博士論文

論文題目

A study on $(\mathfrak{g}, K)$-modules over commutative rings

(可換環上の $(\mathfrak{g}, K)$ 加群の研究)

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1 Introduction

1.1 General background and aims

The theory of $(\mathfrak{g}, K)$-modules over the complex number field $\mathbb{C}$ is an algebraic approach to representation theory of real reductive Lie groups. For a map of pairs $(\mathfrak{q}, M) \to (\mathfrak{g}, K)$, $I^{\mathfrak{g},K}_{\mathfrak{q},M}$ is a right adjoint functor to the forgetful functor from the category of $(\mathfrak{g}, K)$-modules to that of $(\mathfrak{q}, M)$-modules. The functor $I^{\mathfrak{g},K}_{\mathfrak{q},M}$ and its derived functor have been a significant construction of $(\mathfrak{g}, K)$-modules over $\mathbb{C}$. In particular, they include an algebraic analog of real parabolic inductions, and produce the so-called $A_{\mathfrak{q}}(\lambda)$-modules which are discrete series representations of real semisimple Lie groups in special cases.

For studies of $(\mathfrak{g}, K)$-modules over $\mathbb{C}$, it is crucial that rational representations of compact Lie groups or corresponding reductive algebraic groups have the complete reducibility. For example, this guarantees the following results:

- The coordinate rings of complex reductive algebraic groups are cosemisimple ([S] Chapter XIV and XV).
- Every K-module is both injective and projective ([KV] Lemma 2.4).
- Irreducible representations of K form a projective family of generators of the category of K-modules.
- Let $K_{\mathbb{R}}$ be a maximal compact subgroup of K. Then $K_{\mathbb{R}}$-finite distributions on $K_{\mathbb{R}}$ form a convolution algebra $R(K_{\mathbb{R}})$ called the Hecke algebra. This is approximately unital, and the categories of approximately unital $R(K_{\mathbb{R}})$-modules and locally finite representations of $K_{\mathbb{R}}$ are isomorphic. Moreover, if we are given a Harish-Chandra pair $(\mathfrak{g}, K)$, there is an approximately unital algebra $R(\mathfrak{g}, K)$ also called the Hecke algebra such that the categories of $(\mathfrak{g}, K)$-modules and approximately unital $R(\mathfrak{g}, K)$-modules are isomorphic ([KV] I.4 Theorem). It particularly implies that $(\mathfrak{g}, K)$-modules can be treated like modules over rings.
- The right adjoint functor $I^{\mathfrak{g},K}_{\mathfrak{q},M}$ can be constructed as a Hom-type adjoint functor.

- We can define the base-change type construction of $(\mathfrak{g}, K)$-modules from small pairs $(\mathfrak{g}, M)$ known as the Bernstein functor (see [KV] (2.8) and [J1] 1.4.2).
- We are able to get the so-called standard projective and injective resolutions of $(\mathfrak{g}, K)$-modules from the Koszul complexes ([KV] Theorem 2.122). It enables us to compute the derived functor modules.

For deeper studies the highest weight theory is also important. From one viewpoint what cause this phenomenon are weight space decompositions of representations of tori and the structure of root systems.

Recently, G. Harder, F. Januszewski, and the author have proposed analogs of the homological construction of $\mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M}$ to replace $\mathbb{C}$ by commutative rings k for base rings to focus on integral and rational structures of real reductive groups and their representations ([Har], [J1], [J2], [H1], and [H2] for example). J. Bernstein et al. also introduced contraction families as pairs over the polynomial ring $\mathbb{C}[z]$ in [BHS]. These are all regarded as a part of the theory of $(\mathfrak{g}, K)$-modules over commutative rings.

Januszewski has been studying it when the base field is of characteristic 0, and the groups K, M are reductive. He found that similar constructions work in this setting ([J1], [J2]). For instance, he found an approximately unital ring $R(\mathfrak{g}, K)$ such that the category of approximately unital $R(\mathfrak{g}, K)$-modules is isomorphic to that of $(\mathfrak{g}, K)$-modules over k ([J1] Theorem 1.5, see also [KV] I.4 Theorem). He also constructed the standard projective and injective resolutions of $(\mathfrak{g}, K)$-modules ([J1] 1.4.4).

In a view from homological algebra, this cannot be generalized in a straightforward way when the base ring is no longer a field since the complete reducibility fails. For the integral case, G. Harder suggested to consider the complex

$$\mathrm{Hom}_K(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}), -)$$

for an integral analog of the $(\mathfrak{g}, K)$-cohomology modules. In [H1] and [H2], the author constructed the functor $I^{\mathfrak{g},K}_{\mathfrak{q},M}$ and its derived functor in the usual sense over an arbitrary commutative ring. The arguments of [H1] heavily rely on generalities on categories, especially, closed symmetric monoidal categories. Though we know the existence of the functor, we did not understand what they actually produced.

Studies of the functor $I^{\mathfrak{g},K}_{\mathfrak{q},M}$ consist of three steps:

(A) Construct $I^{\mathfrak{g},K}_{\mathfrak{q},M}$ and its derived functor, or prove their existence.

(B) Find pairs and suitable $(\mathfrak{q}, M)$-modules which are meaningful to representation theory of real reductive groups.

(C) Study the resulting $(\mathfrak{g}, K)$-modules from the functor $I^{\mathfrak{g},K}_{\mathfrak{q},M}$.

Generalities on Part (A) were established by [H1] and [H2] as mentioned above. Part (B) is well-studied in principle when the base is $\mathbb{C}$ (see [KV]). The pair

$(\mathfrak{g}, K)$ may be usually the pair associated to a real reductive group $G_{\mathbb{R}}$. Typical examples of $(\mathfrak{q}, M)$ are pairs associated to real parabolic subgroups of $G_{\mathbb{R}}$ and the parabolic subpairs which are stable under the Cartan involution. We also have subsequent ones like Levi subpairs. When we work on their integral models, there may be many choices of their integral forms. Part (C) must also depend on such choices.

For another direction in Part (B), J. Bernstein et al. propose the contraction families in [BHS] as pairs over the polynomial ring $\mathbb{C}[z]$. Let $(\mathfrak{g}, K)$ be a pair over $\mathbb{C}$, equipped with a K-equivariant involution θ of $\mathfrak{g}$. Write $\mathfrak{g}^{\theta=1}$ (resp. $\mathfrak{g}^{\theta=-1}$) for the eigenspace of θ with eigenvalue 1 (resp. -1). Assume that the Lie algebra $\mathfrak{k}$ of K is contained in $\mathfrak{g}^{\theta=1}$. Then $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z]$ is a Lie algebra over $\mathbb{C}[z]$ for the bracket defined summandwisely by

$$[\eta z^m, \xi z^n] = \begin{cases} [\eta, \xi]\, z^{m+n+1} & (\eta, \xi \in \mathfrak{g}^{\theta=-1}) \\ [\eta, \xi]\, z^{m+n} & (\text{otherwise}). \end{cases}$$

The contraction family is a pair over $\mathbb{C}[z]$ consisting of $\tilde{\mathfrak{g}}$ and $K \otimes \mathbb{C}[z]$. Moreover, they construct algebraic families of groups over the projective line $\mathbb{P}^1$ associated to classical symmetric pairs to extend $(\tilde{\mathfrak{g}}, K \otimes \mathbb{C}[z])$ to a pair over $\mathbb{P}^1$. In loc. cit., they studied Harish-Chandra modules over the contraction family associated to the Lie group SU(1,1). In fact, they gave the classification result of generically irreducible admissible representations by weights (loc. cit. Lemma 4.4.3, see also Theorem 4.9.3).

The purpose of this paper is to work on (B) and (C) in both abstact and computational ways. Supplementarily, we also work again on Part (A) to relax the definition of pairs in [H1]. Fix k as a ground commutative ring.

Condition 1.1.1. (1) A k-module V is said to satisfy Condition 1.1.1 (1) if for any flat commutative k-algebra R, the canonical homomorphism

$$\mathrm{Hom}_k(V, k) \otimes R \to \mathrm{Hom}_k(V, R)$$

is an isomorphism.

(2) A k-module V is said to satisfy Condition 1.1.1 (2) if for any k-module W and any flat commutative k-algebra R , the canonical homomorphism

$$\mathrm{Hom}_k(V, W) \otimes R \to \mathrm{Hom}_k(V, W \otimes R)$$

is an isomorphism.

Example 1.1.2. Finitely presented k-modules V satisfy Condition 1.1.1 (2).

Condition 1.1.3. Let K be a flat affine group scheme over k. Write I_e for the kernel of the counit of the coordinate ring of K. Then K is said to satisfy Condition 1.1.3 if the k-modules I_e/I_e^2 and its dual $\mathfrak{k} = \mathrm{Hom}_k(I_e/I_e^2, k)$ enjoy Condition 1.1.1 (1) and (2) respectively.

Example 1.1.4. If k is Noetherian, and I_e/I_e^2 is finitely generated then $\mathfrak{k}$ is also finitely generated. In particular, both I_e/I_e^2 and $\mathfrak{k}$ satisfy Condition 1.1.1 (2).

Notation 1.1.5. For a flat affine group scheme satisfying Condition 1.1.3, its Lie algebra will be denoted by the corresponding small German letter.

A pair consists of a flat affine group scheme K satisfying Condition 1.1.3 and a k-algebra $\mathcal{A}$ with a K-action ϕ, equipped with a K-equivariant Lie algebra homomorphism $\psi : \mathfrak{k} \to \mathcal{A}$. Moreover, a pair is demanded to satisfy the equality $d\phi(\xi) = [\psi(\xi), -]$ for any $\xi \in \mathfrak{k}$, where $d\phi$ is the differential representation of ϕ. The point of modification from [H1] is on the condition of I_e/I_e^2. In loc. cit., we required that I_e/I_e^2 is finitely generated and projective ([H1] Condition 2.2.2). For a pair $(\mathcal{A}, K)$, an $(\mathcal{A}, K)$-module is a K-module, equipped with a K-equivariant $\mathcal{A}$-module structure such that the two induced actions of $\mathfrak{k}$ coincide ([H1]). We consider a version to replace algebras $\mathcal{A}$ by Lie algebras $\mathfrak{g}$. Remark that in this paper, we do not discuss differential graded modules like loc. cit. Then the same arguments as [H1] and [H2] still work.

Lemma 1.1.6. *Let $(\mathcal{A}, K) \to (\mathcal{B}, L)$ be a map of pairs in the sense above. Then we have a forgetful functor $\mathcal{F}^{\mathcal{A},K}_{\mathcal{B},L}$ from the category of $(\mathcal{B}, L)$-modules to that of $(\mathcal{A}, K)$-modules, and it admits a right adjoint functor $I^{\mathcal{B},L}_{\mathcal{A},K}$.*

1.2 Base change

In [J2], Januszewski discussed the behavior of $\mathrm{Ext}^\bullet_{\mathfrak{g},K}$ and the functor $I^{\mathfrak{g},K}_{\mathfrak{q},M}$ under extensions k'/k of fields, and proved the base change formulas

$$\mathrm{Hom}_{\mathfrak{g},K}(X, -) \otimes k' \cong \mathrm{Hom}_{\mathfrak{g}\otimes k', K\otimes k'}(X \otimes k', - \otimes k')$$
$$\mathrm{Ext}^\bullet_{\mathfrak{g},K}(-, -) \otimes k' \cong \mathrm{Ext}_{\mathfrak{g}\otimes k', K\otimes k'}(- \otimes k', - \otimes k')$$
$$\mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M}(V) \otimes k' \simeq \mathbb{R}I^{\mathfrak{g}\otimes k', K\otimes k'}_{\mathfrak{q}\otimes k', M\otimes k'}(V \otimes k')$$
$$H^\bullet(\mathfrak{g}, K, -) \otimes k' \cong H^\bullet(\mathfrak{g} \otimes k', K \otimes k', - \otimes k')$$

under suitable finiteness conditions (see [J2] for details). This can be regarded as Part (C). He also considered rational forms of cohomological inductions (Part (B) and Part (C), see [J2] 7.1).

In [Jant], base change formulas of representations of affine group schemes K over commutative rings k are discussed.

Notation 1.2.1. If $M \to K$ is a homomorphism between flat affine group schemes over k, let us denote the right adjoint functor to the forgetful functor from the category of K-modules to that of M-modules by Ind^K_M.

Let K be an affine group scheme, and V be a K-module which is finitely generated and projective as a K-module. According to [Jant] I.2.10, we have $\mathrm{Hom}_K(V, -) \otimes k' \cong \mathrm{Hom}_{K\otimes k'}(V \otimes k', - \otimes k')$. Moreover, if k' is finitely generated and projective as a k-module then the isomorphism above holds for small colimits of such V. For a homomorphism $M \to K$ between flat affine group schemes over k, the group cohomology $H^\bullet(K, -)$ and the cohomology functor $R^n \mathrm{Ind}^K_M(-)$ respects flat base changes (loc. cit. Proposition I.4.13).

Our goal in this paper for an abstract approach is to establish these isomorphisms along flat homomorphisms from Noetherian rings.

1.3 Representation theory of diagonalizable groups

One of the difficulties of representation theory of flat affine group schemes over commutative rings k lies in the failure of complete reducibility which involves the structures of k. For instance, if k is the ring $\mathbb{Z}$ of integers, the trivial representation $\mathbb{Z}$ of a flat affine group scheme over $\mathbb{Z}$ has a proper subrepresentation $2\mathbb{Z}$. In categorical viewpoints, we may rephrase it as follows: There may not be sufficiently simple families of generators of the category of representations of flat affine group schemes K at least for applications to explicit computations of $\mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M}$ even when K are split reductive groups over $\mathbb{Z}$. We may not have suitable analogs of the notion of K-type. According to [Jant], this difficulty does not occur for diagonalizable groups T over an arbitrary commutative ring k. That is, every T-module V is decomposed into the direct sum of the maximal submodules V_λ on which T act as characters λ, and for T-modules V, V', the set of T-homomorphisms $V \to V'$ is isomorphic to the direct product of that of k-homomorphisms $V_\lambda \to V'_\lambda$. See [Jant] I.2.11 for details.

Our goal for a computational approach is to provide examples of integral forms of Harish-Chandra pairs of the finite covering groups of PU(1,1) and their subpairs to compute the functor $I^{\mathfrak{g},K}_{\mathfrak{q},M}$ explicitly in order to find integral analogs and different new outcomes. For this, we introduce the Hecke algebras of pairs $(\mathfrak{g}, K)$ with K diagonalizable, and interpret [Jant] I.2.11 into homological results on $(\mathfrak{g}, K)$-modules. We also discuss contraction analogs.

As an application of the studies of flat base changes and our homological interpretations of [Jant] I.2.11, we study the algebraic Borel-Weil-Bott induction over $\mathbb{Z}$. For a split reductive group G, choose a split maximal torus and a Borel subgroup $T \subset B$. Then the geometric Borel-Weil-Bott induction can be thought of as the cohomology representations of characters λ of B with respect to the induction from the category of B-modules to that of G-modules. They are studied in [Jant] when the base is a field. The algebraic Borel-Weil-Bott induction is its counterpart in the theory of Harish-Chandra modules ([KV]). For another direction of the Borel-Weil-Bott induction, Januszewski discuss the cases of nonsplit reductive groups G over fields of characteristic 0 ([J1] Theorem 1.9).

1.4 Main Results I

In this paper, our pairs $(\mathcal{A}, K)$ we mainly consider arise from their versions $(\mathfrak{g}, K)$ for Lie algebras through $\mathcal{A} = U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Therefore we write $(\mathfrak{g}, K)$-modules for $(U(\mathfrak{g}), K)$-modules.

Notation 1.4.1. For a pair $(\mathfrak{g}, K)$ over k, denote the category of $(\mathfrak{g}, K)$-modules by $(\mathfrak{g}, K)$-mod.

We next introduce the functors of flat base changes. Let $k \to k'$ be a flat homomorphism of commutative rings, and $(\mathfrak{g}, K)$ be a pair over k.

Lemma A (Proposition 3.1.1)**.** (1) The Lie algebra $\mathfrak{g} \otimes k'$ and the affine group scheme $K \otimes k'$ over k' naturally form a pair $(\mathfrak{g} \otimes k', K \otimes k')$ over k'.

(2) The extension and the restriction of scalars of modules extend to an adjunction

$$- \otimes_k k' : (\mathfrak{g}, K)\text{-mod} \leftrightarrows (\mathfrak{g} \otimes k', K \otimes k')\text{-mod} : \mathrm{Res}^k_{k'}.$$

Assume k to be Noetherian. We compare a relation of Hom modules and flat base changes.

Theorem B (Flat base change theorem, Theorem 3.1.6)**.** Suppose that $\mathfrak{g}$ is finitely generated as a k-module. Then for any finitely generated $(\mathfrak{g}, K)$-module X, we have a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{g},K}(X, -) \otimes k' \cong \mathrm{Hom}_{\mathfrak{g}\otimes k', K\otimes k'}(X \otimes k', - \otimes k').$$

Theorem C (Theorem 3.1.7)**.** Let $k \to k'$ be a flat ring homomorphism, and $(\mathfrak{q}, M) \to (\mathfrak{g}, K)$ be a map of pairs. Suppose that the following conditions are satisfied:

(i) $\mathfrak{k} \oplus \mathfrak{q} \to \mathfrak{g}$ is surjective.

(ii) $\mathfrak{q}$ and $\mathfrak{g}$ are finitely generated as k-modules.

Then we have an isomorphism

$$(I^{\mathfrak{g},K}_{\mathfrak{q},M} V) \otimes_k k' \cong I^{\mathfrak{g}\otimes_k k', K\otimes_k k'}_{\mathfrak{q}\otimes_k k', M\otimes_k k'}(V \otimes_k k').$$

Example 1.4.2. Let k be the ring $\mathbb{Z}$ of integers, and k' be the field $\mathbb{Q}$ of rational numbers. Then Theorem C asserts that $I^{\mathfrak{g},K}_{\mathfrak{q},M}(V)$ is a $\mathbb{Z}$-form (with torsions) of $I^{\mathfrak{g}\otimes\mathbb{Q}, K\otimes\mathbb{Q}}_{\mathfrak{q}\otimes\mathbb{Q}, M\otimes\mathbb{Q}}(V \otimes \mathbb{Q})$.

The condition (i) of Theorem C is satisfied in the following cases:

Example 1.4.3 (The Zuckerman functor)**.** The Lie algebra $\mathfrak{q}$ is equal to $\mathfrak{g}$, and the map $\mathfrak{q} \to \mathfrak{g}$ is the identity. In this case, $\Gamma = I^{\mathfrak{g},K}_{\mathfrak{g},M}$ is called the Zuckerman functor.

Example 1.4.4. The pair $(\mathfrak{g}, K)$ is trivial. In other words, $\mathfrak{g}$ is the zero Lie algebra 0, and K is the trivial group scheme $\mathrm{Spec}\, k$. In this case, the functor $I^{0,\mathrm{Spec}\, k}_{\mathfrak{q},M}$ will be denoted by $H^0(\mathfrak{q}, M, -)$.

Example 1.4.5 (The algebraic Borel-Weil induction)**.** Let G be a split reductive group over $\mathbb{Z}$. Fix a maximal split torus T of G, and a positive root system of the Lie algebra $\mathfrak{g}$ of G. Write $\bar{\mathfrak{b}}$ for the Lie subalgebra of $\mathfrak{g}$ corresponding to the negative roots. Then we have a map $(\bar{\mathfrak{b}}, T) \to (\mathfrak{g}, G)$ of pairs. The corresponding functor $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}$ is called the Borel-Weil induction. Its derived functor is called the Borel-Weil-Bott induction.

We also have its derived version:

Notation 1.4.6. Let $(\mathfrak{g}, K)$ be a pair. Then denote the unbounded derived category of $(\mathfrak{g}, K)$-modules and its full subcategory spanned by complexes cohomologically bounded below by $D(\mathfrak{g}, K)$ and $D^+(\mathfrak{g}, K)$ respectively.

Theorem D (Theorem 3.1.10)**.** Let $k \to k'$ be a flat ring homomorphism, and $(\mathfrak{q}, M) \to (\mathfrak{g}, K)$ be a map of pairs. Suppose that the following conditions are satisfied:

(i) $\mathfrak{k} \oplus \mathfrak{q} \to \mathfrak{g}$ is surjective.

(ii) $\mathfrak{q}$ and $\mathfrak{g}$ are finitely generated as k-modules.

Then we have an equivalence

$$(\mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M}-) \otimes_k k' \simeq \mathbb{R}I^{\mathfrak{g}\otimes k', K\otimes k'}_{\mathfrak{q}\otimes k', M\otimes k'}(- \otimes_k k')$$

on $D^+(\mathfrak{q}, M)$.

In view of Theorem D, the cohomology modules of $I^{\mathfrak{g},K}_{\mathfrak{q},M}$ over $\mathbb{Z}$ are $\mathbb{Z}$-forms of those over $\mathbb{Q}$ via the base change $\otimes\mathbb{Q}$ under the suitable conditions. As mentioned in the introduction of [H1], it is an expected new phenomenon that the cohomology involve torsions. We give an example in Part II.

It will be convenient to consider an unbounded analog of Theorem D. In fact, then we can use infinite homotopy colimits. They are needed when we consider the homotopy descents for instance ([He]). The idea of descents and its applications to number theory have already appeared in [J2]. For the proof of Theorem D, we do not have a standard resolution. Instead we prove that $\otimes k'$ sends injective objects to acyclic objects with respect to $I^{\mathfrak{g}\otimes k', K\otimes k'}_{\mathfrak{q}\otimes k', M\otimes k'}$. Then we see the base change formula of complexes degreewisely. Therefore the argument does not extend literally to the unbounded case. To establish an unbounded analog, we replace the unbounded derived categories. For a pair $(\mathfrak{g}, K)$ over a Noetherian ring k, write the stable derived category of $(\mathfrak{g}, K)$-modules by $\operatorname{Ind}\operatorname{Coh}(\mathfrak{g}, K)$ in the sense of [Kr]. In terms of higher categories, this can be thought of as the ind-completion (see [L1]) of the ∞-category $\operatorname{Coh}(\mathfrak{g}, K)$ of cohomologically bounded complexes whose cohomologies are finitely generated as $(\mathfrak{g}, K)$-modules.

Let $k \to k'$ be a flat homomorphism of Noetherian rings, and $(\mathfrak{q}, M) \to (\mathfrak{g}, K)$ be a map of pairs over k. Then we can define the ind-analogs of the functors above:

$$- \otimes k' : \operatorname{Ind}\operatorname{Coh}(\mathfrak{g}, K) \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{g} \otimes k', K \otimes k')$$

$$- \otimes k' : \operatorname{Ind}\operatorname{Coh}(\mathfrak{q}, M) \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{q} \otimes k', M \otimes k')$$

$$I^{\mathfrak{g},K,\mathrm{ind}}_{\mathfrak{q},M} : \operatorname{Ind}\operatorname{Coh}(\mathfrak{q}, M) \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{g}, K)$$

$$I^{\mathfrak{g}\otimes k', K\otimes k', \mathrm{ind}}_{\mathfrak{q}\otimes k', M\otimes k'} : \operatorname{Ind}\operatorname{Coh}(\mathfrak{q} \otimes k', M \otimes k') \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{g} \otimes k', K \otimes k').$$

Theorem E (Theorem 3.3.4)**.** There is a canonical equivalence

$$I^{\mathfrak{g},K,\mathrm{ind}}_{\mathfrak{q},M}(-)\otimes k' \to I^{\mathfrak{g}\otimes k',K\otimes k',\mathrm{ind}}_{\mathfrak{q}\otimes k',M\otimes k'}(-\otimes k').$$

Moreover, it restricts to the equivalence $\mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M}(-)\otimes k' \simeq \mathbb{R}I^{\mathfrak{g}\otimes k',K\otimes k'}_{\mathfrak{q}\otimes k',M\otimes k'}(-\otimes k')$ of Theorem D under the identifications

$$\operatorname{Ind}\operatorname{Coh}(\mathfrak{q},M)^+ \simeq D(\mathfrak{q},M)^+$$

$$\operatorname{Ind}\operatorname{Coh}(\mathfrak{g}\otimes k',K\otimes k')^+ \simeq D(\mathfrak{g}\otimes k',K\otimes k')^+$$

This reduces to a base change formula for $D(\mathfrak{g},K)$ in special cases by the following assertion:

Proposition F (Proposition 3.3.5)**.** Suppose that k is a field of characteristic 0, $(\mathfrak{g},K)$ be a pair with K reductive and $\dim\mathfrak{g}<+\infty$. Then the embedding $\mathrm{Coh}(\mathfrak{g},K)\to D(\mathfrak{g},K)$ induces an equivalence $\operatorname{Ind}\operatorname{Coh}(\mathfrak{g},K)\simeq D(\mathfrak{g},K)$.

We also show a finite analog of Theorem B, Theorem C, and Theorem D without their conditions (i) and (ii). This is rather a straightforward generalization of [J2] Corollary 2.2 and Theorem 2.5.

Notation 1.4.7. For a category $\mathcal{C}$, its opposite category will be denoted by $\mathcal{C}^{op}$.

Variant G (Variant 3.2.12, Variant 3.2.13, Lemma 3.2.15)**.** Let $(\mathfrak{q},M)\to(\mathfrak{g},K)$ be a map of pairs over a commutative ring k, and $k\to k'$ be a ring homomorphism. Assume that k' is finitely generated and projective as a k-module.

(1) There is a canonical isomorphism $\mathrm{Hom}_{\mathfrak{g},K}(-,-)\otimes k' \cong \mathrm{Hom}_{\mathfrak{g}\otimes k',K\otimes k'}(-\otimes k',-\otimes k')$ on $(\mathfrak{g},K)\text{-mod}^{op}\times(\mathfrak{g},K)\text{-mod}$.

(2) There is a natural isomorphism

$$(I^{\mathfrak{g},K}_{\mathfrak{q},M}-)\otimes_k k' \cong I^{\mathfrak{g}\otimes_k k',K\otimes_k k'}_{\mathfrak{q}\otimes_k k',M\otimes_k k'}(-\otimes_k k').$$

(3) There is a natural equivalence of the functors on the unbounded derived category of $(\mathfrak{q},M)$-modules:

$$\mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M}(-)\otimes k' \simeq \mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M}(-\otimes k').$$

A typical application is to add $\sqrt{-1}$ to the given ring. In fact, we will need $\sqrt{-1}$ (and other fractions) to make integral forms of compact Lie groups to be split. For instance, the special orthogonal group $\mathrm{SO}(2)\cong\operatorname{Spec}\mathbb{Z}[x,y]/(x^2+y^2-1)$ is isomorphic to the split torus of rank 1 after the base change to $\mathbb{Z}\left[\sqrt{-1},\frac{1}{2}\right]$.

Notation 1.4.8. Let $(\mathfrak{h},K)\to(\mathfrak{g},K)$ be a map of pairs over a commutative ring with $K\to K$ being the identity. The left and right adjoint functors to the forgetful functor from the category of $(\mathfrak{g},K)$-modules to that of $(\mathfrak{h},K)$-modules will be denoted by $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{h}}$ and $\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{h}}$ respectively.

Our strategy of the proofs of Theorem B and Variant G is to use general arguments on generators to reduce them to the group case through the induction $\mathrm{ind}_{\mathfrak{k}}^{\mathfrak{g}} : K\text{-mod} \to (\mathfrak{g}, K)\text{-mod}$. The remaining assertion is then a version of [Jant] I.2.10 for flat affine group schemes. Theorem C is basically obtained by formal arguments of adjunctions. Remark that we have to analyze the resulting bijections since the inverse map of Theorem B is not canonical.

Finally, we discuss flat base changes of pro. Unlike the case $k = \mathbb{C}$, it should be difficult in general since the internal Hom of K-mod is quite complicated. In this paper, we find a practically nice setting to imitate the description of [KV] Proposition 5.96. Let G be a real reductive group, $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$ be the associated pair over $\mathbb{C}$ to G, and $(\mathfrak{q}_{\mathbb{C}}, (K_L)_{\mathbb{C}})$ be a θ-stable parabolic subpair, where θ is the Cartan involution. Let $\bar{\mathfrak{u}}_{\mathbb{C}}$ be the opposite nilradical to $\mathfrak{q}_{\mathbb{C}}$. Write h for the element of the Cartan subalgebra corresponding to the half sum of roots of the nilradical $\mathfrak{u}_{\mathbb{C}}$ of $\mathfrak{q}_{\mathbb{C}}$ ([KV] Proposition 4.70).

