On the p-adic elliptic polylogarithm for CM-elliptic curves

(虚数乗法を持つ楕円曲線のp進楕円ポリログについて)

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ON THE p-ADIC ELLIPTIC POLYLOGARITHM FOR CM-ELLIPTIC CURVES

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ABSTRACT. The purpose of this paper is to calculate the p-adic realization of the elliptic polylogarithm for a CM-elliptic curve $\mathbf{E}$ over $\mathbb{Q}$, following the method of Beilinson and Levine [BL] as explained by Huber and Kings [HK], and Wildeshaus [W1] [W2]. We will calculate the specialization of this object at torsion points, and will prove that it is related to special values of elliptic analogues of p-adic polylogarithmic functions.

0. INTRODUCTION

0.1. **Introduction.** The elliptic polylogarithm sheaf, first constructed by Beilinson and Levine [BL], is a pro-variation of mixed Hodge structures on an elliptic curve minus the identity element. The strength of their construction is that it applies to any reasonable theory of mixed sheaves. In particular, the elliptic polylogarithm sheaf is conjectured to be *motivic*. The case in which the elliptic curve has complex multiplication, the specialization of elliptic polylogarithms at torsion points is related to the special values of the complex L-function, as predicted by the Beilinson conjecture [Bei2].

The purpose of this paper is to calculate the p-adic realization of the elliptic polylogarithm sheaf for a CM elliptic curve $\mathbf{E}$ over $\mathbb{Q}$. We will show that the specialization at torsion points of this sheaf is related to the one-variable p-adic L-function associated to $\mathbf{E}$.

Let $\mathbf{E}$ be an elliptic curve over $\mathbb{Q}$ with complex multiplication by the integer ring $\mathcal{O}_{\mathbf{K}}$ of a imaginary quadratic field $\mathbf{K}$. We fix a prime $p > 3$ such that $p = \mathfrak{p}\mathfrak{p}^*$ splits in $\mathbf{K}$. This implies that $\mathbf{E}$ has good *ordinary* reduction over the primes above p.

We let $\psi = \psi_{\mathbf{E}/\mathbf{K}}$ be the Größencharakter of $\mathbf{K}$ associated to $\mathbf{E}/\mathbf{K}$, with conductor $\mathfrak{f}$ prime to p. Let $\mathbf{F} = \mathbf{K}(\mathfrak{f}) = \mathbf{K}(\mathbf{E}[\mathfrak{f}])$ be the ray class field of $\mathbf{K}$ modulo $\mathfrak{f}$, and let $\mathbf{F}_\infty = \mathbf{K}(\mathfrak{f}\mathfrak{p}^\infty)$.

We fix a prime $\mathfrak{P}$ of $\mathbf{F}$ above $\mathfrak{p}$, and we let $\mathbf{F}_{\mathfrak{P}}$ be the completion of $\mathbf{F}$ at $\mathfrak{P}$. We fix an absolutely unramified extension K of $\mathbf{F}_{\mathfrak{P}}$ with ring of integers $\mathcal{O}_K$. Fix a smooth model E over $\mathcal{O}_K$ of $E_K = \mathbf{E} \otimes_{\mathbb{Q}} K$.

We define $\mathscr{H} = H^1(E, K(1))$ to be the *geometric syntomic cohomology* of E with coefficients in $K(1)$. See Definition 1.12 for the precise definition. It is a two-dimensional K-vector space with filtration $F^\bullet$ and Frobenius φ,

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and it splits in to the direct sum of two one-dimensional filtered Frobenius modules $\mathscr{H} = \mathscr{H}(\hat{\omega}) \oplus \mathscr{H}(\hat{\eta})$ with fixed basis $\hat{\omega}$ and $\hat{\eta}$.

Let $S(E)$ be the category of *syntomic coefficients* on E, defined in Definition 1.5. It is a rough p-adic analogue of the category of variation of mixed Hodge structures. The *elliptic logarithmic sheaf* $\mathcal{L}og$ is a pro-object in $S(E)$. One of the main features of this object is the *splitting principle*:

Proposition (= Corollary 2.12). *For a torsion point $x \in E(\mathcal{O}_K)$ of order prime to p, we have*

$$x^*\mathcal{L}og = \prod_{n\geq 0} \mathrm{Sym}^n \mathscr{H}.$$

Let $U = E \setminus O$, where O is the identity element of E. Let $\widehat{\mathscr{H}}$ be the dual of $\mathscr{H}$, and denote by the same symbol the pull back of $\widehat{\mathscr{H}}$ to U. Let $H^1_{\mathrm{syn}}(U, \widehat{\mathscr{H}} \otimes \mathcal{L}og(1))$ be the absolute syntomic cohomology with coefficients in $\mathcal{L}og$ defined in Definition 1.21. The *elliptic polylogarithm* is an element in

$$\mathrm{pol} \in H^1_{\mathrm{syn}}(U, \widehat{\mathscr{H}} \otimes \mathcal{L}og(1)),$$

where $H^1_{\mathrm{syn}}(U, \widehat{\mathscr{H}} \otimes \mathcal{L}og(1))$ is the *absolute syntomic cohomology* (See Definition 1.21 for the definition of absolute syntomic cohomology). The precise definition of pol is given in Definition 3.3. The significance of this element is that it is conjectured to be motivic of origin. In other words, it is conjectured to be the image by a suitable regulator map of the *motivic elliptic polylogarithm* in the corresponding motivic cohomology group.

The pull back of pol by $x \in U(\mathcal{O}_K)$ as above gives an element

$$x^*\mathrm{pol} \in H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes x^*\mathcal{L}og(1)) = \prod_{n\geq 0} H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathrm{Sym}^n \mathscr{H}(1)).$$

The module $\mathscr{H}(\hat{\omega})^{\otimes j-1}$ is a direct summand of $\mathscr{H} \otimes \mathrm{Sym}^j \widehat{\mathscr{H}}$.

Fix a proper ideal $\mathfrak{a} = (a)$ of $\mathcal{O}_{\mathbf{K}}$ prime to $\mathfrak{fp}$, and let $U_a = E \setminus E[\mathfrak{a}]$. We define the *modified elliptic polylogarithm* by

$$\mathrm{pol}_a = (\mathbf{N}\mathfrak{a})\,\mathrm{pol} - [a]^*\,\mathrm{pol}.$$

Here, $[a] : U_a \to U$ is the multiplication by a.

Let $\Theta_{\mathbf{E},\mathfrak{a}}$ be the rational function on $\mathbf{E}$ with a zero of degree $12\mathbf{N}\mathfrak{a}$ at O, and poles of degree 12 at each point in $E[\mathfrak{a}] \setminus O$ (See Definition 4.1). In calculating the element pol_a explicitly, we will prove that there exists a unique system of over convergent functions $D_{\mathfrak{a},j}$ on U_{aK} for $j \geq 1$ satisfying the differential equations

$$dD_{\mathfrak{a},1} = \left(\frac{\varphi}{p} - 1\right) \frac{d\Theta_{\mathbf{E},\mathfrak{a}}}{\Theta_{\mathbf{E},\mathfrak{a}}}$$
$$dD_{\mathfrak{a},j+1} = -D_{\mathfrak{a},j}\,\omega.$$

Here, φ is the Frobenius induced from the *multiplication by* $\psi_{\mathbf{E}/\mathbf{K}}(\mathfrak{p})$ *map* on $\mathbf{E}$. For each j, we will call the function $D_{\mathfrak{a},j}$ the *elliptic polylogarithmic function* (of weight j). This is the elliptic analogue of the p-adic polylogarithmic function $\ell_j^{(p)}$ defined by Coleman [Co].

Definition . Let x be a non-zero torsion point of $E(\mathcal{O}_K)$ of order prime to $\mathfrak{a}p$. For $j \geq 1$, we define the map $h_{x,j}$ to be the composition of

$$H^1_{\mathrm{syn}}(U_{\mathfrak{a}}, \varpi^* \widehat{\mathscr{H}} \otimes \mathcal{L}og(1)) \xrightarrow{x^*} \prod_{n\geq 0} H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathrm{Sym}^n \mathscr{H}(1))$$

with the projection

$$H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathrm{Sym}^j \mathscr{H}(1)) \longrightarrow H^1_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H}(\hat{\omega})^{\otimes j-1}(1))$$

and the canonical isomorphism

$$H^1_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H}(\hat{\omega})^{\otimes j-1}(1)) \xrightarrow{\cong} K\,\hat{\omega}^{\otimes j-1}(1).$$

We prove the following:

Theorem (= Theorem 5). *Let x be a non-zero torsion point of $E(\mathcal{O}_K)$ of order prime to $\mathfrak{a}p$. For any integer $j \geq 1$, we have*

$$h_{x,j}(12\,\mathrm{pol}_{\mathfrak{a}}) = D_{\mathfrak{a},j}(x)\,\hat{\omega}^{\otimes j-1}(1).$$

Here, $D_{\mathfrak{a},j}(x)$ is the value of the function $D_{\mathfrak{a},j}$ at x.

This result is a generalization of the result of Coleman-de Shalit ([CdS], the case when $j = 1$). Since the object pol is conjectured to be motivic of origin, this result gives indication that the the analogue of the results of Gros-Kurihara [GK] and Gros [G] in the cyclotomic case holds in the elliptic situation. In particular, the values $D_{\mathfrak{a},j}(x)$ should have arithmetic meaning.

Along this line of thought, we will prove in Proposition 5.16 that there is some relation between the elliptic polylogarithmic function and the one-variable p-adic L-function of $\mathbf{E}$.

CONTENTS

0.2. **Overview.** The p-adic cohomology theory that we are going to use is rigid syntomic cohomology defined by Amnon Besser, extended in our case to deal with coefficients. The first section of this paper is devoted to the construction of the formalism of rigid syntomic cohomology with coefficients.

Let K be a finite unramified extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$. Let X be a scheme smooth and of finite type over $\mathcal{O}_K$, with smooth compactification $\overline{X}$, such that the complement D is a normal crossing divisor relative to $\mathcal{O}_K$.

We will first define the category $S^{\mathrm{ad}}(X)$ of admissible syntomic coefficients on X (Definition 1.8), which is a rough p-adic analogue of the category of variation of mixed Hodge structures.

Next, for an object $\mathscr{M} \in S^{\mathrm{ad}}(X)$, we will define the *geometric syntomic cohomology*, denoted as $H^i(X, \mathscr{M})$ (Definition 1.12), and *absolute syntomic cohomology*, denoted as $H^i_{\mathrm{syn}}(X, \mathscr{M})$ (Definition 1.21). The geometric syntomic cohomology is the p-adic analogue of Betti cohomology (with the mixed Hodge structure), and absolute syntomic cohomology is the p-adic analogue of Beilinson-Deligne cohomology.

The main theorem in the first part of this paper is Theorem 1, which gives a canonical isomorphism

$$\mathrm{Ext}^1_{S(X)}(K(0), \mathscr{M}) \xrightarrow{\cong} H^1_{\mathrm{syn}}(X, \mathscr{M})$$

for $\mathscr{M}$ in $S^{\mathrm{ad}}(X)$. This is a p-adic analogue of the classical fact that the first Deligne cohomology is isomorphic to the extension group of variation of mixed Hodge structures.

Starting from the second section, we will deal with the elliptic curve mentioned in the introduction. Let $E_K = \mathbf{E} \otimes_{\mathbb{Q}} K$, and let E be a smooth model of E_K over $\mathcal{O}_K$.

In the second section, we will first define the logarithmic sheaf $\mathcal{L}og$ over E, which is a pro-object in the category $S^{\mathrm{ad}}(E)$ (Definition 2.6). Then we will calculate its geometric syntomic cohomology (Proposition 2.8), and will prove the splitting principle (Corollary 2.12).

In the third section, we will define the p-adic elliptic polylogarithm. The definition is parallel to that of the classical case, following the papers [BL], [HK] and [W2].

In the fourth section, we will review the construction of the one-variable p-adic L-function $L_{p,\mathfrak{a}}$ associated to the elliptic curve $\mathbf{E}$ as above. This section is included for the convenience of the reader, and does not contain any new results.

The last section is devoted to the proof of Theorem 5. The main ingredient is the explicit calculation of the partial polylogarithmic element $\widetilde{\mathrm{pol}}_a$ in $S(U_a)$. Since the morphism $h_{x,j}$ maps pol_a through $\widetilde{\mathrm{pol}}_a$, this is enough to prove our main theorem.

As in the classical case, there is the *rigidity principle* (Proposition 5.11). In other words, the extension class of $\widetilde{\mathrm{pol}}_a$ is uniquely determined by its underlying coherent module with connection. This principle allows much easier calculation of $\widetilde{\mathrm{pol}}_a$.

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1. The formalism of rigid syntomic cohomology

1.1. The category of syntomic coefficients.

In this section, we will first define the category of syntomic data $\mathscr{D}_K$ (Definition 1.1). This is a set of data which is necessary to define syntomic cohomology. Then, for $\mathfrak{X}$ in $\mathscr{D}_K$, we will define the category of syntomic coefficients (Definition 1.5), which is a rough p-adic analogue of the category of variation of mixed Hodge structures. In Definition 1.8, we will define the notion of an admissible syntomic coefficient, which is necessary to define absolute syntomic cohomology in Section 1.2.

Then, in Definition 1.12, we will give the definition of *geometric syntomic cohomology*, which is a p-adic analogue of singular cohomology. The geometric syntomic cohomology has a structure of a filtered Frobenius module, much in the same way that singular cohomology has a mixed Hodge structure.

In Example 1.13, we will give calculations of geometric syntomic cohomology in some special cases. Finally, we will prove some basic properties of this cohomology.

Definition 1.1. We define the category of syntomic data $\mathscr{D}_K$ as follows. The object in this category is a triple $\mathfrak{X} = (X, \overline{X}, \phi_X)$, where

(i) X is a smooth scheme over $\mathcal{O}_K$ of finite type.
(ii) $\overline{X}$ is a smooth projective compactification of X such that the complement D is a strict normal crossing divisor relative to $\mathcal{O}_K$.
(iii) Let $\overline{\mathcal{X}}$ be the formal completion of $\overline{X}$ with respect to the special fiber. Then $\phi_X : \overline{\mathcal{X}} \to \overline{\mathcal{X}}$ is a lifting of the absolute Frobenius of $\overline{X}_k$.

The morphism $f : \mathfrak{X} \to \mathfrak{Y}$ in this category is a morphism $f : \overline{X} \to \overline{Y}$ of schemes over $\mathcal{O}_K$ such that $f(X) \subset Y$ and the map induced on the formal completion is compatible with the Frobenius.

Throughout this section, fix a syntomic datum $\mathfrak{X}$ in $\mathscr{D}_K$. We denote by $X_K = X \otimes K$ the generic fiber of X, by $X_k = X \otimes k$ the special fiber of X, by X_K^{an} the rigid analytic space associated to X_K, by $\mathcal{X}$ the formal completion of X with respect to the special fiber, and by $\mathcal{X}_K$ the rigid analytic space associated to $\mathcal{X}$. We will use the same notations for $\overline{X}$. By ([Ber1] Proposition 0.3.5), there is an isomorphism $\overline{\mathcal{X}}_K \simeq \overline{X}_K^{\mathrm{an}}$.

Let $j : \mathcal{X}_K \hookrightarrow \overline{\mathcal{X}}_K$ be the natural inclusion. We say that a subset $V \subset \overline{\mathcal{X}}_K$ is a strict neighborhood of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$, if

$$\overline{\mathcal{X}}_K = V \cup (\overline{\mathcal{X}}_K \setminus \mathcal{X}_K)$$

is an admissible covering of $\overline{\mathcal{X}}_K$.

For any abelian sheaf M^0 on V, we let

$$j_V^\dagger M^0 = \varinjlim_{V' \subset V} \alpha_{VV'*} \alpha_{VV'}^* M^0,$$

where the limit is taken with respect to the strict neighborhoods V' of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$ with inclusion $\alpha_{VV'} : V' \hookrightarrow V$. If M^0 has a structure of a $\mathcal{O}_V$-module, then $j_V^\dagger M^0$ has a structure of a $j_V^\dagger \mathcal{O}_V$-module, and the functors $\alpha_{VV'*}$ and $\alpha_{VV'}^*$ give an equivalence between the category of $j_V^\dagger \mathcal{O}_V$-modules and the category of $j_{V'}^\dagger \mathcal{O}_{V'}$-modules.

Definition 1.2. Let V be a strict neighborhood of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$, and let $\alpha_V : V \hookrightarrow \overline{\mathcal{X}}_K$ be the inclusion. We define $j^\dagger$ to be the functor

$$j^\dagger M^0 = \alpha_{V*} j_V^\dagger M^0$$

with values in the category of sheaves on $\overline{\mathcal{X}}_K$.

If M^0 is a $j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_K}$-module with connection ∇^0, we denote by $\phi_X^* M^0$ the $j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_K}$-module with connection which is defined to be the inverse image of M^0 with respect to the Frobenius.

Definition 1.3. A *Frobenius structure* on M^0 is a horizontal isomorphism $\Phi_{\mathscr{M}} : \phi_X^* M^0 \simeq M^0$.

Remark 1.4. The category of $j^\dagger \mathcal{O}_{\overline{\mathcal{X}}_K}$-modules with overconvergent integral connection and Frobenius structure is a *realization*, in the sense of [Ber1] p.68, of the category $F\text{-Isoc}^\dagger(X_k/K)$ of overconvergent F-isocrystals on X_k.

Suppose M is a coherent $\mathcal{O}_{\overline{X}_K}$-module of finite rank with integrable logarithmic connection

$$\nabla : M \to M \otimes \Omega^1_{\overline{X}_K}(\log D).$$

We define $\mathbf{p}_{\mathrm{rig}}(M, \nabla) = (M_{\mathrm{rig}}, \nabla_{\mathrm{rig}})$ to be the pair such that: M_{rig} is the coherent $j^\dagger\mathcal{O}_{\overline{\mathcal{X}}_K}$-module $j^\dagger(M|_{X_K})^{\mathrm{an}}$, and $\nabla_{\mathrm{rig}} : M_{\mathrm{rig}} \to M_{\mathrm{rig}} \otimes \Omega^1_{\overline{\mathcal{X}}_K}$ is the connection on M_{rig} induced from ∇. Then $\mathbf{p}_{\mathrm{rig}}$ is an exact functor from the category of coherent $\mathcal{O}_{\overline{X}_K}$-modules of finite rank with integrable logarithmic connection along D to the category of coherent $j^\dagger\mathcal{O}_{\overline{\mathcal{X}}_K}$-modules with integrable connection.

Definition 1.5. We define the category of *syntomic coefficients* on $\mathfrak{X}$ to be the category $S(\mathfrak{X})$ defined as follows:

The objects of $S(\mathfrak{X})$ consist of the 4-uple $\mathscr{M} = (M, \nabla, F^\bullet, \Phi_{\mathscr{M}})$ where

(i) M is a coherent $\mathcal{O}_{\overline{X}_K}$-module of finite rank.

(ii) ∇ is an integrable connection on M with logarithmic poles along D.

(iii) $F^\bullet$ is a descending exhaustive separated filtration by sub $\mathcal{O}_{\overline{X}_K}$-modules on M, called the *Hodge filtration*, satisfying Griffith's transversality

$$\nabla(F^m M) \subset F^{m-1} M \otimes \Omega^1_{\overline{X}_K}(\log D).$$

(iv) $\Phi_{\mathscr{M}} : \phi^*_X M_{\mathrm{rig}} \simeq M_{\mathrm{rig}}$ is a Frobenius structure on M_{rig}.

The morphisms in this category are homomorphisms of underlying $\mathcal{O}_{\overline{X}_K}$-modules which are compatible with the above structures.

Remark 1.6.

a) For any object $\mathscr{M} = (M, \nabla, F^\bullet, \Phi_{\mathscr{M}})$ in $S(\mathfrak{X})$, we say that the pair (M, ∇) is the *underlying coherent module with connection* of $\mathscr{M}$.

b) By ([Ber1] Theorem 2.5.7), condition (iv) above ensures that the connection ∇_{rig} on M_{rig} is *overconvergent* (*loc. cit. Definition 2.2.5*).

Definition 1.7. For each integer j, we define the Tate object $K(j)$ to be the object $K(j) = (M, \nabla, F^\bullet, \Phi_{\mathscr{M}})$ in $S(\mathfrak{X})$ defined as follows:

(i) $M = \mathcal{O}_{\overline{X}_K} e_j$ is a free $\mathcal{O}_{\overline{X}_K}$-module of rank one with generator e_j.

(ii) The connection ∇ is defined by $\nabla(e_j) = 0$.

(iii) The Hodge filtration $F^\bullet$ is defined by $F^{-j}M = M$ and $F^{-j+1}M = 0$.

(iv) We have $\phi^*_X M_{\mathrm{rig}} = M_{\mathrm{rig}}$. The Frobenius morphism is defined to be $\Phi_{\mathscr{M}}(e_j) = p^{-j} e_j$.

