J. Math. Sci. Univ. Tokyo
**26** (2019), 141–199.

# *A Singular Perturbation Problem for Heteroclinic Solutions to the FitzHugh-Nagumo Type Reaction-Diffusion System with Heterogeneity*

By Takashi KAJIWARA and Kazuhiro KURATA

**Abstract.** In a previous paper, the first author considered the variational problems for heteroclinic solutions to the FitzHugh-Nagumo type reaction-diffusion system involving heterogeneity $\mu(x)$ and proved the existence of the minimizers. However, the precise location of the transition layer of the minimizers was not clear in the paper.

In this paper, we consider the same problems as the singular perturbation problems. Then we prove that the minimizer has exactly one transition layer near the minimum point of $\mu(x)$ by using the first order energy expansion. Moreover, we derive the more precise energy asymptotic expansion.

## 1. Introduction and Main Results

In this paper, motivated by Chen, Kung and Morita [4], we consider the heteroclinic solution to the following problems involving heterogeneity $\mu(x)$:

$$(1.1)\qquad \begin{cases} -du''(x) = \mu(x)(f(u(x)) - u(x)/\gamma) - v(x) + u(x)/\gamma, & x \in \mathbb{R}, \\ -v''(x) = u(x) - \gamma v(x), & x \in \mathbb{R}, \end{cases}$$

or

$$(1.2)\qquad \begin{cases} -du''(x) = \mu(x)f(u(x)) - v(x), & x \in \mathbb{R}, \\ -v''(x) = u(x) - \gamma v(x), & x \in \mathbb{R}, \end{cases}$$

with

$$(1.3)\qquad (u(x), v(x)) \to (\pm a_\gamma, \pm a_\gamma/\gamma), \qquad x \to \pm\infty,$$

where $d > 0$, $\gamma > 1$, $f(s) = s - s^3$, $a_\gamma = \sqrt{1 - 1/\gamma}$ and $\mu$ is a function satisfying the following conditions:

2010 *Mathematics Subject Classification.* Primary 35J50; Secondary 35K57, 35B40.
Key words: Variational problem, FitzHugh-Nagumo type reaction diffusion systems, heteroclinic solution, singular perturbation problem.

($\mu$1) There exist $\mu_0 > 0$ and $x_0 \in \mathbb{R}$ such that $\mu_0 = \mu(x_0) \le \mu(x) \le 1$ holds for all $x \in \mathbb{R}$. Moreover, $\mu \not\equiv 1$.

($\mu$2) $1 - \mu \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ and $\mu(x) \to 1$ as $|x| \to \infty$.

We note that Chen, Kung and Morita [4] treated the case $\mu \equiv 1$.

(1.1) and (1.2) arise in the FitzHugh-Nagumo type reaction-diffusion system (FHN RD system, in short). The FHN RD system was introduced in physiology, which essentially describes neural excitability. This system has also been studied mathematically as a model which generates complex patterns. A typical FHN RD system is given in the following form:

$$
\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = d\Delta u(x,t) + f(u(x,t)) - v(x,t), & x \in \Omega,\ t > 0, \\ \tau \dfrac{\partial v}{\partial t}(x,t) = D\Delta v(x,t) + u(x,t) - \gamma v(x,t), & x \in \Omega,\ t > 0, \end{cases} \tag{1.4}
$$

where $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ is a domain, $d, D, \tau, \gamma$ are positive constants and $f(s) = s - s^3$. In particular, we treat the steady state problem of (1.4) with $\Omega = \mathbb{R}$ in this paper.

There are many works to study on stationary solutions to (1.4). In the case $N = 1$ and $\Omega = \mathbb{R}$, Klaasen and Troy [12] constructed a pulse solution and a periodic solution. Chen and Choi [3] constructed a pulse solution in the different parameter ranges. Moreover, Chen, Kung and Morita [4] constructed a heteroclinic solution by a variational approach. Reinecke and Sweers [17] constructed a positive radially symmetric solution for the steady state problem of (1.4) for the case $N \ge 1$ and $\Omega = \mathbb{R}^N$. Chen and Tanaka [5] extended the results of [17] under weaker assumptions. In addition, Wei and Winter [19] constructed a standing wave cluster solution which has multiple peaks with a specific geometric pattern. In the case that $\Omega$ is bounded in $\mathbb{R}^N$ $(N \ge 1)$, Oshita [16] or Dancer and Yan [6] focused on the variational structure of (1.4) and showed that the minimizer of the variational problem corresponding to the stationary problem of (1.4) oscillates rapidly when $d > 0$ is small. We note that Oshita [16] treated the Neumann boundary condition and Dancer and Yan [6] treated the Dirichlet boundary condition. For other works, see e.g. [7, 15, 18].

Before we state our results, we shall recall the strategy in [4] which treated the case $\mu \equiv 1$. In the case $\mu \equiv 1$, (1.1) and (1.2) become the following problem:

$$(1.5)\qquad \begin{cases} -du''(x) = f(u(x)) - v(x), & x \in \mathbb{R}, \\ -v''(x) = u(x) - \gamma v(x), & x \in \mathbb{R}. \end{cases}$$

Chen, Kung and Morita assumed $\gamma > 1$ and hence (1.4) has three constant stationary solutions $(-a_\gamma, -a_\gamma/\gamma)$, $(0,0)$ and $(a_\gamma, a_\gamma/\gamma)$, where $a_\gamma = \sqrt{1 - 1/\gamma}$ is the positive root of

$$(1.6)\qquad f(a_\gamma) = \frac{a_\gamma}{\gamma}.$$

In this paper, we shall call a solution to (1.5) satisfying (1.3) a *heteroclinic solution*. To obtain the heteroclinic solution, they introduced some notations. Let $\hat{v} \in C^\infty(\mathbb{R})$ be an odd function satisfying

$$\hat{v}(x) = \begin{cases} a_\gamma/\gamma, & x > 1, \\ -a_\gamma/\gamma, & x < -1, \end{cases}$$

and define $\hat{u} \in C^\infty(\mathbb{R})$ as follows:

$$\hat{u}(x) = -\hat{v}''(x) + \gamma \hat{v}(x).$$

We note that $\hat{u}$ is an odd function and satisfies

$$\hat{u}(x) = \begin{cases} a_\gamma, & x > 1, \\ -a_\gamma, & x < -1. \end{cases}$$

They proposed the following energy functional $J_0(\psi)$ corresponding to (1.5) with (1.3) :

$$J_0(\psi) = \int_{\mathbb{R}} \left[ \frac{\theta^2}{2} \left| u' \right|^2 + \frac{1}{4}(u^2 - a_\gamma^2)^2 + \frac{1}{2}\left( v' - \frac{u'}{\gamma} \right)^2 + \frac{\gamma}{2}\left( v - \frac{u}{\gamma} \right)^2 \right] dx,$$

where $\theta^2 = d - 1/\gamma^2$, $u = \hat{u} + \psi$, $v = \hat{v} + \mathcal{L}\psi$ and $\mathcal{L} :\ L^2(\mathbb{R}) \to H^2(\mathbb{R})$ is the inverse operator of $(-d^2/dx^2 + \gamma)$. They showed that if $\theta^2 = d - 1/\gamma^2 > 0$, then the minimizing problem

$$\sigma_0 = \inf \left\{ J_0(\psi) :\ \psi \in H^1(\mathbb{R}) \right\}$$

has a minimizer $\psi_0 \in H^1(\mathbb{R})$ and then $(u, v) = (\hat{u} + \psi_0, \hat{v} + \mathcal{L}\psi_0)$ is a heteroclinic solution to (1.5).

Now we consider (1.1) and (1.2). We say that $(u, v)$ is a heteroclinic solution to (1.1) or (1.2) if $(u, v)$ is a solution to (1.1) or (1.2) satisfying (1.3). We note that (1.1) and (1.2) also have variational structures. The energy functionals corresponding to these problems are defined as follows:

$$\begin{aligned}\bar{J}_\theta(\psi) = \int_{\mathbb{R}} \Bigg[ & \frac{\theta^2}{2}|u'|^2 + \frac{\mu(x)}{4}(u^2 - a_\gamma^2)^2 \\ & + \frac{1}{2}\left(v' - \frac{u'}{\gamma}\right)^2 + \frac{\gamma}{2}\left(v - \frac{u}{\gamma}\right)^2 \Bigg] dx,\end{aligned} \tag{1.7}$$

$$\begin{aligned}\tilde{J}_\theta(\psi) = \int_{\mathbb{R}} \Bigg[ & \frac{\theta^2}{2}|u'|^2 + \frac{\mu(x)}{4}(u^2 - a_\gamma^2)^2 \\ & + \frac{1}{2}\left(v' - \frac{u'}{\gamma}\right)^2 + \frac{\gamma}{2}\left(v - \frac{u}{\gamma}\right)^2 + \frac{1-\mu(x)}{2\gamma}u^2 \Bigg] dx.\end{aligned} \tag{1.8}$$

Kajiwara [10] proved that under the assumptions ($\mu$1) and ($\mu$2) for $\mu(x)$, the following minimizing problems have minimizers $\bar{\psi}_\theta$ and $\tilde{\psi}_\theta$, respectively:

$$\bar{\sigma}(\theta, \gamma) = \inf\left\{\bar{J}_\theta(\psi) : \ \psi \in H^1(\mathbb{R})\right\}, \tag{1.9}$$

$$\tilde{\sigma}(\theta, \gamma) = \inf\left\{\tilde{J}_\theta(\psi) : \ \psi \in H^1(\mathbb{R})\right\}. \tag{1.10}$$

Moreover, one can see that $(\bar{u}_\theta, \bar{v}_\theta) = (\hat{u} + \bar{\psi}_\theta, \hat{v} + \mathcal{L}\bar{\psi}_\theta)$ and $(\tilde{u}_\theta, \tilde{v}_\theta) = (\hat{u} + \tilde{\psi}_\theta, \hat{v} + \mathcal{L}\tilde{\psi}_\theta)$ are the heteroclinic solutions to (1.1) and (1.2), respectively. However, their precise profiles, for example, the number and the location of the transition layers of $\bar{u}_\theta$ or $\tilde{u}_\theta$, were not clear in [10]. The purpose of this paper is to clarify the profile of $u_\theta$ in the singular perturbation problems as $\theta \to 0$ with $1/\gamma = o(\theta)$.

Our first main results are following:

THEOREM 1. *Assume that $\gamma > 1$, $\theta^2 = d - 1/\gamma^2 > 0$, $1/\gamma = o(\theta)$ and $\mu(x)$ satisfies ($\mu$1) and ($\mu$2). Let $\psi_\theta \in H^1(\mathbb{R})$ be a minimizer of (1.9) or (1.10), $u_\theta = \hat{u} + \psi_\theta$ and $v_\theta = \hat{v} + \mathcal{L}\psi_\theta$. Then for sufficiently small $\theta > 0$, there exists the unique point $x_\theta \in \mathbb{R}$ such that $u_\theta(x_\theta) = 0$.*

Moreover, we set

$$M = \{x \in \mathbb{R} : \ \mu(x) = \mu_0\} \neq \emptyset.$$

Then we obtain the further information of $x_\theta$.

THEOREM 2. *Assume that $\gamma > 1$, $\theta^2 = d - 1/\gamma^2 > 0$, $1/\gamma = o(\theta)$ and $\mu(x)$ satisfies ($\mu$1) and ($\mu$2). Let $\psi_\theta \in H^1(\mathbb{R})$ be a minimizer of (1.9) or (1.10), $u_\theta = \hat{u} + \psi_\theta$ and $x_\theta$ be the point defined in Theorem 1. Then* $\mathrm{dist}(x_\theta, M) \to 0$ *as* $\theta \to 0$.

With these theorems, we can reveal the asymptotic behavior of $u_\theta$. Namely, we can see that $u_\theta$ has exactly one transition layer with $O(\epsilon)$ near the minimum point of $\mu(x)$ for small $\theta > 0$. These results generalize the results for the Allen-Cahn equation. For the Allen-Cahn equation, Nakashima [14] considered the following problem with heterogeneity $h(x)$:

$$\begin{cases} -du''(x) = h(x)f(u(x)), \quad x \in (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{1.11}$$

Here the author assumes that $h(x)$ is a positive smooth function in $(0,1)$ and there exists $x_* \in (0,1)$ such that $h(x_*) = \min h(x)$ and $h''(x_*) > 0$. Then the author constructed the solution to (1.11) which has a transition layer near $x_*$ for sufficiently small $d > 0$. We note that Matsuzawa [13] studied (1.11) without a non-degenerate assumption on $h(x)$. Ei and Matsuzawa [8] considered (1.11) on $\mathbb{R}$. They assumed that $d > 0$ is small and $h(x) \in C(\mathbb{R})$ has an interval $I$ such that $h(x) = \min_{y \in \mathbb{R}} h(y)$ for all $x \in I$. Then they showed the transition layer tends to stay in the center of $I$ from the viewpoint of dynamics. Roughly speaking, these results show that the solution to (1.11) tends to transit near the minimum point of $h(x)$.

