Combinatorics of Constructible Complexes

（和訳　構成可能複体の組合せ論）

八　森　正　泰

Combinatorics of Constructible Complexes

Masahiro Hachimori

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Masahiro Hachimori

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Abstract

This thesis treats combinatorial decomposition properties of simplicial complexes. Among them, famous properties are the following:

$$\text{vertex decomposable} \Rightarrow \text{shellable} \Rightarrow \text{constructible} \Rightarrow \text{Cohen-Macaulay}$$
$$\text{extendablly shellable} \Rightarrow \text{shellable} \Rightarrow \text{partitionable}$$

Among them, shellability as well as Cohen-Macaulayness is the most famous and important property with many applications in combinatorics, such as the Upper Bound Conjecture of polytopes and spheres. The main object we discuss in this thesis is its relaxation, constructibility. Though constructibility appeared in 1972 paper of Hochster, its serious study has not been done until this thesis. One of the aims of this thesis is to reveal the properties of constructibility. One of the results is to show the existence of non-constructible PL-balls and PL-spheres in every dimension at least 3, which solves an open problem of 1978 paper of Danaraj and Klee.

The combinatorial decomposition properties discussed in this thesis is not only constructibility but also shellability and vertex decomposability. In Chapter 3, our results on non-constructible balls and spheres provide strengthenings of the results on shellability, Furch's non-shellable ball construction and Lickorish's condition of non-shellability of spheres. Moreover, we give the solution of the conjecture of Hetyei on shellability of certain cubical decompositions of spheres. Analogously to the discussion of non-constructibility, new conditions of balls and spheres which imply non-vertex decomposability are given.

We also treat the decision problem of constructibility. Decision problems of combinatorial decomposition properties are important problem for computational aims as well as for understanding the properties of combinatorial decompositions. The decision problem of shellability is studied hard by researchers but currently known result is only the case of 2-dimensional pseudomanifolds that shellability of 2-pseudomanifolds can be determined in $O(\#\text{facets})$ time, given by Danaraj and Klee in 1978. In Chapter 4 we try the decision problem of constructibility, and our result is that constructibility of triantulations of 3-dimensional balls with at most two interior vertices can be determined in $O(\#\text{facets})$ time. This is the first nontrivial result on decision problems of combinatorial decomposition properties in 3-dimensional case. This result has applications to determine constructibilities of some triangulations of balls known as non-shellable 3-balls: Bing's ball is determined to be non-constructible and Rudin's, Grünbaum's and Ziegler's balls are shown to be constructible.

The last topic we treat in this thesis is the case of 2-dimensional simplicial complexes. In the case of 2-dimensional pseudomanifolds, it is known that all the properties mentioned in the beginning are equivalent. Especially they are topological properties because Cohen-Macaulayness is. On the other hand, all the properties are known to be different in 3- and higher dimensional cases as discussed in the former chapters, and the properties except for Cohen-Macaulayness is known to be non-topological. The case of general 2-dimensional simplicial complexes can be seen to be lying between them. In Chapter 5 we give several examples which shows the difference of all combinatorial decomposition properties already exists in 2-dimensional simplicial complexes. Especially the result contains a strengthening of formerly known result: every triangulation of the dunce hat was known to be non-shellable, but we show that it is non-constructible. Additionally we show that shellability is not topological for 2-dimensional simplicial complexes.

Preface

Everything started from one book. I happened to buy the textbook "Lectures on Polytopes" [98] written by Prof. Günter M. Ziegler, at the university bookstore about five years ago. I bought it only because the figures (especially of permutahedra and of zonotopal tilings) interested me, but the book turned out to be a very good introduction to the world of polytopes, starting from fundamentals and containing many recent results. Among the many topics, especially Lecture 8 on shellability attracted my interest. Because the shellability is a concept which formalizes a very natural construction of objects by adding cells (= facets) one by one, it is conceivable that triangulations and polytopal decompositions of balls and spheres are shellable. But surprisingly, many counterexamples, that is, non-shellable decompositions of balls and spheres are known. One of such examples, Danzer's cube, is described in the book. I read that part repeatedly and spent much time imaging what is happening on the ball, and then proceeded to other non-shellable balls according to the references in the book. Still the difference between shellable decompositions and non-shellable decompositions was a big mystery to me, and I have been thinking of this for years.

Soon I decided shellability should be the theme of my doctoral study. What I had in my mind at that time was to give some characterizations of non-shellability, though this aim has not been achieved yet. During the study, I fell to thinking that why shellings can add only one facet in one step: what will happen if we allow a lump of facets to be added at each step? After formulating this "generalized" definition of shelling, I thought that I had seen the same formulation somewhere before. I was right. It is given in the book "Cohen-Macaulay Rings" by Profs. Bruns and Herzog [26], named "constructibility." (This concept turned out to go back to a 1972 paper of Hochster [49].) From this point my main interest shifted to constructibility.

Very few works, however, have been done about constructibility. All I could find were a few papers, each of which made a few statements about constructibility, so I decided to study constructibility myself. I started with the problem whether or not there are non-constructible triangulations of 3-balls or spheres analogous to the case of shellability.

My first attempt was to show that every triangulated 3-ball is constructible, which failed as is observed in this thesis. At the same time, I also tried to check whether currently known non-shellable 3-balls are constructible or not. First, I made a paper model of Ziegler's 3-ball (with 10 vertices and 21 facets) and observed that it is constructible. The next targets were Grünbaum's 3-ball (with 14 vertices and 29 facets) and Rudin's 3-ball (with 14 vertices and 41 facets). But there were some problems: for Grünbaum's ball, the coordinates of vertices were not known at that time, and for Rudin's, the number of facets was too large to make a paper model. So instead of paper models, I attempted computer calculations to determine their constructibility by checking all the possible divisions of these two balls. It was found that Grünbaum's ball was constructible, but the constructibility of Rudin's ball remained undetermined: the amount of the possible divisions is too large even for a computer calculation. This led me to consider the algorithmic aspect of combinatorial decompositions, that is, decision problems. After a while, I succeeded in finding an efficient algorithm under a condition that was restricted but valid for Rudin's ball. This result finally showed that Rudin's ball is also constructible. Later, I found a description in Provan and Billera [74] that both of the balls are constructible. If I had known this at the beginning, Chapter 4 of this thesis would not exist now because my original motivation of the work was to know the constructibility of Rudin's ball, though the results contain more than that.

While dealing with the balls on computer, I happened to find a typographical error in the facet list of Grünbaum's ball in the paper of Danaraj and Klee [32]. I sent an e-mail to Prof. Victor Klee, and then Prof. Branco Grünbaum, who heard from Prof. Klee, kindly sent me the correct list, and added a possible set of coordinates for the vertices to realize the ball in E^3. Prof. Ziegler also checked this typographical error for me at almost the same time. I thank them very much, especially Prof. Grünbaum for his efforts to reconstruct the list of facets from his hand-made model more than twenty years after the birth of the ball.

On the other hand, I also managed to show Furch's knotted hole ball is not constructible. Thus my first problem was solved. By that time, Prof. Ziegler had given me many useful suggestions since I sent him an e-mail for the first time. I asked for his advice because he was the author of my favorite textbook. Although I was a complete stranger to him, he kindly sent me informative replies from time to time, which encouraged me very much.

The existence of non-constructible spheres remained to be a question even after I proved the existence of non-constructible balls. The answer suddenly came to me in the autumn of 1998, and I wrote to Prof. Ziegler my proof of the existence of non-constructible 3-spheres. He had a lecture in a fall school on topological combinatorics and introduced the proof there, and brought me a more elegant way to show the statement which was suggested by Prof. Robin Forman in discussions there. I was very impressed by the simplicity of the improved proof. When several new ideas were added by further discussions with Prof. Ziegler, such as improving proofs, extending the arguments to vertex decomposability and giving several examples to show the bounds of the statements, he and I decided to write a paper [45] together. This is my first joint paper, and the results are included in Chapter 3 of this thesis.

In April 1999, after Prof. Ziegler and I finished writing our joint paper, there was a meeting, "Geometric and Topological Combinatorics", in Oberwolfach, Germany. Prof. Ziegler was one of the organizers, and it was very kind of him to include my name in the invitation list and gave me a great opportunity to spend a whole week at Mathematishes Forschungsinstitut Oberwolfach. The very comfortable stay (except for the terrible thunder storm) at the institute and the stimulating leading-edge talks of mathematics gave me an idea to extend the results of non-constructible spheres to solve a conjecture mentioned by Prof. Ziegler while we were working on the joint paper. However, unfortunately (or fortunately) Prof. Ehrenborg pointed out a crucial error in my argument. We spent a whole day to overcome the problem, and finally reached a new idea of a seemingly right definition of the bridge index for tangles which strongly relates to the constructibility of spheres. This result did not fully solve the original conjecture (though later we get very close to the conjecture), but it did solve Prof. Gábor Hetyei's conjecture on shellability, which was in Prof. Ehrenborg's mind throughout our discussion. This idea later developed into a joint paper [36] and the results are also included in Chapter 3 of this thesis.

Chapter 5 is a very recent work, inspired by Prof. Michelle Wachs's talk in the problem session in Oberwolfach. (This work of hers can be found in [92].) Until then, I considered the problems only in the world of pseudomanifolds and I was convinced that the case of two-dimensional pseudomanifolds is too simple. I never thought that there were still questions to be answered for two-dimensional simplicial complexes. In her study of "obstructions to shellability", however, even the case of two-dimensional simplicial complexes needs very complicated arguments, and I learned that, apart from the restricted case of pseudomanifolds two-dimensional world is far from being simple when it comes to the general case. This made me realize that there are much to think about two-dimensional simplicial complexes, and noticed that I did not know whether there are two-dimensional complexes which are, for example, Cohen-Macaulay but not constructible, constructible but not shellable, and shellable but not vertex decomposable. I started to prove that every shellable 2-dimensional simplicial complex is extendably shellable. About six months later, I came across a counterexample to this problem, and counterexamples to other problems were time constructed as its variants at the same time. Although, as Prof. Ziegler pointed out later, Anders Björner studied the same things years ago and achieved many examples (written in [14], [16] and [82]) some of which are similar to mine, the chapter still contains newly derived results. In this study the discussions with Fumihiko Takeuchi helped me study the problems and the seminar at Science University of Tokyo held by Prof. Ryuichi Hirabayashi and Prof. Yoshiko Ikebe were also very helpful for me to get the results.

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I would like to thank to Prof. Anders Björner, Prof. Gil Kalai and Prof. Ziegler, for organizing the Oberwolfach meeting, and Mathematisches Forschungsinstitut Oberwolfach for its hospitality. I am also grateful to the members of the meeting who gave me valuable suggestions and useful information as well as encouragement, although I cannot write all the names here.

There are many other people who helped my work by giving suggestions and information directly to me or indirectly via Prof. Ziegler or via Prof. Ehrenborg. Especially, I wish to give thanks to Prof. Robin Forman, Prof. Gábor Hetyei, Prof. Margaret Readdy, Prof. Anders Björner, Prof. Branco Grünbaum and Prof. Victor Klee.

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I am indebted to the anonymous referees of my formerly published paper [43], who suggested me to simplify the arguments by the use of the facts known in knot theory and topology. Their comments not only improved the paper but also reminded me of the importance of simplified arguments, which made my work progress further.

I benefitted from attending the seminar chaired by Prof. Ryuichi Hirabayashi and Prof. Yoshiko Ikebe at the Science University of Tokyo since last spring and from the discussions about my work, especially the materials in Chapter 5. Discussions at another seminar with Prof. Kenji Kashiwabara, Prof. Tetsuya Abe, Fumihiko Takeuchi, Takayuki Ishizeki have also given rise to several improvements. My special thanks are due to Fumihiko Takeuchi, who read the manuscript of this thesis very carefully and made great efforts to improve my writing, as well as giving me many suggestions and information.

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Masahiro Hachimori

Contents

Chapter 1

Introduction

1.1 Introduction

This thesis studies combinatorics of cell complexes, especially simplicial complexes. Simplicial (or cell) complexes appear everywhere in combinatorics or topology: the boundaries of (simplicial) polytopes, triangulations of manifolds, or even in subtle ways such as the set of chains in a partially ordered set, monotone properties of graphs. Simplicial complexes and cell complexes play a fundamental role in topology, so it is natural that the combinatorial objects with related cell complexes can be treated from a topological viewpoint. Such studies are sometimes mentioned as *topological combinatorics* or *topological methods in combinatorics*. A review of this field can be found in Björner [15].

Though the properties of simplicial complexes are sometimes treated topologically, there are many properties which are not topological. Here "topological" means that the properties are determined by its topological property. Combinatorial decomposition properties, the main subject of this thesis, are such non-topological properties. For example, shellability, the most famous combinatorial decomposition property, is not invariant if we triangulate a 3-dimensional ball in a different way. Such properties can not be discussed only from a topological view, but need some combinatorial arguments. For example, Björner [12], Björner and Wachs [20, 21] supplied such combinatorial methods for shellability, such as the lexicographic labeling on the face poset of cell complexes.

In spite of this strong combinatorial flavor, still topological properties affect on combinatorial decompositions. The firstly found non-shellable triangulations of spheres were of non-PL spheres (in dimensions $d \geq 5$), the non-shellability followed from the fact that shellable triangulations must be PL. And after a long search of non-shellable triangulations of 3-spheres, the answer was given from combinatorial topology, Lickorish's construction using an embedded knot [57]. The relation between knots and shellability was also discussed by Armentrout in his paper [3] who showed a relation between shellability and "link property" in other paper [2].

Many interesting and important fact can be derived from the nice "harmony" of combinatorics and topology. Our standpoint of this thesis lies here. We try in this thesis to use results of combinatorial topology for combinatorics of decomposition properties. (So our method is "combinatorial topological combinatorics.") Especially our main interest is in *constructibility* which is a generalized concept of shellability. Though both shellability and constructibility are defined purely in combinatorial way, it seems that constructibility has more topological flavor. In some sense, constructibility can be seen as a topological relaxation of shellability, and this fits well to our method.

Already combinatorial methods made a good progress in topology, so it is natural that topological methods in combinatorics also work well.

Though the enormous number of studies have been done for shellability and some for vertex decomposability, constructibility seemed not have been treated seriously enough. It only appeared in Stanley [83], Hochster [49], Björner [15], and was mentioned shortly in Danaraj and Klee [32], Provan and Billera [74], Björner [12] to the authors knowledge. (Zeeman [96, Ch. 3] has the same construction restricted to manifolds, and B-constructibility and S-constructibility of Mandel [62], also mentioned in Bachem and Kern [4], is its generalization for cellularly decomposed manifolds.) So many fundamental questions have been left open around constructibility, for example, the existence of non-constructible triangulations of balls or spheres were not discussed in anywhere. This thesis is the first serious study of constructibility, which is a compilation of the papers Hachimori [43], Hachimori and Ziegler [45], Ehrenborg and Hachimori [36] and Hachimori [44] together with some new materials that have not been published yet.

After this chapter of introduction, this thesis starts from Chapter 2 which is a review of some preliminaries of simplicial complexes, combinatorial decomposition properties, and some combinatorial topology. Some fundamental facts of combinatorial decompositions will also be reviewed. Those who know well about the terminologies used in this thesis can skip this chapter and return to recall the precise definitions, terminologies and fundamental propositions when needed in the following chapters.

Chapter 3 treats the relation between combinatorial decompositions of balls and spheres and certain knots embedded in them. In the case of 2-dimensional pseudomanifolds, constructibility is equivalent to the property to be homeomorphic to a ball or sphere. (This will be discussed in Chapter 2.) But we show in this chapter that non-constructible balls and spheres exist in three and higher dimensional cases, different from the case of 2-dimensional pseudomanifolds. Especially, the existence of non-constructible 3-sphere shown in this chapter solves an open problem suggested in Danaraj and Klee [32]. The main result of this chapter is the following which implies the existence of nonconstructible triangulations of 3-balls and spheres.

Main Theorem of Chapter 3.

- *A triangulated 3-ball with a knotted spanning arc consisting of*

$$\begin{cases} \text{at most } 2 \textit{ edges} & \textit{is not constructible,} \\ 3 \textit{ edges} & \textit{can be shellable, but not vertex decomposable,} \\ 4 \textit{ edges} & \textit{can be vertex decomposable.} \end{cases}$$

- *A triangulated 3-sphere or 3-ball with a knot consisting of*

$$\begin{cases} 3 \textit{ edges} & \textit{is not constructible,} \\ 4 \text{ or } 5 \textit{ edges} & \textit{can be shellable, but not vertex decomposable,} \\ 6 \textit{ edges} & \textit{can be vertex decomposable.} \end{cases}$$

- *A triangulated 3-sphere or 3-ball with a knot K consisting of*

$$\begin{cases} \text{at most } \quad b(K) - 1 \textit{ edges} & \textit{is not constructible,} \\ \text{at most } 2 \cdot b(K) - 1 \textit{ edges} & \textit{is not shellable,} \\ \text{at most } 3 \cdot b(K) - 1 \textit{ edges} & \textit{is not vertex decomposable,} \end{cases}$$

  where $b(K)$ is the bridge index of the knot K.

This provides generalizations of the previously known results: the construction of non-shellable triangulations of 3-balls of Furch [38] or Bing [10], Lickorish's construction of non-shellable spheres [57], Armentrout's result on cell partitionings of 3-spheres. It also gives a solution of Hetyei's conjecture about shellability of certain cubical decompositions of spheres. This chapter contains a joint work with Günter M. Ziegler and that with Richard Ehrenborg.

Chapter 4 has a more combinatorial flavor in the setting of the problem: decision problems. This problem asks whether there are efficient algorithms to decide if a given simplicial complex has some property or not. The decision problems of combinatorial decomposition properties are challenging problems in which almost no result is known currently. The only result the one given by Danaraj and Klee [33] that shellability of 2-dimensional pseudomanifolds can be decided in linear order time complexity. This chapter treats this decision problem for constructibility. Our setting is under the condition that the simplicial complex to be calculated is a triangulation of a 3-ball and it has very few vertices in its interior. The main result in this chapter is the following.

Main Theorem of Chapter 4.
If a triangulated 3-ball has at most two interior vertices, then its constructibility can be decided in $O(\#facets)$ time.

The topological properties we use in this chapter are very primitive: that the simplicial complexes that appears in the decompositions are always homeomorphic to 3-dimensional balls, and some properties of triangulations of a 2-ball. But we can see there how these primitive topological observations become powerful tools in the study.

In the last Chapter 5, we study the case of 2-dimensional simplicial complexes. For pseudomanifolds in dimension 2, combinatorial decomposition properties — vertex decomposability, extendable shellability, shellability, constructibility, and Cohen-Macaulayness — are all equivalent. This also implies that these decomposition properties are topological for 2-dimensional pseudomanifolds. But all or them are different in three and higher dimensions and they are not topological (except for Cohen-Macaulayness). Thus 2-dimensional pseudomanifolds have extremely nice properties which are never true in higher dimensional cases. The problem arising here is how the situation changes if we move to the general cases: general 2-dimensional simplicial complexes. Formerly, examples which are Cohen-Macaulay but not shellable (Stanley [87]), and shellable but not extendably shellable (Björner [14]) have been found. What we show in this chapter is the following.

Main Theorem of Chapter 5. *There are 2-dimensional simplicial complexes which are*

- *Cohen-Macaulay but not constructible,*
- *constructible but not shellable,*
- *shellable but not vertex decomposable,*
- *shellable but not extendably shellable.*

Each statement is given by presenting examples of the property. This shows that the gaps between each combinatorial decomposition property exists even in 2-dimensional simplicial complexes. Moreover, we show an example of 2-dimensional simplicial complex which is not shellable but it has a shellable subdivision, showing that shellability is not topological for general 2-dimensional simplicial complexes, contrary to the case of 2-dimensional pseudomanifolds.

1.2 History and story of combinatorial decompositions

Among combinatorial decomposition properties, *shellability* is the most popular one and it has a very long history. According to Ziegler [98], the root of shellability is in 1852, in the work of Schläfli [79] calculating the Euler-Poincaré formula for d-dimensional polytopes. But in his work, shellability of the boundary of a polytope is assumed without proof, which turned out to be non-trivial at all. In 1924, Furch showed in his paper [38] a construction of non-shellable triangulations of 3-balls using knots, and after that many constructions of non-shellable triangulations were discovered by Newman [71], Rudin [78], Bing [10], Grünbaum (unpublished, see [32] or [43]), and so on. These are reviewed in Ziegler's paper [99] where his minimum non-shellable triangulation of a 3-ball using only 10 vertices and 21 facets is presented.

These many studies on shellability of triangulations of 3-balls are related to the famous Poincaré Conjecture stating that every simply connected compact 3-manifolds (without boundary) are 3-spheres. One way of attacking this conjecture is to show that every "fake cube", a manifold with boundary derived from a simply connected 3-manifold by removing a 3-ball, is a "real cube". Shellability is one property which assures the "fake cube" to be a "real cube," because it is known that triangulations of manifolds can be shellable only if the manifolds are PL homeomorphic to balls (if with non-empty boundary) or spheres (if without boundary). This line of study goes to find properties similar but weaker than shellability with the same property, such as collapsibility or sequential unicoherency. These attempts to characterize the 3-sphere from cell partitioning is described in Bing [9, 10, 11], and also seen in Vince [89].

After many discovery of non-shellable triangulations of 3-balls, non-shellable triangulations of 3-spheres were also constructed by Lickorish [57]. As for non-simplicial cases, Vince [90] constructed a non-shellable pseudosimplicial decomposition of 3-spheres, and Armentrout [2, 3] a non-shellable cell partitioning of 3-spheres.

In spite of these negative results, Brugesser and Mani [25] finally gave a proof to the fact that the boundary of a polytope is always shellable, after 120 years from Schläfli's work. This work not only supplied a simple combinatorial proof of Euler-Poincaré formula for high-dimensional polytopes, but it also had a striking application on polytope theory: the solution of Upper Bound Conjecture of polytopes. This conjecture by Mozkin [69] claims that a d-dimensional polytope with n vertices has the maximum number of faces when it is a cyclic polytope. After many attempts of solving this conjecture, the final answer was derived by McMullen [65] which uses induction argument along shellings of the boundaries of polytopes.

After this, the shellability has become a fundamental tool for the study of polytopes with many applications. For example, Stanley [86] showed the nonnegativity of cd-index of polytopes by using S-shellability, a modified version of shellability, and also Billera and Ehrenborg [7] uses shellability of polytopes to calculate cd-index of Eulerian posets. Moreover, applications to computational geometry is becoming popular, for example Seidel [81] uses line-shelling for the construction of convex hulls.

Though the Upper Bound Conjecture (now is a theorem) for polytopes was solved, a generalized conjecture, Upper Bound Conjecture for triangulations of spheres remained open because of the possible existence of non-shellable triangulations of spheres. For this, Stanley introduced the concept of the face ring, or the Stanley-Reisner ring, on simplicial complexes and showed that the Upper Bound Conjecture is true if the face ring of triangulations of spheres are *Cohen-Macaulay* [83]. One property which assures Cohen-Macaulayness was *constructibility*, a combinatorial decomposition property generalized from shellability, introduced by Hochster [49]. At that time it was not known whether or not there are non-constructible triangulations of spheres, but later Edwards [35] showed the Double-Suspension Theorem, the double-suspension of certain homology 3-sphere is homeomorphic to the 5-dimensional sphere, which leads to the existence of non-PL spheres in dimensions $d \geq 5$, (later this double suspension theorem is generalized to any homology 3-spheres by Cannon [27]) which assures the existence of non-constructible (thus non-shellable) triangulations of spheres. But independently from this pessimistic event, Reisner [75] showed a characterization of Cohen-Macaulayness which implies that all triangulations of spheres are Cohen-Macaulay, Stanley's method for Upper Bound Conjecture for spheres completed affirmatively [84].

Extendable shellability, introduced by Danaraj and Klee [32], is related to the decision problem of shellability. Extendably shellable means that every partial shelling can be extended to a complete shelling, thus a shelling of an extendably shellable complex can be constructed easily. Thus if one use the shelling property in a design of an algorithm for some computation, it is desired to be not only shellable but extendably shellable. In spite of this need of extendability, very few is known about extendable shellability. Even it is not known whether all skeletons of a simplex are extendably shellable or not. What is currently known is that every triangulation of a 2-ball or a 2-sphere is extendably shellable as shown in Danaraj and Klee [32] (thus the boundary of a 3-polytope is always extendably shellable), but the boundaries of "almost all" 4-polytopes are not extendably shellable as shown by Ziegler [99].

Vertex decomposability, a stronger concept than shellability, was introduced by Provan and Billera [74] (also in Billera and Provan [8]) in relation with the Hirsch Conjecture. The Hirsch Conjecture claims that the diameter of the graph of a d-polytope with n facets is at most $n - d$. The property of vertex decomposability is that if the dual simplicial complex of the boundary of a simple d-polytope is vertex decomposable, then the polytope satisfies the Hirsch Conjecture. But it turned out that not every polytopes has a vertex decomposable boundary, see Klee and Kleinschmidt [53].

In closing this introduction, we show the conceivably most oldest example of non-shellable cell decomposition of a 3-ball. (It is very unfortunate that it is not a polytopal decomposition.)

This is a puzzle called "Burr puzzle" described in Martin Gardner's Scientific American column, "Mathematical Games" in Jan. 1978, which can be found in a book "Penrose Tiles to Trapdoor Ciphers" [39]. This puzzle is made of six pieces of the right figure which are assembled into the left figure. This is an old kind of puzzle which challenges people to disassemble to pieces or to assemble into the original shape. This object has an extremely interesting property: the whole is homeomorphic to a 3-ball, but the removal of every one piece produces an object which is not homeomorphic to a 3-ball. As is seen later, shellable cell partitionings of a manifold with a non-empty boundary should be homeomorphic to a ball in every step, which implies that this cell partitioning is not shellable. Though this example is not belonging to the class of cell complexes we treat in this thesis (i.e., polytopal complexes), but at least it gives us an insight how non-shellable cell decompositions are possible. For example, the reasoning of non-shellability of Danzer's cube described in Ziegler's textbook [98, p.238] (or in Ziegler [99] with a more beautiful picture) is almost the same in the last step.

According to Gardner, this puzzle is published at least in 1857 in a puzzle book "The Magician's Own Book" written anonymously, and its origin is much older. Really many people have played with this kind of puzzles without knowing they are examples of non-shellable balls...

Chapter 2

Preliminaries

2.1 Simplicial, polytopal and cell complexes

Basic definitions

The main objects treated in this thesis are simplicial complexes.

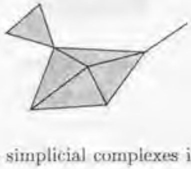
Definition 2.1. A *simplicial complex* C is a set of simplices in some Euclidean space such that

(i) if $\sigma \in C$ and τ is a face of σ, then $\tau \in C$, and

(ii) if $\sigma, \tau \in C$, then $\sigma \cap \tau$ is a face of both σ and τ.

Especially, the empty set $\emptyset$ is always contained in a simplicial complex if the simplicial complex is not empty.

The members of a simplicial complex is called *faces*, or *k-faces* if the dimension is k. 0-faces are *vertices*, 1-faces are *edges*, and the maximal faces in inclusion relation are *facets*. A *k-skeleton* of a simplicial complex is a subcomplex made of all the faces of the complex whose dimension is at most k. The *dimension* of a simplicial complex is the maximum dimension of its facets. The following is an example of a simplicial complex of dimension 2 embedded in E^2.

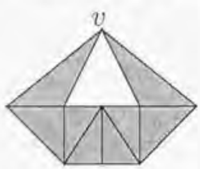


There is another way to define simplicial complexes in a viewpoint of set families: a set family is a simplicial complex if it is closed under taking subsets. Here a set of vertices in one face (in Definition 2.1) corresponds to a set in a family. Simplicial complexes defined in this way are especially called *abstract simplicial complexes*, but both definitions are in fact equivalent, because it is known that every abstract simplicial complex of dimension d (i.e., the size of a set in the family is at most $d+1$) can be realized in the sense of Definition 2.1 in $(2d+1)$-dimensional Euclidean space. This equivalence allows us to use non-straight simplices as in the following figure instead of real simplices in Definition 2.1.