Suppose $\mathscr{M} = (M, \nabla, F^\bullet, \Phi_{\mathscr{M}})$ is an object in $S(\mathfrak{X})$. We denote by $\mathrm{DR}^\bullet_{\mathrm{dR}}(\mathscr{M})$ the de Rham complex

$$\cdots \to M \xrightarrow{\nabla} M \otimes \Omega^1_{\overline{X}_K}(\log D) \xrightarrow{\nabla} M \otimes \Omega^2_{\overline{X}_K}(\log D) \xrightarrow{\nabla} \cdots$$

where M is at degree 0. This complex is a filtered complex with filtration given on each term by

$$F^m \mathrm{DR}^q_{\mathrm{dR}}(\mathscr{M}) = F^{m-q} M \otimes \Omega^q_{\overline{X}_K}(\log D).$$

Similarly, we denote by $\mathrm{DR}^\bullet_{\mathrm{rig}}(\mathscr{M})$ the complex

$$\cdots \to M_{\mathrm{rig}} \xrightarrow{\nabla} M_{\mathrm{rig}} \otimes \Omega^1_{\overline{\mathcal{X}}_K} \xrightarrow{\nabla} M_{\mathrm{rig}} \otimes \Omega^2_{\overline{\mathcal{X}}_K} \xrightarrow{\nabla} \cdots .$$

We define the de Rham cohomology $H^i_{\mathrm{dR}}(\mathfrak{X}, \mathscr{M})$ of $\mathfrak{X}$ with coefficients in $\mathscr{M}$ by

$$H^i_{\mathrm{dR}}(\mathfrak{X}, \mathscr{M}) = \mathbb{R}^i\Gamma(\overline{X}_K, \mathrm{DR}^\bullet_{\mathrm{dR}}(\mathscr{M})).$$

The de Rham cohomology has a filtration, called the *Hodge filtration*, induced from the spectral sequence ([Del1] 1.4.5)

$$E_1^{i,j} = \mathbb{R}^{i+j}\Gamma(\overline{X}_K, \mathrm{Gr}^i_F(\mathrm{DR}^\bullet_{\mathrm{dR}}(\mathscr{M}))) \Rightarrow H^{i+j}_{\mathrm{dR}}(\mathfrak{X}, \mathscr{M}). \tag{1.1}$$

We define the rigid cohomology $H^i_{\mathrm{rig}}(\mathfrak{X}, \mathscr{M})$ of $\mathfrak{X}$ with coefficients in $\mathscr{M}$ by

$$H^i_{\mathrm{rig}}(\mathfrak{X}, \mathscr{M}) = \mathbb{R}^i\Gamma(\overline{\mathcal{X}}_K, \mathrm{DR}^\bullet_{\mathrm{rig}}(\mathscr{M})).$$

This K-vector space has a σ-linear endomorphism ϕ, called the *Frobenius*, defined to be the composition

$$\begin{aligned} \phi : \mathbb{R}^i\Gamma(\overline{\mathcal{X}}_K, \mathrm{DR}^\bullet_{\mathrm{rig}}(\mathscr{M})) &\to \mathbb{R}^i\Gamma(\overline{\mathcal{X}}_K, \mathrm{DR}^\bullet_{\mathrm{rig}}(\phi^*_X \mathscr{M})) \\ &\xrightarrow{\Phi_{\mathscr{M}}} \mathbb{R}^i\Gamma(\overline{\mathcal{X}}_K, \mathrm{DR}^\bullet_{\mathrm{rig}}(\mathscr{M})). \end{aligned} \tag{1.2}$$

We define a natural homomorphism

$$\theta : H^i_{\mathrm{dR}}(\mathfrak{X}, \mathscr{M}) \to H^i_{\mathrm{rig}}(\mathfrak{X}, \mathscr{M}) \tag{1.3}$$

as follows: Let $i : X_K \hookrightarrow \overline{X}_K$ be the inclusion. The natural injection $\mathrm{DR}^\bullet_{\mathrm{dR}}(\mathscr{M}) \hookrightarrow i_*(M|_{X_K} \otimes \Omega^\bullet_{X_K})$ induces the map

$$\mathbb{R}^i\Gamma(\overline{X}_K, \mathrm{DR}^\bullet_{\mathrm{dR}}(\mathscr{M})) \to \mathbb{R}^i\Gamma(X_K, M|_{X_K} \otimes \Omega^\bullet_{X_K}), \tag{1.4}$$

and the morphism $p : X_K^{\mathrm{an}} \to X_K$ induces

$$\mathbb{R}^i\Gamma(X_K, M|_{X_K} \otimes \Omega^\bullet_{X_K}) \to \mathbb{R}^i\Gamma(X_K^{\mathrm{an}}, (M|_{X_K})^{\mathrm{an}} \otimes \Omega^\bullet_{X_K^{\mathrm{an}}}). \tag{1.5}$$

There exists a canonical epimorphism $(M|_{X_K})^{\mathrm{an}} \to j^\dagger(M|_{X_K})^{\mathrm{an}} = M_{\mathrm{rig}}$ which induces the morphism

$$\mathbb{R}^i\Gamma(X_K^{\mathrm{an}}, (M|_{X_K})^{\mathrm{an}} \otimes \Omega^\bullet_{X_K^{\mathrm{an}}}) \to \mathbb{R}^i\Gamma(X_K^{\mathrm{an}}, \mathrm{DR}^\bullet_{\mathrm{rig}}(\mathscr{M})). \tag{1.6}$$

Finally, since X_K^{an} is a strict neighborhood of $\mathcal{X}_K$ in $\overline{\mathcal{X}}_K$ ([Ber1] Exemples (1.2.4)(ii)), there is a natural isomorphism

$$\mathbb{R}^i\Gamma(\overline{\mathcal{X}}_K, \mathrm{DR}^\bullet_{\mathrm{rig}}(\mathscr{M})) \simeq \mathbb{R}^i\Gamma(X_K^{\mathrm{an}}, \mathrm{DR}^\bullet_{\mathrm{rig}}(\mathscr{M})). \tag{1.7}$$

We define θ of (1.3) to be the composition of the maps (1.4), (1.5), (1.6) and (1.7).

Definition 1.8. We define the category $S^{\mathrm{ad}}(\mathfrak{X})$ of *admissible syntomic coefficients* on $\mathfrak{X}$ to be the full subcategory of $S(\mathfrak{X})$ consisting of objects $\mathscr{M}$ satisfying the following:

(i) The spectral sequence (1.1) degenerates at E_1.
(ii) The homomorphism θ of (1.3) is an isomorphism.

(iii) The K-vector space $H^i_{\rm rig}(\mathfrak{X}, \mathscr{M})$, with the action of Frobenius ϕ defined in (1.2) and the filtration induced through the isomorphism θ from the Hodge filtration on $H^i_{\rm dR}(\mathfrak{X}, \mathscr{M})$, is a weakly admissible filtered Frobenius module in the sense of Fontaine ([Fon1] 4.1.4).

Remark 1.9.

a) The category $S(\mathfrak{X})$ is *not abelian*, because the morphisms are not necessarily strict with respect to the filtration. We will regard the category as an *exact category* by taking for exact sequences any sequence

$$0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$$

of objects in $S(\mathfrak{X})$ such that the sequence of underlying $\mathcal{O}_{\overline{X}_K}$-modules is exact, and the morphisms are strictly compatible with the Hodge filtration.

b) The categories $S(\mathfrak{X})$ and $S^{\rm ad}(\mathfrak{X})$ are independent of the choice of ϕ_X up to canonical equivalence of categories. This is a direct consequence of [Ber1] Proposition 2.2.17.

c) For $\mathfrak{X} = (\operatorname{Spec}\mathcal{O}_K, \operatorname{Spec}\mathcal{O}_K, \sigma)$, the category $S(\mathcal{O}_K) = S(\mathfrak{X})$ is equivalent to the category of *filtered Frobenius modules* of finite rank over K. The category $S^{\rm ad}(\mathcal{O}_K)$ is equivalent to the category of *weakly admissible filtered Frobenius modules* over K. The latter category is an abelian category.

Definition 1.10. We say that a morphism between filtered K-vector spaces $f : M \to N$ is *strict*, if it satisfies $f(F^iM) = \operatorname{Im} f \cap F^iN$.

Remark 1.11. Let $I^\bullet$ be any filtered flasque resolution of $\mathrm{DR}^\bullet_{\rm dR}(\mathscr{M})$, and let $C^\bullet = \Gamma(\overline{X}_K, I^\bullet)$. By [Del1] Proposition (1.3.2), the condition of Definition 1.8 (i) is satisfied if and only if the differentials of $C^\bullet$ are strict.

Definition 1.12. For $\mathscr{M}$ in $S(\mathfrak{X})$ such that θ is an isomorphism, we define the *geometric syntomic cohomology*, denoted by $H^i(\mathfrak{X}, \mathscr{M})$, to be the filtered Frobenius module defined in Definition 1.8 (iii).

The geometric syntomic cohomology $H^i(\mathfrak{X}, \mathscr{M})$ defined above is independent up to canonical isomorphism of the choice of the Frobenius ϕ_X on $\overline{\mathcal{X}}$ through the canonical equivalence of categories in Remark 1.9 b).

Example 1.13. In the following cases, the Tate objects $K(j)$ are all admissible.

1. If $\mathcal{O}_K = (\operatorname{Spec}\mathcal{O}_K, \operatorname{Spec}\mathcal{O}_K, \sigma)$, then we have
$$\begin{cases} H^0(\mathcal{O}_K, K(j)) = K(j) & \\ H^i(\mathcal{O}_K, K(j)) = 0 & i \geq 1. \end{cases}$$

2. If $\mathbb{A}^1 = (\mathbb{A}^1_{\mathcal{O}_K}, \mathbb{P}^1_{\mathcal{O}_K}, \phi)$, where ϕ is any lifting of the Frobenius, we have
$$\begin{cases} H^0(\mathbb{A}^1, K(j)) = K(j) & \\ H^i(\mathbb{A}^1, K(j)) = 0 & i \geq 1. \end{cases}$$

3. If $\mathbb{G}_m = (\mathbb{G}_{m,\mathcal{O}_K}, \mathbb{P}^1_{\mathcal{O}_K}, \phi)$, we have

$$\begin{cases} H^0(\mathbb{G}_m, K(j)) = K(j) \\ H^1(\mathbb{G}_m, K(j)) = K(j-1) \\ H^i(\mathbb{G}_m, K(j)) = 0 & i \geq 2. \end{cases}$$

Lemma 1.14. *For any exact sequence*

$$0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$$

in $S^{\mathrm{ad}}(\mathfrak{X})$, there is an associated long exact sequence

$$\cdots \to H^i(\mathfrak{X}, \mathscr{M}') \to H^i(\mathfrak{X}, \mathscr{M}) \to H^i(\mathfrak{X}, \mathscr{M}'') \to \cdots$$

in $S^{\mathrm{ad}}(\mathcal{O}_K)$ obtained by pasting together the sequences for de Rham and rigid cohomology.

Proof. The long exact sequence exists for both de Rham and rigid cohomology. Since this sequence is compatible with the comparison isomorphism θ, we can glue the sequences to obtain a sequence of filtered Frobenius modules such that the sequence of underlying K-vector spaces is exact. The fact that the maps are strictly compatible with the Hodge filtration comes from the assumption that the cohomologies are weakly admissible filtered modules ([Fon1] Proposition 4.2.1). □

Remark 1.15. Occasionally in this paper, we will claim that a morphism of filtered Frobenius module is strict with respect to the filtration without the weakly admissible hypothesis. In the special cases considered in this paper, this follows from the fact that the morphism underlies a morphism of mixed Hodge structures, hence is strict with respect to the filtration. It can also be proved by direct computation.

Definition 1.16. Suppose $f : \mathfrak{Y} \to \mathfrak{X}$ is a morphism between syntomic data, and let $\mathscr{M} = (M, \nabla, F^\bullet, \Phi)$ be an object in $S(\mathfrak{X})$. Then $f^*\mathscr{M}$ is defined to be the object

$$f^*\mathscr{M} = (f^*M, f^*\nabla, f^*F^\bullet, f^*\Phi)$$

in $S(\mathfrak{Y})$.

From the definition, we have:

Lemma 1.17. *Let $\pi : \mathfrak{X} \to \mathcal{O}_K$ be the structure morphism, and let $\mathscr{H}$ be an object in $S(\mathcal{O}_K)$. Then there exists a canonical isomorphism*

$$H^i(\mathfrak{X}, \pi^*\mathscr{H}) \simeq H^i(\mathfrak{X}, K(0)) \otimes \mathscr{H}.$$

1.2. Absolute syntomic cohomology.

Let $\mathfrak{X} = (X, \overline{X}, \phi_X)$ be a syntomic datum in $\mathscr{D}_K$, and let $\mathscr{M}$ be an object in $S^{\mathrm{ad}}(\mathfrak{X})$. The purpose of this section is to define the *absolute syntomic cohomology* of $\mathfrak{X}$ with coefficients in $\mathscr{M}$ (Definition 1.21). This is a p-adic analogue of Beilinson-Deligne cohomology. We will first prepare some notations.

Let I be a finite set, and let $\mathscr{U} = \{\overline{U}_i\}_{i \in I}$ be a covering of $\overline{X}$ by Zariski open sets. We put $\overline{U}_{i_0 \cdots i_n K} = \bigcap_{0 \leq j \leq n} \overline{U}_{i_j K}$. Let $\mathrm{DR}^\bullet_{\mathrm{dR}}(\mathscr{M})$ and $\mathrm{DR}^\bullet_{\mathrm{rig}}(\mathscr{M})$ be the complexes associated to $\mathscr{M}$ defined in Section 1.1.

Denote by $\overline{j}_{i_0 \cdots i_n}$ the inclusion

$$\overline{j}_{i_0 \cdots i_n} : \overline{U}_{i_0 \cdots i_n K} \hookrightarrow \overline{X}_K.$$

We define $C^\bullet_{\mathrm{dR}}(\mathscr{U}, \mathscr{M})$ to be the simple complex associated to the double complex

$$\begin{array}{ccccc}
\prod_i \overline{j}_{i*} \mathrm{DR}^0_{\mathrm{dR}}(\mathscr{M}) & \longrightarrow & \prod_{i_0 i_1} \overline{j}_{i_0 i_1 *} \mathrm{DR}^0_{\mathrm{dR}}(\mathscr{M}) & \longrightarrow & \\
\nabla \big\downarrow & & \nabla \big\downarrow & & \\
\prod_i \overline{j}_{i*} \mathrm{DR}^1_{\mathrm{dR}}(\mathscr{M}) & \longrightarrow & \prod_{i_0 i_1} \overline{j}_{i_0 i_1 *} \mathrm{DR}^1_{\mathrm{dR}}(\mathscr{M}) & \longrightarrow & , \\
\nabla \big\downarrow & & \nabla \big\downarrow & &
\end{array}$$

where the horizontal complexes are the Čech complexes. This complex has a filtration induced from the filtration on $\mathrm{DR}^\bullet_{\mathrm{dR}}(\mathscr{M})$, and we have a filtered quasi-isomorphism

$$\mathrm{DR}^\bullet_{\mathrm{dR}}(\mathscr{M}) \xrightarrow{\simeq} C^\bullet_{\mathrm{dR}}(\mathscr{U}, \mathscr{M}). \tag{1.8}$$

Next, let $U_i = \overline{U}_i \cap X$, and let $\mathcal{U}_i$ be the formal completion of U_i with respect to the special fiber. Let $\mathcal{U}_{i_0 \cdots i_n K} = \bigcap_{0 \leq j \leq n} \mathcal{U}_{i_j K}$. We denote by $j_{i_0 \cdots i_n}$ the inclusion

$$j_{i_0 \cdots i_n} : \mathcal{U}_{i_0 \cdots i_n K} \hookrightarrow \overline{\mathcal{X}}_K.$$

We define $C^\bullet_{\mathrm{rig}}(\mathscr{U}, \mathscr{M})$ to be the simple complex associated to the double complex

$$\begin{array}{ccccc}
\prod_i j_i^\dagger \mathrm{DR}^0_{\mathrm{rig}}(\mathscr{M}) & \longrightarrow & \prod_{i_0 i_1} j^\dagger_{i_0 i_1} \mathrm{DR}^0_{\mathrm{rig}}(\mathscr{M}) & \longrightarrow & \\
\nabla \big\downarrow & & \nabla \big\downarrow & & \\
\prod_i j_i^\dagger \mathrm{DR}^1_{\mathrm{rig}}(\mathscr{M}) & \longrightarrow & \prod_{i_0 i_1} j^\dagger_{i_0 i_1} \mathrm{DR}^1_{\mathrm{rig}}(\mathscr{M}) & \longrightarrow & . \\
\nabla \big\downarrow & & \nabla \big\downarrow & &
\end{array}$$

By [Ber1] Proposition 2.1.8, we have a quasi-isomorphism

$$\mathrm{DR}^\bullet_{\mathrm{rig}}(\mathscr{M}) \xrightarrow{\simeq} C^\bullet_{\mathrm{rig}}(\mathscr{U}, \mathscr{M}). \tag{1.9}$$

We let

$$R^\bullet_{\mathrm{dR}}(\mathscr{U}, \mathscr{M}) = \Gamma(\overline{X}_K, C^\bullet_{\mathrm{dR}}(\mathscr{U}, \mathscr{M})) \text{ and}$$
$$R^\bullet_{\mathrm{rig}}(\mathscr{U}, \mathscr{M}) = \Gamma(\overline{\mathcal{X}}_K, C^\bullet_{\mathrm{rig}}(\mathscr{U}, \mathscr{M})).$$

Note that

$$\Gamma(\overline{X}_K, \overline{j}_{i_0 \cdots i_n *} \mathrm{DR}^r_{\mathrm{dR}}(\mathscr{M})) = \Gamma(\overline{U}_{i_0 \cdots i_n K}, \mathrm{DR}^r_{\mathrm{dR}}(\mathscr{M})).$$

The complex $R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M})$ has a filtration defined on each term by

$$F^m\Gamma(\overline{U}_{i_0\cdots i_n K},\mathrm{DR}^r_{\mathrm{dR}}(\mathscr{M}))=\Gamma(\overline{U}_{i_0\cdots i_n K},F^m\,\mathrm{DR}^r_{\mathrm{dR}}(\mathscr{M})).$$

Lemma 1.18. *Suppose $\mathscr{U}=\{\overline{U}_i\}$ is a covering by affine open sets. We have canonical and functorial isomorphisms*

$$\begin{aligned}H^i(F^mR^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M}))&\simeq F^mH^i_{\mathrm{dR}}(\mathscr{U},\mathscr{M})\ \text{and}\\ H^i(R^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M}))&\simeq H^i_{\mathrm{rig}}(\mathscr{U},\mathscr{M}).\end{aligned}\tag{1.10}$$

Proof. By assumption, $\overline{U}_{i_0\cdots i_n K}$ are affine open subschemes of $\overline{X}_K$. Hence the complex $C^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M})$ is a resolution of $\mathrm{DR}^\bullet_{\mathrm{dR}}(\mathscr{M})$ by objects which are acyclic with respect to the functor $\Gamma(\overline{X}_K,-)$. We therefore have a canonical and functorial isomorphism

$$H^i(R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M}))\simeq H^i_{\mathrm{dR}}(\mathscr{U},\mathscr{M}).$$

Since $\mathscr{M}$ is admissible, the differentials of $R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M})$ are strict with respect to the filtration (See Remark 1.11). This implies that

$$H^i(F^mR^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M}))=F^mH^i(R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M}));$$

hence we have the first isomorphism of (1.10).

Next, since the inclusion $X\hookrightarrow\overline{X}$ is an affine morphism, $U_i=\overline{U}_i\cap X$ is an affine scheme. From [Ber1] Proposition 2.5.2, the complex $C^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M})$ is a resolution of the complex $\mathrm{DR}^\bullet_{\mathrm{rig}}(\mathscr{M})$ by objects which are acyclic with respect to the functor $\Gamma(\overline{\mathcal{X}}_K,-)$; hence the second isomorphism of (1.10). $\square$

There are canonical homomorphisms of complexes of K-vector spaces

$$\phi_{\mathscr{U}}:K\otimes_{\sigma,K}R^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M})\to R^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M}),$$

induced from ϕ_X and $\Phi_{\mathscr{M}}$, and

$$\theta_{\mathscr{U}}:R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M})\to R^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M}),$$

induced from the morphism

$$\begin{aligned}\Gamma(\overline{U}_{i_0\cdots i_n K},\mathrm{DR}^r_{\mathrm{dR}}(\mathscr{M}))&\to\Gamma(U^{\mathrm{an}}_{i_0\cdots i_n K},j^\dagger_{i_0\cdots i_n}\mathrm{DR}^r_{\mathrm{rig}}(\mathscr{M}))\\&=\Gamma(\overline{\mathcal{X}}_K,j^\dagger_{i_0\cdots i_n}\mathrm{DR}^r_{\mathrm{rig}}(\mathscr{M})).\end{aligned}$$

Definition 1.19. We define the complex $R^\bullet_{\mathrm{syn}}(\mathscr{U},\mathscr{M})$ by

$$R^\bullet_{\mathrm{syn}}(\mathscr{U},\mathscr{M})=\mathrm{Cone}(F^0R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M})\to R^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M}))[-1],$$

where the map is defined by $y\mapsto(1-\phi_{\mathscr{U}})(1\otimes\theta_{\mathscr{U}}(y))$.