In the proof of Theorems 1 and 2, the following estimates on $\bar{\sigma} = \bar{\sigma}(\theta, \gamma)$ and $\tilde{\sigma} = \tilde{\sigma}(\theta, \gamma)$ play important roles:

PROPOSITION 1. *Assume that $\gamma > 1$, $\theta^2 = d - 1/\gamma^2 > 0$, $1/\gamma = o(\theta)$ as $\theta \to 0$ and $\mu(x)$ satisfies ($\mu$1) and ($\mu$2). Then the following estimate holds:*

$$\sigma(\theta, \gamma) = a_\gamma^3 \sqrt{\mu_0} c_* \theta + o(\theta) \qquad as \ \ \theta \to 0,$$

*where $\sigma$ represents $\bar{\sigma}$ or $\tilde{\sigma}$ and $c_*$ is the positive constant defined as follows:*

$$c_* = \int_{-1}^{1} \sqrt{\frac{(1-s^2)^2}{2}}\, ds = \frac{2\sqrt{2}}{3}. \tag{1.12}$$

In this paper, we also obtain more accurate estimates of $\bar{\sigma}$ and $\tilde{\sigma}$ under the additional assumptions ($\mu$2') and an additional relation between $\gamma$ and $\theta$:

($\mu$2') $\mu \in C^2(\mathbb{R})$ and there exists a constant $C > 0$ such that $|\mu''(x)| < C$ holds for all $x \in \mathbb{R}$.

Our second main results are the following energy asymptotic expansion:

THEOREM 3. *Assume that $\gamma > 1$, $\theta^2 = d - 1/\gamma^2 > 0$, $1/\gamma = o(\theta)$, and $\mu(x)$ satisfies ($\mu$1), ($\mu$2) and ($\mu$2').*

*(1) Assume $\theta^2 \ll 1/\gamma \ll \theta$. Then the following inequalities hold:*

$$0 \le \bar{\sigma}(\theta,\gamma) - a_\gamma^3\sqrt{\mu_0}c_*\theta \le \frac{a_\gamma^3\sqrt{\mu_0}}{2\theta\gamma^2}A + o\left(\frac{1}{\theta\gamma^2}\right),$$

$$0 \le \tilde{\sigma}(\theta,\gamma) - \left\{a_\gamma^3\sqrt{\mu_0}c_*\theta + \frac{a_\gamma^2}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))\,dx\right\} \le \frac{a_\gamma^3\sqrt{\mu_0}}{2\theta\gamma^2}A + o\left(\frac{1}{\theta\gamma^2}\right),$$

*where $c_*$ is defined in (1.12),*

$$A = \int_0^\infty \int_0^\infty (|y+z| - |y-z|)\, B(y)B(z)\, dydz, \tag{1.13}$$

$$B(y) = U_0(y)(1 - U_0(y)^2), \tag{1.14}$$

*and*

$$U_0(x) = \tanh(x/\sqrt{2}). \tag{1.15}$$

*(2) Assume $\theta^2 \ll 1/\gamma \ll \theta^{4/3}$. Then the following estimate holds:*

$$\bar{\sigma}(\theta,\gamma) = a_\gamma^3\sqrt{\mu_0}c_*\theta + \frac{a_\gamma^3\sqrt{\mu_0}}{2\theta\gamma^2}A + o\left(\frac{1}{\theta\gamma^2}\right), \tag{1.16}$$

*(3) Assume* $\theta^2 \ll 1/\gamma \ll \theta^{3/2}$*. Then the following estimate holds:*

$$\tilde{\sigma}(\theta,\gamma) = a_\gamma^3\sqrt{\mu_0}c_*\theta + \frac{a_\gamma^2}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))\,dx + \frac{a_\gamma^3\sqrt{\mu_0}}{2\theta\gamma^2}A + o\left(\frac{1}{\theta\gamma^2}\right). \tag{1.17}$$

Remark 1. The formula $\theta^2 \ll 1/\gamma \ll \theta$ means that $\theta^2 = o(1/\gamma)$ and $1/\gamma = o(\theta)$ hold as $\theta \to 0$ and $1/\gamma \to 0$.

Remark 2. We conjecture that (1.16) and (1.17) hold under the weaker assumption $\theta^2 \ll 1/\gamma \ll \theta$. We need stronger technical assumptions to control the behaviors of $U_\theta(y)$ as in Lemma 18 (see Section 5).

In the energy expansions (1.16) and (1.17), we can see that the leading term corresponds to the Allen-Cahn energy (see Lemmas 1 and 2) and the second term of (1.16) or third term of (1.17) represents the non-local effect of the FHN RD system. The upper estimate is obtained by substituting an appropriate test function into $\bar{J}(\psi)$ or $\tilde{J}(\psi)$. For the lower estimate, it is necessary to analyze the behavior of the minimizers in details.

We add some comments on the case $\mu \equiv 1$. In this case, we can check that the same statements of Theorem 1 or (1) and (2) of Theorem 3 hold. Moreover, we may assume $x_\theta = 0$ since (1.5) is invariant under translations of $u$. On the other hand, Chen, Kung and Morita [4] showed that if we take $\gamma > 1$ large enough for a given $d > 0$, then one can construct the odd solution $u_d$ to (1.5) and (1.3) which is positive on $(0,\infty)$ by the sub-supersolution method. They also showed the uniqueness of such a solution under the same assumption in [4]. In addition, Kajiwara [11] showed that we can take $\gamma > 1$ independent of $d > 0$ for the statement in [4] to be true. From the uniqueness of the solution, we readily see that $u_\theta \equiv u_d$ under the assumptions that $\mu \equiv 1$, $d - 1/\gamma^2 > 0$ and $\theta^2 = d - 1/\gamma^2$ is small, where $u_\theta$ is the solution obtained by Theorem 1. This implies that $u_d$ can be characterized from the variational viewpoint.

This paper is organized as follows. In Section 2, we prepare some basic lemmas. In Section 3, we prove the upper estimates of energies (Propositions 2 and 3). In particular, we give a proof of Proposition 1 (see Lemma 2 and Proposition 2). In Section 4, we first give a simple proof of the existence of the minimizers of (1.9) and (1.10) with Proposition 2. Then we prove Theorems 1 and 2. We note that we use Proposition 2 also in the proof of the theorems. In Section 5, we show the lower estimates of energies. Section 5 consists of four parts. In Subsection 5.1, we introduce some notations and prepare some useful lemmas. In Subsection 5.2, we show some key lemmas on the behavior of $U_\theta$. The lemmas presented in the subsection play important roles in obtaining the lower estimates. In Subsection 5.3, we present some auxiliary lemmas to reduce the amount of calculation. In Subsection 5.4, we prove Theorem 3.

## 2. Basic Lemma

In this section, we collect some lemmas to show our main theorems. Since the next lemma is well-known, we omit the proof.

LEMMA 1 ([2]). *Let $E(U)$ be as follows:*

$$E(U) = \int_{\mathbb{R}} \left[ \frac{1}{2} \left| U'(x) \right| + W_0(U(x)) \right] dx,$$

*where $W_0(s) = (s^2 - 1)^2/4$. Then the following identity holds:*

$$c_* = \inf \left\{ E(U) : \ U \in H^1_{loc}(\mathbb{R}), \ \lim_{x \to \pm\infty} U(x) = \pm 1 \right\},$$

*where $c_*$ is the same constant defined in (1.12). Moreover, $U_0 = \tanh(x/\sqrt{2})$ attains the minimum of $E(U)$, that is,*

$$E(U_0) = c_*.$$

REMARK 3. It is well known that $U_0 = \tanh(x/\sqrt{2})$ is the unique solution to

$$\begin{cases} -U_0''(y) = f(U_0(y)), & y \in \mathbb{R}, \\ U_0(0) = 0, \\ U_0(y) \to \pm 1, & y \to \pm\infty. \end{cases} \tag{2.1}$$

We also have the following characterization of $c_*$:

$$\begin{aligned}\frac{1}{2}c_* = \inf \Bigg\{ \int_0^\infty \frac{1}{2}\left|U'(x)\right|^2 + \frac{1}{4}\left(U(x)-1\right)^2\, dx; \\ U \in H^1_{loc}([0,\infty)),\, U(0)=0,\, \lim_{x\to\infty} U(x) = 1 \Bigg\} \\ = \inf \Bigg\{ \int_{-\infty}^0 \frac{1}{2}\left|U'(x)\right|^2 + \frac{1}{4}\left(U(x)-1\right)^2\, dx; \\ U \in H^1_{loc}((-\infty,0]),\, U(0)=0,\, \lim_{x\to-\infty} U(x) = -1 \Bigg\}.\end{aligned}$$

The next lemma immediately follows from Lemma 1.

LEMMA 2. *The following inequality holds:*

$$\sigma(\theta,\gamma) \geq a_\gamma^3 \sqrt{\mu_0} c_* \theta \quad \textit{for any } (\theta,\gamma) \in (0,\infty)\times(1,\infty),$$

*where* $\sigma$ *represents* $\bar{\sigma}$ *or* $\tilde{\sigma}$*.*

PROOF. Note that

$$\begin{aligned}\sigma(\theta,\gamma) \geq \inf\Bigg\{ \int_{\mathbb{R}} \left[ \frac{\theta^2}{2}\left|u'(x)\right|^2 + \frac{\mu_0}{4}(u^2-a_\gamma^2)^2 \right] dx : \\ u \in H^1_{loc}(\mathbb{R}),\, \lim_{x\to\pm\infty} u(x) = \pm a_\gamma \Bigg\}.\end{aligned}$$

By the scaling argument and Lemma 1, we have

$$\begin{aligned}\inf \Bigg\{ \int_{\mathbb{R}} \left[ \frac{\theta^2}{2}\left|u'(x)\right|^2 + \frac{\mu_0}{4}(u^2-a_\gamma^2)^2 \right] dx : u \in H^1_{loc}(\mathbb{R}),\, \lim_{x\to\pm\infty} u(x) = \pm a_\gamma \Bigg\} \\ = a_\gamma^3 \sqrt{\mu_0} c_* \theta.\end{aligned}$$

Thus we conclude the desired estimate. □

The next lemma is well-known, but we present in the following form, which will be used later.

LEMMA 3. *Let* $c$ *and* $d$ *be constants such that* $-a_\gamma < c < d < a_\gamma$*. Assume that a function* $u \in H^1_{loc}(\mathbb{R})$ *has a transition from* $c$ *to* $d$ *on the interval* $[x_1, x_2]$*, namely* $u$ *satisfies both* *(i)* $u(x_1) = c$ *and* $u(x_2) = d$ *or* $u(x_1) = d$ *and* $u(x_2) = c$

*and*

*(ii)* $u(x) \in (c, d)$ *for all* $x \in (x_1, x_2)$*.*

*Then it follows that*

$$\int_{x_1}^{x_2} \left[ \frac{\theta^2}{2} \left| u' \right|^2 + \frac{\mu(x)}{4} \left( u^2 - a_\gamma^2 \right)^2 \right] dx \geq \theta K(c, d)(d - c) \sqrt{\frac{\mu_0}{2}},$$

*where* $K(c, d)$ *is defined as*

$$K(c, d) = \min \left\{ \left( a_\gamma^2 - c^2 \right), \left( a_\gamma^2 - d^2 \right) \right\}.$$

PROOF. From the fundamental theorem of calculus and Hölder's inequality, we have

$$d - c = |u(x_2) - u(x_1)| \leq \left( \int_{x_1}^{x_2} \left| u' \right|^2 dx \right)^{1/2} |x_2 - x_1|^{1/2}.$$

Thus we can see

$$\frac{\theta^2}{2} \int_{x_1}^{x_2} \left| u' \right|^2 dx \geq \frac{\theta^2}{2} \cdot \frac{(d - c)^2}{|x_2 - x_1|}.$$

On the other hand, we have

$$\int_{x_1}^{x_2} \frac{\mu(x)}{4} \left( u^2 - a_\gamma^2 \right)^2 dx \geq |x_2 - x_1| \frac{\mu_0}{4} K(c, d)^2.$$

From the above inequalities, we obtain

$$\begin{aligned} &\int_{x_1}^{x_2} \left[ \frac{\theta^2}{2} \left| u' \right|^2 + \frac{\mu(x)}{4} \left( u^2 - a_\gamma^2 \right)^2 \right] dx \\ &\geq 2 \sqrt{\frac{\theta^2}{2} \frac{(d - c)^2}{|x_2 - x_1|} |x_2 - x_1| \frac{\mu_0}{4} K(c, d)^2} \\ &= \theta K(c, d)(d - c) \sqrt{\frac{\mu_0}{2}}. \end{aligned}$$

Thus we conclude the statement. □

The next lemma gives the representation of the Green function and its estimate.