Let k be a Noetherian subring of $\mathbb{C}$, and $(\mathfrak{q}, K_L) \subset (\mathfrak{g}, K)$ be a k-form of $(\mathfrak{q}_{\mathbb{C}}, (K_L)_{\mathbb{C}}) \subset (\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$. Assume that there is a complementary K_L-stable subalgebra $\bar{\mathfrak{u}} \subset \mathfrak{g}$ to $\mathfrak{q}$ which is a k-form of $\bar{\mathfrak{u}}_{\mathbb{C}}$. Moreover, suppose that the following conditions are satisfied:

(i) There is a free basis of $\mathfrak{q}$.

(ii) There is a free basis $\{E_{\alpha_i}\}$ of $\bar{\mathfrak{u}}$ consisting of root vectors of $\bar{\mathfrak{u}}_{\mathbb{C}}$.

(iii) The $(K_L)_{\mathbb{C}}$-orbit of h is contained in the Cartan subalgebra. This is satisfied when the Levi subgroup of G corresponding to $(\mathfrak{q}_{\mathbb{C}}, (K_L)_{\mathbb{C}})$ belongs to the Harish-Chandra class in the sense of [KV] Definition 4.29.

Theorem H (Proposition 4.1.3, Proposition 4.2.2, Proposition 4.2.5)**.** Let Z be a torsion-free $(\mathfrak{q}, K_L)$-module. Moreover, assume that $Z \otimes \mathbb{C}$ is admissible and that h acts on it as a scalar.

(1) The enveloping algebra $U(\bar{\mathfrak{u}})$ is decomposed into a direct sum $U(\bar{\mathfrak{u}}) = \oplus_{\mathcal{O}} U(\bar{\mathfrak{u}})_{\mathcal{O}}$ of K_L-submodules $U(\bar{\mathfrak{u}})_{\mathcal{O}}$ which are free of finite rank as k-modules.

(2) There is an isomorphism of K_L-modules

$$\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(Z) \cong \oplus \mathrm{Hom}_k(U(\bar{\mathfrak{u}})_{\mathcal{O}}, Z).$$

In particular, it enjoys the base change formula

$$\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(Z) \otimes \mathbb{C} \cong \mathrm{pro}_{\mathfrak{q}_{\mathbb{C}}}^{\mathfrak{g}_{\mathbb{C}}}(Z \otimes \mathbb{C}).$$

Suppose that we have a semidirect product $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ which is compatible with the Levi decomposition $\mathfrak{q}_{\mathbb{C}} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_{\mathbb{C}}$. Assume also that $\mathfrak{u}$ is free of rank $r < \infty$. For an $(\mathfrak{l}, K_L)$-module λ on k, (temporarily) define $A_{\mathfrak{q}}(\lambda)$ as $R^{\dim(\mathfrak{u}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}})} \Gamma \, \mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(\lambda \otimes \wedge^r \mathfrak{u})$. Then we obtain the base change formula of $A_{\mathfrak{q}}(\lambda)$ along $k \to \mathbb{C}$ by combining Theorem H and Theorem D.

1.5 Main Results II

In Part II, we start with a reformulation of [Jant] I.2.11 (see 1.3) into a suitable form for our setting.

Theorem 1.5.1. *Let T be a diagonalizable group over a commutative ring k, and Λ be the group of characters of T.*

(1) There is an approximately unital ring $R(T)$ such that the categories of approximately unital $R(T)$-modules and T-modules are isomorphic.

(2) Let k_λ be the T-module attached to $\lambda \in \Lambda$ whose underlying k-module is k. Then $\{k_\lambda\}_{\lambda\in\Lambda}$ is a family of projective generators of the category of T-modules. Moreover, it satisfies Schur's lemma:

$$\mathrm{Hom}_T(k_\lambda, k_{\lambda'}) = \begin{cases} k & (\lambda = \lambda') \\ 0 & (\lambda \neq \lambda') \end{cases}$$

for characters $\lambda, \lambda' \in \Lambda$.

We also establish its relative versions:

Theorem 1.5.2. *Let $(\mathcal{A}, T)$ be a weak pair over a commutative ring k. Suppose that T is diagonalizable. Then there is an approximately unital ring $\mathcal{A}\sharp R(T)$ such that the categories of approximately unital $\mathcal{A}\sharp R(T)$-modules and weak $(\mathcal{A}, T)$-modules are isomorphic.*

Theorem 1.5.3. *Let $(\mathcal{A}, T)$ be a pair over a commutative ring k. Suppose that $T = T^n$ is a split torus of rank n. Then there is an approximately unital ring $R(\mathcal{A}, T)$ such that the categories of approximately unital $R(\mathcal{A}, T)$-modules and $(\mathcal{A}, T)$-modules are isomorphic.*

We next consider integral models of pairs of the finite covering groups of PU(1,1). Fix a positive integer $n > 0$. Then the pair associated to the n-cover of PU(1,1) is given as follows:

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$$

$$T^1 = \mathrm{Spec}\, \mathbb{C}\left[t^{\pm 1}\right]$$

$$\mathrm{Lie}\, T^1 \cong \mathbb{C} \to \mathfrak{sl}_2; 1 \mapsto \frac{n}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover, T^1 acts on $\mathfrak{sl}_2$ by

$$t \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = t^n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$t \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = t^{-n} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$t \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $t \in T^1$. Then the pair associated to a standard minimal parabolic subgroup of the n-cover of PU(1,1) is given by

$$\mathfrak{q}_{\mathbb{C}} = \left\{ \begin{pmatrix} a & -a+b \\ a+b & -a \end{pmatrix} : a, b \in \mathbb{C} \right\}$$

and

$$M_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}\left[t^{\pm 1}\right]/(t^n - 1) = \operatorname{Ker}(T^1 \to T^1; t \mapsto t^n).$$

We also give Borel subalgebras $\mathfrak{b}_{\mathbb{C}}$ and $\bar{\mathfrak{b}}_{\mathbb{C}}$ stable under the Cartan involution as

$$\mathfrak{b}_{\mathbb{C}} = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} : a, b \in \mathbb{C} \right\}$$

$$\bar{\mathfrak{b}}_{\mathbb{C}} = \left\{ \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

Definition 1.5.4. A split $\mathbb{Z}$-form of $(\mathfrak{sl}_2, T^1)$ is a pair $(\mathfrak{g}, T^1)$ over $\mathbb{Z}$ together with a T^1-equivariant Lie algebra homomorphism $\alpha : \mathfrak{g} \to \mathfrak{sl}_2$ such that the following conditions are satisfied:

(i) $\mathfrak{g}$ is free of finite rank as a $\mathbb{Z}$-module.

(ii) The $\mathbb{C}$-linear extension $\mathfrak{g} \otimes \mathbb{C} \to \mathfrak{sl}_2$ is an isomorphism.

(iii) The given map $\psi : \operatorname{Lie} T^1 = \mathfrak{t}^1 \to \mathfrak{g}$ is one-to-one onto the 0-weight space $\mathfrak{g}_0$.

(iv) The diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\alpha} & \mathfrak{sl}_2 \\ \uparrow & \nearrow & \\ \mathfrak{t}^1 & & \end{array}$$

commutes, where the upper diagonal arrow $\mathfrak{t}^1 = \operatorname{Lie} T^1 \cong \mathbb{Z} \to \mathfrak{sl}_2$ is given by

$$1 \mapsto \frac{n}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, α gives rise to an isomorphism $(\mathfrak{g} \otimes \mathbb{C}, T^1) \cong (\mathfrak{sl}_2, T^1)$.

Theorem 1.5.5 (Classification)**.** *For a positive integer m and $q \in \mathbb{C}^\times$, define a split $\mathbb{Z}$-form $(\mathfrak{g}_{n,m}, T^1, \alpha)$ as follows:*

- $\mathfrak{g}_{n,m}$ *has a free $\mathbb{Z}$-basis $\{E, F, H\}$;*

- *The Lie bracket of* $\mathfrak{g}_{n,m}$ *is defined by*

$$[H, E] = nE$$
$$[H, F] = -nF$$
$$[E, F] = mH;$$

- *The split torus* $T^1 = \operatorname{Spec} \mathbb{Z}\left[t^{\pm 1}\right]$ *acts on* $\mathfrak{g}_{n,m}$ *by*

$$t \cdot E = t^n E$$
$$t \cdot F = t^{-n} F$$
$$t \cdot H = H,$$

  where $t \in T^1$*;*

- *The* T^1*-equivariant Lie algebra homomorphism* $\mathfrak{t}^1 \cong \mathbb{Z} \to \mathfrak{g}_{n,m}$ *is given by* $1 \mapsto H$*.*

- *The realization homomorphism* $\alpha : \mathfrak{g}_{n,m} \to \mathfrak{sl}_2$ *is defined as*

$$\alpha(E) = q \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\alpha(F) = \frac{nm}{2q} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$\alpha(H) = \frac{n}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This gives rise to a bijection between the set of isomorphism classes of split $\mathbb{Z}$*-forms and* $\mathbb{C}^\times/\{\pm 1\} \times \mathbb{Z}_{>0}$*, where* $\mathbb{Z}_{>0}$ *is the set of positive integers.*

Convention 1.5.6. In Part II, we sometimes consider a pair over a commutative ring k as that over k' for (flat) k-algebras k' by the base change. We frequently abbreviate $\otimes_k k'$ when the base is clear.

We fix a $\mathbb{Z}$-form $(\mathfrak{g}_{n,m}, T^1, \alpha)$. Then the maximal integral models $\mathfrak{b}$ and $\bar{\mathfrak{b}}$ of $\mathfrak{b}_{\mathbb{C}}$ and $\bar{\mathfrak{b}}_{\mathbb{C}}$ are given by

$$\mathfrak{b} = \alpha^{-1}(\mathfrak{b}_{\mathbb{C}}) = \mathbb{Z}E \oplus \mathbb{Z}H$$
$$\bar{\mathfrak{b}} = \alpha^{-1}(\bar{\mathfrak{b}}_{\mathbb{C}}) = \mathbb{Z}F \oplus \mathbb{Z}H.$$

In particular, these are independent of choice of q. For a character λ of T^1, think of the modules $\operatorname{ind}^{\mathfrak{g}}_{\mathfrak{b}}(k_\lambda)$ and $\operatorname{pro}^{\mathfrak{g}}_{\bar{\mathfrak{b}}}(k_\lambda)$ as k-analogs of the (limit of) discrete series representations.

Theorem 1.5.7. *Let λ be an integer. Then we have the following descriptions:*

$$\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}_{n,m}}(k_\lambda) = \oplus_{p\geq 0} k y_{\lambda+np}$$

$$\mathrm{pro}_{\mathfrak{b}}^{\mathfrak{g}_{n,m}}(k_\lambda) = \oplus_{p\geq 0} k y^{\lambda+np}$$

$$y_{\lambda-n} = 0$$

$$E y_{\lambda+np} = y_{\lambda+n(p+1)};$$

$$F y_{\lambda+np} = -\frac{1}{2} mp(np-n+2\lambda) y_{\lambda+n(p-1)};$$

$$H y_{\lambda+np} = (\lambda+np) y_{\lambda+np}$$

$$t \cdot y_{\lambda+np} = t^{\lambda+np} y_{\lambda+np};$$

$$y^{\lambda-n} = 0;$$

$$E y^{\lambda+np} = -\frac{1}{2} m(p+1)(np+2\lambda) y^{\lambda+n(p+1)};$$

$$F y^{\lambda+np} = y^{\lambda+n(p-1)};$$

$$H y^{\lambda+np} = (\lambda+np) y^{\lambda+np};$$

$$t \cdot y^{\lambda+np} = t^{\lambda+np} y^{\lambda+np},$$

where $t \in T^1$.

We next consider integral and fractional analogs of the real parabolic induction. For our interests to find out both analogs to results over $\mathbb{C}$ and new phenomena, let us concentrate on the case $q = \frac{1}{2}$. Regard $\mathfrak{g}_{n,m}$ as a pair over $\mathbb{Z}[1/2nm]$, and set a subpair $(\mathfrak{q}, M) \subset (\mathfrak{g}_{n,m}, T^1)$ as

$$\mathfrak{q} = \alpha^{-1}(\mathfrak{q}_{\mathbb{C}}) = \mathbb{Z}[1/2nm](-2nmE + F + 2mH) \oplus \mathbb{Z}[1/2nm](2nmE + F)$$

$$M = \operatorname{Spec} \mathbb{Z}[1/2nm][t]/(t^n - 1).$$

Fix a $\mathbb{Z}[1/2nm]$-algebra k. We can now find k-analogs of [KV] Lemma 11.47 and Proposition 11.52:

Theorem 1.5.8. *Regard $(\mathfrak{q}, M) \to (\mathfrak{g}_{n,m}, T^1)$ as a map of pairs over k. Then the functor $I_{\mathfrak{q},M}^{\mathfrak{g}_{n,m},T^1}$ is exact.*

We define the structure of a $(\mathfrak{q}, M)$-module on $k = k_{\epsilon,\mu}$ for $\epsilon \in \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\}$ and $\mu \in k$ by

$$(-2nmE + F + 2mH) \cdot 1 = 0;$$

$$(2nmE + F) \cdot 1 = \mu;$$

$$t \cdot 1 = t^{n\epsilon},$$

where $t \in M$.

Theorem 1.5.9 (Fractional models of principal series representations)**.** *The $(\mathfrak{g}_{n,m}, T^1)$-module $I_{\mathfrak{q},M}^{\mathfrak{g}_{n,m},T^1}(k_{\epsilon,\mu})$ is free over k. Moreover, there is a basis $\{w^{n(p+\epsilon)} : p \in \mathbb{Z}\}$ such that the action of $(\mathfrak{g}_{n,m}, T^1)$ is given by*

$$Ew^{n(p+\epsilon)} = (\frac{1}{4nm}\mu + \frac{1}{2}(p+\epsilon))w^{n(p+1+\epsilon)};$$

$$Fw^{n(p+\epsilon)} = (\frac{1}{2}\mu - nm(p+\epsilon))w^{n(p-1+\epsilon)};$$

$$Hw^{n(p+\epsilon)} = n(p+\epsilon)w^{n(p+\epsilon)};$$

$$t \cdot w^{n(p+\epsilon)} = t^{n(p+\epsilon)}w^{n(p+\epsilon)},$$

where $t \in T^1$.

The usual proofs for $k = \mathbb{C}$ work in this setting. The point is the Iwasawa decomposition $\mathfrak{t}^1 \oplus \mathfrak{q} \cong \mathfrak{g}_{n,m}$ over our ring k. However, this isomorphism fails when we work over $\mathbb{Z}$. As a result, the resulting modules get possibly smaller. Define a subpair $(\mathfrak{q}, M)$ over $\mathbb{Z}$ as

$$\mathfrak{q} = \mathbb{Z}(-2nmE + F + 2mH) \oplus \mathbb{Z}(2nmE + F)$$

$$M = \operatorname{Spec} \mathbb{Z}[t]/(t^n - 1).$$

Notation 1.5.10. Let us denote the 2-adic valuation on $\mathbb{Z}$ by ord_2. That is, for a (nonzero) integer a, $\operatorname{ord}_2 a$ is the highest exponent M such that 2^M divides a.

Theorem 1.5.11. *For $\epsilon \in \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\}$ and $\mu \in \mathbb{Z}$, put the structure of a $(\mathfrak{q}, M)$-module into $\mathbb{Z} = \mathbb{Z}_{\epsilon,\mu}$ by*

$$(-2nmE + F + 2mH) \cdot 1 = 0;$$

$$(2nmE + F) \cdot 1 = \mu;$$

$$t \cdot 1 = t^{n\epsilon},$$

where $t \in M$. Then $I_{\mathfrak{q},M}^{\mathfrak{g}_{n,m},T^1}(\mathbb{Z}_{\epsilon,\mu})$ is nonzero if and only if $\frac{1}{2nm}\mu + \epsilon \in \mathbb{Z}$. Moreover, if $\frac{1}{2nm}\mu + \epsilon \in \mathbb{Z}$, there is a canonical isomorphism

$$I_{\mathfrak{q},M}^{\mathfrak{g}_{n,m},T^1}(\mathbb{Z}_{\epsilon,\mu}) = \oplus_{p \leq -\frac{1}{2nm}\mu - \epsilon} \mathbb{Z}2^{M_p} w^{n(p+\epsilon)} \subset I_{\mathfrak{q},M}^{\mathfrak{g}_{n,m},T^1}(\mathbb{Z}[1/2nm]_{\epsilon,\mu}),$$

where for each integer $p \leq -\frac{1}{2nm}\mu - \epsilon$,

$$M_p = \max\{-\sum_{l=0}^{s} \operatorname{ord}_2(\frac{1}{4nm}\mu + \frac{1}{2}(l+p+\epsilon)) : 0 \leq s \leq -(p + \frac{1}{2nm}\mu + \epsilon + 1)\} \cup \{0\}.$$

Remark 1.5.12. There are other choices of $\mathbb{Z}$-forms of $\mathfrak{q}_{\mathbb{C}}$. In fact, a $\mathbb{Z}$-form is determined by the "Levi part" and a submodule of $\mathbb{Z}(-2nmE+F+2mH)$ as a nilradical. For instance, the maximal $\mathbb{Z}$-form is

$$\mathbb{Z}(-2nmE+F+2mH)\oplus\mathbb{Z}(-2nE+H).$$

For each choice, we can think of $\mu\in\mathbb{Z}$ as the parameter $\mu\in\mathbb{C}$ for $(\mathfrak{q}_{\mathbb{C}},T^1)\subset(\mathfrak{sl}_2,T^1)$ via α. In other words, to fix a $\mathbb{Z}$-form of $\mathfrak{q}_{\mathbb{C}}$ is to fix a $\mathbb{Z}$-form $\mathbb{Z}_{\epsilon,\mu}$ of the $(\mathfrak{q}_{\mathbb{C}},T^1)$-module $\mathbb{C}_{\epsilon,\mu}$ for $\mu\in\mathbb{C}$ in our context. Formally, Theorem 1.5.11 is independent of the choice of $\mathbb{Z}$-forms except the coefficient of μ.

We can find the cases where lower weights vanish and the "full part" survives. Put $q=nm$ to set

$$\mathfrak{q}'=\mathbb{Z}(-E+2nmF+2mH)\oplus\mathbb{Z}(E+2nmF).$$

Similarly, put $q=n$ and $m=2n$ to define

$$\mathfrak{q}''=\mathbb{Z}(-E+F+2H)\oplus\mathbb{Z}(E+F).$$

Then $(\mathfrak{q}',M)$ (resp. $(\mathfrak{q}'',M)$) is a subpair of $(\mathfrak{g}_{n,m},T^1)$ (resp. $(\mathfrak{g}_{n,2n},T^1)$) over $\mathbb{Z}$. In both cases, define $k_{\epsilon,\mu}$ in a similar way.

Theorem 1.5.13. *Let k be a $\mathbb{Z}[1/2nm]$-algebra, $\mu\in k$, and $\epsilon\in\{0,\frac{1}{n},\frac{2}{n},\cdots,\frac{n-1}{n}\}$. Then $(\mathfrak{g}_{n,m},T^1)$-module $I^{\mathfrak{g}_{n,m},T^1}_{\mathfrak{q}',M}(k_{\epsilon,\mu})$ is free over k. Moreover, there is a basis $\{(w')^{n(p+\epsilon)}:p\in\mathbb{Z}\}$ such that the action of $(\mathfrak{g}_{n,m},T^1)$ is given by*

$$E(w')^{n(p+\epsilon)}=(\frac{1}{2}\mu+nm(p+\epsilon))(w')^{n(p+1+\epsilon)};$$

$$F(w')^{n(p+\epsilon)}=(\frac{1}{4nm}\mu-\frac{1}{2}(p+\epsilon))(w')^{n(p-1+\epsilon)};$$

$$H(w')^{n(p+\epsilon)}=n(p+\epsilon)(w')^{n(p+\epsilon)};$$

$$t\cdot(w')^{n(p+\epsilon)}=t^{n(p+\epsilon)}(w')^{n(p+\epsilon)},$$

where $t\in T^1$.

Theorem 1.5.14. *Let $\epsilon\in\{0,\frac{1}{n},\frac{2}{n},\cdots,\frac{n-1}{n}\}$ and $\mu\in\mathbb{Z}$. Then $I^{\mathfrak{g}_{n,m},T^1}_{\mathfrak{q}',M}(\mathbb{Z}_{\epsilon,\mu})$ is nonzero if and only if $\frac{1}{2nm}\mu-\epsilon\in\mathbb{Z}$. Moreover, if $\frac{1}{2nm}\mu-\epsilon\in\mathbb{Z}$ then there is a canonical isomorphism*

$$I^{\mathfrak{g}_{n,m},T^1}_{\mathfrak{q},M}(\mathbb{Z}_{\epsilon,\mu})=\oplus_{p\geq\frac{1}{2nm}\mu-\epsilon}\mathbb{Z}2^{N_p}(w')^{n(p+\epsilon)},$$

where for each integer $p\geq\frac{1}{2nm}\mu-\epsilon$,

$$N_p=\max\{-\sum_{l=0}^{s}\mathrm{ord}_2(\frac{1}{4nm}\mu+\frac{1}{2}(l-p-\epsilon)):0\leq s\leq p-\frac{1}{2nm}\mu+\epsilon-1\}\cup\{0\}.$$

Theorem 1.5.15. *Let k be a $\mathbb{Z}[1/2]$-algebra, $\mu \in k$, and $\epsilon \in \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\}$. Then $(\mathfrak{g}_{n,2n}, T^1)$-module $I^{\mathfrak{g}_{n,2n},T^1}_{\mathfrak{q}'',M}(k_{\epsilon,\mu})$ is free over k. Moreover, there is a basis $\{(w'')^{n(p+\epsilon)} : p \in \mathbb{Z}\}$ such that the action of $(\mathfrak{g}_{n,2n}, T^1)$ is given by*

$$E(w'')^{n(p+\epsilon)} = (\frac{1}{2}\mu + n(p+\epsilon))(w'')^{n(p+1+\epsilon)};$$

$$F(w'')^{n(p+\epsilon)} = (\frac{1}{2}\mu - n(p+\epsilon))(w'')^{n(p-1+\epsilon)};$$

$$H(w'')^{n(p+\epsilon)} = n(p+\epsilon)(w'')^{n(p+\epsilon)};$$

$$t \cdot (w'')^{n(p+\epsilon)} = t^{n(p+\epsilon)}(w'')^{n(p+\epsilon)},$$

where $t \in T^1$.

Theorem 1.5.16. *Let $\epsilon \in \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\}$ and $\mu \in \mathbb{Z}$. Then $I^{\mathfrak{g}_{n,2n},T^1}_{\mathfrak{q}'',M}(\mathbb{Z}_{\epsilon,\mu})$ vanishes if and only if μ is odd. Moreover, if μ is even, there is a canonical isomorphism*

$$I^{\mathfrak{g}_{n,2n},T^1}_{\mathfrak{q}'',M}(\mathbb{Z}_{\epsilon,\mu}) \cong \oplus_{p\in\mathbb{Z}}\mathbb{Z}(w'')^{n(p+\epsilon)}.$$

We also have a contraction analog. Consider the contraction family $(\widetilde{\mathfrak{sl}}_2, T^1)$. Set

$$\tilde{\mathfrak{b}}_{\mathbb{C}} = \mathbb{C}[z]\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C}[z]\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \subset \widetilde{\mathfrak{sl}}_2$$

$$\tilde{\bar{\mathfrak{b}}}_{\mathbb{C}} = \mathbb{C}[z]\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{C}[z]\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \subset \widetilde{\mathfrak{sl}}_2$$

$$X = \begin{pmatrix} z & -z \\ 1 & -z \end{pmatrix} \in \widetilde{\mathfrak{sl}}_2$$

$$Y = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \in \widetilde{\mathfrak{sl}}_2$$

$$\tilde{\mathfrak{q}} = \mathbb{C}[z]X \oplus \mathbb{C}[z]Y$$

$$M = \operatorname{Spec}\mathbb{C}[z][t]/(t^n - 1).$$

Then $(\tilde{\mathfrak{b}}_{\mathbb{C}}, T^1)$, $(\tilde{\bar{\mathfrak{b}}}_{\mathbb{C}}, T^1)$, and $(\tilde{\mathfrak{q}}, M)$ are subpairs of $(\widetilde{\mathfrak{sl}}_2, T^1)$.

Theorem 1.5.17. *Let λ be an integer. Then we have the following descriptions:*

$$\operatorname{ind}^{\widetilde{\mathfrak{sl}}_2}_{\tilde{\bar{\mathfrak{b}}}_{\mathbb{C}}}(\mathbb{C}[z]_\lambda) = \oplus_{p\geq 0}\mathbb{C}[z]\, y_{\lambda+np}$$

$$\operatorname{pro}^{\widetilde{\mathfrak{sl}}_2}_{\tilde{\mathfrak{b}}_{\mathbb{C}}}(\mathbb{C}[z]_\lambda) = \oplus_{p\geq 0}\mathbb{C}[z]\, y^{\lambda+np}$$

$$y_{\lambda-n} = 0$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y_{\lambda+np} = y_{\lambda+n(p+1)};$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y_{\lambda+np} = -\frac{1}{n} pz(np - n + 2\lambda) y_{\lambda-n(p-1)};$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y_{\lambda+np} = \frac{2}{n}(\lambda + np) y_{\lambda+np}$$

$$t \cdot y_{\lambda+np} = t^{\lambda+np} y_{\lambda+np};$$

$$y^{\lambda-n} = 0;$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y^{\lambda+np} = -\frac{z}{n}(p+1)(np + 2\lambda) y^{m+n(p+1)};$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y^{\lambda+np} = y^{\lambda+n(p-1)};$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y^{\lambda+np} = \frac{2}{n}(\lambda + np) y^{\lambda+np};$$

$$t \cdot y^{\lambda+np} = t^{\lambda+np} y^{\lambda+np},$$

where $t \in T^1$*.*

For $\epsilon \in \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\}$ and $\mu \in \mathbb{C}\left[z^{\pm 1}\right]$, $\mathbb{C}\left[z^{\pm 1}\right] = \mathbb{C}\left[z^{\pm 1}\right]_{\epsilon,\mu}$ is a $(\tilde{\mathfrak{q}}, M)$-module for

$$X \cdot 1 = 0;$$

$$Y \cdot 1 = \mu;$$

$$t \cdot 1 = t^{n\epsilon},$$

where $t \in M$. Notice that if $\mu \in \mathbb{C}[z]$, it restricts to $\mathbb{C}[z]$, which the resulting module will be denoted by $\mathbb{C}[z]_{\epsilon,\mu}$.