We say that a covering $\mathscr{V}=\{\overline{V}_j\}_{j\in J}$ is a *refinement* of $\mathscr{U}=\{\overline{U}_i\}_{i\in I}$ if there exists a map $\tau:J\to I$ such that $\overline{V}_j\subset\overline{U}_{\tau(j)}$. By fixing such a τ, we have natural morphisms of complexes

$$\tau_{\mathrm{dR}}:F^0R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M})\to F^0R^\bullet_{\mathrm{dR}}(\mathscr{V},\mathscr{M})$$
$$\tau_{\mathrm{rig}}:R^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M})\to R^\bullet_{\mathrm{rig}}(\mathscr{V},\mathscr{M})$$

defined as the restriction with respect to the inclusion $\overline{V}_j \subset \overline{U}_{\tau(j)}$. This induces a map on the cone

$$\tau_{\mathrm{syn}} : R^\bullet_{\mathrm{syn}}(\mathscr{U}, \mathscr{M}) \to R^\bullet_{\mathrm{syn}}(\mathscr{V}, \mathscr{M}).$$

We have the following:

Lemma 1.20. *Let $\mathscr{V}$ be a refinement of $\mathscr{U}$ as above. The map τ_{syn} is independent up to homotopy on the choice of τ.*

Proof. Suppose $\tau' : J \to I$ is another map satisfying $\overline{V}_j \subset \overline{U}_{\tau'(j)}$. We define the map

$$\prod_{i_0 \cdots i_n} \Gamma(\overline{U}_{i_0 \cdots i_n K}, F^0 \mathrm{DR}^\bullet_{\mathrm{dR}}(\mathscr{M})) \xrightarrow{h_{\mathrm{dR}}} \prod_{j_0 \cdots j_{n-1}} \Gamma(\overline{V}_{j_0 \cdots j_{n-1} K}, F^0 \mathrm{DR}^\bullet_{\mathrm{dR}}(\mathscr{M}))$$

by associating to s the element

$$h_{\mathrm{dR}}(s)_{j_0 \cdots j_{n-1}} = \sum_{k=0}^{n} (-1)^k \operatorname{res}_k(s_{\tau(j_0) \cdots \tau(j_k) \tau'(j_k) \cdots \tau'(j_{n-1})}),$$

where res_k is the restriction map with respect to the inclusion

$$\overline{V}_{j_0 \cdots j_{n-1}} \subset \overline{U}_{\tau(j_0) \cdots \tau(j_k) \tau'(j_k) \cdots \tau(j_{n-1})}.$$

This gives rise to a map

$$h_{\mathrm{dR}} : F^0 R^\bullet_{\mathrm{dR}}(\mathscr{U}, \mathscr{M}) \longrightarrow F^0 R^{\bullet-1}_{\mathrm{dR}}(\mathscr{V}, \mathscr{M}),$$

which by [Mi] III Lemma 2.1 satisfies $dh_{\mathrm{dR}} + h_{\mathrm{dR}} d = \tau'_{\mathrm{dR}} - \tau_{\mathrm{dR}}$. By the same method, we can construct a homotopy

$$h_{\mathrm{rig}} : R^\bullet_{\mathrm{rig}}(\mathscr{U}, \mathscr{M}) \longrightarrow R^{\bullet-1}_{\mathrm{rig}}(\mathscr{V}, \mathscr{M})$$

satisfying $dh_{\mathrm{rig}} + h_{\mathrm{rig}} d = \tau'_{\mathrm{rig}} - \tau_{\mathrm{rig}}$. By construction, the diagram

$$\begin{array}{ccc}
F^0 R^i_{\mathrm{dR}}(\mathscr{U}, \mathscr{M}) & \xrightarrow{(1-\phi)(1\otimes\theta)} & R^i_{\mathrm{rig}}(\mathscr{U}, \mathscr{M}) \\
{\scriptstyle h_{\mathrm{dR}}}\downarrow & & {\scriptstyle h_{\mathrm{rig}}}\downarrow \\
F^0 R^{i-1}_{\mathrm{dR}}(\mathscr{V}, \mathscr{M}) & \xrightarrow{(1-\phi)(1\otimes\theta)} & R^{i-1}_{\mathrm{rig}}(\mathscr{V}, \mathscr{M})
\end{array}$$

is commutative; hence h_{dR} and h_{rig} give rise to a homotopy

$$h_{\mathrm{syn}} : R^\bullet_{\mathrm{syn}}(\mathscr{U}, \mathscr{M}) \longrightarrow R^{\bullet-1}_{\mathrm{syn}}(\mathscr{V}, \mathscr{M})$$

satisfying $dh_{\mathrm{syn}} + h_{\mathrm{syn}} d = \tau'_{\mathrm{syn}} - \tau_{\mathrm{syn}}$. □

As in loc. cit. III Remark 2.2 (a), we define $J_{\mathfrak{X}}$ to be the set of *finite* coverings of $\overline{X}$ modulo the equivalence relation: $\mathscr{U} \sim \mathscr{V}$ if each is a refinement of the other. This set is *filtered* with respect to refinements. By Lemma 1.20, the correspondence $\mathscr{U} \mapsto H^i(R^\bullet_{\mathrm{syn}}(\mathscr{U}, \mathscr{M}))$ factors through $J_{\mathfrak{X}}$.

Definition 1.21. We define the *absolute syntomic cohomology* $H^i_{\mathrm{syn}}(\mathfrak{X}, \mathscr{M})$ of $\mathfrak{X}$ with coefficients in $\mathscr{M}$ by

$$H^i_{\mathrm{syn}}(\mathfrak{X}, \mathscr{M}) = \varinjlim_{\mathscr{U} \in J_{\mathfrak{X}}} H^i(R^\bullet_{\mathrm{syn}}(\mathscr{U}, \mathscr{M})),$$

where the limit is taken with respect to refinements.

Remark 1.22.

a) If the Tate object $K(j)$ is admissible, the absolute syntomic cohomology $H^i_{\mathrm{syn}}(\mathfrak{X}, K(j))$ is isomorphic to the rigid syntomic cohomology $H^i_{\mathrm{syn}}(X, j)$ defined by Amnon Besser ([Bes] Definition 6.1).

b) Let $I_{\mathfrak{X}}$ be the subset of $J_{\mathfrak{X}}$ consisting of *affine* coverings. This is also a filtered set. Since $I_{\mathfrak{X}}$ is a final subset of $J_{\mathfrak{X}}$, the definition of absolute syntomic cohomology does not change if we take the limit over $I_{\mathfrak{X}}$ instead of $J_{\mathfrak{X}}$.

Lemma 1.23. *We have a long exact sequence*

$$\begin{aligned} \cdots \to F^0 H^{i+1}_{\mathrm{dR}}(\mathfrak{X}, \mathscr{M}) &\xrightarrow{(1-\phi)\theta} H^{i+1}_{\mathrm{rig}}(\mathfrak{X}, \mathscr{M}) \\ &\to H^i_{\mathrm{syn}}(\mathfrak{X}, \mathscr{M}) \to \cdots \end{aligned}$$

for any $\mathscr{M}$ in $S^{\mathrm{ad}}(\mathfrak{X})$.

Proof. Let $\mathscr{U}$ be an affine covering in $I_{\mathfrak{X}}$. By definition of a cone, we have the long exact sequence

$$\begin{aligned} \cdots \to H^{i+1}(F^0 R^\bullet_{\mathrm{dR}}(\mathscr{U}, \mathscr{M})) &\to H^{i+1}(R^\bullet_{\mathrm{rig}}(\mathscr{U}, \mathscr{M})) \\ &\to H^i(R^\bullet_{\mathrm{syn}}(\mathscr{U}, \mathscr{M})) \to \cdots. \end{aligned}$$

Since the category $I_{\mathfrak{X}}$ is filtered, the direct limit of the above sequence over $I_{\mathfrak{X}}$ is also exact. The lemma now follows from Remark 1.22 b), Lemma 1.18, and the definition of absolute syntomic cohomology. □

Lemma 1.24. *Suppose $\mathscr{M}$ is an object in $S^{\mathrm{ad}}(\mathfrak{X})$. We have the following exact sequence:*

$$\begin{aligned} 0 \to H^1_{\mathrm{syn}}(\mathcal{O}_K, H^i(\mathfrak{X}, \mathscr{M})) &\to H^{i+1}_{\mathrm{syn}}(\mathfrak{X}, \mathscr{M}) \\ &\to H^0_{\mathrm{syn}}(\mathcal{O}_K, H^{i+1}(\mathfrak{X}, \mathscr{M})) \to 0. \end{aligned}$$

Proof. By definition, we have $H^i(\mathfrak{X}, \mathscr{M}) = H^i_{\mathrm{rig}}(\mathfrak{X}, \mathscr{M})$ with the filtration induced through the isomorphism

$$\theta : H^i_{\mathrm{dR}}(\mathfrak{X}, \mathscr{M}) \xrightarrow{\cong} H^i_{\mathrm{rig}}(\mathfrak{X}, \mathscr{M})$$

from the Hodge filtration on de Rham cohomology. Let $\mathscr{U} = \{\operatorname{Spec} \mathcal{O}_K\}$ be the unique element in $J_{\mathcal{O}_K}$. By the definition of absolute syntomic cohomology, for any $\mathscr{N} = (N, F^\bullet, \phi) \in S(\mathcal{O}_K)$, we have

$$R^\bullet_{\mathrm{syn}}(\mathscr{U}, \mathscr{N}) = [F^0 N \xrightarrow{(1-\phi)} N \to 0 \to \cdots \to 0 \to \cdots]. \tag{1.11}$$

By taking $H^i(\mathfrak{X}, \mathscr{M})$ and $H^{i+1}(\mathfrak{X}, \mathscr{M})$ for $\mathscr{N}$, the assertion of the lemma follows from Lemma 1.23. □

Example 1.25.

1. Let $\mathcal{O}_K = (\operatorname{Spec}\mathcal{O}_K, \operatorname{Spec}\mathcal{O}_K, \sigma)$. By (1.11), we have

$$\begin{cases} H^0_{\mathrm{syn}}(\mathcal{O}_K, K(j)) = 0 & j \neq 0 \\ H^0_{\mathrm{syn}}(\mathcal{O}_K, K(j)) = \mathbb{Q}_p e_j & j = 0, \end{cases} \qquad \begin{cases} H^1_{\mathrm{syn}}(\mathcal{O}_K, K(j)) = K e_j & j > 0 \\ H^1_{\mathrm{syn}}(\mathcal{O}_K, K(j)) = \mathbb{Q}_p e_j & j = 0 \\ H^1_{\mathrm{syn}}(\mathcal{O}_K, K(j)) = 0 & j < 0, \end{cases}$$

   and $H^i_{\mathrm{syn}}(\mathcal{O}_K, K(j)) = 0$ for $i \geq 2$.

2. Let $\mathbb{A}^1 = (\mathbb{A}^1_{\mathcal{O}_K}, \mathbb{P}^1_{\mathcal{O}_K}, \phi_X)$ be as before. The above lemma and the calculation in Example 1.13 give the isomorphisms

$$H^0_{\mathrm{syn}}(\mathbb{A}^1, K(j)) \simeq H^0_{\mathrm{syn}}(\mathcal{O}_K, K(j))$$

   and

$$H^1_{\mathrm{syn}}(\mathbb{A}^1, K(j)) \simeq H^1_{\mathrm{syn}}(\mathcal{O}_K, K(j)).$$

Lemma 1.26. *Let $\mathfrak{Y}$ and $\mathfrak{X}$ be objects in $\mathscr{D}_K$, and let $f : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of syntomic data. Let $\mathscr{M}$ be an object in $S^{\mathrm{ad}}(\mathfrak{X})$, and suppose that $f^*\mathscr{M}$ is in $S^{\mathrm{ad}}(\mathfrak{Y})$. Then there is a functorial morphism*

$$f^* : H^i_{\mathrm{syn}}(\mathfrak{X}, \mathscr{M}) \longrightarrow H^i_{\mathrm{syn}}(\mathfrak{Y}, f^*\mathscr{M}). \tag{1.12}$$

Proof. Let $\mathscr{U} = \{\overline{U}_i\}_{i \in I}$ be a covering in $J_{\mathfrak{X}}$. Then $f^{-1}(\mathscr{U}) = \{f^{-1}(\overline{U}_i)\}_{i \in I}$ is a covering in $J_{\mathfrak{Y}}$. Since the diagram

$$\begin{array}{ccc} F^0 R^\bullet_{\mathrm{dR}}(\mathscr{U}, \mathscr{M}) & \longrightarrow & R^\bullet_{\mathrm{rig}}(\mathscr{U}, \mathscr{M}) \\ {\scriptstyle f^*}\big\downarrow & & {\scriptstyle f^*}\big\downarrow \\ F^0 R^\bullet_{\mathrm{dR}}(f^{-1}(\mathscr{U}), f^*\mathscr{M}) & \longrightarrow & R^\bullet_{\mathrm{rig}}(f^{-1}(\mathscr{U}), f^*\mathscr{M}) \end{array}$$

is commutative, f^* induces a morphism on the cone

$$f^* : H^i(R^\bullet_{\mathrm{syn}}(\mathscr{U}, \mathscr{M})) \longrightarrow H^i(R^\bullet_{\mathrm{syn}}(f^{-1}(\mathscr{U}), f^*\mathscr{M})).$$

The same argument as in the proof of Lemma 1.20 shows that f^* is compatible with restriction by refinements; hence we can pass to the limit. The map f^* of (1.12) is defined to be the composition

$$\begin{aligned} H^i_{\mathrm{syn}}(\mathfrak{X}, \mathscr{M}) &= \varinjlim_{\mathscr{U} \in J_{\mathfrak{X}}} H^i(R^\bullet_{\mathrm{syn}}(\mathscr{U}, \mathscr{M})) \\ &\to \varinjlim_{\mathscr{U} \in J_{\mathfrak{X}}} H^i(R^\bullet_{\mathrm{syn}}(f^{-1}(\mathscr{U}), f^*\mathscr{M})) \\ &\to \varinjlim_{\mathscr{V} \in J_{\mathfrak{Y}}} H^i(R^\bullet_{\mathrm{syn}}(\mathscr{V}, f^*\mathscr{M})) = H^i_{\mathrm{syn}}(\mathfrak{Y}, f^*\mathscr{M}). \end{aligned}$$

□

1.3. **Local calculation of the category of syntomic coefficients.**

The purpose of this section is to give a local characterization of the category of syntomic coefficients. We will show that the category can be constructed locally (Lemma 1.28).

Let $\mathfrak{X} = (X, \overline{X}, \phi_X)$ be a syntomic datum, and let $U = \operatorname{Spec} B$ be an affine open subscheme of $\overline{X}$. Then U_K^{an} is a strict neighborhood of $\mathcal{U}_K$ in $\overline{\mathcal{X}}_K$. By [Ber1] Lemma (2.5.4), there exists strict neighborhoods V' and V of $\mathcal{U}_K$ in U_K^{an} such that $\phi_X(V') \subset V$. Denote by $\phi_{V'}$ the restriction of ϕ_X to V'. This map induces a morphism of ringed spaces $\phi_{V'} : (V', j_{V'}^\dagger \mathcal{O}_{V'}) \to (V, j_V^\dagger \mathcal{O}_V)$. Through the identification of [Ber1] (2.5.1.1), this gives rise to a homomorphism of rings $\phi_U^* : B^\dagger \otimes K \to B^\dagger \otimes K$. This homomorphism is independent of the choice of V and V'.

Let $\mathscr{U} = \{\overline{U}_i\}_{i \in I} \in I_{\mathfrak{X}}$ be an affine open covering of $\overline{X}$. Since the inclusion $X \hookrightarrow \overline{X}$ is affine, the set $\{U_i\}_{i \in I}$, where $U_i = \overline{U}_i \cap X$, is an affine covering of X. Let $B_i = \Gamma(U_i, \mathcal{O}_{U_i})$, and take V_i and V_i' as in the previous paragraph. Then $V_i \cap V_j$ and $V_i' \cap V_j'$ are strict neighborhoods of $\mathcal{U}_i \cap \mathcal{U}_j$ in $\overline{\mathcal{X}}_K$, and the commutative diagram

$$\begin{array}{ccc} V_i' \cap V_j' & \xrightarrow{\phi_{V_i' \cap V_j'}} & V_i \cap V_j \\ \downarrow & & \downarrow \\ V_i' & \xrightarrow{\phi_{V_i'}} & V_i \end{array}$$

gives rise to a commutative diagram of morphism of rings

$$\begin{array}{ccc} B_{ij}^\dagger \otimes K & \xleftarrow{\phi_{ij}^*} & B_{ij}^\dagger \otimes K \\ \uparrow & & \uparrow \\ B_i^\dagger \otimes K & \xleftarrow{\phi_i^*} & B_i^\dagger \otimes K, \end{array}$$

where $\phi_i^* = \phi_{U_i}^*$, $\phi_{ij}^* = \phi_{U_i \cap U_j}^*$ and $B_{ij} = \Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j})$.

Let $A_i = \Gamma(\overline{U}_i, \mathcal{O}_{\overline{U}_i})$ and $A_{ij} = \Gamma(\overline{U}_i \cap \overline{U}_j, \mathcal{O}_{\overline{U}_i \cap \overline{U}_j})$. The inclusion $U_i \hookrightarrow \overline{U}_i$ induces a morphism of rings $A_i \to B_i$. Let $\Omega^1_{A_i \otimes K}(\log D)$ be the sub $A_i \otimes K$-module of $\Omega^1_{B_i \otimes K}$ generated over $\Omega^1_{A_i \otimes K}$ by elements of the form $d \log f$ for $f \in (B_i \otimes K)^\times$.

Definition 1.27. Let the notations as above. We define $S(\mathscr{U}, \mathfrak{X})$ to be the category consisting of the 4-uple $(M_i, \nabla_i, F_i^\bullet, \epsilon_{ij}, \Phi_i)$ such that

(i) M_i is a projective $(A_i \otimes K)$-module of finite type with an integrable connection

$$\nabla_i : M_i \to M_i \otimes_{A_i \otimes K} \Omega^1_{A_i \otimes K}(\log D).$$

(ii) $F_i^\bullet$ is a descending exhaustive separated filtration on M_i by sub $(A_i \otimes K)$-modules satisfying

$$\nabla_i(F_i^m M_i) \subset F_i^{m-1} M_i \otimes \Omega^1_{A_i \otimes K}(\log D).$$

(iii) ϵ_{ij} is an $(A_{ij} \otimes K)$-linear isomorphism

$$\epsilon_{ij} : (A_{ij} \otimes K) \otimes_{(A_i \otimes K)} M_i \simeq (A_{ij} \otimes K) \otimes_{(A_j \otimes K)} M_j$$

compatible with ∇_i, preserving $F_i^\bullet$, and satisfying the cocycle condition.

(iv) Let $M_i^0 = (B_i^\dagger \otimes K) \otimes_{(A_i \otimes K)} M_i$. Φ_i is a $(B_i^\dagger \otimes K)$-linear isomorphism

$$\Phi_i : (B_i^\dagger \otimes K) \otimes_{\phi_{i}^*,(B_i^\dagger \otimes K)} M_i^0 \to M_i^0$$

compatible with ∇_i and ϵ_{ij}.

A morphism in this category $f : (M_i, \nabla_i, F_i^\bullet, \epsilon_{ij}, \Phi_i) \to (N_i, \nabla_i, F_i^\bullet, \epsilon_{ij}, \Phi_i)$ is a collection $\{f_i\}$ of $(A_i \otimes K)$-linear morphisms $f_i : M_i \to N_i$ which are compatible with ∇_i, $F_i^\bullet$, ϵ_{ij} and Φ_i.

We define the functor $P_{\mathscr{U}} : S(\mathfrak{X}) \to S(\mathscr{U}, \mathfrak{X})$ from the category of syntomic coefficients on $\mathfrak{X}$ to $S(\mathscr{U}, \mathfrak{X})$ by associating to any syntomic coefficient $\mathscr{M} = (M, \nabla, F^\bullet, \Phi_M)$ the object $(M_i, \nabla_i, F_i^\bullet, \epsilon_{ij}, \Phi_i)$, where $M_i = \Gamma(\overline{U}_{iK}, M)$, the connection ∇_i is induced from the connection on M, the Hodge filtration is given by $F_i^m M_i = \Gamma(\overline{U}_{iK}, F^m M)$, the isomorphisms ϵ_{ij} is the natural identification, and Φ_i is induced from the Frobenius Φ.

Lemma 1.28. *The functor $P_{\mathscr{U}}$ defined above gives an equivalence of categories.*

Proof. The statement of the lemma without the Frobenius is classical. The lemma follows from [Ber1] Propositions (2.1.12) and (2.5.2). □

1.4. Absolute syntomic cohomology and extensions in $S(\mathfrak{X})$.

The purpose of this section is to prove the following theorem:

Theorem 1. *Suppose $\mathfrak{X}$ is a syntomic datum in $\mathscr{D}_K$, and let $\mathscr{M}$ be an object in $S^{\mathrm{ad}}(\mathfrak{X})$. Then there is a canonical and functorial isomorphism*

$$\eta : \mathrm{Ext}^1_{S(\mathfrak{X})}(K(0), \mathscr{M}) \xrightarrow{\ \simeq\ } H^1_{\mathrm{syn}}(\mathfrak{X}, \mathscr{M}).$$

As in the previous section, let $\mathscr{U}$ be an affine covering in $I_{\mathfrak{X}}$. Define the group $\check{H}^i_{\mathrm{syn}}(\mathscr{U}, \mathscr{M})$ by

$$\check{H}^i_{\mathrm{syn}}(\mathscr{U}, \mathscr{M}) = H^i(R^\bullet_{\mathrm{syn}}(\mathscr{U}, \mathscr{M})).$$

The proof of Lemma 1.18 show that there is a canonical and functorial isomorphism

$$\check{H}^i_{\mathrm{syn}}(\mathscr{U}, \mathscr{M}) \xrightarrow{\ \simeq\ } H^i_{\mathrm{syn}}(\mathfrak{X}, \mathscr{M}).$$

Theorem 1 follows from the following proposition.

Proposition 1.29. *The canonical and functorial map*

$$\mathrm{Ext}^1_{S(\mathfrak{X})}(K(0),\mathscr{M}) \longrightarrow \check{H}^1_{\mathrm{syn}}(\mathscr{U},\mathscr{M}) \tag{1.13}$$

is an isomorphism.

Proof. Let $\mathscr{N}$ be an object in $S(\mathfrak{X})$ such that

$$0 \to \mathscr{M} \to \mathscr{N} \to K(0) \to 0$$

is exact. Let $(N_i, \nabla_i, F_i^\bullet, \epsilon_{ij}, \Phi_i)$ be the image of $\mathscr{N}$ with respect to the functor $P_{\mathscr{U}}$ defined in the previous section. From the fact that the underlying morphisms are strict with respect to the filtration, there is an exact sequence of A_{iK}-modules

$$0 \to F^0 M_i \to F^0 N_i \xrightarrow{\pi_i} A_{iK} e_0 \to 0.$$

Since A_{iK} is free, the above equation is split. Let ι_i be such that $\pi_i \circ \iota_i = \mathrm{id}$. For $\tilde{e}_i = \iota_i(e_0)$, we have

$$N_i = M_i \bigoplus A_{iK}\tilde{e}_i$$

as a filtered A_{iK}-module.

Let $a_i = \nabla_i(\tilde{e}_i) \in M_i$, $b_{ij} = \tilde{e}_j - \epsilon_{ij}(\tilde{e}_i) \in A_{ijK} \otimes M_i$, and $c_i = \Phi_i(1 \otimes \tilde{e}_i) - \tilde{e}_i \in B^\dagger_{iK} \otimes M_i$. Through the identification $M_i = \Gamma(\overline{U}_{iK}, M)$, $A_{ijK} \otimes M_i = \Gamma(\overline{U}_{ijK}, M)$, and $B^\dagger_{iK} \otimes M_i = \Gamma(\overline{\mathcal{X}}_K, j_i^\dagger M_{\mathrm{rig}})$, the triple (a_i, b_{ij}, c_i) defines an element in $Z^1(R^\bullet_{\mathrm{syn}}(\mathscr{U},\mathscr{M}))$.

This correspondence gives the map (1.13). That this map is an isomorphism follows from Lemma 1.28, which asserts that giving an object in $S(\mathfrak{X})$ is equivalent to giving local descent data. □

Next, suppose $\mathfrak{X} = (X, \overline{X}, \phi)$ is a syntomic datum as above. We define the category $M(\mathfrak{X})$ to be the category consisting of the pair (M, ∇), where M is a coherent $\mathcal{O}_{\overline{X}_K}$-module of finite rank on $\overline{X}_K$, and ∇ is an integrable connection with logarithmic singularities along D. There is a natural forgetful functor $\mathrm{For}: S(\mathfrak{X}) \to M(\mathfrak{X})$ defined by associating to $\mathscr{M} = (M, \nabla, F^\bullet, \Phi)$ the underlying coherent module with connection (M, ∇) (See Remark 1.6).

