D_{2d}}^2 + Cd^{-4} \|e\|_{D_{2d}}^2 \\ &\quad + C (d^2 \|\nabla \zeta_t\|_{D_d}^2 + \|\zeta_t\|_{D_{2d}}^2) + CG (\|\zeta_t\|_{D_{2d}} + \|e_t\|_{D_{2d}}), \end{aligned}$$

where G is defined by (2.40). Integrating this inequality over I_d and taking square roots, we have

$$\begin{aligned} \|\omega e_t\|_{Q_{h,T}} &\leq \kappa_d \|e(0)\|_{1,D_d} + \varepsilon_2 \|e_t\|_{D_{2d}} + C_{\varepsilon_2} d^{-1} \|\nabla e\|_{Q_{3d}} \\ &\quad + C (d \|\zeta_t\|_{1,Q_{3d}} + \|\zeta_t\|_{Q_{3d}}) + Ch^{1/2} d^{-1/2} \|e_t\|_{Q_{3d}} + Cd^{-2} \|e\|_{Q_{3d}} \\ &\quad + C [\mathcal{G}(z, Q_{3d}) (\|\zeta_t\|_{Q_{3d}} + \|e_t\|_{Q_{3d}})]^{1/2}. \end{aligned} \quad (2.45)$$

Now, multiplying an arbitrary small number $\varepsilon_3 > 0$ to (2.45) and adding it to (2.41), we can obtain

$$\begin{aligned} \varepsilon_3 \|\omega e_t\|_{Q_{h,T}} + d^{-1} (\|\omega \nabla e\|_{Q_{h,T}} + \|\omega e\|_{Q_{h,T}}) &\leq C \kappa_d (\|e(0)\|_{1,D_d} + d^{-1} \|e(0)\|_{D_d}) \end{aligned}$$

$$\begin{aligned}
& + \varepsilon_1 \|\omega e_t\|_{Q_{h,T}} + \varepsilon_2 \varepsilon_3 \|e_t\|_{Q_{3d}} + \varepsilon_3 C_{\varepsilon_2} d^{-1} \|\nabla e\|_{Q_{3d}} \\
& + C (d \|\zeta_t\|_{1,Q_{3d}} + \|\zeta_t\|_{Q_{3d}} + d^{-1} \|\zeta\|_{1,Q_{3d}} + d^{-2} \|\zeta\|_{Q_{3d}}) \\
& + C (hd^{-1})^{1/2} (\|e_t\|_{Q_{3d}} + d^{-1} \|e\|_{1,Q_{3d}}) + C d^{-2} \|e\|_{Q_{3d}} \\
& + C [\mathcal{G}(z, Q_{3d}) (\|\zeta_t\|_{Q_{3d}} + d^{-2} \|\zeta\|_{Q_{3d}} + \|e_t\|_{Q_{3d}} + d^{-2} \|e\|_{Q_{3d}})]^{1/2}.
\end{aligned}$$

Letting $\varepsilon_3 = 2\varepsilon_1$, we can kick-back the term $\|\omega e_t\|_{Q_{h,T}}$ and we can derive

$$\begin{aligned}
& \varepsilon_1 \|\omega e_t\|_{Q_{h,T}} + d^{-1} (\|\omega \nabla e\|_{Q_{h,T}} + \|\omega e\|_{Q_{h,T}}) \\
& \leq C \kappa_d (\|e(0)\|_{1,D_d} + d^{-1} \|e(0)\|_{D_d}) \\
& \quad + \varepsilon_1 \varepsilon_2 \|e_t\|_{Q_{3d}} + \varepsilon_1 C_{\varepsilon_2} d^{-1} \|\nabla e\|_{Q_{3d}} \\
& \quad + C (d \|\zeta_t\|_{1,Q_{3d}} + \|\zeta_t\|_{Q_{3d}} + d^{-1} \|\zeta\|_{1,Q_{3d}} + d^{-2} \|\zeta\|_{Q_{3d}}) \\
& \quad + C (hd^{-1})^{1/2} (\|e_t\|_{Q_{3d}} + d^{-1} \|e\|_{1,Q_{3d}}) + C d^{-2} \|e\|_{Q_{3d}} \\
& \quad + C [\mathcal{G}(z, Q_{3d}) (\|\zeta_t\|_{Q_{3d}} + d^{-2} \|\zeta\|_{Q_{3d}} + \|e_t\|_{Q_{3d}} + d^{-2} \|e\|_{Q_{3d}})]^{1/2}
\end{aligned}$$

for arbitrary positive numbers ε_1 and ε_2 . Replacing $3d$ by d , we can establish the desired estimate (2.31). \square

We turn to Lemma 2.4.2. In order to address the gap terms, the estimates of Γ on the tubular neighborhood $T(\varepsilon)$ are essential. To state them, we introduce the parabolic dyadic decomposition according to [94]. Let $d_0 := 2 \max\{\text{diam } \Omega, 1\}$ and $d_j = 2^{-j} d_0$ for $j \in \mathbb{N}$. We fix $J_* \in \mathbb{N}$ such that $C_* h \leq d_{J_*} \leq 2C_* h$ for some $C_* \geq 1$, which is determined later independently of h . By definition, $J_* \approx |\log h|$. We remark that

$$\sum_{j=0}^{J_*} \left(\frac{h}{d_j} \right)^r \leq C \quad (2.46)$$

for $r > 0$, where C depends only on r and $\text{diam } \Omega$. Let $\rho(x, t) := \max\{|x - x_0|, \sqrt{t}\}$ and

$$\Omega_{h,j} = \{x \in \Omega_h \mid d_j \leq |x - x_0| \leq 2d_j\}, \quad \Omega_{h,*} = \{x \in \Omega_h \mid |x - x_0| \leq d_{J_*}\}, \quad (2.47)$$

$$Q_{h,j} = \{(x, t) \in Q_{h,T} \mid d_j \leq \rho(x, t) \leq 2d_j\}, \quad Q_{h,*} = \{(x, t) \in Q_{h,T} \mid \rho(x, t) \leq d_{J_*}\}. \quad (2.48)$$

Then, it is clear that

$$\Omega_h = \left(\bigcup_{j=1}^{J_*} \Omega_{h,j} \right) \cup \Omega_{h,*}, \quad Q_{h,T} = \left(\bigcup_{j=1}^{J_*} Q_{h,j} \right) \cup Q_{h,*}.$$

We also set $\Omega'_{h,j} = \Omega_{h,j-1} \cup \Omega_{h,j} \cup \Omega_{h,j+1}$ and $Q'_{h,j} = Q_{h,j-1} \cup Q_{h,j} \cup Q_{h,j+1}$ for later use. We can now state the following gap estimates.

Lemma 2.5.2. *Let $T \leq 1$, $p \in [1, \infty]$, $l \in \mathbb{N}_0$, and $\alpha \in \mathbb{N}_0^N$. Then, we have*

$$\|\partial_t^l \partial_x^\alpha \tilde{\Gamma}\|_{L^p((T(\varepsilon) \times (0, T)) \cap Q_{h,j})} \leq C h^{\frac{2}{p}} d_j^{\frac{1}{p} - (1 - \frac{1}{p})N - |\alpha| - 2l}, \quad (2.49)$$

$$\|\partial_t^l \partial_x^\alpha \tilde{\Gamma}\|_{L^p((\partial\Omega_h \times (0, T)) \cap Q_{h,j})} \leq C d_j^{\frac{1}{p} - (1 - \frac{1}{p})N - |\alpha| - 2l}. \quad (2.50)$$

Moreover, the same estimates hold on $Q_{h,*}$ with d_j replaced by d_{J_*} .

Proof. We show the first inequality (2.49) for $Q_{h,j}$. By the Hölder inequality and the local stability of the extension (2.20), we have

$$\|\partial_t^l \partial_x^\alpha \tilde{\Gamma}\|_{L^p((T(\varepsilon) \times (0, T)) \cap Q_{h,j})} \leq C (\varepsilon d_j^{N+1})^{1/p} \sum_{|\beta| \leq |\alpha|} \|\partial_t^l \partial_x^\beta \Gamma\|_{L^\infty(\hat{Q}_{h,j})},$$

where $\hat{Q}_{h,j} = [(\Omega \cap T(\varepsilon)) \times (0, T)] \cap Q'_{h,j}$. Since $\partial_t^l \partial_x^\beta \Gamma$ is expressed as

$$\partial_t^l \partial_x^\beta \Gamma(x, t) = \int_{\text{supp } \bar{\delta}} \partial_t^l \partial_x^\beta G(x, y; t) \bar{\delta}(y) dy,$$

we can obtain

$$\|\partial_t^l \partial_x^\beta \Gamma\|_{L^\infty(\bar{Q}_{h,j})} \leq C d_j^{-N-2l-|\beta|} \leq C d_j^{-N-2l-|\alpha|}$$

for $|\beta| \leq |\alpha|$, from (2.25) and $\text{supp } \bar{\delta} \cap T(\varepsilon) = \emptyset$. Noting that $\varepsilon \approx h^2$, we can obtain (2.49). The proof of (2.50) is similar since

$$\|\partial_t^l \partial_x^\alpha \tilde{\Gamma}\|_{L^p((\partial\Omega_h \times (0,T)) \cap Q_{h,j})} \leq C d_j^{(N+1)/p} \sum_{|\beta| \leq |\alpha|} \|\partial_t^l \partial_x^\beta \Gamma\|_{L^\infty(\bar{Q}_{h,j})},$$

holds. Hence we can complete the proof. \square

Now, we are ready to show Lemma 2.4.2. We use the dyadic decomposition $\Omega_{h,j}$ and $Q_{h,j}$. In the proof below, and thereafter, we write $\sum_{j,*}$ when the summation includes the integration over $Q_{h,*}$. If it is not included, we denote the summation by \sum_j .

Proof of Lemma 2.4.2. By the Hölder inequality, we have

$$\|F_t\|_{L^1(Q_{h,T})} = \sum_{j,*} \|F_t\|_{L^1(Q_{h,j})} \leq C \sum_{j,*} d_j^{\frac{N}{2}+1} \|F_t\|_{Q_{h,j}}.$$

On the innermost set $Q_{h,*}$, a standard energy argument implies

$$\|F_t\|_{Q_{h,*}} \leq C (\|\Gamma_t\|_{Q_T} + \|\Gamma_{h,t}\|_{Q_{h,T}}) \leq C h^{-\frac{N}{2}-1}.$$

Therefore, noting that $d_{J_*} \approx h$, we have

$$\|F_t\|_{L^1(Q_{h,T})} \leq C + C \sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q_{h,j}}, \quad (2.51)$$

and similarly,

$$\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))} \leq C h + C \sum_j d_j^{\frac{N}{2}+1} \|F\|_{1,Q_{h,j}}. \quad (2.52)$$

Let $Q_{h,j}$ be an arbitrary parabolic annulus. We substitute $z = \Gamma$, $z_h = \Gamma_h$, $d = d_j$, and

$$Q = \Omega_{h,j} \times [0, d_j^2] \quad \text{or} \quad Q = \{x \in \Omega_h \mid |x - x_0| < d_j\} \times [d_j^2, 4d_j^2]$$

into (2.31). Then, we have

$$\begin{aligned} & \varepsilon_1 \|F_t\|_{Q_{h,j}} + d_j^{-1} \|F\|_{1,Q_{h,j}} \\ & \leq C(I_j + X_j) + \varepsilon_1 \varepsilon_2 \|F_t\|_{Q'_{h,j}} + \varepsilon_1 C_{\varepsilon_2} d_j^{-1} \|\nabla F\|_{Q'_{h,j}} \\ & \quad + C(h d_j^{-1})^{1/2} \left(\|F_t\|_{Q'_{h,j}} + d_j^{-1} \|F\|_{1,Q'_{h,j}} \right) + C d_j^{-2} \|F\|_{Q'_{h,j}} \\ & \quad + C \left[\mathcal{G}(\Gamma, Q'_{h,j}) \left(\|\zeta_t\|_{Q'_{h,j}} + d_j^{-2} \|\zeta\|_{Q'_{h,j}} + \|F_t\|_{Q'_{h,j}} + d_j^{-2} \|F\|_{Q'_{h,j}} \right) \right]^{1/2}, \end{aligned} \quad (2.53)$$

for arbitrary ε_1 and ε_2 , where

$$\begin{aligned} I_j &= \|F(0)\|_{1,\Omega'_{h,j}} + d_j^{-1} \|F(0)\|_{\Omega'_{h,j}}, \\ X_j &= d_j \|\zeta_t\|_{1,Q'_{h,j}} + \|\zeta_t\|_{Q'_{h,j}} + d_j^{-1} \|\zeta\|_{1,Q'_{h,j}} + d_j^{-2} \|\zeta\|_{Q'_{h,j}}, \end{aligned}$$

and $\zeta = \tilde{\Gamma} - I_h \tilde{\Gamma}$. The estimates of I_j and X_j are the same as in [94] and thus we have

$$I_j \leq C(h^{-1-N} + h^{-N} d_j^{-1}) d_j^{N/2} e^{-cd_j/h} \quad (2.54)$$

$$X_j \leq C \left(h^{k+1} d_j^{-\frac{N}{2}-k-2} + h^k d_j^{-\frac{N}{2}-k-1} \right) \quad (2.55)$$

from (2.22) and (2.25).

For the estimate of gap terms $\mathcal{G}(\Gamma, Q'_{h,j})$, we recall Lemma 2.5.2. Since

$$\mathcal{G}(\Gamma, Q'_{h,j}) = h^{1/2} \left(\|\nabla \tilde{\Gamma}\|_{(\partial\Omega_h \times (0,T)) \cap Q'_{h,j}} + \|D^2 \tilde{\Gamma}\|_{(T(\varepsilon) \times (0,T)) \cap Q'_{h,j}} \right) + \|\tilde{\Gamma}_t - A\tilde{\Gamma}\|_{Q'_{h,j} \setminus Q_T},$$

Lemma 2.5.2 yields

$$\mathcal{G}(\Gamma, Q'_{h,j}) \leq C \left(h^{\frac{1}{2}} d_j^{-\frac{1}{2} - \frac{N}{2}} + h^{\frac{3}{2}} d_j^{-\frac{3}{2} - \frac{N}{2}} + h d_j^{-\frac{3}{2} - \frac{N}{2}} \right) \leq C \left(h^{\frac{1}{2}} d_j^{-\frac{1}{2} - \frac{N}{2}} + h d_j^{-\frac{3}{2} - \frac{N}{2}} \right). \quad (2.56)$$

Now, multiplying $d_j^{\frac{N}{2}+1}$ to (2.53) and integrating with respect to j , we have

$$\begin{aligned} & \varepsilon_1 \sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}} \|F\|_{1,Q_{h,j}} \\ & \leq C + \varepsilon_1 \varepsilon_2 \sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q'_{h,j}} + \varepsilon_1 C_{\varepsilon_2} \sum_j d_j^{\frac{N}{2}} \|F\|_{1,Q'_{h,j}} \\ & \quad + C C_*^{-1/2} \left[\sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q'_{h,j}} + \sum_j d_j^{\frac{N}{2}} \|F\|_{1,Q'_{h,j}} \right] \\ & \quad + C \sum_j d_j^{\frac{N}{2}+1} \left(h^{\frac{1}{4}} d_j^{-\frac{1}{4} - \frac{N}{4}} + h^{\frac{1}{2}} d_j^{-\frac{3}{4} - \frac{N}{4}} \right) \left(h^{k+1} d_j^{-\frac{N}{2}-k-2} + \|F_t\|_{Q'_{h,j}} + d_j^{-2} \|F\|_{Q'_{h,j}} \right)^{1/2} \\ & \quad + C \sum_j d_j^{N/2-1} \|F\|_{Q'_{h,j}} \end{aligned} \quad (2.57)$$

together with (2.54), (2.55), (2.56), and (2.46). Since $\|F_t\|_{Q_{h,*}} + h^{-1} \|F\|_{1,Q_{h,*}} \leq C h^{-\frac{N}{2}-1}$, we have

$$\sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q'_{h,j}} \leq 2 \sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q_{h,j}} + C, \quad (2.58)$$

$$\sum_j d_j^{\frac{N}{2}} \|F\|_{1,Q'_{h,j}} \leq 2 \sum_j d_j^{\frac{N}{2}} \|F\|_{1,Q_{h,j}} + C. \quad (2.59)$$

Moreover, (2.46) yields

$$\sum_j d_j^{\frac{N}{2}+1} \left(h^{\frac{1}{4}} d_j^{-\frac{1}{4} - \frac{N}{4}} + h^{\frac{1}{2}} d_j^{-\frac{3}{4} - \frac{N}{4}} \right) \left(h^{k+1} d_j^{-\frac{N}{2}-k-2} \right)^{1/2} = \sum_j \left((h d_j^{-1})^{\frac{k}{2} + \frac{4}{3}} d_j^{\frac{1}{2}} + (h d_j^{-1})^{\frac{k}{2}+1} d_j^{\frac{1}{4}} \right) \leq C$$

and the Hölder and the Young inequality gives

$$\begin{aligned} & \sum_j d_j^{\frac{N}{2}+1} \left(h^{\frac{1}{4}} d_j^{-\frac{1}{4} - \frac{N}{4}} + h^{\frac{1}{2}} d_j^{-\frac{3}{4} - \frac{N}{4}} \right) \left(\|F_t\|_{Q'_{h,j}} + d_j^{-2} \|F\|_{Q'_{h,j}} \right)^{1/2} \\ & \leq C h^{1/4} \left[C + 2 \sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}-1} \|F\|_{Q'_{h,j}} \right]^{1/2} \\ & \leq C + \frac{\varepsilon_1}{4} \sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q_{h,j}} + C \sum_j d_j^{\frac{N}{2}-1} \|F\|_{Q'_{h,j}} \end{aligned}$$

Substituting these estimates into (2.57) and letting $\varepsilon_2 = 1/8$, we have

$$\begin{aligned} & \varepsilon_1 \sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}} \|F\|_{1,Q_{h,j}} \\ & \leq C + \frac{\varepsilon_1}{2} \sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q_{h,j}} + C \varepsilon_1 \sum_j d_j^{\frac{N}{2}} \|F\|_{1,Q_{h,j}} \\ & \quad + C C_*^{-1/2} \left[\sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}} \|F\|_{1,Q_{h,j}} \right] + C \sum_j d_j^{\frac{N}{2}-1} \|F\|_{Q'_{h,j}}. \end{aligned}$$

Then, making ε_1 and C_*^{-1} small enough independently of h , we can kick-back several terms and obtain

$$\sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}} \|F\|_{1,Q_{h,j}} \leq C + C \sum_j d_j^{\frac{N}{2}-1} \|F\|_{Q'_{h,j}}. \quad (2.60)$$

Similarly, we can obtain

$$\sum_j d_j^{\frac{N}{2}+2} \|F_t\|_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}+1} \|F\|_{1,Q_{h,j}} \leq Ch^{1/2} + C \sum_j d_j^{\frac{N}{2}} \|F\|_{Q'_{h,j}} \quad (2.61)$$

by multiplying $d_j^{\frac{N}{2}+2}$ to (2.53).

The L^2 -norm of F can be addressed by the duality argument as in [94, Lemma 4.2] and we can obtain the estimate (2.65) below, where $Q''_{h,j} = Q'_{h,j-1} \cup Q'_{h,j} \cup Q'_{h,j+1}$. We postpone the proof of (2.65) (see Lemma 2.5.3 below) and here we complete the proof of Lemma 2.4.2. Multiplying $d_j^{\frac{N}{2}-1}$ to (2.65) and summing up, we have

$$\begin{aligned} & \sum_j d_j^{\frac{N}{2}-1} \|F\|_{Q'_{h,j}} \\ & \leq C + C \sum_i (h^2 \|F_t\|_{Q_{h,i}} + h \|F\|_{1,Q_{h,i}}) \sum_j d_j^{\frac{N}{2}-1} \min \left\{ \left(\frac{d_i}{d_j} \right)^{\frac{N}{2}+1}, \left(\frac{d_j}{d_i} \right)^{\frac{N}{2}+1} \right\} \\ & \quad + CC_*^{-1} \left(\sum_j d_j^{\frac{N}{2}-1} \|F\|_{Q''_{h,j}} + \sum_j d_j^{\frac{N}{2}} \|F\|_{1,Q''_{h,j}} \right) + CC_*^{-1} \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}, \end{aligned}$$

owing to (2.46) and $h \leq C_*^{-1} d_j$. Since d_j is a geometric sequence, we can observe

$$\sum_{j \geq i} d_j^\alpha \leq C d_i^\alpha, \quad \sum_{j \leq i} d_j^{-\alpha} \leq C d_i^{-\alpha}$$

for $\alpha > 0$. This implies

$$\sum_j d_j^{\frac{N}{2}-1} \min \left\{ \left(\frac{d_i}{d_j} \right)^{\frac{N}{2}+1}, \left(\frac{d_j}{d_i} \right)^{\frac{N}{2}+1} \right\} \leq C d_i^N,$$

and thus we have

$$\begin{aligned} \sum_j d_j^{\frac{N}{2}-1} \|F\|_{Q'_{h,j}} & \leq C + CC_*^{-1} \left(\sum_j d_j^{\frac{N}{2}-1} \|F\|_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}} \|F\|_{1,Q_{h,j}} \right) \\ & \quad + C \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}, \end{aligned} \quad (2.62)$$

together with (2.58) and (2.59). Substituting (2.62) into (2.60) and again letting C_* large enough to kick-back the summation in (2.62), we have

$$\sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q_{h,j}} \leq C + C \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}.$$

Recalling (2.51), we can achieve

$$\|F_t\|_{L^1(Q_{h,T})} \leq C + C \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}. \quad (2.63)$$

We repeat this argument. Substituting (2.65) into (2.61), and calculating in the same way as above, we have

$$\sum_j d_j^{\frac{N}{2}+1} \|F\|_{1,Q_{h,j}} \leq Ch^{1/2} + Ch |\log h| \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}.$$

Substituting this into (2.52) we have

$$\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))} \leq Ch^{1/2} + Ch |\log h| \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))},$$

which yields

$$\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))} \leq Ch^{1/2} \quad (2.64)$$

for sufficiently small h . Together with (2.63) and (2.64), we have

$$\|F_t\|_{L^1(Q_{h,T})} \leq C$$

uniformly in h .