Theorem 1.5.18. *(1) There is a free* $\mathbb{C}\left[z^{\pm 1}\right]$*-basis* $\{w^{n(p+\epsilon)} : p \in \mathbb{Z}\}$ *of* $I^{\widetilde{\mathfrak{sl}}_2, T^1}_{\tilde{\mathfrak{q}}, M}(\mathbb{C}\left[z^{\pm 1}\right]_{\epsilon,\mu})$ *such that* $\widetilde{\mathfrak{sl}}_2$ *and* T^1 *act on* $I^{\widetilde{\mathfrak{sl}}_2, T^1}_{\tilde{\mathfrak{q}}, M}(\mathbb{C}\left[z^{\pm 1}\right])$ *as*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} w^{n(p+\epsilon)} = (\frac{1}{2z}\mu + p + \epsilon) w^{n(p+1+\epsilon)};$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} w^{n(p+\epsilon)} = (\frac{1}{2}\mu - z(p+\epsilon)) w^{n(p-1+\epsilon)};$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w^{n(p+\epsilon)} = 2(p+\epsilon) w^{n(p+\epsilon)};$$

$$t \cdot w^{n(p+\epsilon)} = t^{n(p+\epsilon)} w^{n(p+\epsilon)},$$

where $t \in T^1$*.*

(2) Suppose that $\mu \in \mathbb{C}[z]$ *with a nonzero constant term. Then* $I^{\widetilde{\mathfrak{sl}}_2, T^1}_{\tilde{\mathfrak{q}}, M}(\mathbb{C}\left[z^{\pm 1}\right]_{\epsilon,\mu})$ *vanishes.*

(3) Assume $\mu \in z\mathbb{C}[z]$. *Then* $\oplus_{p\in\mathbb{Z}}\mathbb{C}[z]\,w^{n(p+\epsilon)}$ *is a* $(\widetilde{\mathfrak{sl}}_2, T^1)$*-submodule of* $I^{\widetilde{\mathfrak{sl}}_2,T^1}_{\tilde{\mathfrak{q}},M}(\mathbb{C}\left[z^{\pm 1}\right]_{\epsilon,\mu})$ *over* $\mathbb{C}[z]$. *Moreover, we have an isomorphism*

$$\oplus_{p\in\mathbb{Z}}\mathbb{C}[z]\,w^{n(p+\epsilon)} \cong I^{\widetilde{\mathfrak{sl}}_2,T^1}_{\tilde{\mathfrak{q}},M}(\mathbb{C}[z]_{\epsilon,\mu}).$$

Finally, we discuss the algebraic Borel-Weil-Bott induction over $\mathbb{Z}$. Let G be a split reductive group over $\mathbb{Z}$, and $T \subset G$ be a split maximal torus. Choose a positive system Δ^+ of the root system of G. Let $\bar{\mathfrak{b}}$ be the Borel subalgebra corresponding to $-\Delta^+$, and $\bar{\mathfrak{n}}$ be its nilradical.

Theorem 1.5.19 (The algebraic Borel-Weil theorem over $\mathbb{Z}$)**.** *Let* λ *be a dominant character of* T. *Then the* G*-module* $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda)$ *exhibits the maximal* $\mathbb{Z}$*-form of the irreducible representation* $V(\lambda)$ *of* G *over the field* $\mathbb{Q}$ *of rational numbers with highest weight* λ *among those whose highest weight subspaces are* $\mathbb{Z} \subset \mathbb{Q}$ *for the given embedding to* $V(\lambda)$.

Example 1.5.20. Suppose that G is simply connected. Then $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda)$ coincides with the admissible lattice $M_{\max}$ in [Hum] 27.3 Proposition.

For the computations of the derived functor over $\mathbb{Z}$, the new situation is the long exact sequence

$$0 \to I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda) \to I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Q}_\lambda) \to I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}((\mathbb{Q}/\mathbb{Z})_\lambda) \to R^1 I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda) \to R^1 I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Q}_\lambda) \to \cdots.$$

Note that the construction of [KV] (2.124) does not supply an injective resolution of $\mathbb{Z}_\lambda$. On the other hand, we see that it works for $(\mathbb{Q}/\mathbb{Z})_\lambda$. As a result, we obtain the following vanishing theorem

Theorem 1.5.21. *The cohomology* $R^i I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda)$ *vanishes if* $i > |\Delta^+| + 1$.

The remarkable point is that we have a torsion module at the degree $|\Delta^+|+1$ due to the flat base change theorem.

We also see the appearance of torsions for $G = \mathrm{SL}_2$:

Theorem 1.5.22. *For any integer* λ*, the counit* $I^{\mathfrak{sl}_2,\mathrm{SL}_2}_{\bar{\mathfrak{b}},T^1}((\mathbb{Q}/\mathbb{Z})_\lambda) \to (\mathbb{Q}/\mathbb{Z})_\lambda$ *is surjective.*

This implies that if choose T^1 to be the diagonal subgroup, $\bar{\mathfrak{b}}$ to be the subalgebra of lower triangular matrices, and λ to be negative then $R^1 I^{\mathfrak{sl}_2,\mathrm{SL}_2}_{\bar{\mathfrak{b}},T^1}(\mathbb{Z}_\lambda)$ has infinitely many nonzero torsion elements.

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Part I
Flat base change formulas

2 Comodules

2.1 Generalities on comodules

In this section, let (C, Δ, ϵ) be a coalgebra over a commutative ring k. It is easy to formulate the base change adjunction of comodules:

Proposition 2.1.1. *Let k' be a k-algebra.*

(1) The k'-module $C \otimes k'$ is a coalgebra over k' for k'-module homomorphisms induced from the composite arrows

$$C \to C \otimes C \to (C \otimes C) \otimes k' \cong (C \otimes k') \otimes_{k'} (C \otimes k')$$

$$C \to k \to k'.$$

(2) A $C \otimes k'$-comodule W is a C-comodule for

$$W \to W \otimes_{k'} (C \otimes k') \cong W \otimes C,$$

and for a C-comodule V we get a $C \otimes k'$-comodule $V \otimes k'$ for the k'-module homomorphism induced from

$$V \to V \otimes C \to (V \otimes k') \otimes_{k'} (C \otimes k').$$

Moreover, these give rise to an adjunction

$$\mathrm{Hom}_C(V, W) \cong \mathrm{Hom}_{C \otimes k'}(V \otimes k', W).$$

Notation 2.1.2. For a coalgebra C, let us denote the category of C-comodules by C-comod.

In the rest, assume that C is flat over k. We note general constructions of comodules. Let V be a C-comodule, V_0 be a k-submodule, and S be a subset of V.

Construction 2.1.3. Define $\mathcal{I}_{V,V_0}$ as the full subcategory of the overcategory $C\text{-comod}_{/V}$ spanned by subcomodules of V contained in V_0, and V_0° be the colimit of the canonical functor $\mathcal{I}_{V,V_0} \to C$-comod.

Proposition 2.1.4. *(1) The category $\mathcal{I}_{V,V_0}$ is filtered.*

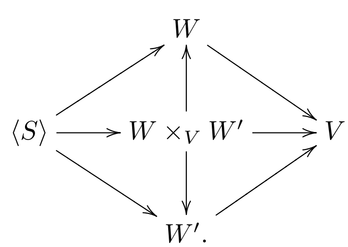
(2) The comodule V_0° exhibits the maximal subcomodule of V contained in V_0.

proof. To prove (1), suppose that we are given two comodules $W, W' \subset V_0$. Then the image of the summation $W \oplus W' \to V$ belongs to $\mathcal{I}_{V,V_0}$. Since $\mathcal{I}_{V,V_0}$ is a diagram of subobjects of a fixed object of a category, the other condition automatically follows. Part (2) now follows since filtered colimits of k-modules are exact. □

Construction 2.1.5. Define $\mathcal{J}_{V,S}$ as the full subcategory of the overcategory $C\text{-comod}_{/V}$ spanned by subcomodules of V containing S, and set $\langle S \rangle$ as the limit of the canonical diagram $\mathcal{J}_{V,S} \to C$-comod.

Proposition 2.1.6. *The comodule $\langle S \rangle$ exhibits the smallest subcomodule of V containing S.*

proof. Choose a vertex $S \subset W \subset V$ of $C\text{-comod}_{/V}$, and denote the composite arrow $\langle S \rangle \to W \to V$ by i. Observe that i is independent of the choice of W. In fact, take another object $S \subset W' \subset V$. Since monomorphisms are stable under pullbacks, $W \times_V W'$ is a subcomodule of W, W' containing S. The resulting commutative diagram



shows the independence.

We next prove that the map $\langle S \rangle \to V$ is a monomorphism. Suppose that we are given two homomorphisms $U \overset{f}{\underset{g}{\rightrightarrows}} \langle S \rangle \xrightarrow{i} V$ such that $i \circ f = i \circ g$. Let us denote the canonical projection $\langle S \rangle \to W$ by p_W, and the inculsion $W \hookrightarrow V$ by i_W. The equality

$$i_W \circ p_W \circ f = i \circ f = i \circ g = i_W \circ p_W \circ g$$

implies $p_W \circ f = p_W \circ g$. Therefore these equal maps form a cone over $\mathfrak{J}_{V,S}$ whose vertex is U. Moreover, the two maps $U \overset{f}{\underset{g}{\rightrightarrows}} \langle S \rangle$ are morphisms of cones. Since $\langle S \rangle$ is terminal, the two arrows are equal.

Finally, we prove that $\langle S \rangle$ is the minimum. In fact, if we are given a subcomodule $S \subset W \subset V$, then we have a commutative diagram

$$\begin{array}{ccc} \langle S \rangle & \longrightarrow & V \\ & \searrow \quad \nearrow & \\ & W & \end{array}$$

by definition. Since the upper horizontal and the upper right diagonal arrows are injective, so is the rest. □

Example 2.1.7 ([Hum] 27 Exercise 8)**.** Set C as the coordinate ring of the affine group scheme SL_2 over $\mathbb{Z}$. Let V be an irreducible representation of SL_2 over $\mathbb{Q}$ with $\dim V = n+1$, and v_n be a highest weight vector of V. Then $V^m := \langle v_n \rangle \subset V$ is described as follows:

$$V^m = \oplus_{i=0}^n \mathbb{Z} v_{n-2i}$$

$$Ev_{n-2i} = (n-i+1)v_{n-2i+2}$$
$$Fv_{n-2i} = (i+1)v_{n-2i-2}.$$

Proposition 2.1.8 ([Jant] I.2.13)**.** *For an element $v \in V$, the comodule $\langle v \rangle$ is contained in a finitely generated k-module.*

This leads us to a categorical conclusion for comodules. To state it, we prepare some general teminologies and facts. For our applications, we may restrict ourselves to abelian categories if necessary. For general references, see [AR] and [Bor].

Definition 2.1.9. Let $\mathcal{A}$ be a locally small cocomplete abelian category. Then a small set G of objects of $\mathcal{A}$ is called a family of generators if the following equivalent conditions are satisfied:

(a) Maps $f, g : X \to Y$ satisfying $f \circ e = g \circ e$ for any $Q \in G$ and $e \in \mathrm{Hom}(Q, X)$ are equal.

(b) For every object $X \in \mathcal{A}$, the morphism $\coprod_{\substack{Q \in G \\ e \in \mathrm{Hom}(Q,X)}} Q \overset{(e)}{\to} X$ is epic.

(c) If we are given a monomorphism $i : X \to Y$ which is not an isomorphism, there exists a map $Q \to Y$ with $Q \in G$ that does not factors through i.

(d) A morphism $X \to Y$ in $\mathcal{A}$ is an isomorphism if and only if for any member $Q \in G$, the induced map $\mathrm{Hom}(Q, X) \to \mathrm{Hom}(Q, Y)$ is a bijection.

The next fact is obvious by definition:

Lemma 2.1.10. *A functor between locally small (cocomplete abelian) categories with a faithful right adjoint functor respects families of generators.*

Definition 2.1.11. Let $\mathcal{C}$ be a locally small category with small filtered colimits. Then an object $A \in \mathcal{C}$ is said to be compact if for any small filtered diagram $Y_\bullet$ of $\mathcal{C}$, the induced map

$$\varinjlim \operatorname{Hom}(A, Y_\bullet) \to \operatorname{Hom}(A, \varinjlim Y_\bullet)$$

is a bijection.

Definition 2.1.12. A locally small cocomplete abelian category is compactly generated if it admits a small set of compact generators.

We have a nontrivial consequence from characterizations of compactly generated categories:

Lemma 2.1.13 ([AR] Remark 1.9, the proof of Theorem 1.11)**.** *Let $\mathcal{A}$ be a compactly generated (abelian) category with a small set G of compact generators. Then compact objects of $\mathcal{A}$ are generated by G under finite colimits.*

These are used in 3.2 and 4.1 as key techniques. We now go back to comodules.

Corollary 2.1.14. *If k is Noetherian, the category C-*comod *is compactly generated. In other words, every comodule is the union of its finitely generated subcomodules.*

proof. The assertions follow from Proposition 2.1.8. Note that for a C-comodule V, the following conditions are equivalent ([Hov] Proposition 1.3.3):

(a) V is compact in C-comod;

(b) V is compact as a k-module;

(c) V is a finitely presented k-module.

□

Corollary 2.1.15. *Suppose that k is a PID. Then indecomposable comodules which are free of finite rank over k form a family of generators of C-*comod.

proof. According to the proof of [KGTL] Proposition 1.2, subcomodules of direct sums of finite copies of C form a family of generators of C-comod. In view of Lemma 2.1.8, we may restrict the members of the families to torsion-free finitely generated subcomodules. The assertion is now obvious. □

The next lemma is used in the end of Part I:

Lemma 2.1.16. *Let $k \to k'$ be an injective homomorphism of commutative rings, C be a flat coalgebra over k, and V be a C-comodule. Suppose that the following conditions are satisfied:*

(i) V is flat as a k-module.

(ii) There is a decomposition $V \cong \oplus_{\mathcal{O}} V_{\mathcal{O}}$ as a k-module.

(iii) $V \otimes k' \cong \oplus_{\mathcal{O}} V_{\mathcal{O}} \otimes k'$ is a direct sum of $C \otimes k'$-subcomodules of $V \otimes k'$.

Then each of $V_{\mathcal{O}}$ is a subcomodule of V, and $V \cong \oplus_{\mathcal{O}} V_{\mathcal{O}}$ exhibits a decomposition as a C-comodule.

proof. According to (ii), we have an isomorphism $V \otimes C \cong \oplus_{\mathcal{O}} V_{\mathcal{O}} \otimes C$. It will suffice to show that the coaction respects each $\mathcal{O}$-component. Take the base change along $k \to k'$ to obtain a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & V \otimes C \\ \downarrow & & \downarrow \\ V \otimes k' & \longrightarrow & (V \otimes k') \otimes_{k'} (C \otimes k'). \end{array}$$

Since V and C are flat, the vertical arrows are injective. Therefore the assertion is reduced to $k = k'$, and it is equivalent to (iii). $\square$

2.2 Representations of flat affine group schemes and $(\mathfrak{g}, K)$-modules

Let H be a commutative Hopf algebra, and write $K = \operatorname{Spec} H$. For a k-module V and a k-algebra R, set $\operatorname{Aut}_R(V \otimes R)$ as the group of automorphisms of the R-module $V \otimes R$. This determines a group k-functor $\operatorname{Aut}(V) : \mathcal{CA}\mathrm{lg}_k \to \mathcal{G}\mathrm{rp}; R \mapsto \operatorname{Aut}_R(V \otimes R)$, where $\mathcal{CA}\mathrm{lg}_k$ (resp. $\mathcal{G}\mathrm{rp}$) is the category of commutative k-algebras (resp. groups). We also write $\mathcal{CA}\mathrm{lg}_{k,\mathrm{flat}}$ for the full subcategory of $\mathcal{CA}\mathrm{lg}_k$ spanned by flat k-algebras. Note that $\mathcal{CA}\mathrm{lg}_{k,\mathrm{flat}}$ is stable under $\otimes$. Recall that a representation of K is a k-module V, equipped with a homomorphism $K \to \operatorname{Aut}(V)$ of group k-functors. Equivalently, a representation is a k-module, equipped with an R-linear group action of $K(R)$ on $V \otimes R$ for each k-algebra R such that for $f : R \to R'$ and $g \in K(R)$ the diagram

$$\begin{array}{ccc} V \otimes R & \xrightarrow{\nu(g)} & V \otimes R \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ V \otimes R' & \xrightarrow[\nu(f \circ g)]{} & V \otimes R' \end{array}$$

commutes. A k-module homomorphism $f : V \to V'$ of representations of K is said to be a K-homomorphism if for all k-algebras R, the diagrams

$$\begin{array}{ccc} V \otimes R & \xrightarrow{\nu(g)} & V \otimes R \\ {\scriptstyle f\otimes id_R}\downarrow & & \downarrow{\scriptstyle f\otimes id_R} \\ V' \otimes R & \xrightarrow[\nu'(g)]{} & V' \otimes R \end{array}$$

commute. Set K-mod as the category of representations of K. If K is flat over k, define K-$\mathrm{mod}_{\mathrm{flat}}$ in a similar way.

Lemma 2.2.1. *The categories* K-mod *and* H-comod *are an isomorphic. Moreover, if K is flat, these are also isomorphic to* K-$\mathrm{mod}_{\mathrm{flat}}$.

proof. See [Wa] Theorem 3.2 for the first assertion. In view of its proof, the coaction of H is recovered by the actions of the valued point groups $K(k)$, $K(H)$, and $K(H \otimes H)$. Therefore the same argument proves H-comod $\cong K$-$\mathrm{mod}_{\mathrm{flat}}$ if H is flat. □

Suppose next that H is flat over k. Though we have a general description of the internal Hom of the symmetric monoidal category H-comod ([Hov] Theorem 1.3.1), it is usually too complicated to compute in practice. Here we give a better realization in a special case:

Proposition 2.2.2. *Let K be an affine group scheme, and V, V' be K-modules.*

(1) If V is finitely generated and projective as a k-module, there is a natural K-action on $\mathrm{Hom}(V, V')$. *Moreover, the standard adjunction* $\mathrm{Hom}_k(- \otimes V, V') \cong \mathrm{Hom}_k(-, \mathrm{Hom}(V, V'))$ *restricts to*

$$\mathrm{Hom}_K(- \otimes V, V') \cong \mathrm{Hom}_K(-, \mathrm{Hom}(V, V')).$$

(2) Suppose that K is flat. If V satisfies Condition 1.1.1 (2), there is a natural K-action on $\mathrm{Hom}(V, V')$. *Moreover, the standard adjunction* $\mathrm{Hom}_k(- \otimes V, V') \cong \mathrm{Hom}_k(-, \mathrm{Hom}(V, V'))$ *restricts to*

$$\mathrm{Hom}_K(- \otimes V, V') \cong \mathrm{Hom}_K(-, \mathrm{Hom}(V, V')).$$

proof. Suppose that V is finitely generated and projective as a k-module. Then for any k-algebra R, we have a canonical isomorphism $\mathrm{Hom}(V, V') \otimes R \cong \mathrm{Hom}(V, V' \otimes R) \cong \mathrm{Hom}_R(V \otimes R, V' \otimes R)$. Under this identification, we put a $K(R)$-action on $\mathrm{Hom}(V, V') \otimes R$ by

$$(\nu(g)f)(v) = \nu_{V'}(g)f(\nu_V(g^{-1})v).$$

Running through all R, we obtain $\mathrm{Hom}(V, V') \in K$-mod. We can see the adjunction in the usual way. If K is flat, we may restrict R to be flat ones (Lemma 2.2.1). Then the same argument works for V with Condition 1.1.1 (2). □

According to Lemma 2.2.1 and Proposition 2.2.2 (2), the arguments of [H1] still work for pairs in the sense of the end of 1.1. Therefore Lemma 1.1.6 follows.

Proposition 2.2.3. *Let $(\mathfrak{g}, K)$ be a pair over a commutative ring k.*

(1) For $(\mathfrak{g}, K)$-modules V and V', the tensor product $V \otimes W$ is a $(\mathfrak{g}, K)$-module for the tensor representation of K and

$$\pi_{V\otimes V'}(x)(v \otimes v') = \pi_V(x)v \otimes v' + v \otimes \pi_{V'}(x)v',$$

where $v \otimes v' \in V \otimes V'$, $x \in \mathfrak{g}$, and π_V (resp. $\pi_{V'}$) denotes the action of $\mathfrak{g}$ on V (resp. V').

(2) The category $(\mathfrak{g}, K)$-mod is closed symmetric monoidal for (1). Moreover, the closed structure is compatible with that of K-mod.

Notation 2.2.4. The internal Hom of the symmetric monoidal category K-mod will be denoted by $F(-,-)$.

Proof of Proposition 2.2.3. It is easy to see that the tensor product $V \otimes V'$ of (1) is a module over both K and $\mathfrak{g}$. Apply $- \otimes V'$ and $- \otimes V$ to the K-equivariant maps

$$\mathfrak{g} \otimes V \to V$$
$$\mathfrak{g} \otimes V' \to V'$$

respectively. Since K-mod is symmetric monoidal, we have two K-equivariant maps from $\mathfrak{g} \otimes V \otimes V'$ to $V \otimes V'$. Since their sum coincides with $\pi_{V\otimes V'}$, it is also K-equivariant. The actions of $\mathfrak{k}$ coincide from the Leibnitz rule of the differential representations for the tensor product. To see that this defines a symmetric monoidal category, it will suffice to show that the constraints of associativity and symmetry of K-mod respect the $\mathfrak{g}$-actions. This is obvious.

Recall that we have a K-equivariant k-homomorphism $\mathfrak{g} \to U(\mathfrak{g}) \otimes U(\mathfrak{g}); x \mapsto x \otimes 1 - 1 \otimes x$ (regard $\mathfrak{g} \cong \mathfrak{g} \otimes k \cong k \otimes \mathfrak{g}$). For $(\mathfrak{g}, K)$-modules V, V', define $\pi = \pi_{F(V,V')} : \mathfrak{g} \otimes F(V, V') \to F(V, V')$ by the following composite arrows:

$$\begin{aligned} \mathfrak{g} \otimes F(V, V') \otimes V &\to U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes F(V, V') \otimes V \\ &\cong U(\mathfrak{g}) \otimes F(V, V') \otimes U(\mathfrak{g}) \otimes V \\ &\overset{\pi_V}{\to} U(\mathfrak{g}) \otimes F(V, V') \otimes V \\ &\to U(\mathfrak{g}) \otimes V' \\ &\overset{\pi_{V'}}{\to} V'. \end{aligned}$$

This is K-equivariant by definition. To see that this is a $\mathfrak{g}$-action, we see that the two maps

$$\mathfrak{g} \otimes \mathfrak{g} \otimes F(V, V') \rightrightarrows F(V, V')$$

coincide. If we write $f \otimes v \mapsto f(v)$ for the counit $F(V, V') \otimes V \to V'$, it is equivalent to

$$(\pi([x, y])f)(v) = (\pi(x)(\pi(y)f))(v) - (\pi(y)(\pi(x)f))(v)$$

for $x, y \in \mathfrak{g}$ and $f \in F(V, V')$. Observe that $\pi_{F(V,V')}$ is characterized by the equality

$$(\pi(x)f)(v) = \pi_{V'}(x)f(v) - f(\pi_V(x)v)$$

by definition, and thus

$$\begin{aligned}(\pi(x)(\pi(y)f))(v) &= \pi(x)(\pi(y)f)(v) - (\pi(y)f)(\pi(x)v) \\ &= \pi(x)\pi(y)f(v) - \pi(x)f(\pi(y)v) - \pi(y)f(\pi(x)v) + f(\pi(y)\pi(x)v).\end{aligned}$$

The assertion now follows from the formal computation

$$\begin{aligned}(\pi([x,y])f)(v) &= \pi([x,y])f(v) - f(\pi([x,y])v) \\ &= \pi(x)\pi(y)f(v) - \pi(y)\pi(x)f(v) - f(\pi(x)\pi(y)v) + f(\pi(y)\pi(x)v) \\ &= (\pi(x)(\pi(y)f))(v) - (\pi(y)(\pi(x)f))(v).\end{aligned}$$

We next show that $F(V, V')$ is a $(\mathfrak{g}, K)$-module. Since V, V' are $(\mathfrak{g}, K)$-modules, the action π can be rewritten as

$$(\pi(\xi)f)(v) = \pi(\xi)f(v) - f(\pi(\xi)v) = d\nu(\xi)f(v) - f(d\nu(\xi)v).$$

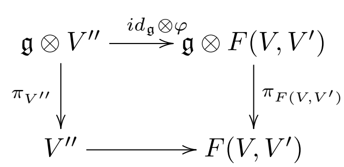
for $\xi \in \mathfrak{k}$. Since the counit $F(V, V') \otimes V \to V'$ is $\mathfrak{k}$-equivariant with respect to the differential representations, we have

$$(\pi(\xi)f)(v) = d\nu(\xi)f(v) - f(d\nu(\xi)v) = (d\nu(\xi)f)(v).$$

Finally, we prove that $F(V, V')$ exhibits the closed structure. Let V'' be another $(\mathfrak{g}, K)$-module, and $\varphi : V'' \to F(V, V')$ be a K-module homomorphism. It will suffice to show that φ is $\mathfrak{g}$-equivariant if and only if the composition $\Phi : V'' \otimes V \to F(V, V') \otimes V \to V'$ is $\mathfrak{g}$-equivariant. Observe that the following conditions are equivalent:

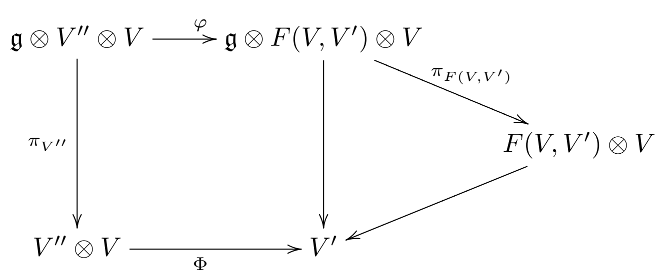
(a) φ is $\mathfrak{g}$-equivariant;

(b) The diagram



commutes;

(c) The diagram



commutes.

One can also rewrite (c) as

$$\Phi(\pi(x)v''\otimes v) = \pi(x)\varphi(v'')(v) - \varphi(v'')(\pi(x)v) = \pi(x)\Phi(v''\otimes v) - \Phi(v''\otimes \pi(x)v)$$

which is equivalent to saying that Φ is $\mathfrak{g}$-equivariant. This completes the proof. □

Definition 2.2.5. Let $(\mathfrak{g}, K)$ be a pair over a commutative ring k, and V be a $(\mathfrak{g}, K)$-module. Then set $V^c = F(V, k)$.

Suppose that we are given a map $(\mathfrak{q}, K) \to (\mathfrak{g}, K)$ of pairs which is the identity on K.

Corollary 2.2.6. *For a $(\mathfrak{q}, K)$-module W and a $(\mathfrak{g}, K)$-module V, there is a natural isomorphism $\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}} W \otimes V \cong \operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(W \otimes \mathcal{F}_{\mathfrak{g},K}^{\mathfrak{q},K}(V))$.*

proof. For a $(\mathfrak{g}, K)$-module X, we have a natural bijection

$$\begin{aligned}
\operatorname{Hom}_{\mathfrak{g},K}(\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}} W \otimes V, X) &\cong \operatorname{Hom}_{\mathfrak{g},K}(\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}} W, F(V, X)) \\
&\cong \operatorname{Hom}_{\mathfrak{q},K}(W, \mathcal{F}_{\mathfrak{g},K}^{\mathfrak{q},K}(F(V, X))) \\
&\cong \operatorname{Hom}_{\mathfrak{q},K}(W \otimes \mathcal{F}_{\mathfrak{g},K}^{\mathfrak{q},K}(V), \mathcal{F}_{\mathfrak{g},K}^{\mathfrak{q},K}(X)) \\
&\cong \operatorname{Hom}_{\mathfrak{g},K}(\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(W \otimes \mathcal{F}_{\mathfrak{g},K}^{\mathfrak{q},K}(V)), X).
\end{aligned}$$

The assertion now follows from the Yoneda lemma. □

Corollary 2.2.7 (The easy duality)**.** *There is a natural isomorphism $\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(W)^c \cong \operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}}(W^c)$ for a $(\mathfrak{q}, K)$-module W.*

proof. For a $(\mathfrak{g}, K)$-module V, we have

$$\begin{aligned}
\operatorname{Hom}_{\mathfrak{g},K}(V, \operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(W)^c) &\cong \operatorname{Hom}_{\mathfrak{g},K}(V \otimes \operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}} W, k) \\
&\cong \operatorname{Hom}_{\mathfrak{g},K}(\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(V \otimes W), k) \\
&\cong \operatorname{Hom}_{\mathfrak{q},K}(V \otimes W, k) \\
&\cong \operatorname{Hom}_{\mathfrak{q},K}(V, W^c) \\
&\cong \operatorname{Hom}_{\mathfrak{g},K}(V, \operatorname{pro}_{\mathfrak{k}}^{\mathfrak{g}}(W^c)).
\end{aligned}$$

The assertion now follows from the Yoneda Lemma. □

3 The Flat Base Change Theorems

3.1 The main statements

We start with the definition of the base change functor. Let $k \to k'$ be a homomorphism of commutative rings. For an algebra $\mathcal{A}$ over k, an $\mathcal{A} \otimes k'$-module W is an $\mathcal{A}$-module for

$$\mathcal{A} \otimes_k W \cong (\mathcal{A} \otimes k') \otimes_{k'} W \to W.$$

Conversely, if we are given an $\mathcal{A}$-module V, $V \otimes k'$ is an $\mathcal{A} \otimes k'$-module for

$$(\mathcal{A} \otimes k') \otimes_{k'} (V \otimes k') \cong (\mathcal{A} \otimes V) \otimes k' \to V \otimes k'.$$

These form an adjunction

$$\mathrm{Hom}_{\mathcal{A}}(V, W) \cong \mathrm{Hom}_{\mathcal{A} \otimes k'}(V \otimes k', W).$$

Similarly, if we are given a flat affine group scheme K over k, we have the base change adjunction (Proposition 2.1.1 (2)). In terms of k-functors, they are described as follows: For k' a k-algebra, we have

$$\begin{aligned} K(R) &\to (K \otimes k')(R \otimes k') \\ &\to \mathrm{Aut}_{R \otimes k'}(W \otimes_{k'} (R \otimes k')) \\ &\cong \mathrm{Aut}_{R \otimes k'}(W \otimes R) \\ &\to \mathrm{Aut}_R(W \otimes R) \end{aligned}$$

$$\begin{aligned} (K \otimes k')(R) &= \mathrm{Hom}_{k'}(k[K] \otimes k', R) \\ &\cong \mathrm{Hom}_k(k[K], R) \\ &\to \mathrm{Aut}_R(V \otimes R) \\ &\cong \mathrm{Aut}_R((V \otimes k') \otimes_{k'} R). \end{aligned}$$

Hence the differential representations are compatible with the restrictions and the flat base changes. That is, let k' be a flat k-algebra.