A proof similar to Theorem 1 gives the following proposition:

Proposition 1.30. *Suppose $\mathfrak{X}$ is a syntomic datum in $\mathscr{D}_K$, and let $\mathscr{M}$ be an object in $S(\mathfrak{X})$ such that the morphism θ in Definition 1.3 is an isomorphism. Then the canonical and functorial map*

$$\eta: \mathrm{Ext}^1_{M(\mathfrak{X})}(K(0),\mathscr{M}) \longrightarrow H^1(\mathfrak{X},\mathscr{M})$$

is an isomorphism.

Suppose $\mathfrak{X}$ is a syntomic datum in $\mathscr{D}_K$, and let $\mathscr{M}$ be an object in $S^{\mathrm{ad}}(\mathfrak{X})$. From the definition, we have the following Lemma:

Lemma 1.31. *Let*

$$H^1_{\mathrm{syn}}(\mathfrak{X},\mathscr{M}) \longrightarrow H^0_{\mathrm{syn}}(\mathcal{O}_K, H^1(\mathfrak{X},\mathscr{M})) \hookrightarrow H^1(\mathfrak{X},\mathscr{M})$$

be the morphism induced from the second morphism of the exact sequence defined in Lemma 1.24. Then the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Ext}^1_{S(\mathfrak{X})}(K(0),\mathscr{M}) & \xrightarrow{\mathrm{For}} & \mathrm{Ext}^1_{M(\mathfrak{X})}(K(0),\mathscr{M}) \\
\eta\downarrow\simeq & & \eta\downarrow\simeq \\
H^1_{\mathrm{syn}}(\mathfrak{X},\mathscr{M}) & \longrightarrow & H^1(\mathfrak{X},\mathscr{M}).
\end{array}$$

1.5. **The Gysin exact sequence.**

The purpose of this section is to prove the Gysin exact sequence for syntomic cohomology in the very special case needed in our paper.

Let $\mathfrak{X}=(X,\overline{X},\phi_X)$ be a syntomic datum in $\mathfrak{D}_{\mathrm{syn},K}$, such that $\overline{X}$ is a curve over $\mathcal{O}_K$. Assume in addition that for $D=\overline{X}\setminus X$, the Frobenius ϕ_X induces a morphism ϕ_D on D such that for the syntomic datum $\mathfrak{D}=(D,D,\phi_D)$, the inclusion induces a morphism of syntomic data $i:\mathfrak{D}\hookrightarrow\mathfrak{X}$. Let $\overline{\mathfrak{X}}=(\overline{X},\overline{X},\phi_X)$ and let $j:\mathfrak{X}\hookrightarrow\overline{\mathfrak{X}}$ be the inclusion. We have the following:

Theorem 2 (Gysin Exact Sequence). *Let $\mathscr{M}$ be a admissible syntomic coefficient on $\overline{\mathfrak{X}}$, such that the pull-backs of $\mathscr{M}$ to $\mathfrak{X}$ and $\mathfrak{D}$ are also admissible. Then there exists an isomorphism*

$$H^0(\mathfrak{X},j^*\mathscr{M})\simeq H^0(\overline{\mathfrak{X}},\mathscr{M}) \tag{1.14}$$

and an exact sequence

$$0\to H^1(\overline{\mathfrak{X}},\mathscr{M})\to H^1(\mathfrak{X},j^*\mathscr{M})\to H^0(\mathfrak{D},i^*\mathscr{M})(-1)\to H^2(\overline{\mathfrak{X}},\mathscr{M})\to 0$$

in $S(\mathcal{O}_K)$.

Before proving this theorem, we prepare some notations.

Let $\mathscr{U}=\{\overline{U}_i\}_{i\in I}$ be a finite covering of $\overline{X}$ as in Section 1.2. We take the covering $\overline{U}_i$ small enough so that the $\overline{U}_i$ are affine, and there exists local parameters t_i on $\overline{U}_i$ such that for $D\cap\overline{U}_i\neq\emptyset$, the schemes $D\cap\overline{U}_i$ are defined by the equation $t_i=0$. We use the notation of Section 1.3.

We define the complexes $R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M})$ and $R^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M})$ as in Section 1.2. The i-th cohomology of the two complexes are respectively isomorphic to $H^i_{\mathrm{dR}}(\overline{\mathfrak{X}},\mathscr{M})$ and $H^i_{\mathrm{rig}}(\overline{\mathfrak{X}},\mathscr{M})$.

We define the complex $R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M}(\log D))$ by

$$R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M}(\log D)):=R^\bullet_{\mathrm{dR}}(\mathscr{U},j^*\mathscr{M}). \tag{1.15}$$

By definition, the cohomology of this complex is isomorphic to $H^i_{\mathrm{dR}}(\mathfrak{X},j^*\mathscr{M})$.

Let $M_{i_0\cdots i_n}=\Gamma(\overline{U}_{i_0\cdots i_n,K},M)$ and

$$M^\dagger_{i_0\cdots i_n}:=M_{i_0\cdots i_n}\otimes_{A_{i_0\cdots i_n K}}A^\dagger_{i_0\cdots i_n K}. \tag{1.16}$$

We define the complex $R^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M}(\log D))$ to be the single complex associated to the double complex

$$\begin{array}{ccccc} & \prod_i M_i^\dagger & \longrightarrow & \prod_{i_0,i_1} M^\dagger_{i_0 i_1} & \longrightarrow \\ (1.17) & \nabla\big\downarrow & & \nabla\big\downarrow & \\ & \prod_i M_i^\dagger \otimes \Omega^1_{A_i}(\log D) & \longrightarrow & \prod_{i_0,i_1} M^\dagger_{i_0 i_1} \otimes \Omega^1_{A_{i_0 i_1}}(\log D) & \longrightarrow . \end{array}$$

There is a natural morphism of complexes

$$R^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M}(\log D)) \longrightarrow R^\bullet_{\mathrm{rig}}(\mathscr{U},j^*\mathscr{M}). \tag{1.18}$$

Proposition 1.32. *The morphism* (1.18) *is a quasi-isomorphism.*

Proof. This is Theorem 4.2.2 of [T]. □

There is a commutative diagram of complexes

$$\begin{array}{cccc} & R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M}) & \xrightarrow{j^*_{\mathrm{dR}}} & R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M}(\log D)) \\ (1.19) & \big\downarrow & & \big\downarrow \\ & R^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M}) & \xrightarrow{j^*_{\mathrm{rig}}} & R^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M}(\log D)). \end{array}$$

We define a morphisms

$$\begin{aligned} \mathrm{Res}_{\mathfrak{D}/\mathfrak{X}} &: R^\bullet_{\mathrm{dR}}(\mathscr{U},\mathscr{M}(\log D)) \to R^\bullet_{\mathrm{dR}}(\mathscr{U},i^*\mathscr{M})(-1)[-1] \\ \mathrm{Res}_{\mathfrak{D}/\mathfrak{X}} &: R^\bullet_{\mathrm{rig}}(\mathscr{U},\mathscr{M}(\log D)) \to R^\bullet_{\mathrm{rig}}(\mathscr{U},i^*\mathscr{M})(-1)[-1] \end{aligned} \tag{1.20}$$

of complexes of K-vector spaces by

$$f(t_i)\frac{dt_i}{t_i} \mapsto f(0), \tag{1.21}$$

in degree 1, and 0 otherwise.

Proposition 1.33. *With the notation as above, the morphisms* $\mathrm{Res}_{\mathfrak{D}/\mathfrak{X}}$ *induces quasi-isomorphisms*

$$\begin{aligned} \mathrm{Res}_{\mathfrak{D}/\mathfrak{X}}[-1] &: \mathrm{Cone}(j^*_{\mathrm{dR}})[-1] \to R^\bullet_{\mathrm{dR}}(\mathbf{I},i^*\mathscr{M})(-1)[-2] \\ \mathrm{Res}_{\mathfrak{D}/\mathfrak{X}}[-1] &: \mathrm{Cone}(j^*_{\mathrm{rig}})[-1] \to R^\bullet_{\mathrm{rig}}(\mathbf{I},i^*\mathscr{M})(-1)[-2]. \end{aligned} \tag{1.22}$$

Proof. The case of de Rham cohomology is standard. The case for rigid cohomology follows from [T] Proposition 4.3.1. □

Proof of Theorem 2. The condition on the finite covering $\mathscr{U}$ that we imposed is cofinal in $I_{\mathfrak{X}}$. Hence the proof of the theorem follows from lemma 1.18. □

Let $\mathbf{E} = \mathbf{E}_{\mathbb{Q}}$ be an elliptic curve over $\mathbb{Q}$, with complex multiplication by the integer ring $\mathcal{O}_{\mathbf{K}}$ of an imaginary quadratic field $\mathbf{K}$. In this case $\mathbf{K}$ is of class number 1. We fix an isomorphism

$$[-] : \mathcal{O}_{\mathbf{K}} \simeq \mathrm{End}(\mathbf{E}_{\mathbf{K}})$$

such that $[a]^*\omega = a\omega$ for $a \in \mathcal{O}_{\mathbf{K}}$. Let $I_{\mathbf{K}}$ be the group of idèles of $\mathbf{K}$, and let

$$\psi = \psi_{\mathbf{E}/\mathbf{K}} : I_{\mathbf{K}} \to \mathbf{K}^\times$$

be the Größencharakter of $\mathbf{E}_{\mathbf{K}} = \mathbf{E} \otimes_{\mathbb{Q}} \mathbf{K}$. We denote by $\mathfrak{f}$ the conductor of ψ. We will often view ψ as a character on the group of fractional ideals of $\mathbf{K}$ prime to $\mathfrak{f}$.

We assume in addition that p splits in $\mathcal{O}_{\mathbf{K}}$, in the form $p = \mathfrak{p}\mathfrak{p}^*$. This implies that $\mathbf{E}$ has good *ordinary* reduction at p. We will view $\mathbf{K}$ as a subfield of K through the inclusion

$$\mathbf{K} \hookrightarrow \mathbf{K}_{\mathfrak{p}} \xrightarrow{\simeq} \mathbb{Q}_p \hookrightarrow K.$$

Let E_K be the base extension of $\mathbf{E}$ by K. Since $\mathbf{E}$ has good reduction over p, there exists a smooth model E of E_K over $\mathcal{O}_K$. As in Section 1.1, we let E_k be the special fiber of E, $\mathcal{E}$ be the formal completion of E with respect to the special fiber, and $\mathcal{E}_K$ the associated rigid analytic space over K.

Let $\pi = \varphi(\mathfrak{p})$, which is a generator of $\mathfrak{p}$. The endomorphism $[\pi] : E \to E$ over $\mathcal{O}_K$ is a lifting of the *relative Frobenius* of E_k ([Si2] Corollary 5.4). We denote by ϕ_E the lifting of the *absolute Frobenius* of E_k induced from $[\pi]$ on the formal completion $\mathcal{E}$. The triple $\mathfrak{E} = (E, E, \phi_E)$ is a syntomic datum in $\mathscr{D}_{\mathrm{syn},K}$.

2.1. The definition of $\mathcal{L}og^{(1)}$.

In this section, we will first formally define the syntomic coefficient $\mathcal{L}og^{(1)}$ in $S(\mathfrak{E})$. Then we will calculate $\mathcal{L}og^{(1)}$ explicitly.

Since $\mathfrak{E}$ is a syntomic datum which is associated to a variety over a global field, and since it is proper, the Tate object $K(j)$ in $S(\mathfrak{E})$ is admissible. The geometric syntomic cohomology $H^1(\mathfrak{E}, K(0))$ is a K-vector space of dimension 2, with filtration

$$H^1(\mathfrak{E}, K(0)) = F^0H^1(\mathfrak{E}, K(0)) \supset F^1H^1(\mathfrak{E}, K(0)) \supset F^2H^1(\mathfrak{E}, K(0)) = 0.$$

We take a basis ω and η of $H^1(\mathfrak{E}, K(0))$ such that $\omega \in F^1H^1(\mathfrak{E}, K(0))$, and the action of ϕ_E^* is given by

$$\phi_E^*\omega = \pi\omega$$
$$\phi_E^*\eta = \pi^*\eta.$$

We let $\widehat{\mathscr{H}} = H^1(\mathfrak{E}, K(0))$ and $\mathscr{H} = H^1(\mathfrak{E}, K(1))$, regarded as objects in $S^{\mathrm{ad}}(\mathcal{O}_K)$. They are duals to each other in this category. We denote by $\check{\omega}$ and $\check{\eta}$ the basis of $\mathscr{H}$ dual to ω and η.

Definition 2.1. Let $\varpi : E \to \operatorname{Spec} \mathcal{O}_K$ be the structure morphism. The object $\varpi^*\mathscr{H} = (H, \nabla, F^\bullet, \Phi)$ in $S(\mathfrak{E})$ defined in Definition 1.16 is given as follows:

(i) H is the free $\mathcal{O}_{E_K}$-module $H = \mathcal{O}_{E_K}\check{\omega} \bigoplus \mathcal{O}_{E_K}\check{\eta}$ of rank two with basis $\check{\omega}$ and $\check{\eta}$.

(ii) The connection is given by $\nabla(\check{\omega}) = \nabla(\check{\eta}) = 0$.

(iii) The filtration is given by

$$H = F^{-1}H \supset F^0 H = \mathcal{O}_{E_K}\check{\eta} \supset F^1 H = 0.$$

(iv) We have $\phi_E^* H = H$. The Frobenius is given by

$$\Phi(\check{\omega}) = \frac{1}{\pi}\check{\omega}$$
$$\Phi(\check{\eta}) = \frac{1}{\pi^*}\check{\eta}.$$

Lemma 2.2. *The geometric syntomic cohomology of $\varpi^*\mathscr{H}$ is given as follows:*

$$H^0(\mathfrak{E}, \varpi^*\mathscr{H}) = \mathscr{H}$$
$$H^1(\mathfrak{E}, \varpi^*\mathscr{H}) = \mathscr{H} \otimes \widehat{\mathscr{H}}$$
$$H^2(\mathfrak{E}, \varpi^*\mathscr{H}) = \mathscr{H}(1).$$

Proof. This follows from Lemma 1.17. □

To ease the notation, we will omit the symbol ϖ^* in what follows.

The short exact sequence in Lemma 1.24 and the calculation above gives the exact sequence

$$0 \to H^1_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H}) \to H^1_{\mathrm{syn}}(\mathfrak{E}, \mathscr{H}) \to H^0_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H} \otimes \widehat{\mathscr{H}}) \to 0.$$

Through the canonical and functorial isomorphism of Theorem 1, the above exact sequence reads

$$0 \to \operatorname{Ext}^1_{S(\mathcal{O}_K)}(K(0), \mathscr{H}) \xrightarrow{\varpi^*} \operatorname{Ext}^1_{S(\mathfrak{E})}(K(0), \mathscr{H})$$
$$\xrightarrow{u} \operatorname{Hom}_{S(\mathcal{O}_K)}(\mathcal{O}_K, \mathscr{H} \otimes \widehat{\mathscr{H}}) \to 0.$$

Let $e : \operatorname{Spec} \mathcal{O}_K \to E$ be the identity element of E. The sequence above is split by $e^* : S(\mathfrak{E}) \to S(\mathcal{O}_K)$.

We can now formally define the object $\mathcal{L}og^{(1)}$ in $S(\mathfrak{E})$.

Definition 2.3. Let $\mathcal{L}og^{(1)}$ be the unique extension

$$0 \to \mathscr{H} \to \mathcal{L}og^{(1)} \to K(0) \to 0$$

such that:

(i) The image of $\mathcal{L}og^{(1)}$ with respect to u is the standard morphism
$$K(0) \to \mathscr{H} \otimes \widehat{\mathscr{H}} = \mathrm{End}(\mathscr{H}).$$
(ii) The pull-back of $\mathcal{L}og^{(1)}$ by e^* is split.

Next, we will explicitly give the shape of the object $\mathcal{L}og^{(1)}$ in $S(\mathfrak{E})$, using the equivalence of categories in Lemma 1.28. Let $\mathscr{U} = \{\overline{U}_i\}_{i\in I}$ be a finite affine open covering of E, and let $A_i = \Gamma(\overline{U}_i, \mathcal{O}_{\overline{U}_i})$ and $A_{iK} = A_i \otimes K$. We choose the covering so that there exists an unique covering $\overline{U}_0$ such that the unit element O of E is in $\overline{U}_0$. We assume that there exists a local parameter t_0 on $\overline{U}_0$ such that O is defined by $t_0 = 0$.

We let $\boldsymbol{\omega}_i$, $\boldsymbol{\eta}_i$ and $\boldsymbol{\alpha}_{ij}$ such that

$$(\oplus_i\boldsymbol{\omega}_i, 0), (\oplus_i\boldsymbol{\eta}_i, \oplus_{ij}\boldsymbol{\alpha}_{ij}) \in \bigoplus_i \Gamma(\overline{U}_{iK}, \Omega^1_{E_K}) \times \bigoplus_{ij} \Gamma(\overline{U}_{ijK}, \mathcal{O}_{E_K}) \tag{2.1}$$

represents the element ω and η in $H^1(\mathfrak{E}, K(0))$.

Let $\phi_i^* : A^\dagger_{iK} \to A^\dagger_{iK}$ and $\phi_{ij}^* : A^\dagger_{ijK} \to A^\dagger_{ijK}$ be the morphisms of rings induced from the Frobenius ϕ^*. There are also morphisms of modules ϕ_i^* : $\Omega^1_{A_i^\dagger} \otimes K \to \Omega^1_{A_i^\dagger} \otimes K$ and $\phi_{ij}^* : \Omega^1_{A^\dagger_{ij}} \otimes K \to \Omega^1_{A^\dagger_{ij}} \otimes K$ induced from the Frobenius.

Since the above maps induce the Frobenius on $\widehat{\mathscr{H}}$, there exists some a_i and b_i in $A^\dagger_{iK}$ such that

$$\begin{aligned}
\phi_i^*(\boldsymbol{\omega}_i) &= \pi\boldsymbol{\omega}_i + da_i \\
\phi_i^*(\boldsymbol{\eta}_i) &= \pi^*\boldsymbol{\eta}_i + db_i \\
0 &= a_j - a_i \\
\phi_{ij}^*(\boldsymbol{\alpha}_{ij}) &= \pi^*\boldsymbol{\alpha}_{ij} + b_j - b_i.
\end{aligned}$$

From the above equation, (a_i) defines an element in $\Gamma(\mathcal{E}_K, \mathcal{O}_{\mathcal{E}_K}) = K$. Without loss of generality, we can take $a_i = 0$.

We let $v_i' = (1/\pi^*)b_i$, and $v_i = v_i' - v_0'(0)$.

Definition 2.4. Let $(L_i^{(1)}, \nabla_i, F_i^\bullet, \epsilon_{ij}, \Phi_i)$ be an object in $S(\mathscr{U}, \mathfrak{E})$ given as follows:

(i) $L_i^{(1)}$ is the free A_{iK}-module $L_i^{(1)} = A_{iK}e_i \bigoplus A_{iK}\hat{\omega} \bigoplus A_{iK}\hat{\eta}$.
(ii) The connection is given by $\nabla_i(\hat{\omega}) = \nabla_i(\hat{\eta}) = 0$, and $\nabla(e_i) = \hat{\omega} \otimes \boldsymbol{\omega}_i + \hat{\eta} \otimes \boldsymbol{\eta}_i$.
(iii) The filtration is given by
$$L_i^{(1)} = F^{-1}L_i^{(1)} \supset F^0L^{(1)} = A_{iK}e_i \bigoplus A_{iK}\hat{\eta} \supset F^1L^{(1)} = 0.$$
(iv) The isomorphism
$$\epsilon_{ij} : A_{ijK} \otimes_{A_{iK}} L_i^{(1)} \simeq A_{ijK} \otimes_{A_{jK}} L_j^{(1)}$$
is given by $\epsilon_{ij}(\hat{\omega}) = \hat{\omega}$, $\epsilon_{ij}(\hat{\eta}) = \hat{\eta}$, and $\epsilon_{ij}(e_i) = e_j - \boldsymbol{\alpha}_{ij}\hat{\eta}$.

(v) The Frobenius Φ_i on $L_i^{(1)}$ is given by

$$\Phi_i(1 \otimes \hat{\omega}) = \frac{1}{\pi}\hat{\omega}$$
$$\Phi_i(1 \otimes \hat{\eta}) = \frac{1}{\pi^*}\hat{\eta},$$

and $\Phi_i(1 \otimes e_i) = e_i + v_i\hat{\eta}$.

Proposition 2.5. *Let $P_{\mathscr{U}} : S(\mathfrak{E}) \to S(\mathscr{U}, \mathfrak{E})$ be the functor in Lemma 1.28. Then we have $P_{\mathscr{U}}(\mathcal{L}og^{(1)}) \simeq (L_i^{(1)}, \nabla_i, F_i^\bullet, \epsilon_{ij}, \Phi_i)$.*

Proof. By definition, the object $P_{\mathscr{U}}(\mathcal{L}og^{(1)})$ is an extension of $K(0)$ by $\mathscr{H}$. It is sufficient to check the characterizations (i) and (ii) of $\mathcal{L}og^{(1)}$. □

2.2. The definition of the elliptic logarithmic sheaf.

In this section, we will define the elliptic logarithmic sheaf $\mathcal{L}og^{(n)}$ in $S(\mathfrak{E})$. Then, we will calculate its geometric syntomic cohomology.