Later in Chapter 3, some examples of 3-dimensional simplicial complexes (triangulations of 3-spheres) are not embedded in E^3, but we need not worry about it because those examples are surely embeddable in E^7.

A simplicial complex is *pure* if all its facets have the same dimension d. A pure simplicial complex is *strongly connected* if any two facets F and G have a sequence $F = F_1, F_2, \ldots, F_k = G$ of facets such that F_i and F_{i+1} has a common $(d-1)$-face, for each $1 \leq i \leq k-1$. A *pseudomanifold* is a pure simplicial complex which is strongly connected and every $(d-1)$-face is contained in at most two facets. For a set A of simplices, the *closure* $\overline{A}$ of A is the minimum simplicial complex which contains A, that is, $\overline{A}$ is the set of all the faces of simplices of A.

An example of a simplicial complex is a triangulation of a (compact connected) manifold (with boundary). A triangulation of a manifold is pure, strongly connected, and in fact is a pseudomanifold. But a pseudomanifold is not always a triangulation of a manifold, as the following example shows, where the neighbourhood of v is not homeomorphic to a ball.



The *boundary complex* ∂C of a pure d-dimensional simplicial complex C is the closure of $(d-1)$-dimensional faces which belongs to only one facet. Usually this term is used for pseudomanifold cases, but we also use for general cases. In the triangulation of a manifold, the boundary complex corresponds to the boundary of the manifold. The *interior* $\mathring{C}$ of C is $C \setminus \partial C$.

In Chapters 3 and 4, our main interest is in the case of pseudomanifolds. In fact, as will be shown in the next section, pseudomanifolds with certain combinatorial decomposition properties become triangulations of balls or spheres, so what we really discuss in the chapters is the case of triangulations of balls or spheres. This emphasis on these special cases is by two reasons. The first reason is historical: the study of triangulations of manifolds has a long history and many studies have been done, and the study of combinatorial decomposition properties (shellability or constructibility, introduced in the next section) of triangulations of balls or spheres had an importance in relation with the Poincaré Conjecture. Also one source of problems comes from the study of polytopes in combinatorics, which is a special case of triangulations (or polytopal decompositions) of spheres. The second reason is that the topology of pseudomanifolds (balls or spheres) is known well. In combinatorial decomposition properties such as shellability or constructibility, the topology will be preserved recursively in the decomposition, and this sometimes assures a good property. The difference between the case of pseudomanifolds and that of general simplicial complexes will be presented in Chapter 5.

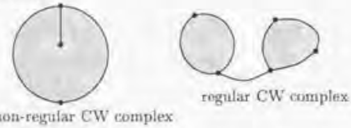
A *polytopal (polyhedral) complex* is a set C of polytopes satisfying

- if $P \in C$ and Q is a face of P, then $Q \in C$, and
- if $P, Q \in C$, then $P \cap Q$ is a face of both P and Q.

For the definitions and properties of polytopes, see Ziegler [98]. Here, a simplicial complex is a special case of polytopal complexes, the case when every polytope is a simplex, so this definition of polytopal complexes is a generalization of Definition 2.1 of simplicial complexes. If every polytope in a polytopal complex is combinatorially equivalent to a cube, then the complex is a *cubical complex*.

An example of a polytopal complex is the *boundary complex* of a polytope, the set of the faces of the polytope except for the polytope itself. The boundary complex of a simplicial polytope is a simplicial complex and that of a cubical polytope is a cubical complex.

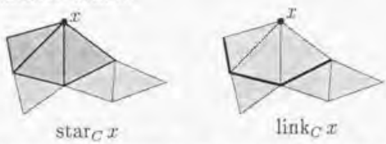
The most general definition in this line seems to be regular CW complexes, see Björner [13, 15]. A (finite) *CW complex* $\mathcal{H}$ is a Hausdorff topological space $X_{\mathcal{H}}$ with a certain kind of cellular decomposition $K = \bigcup_{i=0}^{d} K^i$ such that (i) K^0 is a discrete space of finite points, each point is a 0-cell, and (ii) K^n is obtained by attaching a finite disjoint family of n-balls (n-cells) to K^{n-1} such that each n-cell D_n has a characteristic map $\phi_i : D_n \to K^n$ such that its restriction to the boundary of D_n is a continuous map into K^{n-1} and the restriction to the interior of D_n is a homeomorphism. (The condition of weak topology for CW complexes is not needed here because we are considering a finite case.) A *regular CW complex* is a CW complex such that each cell has a characteristic map which is a homeomorphism. For definitions and further discussions about CW complexes, the reader is recommended to consult textbooks of topology such as Bredon [24], Massey [63], etc. The following figure shows a non-regular CW complex and a regular CW complex.



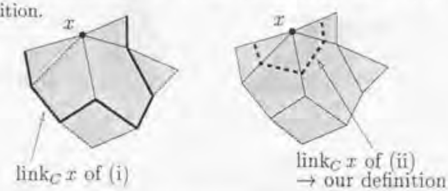
This concept of regular CW complexes is used in many places as a combinatorial object corresponding to polytopal complexes, for example in oriented matroid theory (Björner et. al. [18]), but in this thesis we do not need this because our main interest is in simplicial complexes. But polytopal cases will be discussed sometimes.

The set of faces of a simplicial complex C forms a poset (partially ordered set) ordered by inclusion relation (of the closure), called a *face poset* of C. This face poset has the empty set as its bottom element $\hat{0}$ and the artificial element $\hat{1}$ (regarded as the simplicial complex itself) as the top element. If two simplicial complexes have an isomorphic face poset, then they are *combinatorially equivalent*.

For a simplicial complex C, the *star* $\operatorname{star}_C \sigma$ of a face $\sigma \in C$ is the simplicial complex that contains all faces of facets of C that contain σ, and the *link* $\operatorname{link}_C \sigma$ is the subcomplex of $\operatorname{star}_C \sigma$ that do not intersect with σ.



In the polytopal case, there are two ways to define the link. One way is (i) just the same as simplicial case. This definition is used in Ziegler [98]. The other way is (ii) to define as a polytopal complex which is combinatorially equivalent to the "face figure" of σ in C, i.e., a polytopal complex whose face poset is isomorphic to the subposet of the face poset of C induced by the elements τ satisfying $\sigma \leq \tau$, that is, the upper ideal of σ. In this thesis we choose the latter definition.



These two definitions differs in general but they coincide in the case of simplicial complexes.

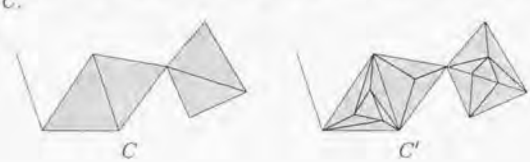
Some operations

For two simplices σ and τ not providing common vertices, the *join* of σ and τ is a simplex with vertices of σ and τ. The join $v * C$ of a vertex v and a simplicial complex C is the set of simplices $\{v * \sigma : \sigma \in C\}$ (not a simplicial complex), and the join of two simplicial complexes C and D is the simplicial complex $C * D = \{\sigma * \tau : \sigma \in C \text{ and } \tau \in D\}$. A *pyramid* over a simplicial complex C is $\overline{v} * C$, a join of C and a 0-ball, and a *suspension* of a simplicial complex D is $\overline{\{v, w\}} * D$, a join of D and a 0-sphere.
The suspension of D is denoted by ΣD.

In this figure, the left figure is a pyramid over a square (this makes a solid pyramid) and the right figure is a suspension of a circle made of 4 edges (this makes the boundary of an octahedron).

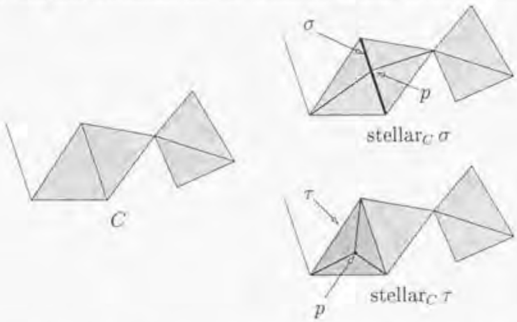
(In usual treatment in PL topology, the join operation is only allowed in the case the two simplices are *joinable*, i.e., each vertices of the involved simplices is the vertex of their convex hull and no intersection occurred by this operation. But in our context, we are not interested in a fixed embedding and only require just the existence of a possible embedding. In other words, our main interest is in the combinatorial structure which can be read from abstract simplicial complexes. So in our situation, we can perform the join operation in an abstract setting and then embed it in some Euclidean space. This is why we omitted the requirement of joinability.)

If a simplicial complex C' has an embedding in E^n, in which other simplicial complex C is already embedded, such that every face of C is a union of some faces of C', then C' is a *subdivision* of C.

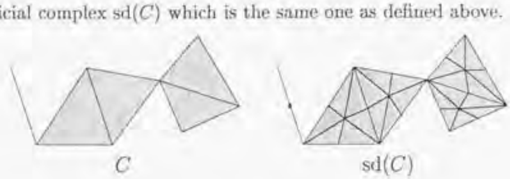


A *stellar subdivision* $\mathrm{stellar}_C\,\sigma$ is a special kind of subdivisions, $\mathrm{stellar}_C\,\sigma = (C \setminus \mathrm{star}_C\,\sigma) \cup (\overline{p} * \mathrm{link}_C\,\sigma)$, where p is a new vertex. This stellar subdivision can be realized by taking a relative interior point p in the face σ and then construct a minimum subdivision of C which contain p as a vertex.

(For a polytopal complex, we should use the definition of links in the way of (i) in p. 12.)



A *barycentric subdivision* $\mathrm{sd}(C)$ is a subdivision of C made by repeated stellar subdivisions: first stellarly subdivide C by all the d-faces of C (where d is $\dim C$), then by all the $(d-1)$-faces, $\cdots$, and lastly by the 1-faces. (The resulting subdivision is unique.) This also can be defined via face posets: in the face poset of C minus the top element C itself and the bottom element $\emptyset$, we associate a simplex $v_{\sigma_{i_1}} v_{\sigma_{i_2}} \cdots v_{\sigma_{i_k}}$ to each chain $\sigma_{i_1} \le \sigma_{i_2} \le \cdots \le \sigma_{i_k}$. Then we get a simplicial complex $\mathrm{sd}(C)$ which is the same one as defined above.



The number of faces: f-vectors and h-vectors

For a d-dimensional simplicial (polytopal) complex C, we denote the number of i-dimensional faces of C by $f_i(C)$, and $f(C) = (f_{-1}(C), f_0(C), f_1(C), \ldots, f_d(C))$ is called the *f-vector* of C. We associate a generating polynomial, *f-polynomial*, to the f-vector,

$$f(C,x) = f_{-1}(C)x^{d+1} + f_0(C)x^d + \cdots + f_{d-1}(C)x + f_d(C).$$

From this polynomial another invariant, *h-vector* is defined to be the coefficients of $f(C, x-1)$, that is,

$$f(C,x-1) = h_0(C)x^{d+1} + h_1(C)x^d + \cdots + h_d(C)x + h_{d+1}(C).$$

(Be careful that f-vector is indexed by $\{-1, 0, \ldots, d\}$ but h-vector is indexed by $\{0, 1, \ldots, d+1\}$.) The polynomial $f(C, x-1)$ is called an h-polynomial and denoted by $h(C, x)$.

These two vectors are related by a linear transform and the knowledge of one of the vectors determines uniquely the other. The explicit formula to derive h-vector from f-vector is as follows:

$$h_k(C) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}(C).$$

Especially,

$$\begin{aligned}
h_0(C) &= 1, \\
h_1(C) &= f_0(C) - (d+1), \\
h_{d+1}(C) &= f_d(C) - f_{d-1}(C) + \cdots + (-1)^d f_0(C) + (-1)^{d+1} f_{-1}(C) \\
&= (-1)^d \tilde{\chi}(C),
\end{aligned}$$

where $\tilde{\chi}(C)$ is *the reduced Euler characteristics*,

$$\tilde{\chi}(C) = -f_{-1}(C) + f_0(C) - \cdots + (-1)^d f_d(C).$$

In spite of the equivalence of f-vectors and h-vectors, there are some cases where using h-vectors are preferred than f-vectors. For example, the boundary complexes of polytopes and also triangulated spheres satisfies a set of equations called Dehn-Sommerville equations (for example Bayer and Billera [5], Ziegler [98]):

$$f_{k-1} = \sum_{i=k}^{d+1} (-1)^{d+1-i} \binom{i}{k} f_{i-1}, \qquad (0 \le k \le \lfloor \frac{d+1}{2} \rfloor)$$

but these equations written in term of h-vectors are simply:

$$h_k = h_{d-k+1}. \qquad (0 \le k \le \lfloor \frac{d+1}{2} \rfloor)$$

Moreover, h-vectors have a combinatorial and algebraic interpretation which we will review in Section 2.3.4.

As noted above, the top element of h-vector, $h_{d+1}(C)$, equals to $(-1)^d \tilde{\chi}(C)$. This means that $h_{d+1}(C)$ is a topological invariant because the reduced Euler characteristics has the following formula:

$$\tilde{\chi}(C) = \sum_{i=-1}^{d} (-1)^i \dim \tilde{H}_i(C),$$

where $\tilde{H}_i$'s are the reduced homology groups. A topological space is *contractible* if it has the homotopy type of one point. We also say that C is contractible if $|C|$ $(= \bigcup_{\sigma \in C} \sigma$, the underlying space of C) is contractible. Because the reduced homology groups are invariant by homotopy, contractible space has $\tilde{H}_i(C) = 0$ for all i. Thus we have $h_{d+1}(C) = 0$ if C is contractible. (This fact will be used in Chapter 5.)

2.2 Balls, spheres, and manifolds

This section provides basic preliminaries on topology, especially on PL (piecewise linear) topology. We just present here some properties which we need in this thesis without proof. For their proofs and further discussions, we recommend Zeeman [96], Hudson [51] or Bing [11].

For a simplicial (polytopal) complex C, the *underlying space* (or *geometric realization*) $|C|$ is the union $\bigcup_{\sigma\in C}\sigma$ of all the simplices belonging to C. If the underlying space is homeomorphic to a manifold M (with boundary), then the simplicial complex is a *triangulation of* M. Throughout this thesis, a *d-ball* or a *d-sphere* is a short for a triangulation of the d-dimensional ball or d-dimensional sphere, respectively. A triangulation of a manifold is a pseudomanifold with an additional condition that the neighbourhood of every point in the underlying space is homeomorphic to a full-ball or a half-ball.

In the case of a polytopal complex, a polytopal complex whose underlying space is homeomorphic to a manifold M is a *polytopal decomposition of* M. A *polytopal ball (sphere)* is short for a polytopal decomposition of a ball (sphere).

A d-dimensional ball (as a topological space, not a triangulation) is *PL* if there is a piecewise linear homeomorphism between the ball and a d-dimensional simplex, and a d-dimensional sphere is *PL* if there is a piecewise linear homeomorphism between the sphere and the boundary of a d-dimensional simplex. We say a PL-d-ball (or simply a PL-ball), and a PL-d-sphere (a PL-sphere) for short. A triangulated manifold is called a *combinatorial manifold* if the link of each vertex is a PL-ball or a PL-sphere.

The following propositions are fundamental in PL topology.

Proposition 2.2. *[96, Lemma 9]*
A triangulation of a ball or a sphere is PL if and only if the link of each vertex is a PL-ball or a PL-sphere, i.e., if it is combinatorial. ∎

Proposition 2.3. *[96, Corollary to Theorem 2]*
If two PL-d-balls meet by a PL-$(d-1)$-ball which lies in their boundaries, then the union is again a PL-d-ball. ∎

Proposition 2.4. *[96, Follows from Theorem 2]*
If two PL-d-balls meet by their whole boundaries, then their union is a PL-d-sphere. ∎

Proposition 2.5. *[96, Theorem 3]*
If we remove a PL-d-ball from a PL-d-sphere, the closure of the rest is a PL-d-ball. ∎

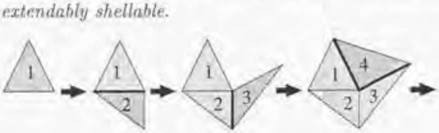
Proposition 2.6. *[96, Corollary to Lemma 8]*
The join of a PL-p-ball and a PL-q-ball is a PL-$(p+q+1)$-ball, and the join of a PL-p-sphere and a PL-q-sphere is a PL-$(p+q+1)$-sphere. Especially, a pyramid over a PL-d-ball is a PL-$(d+1)$-ball, and a suspension of a PL-d-sphere is a PL-$(d+1)$-sphere. ∎

Further, it is known that all 2- and 3-balls and spheres are PL, but there are non-PL 5-spheres. (It is not known whether there are non-PL 4-spheres or not.)

2.3 Combinatorial decomposition properties

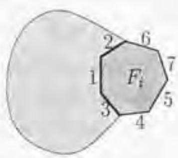
2.3.1 Shellability

Definition 2.7. An ordering of the facets $F_1, F_2, \ldots, F_t$ of a d-dimensional simplicial complex is a *shelling* if $(\overline{F}_1 \cup \cdots \cup \overline{F}_{i-1}) \cap \overline{F}_i$ is a pure $(d-1)$-dimensional simplicial complex, for $2 \leq i \leq t$. A simplicial complex is *shellable* if it admits a shelling. Moreover, if every partial shelling (i.e., an ordering of a subset of facets satisfying the condition) extends to a complete shelling, it is called *extendably shellable*.



For polytopal complexes, there are several types of definitions of shellability all of which generalize the above definition, but the following is now the standard definition because it has a very nice recursion.

Definition 2.8. (Björner and Wachs [21] Björner [13], etc.)
An ordering of the facets $F_1, F_2, \ldots, F_t$ of a d-dimensional polytopal complex is a *shelling* if $(\overline{F}_1 \cup \cdots \cup \overline{F}_{i-1}) \cap \overline{F}_i$ is $(d-1)$-dimensional and has a shelling which extends to a shelling of the boundary of F_i.



There are some other versions for example: (i) only require the intersection is shellable (Brugesser-Mani [25]), (ii) even only require that it is a ball or sphere (Danaraj-Klee [31], "weak shellability"), (iii) require that $F_1 \cup F_2 \cup \ldots \cup F_i$ is a ball for every step (Ewald [37]) except for the last step, and (iv) a custom-tailored version for the application to cd-index, S-shellability ("S" for "spherical") of Stanley [86]. It is known that these are all equivalent in simplicial cases. (But (iii) requires that the realization is a ball or sphere, and (iv) requires to be a sphere.) Another essentially same variation is by indexing the facets in the reverse way; this seems to be familiar among topologists, for example Bing [10].

Although the above definitions of shellability requires that shellable complexes should be pure (easily shown), there is a non-pure version of definition by Björner and Wachs [22, 23]. This non-pure version of shellability, which includes the pure case as a special case, is now assumed to be the standard definition, but we will not use it in this thesis.

One property of shellability is that it is inherited by links:

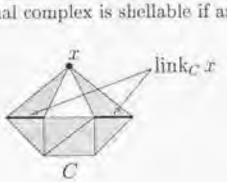
Proposition 2.9. *(Björner[12])*
Every link of a shellable simplicial (polytopal) complex is shellable.

(Be careful that this may no longer be true if we use the first version of the definition of the link in the polytopal case; this is why we chose the second way of definition in p. 12. See Exercise 8.4 of Ziegler [98].)

Proof. A shelling of the complex induces a shelling of each link. ∎

Another way to show this is by using the results of Björner and Wachs [21]: a polytopal complex is shellable if and only if its face poset is CL-shellable ("CL" = "chainwise lexicographic"), and every interval of a CL-shellable poset is CL-shellable.

By this proposition, we can easily check that the following figure is not shellable because $\mathrm{link}_C\, x$ is not shellable. (1-dimensional complex is shellable if and only if it is connected.)



The most interesting and mysterious fact around shellability seems to be the existence of non-shellable triangulations of 3-balls and 3-spheres. This is a surprising fact compared to the fact that every shellable pseudomanifolds are homeomorphic to balls or spheres (this will be shown in Section 2.3.2 in a stronger form), and that the converse for 2-dimensional pseudomanifolds is also true (see Section 2.5). Moreover, the following important theorem is shown by Brugesser and Mani [25], which has many applications in combinatorics of polytopes such as McMullen's Upper Bound Theorem of polytopes [65].

Theorem 2.10. *(Brugesser-Mani [25])*
Every boundary complex of a polytope is shellable. ∎

The proof is done by using the so-called "line shelling".

A number of non-shellable triangulations of 3-balls are reviewed in Ziegler [99]. Currently known construction of non-shellable balls seems to be only of two types: one uses knots, and the other constructs directly a situation that a removal of every one facet corrupt the ball-ness. Such triangulations of balls that no facet can be removed without corrupting ball-ness are especially called *strongly non-shellable* [99].

The former construction, a construction using knots is the oldest one: Furch's knotted hole ball. This one will be treated in Section 3.1 of Chapter 3. The rest can be grouped in the latter construction, though finding some nice reasoning of their non-shellability seems an interesting open problem. An indirect but good-for-understanding example is the Danzer cube described in Ziegler [98, 99], a triangulation of a cube involving a special link made of 12 edges. The small examples of triangulated 3-balls, Rudin [78], Grünbaum (unpublished, see Danaraj and Klee [32] and Hachimori [43].) and Ziegler [99], exhibit a concrete example of triangulation with the property that no facet can be removed without losing ball-ness. (Thus these three examples are strongly non-shellable.) The list of the facets of these triangulations are the following:

Rudin's 3-ball (with 14 vertices and 41 facets):

3 4 7 11	4 5 8 12	5 6 9 13	6 3 10 14	3 4 7 12	4 5 8 13
5 6 9 14	6 3 10 11	4 7 11 12	5 8 12 13	6 9 13 14	3 10 14 11
4 8 11 12	5 9 12 13	6 10 13 14	3 7 14 11	11 12 13 14	7 11 12 13
8 12 13 14	9 13 14 11	10 14 11 12	3 7 12 13	4 8 13 14	5 9 14 11
6 10 11 12	3 9 12 13	4 10 13 14	5 7 14 11	6 8 11 12	1 3 9 13
2 4 10 14	1 5 7 11	2 6 8 12	1 3 7 13	2 4 8 14	1 5 9 11
2 6 10 12	1 7 11 13	2 8 12 14	1 9 13 11	2 10 14 12	

Grünbaum's 3-ball (with 14 vertices and 29 facets):

1 2 3 7	1 2 4 8	1 2 7 8	1 3 5 7	1 4 8 10	1 5 6 13
1 5 7 13	1 6 11 13	1 7 8 10	1 7 11 13	2 3 7 9	2 4 6 8
2 5 6 14	2 5 12 14	2 6 8 14	2 7 8 9	2 8 12 14	3 5 7 9
4 6 8 10	5 6 13 14	5 7 9 13	5 12 13 14	6 8 10 14	6 11 13 14
7 8 9 13	7 8 10 14	7 8 13 14	7 11 13 14	8 12 13 14	

Ziegler's 3-ball (with 10 vertices and 21 facets):

1 2 3 4	1 2 5 6	2 3 6 7	3 4 7 8	4 1 8 5	1 5 6 9
1 6 2 9	1 2 4 9	1 4 8 9	1 8 5 9	2 5 6 10	2 6 7 10
2 7 3 10	2 3 1 10	2 1 5 10	3 6 7 8	3 2 4 8	3 2 6 8
4 5 7 8	4 1 3 7	4 1 5 7			

(The list of Rudin's and Grünbaum's 3-balls are taken from Danaraj-Klee [32], where the typographical error of the 9th facet in the latter is suitably corrected, see [43]. The list of Ziegler's 3-ball is taken from his own paper [99].)

All of these three examples have a geometric realization in E^3, with all vertices on their boundaries. Rudin's ball even has a convex realization, while the rest two seems to have only non-convex realizations. Another example of non-shellable triangulation of a 3-ball, Bing's house with two rooms which will be described in Section 4.4, also has the same property: all the vertices on its boundary and the removal of any one facet corrupting its ball-ness.

The only known way of the construction of non-shellable triangulations of 3-spheres was shown by Lickorish [57]. (If we do not require to be a triangulation, pseudosimplicial decomposition is shown in Vince [90] and cell partitionings in Armentrout [2, 3].) This uses an embedded knot and its non-shellability is shown by using the idea of collapsing and counting the number of generators needed to represent the fundamental group of the knot complement. This non-shellable sphere will be treated in Section 3.3 with a proof of its non-shellability by a different way via constructibility argument.

About extendable shellability, what we remark here is the following.

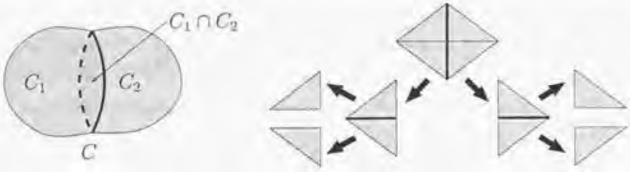
- All the triangulation of 2-balls and 2-spheres are extendably shellable. (Shown later in Section 2.5.) But not all shellable 2-dimensional simplicial complexes are extendably shellable. (See Section 5.3.)
- There are simplicial 4-polytopes whose boundary complexes are not extendably shellable. (Shown in Ziegler [99]. This implies that extendable shellability is strictly stronger than shellability.)

2.3.2 Constructibility

Definition 2.11. A pure d-dimensional simplicial complex C is *constructible* if

(i) C is a simplex, or

(ii) there are two d-dimensional constructible simplicial complexes C_1 and C_2 such that $C_1 \cup C_2 = C$ and that $C_1 \cap C_2$ is a $(d-1)$-dimensional constructible simplicial complex.



This concept of constructibility was first formulated by Hochster [49], and appears in Stanley [83], Björner [15], etc. The relation between shellability and constructibility can be seen from the following reformulation of shellability of simplicial complexes.

Definition 2.12. (Reformulation of Definition 2.7.)
A pure d-dimensional simplicial complex C is shellable if

(i) C is a simplex, or

(ii) there is a d-dimensional shellable simplicial complexes C_1 and a d-simplex C_2 such that $C_1 \cup C_2 = C$ and that $C_1 \cap C_2$ is a $(d-1)$-dimensional shellable simplicial complex.

The equivalence of Definitions 2.7 and 2.12 is easy to see. From this, one can observe that constructibility is a relaxation of shellability, that is, if we restrict C_2 to be a simplex in the Definition 2.11 of constructibility, we have Definition 2.12 of shellability.

The version for polytopal complexes is as follows.

Definition 2.13. A pure d-dimensional polytopal complex C is *constructible* if

(i) C is a polytope, or

(ii) there are two d-dimensional constructible polytopal complexes C_1 and C_2 such that $C_1 \cup C_2 = C$ and that $C_1 \cap C_2$ is a $(d-1)$-dimensional constructible polytopal complex.

This polytopal version is also a relaxation of shellability for polytopal cases. Also this definition includes the simplicial version of the definition above. If we use this definition for regular CW complexes, we should additionally require that the boundary complex of each cell is constructible, see Mandel [62] or Hachimori [42]. (For polytopal case, we do not need this treatment because the boundary complex of a polytopes is constructible.)

Constructibility is inherited by links, as same as the case of shellability.

Proposition 2.14. *(Björner [12, 15])*
Every link of a constructible polytopal complex is constructible.

