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1 Introduction

In this paper, we give the higher dimensional versions of the good reduction criterion for hyperbolic curves given by Oda and Tamagawa. We start from the definition of a hyperbolic curve.

Definition 1.1. 1. Let S be a scheme and let X be a scheme over S. We shall say that X is a *proper hyperbolic curve* over S if the structure morphism $X \to S$ is smooth, proper, geometrically connected, and of relative dimension one over S, each of whose geometric fiber is of genus ≥ 2.

2. Let S be a scheme and let X be a scheme over S. Let D be a divisor of X finite etale over S. We shall say that the pair (X, D) is a *hyperbolic curve* over S if the structure morphism $X \to S$ is smooth, proper, geometrically connected, and of relative dimension one over S, each of whose geometric fiber is of genus g and the morphism $D \to S$ is finite etale of degree n, which satisfies the inequality $2g + n - 2 > 0$. If $n = 0$ (resp. $n > 0$), we call the number $2g$ (resp. $2g + n - 1$) the first Betti number of the curve.

We can consider a proper hyperbolic curve X as a hyperbolic curve $(X, \emptyset)$.

Next, we recall the definition of good reduction of varieties. Let S be the spectrum of a discrete valuation ring O_K. We denote the generic point of S by η and the closed point of S by σ. Let $K = \kappa(\eta)$ be the fractional field of O_K, $k = \kappa(\sigma)$ the residue field of O_K, and p the characteristic of k.

Definition 1.2. 1. Let $X \to \operatorname{Spec} K$ be a proper smooth morphism of schemes. We say that X has good reduction if there exists a proper smooth S-scheme $\mathfrak{X}$ whose generic fiber $\mathfrak{X}_\eta$ is isomorphic to X over K. We refer to $\mathfrak{X}$ as a *smooth model* of X.

2. Let $(\bar{X}, D) \to \operatorname{Spec} K$ be a hyperbolic curve. We say that $(\bar{X}, D)$ has good reduction if there exists a hyperbolic curve $(\bar{\mathfrak{X}}, \mathfrak{D}) \to S$ whose generic fiber $(\bar{\mathfrak{X}}_\eta, \mathfrak{D}_\eta)$ is isomorphic to $(\bar{X}, D)$ over K. We refer to $(\bar{\mathfrak{X}}, \mathfrak{D})$ as a *smooth model* of $(\bar{X}, D)$.

It is an important problem in arithmetic geometry to know criteria for X to have good reduction. Various criteria for good reduction in terms of Galois representations have been established for certain class of varieties. Néron, Ogg, and Shafarevich established a criterion in the case of elliptic curves, and Serre and Tate generalized it to the case of abelian varieties [ST]. Their criterion claims that an abelian variety has good reduction if and only if the action of the inertia subgroup of K on its first l-adic etale cohomology is trivial for some prime $l \neq p$.

As a non-abelian version of the above result, Oda showed that a proper hyperbolic curve has good reduction if and only if the outer action of the inertia subgroup of K on its pro-l fundamental group is trivial [Oda1] [Oda2].

To state Oda's result precisely, we fix some notations. For a profinite group G and p as above (resp. a prime number l), we denote the pro-p' (resp. pro-l) completion of G, which is defined to be the limit of the projective system of quotient groups of G with finite order prime to p (resp. with finite l-power order) by $G^{p'}$ (resp. G^l). Here, if $p = 0$, we regard that every finite group has order prime to 0.

Let K^{sep} be the separable closure of K, $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ the absolute Galois group of K, and I_K its inertia subgroup. (Note that I_K, as a subgroup of G_K, depends on the choice of a prime ideal in the integral closure of O_K in K^{sep} over the maximal ideal of O_K, but it is independent of this choice up to conjugation.) Let $(\bar{X}, D)$ be a hyperbolic curve over K. Write X for the complement $\bar{X} \setminus D$. Then the pro-p' (resp. pro-l) completion $\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^{p'}$ (resp. $\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l$) of the geometric etale fundamental group $\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})$ (with base point $\bar{t}$) admits a continuous homomorphism

$$G_K \to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^{p'}) := \mathrm{Aut}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^{p'})/\mathrm{Inn}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^{p'}) \tag{1}$$

$$(\text{resp. } G_K \to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l) := \mathrm{Aut}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l)/\mathrm{Inn}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l)), \tag{2}$$

which we call the outer Galois representations. Oda and Tamagawa gave the following criterion.

