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Discrepancies of p-Cyclic Quotient Varieties

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Abstract. We consider the quotient variety associated to a linear representation of the cyclic group of order p in characteristic $p > 0$. We estimate the minimal discrepancy of exceptional divisors over the singular locus. In particular, we give criteria for the quotient variety being terminal, canonical and log canonical. As an application, we obtain new examples of non-Cohen-Macaulay terminal singularities, adding to examples recently announced by Totaro.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$ and $G = \mathbb{Z}/p$ the cyclic group of order p. Suppose that G linearly acts on the d-dimensional affine space $V = \mathbb{A}_k^d$. Let $X := V/G$ be the quotient variety. This variety is factorial (see [CW11, Th. 3.8.1]), but not necessarily Cohen-Macaulay. We are interested in singularities of X from the viewpoint of the minimal model program.

To state our main results, we recall basic notions concerning singularities and introduce some notation. Let us consider a modification (proper birational morphism) $f\colon Y \to X$ such that Y is normal, and the exceptional locus $\mathrm{Exc}\,(f) \subset Y$ and the preimage $f^{-1}(X_{\mathrm{sing}})$ of X_{sing} are both of pure dimension $d-1$. We will call such a morphism an *admissible modification.* Note that the last condition implies $f^{-1}(X_{\mathrm{sing}}) \subset \mathrm{Exc}\,(f)$. Note also that an arbitrary modification $Y \to X$ can be altered into an admissible one by blowup and normalization. For an admissible modification $f\colon Y \to X$, we define the *relative canonical divisor* $K_{Y/X}$ in the usual way, which is a Weil divisor with support contained in $\mathrm{Exc}\,(f)$. Let $\mathrm{Exc}\,(f) = \bigcup_{i \in \mathcal{E}_f} E_i$ and

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$f^{-1}(X_{\mathrm{sing}}) = \bigcup_{i\in\mathcal{S}_f} E_i$ be the decompositions into irreducible components with $\mathcal{S}_f \subset \mathcal{E}_f$ and write $K_{Y/X} = \sum_{i\in\mathcal{E}_f} a_i E_i$, where a_i are integers called *discrepancies*. We define

$$\delta(X) := \mathrm{discrep}\,(\mathrm{center} \subset X_{\mathrm{sing}}; X) = \inf_f \min_{i\in\mathcal{S}_f} a_i.$$

Here f runs over admissible modifications of X. We have either $\delta(X) \geq -1$ or $\delta(X) = -\infty$ [KM98, Cor. 2.31]. We say that X is *terminal* (resp. *canonical, log canonical*) if $\delta(X) > 0$ (resp. ≥ 0, ≥ -1). Note that since our variety X is factorial and hence $\delta(X)$ is an integer if not $-\infty$, that X is canonical is equivalent to that X is log terminal (meaning $\delta(X) > -1$).

We will estimate $\delta(X)$ in terms of the given G-representation V. For each integer i with $1 \leq i \leq p$, there exists a unique indecomposable representation of G over k; we denote it as V_i. The representation V decomposes into indecomposable ones: $V = \bigoplus_{\lambda=1}^{l} V_{d_\lambda}$ with $1 \leq d_\lambda \leq p$ and $\sum d_\lambda = d$. The decomposition is unique up to permutation of direct summands. We define an invariant D_V by

$$D_V := \sum_{\lambda=1}^{l} \frac{(d_\lambda - 1)d_\lambda}{2}.$$

We easily see that $D_V = 0$ if and only if the G-action is trivial, and also that $D_V = 1$ if and only if a generator of G is a pseudo-reflection (that is, the fixed point locus V^G has codimension one). In these cases, the quotient variety X is again isomorphic to $\mathbb{A}_k^d$. Excluding these cases, we assume in what follows that $D_V \geq 2$. The fixed point locus V^G is then an l-dimensional linear subspace of V. The quotient morphism $V \to X$ is étale outside V^G and the image of V^G is exactly the singular locus X_{sing} of X. The invariant D_V is clearly additive with respect to direct sums. For indecomposable representations of small dimensions, we have

$$D_{V_1} = 0,\ D_{V_2} = 1,\ D_{V_3} = 3,\ D_{V_4} = 6,\ D_{V_5} = 10.$$

From [ES80], X is Cohen-Macaulay if and only if $d - l = \operatorname{codim} V^G \leq 2$. If $D_V = 2$, then $V = V_2^{\oplus 2} \oplus V_1^{\oplus(d-4)}$ for some n and X is Cohen-Macaulay. Note that $V_1^{\oplus n}$ is the n-dimensional trivial representation. If $D_V = 3$, then V is either $V_3 \oplus V_1^{\oplus(d-3)}$ or $V_2^{\oplus 3} \oplus V_1^{\oplus(d-6)}$; X is Cohen-Macaulay in the former and not Cohen-Macaulay in the latter. If $D_V \geq 4$, then X is never Cohen-Macaulay.