- If we are given a $K \otimes k'$-module W, the differential representation on the restriction of W to K coincides with

$$\mathfrak{k} \otimes W \cong (\mathfrak{k} \otimes k') \otimes_{k'} W \to W;$$

- For a K-module V, the differential representation of $K \otimes k'$ on the $K \otimes k'$-module $V \otimes k'$ is induced from

$$\mathfrak{k} \otimes V \to V \to V \otimes k'$$

  by the universality of the base change.

We now obtain the following consequence from these functorial constructions:

Proposition 3.1.1. *Let $(\mathfrak{g}, K)$ be a pair over k, and k' be a flat k-algebra. Then we have an adjunction of the base change*

$$- \otimes_k k' : (\mathfrak{g}, K)\text{-mod} \leftrightarrows (\mathfrak{g} \otimes k', K \otimes k')\text{-mod} : \mathrm{Res}^k_{k'}.$$

Remark 3.1.2. If K is smooth over k, the base change makes sense for all k' since the smoothness is stable under arbitrary base changes.

Remark 3.1.3. For a weak pair $(\mathfrak{g}, K)$ in the sense of [H1], the base change of weak $(\mathfrak{g}, K)$-modules always makes sense even if K does not satisfy Condition 1.1.3.

Corollary 3.1.4. *Let $(\mathfrak{q}, M) \to (\mathfrak{g}, K)$ be a map of pairs over k, k' be a flat k-algebra, and V be a $(\mathfrak{q} \otimes k', M \otimes k')$-module. Then there is an isomorphism*

$$I_{\mathfrak{q},M}^{\mathfrak{g},K}(\mathrm{Res}_{k'}^{k}(V)) \cong \mathrm{Res}_{k'}^{k}(I_{\mathfrak{q}\otimes k', M\otimes k'}^{\mathfrak{g}\otimes k', K\otimes k'}(V)).$$

In particular, if W is a $(\mathfrak{q}, M)$-module,

$$I_{\mathfrak{q},M}^{\mathfrak{g},K}(\mathrm{Res}_{k'}^{k}(W \otimes k')) \cong \mathrm{Res}_{k'}^{k}(I_{\mathfrak{q}\otimes k', M\otimes k'}^{\mathfrak{g}\otimes k', K\otimes k'}(W \otimes k')).$$

proof. Pass to the right adjoints of $(- \otimes k') \circ \mathcal{F}_{\mathfrak{g},K}^{\mathfrak{q},M} \cong \mathcal{F}_{\mathfrak{g}\otimes k', K\otimes k'}^{\mathfrak{q}\otimes k', M\otimes k'} \circ (- \otimes k')$: For any $(\mathfrak{g}, K)$-module X, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g},K}(X, I_{\mathfrak{q},M}^{\mathfrak{g},K}(\mathrm{Res}_{k'}^{k}(V))) &\cong \mathrm{Hom}_{\mathfrak{q},M}(X, \mathrm{Res}_{k'}^{k}(V)) \\
&\cong \mathrm{Hom}_{\mathfrak{q}\otimes k', M\otimes k'}(X \otimes k', V) \\
&\cong \mathrm{Hom}_{\mathfrak{g}\otimes k', K\otimes k'}(X \otimes k', I_{\mathfrak{q}\otimes k', M\otimes k'}^{\mathfrak{g}\otimes k', K\otimes k'}(V)) \\
&\cong \mathrm{Hom}_{\mathfrak{g},K}(X, \mathrm{Res}_{k'}^{k}(I_{\mathfrak{q}\otimes k', M\otimes k'}^{\mathfrak{g}\otimes k', K\otimes k'}(V))).
\end{aligned}$$

The assertion now follows from the Yoneda lemma. □

Construction 3.1.5 (The comparison natural transform)**.** Let $(\mathfrak{q}, M) \to (\mathfrak{g}, K)$ be a map of pairs over k, and k' be a flat k-algebra. Then applying $I_{\mathfrak{q},M}^{\mathfrak{g},K}$ to the unit of Proposition 3.1.1, we obtain a natural transform

$$I_{\mathfrak{q},M}^{\mathfrak{g},K}(-) \to I_{\mathfrak{q},M}^{\mathfrak{g},K}(\mathrm{Res}_{k'}^{k}(- \otimes k')) \cong \mathrm{Res}_{k'}^{k}(I_{\mathfrak{q}\otimes k', M\otimes k'}^{\mathfrak{g}\otimes k', K\otimes k'}(- \otimes k')).$$

Pass to the adjunction of Proposition 3.1.1 to get

$$I_{\mathfrak{q},M}^{\mathfrak{g},K}(-) \otimes k' \to I_{\mathfrak{q}\otimes k', M\otimes k'}^{\mathfrak{g}\otimes k', K\otimes k'}(- \otimes k')$$

which will be referred to as $\iota = \iota_{k,k'}$.

In the rest of this section, assume k to be Noetherian.

Theorem 3.1.6 (Flat base change theorem)**.** *Let k' be a flat k-algebra, and $(\mathfrak{g}, K)$ be a pair over k with $\mathfrak{g}$ finitely generated over k. Then for any finitely generated $(\mathfrak{g}, K)$-module X, we have an isomorphism*

$$\mathrm{Hom}_{\mathfrak{g},K}(X, -) \otimes k' \cong \mathrm{Hom}_{\mathfrak{g}\otimes k', K\otimes k'}(X \otimes k', - \otimes k').$$

Theorem 3.1.7. *Let $k \to k'$ be a flat ring homomorphism, and $(\mathfrak{q}, M) \to (\mathfrak{g}, K)$ be a map of pairs. Suppose that the following conditions are satisfied:*

(i) $\mathfrak{k} \oplus \mathfrak{q} \to \mathfrak{g}$ *is surjective.*

(ii) $\mathfrak{q}$ *and* $\mathfrak{g}$ *are finitely generated as k-modules.*

Then $\iota : (I_{\mathfrak{q},M}^{\mathfrak{g},K} -) \otimes_k k' \to I_{\mathfrak{q}\otimes_k k', M\otimes_k k'}^{\mathfrak{g}\otimes_k k', K\otimes_k k'}(- \otimes_k k')$ (Construction 3.1.5) is an isomorphism.

We also have its derived version:

Definition 3.1.8. Let k be a Noetherian ring, and $(\mathfrak{g}, K)$ be a pair. Suppose that $\mathfrak{g}$ is finitely generated. Set $\mathrm{Coh}(\mathfrak{g}, K)$ as the full subcategory of the derived category $D(\mathfrak{g}, K)$ spanned by cohomologically bounded complexes with finitely generated cohomologies.

Theorem 3.1.9. *Let k be a Noetherian ring, $(\mathfrak{g}, K)$ be a pair, and k' be a flat k-algebra. Then the flat base change theorem*

$$\mathbb{R}\operatorname{Hom}_{\mathfrak{g},K}(-,-) \otimes k' \simeq \mathbb{R}\operatorname{Hom}_{\mathfrak{g},K}(-,- \otimes k')$$

holds on $\mathrm{Coh}(\mathfrak{g}, K)^{op} \times D(\mathfrak{g}, K)^+$.

Theorem 3.1.10. *Let $k \to k'$ be a flat ring homomorphism, and $(\mathfrak{q}, M) \to (\mathfrak{g}, K)$ be a map of pairs. Suppose that the following conditions are satisfied:*

(i) $\mathfrak{k} \oplus \mathfrak{q} \to \mathfrak{g}$ is surjective.

(ii) $\mathfrak{q}$ and $\mathfrak{g}$ are finitely generated as k-modules.

Then we have an equivalence

$$(\mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M}-) \otimes_k k' \simeq \mathbb{R}I^{\mathfrak{g}\otimes k', K\otimes k'}_{\mathfrak{q}\otimes k', M\otimes k'}(- \otimes_k k')$$

on $D^+(\mathfrak{q}, M)$.

For a simple application, we can prove the algebraic Borel-Weil theorem over fields of characteristic 0. Suppose that k is a field of characteristic 0. Let G be a split reductive group. Fix a maximal split torus T of G and a positive root system of the Lie algebra $\mathfrak{g}$ of G. Write $\bar{\mathfrak{b}}$ for the Lie subalgebra of $\mathfrak{g}$ corresponding to the negative roots.

Proposition 3.1.11. *Let λ be a dominant character of T. There is an isomorphism $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(k_\lambda) \otimes \bar{k} \cong I^{\mathfrak{g}\otimes\bar{k}, G\otimes\bar{k}}_{\bar{\mathfrak{b}}\otimes\bar{k}, T\otimes\bar{k}}(\bar{k}_\lambda)$, where $\bar{k}$ is the algebraic closure of k. In particular, $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(k_\lambda)$ is an absolutely irreducible representation of G.*

Denote the coordinate ring of G by $\mathcal{O}(G)$.

Corollary 3.1.12. *The homomorphism of coalgebras $\oplus_\lambda \operatorname{End}_k(I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(k_\lambda)) \to \mathcal{O}(G)$ is an isomorphism, where λ runs through all dominant characters of T.*

proof. Passing to the base change along $k \to \bar{k}$, we may assume that k is algebraically closed. Then the assertion follows from the algebraic Peter-Weyl theorem. □

Corollary 3.1.13 ([Ti])**.** *The absolutely irreducible representations $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(k_\lambda)$ of G form a complete list of irreducible representations of G.*

3.2 Proof of the theorems

In this section, let k be a Noetherian ring.

Lemma 3.2.1. *Let C be a flat coalgebra, and V, X, Y be C-comodules. Suppose that we are given a commutative diagram of k-modules*

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle g} \searrow \quad \nearrow {\scriptstyle i} & \\ & X, & \end{array}$$

where i is injective. If the maps f and i intertwine the coactions of C, then so does g.

proof. Consider the diagram

$$\begin{array}{ccc} V & \xrightarrow{\rho_V} & V \otimes C \\ {\scriptstyle g}\downarrow & & \downarrow \\ X & \xrightarrow{\rho_X} & X \otimes C \\ {\scriptstyle i}\downarrow & & \downarrow \\ Y & \xrightarrow{\rho_Y} & Y \otimes C, \end{array}$$

where ρ_V (resp. ρ_X, ρ_Y) denotes the coaction of C on V (resp. X, Y). Notice that $i \otimes id_C$ is injective since C is flat. Therefore the equality

$$\begin{aligned} (i \otimes id_C) \circ (g \otimes id_C) \circ \rho_V &= (f \otimes id_C) \circ \rho_V \\ &= \rho_Y \circ f \\ &= \rho_Y \circ i \circ g \\ &= (i \otimes id_C) \circ \rho_X \circ g \end{aligned}$$

implies $(g \otimes id_C) \circ \rho_V = \rho_X \circ g$. $\square$

Lemma 3.2.2. *Let $(\mathfrak{g}, K)$ be a pair over a commutative ring k. Then a k-submodule V' of a $(\mathfrak{g}, K)$-module V is a subobject in* $(\mathfrak{g}, K)$-mod *if and only if it is a submodule over both $\mathfrak{g}$ and K.*

proof. The "only if" direction is obvious. Notice that the two actions of $\mathfrak{k}$ induced from those of $\mathfrak{g}$ and K are compatible with restriction. To prove the "if" direction, it will therefore suffice to prove that the action $\mathfrak{g} \otimes V' \to V'$ is K-equivariant. This follows by application of Lemma 3.2.1 to the diagram

$$\begin{array}{ccc} \mathfrak{g} \otimes V' & \longrightarrow & V' \\ \downarrow & & \downarrow \\ \mathfrak{g} \otimes V & \longrightarrow & V. \end{array}$$

$\square$

Lemma 3.2.3. *Let $\mathfrak{g}$ be a finitely generated Lie algebra over k. Then the enveloping algebra $U(\mathfrak{g})$ is left and right Noetherian.*

proof. The assertion follows since the enveloping algebra is by definition a quasi-commutative filtered algebra whose associated graded algebra is generated by $\mathfrak{g}$. □

Recall that a Grothendieck abelian category is said to be locally Noetherian if every object is presented by a filtered colimit of Noetherian objects.

Proposition 3.2.4. *Let k be a Noetherian ring, and $(\mathfrak{g}, K)$ be a pair over k. If $\mathfrak{g}$ is a finitely generated k-module, the category* $(\mathfrak{g}, K)$-mod *is locally Noetherian. Moreover, for a $(\mathfrak{g}, K)$-module V, the following conditions are equivalent:*

(a) V is Noetherian;

(b) V is compact;

(c) V is finitely generated as a $U(\mathfrak{g})$-module.

proof. Let V be a $(\mathfrak{g}, K)$-module. From Lemma 3.2.3, (c) implies (a). Conversely, if V is a Noetherian object, there exists a maximal finitely generated $(\mathfrak{g}, K)$-submodule $V' \subset V$. Choose a finite set S of generators of V'. Let v be an arbitrary element of V. Then we obtain a K-submodule $V_0 := \langle S, v\rangle$ which is finitely generated as a k-module (Proposition 2.1.8). Since the $\mathfrak{g}$-submodule generated by V_0 is the image of the map

$$U(\mathfrak{g}) \otimes V_0 \to U(\mathfrak{g}) \otimes V \to V,$$

it is a $(\mathfrak{g}, K)$-submodule containing V' (Lemma 3.2.3). The maximality therefore implies $V = V'$. Hence (c) follows. Moreover, Corollary 2.1.14 then implies that $(\mathfrak{g}, K)$-mod is locally Noetherian. The equivalence of (a) and (b) is a consequence of generalities on locally Noetherian abelian categories. □

Definition 3.2.5. Let B be a bialgebra. An element v of a B-comodule (V, ρ) is B-invariant if $\rho(v) = v \otimes 1$. We denote the k-submodule of invariant elements by V^B. In other words, V^B is the equalizer of the coaction ρ and $id_V \otimes 1 : V \to V \otimes B$. If B is the coordinate ring of an affine group scheme K, we will denote V^B by $H^0(K, V)$.

Proposition 3.2.6 ([Jant] I.2.10)**.** *Let V be a B-comodule, and W be a k-module. Then:*

(1) W is a B-comodule for $w \mapsto w \otimes 1$. This is called a trivial comodule.

(2) There is a natural bijection $\mathrm{Hom}_B(W, V) \cong \mathrm{Hom}_k(W, V^B)$.

(3) We have a natural identification $V^B = \mathrm{Hom}_B(k, V)$.

proof. Regard k as a coalgebra over k. Then W is a comodule over k in the obvious way. Since the given map $k \to B$ is a homomorphism of coalgebras, it induces a coaction of B on W which coincides with (1).

Part (2) is obvious by definition: Every B-comodule homomorphism $f : W \to V$ is valued in V^B. Then (3) is obtained by applying $W = k$. □

Variant 3.2.7. Let $(\mathfrak{g}, K)$ be a pair over a commutative ring k, and V be a $(\mathfrak{g}, K)$-module. Then $H^0(\mathfrak{g}, K, V)$ is naturally identified with the intersection of $H^0(K, V)$ and the $\mathfrak{g}$-invariant part of V.

Lemma 3.2.8 ([Jant] I.2.10)**.** *Let B be a bialgebra, V be a B-comodule over a commutative ring k, and k' be a flat k-algebra. Then we have*

$$V^B \otimes k' \cong (V \otimes k')^{B \otimes k'}.$$

proof. Think of V^B as $\mathrm{Ker}(\rho - id_V \otimes 1 : V \to V \otimes B)$. □

Corollary 3.2.9. *Let K be a flat affine group scheme, and V, V' be K-modules. Then there is a canonical isomorphism $H^0(K, F(V, V')) \cong \mathrm{Hom}_K(V, V')$.*

proof. It follows from the natural identification

$$H^0(K, F(V, V')) = \mathrm{Hom}_K(k, F(V, V')) \cong \mathrm{Hom}_K(V, V').$$

□

Corollary 3.2.10. *Let K be a flat affine group scheme, and Q be a representation of K. Suppose that Q is finitely presented as a k-module. Then $\mathrm{Hom}_K(Q, -)$ satisfies the flat base change formula: For any flat k-algebra k', there is a canonical isomorphism*

$$\mathrm{Hom}_K(Q, -) \otimes k' \cong \mathrm{Hom}_{K \otimes k'}(Q \otimes k', - \otimes k').$$

proof. It is immediate from Corollary 3.2.9, Proposition 2.2.2 (2), and Lemma 3.2.8. In fact, we have a natural isomorphism

$$\begin{aligned}
\mathrm{Hom}_K(Q, -) \otimes k' &\cong H^0(K, \mathrm{Hom}_k(Q, -)) \otimes k' \\
&\cong H^0(K \otimes k', \mathrm{Hom}_k(Q, -) \otimes k') \\
&\cong H^0(K \otimes k', \mathrm{Hom}_{k'}(Q \otimes k', - \otimes k')) \\
&\cong \mathrm{Hom}_{K \otimes k'}(Q \otimes k', - \otimes k').
\end{aligned}$$

□

Variant 3.2.11. Let K be a flat affine group scheme over a commutative ring k, and k' be a k-algebra which is finitely generated and projective as a k-module. Then we have a natural isomorphism

$$\mathrm{Hom}_K(-, -) \otimes k' \cong \mathrm{Hom}_{K \otimes k'}(- \otimes k', - \otimes k')$$

on $K\text{-mod}^{op} \times K\text{-mod}$.

proof. Replacing k' by a finitely generated and projective k-module W, we may prove

$$\mathrm{Hom}_K(-,-)\otimes W \cong \mathrm{Hom}_K(-,-\otimes W).$$

Here W is regarded as a trivial K-module. It reduces to the cases where W is free of finite rank by passing to retracts. Then the assertion follows since $\mathrm{Hom}_K(-,-)$ is additive in the second variable. □

Proof of Theorem 3.1.6. Let S be the collection of objects X of $(\mathfrak{g},K)$-mod such that $\mathrm{Hom}_{\mathfrak{g},K}(X,-)$ satisfies the flat base change formula. Recall that $(\mathfrak{g},K)$-mod is compactly generated, whose compact objects are the finitely generated $(\mathfrak{g},K)$-modules (Proposition 3.2.4). Since S is closed under formation of finite colimits, it will suffice to show $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{k}} Q\in S$, where Q is a K-module which is finitely generated as a k-module. For any $(\mathfrak{g},K)$-module W, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g},K}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{k}} Q,W)\otimes k' &\cong \mathrm{Hom}_K(Q,W)\otimes k'\\
&\cong \mathrm{Hom}_{K\otimes k'}(Q\otimes k',W\otimes k')\\
&\cong \mathrm{Hom}_K(Q,\mathrm{Res}^k_{k'}(W\otimes k'))\\
&\cong \mathrm{Hom}_{\mathfrak{g},K}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{k}} Q,\mathrm{Res}^k_{k'}(W\otimes k'))\\
&\cong \mathrm{Hom}_{\mathfrak{g}\otimes k',K\otimes k'}(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{k}} Q\otimes k',W\otimes k')
\end{aligned}$$

(see Corollary 3.2.10). This completes the proof. □

Proof of Theorem 3.1.7. According to Lemma 2.1.10, Proposition 3.2.4, and Definition 2.1.9 (d), it will suffice to show that for any finitely $(\mathfrak{g},K)$-module V and a $(\mathfrak{q},M)$-module W, the k'-homomorphism induced from ι in Construction 3.1.5

$$\mathrm{Hom}_{\mathfrak{g}\otimes k',K\otimes k'}(V\otimes k', I^{\mathfrak{g},K}_{\mathfrak{q},M}(W)\otimes k')\to \mathrm{Hom}_{\mathfrak{g}\otimes k',K\otimes k'}(V\otimes k', I^{\mathfrak{g}\otimes k',K\otimes k'}_{\mathfrak{q}\otimes k',M\otimes k'}(W\otimes k'))$$

is a bijection.

On the other hand, notice that the assumption (i) implies that the forgetful functor $\mathcal{F}^{\mathfrak{q},M}_{\mathfrak{g},K}$ respects compact objects. We therefore have a bijection

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}\otimes k',K\otimes k'}(V\otimes k', I^{\mathfrak{g},K}_{\mathfrak{q},M}(W)\otimes k') &\cong \mathrm{Hom}_{\mathfrak{g},K}(V, I^{\mathfrak{g},K}_{\mathfrak{q},M}(W))\otimes k'\\
&\cong \mathrm{Hom}_{\mathfrak{q},M}(V,W)\otimes k'\\
&\cong \mathrm{Hom}_{\mathfrak{q}\otimes k',M\otimes k'}(V\otimes k',W\otimes k')\\
&\cong \mathrm{Hom}_{\mathfrak{g}\otimes k',K\otimes k'}(V\otimes k', I^{\mathfrak{g}\otimes k',K\otimes k'}_{\mathfrak{q}\otimes k',M\otimes k'}(W\otimes k')).
\end{aligned}$$

The assertion is reduced to showing that these two arrows coincide.

Observe that the adjunction of $(\mathcal{F}^{\mathfrak{q}\otimes k',M\otimes k'}_{\mathfrak{g}\otimes k',K\otimes k'}, I^{\mathfrak{g}\otimes k',K\otimes k'}_{\mathfrak{q}\otimes k',M\otimes k'})$ is k'-linear since so is $\mathcal{F}^{\mathfrak{q}\otimes k',M\otimes k'}_{\mathfrak{g}\otimes k',K\otimes k'}$, and the adjunctions are described by units and counits which are k'-homomorphisms by definition. Therefore the bijection

$$\mathrm{Hom}_{\mathfrak{q}\otimes k',M\otimes k'}(V\otimes k',W\otimes k')\cong \mathrm{Hom}_{\mathfrak{g}\otimes k',K\otimes k'}(V\otimes k', I^{\mathfrak{g}\otimes k',K\otimes k'}_{\mathfrak{q}\otimes k',M\otimes k'}(W\otimes k'))$$

is k'-linear. It implies that the sequence of bijections above is k'-linear. Hence we may restrict the maps along

$$\mathrm{Hom}_{\mathfrak{g},K}(V, I^{\mathfrak{g},K}_{\mathfrak{q},M}(W)) \to \mathrm{Hom}_{\mathfrak{g}\otimes k',K\otimes k'}(V\otimes k', I^{\mathfrak{g},K}_{\mathfrak{q},M}(W)\otimes k').$$

In this case, for $f \in \mathrm{Hom}_{\mathfrak{g},K}(V, I^{\mathfrak{g},K}_{\mathfrak{q},M}(W))$, $f\otimes 1 \in \mathrm{Hom}_{\mathfrak{g}\otimes k',K\otimes k'}(V\otimes k', I^{\mathfrak{g},K}_{\mathfrak{q},M}(W)\otimes k')$ goes to the element in

$$\mathrm{Hom}_{\mathfrak{g}\otimes k',K\otimes k'}(V\otimes k', I^{\mathfrak{g}\otimes k',K\otimes k'}_{\mathfrak{q}\otimes k',M\otimes k'}(W\otimes k')) \cong \mathrm{Hom}_{\mathfrak{g},K}(V, I^{\mathfrak{g},K}_{\mathfrak{q},M}(\mathrm{Res}^k_{k'}(W\otimes k')))$$

described as

$$V \xrightarrow{f} I^{\mathfrak{g},K}_{\mathfrak{q},M}(W) \to I^{\mathfrak{g},K}_{\mathfrak{q},M}(\mathrm{Res}^k_{k'}(W\otimes k')).$$

This coincide with $\iota \circ (f\otimes 1)$ by definition of ι. This completes the proof. □

Notice that for a finitely generated and projective k-module W, the functor $-\otimes W$ respects small limits of k-modules. Hence similar arguments work in the finite setting:

Variant 3.2.12. Let $(\mathfrak{g}, K)$ be a pair over a commutative ring k, and $k \to k'$ be a ring homomorphism. Assume that k' is finitely generated and projective as a k-module. Then we have an isomorphism

$$\mathrm{Hom}_{\mathfrak{g},K}(-,-)\otimes k' \cong \mathrm{Hom}_{\mathfrak{g}\otimes k',K\otimes k'}(-\otimes k', -\otimes k')$$

on $(\mathfrak{g}, K)$-mod$^{op} \times (\mathfrak{g}, K)$-mod.

Variant 3.2.13. Let $(\mathfrak{q}, M) \to (\mathfrak{g}, K)$ be a map of pairs over a commutative ring k, and $k \to k'$ be a ring homomorphism. Assume that k' is finitely generated and projective as a k-module. Then the map

$$\iota : (I^{\mathfrak{g},K}_{\mathfrak{q},M}-)\otimes_k k' \to I^{\mathfrak{g}\otimes_k k',K\otimes_k k'}_{\mathfrak{q}\otimes_k k',M\otimes_k k'}(-\otimes_k k')$$

is an isomorphism.

To prove the derived base change theorems, we need to deal with injective and acyclic objects. Recall that if we are given a Grothendieck abelian category $\mathcal{A}$ and its family C of generators, an object $X \in \mathcal{A}$ is injective if and only if it has a right lifting property with respect to monomorphisms to members of C. In particular, if $\mathcal{A}$ is locally Noetherian, the following conditions are equivalent:

(a) X is injective;

(b) X has a right lifting property with respect to monomorphisms to members of C;

(c) X has a right lifting property with respect to monomorphisms between Noetherian objects.