Proposition 1.3. ([Oda1][Oda2][Tama] section 5) The following are equivalent.

1. $(\bar{X}, D)$ has good reduction.
2. The outer action $I_K \to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^{p'})$ defined by (1) is trivial.
3. There exists a prime number $l \neq p$ such that the outer action $I_K \to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l)$ defined by (2) is trivial.

Oda and Tamagawa's criterion can be regarded as a result in anabelian geometry. Indeed, a hyperbolic curve is a typical example of anabelian variety, i.e., a variety which is determined by its outer Galois representation $G_K \to \mathrm{Out}\, \pi_1(X \otimes K^{\mathrm{sep}}, \bar{t})$ (under suitable assumption on K), by the solution of Grothendieck conjecture due to Tamagawa [Tama] and Mochizuki [Moch]. Therefore it would be natural to expect that we can read off the information on the reduction of X from its outer Galois representation.

For the purpose of giving the higher dimensional versions of this criterion, we give the definition of a hyperbolic polycurve.

Definition 1.4. Let S be a scheme and X a scheme over S.

1. We shall say that X is a *hyperbolic polycurve* (of relative dimension n) over S if there exists a positive integer n and a (not necessarily unique) factorization of the structure morphism $X \to S$

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = S \tag{3}$$

   such that, for each $i \in \{1, \ldots, n\}$, there exists a hyperbolic curve $(\bar{X}_i, D_i) \to X_{i-1}$ (cf. Definition 1.1) and that the scheme $\bar{X}_i \setminus D_i$ is isomorphic to X_i over X_{i-1}. We refer to the above factorization of $X \to S$ as a *sequence of parametrizing morphisms.*

2. For a hyperbolic polycurve $X \to S$, the following are equivalent.
   (a) The morphism $X \to S$ is proper.
   (b) For any sequence of parametrizing morphisms of $X \to S$

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = S,$$

   each morphism $X_i \to X_{i-1}$ is proper for $1 \leq i \leq n$.
   (c) There exists a sequence of parametrizing morphisms of $X \to S$

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = S$$

   such that each morphism $X_i \to X_{i-1}$ is proper for $1 \leq i \leq n$.

   We call such $X \to S$ a proper hyperbolic polycurve.

3. Let X be a hyperbolic polycurve (resp. proper hyperbolic polycurve) of relative dimension n over S. For a sequence of parametrizing morphisms of $X \to S$

$$\mathcal{S} : X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = S, \tag{4}$$

   we call the maximum of the first Betti numbers (resp. genera) of fibers of $X_i \to X_{i-1}$ the maximal first Betti number $b_{\mathcal{S}}$ (resp. genus $g_{\mathcal{S}}$) of $\mathcal{S}$. We call the minimum of the first Betti numbers (resp. genera) of sequences of parametrizing morphisms of X the maximum first Betti number b_X (genus g_X) of X.

The class of hyperbolic polycurves are considered to be anabelian. Indeed, the Grothendieck conjecture holds for hyperbolic polycurves of dimension up to 4 on suitable assumption on K [Moch] [Ho]. Moreover, when X is a strongly hyperbolic Artin neighbourhood ([SS] Definition 6.1) and K is finitely generated over $\mathbb{Q}$, the Grothendieck conjecture holds in any dimension [SS]. Thus it is expected that there exists a good reduction criterion for hyperbolic polycurves which is analogous to the one by Oda and Tamagawa.