In [Yas14, Propositions 6.6 and 6.9], the author proved the following result.

Theorem 1.1. *If $D_V \geq p$, then X is canonical.*

This result provided the first example of log terminal (even canonical) but not Cohen-Macaulay singularities in all positive characteristics (recall that log terminal singularities in characteristic zero are always Cohen-Macaulay). For instance, X has such singularities when $V = V_2^{\oplus 3}$ in characteristic two or $V = V_4$ in characteristic five. Later, further examples of non-Cohen-Macaulay log terminal or canonical singularities [GNT, CT, Kov, Ber] were found. The above theorem also shows that when $V = V_3$ in characteristic $p \geq 5$, X is Cohen-Macaulay (in fact, a hypersurface) but not log terminal (recall that quotient singularities in characteristic zero are always log terminal). It should be mentioned that Hacon and Witaszek [HW] proved that in sufficiently large characteristics, three-dimensional log terminal singularities are Cohen-Macaulay.

The aim of this paper is to strengthen the above theorem. For a positive integer j with $p \nmid j$, we define

$$\mathrm{sht}_V(j) := \sum_{\lambda=1}^{l} \sum_{i=1}^{d_\lambda - 1} \left\lfloor \frac{ij}{p} \right\rfloor .$$

Here $\lfloor \cdot \rfloor$ denotes the round down of rational numbers to integers. The following two theorems are our main results:

Theorem 1.2. *Suppose that $D_V \geq 2$. Then $D_V < p-1$ if and only if $\delta(X) = -\infty$. If $D_V \geq p-1$, then*

$$\delta(X) = d - 1 - l - \max_{1 \leq s \leq p-1} \{s - \mathrm{sht}_V(s)\} \tag{1.1}$$

$$= D_V - 1 - \max_{1 \leq s \leq p-1} \{\mathrm{sht}_V(p-s) + s\}. \tag{1.2}$$

Theorem 1.3. *Suppose that $D_V \geq 2$. Then*

$$\delta(X) \leq D_V - p. \tag{1.3}$$

If $D_V \geq p$, then we also have

$$\delta(X) \geq \frac{2D_V}{p} - 2. \tag{1.4}$$

As a direct consequence of these, we obtain:

Corollary 1.4. *Suppose that $D_V \geq 2$. Then X is terminal (resp. canonical, log canonical) if and only if $D_V > p$ (resp. $D_V \geq p$, $D_V \geq p-1$).*

For instances, in the cases where $V = V_2^{\oplus 3}$ in characteristic two and $V = V_4$ in characteristic five, the quotient variety X is terminal but not Cohen-Macaulay. The dimension four is the smallest possible, because we consider linear actions and the fixed-point locus V^G is always of positive dimension. Note that Totaro [Tot, Cor. 2.2] had earlier announced construction of terminal but not Cohen-Macaulay singularities by using homogeneous varieties with nonreduced stabilizers. These singularities have dimension ≥ 8. The current work was motivated by his result. Later, inspired by construction of the author, Totaro constructed an example of a three-dimensional terminal but not Cohen-Macaulay singularity in characteristic two, by considering a non-linear action of $\mathbb{Z}/2$ (see [Tot, Theorem 5.1]).

In the next section, we review basics of motivic integration and give an expression of $\delta(X)$ in terms of motivic integration (Proposition 2.1). In Section 3, we prove Theorems 1.2 and 1.3.

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2. Motivic Integration

In this section, after briefly recalling basics of motivic integration, we prove a result expressing $\delta(X)$ in terms of motivic integration, Proposition 2.1, which will be necessary in Section 3.