The estimate for $\|tF_{tt}\|_{L^1(Q_{h,T})}$ is similar. Indeed, we can replace F by F_t in the above argument and obtain

$$\sum_j d_j^{\frac{N}{2}+3} \|F_{tt}\|_{Q_{h,j}} + \sum_j d_j^{\frac{N}{2}+2} \|F_t\|_{1,Q_{h,j}} \leq C + C \sum_j d_j^{\frac{N}{2}+1} \|F_t\|_{Q'_{h,j}}.$$

We have already addressed the last term and thus we can achieve

$$\|tF_{tt}\|_{L^1(Q_{h,T})} \leq C + C \sum_j d_j^{\frac{N}{2}+3} \|F_{tt}\|_{Q_{h,j}} \leq C$$

uniformly in h . Hence we can complete the proof. \square

Remark 2.1. We did not use the strong super-approximation property [94, Section 5] (see also [98, page 388]). Therefore, we presented an alternative proof for stability and analyticity estimates for the finite element approximation of parabolic problems.

Remark 2.2. If we can establish the estimate

$$\|F_t\|_{L^1(0,T;W^{1,1}(\Omega_h))} \leq Ch |\log h|^{\underline{k}},$$

where $\underline{k} = 1$ if $k = 1$ and $\underline{k} = 0$ otherwise, in place of (2.52), then we can also obtain the L^∞ -error estimate for the problem (2.10). However, we cannot obtain the above estimate at present.

We show the L^2 -estimate that we admitted in the proof above. We note that the second line of (2.65) below does not appear when $\Omega_h = \Omega$.

Lemma 2.5.3. *There exists $C > 0$ independent of C_* , h , and j that satisfies*

$$\begin{aligned} \|F\|_{Q'_{h,j}} &\leq Ch^2 d_j^{-\frac{N}{2}-1} + C \sum_i (h^2 \|F_t\|_{Q_{h,i}} + h \|F\|_{1,Q_{h,i}}) \min \left\{ \left(\frac{d_i}{d_j} \right)^{\frac{N}{2}+1}, \left(\frac{d_j}{d_i} \right)^{\frac{N}{2}+1} \right\} \\ &\quad + Ch d_j^{-\frac{N}{2}+\frac{1}{2}} + Ch \left(d_j^{-1} \|F\|_{Q''_{h,j}} + \|F\|_{1,Q''_{h,j}} \right) + Ch d_j^{-\frac{N}{2}} \|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}, \end{aligned} \quad (2.65)$$

where $Q''_{h,j} = Q'_{h,j-1} \cup Q'_{h,j} \cup Q'_{h,j+1}$.

Proof. In this proof, we denote the space-time inner products by $[\cdot, \cdot]$. For example,

$$[u, v]_{Q_{h,T}} = \iint_{Q_{h,T}} u(x, t) v(x, t) dx dt, \quad a_{Q_{h,T}}[u, v] = \iint_{Q_{h,T}} (\nabla_x u \cdot \nabla_x v + uv) dx dt.$$

We recall that

$$\|F\|_{Q'_{h,j}} = \sup\{[\phi, F]_{Q_{h,T}} \mid \phi \in C_0^\infty(\mathbb{R}^{N+1}), \text{supp } \phi \subset Q'_{h,j}, \|\phi\|_{Q'_{h,j}} = 1\}.$$

We fix such $\phi \in C_0^\infty(Q'_{h,j})$ and consider the dual parabolic problem

$$\begin{cases} -\partial_t w + Aw = \phi, & \text{in } Q_T, \\ \partial_n w = 0, & \text{on } \partial\Omega \times (0, T), \\ w(T) = 0, & \text{in } \Omega. \end{cases}$$

Then, we state

$$[\phi, F]_{Q_{h,T}} = (\tilde{w}(0), F(0))_{\Omega_h} + \sum_{l=0}^6 D_l, \quad (2.66)$$

where

$$\begin{aligned}
D_0 &= [\tilde{w} - w_h, F_t]_{Q_{h,T}} + a_{Q_{h,T}}[\tilde{w} - w_h, F], \\
D_1 &= [\tilde{w} - w_h, \tilde{\Gamma}_t + A\tilde{\Gamma}]_{Q_{h,T} \setminus Q_T}, & D_2 &= [\tilde{w} - w_h, \partial_{n_h} \tilde{\Gamma}]_{\partial\Omega_h \times (0,T)}, \\
D_3 &= [\phi + \tilde{w}_t - A\tilde{w}, F]_{Q_{h,T} \setminus Q_T}, & D_4 &= [-\partial_{n_h} \tilde{w}, F]_{\partial\Omega_h \times (0,T)}, \\
D_5 &= [\tilde{w}_t, \tilde{\Gamma}]_{Q_{h,T} \setminus Q_T} - [w_t, \Gamma]_{Q_T \setminus Q_{h,T}}, \\
D_6 &= a_{Q_T \setminus Q_{h,T}}[w, \Gamma] - a_{Q_{h,T} \setminus Q_T}[\tilde{w}, \tilde{\Gamma}].
\end{aligned}$$

for arbitrary $w_h \in V_h$. We present the outline of its proof. Noting that $\phi|_{Q_T \setminus Q_{h,T}} \equiv 0$, we have

$$\begin{aligned}
[\phi, F]_{Q_{h,T}} &= [\phi, \tilde{F}]_{Q_T} + [\phi, F]_{Q_{h,T} \setminus Q_T} \\
&= [-w_t, \tilde{F}]_{Q_T} + a_{Q_T}[w, \tilde{F}] + [\phi, F]_{Q_{h,T} \setminus Q_T}
\end{aligned}$$

from identity (2.9). Again applying (2.9), integrating by parts both in time and space, and recalling the asymptotic Galerkin orthogonality (2.26), we have

$$[\phi, F]_{Q_{h,T}} = (\tilde{w}(0), F(0))_{\Omega_h} + D_0 + D_3 + D_4 - [w_h, \tilde{\Gamma}_t + A\tilde{\Gamma}]_{Q_{h,T} \setminus Q_T} - [w_h, \partial_{n_h} \tilde{\Gamma}]_{\partial\Omega_h \times (0,T)}$$

for arbitrary $w_h \in V_h$. Adding the null terms

$$[\tilde{w}, \tilde{\Gamma}_t + A\tilde{\Gamma}]_{Q_{h,T} \setminus Q_T} - [\tilde{w}, \tilde{\Gamma}_t + A\tilde{\Gamma}]_{Q_{h,T} \setminus Q_T} + [w, \Gamma_t + A\Gamma]_{Q_T \setminus Q_{h,T}} (= 0)$$

to the right hand side, we can obtain (2.66).

By estimating each terms in (2.66), we show (2.65). The treatment of $(\tilde{w}(0), F(0))_{\Omega_h}$ is the same as in [94, Lemma 4.2] and we have

$$|(\tilde{w}(0), F(0))_{\Omega_h}| \leq Ch^2 d_j^{-\frac{N}{2}-1}. \quad (2.67)$$

For the estimates of D_l , we choose $w_h = \tilde{I}_h \tilde{w}$. Then, D_0 can be addressed as in [94] and we have

$$\begin{aligned}
|D_0| &\leq \sum_{i,*} \left(\|\tilde{w} - \tilde{I}_h \tilde{w}\|_{Q_{h,i}} \|F_t\|_{Q_{h,i}} + \|\tilde{w} - \tilde{I}_h \tilde{w}\|_{1,Q_{h,i}} \|F\|_{1,Q_{h,i}} \right) \\
&\leq C \sum_{i,*} \left(h^2 \|F_t\|_{Q_{h,i}} + h \|F\|_{1,Q_{h,i}} \right) \min \left\{ \left(\frac{d_j}{d_i} \right)^{\frac{N}{2}+1}, \left(\frac{d_i}{d_j} \right)^{\frac{N}{2}+1} \right\}
\end{aligned} \quad (2.68)$$

since $\|\tilde{w}\|_{2,Q_{h,i}} \leq C \min\{(d_j d_i^{-1})^{N/2+1}, (d_i d_j^{-1})^{N/2+1}\}$ owing to (2.20) and (2.25). In order to address other terms, we set $Q''_{h,j} := Q''_{h,j-1} \cup Q''_{h,j} \cup Q''_{h,j+1}$. We decompose D_1 as

$$D_1 = [\tilde{w} - \tilde{I}_h \tilde{w}, \tilde{\Gamma}_t + A\tilde{\Gamma}]_{Q''_{h,j} \setminus Q_T} + [\tilde{w} - \tilde{I}_h \tilde{w}, \tilde{\Gamma}_t + A\tilde{\Gamma}]_{Q_{h,T} \setminus (Q_T \cup Q''_{h,j})} =: D_{1,1} + D_{1,2}.$$

Since $\|\tilde{w}\|_{2,Q_{h,T}} \leq C \|\phi\|_{Q_{h,T}} = C$ by the standard energy estimate, we have

$$|D_{1,1}| \leq Ch^2 \|\tilde{\Gamma}_t + A\tilde{\Gamma}\|_{(T(\varepsilon) \times (0,T)) \cap Q''_{h,j}} \leq Ch^3 d_j^{-\frac{N}{2}-\frac{3}{2}},$$

together with (2.49). Further, from the gap estimate (2.49), we have

$$\|\tilde{\Gamma}_t + A\tilde{\Gamma}\|_{L^1(T(\varepsilon) \times (0,T))} \leq C \sum_{i,*} h^2 d_i^{-1} \leq Ch,$$

which implies

$$|D_{1,2}| \leq Ch^3 \|w\|_{W^{2,\infty}(Q_T \setminus Q''_{h,j})}$$

together with Lemma 2.3.1 and (2.20). Since we can write

$$w(x, t) = \int_t^T \int_{\Omega} G(x, y; s - t) \phi(y, s) dy ds, \quad (2.69)$$

the Gaussian estimate (2.25) and the assumption $\text{supp } \phi \subset Q'_{h,j}$ yield

$$\|w\|_{W^{m,\infty}(Q_T \setminus Q''_{h,j})} \leq C|Q'_{h,j}|^{1/2} d_j^{-N-m} \leq d_j^{-\frac{N}{2}+1-m}. \quad (2.70)$$

Therefore, we have

$$|D_1| \leq Ch^3 d_j^{-\frac{N}{2}-\frac{3}{2}} + Ch^3 d_j^{-\frac{N}{2}-1} \leq Ch^2 d_j^{-\frac{N}{2}-1} \quad (2.71)$$

since $h d_j^{-1} \leq C_*^{-1} \leq 1$.

The estimate of D_2 is similar. Indeed, we divide D_2 into two parts $D_2 = D_{2,1} + D_{2,2}$, where

$$D_{2,1} = [\tilde{w} - w_h, \partial_{n_h} \tilde{\Gamma}]_{(\partial\Omega_h \times (0,T)) \cap Q''_{h,j}}, \quad D_{2,2} = [\tilde{w} - w_h, \partial_{n_h} \tilde{\Gamma}]_{(\partial\Omega_h \times (0,T)) \setminus Q''_{h,j}}.$$

Recalling the scaled trace inequality

$$\|\psi\|_{(\partial\Omega_h \times (0,T)) \cap Q''_{h,j}} \leq C d_j^{1/2} (d_j^{-1} \|\psi\|_{Q''_{h,j}} + \|\psi\|_{1,Q''_{h,j}}), \quad (2.72)$$

we have

$$\|\tilde{w} - \tilde{I}\tilde{w}\|_{(\partial\Omega_h \times (0,T)) \cap Q''_{h,j}} \leq C d_j^{1/2} h (h d_j^{-1} + 1) \|w\|_{2,Q_T} \leq Ch d_j^{1/2}.$$

Noting that $\nabla \Gamma \cdot n = 0$ on $\partial\Omega$, we have

$$\begin{aligned} & \|\partial_{n_h} \tilde{\Gamma}\|_{(\partial\Omega_h \times (0,T)) \cap Q''_{h,j}} \\ & \leq \|\nabla \tilde{\Gamma} \cdot (n_h - n \circ \pi)\|_{(\partial\Omega_h \times (0,T)) \cap Q''_{h,j}} + \|[\nabla \tilde{\Gamma} - (\nabla \Gamma) \circ \pi] \cdot (n \circ \pi)\|_{(\partial\Omega_h \times (0,T)) \cap Q''_{h,j}} \\ & \leq Ch \left(\|\nabla \tilde{\Gamma}\|_{(\partial\Omega_h \times (0,T)) \cap Q''_{h,j}} + \|\tilde{\Gamma}\|_{2,(T(\varepsilon) \times (0,T)) \cap Q''_{h,j}} \right), \end{aligned}$$

from (2.13) and (2.18). Moreover, the gap estimates (2.49) and (2.50) give

$$\|\partial_{n_h} \tilde{\Gamma}\|_{(\partial\Omega_h \times (0,T)) \cap Q''_{h,j}} \leq Ch d_j^{-\frac{N}{2}-\frac{1}{2}}.$$

Hence we have

$$|D_{2,1}| \leq Ch^2 d_j^{-\frac{N}{2}}.$$

Next we state

$$\|\partial_{n_h} \tilde{\Gamma}\|_{L^1(\partial\Omega_h \times (0,T))} \leq Ch |\log h|. \quad (2.73)$$

Indeed, by the same argument as above, we have

$$\|\partial_{n_h} \tilde{\Gamma}\|_{L^1(\partial\Omega_h \times (0,T))} \leq Ch \|\nabla \tilde{\Gamma}\|_{L^1(\partial\Omega_h \times (0,T))} + C \|D^2 \tilde{\Gamma}\|_{L^1(T(\varepsilon) \times (0,T))},$$

and the gap estimates (2.49) and (2.50) yield

$$\begin{aligned} \|\nabla \tilde{\Gamma}\|_{L^1(\partial\Omega_h \times (0,T))} & \leq C \sum_{i,*} d_i^0 \leq Ch |\log h|, \\ \|D^2 \tilde{\Gamma}\|_{L^1(T(\varepsilon) \times (0,T))} & \leq C \sum_{i,*} h^2 d_i^{-1} \leq Ch, \end{aligned}$$

which lead to (2.73). Therefore, we have

$$|D_{2,2}| \leq Ch^2 \|w\|_{W^{2,\infty}(Q_T \setminus Q''_{h,j})} \|\partial_{n_h} \tilde{\Gamma}\|_{L^1(\partial\Omega_h \times (0,T))} \leq Ch^3 |\log h| d_j^{-\frac{N}{2}-1},$$

and thus

$$|D_2| \leq Ch^2 d_j^{-\frac{N}{2}-1}. \quad (2.74)$$

We divide D_3 into $D_3 = D_{3,1} + D_{3,2}$, where

$$D_{3,1} = [-\tilde{w}_t + A\tilde{w} - \phi, F]_{Q''_{h,j} \setminus Q_T}, \quad D_{3,2} = [-\tilde{w}_t + A\tilde{w}, F]_{Q_{h,T} \setminus (Q_T \cup Q''_{h,j})}.$$

From the energy estimates, $\|-\tilde{w}_t + A\tilde{w} - \phi\|_{Q_{h,T}} \leq C$. Moreover, from (2.17) and the scaled trace inequality (2.72), we have

$$\|F\|_{Q_{h,i}'' \setminus Q_T} \leq C(hd_j^{-1}\|F\|_{Q_{h,j}'''} + h\|\nabla F\|_{Q_{h,j}''}).$$

Thus we have

$$|D_{3,1}| \leq C(hd_j^{-1}\|F\|_{Q_{h,j}'''} + h\|\nabla F\|_{Q_{h,j}''}).$$

The expression (2.69) and the Gaussian estimate (2.25) yield

$$\|-\tilde{w}_t + A\tilde{w}\|_{L^\infty(Q_{h,T} \setminus (Q_T \cup Q_{h,j}''))} \leq Cd_j^{-\frac{N}{2}-1}$$

and (2.15) implies

$$\|F\|_{L^1(Q_{h,T} \setminus (Q_T \cup Q_{h,j}''))} \leq Ch^2\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}.$$

Hence we have

$$|D_{3,2}| \leq Ch^2d_j^{-\frac{N}{2}-1}\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))},$$

which yields

$$|D_3| \leq C(hd_j^{-1}\|F\|_{Q_{h,j}'''} + h\|\nabla F\|_{Q_{h,j}''}) + Ch^2d_j^{-\frac{N}{2}-1}\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))}. \quad (2.75)$$

Similarly, we can observe

$$|D_4| \leq C(hd_j^{-1}\|F\|_{Q_{h,j}'''} + h\|\nabla F\|_{Q_{h,j}''}) + Chd_j^{-\frac{N}{2}}\|F\|_{L^1(0,T;W^{1,1}(\Omega_h))} \quad (2.76)$$

owing to (2.16) and the trace inequality with scaling (2.72). Here, in order to address $\partial_{n_h}\tilde{w}$, we performed the same calculation as in the treatment of $\partial_{n_h}\tilde{\Gamma}$ above.

The treatment of D_5 and D_6 is the same as above. Indeed, we have

$$|D_5| \leq |[\tilde{w}_t, \tilde{\Gamma}]_{(T(\varepsilon) \times (0,T)) \cap Q_{h,j}''}| + |[\tilde{w}_t, \tilde{\Gamma}]_{(T(\varepsilon) \times (0,T)) \setminus Q_{h,j}''}| =: D_{5,1} + D_{5,2},$$

with the estimates

$$D_{5,1} \leq C\|w_t\|_{Q_T}\|\tilde{\Gamma}\|_{(T(\varepsilon) \times (0,T)) \cap Q_{h,j}'''} \leq Chd_j^{-\frac{N}{2}+\frac{1}{2}}$$

from the gap estimate (2.49) and the energy estimate, and

$$D_{5,2} \leq C\|w_t\|_{L^\infty(Q_T \setminus Q_{h,j}'')} \|\tilde{\Gamma}\|_{L^1(T(\varepsilon) \times (0,T))} \leq Ch^2d_j^{-\frac{N}{2}-1}$$

from (2.49) and the expression (2.69). Thus we have

$$|D_5| \leq Chd_j^{-\frac{N}{2}+\frac{1}{2}}. \quad (2.77)$$

Furthermore, we can write $|D_6| \leq D_{6,1} + D_{6,2}$, where

$$D_{6,1} = \|\tilde{w}\|_{1,(T(\varepsilon) \times (0,T)) \cap Q_{h,j}'''} \|\tilde{\Gamma}\|_{1,(T(\varepsilon) \times (0,T)) \cap Q_{h,j}'''}$$

and

$$D_{6,2} = \|\tilde{w}\|_{W^{1,\infty}((T(\varepsilon) \times (0,T)) \setminus Q_{h,j}'')} \|\tilde{\Gamma}\|_{W^{1,1}((T(\varepsilon) \times (0,T)) \setminus Q_{h,j}'')}.$$

From (2.14) and gap estimate (2.49), we have

$$D_{6,1} \leq Ch\|w\|_{2,Q_T}\|\tilde{\Gamma}\|_{1,(T(\varepsilon) \times (0,T)) \cap Q_{h,j}'''} \leq Ch^2d_j^{-\frac{N}{2}-\frac{1}{2}}.$$

Also, (2.70) and (2.49) yield

$$D_{6,2} \leq C\|w\|_{W^{1,\infty}(Q_T \setminus Q_{h,j}'')} \sum_{i,*} \|\tilde{\Gamma}\|_{W^{1,1}((T(\varepsilon) \times (0,T)) \cap Q_{h,i})} \leq Ch^2|\log h|d_j^{-\frac{N}{2}}.$$

Thus we have

$$|D_6| \leq Chd_j^{-\frac{N}{2}+\frac{1}{2}}. \quad (2.78)$$

Summarizing (2.67), (2.68), (2.71), (2.74), (2.75), (2.76), (2.77), and (2.78), we can obtain (2.65), since we can replace $Q_{h,j}'''$ by $Q_{h,j}''$ in (2.75) and (2.76). \square

2.6 Proof of maximal regularity

At this stage, we can show Theorem 2.2.2 for $T \leq 1$.

Proof of Theorem 2.2.2. As discussed in the last part of Section 2.4, we can assume $T \leq 1$. Moreover, it suffices to show (2.12) for the case $p = q$ by the general theory of maximal regularity (cf. [33, Theorem 4.2]).

Let us recall that $u_h \in C^0([0, T]; V_h)$ is the solution of

$$\begin{cases} (u_{h,t}(t), v_h)_{\Omega_h} + a_{\Omega_h}(u_h(t), v_h) = (f_h(t), v_h)_{\Omega_h}, & \forall v_h \in V_h, \\ u_h(0) = 0, \end{cases}$$

for given $f_h \in L^p(0, T; V_h)$. Thus we have an expression

$$A_h u_h(t) = \int_0^t A_h e^{-(t-s)A_h} f_h(s) ds,$$

which implies

$$(-A_h u_h)(x, t) = \int_0^t \int_{\Omega_h} \partial_t \Gamma_{x,h}(y, t-s) f_h(y, s) dy ds =: (\partial_t \Gamma_{x,h} * f_h)(x, t), \quad (x, t) \in Q_{h,T},$$

where $\Gamma_{x,h}$ is the discretized regularized Green's function defined by (2.24) for $x_0 = x \in \Omega_h$. Therefore, maximal regularity is equivalent to the $L^p(Q_{h,T})$ -boundedness of the convolution operator associated with $\partial_t \Gamma_{x,h}$. Moreover, Lemma 2.4.2 yields

$$\|\partial_t \Gamma_{x,h} * f_h\|_{L^p(Q_{h,T})} \leq C \|f_h\|_{L^p(Q_{h,T})} + \|\partial_t \tilde{\Gamma}_x * f_h\|_{L^p(Q_{h,T})},$$

where Γ_x is regularized Green's function defined by (2.23) with respect to $x_0 = x \in \Omega_h$, $\tilde{\Gamma}_x$ is its appropriate extension to Ω_h , and

$$(\partial_t \tilde{\Gamma}_x * f_h)(x, t) := \int_0^t \int_{\Omega_h} \partial_t \tilde{\Gamma}_x(y, t-s) f_h(y, s) dy ds, \quad (x, t) \in Q_{h,T}.$$

Thus, what remains to show is

$$\|\partial_t \tilde{\Gamma}_x * f_h\|_{L^p(Q_{h,T})} \leq C \|f_h\|_{L^p(Q_{h,T})}, \quad \forall f_h \in L^p(Q_{h,T}) \quad (2.79)$$

uniformly with respect to h .