Proof. Let C be a constructible polytopal complex and τ a face of C. We use induction on the number of facets of C. The case of a simplex C is trivial, so we write C as a union of two constructible complexes C_1 and C_2. If τ is contained in only one of C_1 and C_2, say in C_1, then $\operatorname{link}_C \tau = \operatorname{link}_{C_1} \tau$ is constructible by induction. If τ is contained in $C_1 \cap C_2$, then

(i) $(\operatorname{link}_C \tau) \cap C_1 = \operatorname{link}_{C_1} \tau =: L_1$,

(ii) $(\operatorname{link}_C \tau) \cap C_2 = \operatorname{link}_{C_2} \tau =: L_2$,

(iii) $L_1 \cap L_2 = (\operatorname{link}_{C_1} \tau) \cap (\operatorname{link}_{C_2} \tau) = \operatorname{link}_{C_1 \cap C_2} \tau$, and

(iv) $L_1 \cup L_2 = \operatorname{link}_C \tau$.

These observations imply by induction that $\operatorname{link}_C \tau$ is constructible. ∎

Again we remark here that this proposition for polytopal (non-simplicial) case does not hold if we define links by the faces of $\operatorname{star}_C \sigma$ not containing σ (the way of definition (i) in p. 12), as remarked after Proposition 2.9.

We also have the following property of constructible complexes. The proof is omitted because it is obvious.

Proposition 2.15. *Constructible polytopal complexes are strongly connected.* ∎

For the case of pseudomanifolds, constructibility assures a stronger property for the topology of the underlying space.

Proposition 2.16. *(Zeeman [96], Björner [15])*
A d-dimensional constructible simplicial (or polytopal) complex in which any $(d-1)$-face is contained in at most two facets is a PL-d-ball or a PL-d-sphere.

Proof. This is by induction on the size of facets and on the dimension. First, a simplex is a PL-ball by definition, and this makes the induction base.

Let C be a constructible complex with the property that each $(d-1)$-face is contained at most two facets, and assume that C is not a simplex. Then there are two constructible complexes C_1 and C_2 satisfying the condition (ii) of Definition 2.13. Here both C_1 and C_2

satisfy the condition that each $(d-1)$-face is contained in at most two facets, thus by induction, both are PL-d-balls or PL-d-spheres. Moreover, $C_1 \cap C_2$ is contained in the boundaries of both balls because of the requirement that C also satisfies that each $(d-1)$-face is contained at most two facets. (This means that C_1 and C_2 were PL-d-balls, not spheres.) Because the boundary of a d-ball is a $(d-1)$ sphere, $C_1 \cap C_2$ also satisfies that each $(d-2)$-face is contained at most two facets. Also by induction hypothesis, $C_1 \cap C_2$ is a PL-$(d-1)$-ball or PL-$(d-1)$-sphere. Now the statement follows from Propositions 2.3 and 2.4, according to whether $C_1 \cap C_2$ is a ball or a sphere. ∎

Because shellable complexes are constructible, we have the following corollary.

Corollary 2.17. *A shellable pseudomanifold is a PL-ball or a PL-sphere.* ∎

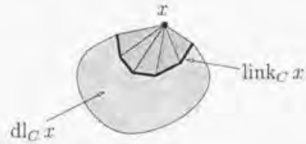
2.3.3 Vertex decomposability

A *deletion* $\mathrm{dl}_C \sigma$ of a face σ of a simplicial complex C is the simplicial complex $\{\tau : \tau \in C \text{ and } \sigma \not\subseteq \tau\}$.

Definition 2.18. A pure d-dimensional simplicial complex C is *vertex decomposable* if

(i) C is a simplex (including the case that C is $\{\emptyset\}$), or

(ii) there is a vertex x such that $\mathrm{link}_C\, x$ and $\mathrm{dl}_C\, x$ are vertex decomposable simplicial complexes.



The vertex x in the definition is called a *shedding vertex.*

This definition is introduced by Provan and Billera [74] in relation with Hirsch conjecture. (See also Billera and Provan [8].)

There is also more general concept, k-decomposability.

Definition 2.19. A pure d-dimensional simplicial complex C is *k-decomposable* if

(i) C is a simplex, or

(ii) there is a face σ with $\dim \sigma \leq k$ such that $\mathrm{link}_C\, \sigma$ and $\mathrm{dl}_C\, x$ are k-decomposable simplicial complexes.

Naturally the following implications hold:

vertex decomposable = 0-decomposable $\Rightarrow$ 1-decomposable $\Rightarrow \cdots \Rightarrow$ d-decomposable.

Moreover, Provan and Billera [74] shows that d-decomposability is equivalent to shellability. (The fact that vertex decomposability implies shellability can be shown directly using induction on the number of facets and the dimension.)

The important property of vertex decomposability is the following.

Theorem 2.20. *(Provan-Billera [74])*
If C is a d-dimensional vertex decomposable simplicial complex, then

$$\operatorname{diam} C \leq f_k(C) - \binom{d+1}{k+1}, \quad \textit{for } 0 \leq k \leq d.$$

∎

Here, $\operatorname{diam} C$ is the diameter of the graph in which vertices are facets of C and two vertices are connected by an edge if the corresponding two facets have a common $(d-1)$-face. Consequently, every vertex decomposable simplicial complex satisfies the Hirsch conjecture (in a dual sense), that is, $\operatorname{diam} C$ is at most $\#\{\text{facets}\} - (d+1)$.
(The Hirsch conjecture states that the diameter of the edge graph of a polytope P (a graph made of edges and vertices of the polytope) is at most $\#\{\text{facets of } P\} - \dim P$, and the vertex decomposable simplicial complexes above corresponds to the boundary complex of the dual (or the polar) of the polytope.)

The Hirsch conjecture for polytopes is still open, but there are simplicial spheres which fails to satisfy the conjecture, 27-sphere with 56 vertices and more than 8000 simplices by Walkup [93], and also such example of a 3-sphere is given in Mani and Walkup [66]. There is a non-vertex decomposable 4-polytope made by Lockeberg [60] but still satisfying the Hirsch conjecture, see Klee and Kleinschmidt [53].

Remark. The facet list of Lockeberg's 4-polytope described in the paper of Klee and Kleinschmidt [53] seems to contain a typographical error. The facet "aejk" (the 43rd facet) should be "aehk" in order to make this simplicial complex to be a sphere.

2.3.4 Other properties

Cohen-Macaulayness

Cohen-Macaulayness is one of the most famous properties of simplicial complexes and has been studied by many researchers. To be precise, this is not a kind of combinatorial decomposition properties, but we introduce this concept as one of combinatorial decomposition properties because this is in a sense a topological relaxation of combinatorial decomposition properties and we can not avoid this in the study of this field.

Usually Cohen-Macaulayness is defined in terms of face rings (or Stanley-Reisner rings) that a simplicial complex is Cohen-Macaulay if its face ring is Cohen-Macaulay, but we define here in terms of reduced homology groups of links which is the characterization of Cohen-Macaulayness by Reisner [75]. For the original algebraic definition, see for example Stanley [86] or Hibi [48].

Definition 2.21. A simplicial complex is *Cohen-Macaulay* if $\tilde{H}_i(\operatorname{link}_C \sigma) = 0$ except $i = \dim \operatorname{link}_C \sigma$ for any face σ of C, where $\tilde{H}_i$ is the reduced homology group over a ring R.

Remark. This Cohen-Macaulayness depends on the choice of R. In this thesis, we assume that the ring R is always $\mathbb{Z}$. It is known that Cohen-Macaulayness over $\mathbb{Z}$ is stronger than to be Cohen-Macaulay over any field, see Björner [15, p. 1855].

The following property is known.

Proposition 2.22. *(Munkres [70, Corollary 3.4])*
A simplicial complex C is Cohen-Macaulay if and only if $|C|$ satisfies that $\tilde{H}_i(|C|) = 0 = H_i(|C|, |C| \setminus p)$ for all $p \in |C|$ and $i < \dim C$, where H_i denotes the singular relative homology group and $\tilde{H}_i$ denotes the singular reduced homology group. Thus Cohen-Macaulayness is topological, i.e., if the underlying spaces of C and C' are homeomorphic and C is Cohen-Macaulay, then C' is also Cohen-Macaulay. ∎

Though the proof in Munkres [70] is written in terms of cohomology over a field, the same argument can be used for homology over $\mathbb{Z}$. The following proof is the same as the original except for replacing cohomology by homology.

Proof. If $\operatorname{link}_C \sigma \neq \emptyset$, then

$$\begin{aligned} H_j(|C|, |C| \setminus p) &\simeq H_j(|\operatorname{star}_C \sigma|, |\operatorname{star}_C \sigma| \setminus p) \\ &\simeq H_j(|\operatorname{star}_C \sigma|, |\partial\sigma * \operatorname{link}_C \sigma|) \\ &\simeq \tilde{H}_{j-1}(|\partial\sigma * \operatorname{link}_C \sigma|) \quad \cdots (*) \\ &\simeq \tilde{H}_{j-\dim\sigma-1}(|\operatorname{link}_C \sigma|), \end{aligned}$$

and if $\operatorname{link}_C \sigma = \emptyset$, then

$$\begin{aligned}
H_j(|C|, |C| \setminus p) &\simeq H_j(|\sigma|, |\sigma| \setminus p) \\
&\simeq H_j(|\sigma|, |\partial\sigma|) \\
&\simeq \tilde{H}_{j-1}(|\partial\sigma|) \quad \cdots (*) \\
&\simeq \begin{cases} \mathbb{Z} & \text{if } j = \dim\sigma \\ 0 & \text{if } j \neq \dim\sigma \end{cases} \\
&\simeq \tilde{H}_{j-\dim\sigma-1}(\emptyset).
\end{aligned}$$

Here, both $(*)$ are implied by the long exact sequence

$$\begin{array}{ccccccc}
\cdots & \to & H_{i+1}(X, A) & \to & \tilde{H}_i(A) & \to & \tilde{H}_i(X) \\
 & \to & H_i(X, A) & \to & \tilde{H}_{i-1}(A) & \to & \tilde{H}_{i-1}(X) \\
 & \to & \cdots & & & & \\
 & \to & H_0(X, A) & \to & \tilde{H}_{-1}(A) & \to & \tilde{H}_{-1}(X) \\
 & \to & 0 & & & &
\end{array}$$

for $X \supset A$ and $A \neq \emptyset$.

Thus we have

$$H_j(|C|, |C| \setminus p) \simeq \tilde{H}_{j-\dim\sigma-1}(|\operatorname{link}_C \sigma|) \quad (= \tilde{H}_{j-\dim\sigma-1}(\operatorname{link}_C \sigma)), \quad \cdots (**)$$

for all j and $p \in \mathring{\sigma}$.

Now we show that the following conditions are equivalent.

(i) $\tilde{H}_i(|C|) = 0 = H_i(|C|, |C| \setminus p)$ for $i < \dim C$ and $p \in |C|$,

(ii) $\tilde{H}_i(\operatorname{link}_C \sigma) = 0$ except $i = \dim \operatorname{link}_C \sigma$ for any face σ of C.

The first remark is that both conditions imply that C is pure. That (i) implies the purity of C follows from the fact that $H_k(|C|, |C| \setminus p) \simeq \mathbb{Z}$ if $p \in \mathring{\sigma}$ and σ is a k-dimensional facet. That (ii) implies the purity of C is verified as follows: Let C satisfy (ii) but non-pure, and D the subcomplex of C generated by the facets whose dimension is less than $\dim C$. Then if we take a facet of $C \cap D$ to be σ, then $\operatorname{link}_C \sigma$ is disconnected but its dimension is at least one. This contradicts the condition of (ii) because disconnected complex Δ with dimension at least one has $\tilde{H}_0(\Delta) \neq 0$.
Now because C is pure, $\dim \operatorname{link}_C \sigma + \dim\sigma + 1 = \dim C$ in both (i) and (ii). The condition $\tilde{H}_i(\operatorname{link}_C \sigma) = 0$ for $i < \dim \operatorname{link}_C \sigma$ of (ii) is equivalent to the condition $\tilde{H}_{i-\dim\sigma-1}(\operatorname{link}_C \sigma) = 0$ for $i < \dim C$ and $\sigma \neq \emptyset$, and this is equivalent to the condition $H_i(|C|, |C| \setminus x) = 0$ for $i < \dim C$ from $(**)$. For $\sigma = \emptyset$, the condition is equivalent to the condition $H_i(|C|) = 0$ for $i < \dim C$. Thus (i) and (ii) are equivalent.

∎

Especially, triangulations of balls and spheres are all Cohen-Macaulay (over $\mathbb{Z}$).

The typical property of Cohen-Macaulayness is the following.

Proposition 2.23. *The h-vectors of Cohen-Macaulay simplicial complexes are nonnegative.* ∎

For the proof of this proposition, see for example Stanley [86]. The statement follows from the fact that h-vectors correspond to the coefficients of Hilbert series of the face ring of C.

The class of constructible simplicial complexes is an important subclass of Cohen-Macaulay simplicial complexes.

Proposition 2.24. *A constructible simplicial complex is Cohen-Macaulay.*

Proof. (Hibi [48, Lemma 23.6])
This was originally proved by Hochster [49] in terms of face rings, but Reisner's characterization (Definition 2.21) makes the proof very easy.

The proof is by induction on the number of facets and the dimension. If a d-dimensional constructible simplicial complex C is a simplex, then it is Cohen-Macaulay because all of the reduced homology groups of a ball are 0.

If C is not a simplex, then there are two constructible simplicial complexes C_1 and C_2 with $C_1 \cap C_2$ is a $(d-1)$-dimensional constructible complex and $C_1 \cup C_2 = C$. The reduced homology groups of C, C_1, C_2 have the following "reduced Mayer-Vietoris exact sequence":

$$\begin{array}{llllll}
\cdots & \to & \tilde{H}_i(C_1 \cap C_2) & \to & \tilde{H}_i(C_1) \oplus \tilde{H}_i(C_2) & \to & \tilde{H}_i(C_1 \cup C_2) \\
& \to & \tilde{H}_{i-1}(C_1 \cap C_2) & \to & \tilde{H}_{i-1}(C_1) \oplus \tilde{H}_{i-1}(C_2) & \to & \tilde{H}_{i-1}(C_1 \cup C_2) \\
& \to & \cdots & & & & \\
& \to & \tilde{H}_0(C_1 \cap C_2) & \to & \tilde{H}_0(C_1) \oplus \tilde{H}_0(C_2) & \to & \tilde{H}_0(C_1 \cup C_2) \\
& \to & \tilde{H}_{-1}(C_1 \cap C_2) & \to & \tilde{H}_{-1}(C_1) \oplus \tilde{H}_{-1}(C_2) & \to & \tilde{H}_{-1}(C_1 \cup C_2) \\
& \to & 0 & & & &
\end{array}$$

Here by induction hypothesis, $\tilde{H}_i(C_1)$, $\tilde{H}_i(C_2)$ are 0 for all $i \leq d-1$ and $\tilde{H}_i(C_1 \cap C_2)$ are 0 for all $i \leq d-2$, the above exact sequence implies that $\tilde{H}_i(C)$ $(= \tilde{H}_i(\mathrm{link}_C\, \emptyset))$ is 0 for all $i \leq d-1$.

For the links $\mathrm{link}_C\, \sigma$ with $\sigma \neq \emptyset$, the reduced homology groups of the link of the dimensions less than $\dim \mathrm{link}_C\, \sigma$ disappear from the induction hypothesis on the dimension because $\mathrm{link}_C\, \sigma$ is constructible from Proposition 2.14 and has a smaller dimension. ∎

Partitionability

Definition 2.25. A simplicial complex C is *partitionable* if the set of faces of C is partitioned into the sets of the form $\{\tau : \phi(\sigma) \subseteq \tau \subseteq \sigma\}$, where σ is a facet of C and $\phi(\sigma)$ is a face of σ.

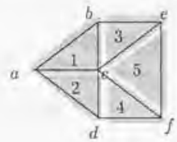
In the term of face posets, a simplicial complex is partitionable if its face poset (minus the top element) can be partitioned into intervals whose tops are facets.

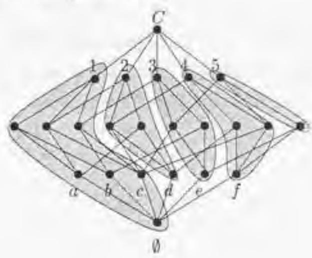
One class of partitionable simplicial complexes are shellable simplicial complexes in which a shelling $F_1, F_2, \ldots, F_t$ of a shellable simplicial complex C induces a partition in a natural way:

(i) For F_1, we set $\phi(F_1) = \emptyset$.

(ii) For F_i with $i \geq 2$, we set $\phi(F_i)$ to be the unique minimal face R_i of F_i which is not contained in $F_1 \cup F_2 \cup \cdots \cup F_{i-1}$.

The following figure shows the partition induced by a shelling.





For a pure partitionable simplicial complex, there is a combinatorial interpretation of h-vectors, that is, we have the following proposition.

Proposition 2.26. *For a partitionable simplicial complex, we have*

$$h_i(C) = \#\{\sigma : \dim \phi(\sigma) = i - 1\}.$$

∎

For example in the above figure, $f(C) = (1, 6, 10, 5)$ and $h(C) = (1, 3, 1, 0)$, this coincides with the numbers $\#\{i : \dim \phi(\sigma) = i - 1\}$. The proof is just by counting the faces contained in each intervals using the fact that each interval is a boolean lattice. (See for example Ziegler [98], Stanley [86], Kleinschmidt and Onn [54], etc.)

From the way how a shelling induces a partition described above, h-vectors can be calculated easily from a shelling $F_1, F_2, \ldots, F_t$ of C: $h_k(C)$ is the number of i's such that the minimum face of F_i which is not contained in F_j with $j \leq i - 1$ has dimension $k - 1$. (See

Ziegler [98, p.247].) For this, we need only one arbitrary shelling because h-vector is already determined by its f-vector and we get the same answer no matter how we take a shelling.

A consequence of the calculation of h-vectors from the partition as above is that h-vectors of partitionable simplicial complexes are non-negative. This mysterious coincidence with the fact that Cohen-Macaulay simplicial complexes have non-negative h-vectors leads us to the conjecture of Garsia [40] and Stanley [85]: Cohen-Macaulay simplicial complexes are partitionable. Some related results are shown such as Duval and Zhang [34], but the problem is still open. (The converse direction is not true: There are partitionable simplicial complexes which are not Cohen-Macaulay, see Stanley [86, p.85].)

Important study about partitionability was done in Kleinschmidt and Onn [54]. They showed that polyhedral fans and oriented matroid polytopes are signable, which in the simplicial case means that they are partitionable. Both of these two classes are not known to be shellable or not, but their result shows the partitionability not using shellability. Moreover, their arguments can be used to show the upper bound property not using shellability or even Cohen-Macaulayness. Further study can be found in Onn [73]. Thus partitionability is a very useful and important property, but we do not treat in this thesis except for the calculation of h-vectors in Chapter 5.

Simplicial collapsing

Definition 2.27. If a face G of a simplicial complex C is contained in only one face F and $\dim G = \dim F - 1$, then we write $C \searrow^{s(e)} (C - \{G, F\})$ and we call this operation an *elementary simplicial collapse*. (This only happens when F is a facet. Such face G is called *free*.)

If a simplicial complex C' is derived by a sequence of elementary simplicial collapse from C, C *simplicially collapses to* C' and denoted by $C \searrow^{s} C'$. Especially if C' is one vertex, then C is *simplicially collapsible*.

If C has a subdivision which is simplicially collapsible, then C is *(polyhedrally) collapsible*, denoted by $C \searrow C'$.

There is another but equivalent way to define collapsing: a face G is free if it is contained in only one facet, and an elementary simplicial collapsing removes all the faces which contains G, for example Björner [15] or Welker [95].

Collapsing is a fundamental tool in combinatorial topology, for example in the regular neighborhood theory. As is easily observed, each elementary simplicial collapse is a strong deformation retract which is performed by a combinatorial operation, especially it preserve the homotopy type of the underlying space. So collapsing is a combinatorial analogy of homotopy equivalence. In particular, a collapsible simplicial complex has a contractible (i.e., homotopy equivalent to one point) underlying space.

The relation between shellability and simplicial collapsibility is not clear because collapsible simplicial complexes are always contractible but shellable simplicial complexes can have non-zero homology. But if we restrict to the contractible case, they certainly have a strong relation.

Proposition 2.28. *A shellable contractible simplicial complex is simplicially collapsible.*

The proof is by performing elementary simplicial collapses in the reverse way of a shelling.

∎

2.4 The hierarchy of combinatorial decomposition properties

Summarizing the relations among the combinatorial decomposition properties introduced in the last section, we re-state the following proposition.

Proposition 2.29. *For simplicial complexes, the following implications hold:*

- *vertex decomposable* $\Rightarrow$ *shellable* $\Rightarrow$ *constructible* $\Rightarrow$ *Cohen-Macaulay,*
- *extendably shellable* $\Rightarrow$ *shellable,*
- *shellable* $\Rightarrow$ *partitionable,*
- *contractible and shellable* $\Rightarrow$ *simplicially collapsible,*

and for polytopal complexes, we have

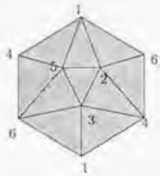
- *extendably shellable* $\Rightarrow$ *shellable* $\Rightarrow$ *constructible.*

∎

All of the implications in the proposition is strict:

- There are shellable but not vertex decomposable simplicial complexes. For example, the existence of polytopes with non-vertex decomposable boundary is known. (Lockeberg's polytope, see Klee-Kleinschmidt [53].)
- There are constructible but not shellable simplicial complexes. For instance, Rudin's ball, Grünbaum's ball, Ziegler's ball are such example, see Proposition 4.6.
- There are Cohen-Macaulay but not constructible simplicial complexes. For this, triangulations of balls and spheres which are not constructible will be shown in Chapter 3. Also homology spheres which are not homeomorphic to spheres (e.g., Poincaré sphere) are Cohen-Macaulay but not constructible.
- There are shellable simplicial complexes that are not extendably shellable. Such examples are shown in Ziegler [98] and Ziegler [99]. He showed that the boundary complexes of almost all 4-polytopes are not extendably shellable while all the boundary complexes of polytopes are shellable.

- There are partitionable but not shellable simplicial complexes. For example, the following triangulation of the projective plane is not shellable (because it is not Cohen-Macaulay (over $\mathbb{Z}$)) but partitionable.



- There are simplicially collapsible but not shellable complexes.

In the case of pseudomanifolds, the first line of implications is refined as follows.

Proposition 2.30. *For pseudomanifolds, the following implications hold:*

- *vertex decomposable* $\Rightarrow$ *shellable* $\Rightarrow$ *constructible* $\Rightarrow$ *PL-balls or PL-spheres* $\Rightarrow$ *balls or spheres* $\Rightarrow$ *Cohen-Macaulay.*

∎

As same as above, these refined implications are also strict.

- There are PL-balls and PL-spheres which are not constructible. This will be shown in Chapter 3.
- There are balls and spheres which are not PL. The existence of non-PL spheres follows from Edwards' "double suspension theorem" [35] or its generalized version by Cannon [27].
- There are Cohen-Macaulay simplicial complexes which are not balls or spheres: homology spheres are Cohen-Macaulay.

2.5 The case of 2-dimensional pseudomanifolds

In the case of 2-dimensional pseudomanifolds, many good properties are known to hold which are never true in general. Among them, we see in this section that all the inverse implications of Proposition 2.30 holds for 2-dimensional pseudomanifolds, and that combinatorial decomposition properties are topological property.

We start from the most important proposition, which we will use many times throughout this thesis. This is a classical result.

Proposition 2.31. *All polytopal 2-balls and 2-spheres are shellable (thus constructible).*

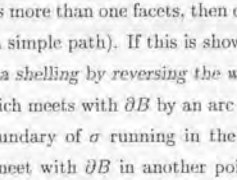
Proof. We show that all 2-balls are shellable. The case of 2-spheres follows immediately: Choose one facet σ of a 2-sphere and remove it, then the remained 2-ball (2-sphere minus 2-ball is a 2-ball) is shellable and its shelling extends to that of whole 2-sphere by just adding σ in the end of the shelling.

To show the shellability of a 2-ball, we construct its shelling in a reverse way. What we show in the sequel is that if B has more than one facets, then every 2-ball B has a facet σ such that $\overline{(B-\sigma)} \cap \sigma$ is an arc (i.e., a simple path). If this is shown, then we successively remove such σ from the ball and we get a shelling by reversing the way of this removal sequence.

For this, we find a facet σ which meets with ∂B by an arc and starting from one endpoint of the arc and following the boundary of σ running in the interior of B, either we reach another endpoint of the arc or meet with ∂B in another point. In the former case, we are done. In the latter case, the arc divide B into two balls and we use the following claim.

Claim. If a 2-ball $D \subset B$ has the property that $(\partial D - \partial B)$ is an arc k contained in the boundary of one facet of B, then D contains a facet τ such that $(\partial\tau - \partial B)$ is an arc.

The claim is shown by induction on the size of D. First, if D has only one facet, then the claim above is trivially true. If D has more than one facets, we choose one facet τ' of D such that τ' meets with ∂B by at least one arc. If $(\partial\tau' - \partial B)$ is one arc, then we are done. Else, it consists of at least two arcs, so we can take k' from one of the arcs which is different from k. Then k' divides B into two balls D_1 and D_2 such that D_1 is contained in D. Now, D_1 satisfies the condition of the claim and has smaller number of facets than D, and the induction hypothesis implies that D_1 (thus D) contains a facet τ with the property we need.



∎

The following proposition follows immediately from the claim in the proposition above. This proposition for the special case in simplicial 2-balls plays the key role in Chapter 4.

Proposition 2.32. *Let a simplicial 2-ball B have a spanning edge which divides B into two 2-balls B_1 and B_2. If B_i has no interior vertices, then it has a facet (2-simplex) with two edges in ∂B, for $i = 1, 2$.*

Proof. The claim in the proof of Proposition 2.31 assures the existence of a facet τ such that $(\overline{B_i - \tau}) \cap \tau$ is an arc. But by assumption, B_i has no interior vertices, which means that the arc $(\overline{B_i - \tau}) \cap \tau$ is a spanning edge, which means that τ has two edges in ∂B. ∎

Proposition 2.31 together with Corollary 2.17 concludes that shellability of 2-pseudomanifolds is determined by their topology, that is, a 2-pseudomanifold is shellable if and only if it is a 2-ball or a 2-sphere.

Corollary 2.33. *A 2-pseudomanifold is shellable if and only if it is a 2-ball or a 2-sphere.* ∎

Or equivalently:

Corollary 2.34. *A 2-pseudomanifold is constructible if and only if it is a 2-ball or a 2-sphere.* ∎

Thus we can conclude that:

Corollary 2.35. *A constructible 2-pseudomanifold is shellable.* ∎

Also Proposition 2.31 implies the following corollary.

Corollary 2.36. *All 2-balls and 2-spheres are extendably shellable.* ∎

For vertex decomposability, we also have the following proposition, shown by Provan and Billera [74].

Proposition 2.37. *All 2-balls and 2-spheres are vertex decomposable.*

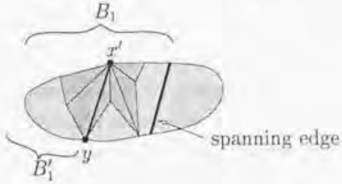
Proof. It is enough to show the case of 2-balls, because in the case of 2-spheres, every vertex x of a 2 sphere S can be taken as the first shedding vertex making the remaining $\mathrm{dl}_S\, x$ to be a 2-ball.

For a 2-ball B, we show in the following that there always exists a vertex x in ∂B such that no spanning edge is incident to x. If this is shown, such a vertex x becomes a shedding vertex by an induction argument because in this case $\mathrm{link}_B\, x$ is a connected 1-complex which is easily shown to be vertex decomposable, and $\mathrm{dl}_B\, x$ is a 2-ball and its vertex decomposability is shown by induction.

To show the existence of such a vertex x, we first choose a vertex x' arbitrary from ∂B. If this satisfies the condition, we are done. Else x' is incident to a spanning edge. For such a case, we show the following claim.

Claim. If a 2-ball B has a spanning edge which divides B into two 2-balls B_1 and B_2, then each divided two ball B_i has a vertex x in ∂B such that x is incident to no spanning edge.