In [Nag], we studied good reduction criterion of proper hyperbolic polycurves under some assumptions. In this paper, we improve the main theorem of [Nag] and discuss also non-proper cases. The main results of this paper are as follows:

Theorem 1.5. Let K be a discrete valuation field with valuation ring O_K with residual characteristic $p \geq 0$. Let I_K be an inertia subgroup of the absolute Galois group G_K of K. Let X be a proper hyperbolic polycurve over K and g_X the maximum genus of X [cf. Definition 1.4]. Consider the following conditions.

(A) X has good reduction.

(B) The outer Galois representation $I_K \to \mathrm{Out}(\pi_1(X \times_{\mathrm{Spec}\, K} \mathrm{Spec} K^{\mathrm{sep}}, \bar{x})^{p'})$ of I_K is trivial.

Then, we have the following.

1. (A) implies (B).
2. If we assume that $p = 0$, (B) implies (A).
3. If we assume that $p > 2g_X + 1$ and that the dimension of X is 2, (B) implies (A).
4. If we assume that X has a K-rational point x, that the Galois representation $I_{K(x)} \to \mathrm{Aut}(\pi_1(X \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, K^{\mathrm{sep}}, \bar{x})^{p'})$ induced by x is trivial, and that $p > 2g_X + 1$, then (A) holds.

Theorem 1.6. Let K, O_K, and I_K be as in Theorem 1.5. Let X be a hyperbolic polycurve over K with a sequence of parametrizing morphisms

$$\mathcal{S} : X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}\, K. \tag{5}$$

We write $b_{\mathcal{S}}$ for the maximal first Betti number of $\mathcal{S}$ [cf. Definition 1.4]. Consider the following conditions.

(A) There exists a hyperbolic polycurve $\mathfrak{X} \to \mathrm{Spec}\, O_K$ with a sequence of parametrizing morphisms

$$\mathfrak{X} = \mathfrak{X}_n \to \mathfrak{X}_{n-1} \to \ldots \to \mathfrak{X}_1 \to \mathfrak{X}_0 = \mathrm{Spec}\, O_K \tag{6}$$

whose generic fiber is isomorphic to $\mathcal{S}$.

(B) The outer Galois representation $I_K \to \mathrm{Out}(\pi_1(X \times_{\mathrm{Spec}\, K} \mathrm{Spec} K^{\mathrm{sep}}, \bar{x})^{p'})$ of I_K is trivial.

Then, we have the following.

1. If we assume that $p = 0$ or $p > b_{\mathcal{S}} + 1$, (A) implies (B).
2. If we assume that $p = 0$, (B) implies (A).
3. If we assume that $p > b_{\mathcal{S}} + 1$ and that the dimension of X is 2, (B) implies (A).

Also, we can show a higher dimensional version of the good reduction criterion described by the outer Galois representation in general if we assume a very strong condition on $b_{\mathcal{S}}$ and p.

Theorem 1.7. Let K, O_K, I_K, X, $\mathcal{S}$, $b_{\mathcal{S}}$, and n be as in Theorem 1.6. Assume that $n \geq 3$. Define a function $f_{b_{\mathcal{S}}}(m)$ for $m \geq 3$ in the following way;

- For $m = 3$, $f_{b_{\mathcal{S}}}(3) = 2^{b_{\mathcal{S}}^2}$.
- For $m \geq 3$,

$$f_{b_{\mathcal{S}}}(m+1) = (f_{b_{\mathcal{S}}}(m)) \times (2^{b_{\mathcal{S}}^2 \times f_{b_{\mathcal{S}}}(m)^2})^{f_{b_{\mathcal{S}}}(m)}.$$

Consider the conditions (A) and (B) in Theorem 1.6. Then, if $p > 2^{b_{\mathcal{S}} \times f_{b_{\mathcal{S}}}(n)}$, (B) implies (A).