Let

$$(\partial_t \Gamma_x * f)(x, t) := \int_0^t \int_{\Omega} \partial_t \Gamma_x(y, t-s) f(y, s) dy ds, \quad (x, t) \in Q_T$$

for $f \in L^p(Q_T)$. Then, from the argument in [47, pp. 685–686], we have

$$\|\partial_t \Gamma_x * f\|_{L^p(Q_T)} \leq C \|f\|_{L^p(Q_T)}, \quad \forall f \in L^p(Q_T)$$

uniformly with respect to h for $p \in (1, \infty)$. Now, we show (2.79). For $f_h \in L^p(0, T; V_h)$, let $\tilde{f}_h \in L^p(Q_T)$ be the zero-extension of f_h . Then,

$$(\partial_t \tilde{\Gamma}_x * f_h)(x, t) = (\partial_t \Gamma_x * \tilde{f}_h)(x, t) + \Phi(x, t)$$

for $(x, t) \in Q_{h,T}$, where

$$\Phi(x, t) = \int_0^t \int_{\Omega_h \setminus \Omega} \partial_t \tilde{\Gamma}_x(y, t-s) f_h(y, s) dy ds.$$

Thus, we have

$$\|\partial_t \tilde{\Gamma}_x * f_h\|_{L^p(Q_{h,T})} \leq C \|f_h\|_{L^p(Q_{h,T})} + \|\partial_t \Gamma_x * \tilde{f}_h\|_{L^p(Q_{h,T} \setminus Q_T)} + \|\Phi\|_{L^p(Q_{h,T})}. \quad (2.80)$$

As in the proof of the Young inequality for convolution operators, one can see

$$\begin{aligned} \|\partial_t \Gamma_x * \tilde{f}_h\|_{L^p(Q_{h,T} \setminus Q_T)} &\leq \max_{x \in \Omega_h \setminus \Omega} \left(\iint_{Q_T} |\partial_t \Gamma_x(y, s)| dy ds \right)^{1/p'} \\ &\quad \times \max_{y \in \Omega} \left(\iint_{Q_{h,T} \setminus Q_T} |\partial_t \Gamma_x(y, t)| dx dt \right)^{1/p} \|f_h\|_{L^p(Q_{h,T})} \end{aligned}$$

and

$$\begin{aligned} \|\Phi\|_{L^p(Q_{h,T})} &\leq \max_{x \in \Omega_h} \left(\iint_{Q_{h,T} \setminus Q_T} |\partial_t \tilde{\Gamma}_x(y, s)| dy ds \right)^{1/p'} \\ &\quad \times \max_{y \in \Omega_h \setminus \Omega} \left(\iint_{Q_{h,T}} |\partial_t \tilde{\Gamma}_x(y, t)| dx dt \right)^{1/p} \|f_h\|_{L^p(Q_{h,T})}, \end{aligned}$$

where p' fulfills $1/p + 1/p' = 1$. Here, we should discuss the integrability of $\partial_t \Gamma_x(y, t)$ with respect to $(x, t) \in Q_{h,T}$. Since the elliptic operator $-\Delta + I$ with the Neumann boundary condition generates a bounded semigroup in $C^0(\bar{\Omega})$, we have

$$\sup_{y \in \Omega} |\partial_t \Gamma_x(y, t)| \leq C \sup_{y \in \Omega} |(-\Delta + I)\bar{\delta}_x(y)| \leq Ch^{-N-2} \quad (2.81)$$

uniformly with respect to $x \in \Omega_h$. Therefore, the function $(x, t) \mapsto \partial_t \Gamma_x(y, t)$ is bounded for each $y \in \Omega$ and thus integrable.

We here address $\iint_{Q_{h,T} \setminus Q_T} |\partial_t \Gamma_x(y, t)| dx dt$ only. As in (2.48), we define $Q_{h,j}(y)$ and $Q_{h,*}(y)$ as the parabolic dyadic decomposition centered at $(y, 0)$, i.e.,

$$Q_{h,j}(y) := \{(x, t) \in Q_{h,T} \mid d_j \leq \rho_y(x, t) \leq 2d_j\}, \quad Q_{h,*}(y) := \{(x, t) \in Q_{h,T} \mid \rho_y(x, t) \leq d_{J_*}\},$$

where $\rho_y(x, t) = \max\{|x - y|, \sqrt{t}\}$. Then, as discussed in the proof of Lemma 2.5.2, for $(x, t) \in Q_{h,j}(y)$, we have

$$|\partial_t \Gamma_x(y, t)| \leq Cd_j^{-N-2},$$

which implies

$$\iint_{Q_{h,j}(y) \setminus Q_T} |\partial_t \Gamma_x(y, t)| dx dt \leq h^2 d_j^{-1}.$$

On the innermost set $Q_{h,*}(y)$, (2.81) yields

$$\iint_{Q_{h,*}(y) \setminus Q_T} |\partial_t \Gamma_x(y, t)| dx dt \leq C|Q_{h,*}(y) \setminus Q_T| h^{-N-2} \leq Ch.$$

Therefore, we obtain

$$\iint_{Q_{h,T} \setminus Q_T} |\partial_t \Gamma_x(y, t)| dx dt \leq Ch + C \sum_j h^2 d_j^{-1} \leq Ch$$

owing to (2.46), where the constant C is independent of y and h . The treatment of the other terms is similar and we can derive

$$\begin{aligned} \max_{x \in \Omega_h \setminus \Omega} \iint_{Q_T} |\partial_t \Gamma_x(y, s)| dy ds &\leq C|\log h|, \\ \max_{x \in \Omega_h} \iint_{Q_{h,T} \setminus Q_T} |\partial_t \tilde{\Gamma}_x(y, s)| dy ds &\leq Ch, \\ \max_{y \in \Omega_h \setminus \Omega} \iint_{Q_{h,T}} |\partial_t \tilde{\Gamma}_x(y, t)| dx dt &\leq C|\log h|, \end{aligned}$$

with the constant C independent of h . Consequently, we have

$$\|\partial_t \Gamma_x * \tilde{f}_h\|_{L^p(Q_{h,T} \setminus Q_T)} + \|\Phi\|_{L^p(Q_{h,T})} \leq C\|f_h\|_{L^p(Q_{h,T})} \quad (2.82)$$

for $p \in (1, \infty)$. Substituting (2.82) into (2.80), we can obtain (2.79). Hence we can complete the proof. \square

Chapter 3

Energy dissipative numerical schemes for gradient flows of planar curves

Abstract

In this chapter, we develop an energy dissipative numerical scheme for gradient flows of planar curves, such as the curvature flow and the elastic flow. Our study presents a general framework for solving such equations. To discretize the time variable, we use a similar approach to the discrete partial derivative method, which is a structure-preserving method for gradient flows of graphs. For the approximation of curves, we use B-spline curves. Owing to the smoothness of B-spline functions, we can directly address higher order derivatives. Moreover, since B-spline curves require few degrees of freedom, we can reduce the computational cost. In the last part of the chapter, we present some numerical examples of the elastic flow, which exhibit topology-changing solutions and more complicated evolution. Videos illustrating our method are available on YouTube.

This chapter is based on the published paper [66]:

- T. Kemmochi. Energy dissipative numerical schemes for gradient flows of planar curves. *BIT Numer. Math.*, vol. 57, no. 4, pp. 991–1017, 2017.

3.1 Introduction

In this chapter, we consider numerical methods for the computation of the L^2 -gradient flow of a planar curve:

$$\mathbf{u}_t = -\operatorname{grad} E(\mathbf{u}), \quad t > 0, \quad (3.1)$$

where \mathbf{u} is a time-dependent planar curve, E is an energy functional, and $\operatorname{grad} E$ is the Fréchet derivative with respect to the L^2 -structure with line integral ds . The gradient flow (3.1) is energy dissipative, since

$$\frac{d}{dt} E[\mathbf{u}] = - \int |\operatorname{grad} E(\mathbf{u})|^2 ds \leq 0.$$

Typical examples are the curvature flow (the curve shortening flow)

$$\mathbf{u}_t = \kappa \quad (3.2)$$

and the elastic flow (the Willmore flow)

$$\mathbf{u}_t = -2\varepsilon^2 \left(\nabla_s^2 \kappa + \frac{1}{2} |\kappa|^2 \kappa \right) + \kappa \quad (3.3)$$

with energy functionals

$$E[\mathbf{u}] = \int ds, \quad \text{and} \quad E[\mathbf{u}] = \varepsilon^2 \int |\boldsymbol{\kappa}|^2 ds + \int ds,$$

respectively, where $\boldsymbol{\kappa}$ is the curvature vector, and ∇_s is the normal component of the arc-length derivative (Example 3.1). Note that the elastic flow is a fourth-order nonlinear evolution equation.

We consider a dissipative numerical scheme for (3.1), i.e., a scheme which has the discrete energy dissipation property $E[\mathbf{u}_h^{n+1}] \leq E[\mathbf{u}_h^n]$ at each time step. In general, a numerical method that retains a certain property for a target equation is called structure-preserving. It is known that the numerical solutions obtained by these methods are not only physically realistic but also have the advantage of numerical stability (cf. [45, 57]). In particular, structure-preserving methods are suitable for computations over long time intervals.

We explain our motivation to concentrate on the energy dissipation of the gradient flow (3.1) in more detail. As is well known, the (classical) solution of the curvature flow (3.2) blows up in finite time (we set T). In particular, if the initial curve is self-crossing, then a cusp appears at the time T . On the other hand, the elastic flow (3.3), which is a regularized version of (3.2), has a unique global solution (see [36]). Moreover, as $\varepsilon \downarrow 0$, the solution of (3.3) converges to that of (3.2) in each time interval $(0, T') \subset (0, T)$ (see [11]). Note that the elastic flow formally degenerates to the curvature flow as $\varepsilon \downarrow 0$. We are interested in determining the limit of the elastic flow as $\varepsilon \downarrow 0$ after time T . Therefore, the structure-preserving method is effective for numerical investigation of the long-time behavior of the elastic flow with self-crossing initial curves.

There are many works that consider the numerical computation of equations (3.2) and (3.3) (for example, [8, 9, 28, 29, 36]). However, none of them explicitly consider the discrete energy dissipation property. Although some numerical examples in these works seem to be dissipative, no mathematical evidence is given. Moreover, all of them use the P1-finite element method, which approximates solutions by polygonal curves. Therefore, since they cannot handle higher order derivatives directly, it is necessary to consider the mixed formulation for fourth-order equations.

For gradient flows of graphs, there are some general frameworks to construct dissipative numerical schemes. In [44, 45], a structure-preserving finite difference scheme, the discrete variational derivative method, is proposed to approximate the solution of the equation

$$u_t = (-1)^{s+1} \left(\frac{\partial}{\partial x} \right)^{2s} \frac{\delta E}{\delta u} \quad (3.4)$$

over a bounded interval in \mathbb{R} . Here, $\delta E/\delta u$ is the Fréchet derivative or the first variation of the energy functional E with respect to the usual L^2 -structure, i.e., the L^2 space with respect to the Lebesgue measure. Finite element schemes, called discrete partial derivative methods (DPDM), for the same problems are presented in [85]. The main idea of these studies is to discretize the chain rule. In [85], discrete partial derivatives are introduced and a discretization of the chain rule is obtained. A similar approach is used in our scheme. See also [2] for local discontinuous Galerkin schemes for the above problems.

In the present chapter, we apply the idea of DPDM to the discretization of the time variable. We obtain the discrete chain rule formula with respect to the $L^2(ds)$ -structure. In contrast to problems such as (3.4), our problem (3.1) is accompanied by the line element ds , which increases the complexity of the problem. Due to the inclusion of the line element, we cannot use the DPDM for (3.1). Therefore, we will present a generalization of the DPDM and derive a new scheme for weak forms of general gradient flows (Scheme 3.2).

For the approximation of curves, we use B-spline curves (cf. [39, 89]). The B-spline approach (also called NURBS in general) is widely used to compute the solution to large-scale deformation problems, such as fluid-structure interaction (see, e.g., [10]). This method is called isogeometric analysis [25]. It is known that B-spline curves requires few degrees of freedom (DOF), which helps us to reduce DOF in our scheme.

Moreover, it is worth emphasizing that a B-spline curve of degree p (Definition 3.2) has a C^{p-1} -parametrization. Hence we can directly address higher order derivatives, and we can derive the fully discretized scheme by the Galerkin method without mixed formulation (Scheme 3.3). This procedure is independent of the expression of the energy functional, and thus, our scheme gives a general framework

for the approximation of gradient flows (3.1). We can also obtain the discrete energy dissipation with our scheme (Lemma 3.3.1). In this study, we do not consider solvability or error estimates. Nevertheless, we try to investigate the convergence rates numerically for the circular solutions in Section 3.4 (see Example 3.2).

This chapter is structured as follows. In Section 3.2, we present some necessary definitions and notation for gradient flows of planar curves (Subsection 3.2.1), illustrate the use of the DPDM (Subsection 3.2.2), and give a brief introduction to B-spline curves (Subsection 3.2.3). In Section 3.3, we derive our energy dissipative scheme (Scheme 3.3) under the framework of the DPDM, and introduce the discrete energy dissipation property (Lemma 3.3.1). Finally, we present some numerical examples of our scheme in Section 3.4 in the context of the elastic flow (3.3). Topology-changing solutions and more complicated evolution are reported, which have not previously been shown in the literature. Videos illustrating our method are available on YouTube¹. We conclude this chapter with conclusion and future work in Section 3.5.

3.2 Preliminaries

3.2.1 Geometric gradient flows for planar curves

We first introduce the notation on the geometric gradient flows of planar curves. Let $\mathbf{H}_\pi^1 = \{\mathbf{u} = \mathbf{u}(\zeta) \in H^1(0, 1; \mathbb{R}^2) \mid \mathbf{u}(0) = \mathbf{u}(1)\}$ and $\mathbf{H}_\pi^m = \{\mathbf{u} \in H^m(0, 1; \mathbb{R}^2) \mid \mathbf{u}' \in \mathbf{H}_\pi^{m-1}\}$ for $m \in \mathbb{N}$, $m \geq 2$, where the space $H^m(0, 1; \mathbb{R}^2) = (H^m(0, 1))^2$ is the m -th order Sobolev space of L^2 -type. Here, and hereafter, we denote the variable of the representation parameter of curves by $\zeta \in (0, 1)$. Note that the space \mathbf{H}_π^m is embedded into the space of the planar closed curves with C^{m-1} -parametrization. We define an energy functional $E: \mathbf{H}_\pi^m \rightarrow \mathbb{R}$ as

$$E[\mathbf{u}] = \int F(\mathbf{u}, \mathbf{u}', \dots, \mathbf{u}^{(m)}) ds, \quad \mathbf{u} \in \mathbf{H}_\pi^m, \quad (3.5)$$

where $F: (\mathbb{R}^2)^{m+1} \rightarrow \mathbb{R}$ is the energy density function, and $ds = ds(\mathbf{u})$ is the line element of the curve \mathbf{u} . Let $\text{grad } E$ be the Fréchet derivative of the functional E in the topology of $L^2(ds)$. That is,

$$\langle \text{grad } E(\mathbf{u}), \mathbf{v} \rangle = \int \text{grad } E(\mathbf{u}) \cdot \mathbf{v} ds(\mathbf{u}).$$

Then, the gradient flow for E is represented by the following evolution equation:

$$\mathbf{u}_t = -\text{grad } E(\mathbf{u}), \quad t > 0. \quad (3.6)$$

We now present examples of such gradient flows.

Example 3.1. (i) (Curvature flow) If $E[\mathbf{u}] = \int ds$, then equation (3.6) is the curvature flow

$$\mathbf{u}_t = \boldsymbol{\kappa}, \quad (3.7)$$

where $\boldsymbol{\kappa} = \mathbf{u}_{ss}$ is the curvature vector, s is the arc-length parameter, and \mathbf{f}_s expresses the arc-length derivative for $\mathbf{f} \in \mathbf{H}_\pi^1$.

(ii) (Elastic flow) If $E[\mathbf{u}] = \varepsilon^2 \int |\boldsymbol{\kappa}|^2 ds + \int ds$ with a constant $\varepsilon > 0$, then equation (3.6) is the elastic flow (or Willmore flow)

$$\mathbf{u}_t = -2\varepsilon^2 \left(\nabla_s^2 \boldsymbol{\kappa} + \frac{1}{2} |\boldsymbol{\kappa}|^2 \boldsymbol{\kappa} \right) + \boldsymbol{\kappa}, \quad (3.8)$$

where $\nabla_s \mathbf{v} = (\mathbf{v}_s, \boldsymbol{\tau})\boldsymbol{\tau} - \mathbf{v}_s$, and $\boldsymbol{\tau} = \mathbf{u}_s$.

In this chapter, we focus on the energy dissipation property, which is given as follows:

$$\frac{d}{dt} E[\mathbf{u}] = \int \text{grad } E(\mathbf{u}) \cdot \mathbf{u}_t ds = - \int |\mathbf{u}_t|^2 ds \leq 0.$$

¹URL: https://www.youtube.com/playlist?list=PLMF3dSqWEii691oXvCtgDCI4PYijq_4L3

3.2.2 Discrete partial derivative method

In this subsection, we introduce the DPDM, which was first presented in [85]. Let $E: H_\pi^2(0, 1) \rightarrow \mathbb{R}$ be an energy functional that is defined as

$$E[u] = \int_0^1 G(u_\zeta, u_{\zeta\zeta}) d\zeta, \quad u = u(\zeta) \in H_\pi^2(0, 1), \quad (3.9)$$

where $G = G(p, q): \mathbb{R}^2 \rightarrow \mathbb{R}$ is the energy density function. Although we can consider more general energy functionals and density functions, we consider energy functionals E such as (3.9) for simplicity. Let us denote the first variation of E by $\delta E/\delta u$, i.e.,

$$\frac{\delta E}{\delta u} = -\frac{\partial}{\partial \zeta} G_p(u_\zeta, u_{\zeta\zeta}) + \left(\frac{\partial}{\partial \zeta} \right)^2 G_q(u_\zeta, u_{\zeta\zeta}).$$

Then, the L^2 -gradient flow for the energy functional E is written as the evolution equation

$$u_t = -\frac{\delta E}{\delta u}, \quad t > 0 \quad (3.10)$$

with an initial condition.

We present the derivation of the energy dissipation property of the flow (3.10) for comparison with DPDM, which is illustrated later. From the chain rule for G , we have

$$\frac{\partial}{\partial t} G(u_\zeta, u_{\zeta\zeta}) = G_p(u_\zeta, u_{\zeta\zeta}) u_{\zeta,t} + G_q(u_\zeta, u_{\zeta\zeta}) u_{\zeta\zeta,t}, \quad (3.11)$$

which implies

$$\frac{d}{dt} E[u] = \int_0^1 [G_p(u_\zeta, u_{\zeta\zeta}) u_{\zeta,t} + G_q(u_\zeta, u_{\zeta\zeta}) u_{\zeta\zeta,t}] d\zeta \quad (3.12)$$

for $u \in C^1([0, T]; H_\pi^2(0, 1))$. Integrating the right hand side by parts and applying the periodic boundary condition, we obtain

$$\begin{aligned} \frac{d}{dt} E[u] &= \int_0^1 \left[-\frac{\partial}{\partial \zeta} G_p(u_\zeta, u_{\zeta\zeta}) + \left(\frac{\partial}{\partial \zeta} \right)^2 G_q(u_\zeta, u_{\zeta\zeta}) \right] u_t d\zeta \\ &= \int_0^1 \frac{\delta E}{\delta u} u_t d\zeta. \end{aligned} \quad (3.13)$$

Finally, substituting the equation of the gradient flow (3.10), we can derive

$$\frac{d}{dt} E[u] = - \int_0^1 |u_t|^2 d\zeta \leq 0, \quad (3.14)$$

which is nothing less than the desired energy dissipation property.