The proof of the claim is by induction on the number of facets of B_i. If B_i has only one facet, then the statement is clear. If not, take a vertex x' on $B_i \cap \partial B$ different from the two endpoints of the spanning edge dividing B_1 and B_2. If x' is not incident with any spanning edge of B, then we are done. Else there is a spanning edge $x'y$. If we take this spanning edge as the first spanning edge, then it divides B into B_1' and B_2' such that B_1' is smaller than B_1 and B_1' has the vertex we are looking for by the induction hypothesis.



∎

The rest reverse implications of Proposition 2.31 are "spheres or balls ⇒ PL-spheres or balls" and "Cohen-Macaulay ⇒ spheres or balls," and these two implications also known to hold. The former follows from the fact that triangulated non-PL spheres do not exist in dimensions at most three. (This is open for dimension 4.) The latter follows from the classification of 2-surfaces: there is no 2-surfaces with $\tilde{H}_1 = 0$ except for balls and spheres.

In summary, we have the following theorem.

Theorem 2.38. *For 2-pseudomanifolds, vertex decomposability, extendable shellability, shellability, constructibility, being 2-balls or 2-spheres, being PL-2-balls or PL-2-spheres, and Cohen-Macaulayness, are all equivalent. Moreover, these properties are topological.* ∎

2.6 Knots and tangles

This section provides a brief introduction to knots and tangles. For further study, there are many textbooks on knots, for example, Lickorish [58], Livingston [59], Rolfsen [76], etc.

A *knot* is a simple closed tame arc contained in some 3-dimensional manifold (with boundary) M^3, where *tame* means that it is piecewise linear. In this thesis, we always treat the case where M^3 is a 3-ball B^3 or a 3-sphere S^3. There are several ways to define *knot equivalence* for example using Reidemeister moves, ambient isotopy, or homeomorphism of M^3. In this thesis, we use the most primitive way for the definition of knot equivalence as follows.

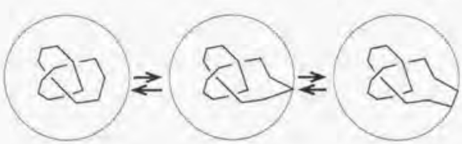
Definition 2.39. Let k and k' are two simple closed piecewise linear arcs. If $k = p_0p_1 \cdots p_ip_{i+1} \cdots p_tp_0$ and $k' = p_0p_1 \cdots p_iqp_{i+1} \cdots p_tp_0$ and the triangle p_iqp_{i+1} does not intersect with other part of the arc, then k and k' are related by an *elementary move*. If two knots are related by a sequence of elementary moves, then these two knots are *equivalent*. We also say that these two knots are of the same *type*, and the representative of the equivalence class is mentioned as a *knot type*.

It can be deduced from this definition that if the difference between two knots bounds a disk, then they are equivalent.

If the knot itself bounds a disk then it is trivial.

Definition 2.40. A knot k is *trivial* or is an *unknot* if there is a disc (2-ball) in M^3 whose boundary is k. If not, k is *knotted*.

The reason we use this definition is because we want to use the equivalence relation in a slightly generalized way than is used in usual contexts. That is, if $M^3 = B^3$, we allow some parts of the knot to go onto the boundary or into the interior during the sequence of elementary moves, while usually the knots are required to be in the interior of M^3 all the time.



The modifications in the above figure are not equivalent if we defined the equivalence relation by ambient isotopy or homeomorphism of M^3, but are equivalent by our definition. But we should note that this does not contain a radical change between the usual definition and our definition. In fact, all the properties and techniques are valid for our definition: what we should do to apply usual argument is just to move the arcs on the boundary into the interior by slightly perturbing the arc.

A *spanning arc* is a simple tame arc contained in a 3-ball B^3 with its two end points lying in the boundary. If this arc is made of one edge of a triangulation of B^3 and contained in the interior, then it is especially called a *spanning edge*. Again here we allow some part of spanning arcs to be contained in the boundary of B^3 while usual treatment requires the arc except for the endpoints to be contained in the interior, as same as the knot case above.

Let us imagine to join two endpoints of a spanning arc by a simple tame arc lying in the boundary of B^3 to get a knot in B^3. Now suppose we make two knots k and k' from one spanning arc in this way by joining differently in the boundary of B^3. Then what we have are two knots which are equivalent to each other. This fact can be easily shown from the fact that every simple closed curve in a 2-sphere S^2 is not knotted, i.e., bounds a disc. In our situation, two knots k and k' differs only in the part contained in the boundary of B^3. If the endpoints of the spanning arc are a and b and p_1 is a point in k where the segment from a to p_1 is common with k' but from p_1 to the next common point p_2 is different, then two different arcs from p_1 to p_2 together make a simple closed arc which bounds a disk. Then we can perform a sequence of elementary moves from the part of k' to that of k to reach the situation that the arcs from a to p_2 are the same. Repeating this procedure, we finally construct a sequence of elementary moves from k' to k.

This fact that we always get the same type knot no matter how we join two endpoints of a spanning arc enables us to define the knot type of spanning arcs as follows.

Definition 2.41. Two spanning arcs are *equivalent* if the knots derived by joining two endpoints of each spanning arc by a tame simple arc in the boundary of B^3 are equivalent. If the knot is trivial, then the spanning arc is *trivial* or *unknotted*, and otherwise *knotted*. The *type* of a spanning arc is the type of the knot derived from the spanning arc.

There is another way to define the equivalence of knotted spanning arcs using elementary moves as same as the case of knots. For this we use the following two types of elementary moves: we say that two spanning arcs l and l' are related by an *elementary move* if

(i) $l = p_0p_1 \cdots p_ip_{i+1} \cdots p_t$ and $l' = p_0p_1 \cdots p_iqp_{i+1} \cdots p_t$, and the triangle p_iqp_{i+1} does not intersect with other part of the arc, or

(ii) $l = p_0p_1 \cdots p_{t-1}p_t$ and $l = p_0p_1 \cdots p_{t-1}q$, and the triangle $p_{t-1}p_tq$ does not intersect with other part of the arc,

and two spanning arcs are *equivalent* if there is a sequence of elementary moves from l to l'.

That this way of definition is the same as Definition 2.41 is easily seen from the Definition 2.39 of equivalence of knots.

A *tangle* is a mutually disjoint set of knots and spanning arcs in a 3-ball B^3.

For tangles, we define the equivalence relation as before.

Definition 2.42. Two tangles t and t_1 are related by an *elementary move* if

(i) $t = p_0p_1 \cdots p_ip_{i+1} \cdots p_t$ and $t' = p_0p_1 \cdots p_iqp_{i+1} \cdots p_t$, and the triangle p_iqp_{i+1} does not intersect with other part of the tangle, or

(ii) $t = p_0p_1 \cdots p_{t-1}p_t$ and $t = p_0p_1 \cdots p_{t-1}q$, and the triangle $p_{t-1}p_tq$ does not intersect with other part of the tangle,

and two tangles are *equivalent* if there is a sequence of elementary moves from t to t'. The equivalence class is mentioned as the *type* of tangles.

By definition, knots and spanning arcs are special cases of tangles. To define triviality or knottedness, intuitively, we want to define a tangle is trivial when it is equivalent to a set of parallel unknotted spanning arcs and a set of unknots which are not linked. To define this precisely, we need the concept of a *semispanning disc* which is a disc in B^3 whose boundary consists of two arcs, one lies in the boundary of B^3 and one is a spanning arc of B^3. If there is a semispanning disc which has a spanning arc l on the boundary, then l is always trivial, and conversely if l is trivial, then there is a semispanning disc containing l on its boundary. So having a semispanning disc is equivalent to unknottedness. This situation is also said that the spanning arc is *straight*. (This term "straight" do not mean that it is straight geometrically.) For knots, we define *spanning discs* as same as the case of spanning arcs, that is, a spanning disc of a knot k is a disc whose boundary is k. Now we define triviality of tangles as follows.

Definition 2.43. A tangle is *trivial* if its spanning arcs and knots have semispanning discs or spanning discs which are mutually disjoint. Otherwise it is *tangled*.

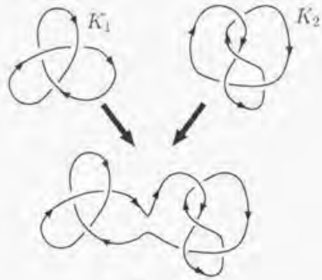
Especially, if a tangle made of only spanning arcs are trivial, they are called *simultaneously straight*.

Remark that the knottedness and triviality for knots or spanning arcs defined above is equivalent to this definition as special cases.

The concept of *links* is lying between that of knots and tangles, that is, a link is a set of tame simple closed arcs in B^3 (or M^3 in general). (Be careful that this "link" is different from the "link" defined on simplicial and polytopal complexes in Section 2.1!)

Borromean link

A simple way to construct complicated knots in S^3 is by taking the *connected sum* of knots. This operation is described in the following figure.



$K_1\#K_2$

Precise description is as follows: Given two knots K_1 and K_2 with fixed orientations embedded in S^3's, remove a small ball from each of both S^3's such that the intersection of the ball and the knot is a trivial spanning arc, resulting two balls with knotted spanning knots of types K_1 and K_2. Then join these two balls by their boundary such that the endpoints of the spanning arcs meets, to get an oriented knot $K_1\#K_2$ in a 3-sphere. It is known that this operation is well-defined, and it is associative, the unique identity is the unknot, and there is no inverse for a nontrivial knot. Moreover, any knot can be decomposed uniquely into *prime knots*, i.e., knots that are not the connected sums of any other two non-trivial knots.

Note that there can be different connected sums of K_1 and K_2 if we do not give orientations to the knots. So the connected sum is not well-defined without the orientation. But we will abuse this concept for non-oriented knots in the following. This will not cause us a trouble because giving different orientations only affect the orientation of each prime knot in the prime knot decomposition.

To see that this connected sum operation produces complicated knots, let us start from the simplest knot, the *trefoil knot* κ. We denote $\kappa^{\#n}$ to be the connected sum of n copies of κ, i.e., $\overbrace{\kappa\#\kappa\#\cdots\#\kappa}^{n \text{ times}}$.

Trefoil knot κ

$\kappa^{\#n}$

One way to measure the complexity of a knot is to see the fundamental group of the complement, i.e., of the space S^3 − (the regular neighbourhood of the knot). This group is called a *knot group*. In our case, the knot group of κ is representable by two generators, while $\kappa^{\#n}$ needs at least $n+1$ generators. (See Goodrick [41].) We will see the same situation for another complexity index, the bridge index, in Chapter 3. There the bridge index of the trefoil knot κ is 2, and that of $\kappa^{\#n}$ is $n+1$. (Although these two indices coincides in this case, they are different in general. The knot complement can be represented by generators of the size of the bridge index, but there is smaller representation in general.)

A generalization of knots and spanning arcs to the high dimensions is the concept of ball pairs and sphere pairs.

Definition 2.44. A *ball pair* is a pair (B_1, B_2) of a d-ball B_1 and a k-ball B_2 such that B_2 is embedded in B_1 and ∂B_2 is contained in ∂B_1. A *sphere pair* is a pair (S_1, S_2) of a d-sphere and a k-sphere such that S_2 is embedded in S_1.

The *standard ball pair* of dimensions d and k is the pair of $\Sigma^{d-k}\Delta_k$ and Δ_k, where Δ_k is the standard simplex and $\Sigma^{d-k}\Delta_k$ is its $(d-k)$-fold suspension, and the *standard sphere pair* is the pair of the boundaries of the standard ball pair of dimensions $d+1$ and $k+1$.

A ball pair or sphere pair is *pair!unknottedunknotted* if there is a homeomorphism to the standard ball pair or the standard sphere pair, and otherwise *pair!knottedknotted*.

The following is known.

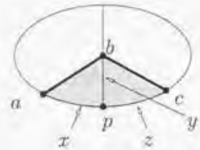
Proposition 2.45. *If there are two unknotted ball pairs (B_1, B_2) and (B'_1, B'_2) of dimensions d and k, the following holds.*

- *If these two unknotted ball pairs meet by a ball-pair (D_1, D_2) of dimensions $d-1$ and $k-1$ such that $D_1 \in \partial B_1 \cap \partial B'_1$ and $D_2 \in \partial B_2 \cap \partial B'_2$, then the ball pair $(B_1 \cup B'_1, B_2 \cup B'_2)$ is an unknotted ball pair.*
- *If these two unknotted ball pairs meet by a sphere-pair (S_1, S_2) of dimensions $d-1$ and $k-1$ such that $S_1 = \partial B_1 = \partial B'_1$ and $D_2 = \partial B_2 = B'_2$, then the sphere pair $(B_1 \cup B'_1, B_2 \cup B'_2)$ is an unknotted sphere pair.*

∎

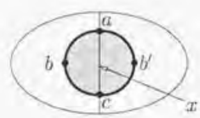
This proposition is shown, for example, in Zeeman [96, Lemmas 18 and 19]. Later in Sections 3.2 and 3.3 we will use this for the case of dimensions 3 and 1: the case of ordinary knots and spanning arcs. For such special cases, the proof is very easy. For the ball pair case, see the following figure. Let us assume that the arc ab in the left 3-ball C_1 and the arc bc

in the right 3-ball C_2 is not knotted and show that the arc abc is not knotted in the 3-ball $C = C_1 \cup C_2$.



For this, let p be a point on $\partial C \cap (C_1 \cap C_2)$, y an arc from b to p contained in $C_1 \cap C_2$, x an arc from a to p contained in $\partial C_1 \cap \partial C$, and z an arc from c to p contained in $\partial C_2 \cap \partial C$. Then x and y together form an arc in ∂C_1 which joins a and b. Because ab is an unknotted spanning arc of C_1, the closed arc ab-byp-pxa is a trivial knot, that is, it bounds a 2-ball. (Here we may assume that the arc byp is the only part of the 2-ball that is contained in ∂C_2.) Similarly bc-czp-pyb is a trivial knot that bounds a 2-ball. The union of the two 2-balls is again a 2-ball, and it proves that the knot ab-bc-czp-pxa, and hence the spanning arc ab-bc, are not knotted.

The sphere pair case is almost the same. See the following figure in which the spanning arc abc in the left 3-ball C_1 and the spanning arc $ab'c$ in the right 3-ball C_2 is unknotted, and we show that the knot $abcb'a$ is not knotted in the sphere $C = C_1 \cup C_2$. (The case that C is a 3-ball also works.)



Take an arc x in $C_1 \cap C_2$ from a to c. (This arc exists since $C_1 \cap C_2$ is a 2-ball or 2-sphere.) the closed curves abc-cxa and $ab'c$-cxa both bound 2-balls. These 2-balls intersect in the curve axc, and hence their union is a 2-ball bounded by the closed arc abc-$cb'a$, which shows that the knot abc-$cb'a$ is trivial.

Chapter 3

Knots and combinatorial decompositions

This chapter treats the case of 3- and higher dimensional pseudomanifolds, especially the case of triangulations of 3-balls and 3-spheres. The main problem considered here is to construct non-constructible triangulations of 3-balls and 3-spheres. Non-shellable triangulations of 3-balls and 3-spheres were known, but the case of non-constructible ones were open. For this we extend the well-known construction of non-shellable 3-balls and spheres using knots, and show many stronger results for combinatorial decompositions. (Another construction of non-constructible 3-balls, not using knots, will appear in Chapter 4.)

Most of the materials of this chapter are from a joint work with Günter M. Ziegler (Sections 3.2 through 3.4 and 3.8) and with Richard Ehrenborg (Sections 3.5 through 3.9).

In Sections 3.1 to 3.3, we discuss the existence of non-constructible 3-balls and 3-spheres starting from an extension of Furch's construction of non-shellable 3-balls: we show that 3-balls having a knotted spanning arc made of at most two edges are not constructible. We also show that non-constructible 3-spheres exist. This solves an open problem in Danaraj and Klee [32]. Non-constructible 3-spheres given in Section 3.3 are 3-spheres containing a knot made of three edges, which were shown to be non-shellable by Lickorish under some additional condition. Thus our result is a strengthening of his result. The existence of non-constructible 3-balls and 3-spheres are extended to the case of higher dimensions in Section 3.4. In Sections 3.5 to 3.7 we discuss extensions of the results of Sections 3.1 to 3.3 by introducing the bridge index of knots and tangles, and conditions which implies non-constructibility are given in terms of the bridge index and the size of knots or tangles contained in the triangulations. This also gives an answer to Hetyei's conjecture on shellability of certain cubical decompositions of spheres. Constructibility of cell partitionings is also discussed in Section 3.9. In Section 3.8, an analogue of the results of Sections 3.1 to 3.7 are given for vertex decomposability. Here conditions which imply non-vertex decomposability are given by the size of knots contained in the triangulations, as same as the case of constructibility but in a weakened way. Thus these results provide a hierarchy of combinatorial decomposition properties measured by the size of knots or knotted spanning arcs contained in the triangulations, which is summarized in the last Section 3.10.

3.1 Furch's knotted hole ball

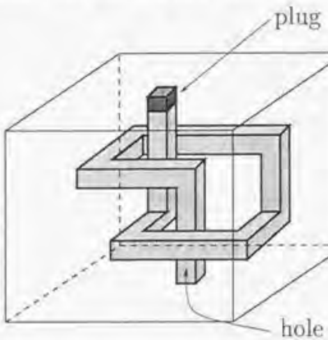
Historically, the first appeared example of a non-shellable triangulation of a 3-ball seems to be Furch's knotted hole ball. (Appears in Furch [38] and also described in Bing [10], Stillwell [88], Ziegler [98], and Ziegler [99].) As the name describes, it uses a special knot embedded in the triangulation to show its non-shellability. The construction of the triangulation is as follows:

(i) First triangulate a 3-ball finely enough.

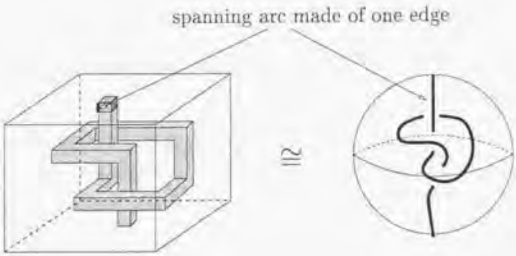
(ii) Starting from a facet which meets the boundary by a 2-face, dig a hole to another side making a knot in the interior of the original ball.

(iii) Stop digging just one step before corrupting the property that the object is a 3-ball, that is, leaving one interior edge to the exit to the opposite side.

This construction is sometimes described in the setting of "pile of cubes," that is, dig a knotted hole from the bottom face of a large pile of cubes to the upper face, and stop digging in the last step and leave one cube as a "plug" of the hole. If we triangulate each cube into 6 simplices, we get a non-shellable triangulated 3-ball. (The cubical complex before triangulation is also a non-shellable cubical 3-ball.) The following figure shows this construction.



The critical fact for the non-shellability of Furch's ball is that this ball has a knotted spanning arc made of one edge ("knotted spanning edge").



In the construction, the knot type of the knotted spanning edge is the type of the knot we chose for the knotted hole. Thus the type of the knotted spanning edge can be chosen arbitrary. Further, we can split the edge into n edges by stellar subdivisions without changing the type of the spanning arc. Thus we have the following proposition.

Proposition 3.1. *(Furch [38])*
Given a knot K and a natural number $n \geq 1$, we can construct a triangulated 3-ball which embeds a knotted spanning arc k, of the same type as K, as a 1-dimensional subcomplex made of n edges. ∎

Furch showed that such triangulated 3-balls with a knotted spanning edge is not shellable.

Theorem 3.2. *(Furch [38])*
A triangulated 3-ball which has a knotted spanning arc made of one edge is not shellable.

The proof is quite simple, shown by induction that every shellable triangulation has no such spanning arc, that is, a 3-simplex has no such spanning arc, and if $\bigcup_{j=1}^{i-1} \overline{F_j}$ has no such spanning arc in a shelling $F_1, \ldots, F_t$, then the next step $(\bigcup_{j=1}^{i-1} \overline{F_j}) \cup \overline{F_i}$ cannot have one, either.

The next section will give precise proof for this theorem in a stronger form.

3.2 Non-constructible 3-balls — 3-balls with a knotted spanning arc

The first generalization of Theorem 3.2 is to the case of constructibility.

Theorem 3.3. *A triangulated 3-ball which has a knotted spanning arc made of one edge is not constructible.*

Because non-constructible complexes are non-shellable, this theorem implies Theorem 3.2. This generalized theorem is shown in Hachimori [43], which firstly showed the existence of non-constructible triangulations of 3-balls. (In the paper, one more example is shown to be non-constructible, which appears in Section 4.4 in Chapter 4. Before the paper, a non-constructible regular CW complex was in Walkup [94], a non-shellable cell complex with only three facets.)

But it turns out that we can be more aggressive for the statement, that is, we have the following theorem in a more generalized way:

Theorem 3.4. *A triangulated 3-ball which has a knotted spanning arc made of at most two edges is not constructible.*

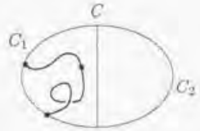
This theorem is shown in Hachimori and Ziegler [45], a joint work of Günter M. Ziegler and the author, and this is the key fact to show the main theorem of that paper: the existence of non-constructible triangulations of 3-spheres. (The paper also contains the materials in Section 3.8 and Section 4.1.)

From here to the next section, we will describe this striking result according to the method shown in the paper. The first step is to prove Theorem 3.4.

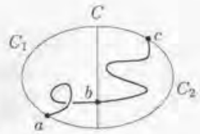
Proof of Theorem 3.4. We show by induction on the number of facets of C that in a constructible triangulation C of a 3-ball, a spanning arc that consists of only two edges ab and bc cannot be knotted. (We may assume that the arc in question has exactly two edges, since an arc consisting of a single edge can be extended by an edge on the boundary. Recall for this that we allow parts of spanning arcs to lie in the boundary of the ball. See Section 2.6.)

If C is a single simplex (tetrahedron), then the arc cannot be knotted. Otherwise C decomposes into two constructible complexes C_1 and C_2 as in Definition 2.11 of constructibility; both C_1 and C_2 are triangulated 3-balls by Proposition 2.16. There are two cases to be considered.

Case 1: The two edges ab and bc are both contained in C_1. They form a spanning arc ab-bc of C_1, which by induction cannot be knotted.



Case 2: One edge ab is contained in C_1 and the other one bc is contained in C_2. C_1 is constructible, so by induction ab is an unknotted spanning arc of C_1, and similarly for the arc bc in C_2.



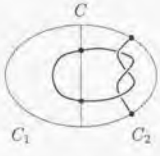
In this case the unknottedness of ab-bc is shown from Proposition 2.45.

∎

Thus we have the following corollary from Proposition 3.1.

Corollary 3.5. *There are non-constructible triangulations of 3-balls.*

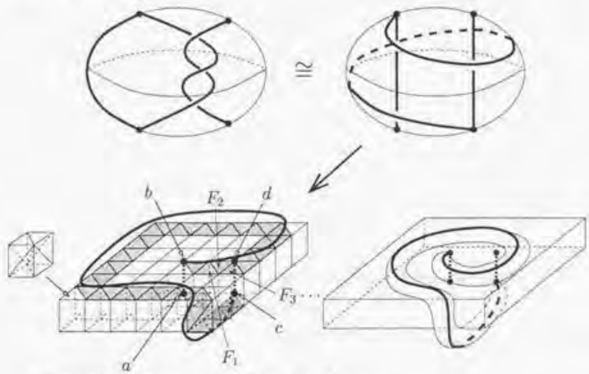
Remark that this theorem is sharp, that is, the existence of a knotted spanning edge with three edges will not lead to non-constructibility. The following figure shows why the proof fails for the case with three edges.



In the figure, the whole ball C has a knotted spanning arc made of three edges, but each subdivided balls C_1 and C_2 need not have one made of less than or equal to three edges. This observation is realized by the following "real" example which is shellable but has a knotted spanning arc made of three edges.

Example 3.6. (A shellable 3-ball with a knotted spanning arc consisting of 3 edges.)

Let C_1 be a pile of $6 \times 6 \times 1$ cubes in which each cube is split into 6 tetrahedra. Then $C := C_1 \cup (b * (\text{gray faces})) = C_1 \cup (b * F_1) \cup (b * F_2) \cup \cdots$ is a shellable 3-ball because C_1 is shellable, and the arc ab-bc-cd is a knotted spanning arc of the 3-ball as is indicated in the upper part of the figures.



$C = (\text{pile of cubes}) \cup (b * F_1) \cup (b * F_2) \cup \cdots$

3.3 Non-constructible 3-spheres — 3-spheres with a knot consisting of 3 edges

Using Theorem 3.4, we can now prove the theorem which shows the existence of non-constructible 3-spheres.

The theorem we will show in this section is that a 3-sphere is not constructible if it has a knot made of only three edges. To conclude from this the existence of non-constructible triangulations of 3-spheres, we need to show that triangulations which embeds a knot in such a way really exist. In fact, we can construct triangulations which embed a knot of any type consisting of any number of edges.

Proposition 3.7. *Given a knot K and a natural number $n \geq 3$, we can construct a triangulated 3-sphere or ball which embeds a knot k, of the same type as K, as a 1-dimensional subcomplex made of n edges.*

Proof. Such construction is well-known in combinatorial topology. We refer here Lickorish's paper [57], but the origin seems much older than it.

We first show the case of 3-spheres and $n = 3$. We prepare a triangulated 3-ball C which has a knotted spanning edge with endpoints a and b, for example by the Furch's "knotted hole" construction described in Section 3.1. Remark that the knot type of the spanning edge can be arbitrarily chosen: for example, if we make a knotted hole such that the hole has the same knot type as K, then we have a knotted spanning edge of the same type as K. Then we make a join over the boundary of C, that is, let $\tilde{C} = C \cup (\partial C * v)$ where v is a newly introduced vertex. The resulting $\tilde{C}$ is a triangulated 3-sphere because the operation of taking join over the boundary can be seen as joining two 3-balls by their boundaries (see Proposition 2.4), and the closed arc $k = a\text{-}b\text{-}v\text{-}a$ is a knot of the same type as K, made of just three edges.

If n is larger than 3, the required triangulation can be obtained by stellarly subdividing the 3-sphere with a knot made of three edges.

A 3-ball embedding such a knot can be obtained by removing one facet from a 3-sphere constructed above. ∎

Now the theorem.

Theorem 3.8. *If a triangulated 3-sphere or 3-ball has a knot made of three edges, then it is not constructible.*

Proof. We will show that in a constructible 3-ball or 3-sphere C every knot consisting of three edges (= "triangle") is trivial.

We use induction on the number of facets. The case of a simplex is clear. Otherwise the complex C can be divided into two constructible complexes C_1 and C_2. From Proposition 2.16

both C_1 and C_2 must be 3-balls. If one of them contains all the three edges of a triangle κ, then κ is trivial by induction. If not, then one of them, say C_1, has two edges ab and bc of κ, and the other one C_2 has the third edge ca of κ. Now ab-bc is a spanning arc of C_1 and ca is a spanning arc of C_2, and both spanning arcs are not knotted from Theorem 3.4. This implies that κ is trivial from Proposition 2.45 (or from the fact that the connected sum of two trivial knots is trivial).

∎

This theorem originally has a different proof, using another property of constructible 3-spheres described later in Section 4.1. This simplified version is brought to us from a comment of Robin Forman.

From Proposition 3.7 and Theorem 3.8, we have the following corollary.

Corollary 3.9. *There are non-constructible triangulations of 3-spheres.*

Danaraj and Klee [32] asks whether every 3-sphere is constructible or not. Thus the corollary above solves this open problem.

Theorem 3.8 generalizes the following theorem proved by Lickorish in two ways.

Theorem 3.10. *(Lickorish [57])*
If a triangulated 3-sphere (or 3-ball) has a knot made of three edges such that the fundamental group of the knot complement has no less than 4, then it is not shellable. ∎

Our Theorem 3.8 extends this theorem of Lickorish from the shellability case to the constructibility case, and also removes the complexity condition of the knot.