Lemma 4. *Assume that $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ and $f(-x) = -f(x)$ holds for all $x \in \mathbb{R}$. Let $w$ be a solution to*

$$\begin{cases} -w''(x) + w(x) = f(x), \quad x \in \mathbb{R}, \\ w \in H^2(\mathbb{R}). \end{cases} \tag{2.2}$$

*Then the following statements hold:*

*(1) $w(-x) = -w(x)$ holds for all $x \in \mathbb{R}$ and $w$ is represented as follows:*

$$w(x) = \frac{1}{2}\int_0^\infty \left(e^{-|x-z|} - e^{-|x+z|}\right) f(z)\,dz \quad x \in \mathbb{R}. \tag{2.3}$$

*(2) The following identity holds:*

$$\|w\|^2_{H^1(\mathbb{R})} = \int_0^\infty \int_0^\infty \left(e^{-|y-z|} - e^{-|y+z|}\right) f(y)f(z)\,dydz. \tag{2.4}$$

Proof. (1) It suffices to show (2.3). By using the Green function $G_0(x,z) = e^{-|x-z|}/2$, we can write $w$ as follows:

$$w(x) = \int_{\mathbb{R}} G_0(x,z)f(z)\,dz = \int_{\mathbb{R}} \frac{1}{2}e^{-|x-z|}f(z)\,dz.$$

Then we calculate as follows:

$$\begin{aligned} w(x) &= \frac{1}{2}\int_0^\infty e^{-|x-z|}f(z)\,dz + \frac{1}{2}\int_{-\infty}^0 e^{-|x-z|}f(z)\,dz \\ &= \frac{1}{2}\int_0^\infty e^{-|x-z|}f(z)\,dz + \frac{1}{2}\int_{\infty}^0 e^{-|x+u|}f(-u)\,(-du) \\ &= \frac{1}{2}\int_0^\infty \left(e^{-|x-z|} - e^{-|x+z|}\right) f(z)\,dz. \end{aligned}$$

Hence we conclude (2.3).

(2) Multiplying (2.2) by $w$ and integrating over $\mathbb{R}$, we obtain

$$\|w\|^2_{H^1(\mathbb{R})} = \int_{\mathbb{R}} w(y)f(y)\,dy.$$

From (1), the right hand side is written as follows:

$$\begin{aligned}\int_{\mathbb{R}} w(y)f(y)\,dy &= \int_{\mathbb{R}}\int_0^\infty \frac{1}{2}\left(e^{-|y-z|} - e^{-|y+z|}\right) f(z)f(y)\,dzdy \\ &= \int_0^\infty \int_0^\infty \frac{1}{2}\left(e^{-|y-z|} - e^{-|y+z|}\right) f(z)f(y)\,dzdy \\ &\quad + \int_{-\infty}^0 \int_0^\infty \frac{1}{2}\left(e^{-|y-z|} - e^{-|y+z|}\right) f(z)f(y)\,dzdy.\end{aligned}$$

By changing variables and using the assumptions on $f$, the second term of the above identity is written as

$$\begin{aligned}&\int_{-\infty}^0 \int_0^\infty \frac{1}{2}\left(e^{-|y-z|} - e^{-|y+z|}\right) f(z)f(y)\,dzdy \\ &= \int_{\infty}^0 \int_0^\infty \frac{1}{2}\left(e^{-|-u-z|} - e^{-|-u+z|}\right) f(z)f(-u)\,dz(-du) \\ &= \int_0^\infty \int_0^\infty \frac{1}{2}\left(e^{-|u-z|} - e^{-|u+z|}\right) f(z)f(u)\,dzdu.\end{aligned}$$

Thus we conclude (2.4). □

## 3. Upper Estimate of Energies

In this section, we give the upper estimates of $\bar{J}_\theta(\psi)$ and $\tilde{J}_\theta(\psi)$ for small $\theta > 0$. Let $U_0$ be the function defined in (1.15). Moreover, we set functions $u_*$, $\psi_*$ and $v_*$ as follows:

$$u_*(x) = a_\gamma U_0\left(\frac{a_\gamma\sqrt{\mu_0}}{\theta}(x - x_0)\right), \tag{3.1}$$

$$\psi_*(x) = u_*(x) - \hat{u}(x), \tag{3.2}$$

$$v_*(x) = \hat{v}(x) + (\mathcal{L}\psi_*)(x). \tag{3.3}$$

REMARK 4. $(u_*, v_*)$ is the unique solution to

$$
\begin{cases}
-\theta^2 u_*''(x) = \mu_0 u_*(x)\left(a_\gamma^2 - u_*(x)^2\right), & x \in \mathbb{R}, \\
-v_*''(x) + \gamma v_*(x) = u_*(x), & x \in \mathbb{R}, \\
u_*(x_0) = 0, \\
(u_*(x), v_*(x)) \to (\pm a_\gamma, \pm a_\gamma/\gamma), & x \to \pm\infty.
\end{cases} \tag{3.4}
$$

Our goal in this section is to show the following propositions:

PROPOSITION 2. *Assume that $\gamma > 1$, $\theta^2 = d - 1/\gamma^2 > 0$, $1/\gamma = o(\theta)$ as $\theta \to 0$ and $\mu$ satisfies ($\mu$1) and ($\mu$2). Then the following inequality holds:*

$$\sigma(\theta, \gamma) \le a_\gamma^3 \sqrt{\mu_0} c_* \theta + o(\theta),$$

*where $\sigma$ represents $\bar{\sigma}$ or $\tilde{\sigma}$ and $c_*$ is defined in (1.12).*

PROPOSITION 3. *Assume that $\gamma > 1$, $\theta^2 = d - 1/\gamma^2 > 0$, $1/\gamma = o(\theta)$ as $\theta \to 0$, $\theta^2 \ll 1/\gamma \ll \theta$ and $\mu$ satisfies ($\mu$1), ($\mu$2) and ($\mu$2'). Then the following inequalities hold:*

$$
\begin{aligned}
\bar{\sigma}(\theta, \gamma) &\le a_\gamma^3 \sqrt{\mu_0} c_* \theta + \frac{a_\gamma^3 \sqrt{\mu_0}}{2\theta\gamma^2} A + O(\gamma^{-3/2}), \\
\tilde{\sigma}(\theta, \gamma) &\le a_\gamma^3 \sqrt{\mu_0} c_* \theta + \frac{a_\gamma^2}{2\gamma} \int_{\mathbb{R}} (1 - \mu(x))\, dx + \frac{a_\gamma^3 \sqrt{\mu_0}}{2\theta\gamma^2} A + O(\gamma^{-3/2}),
\end{aligned}
$$

*where $A$ is defined in (1.13).*

REMARK 5. The assumption $\theta^2 \ll 1/\gamma$ leads to $\gamma^{-3/2} \ll 1/(\theta\gamma^2)$.

We treat only $\tilde{J}_\theta(\psi)$ since it suffices to show the estimate of $\tilde{\sigma}(\theta, \gamma)$ for the proof of Propositions 2 and 3. For simplicity, we write $a$, $J_\theta(\psi)$ and $\sigma(\theta, \gamma)$ instead of $a_\gamma$, $\tilde{J}_\theta(\psi)$ and $\tilde{\sigma}(\theta, \gamma)$, respectively. Propositions 2 and 3 are proved by calculating $J_\theta(\psi_*)$. For reader's convenience, we recall $J_\theta(\psi_*)$:

$$
\begin{aligned}
J_\theta(\psi_*) = \int_{\mathbb{R}} \Bigg[ &\frac{\theta^2}{2} \left|u_*'\right|^2 + \frac{\mu(x)}{4}(u_*^2 - a^2)^2 \\
&+ \frac{1}{2}\left(v_*' - \frac{u_*'}{\gamma}\right)^2 + \frac{\gamma}{2}\left(v_* - \frac{u_*}{\gamma}\right)^2 + \frac{1 - \mu(x)}{2\gamma} u_*^2 \Bigg] dx.
\end{aligned}
$$

We now calculate each term of $J_\theta(\psi_*)$ in Lemmas 5–9. For simplicity, we define $J_\theta^{(i)}(\psi)$ $(i = 1, 2, 3, 4, 5)$ as follows:

$$J_\theta^{(1)}(\psi) = \int_{\mathbb{R}} \frac{\theta^2}{2} \left|u'\right|^2 dx, \tag{3.5}$$

$$J_\theta^{(2)}(\psi) = \int_{\mathbb{R}} \frac{\mu(x)}{4} (u^2 - a^2)^2 dx, \tag{3.6}$$

$$J_\theta^{(3)}(\psi) = \int_{\mathbb{R}} \frac{1}{2} \left(v' - \frac{u'}{\gamma}\right)^2 dx, \tag{3.7}$$

$$J_\theta^{(4)}(\psi) = \int_{\mathbb{R}} \frac{\gamma}{2} \left(v - \frac{u}{\gamma}\right)^2 dx, \tag{3.8}$$

$$J_\theta^{(5)}(\psi) = \int_{\mathbb{R}} \frac{1 - \mu(x)}{2\gamma} u^2 dx, \tag{3.9}$$

where $u$, $v$ are defined by $u = \hat{u} + \psi,\ v = \hat{v} + \mathcal{L}\psi$. We begin with an estimate of $J_\theta^{(1)}(\psi_*)$.

LEMMA 5. *Let $u_*$ and $\psi_*$ be functions defined in (3.1) and (3.2), respectively. Then $J_\theta^{(1)}(\psi_*)$ defined in (3.5) is calculated as follows:*

$$J_\theta^{(1)}(\psi_*) = \frac{a^3\theta\sqrt{\mu_0}}{2} \int_{\mathbb{R}} \left|U_0'(x)\right|^2 dx,$$

*where $U_0$ is defined in (1.15).*

PROOF. Since we can see that

$$u_*'(x) = \frac{a^2\sqrt{\mu_0}}{\theta} U_0' \left(\frac{a\sqrt{\mu_0}(x - x_0)}{\theta}\right),$$

we calculate as follows:

$$\begin{aligned} J_\theta^{(1)}(\psi_*) &= \frac{\theta^2}{2} \left(\frac{a^2\sqrt{\mu_0}}{\theta}\right)^2 \int_{\mathbb{R}} \left|U_0' \left(\frac{a\sqrt{\mu_0}(x - x_0)}{\theta}\right)\right|^2 dx \\ &= \frac{\theta^2}{2} \frac{a^4\mu_0}{\theta^2} \int_{\mathbb{R}} \left|U_0'(y)\right|^2 \left(\frac{\theta}{a\sqrt{\mu_0}} dy\right) \\ &= \frac{a^3\theta\sqrt{\mu_0}}{2} \int_{\mathbb{R}} \left|U_0'(y)\right|^2 dy. \ \square \end{aligned}$$

We next give some estimates of $J_\theta^{(2)}(\psi_*)$.

Lemma 6. *Let $u_*$ and $\psi_*$ be functions defined in (3.1) and (3.2), respectively. Then the following statements hold:*

*(1) Let $\mu$ be a function satisfying ($\mu$1) and ($\mu$2). Then $J_\theta^{(2)}(\psi_*)$ defined in (3.6) is calculated as follows:*

$$J_\theta^{(2)}(\psi_*) = \frac{a^3\sqrt{\mu_0}\theta}{4}\int_{\mathbb{R}} \left(U_0(y)^2 - 1\right)^2 dy + o(\theta) \quad \text{as } \theta \to 0,$$

*where $U_0$ is defined in (1.15).*

*(2) Let $\mu$ be a function satisfying ($\mu$1), ($\mu$2) and ($\mu$2'). Then $J_\theta^{(2)}(\psi_*)$ is calculated as follows:*

$$J_\theta^{(2)}(\psi_*) = \frac{a^3\sqrt{\mu_0}\theta}{4}\int_{\mathbb{R}} \left(U_0(y)^2 - 1\right)^2 dy + O(\theta^3) \quad \text{as } \theta \to 0.$$

Proof. From the definition of $J_\theta^{(2)}(\psi_*)$, we have

$$\begin{aligned} J_\theta^{(2)}(\psi_*) &= \frac{1}{4}\int_{\mathbb{R}} \mu(x)\left\{a^2 U_0\left(\frac{a\sqrt{\mu_0}(x-x_0)}{\theta}\right)^2 - a^2\right\}^2 dx \\ &= \frac{a^4}{4}\int_{\mathbb{R}} \left[\mu\left(x_0 + \frac{\theta y}{a\sqrt{\mu_0}}\right)(U_0(y)^2-1)^2\right]\left(\frac{\theta}{a\sqrt{\mu_0}}dy\right) \\ &= \frac{a^3\theta}{4\sqrt{\mu_0}}\int_{\mathbb{R}} \mu\left(x_0 + \frac{\theta y}{a\sqrt{\mu_0}}\right)(U_0(y)^2-1)^2\, dy. \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{\sqrt{\mu_0}}\mu\left(x_0 + \frac{\theta y}{a\sqrt{\mu_0}}\right) &= \frac{1}{\sqrt{\mu_0}}\left\{\mu(x_0) - \mu(x_0) + \mu\left(x_0 + \frac{\theta y}{a\sqrt{\mu_0}}\right)\right\} \\ &= \sqrt{\mu_0}\left\{1 - \frac{1}{\mu_0}\left(\mu\left(x_0 + \frac{\theta y}{a\sqrt{\mu_0}}\right) - \mu(x_0)\right)\right\}, \end{aligned}$$

we obtain the following:

$$\begin{aligned} J_\theta^{(2)}(\psi_*) =& \frac{a^3\theta\sqrt{\mu_0}}{4}\left[\int_{\mathbb{R}}(U_0(y)^2-1)^2\, dy \right.\\ &\left. + \frac{1}{\mu_0}\int_{\mathbb{R}}\left\{\mu\left(x_0 + \frac{\theta y}{a\sqrt{\mu_0}}\right) - \mu(x_0)\right\}(U_0(y)^2-1)^2\, dy\right]. \end{aligned} \tag{3.10}$$

Now we assume that $\mu$ satisfies ($\mu$1) and ($\mu$2). Then from the dominated convergence theorem, the second term of the above equation tends to 0 as $\theta \to 0$. Thus we have

$$J_\theta^{(2)}(\psi_*) = \frac{a^3\theta\sqrt{\mu_0}}{4}\int_{\mathbb{R}}(U_0(y)^2-1)^2\,dy + o(\theta).$$

Hence we conclude the statement of (1).