Remark 1.8. The main result of [Nag] is described as follows: Let K, O_K, and I_K as in Theorem 1.5. Let X be a proper hyperbolic polycurve over K which has a sequence of parametrizing morphisms such that each step $X_i \to X_{i-1}$ has a section. Write g_X for the minimum of maximal genera of such sequences of parametrizing morphisms of X [cf. Definition 1.4]. Consider the condition (A) in Theorem 1.5 and

(B)' For any closed point x of X, and for any choice of valuation ring $O_{K(x)}$ of the residual field $K(x)$ of x over $O_{K(x)}$, the action of inertia subgroup $I_{K(x)}$ of $O_{K(x)}$ on $\pi_1(X \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, K(x)^{\mathrm{sep}}, \bar{x})^{p'}$ is trivial.
Then (A) implies (B)'. If $p = 0$ or $p > 2g + 1$, (B)' implies (A).

This result is weaker than Theorem 1.5 because we need to assume that each $X_i \to X_{i-1}$ has a section and that the condition (B)' is stronger than the condition (B) in Theorem 1.5 or the condition given in Theorem 1.5.4.

To prove the implication (B) or (B)' $\Rightarrow$ (A) by induction on dimension of X, we need the homotopy exact sequence of geometric fundamental groups for smooth fibrations $X_i \to X_{i-1} (2 \leq i \leq n)$. In the previous paper [Nag], we constructed homotopy exact sequences of Tannakian fundamental groups of certain categories of smooth $\mathbb{Q}_l$-sheaves, and to do so, we needed the existence of sections of the above fibrations. Also we needed the assumption (B)' stronger than (B) because we used a criterion for smoothness of $\mathbb{Q}_l$-sheaves which is due to Drinfeld [Dri].

In this paper, we use different arguments from those of [Nag] to obtain stronger results. Since the implication (A) $\Rightarrow$ (B) follows from the standard argument of specialization, we explain key ingredients of the proof of the implication (B) $\Rightarrow$ (A) (assuming the condition on p, g_X, and $b_{\mathcal{S}}$ in the assertions), which enables us to improve the result of [Nag].

1. Comparison of inertia groups and centralizers

   If $p = 0$, we will translate the outer Galois action of the inertia group into the action of the centralizer subgroup $Z_\Pi(\Delta)$ of geometric etale fundamental group Δ in etale fundamental group Π by using the decomposition $\Pi \cong \Delta \times Z_\Pi(\Delta)$. This decomposition can be obtained from homotopy exact sequences of geometric etale fundamental groups (which exists since $p = 0$), which shows that Δ is center-free, and the hypothesis on the outer Galois action. Using this technique, we can prove the implication (B) $\Rightarrow$ (A).

2. Intermediate quotient group

   Note that we do not have appropriate homotopy exact sequences associated to fibrations $X_i \to X_{i-1} (2 \leq i \leq n)$ if $p > 0$. In fact, the functor of taking pro-p' completion (of profinite groups) is not an exact functor. Moreover, if the characteristic of K is positive, we do not have necessarily a fibration exact sequence of (full) etale fundamental groups. In this paper, we consider an intermediate quotient group of geometric etale fundamental groups (for which we will write $\Delta^{(l,p')}$) between pro-p' completion and pro-l completion, for which we can obtain the homotopy exact sequence. If the dimension of X is 2, we can show the implication (B) $\Rightarrow$ (A) by using the center-freeness of $\Delta^{(l,p')}$ and the same argument as 1. Also, if X admits a closed point x, we can compare the action of inertia group $I_{K(x)}$ at x and that of the inertia group which we consider in the induction step, and so we can prove Theorem 1.5.4.

3. Further intermediate quotient group

   If the dimension of X is equal to or greater than 3, we do not know if the group $\Delta^{(l,p')}$ in 2. is center-free. However, if p is big enough, we can take a certain quotient $\overline{\Delta}$ which is center-free and for which there exists the homotopy exact sequence. Thus we can prove the implication (B) $\Rightarrow$ (A) in higher dimensional case if p is big enough.

In the appendices of this paper, we give three interesting examples of hyperbolic polycurves which shows that the anabelianity of hyperbolic polycurves is weaker than that of hyperbolic curves in some sense.

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