Lickorish himself mentions in his paper that the complexity condition cannot be removed from his theorem. His method fails for simple (= not complicated enough) knots, for example, a trefoil knot or a connected sum of two trefoil knots is not enough. On the other hand, our theorem guarantees non-constructibility, thus non-shellability, of such 3-spheres with a knot of *any* type. For instance, we conclude that the 3-sphere with a trefoil knot made of three edges, for which Lickorish's method does not work, is not shellable.

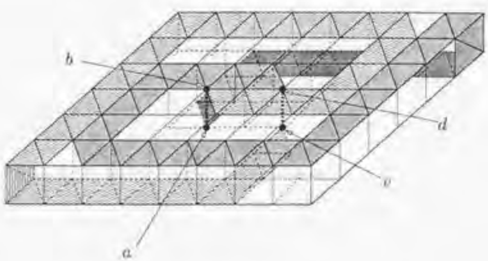
But we should remark that this *does not* mean that our method is more powerful than Lickorish's. In fact, Theorem 3.10 above is a corollary to his original theorem. The original statement is much stronger: it guarantees a property that any facet removal produces a 3-ball which is not simplicially collapsible from the assumptions. (The fact that shellable balls are simplicially collapsible implies the theorem from this.) Not to be simplicially collapsible is a very strong property (see the remark below), and this is why Theorem 3.10 needs the additional condition of knot complexity. The reason why our Theorem 3.8 does not need the complexity condition is because we attacked to constructibility, which is closer to shellability than simplicial collapsibility.

Remark. Not to be simplicially collapsible is stronger than not to be shellable. For example, Rudin's ball is not shellable but is simplicially collapsible because of the theorem of Chillingworth [28, 29], and also Lickorish and Martin [56] shows that a 3-ball with a knotted spanning edge can be simplicially collapsible. (This latter paper is the source of the comment of Lickorish that the complexity condition of the knot cannot be removed from Theorem 3.10.)

For constructibility, it is not known whether constructible 3-balls are always simplicially collapsible or not. But non-constructible but simplicially collapsible examples exist: simplicially collapsible 3-balls with a knotted spanning edge!

In concluding this section, we give an example which shows that the number "three" of the edges of the knot in Theorem 3.8 is sharp.

Example 3.11. (A shellable 3-ball and 3-sphere with a knot consisting of 4 edges.)
This example arises in the same line of construction as Example 3.6. Let C_1 be a pile of $8\times 6\times 1$ cubes in which each cube is split into 6 tetrahedra as before. Then the 3-ball $C_2 = C_1 \cup (b *$ (slashed faces minus the face incident to b)) $\cup$ ($d *$ (gray faces minus the face incident to d)) has a knot *ab-bc-cd-da*. This knot *ab-bc-cd-da* is not trivial because *ab-bc-cd* is a non-trivial knotted spanning arc. (It makes a trefoil knot.) Its shellability is easily seen as in Example 3.6. To get a 3-sphere with a knot consisting of 4 edges, we have only to take a cone over the boundary of C_2, that is, $C := C_2 \cup (v * \partial C_2)$. The shelling of C_2 can be trivially extended to that of C because ∂C_2 is shellable since it is a 2-sphere.



3.4 The case of higher dimensions

In the previous sections, we showed the existence of non-constructible 3-spheres and 3-balls. This directly implies the existence of non-constructible d-spheres and d-balls, for $d \geq 3$.

Corollary 3.12. *There exist non-constructible d-balls and d-spheres, for $d \geq 3$.*

Proof. If C is a constructible complex, then the link of any face of C is always constructible from Proposition 2.14. This shows the following immediately.

- If C is a non-constructible $(d-1)$-ball, then the pyramid over C is a non-constructible d-ball.
- If C is a non-constructible $(d-1)$-sphere, then the suspension ΣC is a non-constructible d-sphere.

These show the statement together with Corollaries 3.5 and 3.9. ∎

The constructions used in the above proof always produce PL-balls or PL-spheres from Proposition 2.6. (Note that every triangulated 3-balls are PL.) Thus what we showed is a stronger statement: there exist non-constructible PL-d-balls and PL-d-spheres in all dimensions $d \geq 3$.

It is known that there are non-PL spheres in dimensions $d \geq 5$: if H is a homology 3-sphere which is not homeomorphic to a 3-sphere, then its double suspension $\Sigma^2 H$ is homeomorphic to a 5-sphere. (This "double suspension theorem" is first shown by Edwards [35] for a certain type of homology sphere, and later generalized to any homology sphere by Cannon [27].) But $\Sigma^2 H$ is not PL. The non-PLness of $\Sigma^2 H$ can be seen from the fact that a sphere is a PL-sphere if and only if it is a combinatorial manifold, i.e., it has a triangulation with the property that every link of a vertex is a PL-sphere (Proposition 2.2). In $\Sigma^2 H$, the links of the two vertices used in the second suspension are ΣH which is not a PL-sphere. Early discussion about the non-PLness of $\Sigma^2 H$ can be found in Curtis and Zeeman [30].

From Proposition 2.16, we already know that constructible d-balls and d-spheres are always PL, so the existence of non-constructible triangulations of d-balls and d-spheres, for $d \geq 5$, were already known to us, independent to the theory developed in this chapter. But the cases of dimension 3 and 4, and the PL cases for all dimensions $d \geq 3$ are firstly shown thanks to our theory.

Remark. Recently Björner and Lutz [19] (also Lutz [61]) constructed a series of very compact triangulations of non-PL spheres, which has only $d + 13$ vertices for dimension d. This construction is based on their small (with 16 vertices, conjectured to be the smallest) triangulation of Poincaré sphere and "one-point-suspension". This example (18 vertices for 5-sphere) is currently the smallest non-constructible triangulation of spheres. On the other hand, what the author achieved from Theorem 3.8 is an example with 381 vertices and 1928 facets, though this one has an additional property to be PL and having lower dimension. (Probably we can slightly reduce the size than this, but far from the example of Björner and Lutz.)

3.5 Bridge index of knots and tangles (I)

As is shown by Examples 3.6 and 3.11, the numbers of edges in Theorems 3.4 and 3.8 are both sharp. But there is a possibility to extend them further by introducing a condition for the complexity of the knot. The idea of using the condition of the complexity of the knot is seen in several papers: for example, Armentrout [2] and Lickorish [57] for shellability, and Goodrick [41] for simplicial collapsibility (also see Bing's article [10]).

The measure we use is the bridge index (or the bridge number) of knots, and our goal is to show that if a 3-sphere C has a knot K with $b(K) > e(K)$, where $b(K)$ is the bridge index and $e(K)$ is the number of edges K is made of, then C is not constructible.

All the results we will give here from this section through the end of this chapter is taken from Ehrenborg and Hachimori [36], a joint work of Richard Ehrenborg and the author.

The bridge index is already used for the shellability of cell decompositions in Armentrout [2]. The idea to use this complexity index for our constructibility argument is brought by Günter M. Ziegler inspired from the fact that the knot in our previously given Example 3.11 is in a "2-bridge position".

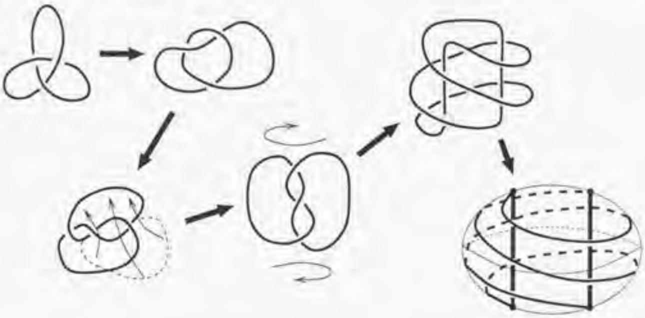
We start from reviewing the definition of bridge index for knots.

Definition 3.13. A knot K in B^3 is in an *n-bridge position* if it is the union of n simultaneously straight spanning arcs which are contained in the interior of B^3 and some other arcs contained in the boundary of B^3. The *bridge index* $b(K)$ of K is the minimum number m such that there is a knot κ in B^3 in an m-bridge position which is equivalent to K.

If a knot K is in a 3-sphere S^3, then take a 3-ball B^3 in S^3 which contains K, and define the bridge index with respect to B^3.

This bridge index is first introduced by Schubert [80] and many properties are discussed in it. There are several different definitions for bridge index, see for example Livingston [59], Rolfsen [76] or Adams [1], which are, of course, all equivalent. The definition we give here is the one used in Armentrout [2], Goodrick [41] and Lickorish and Martin [56].

The unknot is the unique knot with the bridge index 1. It is easy to check that the trefoil knot has the bridge index 2 from the following figure because it has a 2-bridge position embedding and it has larger bridge index than 1 since it is knotted. (There are many other knots with the bridge index 2.)



2-bridge position

Moreover, Schubert showed the following.

Proposition 3.14. *For two knots K_1 and K_2, $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$.* ∎

Thus, any positive number b has knots with their bridge indices equal to b. For example, $(b-1)$-fold connected sum of trefoils has the bridge index b. (0-fold connected sum is the unknot.)

If there is a knot in a constructible 3-sphere or a 3-ball and if we repeat the divisions according to the definition of constructibility, the knot will be decomposed into pieces of tangles. So what we have to discuss is the relation between the bridge index of the original knot and that of tangles that the knot will be decomposed into. For this, we should first extend the definition of the bridge index which can be used for tangles.

Definition 3.15. Let T be a tangle in a 3-ball B^3. The tangle T is in an *m-bridge position* if T is the union of simultaneously straight m spanning arcs in the interior of B^3 and some other simple arcs contained in the boundary of B^3. Every connected component is required to have at least one spanning arc, so a closed arc on the boundary or a simple arc which is realized by one arc on the boundary is prohibited. For a tangle T, we define the *bridge index* $b(T)$ as the minimum positive integer m such that there is a tangle τ in an m-bridge position and τ is equivalent to T.

If a tangle T is in a 3-sphere S^3, (in this case, T is a link) then we take a 3-ball B^3 in C which contains T and define its bridge index with respect to B^3.

A straight spanning arc has the bridge index 1 and a set of simultaneously straight t spanning arcs has the bridge index t. It is easy to see that this definition is the same as Definition 3.13 if the tangle happens to be a knot, so we will use this definition for bridge index from now on.

The key proposition is the following, a kind of subadditivity of the bridge index.

Proposition 3.16. *Let C be a 3-ball (respectively, 3-sphere) and C_1 and C_2 be 3-balls such that $C = C_1 \cup C_2$ and that $C_1 \cap C_2$ is a 2-ball (respectively, 2-sphere). Let T be a tangle of C, T_1 the intersection $T \cap C_1$, and T_2 the topological closure of $T - T_1$. (Hence T_1 and T_2 are tangles of C_1 and C_2, respectively.) Then we have*

$$b(T) \leq b(T_1) + b(T_2).$$

Proof. Consider first the case when C is a 3-sphere. It is possible to choose a 3-ball $C' \subseteq C$ such that T is contained in C', $C'_i = C' \cap C_i$ is a 3-ball for $i = 1, 2$, the tangle T_i is contained in C'_i for $i = 1, 2$ and $C'_1 \cap C'_2$ is a 2-ball in $C_1 \cap C_2$. Now when replacing C, C_1, C_2 by C', C'_1, C'_2 the bridge indices of T, T_1 and T_2 do not change. Hence we can assume that C is a 3-ball.

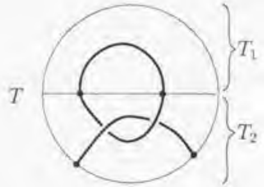
We will construct a tangle τ which is equivalent to the tangle T and is in a $(b(T_1)+b(T_2))$-bridge position. This will prove that $b(T) = b(\tau) \leq b(T_1) + b(T_2)$ which is the claim of the proposition.

The intersection $T_1 \cap T_2$ is a set P of points $\{p_1, p_2, \ldots, p_t\}$ in $C_1 \cap C_2$. Using some elementary deformations, we can assume that all the points of P are lying on the boundary of the disc $C_1 \cap C_2$.

In both balls, we optimize the positions of tangles to achieve the minimum number of the spanning arcs in both embeddings, i.e., we deform the tangle T_i by some sequence of elementary moves into τ_i such that τ_i is in a $b(T_i)$-bridge position in C_i, for $i = 1, 2$. Without loss of generality, we can assume that the endpoints in τ_i are not lying in $C_1 \cap C_2$. Let p'_{ij} be the endpoint of τ_i corresponding to the point p_j of T_i. Then we connect p_j and p'_{ij} by an arc on the boundary of C_i $(i = 1, 2)$ such that $\tau = \tau_1 \cup \tau_2 \cup \{p'_{1j} p_j p'_{2j}\}$ is equivalent to T. That such connection is possible can be easily checked step by step according to the elementary deformations from T_i to τ_i.

Now τ is a tangle in a $(b(T_1) + b(T_2))$-bridge position. Moreover τ is equivalent to T thus proving the desired inequality. ∎

We remark that the requirement in Definition 3.15 that every connected component must have at least one spanning arc is unavoidable in the proof of this proposition. Without it there may be cases that a spanning arc β in τ_2 should be realized by one simple arc on the boundary but an arc $p'_{i_j}p_j$ should cross the arc, making the construction in the proof, thus the statement, fail.



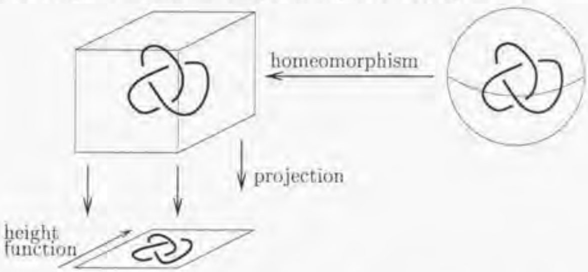
In this figure, $b(T_1) = 1$, $b(T_2) = 2$, and $b(T) = 2$, satisfying the subadditivity. If we do not require that every component has at least one spanning arc, then the bridge indices of T_1 and T_2 become 0 and that of T becomes 1, not satisfying the subadditivity.

3.6 Bridge index of knots and tangles (II)

This section present some other ways to define the bridge index, but this is an additional section which has no further use in this thesis. So the reader can skip this section, but the materials in this section may help the reader imagine what is the bridge index and why the subadditivity of Proposition 3.16 should hold.

In this section, we assume that knots and tangles are embedded in a 3-ball.

Another definition of the bridge index is by counting the local maxima in a projection of a knot into a plane (see for example Livingston [59]). Here, we mean a *projection* of a knot or a tangle the projection of a 3-cube onto a plane, where the cube is an image $f(B^3)$ of the 3-ball in which the knot or tangle is embedded, where f is a homeomorphism. (The term "projection of a tangle" indicates the composite of f and the projection.)



Here, if we choose another homeomorphism f' from B^3 to the same cube, then we get another projection. In the projection, we take a *height function* h along one of the edge of the square, and a *local maximum* of a tangle T is such a point p (in B^3) that p has a small neighborhood N in which $h(p) \geq h(x)$ for all $x \in T \cap N$. (But we do not say the endpoints of spanning arcs of tangles to be local maxima because they have only half-neighborhoods.)

Using this projection, the bridge index of a knot is defined as the minimum of the number of the local maxima of the knot in a projection, where the projection ranges over all the possible choices of the homeomorphism f.

An extension we propose for tangles from this version is the following:

Definition 3.17. The bridge index $b'(T)$ for a tangle T is

$$b'(T) = \min \#\{\text{local maxima of } T\} + \#\{\text{spanning arcs of } T\},$$

where min is taken over all possible projections.

We have one more version, in the same spirit as above, as follows:

Definition 3.18. The bridge index $b''(T)$ for a tangle T is

$$b''(T) = \min \#\{\text{local maxima of } T\},$$

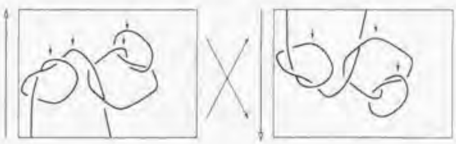
where min is taken over all possible projections in which all the endpoints of the spanning arcs of T lies on the bottom line.

The following two propositions show that the three different definitions of the bridge index are equivalent.

Proposition 3.19. *$b'(T) = b''(T)$ for any tangle T.*

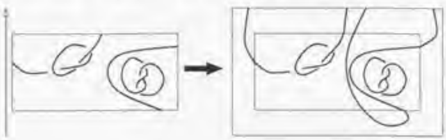
Proof. $[\, b'(T) \leq b''(T)\,]$

Let π be a projection with all endpoints on the bottom line. Let $-\pi$ be a projection, which reverses the height function of π. If the number of local maxima of T in π is $b''(T)$, i.e., π achieves the minimum number of local maxima, then the number of local maxima of T in $-\pi$ is $b''(T) - \#\{\text{spanning arcs of } T\}$. This follows from the fact that in π, $\#\{\text{local maxima}\} = \#\{\text{local minima}\}$ for closed cycles and $\#\{\text{local maxima}\} = \#\{\text{local minima}\} - 1$ for spanning arcs. The inequality follows.



$[\, b''(T) \geq b'(T)\,]$

Let π be a projection of T with $m(T)$ local maxima. We assume that π achieves the minimum, i.e., $b'(T) = m(T) + \#\{\text{spanning arcs of } T\}$. We make another projection π' of T from π as indicated in the following figure. (Extend the spanning arcs in the projection, without increasing the number of local maxima, such that the endpoints of spanning arcs ends in the upper edge of the big square.)



Now $-\pi'$ is a projection satisfying the condition that all the endpoints of spanning arcs lie on the bottom edge, and the number of local maxima of T with the height function $-\pi'$ equals to $m(T) + \#\{\text{spanning arcs of } T\} = b'(T)$, and the inequality follows.

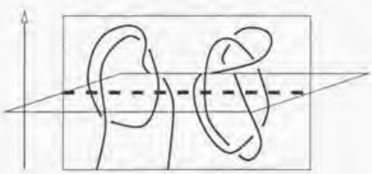
∎

Proposition 3.20. $b''(T) = b(T)$ *for any tangle* T.

Proof. We use the following fact: if a projection of a tangle consists of a set of arcs (without closed cycles, can be intersecting) such that each of the arc has both endpoints at the bottom and has only one local maximum, then there is an embedding of the tangle made of simultaneously straight spanning arcs. And also a set of simultaneously straight spanning arcs can be embedded into a plane without intersection. In the reverse direction, a set of arcs embedded in a plane can be embedded as a set of simultaneously straight spanning arcs, and a set of simultaneously straight spanning arcs has a projection with endpoints at the bottom in which each arc has only one local maximum.

[$b''(T) \geq b(T)$]

For a given projection π of T whose endpoints are at the bottom, we can modify the projection such that it has a horizontal cutting plane so that the tangle above the plane consists of only arcs with one local maximum, and below the plane consists of only arcs with one (or zero) local minimum, as the following figure.



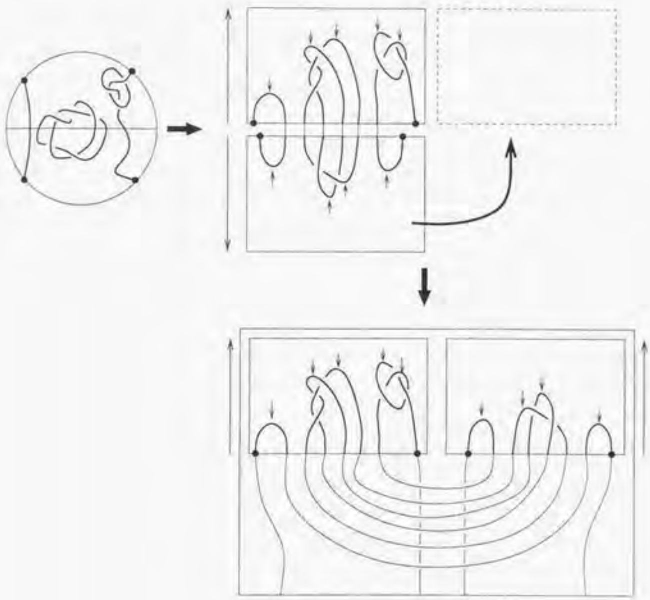
From this, we can construct an embedding of T into a 3-ball which is in a $b''(T)$-bridge position. To do this, we just embed the upper part in the interior of the ball as a set of simultaneously straight spanning arcs, and the lower part on the boundary. (Note that this is always possible. This is because for the embedding of the upper part into the ball, the places of endpoints on the boundary of the ball can be chosen arbitrary.) In this embedding, the number of spanning arcs equals to the number of local maxima in the original projection.

[$b(T) \geq b''(T)$]

From an embedding of T in a $b(T)$-bridge position, write a projection as the figure above such that the upper part is the set of spanning arcs in the interior of the ball and the lower part is the set of arcs on the boundary of the ball. Then the number of local maxima in the projection is $b(T)$.

∎

The way of defining the bridge index by b'' provides a different proof of Proposition 3.16 as the following figure indicates.



3.7 Non-constructibility and the bridge index

Proposition 3.16 provides the promised theorem as follows. Instead of showing for simplicial balls or spheres, we show here for polytopal balls or spheres. This change is not radical (in fact, it is easy to see that Theorems shown in Section 3.2 and 3.3 holds for polytopal cases without any change of discussion), but we state the theorem for polytopal balls and spheres in order to use it for Hetyei's conjecture below.

Theorem 3.21. *Let C be a 3-dimensional polytopal ball or sphere which is constructible. Let T be a tangle contained in the 1-skeleton of the polytopal complex C. Then we have the inequality*

$$b(T) \leq e(T),$$

where $b(T)$ is the bridge index of T and $e(T)$ is the number of edges of T.

Proof. The proof is by induction on C. The induction basis is when C is a 3-dimensional polytope. Then T is a disjoint union of straight spanning arcs and unknots. Let k be the number of components of T. Then $b(T) = k \leq e(T)$, and the induction base is complete.

The induction step is as follows. If C is not a simplex, we have two 3-dimensional complexes C_1 and C_2 satisfying the condition (ii) of Definition 2.11, and from Proposition 2.16 they are 3-balls and $C_1 \cap C_2$ is a 2-ball or sphere. Let $T_1 = T \cap C_1$ and $T_2 = \overline{T - T_1}$. By Proposition 3.16 and the induction hypothesis we obtain

$$b(T) \leq b(T_1) + b(T_2) \leq e(T_1) + e(T_2) = e(T).$$

This completes the induction. ∎

Corollary 3.22. *Let C be a 3-dimensional polytopal ball or sphere. Assume that the 1-skeleton of the complex C contains a knot K such that*

$$e(K) \leq b(K) - 1.$$

Then the polytopal complex C is non-constructible. ∎

One consequence from Corollary 3.22 is the following theorem.

Theorem 3.23. *Given a non-negative integer n there exists a triangulation C of a 3-dimensional sphere or ball such that the n-fold barycentric subdivision $\mathrm{sd}^n(C)$ is non-constructible.*

Proof. Choose a knot K with bridge index larger than or equal to $3 \cdot 2^n + 1$. Let C be a triangulation of a 3-dimensional sphere or ball that contains K on three edges. Such a triangulation can be constructed from Proposition 3.7. Because taking the barycentric subdivision divides each edge into two edges, the knot K contained in $\mathrm{sd}^n(C)$ consists of $3 \cdot 2^n$ edges. From Corollary 3.22, it follows now that the complex $\mathrm{sd}^n(C)$ is non-constructible.

For the case of 3-balls, we can reduce the complexity. For this, we use Proposition 3.1 to make a triangulated 3-ball with a spanning arc made of one edge, then apply Theorem 3.21. The spanning arc in $\mathrm{sd}^n(C)$ is made of 2^n edges, so if the knotted spanning arc has the bridge index at least $2^n + 1$, then the $\mathrm{sd}^n(C)$ is not constructible. ∎

Similar statement for shellability is already shown in Kearton-Lickorish [52] or Lickorish [57]. In fact, the proof of Theorem 3.10 in Lickorish [57] showed the following strong statement.

Theorem 3.10′. *If a triangulated 3-sphere has a knot made of e edges such that the fundamental group of the knot complement has no less than $e+1$, then any removal of one facet gives a 3-ball which is not simplicially collapsible. Thus the triangulation is not shellable.* ∎

But the results for simplicial collapsibility can not be used for constructibility because it is not known whether constructible simplicial balls are always simplicially collapsible or not.

One more application of Corollary 3.22 is the following conjecture of Gábor Hetyei.

Conjecture 3.24 (Hetyei [46, 47]). There exist non-shellable triangulations of d-balls whose cubical barycentric subdivision is again non-shellable.

Here, the *cubical barycentric subdivision* of a simplicial complex C is the abstract cubical complex $\square(C)$ such that

(i) the set of vertices of $\square(C)$ is the set of non-empty faces of C, and

(ii) a face of the cubical complex $\square(C)$ is an interval of the face poset of C.

It is straightforward to see that the cubical barycentric subdivision $\square(C)$ is a cubical complex and that $\square(C)$ is a subdivision of the simplicial complex C. Hence the simplicial complex C and its cubical barycentric subdivision $\square(C)$ have the same geometrical realization. For an example of cubical barycentric subdivision, see the following figure.

Now we can give the affirmative answer to Hetyei's Conjecture 3.24 from Corollary 3.22. Before this, Margaret Readdy already settled this conjecture for dimensions $d \geq 4$: every suspension of a non-shellable sphere satisfies the condition. So the newly derived fact is the remained three-dimensional case. Our solution is also given in a stronger form, for constructibility. In the following proof, the last part for the cases of $d \geq 4$ is essentially the same as her argument.

Theorem 3.25. *Let d be greater than or equal to 3. Then there exists a d-dimensional simplicial PL-sphere C_d such that the cubical barycentric subdivision $\square(C_d)$ is non-constructible.*

Proof. Consider first the case when d is equal to 3. Choose a knot K with bridge index larger than or equal to 7 and let C_3 be a simplicial complex that contains the knot K on three edges. Observe that the complex C_3 is non-constructible. By the same argument as in Theorem 3.23, the cubical complex $\square(C_3)$ is non-constructible.

The remaining part of the proof is by induction on dimension. Let C_d be the suspension of C_{d-1}, that is, $C_d = C_{d-1} \cup (u * C_{d-1}) \cup (v * C_{d-1})$, where u and v are newly introduced vertices. Then we have that $\mathrm{link}_{C_d}(v) = C_{d-1}$, and hence C_d is non-constructible. Observe that $\mathrm{link}_{\square(C_d)}(v) = C_{d-1}$, and hence $\square(C_d)$ is also non-constructible from Proposition 2.14. ∎

Remark. We can also prove Hetyei's conjecture directly from Lickorish's Theorem 3.10′ for non-shellable spheres as follows. Let C be a 3-sphere which has a knot K consisting of three edges. Let us take stellar subdivisions by all the 3-faces of $\square(C)$ and then take stellar subdivisions by all the 2-faces of $\square(C)$ to get a subdivision C' of $\square(C)$. Then we can show that C' is shellable if $\square(C)$ is shellable. Now C' and $\square(C)$ has the same 1-skeleton, especially K consists of 6 edges in C'. From Theorem 3.10′, if the knot complement of a knot made of e edges has no representation by less than e generators in triangulated S^3, the triangulation is not shellable. So we conclude that $\square(C)$ is not shellable if the minimum representation of the knot complement of K needs 7 generators.

But this method can not be used for constructibility.

3.8 Vertex decomposability, shellability and the bridge index

The results shown from Sections 3.2 to 3.7 has an analogue for vertex decomposability, which we will show in this section. (The first half of this section is from Hachimori and Ziegler [45] and the latter part is from Ehrenborg and Hachimori [36], also analogous to the previous sections.)

Theorem 3.26. *If a 3-ball C has a knotted spanning arc consisting of at most 3 edges, then C is not vertex decomposable.*

Proof. We show that every knotted spanning arc consisting of at most 3 edges are not knotted in a vertex decomposable 3-ball C.