Next, we assume that $\mu$ satisfies ($\mu$1), ($\mu$2) and ($\mu$2'). Since $\mu'(x_0)=0$, from Taylor's theorem, we have

$$\mu\left(x_0+\frac{\theta y}{a\sqrt{\mu_0}}\right)-\mu(x_0)=\frac{1}{2}\mu''\left(x_0+\kappa\frac{\theta}{a\sqrt{\mu_0}}y\right)\left(\frac{\theta}{a\sqrt{\mu_0}}y\right)^2$$

for any fixed $y\in\mathbb{R}$, where $\kappa\in(0,1)$ is a constant which depends on $y\in\mathbb{R}$. Since $|\mu''|<C$ on $\mathbb{R}$, we deduce that

$$\left|\mu\left(x_0+\frac{\theta y}{a\sqrt{\mu_0}}\right)-\mu(x_0)\right|\le\frac{C}{2}\frac{\theta^2y^2}{a^2\mu_0}.$$

Thus the second term of (3.10) is estimated as follows:

$$\begin{aligned}&\int_{\mathbb{R}}\left\{\mu\left(x_0+\frac{\theta y}{a\sqrt{\mu_0}}\right)-\mu(x_0)\right\}(U_0(y)^2-1)^2\,dy\\&<\frac{C}{2}\frac{\theta^2}{a^2\mu_0}\int_{\mathbb{R}}y^2\left(U_0(y)^2-1\right)^2\,dy\\&=O(\theta^2).\end{aligned}\tag{3.11}$$

Combining (3.10) and (3.11), we obtain the statement of (2). □

We treat $J_\theta^{(3)}(\psi_*)+J_\theta^{(4)}(\psi_*)$ in Lemmas 7 and 8.

LEMMA 7. *Let $u_*$,$\psi_*$ and $v_*$ be functions defined in (3.1), (3.2) and (3.3), respectively. Then the following identity holds:*

$$J_\theta^{(3)}(\psi_*)+J_\theta^{(4)}(\psi_*)=\frac{a^4\mu_0\sqrt{\gamma}}{2\theta^2\gamma^3}\tilde{J}(\theta,\gamma),$$

*where*

$$\tilde{J}(\theta,\gamma)=\int_0^\infty\int_0^\infty\left[\left(e^{-\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}|s-t|}-e^{-\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}|s+t|}\right)B(s)B(t)\right]dsdt\tag{3.12}$$

*and $B(s)$ is defined in (1.14).*

Proof. Set $w_*$ as follows:

$$w_* = v_* - \frac{u_*}{\gamma}.$$

It is easy to check that $(u_*, v_*)$ satisfies the following equations:

$$-\frac{u_*''(x)}{\gamma} = \frac{\mu_0}{\theta^2\gamma} u_*(x)\left(a^2 - u_*(x)^2\right),$$
$$-v_*''(x) + \gamma\left(v_*(x) - \frac{u_*(x)}{\gamma}\right) = 0.$$

Thus $w_*$ satisfies

$$-w_*''(x) + \gamma w_*(x) = -\frac{\mu_0}{\theta^2\gamma} u_*(x)\left(a^2 - u_*(x)^2\right).$$

Now we set $\tilde{w}_*$ as follows:

$$\tilde{w}_*(y) = w_*\left(x_0 + \frac{y}{\sqrt{\gamma}}\right).$$

Then we can see that

$$\begin{aligned}
&\int_{\mathbb{R}} \left[\frac{1}{2}\left(v_*' - \frac{u_*'}{\gamma}\right)^2 + \frac{\gamma}{2}\left(v_* - \frac{u_*}{\gamma}\right)^2\right] dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \left[\left|w_*'(x)\right|^2 + \gamma\left|w_*(x)\right|^2\right] dx \\
&= \frac{\gamma}{2}\int_{\mathbb{R}} \left[\left|\tilde{w}_*'(y)\right|^2 + \left|\tilde{w}_*(y)\right|^2\right] \frac{1}{\sqrt{\gamma}}\, dy \\
&= \frac{\sqrt{\gamma}}{2}\left\|\tilde{w}_*\right\|_{H^1(\mathbb{R})}^2. \qquad (3.13)
\end{aligned}$$

Moreover, $\tilde{w}_*$ satisfies

$$-\tilde{w}_*''(y) + \tilde{w}_*(y) = -\frac{\mu_0}{\theta^2\gamma^2}\tilde{u}_*(y)\left(a^2 - \tilde{u}_*(y)^2\right),$$

where $\tilde{u}_*(y) = u_*(x_0 + y/\sqrt{\gamma})$. Since $\tilde{u}_*(-y) = -\tilde{u}_*(y)$ holds for all $y \in \mathbb{R}$, we obtain

$$\left\|\tilde{w}_*\right\|_{H^1(\mathbb{R})}^2 = \int_0^\infty \int_0^\infty \left(e^{-|y-z|} - e^{-|y+z|}\right) H(z)H(y)dydz \qquad (3.14)$$

from (2) of Lemma 4, where

$$H(y) = -\frac{\mu_0}{\theta^2\gamma^2}\tilde{u}_*(y)\left(a^2 - \tilde{u}_*(y)^2\right).$$

Now we rewrite $H(y)$ by using $U_0(y)$:

$$\begin{aligned}
H(y) &= -\frac{\mu_0}{\theta^2\gamma^2}\tilde{u}_*(y)\left(a^2 - \tilde{u}_*(y)^2\right) \\
&= -\frac{\mu_0}{\theta^2\gamma^2}u_*\left(x_0 + \frac{y}{\sqrt{\gamma}}\right)\left(a^2 - u_*\left(x_0 + \frac{y}{\sqrt{\gamma}}\right)^2\right) \\
&= -\frac{\mu_0}{\theta^2\gamma^2}aU_0\left(\frac{a\sqrt{\mu_0}}{\theta}\frac{y}{\sqrt{\gamma}}\right)\left(a^2 - a^2U_0\left(\frac{a\sqrt{\mu_0}}{\theta}\frac{y}{\sqrt{\gamma}}\right)^2\right) \\
&= -\frac{a^3\mu_0}{\theta^2\gamma^2}B\left(\frac{a\sqrt{\mu_0}}{\theta}\frac{y}{\sqrt{\gamma}}\right),
\end{aligned}$$

where $B(s)$ is defined in (1.14). Hence by changing variables, we calculate (3.14) as follows:

$$\begin{aligned}
\|\tilde{w}_*\|^2_{H^1(\mathbb{R})} &= \frac{a^6\mu_0^2}{\theta^4\gamma^4}\int_0^\infty\int_0^\infty\left(e^{-\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}|s-t|} - e^{-\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}|s+t|}\right) \\
&\qquad\qquad \times B(s)B(t)\left(\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\right)^2 dsdt \\
&= \frac{a^4\mu_0}{\theta^2\gamma^3}\tilde{J}(\theta,\gamma), \qquad (3.15)
\end{aligned}$$

where $\tilde{J}(\theta,\gamma)$ is defined in (3.12). Combining (3.13) and (3.15), we conclude the statement of the lemma. □

LEMMA 8. *Assume that $\mu$ satisfies ($\mu$1) and ($\mu$2). Let $u_*$ and $v_*$ be functions defined in (3.1) and (3.3). Then the following statements hold true:*

*(1) If $1/\gamma = o(\theta^{6/5})$ as $\theta \to 0$, then*

$$J_\theta^{(3)}(\psi_*) + J_\theta^{(4)}(\psi_*) = o(\theta). \qquad (3.16)$$

*(2) If $\theta^2 \ll 1/\gamma \ll \theta$ as $\theta \to 0$, then*

$$J_\theta^{(3)}(\psi_*) + J_\theta^{(4)}(\psi_*) \leq \frac{a^3\sqrt{\mu_0}}{2\theta\gamma^2} A + O(\gamma^{-3/2}), \tag{3.17}$$

*where $A$ is defined in (1.13).*

(3) *Moreover, by combining (1) and (2), it follows that if $1/\gamma = o(\theta)$, then (3.16) holds.*

Proof. (1) First, we assume that $1/\gamma = o(\theta^{6/5})$. From Lemma 7, we recall that

$$J_\theta^{(3)}(\psi_*) + J_\theta^{(4)}(\psi_*) = \frac{a^4\mu_0\sqrt{\gamma}}{2\theta^2\gamma^3} \tilde{J}(\theta, \gamma). \tag{3.18}$$

It is easy to check that there exists a constant $C_0 > 0$ such that $\tilde{J}(\theta, \gamma) < C_0$. Moreover, we can see

$$\frac{\sqrt{\gamma}}{\theta^2\gamma^3} = \frac{1}{\theta^2} \cdot o(\theta^{5/2 \cdot 6/5}) = o(\theta).$$

Thus we obtain the conclusion of the lemma in the case (1).

(2) Next, we consider the case (2). From the Taylor expansion of $e^{-x}$, we can see that

$$e^{-x} \leq 1 - x + \frac{x^2}{2}$$

and

$$-e^{-x} \leq -1 + x + \frac{x^2}{2}.$$

Thus we obtain that

$$\begin{aligned} &e^{-\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}|y-z|} - e^{-\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}|y+z|} \\ &\leq \frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}(|y+z| - |y-z|) + \frac{1}{2}\left(\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\right)^2 \left(|y-z|^2 + |y+z|^2\right) \\ &\leq \frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\left(|y+z| - |y-z|\right) + \frac{\theta^2\gamma}{a^2\mu_0}(y^2 + z^2). \end{aligned}$$

Thus we have the following inequality:

$$\begin{aligned}\tilde{J}(\theta,\gamma) \leq & \frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\int_0^\infty\int_0^\infty (|y+z|-|y-z|)\,B(y)B(z)\,dydz \\ & + \frac{\theta^2\gamma}{a^2\mu_0}\int_0^\infty\int_0^\infty (y^2+z^2)B(y)B(z)\,dydz\end{aligned}$$

Combining the above inequality with (3.18), we arrive at

$$J_\theta^{(3)}(\psi_*) + J_\theta^{(4)}(\psi_*) \leq \frac{a^3\sqrt{\mu_0}}{2\theta\gamma^2}A + \frac{a^2}{\gamma^{3/2}}\int_0^\infty\int_0^\infty (y^2+z^2)B(y)B(z)\,dydz.$$

As a consequence, we have proved (2).

(3) Any $1/\gamma = o(\theta)$ case is contained in either case (1) or case (2). With attention to $1/(\theta\gamma^2) = o(\theta)$ and $1/\gamma^{3/2} = o(\theta)$, we can see $J_\theta^{(3)}(\psi_*) + J_\theta^{(4)}(\psi_*) = o(\theta)$ even if in case (2). Hence we conclude that (3.16) holds for any $1/\gamma = o(\theta)$. □

Finally we calculate $J_\theta^{(5)}(\psi_*)$.

LEMMA 9. *Assume that $\mu$ satisfies ($\mu$1), ($\mu$2) and ($\mu$2'). Let $u_*$ and $\psi_*$ be functions defined in (3.1) and (3.2). Then $J_\theta^{(5)}(\psi_*)$ defined in (3.9) is calculated as follows:*

$$\begin{aligned}J_\theta^{(5)}(\psi_*) = & \frac{a^2}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))\,dx \\ & + \frac{a\theta}{2\gamma\sqrt{\mu_0}}\left\{\int_{\mathbb{R}}(1-\mu_0)\left(U_0(x)^2-1\right)\,dx + o(1)\right\}.\end{aligned}$$

PROOF. From the definition of $J_\theta^{(5)}(\psi_*)$, we can see that

$$\begin{aligned}(3.19)\qquad & J_\theta^{(5)}(\psi_*) \\ & = \frac{1}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))(u_*(x)^2-a^2)\,dx + \frac{1}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))a^2\,dx \\ & = \frac{1}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))\left\{a^2U_0\left(\frac{a\sqrt{\mu_0}(x-x_0)}{\theta}\right)-a^2\right\}\,dx \\ & \quad + \frac{a^2}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))\,dx\end{aligned}$$

$$= \frac{a^2}{2\gamma}\int_{\mathbb{R}}\left\{1-\mu\left(x_0+\frac{\theta y}{a\sqrt{\mu_0}}\right)\right\}(U_0(y)^2-1)\frac{\theta}{a\sqrt{\mu_0}}\,dy$$
$$+\frac{a^2}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))\,dx$$
$$= \frac{a\theta}{2\gamma\sqrt{\mu_0}}\int_{\mathbb{R}}\left\{1-\mu\left(x_0+\frac{\theta y}{a\sqrt{\mu_0}}\right)\right\}(U_0(y)^2-1)\,dy$$
$$+\frac{a^2}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))\,dx.$$

From the continuity of $\mu$ and $\mu(x_0)=\mu_0$, we easily see that

$$\int_{\mathbb{R}}\left\{1-\mu\left(x_0+\frac{\theta y}{a\sqrt{\mu_0}}\right)\right\}(U_0(y)^2-1)\,dy$$
$$=\int_{\mathbb{R}}(1-\mu_0)(U_0(y)^2-1)\,dy+o(1).$$

Thus we conclude the statement. □

With these lemmas, we prove Propositions 2 and 3.