Definition 2.6. We define $\mathcal{L}og^{(n)}$ to be the n-th symmetric power

$$\mathcal{L}og^{(n)} = \mathrm{Sym}^n \mathcal{L}og^{(1)}.$$

The image of this object with respect to $P_{\mathscr{U}}$ is given by the 5-uple

$$(L_i^{(n)}, \nabla_i, F_i^\bullet, \epsilon_{ij}, \Phi_i),$$

where

(i) $L_i^{(n)}$ is the free A_{iK}-module

$$L_i^{(n)} = \prod_{\substack{r+s+t=n \\ r,s,t\geq 0}} A_{iK}(e_i^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t).$$

(ii) The connection is given by

$$\nabla_i(e_i^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t) = re_i^{(r-1)} \cdot \hat{\omega}^s \cdot \hat{\eta}^t(\hat{\omega} \otimes \boldsymbol{\omega}_i + \hat{\eta} \otimes \boldsymbol{\eta}_i).$$

(iii) The filtration is given by

$$F^m L_i^{(n)} = \prod_{\substack{r+s+t=n \\ -m\geq s \\ r,t\geq 0}} A_{iK}(e_i^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t)$$

(iv) The isomorphism ϵ_{ij} is given by

$$\epsilon_{ij}(e_i^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t) = (e_j - \boldsymbol{\alpha}_{ij}\hat{\eta})^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t.$$

(v) The Frobenius Φ_i on $L_i^{(n)}$ is given by

$$\Phi_i(e_i^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t) = \frac{1}{\pi^{*s}\pi^t}(e_i + v_i\hat{\eta})^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t.$$

There is a morphism of sheaves $\mathcal{L}og^{(n+1)} \to \mathcal{L}og^{(n)}$, which is defined locally by

$$\left(e_i^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t\right) \mapsto \begin{cases} r\left(e_i^{(r-1)} \cdot \hat{\omega}^s \cdot \hat{\eta}^t\right) & r > 0 \\ 0 & r = 0. \end{cases}$$

Through this morphism, we will regard the system $\left(\mathcal{L}og^{(n)}\right)_{n \in \mathbb{N}}$ as a pro-object in the category $S(\mathfrak{E})$.

Remark 2.7. 1. The sheaf $\mathcal{L}og^{(n)}$ fits into the sequence

$$0 \to \operatorname{Sym}^{n+1} \mathscr{H} \to \mathcal{L}og^{(n+1)} \to \mathcal{L}og^{(n)} \to 0.$$

2. We define $\mathcal{L}og^{(0)}$ to be $K(0)$.
3. From the splitting property of $\mathcal{L}og^{(1)}$, we have

$$e^* \mathcal{L}og^{(n)} = \prod_{s=0}^{n} \operatorname{Sym}^s \mathscr{H}$$

for $n \geq 0$. We fix this isomorphism so that

$$e_i^r \cdot \widehat{\omega}^s \cdot \widehat{\eta}^t \mapsto r!\,(\widehat{\omega}^s \cdot \widehat{\eta}^t). \tag{2.2}$$

This ensures that the diagram

$$\begin{array}{ccc} e^* \mathcal{L}og^{(n+1)} & \xrightarrow{\simeq} & \prod_{s=0}^{n+1} \operatorname{Sym}^s \mathscr{H} \\ \downarrow & & \downarrow \\ e^* \mathcal{L}og^{(n)} & \xrightarrow{\simeq} & \prod_{s=0}^{n} \operatorname{Sym}^s \mathscr{H}, \end{array}$$

where the right vertical arrow is the natural projection on the direct summand, is commutative.

Proposition 2.8. *We have*

$$H^0(\mathfrak{E}, \mathcal{L}og^{(n)}) = \operatorname{Sym}^n \mathscr{H}$$
$$H^1(\mathfrak{E}, \mathcal{L}og^{(n)}) = \operatorname{Sym}^{n+1} \mathscr{H}(-1)$$
$$H^2(\mathfrak{E}, \mathcal{L}og^{(n)}) = K(-1).$$

Proof. The proof is by induction on n. If $n = 0$, then $\mathcal{L}og^{(0)} = K(0)$, hence the proposition is immediate from the definition. Next, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{H} \otimes \operatorname{Sym}^n \mathscr{H} & \to & \mathcal{L}og^{(1)} \otimes \operatorname{Sym}^n \mathscr{H} & \to & \operatorname{Sym}^n \mathscr{H} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \operatorname{Sym}^{n+1} \mathscr{H} & \to & \mathcal{L}og^{(n+1)} & \to & \mathcal{L}og^{(n)} & \to & 0 \end{array}$$

where the first two vertical arrows are induced from the multiplication map, and the last one is the canonical inclusion. The horizontal sequences are exact in $S(\mathfrak{E})$. The first boundary morphism of the long exact sequence for geometric syntomic cohomology gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^n \mathscr{H} & \longrightarrow & \widehat{\mathscr{H}} \otimes \mathscr{H} \otimes \mathrm{Sym}^n \mathscr{H} \\ \downarrow{\simeq} & & \downarrow{\mathrm{id} \otimes \mathrm{mult}} \\ H^0(\mathfrak{E}, \mathcal{L}og^{(n)}) & \xrightarrow{\gamma} & \widehat{\mathscr{H}} \otimes \mathrm{Sym}^{n+1} \mathscr{H}. \end{array}$$

This shows that γ is induced by the multiplication map, hence is injective. It follows that

$$\mathrm{Sym}^{n+1} \mathscr{H} = H^0(\mathfrak{E}, \mathrm{Sym}^{n+1} \mathscr{H}) \simeq H^0(\mathfrak{E}, \mathcal{L}og^{(n+1)}).$$

The second boundary morphism of the long exact sequence for geometric syntomic cohomology gives a commutative diagram

$$\begin{array}{ccc} \mathscr{H}(-1) \otimes \mathrm{Sym}^n \mathscr{H} & \xrightarrow{\simeq} & \mathscr{H}(-1) \otimes \mathrm{Sym}^n \mathscr{H} \\ \downarrow & & \downarrow{\mathrm{mult}} \\ H^1(\mathfrak{E}, \mathcal{L}og^{(n)}) & \xrightarrow{\delta} & (\mathrm{Sym}^{(n+1)} \mathscr{H})(-1). \end{array}$$

Hence δ is surjective, hence an isomorphism by reasons of dimension. □

2.3. The pullback of the logarithmic sheaf.

Let the notations be as above. Let a be an element in $\mathcal{O}_{\mathbf{K}}$ prime to p, and let $[a] : \mathbf{E} \to \mathbf{E}$ the endomorphism induced from a. Since $\mathrm{End}(E)$ is commutative, this induces a morphism $[a] : \mathfrak{E} \to \mathfrak{E}$ of syntomic datum.

The purpose of this section is to prove the following:

Proposition 2.9. *There exists an isomorphism*

$$\mathcal{L}og^{(n)} \xrightarrow{\simeq} [a]^* \mathcal{L}og^{(n)}.$$

Since $\mathcal{L}og^{(n)} = \mathrm{Sym}^n \mathcal{L}og^{(1)}$, in order to prove the proposition, it is sufficient to prove that there exists an isomorphism

$$\varrho_a : \mathcal{L}og^{(1)} \xrightarrow{\simeq} [a]^* \mathcal{L}og^{(1)}.$$

We will prove this explicitly (Lemma 2.11). We first prepare some notations.

Let $\{\overline{U}_i\}_{i \in I}$ be an affine covering of E. Let $\overline{V}_i = [a]^{-1}(\overline{U}_i)$. Since $[a]$ is an affine morphism, $\{\overline{V}_i\}_{i \in I}$ is also an affine covering of E. We let $\widetilde{\omega}_i$, $\widetilde{\eta}_i$ be elements in $\Gamma(\overline{U}_{iK}, \Omega^1_{E_K})$, and $\widetilde{\alpha}_{ij}$ be elements in $\Gamma(\overline{U}_{iK}, \mathcal{O}_{E_K})$ such that $(\oplus_i \widetilde{\omega}_i, 0)$ and $(\oplus_i \widetilde{\eta}_i, \oplus_{ij} \widetilde{\alpha}_{ij})$ represents the element ω and η in $H^1(\mathfrak{E}, K(0))$.

The morphism

$$[a]^* : H^1(\mathfrak{E}, K(0)) \longrightarrow H^1(\mathfrak{E}, K(0))$$

is an isomorphism of filtered Frobenius modules, given on the basis by

$$[a]_i^*(\omega_i) = a\widetilde{\omega}_i$$
$$[a]_i^*(\eta_i) = a^* \widetilde{\eta}_i + dT_i$$

for some T_i in C_{iK}, such that

$$[a]_{ij}^*(\boldsymbol{\alpha}_{ij}) = a^*\widetilde{\boldsymbol{\alpha}}_{ij} + (T_j - T_i). \tag{2.3}$$

Lemma 2.10. *Let*

$$\widetilde{b}_i = \frac{1}{a^*}\left([a]_i^*(b_i) + \pi^* T_i - \widetilde{\phi}_i^*(T_i)\right).$$

The action of the Frobenius on $\widetilde{\boldsymbol{\omega}}_i$ and $\widetilde{\boldsymbol{\eta}}_i$ is given by

$$\begin{aligned}
\widetilde{\phi}_i^*(\widetilde{\boldsymbol{\omega}}_i) &= \pi\widetilde{\boldsymbol{\omega}}_i \\
\widetilde{\phi}_i^*(\widetilde{\boldsymbol{\eta}}_i) &= \pi^*\widetilde{\boldsymbol{\eta}}_i + d\widetilde{b}_i \\
\widetilde{\phi}_{ij}^*(\widetilde{\boldsymbol{\alpha}}_{ij}) &= \pi^*(\widetilde{\boldsymbol{\alpha}}_{ij}) + \widetilde{b}_j - \widetilde{b}_i.
\end{aligned}$$

Proof. The compatibility of the Frobenius with $[a]$ gives the equalities $\widetilde{\phi}_i^* \circ [a]_i^* = [a]_i^* \circ \phi_i^*$ and $\widetilde{\phi}_{ij}^* \circ [a]_{ij}^* = [a]_{ij}^* \circ \phi_{ij}^*$. We have

$$\begin{aligned}
\widetilde{\phi}_i^*(\widetilde{\boldsymbol{\omega}}_i) &= (1/a)\widetilde{\phi}_i^* \circ [a]_i^*(\boldsymbol{\omega}_i) \\
&= (1/a)[a]_i^* \circ \phi_i^*(\boldsymbol{\omega}_i) \\
&= (\pi/a)[a]_i^*\boldsymbol{\omega}_i \\
&= \pi\widetilde{\boldsymbol{\omega}}_i,
\end{aligned}$$

$$\begin{aligned}
\widetilde{\phi}_i^*(\widetilde{\boldsymbol{\eta}}_i) &= (1/a^*)\widetilde{\phi}_i^*([a]_i^*(\boldsymbol{\eta}_i) - dT_i) \\
&= (1/a^*)\left([a]_i^*(\widetilde{\phi}_i^*(\boldsymbol{\eta}_i)) - \widetilde{\phi}_i^*(dT_i)\right) \\
&= (1/a^*)\left([a]_i^*(\pi^*\boldsymbol{\eta}_i + db_i) - \widetilde{\phi}_i^*(dT_i)\right) \\
&= \pi^*\widetilde{\boldsymbol{\eta}}_i + (1/a^*)([a]_i^*(db_i) + \pi^* dT_i - \widetilde{\phi}_i^*(dT_i)).
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{b}_j - \widetilde{b}_i &= \frac{1}{a^*}\left([a]_{ij}^*(b_j - b_i) + \pi^*(T_j - T_i) - \widetilde{\phi}_{ij}^*(T_j - T_i)\right) \\
&= \frac{1}{a^*}\left([a]_{ij}^*(\phi_{ij}^*\boldsymbol{\alpha}_{ij} - \pi^*\boldsymbol{\alpha}_{ij}) + \pi^*(T_j - T_i) - \widetilde{\phi}_{ij}^*([a]_{ij}^*\boldsymbol{\alpha}_{ij} - a^*\widetilde{\boldsymbol{\alpha}}_{ij})\right) \\
&= \phi_{ij}^*(\widetilde{\boldsymbol{\alpha}}_{ij}) + \pi^*(T_j - T_i) - \pi^*(\widetilde{\boldsymbol{\alpha}_{ij}} + (T_j - T_i)) \\
&= \phi_{ij}^*(\widetilde{\boldsymbol{\alpha}}_{ij}) - \pi^*\widetilde{\boldsymbol{\alpha}_{ij}}
\end{aligned}$$

as desired. □

We let $\overline{U}_i = \operatorname{Spec} A_i$ and $\overline{V}_i = \operatorname{Spec} C_i$. Then $[a]$ induces homomorphisms $[a]_i^* : A_i \to C_i$. The coefficient $[a]^*\mathcal{L}og^{(1)} = ([a]^*L_i^{(1)}, \nabla_i, F_i^\bullet, \epsilon_{ij}, \Phi_i)$ is given locally on $\overline{V}_i$ as follows:

(i) $[a]^*L_i^{(1)}$ is the free C_{iK}-module $L_i^{(1)} = C_{iK}e_i \bigoplus C_{iK}\hat{\omega} \bigoplus C_{iK}\hat{\eta}$.
(ii) The connection is given by $\nabla_i(\hat{\omega}) = \nabla_i(\hat{\eta}) = 0$, and

$$\nabla(e_i) = \hat{\omega} \otimes [a]_i^*(\boldsymbol{\omega}_i) + \hat{\eta} \otimes [a]_i^*(\boldsymbol{\eta}_i).$$

(iii) The filtration is given by

$$\begin{aligned}[a]^*L_i^{(1)} = F^{-1}([a]^*L_i^{(1)}) \supset F^0([a]^*L^{(1)}) \\ = C_{iK}e_i \bigoplus C_{iK}\hat{\eta} \supset F^1([a]^*L^{(1)}) = 0.\end{aligned}$$

(iv) The isomorphism

$$\epsilon_{ij} : C_{ijK} \otimes_{C_{iK}} [a]^*L_i^{(1)} \simeq C_{ijK} \otimes_{C_{jK}} [a]^*L_j^{(1)}$$

is given by $\epsilon_{ij}(\hat{\omega}) = \hat{\omega}$, $\epsilon_{ij}(\hat{\eta}) = \hat{\eta}$, and $\epsilon_{ij}(e_i) = e_j - [a_{ij}]^*(\alpha_{ij})\hat{\eta}$.

(v) The Frobenius Φ_i on $[a]^*L_i^{(1)}$ is given by

$$\begin{aligned}\Phi_i(1 \otimes \hat{\omega}) &= \frac{1}{\pi}\hat{\omega} \\ \Phi_i(1 \otimes \hat{\eta}) &= \frac{1}{\pi^*}\hat{\eta},\end{aligned}$$

and

$$\Phi_i(1 \otimes e_i) = e_i + \frac{1}{\pi^*}[a]_i^*(v_i)\hat{\eta}.$$

Lemma 2.11. *We define the morphism $\varrho_a : \mathcal{L}og^{(1)} \to [a]^*\mathcal{L}og^{(1)}$ locally on $(\overline{V}_i)$ by*

$$\begin{aligned}\varrho_a(e_i) &= e_i - T_i\hat{\eta} \\ \varrho_a(\hat{\omega}) &= a\hat{\omega} \\ \varrho_a(\hat{\eta}) &= a^*\hat{\eta}.\end{aligned}$$

Then ϱ_a is an isomorphism in $S^{\mathrm{ad}}(\mathfrak{E})$.

Proof. We can check directly that ϱ_a is indeed a morphism of syntomic datum. That ϱ_a is an isomorphism can be checked locally, and the lemma follows from the definition.

□

Corollary 2.12 (Splitting principle). *Let $x : \operatorname{Spec} \mathcal{O}_K \to E$ be a torsion point of E, whose order is prime to p. Then we have*

$$x^*\mathcal{L}og^{(n)} \simeq \prod_{s=0}^{n} \operatorname{Sym}^s \mathscr{H}.$$

Proof. Take $a \in \mathcal{O}_{\mathbf{K}}$ such that the morphism $[a] : E \to E$ is an isogeny which maps x to O. Since $x^* \circ [a]^* = e^*$, we have

$$x^*\mathcal{L}og^{(n)} \overset{\varrho_a}{\simeq} x^*[a]^*\mathcal{L}og^{(n)} = e^*\mathcal{L}og^{(n)} = \prod_{s=0}^{n} \operatorname{Sym}^s \mathscr{H}$$

as desired. □

3. THE ELLIPTIC POLYLOGARITHMIC SHEAF

Let the notations be as in the previous section. Take an ideal $\mathfrak{a} = (a)$ of $\mathcal{O}_{\mathbf{K}}$ prime to $\mathfrak{f}p$ such that that $\mathbf{K}(\mathfrak{a}) \subset K$. Let $\mathbf{U} = \mathbf{E}\backslash O$ and $\mathbf{U}_a = \mathbf{E} \setminus \mathbf{E}[\mathfrak{a}]$. Let $U = E \setminus O$, and $U_a = E \setminus E[\mathfrak{a}]$ be the corresponding schemes over $\mathcal{O}_K$. Then the triples $\mathfrak{U} = (U, E, \phi_E)$ and $\mathfrak{U}_a = (U_a, E, \phi_E)$ are syntomic data over K. In particular, $\mathfrak{U} = \mathfrak{U}_{\mathrm{id}}$. We let $j_a : \mathfrak{U}_a \hookrightarrow \mathfrak{E}$ be the natural inclusion. Let $\mathfrak{D}_a$ be the syntomic datum (D_a, D_a, ϕ_{D_a}) such that $D_a = \mathfrak{E}[\mathfrak{a}]$.

The purpose of this section is to give the definition of the polylogarithmic sheaf $\mathrm{pol}^{(n)}$ in $S(\mathfrak{U})$, and the modified polylogarithmic sheaf $\mathrm{pol}_a^{(n)}$ in $S(\mathfrak{U}_a)$ (Definitions 3.3 and 3.5). The two are related by the equation given in Proposition 3.7

$$\mathrm{pol}_a^{(n)} = (\deg a)\,\mathrm{pol}^{(n)} - [a]^* \,\mathrm{pol}^{(n)}.$$

The Gysin exact sequence in Theorem 2 gives an isomorphism

$$\begin{aligned} H^0(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^* \mathcal{L}og^{(n)}(1)) &= H^0(\mathfrak{E}, \widehat{\mathscr{H}} \otimes \mathcal{L}og^{(n)}(1)) \\ &= \widehat{\mathscr{H}} \otimes \mathrm{Sym}^n \mathscr{H}(1) \end{aligned} \tag{3.1}$$

and the exact sequence

$$\begin{aligned} 0 \to H^1(\mathfrak{E}, \widehat{\mathscr{H}} \otimes \mathcal{L}og^{(n)}(1)) \to H^1(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^* \mathcal{L}og^{(n)}(1)) \\ \to H^0(\mathfrak{D}_a, \widehat{\mathscr{H}} \otimes i_a^* \mathcal{L}og^{(n)}) \xrightarrow{u} H^2(\mathfrak{E}, \widehat{\mathscr{H}} \otimes \mathcal{L}og^{(n)}(1)) \to 0. \end{aligned} \tag{3.2}$$

The calculations of the previous section gives

$$\begin{aligned} H^1(\mathfrak{E}, \widehat{\mathscr{H}} \otimes \mathcal{L}og^{(n)}(1)) &= \widehat{\mathscr{H}} \otimes \mathrm{Sym}^{n+1} \mathscr{H} \\ H^2(\mathfrak{E}, \widehat{\mathscr{H}} \otimes \mathcal{L}og^{(n)}(1)) &= \widehat{\mathscr{H}} \end{aligned}$$

and from Corollary 2.12 and (2.2), we have

$$\begin{aligned} H^0(\mathfrak{D}_a, \widehat{\mathscr{H}} \otimes i_a^* \mathcal{L}og^{(n)}) &= \bigoplus_{\mathbf{E}[\mathfrak{a}]} H^0(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes x^* \mathcal{L}og^{(n)}) \\ &\simeq \bigoplus_{\mathbf{E}[\mathfrak{a}]} \left(\widehat{\mathscr{H}} \otimes \left(\prod_{s=0}^{n} \mathrm{Sym}^s \mathscr{H} \right) \right). \end{aligned}$$

By considering the case $n = 0$, we can see that the map u in (3.2) maps $\widehat{\mathscr{H}} \otimes (\prod_{s=1}^n \mathrm{Sym}^s \mathscr{H})$ to 0. Hence the sequence (3.2) is

$$\begin{aligned} 0 \to \widehat{\mathscr{H}} \otimes \mathrm{Sym}^{n+1} \mathscr{H} \to H^1(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^* \mathcal{L}og^{(n)}(1)) \\ \to \left(\left(\bigoplus_{\mathbf{E}[\mathfrak{a}]} \widehat{\mathscr{H}} \right) / \Delta(\widehat{\mathscr{H}}) \right) \oplus \bigoplus_{\mathbf{E}[\mathfrak{a}]} \left(\widehat{\mathscr{H}} \otimes \left(\prod_{s=1}^{n} \mathrm{Sym}^s \mathscr{H} \right) \right) \\ \to 0. \end{aligned} \tag{3.3}$$

Here, $\Delta : \widehat{\mathscr{H}} \to \bigoplus_{\mathbf{E}[\mathfrak{a}]} \widehat{\mathscr{H}}$ is the diagonal morphism.