DPDM is an energy dissipative numerical scheme for the equation of the form (3.10). The key idea of DPDM is discretization of the chain rule (3.11). Let us discretize the time variable of the gradient flow (3.10) by the finite difference method. We illustrate how to determine the discrete equation of the gradient flow by DPDM. Let u^n be the approximate solution at the n -th step and Δt be the time increment. We first discretize the left hand side of the discrete gradient flow by $(u^{n+1} - u^n)/\Delta t$. Then, the discrete counterpart of dE/dt is

$$\frac{E[u^{n+1}] - E[u^n]}{\Delta t} = \int_0^1 \frac{G(u_\zeta^{n+1}, u_{\zeta\zeta}^{n+1}) - G(u_\zeta^n, u_{\zeta\zeta}^n)}{\Delta t} d\zeta,$$

and thus that of $\partial_t G(u_\zeta, u_{\zeta\zeta})$ is $[G(u_\zeta^{n+1}, u_{\zeta\zeta}^{n+1}) - G(u_\zeta^n, u_{\zeta\zeta}^n)]/\Delta t$. Since $\partial_t u$ is approximated by $(u^{n+1} - u^n)/\Delta t$, we hope that the discrete analogue of (3.11) is the equation of the form

$$\frac{G(u_\zeta^{n+1}, u_{\zeta\zeta}^{n+1}) - G(u_\zeta^n, u_{\zeta\zeta}^n)}{\Delta t} = A \frac{u_\zeta^{n+1} - u_\zeta^n}{\Delta t} + B \frac{u_{\zeta\zeta}^{n+1} - u_{\zeta\zeta}^n}{\Delta t} \quad (3.15)$$

with suitable functions A and B depending on u^n and u^{n+1} . If we can find such A and B , then we have

$$\frac{E[u^{n+1}] - E[u^n]}{\Delta t} = \int_0^1 \left(A \frac{u_\zeta^{n+1} - u_\zeta^n}{\Delta t} + B \frac{u_{\zeta\zeta}^{n+1} - u_{\zeta\zeta}^n}{\Delta t} \right) d\zeta \quad (3.16)$$

$$= \int_0^1 (-\partial_\zeta A + \partial_\zeta^2 B) \frac{u^{n+1} - u^n}{\Delta t} d\zeta \quad (3.17)$$

provided that u^n and u^{n+1} are periodic with respect to ζ . Comparing the last equation with (3.13), we can determine the discrete first variation by the formula

$$\frac{\delta E_d}{\delta(u^{n+1}, u^n)} = -\partial_\zeta A + \partial_\zeta^2 B \quad (3.18)$$

and thus the discrete gradient flow can be written as

$$\frac{u^{n+1} - u^n}{\Delta t} = -(-\partial_\zeta A + \partial_\zeta^2 B) \quad (3.19)$$

for A and B satisfying (3.15). Substituting (3.19) into (3.17), we can obtain the discrete dissipation property

$$\frac{E[u^{n+1}] - E[u^n]}{\Delta t} = - \int_0^1 \left| \frac{u^{n+1} - u^n}{\Delta t} \right|^2 d\zeta \leq 0,$$

which corresponds to equation (3.14).

Based on the above observation, we define the discrete partial derivatives $\partial G_d / \partial(\cdot, \cdot)$ as the functions that satisfy the following relation (corresponding to (3.15)):

$$G(u_\zeta, u_{\zeta\zeta}) - G(v_\zeta, v_{\zeta\zeta}) = \frac{\partial G_d}{\partial(u_\zeta, v_\zeta)}(u_\zeta - v_\zeta) + \frac{\partial G_d}{\partial(u_{\zeta\zeta}, v_{\zeta\zeta})}(u_{\zeta\zeta} - v_{\zeta\zeta}), \quad (3.20)$$

for all $u, v \in H_\pi^2(0, 1)$. Note that the discrete partial derivatives which solve the relation (3.20) may not be unique. When G has a certain decomposition (similar to the equation (11) in [85])

$$G(u_\zeta, u_{\zeta\zeta}) = \sum_l f_l(u_\zeta) g_l(u_{\zeta\zeta}),$$

a method for deriving the discrete partial derivatives is given in [85]. Once the discrete partial derivatives are constructed, we can define the discrete first variation as

$$\frac{\delta E_d}{\delta(u, v)} = -\frac{\partial}{\partial \zeta} \frac{\partial G_d}{\partial(u_\zeta, v_\zeta)} + \left(\frac{\partial}{\partial \zeta} \right)^2 \frac{\partial G_d}{\partial(u_{\zeta\zeta}, v_{\zeta\zeta})}$$

and derive the strong form of the discrete gradient flow

$$\frac{u^{n+1} - u^n}{\Delta t} = -\frac{\delta E_d}{\delta(u^{n+1}, u^n)}, \quad n \in \mathbb{N} \quad (3.21)$$

from the above observation (see (3.18) and (3.19)). Introducing the weak form of equation (3.21), we can derive the scheme of DPDM for the gradient flow (3.10) as follows.

Scheme 3.1 (DPDM for the gradient flow of a graph). Let $n \in \mathbb{N}$ and $u^n \in H_\pi^2(0, 1)$ be given. Find $u^{n+1} \in H_\pi^2(0, 1)$ that satisfies

$$\left(\frac{u^{n+1} - u^n}{\Delta t}, v \right) = - \left(\frac{\partial G_d}{\partial(u_\zeta^{n+1}, u_\zeta^n)}, v_\zeta \right) - \left(\frac{\partial G_d}{\partial(u_{\zeta\zeta}^{n+1}, u_{\zeta\zeta}^n)}, v_{\zeta\zeta} \right), \quad (3.22)$$

for all $v \in H_\pi^2(0, 1)$.

We can now check the discrete energy dissipation property for the weak formulation. Here we give the proof for comparison with our scheme (see Lemma 3.3.1).

Lemma 3.2.1. *Let u^n and u^{n+1} satisfy the relation (3.22). Then, we have*

$$\frac{E[u^{n+1}] - E[u^n]}{\Delta t} = - \left\| \frac{u^{n+1} - u^n}{\Delta t} \right\|_{L^2(0,1)}^2 \leq 0$$

for all $n \in \mathbb{N}$ and $\Delta t > 0$.

Proof. Recall that the discrete partial derivatives satisfy

$$E[u] - E[v] = \int_0^1 \left[\frac{\partial G_d}{\partial(u_\zeta, v_\zeta)}(u_\zeta - v_\zeta) + \frac{\partial G_d}{\partial(u_{\zeta\zeta}, v_{\zeta\zeta})}(u_{\zeta\zeta} - v_{\zeta\zeta}) \right] d\zeta \quad (3.23)$$

by its definition (3.20) (see also (3.16)). Let us write

$$\partial_d u^n = \frac{u^{n+1} - u^n}{\Delta t}.$$

Then, substituting $v = \partial_d u^n$ into the weak form (3.22), we can derive

$$\begin{aligned} \frac{E[u^{n+1}] - E[u^n]}{\Delta t} &= \left(\frac{\partial G_d}{\partial(u_\zeta^{n+1}, u_\zeta^n)}, \partial_d u_\zeta^n \right) + \left(\frac{\partial G_d}{\partial(u_{\zeta\zeta}^{n+1}, u_{\zeta\zeta}^n)}, \partial_d u_{\zeta\zeta}^n \right) \\ &= - \|\partial_d u^n\|_{L^2(0,1)}^2. \end{aligned}$$

Here, the first equality is derived from (3.23). Hence we can complete the proof. \square

Note that the key point is substituting $v = \partial_d u^n$ into (3.22). Therefore, this proof can be performed also in the case of the Galerkin method.

3.2.3 B-spline curves

In our scheme, we use B-spline curves to discretize the solution curves. We say that a set of points $\Xi = \{\xi_1, \xi_2, \dots, \xi_n\} \subset \mathbb{R}$ is a *knot vector* if $\xi_i \leq \xi_{i+1}$ for all i .

Definition 3.1 (B-spline basis functions and B-spline curves). Let $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $\Xi = \{\xi_1, \xi_2, \dots, \xi_n\}$ be a knot vector.

- (i) The i -th B-spline basis function of degree p with respect to Ξ is a piecewise polynomial function $N_{p,i}^\Xi$ that is generated by the following formula:

$$N_{0,i}^\Xi(\xi) = \chi_{[\xi_i, \xi_{i+1})}(\xi), \quad \xi \in \mathbb{R},$$

for $i = 1, 2, \dots, n-1$, and

$$N_{p,i}^\Xi(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{p-1,i}^\Xi(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{p-1,i+1}^\Xi(\xi), \quad \xi \in \mathbb{R},$$

for $i = 1, 2, \dots, n-p-1$ and $p \geq 1$, where χ_I is the characteristic function of $I \subset \mathbb{R}$. Here, if $\xi_{i+p} = \xi_i$ (resp. $\xi_{i+p+1} = \xi_{i+1}$), then the term $(\xi - \xi_i)/(\xi_{i+p} - \xi_i)$ (resp. $(\xi_{i+p+1} - \xi)/(\xi_{i+p+1} - \xi_{i+1})$) is regarded as null.

- (ii) A curve $\mathbf{u}: [a, b] \rightarrow \mathbb{R}^2$ is a *B-spline curve of degree p* if \mathbf{u} is represented by

$$\mathbf{u}(\zeta) = \sum_{i=1}^{n-p-1} N_{p,i}^\Xi(\zeta) \mathbf{P}_i, \quad \zeta \in [a, b],$$

for some knot vector Ξ and $n \in \mathbb{N}$. The coefficient \mathbf{P}_i is called a *control point*.

In fact, if the knot vector is disjoint (i.e., $i \neq j \implies \xi_i \neq \xi_j$), then it is known that $N_{p,i}^\Xi$ is a C^{p-1} -function. For more details on the properties of B-spline functions, we refer the reader to [39, 89, 95].

In the present chapter, we only consider the periodic B-spline functions and curves. Let $[a, b] \subset \mathbb{R}$ be an interval, $p \in \mathbb{N}$, $N \in \mathbb{N}$, and $h = 1/N$. We define a knot vector Ξ by

$$\Xi = \{\xi_i\}_{i=1}^{N+2p+1} = \{a - ph, a - (p-1)h, \dots, b + (p-1)h, b + ph\}, \quad (3.24)$$

and let $N_{p,i} = N_{p,i}^\Xi$ be the corresponding B-spline basis function. In this case, if $N > p$, we can see that

$$\left(\frac{d}{d\zeta}\right)^m N_{p,i}(a) = \left(\frac{d}{d\zeta}\right)^m N_{p,i+N}(b),$$

for $i = 1, 2, \dots, p$ and $m = 0, 1, \dots, p-1$. Therefore, the function

$$B_{p,i}(\zeta) = B_{h,p,i}(\zeta) = \begin{cases} N_{p,i}(\zeta), & \zeta \in [a, \xi_{i+p+1}], \\ N_{p,i+N}(\zeta), & \zeta \in [\xi_{i+N}, b], \\ 0, & \text{otherwise} \end{cases} \quad i = 1, 2, \dots, p \quad (3.25)$$

is a periodic C^{p-1} -function on $[a, b]$. The restriction $N_{p,i}|_{[a,b]}$ for $i > p$ is also C^{p-1} -periodic on $[a, b]$. Thus, we can define a closed B-spline curve as follows.

Definition 3.2 (Periodic B-spline). Let $[a, b] \subset \mathbb{R}$ be an interval, $p \in \mathbb{N}$, $N \in \mathbb{N}$ with $N > p$, and $h = 1/N$. We define a *periodic B-spline basis function of degree p* $B_{p,i} = B_{h,p,i}$ by

$$B_{p,i} = \begin{cases} \text{equation (3.25)} & \text{if } i = 1, 2, \dots, p, \\ N_{p,i}^\Xi|_{[a,b]}, & \text{if } i = p+1, p+2, \dots, N, \end{cases}$$

where Ξ is a knot vector defined by (3.24). We also define a *closed B-spline curve* as a curve $\mathbf{u}: [a, b] \rightarrow \mathbb{R}^2$ expressed by

$$\mathbf{u}(\zeta) = \sum_{i=1}^N B_{p,i}(\zeta) \mathbf{P}_i, \quad \zeta \in [a, b],$$

for some $\{\mathbf{P}_i\}_{i=1}^N \subset \mathbb{R}^2$.

We illustrate the figure of $B_{p,i}$ for $p = 3$ in Figure 3.1. One can observe that each $B_{p,i}$ has a compact support. Indeed, one can show that $\text{supp } N_{p,i}^\Xi = [\xi_i, \xi_{i+p+1}]$ in general. Thus matrices appearing in the Galerkin method are sparse as in the usual FEM.

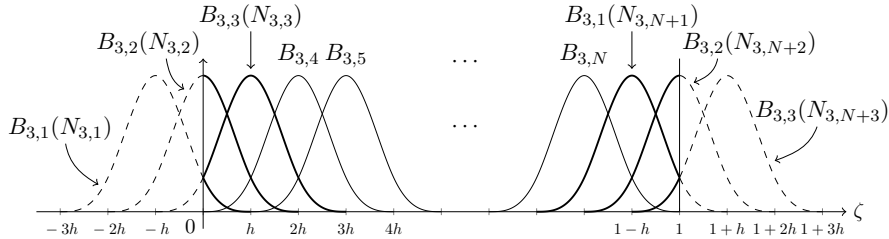


Figure 3.1: Periodic B-spline basis functions $B_{p,i}$ for $p = 3$ on the interval $[0, 1]$.

We also present examples of periodic B-spline curves for $p = 3$ in Figure 3.2. One can see that a smooth curve (Figure 3.2, left) is expressed by only six control points (i.e., DOF is twelve). If one intend to describe such curves by polygonal curves, then hundreds of points are required. We remark that a B-spline curve may not be a C^{p-1} -curve in general. Namely, there may exist $\zeta_0 \in [0, 1]$ such that $\partial_\zeta \mathbf{u}(\zeta_0) = 0$.

3.3 Derivation of an energy dissipative numerical scheme

In this section, we derive a numerical scheme for geometric gradient flows (3.6) for the energy functional E given by (3.5). We first consider the time discretization with an idea similar to DPDM. After that, we approximate the solution curves by B-spline in terms of the Galerkin method.

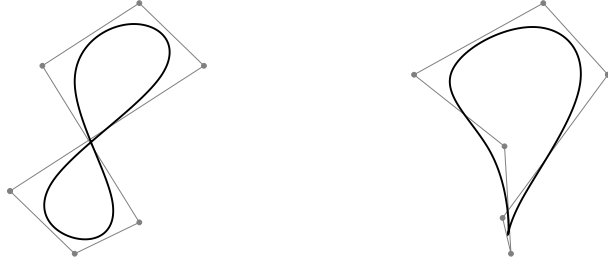


Figure 3.2: Periodic B-spline curves for $p = 3$. Thick lines are B-splines curves and gray-colored points are control points. Thin lines are polygonal curves constructed by connecting control points in order. The right curve may have a corner or a cusp.

Let us discretize the time variable. The definition of the discrete partial derivatives (3.20) is a discrete analogue of the chain rule formula (3.11) for a smooth function u , and it gives the key equation (3.23) for the proof of the discrete energy dissipation, which is itself the discrete counterpart of (3.12). In our case, the corresponding chain rule can be expressed as

$$\frac{d}{dt}E[\mathbf{u}] = \int \text{grad } E(\mathbf{u}) \cdot \mathbf{u}_t ds(\mathbf{u}). \quad (3.26)$$

Here, we denote the line element of \mathbf{u} by $ds(\mathbf{u})$ to emphasize the dependence on \mathbf{u} . Now, we discretize the chain rule (3.26). We first replace the time derivatives with time differences by expressing $\frac{d}{dt}E[\mathbf{u}]$ and \mathbf{u}_t as $E[\mathbf{u}] - E[\mathbf{v}]$ and $\mathbf{u} - \mathbf{v}$, respectively. Moreover, the line element $ds(\mathbf{u})$ should be replaced appropriately. In the original formula (3.26), there is one function \mathbf{u} only. However, in the discretization, there are two functions \mathbf{u} and \mathbf{v} as in (3.20). Therefore, we have some choices to discretize the term $ds(\mathbf{u})$, for example, $ds(\mathbf{u})$, $ds(\mathbf{v})$, and $ds((\mathbf{u} + \mathbf{v})/2)$. Here, we use $ds((\mathbf{u} + \mathbf{v})/2)$. Then, we define a discrete gradient, $\text{grad}_d E: \mathbf{H}_\pi^m \times \mathbf{H}_\pi^m \rightarrow \mathbb{R}^2$, with a function that satisfies the following formula:

$$E[\mathbf{u}] - E[\mathbf{v}] = \int \text{grad}_d E(\mathbf{u}, \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) ds\left(\frac{\mathbf{u} + \mathbf{v}}{2}\right), \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}_\pi^m. \quad (3.27)$$

Thus, according to (3.21), the strong form of the time-discrete problem is written as follows:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = -\text{grad}_d E(\mathbf{u}^{n+1}, \mathbf{u}^n), \quad n \in \mathbb{N}. \quad (3.28)$$

We now discuss how to calculate the discrete gradient $\text{grad}_d E(\mathbf{u}, \mathbf{v})$. The discrete chain rule (3.27) can be expressed as

$$E[\mathbf{u}] - E[\mathbf{v}] = \int_0^1 \left| \frac{\mathbf{u}_\zeta + \mathbf{v}_\zeta}{2} \right| \text{grad}_d E(\mathbf{u}, \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) d\zeta, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}_\pi^m, \quad (3.29)$$

and comparing (3.29) with (3.23), we can derive the relationship between $\text{grad}_d E$ and the discrete first variation $\partial E_d / \partial(\mathbf{u}, \mathbf{v})$ as follows.

$$\left| \frac{\mathbf{u}_\zeta + \mathbf{v}_\zeta}{2} \right| \text{grad}_d E(\mathbf{u}, \mathbf{v}) = \frac{\delta E_d}{\delta(\mathbf{u}, \mathbf{v})}. \quad (3.30)$$

Here $\partial E_d / \partial(\mathbf{u}, \mathbf{v}) = (\partial E_d / \partial(u_1, v_1), \partial E_d / \partial(u_2, v_2))^T$ is a vector-valued function, and each component can be derived as in Subsection 3.2.2. Letting

$$G(\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_m) := F(\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_m) |\mathbf{p}_1|,$$

we can express the energy E by

$$E[\mathbf{u}] = \int_0^1 G(\mathbf{u}, \mathbf{u}_\zeta, \dots, \partial_\zeta^m \mathbf{u}) d\zeta.$$

Thus the discrete first variation is given by

$$\frac{\delta E_d}{\delta(\mathbf{u}, \mathbf{v})} = \sum_{j=0}^m (-1)^j \left(\frac{\partial}{\partial \zeta} \right)^j \frac{\partial G_d}{\partial(\partial_\zeta^j \mathbf{u}, \partial_\zeta^j \mathbf{v})}, \quad (3.31)$$

with the (vector-valued) partial derivatives

$$\frac{\partial G_d}{\partial(\partial_\zeta^j \mathbf{u}, \partial_\zeta^j \mathbf{v})} = \left(\frac{\partial G_d}{\partial(\partial_\zeta^j u_1, \partial_\zeta^j v_1)}, \frac{\partial G_d}{\partial(\partial_\zeta^j u_2, \partial_\zeta^j v_2)} \right)^T, \quad j = 0, 1, \dots, m,$$

which satisfy the relation

$$G(\mathbf{u}, \mathbf{u}_\zeta, \dots, \partial_\zeta^m \mathbf{u}) - G(\mathbf{v}, \mathbf{v}_\zeta, \dots, \partial_\zeta^m \mathbf{v}) = \sum_{j=0}^m \frac{\partial G_d}{\partial(\partial_\zeta^j \mathbf{u}, \partial_\zeta^j \mathbf{v})} \cdot \partial_\zeta^j (\mathbf{u} - \mathbf{v}), \quad (3.32)$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{H}_\pi^m$. Note that, as in the previous case (Subsection 3.2.2), the discrete partial derivative may not be unique. The definition (3.32) is a generalization of (3.20), and thus it implies

$$E[\mathbf{u}] - E[\mathbf{v}] = \int_0^1 \sum_{j=0}^m \frac{\partial G_d}{\partial(\partial_\zeta^j \mathbf{u}, \partial_\zeta^j \mathbf{v})} \cdot \partial_\zeta^j (\mathbf{u} - \mathbf{v}) d\zeta, \quad (3.33)$$

which corresponds to the key equation (3.23). As in Lemma 3.2.1, equation (3.33) plays a crucial role in the proof of the discrete dissipation property for our scheme. See the proof of Lemma 3.3.1 below.

Now, substituting (3.30) and (3.31) into (3.28), we may solve the equation

$$\left| \frac{\mathbf{u}_\zeta^{n+1} + \mathbf{u}_\zeta^n}{2} \right| \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = - \sum_{j=0}^m (-1)^j \left(\frac{\partial}{\partial \zeta} \right)^j \frac{\partial G_d}{\partial(\partial_\zeta^j \mathbf{u}^{n+1}, \partial_\zeta^j \mathbf{u}^n)}, \quad (3.34)$$

instead of solving (3.28). Therefore, the weak form of (3.34) gives our semi-discrete scheme for the gradient flow (3.6). Since the time increment Δt can differ at each step, we denote the time increment from the n -th step to the $(n+1)$ -th step by Δt_n in what follows.

Scheme 3.2 (Semi-discrete scheme for the geometric gradient flow). Find $\mathbf{u}^{n+1} \in \mathbf{H}_\pi^m$ that satisfies

$$\left(\left| \frac{\mathbf{u}_\zeta^{n+1} + \mathbf{u}_\zeta^n}{2} \right| \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t_n}, \mathbf{v} \right) = - \sum_{j=0}^m \left(\frac{\partial G_d}{\partial(\partial_\zeta^j \mathbf{u}^{n+1}, \partial_\zeta^j \mathbf{u}^n)}, \partial_\zeta^j \mathbf{v} \right), \quad \forall \mathbf{v} \in \mathbf{H}_\pi^m$$

for given $\mathbf{u}^n \in \mathbf{H}_\pi^m$, where $G(\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_m) = F(\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_m) |\mathbf{p}_1|$.

We now consider the full discretization of the gradient flow (3.6). Let \mathbf{V}_h^p be the space of closed B-spline curves of degree p on the interval $(0, 1)$ as defined in Definition 3.2. Then, by the Sobolev embedding theorem, $\mathbf{V}_h^p \hookrightarrow \mathbf{H}_\pi^m$ if $p \geq m+1$. Thus, we can derive a fully discretized problem by the Galerkin method.