Proof of Proposition 2. With Lemmas 5, 6, 8, 9 and (3.19), we can estimate $J_\theta(\psi_*)$ as follows:

$$J_\theta(\psi_*) \le a^3\theta\sqrt{\mu_0}\int_{\mathbb{R}}\left[\frac{|U_0'(x)|^2}{2}+\frac{1}{4}\left(U_0(x)^2-1\right)^2\right]dx$$
$$+\frac{a^2}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))\,dx$$
$$+\frac{a\theta}{2\gamma\sqrt{\mu_0}}\int_{\mathbb{R}}(1-\mu_0)\left(U_0(x)^2-1\right)^2\,dx+o(\theta).$$

From Lemma 1 and the assumption on $\gamma$ and $\theta$, we can see

$$J_\theta(\psi_*) \le a^3\sqrt{\mu_0}c_*\theta+o(\theta).$$

Thus we have shown the statement. □

Proof of Proposition 3. With Lemmas 5, 6, 8 and 9, we can estimate $J_\theta(\psi_*)$ as follows:

$$J_\theta(\psi_*) \le a^3\theta\sqrt{\mu_0}\int_{\mathbb{R}}\left[\frac{|U_0'(x)|^2}{2}+\frac{1}{4}\left(U_0(x)^2-1\right)^2\right]dx$$

$$
\begin{aligned}
&+ O(\theta^3) + \frac{a^3\sqrt{\mu_0}}{2\theta\gamma^2}A + O(\gamma^{-3/2}) \\
&+ \frac{1}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))\,dx \\
&+ \frac{a\theta}{2\gamma\sqrt{\mu_0}}\left\{\int_{\mathbb{R}}(1-\mu(x))\left(U_0(x)^2-1\right)^2\,dx + o(1)\right\}.
\end{aligned}
$$

Since $\theta^3 = o(\gamma^{-3/2})$ and $\theta/\gamma = o(\gamma^{-3/2})$, we can see

$$
J_\theta(\psi_*) \le a^3\sqrt{\mu_0}c_*\theta + \frac{1}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))\,dx + \frac{a^3\sqrt{\mu_0}}{2\theta\gamma^2}A + O(\gamma^{-3/2}).
$$

Thus we complete the proof. □

Proof of Proposition 1. We can readily prove the statement from Lemma 2 and Proposition 2. □

## 4. Behavior of the Minimizer

In this section, we will investigate the behavior of the minimizer. As in the previous section, we treat only $\tilde{J}_\theta(\psi)$. For simplicity, we write $a$, $J_\theta(\psi)$ and $\sigma(\theta,\gamma)$ as $a_\gamma$, $\tilde{J}_\theta(\psi)$ and $\tilde{\sigma}(\theta,\gamma)$, respectively.

### 4.1. Existence of the minimizer

We show the existence of a minimizer of (1.10). Although the existence of the minimizer has been already shown in [10], we can show it easier by using the estimate of $\sigma(\theta,\gamma)$. First, we give a lemma to show the existence of a minimizer.

Lemma 10. *Fix $\theta > 0$ small enough. Let $\{\psi_j\}_j$ be a minimizing sequence of the minimizing problem (1.10) and $u_j = \hat{u} + \psi_j$. Moreover, let $\{x_j\}_j$ be a sequence in $\mathbb{R}$ such that $u_j(x_j) = 0$. Then there exists a constant $C_1 > 0$ such that $|x_j| < C_1$ for all $j \in \mathbb{N}$.*

Proof. We prove by contradiction. Namely, we assume that there exists a subsequence $\{x_{j_k}\}_k$ such that $|x_{j_k}| \to \infty$ as $k \to \infty$. By taking a subsequence of $\{x_{j_k}\}_k$ if necessary, we may assume that $x_{j_k} \to \infty$ as $k \to \infty$.

For simplicity, we write $x_{j_k} = x_j$. Let $\delta > 0$ be a small constant and take $j$ large enough such that the following inequalities hold:

$$\mu(x) > 1 - \delta \quad \text{for all} \;\; x \geq x_j,$$
$$J_\theta(\psi_j) \leq a^3 c_*(\sqrt{\mu_0} + \delta)\theta.$$

The existence of $\delta$ is guaranteed by ($\mu$2) and Proposition 2. Now we define $E_i^{(j)}(\psi)$ $(i = 1, 2)$ as follows:

$$E_1^{(j)}(\psi) = \int_{-\infty}^{x_j} \frac{\theta^2}{2} \left|u'(x)\right|^2 + \frac{\mu(x)}{4} \left(u^2 - a^2\right)^2 \, dx,$$
$$E_2^{(j)}(\psi) = \int_{x_j}^{\infty} \frac{\theta^2}{2} \left|u'(x)\right|^2 + \frac{\mu(x)}{4} \left(u^2 - a^2\right)^2 \, dx,$$

where $u = \hat{u} + \psi$. Then we can see that

$$E_1^{(j)}(\psi_j) + E_2^{(j)}(\psi_j) < J_\theta(\psi_j) < a^3 c_*(\sqrt{\mu_0} + \delta)\theta. \tag{4.1}$$

Moreover, if necessary, by taking $\delta > 0$ small enough, we can obtain

$$\begin{aligned}
&\inf\left\{E_1^{(j)}(\psi) \,:\, \psi \in H^1((-\infty, x_j]), u(x_j) = 0,\; u = \hat{u} + \psi\right\} \\
&\geq \inf\left\{\int_{-\infty}^{x_j} \frac{\theta^2}{2}|u'|^2 + \frac{\mu_0}{4}(u^2 - a^2)^2 \, dx \;: \right. \\
&\qquad\qquad \left. u \in H^1{}_{loc}((-\infty, x_j]), u(x_j) = 0, u(x) \to -a \; (x \to -\infty)\right\} \\
&= \frac{1}{2} a^3 \sqrt{\mu_0} c_* \theta,
\end{aligned}$$

and

$$\begin{aligned}
&\inf\left\{E_2^{(j)}(\psi) \,:\, \psi \in H^1([x_j, \infty)), u(x_j) = 0,\; u = \hat{u} + \psi\right\} \\
&\geq \inf\left\{\int_{x_j}^{\infty} \frac{\theta^2}{2}|u'|^2 + \frac{1-\delta}{4}(u^2 - a^2)^2 \, dx : \right. \\
&\qquad\qquad \left. u \in H^1{}_{loc}([x_j, \infty)), u(x_j) = 0, u(x) \to a \; (x \to \infty)\right\} \\
&= \frac{1}{2} a^3 \sqrt{1-\delta} c_* \theta.
\end{aligned}$$

Here we used the Remark 2.1 and the same scaling argument as in the proof of Lemma 2. Hence we obtain

$$\frac{1}{2} a^3 c_* \theta \left(\sqrt{\mu_0} + \sqrt{1-\delta}\right) \leq E_1^{(j)}(\psi_j) + E_2^{(j)}(\psi_j).$$

However, this contradicts (4.1) for small $\theta > 0$. □

We prove the existence of the minimizer.

PROPOSITION 4. *Assume that* $\gamma > 1$, $\theta^2 = d - 1/\gamma^2 > 0$, $1/\gamma = o(\theta)$ *as* $\theta \to 0$ *and* $\mu$ *satisfies (*$\mu$*1) and (*$\mu$*2). Then minimizing problem (1.10) has a minimizer.*

PROOF. Let $\{\psi_j\}_j$ be a minimizing sequence of (1.10) and $u_j = \hat{u} + \psi_j$. From Lemma 10, we may assume that

$$u_j \geq 0 \quad \text{on } (C_1, \infty) \quad \text{for all } \ j \in \mathbb{R},$$

where $C_1$ is defined in Lemma 10. We may assume that $C_1 > 1$. Thus from Proposition 2, we can see

$$\begin{aligned}
\|\psi_j\|^2_{L^2(C_1,\infty)} &= \int_{C_1}^{\infty} (u_j - a)^2 \, dx \\
&\leq \frac{1}{a^2} \int_{C_1}^{\infty} (u_j + a)^2 (u_j - a)^2 \, dx, \\
&\leq \frac{4}{a^2} J_\theta(\psi_j) \\
&\leq 8a\sqrt{\mu_0} c_* \theta.
\end{aligned}$$

Thus there exists a constant $C_2 > 0$ such that

$$\|\psi_j\|_{L^2(C_1,\infty)} < C_2.$$

Similarly we can see

$$\|\psi_j\|_{L^2(-\infty,-C_1)} < C_2.$$

On the other hand, for any $x \in (-C_1, C_1)$, we obtain

$$\begin{aligned}
|u_j(x)| = |u_j(x) - u_j(x_j)| &\leq \int_{-C_1}^{C_1} |u_j'| \, dx \\
&\leq \sqrt{2C_1} \left\|u_j'\right\|_{L^2(\mathbb{R})} \leq \frac{2\sqrt{C_1 J_\theta(\psi_j)}}{\theta}
\end{aligned}$$

by Schwarz's inequality. It follows that

$$\|u_j\|_{L^\infty(-C_1,C_1)} < \hat{C}_2$$

for some $\hat{C}_2 > 0$. Thus we conclude that there exists a constant $\tilde{C}_2$ such that

$$\|\psi_j\|_{L^2(\mathbb{R})} < \tilde{C}_2.$$

Moreover, since it follows that $\{\|\psi_j'\|_{L^2(\mathbb{R})}\}_j$ is uniformly bounded from Proposition 2, there exists $\psi_0 \in H^1(\mathbb{R})$ such that

$$\psi_j \to \psi_0 \quad \text{weakly in} \quad H^1(\mathbb{R}) \quad \text{and} \quad \psi_j \to \psi_0 \quad \text{in} \quad C_{loc}(\mathbb{R}).$$

We define $u_0 = \hat{u} + \psi_0$ and $v_0 = \hat{v} + \mathcal{L}\psi_0$. Then $(u_0, v_0)$ satisfies

$$\lim_{x \to \pm\infty} (u_0(x), v_0(x)) = (\pm a, \pm a/\gamma)$$

since $\psi_0,\ \mathcal{L}\psi_0 \in H^1(\mathbb{R})$. Now we prove $\psi_0$ is a minimizer of (1.10). First, from the lower semicontinuity in $L^2(\mathbb{R})$, we see that

$$\int_{\mathbb{R}} |u_0'|^2 \, dx \leq \liminf_{j \to \infty} \int_{\mathbb{R}} |u_j'|^2 \, dx.$$

Then, from Fatou's lemma, we have

$$\int_{\mathbb{R}} \frac{\mu(x)}{4} \left(u_0^2 - a^2\right)^2 dx \leq \liminf_{j \to \infty} \int_{\mathbb{R}} \frac{\mu(x)}{4} \left(u_j^2 - a^2\right)^2 dx$$

and

$$\frac{1}{2\gamma} \int_{\mathbb{R}} (1 - \mu(x)) u_0^2 \, dx \leq \liminf_{j \to \infty} \frac{1}{2\gamma} \int_{\mathbb{R}} (1 - \mu(x)) u_j^2 \, dx.$$

We set $v_j = \hat{v} + \mathcal{L}\psi_j$ and then we can see

$$v_j - \frac{u_j}{\gamma} \to v_0 - \frac{u_0}{\gamma} \quad \text{weakly in} \quad H^1(\mathbb{R}).$$

Hence it follows that

$$\int_{\mathbb{R}} \left(v_0' - \frac{u_0'}{\gamma}\right)^2 + \left(v_0 - \frac{u_0}{\gamma}\right)^2 dx$$
$$\leq \liminf_{j \to \infty} \int_{\mathbb{R}} \left(v_j' - \frac{u_j'}{\gamma}\right)^2 + \left(v_j - \frac{u_j}{\gamma}\right)^2 dx.$$

As a consequence, we obtain that

$$J_\theta(\psi_0) \leq \liminf_{j\to\infty} J_\theta(\psi_j) = \sigma(\theta,\gamma).$$

This means $\psi_0$ is a minimizer of (1.10) and $(u_0, v_0)$ is a heteroclinic solution to (1.2). □

We next show Theorem 1

### 4.2. Proof of Theorem 1

We now prove Theorem 1. Here we shall show the generalized statement as follows:

THEOREM 4. *Assume that $\gamma > 1$, $\theta^2 = d - 1/\gamma^2 > 0$, $1/\gamma = o(\theta)$ and $\mu(x)$ satisfies ($\mu$1) and ($\mu$2). Let $\psi_\theta \in H^1(\mathbb{R})$ be a minimizer of (1.9) or (1.10), $u_\theta = \hat{u} + \psi_\theta$ and $v_\theta = \hat{v} + \mathcal{L}\psi_\theta$. Then for any $b \in (-a_\gamma, a_\gamma)$, there exists the unique point $x_\theta(b) \in \mathbb{R}$ such that $u_\theta(x_\theta(b)) = b$ by taking $\theta > 0$ small enough if necessary.*

PROOF. We use the notation $a$ instead of $a_\gamma$, for simplicity. From the boundary condition $u_\theta(x) \to \pm a$ as $x \to \pm\infty$, it is clear that for any $b \in (-a, a)$, there exists at least one point $x_\theta = x_\theta(b) \in \mathbb{R}$ such that $u_\theta(x_\theta) = b$. Moreover, we can see that for any $m_0 > 0$, we may assume that

$$u_\theta > b - m_0 \ \text{ on } \ (x_\theta, \infty) \quad \text{ and } \quad u_\theta < b + m_0 \ \text{ on } \ (-\infty, x_\theta) \tag{4.2}$$

by taking $\theta$ small enough if necessary. Indeed, if there exist $m_1 > 0$, $\{\theta_j\}_j$ and $y_j > x_{\theta_j}$ such that $\theta_j \to 0$ $(j \to \infty)$ and $u_{\theta_j}(y_j) < b - m_1$, then there exists $x'_{\theta_j} > y_j$ such that $u(x'_{\theta_j}) = b$. On the other hand, we readily see

$$J_{\theta_j}(\psi_{\theta_j}) > E_3^{(j)}(\psi_j) + E_4^{(j)}(\psi_j) + E_5^{(j)}(\psi_j),$$

where $E_i^{(j)}(\psi)$ $(i = 3,4,5,\ j \in \mathbb{N})$ is defined as follows:

$$E_3^{(j)}(\psi_{\theta_j}) = \int_{-\infty}^{x_{\theta_j}} \frac{\theta^2}{2}\left|u'_{\theta_j}\right|^2 + \frac{\mu(x)}{4}\left(u_{\theta_j}^2 - 1\right)^2 dx,$$

$$E_4^{(j)}(\psi_{\theta_j}) = \int_{x'_{\theta_j}}^{\infty} \frac{\theta^2}{2}\left|u'_{\theta_j}\right|^2 + \frac{\mu(x)}{4}\left(u_{\theta_j}^2 - 1\right)^2 dx,$$

$$E_5^{(j)}(\psi_{\theta_j}) = \int_{x_{\theta_j}}^{x'_{\theta_j}} \frac{\theta^2}{2}\left|u'_{\theta_j}\right|^2 + \frac{\mu(x)}{4}\left(u_{\theta_j}^2 - 1\right)^2 dx.$$

Then we can see

$$E_3^{(j)}(\psi_j) + E_4^{(j)}(\psi_j) \geq a^3\sqrt{\mu_0}c_*\theta_j. \tag{4.3}$$

Indeed, set

$$\bar{u}_{\theta_j}(x) = \begin{cases} u_{\theta_j}(x), & x \leq x_{\theta_j}, \\ u_{\theta_j}(x + x'_{\theta_j} - x_{\theta_j}), & x > x_{\theta_j} \end{cases}$$

and then we have

$$E_3^{(j)}(\psi_j) + E_4^{(j)}(\psi_j) = \int_{\mathbb{R}} \frac{\theta_j^2}{2}\left|\bar{u}'_{\theta_j}\right|^2 + \frac{\mu_0}{4}(\bar{u}_{\theta_j}^2 - 1)^2\,dx.$$

In addition, $\bar{u}_{\theta_j} \in \{u \in H^1_{loc}(\mathbb{R}), \lim_{x\to\pm\infty} u(x) = \pm a\}$ since $u_{\theta_j}(x_{\theta_j}) = u_{\theta_j}(x'_{\theta_j})$. Thus we obtain (4.3) from Lemma 2.

Moreover, from Lemma 3, we deduce that

$$E_5^{(j)}(\psi_j) \geq C_3\theta_j,$$

where $C_3 = K(b - m_1, b)m_1\sqrt{\mu_0/2}$. Thus we obtain

$$J_{\theta_j}(\psi_{\theta_j}) \geq (a^3\sqrt{\mu_0}c_* + C_3)\theta_j,$$

but this contradicts the upper estimate $J_{\theta_j}(\psi_{\theta_j}) < a^3\sqrt{\mu_0}c_*\theta_j + o(\theta_j)$. Thus (4.2) should be true.

Now we set

$$U_\theta(y) = \frac{1}{a}u_\theta\left(\frac{\theta y}{a\sqrt{\mu_0}} + x_\theta\right),$$
$$\Psi_\theta(y) = U_\theta(y) - \frac{1}{a}\hat{u}(y),$$
$$V_\theta(y) = \frac{1}{a}\hat{v}(y) + (\mathcal{L}\Psi_\theta)(y).$$

We note that since $U_\theta(0) = b/a$ for any $\theta > 0$, it follows that

$$U_\theta > \frac{b}{a} - \frac{\delta}{a} \;\text{ on }\; (0,\infty) \quad\text{ and }\quad U_\theta < \frac{b}{a} + \frac{\delta}{a} \;\text{ on }\; (-\infty, 0) \tag{4.4}$$

for small $\delta > 0$ from (4.2). We shall investigate the asymptotic behavior of $U_\theta$ as $\theta \to 0$. We write $J_\theta(\psi_\theta)$ with $U_\theta$ and $V_\theta$:

$$J_\theta(\psi_\theta) = \int_{\mathbb{R}}\left[\frac{\theta^2}{2}\left|u'_\theta\right|^2 + \frac{\mu(x)}{4}(u_\theta^2 - a^2)^2 + \frac{1}{2}\left(v'_\theta - \frac{u'_\theta}{\gamma}\right)^2\right.$$

$$
\begin{aligned}
&+\frac{\gamma}{2}\left(v_\theta-\frac{u_\theta}{\gamma}\right)^2+\frac{1-\mu(x)}{2\gamma}u_\theta^2\Bigg] dx\\
=&\int_{\mathbb{R}}\left[\frac{a^4\theta^2\mu_0}{2\theta^2}\left|U_\theta'\right|^2+\frac{a^4\mu\left((\theta y/(a\sqrt{\mu_0}))+x_\theta\right)}{4}(U_\theta^2-1)^2\right.\\
&+\frac{a^4}{2\theta^2}\left(V_\theta'-\frac{U_\theta'}{\gamma}\right)^2+\frac{a^2\gamma}{2}\left(V_\theta-\frac{U_\theta}{\gamma}\right)^2\\
&\left.+\frac{1-\mu\left((\theta y/(a\sqrt{\mu_0}))+x_\theta\right)}{2\gamma}a^2U_\theta^2\right]\frac{\theta}{a\sqrt{\mu_0}}dy\\
=&a^3\sqrt{\mu_0}\theta\int_{\mathbb{R}}\left[\frac{1}{2}\left|U_\theta'\right|^2+\frac{\mu\left((\theta y/(a\sqrt{\mu_0}))+x_\theta\right)}{4\mu_0}\left(U_\theta^2-1\right)^2\right.\\
&+\frac{1}{2\theta\mu_0}\left(V_\theta'-\frac{U_\theta'}{\gamma}\right)^2+\frac{\gamma}{2a^2\mu_0}\left(V_\theta-\frac{U_\theta}{\gamma}\right)^2\\
&\left.+\frac{1-\mu\left((\theta y/(a\sqrt{\mu_0}))+x_\theta\right)}{2a^2\mu_0\gamma}U_\theta^2\right]dy
\end{aligned}
$$

Thus from Proposition 2, we obtain

$$
\begin{aligned}
E^*(U_\theta):=&\int_{\mathbb{R}}\left[\frac{1}{2}\left|U_\theta'\right|^2+\frac{\mu\left((\theta y/(a\sqrt{\mu_0}))+x_\theta\right)}{4\mu_0}\left(U_\theta^2-1\right)^2\right]dy\\
\le& c_*+o(1). \qquad (4.5)
\end{aligned}
$$

Moreover, since $E^*(U_\theta)\ge E(U_\theta)\ge c_*$ holds, we can see

$$
\int_{\mathbb{R}}\left[\frac{1}{2\theta^2}\left(V_\theta'-\frac{U_\theta'}{\gamma}\right)^2+\frac{\gamma}{2}\left(V_\theta-\frac{U_\theta}{\gamma}\right)^2\right]dy\le o(1) \tag{4.6}
$$

Combining (4.4) and (4.5), we can show that there exists a constant $C_4>0$ such that

$$
\|\Psi_\theta\|_{H^1(\mathbb{R})}<C_4 \tag{4.7}
$$

as in the proof of the boundedness of $\{\psi_j\}_j$ in Proposition 4. We now prove that there exists a positive constant $\tilde{C}_4$ such that

$$
\|\Psi_\theta''\|_{L^2(\mathbb{R})}<\tilde{C}_4. \tag{4.8}
$$

From the definition of $U_\theta$ and $V_\theta$, the following equation is obtained:

$$-\frac{da^3\mu_0}{\theta^2}U_\theta''(y) = \mu\left(\frac{\theta y}{a\sqrt{\mu_0}} + x_\theta\right) f(aU_\theta(y)) - aV_\theta(y).$$

With $f(a) = a/\gamma$, the right hand side is written as follows:

$$\begin{aligned}(\text{r.h.s.}) = \mu\left(\frac{\theta y}{a\sqrt{\mu_0}} + x_\theta\right)(f(aU_\theta(y)) - f(a)) - a\left(V_\theta(y) - \frac{U_\theta(y)}{\gamma}\right)\\ + a\left(\frac{1}{\gamma} - \frac{U_\theta(y)}{\gamma}\right) + \left\{\mu\left(\frac{\theta y}{a\sqrt{\mu_0}} + x_\theta\right) - 1\right\}\frac{a}{\gamma}.\end{aligned}$$

Here we remark that we can prove that $U_\theta$ is uniformly bounded in $L^\infty(\mathbb{R})$ with (4.5) by almost the same argument in the proof of Lemma 2.6 in [10]. Hence there exists a constant $C_5 > 0$ such that

$$\left|\mu\left(\frac{\theta y}{a\sqrt{\mu_0}} + x_\theta\right)(f(aU_\theta) - f(a))\right| \le C_5 a\,|U_\theta - 1|\,.$$

Moreover, we have

$$\begin{aligned}\int_0^\infty \frac{a^2}{\gamma^2}\left|\mu\left(\frac{\theta y}{a\sqrt{\mu_0}} + x_\theta\right) - 1\right|^2 dy &= \frac{a^2}{\gamma^2}\int_{x_\theta}^\infty (1-\mu(z))^2\left(\frac{a\sqrt{\mu_0}}{\theta}dz\right)\\ &\le \frac{a^3\sqrt{\mu_0}}{\gamma^2\theta}\,\|1-\mu\|_{L^2(\mathbb{R})}^2\,.\end{aligned}$$

Note that, by the assumption ($\mu$2), we have $1-\mu \in L^2(\mathbb{R})$, since $1-\mu \in L^1(\mathbb{R})$ and $1-\mu \in L^\infty(\mathbb{R})$. Thus we obtain

$$\begin{aligned}\frac{da^3\mu_0}{\theta^2}\,\|U_\theta''\|_{L^2(0,\infty)} \le a\left(C_5 + \frac{1}{\gamma}\right)\|U_\theta - 1\|_{L^2(0,\infty)} + a\left\|V_\theta - \frac{U_\theta}{\gamma}\right\|_{L^2(0,\infty)}\\ + \left(\frac{a^3\sqrt{\mu_0}}{\gamma^2\theta}\right)^{1/2}\|1-\mu\|_{L^2(\mathbb{R})}\,.\end{aligned}$$

From (4.6), (4.7) and $d/\theta^2 = 1 + o(1)$, we can see

$$\|U_\theta''\|_{L^2(0,\infty)} < \tilde{C}_5$$

holds for some constant $\tilde{C}_5 > 0$. Similarly we can deduce

$$\|U_\theta''\|_{L^2(-\infty,0)} < \tilde{C}_5.$$

As a consequence, we have shown (4.8). Combining (4.7) and (4.8), we can see that there exists $\Psi_* \in H^2(\mathbb{R})$ such that

$$\Psi_\theta \to \Psi_* \text{ weakly in } H^2(\mathbb{R}) \quad \text{and} \quad \Psi_\theta \to \Psi_* \text{ in } C^1_{loc}(\mathbb{R}).$$

Let $U_* = \hat{u}/a + \Psi_*$. Then we show the equation which $U_*$ satisfies. For any $\phi \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned} &\int_{\mathbb{R}} \frac{da^3\mu_0}{\theta^2} U_\theta' \phi' \, dy \\ &= \int_{\mathbb{R}} \left[ \mu \left( \frac{\theta y}{a\sqrt{\mu_0}} + x_\theta \right) (f(aU_\theta(y)) - f(a)) - a \left( V_\theta(y) - \frac{U_\theta(y)}{\gamma} \right) \right. \\ &\qquad \left. + a \left( \frac{U_\theta(y)}{\gamma} - \frac{1}{\gamma} \right) + \left\{ 1 - \mu \left( \frac{\theta y}{a\sqrt{\mu_0}} + x_\theta \right) \right\} \frac{a}{\gamma} \right] \phi \, dy. \end{aligned} \tag{4.9}$$