We denote by X an irreducible variety of dimension d over k. We denote the arc space of X by $J_\infty X$. This space has the motivic measure denoted by μ_X. We let the motivic measure take values in the ring denoted by $\hat{\mathcal{M}}'$ in [Yas14], a version of the completed Grothendieck ring of varieties. The element of $\hat{\mathcal{M}}'$ defined by a variety Z is denoted by $[Z]$. We write the special element $[\mathbb{A}^1_k]$ as $\mathbb{L}$, which is invertible by construction. There exists a ring homomorphism $P\colon \hat{\mathcal{M}}' \to \mathbb{Z}((T^{-1}))$ which sends $[Z]$ to the Poincaré polynomial of Z (see *ibid.*, pp. 1141-1142). We define the *degree* of $\alpha = \sum a_i T^i \in \mathbb{Z}((T^{-1}))$ by $\deg \alpha := \sup\{i \mid a_i \neq 0\}$. For $a \in \hat{\mathcal{M}}'$, we define its *dimension* as $\dim a := \frac{1}{2} \deg P(a)$ so that for a variety Z and an integer n, we have $\dim [Z]\mathbb{L}^n = \dim Z + n$.

For $n \in \mathbb{N}$, let $\pi_n\colon J_\infty X \to J_n X$ be the truncation map to n-jets. A subset $U \subset J_\infty X$ is called *stable* if there exists $n \in \mathbb{N}$ such that $\pi_n(U)$ is a constructible subset of $J_n X$, $U = \pi_n^{-1}\pi_n(U)$ and for every $n' \geq n$, the map $\pi_{n'+1}(U) \to \pi_{n'}(U)$ is a piecewise trivial $\mathbb{A}^d$-fibration. The measure $\mu_X(U)$ of a stable subset U is defined to be $[\pi_n(U)]\mathbb{L}^{-nd}$ for $n \gg 0$. A *measurable subset* of $J_\infty X$ is a subset approximated by a sequence of stable subsets. The measure of a measurable subset is defined as the limit of ones of stable subsets (for details, see [DL02, Appendix], [Seb04, Section 6]).

In what follows, we assume that X is normal and the canonical sheaf ω_X is invertible. The *ω-Jacobian ideal* $\mathcal{J}_X \subset \mathcal{O}_X$ is then defined by $\mathcal{J}_X \omega_X = \operatorname{Im}\left(\bigwedge^d \Omega_{X/k} \to \omega_X\right)$. For an admissible modification $f\colon Y \to X$, let $f_\infty\colon J_\infty Y \to J_\infty X$ be the induced map of arc spaces. For a measurable subset $U \subset J_\infty Y_{\mathrm{sm}} \subset J_\infty Y$, where Y_{sm} denotes the smooth locus, we have the *change of variables formula*,

$$\int_U \mathbb{L}^{-\operatorname{ord} K_{Y/X}}\, d\mu_Y = \int_{f_\infty(U)} \mathbb{L}^{\operatorname{ord} \mathcal{J}_X}\, d\mu_X.$$

Here ord ? denotes the order function associated to a divisor or an ideal sheaf. The proofs of our main results are based on repeated evaluation of dimensions of integrals as above. For this reason, we introduce the following notation: For a measurable subset U of $J_\infty X$ (resp. $J_\infty Y_{\mathrm{sm}}$), we define

$$\nu(U) := \dim\left(\int_U \mathbb{L}^{\operatorname{ord} \mathcal{J}_X}\, d\mu_X\right)$$
$$\left(\text{resp. } \nu(U) := \dim\left(\int_U \mathbb{L}^{-\operatorname{ord} K_{Y/X}}\, d\mu_Y\right)\right),$$

provided that the integral converges. We say that a measurable subset U of $J_\infty X$ or $J_\infty Y_{\mathrm{sm}}$ is *small* if the relevant integral converges. When $U \subset J_\infty Y_{\mathrm{sm}}$, we have $\nu(U) = \nu(f_\infty(U))$.

In what follows, we write $A =_{\text{a.e.}} B$ (resp. $A \subset_{\text{a.e.}} B$) to mean that there exists a measure zero subset C such that $A \setminus C = B \setminus C$ (resp. $A \setminus C \subset B \setminus C$); "a.e." stands for "almost everywhere". We also denote by π_X the truncation map $J_\infty X \to J_0 X = X$ and similarly for π_Y.