Lemma 3.2.14. *Let $(\mathfrak{g}, K)$ be a pair over a Noetherian ring k, and k' be a flat k-algebra. Suppose that $\mathfrak{g}$ is finitely generated over k. If I is an injective $(\mathfrak{g}, K)$-module, so is $\mathrm{Res}^k_{k'}(I\otimes k')$.*

proof. This is an immediate consequence of Theorem 3.1.6. In fact, we have

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g},K}(B, \mathrm{Res}^k_{k'}(I \otimes k')) &\cong \mathrm{Hom}_{\mathfrak{g},K}(B, I) \otimes k' \\ &\twoheadrightarrow \mathrm{Hom}_{\mathfrak{g},K}(A, I) \otimes k' \\ &\cong \mathrm{Hom}_{\mathfrak{g},K}(A, \mathrm{Res}^k_{k'}(I \otimes k')).\end{aligned}$$

□

Proof of Theorem 3.1.9. For a finitely generated $(\mathfrak{g}, K)$-module X, and a complex I concentrated in nonnegative degrees of injective $(\mathfrak{g}, K)$-modules, we have

$$\begin{aligned}\mathbb{R}\,\mathrm{Hom}_{\mathfrak{g},K}(X, I) \otimes k' &\simeq \mathrm{Hom}_{\mathfrak{g},K}(X, I) \otimes k' \\ &\cong \mathrm{Hom}_{\mathfrak{g},K}(X, I \otimes k') \\ &= \mathbb{R}\,\mathrm{Hom}_{\mathfrak{g},K}(X, I \otimes k').\end{aligned}$$

The general case is deduced by passing to shifts and finite colimits of $\mathrm{Coh}(\mathfrak{g}, K)$. This completes the proof. □

Proof of Theorem 3.1.10. Let $I^\bullet$ be a complex bounded below of injective $(\mathfrak{g}, K)$-modules. Since $\mathrm{Res}^k_{k'}$ is exact and conservative on $(\mathfrak{g} \otimes k', K \otimes k')$-mod, so is on $D(\mathfrak{g} \otimes k', K \otimes k')$. Hence each $I^n \otimes k'$ is $I^{\mathfrak{g}\otimes k', K\otimes k'}_{\mathfrak{q}\otimes k', M\otimes k'}$-acyclic (Corollary 3.1.4 and Lemma 3.2.14). Theorem 3.1.7 now implies

$$\begin{aligned}(\mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M} I^\bullet) \otimes_k k' &= I^{\mathfrak{g},K}_{\mathfrak{q},M}(I^\bullet) \otimes k' \\ &\cong I^{\mathfrak{g}\otimes k', K\otimes k'}_{\mathfrak{q}\otimes k', M\otimes k'}(I^\bullet \otimes k') \\ &\simeq \mathbb{R}I^{\mathfrak{g}\otimes k', K\otimes k'}_{\mathfrak{q}\otimes k', M\otimes k'}(I^\bullet \otimes_k k').\end{aligned}$$

This completes the proof. □

Variant G (3) is deduced from the following finite variant of Lemma 3.2.14:

Lemma 3.2.15. *Let* $(\mathfrak{g}, K)$ *be a pair over a commutative ring, and* Q *be a* $(\mathfrak{g}, K)$*-module which is finitely generated and projective as a* k*-module. Then* $- \otimes Q$ *respects injectively fibrant complexes of* $(\mathfrak{g}, K)$*-modules (see [H2]).*

proof. We have a canonical isomorphism $Q \cong \mathrm{Hom}_k(\mathrm{Hom}_k(Q, k), k)$ of $(\mathfrak{g}, K)$-modules (see Proposition 2.2.2, Proposition 2.2.3). Hence we have a natural isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g},K}(-, - \otimes Q) &\cong \mathrm{Hom}_{\mathfrak{g},K}(-, - \otimes \mathrm{Hom}_k(\mathrm{Hom}_k(Q, k), k)) \\ &\cong \mathrm{Hom}_{\mathfrak{g},K}(-, \mathrm{Hom}_k(\mathrm{Hom}_k(Q, k), -)) \\ &\cong \mathrm{Hom}_{\mathfrak{g},K}(- \otimes \mathrm{Hom}_k(Q, k), -).\end{aligned}$$

The assertion now follows since $\mathrm{Hom}_k(Q, k)$ is flat as a k-module. □

3.3 The unbounded derived version

In this section, we replace $D(\mathfrak{g}, K)$ by another ∞-category to establish a generalization of Theorem 3.1.10. Regard $D(\mathfrak{g}, K)$ as the derived ∞-category, and set $\operatorname{Ind}\operatorname{Coh}(\mathfrak{g}, K)$ as the ind-completion of $\operatorname{Coh}(\mathfrak{g}, K)$ in the sense of [L1]. Let $k \to k'$ be a flat ring homomorphism of Noetherian rings, and $(\mathfrak{q}, M) \to (\mathfrak{g}, K)$ be a map of pairs over k. Suppose that the following conditions are satisfied:

(i) $\mathfrak{k} \oplus \mathfrak{q} \to \mathfrak{g}$ is surjective.

(ii) $\mathfrak{q}$ and $\mathfrak{g}$ are finitely generated as k-modules.

Lemma 3.3.1. *The functors*

$$- \otimes k' : D(\mathfrak{g}, K) \to D(\mathfrak{g} \otimes k', K \otimes k')$$
$$- \otimes k' : D(\mathfrak{q}, M) \to D(\mathfrak{q} \otimes k', M \otimes k')$$
$$\mathcal{F}^{\mathfrak{q}, M}_{\mathfrak{g}, K} : D(\mathfrak{g}, K) \to D(\mathfrak{q}, M)$$

respect coherent objects. In particular, they extend to left adjoint functors

$$- \otimes k' : \operatorname{Ind}\operatorname{Coh}(\mathfrak{g}, K) \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{g} \otimes k', K \otimes k')$$
$$- \otimes k' : \operatorname{Ind}\operatorname{Coh}(\mathfrak{q}, M) \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{q} \otimes k', M \otimes k')$$
$$\mathcal{F}^{\mathfrak{q}, M}_{\mathfrak{g}, K} : \operatorname{Ind}\operatorname{Coh}(\mathfrak{g}, K) \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{q}, M).$$

proof. It follows by definition. For $\mathcal{F}^{\mathfrak{q}, M}_{\mathfrak{g}, K}$, use (i). □

Let us denote the resulting right adjoint functors as

$$\operatorname{Res}^{k, \mathrm{ind}}_{k'} : \operatorname{Ind}\operatorname{Coh}(\mathfrak{g} \otimes k', K \otimes k') \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{g}, K)$$
$$\operatorname{Res}^{k, \mathrm{ind}}_{k'} : \operatorname{Ind}\operatorname{Coh}(\mathfrak{q} \otimes k', M \otimes k') \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{q}, M)$$
$$I^{\mathfrak{g}, K, \mathrm{ind}}_{\mathfrak{q}, M} : \operatorname{Ind}\operatorname{Coh}(\mathfrak{q}, M) \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{g}, K).$$

Remark 3.3.2 (The second adjoint functor)**.** Since $\mathcal{F}^{\mathfrak{q}, M}_{\mathfrak{g}, K}$ is a proper left adjoint functor between compactly generated stable ∞-categories, $I^{\mathfrak{g}, K, \mathrm{ind}}_{\mathfrak{q}, M}$ admits a right adjoint functor ([L1] Corollary 5.5.2.9 (1)).

To see the relation of our new right adjoint functors with the classical derived functors, recall that the standard t-structure on $D(\mathfrak{g}, K)$ descends to $\operatorname{Coh}(\mathfrak{g}, K)$, and then extends to $\operatorname{Ind}\operatorname{Coh}(\mathfrak{g}, K)$.

Lemma 3.3.3. *(1) The functors*

$$- \otimes k' : \operatorname{Ind}\operatorname{Coh}(\mathfrak{g}, K) \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{g} \otimes k', K \otimes k')$$
$$- \otimes k' : \operatorname{Ind}\operatorname{Coh}(\mathfrak{q}, M) \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{q} \otimes k', M \otimes k')$$
$$\mathcal{F}^{\mathfrak{q}, M}_{\mathfrak{g}, K} : \operatorname{Ind}\operatorname{Coh}(\mathfrak{g}, K) \to \operatorname{Ind}\operatorname{Coh}(\mathfrak{q}, M)$$

are t-exact.

(2) The functors

$$\mathrm{Res}^{k,\mathrm{ind}}_{k'} : \mathrm{Ind}\,\mathrm{Coh}(\mathfrak{g}\otimes k', K\otimes k') \to \mathrm{Ind}\,\mathrm{Coh}(\mathfrak{g}, K)$$

$$\mathrm{Res}^{k,\mathrm{ind}}_{k'} : \mathrm{Ind}\,\mathrm{Coh}(\mathfrak{q}\otimes k', M\otimes k') \to \mathrm{Ind}\,\mathrm{Coh}(\mathfrak{q}, M)$$

$$I^{\mathfrak{g},K,\mathrm{ind}}_{\mathfrak{q},M} : \mathrm{Ind}\,\mathrm{Coh}(\mathfrak{q}, M) \to \mathrm{Ind}\,\mathrm{Coh}(\mathfrak{g}, K)$$

are left t-exact.

In particular, the adjunctions restrict to the eventually coconnective part.

proof. Part (1) follows since

$$-\otimes k' : \mathrm{Coh}(\mathfrak{g}, K) \to \mathrm{Coh}(\mathfrak{g}\otimes k', K\otimes k')$$

$$-\otimes k' : \mathrm{Coh}(\mathfrak{q}, M) \to \mathrm{Coh}(\mathfrak{q}\otimes k', M\otimes k')$$

$$\mathcal{F}^{\mathfrak{q},M}_{\mathfrak{g},K} : \mathrm{Coh}(\mathfrak{g}, K) \to \mathrm{Coh}(\mathfrak{q}, M)$$

are t-exact. Then (2) is immediate from the generalities on t-structures. $\square$

Recall that for a stable ∞-category $\mathcal{C}$ with a coherent t-structure, there is a canonical equivalence $\mathrm{Ind}\,\mathrm{Coh}(\mathcal{C})^+ \simeq \mathcal{C}^+$ ([BZNP] Proposition 6.3.2). If we restrict the diagram

$$\begin{array}{ccc} \mathrm{Ind}\,\mathrm{Coh}(\mathfrak{g}, K) & \xrightarrow{\otimes k'} & \mathrm{Ind}\,\mathrm{Coh}(\mathfrak{g}\otimes k', K\otimes k') \\ \downarrow & & \downarrow \\ D(\mathfrak{g}, K) & \xrightarrow[\otimes k']{} & D(\mathfrak{g}\otimes k', K\otimes k') \end{array}$$

to the eventually coconnective part, the vertical arrows are equivalences. Passing to the right adjoint, we conclude that $\mathrm{Res}^{k,\mathrm{ind}}_{k'}$ coincides with $\mathrm{Res}^{k}_{k'}$ on $D(\mathfrak{g}\otimes k', K\otimes k')^+$ and $D(\mathfrak{q}\otimes k', M\otimes k')^+$ under the identification. Similarly, we have $I^{\mathfrak{g},K,\mathrm{ind}}_{\mathfrak{q},M}|_{D(\mathfrak{q},M)^+} \simeq \mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M}|_{D(\mathfrak{q},M)^+}$.

Theorem 3.3.4. *The comparison map* $\iota : I^{\mathfrak{g},K,\mathrm{ind}}_{\mathfrak{q},M}(-)\otimes k' \to I^{\mathfrak{g}\otimes k',K\otimes k',\mathrm{ind}}_{\mathfrak{q}\otimes k',M\otimes k'}(-\otimes k')$ *is an equivalence. Moreover, it restricts to the equivalence* $\mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M}(-)\otimes k' \simeq \mathbb{R}I^{\mathfrak{g}\otimes k',K\otimes k'}_{\mathfrak{q}\otimes k',M\otimes k'}(-\otimes k')$ *of Theorem 3.1.10 under the identifications*

$$\mathrm{Ind}\,\mathrm{Coh}(\mathfrak{q}, M)^+ \simeq D(\mathfrak{q}, M)^+$$

$$\mathrm{Ind}\,\mathrm{Coh}(\mathfrak{g}\otimes k', K\otimes k')^+ \simeq D(\mathfrak{g}\otimes k', K\otimes k')^+$$

proof. Since the functors are continuous, we may prove the equivalence on $\mathrm{Coh}(\mathfrak{q}, M)$. Then the assertion is reduced to Theorem 3.1.10 since Construction 3.1.5 is compatible with our ind-setting under the equivalences of the type $\mathrm{Ind}\,\mathrm{Coh}(\mathcal{C})^+ \simeq \mathcal{C}^+$ (recall the compatibility of the adjunctions of $\otimes k'$ and $\mathcal{F}$ in the two settings from Lemma 3.3.1 and the argument below there). $\square$

Finally, suppose that k is a field of characteristic 0, $(\mathfrak{g}, K)$ be a pair with K reductive and $\dim \mathfrak{g} < +\infty$.

Proposition 3.3.5. *The embedding* $\mathrm{Coh}(\mathfrak{g}, K) \to D(\mathfrak{g}, K)$ *induces an equivalence* $\mathrm{Ind}\,\mathrm{Coh}(\mathfrak{g}, K) \simeq D(\mathfrak{g}, K)$*.*

proof. If we are given an arbitrary finitely generated $(\mathfrak{g}, K)$-module V, there is a finite dimensional K-submodule V_0 such that the induced homomorphism $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V_0 \to V$ is surjective. Since its kernel is also finitely generated, we can repeat this procedure to obtain a resolution of V by finitely generated and projective $(\mathfrak{g}, K)$-modules. According to the existence of the standard projective resolution ([J1] 1.4.4), the category $(\mathfrak{g}, K)$-mod has a finite homological dimension. In particular, we may assume the resolution to be bounded by truncations. Moreover, it implies that V is compact in the ∞-category $D(\mathfrak{g}, K)$. Passing to shifts and finite colimits, we can conclude that every coherent complex is compact in $D(\mathfrak{g}, K)$. Since $\mathrm{Coh}(\mathfrak{g}, K)$ generates $D(\mathfrak{g}, K)$ under colimits, the equivalence follows ([L1] Proposition 5.3.5.11, Proposition 5.5.1.9). □

4 Variants for pro

4.1 Computation of pro

Lemma 4.1.1. *Let K be a flat affine group scheme over a Noetherian ring k, and $\{V_{\mathcal{O}}\}_{\mathcal{O}}$ be a set of K-modules. Suppose that for any finitely generated K-module Q, $\mathrm{Hom}_K(Q, V_{\mathcal{O}})$ vanishes for all but finitely many indices $\mathcal{O}$. Then the direct sum $\oplus V_{\mathcal{O}}$ also exhibits a product of $\{V_{\mathcal{O}}\}$ in K-*mod*.*

proof. It is obvious since we have a bijection for any finitely generated K-module Q

$$\mathrm{Hom}_K(Q, \oplus V_{\mathcal{O}}) \cong \oplus \mathrm{Hom}_K(Q, V_{\mathcal{O}}) \cong \prod \mathrm{Hom}_K(Q, V_{\mathcal{O}}).$$

The second one follows from the assumption on $\{V_{\mathcal{O}}\}$. □

We give a characterization of the assumption above in practical settings.

Proposition 4.1.2. *Let k be a Noetherian domain, K be a flat affine group scheme over k, and $V = \oplus V_{\mathcal{O}}$ be a direct sum of K-modules. Denote the fractional field of k by* $\mathrm{Frac}(k)$*.*

(1) If V is torsion-free, and $V \otimes \mathrm{Frac}(k)$ is admissible then for any finitely generated K-module Q, $\mathrm{Hom}_K(Q, V_{\mathcal{O}})$ vanishes for all but finitely many indices $\mathcal{O}$.

(2) If for any finitely generated K-module Q, $\mathrm{Hom}_K(Q, V)$ is finitely generated then $V \otimes \mathrm{Frac}(k)$ is admissible.

(3) Suppose that each of $V_{\mathcal{O}}$ is finitely generated. If for a finitely generated K-module Q, $\mathrm{Hom}_K(Q, V_{\mathcal{O}})$ vanishes for all but finitely many indices $\mathcal{O}$ then $\mathrm{Hom}_K(Q, V)$ is finitely generated.

proof. To see (1), consider a sequence for a finitely generated K-module Q

$$\begin{aligned}\mathrm{Hom}_K(Q, V) &= \oplus \mathrm{Hom}_K(Q, V_{\mathcal{O}}) \\ &\subset \oplus \mathrm{Hom}_{K\otimes \mathrm{Frac}(k)}(Q \otimes \mathrm{Frac}(k), V_{\mathcal{O}} \otimes \mathrm{Frac}(k)) \\ &\cong \mathrm{Hom}_{K\otimes \mathrm{Frac}(k)}(Q \otimes \mathrm{Frac}(k), V \otimes \mathrm{Frac}(k)).\end{aligned}$$

Since $V \otimes \mathrm{Frac}(k)$ is admissible, $\mathrm{Hom}_K(Q, V_{\mathcal{O}} \otimes \mathrm{Frac}(k))$ vanishes for all but finitely many $\mathcal{O}$. Since $V_{\mathcal{O}}$ are torsion-free, the submodules $\mathrm{Hom}_K(Q, V_{\mathcal{O}})$ vanish for almost all $\mathcal{O}$.

Part (2) follows from the flat base change theorem: For any finitely generated K-module Q, we have

$$\dim \mathrm{Hom}_{K\otimes \mathrm{Frac}(k)}(Q\otimes \mathrm{Frac}(k), V\otimes \mathrm{Frac}(k)) = \dim \mathrm{Hom}_K(Q, V)\otimes \mathrm{Frac}(k) < +\infty.$$

Since finite dimensional representations of $K\otimes\mathrm{Frac}(k)$-modules are generated by representations $Q \otimes \mathrm{Frac}(k)$ under finite colimits (Corollary 2.1.14, Proposition 2.1.1), $V \otimes \mathrm{Frac}(k)$ is admissible.

Finally, suppose that $V_{\mathcal{O}}$ are finitely generated. Then for a finitely generated K-module Q, $\mathrm{Hom}_K(Q, V)$ is isomorphic to a direct sum of $\mathrm{Hom}_K(Q, V_{\mathcal{O}})$ along finitely many indices $\mathcal{O}$. Since $V_{\mathcal{O}}$ is finitely generated, so is $\mathrm{Hom}_K(Q, V)$. This completes the proof. □

Proposition 4.1.3. *Let $(\mathfrak{q}, M) \to (\mathfrak{g}, M)$ be an injective map of pairs over a Noetherian ring k, and Z be a $(\mathfrak{q}, M)$-module. Suppose that the map $M \to M$ is the identity. Moreover, assume the following conditions:*

(i) For $x \in \mathfrak{g}$, we have $[x, x] = 0$.

(ii) There is an M-equivariant Lie subalgebra $\bar{\mathfrak{u}} \subset \mathfrak{g}$ such that the summation map $\mathfrak{q} \oplus \bar{\mathfrak{u}} \to \mathfrak{g}$ is an isomorphism of k-modules.

(iii) There are free bases of $\mathfrak{q}$ and $\bar{\mathfrak{u}}$.

(iv) The enveloping algebra $U(\bar{\mathfrak{u}})$ is decomposed into a direct sum $U(\bar{\mathfrak{u}}) = \oplus_{\mathcal{O}} U(\bar{\mathfrak{u}})_{\mathcal{O}}$ of M-submodules $U(\bar{\mathfrak{u}})_{\mathcal{O}}$ which are finitely generated as k-modules.

(v) For any finitely generated M-module Q, $\mathrm{Hom}_M(Q, \mathrm{Hom}(U(\bar{\mathfrak{u}})_{\mathcal{O}}, Z))$ vanishes for all but finitely many $\mathcal{O}$.

Then we have an isomorphism as an M-module

$$\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{q}}(Z) \cong \oplus_{\mathcal{O}} \mathrm{Hom}_k(U(\bar{\mathfrak{u}})_{\mathcal{O}}, Z).$$

In particular, a base change formula along a ring homomorphism $k \to k'$ between Noetherian rings

$$\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{q}}(Z) \otimes k' \cong \mathrm{pro}^{\mathfrak{g}\otimes k'}_{\mathfrak{q}\otimes k'}(Z \otimes k')$$

is valid in the following cases:

(a) $k \to k'$ is flat.

(b) *For any finitely generated $M \otimes k'$-module Q,* $\mathrm{Hom}_{M\otimes k'}(Q, \mathrm{Hom}(U(\bar{\mathfrak{u}})_{\mathcal{O}} \otimes k', Z \otimes k'))$ *vanishes for all but finitely many* $\mathcal{O}$.

Remark 4.1.4. The functor $\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{q}}$ can be regarded as a right adjoint functor to the forgetful functor from the category of weak $(\mathfrak{g}, M)$-modules to that of weak $(\mathfrak{q}, M)$-modules. Therefore the base change functor along arbitrary ring homomorphisms makes sense (Remark 3.1.3).

Proof of Proposition 4.1.3. According to the PBW theorem, we have an isomorphism of M-modules

$$\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{q}}(Z) \cong F(U(\bar{\mathfrak{u}}), Z).$$

The condition (v) and Lemma 4.1.1 imply that $F(U(\bar{\mathfrak{u}}), Z) \cong \oplus_{\mathcal{O}} \mathrm{Hom}(U(\bar{\mathfrak{u}})_{\mathcal{O}}, Z)$. This completes the proof. □

4.2 Examples

Suppose that we are given a reductive pair $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$ over $\mathbb{C}$ and a θ-stable parabolic subpair $(\mathfrak{q}_{\mathbb{C}}, (K_L)_{\mathbb{C}})$ in the sense of [KV], where θ is the Cartan involution. Let $\mathfrak{l}_{\mathbb{C}}$ (resp. $\mathfrak{u}_{\mathbb{C}}$, $\bar{\mathfrak{u}}_{\mathbb{C}}$) denote the Levi part (resp. nilradical, the opposite nilradical) of $\mathfrak{q}$, let $\Delta(\bar{\mathfrak{u}}_{\mathbb{C}}) = \{\alpha_1, \cdots, \alpha_s\}$ be the set of roots in $\bar{\mathfrak{u}}_{\mathbb{C}}$, and $h = h_{\rho(\mathfrak{u}_{\mathbb{C}})}$ be the element of the Cartan subalgebra as in [KV] Proposition 4.70. In particular, we have $\alpha_i(h) < 0$ for $\alpha_i \in \Delta(\bar{\mathfrak{u}}_{\mathbb{C}})$.

Example 4.2.1. Observe that $(\mathfrak{l}_{\mathbb{C}}, (K_L)_{\mathbb{C}}) \subset (\mathfrak{q}_{\mathbb{C}}, (K_L)_{\mathbb{C}})$ are θ-stable subpairs of $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$, where $\mathfrak{l}_{\mathbb{C}}$ is the Levi part of $\mathfrak{q}_{\mathbb{C}}$. Note that $\mathfrak{u}_{\mathbb{C}}$ is also θ-stable. Therefore they associate maps of contraction families over the polynomial ring $\mathbb{C}[z]$

$$(\tilde{\mathfrak{l}}_{\mathbb{C}}, (K_L)_{\mathbb{C}} \otimes \mathbb{C}[z]) \leftarrow (\tilde{\mathfrak{q}}_{\mathbb{C}}, (K_L)_{\mathbb{C}} \otimes \mathbb{C}[z]) \to (\tilde{\mathfrak{g}}_{\mathbb{C}}, K_{\mathbb{C}} \otimes \mathbb{C}[z])$$

in the sense of [BHS]. Define the cohomological induction as

$$\mathbb{R}I^{\tilde{\mathfrak{g}}_{\mathbb{C}}, K_{\mathbb{C}}\otimes\mathbb{C}[z]}_{\tilde{\mathfrak{q}}_{\mathbb{C}}, (K_L)_{\mathbb{C}}\otimes\mathbb{C}[z]} \mathcal{F}^{\tilde{\mathfrak{q}}_{\mathbb{C}}, (K_L)_{\mathbb{C}}\otimes\mathbb{C}[z]}_{\tilde{\mathfrak{l}}_{\mathbb{C}}, (K_L)_{\mathbb{C}}\otimes\mathbb{C}[z]}(- \otimes_{\mathbb{C}[z]} \wedge^{\dim \mathfrak{u}} \tilde{\mathfrak{u}}),$$

where $\mathcal{F}^{\tilde{\mathfrak{q}}_{\mathbb{C}}, (K_L)_{\mathbb{C}}\otimes\mathbb{C}[z]}_{\tilde{\mathfrak{l}}_{\mathbb{C}}, (K_L)_{\mathbb{C}}\otimes\mathbb{C}[z]}$ is the forgetful functor

$$(\tilde{\mathfrak{l}}_{\mathbb{C}}, (K_L)_{\mathbb{C}} \otimes \mathbb{C}[z])\text{-mod} \to (\tilde{\mathfrak{q}}_{\mathbb{C}}, (K_L)_{\mathbb{C}} \otimes \mathbb{C}[z])\text{-mod}.$$

Remark that $\mathrm{pro}^{\tilde{\mathfrak{g}}_{\mathbb{C}}}_{\tilde{\mathfrak{q}}_{\mathbb{C}}}$ is exact (Variant 5.1.6, Corollary 5.1.12). Let Z be a torsion-free $(\tilde{\mathfrak{l}}_{\mathbb{C}}, (K_L)_{\mathbb{C}} \otimes \mathbb{C}[z])$-module with a scalar action of h. If $Z \otimes \mathbb{C}(z)$ is admissible, the cohomological induction enjoys a flat base change formula to the algebraic closure $\overline{\mathbb{C}(z)}$ of the field of rational functions $\mathbb{C}(z)$

$$\begin{aligned}
&\mathbb{R}I^{\tilde{\mathfrak{g}}_{\mathbb{C}}, K_{\mathbb{C}}\otimes\mathbb{C}[z]}_{\tilde{\mathfrak{q}}_{\mathbb{C}}, (K_L)_{\mathbb{C}}\otimes\mathbb{C}[z]} \mathcal{F}^{\tilde{\mathfrak{q}}_{\mathbb{C}}, (K_L)_{\mathbb{C}}\otimes\mathbb{C}[z]}_{\tilde{\mathfrak{l}}_{\mathbb{C}}, (K_L)_{\mathbb{C}}\otimes\mathbb{C}[z]}(- \otimes_{\mathbb{C}[z]} \wedge^{\dim \mathfrak{u}} \tilde{\mathfrak{u}}) \otimes \overline{\mathbb{C}(z)} \\
&\cong \mathbb{R}I^{\mathfrak{g}\otimes\overline{\mathbb{C}(z)}, K_{\mathbb{C}}\otimes\overline{\mathbb{C}(z)}}_{\mathfrak{q}_{\mathbb{C}}\otimes\overline{\mathbb{C}(z)}, (K_L)_{\mathbb{C}}\otimes\overline{\mathbb{C}(z)}} \mathcal{F}^{\mathfrak{q}_{\mathbb{C}}\otimes\overline{\mathbb{C}(z)}, (K_L)_{\mathbb{C}}\otimes\overline{\mathbb{C}(z)}}_{\mathfrak{l}_{\mathbb{C}}\otimes\overline{\mathbb{C}(z)}, (K_L)_{\mathbb{C}}\otimes\overline{\mathbb{C}(z)}} Z \otimes \wedge^{\dim \mathfrak{u}} \mathfrak{u} \otimes \overline{\mathbb{C}(z)})
\end{aligned}$$

(use [KV] Proposition 5.96). Suppose also that the τ-type $Z_\tau \subset Z$ for each irreducible representation τ of $(K_L)_\mathbb{C}$ is free of finite rank over $\mathbb{C}[z]$. Then for any $\mathbb{C}$-algebra homomorphism $\mathbb{C}[z] \to \mathbb{C}$, we have a base change formula

$$\mathrm{pro}_{\tilde{\mathfrak{q}}_\mathbb{C}}^{\tilde{\mathfrak{g}}_\mathbb{C}}(Z) \otimes_{\mathbb{C}[z]} \mathbb{C} \cong \mathrm{pro}_{\tilde{\mathfrak{q}}_\mathbb{C} \otimes_{\mathbb{C}[z]} \mathbb{C}}^{\tilde{\mathfrak{g}}_\mathbb{C} \otimes_{\mathbb{C}[z]} \mathbb{C}}(Z \otimes_{\mathbb{C}[z]} \mathbb{C}).$$

Let k be a Noetherian subring of $\mathbb{C}$, and $(\mathfrak{q}, K_L) \subset (\mathfrak{g}, K)$ be a k-form of $(\mathfrak{q}_\mathbb{C}, (K_L)_\mathbb{C}) \subset (\mathfrak{g}_\mathbb{C}, K_\mathbb{C})$ in the sense that $(\mathfrak{q}_\mathbb{C}, (K_L)_\mathbb{C}) \subset (\mathfrak{g}_\mathbb{C}, K_\mathbb{C})$ is isomorphic to the base change of $(\mathfrak{q}, K_L) \subset (\mathfrak{g}, K)$. Assume that there is a complementary K_L-stable subalgebra $\bar{\mathfrak{u}} \subset \mathfrak{g}$ to $\mathfrak{q}$ which is a k-form of $\bar{\mathfrak{u}}_\mathbb{C}$. Moreover, suppose that the following conditions are satisfied:

(i) There is a free basis of $\mathfrak{q}$.