We note that there exist $\mu_1 \in [\mu_0, 1]$ and $\{\theta_j\}_j$ such that $\mu(\theta_j y/(a\sqrt{\mu_0}) + x_{\theta_j}) \to \mu_1$. By taking $\theta = \theta_j \to 0$, we can deduce that

$$\int_{\mathbb{R}} U_*' \phi' \, dy = \int_{\mathbb{R}} \frac{\mu_1}{\mu_0} f(U_*) \phi \, dy.$$

Hence $U_*$ is the unique solution to

$$\begin{cases} -U_*''(x) = \dfrac{\mu_1}{\mu_0} f(U_*), & x \in \mathbb{R}, \\ U_*(x) \to \pm 1 & x \to \pm\infty. \end{cases}$$

This implies that there exist positive constants $m_*$ and $\delta_*$ such that

$$U_*'(x) > m_* \quad \text{for all } x \in [-\delta_*, \delta_*].$$

Since $U_\theta' \to U_*'$ in $C_{loc}(\mathbb{R})$, we have

$$U_\theta'(x) > \frac{m_*}{2} \quad \text{for all } x \in [-\delta_*, \delta_*].$$

This is equivalent to

$$u_\theta'(x) > \frac{a^2 m_*}{2\theta} \quad \text{for all } x \in \left[ x_\theta - \frac{\theta\delta_*}{a}, x_\theta + \frac{\theta\delta_*}{a} \right].$$

This leads to

$$u_\theta\left( x_\theta + \frac{\theta\delta_*}{a} \right) > b + \frac{am_*\delta_*}{2} \quad \text{and} \quad u_\theta\left( x_\theta - \frac{\theta\delta_*}{a} \right) < b - \frac{am_*\delta_*}{2}$$

for any small $\theta > 0$. Thus we can prove the uniqueness of $x_\theta$ from (4.2) with $m_0 = am_*\delta_*/2$. □

### 4.3. Proof of Theorem 2

We next give the proof of Theorem 2.

Proof of Theorem 2. We prove by contradiction. Namely, we suppose that there exist $\delta_0 > 0$ and $\{\theta_j\}_j$ such that $\theta_j \to 0$ and $\mathrm{dist}(x_{\theta_j}, M) \geq \delta_0$ holds for all $j \in \mathbb{N}$. Set $\mu_1$ as follows:

$$\mu_1 = \inf\left\{\mu(x) : \ \mathrm{dist}(x, M) \geq \frac{\delta_0}{2}\right\}.$$

Then, we have $\mu_1 > \mu_0$. Let $\rho > 0$ be a small constant. We suppose that $u_{\theta_j}$ has a transition from $-a+\rho$ to $a-\rho$ on the interval $I_{\theta_j}(\rho) \subset \mathbb{R}$. We remark $x_\theta \in I_{\theta_j}(\rho)$. We set

$$E_6^{(j)}(\psi_{\theta_j}) = \int_{I_{\theta_j}(\rho)} \left[\frac{\theta_j^2}{2}\left|u'_{\theta_j}\right|^2 + \mu(x)W(u_{\theta_j})\right] dx,$$

where $W(s) = (s^2 - a^2)^2/4$. From Proposition 2, it is easy to see that

$$E_6^{(j)}(\psi_{\theta_j}) \leq J_{\theta_j}(\psi_{\theta_j}) \leq a^3\sqrt{\mu_0}c_*\theta_j + o(\theta_j). \tag{4.10}$$

Now we deduce the lower estimate of $E_6^{(j)}(\psi_{\theta_j})$. First, we estimate $\mu$ on $I_{\theta_j}(\rho)$. From (4.10), we have

$$a^3\sqrt{\mu_0}c_*\theta_j + o(\theta_j) \geq J_\theta(\psi_{\theta_j}) \geq \int_{I_{\theta_j}(\rho)} \mu(x)W(u_{\theta_j})\,dx \geq \frac{\mu_0}{4}\rho^4\left|I_{\theta_j}(\rho)\right|.$$

Thus we see that

$$\left|I_{\theta_j}(\rho)\right| \leq \frac{4}{\mu_0\rho^4}\left(a^3\sqrt{\mu_0}c_*\theta_j + o(\theta_j)\right).$$

Since $x_{\theta_j} \in I_{\theta_j}(\rho)$ and $\mathrm{dist}(x_{\theta_j}, M) > \delta_0$, this means that there exists $j_0 \in \mathbb{N}$ such that $\mathrm{dist}(I_{\theta_j}(\rho), M) > \delta_0/2$ holds for all $j \geq j_0$. Hence we may assume

$$\mu(x) \geq \mu_1 \quad \text{for all } \ x \in I_{\theta_j}(\rho).$$

Next, we estimate the integrand of $E_6^{(j)}(\psi_{\theta_j})$. For any $x \in I_{\theta_j}(\rho)$, we can see the following:

$$\begin{aligned}\frac{{\theta_j}^2}{2}\left|u'_{\theta_j}\right|^2 + \mu(x)W(u_{\theta_j}(x)) &\geq \frac{{\theta_j}^2}{2}\left|u'_{\theta_j}\right|^2 + \mu_1 W(u_{\theta_j}(x)) \\ &\geq 2\sqrt{\frac{{\theta_j}^2}{2}\left|u'_{\theta_j}\right|^2 \mu_1 W(u_{\theta_j}(x))} \\ &= \theta_j \left|u'_{\theta_j}\right| \sqrt{2\mu_1 W(u_{\theta_j}(x))} \\ &= \theta_j \sqrt{\mu_1}\frac{d}{dx}\left\{h(u_\theta(x))\right\},\end{aligned}$$

where $h(s) = \int_0^s \sqrt{2W(t)}\,dt$. As a consequence, we obtain

$$E_6^{(j)}(\psi_{\theta_j}) \geq \theta_j\sqrt{\mu_1}\left\{h(a-\rho) - h(-a+\rho)\right\} = \theta_j\sqrt{\mu_1}\int_{-a+\rho}^{a-\rho}\sqrt{2W(s)}\,ds.$$

With attention to $W(a\tau) = a^4(\tau^2-1)^2/4$ and $c_* = \int_{-1}^1 \sqrt{(1-\tau^2)^2/2}\,d\tau$, we deduce

$$\begin{aligned}E_6^{(j)}(\psi_{\theta_j}) &\geq \theta_j\sqrt{\mu_1}\int_{-1+\rho/a}^{1-\rho/a} a^2\sqrt{\frac{(1-\tau^2)^2}{2}}\,(ad\tau) \\ &= a^3\theta_j\sqrt{\mu_1}\left(c_* - 2\int_{1-\rho/a}^{1}\sqrt{\frac{(1-\tau^2)^2}{2}}\,d\tau\right).\end{aligned}$$

Hence by taking $\rho \to 0$, we obtain

$$E_6^{(j)}(\psi_{\theta_j}) \geq a^3\theta_j\sqrt{\mu_1}c_*.$$

However, it clearly contradicts (4.10). Thus we conclude the statement. □

## 5. Lower Estimate for Energies

In this section, we give a proof for the lower estimates of $\bar{\sigma}(\theta,\gamma)$ and $\tilde{\sigma}(\theta,\gamma)$. For simplicity, we write $a$ as $a_\gamma$. Let $\psi_\theta$ be a minimizer of (1.10), $(u_\theta, v_\theta) = (\hat{u}+\psi_\theta, \hat{v}+\mathcal{L}\psi_\theta)$ and $(U_\theta, V_\theta)$ be the function defined as follows:

$$U_\theta(y) = \frac{1}{a}u_\theta\left(x_\theta + \frac{\theta y}{a\sqrt{\mu_0}}\right), \tag{5.1}$$

$$V_\theta(y) = \frac{1}{a} v_\theta \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right), \tag{5.2}$$

where $x_\theta$ is defined in Theorem 2. From Theorem 4,

$$U_\theta(x) \begin{cases} > 0, & x > 0, \\ < 0, & x < 0 \end{cases} \tag{5.3}$$

holds for small $\theta > 0$. In this section, we always assume that $\theta > 0$ is small enough so that (5.3) holds.

Our goal in this section is to prove the following statement:

THEOREM 5. *Assume that $\gamma > 1$, $\theta^2 = d - 1/\gamma^2 > 0$, $1/\gamma = o(\theta)$, and $\mu(x)$ satisfies ($\mu$1), ($\mu$2) and ($\mu$2').*

*(1) Assume that $\theta^2 \ll 1/\gamma \ll \theta$. Then the following estimate holds:*

$$\tilde{\sigma}(\theta, \gamma) \geq a^3\sqrt{\mu_0} c_* \theta + \frac{a^2}{2\gamma} \int_{\mathbb{R}} (1 - \mu(x))\, dx + o\left( \frac{1}{\theta\gamma^2} \right),$$

*where $c_*$ is defined in (1.12).*

*(2) Assume that $\theta^2 \ll 1/\gamma \ll \theta^{4/3}$. Then the following estimate holds:*

$$\bar{\sigma}(\theta, \gamma) \geq a^3\sqrt{\mu_0} c_* \theta + \frac{a^3\sqrt{\mu_0}}{2\theta\gamma^2} A + o\left( \frac{1}{\theta\gamma^2} \right),$$

*where $A$ is defined in (1.13).*

*(3) Assume that $\theta^2 \ll 1/\gamma \ll \theta^{3/2}$. Then the following estimate holds:*

$$\tilde{\sigma}(\theta, \gamma) \geq a^3\sqrt{\mu_0} c_* \theta + \frac{a^2}{2\gamma} \int_{\mathbb{R}} (1 - \mu(x))\, dx + \frac{a^3\sqrt{\mu_0}}{2\theta\gamma^2} A + o\left( \frac{1}{\theta\gamma^2} \right).$$

Combining Proposition 3 and Theorem 5, we readily see that Theorem 3 follows. We can prove Theorem 5 by calculating the each term of $J_\theta(\psi_\theta)$, where $J_\theta$ represents $\bar{J}_\theta$ or $\tilde{J}_\theta$. However, the calculation is rather complicated and needs some lemmas on the behaviors of $U_\theta$. Therefore we divide this section into four parts. In Subsection 5.1, we introduce some notations and prove useful lemmas. In Subsection 5.2, we show key lemmas on the behavior of $U_\theta$. The lemmas presented in the subsection play important roles in the proof of the lower estimates. In Subsection 5.3, we present some auxiliary lemmas to reduce the amount of calculation. In Subsection 5.4, we prove Theorem 5.

### 5.1. Notations and useful lemmas

We introduce following notations:

$$B_\theta(x) = U_\theta(x) - U_\theta(x)^3, \tag{5.4}$$

$$G_d(x,y) = \frac{1}{2\sqrt{1-1/(d\gamma^2)}} \exp\left\{-\sqrt{1-\frac{1}{d\gamma^2}}\,|x-y|\right\}, \tag{5.5}$$

$$\Gamma(x,y) = G_d\left(\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}x, \frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}y\right) - G_d\left(\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}x, -\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}y\right). \tag{5.6}$$

Here we remark that $G_d(x,y)$ is the Green function corresponding to

$$\begin{cases} -w''(x) + \left(1 - \dfrac{1}{d\gamma^2}\right) w(x) = f(x), & x \in \mathbb{R}, \\ w(x) \to 0, & x \to \pm\infty. \end{cases}$$

The Green function $G_d(x,y)$ appears in the calculation of $J_\theta^{(3)}(\psi_\theta)+J_\theta^{(4)}(\psi_\theta)$, where $J_\theta^{(i)}$ $(i = 1, 2, \cdots, 5)$ are defined in (3.5) – (3.9). We may assume that

$$\frac{1}{2\sqrt{1-1/(d\gamma^2)}} < 1 \tag{5.7}$$

holds under the assumption $1/\gamma = o(\theta)$ since $d = \theta^2 + 1/\gamma^2$.

Now we show some useful lemmas. First, we prove a lemma on $U_\theta(y)$. This lemma has been already shown essentially in the proof of Theorem 1.