Scheme 3.3 (Fully discretized scheme for the geometric gradient flow). Let \mathbf{V}_h^p be the space of closed B-spline curves of degree p on the interval $(0, 1)$ as defined in Definition 3.2 for $N \in \mathbb{N}$, $h = 1/N$, and $p \geq m+1$. Assume $\mathbf{u}_h^n \in \mathbf{V}_h^p$ is given. Find $\mathbf{u}_h^{n+1} \in \mathbf{V}_h^p$ that satisfies

$$\left(\left| \frac{\mathbf{u}_{h,\zeta}^{n+1} + \mathbf{u}_{h,\zeta}^n}{2} \right| \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t_n}, \mathbf{v}_h \right) = - \sum_{j=0}^m \left(\frac{\partial G_d}{\partial(\partial_\zeta^j \mathbf{u}_h^{n+1}, \partial_\zeta^j \mathbf{u}_h^n)}, \partial_\zeta^j \mathbf{v}_h \right), \quad (3.35)$$

for all $\mathbf{v}_h \in \mathbf{V}_h^p$, where $G(\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_m) = F(\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_m) |\mathbf{p}_1|$.

Then, we can establish the discrete energy dissipation property.

Lemma 3.3.1 (Discrete energy dissipation). *Let \mathbf{u}_h^n and \mathbf{u}_h^{n+1} satisfy the relation (3.35). Then, we have*

$$\frac{E[\mathbf{u}_h^{n+1}] - E[\mathbf{u}_h^n]}{\Delta t_n} = - \int_0^1 \left| \frac{\mathbf{u}_{h,\zeta}^{n+1} + \mathbf{u}_{h,\zeta}^n}{2} \right| \left| \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t_n} \right|^2 d\zeta \leq 0.$$

Proof. Substituting $\mathbf{v}_h = (\mathbf{u}_h^{n+1} - \mathbf{u}_h^n)/\Delta t_n$ into the scheme (3.35), we have

$$\begin{aligned} \int_0^1 \left| \frac{\mathbf{u}_{h,\zeta}^{n+1} + \mathbf{u}_{h,\zeta}^n}{2} \right| \left| \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t_n} \right|^2 d\zeta &= - \sum_{j=0}^m \left(\frac{\partial G_d}{\partial(\partial_\zeta^j \mathbf{u}_h^{n+1}, \partial_\zeta^j \mathbf{u}_h^n)}, \frac{\partial_\zeta^j (\mathbf{u}_h^{n+1} - \mathbf{u}_h^n)}{\Delta t_n} \right) \\ &= - \frac{E[\mathbf{u}_h^{n+1}] - E[\mathbf{u}_h^n]}{\Delta t_n}, \end{aligned}$$

by the definition of the partial derivatives (3.32). In the last equality, we applied the discrete chain rule formula (3.33). Hence we can establish the desired assertion. \square

3.4 Numerical examples

In this section, we present some numerical examples of the elastic flow (3.8) with a small parameter ε computed by our scheme (3.35). Videos illustrating the following examples are available on YouTube². The corresponding functional is the elastic energy

$$E[\mathbf{u}] = \varepsilon^2 \int |\kappa|^2 ds + \int ds = \int_0^1 \left(\varepsilon^2 \frac{\det(\mathbf{u}_\zeta, \mathbf{u}_{\zeta\zeta})^2}{|\mathbf{u}_\zeta|^5} + |\mathbf{u}_\zeta| \right) d\zeta, \quad (3.36)$$

where

$$\det(\mathbf{u}_\zeta, \mathbf{u}_{\zeta\zeta}) = \det \begin{pmatrix} u_{1,\zeta} & u_{1,\zeta\zeta} \\ u_{2,\zeta} & u_{2,\zeta\zeta} \end{pmatrix} = u_{1,\zeta} u_{2,\zeta\zeta} - u_{1,\zeta\zeta} u_{2,\zeta}, \quad \mathbf{u}(\zeta) = \begin{pmatrix} u_1(\zeta) \\ u_2(\zeta) \end{pmatrix}.$$

It is known that equation (3.8) has a unique time-global solution (see, e.g., [36, Theorem 3.2]). Therefore, the turning number $|\int \kappa ds|/(2\pi) \in \mathbb{N}$ is invariant.

To calculate the discrete partial derivatives for E , let

$$G_1(\mathbf{u}_\zeta, \mathbf{u}_{\zeta\zeta}) = \frac{\det(\mathbf{u}_\zeta, \mathbf{u}_{\zeta\zeta})^2}{|\mathbf{u}_\zeta|^5}, \quad G_2(\mathbf{u}_\zeta) = |\mathbf{u}_\zeta|$$

for $\mathbf{u} \in \mathbf{H}_\pi^2$. Then, the energy density function for E is $G := \varepsilon^2 G_1 + G_2$. We can easily compute the discrete partial derivatives of G_2 . Indeed, since

$$G_2(\mathbf{u}_\zeta) - G_2(\mathbf{v}_\zeta) = \frac{\mathbf{u}_\zeta + \mathbf{v}_\zeta}{|\mathbf{u}_\zeta| + |\mathbf{v}_\zeta|} \cdot (\mathbf{u}_\zeta - \mathbf{v}_\zeta),$$

we can define

$$\frac{\partial G_{2,d}}{\partial(u_{j,\zeta}, v_{j,\zeta})} = \frac{u_{j,\zeta} + v_{j,\zeta}}{|\mathbf{u}_\zeta| + |\mathbf{v}_\zeta|}, \quad j = 1, 2$$

by (3.32). We can derive the discrete partial derivatives of G_1 in several ways. In the following examples, these derivatives are computed by dividing $G_1(\mathbf{u}) - G_1(\mathbf{v})$ as follows:

$$G_1(\mathbf{u}_\zeta, \mathbf{u}_{\zeta\zeta}) - G_1(\mathbf{v}_\zeta, \mathbf{v}_{\zeta\zeta}) = \frac{\det(\mathbf{u}_\zeta, \mathbf{u}_{\zeta\zeta})^2 - \det(\mathbf{v}_\zeta, \mathbf{v}_{\zeta\zeta})^2}{|\mathbf{u}_\zeta|^5} + \det(\mathbf{v}_\zeta, \mathbf{v}_{\zeta\zeta})^2 \left(\frac{1}{|\mathbf{u}_\zeta|^5} - \frac{1}{|\mathbf{v}_\zeta|^5} \right).$$

The first and the second terms of the right-hand side are calculated as

$$\begin{aligned} \det(\mathbf{u}_\zeta, \mathbf{u}_{\zeta\zeta})^2 - \det(\mathbf{v}_\zeta, \mathbf{v}_{\zeta\zeta})^2 &= [\det(\mathbf{u}_\zeta, \mathbf{u}_{\zeta\zeta}) + \det(\mathbf{v}_\zeta, \mathbf{v}_{\zeta\zeta})] \times [v_{2,\zeta\zeta}(u_{1,\zeta} - v_{1,\zeta}) - v_{1,\zeta\zeta}(u_{2,\zeta} - v_{2,\zeta}) \\ &\quad - u_{2,\zeta}(u_{1,\zeta\zeta} - v_{1,\zeta\zeta}) + u_{1,\zeta}(u_{2,\zeta\zeta} - v_{2,\zeta\zeta})], \end{aligned}$$

and

$$\frac{1}{|\mathbf{u}_\zeta|^5} - \frac{1}{|\mathbf{v}_\zeta|^5} = - \frac{|\mathbf{u}_\zeta|^{10} - |\mathbf{v}_\zeta|^{10}}{|\mathbf{u}_\zeta|^5 |\mathbf{v}_\zeta|^5 (|\mathbf{u}_\zeta|^5 + |\mathbf{v}_\zeta|^5)}$$

²URL: https://www.youtube.com/playlist?list=PLMF3dSqWEii691oXvCtgDCI4PYijq_4L3

$$= -\frac{\sum_{k=0}^4 |\mathbf{u}_\zeta|^{8-2k} |\mathbf{v}_\zeta|^{2k}}{|\mathbf{u}_\zeta|^5 |\mathbf{v}_\zeta|^5 (|\mathbf{u}_\zeta|^5 + |\mathbf{v}_\zeta|^5)} (\mathbf{u}_\zeta + \mathbf{v}_\zeta) \cdot (\mathbf{u}_\zeta - \mathbf{v}_\zeta),$$

respectively. Although we can derive partial derivatives of G_1 with these equations, we omit them.

Before presenting numerical examples, we recall the steady-state solutions for the elastic flow. It is known that steady closed curves of the elastic energy (3.36) are the circle of radius ε , the figure-eight-shaped curve with scale ε , and their multiple versions (see Figure 3.3 and [83, 96]). Their energies are

$$E[\text{circle}] = 4\pi\varepsilon, \quad E[\text{eight-shaped}] \approx \varepsilon \cdot 21.2075,$$

respectively. The exact value of the latter energy is expressed by the elliptic integrals (cf. [90]).

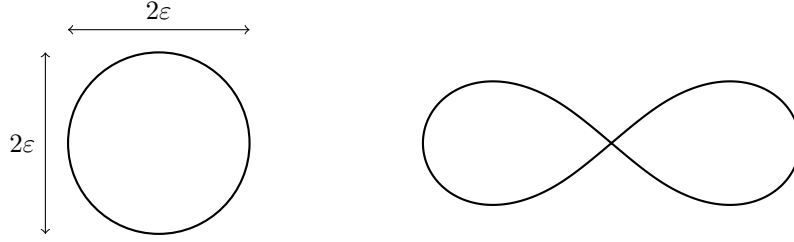


Figure 3.3: Steady states of the elastic energy (3.36).

In our numerical experiences, we calculated numerical solutions until the given maximum time T_{\max} except for the latter part of Example 3.2. For every examples, we solved the nonlinear equation (3.35) by the usual Newton method at each time step. In each Newton method, we stopped the iteration if the relative increment of the sequence of the Newton method is smaller than 10^{-10} , i.e.,

$$\max_i \frac{\|\mathbf{P}_i^{n,(k)} - \mathbf{P}_i^{n,(k-1)}\|_\infty}{\varepsilon} \leq 10^{-10},$$

where $\{\mathbf{P}_i^{n,(k)}\}_k$ is the sequence of candidates of control points expressing the circle \mathbf{u}^n generated by the Newton method with the initial guess $\mathbf{P}_i^{n,(0)} = \mathbf{P}_i^{n-1}$, and $\|\cdot\|_\infty$ is the maximum norm in \mathbb{R}^2 . Here we divide the increment $\|\mathbf{P}_i^{n,(k)} - \mathbf{P}_i^{n,(k-1)}\|_\infty$ by ε since the scale of the steady state is ε . We also apply the similar idea when we determine if the numerical solution is steady state in Example 3.2.

Example 3.2. We first show examples with circular initial curves. In this case, the exact solution is a circle at every time, and the steady state is the circle of radius ε as mentioned above. Figure 3.4(a) shows the evolution of the curve at $t \approx 0, 0.1, \dots, 0.6 = T_{\max}$, with parameters

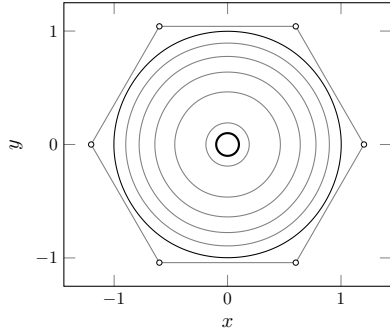
$$p = 3, \quad \varepsilon = 0.1, \quad N = 6, \quad \Delta t = 0.01,$$

where the initial curve is the L^2 -projection of the unit circle. We can observe that six control points are sufficient to express a circle by a B-spline curve. The energy at $t \approx 0.6$ is $E \approx 1.2583$, which approximates the exact value of the energy at the steady state $4\pi\varepsilon \approx 1.2566$. Figure 3.4(b) shows the evolution of the energy. The discrete energy dissipation property is clearly visible. In Figure 3.4(a), one can observe that the curve shrinks like the curvature flow (3.7) until $t \approx 0.5$, and it stops shrinking when the radius approaches ε .

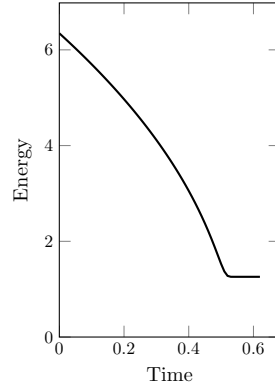
We here investigate the effect of the degree of B-splines p to the accuracy. We computed the numerical solution whose initial curve is the unit circle with $\varepsilon = 0.5$ and $\Delta t = 0.01$ until the solution settles into the steady state. We stopped time evolution when the relative increment of the control points is sufficiently small. More precisely, we determined that the solution was the steady state if

$$\max_i \frac{\|\mathbf{P}_i^n - \mathbf{P}_i^{n-1}\|_\infty}{\varepsilon} \leq \varepsilon_M,$$

where $\{\mathbf{P}_i^n\}_i$ is the set of control points of the circle \mathbf{u}^n and $\varepsilon_M \approx 10^{-16}$ is the machine epsilon. We divided the increment $\|\mathbf{P}_i^n - \mathbf{P}_i^{n-1}\|_\infty$ by ε by the same reason stated above.



(a) Evolution of the circle. The outermost curve and the innermost one are at times $t = 0$, and $t \approx 0.6$, respectively.



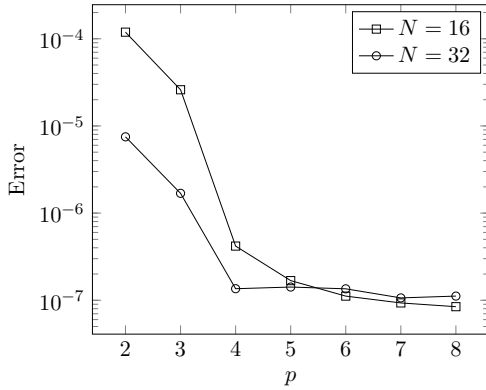
(b) Evolution of the energy.

Figure 3.4: Example 3.2.

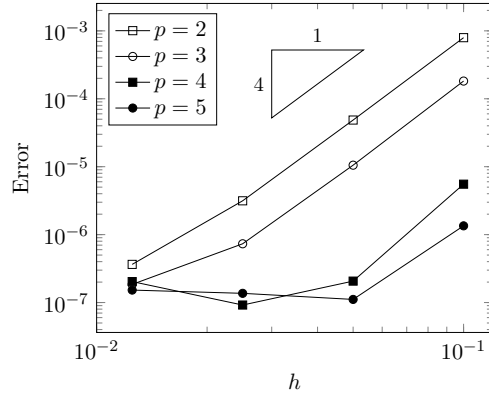
We varied the parameters N and p as $N = 16, 32$ and $p = 2, 3, \dots, 8$, and for each numerical steady state, we calculated the maximum distance from the origin, which we denote by $d_{p,N}$. The result is shown in Figure 3.5(a), which illustrates the error $|d_{p,N} - \varepsilon|$. Moreover, in order to investigate the convergence rates numerically, we computed the numerical solution for $p = 2, 3, 4, 5$ and $N = 10, 20, 40, 80$, and then calculated $d_{p,N}$. The result is plotted in Figure 3.5(b).

From the approximation theory for B-spline functions (see e.g. [95, Theorem 6.25]), we can expect that the convergence rate is $O(h^{p+1})$. Actually, we can see that the error is small for large p . However, we cannot observe that the error decays exponentially with respect to p from Figure 3.5(a). Moreover, for $N = 32$, the error does not decrease for $p \geq 4$ and it remains around $1.0 \cdot 10^{-7}$.

We can not determine the convergence rate from Figure 3.5(b). For both $p = 2$ and $p = 3$, it behaves as $O(h^4)$. Thus something like superconvergence phenomena may be triggered for $p = 2$. For $p \geq 4$ and large N , errors are remaining around $1.0 \cdot 10^{-7}$ as in Figure 3.5(a). We expect that we cannot make the solution more accurate due to accumulation of errors by numerical quadratures and the Newton method.



(a) p v.s. error



(b) h v.s. error

Figure 3.5: Behavior of errors.

From the above results, reasonable choice of p may be $p = 2, 3, 4$ since the solution for $p \geq 5$ is not more accurate than that of $p = 4$ in our environment. Furthermore, in view of cost of computation, we should not make p so large. Indeed, while the size of the matrix appearing in the newton method is independent of p and is $2N$ (see Definition 3.2), its bandwidth depends on p and is $2p + 1$. Thus, in order to balance accuracy and cost, we choose $p = 3$ for the examples in the rest of this section.

In the following examples, we can observe that the energy decreases drastically and that the control points get too close. When these phenomena occurred, we observed that the Newton method failed. To overcome these problems, we adopt adaptive time steps and we remove the control points if adjacent

control points are too close. We do not insert control points to sparse part since B-spline curves can be represented by a few control points. On the other hand, if adjacent control points are too close, then the curve \mathbf{u}_h behaves like a constant function around the corresponding parameter ζ . Thus the derivative $|\partial_\zeta \mathbf{u}_h|$ is too small in the neighborhood of ζ and it may make the equation (3.35) degenerate. That is, there may exist zeros in the diagonal components of the matrix. Hence we apply elimination of control points rather than insertion.

Now we determine appropriate time increment. When the energy is decreasing rapidly, the time increment should be much smaller than the inverse of the gradient of the energy. Thus we define the time increment by

$$\Delta t_n = \min \left\{ \tau, \left| \frac{E[\mathbf{u}_h^n] - E[\mathbf{u}_h^{n-1}]}{\Delta t_{n-1}} \right|^{-2} \right\} \quad (3.37)$$

for $n \geq 1$ with given maximum time increment τ . For $n = 0$, we need another consideration to determine the initial time increment. We should make Δt_0 much smaller than $\left| \frac{d}{dt} E[\mathbf{u}(t)] \right|^{-1}$ with small t . As observed before, we have

$$\frac{d}{dt} E[\mathbf{u}(t)] = - \int \left| -2\varepsilon^2 \left(\nabla_s^2 \boldsymbol{\kappa} + \frac{1}{2} |\boldsymbol{\kappa}|^2 \boldsymbol{\kappa} \right) + \boldsymbol{\kappa} \right|^2 ds.$$

Moreover, as $\varepsilon \downarrow 0$, the solution of the elastic flow converges to that of the curvature flow in short time intervals (see [11]). Therefore, we can approximate the gradient of the energy as

$$\frac{d}{dt} E[\mathbf{u}(t)] \approx - \int |\boldsymbol{\kappa}|^2 ds,$$

provided that ε and t is sufficiently small. According to this observation, we define Δt_0 by

$$\Delta t_0 = \min \left\{ \tau, \left| \int |\boldsymbol{\kappa}(\mathbf{u}_h^0)|^2 ds(\mathbf{u}_h^0) \right|^{-2} \right\}$$

in analogy with (3.37).

We also have to present a criterion for elimination of control points. We can observe, in the following examples, that the Newton method fails if the adjacent control points are too close. At the same time, the (local) scale of the solution curve is approximately ε . That is, the typical size of the part of the curve condensing control points is ε . Therefore, if the control points are uniformly distributed, the distance between two adjacent points are $O(\varepsilon)$. Hence, we remove control points when the distance is smaller than ε^2 , which is much smaller than ε . Note that the energy may increase when control points are eliminated; however, the shape of the curve will be less affected (see Figure 3.6). We here try to justify this operation. In the following examples, initial curves have complicated shapes, which require lots of control points. On the other hand, the steady state are circular and figure-eight-shaped, which are quite simple and require less numbers of control points. Therefore, we should reduce the number of control points throughout the evolution of curves. That is, our elimination procedure may be reasonable in view of shapes of curves.

Example 3.3. The second example is shown in Figure 3.7. The initial curve is figure-eight-shaped. The parameters are

$$\varepsilon = 0.2, \quad N = 12, \quad \tau = 0.01.$$

In this case, the elimination of control points did not occur. Figure 3.7(a) shows the evolution of the curve at $t \approx 0, 0.2, \dots, 1.2$ and Figure 3.7(b) shows the evolution of the energy. The energy at $t \approx 1.2$ is $E \approx 4.2433$, which approximates the exact value of the energy at the steady state ≈ 4.2415 .

In Figure 3.7(a), initially the small loop (the right loop) shrinks faster than the larger one. When the scale of the right loop becomes ε , the loop stops shrinking, and the left one begins to shrink. Finally, the left one also stops shrinking, and the curve approaches the steady state.

Example 3.4. The third example is shown in Figure 3.8. The initial shape of the curve is a cardioid-like curve as shown in Figure 3.8(a). The initial parameters of the curve are

$$\varepsilon = 0.1, \quad N = 12, \quad \tau = 0.005.$$

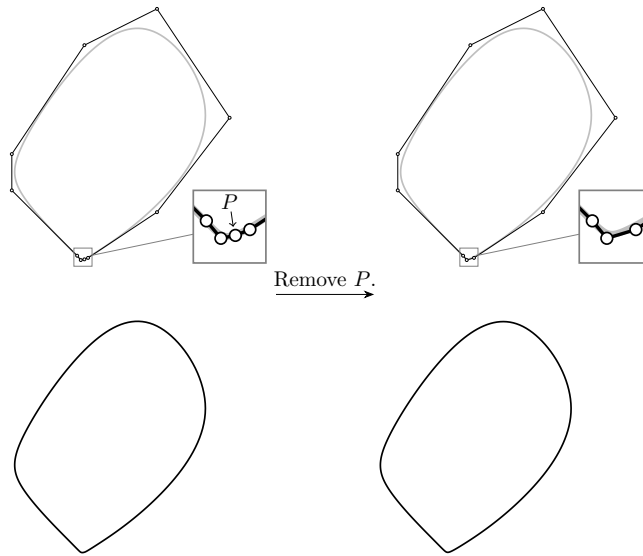


Figure 3.6: Effect of eliminating control points. The left two figures show B-spline curves of degree $p = 3$ with respect to the control points as shown in the upper figure. The right two figures show the B-spline curves of the same degree with respect to the control points except for the point P .