(ii) There is a free basis $\{E_{\alpha_i}\}$ of $\bar{\mathfrak{u}}$ consisting of root vectors of $\bar{\mathfrak{u}}_\mathbb{C}$.

(iii) The $(K_L)_\mathbb{C}$-orbit of h is contained in the Cartan subalgebra.

Proposition 4.2.2. *In this setting, there is a family* $\{U(\bar{\mathfrak{u}})_\mathcal{O}\}$ *of finitely generated* K_L*-submodules of* $U(\bar{\mathfrak{u}})$ *such that*

$$U(\bar{\mathfrak{u}}) = \oplus_\mathcal{O} U(\bar{\mathfrak{u}})_\mathcal{O}.$$

Construction 4.2.3. Let G be the component group $\pi_0((K_L)_\mathbb{C})$ of $(K_L)_\mathbb{C}$. For each $x \in G$, fix a representative $g_x \in (K_L)_\mathbb{C}$ and set $h_x = \mathrm{Ad}(g_x)h$, where Ad is the action of $(K_L)_\mathbb{C}$ on $\mathfrak{g}_\mathbb{C}$. Since the unit component $(K_L)^0_\mathbb{C}$ centralizes h, it is independent of the choice of g_x. In particular, if $g = e$ is the unit then $h_e = h$. Observe next that G acts on the complex vector space $\mathbb{C}^G$ by translation of entries. For a G-orbit $\mathcal{O}$ in $\mathbb{C}^G$, define $U(\bar{\mathfrak{u}})_\mathcal{O}$ as

$$U(\bar{\mathfrak{u}})_\mathcal{O} = \oplus_{\vec{r} \in \mathcal{O}} \oplus_{\substack{\sum n_i \alpha_i(h_x) = r_x \\ \text{for any } x \in G}} k E_{\alpha_1}^{n_1} E_{\alpha_2}^{n_2} \cdots E_{\alpha_s}^{n_s}.$$

We also set

$$U(\bar{\mathfrak{u}})_{\vec{r}} = \oplus_{\substack{\sum n_i \alpha_i(h_x) = r_x \\ \text{for any } x \in G}} k E_{\alpha_1}^{n_1} E_{\alpha_2}^{n_2} \cdots E_{\alpha_s}^{n_s}.$$

Proof of Proposition 4.2.2. The k-modules $U(\bar{\mathfrak{u}})_\mathcal{O}$ are finitely generated by definition. According to the PBW theorem, we have $U(\bar{\mathfrak{u}}) \cong \oplus_\mathcal{O} U(\bar{\mathfrak{u}})_\mathcal{O}$ as a k-module. To see that $U(\bar{\mathfrak{u}})_\mathcal{O}$ is a K_L-submodule, we may assume $k = \mathbb{C}$ (Lemma 2.1.16).

Fix $\vec{r} \in \mathcal{O}$ and exponents $\{n_i\}$ with $\sum n_i \alpha_i(h_x) = r_x$. Let $g \in (K_L)_\mathbb{C}$ in a component $x \in G$, and write

$$\mathrm{Ad}(g) E_{\alpha_1}^{n_1} E_{\alpha_2}^{n_2} \cdots E_{\alpha_s}^{n_s} = \sum_{\vec{r}' \in \mathbb{C}^G} v'_{\vec{r}'}$$

with $v'_{\vec{r}'} \in U(\bar{\mathfrak{u}})_{\vec{r}'}$. Then for any $y \in G$,

$$\begin{aligned}\sum_{\vec{r}' \in \mathbb{C}^G} r'_y v_{\vec{r}'} &= \sum [h_y, v_{\vec{r}'}] \\ &= \left[h_y, \mathrm{Ad}(g) E_{\alpha_1}^{n_1} E_{\alpha_2}^{n_2} \cdots E_{\alpha_s}^{n_s}\right] \\ &= \mathrm{Ad}(g)\left[\mathrm{Ad}(g)^{-1} h_y, E_{\alpha_1}^{n_1} E_{\alpha_2}^{n_2} \cdots E_{\alpha_s}^{n_s}\right] \\ &= \mathrm{Ad}(g)\left[h_{x^{-1}y}, E_{\alpha_1}^{n_1} E_{\alpha_2}^{n_2} \cdots E_{\alpha_s}^{n_s}\right] \\ &= r_{x^{-1}y}\, \mathrm{Ad}(g) E_{\alpha_1}^{n_1} E_{\alpha_2}^{n_2} \cdots E_{\alpha_s}^{n_s} \\ &= r_{x^{-1}y} \sum_{\vec{r}' \in \mathbb{C}^G} v_{\vec{r}'}.\end{aligned}$$

Therefore $v_{\vec{r}'}$ vanishes unless $\vec{r}' = x^{-1} \cdot \vec{r}$. In particular, $\mathrm{Ad}(g) E_{\alpha_1}^{n_1} E_{\alpha_2}^{n_2} \cdots E_{\alpha_s}^{n_s} \in U(\bar{\mathfrak{u}})_{x^{-1}\vec{r}} \subset U(\bar{\mathfrak{u}})_{\mathcal{O}}$. This completes the proof. □

Example 4.2.4. Let p, q be nonnegative integers. Let $p = \sum p_i$ and $q = \sum q_j$ be partitions. Then the diagonal embedding $\mathrm{GL}_p \times \mathrm{GL}_q \to \mathrm{GL}_{p+q}$ gives rise to a pair $(\mathfrak{gl}_{p+q}, \mathrm{GL}_p \times \mathrm{GL}_q)$ over $\mathbb{Z}$. Choose the subgroup of diagonal matrices in GL_{p+q} as a split maximal torus, and Q be a parabolic subgroup of GL_{p+q} whose Levi part is $\prod \mathrm{GL}_{p_i+q_i}$. Then $\mathfrak{q}$ and $\prod(\mathrm{GL}_{p_i} \times \mathrm{GL}_{q_i})$ form a subpair of $(\mathfrak{gl}_{p+q}, \mathrm{GL}_p \times \mathrm{GL}_q)$. Moreover, it is an integral model of the pair associated to $U(p,q)$ and a θ-stable parabolic subpair. Moreover, it enjoys the conditions above.

For (v) in Proposition 4.1.3, let Z be a torsion-free $(\mathfrak{q}, K_L)$-module. See also Proposition 4.1.2 (1).

Proposition 4.2.5 ([KV] Proposition 5.96)**.** *If $Z \otimes \mathbb{C}$ is admissible, and that $\mathfrak{h}$ acts on $Z \otimes \mathbb{C}$ as a scalar then the $(K_L)_{\mathbb{C}}$-module $\oplus \mathrm{Hom}(U(\bar{\mathfrak{u}})_{\mathcal{O}}, Z \otimes \mathbb{C})$ is admissible.*

Part II
Integral structures

5 The Hecke algebras

5.1 The Hecke algebras of diagonalizable groups

Throughout this part we fix a base ring k. We interpret [Jant] I.2.11 into the theory of Hecke algebras to compute the functor I in an extremely special case. Let T be a diagonalizable group over k in the sense of [Jant] I.2.5. Namely, there is an additive group Λ such that $T \cong \operatorname{Spec} k[\Lambda]$ as an affine group scheme, where $k[\Lambda]$ is the group algebra. Notice that if we denote the standard basis of $k[\Lambda]$ by $\{t^\lambda\}_{\lambda \in \Lambda}$, $k[\Lambda]$ is a Hopf algebra for $\Delta(t^\lambda) = t^\lambda \otimes t^\lambda$ and $\epsilon(t^\lambda) = 1$. We will refer to $k[\Lambda]$ as C.

Notation 5.1.1. For each index $\lambda \in \Lambda$, denote the subcomodule kt^λ by k_λ, and let p_λ be the projection $k[\Lambda] \cong \oplus kt^\lambda \to k$ to the λ-component, namely,

$$p_\lambda(t^{\lambda'}) = \begin{cases} 1 & (\lambda = \lambda') \\ 0 & (\lambda \neq \lambda'). \end{cases}$$

Recall that the dual k-module $C^* = \mathrm{Hom}_k(C, k)$ inherits the structure of a k-algebra for

$$C^* \otimes C^* \to (C \otimes C)^* \xrightarrow{\Delta^*} C^*$$

and $\epsilon \in C^*$ as a multiplication and a unit respectively. Explicitly, we have an isomorphism of k-algebras

$$C^* \cong \prod_{\lambda \in \Lambda} kp_\lambda,$$

where the right hand side is a k-algebra by the product of copies of the algebra k.

Notation 5.1.2. The category of C^*-modules will be denoted by C^*-mod.

A C-comodule V is a C^*-module for

$$C^* \otimes V \to C^* \otimes V \otimes C \cong C^* \otimes C \otimes V \to V,$$

where the first arrow (resp. the second isomorphism, the last arrow) is given by the coaction (resp. the switch of the components, the evaluation). In our setting, if we write the coaction as $\rho(v) = \sum v_\lambda \otimes t^\lambda$ then the induced C^*-action is given by

$$p_\lambda \cdot v = v_\lambda.$$

In this way, we obtain a functor

$$C\text{-comod} \to C^*\text{-mod}.$$

Lemma 5.1.3 ([Jant] I.2.11)**.** *For indices $\lambda, \lambda' \in \Lambda$ we have*

$$\mathrm{Hom}_C(k_\lambda, k_{\lambda'}) = \begin{cases} k & (\lambda = \lambda') \\ 0 & (\lambda \neq \lambda'), \end{cases}$$

proof. The case $\lambda = \lambda'$ is obvious. Suppose that $\lambda \neq \lambda'$. Then any C-comodule homomorphism $f : k_\lambda \to k_{\lambda'}$ is zero since

$$f(1) = f(p_\lambda \cdot 1) = p_\lambda f(1) = 0.$$

□

We next set $R(T)$ as the k-submodule of C^* spanned by $\{p_\lambda : \lambda \in \Lambda\}$, i.e., $R(T) = \oplus\, kp_\lambda \subset C^*$. This is a not necessarily unital subalgebra of C^*. We call it the Hecke algebra of C.

Notation 5.1.4. The category of $R(T)$-modules will be denoted by $R(T)$-mod.

Notice that there are canonical functors

$$C\text{-comod} \to C^*\text{-mod} \to R(T)\text{-mod}$$

We call a unital C^*-module (resp. an $R(T)$-module) V rational (resp. approximately unital) if for any $v \in V$ there exists a finite subset $I \subset \Lambda$ such that

$$\sum_{\lambda \in I} p_\lambda v = v.$$

Their categories will be denoted by $C^*\text{-mod}^{\text{rat}}$ and $R(T)\text{-mod}^{\text{un}}$ respectively.

Theorem 5.1.5. *The functors above restrict to isomorphisms of categories*

$$C\text{-comod} \cong C^*\text{-mod}^{\text{rat}} \cong R(T)\text{-mod}^{\text{un}}.$$

proof. It is clear from the definition that rational C^*-modules restrict to approximately unital $R(T)$-modules. Let V be a C-comodule with the coaction ρ, and $v \in V$. Observe that if we write $\rho(v) = \sum_{\lambda \in I} v_\lambda \otimes t^\lambda$ for some finite subset $I \subset \Lambda$ then we get $\sum_{\lambda \in I} p_\lambda v = v$. Therefore we have shown that the functors factor through $C^*\text{-mod}^{\text{rat}}$ and $R(T)\text{-mod}^{\text{un}}$.

Let V be a rational C^*-module or an approximately unital $R(T)$-module, and $v \in V$. Choose a finite subset $I \subset \Lambda$ such that $\sum_{\lambda \in I} p_\lambda v = v$. If an index $\lambda \in \Lambda \backslash I$, we have

$$p_\lambda v = p_\lambda (\sum_{\mu \in I} p_\mu v) = (\sum_{\mu \in I} p_\lambda p_\mu) v = 0.$$

We now construct the inverses. If we are given an approximately unital $R(T)$-module M, it naturally extends to a rational C^*-module by

$$(\sum c_\lambda p_\lambda) v = \sum c_\lambda p_\lambda v$$

which is essentially a finite sum, where $c_\lambda \in k$. For a rational C^*-module V, the structure ρ of a C-comodule on V arises as

$$\rho(v) = \sum_\lambda t^\lambda \otimes p_\lambda v.$$

These functors provide the desired inverses. □

Variant 5.1.6. Let k be a $\mathbb{C}$-algebra, and K be a complex reductive group. In general, for a free k-module W of finite rank, $\operatorname{End} W$ is a coalgebra over k for the canonical isomorphism $\operatorname{End}(W) \cong \operatorname{End}(W)^*$. For each irreducible representation V of K, we have a coalgebra homomorphism

$$\operatorname{End}(V \otimes k') \to \mathcal{O}(K \otimes k').$$

Passing to all isomorphism classes, we get an isomorphism

$$\oplus \operatorname{End}(V \otimes k') \cong \mathcal{O}(K \otimes k')$$

of coalgebras (the Peter-Weyl theorem). Passing to their duals, we obtain an approximately unital ring $R(K \otimes k') \cong \oplus \mathrm{End}(V \otimes k')$ which is compatible with base changes. Moreover, the categories of $K \otimes k'$-modules and approimately unital $R(K \otimes k')$-modules are isomorphic.

Remark 5.1.7 ([GT], [KGTL])**.** It is known that the isomorphism C-comod $\cong$ C^*-mod$^{\mathrm{rat}}$ is valid for coalgebras C which are projective as k-modules.

Corollary 5.1.8 (T-type decomposition, [Jant] I.2.11)**.** *Let V be a C-comodule. For each $\lambda \in \Lambda$, set V_λ as the image of the action of p_λ on V. Then we have a decomposition of the C-comodule*

$$V = \oplus_{\lambda \in \Lambda} V_\lambda.$$

Corollary 5.1.9. *The comodules k_λ form a family of projective generators of the category C-*comod.

Corollary 5.1.10. *The category C-*comod *has enough projectives.*

proof. This follows from [Bor] Proposition 4.6.6. □

Corollary 5.1.11. *A C-comodule V is injective (resp. projective) if and only if each type V_λ is injective (resp. projective) as a k-module.*

proof. Thanks to the action of $R(T)$, each of V_λ is injective (resp. projective) in C-comod if and only if so is in the category of k-modules. The assertion now follows since each V_λ is a retract of V. □

As an application we can introduce the notion of T-finite part:

Corollary 5.1.12. *The embedding*

$$C\text{-comod} \cong R(T)\text{-mod}^{\mathrm{un}} \hookrightarrow R(T)\text{-mod}$$

admits a right exact right adjoint functor $(-)_T$.

proof. Let V be an $R(T)$-module. We say that an element $v \in V$ is T-finite if there is a finite subset $I \subset \Lambda$ such that $\sum_{\lambda \in I} p_\lambda v = v$, and write $V_T \subset V$ for the subset of T-finite elements of V. Then V_T is an approximately unital $R(T)$-submodule of V. Moreover $V_T \subset V$ exhibits an $R(T)$-mod$^{\mathrm{un}}$-colocalization of V. It is proved in a similar way to [KV] Proposition 1.55 that the resulting colocalization functor is exact. □

Corollary 5.1.13. *We have $(C^*)_T = R(T)$.*

Remark 5.1.14. The arguments above work if we replace C by a diagonal coalgebra in the sense of [AJ] Example 1.3.7. It is equivalent to saying that there are free bases $\{t^\lambda\}$ and $\{s^\lambda\}$ such that

$$\Delta(s^\lambda) = t^\lambda \otimes s^\lambda,$$

where Δ is the comultiplication of C. In fact, the coassociativity of Δ implies

$$\Delta(t^\lambda) \otimes s^\lambda = t^\lambda \otimes t^\lambda \otimes s^\lambda.$$

In particular, we have $\Delta(t^\lambda) = t^\lambda \otimes t^\lambda$.

Thanks to Corollary 5.1.9, the projective model structure also exists.

Notation 5.1.15. Let $\mathcal{A}$ be an abelian category, M be its object, and n be an integer. Then we denote the cochain complexes

$$D^n M = \cdots \to 0 \to \overset{-n}{M} = \overset{-n+1}{M} \to 0 \to \cdots$$

$$S^n M = \cdots \to 0 \to \overset{-n}{M} \to 0 \to \cdots$$

by $D^n M$ and $S^n M$ respectively. Notice that we have a natural inclusion $S^{n-1}M \to D^n M$.

Corollary 5.1.16. *There exists a combinatorial model structure on the category of cochain complexes of C-comodules which is described as follows:*

(F) A map is a fibration if and only if it is an epimorphism.

(W) A map is a weak equivalence if and only if it is a quasi-isomorphism.

(C) A map is a cofibration if and only if it is a degreewise split monomorphism with a cofibrant cokernel.

Moreover, the generating cofibrations (resp. trivial cofibrations) are the standard embeddings $S^{p-1}k_\lambda \to D^p k_\lambda$ (resp. $0 \to D^p k_\lambda$), where p runs through all integers.

proof. This is a direct consequence of [CHov] Theorem 5.7: For a locally presentable abelian category $\mathcal{A}$ equipped with a small set G of projective generators, there exists a model structure on the category of cochain complexes of objects of $\mathcal{A}$ such that the following conditions are satisfied:

(F) A morphism is a fibration if and only if it is an epimorphism.

(W) A morphism is a weak equivalence if and only if it is a quasi-isomorphism.

(C) A morphism is a cofibration if and only if it is a degreewise split monomorphism with a cofibrant cokernel.

Moreover, the generating cofibrations (resp. trivial cofibrations) are the standard embeddings $S^{m-1}P \to D^m P$ (resp. $0 \to D^m P$), where m runs through all integers, and P are the members of G. □

5.2 The Hecke algebra in relative settings

Let $T = \operatorname{Spec} k[\Lambda]$ be a diagonalizable group as the previous section.

Lemma 5.2.1. *Let V and V' be T-modules.*

(1) The action of $R(T)$ corresponding to the tensor representation $V \otimes V'$ of T is given by

$$p_\lambda(v \otimes v') = \sum_\mu p_\mu v \otimes p_{\lambda-\mu} v'.$$

(2) The Hom k-module $\mathrm{Hom}(V, V')$ *is an* $R(T)$*-module for*

$$(p_\lambda f)(v) = \sum_{\mu \in \Lambda} p_{\lambda+\mu} f(p_\mu v).$$

Moreover, the T*-finite part* $\mathrm{Hom}(V, V')_T$ *exhibits the closed structure of the symmetric monoidal category* T-mod.

proof. Part (1) is obtained by unwinding the definitions. To see (2), let $\lambda, \lambda' \in \Lambda$, $f \in \mathrm{Hom}(V, V')$, and $v \in V$. Then we have

$$\begin{aligned}
(p_\lambda(p_{\lambda'} f))(v) &= \sum_{\mu \in \Lambda} p_{\lambda+\mu} (p_{\lambda'} f)(p_\mu v) \\
&= \sum_{\mu \in \Lambda} p_{\lambda+\mu} \sum_{\mu' \in \Lambda} p_{\lambda'+\mu'} f(p_{\mu'} p_\mu v) \\
&= \sum_{\mu \in \Lambda} p_{\lambda+\mu} p_{\lambda'+\mu} f(p_\mu v) \\
&= \begin{cases} \sum p_{\lambda+\mu} f(p_\mu v) & (\lambda = \lambda') \\ 0 & (\lambda \neq \lambda'), \end{cases} \\
&= ((p_\lambda p_{\lambda'}) f)(v).
\end{aligned}$$

This completes the proof. $\square$

Notice that the k-algebra $k[\Lambda]$ acts on the dual space $k[\Lambda]^*$ as

$$t^\lambda (\sum_\mu c_\mu p_\mu) = \sum_\mu c_\mu p_{\mu-\lambda}.$$

In particular, it restricts to $R(T) \subset k[\Lambda]^*$. Therefore we have a natural isomorphism

$$R(T) \otimes V \cong R(T) \otimes_{k[\Lambda]} (k[\Lambda] \otimes V)$$

for a k-module V. If we write $\lambda \in \Lambda$ and $v = \sum_\mu t^\mu \otimes v_\mu \in k[\Lambda] \otimes V$. Then the inverse image of $p_\lambda \otimes v$ is $\sum_\mu p_{\lambda-\mu} \otimes v_\mu$.

Suppose next that V is a T-module. Then we have a $k[\Lambda]$-module automorphism $V \otimes k[\Lambda] \cong V \otimes k[\Lambda]$ attached to the identity map of $k[\Lambda]$. Taking the base change $R(T) \otimes_{k[\Lambda]} -$ and switching the factors, we get an isomorphism of k-modules

$$\tau_V : R(T) \otimes V \cong V \otimes R(T);$$

$$p_\lambda \otimes v \mapsto \sum_\mu p_\mu v \otimes p_{\lambda-\mu}.$$

Its inverse is given by

$$v \otimes p_\lambda \mapsto \sum_\mu p_{\lambda+\mu} \otimes p_\mu v.$$

Let $(\mathcal{A}, T)$ be a weak pair, i.e., an algebra object of the symmetric monoidal category T-mod.

Lemma 5.2.2. *The k-module $\mathcal{A} \otimes R(T)$ is a not necessarily unital algebra for the module homomorphism*

$$(\mathcal{A} \otimes R(T)) \otimes (\mathcal{A} \otimes R(T)) \stackrel{id_{\mathcal{A}} \otimes \tau_{\mathcal{A}} \otimes id_{R(T)}}{\to} \mathcal{A} \otimes \mathcal{A} \otimes R(T) \otimes R(T) \to \mathcal{A} \otimes R(T);$$

$$(a \otimes p_\lambda)(b \otimes p_\mu) = ap_{\lambda-\mu} b \otimes p_\mu.$$

The resulting algebra will be referred to as $\mathcal{A}\sharp R(T)$.

proof. It will suffice to check the associativity. Let $a \otimes p_\lambda$, $b \otimes p_\mu$, and $c \otimes p_w$ be homogeneous elements of $\mathcal{A} \otimes R(T)$. Then Lemma 5.2.1 (1) implies

$$\begin{aligned}((a \otimes p_\lambda)(b \otimes p_\mu))(c \otimes p_w) &= (ap_{\lambda-\mu} b \otimes p_\mu)(c \otimes p_w) \\ &= ap_{\lambda-\mu} bp_{\mu-w} c \otimes p_w \\ &= ap_{\lambda-w}(bp_{\mu-w} c) \otimes p_w \\ &= (a \otimes p_\lambda)(bp_{\mu-w} c \otimes p_w) \\ &= (a \otimes p_\lambda)((b \otimes p_\mu)(c \otimes p_w))\end{aligned}$$

This completes the proof. □

We say that an $\mathcal{A}\sharp R(T)$-module is approximately unital if so is it as an $R(T)$-module. The category of approximately unital $\mathcal{A}\sharp R(T)$-modules will be denoted by $\mathcal{A}\sharp R(T)\text{-mod}^{\text{un}}$. The category of weak $(\mathcal{A}, T)$-modules are referred to as $(\mathcal{A}, T)\text{-mod}_w$ (see [H1]).

Corollary 5.2.3. *There is an isomorphism*

$$(\mathcal{A}, T)\text{-mod}_w \cong \mathcal{A}\sharp R(T)\text{-mod}^{\text{un}}.$$

proof. Similar computations to the proof of Lemma 5.2.2 show that a weak $(\mathcal{A}, T)$-module M is an approximately unital $\mathcal{A}\sharp R(T)$-module for

$$(a \otimes p_\lambda)m = a(p_\lambda m).$$

Conversely, if we are given an approximately unital $\mathcal{A}\sharp R(T)$-module M then define an action of $\mathcal{A}$ on M by the essentially finite sum

$$am = \sum_{\lambda \in \Lambda} (a \otimes p_\lambda)m.$$

These correspondences determine the desired isomorphism. □

Remark 5.2.4. The functor $(-)_T$ is compatible with the action of $\mathcal{A}$. Namely, the embedding of $\mathcal{A}\sharp R(T)\text{-mod}^{\text{un}}$ to the category $\mathcal{A}\sharp R(T)\text{-mod}$ of $\mathcal{A}\sharp R(T)$-modules admits an exact right adjoint functor $(-)_T$, and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{A}\sharp R(T)\text{-mod} & \xrightarrow{(-)_T} & \mathcal{A}\sharp R(T)\text{-mod}^{\text{un}} \\ \downarrow & & \downarrow \\ R(T)\text{-mod} & \xrightarrow[(-)_T]{} & R(T)\text{-mod}^{\text{un}} \end{array}$$

We next consider its version for pairs. For applications, consider the split torus $T = T^n$ of rank $n \geq 0$. In other words, put $\Lambda = \mathbb{Z}^n$. Then its Lie algebra $\mathfrak{t} = \mathfrak{t}^n$ has a basis $\{H_1, \cdots, H_n\}$ which acts on $R(T^n)$ by

$$H_i p_\lambda = \lambda_i p_\lambda.$$

If we are given a pair $(\mathcal{A}, T^n)$, the algebra structure of $\mathcal{A}\sharp R(T)$ descends to $\mathcal{A} \otimes_{U(\mathfrak{t}^n)} R(T^n)$. We will refer to it as $R(\mathcal{A}, T^n)$. An $R(\mathcal{A}, T^n)$-module is said to be approximately unital if so is it as an $R(T^n)$-module.

Corollary 5.2.5. *Let $(\mathcal{A}, T^n)$ be a pair. Then the isomorphism of Corollary 5.2.3 restricts to an isomorphism of categories of approximately unital $R(\mathcal{A}, T^n)$-modules and $(\mathcal{A}, T^n)$-modules.*

6 Integral models of representations of split semisimple Lie groups of type A_1

We start with the classification theorem of split $\mathbb{Z}$-forms of $(\mathfrak{sl}_2, T^1)$ in 1.5.

Proof of Theorem 1.5.5. Suppose that we are given a split $\mathbb{Z}$-form $(\mathfrak{g}, T^1, \alpha)$ of $(\mathfrak{sl}_2, T^1)$. Set $H = \psi(1) \in \mathfrak{g}$. Since α is T^1-equivariant, we have a weight decomposition

$$\mathfrak{g} = \mathfrak{g}_{-n} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_n,$$

where each T^1-weight module is free of rank 1. Hence we can find a nonzero complex number q which is unique up to sign and a unique element $E \in \mathfrak{g}$ such that

$$\alpha(\mathfrak{g}_n) = q\mathbb{Z}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$\alpha(E) = q\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Moreover, there is a unique positive integer m such that $[\mathfrak{g}_n, \mathfrak{g}_{-n}] = m\mathbb{Z}H$. We can then find a unique element $F \in \mathfrak{g}_{-n}$ such that $[E, F] = mH$. Since α is a Lie algebra homomorphism, we have

$$\alpha(F) = \frac{nm}{2q}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

To see that this correspondence is injective up to sign of q, suppose that we are given an isomorphism $f : (\mathfrak{g}_{n,m}, T^1, \alpha) \cong (\mathfrak{g}_{n,m'}, T^1, \alpha')$. Since f is T^1-equivariant, f restricts to an isomorphism of n-weight submodules. The argument above shows that the corresponding complex numbers q and q' are equal up to sign. Moreover, we have $f(E) = \frac{q'}{q}E$. The condition (iv) in Definition 1.5.4 implies that $f(H) = H$. Since f is a Lie algebra homomorphism between Lie algebras which are torsion-free as $\mathbb{Z}$-modules, $m = m'$ and $f(F) = \frac{q}{q'}F$. We now conclude that the set of isomorphism classes of split $\mathbb{Z}$-forms are bijective to that of pairs $(m, \pm q)$. $\square$

In this section, fix positive integers $n, m > 0$ to consider the pair $(\mathfrak{g}_{n,m}, T^1)$.