From the definition, we have

$$H^0_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathrm{Sym}^{n+1}\mathscr{H}) = 0 \qquad (n \geq 1)$$

$$H^0_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes (\prod_{s=1}^{n} \mathrm{Sym}^s \mathscr{H})) = H^0_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathscr{H}).$$

Hence by taking $H^0_{\mathrm{syn}}(\mathcal{O}_K, -)$ of (3.3), we have an exact sequence

(3.4)

$$\begin{aligned} 0 &\to H^0_{\mathrm{syn}}(\mathcal{O}_K, H^1(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^*\mathcal{L}og^{(n)}(1))) \\ &\to \bigoplus_{\mathbf{E}[\mathfrak{a}]} H^0_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathscr{H}) \to H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathrm{Sym}^{n+1}\mathscr{H}) \to \cdots \end{aligned}$$

By (3.1), we have

$$H^1_{\mathrm{syn}}(\mathcal{O}_K, H^0(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^*\mathcal{L}og^{(n)}(1))) = H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathrm{Sym}^n \mathscr{H}(1)).$$

Hence the short exact sequence in Lemma 1.24 gives

$$\begin{aligned} 0 &\to H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathrm{Sym}^n \mathscr{H}(1)) \to H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^*\mathcal{L}og^{(n)}(1)) \\ &\to H^0_{\mathrm{syn}}(\mathcal{O}_K, H^1(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^*\mathcal{L}og^{(n)}(1))) \to 0. \end{aligned} \tag{3.5}$$

Combining the sequences (3.4) and (3.5) with the canonical isomorphism

$$\begin{aligned} H^0_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathscr{H}) &= \mathrm{Hom}_{S(\mathcal{O}_K)}(K(0), \widehat{\mathscr{H}} \otimes \mathscr{H}) \\ &= \mathrm{Hom}_{S(\mathcal{O}_K)}(\mathscr{H}, \mathscr{H}), \end{aligned} \tag{3.6}$$

we have a morphism

$$H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^*\mathcal{L}og^{(n)}(1)) \xrightarrow{\tau_a^{(n)}} \bigoplus_{\mathbf{E}[\mathfrak{a}]} \mathrm{Hom}_{S(\mathcal{O}_K)}(\mathscr{H}, \mathscr{H}).$$

Lemma 3.1. *The maps $\tau_a^{(n)}$ given above are surjective.*

Proof. Since both the map in (3.5) and the isomorphism (3.6) are surjective, it is enough to check that the map

$$H^0_{\mathrm{syn}}(\mathcal{O}_K, H^1(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^*\mathcal{L}og^{(n)}(1))) \to \bigoplus_{\mathbf{E}[\mathfrak{a}]} H^0_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathscr{H})$$

in (3.4) is surjective. The last two terms in the sequence (3.4) gives a system

$$\begin{array}{ccc} \bigoplus_{\mathbf{E}[\mathfrak{a}]} H^0_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathscr{H}) & \longrightarrow & H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathrm{Sym}^{n+2}\mathscr{H}) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ \bigoplus_{\mathbf{E}[\mathfrak{a}]} H^0_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathscr{H}) & \longrightarrow & H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathrm{Sym}^{n+1}\mathscr{H}), \end{array}$$

where the right vertical arrow is the zero map. This proves that the horizontal arrows are zero maps, hence that $\tau_a^{(n)}$ is surjective. □

Proposition 3.2. *There exists a unique system of elements*

$$\mathrm{pol}_a^{(n)} \in H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j^*\mathcal{L}og^{(n)}(1)) \qquad (n \geq 1)$$

compatible with the maps

$$H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^*\mathcal{L}og^{(n+1)}(1)) \to H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^*\mathcal{L}og^{(n)}(1)),$$

such that

$$\tau_a^{(n)}(\mathrm{pol}_a^{(n)}) = \bigoplus_{P \in \mathbf{E}[\mathfrak{a}]} (N(P)\,\mathrm{id})$$

in $\bigoplus_{\mathbf{E}[\mathfrak{a}]} \mathrm{Hom}_{S(\mathcal{O}_K)}(\mathscr{H}, \mathscr{H})$, *where* $N(P) = (\mathbf{N}\mathfrak{a} - 1)$ *if* $P = O$ *and* $N(P) = -1$ *otherwise.*

Proof. We define the element $\mathrm{pol}_a^{(n)}$ as follows. By the previous lemma, the map $\tau_a^{(n+1)}$ is surjective. We define

$$p_a^{(n+1)} \in H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j^*\mathcal{L}og^{(n+1)}(1))$$

to be the element such that

$$\tau_a^{(n+1)}(p_a^{(n+1)}) = \bigoplus_{P \in \mathbf{E}[\mathfrak{a}]} (N(P)\,\mathrm{id}).$$

The element $p_a^{(n+1)}$ is determined uniquely up to the kernel of $\tau_a^{(n+1)}$, which is $H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathrm{Sym}^{n+1}\mathscr{H}(1))$ by construction. The first two terms in the sequence (3.5) gives a commutative diagram

$$\begin{tikzcd}
H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathrm{Sym}^{n+1}\mathscr{H}(1)) \arrow[r] \arrow[d] & H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j^*\mathcal{L}og^{(n+1)}(1)) \arrow[d] \\
H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathrm{Sym}^{n}\mathscr{H}(1)) \arrow[r] & H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j^*\mathcal{L}og^{(n)}(1)).
\end{tikzcd}$$

We define $\mathrm{pol}_a^{(n)}$ to be the image of $p_a^{(n+1)}$ with respect to the right vertical arrow. Since the left vertical arrow is the zero map, $\mathrm{pol}_a^{(n)}$ is determined independently of the choice of $p_a^{(n+1)}$. This construction gives the desired result. $\square$

Let η_a be the isomorphism

$$\mathrm{Ext}^1_{S(\mathfrak{U}_a)}(K(0), \widehat{\mathscr{H}} \otimes j_a^*\mathcal{L}og^{(n)}(1)) \simeq H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^*\mathcal{L}og^{(n)}(1))$$

proved in Theorem 1.

Definition 3.3. We define the *elliptic polylogarithm sheaf* in $S(\mathfrak{U})$ to be an object representing the class

$$\eta_{\mathrm{id}}^{-1}(\mathrm{pol}_{\mathrm{id}}^{(n)}) \in \mathrm{Ext}^1_{S(\mathfrak{U})}(K(0), \widehat{\mathscr{H}} \otimes j^*\mathcal{L}og^{(n)}(1)).$$

By abuse of notation, we will denote this element also by $\mathrm{pol}_a^{(n)}$. The object $\mathrm{pol}^{(n)}$ is determined up to unique isomorphism.

Remark 3.4. In [HK] (Definition A.3.3), this element is referred to as a *cohomological polylogarithm*.

Definition 3.5. We define the *modified elliptic polylogarithm sheaf* $\mathrm{pol}_a^{(n)}$ in $S(\mathfrak{U}_a)$ to be any object representing the class

$$\eta_a^{-1}(\mathrm{pol}_a^{(n)}) \in \mathrm{Ext}^1_{S(\mathfrak{U}_a)}(K(0), \widehat{\mathscr{H}} \otimes j_a^* \mathcal{L}og^{(n)}(1)).$$

Lemma 3.6. *The following diagram is commutative.*

$$\begin{array}{ccc} H^1(\mathfrak{U}, \widehat{\mathscr{H}} \otimes j^* \mathcal{L}og^{(n)}(1)) & \longrightarrow & H^0(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes e^* \mathcal{L}og^{(n)}) \\ {\scriptstyle [a]^*}\downarrow & & \downarrow{\scriptstyle \Delta} \\ H^1(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^* \mathcal{L}og^{(n)}(1)) & \longrightarrow & \bigoplus_{\mathbf{E}[\mathfrak{a}]} H^0(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes e^* \mathcal{L}og^{(n)}), \end{array}$$

where the right vertical morphism Δ is the diagonal.

From the above lemma and from the definition of the polylogarithms, we have the following:

Proposition 3.7. *We have*

$$\mathrm{pol}_a^{(n)} = (\deg a)\,\mathrm{pol}^{(n)} - [a]^*\,\mathrm{pol}^{(n)}$$

in $H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes j_a^* \mathcal{L}og^{(n)})$.

4. The review of p-adic L-functions

In this section, we will review the construction of the one variable p-adic L-function associated to the elliptic curve $\mathbf{E}$ of the previous sections. Our main reference for the properties of elliptic curves with complex multiplication will be [R]. We assume that $p \neq 2, 3$.

4.1. The basic definition and properties.

We fix a Weierstraß model of $\mathbf{E}$ over $\mathcal{O}_\mathbf{K}$

$$y^2 = 4x^3 - g_2 x + g_3,$$

and we let

$$\omega = \frac{dx}{y}$$

be the associated invariant differential. Since $\mathbf{K}$ has class number 1 and $\mathbf{E}$ has good reduction over the primes above $p \neq 2, 3$, we can choose a model such that the discriminant $\Delta(\mathbf{E})$ of $\mathbf{E}$ corresponding to this model is prime to p. Let $\mathfrak{a}$ be a non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_\mathbf{K}$ prime to 6, and let a be a generator of $\mathfrak{a}$.

Definition 4.1. Define a rational function on $\mathbf{E}$ by

$$\Theta_{\mathbf{E},\mathfrak{a}} = a^{-12} \Delta(\mathbf{E})^{\mathrm{N}\mathfrak{a}-1} \prod_{P \in E[\mathfrak{a}] \setminus O} (x - x(P))^{-6}.$$

This function has the following properties.

Lemma 4.2.

a) $\Theta_{\mathbf{E},\mathfrak{a}}$ *is independent of the choice of the Weierstraß model.*

b) *The rational function* $\Theta_{\mathbf{E},\mathfrak{a}}$ *is defined over* $\mathbf{K}$.

c) *The divisor of* $\Theta_{\mathbf{E},\mathfrak{a}}$ *is given by*

$$12\mathbf{N}\mathfrak{a}[O] - 12 \sum_{P \in \mathbf{E}[\mathfrak{a}]} [P].$$

Next, denote by $L = L(\mathbf{E}, \omega)$ the lattice of periods, and by

$$\xi(z) = \xi(z; L) : \mathbb{C}/L \xrightarrow{\simeq} \mathbf{E}(\mathbb{C})$$

the standard analytic uniformization given by $\xi(z) = (\wp(z; L), \wp'(z; L))$ for the Weierstraß $\wp$-function $\wp(z; L)$ associated to L. We choose an element Ω in $\mathbb{C}^\times$ such that $L = \Omega\mathfrak{f}$. Then $\xi(\Omega)$ is a $\mathcal{O}_{\mathbf{K}}$ generator of $\mathbf{E}[\mathfrak{f}]$.

We will write $\Theta_{L,\mathfrak{a}} = \Theta_{\mathbf{E},\mathfrak{a}} \circ \xi$.

Definition 4.3. For $k \geq 1$, we define the Eisenstein series

$$E_k(z; L) = \lim_{s \to k} \sum_{\gamma \in L} \frac{(\overline{z} + \overline{\gamma})^k}{|z + \gamma|^{2s}},$$

where the limit means evaluation of the analytic continuation at $s = k$.

Theorem 3. *For every* $k \geq 1$,

$$\left(\frac{d}{dz}\right)^k \Theta_{\mathbf{E},\mathfrak{a}}(z) = 12(-1)^{k-1}(k-1)!(\mathbf{N}\mathfrak{a}E_k(z; L) - \psi(\mathfrak{a})^k E_k(\psi(\mathfrak{a})z, L)).$$

Proof. This is Theorem 7.13 of [R], combined with the equality

$$E_k(z, \mathfrak{a}^{-1}L) = \psi(\mathfrak{a})^k E_k(\psi(\mathfrak{a})z, L)$$

given in the proof of (loc. cit.) Theorem 7.17. □

Definition 4.4. We define the Hecke L-functions associated to powers of $\overline{\psi}$ to be the analytic continuation of the Dirichlet series

$$L(\overline{\psi}^k, s) = \sum \frac{\overline{\psi}(\mathfrak{b})}{\mathbf{N}\mathfrak{b}^s},$$

summing over ideals $\mathfrak{b}$ of $\mathcal{O}_{\mathbf{K}}$ prime to the conductor of $\overline{\psi}^k$. If $\mathfrak{c}$ is an ideal of $\mathcal{O}_{\mathbf{K}}$ prime to $\mathfrak{f}$, we define the partial L-function $L(\overline{\psi}^k, s, \mathfrak{c})$ by the same formula, but with the sum restricted to the ideals of $\mathcal{O}_{\mathbf{K}}$ prime to $\mathfrak{f}$ such that $[\mathfrak{b}, \mathbf{K}(\mathfrak{f})/\mathbf{K}] = [\mathfrak{c}, \mathbf{K}(\mathfrak{f})/\mathbf{K}]$.

Proposition 4.5 (= [R] Proposition 7.15)**.** *Suppose* $v \in \mathbf{K}L/L$ *has exact order* $\mathfrak{f}$. *Then for every* $k \geq 1$,

$$E_k(v; L) = v^{-k}\psi(\mathfrak{c})^k L(\overline{\psi}^k, k, \mathfrak{c}),$$

where $\mathfrak{c} = \mathcal{O}_{\mathbf{K}} v/\Omega$.

4.2. **The construction of the p-adic L-function.**

Let $\mathbf{F} = \mathbf{K}(\mathfrak{f})$ be the ray class field of $\mathbf{K}$ modulo $\mathfrak{f}$, and $\mathbf{F}_\infty = \mathbf{F}(\mathfrak{p}^\infty)$. Let

$$\psi_{\mathfrak{p}} : \mathcal{G} = \mathrm{Gal}(\mathbf{F}_\infty/\mathbf{K}) \longrightarrow \mathbb{C}_p^\times$$

be the p-adic character defined on $\mathcal{G}$ by

$$\psi_{\mathfrak{p}}(\sigma_{\mathfrak{c}}) = \psi(\mathfrak{c}), \qquad (\mathfrak{c}, \mathfrak{f}p) = 1, \qquad \sigma_{\mathfrak{c}} = (\mathfrak{c}, \mathbf{K}(\mathfrak{f}\mathfrak{p}^\infty)/\mathbf{K})$$

The purpose of this section is to define a measure $\mu_{\mathfrak{a}}$ on $\mathcal{G}$ such that for any integer $k \geq 1$, we have

$$\Omega_{\mathfrak{p}}^{-k} \int_{\mathcal{G}} \psi_{\mathfrak{p}}^k(g) d\mu_{\mathfrak{a}}(g) = \Omega^{-k} 12(-1)^{k-1}(k-1)! \left(1 - \frac{\psi(\mathfrak{p})}{p}\right)$$
$$\times \left(\mathbf{N}\mathfrak{a} - \psi(\mathfrak{a})^k\right) L_{\mathfrak{f}}(\bar{\psi}^{-k}, 0).$$

Using this measure, we will define the p-adic L-function of p-adic continuous characters on $\mathcal{G}$.

We fix a set B of ideals of $\mathcal{O}_{\mathbf{K}}$ such that $\sigma_{\mathfrak{c}} = [\mathfrak{c}, \mathbf{F}/\mathbf{K}]$ for $\mathfrak{c} \in B$ is a representative of $\mathrm{Gal}(\mathbf{F}/\mathbf{K})$. Let $G_\infty = \mathrm{Gal}(\mathbf{F}_\infty/\mathbf{F}) \subset \mathcal{G}$. It is enough to define the measure on each coset $\sigma_{\mathfrak{c}} G_\infty$.

Let $\widehat{\mathbf{E}}$ be the formal group associated to $\mathbf{E}$ over $\mathcal{O}_{\mathbf{K}_{\mathfrak{p}}} = \mathbb{Z}_p$ with respect to the parameter $t = -2x/y$. This group is a Lubin-Tate group with respect to the prime element π in $\mathbb{Z}_p$.

Let $W(\overline{\mathbb{F}}_p)$ be the ring of integers of the completion of the maximal unramified extension of $\mathbb{Q}_p$. We denote by σ the Frobenius endomorphism of $W(\overline{\mathbb{F}}_p)$. Let $\widehat{\mathbb{G}}_m$ be the formal multiplicative group over $\mathbb{Z}_p$ with parameter T.

As a special case of [La] Theorem 3.1, we have the following:

Theorem 4. *Let $\Omega_{\mathfrak{p}}$ be a unit of $W(\overline{\mathbb{F}}_p)$, such that $\Omega_{\mathfrak{p}}^\sigma/\Omega_{\mathfrak{p}} = \pi/p$ (Such units exist.) There exists a unique isomorphism*

$$\rho : \widehat{\mathbb{G}}_m \xrightarrow{\ \simeq\ } \widehat{\mathbf{E}}$$

of formal groups over $W(\overline{\mathbb{F}}_p)$ which commutes with the operation of $\mathbb{Z}_p$, and such that

$$(d/dT)\rho(T)|_{T=0} = \Omega_{\mathfrak{p}}.$$

This power series $\rho(T)$ satisfies $\rho^\sigma = \rho \circ [\pi/p]$.

Let $\tau_{\mathfrak{c}}$ be the translation of of $\mathbf{E}$ with respect to the element $\xi(\psi(\mathfrak{c})\Omega) \in \mathbf{E}[\mathfrak{f}] \subset \mathbf{E}(\mathbf{F})$. Then $\Theta_{\mathbf{E},\mathfrak{a}} \circ \tau_{\mathfrak{c}}$ is a rational function on $\mathbf{E}$ defined over $\mathbf{F}$.

Lemma 4.6. *Let $Q_{\mathfrak{a},\mathfrak{c}}(t) \in \mathbf{F}[[t]]$ be the Taylor series expansion of $\Theta_{\mathbf{E},\mathfrak{a}} \circ \tau_{\mathfrak{c}}$. Then the polynomial $Q_{\mathfrak{a},\mathfrak{c}}(t)$ is an element of $\mathcal{O}_{\mathbf{F}_{\mathfrak{p}}}[[t]]^\times$.*

We define the Frobenius morphism φ on $W(\overline{\mathbb{F}}_p)[[T]]$ by

$$\varphi(f(T)) = f^\sigma((1+T)^p - 1)$$

for $f(T)$ in $W(\overline{\mathbb{F}}_p)[[T]]$.

Definition 4.7. We define the polynomial $h_{\mathfrak{a},\mathfrak{c}}(T)$ by

$$h_{\mathfrak{a},\mathfrak{c}}(T) = \left(1 - \frac{\varphi}{p}\right) \log Q_{\mathfrak{a},\mathfrak{c}} \circ \rho(T).$$

Definition 4.8. We define $\mu_{\mathfrak{a},\mathfrak{c}}(x)$ to be the measure on $\mathbb{Z}_p^\times$ which satisfy

$$\int_{\mathbb{Z}_p^\times} (1+T)^x d\mu_{\mathfrak{a},\mathfrak{c}}(x) = h_{\mathfrak{a},\mathfrak{c}}(T).$$

This does not depend on the choice of the representative $\mathfrak{c}$.

The action of G_∞ on $\mathbf{E}[\mathfrak{p}^\infty]$ gives an isomorphism

$$\kappa : \mathcal{G}_\infty \xrightarrow{\simeq} \mathcal{O}_{\mathbf{K}_\mathfrak{p}}^\times = \mathbb{Z}_p^\times.$$

We denote again by $\mu_{\mathfrak{a},\mathfrak{c}}(g)$ the measure on G_∞ which is the pull back of $\mu_{\mathfrak{a},\mathfrak{c}}(x)$ with respect to κ.

Definition 4.9. We extend the measure $\mu_{\mathfrak{a},(1)}$ to a measure $\mu_\mathfrak{a}$ on $\mathcal{G}$ with values in $W(\overline{\mathbb{F}}_p)$ by setting

$$\mu_\mathfrak{a}(\sigma_\mathfrak{c}\sigma) = \mu_{\mathfrak{a},\mathfrak{c}}(\sigma) \qquad (\sigma \in G_\infty). \tag{4.1}$$

Let $\lambda_{\widehat{\mathbf{E}}}(t)$ be the logarithmic map

$$\widehat{\mathbf{E}} \xrightarrow{\simeq} \widehat{\mathbb{G}}_a$$

of $\widehat{\mathbf{E}}$ such that $\lambda'_{\widehat{\mathbf{E}}}(0) = 1$. We define the operator D on $\mathbb{Z}_p[[t]]$ by

$$D = \frac{1}{\lambda'_{\widehat{\mathbf{E}}}(t)} \frac{d}{dt}.$$

Proposition 4.10 (=[R] Proposition 7.20)**.** *Identifying (x, y) both with $(\wp(z; L), \wp'(z; L))$ and with $(x(t), y(t))$ leads to the commutative diagram*

$$\begin{tikzcd}
\mathbf{K}(\wp(z), \wp'(z)) \arrow[d, "d/dz"'] & \mathbf{K}(\mathbf{E}) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & \mathbf{K}(x(t), y(t)) \arrow[r] \arrow[d, "D"'] & \mathbb{Z}_p[[t]] \arrow[d, "D"'] \\
\mathbf{K}(\wp(z), \wp'(z)) & \mathbf{K}(\mathbf{E}) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \mathbf{K}(x(t), y(t)) \arrow[r] & \mathbb{Z}_p[[t]].
\end{tikzcd}$$

Let D_ρ be the operator on $W(\overline{\mathbb{F}}_p)[[T]]$ defined by

$$D_\rho = (1+T)\frac{d}{dT}.$$

This corresponds to the operator $\Omega_\mathfrak{p} D$ through the isomorphism ρ.