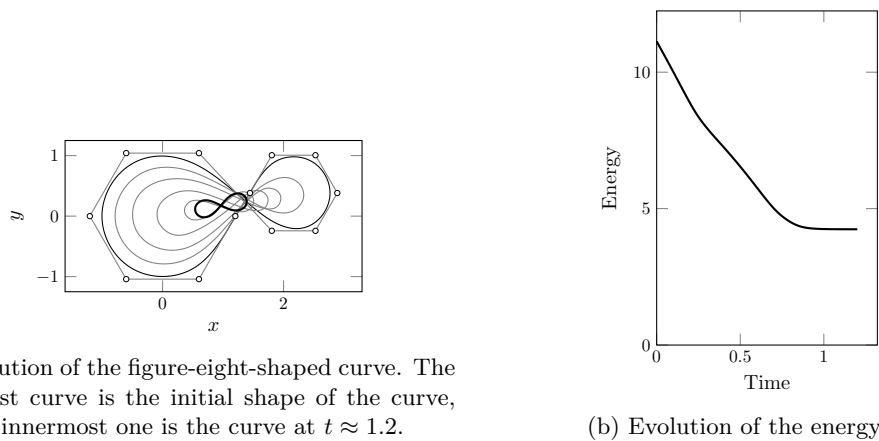
As in the previous cases, the elimination of control points did not occur. Figure 3.8 shows the evolution of the curve at $t \approx 0, 0.2, 0.4, 0.6$ and Figure 3.9 shows the evolution of the energy. In this case, the steady state is a double-looped circle with radius $\varepsilon = 0.1$. Therefore, the energy of the solution at $t \approx 0.6$ ($E \approx 2.5228$) is approximately twice the value of that of Example 3.2.

The behavior of the curve is similar to Example 3.3. That is, initially the smaller loop shrinks until the scale is approximately ε . Then, the larger one shrinks and the curve approaches the steady state.

Example 3.5. This example shows a topology-changing solution. The initial curve is shown in Figure 3.10(a), and Figure 3.10 shows its evolution. Figures 3.11(a) and (b) illustrate the evolution of the energy and the number of control points, respectively. The parameters are

$$\varepsilon = 0.2, \quad N = 20 \text{ (initially)}, \quad \tau = 0.005.$$

One can observe that the topology of the curve changes at around $t = 1.05$ (Figures 3.10(d) and (e)). At the same time, the energy decreases drastically (Figure 3.11(a)), and some control points become



(a) Evolution of the figure-eight-shaped curve. The outermost curve is the initial shape of the curve, and the innermost one is the curve at $t \approx 1.2$.

(b) Evolution of the energy.

Figure 3.7: Example 3.3.

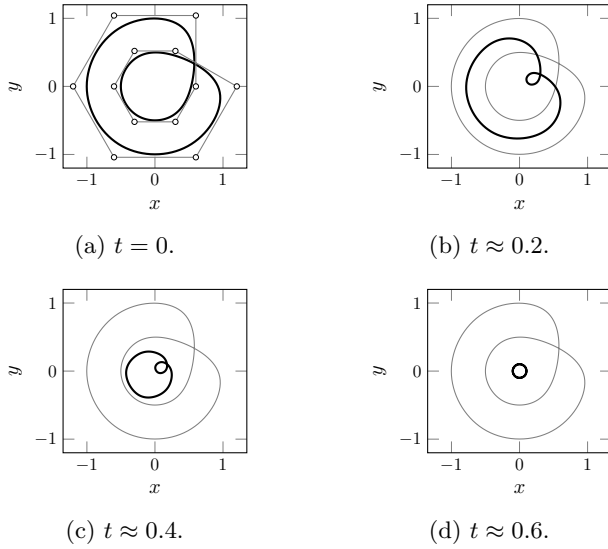


Figure 3.8: Evolution of the curves in Example 3.4.

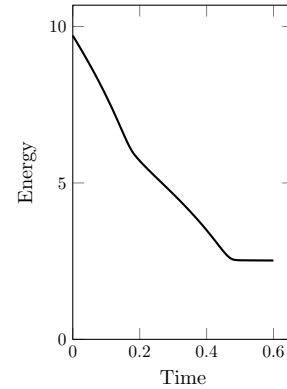


Figure 3.9: Evolution of the energy in Example 3.4.

dense (Figure 3.12). As mentioned earlier, we implement an algorithm that deletes a control point if it is too close to the adjacent point. Therefore, the elimination of control points occurs when the topology changes, and the number of control points finally converges to $N = 10$.

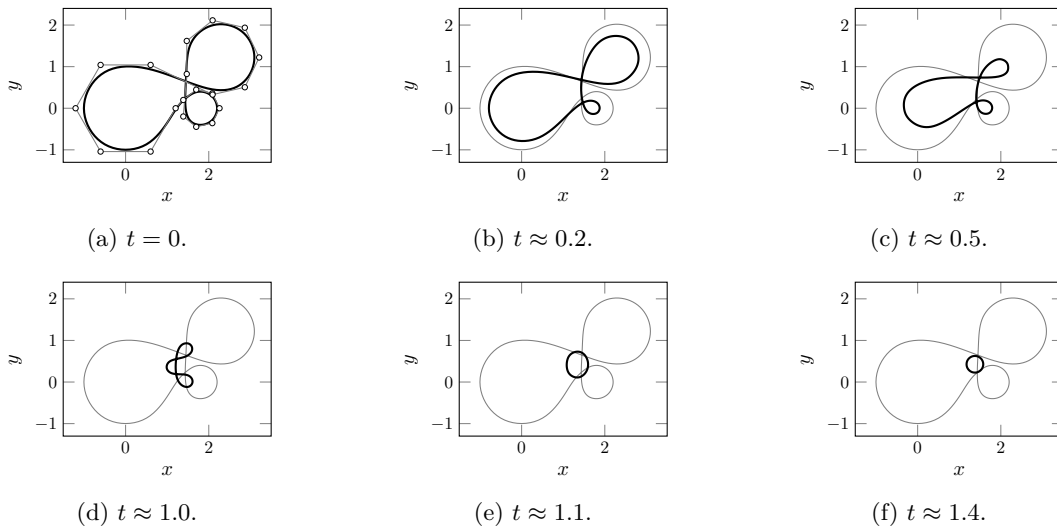


Figure 3.10: Evolution of the curves in Example 3.5.

Example 3.6. The following two examples investigate problems with more complicated solutions. The initial shape of the curve is shown in Figure 3.13(a), and Figure 3.13 shows its evolution. Figures 3.14(a) and (b) illustrate the evolution of the energy and the number of control points, respectively. The parameters are

$$\varepsilon = 0.2, \quad N = 49 \text{ (initially)}, \quad \tau = 0.005.$$

In this example, the topology of the curve changes frequently. For example, the loop in the upper left of the curve disappears at around $t = 1.35$. Since the turning number of the initial curve is zero, we can easily determine before computation that the steady state is a figure-eight-shaped curve. However, the evolution of the curve is quite complicated so that we cannot predict the behavior. When the topology changes, the energy decreases rapidly as in Example 3.5, and the number of control points also decreases at the same time. The final value of N is $N = 12$.

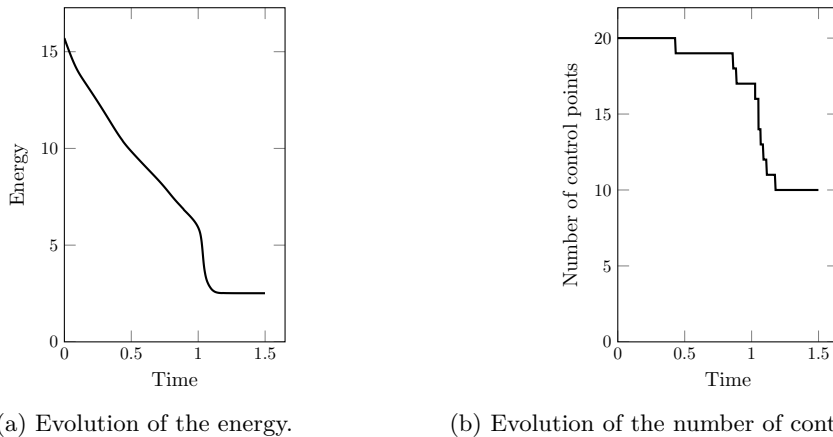


Figure 3.11: Example 3.5.

Example 3.7. This is the last example. The initial curve for the final example is shown in Figure 3.15(a), and Figure 3.15 shows its evolution. Figures 3.16(a) and (b) illustrate the evolution of the energy and the number of control points, respectively. The parameters are

$$\varepsilon = 0.2, \quad N = 54 \text{ (initially)}, \quad \tau = 0.005.$$

The solution displays complicated behavior as in Example 3.6, and the topology changes frequently. Since the turning number of the initial curve is one, the steady state is a circle with radius ε . However, as in the previous example, the evolution is too complicated to predict. The energy and the number of control points decrease drastically when the topology changes, and the final number of control points is $N = 9$.

3.5 Concluding remarks

In the present chapter, we presented a new framework to construct energy dissipative numerical schemes for gradient flows of planar curves. The features of our scheme are:

- We can guarantee the discrete energy dissipation property for any gradient flows.
- We can reduce DOF since B-spline curves requires few DOF.

Owing to these features, our scheme seems to be effective. However, we did not discuss the theoretical convergence rate of our scheme and we could not determine the rate numerically for the circular example. Therefore, the theoretical analysis, such as stability and error estimates, are the most important topics that should be treated for the future.

Our method can be extended to gradient flows of curves in \mathbb{R}^3 directly. We can also extend the method to gradient flows of evolving surfaces in \mathbb{R}^3 in principle by using NURBS surfaces [39, 89]. However, it is not trivial how to choose appropriate patches (i.e., local coordinates) if the surfaces cannot be represented as graphs of functions.

We presented stable numerical examples for the elastic flow. Since we did not impose any conditions for the distribution of control points, dense and sparse parts were observed, which may trigger the failure of the Newton method. In order to overcome the problem, we removed control points in a certain criterion. Although this operation may be justified in view of shapes of curves, it may break the dissipation property. Thus we should study how to control the distribution of control points.

In this chapter, we addressed gradient flows without constraints. Nevertheless, there are important geometric equations with constraints, such as the area-preserving curvature flow and the length-preserving Willmore flow. Of course, our method cannot be applied these equations. However, our strategy may be effective. Namely, we can derive a dissipative scheme by discretizing the chain rule and other ingredients of the proof of dissipativity, such as the Lagrange multiplier. Other than constraints of geometric quantities, topology-preserving flows are also interesting. These problems remain an area for future work.

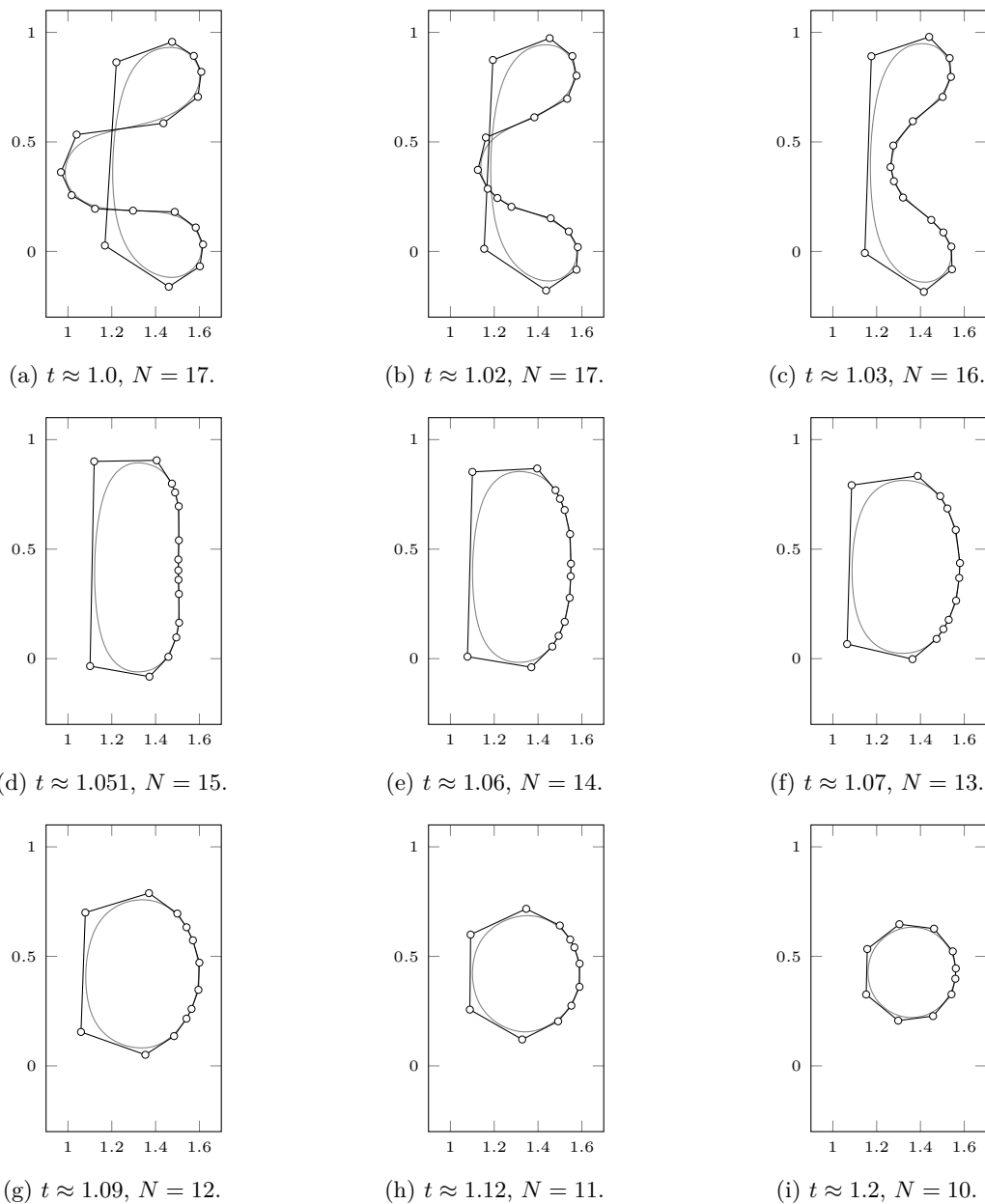


Figure 3.12: Diminution of control points in Example 3.5 at the times described as in the figures. The small circles represent control points, N is the number of control points, and the gray colored curve is the solution.

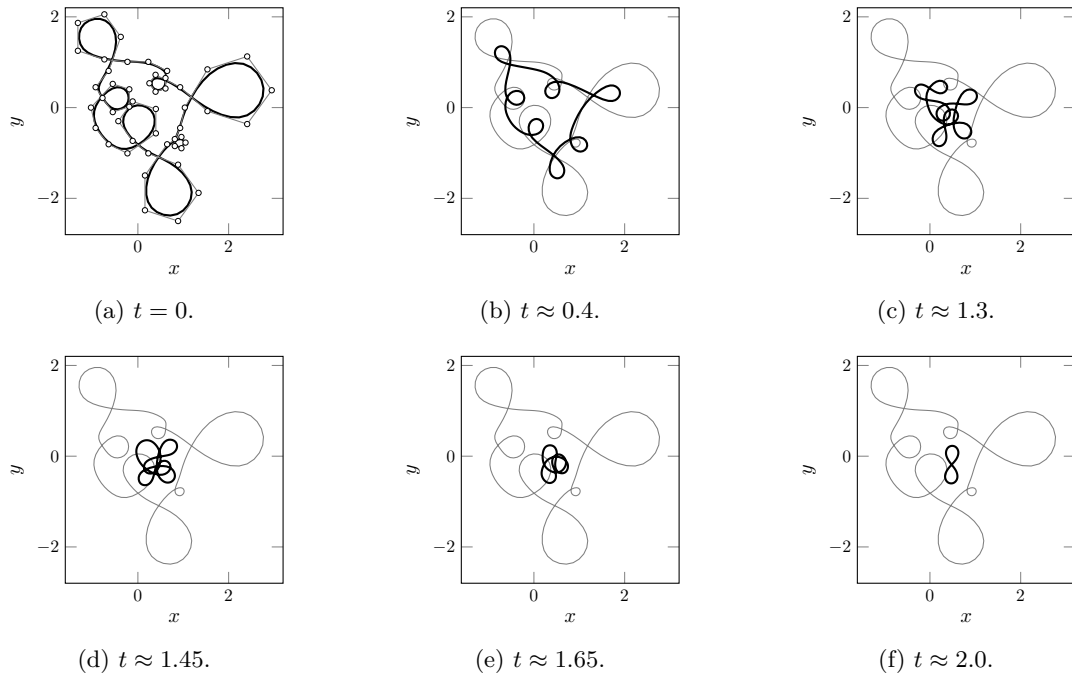
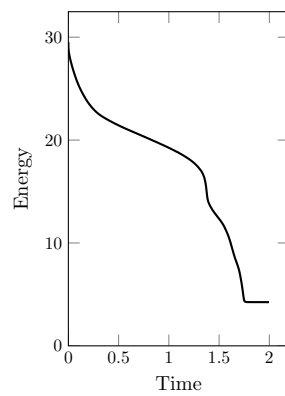
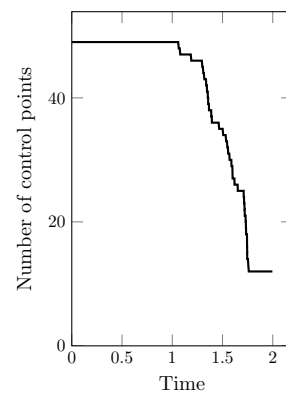


Figure 3.13: Evolution of the curves in Example 3.6.



(a) Evolution of the energy.



(b) Evolution of the number of control points.

Figure 3.14: Example 3.6.

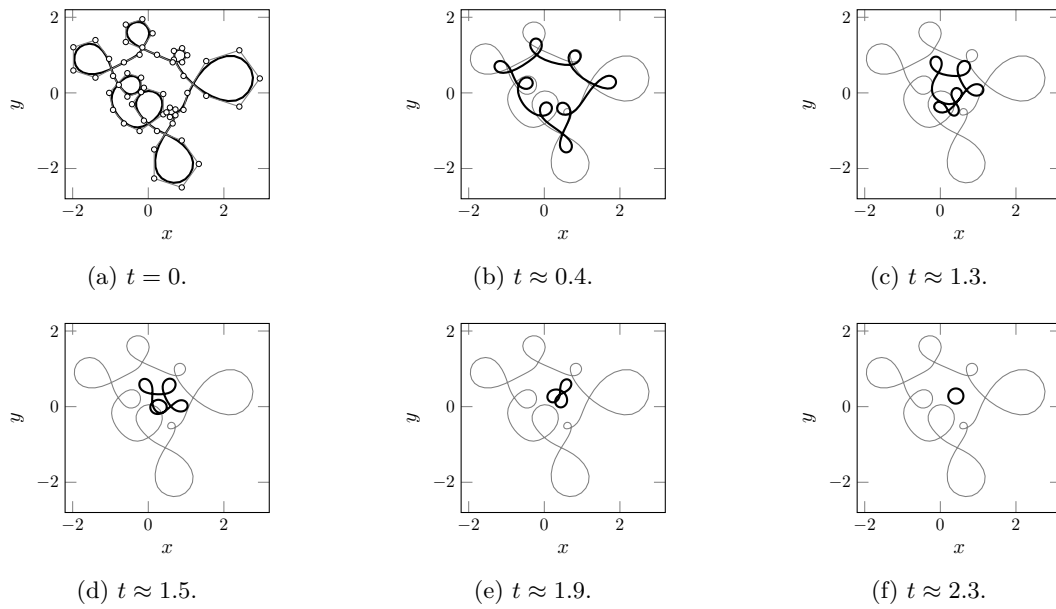
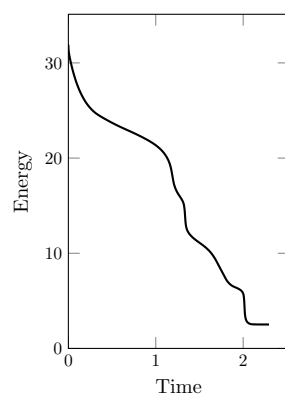
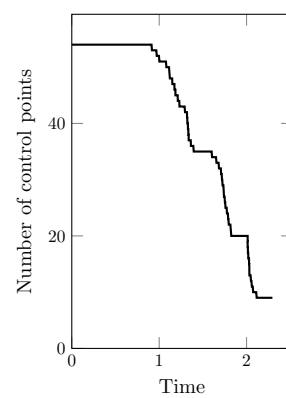


Figure 3.15: Evolution of the curves in Example 3.7.



(a) Evolution of the energy.



(b) Evolution of the number of control points.

Figure 3.16: Example 3.7.

Chapter 4

On the finite element approximation for non-stationary saddle-point problems

Abstract

In this chapter, we present a numerical analysis of the hydrostatic Stokes equations, which are linearization of the primitive equations describing the geophysical flows of the ocean and the atmosphere. The hydrostatic Stokes equations can be formulated as an abstract non-stationary saddle-point problem, which also includes the non-stationary Stokes equations. We first consider the finite element approximation for the abstract equations with a pair of spaces under the discrete inf-sup condition. The aim of this chapter is to establish error estimates for the approximated solutions in various norms, in the framework of analytic semigroup theory. Our main contribution is an error estimate for the pressure with a natural singularity term t^{-1} , which is induced by the analyticity of the semigroup. We also present applications of the error estimates for the finite element approximations of the non-stationary Stokes and the hydrostatic Stokes equations.