6.1 Discrete series representations and their limits over $\mathbb{Z}$

Let k be a commutative ring. In view of the PBW theorem and Lemma 5.2.1 (2), the usual computations over $\mathbb{C}$ work over k (see [KV] (2.12)):

Proof of Theorem 1.5.7. Put $y_{\lambda+np} = E^p \otimes 1$, and define

$$y^{\lambda+np} \in \mathrm{pro}_{\mathfrak{b}}^{\mathfrak{g}_{n,m}}(k_\lambda) \cong \mathrm{Hom}_{U(\mathfrak{b})}(U(\mathfrak{g}_{n,m}), k_\lambda)_{T^1}$$

as

$$y^{\lambda+np}(F^q) = \begin{cases} 1 & (p = q) \\ 0 & (q \neq p). \end{cases}$$

They form free bases. The actions $Fy_{\lambda+np}$ and $Ey^{\lambda+np}$ are computed as

$$\begin{aligned}
(Ey^{\lambda+np})(F^{p+1}) &= y^{\lambda+np}(F^{p+1}E) \\
&= y^{\lambda+np}(EF^{p+1} - \frac{1}{2}nmp(p+1)F^p - m(p+1)HF^p) \\
&= -\frac{1}{2}nmp(p+1) - m(p+1)\lambda
\end{aligned}$$

$$\begin{aligned}
Fy_{\lambda+np} &= FE^p \otimes 1 \\
&= E^pF \otimes 1 - \frac{1}{2}mp(n(p+1) - 2n + 2\lambda)E^{p-1} \otimes 1 \\
&= -\frac{1}{2}mp(np - n + 2\lambda)y_{\lambda+n(p-1)}.
\end{aligned}$$

The rest is obvious by definition. □

Note that $\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}_{n,m}}(k_\lambda)$ is finitely generated as a $U(\mathfrak{g}_{n,m})$-module by definition, and $\mathrm{pro}_{\mathfrak{b}}^{\mathfrak{g}_{n,m}}(k_\lambda)$ is not. However, they still have the same T^1-types. Moreover, each of them is free of finite rank 1 as a k-module. In particular, these representations are admissible in the sense of [BHS].

Remark 6.1.1. Suppose that k is Noetherian. Since we are working with possibly torsion modules, it might be convenient to emphasize that $\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}_{n,m}}(k_\lambda)$ and $\mathrm{pro}_{\mathfrak{b}}^{\mathfrak{g}_{n,m}}(k_\lambda)$ satisfy the following condition on T^1-modules V as an estimation of the size of V: For any finitely generated T^1-module Q, the k-module $\mathrm{Hom}_{T^1}(Q, V)$ is finitely generated. This condition is quite delicate: Suppose that k is an integral domain. Even a finitely generated and torsion-free k-form of an admissible $(\mathfrak{g} \otimes \mathrm{Frac}(k), T^1)$-module is not admissible in general. A commutative model of counterexamples is given as follows: Put $k = \mathbb{Z}$, and consider a pair $(\mathbb{Z}, \mathrm{Spec}\,\mathbb{Z})$. Set $V = \mathbb{Z}[1/2]$, and put an action of the polynomial ring $\mathbb{Z}[x]$ on V by $x = 2^{-1}$. Then V is a finitely generated $\mathbb{Z}[x]$-module, $V \otimes \mathbb{Q}$ is of finite dimension over $\mathbb{Q}$, and torsion free as a $\mathbb{Z}$-module. However, V is not finitely generated over $\mathbb{Z}$. See also Proposition 4.1.2.

Remark 6.1.2. Let $(\mathfrak{g}, K)$ be a pair over a Noetherian domain k. Suppose that $\mathfrak{g}$ is finitely generated as a k-module. Then every $(\mathfrak{g}, K)$-module has a finitely generated $(\mathfrak{g}, K)$-submodule (Proposition 3.2.4). In particular, if we are given a $(\mathfrak{g}, K)$-module V such that $V \otimes \mathrm{Frac}(k)$ is irreducible as a $(\mathfrak{g} \otimes \mathrm{Frac}(k), K \otimes \mathrm{Frac}(k))$-module then there is a finitely generated $(\mathfrak{g}, K)$-submodule $V' \subset V$ such that $V' \otimes \mathrm{Frac}(k) \cong V \otimes \mathrm{Frac}(k)$.

6.2 Integral and fractional models of principal series represe-netations

We next consider models of the real parabolic inductions. Set

$$X = -2nmE + F + 2mH$$

$$Y = 2nmE + F$$
$$\mathfrak{q} = \mathbb{Z}X \oplus \mathbb{Z}Y$$
$$M = \mathrm{Spec}\,\mathbb{Z}[t]/(t^n - 1)$$

to obtain a subpair $(\mathfrak{q}, M) \subset (\mathfrak{g}_{n,m}, T^1)$ over $\mathbb{Z}$. Let k be a $\mathbb{Z}[1/2nm]$-algebra, and regard $(\mathfrak{q}, M)$ and $(\mathfrak{g}_{n,m}, T^1)$ as pairs over k.

Proposition 6.2.1. *The following diagram is 2-commutative:*

$$\begin{array}{ccc} (\mathfrak{q}, M)\text{-mod} & \xrightarrow{I^{\mathfrak{g}_{n,m},T^1}_{\mathfrak{q},M}} & (\mathfrak{g}_{n,m}, T^1)\text{-mod} \\ \downarrow & & \downarrow \\ M\text{-mod} & \xrightarrow[\mathrm{Ind}^{T^1}_M]{} & T^1\text{-mod} \end{array}$$

proof. Observe that the summation of $\mathfrak{q} \subset \mathfrak{g}_{n,m}$ and $\psi : \mathfrak{t}^1 \to \mathfrak{g}_{n,m}$ determines an isomorphism

$$\mathfrak{q} \oplus \mathfrak{t}^1 \cong \mathfrak{g}_{n,m};$$
$$(-\frac{1}{4nm}X + \frac{1}{4nm}Y, \frac{1}{2n}H) \leftarrow E$$
$$(\frac{1}{2}X + \frac{1}{2}Y, -mH) \leftarrow F$$
$$H \leftarrow H.$$

Hence for a $(\mathfrak{q}, M)$-module W and a T^1-module χ, we have

$$\begin{aligned} \mathrm{Hom}_{T^1}(\chi, I^{\mathfrak{g}_{n,m},T^1}_{\mathfrak{q},M}(W)) &\cong \mathrm{Hom}_{\mathfrak{g}_{n,m},T^1}(\mathrm{ind}^{\mathfrak{g}_{n,m}}_{\mathfrak{t}^1}\chi, I^{\mathfrak{g}_{n,m},T^1}_{\mathfrak{q},M}(W)) \\ &\cong \mathrm{Hom}_{\mathfrak{q},M}(\mathrm{ind}^{\mathfrak{g}_{n,m}}_{\mathfrak{t}^1}\chi, W) \\ &\cong \mathrm{Hom}_M(\chi, W); \end{aligned}$$

$$f \mapsto p \circ f,$$

where $p : I^{\mathfrak{g}_{n,m},T^1}_{\mathfrak{q},M}(W) \to W$ is the counit. This proves the assertion. □

Corollary 6.2.2. *The functor* $I_{\mathfrak{q},M}^{\mathfrak{g}_{n,m},T^1}$ *is exact.*

proof. The left exactness follows since $I_{\mathfrak{q},M}^{\mathfrak{g}_{n,m},T^1}$ is right adjoint. We show that it is also right exact. In view of Proposition 5.1.9 and Proposition 6.2.1, it will suffice to solve the lifting problem

$$\begin{array}{ccc} & & \operatorname{Ind}_M^{T^1} X \\ & \nearrow & \downarrow{\scriptstyle \operatorname{Ind}_M^{T^1}(p)} \\ k_\lambda & \longrightarrow & \operatorname{Ind}_M^{T^1} Y, \end{array}$$

where $p : X \to Y$ is a surjection of M-modules, and $\lambda \in \mathbb{Z}$. Pass to the adjunction so that it is equivalent to

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow{\scriptstyle p} \\ k_\lambda & \longrightarrow & Y. \end{array}$$

The dotted arrow now exists from Proposition 5.1.9. □

Corollary 6.2.3. *There is an isomorphism* $\operatorname{Ind}_M^{T^1} \cong R(T^1) \otimes_{R(M)} -$*. In particular,* $I_{\mathfrak{q},M}^{\mathfrak{g}_{n,m},T^1}$ *enjoys every base change, i.e., for a homomorphism* $k \to k'$ *of* $\mathbb{Z}[1/2nm]$*-algebras, there is a canonical isomorphism*

$$I_{\mathfrak{q},M}^{\mathfrak{g}_{n,m},T^1}(-) \otimes_k k' \cong I_{\mathfrak{q}\otimes_k k', M\otimes_k k'}^{\mathfrak{g}_{n,m}\otimes_k k', T^1\otimes_k k'}(- \otimes_k k').$$

The adjoint functor theorem also implies the following property:

Corollary 6.2.4. *The functor* $I_{\mathfrak{q},M}^{\mathfrak{g}_{n,m},T^1}$ *admits a right adjoint functor.*

Theorem 1.5.9 is obtained by use of the Hecke algebras:

Proof of Theorem 1.5.9. Proposition 6.2.1 and its proof imply that the restriction along $R(T^1) \to R(\mathfrak{g}_{n,m}, T^1)$ gives rise to an isomorphism

$$I_{\mathfrak{q},M}^{\mathfrak{g}_{n,m},T^1}(k_{\epsilon,\mu}) = \operatorname{Hom}_{\mathfrak{q},M}(R(\mathfrak{g}_{n,m}, T^1), k_{\epsilon,\mu})_{T^1} \cong \operatorname{Hom}_M(R(T^1), k_{n\epsilon})_{T^1}$$

Set $w^{n(p+\epsilon)} \in (\operatorname{Hom}_{\mathfrak{q},M}(R(\mathfrak{g}_{n,m}, T^1), k_{\epsilon,\mu})_{T^1})_{n(p+\epsilon)}$ as

$$w^{n(p+\epsilon)}(1 \otimes p_\lambda) = \begin{cases} 1 & (\lambda = n(p+\epsilon)) \\ 0 & (\text{otherwise}). \end{cases}$$

Following Corollary 5.2.3, we can compute the actions of $R(T^1)$, E and F as

$$\begin{aligned} (p_\lambda w^{n(p+\epsilon)})(p_{\lambda'}) &= w^{n(p+\epsilon)}(p_\lambda \cdot p_{\lambda'}) \\ &= \begin{cases} 1 & (\lambda = \lambda' = n(p+\epsilon)) \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

$$
\begin{aligned}
(Ew^{n(p+\epsilon)})(p_{n(p+1+\epsilon)}) &= \sum_{\lambda\in\mathbb{Z}} w^{n(p+\epsilon)}((1\otimes p_{n(p+1+\epsilon)})\cdot(E\otimes p_\lambda)) \\
&= w^{n(p+\epsilon)}(E\otimes p_{n(p+\epsilon)}) \\
&= w^{n(p+\epsilon)}((-\frac{1}{4nm}X+\frac{1}{4nm}Y+\frac{1}{2n}H)\otimes p_{n(p+\epsilon)}) \\
&= \frac{1}{4nm}\mu+\frac{1}{2}(p+\epsilon) \\
(Fw^{n(p+\epsilon)})(p_{n(p-1+\epsilon)}) &= \sum_{\lambda\in\mathbb{Z}} w^{n(p+\epsilon)}((1\otimes p_{n(p-1+\epsilon)})\cdot(F\otimes p_\lambda)) \\
&= w^{n(p+\epsilon)}(F\otimes p_{n(p+\epsilon)}) \\
&= w^{n(p+\epsilon)}((\frac{1}{2}X+\frac{1}{2}Y-mH)\otimes p_{n(p+\epsilon)}) \\
&= \frac{1}{2}\mu-nm(p+\epsilon)
\end{aligned}
$$

This completes the proof. □

Remark 6.2.5. The counit $I^{\mathfrak{g}_{n,m},T^1}_{\mathfrak{q},M}(k_{\epsilon,\mu})\to k_{\epsilon,\mu}$ is given by $w^{n(p+\epsilon)}\mapsto 1$.

Essentially new phenomena occur when $2nm$ is not invertible in k. Let $k=\mathbb{Z}$. For parameters $\epsilon\in\{0,\frac{1}{n},\cdots,\frac{n-1}{n}\}$, $\mu\in\mathbb{Z}$ and each integer $\lambda\in\mathbb{Z}$, we have a compatible homomorphism

$$
\begin{aligned}
\mathrm{Hom}_{T^1}(\mathbb{Z}_\lambda, I^{\mathfrak{g}_{n,m},T^1}_{\mathfrak{q},M}(\mathbb{Z}_{\epsilon,\mu})) &\cong \mathrm{Hom}_{\mathfrak{g}_{n,m},T^1}(\mathrm{ind}^{\mathfrak{g}_{n,m}}_{\mathfrak{t}^1}\mathbb{Z}_\lambda, I^{\mathfrak{g}_{n,m},T^1}_{\mathfrak{q},M}(\mathbb{Z}_{\epsilon,\mu})) \\
&\cong \mathrm{Hom}_{\mathfrak{q},M}(\mathrm{ind}^{\mathfrak{g}_{n,m}}_{\mathfrak{t}^1}\mathbb{Z}_\lambda,\mathbb{Z}_{\epsilon,\mu}) \\
&\to \mathrm{Hom}_M(\mathbb{Z}_\lambda,\mathbb{Z}_{\epsilon,\mu})
\end{aligned}
$$

with the sequence of isomorphisms in the proof of Proposition 6.2.1. Since the localization homomorphism $\mathbb{Z}\to\mathbb{Z}[1/2nm]$ is injective, $\mathrm{Hom}_{\mathfrak{q},M}(\mathrm{ind}^{\mathfrak{g}_{n,m}}_{\mathfrak{t}^1}\mathbb{Z}_\lambda,\mathbb{Z}_{\epsilon,\mu})$ is bijective to the set of M-homomorphisms $\mathbb{Z}_\lambda\to\mathbb{Z}_{\epsilon,\mu}$ which (uniquely) extends to a $(\mathfrak{q},M)$-homomorphism $\mathrm{ind}^{\mathfrak{g}_{n,m}}_{\mathfrak{t}^1}\mathbb{Z}_\lambda\to\mathbb{Z}_{\epsilon,\mu}$. For a more explicit description, we analyze the bijection

$$
\mathrm{Hom}_{\mathfrak{q},M}(\mathrm{ind}^{\mathfrak{g}_{n,m}}_{\mathfrak{t}^1}\mathbb{Z}[1/2nm]_\lambda,\mathbb{Z}[1/2nm]_{\epsilon,\mu})\cong\mathrm{Hom}_M(\mathbb{Z}[1/2nm]_\lambda,\mathbb{Z}[1/2nm]_{\epsilon,\mu}).
$$

We may assume that λ is of the form $n(p+\epsilon)$ for some integer $p\in\mathbb{Z}$; Otherwise $\mathrm{Hom}_M(\mathbb{Z}[1/2nm]_\lambda,\mathbb{Z}[1/2nm]_{\epsilon,\mu})=0$. Let $\varphi\in\mathrm{Hom}_M(\mathbb{Z}[1/2nm]_{n(p+\epsilon)},\mathbb{Z}[1/2nm]_{\epsilon,\mu})$. According to the description of the isomorphism $\mathfrak{q}\oplus\mathfrak{t}^1\cong\mathfrak{g}_{n,m}$ over $\mathbb{Z}[1/2nm]$ (see the proof of Proposition 6.2.1), the extension is given by

$$
\begin{aligned}
\varphi(E^{s+1}\otimes 1) &= \varphi((-\frac{1}{4nm}X+\frac{1}{4nm}Y+\frac{1}{2n}H)E^s\otimes 1) \\
&= (\frac{1}{4nm}\mu+\frac{1}{2}(s+p+\epsilon))\varphi(E^s\otimes 1) \\
\varphi(F^{s+1}E^t\otimes 1) &= \varphi((\frac{1}{2}X+\frac{1}{2}Y-mH)(F^sE^t\otimes 1)) \\
&= (\frac{1}{2}\mu+nm(s-t-p-\epsilon))\varphi(F^sE^t\otimes 1)
\end{aligned}
$$

for nonnegative integers $s, t \geq 0$.

Lemma 6.2.6. *We have*

$$\max\{\sum_{l=1}^{s}(1 - \mathrm{ord}_2\, l) : s \geq 0\} = \infty.$$

proof. Put $s = 2^a - 1$ for some nonnegative integer a. Then

$$\begin{aligned}\sum_{l=1}^{s}(1 - \mathrm{ord}_2\, l) &= 2^a - 1 - \sum_{b=1}^{\infty}\left[\frac{2^a - 1}{2^b}\right] \\ &= 2^a - 1 - \sum_{b=1}^{a}(2^{b-1} - 1) \\ &= a.\end{aligned}$$

The assertion now follows. □

Proof of Theorem 1.5.11. Let φ be a nonzero element of $\mathrm{Hom}_M(\mathbb{Z}_{n(p+\epsilon)}, \mathbb{Z}_{\epsilon,\mu})$. Suppose that either of the following conditions fail.

(i) $\frac{1}{2nm}\mu + \epsilon \in \mathbb{Z}$;

(ii) $p \leq -\frac{1}{2nm}\mu - \epsilon$.

Then $\varphi(E^s \otimes 1)$ never vanishes for $s \geq 0$. If (i) is satisfied, and (ii) fails then Lemma 6.2.6 implies

$$\min\{\sum_{l=0}^{s-1}\mathrm{ord}_2(\frac{1}{4nm}\mu + \frac{1}{2}(l + p + \epsilon)) : s \geq 0\} = -\infty.$$

Suppose that (i) fails. Then there exists a prime number q such that $\mathrm{ord}_q(\frac{1}{2nm}\mu + \epsilon) < 0$, where ord_q denotes the q-adic valuation on $\mathbb{Z}$. This implies that for any $l \geq 0$,

$$\begin{aligned}\mathrm{ord}_q(\frac{1}{2}(\frac{1}{2nm}\mu + (l + p + \epsilon))) &= -\mathrm{ord}_q\, 2 + \mathrm{ord}_q(\frac{1}{2nm}\mu + (l + p + \epsilon)) \\ &= -\mathrm{ord}_q\, 2 + \mathrm{ord}_q(\frac{1}{2nm}\mu + \epsilon) < 0,\end{aligned}$$

and

$$\min\{\sum_{l=0}^{s-1}\mathrm{ord}_q(\frac{1}{4nm}\mu + \frac{1}{2}(l + p + \epsilon)) : s \geq 0\} = -\infty.$$

Hence φ never extends to a $(\mathfrak{q}, M)$-homomorphism $\mathrm{ind}_{\mathfrak{t}^1}^{\mathfrak{g}_{n,m}} \mathbb{Z}_{n(p+\epsilon)} \to \mathbb{Z}_{\epsilon,\mu}$.

If (i) and (ii) are satisfied, $s_0 = -\frac{1}{2nm}\mu - \epsilon - p$ is a nonnegative integer, and $\varphi(E^{s_0+1} \otimes 1) = 0$. Since μ is even from (i),

$$\varphi(F^{s+1}E^t \otimes 1) \in \mathbb{Z}\varphi(F^s E^t \otimes 1)$$

for all $s, t \geq 0$. Hence φ has an extension if and only if $\varphi(E^{s'} \otimes 1) \in \mathbb{Z}$ for $1 \leq s' \leq s_0$ in this case. It is characterized as

$$\operatorname{ord}_2 \varphi(1) + \sum_{l=0}^{s'-1} \operatorname{ord}_2(\frac{1}{4nm}\mu + \frac{1}{2}(l + p + \epsilon)) \geq 0$$

for all s'. This completes the proof. $\square$

Theorem 1.5.13-Theorem 1.5.16 are obtained by similar arguments. Note that we have recurrence formulas for the cases of $\mathfrak{q}', \mathfrak{q}''$

$$\varphi(F^{s+1} \otimes 1) = (\frac{1}{4nm}\mu + \frac{1}{2}(s - p - \epsilon))\varphi(F^s \otimes 1)$$

$$\varphi(E^{s+1}F^t \otimes 1) = (\frac{1}{2}\mu + nm(s - t + p + \epsilon))\varphi(E^s F^t \otimes 1)$$

$$\varphi(F^{s+1} \otimes 1) = (\frac{1}{2}\mu + n(s - p - \epsilon))\varphi(F^s \otimes 1)$$

$$\varphi(E^{s+1}F^t \otimes 1) = (\frac{1}{2}\mu + n(s - t + p + \epsilon))\varphi(E^s F^t \otimes 1).$$

Though we cannot take the base change from $\mathbb{Z}$ to the finite field $\mathbb{F}_2$ since M is singular, we can find another situation in an appropriate sense.

Example 6.2.7. Put $n = 2$, $q = 1$, and $m = 1$ in Theorem 1.5.5. We also set

$$\mathfrak{q} = \mathbb{Z}(-E + F + H) \oplus \mathbb{Z}(E + F).$$

Then $\mathfrak{q}$ and M eventually form a subpair of $(\mathfrak{g}_{2,1}, T^1)$ over $\mathbb{F}_2$-algebras k. Since $M \to T^1$ induces an isomorphism of their Lie algebras, $I^{\mathfrak{g},T^1}_{\mathfrak{q},M}(k_{\epsilon,\mu})$ can be computed as the weak version $I^{\mathfrak{g},T^1}_{\mathfrak{q},M,w}(k_{\epsilon,\mu})$ (see [H1]). Hence we obtain

$$I^{\mathfrak{g},T^1}_{\mathfrak{q},M}(k_{\epsilon,\mu}) \cong \operatorname{Hom}_{\mathfrak{q},M}(U(\mathfrak{g}) \otimes R(T^1), k_{\epsilon,\mu})_{T^1}$$

from Corollary 5.2.3.

6.3 Variants for contraction families

Fix a positive integer n, and consider the pair $(\mathfrak{sl}_2, T^1)$ over $\mathbb{C}$ associated to the n-cover of PU(1,1). In this section, we compute analogs of the previous section for the associated contraction family $(\widetilde{\mathfrak{sl}}_2, T^1)$ over $\mathbb{C}[z]$. Recall that the Cartan involution θ of $\mathfrak{sl}_2$ is given by

$$\theta\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\theta\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\theta\left(\left(\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right)\right)=\left(\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right)$$

In view of naturality of the construction of contraction families, the θ-stable parabolic subpairs $(\mathfrak{b}_{\mathbb{C}}, T^1)$ and $(\bar{\mathfrak{b}}_{\mathbb{C}}, T^1)$ of $(\mathfrak{sl}_2, T^1)$ extend to subpairs $(\tilde{\mathfrak{b}}_{\mathbb{C}}, T^1), (\tilde{\bar{\mathfrak{b}}}_{\mathbb{C}}, T^1) \subset (\widetilde{\mathfrak{sl}}_2, T^1)$. Theorem 1.5.17 is deduced from similar computations to Theorem 1.5.7.

On the other hand, the counterpart of real parabolic inductions and principal series representations is nontrivial since $\mathfrak{q}_{\mathbb{C}}$ is not θ-stable. We suggest the $\mathbb{C}[z]$-submodule $\tilde{\mathfrak{q}}$ of $\widetilde{\mathfrak{sl}}_2$ spanned by the two elements

$$X=\left(\begin{array}{cc}z & -z\\ 1 & -z\end{array}\right)$$

$$Y=\left(\begin{array}{cc}0 & z\\ 1 & 0\end{array}\right)$$

as a contraction model of $\mathfrak{q}$. It gives rise to a subpair $(\tilde{\mathfrak{q}}, M)$ with $M = M_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}[z]$. Note that a similar issue to Theorem 1.5.11 appears in this situation. Namely, we only have a decomposition over $\mathbb{C}\left[z^{\pm 1}\right]$:

$$\tilde{\mathfrak{q}} \oplus \mathfrak{t}^1 \cong \widetilde{\mathfrak{sl}}_2;$$

$$\left(\frac{-b+cz}{2z}X + \frac{b+cz}{2z}Y, \frac{2a+b-zc}{2}\left(\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right)\right) \mapsto \left(\begin{array}{cc}a & b\\ c & -a\end{array}\right).$$

Notice also that $\mathbb{C}[z]X$ is an ideal of $\tilde{\mathfrak{q}}$ so that the projection $\tilde{\mathfrak{q}} \to \mathbb{C}[z]Y$ is a Lie algebra homomorphism. With these in mind, we can deduce Theorem 1.5.18 by similar arguments.

Remark 6.3.1. Under the condition of Theorem 1.5.18 (3), it turns out that we have shown the base change formula

$$I^{\widetilde{\mathfrak{sl}}_2, T^1}_{\tilde{\mathfrak{q}}, M}(\mathbb{C}[z]_{\epsilon,\mu}) \otimes_{\mathbb{C}[z]} \mathbb{C}\left[z^{\pm 1}\right] \cong I^{\widetilde{\mathfrak{sl}}_2, T^1}_{\tilde{\mathfrak{q}}, M}(\mathbb{C}\left[z^{\pm 1}\right]_{\epsilon,\mu})$$

without the conditions in Theorem 3.1.7

Remark 6.3.2. The $(\widetilde{\mathfrak{sl}}_2, T^1)$-module $I^{\widetilde{\mathfrak{sl}}_2, T^1}_{\tilde{\mathfrak{q}}, M}(\mathbb{C}\left[z^{\pm 1}\right]_{\epsilon,\mu})$ is generically irreducible in the sense that its base change to the algebraic closure $\overline{\mathbb{C}(z)}$ is irreducible (see [BHS]) if and only if μ does not belong to $2z\mathbb{Z} \pm 2z\epsilon$.

Remark 6.3.3. The construction of $\tilde{\mathfrak{q}}$ is generalized in the following way: Suppose that we are given a complex reductive Lie algebra $\mathfrak{g}$ and a parabolic subalgebra $\mathfrak{q}$ with abelian nilradical $\mathfrak{u}$. Fix a Cartan subalgebra and a Borel subalgebra $\mathfrak{b}$ contained in $\mathfrak{q}$. Then we obtain a Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Write $\bar{\mathfrak{u}}$ for the nilradical of the opposite parabolic subalgebra to $\mathfrak{q}$. Then $\mathfrak{g}$ and $\mathfrak{l}$ form a symmetric pair for the involution θ defined as

$$\theta(x)=\begin{cases} x & (x \in \mathfrak{l}) \\ -x & (x \in \mathfrak{u} \oplus \bar{\mathfrak{u}}). \end{cases}$$

If we write $\tilde{\mathfrak{g}}$ for the associated contraction family, we have an isomorphism $\Phi : \mathfrak{g} \otimes \mathbb{C}\left[z^{\pm 1}\right] \cong \tilde{\mathfrak{g}} \otimes_{\mathbb{C}[z]} \mathbb{C}\left[z^{\pm 1}\right]$ of Lie algebras over $\mathbb{C}\left[z^{\pm 1}\right]$:

$$\Phi(x) = \begin{cases} x & (x \in \mathfrak{q}) \\ z^{-1}x & (x \in \bar{\mathfrak{u}}). \end{cases}$$

For a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the image $\Phi(z\mathbb{C}[z]\mathfrak{h})$ is a Lie subalgebra of $\tilde{\mathfrak{g}}$. Passing to this isomorphism, we can identify Theorem 1.5.18 (1) with Theorem 1.5.9. More specifically, suppose that there is a semidirect decomposition $\mathfrak{h} = \mathfrak{h}_s \oplus \mathfrak{h}_n$ with $\mathfrak{h}_n$ being an ideal, and that the subalgebra $\mathfrak{h}_s$ is contained in $\mathfrak{q}$. Then we can find a larger subalgebra $\Phi(\mathbb{C}[z]\mathfrak{h}_s \oplus z\mathbb{C}[z]\mathfrak{h}_n)$.

Remark 6.3.4. In the previous section, we considered pairs $(\mathfrak{g}_{n,m}, T^1)$. From the perspectives on contraction families, we can regard each of them as a special fiber $z = m$ of the contraction family over $\mathbb{Z}[z]$ associated to $(\mathfrak{g}_{n,1}, T^1)$. Following this idea, we can think that $\mathfrak{q}, \mathfrak{q}' \subset \mathfrak{g}_{n,m}$ are obtained from the construction of Remark 6.3.3. The maximal $\mathbb{Z}$-form in Remark 1.5.12 is obtained by the latter construction in Remark 6.3.3.