PROPOSITION 2.1. *Let $C_r \subset \pi_X^{-1}(X_{\mathrm{sing}})$, $r \in \mathbb{N}$ be a countable collection of small measurable subsets such that $\pi_X^{-1}(X_{\mathrm{sing}}) =_{\text{a.e.}} \bigcup_{r \in \mathbb{N}} C_r$. Then*

$$\delta(X) = d - 1 - \sup_r \nu(C_r).$$

The corresponding result in characteristic zero would be well-known to specialists and is an easy consequence of the existence of log resolution, the change of variables formula and explicit computation of $\int_U \mathbb{L}^{-\operatorname{ord} K_{Y/X}}$ for some small measurable subsets $U \subset J_\infty Y$. Since we work in positive characteristic, we do not know the existence of log resolution. However a result of Reguera [Reg09, Prop. 3.7(vii)], that every stable point determines a divisorial valuation, can take the place of the existence of log resolution. Indeed similar results for MJ-discrepancies have been proved by using it [IR17, Th. 3.18]. The rest of this section is devoted to the proof of the above proposition, which uses only standard arguments. We first prove a few auxiliary results.

LEMMA 2.2. *Let C, C_i, D_i, $i \in \mathbb{N}$ be small measurable subsets of $J_\infty X$.*

(1) *If $C \subset_{\text{a.e.}} \bigcup_{i \in \mathbb{N}} D_i$, then $\nu(C) \le \sup_{i \in \mathbb{N}} \nu(D_i)$.*

(2) *If $\bigcup_{i \in \mathbb{N}} C_i =_{\text{a.e.}} \bigcup_{i \in \mathbb{N}} D_i$, then $\sup_{i \in \mathbb{N}} \nu(C_i) = \sup_{i \in \mathbb{N}} \nu(D_i)$.*

PROOF. For the first assertion, let $D_i' := D_i \setminus \bigcup_{j<i} D_j$. Then the sets D_i' are mutually disjoint and $\bigcup_i D_i' = \bigcup_i D_i$. Therefore

$$\int_C \mathbb{L}^{\operatorname{ord} \mathcal{J}_X} \, d\mu_X = \sum_{i \in \mathbb{N}} \int_{C \cap D_i'} \mathbb{L}^{\operatorname{ord} \mathcal{J}_X} \, d\mu_X.$$

It follows that

$$\nu(C) = \sup_{i \in \mathbb{N}} \nu(C \cap D'_i) \le \sup_{i \in \mathbb{N}} \nu(D'_i) \le \sup_{i \in \mathbb{N}} \nu(D_i).$$

Thus the first assertion holds. For the second assertion, we apply the first assertion to $C = C_i$ for each i to get

$$\nu(C_i) \le \sup_{j \in \mathbb{N}} \nu(D_j).$$

This shows $\sup_i \nu(C_i) \le \sup_i \nu(D_i)$. The opposite inequality is proved by the same argument. □

For $i \in \mathcal{E}_f$ and an integer $b > 0$, we define

$$E_i^\circ := (E_{i,\mathrm{sm}} \cap Y_{\mathrm{sm}}) \setminus \bigcup_{j \in \mathcal{E}_f \setminus \{i\}} E_j \subset Y,$$
$$N_{i,b} := \{\gamma \in \pi_Y^{-1}(E_i^\circ) \mid \mathrm{ord}_{E_i}(\gamma) = b\} \subset J_\infty Y.$$

To emphasize f, we also write $N_{i,b}$ as $N_{i,b}^f$. By a standard computation of motivic integration,

$$\int_{N_{i,b}} \mathbb{L}^{-\mathrm{ord}\, K_{Y/X}}\, d\mu_Y = \mu_Y(N_{i,b})\mathbb{L}^{-a_i b} = [E_i^\circ](\mathbb{L}-1)\mathbb{L}^{-(1+a_i)b}.$$

In particular,

$$\nu(N_{i,b}) = d - (1 + a_i)b.$$

Lemma 2.3. *Let $\mathfrak{M}$ be the set of (isomorphism classes of) admissible modifications $f\colon Y \to X$. We have*

$$\delta(X) = d - 1 - \sup_{f \in \mathfrak{M}, i \in \mathcal{S}_f, b > 0} \nu(N_{i,b}^f).$$

Proof. If X is not log canonical, then the both sides are $-\infty$. If X is log canonical, since $1 + a_i \ge 0$, we have $\nu(N_{i,b}^f) \le \nu(N_{i,1}^f)$ for every $f \in \mathfrak{M}$,