This chapter is based on the published paper [67]:

- T. Kemmochi. On the finite element approximation for non-stationary saddle-point problems. *Japan J. Indust. Appl. Math.*, in press. DOI: <https://doi.org/10.1007/s13160-017-0293-5>

4.1 Introduction

Let $\Omega = (0, 1)^2 \times (-D, 0) \subset \mathbb{R}^3$ be a (shallow) box domain with $D > 0$. We consider the (non-stationary) hydrostatic Stokes equations

$$\begin{cases} u_t - \Delta u + \nabla_H p = 0, & \text{in } \Omega \times (0, T), \\ \operatorname{div}_H \bar{v} = 0, & \text{in } \Omega \times (0, T), \\ u(0) = u_0, & \text{in } \Omega \end{cases} \quad (4.1)$$

for an unknown velocity $u: \Omega \rightarrow \mathbb{R}^2$ and pressure $p: G := (0, 1)^2 \rightarrow \mathbb{R}$, where $\nabla_H p = (\partial_x p, \partial_y p)^T$, $\operatorname{div}_H v = \partial_x v_1 + \partial_y v_2$, and $\bar{v} = \int_{-D}^0 v(\cdot, z) dz$. We impose the boundary conditions

$$\begin{cases} \partial_z u = 0, & \text{on } \Gamma_u := G \times \{0\}, \\ u = 0, & \text{on } \Gamma_b := G \times \{-D\}, \\ u \text{ and } p \text{ are periodic} & \text{on } \Gamma_l := \partial G \times (-D, 0). \end{cases} \quad (4.2)$$

These are the linearized equations of the primitive equations (without the Coriolis force) described as

$$\begin{cases} \partial_t u + (U \cdot \nabla)u - \Delta u + \nabla_H p = 0, & \text{in } \Omega \times (0, T), \\ \partial_z p = 0, & \text{in } \Omega \times (0, T), \\ \operatorname{div} U = 0, & \text{in } \Omega \times (0, T), \\ u(0) = u_0, & \text{in } \Omega \end{cases} \quad (4.3)$$

with boundary conditions

$$\begin{cases} \partial_z u = 0, u_3 = 0, & \text{on } \Gamma_u, \\ U = 0, & \text{on } \Gamma_b, \\ U \text{ and } p \text{ are periodic} & \text{on } \Gamma_l, \end{cases}$$

where $U = (u, u_3): \Omega \rightarrow \mathbb{R}^3$. The primitive equations are derived from the Navier-Stokes equations under the assumption that the vertical motion is much smaller than the horizontal motion, and were first introduced by Lions, Temam, and Wang [79, 80, 81]. This model is considered to describe the geophysical flows of the ocean and the atmosphere.

In this chapter, we consider the finite element approximation of the hydrostatic Stokes problem (4.1) and (4.2) as follows. Find $u_h: (0, T) \rightarrow V_h$ and $p_h: (0, T) \rightarrow Q_h$ satisfying the variational equation

$$\begin{cases} (u_{h,t}, v_h)_\Omega + (\nabla u_h, \nabla v_h)_\Omega - (\operatorname{div}_H \bar{v}_h, p_h)_G = 0, & \forall v_h \in V_h, \\ (\operatorname{div}_H \bar{u}_h, q_h)_G = 0, & \forall q_h \in Q_h, \\ (u_h(0), v_h)_\Omega = (u_0, v_h)_\Omega, & \forall v_h \in V_{h,\sigma}, \end{cases}$$

where $V_h \subset H^1(\Omega)^2$ and $Q_h \subset L_0^2(G)$ are finite-dimensional subspaces with suitable boundary conditions, the bracket $(\cdot, \cdot)_X$ expresses the L^2 -inner product over a domain X , and $V_{h,\sigma} = \{v_h \in V_h \mid (\operatorname{div}_H \bar{v}_h, q_h)_G = 0, \forall q_h \in Q_h\}$. The precise definitions are given in Section 4.4. The aim of this chapter is to establish error estimates for u_h and p_h in the framework of analytic semigroup theory, as preliminaries for the numerical analysis of the primitive equations. Although there are several results available on the finite element method for the steady hydrostatic Stokes equations (e.g., [51, 52, 53, 54]), there are no results for the non-stationary case, to the best of our knowledge.

As with the Navier-Stokes equations, the primitive equations are widely used in numerical computations for atmospheric and oceanic phenomena. Finite element approximations and error estimates are presented in [22, 55, 20, 19] for the steady primitive equations and in [21, 59, 56] for non-stationary problems. In [59] and [56], error estimates for various fully discretized schemes are provided. These estimates have exponential growth in time T (i.e., e^{cT}), since their arguments are based on the discrete Gronwall inequality. Therefore, these results are time-local estimates in a sense.

We are interested in time-global error estimates for finite element approximations of the hydrostatic Stokes and the primitive equations. In contrast to the three-dimensional Navier-Stokes equations, it is known that the three-dimensional primitive equations are globally well-posed in the L^p -settings (see [18] for $p = 2$, and [61] for $p \in (1, \infty)$). In their proofs, the analyticity of the hydrostatic Stokes semigroup plays a crucial role. We can thus expect that the analytic semigroup approach presents an efficient method for the numerical analysis of the primitive equations. Indeed, for the two-dimensional Navier-Stokes equations, time-global error estimates have been obtained via the analytic semigroup approach in [86]. Their results are based on the error estimates for the non-stationary Stokes equations established in [87].

In order to derive error estimates for the finite element approximation, we formulate the hydrostatic Stokes equations as an abstract evolution problem. We define

$$V = \{v \in H^1(\Omega)^2 \mid v|_{\Gamma_b} = 0, v \text{ is periodic on } \Gamma_l\}. \quad (4.4)$$

Additionally, let $Q = L_0^2(G)$ and $H = L^2(\Omega)^2$. Then, a weak form of the hydrostatic Stokes equations (4.1) can be given as a non-stationary saddle-point problem as follows. Find $u: (0, T) \rightarrow V$ and $p: (0, T) \rightarrow Q$ satisfying

$$\begin{cases} (u_t(t), v)_H + a(u(t), v) + b(v, p(t)) = 0, & \forall v \in V, \\ b(u(t), q) = 0, & \forall q \in Q, \end{cases} \quad (4.5)$$

where

$$a(u, v) = \iiint_{\Omega} \nabla u : \nabla v \, dx dy dz, \quad b(v, q) = - \iint_G (\operatorname{div}_H \bar{v}) q \, dx dy$$

for $u, v \in V$ and $q \in Q$. Once we write the hydrostatic Stokes equations as above, we can use the same arguments for error estimates as is the case for the usual Stokes problem in [87]. Then, we can obtain the following error estimates for the velocity:

$$\begin{aligned} \|\nabla(u(t) - u_h(t))\|_{L^2(\Omega)} &\leq Ch t^{-1} \|u_0\|_{L^2(\Omega)}, \\ \|u(t) - u_h(t)\|_{L^2(\Omega)} &\leq Ch^2 t^{-1} \|u_0\|_{L^2(\Omega)}, \end{aligned}$$

under the assumptions stated in Section 4.2 (Assumptions 4.1 and 4.2). In particular, the discrete inf-sup condition (4.11) plays an important role as in the stationary case.

According to [87], we can also obtain an error estimate for the pressure. The estimate presented in [87] is

$$\|p(t) - p_h(t)\|_{L^2} \leq Ch(t^{-1} + t^{-3/2}) \|u_0\|_{L^2}$$

for the two-dimensional Stokes equations

$$\begin{cases} u_t - \Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(0) = u_0 \end{cases} \quad (4.6)$$

with the Dirichlet boundary condition. However, the singularity $t^{-3/2}$ is unnatural from the viewpoint of analytic semigroup theory. Indeed, an optimal order error estimate should be of the form

$$\|p(t) - p_h(t)\|_{L^2} \leq Ch \|\nabla p(t)\|_{L^2},$$

and (4.6) yields

$$\|\nabla p(t)\|_{L^2} \leq \|u_t(t)\|_{L^2} + \|\Delta u(t)\|_{L^2} \leq Ct^{-1} \|u_0\|_{L^2}.$$

In this chapter, we first address the abstract problem (4.5) for bilinear forms $a: V \times V \rightarrow \mathbb{R}$ and $b: V \times Q \rightarrow \mathbb{R}$ defined on Hilbert spaces V and Q , and its Galerkin approximation

$$\begin{cases} (u_{h,t}(t), v_h)_H + a(u_h(t), v_h) + b(v_h, p_h(t)) = 0, & \forall v_h \in V_h, \\ b(u_h(t), q_h) = 0, & \forall q_h \in Q_h, \end{cases} \quad (4.7)$$

for appropriate finite-dimensional subspaces $V_h \subset V$ and $Q_h \subset Q$. The main contribution of this chapter is to derive error estimates for the approximation problem (4.7), both for the velocity u and the pressure p . In particular, we remove the term $t^{-3/2}$ from the error estimate for the pressure. Consequently, our results are a generalization and modification of the results in [87]. After deriving the error estimates for (4.7), we apply the results to error estimates for the finite element approximation of the hydrostatic Stokes equations.

Here, we present the idea of the proof for error estimates in the case of the hydrostatic Stokes equations. Our strategy is similar to that of [87]. Namely, we first rewrite the error in terms of the contour integral of the resolvent and then we reduce the error estimate for the velocity to that of the resolvent problem. The key idea is to establish the V' -error estimate for the resolvent problem as well as the H^1 - and L^2 -error estimates, which are already addressed in [87]. Here, V' denotes the dual space of V defined by (4.4). This estimate coincides with the H^{-1} -error estimate if we impose the Dirichlet boundary condition. Then, we can obtain the V' -error estimate for the time derivative of the velocity of the form

$$\|u_t(t) - u_{h,t}(t)\|_{V'} \leq Ch t^{-1} \|u_0\|_{L^2(\Omega)}.$$

Finally, we can establish an error estimate for the pressure without the term $t^{-3/2}$, with the aid of the discrete inf-sup condition. We shall perform the above procedure in the abstract setting.

The rest of the chapter is organized as follows. In Section 4.2, we introduce an abstract saddle-point problem and its Galerkin approximation, as well as the notation and assumptions in subsection 4.2.1. After that, we introduce the resolvent problems and present some preliminary results in subsection 4.2.2. Our main results are presented in Section 4.3. As mentioned above, we derive the error estimate in the

dual norm for the resolvent problem, and then we establish the error estimate for the evolution equation. We apply these results for the Stokes and the hydrostatic Stokes equations in Section 4.4. For the usual Stokes equations (subsection 4.4.1), error estimates for the velocity are already available. However, the estimate for the pressure presented here is strictly sharper than that of [87]. The error estimates for the non-stationary hydrostatic Stokes equations will be presented in subsection 4.4.2. Finally, we present our conclusions and areas for future works in Section 4.5.

4.2 Preliminaries

4.2.1 Notation and assumptions

Throughout this chapter, except for in the last section, the symbols H , V , and Q denote Hilbert spaces with dense and continuous injections $V \hookrightarrow H \hookrightarrow V'$, where V' is the dual space of V , and $a: V \times V \rightarrow \mathbb{C}$ and $b: V \times Q \rightarrow \mathbb{C}$ are continuous bilinear forms. We assume that a is symmetric for simplicity, and that a is coercive and b satisfies the inf-sup condition:

$$\begin{aligned} \operatorname{Re} a(v, v) &\geq \alpha \|v\|_V, \quad \forall v \in V, \\ \sup_{v \in V} \frac{|b(v, q)|}{\|v\|_V} &\geq \beta_1 \|q\|_Q, \quad \forall q \in Q, \end{aligned} \quad (4.8)$$

for some positive constants α and β_1 . We consider the following abstract non-stationary Stokes problem:

$$\begin{cases} (u_t(t), v)_H + a(u(t), v) + b(v, p(t)) = 0, & \forall v \in V, \\ b(u(t), q) = 0, & \forall q \in Q, \\ u(0) = u_0, \end{cases} \quad (4.9)$$

for $t \in (0, T)$, where $u_0 \in H_\sigma := \overline{V_\sigma}^{\|\cdot\|_H}$ and

$$V_\sigma := \{v \in V \mid b(v, q) = 0, \forall q \in Q\}.$$

We define a linear operator A on H_σ associated with the bilinear form a by

$$\begin{cases} D(A) = \{u \in V_\sigma \mid \exists w \in H_\sigma \text{ s.t. } (w, v)_H = a(u, v), \forall v \in V_\sigma\}, \\ (Au, v)_H = a(u, v), \quad \forall u \in D(A), \quad \forall v \in V_\sigma, \end{cases} \quad (4.10)$$

which is the abstract version of the Stokes operator. By coercivity (4.8) and the usual semigroup theory (e.g., see [88]), the operator $-A$ generates an analytic contraction semigroup e^{-tA} on H_σ . Thus, choosing $v \in V_\sigma$ as a test function in (4.9), we can construct a mild solution by $u(t) = e^{-tA}u_0$ for $t > 0$. Moreover, owing to the inf-sup condition and the closed range theorem (see, e.g., [38]), we can find $p(t) \in Q$ that satisfies

$$b(v, p(t)) = -(u_t(t), v)_H + a(u(t), v), \quad \forall v \in V,$$

for almost all $t \in (0, T)$. The uniqueness of these solutions is clear. Finally, we define a linear operator $B: D(B) \subset Q \rightarrow H$ associated with b by

$$\begin{cases} D(B) = \{q \in Q \mid \exists w \in H \text{ s.t. } (w, v)_H = b(v, q), \forall v \in V\}, \\ (Bq, v)_H = b(v, q), \quad \forall q \in D(B), \quad \forall v \in V. \end{cases}$$

We next consider the Galerkin approximation for (4.9). Let $V_h \subset V$ and $Q_h \subset Q$ be finite-dimensional subspaces. We assume that they have the following properties:

Assumption 4.1. (A-1) [discrete inf-sup condition] There exists $\beta_2 > 0$ such that

$$\sup_{v_h \in V_h} \frac{|b(v_h, q_h)|}{\|v_h\|_V} \geq \beta_2 \|q_h\|_Q, \quad \forall q_h \in Q_h, \quad (4.11)$$

uniformly in $h > 0$.

(A-2) [approximation property (1)] For each $v \in D(A)$, we can find $v_h \in V_h$ satisfying

$$\begin{aligned} \|v_h\|_V &\leq C\|v\|_V, \\ \|v - v_h\|_H &\leq Ch\|v\|_V, \\ \|v - v_h\|_V &\leq Ch\|Av\|_H, \end{aligned} \quad (4.12)$$

where C is independent of h and v .

(A-3) [approximation property (2)] For each $q \in D(B)$, we can find $q_h \in Q_h$ satisfying

$$\|q - q_h\|_Q \leq Ch\|Bq\|_H,$$

where C is independent of h and q .

Remark 4.1. The assumption (A-2) includes the condition on “elliptic regularity”. For example, let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and V_h be the conforming P^1 -finite element space with respect to a shape-regular triangulation of Ω . Then, for every $v \in H^2(\Omega)$, we can construct $v_h \in V_h$ with the error estimate

$$\|v - v_h\|_{H_0^1} \leq Ch\|v\|_{H^2}.$$

However, if A is the Laplace operator defined by

$$(Au, v)_{L^2} = (\nabla u, \nabla v)_{L^2}, \quad u, v \in H_0^1(\Omega),$$

then the error estimate (4.12) does not hold in general. Indeed, (4.12) requires that the condition $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ is satisfied, or equivalently,

$$\|v\|_{H^2} \leq C\|Av\|_{L^2}$$

for all $v \in D(A)$, which is not true for non-convex polygonal domains (see [50]).

We also introduce the discrete “solenoidal” space $V_{h,\sigma}$, defined by

$$V_{h,\sigma} := \{v_h \in V_h \mid b(v_h, q_h) = 0, \forall q_h \in Q_h\}. \quad (4.13)$$

Note that $V_{h,\sigma} \not\subset V_\sigma$ in general. We now formulate the Galerkin (semi-discrete) approximation for the problem (4.9) as follows: find $u_h(t) \in V_h$ and $q_h(t) \in Q_h$ that satisfy

$$\begin{cases} (u_{h,t}(t), v_h)_H + a(u_h(t), v_h) + b(v_h, p_h(t)) = 0, & \forall v_h \in V_h, \\ b(u_h(t), q_h) = 0, & \forall q_h \in Q_h, \\ u_h(0) = P_{h,\sigma}u_0, \end{cases} \quad (4.14)$$

where $P_{h,\sigma}: H \rightarrow V_{h,\sigma}$ is the orthogonal projection. We define the discrete “Stokes operator” A_h by

$$(A_h u_h, v_h)_H = a(u_h, v_h), \quad \forall u_h, v_h \in V_{h,\sigma}.$$

Then, as in the continuous case, the operator $-A_h$ generates an analytic contraction semigroup e^{-tA_h} on $H_{h,\sigma} := (V_{h,\sigma}, \|\cdot\|_H)$, and thus we can construct a unique solution (u_h, p_h) of the equation (4.14) due to the discrete inf-sup condition (4.11).

4.2.2 Finite element method for resolvent problems

We show error estimates for (4.14) via the resolvent estimates, as originally shown in [87] for the non-stationary Stokes problem. Let $\Gamma = \{re^{\pm i(\pi-\delta)} \in \mathbb{C} \mid r \in [0, \infty)\}$ be a path for $\delta \in (0, \pi/2)$, which is oriented so that the imaginary part increases along Γ . Then, since the semigroups e^{-tA} and e^{-tA_h} are analytic, we can write

$$u(t) - u_h(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{t\lambda} [(\lambda + A)^{-1} - (\lambda + A_h)P_{h,\sigma}] u_0 d\lambda. \quad (4.15)$$

Therefore, the error estimate for u_h is reduced to that of resolvent problems:

$$\lambda(w, v)_H + a(w, v) = (g, v)_H, \quad \forall v \in V_\sigma, \quad (4.16)$$

and

$$\lambda(w_h, v_h)_H + a(w_h, v_h) = (g, v_h)_H, \quad \forall v_h \in V_{h,\sigma}, \quad (4.17)$$

for given $g \in H$ and $\lambda \in \Sigma_\delta := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \pi - \delta\}$ with an arbitrarily fixed $\delta \in (0, \pi/2)$. Owing to the closed range theorem, the problem (4.16) is equivalent to the following equation for w and π :

$$\begin{cases} \lambda(w, v)_H + a(w, v) + b(v, \pi) = (g, v)_H, & \forall v \in V, \\ b(w, q) = 0, & \forall q \in Q, \end{cases} \quad (4.18)$$

and (4.17) is also equivalent to the problem

$$\begin{cases} \lambda(w_h, v_h)_H + a(w_h, v_h) + b(v_h, \pi_h) = (g, v_h)_H, & \forall v_h \in V_h, \\ b(w_h, q_h) = 0, & \forall q_h \in Q_h, \end{cases} \quad (4.19)$$

where $w_h \in V_h$ and $\pi_h \in Q_h$ are unknown functions. We assume that equation (4.18) admits the following estimate.

Assumption 4.2. (A-4) For each $g \in H$ and $\lambda \in \Sigma_\delta$, equation (4.18) has a unique solution $(w, \pi) \in V_\sigma \times Q$, which admits the regularity $w \in D(A)$ and $\pi \in D(B)$. Moreover, the following resolvent estimate holds:

$$|\lambda| \|w\|_H + |\lambda|^{1/2} \|w\|_V + \|Aw\|_H + \|B\pi\|_H \leq C \|g\|_H,$$

where C is independent of g and λ .

It is known that assumptions (A-1)–(A-4) allow us to obtain error estimates for the velocity.

Theorem 4.2.1. Let $\delta \in (0, \pi/2)$, and suppose that assumptions (A-1)–(A-4) hold. Then, we have

$$\|[(\lambda + A)^{-1}P_\sigma - (\lambda + A_h)^{-1}P_{h,\sigma}]g\|_V \leq Ch \|g\|_H, \quad (4.20)$$

$$\|[(\lambda + A)^{-1}P_\sigma - (\lambda + A_h)^{-1}P_{h,\sigma}]g\|_H \leq Ch^2 \|g\|_H, \quad (4.21)$$

for any $g \in H$ and $\lambda \in \Sigma_\delta$, where $P_\sigma: H \rightarrow H_\sigma$ is the orthogonal projection and C is independent of h , g , and λ .

An outline of the proof is given below. We refer the reader to [87, Theorems 3.2 and 4.2] for the detailed proof.

Proof. Fix $g \in H$, $\delta \in (0, \pi/2)$, and $\lambda \in \Sigma_\delta$ arbitrarily. Let (w, π) and (w_h, π_h) be the solutions of (4.18) and (4.19), respectively. Choose $v_h \in V_h$ and $q_h \in Q_h$ arbitrarily and consider $w_h - v_h$ and $\pi_h - q_h$. Substituting $\phi_h \in V_h$ and $\psi_h \in Q_h$ into (4.18) and (4.19), we have

$$\begin{aligned} a_\lambda(w_h - v_h, \phi_h) + b(\phi_h, \pi_h - q_h) &= a_\lambda(w - v_h, \phi_h) + b(\phi_h, \pi - q_h) \\ &=: {}_{V'_h} \langle F_h, \phi_h \rangle_{V_h}, \quad \forall \phi_h \in V_h, \end{aligned} \quad (4.22)$$

and

$$b(w_h - v_h, \psi_h) = b(w - v_h, \psi_h) =: {}_{Q'_h} \langle G_h, \psi_h \rangle_{Q_h}, \quad \forall \psi_h \in Q_h, \quad (4.23)$$

where

$$a_\lambda(u, v) = \lambda(u, v)_H + a(u, v)$$

for $u, v \in V$. By the elementary inequality $|s\lambda + t| \geq \sin(\delta/2)(s|\lambda| + t)$ for $\lambda \in \Sigma_{\pi-\delta}$ and $s, t \geq 0$, we can obtain

$$|a_\lambda(v, v)| \geq \alpha_1 \|v\|_{H^1(\Omega)}^2$$

for all $v \in V$, where $\alpha_1 > 0$ is independent of λ . Therefore, equations (4.22) and (4.23), the discrete inf-sup condition (A-1), and the generalized Lax-Milgram theorem (e.g., see [38, Theorem 2.34]) yield

$$|\lambda|^{1/2} \|w_h - v_h\|_H + \|w_h - v_h\|_V + \|\pi_h - q_h\|_Q \leq C \left(\|F_h\|_{V'_h} + \|G_h\|_{Q'_h} \right) \quad (4.24)$$

for some constant $C > 0$, which is independent of h due to (A-1). From the definition of F_h and G_h , we have

$$\|F_h\|_{V'_h} + \|G_h\|_{Q'_h} \leq C \left(|\lambda|^{1/2} \|w - v_h\|_H + \|w - v_h\|_V + \|\pi - q_h\|_Q \right),$$

which implies, together with (4.24), that

$$|\lambda|^{1/2} \|w - w_h\|_H + \|w - w_h\|_V + \|\pi - \pi_h\|_Q \leq C \left(|\lambda|^{1/2} \|w - v_h\|_H + \|w - v_h\|_V + \|\pi - q_h\|_Q \right).$$

Therefore, assumptions (A-2)–(A-4) lead to

$$|\lambda|^{1/2} \|w - w_h\|_H + \|w - w_h\|_V + \|\pi - \pi_h\|_Q \leq Ch \|g\|_H, \quad (4.25)$$

which implies (4.20).