7 The algebraic Borel-Weil-Bott theorem over $\mathbb{Z}$

Let G be a split reductive group over $\mathbb{Z}$, and $T \subset G$ be a split maximal torus. Choose a positive system Δ^+ of the root system of G. Let $\mathfrak{b}$ (resp. $\bar{\mathfrak{b}}$) be the Borel subalgebra corresponding to Δ^+ (resp. the set $-\Delta^+$ of negative roots), and $\mathfrak{n}$ (resp. $\bar{\mathfrak{n}}$) be its nilradical. Suppose that we are given a flat ring k over $\mathbb{Z}$ and a character λ of T. Then Theorem 3.1.7 says $I^{\mathfrak{g}\otimes k, G\otimes k}_{\bar{\mathfrak{b}}\otimes k, T\otimes k}(k_\lambda) \cong I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda) \otimes k$.

7.1 The Borel-Weil theorem over $\mathbb{Z}$

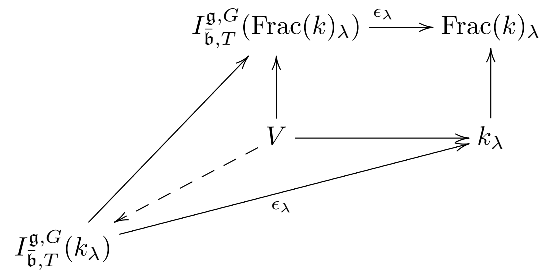
Theorem 7.1.1. *(1) The G-module $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda)$ is free of finite rank as a $\mathbb{Z}$-module.*

(2) The G-module $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(k_\lambda)$ is nonzero if and only if λ is dominant.

(3) The counit $\epsilon_\lambda : I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(k_\lambda) \to k_\lambda$ is surjective if λ is dominant.

(4) If k is an integral domain, $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(k_\lambda)$ is the maximal k-form among those whose highest weight spaces are $k \subset \mathrm{Frac}(k)$ for the given embedding to $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathrm{Frac}(k)_\lambda)$. In other words, if we have a G-submodule V of $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathrm{Frac}(k)_\lambda)$

fitting the following square



then the dotted arrow (uniquely) exists.

(5) *Let λ, λ' be dominant characters of T, and V, V' be nonzero G-submodules of $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda)$ and $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_{\lambda'})$ respectively. Then there is an isomorphism*

$$\mathrm{Hom}_G(V, V') \cong \begin{cases} \mathbb{Z} & (\lambda = \lambda') \\ 0 & (\lambda \neq \lambda'). \end{cases}$$

proof. Since $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda)$ is embedded to the $\mathbb{Q}$-module $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Q}_\lambda)$, $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda)$ is torsion free. Write $V^M(\lambda) = I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda)$. Then (1) is reduced to showing that $V^M(\lambda)_\mu$ is finitely generated for each T-weight μ. According to the algebraic Borel-Weil theorem over $\mathbb{C}$ ([KV] Corollary 4.160), we may assume λ is dominant, and prove it by the descending induction of the level of μ. The level of μ is largest if and only if $\mu = \lambda$. In this case, the highest weight space is contained in $\mathbb{Z}$. In fact, we have a diagram

$$\begin{array}{ccccc} V^M(\lambda)_\lambda & \longrightarrow & V^M(\lambda) & \longrightarrow & \mathbb{Z}_\lambda \\ \downarrow & & \downarrow & & \downarrow \\ V(\lambda)_\lambda & \longrightarrow & V(\lambda) & \longrightarrow & \mathbb{Q}_\lambda \end{array}$$

where $V(\lambda) = I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Q}_\lambda)$. Since the bottom composite arrow is an isomorphism of T-modules, and the vertical arrows are injective, the top composite arrow is also injective. Since $\mathbb{Z}$ is a PID, the assertion follows. If the level of μ is smaller than λ, we have an injective homomorphism

$$\prod_{\alpha \in \Delta^+} E_\alpha : V^M(\lambda)_\mu \to \oplus V^M(\lambda)_{\mu+\alpha},$$

where E_α are root vectors of $\mathfrak{g}$. The induction hypothesis says that the target is finitely generated. Since $\mathbb{Z}$ is Noetherian, $V^M(\lambda)_\mu$ is finitely generated. If the level is small enough, the algebraic Borel-Weil theorem over $\mathbb{C}$ implies that the weight space vanishes. Hence this procedure stops, and (1) follows.

We next prove (2) for $k = \mathbb{Z}$. Recall that $V^M(\lambda) \to V(\lambda)$ is injective. In particular, Proposition 3.1.11 implies the "only if" direction. The flat base change theorem also shows the "if" direction.

To see (2) for every k, notice that $\mathbb{Z} \to k$ is injective. In fact, k is flat over the integral domain $\mathbb{Z}$. Hence $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda) \to I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(k_\lambda)$ is injective. Part (2) for $k = \mathbb{Z}$ thus shows the "if" direction. The converse follows from (2) for $k = \mathbb{Z}$ and the flat base change theorem.

We next prove (3). Suppose that λ is dominant. It is deduced by finding a G-module V with a surjective $(\bar{\mathfrak{b}}, T)$-module homomorphism to k_λ. Passing to base changes, we may assume $k = \mathbb{Z}$:

$$\begin{array}{ccc} V \otimes k & \longrightarrow & I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(k_\lambda) \\ & \searrow \quad \swarrow & \\ & k_\lambda. & \end{array}$$

Let v_λ be the highest weight vector of $V(\lambda)$ with $\epsilon_\lambda(v_\lambda) = 1$, and $V^m(\lambda)$ be the G-submodule generated by v_λ over $\mathbb{Z}$. Notice that $V^m(\lambda)$ is nonzero from (2). Moreover, $V^m(\lambda)$ is a $\mathbb{Z}$-form of $V(\lambda)$ by Proposition 3.1.11. Notice also that the G-module $V^m(\lambda)$ contains $\mathbb{Z}v_\lambda$. Observe that for a nonzero element $q \in \mathbb{Q}$, $qV^m(\lambda)$ is the minimal G-submodule $\langle qv_\lambda \rangle$ of $V(\lambda)$ containing qv_λ since the scalar multiplication by q gives rise to an automorphism $V(\lambda) \cong V(\lambda)$. Let $v \in V^m(\lambda)$, and write $\epsilon_\lambda(v) = q \in \mathbb{Q}$. According to the T-type decomposition, qv_λ belongs to $V^m(\lambda)$. Then we get $V^m(\lambda) \supset qV^m(\lambda)$ from $qV^m(\lambda) = \langle qv_\lambda \rangle$. Since $V^m(\lambda)$ is free of finite rank over $\mathbb{Z}$ (Proposition 2.1.8), q belongs to $\mathbb{Z}$. Hence (3) follows. The assertion (4) is a consequence of (2), (3), and the universal property of $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}$. In fact, $I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(k_\lambda) \to I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathrm{Frac}(k)_\lambda)$ is injective.

Finally, we prove (5). In view of Theorem 3.1.11 and Theorem 3.1.6, we have

$$\mathrm{Hom}_G(V, V') \otimes \mathbb{Q} \cong \mathrm{Hom}_{G\otimes\mathbb{Q}}(V \otimes \mathbb{Q}, V' \otimes \mathbb{Q}).$$

Moreover, the base change to $\mathbb{C}$ and Schur's lemma imply

$$\mathrm{Hom}_{G\otimes\mathbb{Q}}(V \otimes \mathbb{Q}, V' \otimes \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & (\lambda = \lambda') \\ 0 & (\lambda \neq \lambda'). \end{cases}$$

Since V and V' are finitely generated and torsion-free as $\mathbb{Z}$-modules, we have a desired isomorphism. This completes the proof. □

Remark 7.1.2. We have a more precise description of Theorem 7.1.1 (5) when $\lambda = \lambda'$ and $V' = V^M(\lambda)$. Let $v \in V$ be a generator of the highest weight space of V (v exists since $\mathbb{Z}$ is a PID). Then

$$\mathrm{Hom}_{\bar{\mathfrak{b}},T}(V, \mathbb{Z}_\lambda) \hookrightarrow \mathrm{Hom}_{\bar{\mathfrak{b}}\otimes\mathbb{C},T\otimes\mathbb{C}}(V \otimes \mathbb{C}, \mathbb{C}_\lambda) \cong \mathbb{C}.$$

If $\epsilon_\lambda|_V(v) = n \neq 0$, we have $\mathrm{Hom}_{\bar{\mathfrak{b}},T}(V, \mathbb{Z}_\lambda) \cong \frac{1}{n}\mathbb{Z}\epsilon_\lambda|_V$.

Corollary 7.1.3 ([Hum] 27.3 Proposition)**.** *Let λ be an anti-dominant character of T, v_λ be a lowest weight vector of $I_{\mathfrak{b},T}^{\mathfrak{g},G}(\mathbb{Q}_\lambda)$, and V^m be the G-submodule over $\mathbb{Z}$ of $I_{\mathfrak{b},T}^{\mathfrak{g},G}(\mathbb{Q}_\lambda)$ generated by v_λ. Then there is an isomorphism of G-modules*

$$\mathrm{Hom}(V^m, \mathbb{Z}) \cong I_{\bar{\mathfrak{b}},T}^{\mathfrak{g},G}(\mathbb{Z}_{-\lambda}).$$

proof. Let V be a G-submodule of $\mathrm{Hom}_{\mathbb{Z}}(V^m, \mathbb{Q})$ over $\mathbb{Z}$. Identify $\mathrm{Hom}(V^m, \mathbb{Z})$ as a G-submodule of $\mathrm{Hom}_{\mathbb{Z}}(V^m, \mathbb{Q})$ for the ring homomorphism $\mathbb{Z} \to \mathbb{Q}$. Assume $V_{-\lambda} = \mathrm{Hom}(V^m, \mathbb{Z})_{-\lambda}$. Consider a sequence

$$\begin{aligned} \mathrm{Hom}_G(\mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z}), I_{\mathfrak{b},T}^{\mathfrak{g},G}(\mathbb{Q}_\lambda)) &\cong \mathrm{Hom}_{\mathfrak{b},T}(\mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z}), \mathbb{Q}_\lambda) \\ &\supset \mathrm{Hom}_{\mathfrak{b},T}(\mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z}), \mathbb{Z}_\lambda) \\ &\cong \mathrm{Hom}_{\mathfrak{b},T}(\mathbb{Z}_{-\lambda}, V) \\ &\cong V_{-\lambda} \\ &= \mathrm{Hom}(V^m, \mathbb{Z})_{-\lambda}. \end{aligned}$$

Then we obtain a G-homomorphism $\mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z}) \to I_{\mathfrak{b},T}^{\mathfrak{g},G}(\mathbb{Q}_\lambda)$ corresponding to the map

$$v_\lambda^\vee : V^m \to V_\lambda^m \to \mathbb{Z}; v_\lambda \mapsto 1,$$

where the first map is the projection to the weight submodule with weight λ. Its image contains v_λ. Therefore the embedding $V^m \hookrightarrow I_{\mathfrak{b},T}^{\mathfrak{g},G}(\mathbb{Q}_\lambda)$ factors through $\mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$ by definition. Taking its dual, we get

$$V \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(V^m, \mathbb{Z}); v_\lambda^\vee \mapsto v_\lambda^\vee.$$

This shows that $\mathrm{Hom}(V^m, \mathbb{Z})$ exhibits the maximal G-submodule over $\mathbb{Z}$ of the irreducible G-module over $\mathbb{Q}$ with highest weight $-\lambda$ whose highest weight submodule is $\mathbb{Z}v_\lambda^\vee$ (see Corollary 3.1.13). The assertion now follows from Theorem 7.1.1 (4). □

Corollary 7.1.4. *Suppose that k is a PID containing $\mathbb{Z}$, and G be a split reductive group over k. Then for any G-module V which is free of finite rank over k, there exists an embedding $V \to \oplus_{\lambda_i} I_{\bar{\mathfrak{b}},T}^{\mathfrak{g},G}(k_{\lambda_i})$ which extends to an isomorphism*

$$V \otimes \mathrm{Frac}(k) \cong \oplus_{\lambda_i} I_{\bar{\mathfrak{b}},T}^{\mathfrak{g},G}(\mathrm{Frac}(k)_{\lambda_i}).$$

proof. Notice that $H^0(\mathfrak{n}, V)$ is a T-submodule of V. The flat base change theorem says $H^0(\mathfrak{n}, V) \otimes \mathrm{Frac}(k) \cong H^0(\mathfrak{n}, (V \otimes \mathrm{Frac}(k))$. We can find a projection $p : V \otimes \mathrm{Frac}(k) \to (V \otimes \mathrm{Frac}(k))^{\mathfrak{n}}$ which is a $(\bar{\mathfrak{b}}, T)$-homomorphism. In fact, we may assume that $V \otimes \mathrm{Frac}(k)$ is irreducible from the complete reducibility of representations of $G \otimes \mathrm{Frac}(k)$. Then $(V \otimes \mathrm{Frac}(k))^{\mathfrak{n}}$ is the highest weight space of $V \otimes \mathrm{Frac}(k)$ (Theorem 7.1.1). The projection along the weight space decomposition is the desired map. Multiplying an element of k, we may assume $p(V) \subset V^{\mathfrak{n}}$. The map p induces an isomorphism $V \otimes \mathrm{Frac}(k) \cong I_{\bar{\mathfrak{b}},T}^{\mathfrak{g},G}(V^{\mathfrak{n}} \otimes \mathrm{Frac}(k))$.

Choose a basis $\{v_i\}$ of $V^{\mathfrak{n}}$ which consists of weight vectors, and denote the weight of v_i by λ_i. Therefore we have a diagram

$$\begin{array}{ccccc}
V \otimes \mathrm{Frac}(k) & \xrightarrow{\sim} & I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}((V \otimes \mathrm{Frac}(k))^{\mathfrak{n}}) & \xrightarrow{\sim} & \oplus I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathrm{Frac}(k)_{\lambda_i}) \\
\uparrow & & \uparrow & & \uparrow \\
V & \longrightarrow & I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(V^{\mathfrak{n}}) & \xrightarrow{\sim} & \oplus I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(k_{\lambda_i}).
\end{array}$$

This completes the proof. □

In the next section, we give preliminary results to compute the derived functor $\mathbb{R}I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}$.

7.2 pro and Γ

Proposition 7.2.1. *Let $(\mathfrak{q}, M) \subset (\mathfrak{g}, M)$ be an inclusion of pairs over a commutative ring. Suppose that the following conditions are satisfied:*

(i) For $x \in \mathfrak{g}$, we have $[x, x] = 0$.

(ii) $\mathfrak{q}$ and $\mathfrak{g}/\mathfrak{q}$ are free of finite rank.

(iii) M is diagonalizable.

Then the functor $\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{q}}$ is exact.

proof. By definition, $\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{q}}$ is described as

$$\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{q}}(-) \cong \mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), -)_M.$$

The assertion now follows from Corollary 5.1.12 and the PBW theorem ([Bou] Corollary 1.2.7.5). □

Example 7.2.2. In the algebraic Borel-Weil setting as above, $\mathfrak{q} = \bar{\mathfrak{b}}$ and $K = T$ enjoy the conditions of Proposition 7.2.1.

We next give general results on the Zuckerman functor. Let $(\mathfrak{g}, M) \to (\mathfrak{g}, K)$ be a map of pairs over a commutative ring k with $\mathfrak{g} \to \mathfrak{g}$ being the identity. Write $\mathcal{O}(K)$ for the coordinate ring of K.

Proposition 7.2.3. *Let V be a $(\mathfrak{g}, M)$-module.*

(1) Regard $V \otimes \mathcal{O}(K)$ as a $(\mathfrak{k}, M)$-module for the tensor product representation of V and the left regular representation $\mathcal{O}(K)$. Then the invariant part $H^0(\mathfrak{k}, M; V \otimes \mathcal{O}(K))$ is a $(\mathfrak{g}, K)$-module for the right regular action of K and the map

$$\mathfrak{g} \otimes V \otimes \mathcal{O}(K) \to \mathfrak{g} \otimes \mathcal{O}(K) \otimes V \otimes \mathcal{O}(K) \to V \otimes \mathcal{O}(K).$$

The first map is induced from the action of K on $\mathfrak{g}$, and the second arrow is defined by the multiplication of $\mathcal{O}(K)$ and the $\mathfrak{g}$-action on V.

(2) We have a natural isomorphism $I^{\mathfrak{g},K}_{\mathfrak{g},M}(V) \cong H^0(\mathfrak{k}, M; V \otimes \mathcal{O}(K))$.

proof. Let V be a $(\mathfrak{g}, M)$-module. Follow the notations ϕ, ψ, ν, π of pairs and modules in [H1] for a while. Recall that the actions $\tilde{\nu}$ and $\tilde{\pi}$ of the weak $(\mathfrak{g}, K)$-module $\mathrm{Ind}^K_M(V)$ can be described as follows ([Jant] I.3.3): Let R be a flat k-algebra.

- For $g \in K(R)$ and $f \in V \otimes k[K] \otimes R \cong \mathrm{Hom}(K \otimes R, (V \otimes R)_a)$, $(\tilde{\nu}(g)f)(g') = f(g'g)$. In particular, if we embed it to $V \otimes k[K] \otimes R$ to $V \otimes k[K] \otimes R[\epsilon]/(\epsilon^2)$, its differential representation is given by $(d\tilde{\nu}(\xi)f)(g) = f(g\xi)$ for $\xi \in \mathfrak{k}(R)$.
- For $x \in \mathfrak{g} \otimes R$, $(\tilde{\pi}(x)f)(g) = \pi(\phi(g)x)f(g)$.

In view of [H1], it will suffice to show that $d\tilde{\nu}(\xi) = \tilde{\pi}(\xi)$ if and only if for the all f, $\pi(\xi)f(\xi^{-1}-) = f(-)$. In fact, if $d\tilde{\nu}(\xi) = \tilde{\pi}(\xi)$ is satisfied, we have

$$\begin{aligned}\pi(\xi)f(\xi^{-1}g) &= \pi(\xi)f(gg^{-1}\xi^{-1}g)\\ &= \pi(\xi)\pi(g(g^{-1}\xi^{-1}g)g^{-1})f(g)\\ &= f(g).\end{aligned}$$

Conversely, suppose that we have $\pi(\xi)f(\xi^{-1}-) = f(-)$. Then it implies $\pi(\xi)f(-) = f(\xi-)$ (apply $\pi(\xi^{-1})$ and replace ξ by ξ^{-1}). Hence we get

$$f(g\xi) = f(g\xi g^{-1}g) = \pi(g\xi g^{-1})f(g).$$

□

Theorem 7.2.4. *Suppose that k is a PID. Suppose that the following conditions are satisfied:*

(i) For $x \in \mathfrak{g}$, we have $[x, x] = 0$.

(ii) $\psi : \mathfrak{k} \to \mathfrak{g}$ is injective.

(iii) $\mathfrak{k}$ and $\mathfrak{g}/\mathfrak{k}$ are free of finite rank as k-modules.

Then we have an equivalence on $D^+(\mathfrak{g}, M)$

$$\mathbb{R}\Gamma \simeq \mathbb{R}\,\mathrm{Hom}_{\mathfrak{k},M}(k, - \otimes \mathcal{O}(K)).$$

proof. It will suffice to show that if I is an injective $(\mathfrak{g}, M)$-module, $I \otimes \mathcal{O}(K)$ is acyclic for the functor $\mathrm{Hom}_{\mathfrak{k},M}(k, -)$ from the category of $(\mathfrak{k}, M)$-modules to that of k-modules. Recall that we have an isomorphism $\mathcal{O}(K) \cong \varinjlim_{Q \subset \mathcal{O}(K)} Q$, where Q runs through K-submodules which are free of finite rank as a k-module. According to Proposition 3.2.4, we have an isomorphism

$$H^i(\mathfrak{k}, M, I \otimes \mathcal{O}(K)) \cong \varinjlim H^i(\mathfrak{k}, M, I \otimes Q).$$

The assertion is reduced to seeing that this cohomology vanishes if $i > 0$. According to the PBW theorem, $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{k}}$ is exact. In particular, I is injective as a $(\mathfrak{k}, M)$-module. Lemma 3.2.15 implies that $I \otimes Q$ is also injective as a $(\mathfrak{k}, M)$-module. The vanishing now follows. □

Example 7.2.5 ([KV])**.** Let $(\mathfrak{g}, K)$ be a pair over a commutative ring k. Suppose that the following conditions are satisfied:

(i) For any $x \in \mathfrak{g}$, $[x, x] = 0$.

(ii) The map $\psi : \mathfrak{k} \to \mathfrak{g}$ is injective.

(iii) $\mathfrak{k}$ and $\mathfrak{g}/\mathfrak{k}$ are free of finite rank.

(iv) K is diagonalizable.

(v) The K-type $(\mathfrak{g}/\mathfrak{k})_\lambda$ is free of finite rank for any character λ of K.

Then the Koszul complex $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \wedge^\bullet(\mathfrak{g}/\mathfrak{k})$ is a projective resolution of the trivial $(\mathfrak{g}, K)$-module k. Recall that the differential $\partial_n : U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \wedge^{n+1}(\mathfrak{g}/\mathfrak{k}) \to U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \wedge^n(\mathfrak{g}/\mathfrak{k})$ is given by

$$\partial_n(u \otimes x_1 \wedge \cdots \wedge x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} u x_i \otimes x_1 \wedge \overset{\overset{i}{\vee}}{\cdots} \wedge x_{n+1} + \sum_{r<s} (-1)^{r+s} u \otimes \pi([x_r, x_s]) \wedge x_1 \overset{\overset{r,s}{\vee}}{\cdots} \wedge x_{n+1},$$

where $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{k}$ is the projection. Here the exterior algebra $\wedge^\bullet(\mathfrak{g}/\mathfrak{k})$ is defined as the algebra with the relations $x \cdot x = 0$ for all $x \in \mathfrak{g}/\mathfrak{k}$. According to (v), $\wedge^\bullet(\mathfrak{g}/\mathfrak{k})$ is free as a k-module, and admits a natural K-action.

Example 7.2.6. Let $p \leq q$ be nonnegative integers. Then the diagonal embedding $\mathrm{GL}_p \times \mathrm{GL}_q \to \mathrm{GL}_{p+q}$ gives rise to a pair $(\mathfrak{gl}_{p+q}, \mathrm{GL}_p \times \mathrm{GL}_q)$ over $\mathbb{Z}$. Choose the subgroup T of diagonal matrices as a split maximal torus of GL_{p+q}. Then $(\mathfrak{gl}_{p+q}, \mathrm{GL}_p \times \mathrm{GL}_q)$ (resp. $(\mathfrak{gl}_p \oplus \mathfrak{gl}_q, T)$) satisfies the conditions of Theorem 7.2.4 (resp. Example 7.2.5). Let L be the subgroup of matrices whose (i, j)-entries are zero unless

$$(i, j) \in \{(i, i) : 1 \leq i \leq p+q\} \cup \{(i, p+i) : 1 \leq i \leq p\} \cup \{(i, i-p) : p+1 \leq i \leq 2p\},$$

and Q be a parabolic subgroup whose Levi part is L. Then $(\mathfrak{q}, T) \subset (\mathfrak{gl}_{p+q}, \mathrm{GL}_p \times \mathrm{GL}_q)$ is an integral model of the pair associated to $\mathrm{U}(p, q)$ and its θ-stable parabolic pair whose group part is a torus.

We also need additional homological lemmas to see vanishing properties of the Borel-Weil-Bott induction.

Lemma 7.2.7. *Let K be a flat affine group scheme over a PID k, and I be an injective K-module. Then the internal Hom $F(-, I)$ of the symmetric monoidal category of K-modules is exact.*

proof. The left exactness is obvious since the tensor product respects (finite) colimits. The right exactness is reduced to showing that it sends injective maps to surjective maps. Let $A \to B$ be an injective K-homomorphism, and Q be a flat K-module. Then we have

$$\begin{aligned} \mathrm{Hom}_K(Q, F(B, I)) &\cong \mathrm{Hom}_K(Q \otimes B, I) \\ &\twoheadrightarrow \mathrm{Hom}_K(Q \otimes A, I) \\ &\cong \mathrm{Hom}_K(Q, F(A, I)). \end{aligned}$$

The second map is surjective since $Q \otimes A \to Q \otimes B$ is injective. Since flat K-modules generate the category of K-modules (Corollary 2.1.15), $F(B, I) \to F(A, I)$ is surjective. This completes the proof. □

Lemma 7.2.8. *Let $(\mathfrak{g}, K)$ be a pair over a PID k, and I be a $(\mathfrak{g}, K)$-module which is injective as a K-module. Then $F(-, I)$ sends projective $(\mathfrak{g}, K)$-modules to injective $(\mathfrak{g}, K)$-modules.*

proof. Let P be a projective $(\mathfrak{g}, K)$-module. Suppose that we are given $(\mathfrak{g}, K)$-homomorphisms $i : A \to B$ and $f : A \to F(P, I)$ with i injective. Pass to the adjunctions of Proposition 2.2.3 so that it is equivalent to the diagram

$$\begin{array}{ccc} P & \longrightarrow & F(A, I) \\ & \searrow & \uparrow \\ & & F(B, I). \end{array}$$

Since the induced map $F(B, I) \to F(A, I)$ is surjective from Lemma 7.2.7, the dotted arrow exists. Passing to the adjunction again, we conclude that f factors through B. The assertion now follows. □

7.3 The Borel-Weil-Bott theorem over $\mathbb{Z}$

Let us go back to the previous setting. Let λ be a character of T. According to Theorem 3.1.10, we have a canonical isomorphism $R^i I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda) \otimes \mathbb{Q} \cong R^i I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Q}_\lambda)$. We also have a long exact sequence

$$0 \to I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda) \to I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Q}_\lambda) \to I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}((\mathbb{Q}/\mathbb{Z})_\lambda) \to R^1 I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda) \to R^1 I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Q}_\lambda) \to \cdots .$$

This implies that if $R^i I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Q}_\lambda)$ is nonzero then $R^{i-1} I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}((\mathbb{Q}/\mathbb{Z})_\lambda)$ is the torsion part of $R^i I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda)$; Otherwise $R^i I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda)$ is a torsion G-module, and it is the cokernel of $R^{i-1} I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Q}_\lambda) \to R^{i-1} I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}((\mathbb{Q}/\mathbb{Z})_\lambda)$.

In view of Example 7.2.5, Lemma 7.2.8, and Corollary 5.1.11, we obtain the standard injective resolution of $(\mathbb{Q}/\mathbb{Z})_\lambda$ as a $(\bar{\mathfrak{b}}, T)$-module. In particular, the cohomology $R^i I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}((\mathbb{Q}/\mathbb{Z})_\lambda)$ vanishes if $i > |\Delta^+|$. The long exact sequence now tells us the following result:

Theorem 7.3.1. *The cohomology $R^i I^{\mathfrak{g},G}_{\bar{\mathfrak{b}},T}(\mathbb{Z}_\lambda)$ vanishes if $i > |\Delta^+| + 1$.*

Finally, we restrict ourselves to $G = \mathrm{SL}_2$. Then we can choose the subgroup T^1 of diagonal matrices as a maximal split torus. Set

$$\bar{\mathfrak{b}} = \left\{ \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix} : a, b \in \mathbb{Z} \right\}.$$

Theorem 7.3.2. *For any integer λ, the counit $\epsilon_\lambda : I^{\mathfrak{sl}_2, \mathrm{SL}_2}_{\bar{\mathfrak{b}}, T^1}((\mathbb{Q}/\mathbb{Z})_\lambda) \to (\mathbb{Q}/\mathbb{Z})_\lambda$ is surjective.*

proof. Let n be a positive integer such that $\lambda + 2n \geq 0$. Consider the SL_2-module V^m described as

$$V^m = \oplus_{i=0}^{\lambda+2n} \mathbb{Z} v_{\lambda+2n-2i}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v_{\lambda+2n-2i} = (\lambda + 2n - i + 1) v_{\lambda+2n-2i+2}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} v_{\lambda+2n-2i} = (i+1) v_{\lambda+2n-2i-2}.$$

(see Example 2.1.7). Then the map

$$V^m \to (\mathbb{Q}/\mathbb{Z})_\lambda$$

$$v_{\lambda+2n-2i} \mapsto \begin{cases} \frac{1}{n} & (i = n) \\ 0 & (i \neq n) \end{cases}$$

is a $(\bar{\mathfrak{b}}, T^1)$-homomorphism. Passing to all n, we conclude that the image of ϵ_λ runs through all fractions. □

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