$i \in \mathcal{S}_f$, $b > 0$. Therefore

$$\begin{aligned}\delta(X) &= \inf_{f\in\mathfrak{M}, i\in\mathcal{S}_f} a_i \\ &= d-1-\sup_{f\in\mathfrak{M}, i\in\mathcal{S}_f} \nu(N^f_{i,1}) \\ &= d-1-\sup_{f\in\mathfrak{M}, i\in\mathcal{S}_f, b>0} \nu(N^f_{i,b}). \ \square\end{aligned}$$

LEMMA 2.4. *There exists a countable set* $\mathfrak{M}_0 \subset \mathfrak{M}$ *of admissible modifications of* X *such that*

$$\pi_X^{-1}(X_{\mathrm{sing}}) =_{\mathrm{a.e.}} \bigcup_{f\in\mathfrak{M}_0,\, i\in\mathcal{S}_f,\, b>0} f_\infty(N^f_{i,b}).$$

PROOF. The lemma follows from the following two facts:

(1) There exist countably many measurable subsets $C_j \subset \pi_X^{-1}(X_{\mathrm{sing}})$, $j \in \mathbb{N}$ such that $\pi_X^{-1}(X_{\mathrm{sing}}) =_{\mathrm{a.e.}} \bigcup_j C_j$.

(2) Every measurable subset $C \subset \pi_X^{-1}(X_{\mathrm{sing}})$ is almost everywhere covered by $f_\infty(\pi_Y^{-1}(E_i^\circ))$ for countably many f's.

For the first fact, we can for instance take C_j to be the locus of arcs having order $j+1$ along X_{sing} ([DL99, Lemma 4.1], [Seb04, Lemme 4.5.4]). For the second one, we need a result of Reguera [Reg09, Prop. 3.7(vii)] that every stable point of $J_\infty X$ determines a divisorial valuation on the function field, which means that for every irreducible stable subset $C' \subset \pi_X^{-1}(X_{\mathrm{sing}})$, there exist an admissible modification $f\colon Y \to X$ and $i \in \mathcal{S}_f$ such that $f_\infty(\pi_Y^{-1}(E_i^\circ))$ contains the generic point of C'. Let $C \subset \pi_X^{-1}(X_{\mathrm{sing}})$ be a measurable subset. By definition of measurable subsets (see [DL02, Def. A.5]), there exists a sequence of stable subsets which approximate C. In particular there exists a stable subset C' such that $\dim \mu_X(C \triangle C')$ is arbitrarily small, say $< \dim \mu_X(C)$. Here $A\triangle B$ denotes the symmetric difference of A and B, that is, the subset $(A\cup B)\setminus(A\cap B)$. Then $\dim \mu_X(C') = \dim \mu_X(C)$. Replacing C' with $C' \setminus \pi_X^{-1}(X_{\mathrm{sm}}) = C' \cap \pi_X^{-1}(X_{\mathrm{sing}})$, we may further assume

$C' \subset \pi_X^{-1}(X_{\mathrm{sing}})$. Decomposing C' into irreducible stable subsets and applying Reguera's result to each of them, we see that there exist finitely many subsets $B_1, \dots, B_m \subset \pi_X^{-1}(X_{\mathrm{sing}})$ of the form $f_\infty(\pi_Y^{-1}(E_i^\circ))$ such that

$$\dim \mu_X\left(C' \setminus \bigcup_{h=1}^{m} B_h\right) < \dim \mu_X(C') = \dim \mu_X(C).$$

Since

$$C \setminus \bigcup_{h=1}^{m} B_h \subset \left(C' \setminus \bigcup_{h=1}^{m} B_h\right) \cup (C \setminus C'),$$

we have

$$\begin{aligned}\dim \mu_X\left(C \setminus \bigcup_{h=1}^{m} B_h\right) &\leq \max\left\{\dim \mu_X\left(C' \setminus \bigcup_{h=1}^{m} B_h\right), \dim \mu_X(C \setminus C')\right\} \\ &< \dim \mu_X(C).\end{aligned}$$

We then apply this argument to $C \setminus \bigcup_{h=1}^{m} B_h$ instead of C and repeat it. Eventually we get a measure zero subset by removing countably many subsets of the same form from C, equivalently C is almost everywhere covered by countably subsets of this form. Since each subset of the form $f_\infty(\pi_Y^{-1}(E_i^\circ))$ is almost everywhere covered by countably many subsets of the form $f_\infty(N_{i,b}^f)$, the second fact above holds. □

Proof of Proposition 2.1. For any admissible modification f and $i \in \mathcal{S}_f$, since $f_\infty(N_{i,1}^f) \subset_{\text{a.e.}} \bigcup_r C_r$, from Lemma 2.2, we have

$$a_i = d - 1 - \nu(N_{i,1}) \geq d - 1 - \sup_r \nu(C_r).$$

Thus $\delta(X) \geq d - 1 - \sup_r \nu(C_r)$.