Lemma 4.11. *For $k \geq 1$, the measure $\mu_{\mathfrak{a},\mathfrak{c}}(g)$ satisfies*

$$\Omega_\mathfrak{p}^{-k} \int_{G_\infty} \kappa^k(g) d\mu_{\mathfrak{a},\mathfrak{c}}(g) = \Omega^{-k} 12(-1)^{k-1}(k-1)! \left(1 - \frac{\psi(\mathfrak{p})^k}{p}\right) \\ \times \left(\mathbf{N}\mathfrak{a} L_\mathfrak{f}(\psi^{-k}, 0, \mathfrak{c}) - \psi(\mathfrak{a})^k L_\mathfrak{f}(\psi^{-k}, 0, \mathfrak{a}\mathfrak{c})\right).$$

Proof. By definition, we have

$$\int_{G_\infty} \kappa^k(g)d\mu_{\mathfrak{a},\mathfrak{c}}(g) = D_\rho^k h_{\mathfrak{a},\mathfrak{c}}(T)|_{T=0}.$$

Through the isomorphism ρ, we have

$$\begin{aligned}
D_\rho^k h_{\mathfrak{a},\mathfrak{c}}(T)|_{T=0} &= D_\rho^k \left(1 - \frac{\varphi}{p}\right)(\log Q_{\mathfrak{a},\mathfrak{c}} \circ \rho(T))|_{T=0} \\
&= D_\rho^k \left(\log Q_{\mathfrak{a},\mathfrak{c}} \circ \rho(T) - \frac{[p]}{p}(\log Q_{\mathfrak{a},\mathfrak{c}} \circ \rho(T))^\sigma\right)|_{T=0} \\
&= D_\rho^k \left(\log Q_{\mathfrak{a},\mathfrak{c}} \circ \rho(T) - \frac{1}{p}\log Q_{\mathfrak{a},\mathfrak{c}} \circ \rho \circ [\pi](T)\right)|_{T=0} \\
&= \Omega_{\mathfrak{p}}^k \left(1 - \frac{\pi^k}{p}\right) D^k \log Q_{\mathfrak{a},\mathfrak{c}}(t)|_{t=0} \\
&= \Omega_{\mathfrak{p}}^k \left(1 - \frac{\psi(\mathfrak{p})}{p}\right)\left(\frac{d}{dz}\right)^k \log \Theta_{\mathbf{E},L}(z + \psi(\mathfrak{c})\Omega)|_{z=0}.
\end{aligned}$$

Then from Theorem 3 and Proposition 4.5, we have

$$\begin{aligned}
\left(\frac{d}{dz}\right)^k \log \Theta_{\mathbf{E},L}(z + \psi(\mathfrak{c})\Omega)|_{z=0} &= \left(\frac{d}{dz}\right)^k \log \Theta_{\mathbf{E},L}(z)|_{z=\psi(\mathfrak{c})\Omega} \\
&= \Omega^{-k} 12(-1)^{k-1}(k-1)! \\
&\quad \times \left(\mathbf{N}\mathfrak{a} L(\overline{\psi}^k, k, \mathfrak{c}) - \psi^k(\mathfrak{a}) L(\overline{\psi}^k, k, \mathfrak{a}\mathfrak{c})\right).
\end{aligned}$$

The lemma follows from the equality

$$L(\overline{\psi}^k, k, \mathfrak{m}) = L(\psi^{-k}, 0, \mathfrak{m})$$

for any ideal $\mathfrak{m}$ of $\mathcal{O}_{\mathbf{K}}$ prime to $\mathfrak{f}$. □

Proposition 4.12. *For any integer $k \geq 1$, the measure $\mu_{\mathfrak{a}}$ defined in Definition 4.9 satisfies*

$$\begin{aligned}
\Omega_{\mathfrak{p}}^{-k} \int_{\mathcal{G}} \psi_{\mathfrak{p}}^k(g) d\mu_{\mathfrak{a}}(g) &= \Omega^{-k} 12(-1)^{k-1}(k-1)! \left(1 - \frac{\psi(\mathfrak{p})^k}{p}\right) \\
&\quad \times \left(\mathbf{N}\mathfrak{a} - \psi(\mathfrak{a})^k\right) L_{\mathfrak{f}}(\psi^{-k}, 0).
\end{aligned}$$

Proof. By definition, we have

$$\begin{aligned}
\int_{\mathcal{G}} \psi_{\mathfrak{p}}^k(g) d\mu_{\mathfrak{a}}(g) &= \sum_{\mathfrak{c} \in B} \int_{\sigma_{\mathfrak{c}} G_\infty} \psi_{\mathfrak{p}}^k(\sigma) d\mu_{\mathfrak{a}}(\sigma) \\
&= \sum_{\mathfrak{c} \in B} \int_{G_\infty} \kappa^k(\sigma) d\mu_{\mathfrak{a},\mathfrak{c}}(\sigma).
\end{aligned}$$

Hence the proposition follows from Lemma 4.11. □

Definition 4.13. . We define the p-adic L-function of modulus $\mathfrak{f}$ to be the function whose domain of definition is the set of all p-adic continuous characters on $\mathcal{G}$, and which assigns to every ϵ the value

$$L_{p,\mathfrak{a}}(\epsilon) = \int_{\mathcal{G}} \epsilon^{-1}(g) d\mu_{\mathfrak{a}}(g).$$

Proposition 4.12 gives the interpolation property

$$\begin{aligned}\Omega_{\mathfrak{p}}^{j} L_{p,\mathfrak{a}}(\psi_{\mathfrak{p}}^{j}) = \Omega^{j} 12(-1)^{(j+1)}(-(j+1))! \left(1 - \frac{\psi^{-j}(\mathfrak{p})}{p}\right) \\ \times \left(\mathbf{N}\mathfrak{a} - \psi(\mathfrak{a})^{-j}\right) L_{\mathfrak{f}}(\psi^{j}, 0)\end{aligned}$$

for $j \leq -1$.

5. The specialization at torsion points

We keep the notation of the previous sections. In particular, we fix an ideal $\mathfrak{a}$ of $\mathcal{O}_{\mathbf{K}}$ prime to $\mathfrak{f}p$ as in the previous section. Recall that $U_{\mathfrak{a}} = E \setminus E[\mathfrak{a}]$, and that $\mathcal{U}_{\mathfrak{a}K}$ is the rigid analytic space associated to the formal completion of $U_{\mathfrak{a}}$ with respect to the special fiber. In this section, we will prove the existence of the elliptic polylogarithmic functions $D_{\mathfrak{a},j}$ ($j \geq 1$), which are over convergent functions on $\mathcal{U}_{\mathfrak{a}K}$ satisfying the differential equation

$$\begin{aligned} dD_{\mathfrak{a},1} &= \left(\frac{\varphi}{p} - 1\right) \frac{d\Theta_{\mathfrak{E},\mathfrak{a}}}{\Theta_{\mathfrak{E},\mathfrak{a}}} \\ dD_{\mathfrak{a},j+1} &= -D_{\mathfrak{a},j}\omega \qquad (j \geq 1). \end{aligned} \tag{5.1}$$

Here, φ is the Frobenius induced on $\mathcal{U}_{\mathfrak{a}K}$ from the Frobenius ϕ_E on $\mathcal{E}$.

Assume that $K \supset \mathbf{F} = \mathbf{K}(E[\mathfrak{f}])$, and let $x : \operatorname{Spec} \mathcal{O}_K \hookrightarrow E$ be a torsion point in $E(\mathcal{O}_K)$ of order prime to $p\mathfrak{a}$. By Corollary 2.12, there is an isomorphism

$$x^* \mathcal{L}og^{(n)} \simeq \bigoplus_{s=0}^{n} \operatorname{Sym}^s \mathscr{H}.$$

Let $\mathscr{H}(\hat{\omega})$ be the Frobenius filtered module in $S(\mathcal{O}_K)$ defined to be the triple $(H(\hat{\omega}), F^\bullet, \Phi)$ such that

(i) $H(\hat{\omega}) = K\hat{\omega}$ is a one-dimensional K-vector space.
(ii) The filtration is given by

$$H(\hat{\omega}) = F^{-1} H(\hat{\omega}) \supset F^0 H(\hat{\omega}) = 0.$$

(iii) The Frobenius is given by $\Phi(\hat{\omega}) = (1/\pi)\hat{\omega}$.

We define $\mathscr{H}(\hat{\eta})$ to be the Frobenius filtered module $(H(\hat{\eta}), F^\bullet, \Phi)$ in $S(\mathcal{O}_K)$ such that

(i) $H(\hat{\eta}) = K\hat{\eta}$ is a one-dimensional K-vector space.
(ii) The filtration is given by

$$H(\hat{\eta}) = F^0 H(\hat{\eta}) \supset F^1 H(\hat{\eta}) = 0.$$

(iii) The Frobenius is given by $\Phi(\hat{\eta}) = (1/\pi^*)\hat{\eta}$.

By abuse of notation, we will denote by the same symbol the pull backs of $\mathscr{H}(\hat{\omega})$ and $\mathscr{H}(\hat{\eta})$ by the structure morphism ϖ^*.

Definition 5.1. We define the map

$$h_{x,j} : H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H} \otimes \mathcal{L}og^{(n)}(1)) \to H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes \mathscr{H}(\hat{\omega})^{\otimes s}(1))$$

for $0 \leq j \leq n$ to be the composition of

$$H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H} \otimes \mathcal{L}og^{(n)}(1)) \xrightarrow{x^*} H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes (\prod_{s=0}^{n} \mathrm{Sym}^s \mathscr{H})(1))$$

with the projection

$$H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes (\prod_{s=0}^{n} \mathrm{Sym}^s \mathscr{H})(1)) \to H^1_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H}(\omega) \otimes \mathscr{H}(\hat{\omega})^{\otimes (j)}(1)).$$

This does not depend on the choice of the $n \geq j$.

For $j \geq 0$, there is a natural isomorphism

$$H^1_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H}(\omega) \otimes \mathscr{H}(\hat{\omega})^{\otimes j}(1)) \xrightarrow{\cong} K\,\omega \otimes \hat{\omega}^{\otimes j}(1).$$

We will prove our Main Theorem:

Theorem 5. *Let x be a non-zero torsion point of $E(\mathcal{O}_K)$ prime to $\mathfrak{a}\mathfrak{p}$. For $j \geq 0$, we have*

$$\sum_{\mathfrak{c} \in B} h_{j,x(\mathfrak{c})} \left(12\, \mathrm{pol}_a^{(n)}\right) = D_{\mathfrak{a},j}\, \omega \otimes \hat{\omega}^{(j)}(1).$$

5.1. **The partial logarithmic sheaf.**

In this section, we will define the *partial logarithmic sheaf*. This is a quotient of the elliptic logarithmic sheaf, such that the underlying coherent $\mathcal{O}_{E_K}$-module is free. This makes calculations of polylogarithms much simpler. Since the map $h_{x,j}$ factors through this quotient, this is very useful for the proof of the main theorem.

Definition 5.2. We define $\widetilde{\mathcal{L}og}^{(n)}$ to be an object in $S(\mathfrak{E})$ given by the 4-uple $(\widetilde{L}^{(n)}, \nabla, F^\bullet, \Phi)$, where

(i) $\widetilde{L}^{(n)}$ is the free $\mathcal{O}_{E_K}$-module

$$\widetilde{L}^{(n)} = \prod_{\substack{r+s=n \\ r,s \geq 0}} \mathcal{O}_{E_K}(e^{\otimes r} \otimes \hat{\omega}^{\otimes s})$$

(ii) The connection is given by

$$\nabla(e^{\otimes r} \otimes \hat{\omega}^{\otimes s}) = r e^{\otimes (r-1)} \otimes \hat{\omega}^{\otimes (s+1)} \otimes \boldsymbol{\omega},$$

where $\omega \in H^0(E_K, \Omega^1_E) \subset \widehat{\mathscr{H}}$.

(iii) The filtration is given by

$$F^m \widetilde{L}^{(n)} = \prod_{\substack{r+s=n \\ -m \geq s \geq 0 \\ r \geq 0}} \mathcal{O}_{E_K}(e^{\otimes r} \otimes \hat{\omega}^{\otimes s})$$

(iv) The Frobenius Φ on $L^{(n)}$ is given by

$$\Phi_I(e^{\otimes r} \otimes \hat{\omega}^{\otimes s}) = \frac{1}{\pi^{*s}} e^{\otimes r} \otimes \hat{\omega}^{\otimes s}.$$

There is a morphism of sheaves $\widetilde{\mathcal{L}og}^{(n+1)} \to \widetilde{\mathcal{L}og}^{(n)}$, which is defined by

$$e^{\otimes r} \otimes \hat{\omega}^{\otimes s} \mapsto \begin{cases} re^{\otimes(r-1)} \otimes \hat{\omega}^{\otimes s} & r > 0 \\ 0 & r = 0. \end{cases}$$

Through this morphism, we will regard the system $(\widetilde{\mathcal{L}og}^{(n)})_{n \in \mathbb{N}}$ as a pro-object in the category $S(\mathfrak{E})$. We will call this the *partial logarithm sheaf*. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{L}og^{(n+1)} & \longrightarrow & \mathcal{L}og^{(n)} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{L}og}^{(n+1)} & \longrightarrow & \widetilde{\mathcal{L}og}^{(n)} \end{array}$$

where the vertical maps are induced from the natural projections.

Lemma 5.3. *The partial logarithmic sheaf satisfies the following properties:*

a) *For any isogeny $[a] : E \to E$ of degree prime to p, there is an isomorphism*

$$\widetilde{\mathcal{L}og}^{(n)} \xrightarrow{\simeq} [a]^* \widetilde{\mathcal{L}og}^{(n)}.$$

b) *For any torsion point $x : \operatorname{Spec} \mathcal{O}_K \to E$ of E of order prime to p, we have the splitting principle*

$$x^* \widetilde{\mathcal{L}og}^{(n)} \xrightarrow{\simeq} \prod_{s=0}^{n} \mathscr{H}(\hat{\omega})^{\otimes s}.$$

The isomorphism is normalized by

$$e^{\otimes r} \otimes \hat{\omega}^{\otimes s} \mapsto r!\, \hat{\omega}^{\otimes s}.$$

By functoriality, for any torsion point $x : \operatorname{Spec} \mathcal{O}_K \to E$ of $E[\mathfrak{f}]$, the following diagram is commutative:

$$\begin{array}{ccc} H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes \mathcal{L}og^{(n)}) & \xrightarrow{x^*} & H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}} \otimes (\prod_{s=0}^{n} \mathrm{Sym}^s \mathscr{H})) \\ {\scriptstyle u}\downarrow & & \downarrow \\ H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}) & \xrightarrow{x^*} & H^1_{\mathrm{syn}}(\mathcal{O}_K, \widehat{\mathscr{H}}(\omega) \otimes (\prod_{s=0}^{n} \mathscr{H}(\hat{\omega})^{\otimes s})) \end{array}$$

Here, the second vertical arrow is the projection onto the direct summand.

Hence from this fact, in order to calculate the image of $\mathrm{pol}_a^{(n)}$ with respect to the morphism $h_{x,j}$, we need only to know the image of $\mathrm{pol}_a^{(n)}$ with respect to the map u. The rest of this section is devoted to explicitly representing the object $\widetilde{\mathrm{pol}}_a^{(n)} := u(\mathrm{pol}_a^{(n)})$.

5.2. **The cohomology of the partial logarithmic sheaf.**

Definition 5.4. We define $\mathscr{M}(\hat{\eta})^{(n)}$ to be the object in $S(\mathfrak{E})$ such that the image with respect to $P_{\{\overline{U}_i\}}$ is given by the 5-uple

$$(M(\hat{\eta})_i^{(n)}, \nabla_i, F_i^\bullet, \epsilon_{ij}, \Phi_i),$$

where

(i) $M(\hat{\eta})_i^{(n)}$ is the free A_{iK}-module

$$M(\hat{\eta})_i^{(n)} = \prod_{\substack{r+s+t=n\\ r,s\geq 0\\ t\geq 1}} A_{iK}(e_i^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t)$$

(ii) The connection is given by

$$\nabla_i(e_i^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t) = r \otimes e_i^{(r-1)} \cdot \hat{\omega}^s \cdot \hat{\eta}^t(\hat{\omega} \otimes \boldsymbol{\omega}_i + \hat{\eta} \otimes \boldsymbol{\eta}_i)$$

(iii) The filtration is given by

$$F^m M(\hat{\eta})_i^{(n)} = \prod_{\substack{r+s+t=n\\ -m\geq s\geq 0\\ r\geq 0\\ t\geq 1}} A_{iK}(e_i^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t)$$

(iv) The isomorphism ϵ_{ij} is given by

$$\epsilon_{ij}(e_i^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t) = (e_j - \alpha_{ij}\hat{\eta})^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t.$$

(v) The Frobenius Φ_i on $M(\hat{\eta})_i^{(n)}$ is given by

$$\Phi_i(e_i^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t) = \frac{1}{\pi^{*s}\pi^t}(e_i + v_i\hat{\eta})^r \cdot \hat{\omega}^s \cdot \hat{\eta}^t.$$

There is an exact sequence

$$0 \to \mathscr{M}(\hat{\eta})^{(n)} \to \mathcal{L}og^{(n)} \to \widetilde{\mathcal{L}og}^{(n)} \to 0 \tag{5.2}$$

Let

$$\mathscr{H}(\hat{\eta})^{(n)} = \prod_{j=1}^{n} \mathscr{H}(\hat{\omega})^{\otimes(n-j)} \otimes \mathscr{H}(\hat{\eta})^{\otimes j}.$$

Lemma 5.5. *We have*

$$H^0(\mathfrak{E}, \mathscr{M}(\hat{\eta}^{(n)})) = \mathscr{H}(\hat{\eta})^{(n)}$$
$$H^1(\mathfrak{E}, \mathscr{M}(\hat{\eta}^{(n)})) = \mathscr{H}(\hat{\eta})^{(n+1)}(-1)$$
$$H^2(\mathfrak{E}, \mathscr{M}(\hat{\eta}^{(n)})) = \mathscr{H}(\hat{\eta})(-1)$$

Proof. The proof follows that of Proposition 2.8. We prove the lemma by induction on n. If $n = 1$, then the lemma is trivial. Suppose the lemma holds for $n > 1$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{H} \otimes \mathscr{H}(\hat{\eta})^{(n)} & \to & \mathcal{L}og^{(1)} \otimes \mathscr{H}(\hat{\eta})^{(n)} & \to & \mathscr{H}(\hat{\eta})^{(n)} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{H}(\hat{\eta})^{(n+1)} & \to & \mathscr{M}(\hat{\eta})^{(n+1)} & \to & \mathscr{M}(\hat{\eta})^{(n)} & \to & 0, \end{array}$$

where the first two vertical arrows are induced from the multiplication map, and the last one is the canonical inclusion. The horizontal sequences are exact in $S(\mathfrak{E})$. The first boundary morphism of the long exact sequence for geometric syntomic cohomology gives a commutative diagram

$$\begin{tikzcd} \mathscr{H}(\hat{\eta})^{(n)} \arrow[r] \arrow[d, "\simeq"] & \widehat{\mathscr{H}} \otimes \mathscr{H} \otimes \mathscr{H}(\hat{\eta})^{(n)} \arrow[d, "\mathrm{id}\otimes\mathrm{mult}"] \\ H^0(\mathfrak{E}, \mathscr{M}(\hat{\eta})^{(n)}) \arrow[r, "\gamma"] & \widehat{\mathscr{H}} \otimes \mathscr{H}(\hat{\eta})^{(n)}. \end{tikzcd}$$

This shows that γ is induced by the multiplication map, hence is injective. It follows that

$$\mathscr{H}(\hat{\eta})^{(n+1)} = H^0(\mathfrak{E}, \mathscr{H}(\hat{\eta})^{(n+1)}) \simeq H^0(\mathfrak{E}, \mathscr{M}(\hat{\eta})^{(n+1)}).$$

The second boundary morphism of the long exact sequence for geometric syntomic cohomology gives a commutative diagram

$$\begin{tikzcd} \mathscr{H}(-1) \otimes \mathscr{H}(\hat{\eta})^{(n)} \arrow[r, "\simeq"] \arrow[d] & \mathscr{H}(-1) \otimes \mathscr{H}(\hat{\eta})^{(n)} \arrow[d, "\mathrm{mult}"] \\ H^1(\mathfrak{E}, \mathscr{M}(\hat{\eta})^{(n)}) \arrow[r, "\delta"] & \mathscr{H}(\hat{\eta})^{(n+1)}(-1). \end{tikzcd}$$

Hence δ is surjective, hence an isomorphism by reasons of dimension. □

Lemma 5.6. *We have*

$$H^2(\mathfrak{E}, \mathcal{L}og^{(n)}) \xrightarrow{\simeq} H^2(\mathfrak{E}, \widetilde{\mathcal{L}og}^{(n)}).$$

Proof. The surjection follows from the long exact sequence associated to the short exact sequence (5.2). The injection will be proved by induction on n. If $n = 0$, then

$$\mathcal{L}og^{(0)} \simeq \widetilde{\mathcal{L}og}^{(0)} \simeq K(0),$$

hence there is nothing to prove. Suppose the statement is true for n. We have a diagram

$$\begin{tikzcd} H^2(\mathfrak{E}, \mathcal{L}og^{(n+1)}) \arrow[r, "\simeq"] \arrow[d] & H^2(\mathfrak{E}, \mathcal{L}og^{(n)}) \arrow[r] \arrow[d, "\simeq"'] & 0 \\ H^2(\mathfrak{E}, \widetilde{\mathcal{L}og}^{(n+1)}) \arrow[r] & H^2(\mathfrak{E}, \widetilde{\mathcal{L}og}^{(n)}) \arrow[r] & 0, \end{tikzcd}$$

where the top horizontal arrow is an isomorphism from the proof of Proposition 2.8, and the right vertical arrow is an isomorphism from the induction hypothesis. This proves that the left vertical arrow is injective, hence the lemma. □

Lemma 5.7. *We have*

$$\dim_K H^0(\mathfrak{E}, \widetilde{\mathcal{L}og}^{(n)}) = 1.$$

In particular, the sequence

$$0 \to H^0(\mathfrak{E}, \mathscr{M}(\hat{\eta})^{(n)}) \to H^0(\mathfrak{E}, \mathcal{L}og^{(n)}) \to H^0(\mathfrak{E}, \widetilde{\mathcal{L}og}^{(n)}) \to 0$$

is exact, and we have

$$H^0(\mathfrak{E}, \widetilde{\mathcal{L}og}^{(n)}) \xrightarrow{\cong} \mathscr{H}(\hat{\omega})^{\otimes n}.$$

Proof. The second statement follows from the first, since by Lemma 5.5 and Proposition 2.8, we have

$$\dim_K H^0(\mathfrak{E}, \mathscr{M}(\hat{\eta})^{(n)}) = n$$
$$\dim_K H^0(\mathfrak{E}, \mathcal{L}og^{(n)}) = n + 1.$$

For the first statement, let

$$f = \sum_{r=0}^{n} f_r e^{\otimes r} \otimes \hat{\omega}^{\otimes n-r}$$

be an element in $H^0(\mathfrak{E}, \widetilde{\mathcal{L}og}^{(n)})$. By definition, $\nabla(f) = 0$, hence

$$\nabla(f) = \sum_{r=0}^{n} df_r e^{\otimes r} \otimes \hat{\omega}^{\otimes n-r} + \sum_{r=0}^{n} r f_r e^{\otimes(r-1)} \otimes \hat{\omega}^{\otimes(n-r+1)} \otimes \omega.$$

This holds if and only if

$$\begin{cases} f_r \in K & r = n \\ f_r = 0 & r < n. \end{cases}$$

Hence we have the proof of the lemma. □

From the above lemma, the natural map

$$H^1(\mathfrak{E}, \mathscr{M}(\hat{\eta})^{(n)}) \longrightarrow H^1(\mathfrak{E}, \mathcal{L}og^{(n)})$$

is injective. We denote by $\widetilde{\mathscr{H}}^{(n)}$ the cokernel of this map. By Proposition 2.8 and Lemma 5.5,

$$H^1(\mathfrak{E}, \mathscr{M}(\hat{\eta})^{(n)}) = \mathscr{H}(\hat{\eta})^{(n)}(-1)$$
$$H^1(\mathfrak{E}, \mathcal{L}og^{(n)}) = \mathrm{Sym}^{(n+1)} \mathscr{H}(-1),$$

and the natural map is the natural inclusion. Hence we have

$$\widetilde{\mathscr{H}}^{(n)} = \mathscr{H}(\hat{\omega})^{\otimes n}(-1).$$

Combining this fact with Lemma 5.6 and Lemma 5.5, the long exact sequence associated to the short exact sequence (5.2) yields the following:

Lemma 5.8. *The sequence*

$$0 \to \mathscr{H}(\omega) \otimes \mathscr{H}(\hat{\omega})^{\otimes n} \to H^1(\mathfrak{E}, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1)) \to \mathscr{H}(\omega) \otimes \mathscr{H}(\hat{\eta}) \to 0$$

is exact.