First we observe that if x is a shedding vertex of a vertex decomposable d-ball, then x lies in the boundary. Furthermore, every vertex y adjacent to x is either in the interior of C, or the edge xy is contained in the boundary of C. This is because the deletion $\mathrm{dl}_C\, x$ must be a 3-ball, and the link of x is a 2-ball.

Again we use induction on the number of facets. If the spanning arc is made of 1 or 2 edges, then it is not knotted by Theorem 3.4. So we can assume that the spanning arc is made of 3 edges, where the first and last edge do not lie in the boundary of the ball. Thus if the arc is *ab-bc-cd*, the edges *ab* and *cd* lie in the interior of C. In particular, b and c are not shedding vertices.

The vertex a also cannot be a shedding vertex: otherwise *bc-cd* is a 2-edge knotted spanning arc in the 3-ball $\mathrm{dl}_C\, a$ (to verify this we use an argument as in the proof of Theorem 3.4), and thus $\mathrm{dl}_C\, a$ is not constructible (not even shellable) by Theorem 3.4. Similarly d cannot be a shedding vertex.

Thus x must be taken different from $\{a, b, c, d\}$. In this case, however, $\mathrm{dl}_C\, x$ has a knotted spanning arc with 3 edges and has a smaller number of facets than C, contradicting the induction hypothesis. ∎

For example, we can observe (directly from the figure) that the shellable 3-ball shown in Example 3.6 is not vertex decomposable.

As same as Theorem 3.4, the number "3" of edges in the knotted spanning arc is best possible, because there are vertex decomposable 3-balls that have a knotted spanning arc with 4 edges.

Example 3.27. (A vertex decomposable 3-ball with a knotted spanning arc made of 4 edges.) In the figure of Example 3.6, $C' = C_1 \cup (v * (\text{gray faces}))$, where v is a newly introduced vertex, has a knotted spanning arc *ab-bv-vc-cd* with 4 edges. This 3-ball C' is vertex decomposable. (One can take v as the first shedding vertex.)

As in the case of constructibility in Section 3.3, from Theorem 3.26 we get a result for knots in vertex decomposable 3-spheres resp. 3-balls.

Theorem 3.28. *If a 3-sphere or a 3-ball C has a knot which consists of at most 5 edges, then C is not vertex decomposable.*

Proof. We use Theorem 3.26 and induction on the number of facets.

If C is a simplex, the statement obviously holds. Let C be vertex decomposable, let x be a shedding vertex of C and let κ be a knot with at most 5 edges. If x is a vertex of κ, then $\mathrm{dl}_C\, x$ has a knotted spanning arc with at most 3 edges, contradicting to Theorem 3.26. Otherwise $\mathrm{dl}_C\, x$ has a knot κ with at most 5 edges, contradicting to the induction hypothesis. ∎

The number of edges in this theorem is again best possible, as is shown in the following example.

Example 3.29. (A vertex decomposable 3-ball and 3-sphere with a knot consisting of 6 edges.) In the figure of Example 3.11, $C_2'' = C_1 \cup (v * \text{(slashed faces)}) \cup (w * \text{(gray faces)}$, where v and w are newly introduced vertices, has a knot *ab-bv-vc-cd-dw-wa* with 6 edges, and this 3-ball is vertex decomposable. From this 3-ball, we can construct a vertex decomposable 3-sphere by taking a cone over its boundary, namely, $C' = C_2' \cup (u * \partial C_2')$.

For the bridge index version, we provide improved bounds for both of shellability and vertex decomposability cases.

What we use is the following lemma on the bridge index of tangles.

Lemma 3.30. *Let C be a 3-ball and T be a tangle in C, and let $C_1 \cup C_2 = C$ and $T_1 \cup T_2 = T$ be the decomposition assumed in Proposition 3.16. If $b(T_2) = 1$, then*

(i) *if $T_1 \cap T_2$ consists of two points, then $b(T) \leq b(T_1)$.*

(ii) *if $T_1 \cap T_2$ is one point, then $b(T) = b(T_1)$.*

(iii) *if $T_1 \cap T_2 = \emptyset$, then $b(T) = b(T_1) + 1$.* ∎

The proof is almost trivial, so we omit describing it.

In the case of shellable simplicial complexes, we have the following theorem.

Theorem 3.31. *Let C be a 3-dimensional simplicial ball or sphere which is shellable. Let K be a knot contained in the 1-skeleton of the simplicial complex C. Then we have the inequality*

$$2 \cdot b(K) \leq e(K).$$

Proof. We may assume that K is not the unknot.

If C is shellable, there is a shelling $F_1, F_2, \dots, F_n$ such that $(F_1 \cup \cdots \cup F_{j-1}) \cap F_j$ is a shellable 2-complex on ∂F_j. Especially, $F_1 \cup \cdots \cup F_{j-1}$ and F_j are 3-balls and $(F_1 \cup \cdots \cup F_{j-1}) \cap F_j$ is a 2-ball, for $2 \leq j \leq n$.

Let $C_1^{(n+1)} = C$, $C_1^{(i)} = F_1 \cup \cdots \cup F_{i-1}$, and $C_2^{(i)} = F_i$. Let $T_1^{(n+1)} = K$, $T_1^{(i)} = T_1^{(i+1)} \cap C_1^{(i)}$, and $T_2^{(i)} = \overline{T_1^{(i+1)} - T_1^{(i)}}$. ($T_1^{(1)} = \emptyset$.) Note that $C_1^{(i+1)} = C_1^{(i)} \cup C_2^{(i)}$ and $T_1^{(i+1)} = T_1^{(i)} \cup T_2^{(i)}$ are decompositions described in Proposition 3.16.

Because $C_2^{(i)}$ is a 3-simplex and $C_1^{(i)} \cap C_2^{(i)}$ is a pure 2-subcomplex on its boundary, the possible cases are the following.

(1) $T_2^{(i)}$ in $C_2^{(i)}$ is an arc made of two edges and $T_1^{(i)} \cap T_2^{(i)}$ consists of two points.

(2) $T_2^{(i)}$ in $C_2^{(i)}$ is an arc made of two edges and $T_1^{(i)} \cap T_2^{(i)}$ is one point.

(3) $T_2^{(i)}$ in $C_2^{(i)}$ is an arc made of two edges and $T_1^{(i)} \cap T_2^{(i)}$ is empty.

(4) $T_2^{(i)}$ in $C_2^{(i)}$ is an edge and $T_1^{(i)} \cap T_2^{(i)}$ consists of two points.

(5) $T_2^{(i)}$ in $C_2^{(i)}$ is an edge and $T_1^{(i)} \cap T_2^{(i)}$ is one point.

(6) $T_2^{(i)}$ in $C_2^{(i)}$ is an edge and $T_1^{(i)} \cap T_2^{(i)}$ is empty.

(7) $T_2^{(i)}$ in $C_2^{(i)}$ is empty.

(8) $T_2^{(i)}$ in $C_2^{(i)}$ is made of two disjoint edges. (This only occurs when $i = 1$.)

(9) $T_2^{(i)}$ in $C_2^{(i)}$ is an arc made of three edges. (This only occurs when $i = 1$.)

Ranging i from 1 to n, we denote by n_k the number of i's such that i-th step is of type (k).

For the types from (1) to (6), because $T_2^{(i)}$ is a trivial spanning arc, $b(T_2^{(i)}) = 1$. So Lemma 3.30 shows that the types (3) and (6) increase the bridge index by one and others do not, when increasing the index i from 1 to n. In the cases (8) and (9), they increase the bridge index by two and one, respectively. Thus we have

$$b(K) \leq n_3 + n_6 + 2 \cdot n_8 + n_9.$$

On the other hand, the types (1) and (4) decrease the Euler characteristic of the tangle by one, the types (3), (6) and (9) increase by one, the type (8) increase by two, and others make no change. Thus we have

$$n_1 + n_4 = n_3 + n_6 + 2 \cdot n_8 + n_9,$$

because both $T_1^{(n+1)} = K$ and $T_1^{(1)} = \emptyset$ have the Euler characteristic 0.

Hence we have

$$\begin{aligned}
e(K) &= 2 \cdot (n_1 + n_2 + n_3) + n_4 + n_5 + n_6 + 2 \cdot n_8 + 3 \cdot n_9 \\
&\geq n_1 + n_3 + n_4 + n_6 + 2 \cdot n_8 + n_9 \\
&= 2 \cdot (n_3 + n_6 + 2 \cdot n_8 + n_9) \\
&\geq 2 \cdot b(K).
\end{aligned}$$

∎

(Note that the proof of the following theorem is only valid for the case of simplicial complexes, not for general polytopal complexes. For the polytopal cases, see Theorem 3.31′ in page 74.)

We can show the following theorem for the case of vertex decomposability in a similar way.

Theorem 3.32. *Let C be a 3-dimensional simplicial ball or sphere which is vertex decomposable. Let K be a knot contained in the 1-skeleton of the simplicial complex C. Then we have the inequality*

$$3 \cdot b(K) \leq e(K).$$

Proof. If C is vertex decomposable, there is a sequence of shedding vertices $x_n, x_{n-1}, \ldots, x_1$ of C. Let $C_1^{(n+1)} = C$, $C_1^{(i)} = \mathrm{dl}_{C_1^{(i+1)}} x_i$, and $C_2^{(i)} = x_i * \mathrm{link}_{C_1^{(i+1)}}(x_i)$. Let $T_1^{(n+1)} = K$, $T_1^{(i)} = T_1^{(i+1)} \cap C_1^{(i)}$, and $T_2^{(i)} = \overline{T_1^{(i+1)} - T_1^{(i)}}$. ($T_1^{(1)} = \emptyset$.) Observe that $C_1^{(i+1)} = C_1^{(i)} \cup C_2^{(i)}$ and $T_1^{(i+1)} = T_1^{(i)} \cup T_2^{(i)}$ are decompositions described in Proposition 3.16.

By considering the fact that $C_2^{(i)}$ is a star with the center vertex x_i, we observe that there are types of the decomposition as described in the proof of Theorem 3.31. But this time, the type (4) does not occur, that is, $n_4 = 0$.

For the cases from (1) to (6), because $C_2^{(i)}$ is a vertex decomposable 3-ball, Theorem 3.26 shows that $T_2^{(i)}$ is a trivial spanning arc, hence $b(T_2^{(i)}) = 1$. So Lemma 3.30 shows that types (3) and (6) increase the bridge index by one and others do not, when increasing i from 1 to n. For the cases (8) and (9), they increase the bridge index by two and one, respectively. Thus we have

$$b(K) \leq n_3 + n_6 + 2 \cdot n_8 + n_9.$$

On the other hand, the type (1) decreases the Euler characteristic of the tangle by one, the types (3), (6) and (9) increase by one, the type (8) increase by two, and others make no change. Thus we have

$$n_1 = n_3 + n_6 + 2 \cdot n_8 + n_9,$$

because both $T_1^{(n+1)} = K$ and $T_1^{(1)} = \emptyset$ have the Euler characteristic 0.

Hence we have

$$\begin{aligned} e(K) &= 2 \cdot (n_1 + n_2 + n_3) + 1 \cdot (n_5 + n_6) + 2 \cdot n_8 + 3 \cdot n_9 \\ &\geq 2 \cdot n_1 + n_3 + n_6 + 2 \cdot n_8 + n_9 \\ &= 3 \cdot (n_3 + n_6 + 2 \cdot n_8 + n_9) \\ &\geq b(K). \end{aligned}$$

■

Theorem 3.31 is stated for simplicial decompositions, not for polytopal cases. This is because Lemma 3.30 allows us to add only one arc in one step. To overcome this restriction to state the theorem for polytopal decompositions, some generalization of the lemma is needed. The key for such a generalization is a strengthening of simultaneous straightness, introduced by Richard Ehrenborg in our joint work. The definition is as follows: Let a set of spanning arcs is in a 3-ball C and B is a 2-ball in the boundary ∂C of C. Then the spanning arcs are *simultaneously straight with respect to* B if the arcs have mutually disjoint semispanning discs each of which avoids the interior of B.

The following lemma is a generalization of Lemma 3.30.

Lemma 3.30′. *Let $C = C_1 \cup C_2$ and $T = T_1 \cup T_2$ be the decomposition of a 3-ball and a tangle as in Proposition 3.16. Moreover we assume that T_2 is simultaneously straight with respect to $C_1 \cap C_2$. Assume that T_2 have*

- *number a of arcs each of which intersects with T_1 in two points,*
- *number b of arcs each of which intersects with T_1 in one point, and*
- *number c of arcs each of which intersects with T_1 in zero points.*

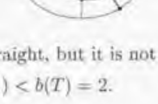
If T_2 is simultaneously straight with respect to $C_1 \cap C_2$, then we have

$$b(T) \leq b(T_1) + c.$$

Proof. Because T_2 is simultaneously straight with respect to $C_1 \cap C_2$, the arcs of T_2 have mutually disjoint semispanning discs avoiding the interior of $C_1 \cap C_2$. Along these semispanning discs, we can move the arcs onto $\partial C_2 \setminus C_1$ by elementary moves. Thus we can assume without loss of generality that the arcs of T_2 are all on the boundary of C.

Now take a tubular neighborhood $N(k_i)$ for each arc k_i of T_2. If we take the neighborhoods small enough, then they are mutually disjoint and also disjoint from the arcs of T_1. Define $C^\circ = \overline{C - \bigcup N(k_i)}$ and consider to add $N(k_i)$ one by one to C°. It is easy to observe that each step satisfies the condition of Lemma 3.30, and the inequality follows. ∎

Note that this lemma is not valid for the case T_2 is just simultaneously straight. For this, see the following figure.



In this figure, T_2 is simultaneously straight, but it is not simultaneously straight with respect to $C_1 \cap C_2$. In this example, $1 = b(T_1) < b(T) = 2$.

Now the generalized theorem.

Theorem 3.31′. *Let C be a 3-dimensional polytopal ball or sphere which is shellable. Let K be a knot contained in the 1-skeleton of the simplicial complex C. Then we have the inequality*

$$2 \cdot b(K) \leq e(K).$$

Proof. We may assume that K is not the unknot. Since C is shellable there is an ordering of the facets $F_1, F_2, \ldots, F_n$ (i.e., a shelling) such that $(F_1 \cup \cdots \cup F_{j-1}) \cap F_j$ is a shellable 2-complex on ∂F_j.

Set $C_1^{(i)}$, $C_2^{(i)}$, $T_1^{(i)}$ and $T_2^{(i)}$ as in the proof of Theorem 3.31. Observe that $T_2^{(i)}$ is in $\partial C_2^{(i)} \setminus C_1^{(i)}$. This assures that $T_2^{(i)}$ is simultaneously straight with respect to $C_1^{(i)} \cap C_2^{(i)}$, that is, the condition of Lemma 3.30′ is satisfied for each i. Let a_i, b_i and c_i be the number of arcs of $T_2^{(i)}$ described in Lemma 3.30′. Then the lemma shows that

$$b(T_1^{(i+1)}) \leq b(T_1^{(i)}) + c_i.$$

Because $b(T_1^{(1)}) = b(\emptyset) = 0$, we have

$$b(K) \leq \sum_{i=1}^{n} c_i.$$

On the other hand, since the Euler characteristic of the tangle increases by $c_i - a_i$ as i increases, and both $T_1^{(1)} = \emptyset$ and $T_1^{(n+1)} = K$ have Euler characteristic 0, we have

$$\sum_{i=1}^{n} (c_i - a_i) = 0.$$

Hence we have

$$e(K) \geq \sum_{i=1}^{n} (a_i + b_i + c_i) \geq \sum_{i=1}^{n} (a_i + c_i) = 2 \cdot \sum_{i=1}^{n} c_i \geq 2 \cdot b(K).$$

∎

3.9 Compatible and weakly compatible knots

The method we used for Theorem 3.21 can be used for the dual setting, in the same setting as Armentrout's paper [2]. What he discussed is the relation between shellability of a simple cell partitioning and knots contained in it in general position, i.e., knots intersecting with only 3- and 2-cells and the intersection with 2-cells are disjoint union of points. To describe his results, we need the definitions of compatibility of knots.

Definition 3.33. A knot K is *compatible* with a cell partitioning C if each 3-cell in C intersects with K by an empty set or one segment.

A knot K is *weakly compatible* with C if each 3-cell in C intersects with K by an empty set or a simultaneously straight spanning arcs of the cell.

Armentrout's results are the following relations between the bridge index of the knot K contained in a simple cell partitioning C in a general position and the number $p(K)$ of segments which K is decomposed into by the partitioning C.

- If K is compatible with C and $b(K) > 2p(K)$, then C is not shellable. (Theorem 1 of [2])
- If K is weakly compatible with C and $b(K) > p(K)$, then C is not shellable. (Theorem 3 of [2])

The following theorem extends the latter one into constructibility. The proof is essentially the same as Theorem 3.21.
(Constructibility of cell partitioning can be defined that C is constructible if (i) C has only one 3-cell or (ii) there are two constructible parts C_1 and C_2 such that $C_1 \cap C_2$ is a $(d-1)$-dimensional ball or sphere.)

Theorem 3.34. *If C is a constructible (not necessarily simple)cell partitioning of a 3-sphere or a 3-ball and C contains a tangle T which is weakly compatible with C then*

$$b(T) \leq p(T).$$

Proof. The proof is by induction on the number of facets of C. If C has only one 3-cell, then T is a set of simultaneously straight spanning arcs. In this case $b(T)$ and $p(T)$ are both equal to the number of spanning arcs of T. Hence the induction base is complete.

The induction step is the same as Theorem 3.21. Because C is constructible, we have a partition of C into C_1 and C_2 which are constructible cell partitionings, both are 3-balls and their intersection is a 2-ball or sphere. Let $T_1 = T \cap C_1$ and $T_2 = \overline{T - T_1}$. By Proposition 3.16 and the induction hypothesis we obtain

$$b(T) \leq b(T_1) + b(T_2) \leq p(T_1) + p(T_2) = p(T).$$

This completes the induction. ∎

Corollary 3.35. *If C is a constructible cell partitioning of a 3-ball or a 3-sphere and C contains a knot K which is weakly compatible with C then*

$$b(K) \leq p(K).$$

Thus, if $b(K) > p(K)$, then C is not constructible. ∎

Armentrout's Theorem 3 in Armentrout [2] was shown as a consequence of his Theorem 1. His Theorem 1 can be reproved by a very simple proof using a similar method to Theorem 3.31, which is different from his original proof.

Theorem 3.36. *(Theorem 1 of Armentrout [2])*
If C is a shellable cell partitioning of a 3-dimensional ball or sphere and C contains a knot K which is weakly compatible with C then

$$b(K) \leq 2 \cdot p(K).$$

Proof. As same as in the proof of Theorem 3.31, there is an ordering of the facets $F_1, F_2, \ldots, F_n$ such that $(F_1 \cup \cdots \cup F_{j-1}) \cap F_j$ is a shellable 2-complex on ∂F_j, and we define $C_1^{(i)}$, $C_2^{(i)}$, $T_1^{(i)}$ and $T_2^{(i)}$ in the same way.

In this case, the possible case of $C_2^{(i)}$ are classified as follows.

(1) $T_2^{(i)}$ in $C_2^{(i)}$ is an arc and $T_1^{(i)} \cap T_2^{(i)}$ consists of two points.

(2) $T_2^{(i)}$ in $C_2^{(i)}$ is an arc and $T_1^{(i)} \cap T_2^{(i)}$ is one point.

(3) $T_2^{(i)}$ in $C_2^{(i)}$ is an arc and $T_1^{(i)} \cap T_2^{(i)}$ is empty.

(4) $T_1^{(i)} \cap T_2^{(i)}$ is empty.

We denote by n_k the number of i's such that i-th step is of the type (k).

From the condition of compatibility of K, $T_2^{(i)}$ in $C_2^{(i)}$ in types (1), (2) and (3) is a trivial spanning arc, so $b(T_2^{(i)}) = 1$. Hence Lemma 3.30 shows that type (3) decreases the bridge index by 1 and others do not. Thus we have

$$b(K) \leq n_3.$$

On the other hand, the calculation of Euler characteristic shows that

$$n_1 = n_3.$$

Hence we have

$$\begin{aligned} p(K) &= n_1 + n_2 + n_3 \\ &\geq n_1 + n_3 \\ &\geq 2 \cdot n_3 \\ &\geq 2 \cdot b(K). \end{aligned}$$

∎

3.10 The hierarchy of combinatorial decomposition properties and the conjectured bound

Upto the last section we have exhibited the following hierarchy of combinatorial decomposition properties according to the existence of knots of small size.

Theorem 3.37. *A 3-ball with a knotted spanning arc consisting of*

$$\begin{cases} \textit{at most } 2 \textit{ edges} & \textit{is not constructible,} \\ 3 \textit{ edges} & \textit{can be shellable, but not vertex decomposable,} \\ 4 \textit{ edges} & \textit{can be vertex decomposable.} \end{cases}$$

A 3-sphere or 3-ball with a knot consisting of

$$\begin{cases} 3 \textit{ edges} & \textit{is not constructible,} \\ 4 \textit{ or } 5 \textit{ edges} & \textit{can be shellable, but not vertex decomposable,} \\ 6 \textit{ edges} & \textit{can be vertex decomposable.} \end{cases}$$

∎

Theorem 3.38. *A 3-sphere or 3-ball with a knot K consisting of*

$$\begin{cases} \textit{at most } \quad b(K)-1 \textit{ edges} & \textit{is not constructible,} \\ \textit{at most } 2\cdot b(K)-1 \textit{ edges} & \textit{is not shellable,} \\ \textit{at most } 3\cdot b(K)-1 \textit{ edges} & \textit{is not vertex decomposable.} \end{cases}$$

∎

For Theorem 3.38, a bound for the number of edges of a knot possibly contained in a complex with combinatorial decomposition properties is the following, which was first pointed out by Günter M. Ziegler.

Proposition 3.39. *There are shellable 3-balls and 3-spheres which has a knot K with $e(K) = 2\cdot b(K)$, and vertex decomposable 3-balls and 3-spheres which has a knot K' with $e(K') = 3\cdot b(K')$.*

Proof. The construction is the same as Examples 3.6 and 3.11. In fact, the examples shown there are knotted spanning arcs and knots of bridge index 2, in a 2-bridge position. To make higher bridge index examples, we have only to prepare a big enough pile of cubes with height 1, then chose k vertical edges and join their endpoints by suitable corridors of width 1 on the boundary of the pile, and lastly add edges along the corridors using the same technique used in Example 3.11 or 3.29. 3-spheres are derived by making a cone over the boundary as before. ∎

Thus the case of shellability and vertex decomposability achieve the sharp bound.

In discussions among Günter M. Ziegler, Richard Ehrenborg and I, we conjectured the above bound for shellability is the sharp bound also for the constructibility case.

Conjecture 3.40.

- If a 3-ball or a 3-sphere has a knots K with $e(K) \leq 2 \cdot b(K) - 1$, then it is not constructible.

The case of $b(K) = 2$ is already solved by Theorems 3.8.

Chapter 4

Deciding constructibility — the case of 3-balls

The decision problem of combinatorial decomposition properties are one of the challenging problems in the study of this field. The importance of this problem is mentioned in the review paper [32] of Danaraj and Klee, who showed the linear-time solvability of shellability of 2-pseudomanifolds in Danaraj and Klee [33]. Except for their initial study, almost nothing is done for this algorithmic problem. Some exceptions are solvability of Cohen-Macaulayness by Garsia [40] and NP-ness of partitionability by Noble [72].

In this chapter, we try the decision problem of constructibility. Because the case of 2-pseudomanifolds are already solved (constructibility of 2-pseudomanifolds is equivalent to shellability), our interest is in the case of 3-pseudomanifolds.

In this chapter, our stand point is that the complexity of algorithms for simplicial complexes should be measured by the order of $\#\{\text{facets}\} \times \log(\#\{\text{vertices}\})$. This is because simplicial complexes can be represented by a list of facets such as

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 8 & 11 \\ \cdots\cdots \end{array}$$

where each number indicates the index of a vertex and each row indicates that there is a facet with the vertices listed in the row. In this example, there are facets with vertices $\{1,2,3,4,5\}$, $\{1,3,4,5,6\}$, $\{1,4,5,8,11\}$, and so on. This way of presenting the data is the smallest one which can be used for general simplicial complexes, and the size of bits to be used is $O(\#\text{facets} \times \log(\#\text{vertices}))$.

Section 4.1 gives a result which shows that the decision problem of 3-spheres is reduced to the problem of 3-balls. Thus we consider the problem of 3-balls after Section 4.2. In Section 4.2 we introduce a notion of reduced 3-balls which is the key concept in the following discussion, and Section 4.3 gives a characterization of constructibility in the case of 3-balls without interior vertices. An application of this result is given in Section 4.4 and Bing's 3-ball, formerly known to be non-shellable, is shown to be non-constructible. Section 4.5 discuss how to extend the result of Section 4.3, and give a generalization of the characterization of constructibility allowing the existence of upto two interior vertices. In Section 4.6, however, we give an example which shows that the same extension can not be made for more than two interior vertices. In the last Section 4.7 we give an algorithm for the decision problem of constructibility of 3-balls with at most two interior vertices using the result of Section 4.5. The algorithm runs in $O(\#\text{facets})$ time and this shows the polynomial time solvability of constructibility in this special case.

4.1 3-balls and 3-spheres

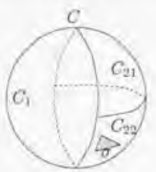
Our first result shown in this chapter is that the decision problem of constructibility of 3-spheres is reduced to that of constructibility of 3-balls.

Theorem 4.1. *Let C be a triangulation of a 3-sphere and σ any facet of C. Then C is constructible if and only if the 3-ball $C \setminus \sigma$ is constructible.*

Proof. The "if" part is trivial, so we show the "only if" part. Let C be constructible. Then by definition there are two constructible 3-balls C_1 and C_2 such that $C_1 \cup C_2 = C$, and $C_1 \cap C_2$ is a constructible 2-sphere. We may assume that σ is contained in C_2. If $C_2 = \overline{\sigma}$, then we are done. Otherwise C_2 is the union of two constructible 3-balls C_{21} and C_{22} that satisfy the conditions for constructibility. We may assume that C_{22} contains σ. We define $C_1' := C_1 \cup C_{21}$ and $C_2' := C_{22}$. Then

(i) C_2' is a constructible 3-ball by definition.

(ii) $C_1' \cap C_2' = \partial C_2' = \partial C_{22}$ is constructible because it is a 2-sphere.



(iii) $C_1' = C_1 \cup C_{21}$, where both C_1 and C_{21} are constructible 3-balls by definition. Their intersection $C_1 \cap C_{21} = \overline{\partial C_{21} \setminus (C_{21} \cap C_{22})}$ is a constructible 2-ball, since removal of a 2-ball from a 2-sphere always leaves a 2-ball, and all 2-balls are constructible. Thus C_1' is a constructible 3-ball.

So C_1' and C_2' instead of C_1 and C_2 satisfy the definition of constructibility. Continuing this argument, the number of facets of C_2 is reduced until C_2 has only the one facet σ, showing that $C \setminus \sigma$ is constructible. ∎

This theorem is from a joint work with Günter M. Ziegler [45]. Originally this consisted a part of the proof of Theorem 3.8 in Section 3.3 until the simplified proof as shown there was given. The original argument of Theorem 3.8 using Theorem 4.1 is as follows: If a 3-sphere has a knot made of three edges, we remove a facet which meets with the knot by one edge. Then we get a 3-ball with a knotted spanning arc made of two edges, which implies that the 3-ball is not constructible from Theorem 3.4. The non-constructibility of the ball implies that the 3-sphere is not constructible from Theorem 4.1. For the case of a 3-ball containing the knot made of three edges, we make a cone over its boundary to get a 3-sphere with a knot made of three edges. If the ball is constructible, then the sphere we get is also constructible. This is a contradiction which completes the proof of Theorem 3.8.

Assume that we have an algorithm to decide the constructibility of 3-balls. If we are given a 3-sphere, then we remove one facet chosen arbitrary to get a 3-ball. From Theorem 4.1, the answer for the 3-ball is precisely the answer to the original 3-sphere. The order of the running time of the algorithm is not changed. So the time complexity of the problem for 3-spheres is at most that for 3-balls.