Let $\mathfrak{M}_0$ be a countable set of admissible modifications of X as in Lemma 2.4. Then

$$\bigcup_r C_r =_{\text{a.e.}} \bigcup_{f \in \mathfrak{M}_0, i \in \mathcal{S}_f, b > 0} f_\infty(N_{i,b}^f)$$

and from Lemma 2.2,

$$\sup_r \nu(C_r) = \sup_{f \in \mathfrak{M}_0, i \in \mathcal{S}_f, b > 0} \nu(N_{i,b}^f).$$

This equality together with Lemma 2.3 shows

$$\begin{aligned}\delta(X) &= d-1-\sup_{f\in\mathfrak{M},i\in\mathcal{S}_f,b>0}\nu(N^f_{i,b})\\ &\le d-1-\sup_{f\in\mathfrak{M}_0,i\in\mathcal{S}_f,b>0}\nu(N^f_{i,b})\\ &= d-1-\sup_r \nu(C_r). \ \square\end{aligned}$$

3. Proof of Theorems 1.2 and 1.3

In this section, we prove Theorems 1.2 and 1.3, applying Proposition 2.1 to a special choice of small measurable subsets for which the values of ν were explicitly computed in [Yas14].

Let $V = \mathbb{A}^d_k$ and $X = V/G$ be as in Introduction. We assume $D_V \geq 2$.

PROPOSITION 3.1. *Let* $J := \{j \in \mathbb{Z} \mid j > 0, p \nmid j\} \cup \{0\}$. *There exist measurable subsets* $M_j \subset \pi_X^{-1}(X_{\mathrm{sing}})$, $j \in J$ *such that* $\bigcup_{j\in J} M_j =_{\text{a.e.}} \pi_X^{-1}(X_{\mathrm{sing}})$ *and*

$$\int_{M_j} \mathbb{L}^{\operatorname{ord}\mathcal{J}_X}\, d\mu_X = \begin{cases}(\mathbb{L}-1)\mathbb{L}^{l+j-1-\lfloor j/p\rfloor - \mathrm{sht}_V(j)} & (j>0)\\ \mathbb{L}^l & (j=0).\end{cases}$$

PROOF. This follows from computation in Proof of [Yas14, Prop. 6.9] and a version of the change of variables formula, *ibid.*, Theorem 5.20. For $j > 0$, we take M_j to be the image of $\mathcal{J}_{\infty,j}\mathcal{X}$, the space of twisted arcs with ramification jump j (for the definition, see *ibid.*, Definition 3.11). A point of $\mathcal{J}_{\infty,j}\mathcal{X}$ corresponds to a G-equivariant morphism $\operatorname{Spec}\mathcal{O}_L \to V$, where L is a Galois extension of $k((t))$ with Galois group G and ramification jump j, and $\mathcal{O}_L$ is the integral closure of $k[[t]]$ in L. The closed point of $\operatorname{Spec}\mathcal{O}_L$ maps into V^G. This implies $M_j \subset \pi_X^{-1}(X_{\mathrm{sing}})$. For $j = 0$, we take M_0 to be the intersection of the image of $\mathcal{J}_{\infty,0}\mathcal{X}$ and $\pi_X^{-1}(X_{\mathrm{sing}})$. Since the map $\bigsqcup_{j\in J}\mathcal{J}_{\infty,j}\mathcal{X} \to J_\infty X$ is almost bijective (*ibid.*, Proposition 3.17), these sets M_j almost cover $\pi_X^{-1}(X_{\mathrm{sing}})$. $\square$