Proposition 5.9. *We have*

$$H^0_{\mathrm{syn}}(\mathcal{O}_K, H^1(\mathfrak{E}, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))) = 0$$
$$H^1_{\mathrm{syn}}(\mathcal{O}_K, H^1(\mathfrak{E}, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))) = H^1_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H}(\omega) \otimes \mathscr{H}(\hat{\omega})^{\otimes n})$$

for $n \geq 2$.

5.3. **The partial polylogarithmic sheaf.**

We define the map

$$\widetilde{\tau}_{\mathfrak{a}}^{(n)} : H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1)) \to \bigoplus_{\mathbf{E}[\mathfrak{a}]} \mathrm{Hom}_{S(\mathcal{O}_K)}(\mathscr{H}(\hat{\omega}), \mathscr{H}(\hat{\omega}))$$

in a manner similar to that of $\tau_{\mathfrak{a}}^{(n)}$. Namely, it is the composition of the map

(5.3)
$$H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1)) \to H^0_{\mathrm{syn}}(\mathcal{O}_K, H^1(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))),$$

which is the edge morphism of Leray's spectral sequence in Lemma 1.24, with the map

(5.4)
$$\begin{aligned} H^0_{\mathrm{syn}}(\mathcal{O}_K, H^1(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))) &\to \bigoplus_{\mathbf{E}[\mathfrak{a}]} H^0_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H}(\omega) \otimes \mathscr{H}(\hat{\omega})) \\ &= \bigoplus_{\mathbf{E}[\mathfrak{a}]} \mathrm{Hom}_{S(\mathcal{O}_K)}(\mathscr{H}(\hat{\omega}), \mathscr{H}(\hat{\omega})) \end{aligned}$$

induced from the Gysin exact sequence.

By functoriality, we have the following commutative diagram:

$$\begin{array}{ccc} H^1_{\mathrm{syn}}(\mathfrak{U}_a, \widehat{\mathscr{H}} \otimes \mathcal{L}og^{(n)}(1)) & \xrightarrow{\tau_{\mathfrak{a}}^{(n)}} & \bigoplus_{\mathbf{E}[\mathfrak{a}]} \mathrm{Hom}_{S(\mathcal{O}_K)}(\mathscr{H}, \mathscr{H}) \\ \downarrow & & \downarrow \\ H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1)) & \xrightarrow{\widetilde{\tau}_{\mathfrak{a}}^{(n)}} & \bigoplus_{\mathbf{E}[\mathfrak{a}]} \mathrm{Hom}_{S(\mathcal{O}_K)}(\mathscr{H}(\hat{\omega}), \mathscr{H}(\hat{\omega})). \end{array}$$

Hence $\widetilde{\mathrm{pol}}_a^{(n)}$ is an element in $H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))$ which satisfy

$$\widetilde{\tau}_a^{(n)}(\widetilde{\mathrm{pol}}_a^{(n)}) = \oplus_{P \in \mathbf{E}[\mathfrak{a}]}(N(P)\,\mathrm{id}) \tag{5.5}$$

for $N(P) = (\deg[a] - 1)$ if $P = O$ and $N(P) = -1$ otherwise.

Proposition 5.10. *The system of elements*

$$\widetilde{\mathrm{pol}}_a^{(n)} \in H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))$$

is uniquely determined by (5.5).

Proof. Since by Proposition 5.9, we have

$$H^0_{\mathrm{syn}}(\mathcal{O}_K, H^1(\mathfrak{E}, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))) = 0.$$

Hence (5.4) is injective, and the kernel of $\tau_a^{(n)}$ is the kernel of (5.3), which is

$$H^1_{\mathrm{syn}}(\mathcal{O}_K, H^0(\mathfrak{E}, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))).$$

By Lemma 5.7, we have

$$H^0(\mathfrak{E}, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1)) \xrightarrow{\cong} \mathscr{H}(\hat{\omega})^{\otimes n}. \tag{5.6}$$

We have a commutative diagram

$$\begin{array}{ccc} H^1_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H}(\hat{\omega})^{\otimes(n+1)}) & \longrightarrow & H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n+1)}(1)) \\ \downarrow & & \downarrow \\ H^1_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H}(\hat{\omega})^{\otimes n}) & \longrightarrow & H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1)) \end{array}$$

induced from the projection, and the compatibility with the Frobenius implies that the left vertical arrow is the zero map. This shows that $\widetilde{\mathrm{pol}}_a^{(n)}$ is uniquely determined by (5.5). □

The system of elements

$$\widetilde{\mathrm{pol}}_a^{(n)} \in H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))$$

corresponds to a system of objects in $S(\mathfrak{U}_a)$ through the isomorphism

$$\mathrm{Ext}^1_{S(\mathfrak{U}_a)}(K(0), \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1)) \xrightarrow{\eta} H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))$$

defined in Theorem 1. By abuse of notation, we denote this system of objects by the same symbol $\widetilde{\mathrm{pol}}_a^{(n)}$.

Let $M^\nabla(U_a)$ be the category of coherent $\mathcal{O}_{E_K}$-modules with logarithmic poles along D_a. A similar argument as in Theorem 1 shows that there exists an isomorphism

$$\eta : \mathrm{Ext}^1_{M(U_a)}(K(0), M) \longrightarrow H^1_{\mathrm{dR}}(U_a, M) \tag{5.7}$$

for any object M in $M^\nabla(U_a)$. For any object $\mathscr{M} = (M, \nabla, F^\bullet, \Phi)$ in $S^{\mathrm{ad}}(\mathfrak{U}_a)$, by definition, we have $H^1_{\mathrm{dR}}(U_a, M) \simeq H^1(\mathfrak{U}_a, \mathscr{M})$. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_{S(\mathfrak{U}_a)}(K(0), \mathscr{M}) & \longrightarrow & \mathrm{Ext}^1_{M(U_a)}(K(0), M) \\ \simeq\downarrow & & \simeq\downarrow \\ H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{M}) & \longrightarrow & H^1(\mathfrak{U}_a, \mathscr{M}), \end{array}$$

where the top horizontal arrow is the functor forgetting $F^\bullet$ and Φ, and the bottom horizontal arrow is the one induced from the edge morphism of Leray's spectral sequence

$$H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{M}) \longrightarrow H^0_{\mathrm{syn}}(\mathcal{O}_K, H^1(\mathfrak{U}_a, \mathscr{M})) \hookrightarrow H^1(\mathfrak{U}_a, \mathscr{M}).$$

Proposition 5.11 (Rigidity). *The objects $\widetilde{\mathrm{pol}}_a^{(n)}$ in $S(U_a)$ are uniquely determined by its underlying $\mathcal{O}_{E_K}$-module with logarithmic connection.*

Proof. In order to prove the proposition, it is necessary to prove that the bottom horizontal arrow is *injective* for the system of objects $\mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1)$ in $S^{\mathrm{ad}}(\mathfrak{U}_a)$. This is already proved in the proof of Proposition 5.10, hence we have the desired result. □

Definition 5.12. We define the object $(\widetilde{P}_{\mathfrak{a}}^{(n)}, \nabla)$ in $M^\nabla(U_a)$ as follows:

(i) $\widetilde{P}_{\mathfrak{a}}^{(n)}$ is the free $\mathcal{O}_{E_K}$-module

$$\widetilde{P}_{\mathfrak{a}}^{(n)} = \mathcal{O}_{E_K} e' \bigoplus \mathcal{O}_{E_K} \omega \otimes \widetilde{L}^{(n)}(1).$$

(ii) The connection is given by

$$\nabla(e') = \omega \otimes e^{\otimes(n-1)} \otimes \hat{\omega}(1) \otimes \frac{d\Theta_{\mathbf{E},\mathfrak{a}}}{\Theta_{\mathbf{E},\mathfrak{a}}}.$$

Proposition 5.13. *The image of the extension class of the object $(\widetilde{P}_{\mathfrak{a}}^{(n)}, \nabla)$ through the isomorphism*

$$\mathrm{Ext}^1_{M^\nabla(U_a)}(K(0), \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1)) \xrightarrow{\simeq} H^1(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))$$

is the underlying $\mathcal{O}_{E_K}$-module with connection of the element $12\,\widetilde{\mathrm{pol}}_a^{(n)}$.

Proof. The divisor of the function $\Theta_{\mathbf{E},\mathfrak{a}}$ is

$$12\mathbf{N}\mathfrak{a}[O] - 12 \sum_{P \in \mathbf{E}[\mathfrak{a}]} [P]$$

(See Lemma 4.2). Hence the residue of the differential is

$$\mathrm{Res}_{P \in \mathbf{E}[\mathfrak{a}]} \frac{d\Theta_{\mathbf{E},\mathfrak{a}}}{\Theta_{\mathbf{E},\mathfrak{a}}} = 12N(P)$$

for $N(P) = (\mathbf{N}\mathfrak{a} - 1)$ if $P = O$ and $N(P) = -1$ otherwise. Hence the image by η of $(\widetilde{P}_a^{(n)}, \nabla)$ are elements in $H^1(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))$ which maps with respect to the residue map

$$H^1(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)})) \longrightarrow \bigoplus_{\mathbf{E}[\mathfrak{a}]} \mathscr{H}(\omega) \otimes \mathscr{H}(\hat{\omega})$$

to the element

$$\bigoplus_{P \in \mathbf{E}[\mathfrak{a}]} \left(12N(P)\, \omega \otimes e^{\otimes(n-1)} \otimes \hat{\omega} \right).$$

This is the characterization of the element $12\,\widetilde{\mathrm{pol}}_a^{(n)}$, hence we have the desired assertion. □

We define a filtration on $\widetilde{P}_a^{(n)}$ by

$$\begin{cases} F^m \widetilde{P}_a^{(n)} = \mathcal{O}_{E_K} e' \bigoplus (\mathcal{O}_{E_K} \omega \otimes F^m \widetilde{L}^{(n)}(1)) & m \leq 0 \\ F^m \widetilde{P}_a^{(n)} = (\mathcal{O}_{E_K} \omega \otimes F^m \widetilde{L}^{(n)}(1)) & m > 0. \end{cases}$$

By the previous proposition and from the fact that $\widetilde{\mathrm{pol}}_a^{(n)}$ exists, there exists a unique Frobenius structure Φ on $\widetilde{P}_a^{(n)}$ which makes $(\widetilde{P}_a^{(n)}, \nabla, F^\bullet, \Phi)$ an object of $S(\mathfrak{U})$.

Let $A = \Gamma(U_a, \mathcal{O}_{U_a})$. The Frobenius Φ on $\widetilde{P}_a^{(n)}$ be given by

$$\Phi(e') = e' + \sum_{j=0}^{n} \frac{1}{(n-j)!} D_{\mathfrak{a},j} \omega \otimes e^{\otimes(n-j)} \otimes \hat{\omega}^{\otimes j}(1).$$

for some functions $D_{\mathfrak{a},j} \in A_K^\dagger$ for $j \geq 0$. The compatibility of the Frobenius with the connection gives

$$\begin{aligned} \Phi \circ \nabla(e') &= \Phi \left(\omega \otimes e^{\otimes(n-1)} \otimes \hat{\omega}(1) \otimes \frac{d\Theta_{\mathbf{E},\mathfrak{a}}}{\Theta_{\mathbf{E},\mathfrak{a}}} \right) \\ &= \omega \otimes e^{\otimes(n-1)} \otimes \hat{\omega}(1) \otimes \frac{\varphi}{p} \left(\frac{d\Theta_{\mathbf{E},\mathfrak{a}}}{\Theta_{\mathbf{E},\mathfrak{a}}} \right) \end{aligned}$$

and

$$\begin{aligned} \nabla \circ \Phi(e') &= \nabla \left(e' + \sum_{j=0}^{n} \frac{1}{(n-j)!} D_{\mathfrak{a},j}\, \omega \otimes e^{\otimes(n-j)} \otimes \hat{\omega}^{\otimes j}(1) \right) \\ &= \omega \otimes e^{\otimes(n-1)} \otimes \hat{\omega}(1) \otimes \frac{d\Theta_{\mathbf{E},\mathfrak{a}}}{\Theta_{\mathbf{E},\mathfrak{a}}} \\ &\quad + \sum_{j=0}^{n} \frac{1}{(n-j)!} dD_{\mathfrak{a},j}\, \omega \otimes e^{\otimes(n-j)} \otimes \hat{\omega}^{\otimes j}(1) \\ &\quad + \sum_{j=0}^{n} \frac{1}{(n-j-1)!} D_{\mathfrak{a},j}\, \omega \otimes e^{\otimes(n-j-1)} \otimes \hat{\omega}^{\otimes(j+1)}(1) \otimes \omega \end{aligned}$$

Comparing the two equations, we get the differential equations

$$dD_{\mathfrak{a},0} = 0$$
$$dD_{\mathfrak{a},1} + D_{\mathfrak{a},0}\,\boldsymbol{\omega} = \left(\frac{\varphi}{p} - 1\right)\frac{d\Theta_{\mathbf{E},\mathfrak{a}}}{\Theta_{\mathbf{E},\mathfrak{a}}}$$
$$dD_{\mathfrak{a},j+1} + D_{\mathfrak{a},j}\,\boldsymbol{\omega} = 0 \qquad (1 \leq j \leq n-1).$$

Since the $dD_{\mathfrak{a},0} = 0$, the function $D_{\mathfrak{a},0}$ is a constant function in $A_K^\dagger$. There function

$$\tilde{D}_{\mathfrak{a},1} = \left(\frac{\varphi}{p} - 1\right)\log\Theta_{\mathbf{E},\mathfrak{a}}$$

in $A_K^\dagger$ satisfies

$$\left(\frac{\varphi}{p} - 1\right)\frac{d\Theta_{\mathbf{E},\mathfrak{a}}}{\Theta_{\mathbf{E},\mathfrak{a}}}.$$

Since a non-zero constant multiple of $\boldsymbol{\omega}$ is *not* integrable in $A_K^\dagger$, this implies that $D_{\mathfrak{a},0} = 0$.

Definition 5.14. We call the functions $D_{\mathfrak{a},j}$ ($j \geq 1$) the *p-adic elliptic polylogarithmic functions* of weight j with respect to the ideal $\mathfrak{a}$.

Since the functions $D_{\mathfrak{a},j}$ are elements of $A_K^\dagger$, they are over convergent functions on $\mathcal{U}_{\mathfrak{a}K}$. They are the *unique* system of over convergent functions satisfying the differential equations

$$dD_{\mathfrak{a},1} = \left(\frac{\varphi}{p} - 1\right)\frac{d\Theta_{\mathbf{E},\mathfrak{a}}}{\Theta_{\mathbf{E},\mathfrak{a}}}$$
$$dD_{\mathfrak{a},j+1} = -D_{\mathfrak{a},j}\boldsymbol{\omega} \qquad (j \geq 1).$$

5.4. **The proof of the main theorem.**

In this section, we will give the proof of Theorem 5. We take a non-zero element $x \in E[\mathfrak{f}]$. As mentioned at the end of Section 5.1, it is sufficient to calculate the image of the object

$$12\,\widetilde{\mathrm{pol}}_a^{(n)} \in H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1))$$

with respect to the specialization map

$$H^1_{\mathrm{syn}}(\mathfrak{U}_a, \mathscr{H}(\omega) \otimes \widetilde{\mathcal{L}og}^{(n)}(1)) \to H^1_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H}(\omega) \otimes \prod_{0\leq j\leq n} \mathscr{H}(\bar{\omega})^{\otimes j}).$$

The map induced on the corresponding extension classes the pull pack map x^*, hence the extension class of

$$x^*(12\widetilde{\mathrm{pol}}_a^{(n)}) \in H^1_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H}(\omega) \otimes \prod_{0\leq j\leq n} \mathscr{H}(\bar{\omega})^{\otimes j})$$

corresponds to the filtered Frobenius module $(x^*\widetilde{P}_a^{(n)}, F^\bullet, \Phi)$ in $S(\mathcal{O}_K)$ given as follows:

(i) We have

$$\widetilde{P}_a^{(n)} = Ke' \bigoplus \prod_{j=0}^{m} K\omega \otimes \hat{\omega}^{\otimes j}(1).$$

(ii) The filtration is given by

$$\begin{cases} F^m(x^*\widetilde{P}_a^{(n)}) = Ke' \bigoplus \prod_{j\le -m-1} K(\omega \otimes \hat{\omega}^{\otimes j}(1)) & m \le 0 \\ F^m(x^*\widetilde{P}_a^{(n)}) = 0 & m > 0 \end{cases}$$

(iii) The Frobenius Φ on $x^*\widetilde{P}_a^{(n)}$ is given by

$$\Phi(e') = e' + \sum_{j=0}^{n} D_{\mathfrak{a},j}(x)\, \omega \otimes \hat{\omega}^{\otimes j}(1).$$

Here, $D_{\mathfrak{a},j}(x)$ is the value of the function $D_{\mathfrak{a},j}$ at the torsion point x.

Since the isomorphism

$$H^1_{\mathrm{syn}}(\mathcal{O}_K, \mathscr{H}(\omega) \otimes \prod_{0\le j\le n} \mathscr{H}(\hat{\omega})^{\otimes j}(1)) = \prod_{0\le j\le n} K\,\omega \otimes \hat{\omega}^{\otimes j}(1)$$

is given by the Frobenius structure, one can see from the explicit shape of $(x^*\widetilde{P}_a^{(n)}, F^\bullet, \Phi)$ that

$$x^*(12\,\mathrm{pol}_a^{(n)}) = \prod_{1\le j\le n} D_{\mathfrak{a},j}(x)\, \omega \otimes \hat{\omega}^{\otimes j}(1).$$

Hence Theorem 5 follows.

5.5. The relation to the p-adic L-function.

In the last section, we will give the relation of the polylogarithmic functions $D_{\mathfrak{a},j}$ with the p-adic L-function of $\mathbf{E}$.

Definition 5.15. We extend the system of functions $D_{\mathfrak{a},j}$ $(j \ge 1)$ to integers $j \le 0$ by placing

$$dD_{\mathfrak{a},j+1} = -D_{\mathfrak{a},j}\omega \qquad (j \in \mathbb{Z}).$$

Then $D_{\mathfrak{a},j}$ are rigid analytic functions on $\mathcal{U}_{\mathfrak{a}K}$ (See [CdS] proof of Lemma (75)). Fix ξ, Ω, and $\Omega_{\mathfrak{p}}$ as in Section 4. Then $x = \xi(\Omega)$ is a $\mathcal{O}_{\mathbf{K}}$-module generator of $\mathbf{E}[\mathfrak{f}]$. We fix a set of ideals B of $\mathcal{O}_{\mathbf{K}}$ prime to $\mathfrak{f}p$ such that $\sigma_{\mathfrak{c}} = (\mathfrak{c}, \mathbf{F}/\mathbf{K})$ for $\mathfrak{c} \in B$ is a set of representative for the group $\mathrm{Gal}(\mathbf{F}/\mathbf{K})$. Let $x_{\mathfrak{c}} = \sigma_{\mathfrak{c}}(x) \in \mathbf{E}[\mathfrak{f}]$.

We have the following Proposition:

Proposition 5.16. *For each integer $j \le -1$, we have*

$$\sum_{\mathfrak{c}\in B} D_{\mathfrak{a},j+1}(x_{\mathfrak{c}}) = (-1)^{j+1}\Omega_{\mathfrak{p}}^{j} L_{p,\mathfrak{a}}(\psi_{\mathfrak{p}}^{j}).$$

Proof. Since ω is a translation invariant differential, the value of the function $D_{\mathfrak{a},j+1}$ for $j \leq -1$ at x_ϵ is *precisely* the value of the function

$$(-1)^{j+1}\left((1+T)\frac{d}{dT}\right)^{-j} h_{\mathfrak{a},\epsilon}(T)$$

at $T = 0$. By definition, this is equal to

$$(-1)^{j+1}\int_{G_\infty} \kappa(g)^{-j} d\mu_{\mathfrak{a},\epsilon}(g).$$

Hence the proposition follows from the construction of the p-adic L-function, given in Definition 4.13. $\square$

We conjecture the following:

Conjecture . *For each integer $j \geq 0$, we have*

$$\sum_{\epsilon \in B} D_{\mathfrak{a},j+1}(x_\epsilon) = (-1)^{j+1}\Omega_{\mathfrak{p}}^j L_{p,\mathfrak{a}}(\psi_{\mathfrak{p}}^j).$$

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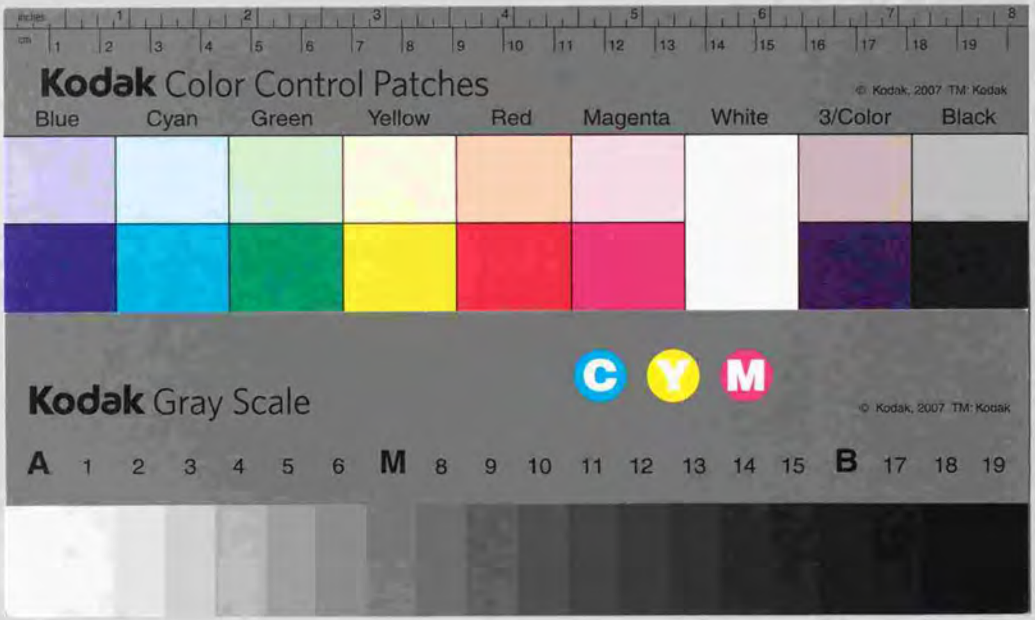
Kodak Color Control Patches
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Blue Cyan Green Yellow Red Magenta White 3/Color Black
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C Y M
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A 1 2 3 4 5 6 M 8 9 10 11 12 13 14 15 B 17 18 19