As is shown in Proposition 2.16, constructible 3-pseudomanifolds are just 3-balls and 3-spheres. Now the 3-sphere case is reduced to the 3-ball case, if we want to try to decide the constructibility of a given 3-pseudomanifold with an information of its topology, what we need is an efficient algorithm for the decision problem of constructibility of 3-balls.

Remark. If we *know* the topology of simplicial complexes a priori, the situation is as above. But usually to know the topology of a given simplicial complexes is a very difficult problem. For example, it is known that the problem to decide whether any two manifolds are homeomorphic or not is undecidable if the dimension is 4 and higher. (For example, see Stillwell [88].) Moreover, to decide whether a manifold is a sphere or not is also undecidable in dimensions starting from 5. (See Volodin, Kuznetsov and Fomenko [91].) The problem to decide if a triangulated manifold is a 3-ball or not is shown to be decidable by Rubinstein [77] using normal surface theory (it is written that Haken already had the result before this), but still the algorithm is far from efficient and the time complexity is not known. But there are many cases in which we know a priori that the simplicial complexes are balls or spheres, for example the data are produced from triangulating balls, spheres, or polytopes. For such cases, we can use algorithms specialized for balls or spheres without worrying about how to decide the topology.

4.2 Reduced balls

From this section, we describe how to decide constructibility of 3-balls and our goal is to give an algorithm to decide constructibility of a given 3-ball under the condition that the number of vertices contained in the interior of the ball is at most 2. The algorithm runs in $O(\#\{\text{facets}\})$ time.

Our method relies on the following definition of reducedness of balls.

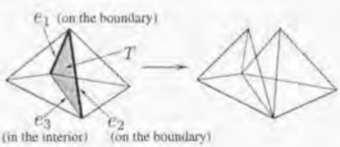
Definition 4.2. A *reduced* d-ball C is a d-ball in which every $(d-1)$-face in the interior $\mathring{C}$ of C has more than one $(d-2)$-faces in $\mathring{C}$. Equivalently, a d-ball is reduced if every $(d-1)$-face in $\mathring{C}$ has at least $(d-1)$ of its $(d-2)$-faces on the boundary ∂C of C.

In particular, a reduced 3-ball is a 3-ball in which every 2-face in the interior has at most 1 edge on the boundary.

To see the importance of this concept, consider the following two operations applied for a given d-ball.

(I) If T is a $(d-1)$-face contained in $\mathring{C}$ and all of its $(d-2)$-faces are in ∂C, then divide the ball C into two balls C_1 and C_2 by T.

(II) If T is a $(d-1)$-face contained in $\mathring{C}$ and $d-1$ of its $(d-2)$-faces are in ∂C, then split T as the following figure. (Let us call the resulting ball C'.)



For these operations for a 3-ball C, we have the following proposition.

Proposition 4.3. *For the two operations above,*

(I) *C is constructible if and only if both C_1 and C_2 are constructible.*

(II) *C is constructible if and only if C' is constructible.*

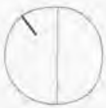
Proof. Claim (I) is trivial.
[if part of (II)]
Let C' be constructible. Then there are two d-balls C_1' and C_2' satisfying the condition of Definition 2.11. Let us divide C into C_1 and C_2 such that C_i and C_i' have the same set of

facets, for $i = 1, 2$. If $C_1 \cap C_2$ does not contain T, then one of C_1 and C_2 contains T and the constructibility of C can be shown by induction on the size of the ball. If $C_1 \cap C_2$ contains T, then $C_1 \cap C_2 = (C_1' \cap C_2') \cup \overline{T}$ is constructible by (I) for dimension $d-1$, which shows the constructibility of C.

[only if part of (II)]

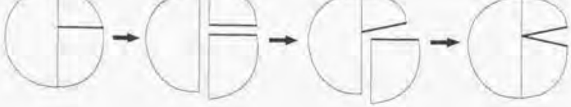
Let C be constructible. Then there are two d-balls C_1 and C_2 satisfying the condition of Definition 2.11. There are 3 cases.

- $C_1 \cap C_2$ intersect with T by a face with dimension less than $(d-1)$ (including $\emptyset$).
  In this case, one of C_1 and C_2 contains T such that only one $(d-1)$-face of T is in the interior and the rest are on the boundary, and the constructibility of C' is shown by induction on the size of the ball.



- $C_1 \cap C_2$ contains T.
  If we divide C' into C_1' and C_2' such that C_i and C_i' have the same set of facets, $i = 1, 2$. Then C_1' and C_2' are constructible by definition and $C_1' \cap C_2' = \overline{(C_1 \cap C_2) - \overline{T}}$ is also constructible from (I) for $(d-1)$-dimensional balls.

- $C_1 \cap C_2$ intersect with T by a $(d-1)$-face.
  Let us assume that C_2 contains T. Here T has all of its proper faces on ∂C_2, thus it divides C_2 into two balls C_{21} and C_{22}. (Here we assume that the vertices to be split in the operation (II) for C are made to be different in this division.) From (I) for $(d-1)$-dimensional balls, C_{21} and C_{22} are constructible. Now observe that $C' = C_1 \cup C_{21} \cup C_{22}$. Let us define $B := C_1 \cap C_2$, $B_1 := C_1 \cap C_{21}$, and $B_2 := C_1 \cap C_{22}$. Here, $B = B_1 \cup B_2$ and $B_1 \cap B_2$ is a $(d-2)$-simplex. Because B is constructible by definition, (I) for dimension $d-1$ assures that B_1 and B_2 are constructible. Thus, $C_1 \cup C_{21}$ is constructible, and $C' = (C_1 \cup C_{21}) \cup C_{22}$ is constructible. The following figure shows this construction.



∎

Thus if we apply these two operations for a given d-ball, we finally get a set of reduced d-balls, that is, "reduced" means that we can not apply both of the operations above. Because the operations preserve the constructibility, we can decide the constructibility of C from the constructibility of the reduced d-balls. So characterizations of the constructibility of reduced d-balls can be used for the decision of constructibility of d-balls.

4.3 3-balls with no interior vertices

First we remark that the two operations introduced in the previous section preserve the number of vertices contained in the interior of the ball. Our first step is the case of 3-balls with no interior vertices, and correspondingly what we show in this section is a characterization of constructibility of reduced 3-balls with no interior vertices.

Proposition 4.4. *If a reduced 3-ball has no interior vertices, then it is constructible if and only if it is a simplex, or equivalently, if and only if it has no spanning edge.*

Proof. Let C be a reduced 3-ball with no interior vertices which is constructible. Assume that C is not a simplex. Then from the Definition 2.11, there are two subcomplexes C_1 and C_2 satisfying the condition. In particular, $C_1 \cap C_2$ is a 2-ball from Proposition 2.16 without interior vertices. Here, $C_1 \cap C_2$ should be made of the 2-faces contained in the interior of C, thus at most one of the edges of each 2-face is on the boundary. But this is impossible from Proposition 2.32. ∎

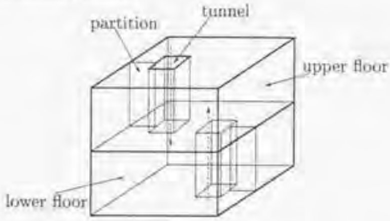
This proposition provides a very easy algorithm to decide constructibility for this case, that is, apply the reduction operation as possible and if we can divide the 3-ball into disjoint set of simplices then the ball is constructible, and if we get stuck before that then it is not constructible. We remark that we need no backtracking in this process. Corresponding algorithm to implement this procedure is as follows: list up all the 2-faces and mark the edges which are on the boundary. Then pick up a 2-face whose two edges are marked and mark the third edge. After repeating this, all the edges are marked if and only if the 3-ball is constructible.

The time complexity of this algorithm will be discussed in Section 4.7, and will be shown to be $O(\#\text{facets})$.

4.4 Bing's house with two rooms

Other than algorithmic application, Proposition 4.4 can be used for a construction of non-constructible 3-balls which is completely different from that of Chapter 3.

Example 4.5. The example of 3-ball we will describe here is called "Bing's house with 2 rooms". This example is shown in [10]. This is known to be a non-shellable 3-ball, but here we show that it is non-constructible, either.



Bing's house with two rooms

This is a house with 2 rooms as above, the walls are made out of one layer of bricks (cubes), one enters to the lower floor through a tunnel from the roof and to the upper floor through a tunnel from below. After constructing such an object C with cubes, we triangulate the cubes as follows. Let us order the vertices as follows. First list vertices v such that there is a cube D in which v is a connected component of $D \cap \partial C$. (The vertices on the inside corners of C.) Next list the vertices which is not listed yet and is on an edge that is a connected component of $D \cap \partial C$, for some cube D. Last list the remainder. Then we triangulate each 2-face such that the first vertex in the list is contained in the added diagonal. Finally we triangulate each cube into six simplices by taking cones from the first vertex to the six triangles contained in the 2-faces of the cube which do not contain the vertex. (This triangulation is a "pulling triangulation", made by pulling the vertices in the listed order. The concept "pulling" is described in [55, Sec 14.2].)

In this systematic triangulation, we can see that each facet intersects with ∂C in a disconnected set, and this is the reason why C is not shellable. Moreover, we can also see that there is no triangle in the interior of C such that 2 or 3 of its edges are on ∂C, that is, C is reduced. So from Proposition 4.4, C is not constructible. (Because all of the vertices of C are on ∂C, the condition for Proposition 4.4 is satisfied.)

This is a thickened example of special spines mentioned in Section 5.1. In fact, the same can be shown for other examples of special spines of 3-cubes, for example the "house with one room". These non-constructible 3-balls have no knotted spanning edges, but it has spanning edges in everywhere.

Other than the above Bing's house and its relatives, there are many cases satisfying the condition to be a 3-ball and having no interior vertices. For example, the three non-shellable 3-balls mentioned in Section 2.3.1 have no interior vertices. By a computer calculation using our method, we can very easily check their constructibility.

Proposition 4.6. *Rudin's ball, Grünbaum's ball, and Ziegler's ball are constructible.* ∎

Remark. The constructibility of Rudin's ball and Grünbaum's ball is already commented in Provan and Billera [74], without mentioning how to check it. (My attempt, before noticing that the constructibility of the balls is already known, to check Rudin's ball without using our method failed, because the number of facets 41 was too many to enumerate all the possible divisions.) The constructibility of Ziegler's ball was already known to me before I made a computer calculation, because I made a paper model of the ball to check it.

On the other hand, Bing's house needs much more facets, over 1500, and it is far from direct computer calculation or hand calculation. But our method also works easily for such big examples to check on computers.

4.5 3-balls with few interior vertices

This section extends the result of Section 4.3 to the cases with a few interior vertices. The first extension is for the case with just one interior vertex.

Proposition 4.7. *If a reduced 3-ball C has only one vertex v in the interior, then it is constructible if and only if it has no spanning edge.*

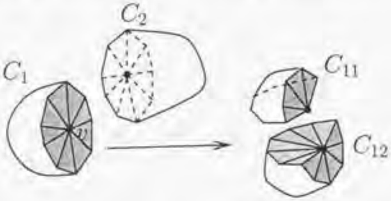
(This is equivalent to saying that C is constructible if and only if C is a star with a center v.)

Proof. The "if" part is trivial because a reduced 3-ball with only one interior vertex v without spanning edge must be a star with a center v, and a 3-dimensional star is constructible because 2-spheres are shellable. So we only have to show the "only if" part.

Let C be constructible. Because it is not a simplex, there are two subcomplexes C_1 and C_2 satisfying the condition of Definition 2.11. Here, C_1 and C_2 are constructible 3-balls and $C_1 \cap C_2 = B$ is a 2-ball. Because B is made of 2-faces in $\overset{\circ}{C}$ and can have at most one interior vertex, from Proposition 2.32 together with the condition of reducedness of C, it has no spanning edge. Thus B must be a 2-dimensional star with a center v.

Now remark that both of the 3-balls C_i $(i = 1, 2)$ are constructible and $\overline{\partial C_i - \partial C}$ is a star of v.

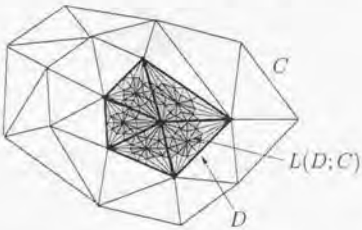
Because C_i is constructible, if it is not a simplex, it will be divided again into two 3-balls C_{i1} and C_{i2} such that $C_{i1} \cap C_{i2} = B'$ is a 2-ball made of 2-faces in $\overset{\circ}{C}$. B' does not have interior vertices. If the boundary of B' is completely contained in ∂C, then Proposition 2.32 concludes that there is a 2-face in B' with two edges in ∂C, contradicting the reducedness of C. Thus $\partial B'$ must contain edges not in ∂C. These edges must be taken from $\overset{\circ}{B}$ and there are no choice other than to take two edges from $\overset{\circ}{B}$, both incident to v. Now if B' has a spanning edge e not containing v, then it divides B' into two 2-balls the boundary of one of which is made of e and those of ∂C. By Proposition 2.32, there must be a 2-face whose two edges are in $\partial B'$, which is impossible from reducedness of C. Thus all the spanning edge of e must contain v, which shows that all the interior edges of B' is incident to v.



Here again the 3-balls C_{ik} $(k = 1, 2)$ are constructible and $\overline{\partial C_{ik} - \partial C}$ is a star of v. Thus if C_{ik} is not a simplex, we can do the same argument as above for C_{ik}. Continuing this argument, we finally have all the balls divided into simplices and then conclude that all the interior edges of the cutting faces, equivalently all the interior edges of C, must be incident with v, which shows that C has no spanning edges. ∎

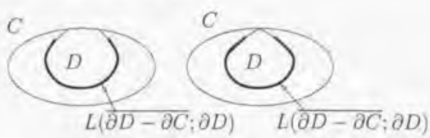
From this, it seems that having spanning edges is bad for constructibility under some conditions. One more extension up to the case with two interior vertices can be achieved in this line, but more complicated argument is needed.

Before describing it, we introduce one technical definition. In the following, for a pair of simplicial complexes $C \supset D$, we denote by $L(D; C)$ the set $\left|\{\sigma \in \mathrm{sd}^2(D) : |\sigma| \cap |\overline{(C - D)}| = \emptyset\}\right|$, where $\mathrm{sd}^2(D)$ is the second barycentric subdivision of D.



We remark that $L(D; C)$ is just a point set but we can associate the cell complex structure from C, that is, a cell complex $\{\sigma \cap L(D; C) : \sigma \in C\} \cup \{\sigma \cap \partial L(D; C) : \sigma \in C\}$. In the following, we treat $L(D; C)$ as a cell complex in this sense and use the terms as if they are simplicial complexes. It is easy to see that this cell complex structure has almost the same property as D, for example, Proposition 2.32 holds for $L(D; C)$ when $|L(D; C)|$ is a 2-ball.

To show the following lemmas and proposition, we see the shapes of $L(\overline{\partial D - \partial C}; \partial D)$ instead of those of $\partial D - \partial C$, for a pair of 3-balls $C \supset D$. We use this trick in order to avoid the singular case. For this, see the following figure.



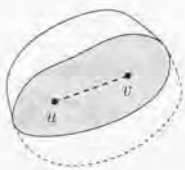
In this figure, the right figure is a singular case of the left, but the shape of $L(\overline{\partial D - \partial C}; \partial D)$ is the same, i.e., both are 1-balls.

We use four lemmas to show Proposition 4.12.

Lemma 4.8. *Let D be a 3-ball that is a subcomplex of a reduced 3-ball C, and assume that C has two interior vertices u and v. If D satisfies:*

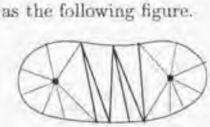
- *D has no vertices in its interior,*
- *$L(\overline{\partial D - \partial C}; \partial D)$ is a 2-ball and contains two vertices u and v in its interior,*
- *D contains a spanning edge of C, and*
- *u and v are not joined by an edge in D,*

then D is not constructible.



Proof. Assume that there are constructible 3-balls which satisfy all the conditions of the statement. Let D^* be the smallest one among these. It is easy to see that D^* is not a simplex.

Remark that $L(\overline{\partial D^* - \partial C}; \partial D^*)$ has spanning edges because u and v are not joined by an edge in D. Because C is reduced, the 2-faces of $L(\overline{\partial D^* - \partial C}; \partial D^*)$ have at most one edge on the boundary. Thus from Proposition 2.32, the spanning edges of $L(\overline{\partial D^* - \partial C}; \partial D^*)$ have the interior vertices on each side as the following figure.



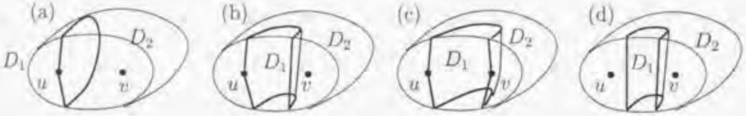
Now because D^* is a constructible 3-ball, there are two constructible 3-balls D_1 and D_2 satisfying the condition of Definition 2.11. Again from Proposition 2.32, the possibility of the 2-ball $D_1 \cap D_2$ is restricted. It is easy to observe that there are only the following cases.

(a) $D_1 \cap D_2$ contains one of u and v on its boundary and contains no spanning edges of C.

(b) The boundary of $D_1 \cap D_2$ contains one interior vertex and one spanning edge of $D_1 \cap D_2$.

(c) $D_1 \cap D_2$ contains both u and v on its boundary.

(d) $D_1 \cap D_2$ does not contain u and v, and contains two spanning edges of C on its boundary.



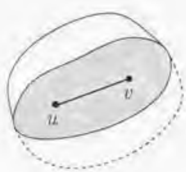
In the figure, D_2 of (a), D_2 of (b), D_2 of (c), and D_2 of (d) satisfy the first two and the last condition in the statement. In (a), D_1 can not contain spanning edges of C, so D_2 contains spanning edges. In (b) and (d), $D_1 \cap D_2$ contains spanning edges of C, thus D_2 contains spanning edges of C. In (c), $D_1 \cap D_2$ contains spanning edges of C because u and v are not joined by an edge in D, so D_2 has spanning edges. Thus in all cases, D_2 satisfies all the conditions of the statement, contradicting the minimality of D^*. ∎

Remark. For the last condition of the nonexistence of the edge uv in D, we note that if uv is in $\mathring{D}$, the proof fails because D_2 in (c) can have no spanning edges of C if uv is in $D_1 \cap D_2$. But if uv is in ∂D, we have the next lemma.

Lemma 4.9. *Let D be a 3-ball that is a subcomplex of a reduced 3-ball C, and assume that C has two interior vertices u and v. If D satisfies:*

- *D has no vertices in its interior,*
- *$L(\overline{\partial D - \partial C}; \partial D)$ is a 2-ball and contains two vertices u and v in its interior,*
- *D contains a spanning edge of C, and*
- *the edge uv is in $\partial D - \partial C$.*

then D is not constructible.



Proof. Assume that there are constructible 3-balls which satisfy all the conditions of the statement. Let D^* be the smallest one among these. It is easy to see that D^* is not a simplex.

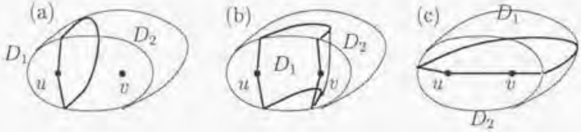
In this case, $L(\overline{\partial D^* - \partial C}; \partial D^*)$ does not have spanning edges.

Because D^* is a constructible 3-ball, there are two constructible 3-balls D_1 and D_2 satisfying the condition of Definition 2.11. It is easy to observe that there are only the following cases.

(a) $D_1 \cap D_2$ contains one of u and v on its boundary and contains no spanning edges of C.

(b) $D_1 \cap D_2$ contains both u and v on its boundary but the edge uv.

(c) $D_1 \cap D_2$ contains the edge uv on its boundary

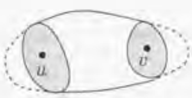


Here D_2 in (a), and one of D_1 and D_2 in (c) satisfy the conditions of the statement, contradicting to the minimality of D^*. And in (b), D_2 satisfies the conditions of Lemma 4.8, contradicting its constructibility. ∎

Lemma 4.10. *Let D be a 3-ball that is a subcomplex of a reduced 3-ball C, and assume that C has two interior vertices u and v. If D satisfies:*

- *D has no vertices in its interior,*
- *$L(\overline{\partial D - \partial C}; \partial D)$ is a disjoint union of two 2-balls each of which contains one of u and v in its interior, and*
- *D has a spanning edge of C,*

then D is not constructible.



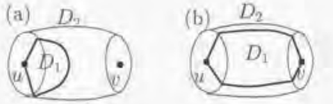
Proof. Assume that there are constructible 3-balls which satisfy all the conditions in the statement. Let D^* be the smallest one among these. It is easy to see that D^* is not a simplex.

From Proposition 2.32, the 2-balls which are the components of $L(\overline{\partial D - \partial C}; \partial D)$ can have no spanning edge, thus stars of u and v, respectively.

Now D^* is a constructible 3-ball, it is divided into two constructible 3-balls D_1 and D_2 as Definition 2.11. The 2-ball $D_1 \cap D_2$ is the following:

(a) $D_1 \cap D_2$ has one of u and v on the boundary and has no spanning edge of C.

(b) $D_1 \cap D_2$ has both u and v on the boundary.



In the case (a), D_2 satisfies all the conditions of the statement, contradicting the minimality of D^*. For the case (b), if $D_1 \cap D_2$ does not contain the edge uv (or the edge uv does not exist in D from the first), $D_1 \cap D_2$ contains spanning edges of C and at least one of D_1 and D_2 does not contain the edge uv, so one of D_1 and D_2 satisfies the conditions of Lemma 4.8, contradicting the constructibility of D_1 and D_2. And if $D_1 \cap D_2$ contains the edge uv, then at least one of D_1 and D_2 contains spanning edges of C, thus Lemma 4.9 concludes that at least one of D_1 and D_2 is not constructible, again lead to a contradiction. ∎

Lemma 4.11. *Let D be a 3-ball that is a subcomplex of a reduced 3-ball C, and assume that C has two interior vertices u and v. If D satisfies:*

- *D has u on its boundary and v in its interior,*
- *$L(\overline{\partial D - \partial C}; \partial D)$ is a 2-ball which contains v in its interior, and*
- *D has a spanning edge,*

then D is not constructible.

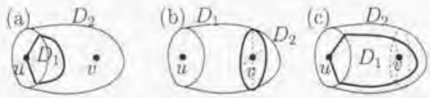
Proof. Assume that there are constructible 3-balls which satisfy all the conditions in the statement. Let D^* be the smallest one among these. It is easy to see that D^* is not a simplex.

By the same observation as in the proof of the previous lemma, $L(\overline{\partial D - \partial C}; \partial D)$is a star of v. Because D^* is a constructible 3-ball, it is divided into two constructible 3-balls D_1 and D_2 as Definition 2.11. From Proposition 2.32, the possibility of the 2-ball $D_1 \cap D_2$ is only the following.

(a) $D_1 \cap D_2$ contains u and no spanning edge.

(b) $D_1 \cap D_2$ contains v and no spanning edge.

(c) $D_1 \cap D_2$ contains both u and v.



In the case (a), D_2 satisfies the conditions of the statement, contradicting the minimality of D^*. In the case (b), D_1 satisfies the conditions of Lemma 4.10, contradicting the constructibility of D_1. In the case (c), if the edge uv is not contained in $D_1 \cap D_2$ (or it does not exist in D from the first), then $D_1 \cap D_2$ contains spanning edges of C and at least one of D_1 and D_2 does not contain uv, thus satisfies the conditions of Lemma 4.8, contradicting its constructibility. And if the edge uv exists in $D_1 \cap D_2$, then at least one of D_1 and D_2 contains spanning edges of C, so it satisfies the conditions of Lemma 4.9, contradicting the constructibility of D_1 and D_2. ∎

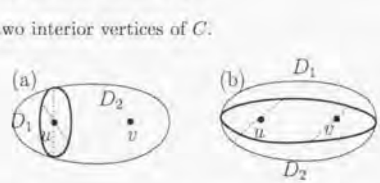
Now we can show the following proposition.

Proposition 4.12. *If a reduced 3-ball has exactly two interior vertices, then it is constructible if and only if it has no spanning edges.*

Proof.
[if part]
Let C be a reduced 3-ball with two interior vertices u and v which has no spanning edges. Then the facets of C can be only of two types: (i) one edge and its two end vertices are in ∂C and the rest are in $\mathring{C}$, and (ii) one 2-face and its proper faces are in ∂C and the rest are in $\mathring{C}$. From this we conclude that every facet of C belongs to either $\operatorname{star} u$ or $\operatorname{star} v$, thus $C = \operatorname{star} u \cup \operatorname{star} v = \operatorname{star} u \cup \overline{(\operatorname{star} v - \operatorname{star} u)}$. Here we also observe that there is an edge between u and v, so v lies in $\partial(\operatorname{star} u)$. Hence $\overline{(\operatorname{star} v - \operatorname{star} u)} = v * (\text{2-ball})$ and it is constructible because 2-balls are constructible. Also $\operatorname{star} u = u * (\text{2-sphere})$ is constructible, and $(\operatorname{star} u) \cap \overline{(\operatorname{star} v - \operatorname{star} u)} = (\text{2-ball})$ is constructible, thus C is constructible.
[only if part]
Let C be a reduced 3 ball with two interior vertices u and v, and assume that it is constructible and has spanning edges. Then there are two constructible 3-balls C_1 and C_2 satisfying the condition of Definition 2.11. Here $D_1 \cap D_2$ is a 2-ball contained in the interior of C. From the reducedness of C and Proposition 2.32, there are only two possibility for $D_1 \cap D_2$.

(a) $D_1 \cap D_2$ contains one interior vertex of C.

(b) $D_1 \cap D_2$ contains two interior vertices of C.



In the case (a), D_2 satisfies the conditions of Lemma 4.11. In the case (b), if the edge uv is not contained in $D_1 \cap D_2$, then $D_1 \cap D_2$ contains spanning edges of C and one of D_1

and D_2 does not contain uv, so it satisfies the conditions of Lemma 4.8. And if the edge uv is contained in $D_1 \cap D_2$, then at least one of D_1 and D_2 has spanning edges of C, thus satisfies the conditions of Lemma 4.9. Thus in all the cases, at least one of D_1 and D_2 is not constructible. A contradiction. ∎

We finally come to state the following theorem, summarizing Propositions 4.4, 4.7 and 4.12.

Theorem 4.13. *If a reduced 3-ball has at most two interior vertices, then it is constructible if and only if it has no spanning edges.* ∎

The next section provides an example which shows that the number "two" of interior vertices in this theorem is sharp.

4.6 3-balls with many interior vertices

From the results of the last section, one may think that we can give a characterization of constructibility for arbitrary reduced 3-balls in a similar way, but the situation already is different at all in the case with three interior vertices.

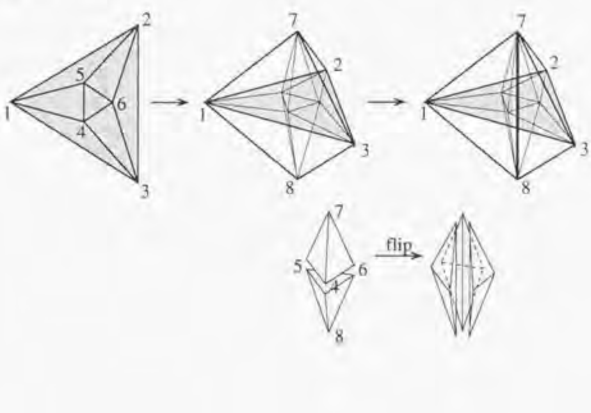
Theorem 4.14. *There are shellable reduced 3-balls with spanning edges. Such example can be constructed with only three interior vertices.*

Proof. The following is a list of facets of an example of such 3-balls with 8 vertices and 15 facets.