For a positive integer j with $p \nmid j$, we write $j = np + s$ with $n \geq 0$ and $1 \leq s \leq p-1$. Since $\mathrm{sht}_V(j) = D_V n + \mathrm{sht}_V(s)$, we have

$$\nu(M_j) = l + (p-1-D_V)n + s - \mathrm{sht}_V(s). \tag{3.1}$$

Note that since $\mathrm{sht}_V(1) = 0$,

$$\nu(M_1) = l + 1 > l = \nu(M_0). \tag{3.2}$$

LEMMA 3.2. *We have*

$$s - \mathrm{sht}_V(s) = \mathrm{sht}_V(p - s) + s + d - l - D_V.$$

PROOF. The proof here is taken from [Yas14, Proof of Prop. 6.36]. We have

$$\begin{aligned}
&\mathrm{sht}_V(p - s) + s + d - l - D_V \\
&= s + \left(\sum_{\lambda=1}^{l} \sum_{i=1}^{d_\lambda - 1} i + \left\lfloor -\frac{is}{p} \right\rfloor \right) + d - l - D_V \\
&= s + \left(\sum_{\lambda=1}^{l} \sum_{i=1}^{d_\lambda - 1} - \left\lfloor \frac{is}{p} \right\rfloor - 1 \right) + d - l \\
&= s - \mathrm{sht}_V(s). \ \square
\end{aligned}$$

LEMMA 3.3. *We have*

$$\mathrm{sht}_V(s) \leq \frac{(s-1)D_V}{p}.$$

PROOF. When i varies from 1 to $d_\lambda - 1$, the differences $\frac{is}{p} - \left\lfloor \frac{is}{p} \right\rfloor$ take $d_\lambda - 1$ distinct values in $\{1/p, \dots, (p-1)/p\}$. Therefore

$$\sum_{i=1}^{d_\lambda - 1} \left\lfloor \frac{is}{p} \right\rfloor \leq \sum_{i=1}^{d_\lambda - 1} \frac{is}{p} - \sum_{j=1}^{d_\lambda - 1} \frac{j}{p} = \frac{s-1}{p} \sum_{i=1}^{d_\lambda - 1} i = \frac{(s-1)(d_\lambda - 1)d_\lambda}{2p}.$$

The lemma follows by taking the sum over λ. $\square$

PROOF OF THEOREM 1.2. From Propositions 2.1 and 3.1, $\delta(X) = -\infty$ if and only if $\nu(M_j)$ are not bounded above. In turn, from (3.1), this is equivalent to that $D_V < p-1$, which proves the first assertion. Equality (1.1) follows again from Propositions 2.1 and 3.1 and from formulas (3.1) and (3.2). Equality (1.2) follows from Lemma 3.2. $\square$

PROOF OF THEOREM 1.3. If $\delta(X) = -\infty$, then (1.3) is trivial. We may suppose that $\delta(X) \neq -\infty$, which is, from Theorem 1.2, equivalent to $D_V \geq p-1$. Since

$$\mathrm{sht}_V(p-(p-1)) + (p-1) = p-1,$$

again from Theorem 1.2, we have

$$\delta(X) \leq D_V - p.$$

On the other hand, from Theorem 1.2 and Lemma 3.3,

$$\begin{aligned}\delta(X) &= D_V - 1 - \max_{1\leq s\leq p-1}\{\mathrm{sht}_V(p-s)+s\}\\ &\geq D_V - 1 - \max_{1\leq s\leq p-1}\left\{\frac{p-s-1}{p}D_V + s\right\}\\ &= \frac{D_V}{p} - 1 - \max_{1\leq s\leq p-1}\left\{s\left(1-\frac{D_V}{p}\right)\right\}.\end{aligned}$$

If $D_V \geq p$, then

$$= \frac{D_V}{p} - 1 - \left(1-\frac{D_V}{p}\right) = \frac{2D_V}{p} - 2. \ \square$$

REMARK 3.4. The additive action of the fixed point part V^G on V commutes with the G-action. Therefore X as well as its jet schemes and the arc space inherit the V^G-action. Using this structure, we can show that for a closed subset $C \subset X_{\mathrm{sing}}$, $\nu(M_j \cap \pi_X^{-1}(C)) = \nu(M_j) - l + \dim C$ with M_j as in Proposition 3.1. By the same argument as above, we can prove that if $D_V \geq p-1$, then

$$\frac{2D_V}{p} - 2 + l - \dim C \leq \mathrm{discrep}\,(\mathrm{center} \subset C; X) \leq D_V - p + l - \dim C.$$

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