The error estimate (4.21) is demonstrated by the standard duality argument. Consider the dual problem

$$\begin{cases} \lambda(\phi, \zeta)_H + a(\phi, \zeta) + b(\phi, \eta) = (\phi, w - w_h)_H, & \forall \phi \in V, \\ b(\zeta, \psi) = 0, & \forall \psi \in Q. \end{cases} \quad (4.26)$$

The solution $(\zeta, \eta) \in V \times Q$ has the estimate

$$|\lambda| \|\zeta\|_H + |\lambda|^{1/2} \|\zeta\|_V + \|A\zeta\|_H + \|B\eta\|_H \leq C \|v - v_h\|_H, \quad (4.27)$$

from assumption (A-4). Substituting $\phi = w - w_h$ into (4.26) and recalling equations (4.18) and (4.19), we have

$$\|w - w_h\|_H^2 \leq Ch \|g\|_H \times \left(|\lambda|^{1/2} \|\zeta - \zeta_h\|_H + \|\zeta - \zeta_h\|_V + \|\eta - \eta_h\|_Q \right)$$

for any $\zeta_h \in V_h$ and $\eta_h \in Q_h$. Hence, together with (A-2), (A-3), and (4.27), we obtain (4.21). \square

4.3 Abstract results

This section is devoted to the error estimates for the abstract Stokes problems.

Theorem 4.3.1. *Let $u_0 \in H_\sigma$ and let (u, p) and (u_h, p_h) be the solutions of (4.9) and (4.14), respectively. Assume that (A-1)–(A-4) hold. Then, we have the following error estimates:*

$$\|u(t) - u_h(t)\|_V \leq Ch t^{-1} \|u_0\|_H, \quad (4.28)$$

$$\|u(t) - u_h(t)\|_H \leq Ch^2 t^{-1} \|u_0\|_H, \quad (4.29)$$

$$\|u_t(t) - u_{h,t}(t)\|_{V'} \leq Ch t^{-1} \|u_0\|_H, \quad (4.30)$$

$$\|p(t) - p_h(t)\|_Q \leq Ch t^{-1} \|u_0\|_H, \quad (4.31)$$

for all $t \in (0, T)$, where each constant C depends only on the constants appearing in assumptions (A-1)–(A-4), but is independent of h , u_0 , t , and T .

Remark 4.2. The error estimates for the velocity (4.28) and (4.29) are given in [87, Theorems 3.1 and 4.1]. Although there is an error estimate for the pressure in [87], our result, (4.31), is strictly sharper. Indeed, it was shown that the estimate

$$\|p(t) - p_h(t)\|_Q \leq Ch \left(t^{-1} + t^{-3/2} \right) \|u_0\|_H$$

holds in [87, Theorem 5.1].

The estimates (4.28) and (4.29) are the consequence of the resolvent estimates (4.20) and (4.21), and the estimate for the pressure (4.31) is obtained from the discrete inf-sup condition (4.11). To show (4.30), we need another resolvent estimate.

Lemma 4.3.1. *Let $\delta \in (0, \pi/2)$. Suppose that assumptions (A-1)–(A-4) hold. Then, we have*

$$\left\| \left[(\lambda + A)^{-1} P_\sigma - (\lambda + A_h)^{-1} P_{h,\sigma} \right] g \right\|_{V'} \leq Ch |\lambda|^{-1} \|g\|_H,$$

for any $g \in H$ and $\lambda \in \Sigma_\delta$, where C is independent of h , g , and λ .

Proof. Fix $g \in H$, $\delta \in (0, \pi/2)$, and $\lambda \in \Sigma_\delta$ arbitrarily, and let (w, π) and (w_h, π_h) be the solutions of (4.18) and (4.19), respectively. It is sufficient to show that

$$(w - w_h, v)_H \leq Ch|\lambda|^{-1} \|g\|_H \|v\|_V \quad (4.32)$$

holds for arbitrary $v \in V$, since $\|F\|_{V'} = \sup_{v \in V} (F, v)_H / \|v\|_V$ for $F \in H \hookrightarrow V'$. Fix $v \in V$ and choose $v_h \in V_h$ as in (A-2). Then,

$$(w - w_h, v)_H = (w - w_h, v - v_h)_H + (w - w_h, v_h)_H =: I_1 + I_2.$$

Since a standard energy method yields $\|w\|_H + \|w_h\|_H \leq C|\lambda|^{-1} \|g\|_H$, we have

$$|I_1| \leq C|\lambda|^{-1} \|g\|_H \cdot h \|v\|_V$$

from assumption (A-2). Moreover, equations (4.18) and (4.19) imply

$$\lambda I_2 = -a(w - w_h, v_h) - b(v_h, \pi - \pi_h).$$

Together with (4.25), we have

$$|I_2| \leq Ch|\lambda|^{-1} \|g\|_H,$$

which gives (4.32). Hence we can complete the proof. \square

Proof of Theorem 4.3.1. (1) Proof of (4.28) and (4.29). The derivation of these estimates was originally presented in [87]. Indeed, one can check (4.28) and (4.29) directly from equation (4.15) and Theorem 4.2.1.

(2) Proof of (4.30). From (4.15) and Lemma 4.3.1, we have

$$\|u_t(t) - u_{h,t}(t)\|_{V'} \leq \int_{\Gamma} |\lambda e^{t\lambda}| \cdot Ch|\lambda|^{-1} \|u_0\|_H |d\lambda| \leq Ch t^{-1} \|u_0\|_H,$$

where $\Gamma = \partial\Sigma_\delta$ for an arbitrary $\delta \in (0, \pi/2)$.

(3) Proof of (4.31). Fix $q_h \in Q_h$ arbitrarily. Then, by the discrete inf-sup condition (4.11), we have

$$\beta_2 \|p_h(t) - q_h\|_Q \leq \sup_{v_h \in V_h} \frac{b(v_h, p_h(t) - q_h)}{\|v_h\|_V}.$$

The equations (4.9) and (4.14) yield

$$b(v_h, p_h(t) - q_h) = b(v_h, p(t) - q_h) + (u_t(t) - u_{h,t}(t), v_h)_H + a(u(t) - u_h(t), v_h),$$

which leads to

$$\|p_h(t) - q_h\|_Q \leq C (\|p(t) - q_h\|_Q + h t^{-1} \|u_0\|_H) \|v_h\|_V$$

from (4.28) and (4.30). Therefore, noting $p - p_h = (p - q_h) + (q_h - p_h)$, we have

$$\|p(t) - p_h\|_Q \leq C \|p(t) - q_h\|_Q + Ch t^{-1} \|u_0\|_H.$$

Finally, choosing $q_h \in Q_h$ as in assumption (A-3), we obtain

$$\|p(t) - q_h\|_Q \leq Ch \|Bp(t)\|_H \leq Ch t^{-1} \|u_0\|_H,$$

since $Bp = -u_t - Au$ and $u(t) = e^{-tA} u_0$. This completes the proof. \square

Remark 4.3. Throughout this section, we have considered the homogeneous problem (4.9). We now consider the inhomogeneous problem

$$\begin{cases} (u_t, v)_H + a(u, v) + b(v, p) = \langle f, v \rangle_{V, V'}, & \forall v \in V, \\ b(u, q) = 0, & \forall q \in Q, \end{cases}$$

with an external force $f: (0, T) \rightarrow V'$. If $f \in C^\theta([0, T]; H)$ for some $\theta \in (0, 1]$, then we have

$$\|u(t) - u_h(t)\|_H + h \|u(t) - u_h(t)\|_V \leq Ch (t^{-1} \|u_0\|_H + t^\theta \|f\|_{C^\theta([0, T]; H)} + \|f(t)\|_H)$$

by the same argument as in [42, § 5]. However, we cannot extend this result to the V' -error estimate and the pressure estimate at present.

4.4 Applications

In this section, we apply Theorem 4.3.1 to the non-stationary Stokes and the hydrostatic Stokes problem. Hereafter, $L^2(\Omega)$ and $H^s(\Omega)$ denote the Lebesgue and Sobolev spaces, respectively.

4.4.1 Non-stationary Stokes equation

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a convex polygonal or polyhedral domain. We consider the non-stationary Stokes equations in Ω with the homogeneous Dirichlet boundary condition:

$$\begin{cases} (u_t(t), v) + (\nabla u(t), \nabla v) - (\operatorname{div} v, p(t)) = 0, & \forall v \in H_0^1(\Omega)^d, \\ (\operatorname{div} u(t), q) = 0, & \forall q \in L_0^2(\Omega), \\ u(0) = u_0, \end{cases} \quad (4.33)$$

where $u_0 \in L_\sigma^2(\Omega)$. Here, we use the usual notation

$$\begin{aligned} H_0^1(\Omega) &= \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = 0\}, \\ L_0^2(\Omega) &= \{q \in L^2(\Omega) \mid \int_\Omega q = 0\}, \\ H_{0,\sigma}^1(\Omega) &= \{v \in H_0^1(\Omega)^d \mid \operatorname{div} v = 0\}, \\ L_\sigma^2(\Omega) &= \overline{H_{0,\sigma}^1(\Omega)}^{\|\cdot\|_{L^2}}, \end{aligned}$$

and let (\cdot, \cdot) denote the L^2 -inner product over Ω . Let $H = L^2(\Omega)^d$, $V = H_0^1(\Omega)^d$, $Q = L_0^2(\Omega)$, $a(u, v) = (\nabla u, \nabla v)$, and $b(v, q) = -(\operatorname{div} v, q)$. Then, $H_\sigma = L_\sigma^2(\Omega)$ and $V_\sigma = H_{0,\sigma}^1(\Omega)$. We also define the operators A and B as in Section 4.2. The domain of A coincides with $H^2(\Omega) \cap V_\sigma$ and the estimate

$$\|v\|_{H^2} \leq C \|Av\|_{L^2}, \quad \forall v \in D(A), \quad (4.34)$$

holds since Ω is a convex polygon or polyhedron (see [64] for the 2D case and [27] for the 3D case). Hence, assumption (A-4) holds.

Next, we consider the finite element approximation of (4.33). Let \mathcal{T}_h be a conforming triangulation (or tetrahedralization) of Ω with parameter $h = \max_{K \in \mathcal{T}_h} \operatorname{diam} K$. We define the pair of approximation spaces (V_h, Q_h) as the P^1 -bubble- P^1 element (MINI element) or P^2 - P^1 element (Taylor-Hood) with respect to \mathcal{T}_h . For the precise definition, see, e.g., [38]. We set $V_{h,\sigma}$ as in (4.13). If \mathcal{T}_h is shape-regular and quasi-uniform, then we can check that assumptions (A-1)–(A-3) hold, together with (4.34) (see, e.g., [16, 38]).

The approximation scheme is as follows. Find $(u_h(t), p_h(t)) \in V_h \times Q_h$ which satisfies

$$\begin{cases} (u_{h,t}(t), v_h) + (\nabla u_h(t), \nabla v_h) - (\operatorname{div} v_h, p_h(t)) = 0, & \forall v_h \in V_h, \\ (\operatorname{div} u_h(t), q_h) = 0, & \forall q_h \in Q_h, \\ u_h(0) = P_{h,\sigma} u_0, \end{cases} \quad (4.35)$$

where $P_{h,\sigma}$ is the L^2 -projection onto $V_{h,\sigma}$, as in (4.14). Then, since we have already checked that assumptions (A-1)–(A-4) hold, we can state the following error estimates.

Theorem 4.4.1. *Let Ω be a convex polygonal or polyhedral domain, \mathcal{T}_h be a shape-regular and quasi-uniform triangulation of Ω , and (V_h, Q_h) be the pair of finite elements mentioned above. Let (u, p) and (u_h, p_h) be the solutions of (4.33) and (4.35), respectively, for the initial value $u_0 \in L_\sigma^2(\Omega)$. Then, we have the following error estimates:*

$$\begin{aligned} \|u(t) - u_h(t)\|_{H^1} &\leq C h t^{-1} \|u_0\|_{L^2}, \\ \|u(t) - u_h(t)\|_{L^2} &\leq C h^2 t^{-1} \|u_0\|_{L^2}, \\ \|u_t(t) - u_{h,t}(t)\|_{H^{-1}} &\leq C h t^{-1} \|u_0\|_{L^2}, \\ \|p(t) - p_h(t)\|_{L^2} &\leq C h t^{-1} \|u_0\|_{L^2}, \end{aligned}$$

for all $t > 0$, where each constant C is independent of h , u_0 , t , and T .

Remark 4.4. For the MINI element, the convergence rate is optimal. However, for the Taylor-Hood element, it is not. For optimal order error estimates with higher order elements, our argument requires higher regularity for the solution of the resolvent problem (4.16), which is impossible when $g \in H$ has no more regularity. We leave the investigation of the optimal order error estimates for higher order elements as an area for future work.

4.4.2 Non-stationary hydrostatic Stokes equation

The second example is the hydrostatic Stokes problem, which is a linearized form of the primitive equations. Let $G = (0, 1) \subset \mathbb{R}^2$ and $\Omega = G \times (-D, 0) \subset \mathbb{R}^3$ with $D > 0$. The unknown functions of the hydrostatic Stokes equations are the horizontal velocity $u: \Omega \times (0, T) \rightarrow \mathbb{R}^2$ and the surface pressure $p: G \times (0, T) \rightarrow \mathbb{R}$. The equations are given by

$$\begin{cases} u_t - \Delta u + \nabla_H p = 0, & \text{in } \Omega \times (0, T), \\ \operatorname{div}_H \bar{u} = 0, & \text{in } \Omega \times (0, T), \\ u(\cdot, 0) = u_0, & \text{in } \Omega, \end{cases} \quad (4.36)$$

with boundary conditions

$$\begin{cases} \partial_z u = 0, & \text{on } \Gamma_u := G \times \{0\}, \\ u = 0, & \text{on } \Gamma_b := G \times \{-D\}, \\ u \text{ is periodic} & \text{on } \Gamma_l := \partial G \times (-D, 0), \end{cases} \quad (4.37)$$

where $\nabla_H q = (\partial_x q, \partial_y q)^T$, $\operatorname{div}_H v = \partial_x v_1 + \partial_y v_2$, and $\bar{v}(x, y) = \int_{-D}^0 v(x, y, z) dz$.

Let $H = L^2(\Omega)^2$, $V = \{v \in H^1(\Omega)^2 \mid v|_{\Gamma_b} = 0 \text{ and } v|_{\Gamma_l} \text{ is periodic}\}$, $Q = L_0^2(G)$, $a(u, v) = (\nabla u, \nabla v)_\Omega$ for $u, v \in V$, and $b(v, q) = -(\operatorname{div}_H \bar{v}, q)_G$ for $(v, q) \in V \times Q$, where $(\cdot, \cdot)_X$ denotes the L^2 -inner product over a domain X . Then, the weak formulation of the problem (4.36) and (4.37) is described by equation (4.9), i.e.,

$$\begin{cases} (u_t(t), v)_\Omega + (\nabla u, \nabla v)_\Omega - (\operatorname{div}_H \bar{v}, p)_G = 0, & \forall v \in V, \\ (\operatorname{div}_H \bar{u}, q)_G = 0, & \forall q \in Q. \end{cases}$$

Thus, we can construct a finite element scheme for the hydrostatic Stokes equations. Let \mathcal{T}_h be a tetrahedralization (a set of open tetrahedra) of Ω with $h = \max_{K \in \mathcal{T}_h} \operatorname{diam} K$, and let $\tilde{\mathcal{T}}_h$ be the triangulation of Γ_u induced by \mathcal{T}_h . Namely,

$$\tilde{\mathcal{T}}_h = \{T \subset \Gamma_u \mid \exists K \in \mathcal{T}_h \text{ s.t. } \bar{T} = \Gamma_u \cap \partial K\},$$

where \bar{T} is the closure in \mathbb{R}^2 . We suppose that

- V_h is the space of P^2 -finite elements or P^1 -bubble finite elements with respect to \mathcal{T}_h , and each $v_h \in V_h$ vanishes on Γ_b and is periodic on Γ_l ,
- Q_h is the space of P^1 -finite elements with respect to $\tilde{\mathcal{T}}_h$ and $\int_G q_h = 0$ for each $q_h \in Q_h$.

Then, we can introduce the finite element approximations for (4.36) and (4.37) as follows. Find $u_h: (0, T) \rightarrow V_h$ and $p_h: (0, T) \rightarrow Q_h$ which satisfy

$$\begin{cases} (u_{h,t}(t), v_h)_\Omega + (\nabla u_h, \nabla v_h)_\Omega - (\operatorname{div}_H \bar{v}_h, p_h)_G = 0, & \forall v_h \in V_h, \\ (\operatorname{div}_H \bar{u}_h, q_h)_G = 0, & \forall q_h \in Q_h, \\ (u_h(0), v_h)_\Omega = (u_0, v_h)_\Omega, & \forall v_h \in V_{h,\sigma}, \end{cases} \quad (4.38)$$

where $V_{h,\sigma}$ is defined by (4.13).

In order to discuss the error estimates in the framework of Theorem 4.3.1, we should check the conditions (A-1)–(A-4). Let A be the operator defined by (4.10), which is called the hydrostatic Stokes operator in the present case. Then, it is known that $0 \in \rho(A)$, A generates a bounded analytic semigroup on H_σ , and A satisfies the regularity property

$$\|v\|_{H^2(\Omega)} \leq C \|Av\|_{L^2(\Omega)}, \quad \forall v \in D(A).$$

We refer to [61, Theorem 3.1] for the proof. Therefore, we can check that conditions (A-2)–(A-4) hold.

Finally, we confirm (A-1) holds by introducing a prismatic mesh.

Definition 4.1. We say that a tetrahedralization \mathcal{T}_h of Ω is *prismatic* if the following condition holds: for each $K \in \mathcal{T}_h$, there exists $T \in \tilde{\mathcal{T}}_h$ such that

$$K \subset P_T := \{(x, y, z) \in \Omega \mid (x, y, 0) \in T\},$$

where $\tilde{\mathcal{T}}_h$ is the triangulation of Γ_u induced by \mathcal{T}_h .

We can construct such a mesh by the following procedure.

1. Triangulate the surface Γ_u and denote the triangulation by $\tilde{\mathcal{T}}_h$.
2. Construct a prism P_T in Ω for each $T \in \tilde{\mathcal{T}}_h$.
3. Decompose each prism P_T into tetrahedra so that the set of tetrahedra becomes a conforming tetrahedralization of Ω .

In [23], it is proved that the pair (V_h, Q_h) mentioned above satisfies the discrete inf-sup condition (4.11), provided that the mesh is prismatic. Indeed, if the mesh is prismatic, then we can extend a function $q_h \in Q_h$ naturally to a piecewise linear function over the mesh \mathcal{T}_h , and thus we can use the usual inf-sup condition for the MINI element or Taylor-Hood element. Hence, we can confirm (A-1) holds.

Therefore, we can apply Theorem 4.3.1 and obtain the following error estimates.

Theorem 4.4.2. *Let Ω , \mathcal{T}_h , and (V_h, Q_h) be as described above. Assume that the mesh \mathcal{T}_h is shape-regular, quasi-uniform, and prismatic. Let (u, p) and (u_h, p_h) be the solutions of (4.36) and (4.38), respectively, for the initial value $u_0 \in L^2(\Omega)^2$ satisfying $\operatorname{div}_H \bar{u}_0 = 0$ in the distributional sense. Then, we have the following error estimates:*

$$\begin{aligned} \|u(t) - u_h(t)\|_{H^1(\Omega)} &\leq Cht^{-1} \|u_0\|_{L^2(\Omega)}, \\ \|u(t) - u_h(t)\|_{L^2(\Omega)} &\leq Ch^2 t^{-1} \|u_0\|_{L^2(\Omega)}, \\ \|u_t(t) - u_{h,t}(t)\|_{V'} &\leq Cht^{-1} \|u_0\|_{L^2(\Omega)}, \\ \|p(t) - p_h(t)\|_{L^2(G)} &\leq Cht^{-1} \|u_0\|_{L^2(\Omega)}, \end{aligned}$$

for all $t > 0$, where each constant C is independent of h , u_0 , t , and T , and V' is the dual space of $V = \{v \in H^1(\Omega)^2 \mid v|_{\Gamma_b} = 0 \text{ and } v|_{\Gamma_l} \text{ is periodic}\}$.

4.5 Concluding remarks

In the present chapter, we considered the abstract non-stationary saddle-point problem (4.9) and its finite element approximation (4.14). Our main contribution (Theorem 4.3.1) is the derivation of error estimates for the velocity and the pressure in various norms. In particular, the error estimate for the pressure with the optimal singularity (i.e., the term t^{-1}) is a new result. We then applied this result to establish error estimates for the finite element approximation for the non-stationary Stokes and the hydrostatic Stokes equations. However, as mentioned in Remark 4.3, we have not obtained the error estimates for the pressure for inhomogeneous problems. Moreover, the convergence rate is not optimal for finite elements of higher degree (Remark 4.4). Furthermore, we should consider a numerical analysis for the primitive equations (4.3) in the framework of analytic semigroup theory, as performed in [86] for the two-dimensional Navier-Stokes equations. These problems remain an area for future work.

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