1257	2367	1347	1457	2567	3467
1258	2368	1348	1458	2568	3468
4578	4678	5678			

The vertices "4", "5" and "6" are interior vertices, and the edge "78" is the spanning edge.

This example is constructed as follows. First we take a triangulated 2-ball as the following figure and form a bipyramid by introducing two new vertices "7" and "8". Then replace the two tetrahedra "4567" and "4568" by three tetrahedra "4578", "4678" and "5678". (This operation is a 'flip'.) It is easy to check "78" is a spanning edge and the ball is reduced, but this example is shellable. ∎



Also the converse is not true for the case with many interior vertices.

Theorem 4.15. *There are nonconstructible reduced 3-balls with no spanning edges.*

Proof. By Theorem 4.1, a 3-sphere C is constructible if and only if $C \setminus \sigma$ is constructible for any facet σ of C, and Theorem 3.8 assures the existence of nonconstructible 3-sphere. If we take C to be non-constructible, then $C \setminus \sigma$ is nonconstructible. And such 3-balls derived from 3-spheres by removing one facet clearly is reduced and contains no spanning edges. ∎

4.7 The algorithm

As is mentioned in Section 4.3, the characterization of constructibility of reduced balls derived in this chapter can be used for giving a simple algorithm to decide constructibility of (not necessarily reduced) 3-balls. The algorithm first applies the two reduction operations introduced in Section 4.2, and then check the constructibility of derived reduced balls.

However since producing reduced balls literally is not efficient, so we give an efficient algorithm which implement the way described above.

In our algorithm, we use Theorem 4.13 in the last step to get the answer, that is, we check whether the ball has spanning edges or not. This property, having spanning edges or not, is not affected by the operation (I) of dividing by a triangle, so we need not do this operation.

As for the operation (II) of splitting a triangle, what we really need to do is to make the third edges appear on the boundary of the ball. So instead of doing the real splitting, we just keep in mind that the edge is now on the boundary by marking the edge.

From these observations, we propose the following algorithm.

Algorithm

Given: a list of facets of a triangulated 3-ball C which has at most two interior vertices.

Step 0: Calculate the boundary complex ∂C of C, and make a list L of edges of C in which the edges on ∂C is marked. Also check which vertices are in the interior.

Step 1: List up all the 2-faces of C and if there is a 2-face whose 2 edges are marked in L, then mark the third edge. Repeat this step while there are such 2-faces.

Step 2: Check the edges which are not marked in L. If there is an unmarked edge with no interior vertex, then C is not constructible. Otherwise C is constructible.

To make this algorithm run efficiently, we should keep the list of 2-faces and the list of edges, and marking should be done on the list of edges. Marking the edges should be reflected to the list of 2-faces, that is, each 2-face should be linked to each edge and vice versa. We also should avoid unnecessary checking in Step 1, so we make a partitioning of the list of 2-faces into 4 parts by how many edges are marked. (In fact, the list of 2-faces with three edges marked is not needed.) While Step 1 is repeatedly executed, the marking of one edge will change the situation of the 2-faces which contain the edge. So as soon as one edge is newly marked, we move the affected 2-faces into suitable partition. By this careful treatment, each 2-face will be checked at most 3 times, thus the total number of repetition of Step 1 is bounded by $3 \times f_2$, where f_i is the number of i-faces.

In Step 0, calculation of the boundary complex can be done in $O(f_3)$. This can be done in a very simple way: First produce four 2-faces from one facet and list all of them without considering the multiplication. Then, to remove multiplication, check the list of 2-faces from top to bottom and if the 2-face is new then do nothing and it appeared already then remove it. This can be performed if the vertices in each 2-face is sorted and use a large table to indicate the 2-faces already appeared. The list of edges can also be made in the same way.

Here we remark that the number of faces satisfies certain equations from Dehn-Sommerville equations for spheres, see p. 15. For the case of 3-balls, $O({f_0}^2) = O(f_1) = O(f_2) = O(f_3)$. So each operations of making lists can be done in $O(f_3)$ and the number of repetitions in Step 1 is also in $O(f_3)$, we conclude that the algorithm runs in $O(f_3)$ time.

Chapter 5

The hierarchy of combinatorial decompositions of 2-dimensional simplicial complexes

From the drastic difference between the simplicity of 2-pseudomanifolds shown in Section 2.5 and the complexity of 3- and higher dimensional pseudomanifolds shown in Chapter 3, one may think that the 2-dimensional world is very simple and 3-dimensional world is complicated. This is not wrong but it is a big mistake if one is tempted to include general 2-dimensional simplicial complexes other than pseudomanifolds to his "2-dimensional world." In this chapter we show several examples which show that the 2-dimensional world becomes already complicated enough if general simplicial complexes are considered.

This may be related to the fact that some class of 2-dimensional simplicial complexes are spines of 3-dimensional manifolds, and 3-manifolds can be reconstructed from such spines. Thus the topology of general 2-dimensional simplicial complexes is complicated as 3-dimensional manifolds, far from the case of 2-pseudomanifolds.

After reviewing the formerly known result that every triangulation of the dunce hat is not shellable in Section 5.1, in Section 5.2 we extend the result and show that every triangulation of the dunce hat is not constructible. This shows the existence of 2-dimensional simplicial complexes which are Cohen-Macaulay but not constructible. In the next Section 5.3 we give a modified example of the dunce hat and show the example is shellable but not extendably shellable. In Section 5.4 we use this example to construct an example which is constructible but not shellable, and in Section 5.5 an example which is non-shellable but has a shellable subdivision.

5.1 Cohen-Macaulay but not shellable 2-complexes

By seeing Proposition 2.31, one may guess stronger implication: every Cohen-Macaulay 2-complex may be shellable. But unfortunately it isn't. The following fact is known.

Proposition 5.1. *(Stanley [87, p.84])*
Every triangulation of the "dunce hat" is not shellable, while it is Cohen-Macaulay.

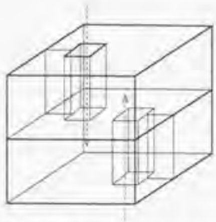
Dunce hat

The *dunce hat* is shown in the figure above. Here, three edges of the triangle is identified as is indicated by the arrows.

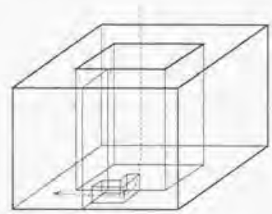
Proof. The argument to prove this proposition is as follows: that the dunce hat D is contractible (for example see Zeeman [97]) means that D has $\tilde{H}_i(D) = 0$ for all i, which means that $h_3(D) = 0$. If D is shellable, every facet F_i of its shelling $F_1, \ldots, F_t$ should satisfy $(F_1 \cup \cdots \cup F_{i-1}) \cap F_i \neq \partial F_i$ because $h_3(D) = 0$ (see Proposition 2.26), but this is impossible because Δ has no boundary.

That D is Cohen-Macaulay is verified by checking the links one by one. For 2-faces and 1-faces, there is nothing to be checked especially. For 0-faces, their links are 1-dimensional connected complex, so $\tilde{H}_{-1} = \tilde{H}_0 = 0$. For the empty face, its link is the whole D and $\tilde{H}_i(D) = 0$ for all i because D is contractible. Thus D is Cohen-Macaulay. ∎

This example, the dunce hat, arises from the study of simplicial collapsing in combinatorial topology. In the context of combinatorial topology, if M_2 is obtained from M_1 by a sequence of elementary collapse, M_2 is called a *spine* of M_1. The dunce hat is a famous example of a spine of a 3-cube which is not collapsible. That every triangulation of the dunce hat does not simplicially collapses to a point is easily seen because there is nowhere to start with: it has no boundary, thus there are no free faces. Other such examples, spines of 3-cubes which are not collapsible to a point, are known for example, "Bing's house with two rooms", "house with one room"(or "abalone"). (See Hog-Angeloni and Metzler [50], Matveev and Rolfsen [64], etc.)

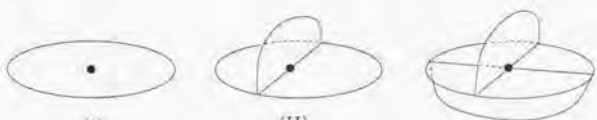


Bing's house with two rooms



House with one room (or abalone)

Especially, the two examples shown in the picture above have the property that the singularities are in general position. More precisely, the neighborhood of each point is one of the following.



(I) (II) (III)

In this figure, (II) and (III) are singularities. A 2-simplicial complex is a *standard polyhedron* or *special polyhedron* if the neighborhood of each point is one of the above and that the singularities form a cell complex, that is, the points (II) forms a disjoint set of open arcs. (It should be noted that in combinatorial topology "polyhedron" is used for the geometric realization of a simplicial complex, different from the term of combinatorics which means an unbounded polytope.) So we can say that "Bing's house with two rooms" and the "house with one room" are spines of 3-cubes which is at the same time special polyhedra. Such spines are called *standard spines* or *special spines*. These standard spines are extensively studied in relation with 3-manifolds, for example Benedetti and Petronio [6], Hog-Angeloni and Metzler [50], Matveev and Rolfsen [64].

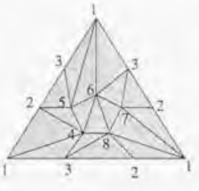
In the next section, we want to discuss a class of 2-simplicial complexes slightly larger than that of standard polyhedra.

Definition 5.2. A 2-simplicial complex is a *near-standard polyhedron* if the neighborhood of each point is one of the above three types except for one point. The exceptional point is the *non-standard point* and the rest are *standard points*.

For example, the dunce hat is not a standard polyhedron, but is a near-standard polyhedron, see p. 107.

5.2 Dunce hat is not constructible

Now the dunce hat is Cohen-Macaulay but not shellable, it is natural to ask whether it is constructible or not. The following is an example of a triangulation of the dunce hat.



If one examines the constructibility of this example by computers or by hands, he will figure out that it is not constructible. In fact, we can show that *any* triangulations of the dunce hat are not constructible, which gives a different proof of Proposition 5.1. For this, we need the following lemmas.

Lemma 5.3. *If a simplicial complex C is the union of its subcomplexes C_1 and C_2, i.e., $C = C_1 \cup C_2$, then*

$$\begin{aligned}(-1)^{\dim C} h_{\dim C+1}(C) &= (-1)^{\dim C_1} h_{\dim C_1+1}(C_1) + (-1)^{\dim C_2} h_{\dim C_2+1}(C_2) \\ &\quad - (-1)^{\dim C_1\cap C_2} h_{\dim C_1\cap C_2+1}(C_1 \cap C_2).\end{aligned}$$

Proof. From the definition of h-vector,

$$(-1)^{\dim C} h_{\dim C+1}(C) = -f_{-1}(C) + f_0(C) - f_1(C) + \cdots + (-1)^{\dim C} f_{\dim C}.$$

The statement follows immediately from this.
(This is just the same to say that the reduced Euler characteristic is a valuation.) ∎

For a 2-dimensional simplicial complex, the link of a vertex is a 1-dimensional simplicial complex, i.e., a graph. We say a vertex is *splittable* if the graph appeared as its link has a bridge, where a *bridge* is a vertex of a graph such that the removal of the vertex increases the number of connected components of the graph.

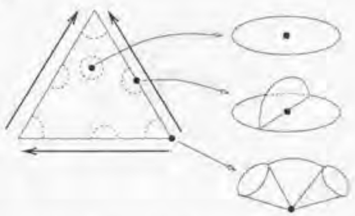
An easily observed fact is the following lemma.

Lemma 5.4. *Assume that C is a pure 2-dimensional simplicial complex and it is divided into two parts C_1 and C_2 such that $C_1 \cap C_2$ is a 1-dimensional simplicial complex and $C_1 \cup C_2 = C$. If C has at most one splittable vertex, then $C_1 \cap C_2$ is a graph with cycles, i.e., it is not a tree. Especially, $h_2(C_1 \cap C_2) > 0$.*

Proof. If a graph is a tree, then it has at least two end vertices. But in $C_1 \cap C_2$, a vertex is an end vertex only if it is a splittable vertex in C. Thus the graph $C_1 \cap C_2$ can not be a tree.

The inequality $h_2(C_1 \cap C_2) > 0$ is easily deduced from the calculation of h-vectors of 1-dimensional complexes that $h_2 = \#\text{edges} - \#\text{vertices} + 1$. ∎

For example, the standard points are not splittable. In the dunce hat, the neighborhood of each point is as follows:



So there is only one point which is splittable (the non-standard point at the top of the hat), and the other points are all standard. Thus any division of the dunce hat into two parts always produces a graph with positive h_2 in their intersection.

What we show is the following proposition.

Proposition 5.5. *A contractible 2-dimensional simplicial complex C with at most one splittable vertex is not constructible. Especially, any triangulations of a contractible near-standard polyhedron are all non-constructible.*

Proof. Let C be constructible. Then there are two subcomplexes C_1 and C_2 satisfying the condition of constructibility. Here, $\dim C_1 = \dim C_2 = 2$ and $\dim C_1 \cap C_2 = 1$. We observe that:

- From Lemma 5.3, we have $h_3(C) = h_3(C_1) + h_3(C_2) + h_2(C_1 \cap C_2)$.
- From Lemma 5.4, $h_2(C_1 \cap C_2) > 0$.
- Because C is contractible, $h_3(C) = 0$.

Hence one of $h_3(C_1)$ and $h_3(C_2)$ should be negative. This contradicts the fact that C_1 and C_2 are constructible because Cohen-Macaulay complexes has non-negative h-vectors from Proposition 2.23.

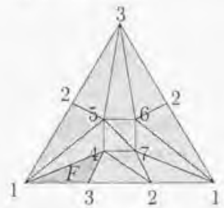
∎

Since the dunce hat is a contractible near-standard polyhedron, we have the following corollary.

Corollary 5.6. *Any triangulation of the dunce hat is not constructible.* ∎

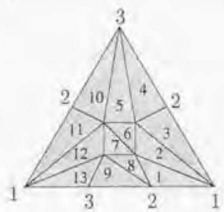
5.3 Shellable but not extendably shellable 2-complexes

Slight modification of the dunce hat gives a very interesting example. See the following figure. (The vertices with the same labeling are identified.)



In this example D', the boundary is just one edge $\{1,3\}$. On the other hand, counting the number of faces shows that $f(D') = (1, 7, 19, 13)$, thus $h(D') = (1, 4, 8, 0)$. So $h_3(D') = 0$. (This can be shown from the fact that D' is contractible.) Again this shows that every facet F_i in a shelling $F_1, F_2, \ldots, F_t$ should satisfy $(F_1 \cup \cdots \cup F_{i-1}) \cap F_i \neq \partial F_i$. Thus the only facet which can be chosen for the last facet is the facet F indicated in the figure. That there is a shelling can be checked easily as indicated in the figure below. So we conclude that this simplicial complex D' is shellable and that all its shelling ends at the unique facet F.

This property that all shelling ends at one unique facet implies that the simplicial complex D' is *not* extendably shellable because every partial shelling which starts from F will not extend to the whole shelling of D'.



Proposition 5.7. *There are 2-dimensional simplicial complexes which are shellable but not extendably shellable.* ∎

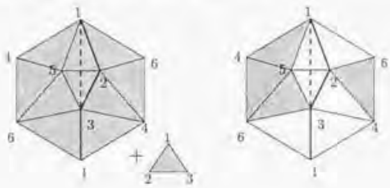
By taking a careful look of the triangulation above, one will notice that it is not vertex decomposable, i.e., any deletion of one vertex produces a non-shellable complex. Thus we have shown the following at the same time.

Proposition 5.8. *There are 2-dimensional simplicial complexes which are shellable but not vertex decomposable.* ∎

Remark. The existence of shellable but not extendably shellable 2-complexes is not new. Björner [14, p.277, Exercise 7.37] shows a smaller example:

$$123, 125, 126, 134, 136, 145, 234, 235, 246, 356, 456.$$

This is a triangulation of the projective plane plus one additional facet "123". This example is shellable in this order of facets, however 145-456-246-356 is a partial shelling but this will not extend further. The f-vector of this example is $(1, 6, 15, 11)$.
(For this, I thank Günter M. Ziegler for the information of this example, and also Fumihiko Takeuchi for letting me know the example of the partial shelling can not be extended. Björner and Eriksson [16] also has a reference to this example.)



Remark. If we take a barycentric subdivision of D', then it is vertex decomposable because the barycentric subdivision of a shellable complex is vertex decomposable, see for example Björner and Wachs [22]. On the other hand, it is still not extendably shellable. The author does not know whether there are extendably shellable but not vertex decomposable 2-complexes or not.

Remark. After the remark above was written, Moriyama and Takeuchi [68] answered the question. That is, they made two 2-dimensional examples which are extendably shellable but not vertex decomposable. Both of their examples consist of 6 vertices and 10 facets.

Remark. Simon [82] showed an example of 2-dimensional shellable simplicial complex with 7 vertices and 14 facets:

$$123, 134, 125, 147, 157, 234, 235\ 247, 357, 267, 367, 246, 356, 456,$$

which has the property that every shelling ends by "456" as same as our example D'. Although this one has a larger f-vector $(1, 7, 20, 14)$ than our D', it has precisely the same property as D', that is, $h_3 = 0$ (in fact it is contractible) while it has only one edge "45" as its boundary. Thus this example can be seen as a variant of our example D'. (I thank Fumihiko Takeuchi for letting me know about this paper.)

In Simon [82], the following 2-dimensional simplicial complex with 6 vertices and 10 facets is also given:

$$123, 234, 134, 146, 156, 125, 235, 246, 356, 456.$$

This example is shellable but not vertex decomposable, so Proposition 5.8 is not a new result.

Though the latter example was given as an example of shellable but not vertex decomposable one, it is also an example of shellable but not extendably shellable simplicial complex which is even smaller than Björner's examples remarked in the preceding page. This fact is pointed out by Moriyama and Takeuchi below.

Remark. Recently Moriyama [67] and Moriyama and Takeuchi [68] made an attempt to enumerate small 2-dimensional shellable simplicial complexes, and found out many examples of shellable but not extendably shellable ones. Among them, two examples have only 6 vertices and 9 facets (both has f-vectors $(1, 6, 14, 9)$), even smaller than Simon's example in the remark above. The lists of facets of these examples are the following:

$$\text{1-V6F9: } 124, 126, 134, 135, 245, 256, 346, 356, 456,$$

and

$$\text{2-V6F9: } 123, 126, 135, 234, 245, 256, 346, 356, 456.$$

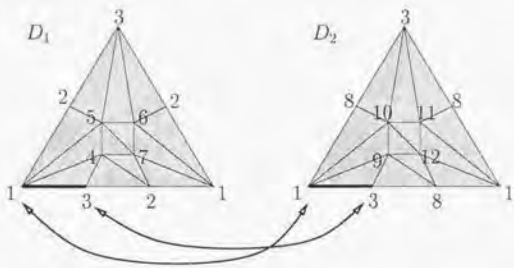
These examples are shown by computer enumeration to be the minimum among the examples which are shellable but not extendably shellable simplicial complexes.

These two examples are contractible and the reasoning of their non-extendable shellability are similar to our D', though the boundaries of both of them consist of two edges.

5.4 Constructible but not shellable 2-complexes

The next question is whether there are constructible 2-complexes which are not shellable. The answer is yes.

An example arises from the example shown in the previous section. Let D_1 and D_2 be two copies of D' in the previous section and D'' the simplicial complex derived by joining them by the edge $\{1,3\}$.



It is easy to check that D'' is constructible, because D_1 and D_2 are shellable and $D_1 \cap D_2 = \{1,3\}$. But D'' is not shellable. The non-shellability is shown as same as the case of the dunce hat: D'' has no boundary but $h_3 = 0$. (In this example, $f = (1, 12, 37, 26)$ and $h = (1, 9, 16, 0)$. The fact that $h_3 = 0$ is also deduced from the fact that D'' is contractible.)

Thus we have the following proposition.

Proposition 5.9. *There are 2-dimensional simplicial complexes which are constructible but not shellable.* ∎

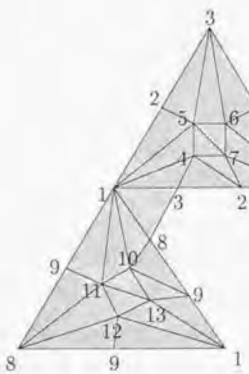
Remark. This example has two splittable vertices, 1 and 3.

5.5 Subdivisions and shellability

One more question arises from Proposition 2.31: whether shellability is topological for general 2-dimensional simplicial complexes or not. We have the following proposition which answers this question.

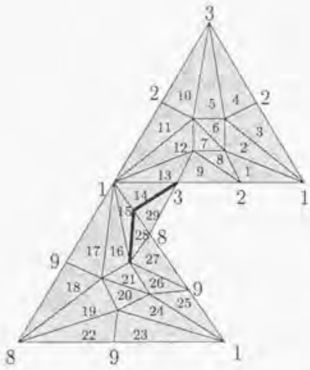
Proposition 5.10. *There are 2-dimensional simplicial complexes which are not shellable but has a shellable subdivision, that is, shellability for 2-dimensional simplicial complexes is not a topological property.*

Proof. An example is the following, namely, two copies of D' of Section 5.3 are joined by a triangle.



The example shown in the figure is not shellable: its h-vector is $(1, 10, 16, 0)$ (f-vector is $(1, 13, 39, 27)$) and $h_3 = 0$. (This again can be calculated from the fact that this example is contractible.) This shows that the 2-face $\{1, 3, 8\}$ must be the last facet in every shelling because the edge $\{3, 8\}$ is the only edge on the boundary, but it cannot be the last one because removing it gives a non-shellable complex because the link of the vertex 1 becomes disconnected. Thus no facet can be the last facet in a shelling, which means that the complex is not shellable.

But let us stellarly subdivide it as follows.



Then it becomes shellable: the small numbering in the figure shows a possible example of its shelling. ∎

As for stellar subdivisions of 2-dimensional simplicial complexes, the following theorem is known.

Theorem 5.11. *(Ewald [37])*
Let C and C' be two simplicial complexes such that $|C| \simeq |C'|$, then there is a simplicial complex C'' which is a common stellar subdivision of C and C'.

There are two types of stellar subdivisions for 2-dimensional simplicial complexes: (i) p is taken in the interior of a 2-face, and (ii) p is taken in the interior of a 1-face. Brugesser and Mani has shown the following theorem.

Theorem 5.12. *(Brugesser and Mani [25]) Every stellar subdivision of a shellable d-dimensional simplicial complex is again shellable.*

Thus if the reverse operations of stellar subdivisions of the above two types preserve shellability, shellability becomes topological property, but the example above shows that the type (ii) will not preserve shellability in general. However the type (i) preserves shellability as follows.

Proposition 5.13. *Let C be a 2-dimensional simplicial complex and σ be a 2-face of C. Let p be an interior point of σ and C' be the stellar subdivision of C by p. If C' is shellable, C is shellable.*

Proof. Let the facet $\sigma = abc$ be divided into three facets $\sigma_1 = abp$, $\sigma_2 = bcp$ and $\sigma_3 = cap$ by the stellar subdivision. Let $\pi' : \tau_1, \tau_2, \ldots, \tau_k, \sigma_1, \tau_{k+1}, \ldots, \tau_l, \sigma_2, \tau_{l+1}, \ldots, \tau_m, \sigma_3, \tau_{m+1}, \ldots, \tau_t$

be a shelling of C'. (Permute a, b and c if needed.) We will show that an ordering of facets $\pi : \tau_1, \tau_2, \dots, \tau_k, \tau_{k+1}, \dots, \tau_l, \sigma, \tau_{l+1}, \tau_m, \tau_{m+1}, \dots, \tau_t$ is a shelling of C.

Let us check that each facet satisfies the condition of shelling in this new ordering. Let us denote by $\mathrm{Prev}_\pi(\tau_i)$ the union $\tau_1 \cup \tau_2 \cup \cdots, \tau_{i-1}$ in an ordering π of facets.

- First, τ_1 to τ_k satisfies the condition of shelling is obvious because there is no difference from the shelling π' of C'.

- Because $\mathrm{Prev}_{\pi'}(\sigma_1) \cap \sigma_1$ is an edge ab and the edges ap and bp will not appear in $\mathrm{Prev}_{\pi'}(\tau_i)$ while $i \leq l$, $\mathrm{Prev}_{\pi'}(\tau_i) \cap \tau_i = \mathrm{Prev}_\pi(\tau_i) \cap \tau_i$ for $k+1 \leq i \leq l$. Thus τ_{k+1} to τ_l satisfies the condition of shelling.

- The edge ab is in $\mathrm{Prev}_\pi(\sigma)$ because $\mathrm{Prev}_\pi(\sigma_1) = \mathrm{Prev}_{\pi'}(\sigma_1)$ contains it. Thus if σ do not satisfy the condition of shelling in π, then there is only one possibility: the vertex b is contained in $\mathrm{Prev}_\pi(\sigma)$ but two edges bc and ca are not contained in it. But if so, in π', $\mathrm{Prev}_{\pi'}(\sigma_2)$ should be the union of the edge ap and the vertex b, contradicting that π' is a shelling. So σ satisfies the condition of shelling.

- From τ_{l+1} to τ_m, the difference between $\mathrm{Prev}_{\pi'}(\tau_i) \cap \tau_i$ and $\mathrm{Prev}_\pi(\tau_i) \cap \tau_i$ is that the edge ca is always contained in the latter but it may not be contained in the former. But this difference will not corrupt the condition of shelling. Thus τ_{l+1} to τ_m satisfies the condition of shelling.

- From τ_{m+1} to τ_t, $\mathrm{Prev}_{\pi'}(\tau_i) \cap \tau_i = \mathrm{Prev}_\pi(\tau_i) \cap \tau_i$, so the condition of shelling is satisfied.

Thus π is a shelling of C, which shows that C is shellable. ∎

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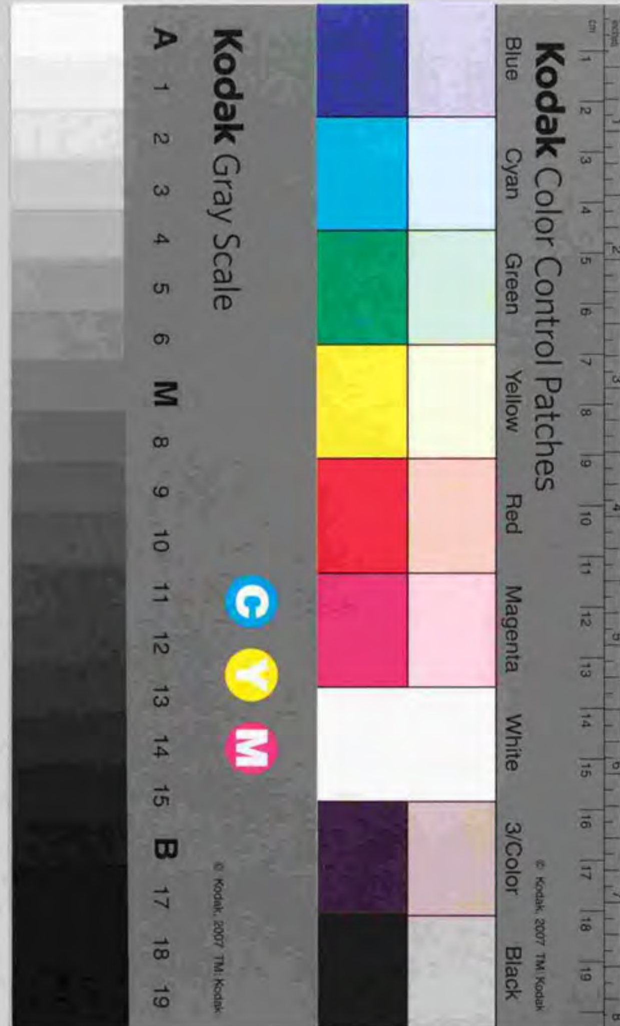
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