LEMMA 11. *Let $\psi_\theta$ be a minimizer of (1.9) or (1.10), $u_\theta = \hat{u} + \psi_\theta$ and $U_\theta$ be defined in (5.1). Then $U_\theta \to U_0$ in $C^1_{loc}(\mathbb{R})$ as $\theta \to 0$.*

PROOF. We recall that, for any $\phi \in C_c^\infty(\mathbb{R})$, $U_\theta$ satisfies the following identity.

$$\begin{aligned} &\int_{\mathbb{R}} \frac{da^3\mu_0}{\theta^2} U_\theta' \phi'\, dy \\ &= \int_{\mathbb{R}} \left[\mu\left(\frac{\theta y}{a\sqrt{\mu_0}} + x_\theta\right)(f(aU_\theta(y)) - f(a)) - a\left(V_\theta(y) - \frac{U_\theta(y)}{\gamma}\right)\right. \\ &\qquad \left. + a\left(\frac{U_\theta(y)}{\gamma} - \frac{1}{\gamma}\right) + \left\{1 - \mu\left(\frac{\theta y}{a\sqrt{\mu_0}} + x_\theta\right)\right\}\frac{a}{\gamma}\right]\phi\, dy. \end{aligned} \tag{4.9}$$

We note it follows that $\mu(\theta y/(a\sqrt{\mu_0}) + x_\theta) \to \mu_0$ as $\theta \to 0$ from Theorem 2. Moreover, we remark that $U_0$ is the unique solution to (2.1). Thus we can conclude the statement as in the proof of Theorem 1 (see the proof of Theorem 4). □

Next, we show some lemmas on $B_\theta(y)$ and $\Gamma(x,y)$. We choose $\bar{\delta}$ small such that $(1-2\bar{\delta})^2 > 2/3$. Then there exists a positive constant $R$ such that

$$1 - \bar{\delta} < U_0(x) < 1 \quad \text{for all } \ x \geq R. \tag{5.8}$$

LEMMA 12. *Let $B(y)$ and $B_\theta(y)$ be functions defined in (1.14) and (5.4). Then the following holds for sufficiently small $\theta > 0$:*

$$B(y) - B_\theta(y) \begin{cases} > 0, & \text{if } \ U_0(y) < U_\theta(y) \ \text{and} \ |y| \geq R, \\ < 0, & \text{if } \ U_0(y) > U_\theta(y) \ \text{and} \ |y| \geq R. \end{cases} \tag{5.9}$$

*Moreover, there exists a positive constant $C$ such that*

$$|B(y) - B_\theta(y)| < C\,|U_0(y) - U_\theta(y)|\,. \tag{5.10}$$

PROOF. From the definition of $B(y)$ and $B_\theta(y)$, it is easy to check that

$$\begin{aligned} B(y) - B_\theta(y) &= U_0(y) - U_0(y)^3 - \left(U_\theta(y) - U_\theta(y)^3\right) \\ &= \left(U_0(y) - U_\theta(y)\right)\left\{1 - \left(U_0(y)^2 + U_0(y)U_\theta(y) + U_\theta(y)^2\right)\right\}. \end{aligned}$$

From the equation, we can derive (5.10) since $U_0$ and $U_\theta$ are uniformly bounded.

Recall that $\bar{\delta}$ is the constant such that $(1-2\bar{\delta})^2 > 2/3$ and $R$ is the constant defined in (5.8). Then we may assume that $1 - \bar{\delta} < U_\theta(R)$ for sufficiently small $\theta > 0$. Moreover, we can prove that

$$U_\theta(x) > 1 - 2\bar{\delta} \quad \text{for all } \ x > R$$

similarly as (4.2). Hence we obtain

$$1 - \left(U_0(y)^2 + U_0(y)U_\theta(y) + U_\theta(y)^2\right) < -1 \quad \text{for all } \ y > R.$$

Thus we conclude

$$B(y) - B_\theta(y) \begin{cases} > 0, & U_0(y) < U_\theta(y) \text{ and } y \geq R, \\ < 0, & U_0(y) > U_\theta(y) \text{ and } y \geq R. \end{cases}$$

For $y < -R$, we can prove similarly. □

LEMMA 13. *Let $\Gamma(x, y)$ be a function defined in (5.6). Then the following inequalities hold for small $\theta > 0$:*

$$\Gamma(x, y) \begin{cases} > 0, & x > 0, \ y > 0, \\ < 0, & x > 0, \ y < 0, \end{cases} \tag{5.11}$$

$$\int_{\mathbb{R}} |\Gamma(x, y) B(x)| \, dx \leq \frac{4\theta\sqrt{\gamma}}{a\sqrt{\mu_0}} \int_{\mathbb{R}} |x| \, B(x) \, dx \quad \text{for all } y \in \mathbb{R}, \tag{5.12}$$

*where $B(x)$ is defined in (1.14).*

PROOF. It is obvious that (5.11) holds.

We note that for any $c, d > 0$,

$$\left| e^{-c} - e^{-d} \right| \leq |d - c| + \frac{1}{2} |d - c|^2$$

holds. Noting (5.7), we have

$$\begin{aligned} |\Gamma(x, y)| &\leq \left| G_d \left( \frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}} x, \frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}} y \right) - G_d \left( \frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}} x, -\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}} y \right) \right| \\ &\leq \frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}} \big| \, |x - y| - |x + y| \, \big| + \frac{1}{2} \frac{\theta^2 \gamma}{a^2 \mu_0} \left( |x - y| - |x + y| \right)^2 \\ &\leq \frac{2\theta\sqrt{\gamma}}{a\sqrt{\mu_0}} |x| + \frac{2\theta^2\gamma}{a^2\mu_0} |x|^2 . \end{aligned}$$

Thus we see that

$$\begin{aligned} \int_{\mathbb{R}} |\Gamma(x, y) B(x)| \, dx &\leq \frac{2\theta\sqrt{\gamma}}{a\sqrt{\mu_0}} \int_{\mathbb{R}} |x B(x)| \, dx + \frac{2\theta^2\gamma}{a^2\mu_0} \int_{\mathbb{R}} \left| x^2 B(x) \right| \, dx \\ &\leq \frac{4\theta\sqrt{\gamma}}{a\sqrt{\mu_0}} \int_{\mathbb{R}} |x| \, B(x) \, dx \end{aligned}$$

for sufficiently small $\theta$. Thus we conclude the statement. □

### 5.2. Key lemmas on the behavior of $U_\theta$

In this subsection, we prove some lemmas, which reveal the dependency of $U_\theta$ on $\theta$ and $\gamma$.

The next lemma gives the uniform estimate of $V_\theta - U_\theta/\gamma$. Moreover, this lemma is used in the proof of Lemma 15.

LEMMA 14. *Assume $\theta^2 \ll 1/\gamma \ll \theta$. Let $\psi_\theta$ be a minimizer of (1.9) or (1.10), $u_\theta = \hat{u} + \psi_\theta$ and $(U_\theta, V_\theta)$ be defined in (5.1) and (5.2). Then there exists a positive constant $C$ such that*

$$\left\| V_\theta - \frac{U_\theta}{\gamma} \right\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\gamma^{3/4}}$$

PROOF. Let $J_\theta^{(i)}(\psi)$ $(i = 1, 2, \ldots, 5)$ be functionals defined in (3.5) – (3.9). Then we can see that

$$J_\theta^{(1)}(\psi_\theta) + J_\theta^{(2)}(\psi_\theta) \geq a^3\sqrt{\mu_0}c_*\theta.$$

Hence we obtain

$$J_\theta^{(i)}(\psi_\theta) \leq \frac{a}{2\gamma}\int_{\mathbb{R}}(1-\mu(x))\,dx + \frac{a^3\sqrt{\mu_0}}{2\theta\gamma^2}A + o\left(\frac{1}{\theta\gamma^2}\right) \qquad (i = 3, 4)$$

from Proposition 3 and the positivity of $J_\theta^{(i)}(\psi)$ $(i = 3, 4, 5)$.

Now we shall rewrite $J^{(i)}(\psi_\theta)$ $(i = 3, 4)$ with $U_\theta$ and $V_\theta$:

$$\begin{aligned} J_\theta^{(3)}(\psi_\theta) =& \frac{1}{2}\int_{\mathbb{R}} \left( v_\theta'(x) - \frac{u_\theta'(x)}{\gamma} \right)^2 dx, \\ =& \frac{a^2\mu_0}{2\theta^2}\int_{\mathbb{R}} \left\{ aV_\theta'\left(\frac{a\sqrt{\mu_0}(x-x_\theta)}{\theta}\right) - \frac{1}{\gamma}aU_\theta'\left(\frac{a\sqrt{\mu_0}(x-x_\theta)}{\theta}\right) \right\}^2 dx \\ =& \frac{a^3\sqrt{\mu_0}}{2\theta}\int_{\mathbb{R}} \left( V_\theta'(y) - \frac{U_\theta'(y)}{\gamma} \right)^2 dy, \end{aligned}$$

$$\begin{aligned} J_\theta^{(4)}(\psi_\theta) =& \frac{\gamma}{2}\int_{\mathbb{R}} \left( v_\theta(x) - \frac{u_\theta(x)}{\gamma} \right)^2 dx, \\ =& \frac{\gamma}{2}\int_{\mathbb{R}} \left\{ aV_\theta\left(\frac{a\sqrt{\mu_0}(x-x_\theta)}{\theta}\right) - \frac{1}{\gamma}aU_\theta\left(\frac{a\sqrt{\mu_0}(x-x_\theta)}{\theta}\right) \right\}^2 dx \\ =& \frac{a\gamma\theta}{2\sqrt{\mu_0}}\int_{\mathbb{R}} \left( V_\theta(y) - \frac{U_\theta(y)}{\gamma} \right)^2 dy, \end{aligned}$$

As a consequence, we obtain the following inequalities:

$$\int_{\mathbb{R}} \left( V_\theta'(y) - \frac{U_\theta'(y)}{\gamma} \right)^2 dy \le \frac{2\theta}{a^3\sqrt{\mu_0}} \cdot \frac{a}{2\gamma} M_1 \le M_2 \frac{\theta}{\gamma},$$
$$\int_{\mathbb{R}} \left( V_\theta(y) - \frac{U_\theta(y)}{\gamma} \right)^2 dy \le \frac{2\sqrt{\mu_0}}{a\gamma\theta} \cdot \frac{a}{2\gamma} M_1 \le \frac{M_3}{\theta\gamma^2},$$

where $M_i$ $(i = 1, 2, 3)$ are positive constants. Therefore, we have

$$\begin{aligned}\left\| V_\theta - \frac{U_\theta}{\gamma} \right\|_{L^\infty(\mathbb{R})} &\le \left\| V_\theta' - \frac{U_\theta'}{\gamma} \right\|_{L^2(\mathbb{R})}^{1/2} \cdot \left\| V_\theta - \frac{U_\theta}{\gamma} \right\|_{L^2(\mathbb{R})}^{1/2} \\ &\le M_2^{1/4} \cdot M_3^{1/4} \cdot \frac{\theta^{1/4}}{\gamma^{1/4}} \cdot \frac{1}{\gamma^{1/2}} \cdot \frac{1}{\theta^{1/4}} \le \frac{C}{\gamma^{3/4}}.\end{aligned}$$

Here we used the interpolation inequality

$$\|u\|_{L^\infty(\mathbb{R})} \le \|u'\|_{L^2(\mathbb{R})}^{1/2} \|u\|_{L^2(\mathbb{R})}^{1/2} \quad \text{for any } u \in H^1(\mathbb{R}).$$

Thus we conclude the statement. □

The next lemma shows the behavior of $U_\theta(y)$ as $y \to \pm\infty$. This lemma is used in the proof of Lemma 17.

LEMMA 15. *Assume $\theta^2 \ll 1/\gamma \ll \theta$. Let $\psi_\theta$ be a minimizer of (1.9) or (1.10), $u_\theta = \hat{u} + \psi_\theta$ and $U_\theta$ be defined in (5.1). There exists positive constants $C$ and $\delta_1$ such that*

$$|U_\theta(y) - 1| \le \frac{C}{\gamma^{3/4}} + Ce^{-\delta_1 y} \quad \textit{for all } y \ge 0, \tag{5.13}$$

$$|U_\theta(y) + 1| \le \frac{C}{\gamma^{3/4}} + Ce^{\delta_1 y} \quad \textit{for all } y \le 0. \tag{5.14}$$

PROOF. It suffices to show (5.13). Fix $y \ge 0$. We then derive the equation $U_\theta$ should satisfy.

$$\begin{aligned}-U_\theta''(y) =& -\frac{\theta^2}{a^3\mu_0} u_\theta'' \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right) \\ =& \frac{\theta^2}{a^3\mu_0 d} \left[ \mu \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right) f \left( u_\theta \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right) \right) \right. \\ & \left. - v_\theta \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right) \right]\end{aligned}$$

$$
\begin{aligned}
&=\frac{\theta^2}{a^3\mu_0 d}\left[\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu_0}}\right) f\left(aU_\theta(y)\right)-aV_\theta(y)\right]\\
&=\frac{\theta^2}{a^3\mu_0 d}\left[\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu_0}}\right)\left(aU_\theta(y)-(aU_\theta(y))^3\right)\right.\\
&\qquad\qquad\left.-a\left(V_\theta(y)-\frac{U_\theta(y)}{\gamma}\right)-\frac{a}{\gamma}U_\theta(y)\right].
\end{aligned}
$$

We rewrite the right hand side with the relation $a-a^3=a/\gamma$:

$$
\begin{aligned}
\text{(r.h.s.)}=\frac{\theta^2}{a^3\mu_0 d}\Bigg[&\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu_0}}\right)\left(a^3U_\theta-a^3U_\theta^3\right)-a\left(V_\theta-\frac{U_\theta}{\gamma}\right)\\
&-\left(1-\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu_0}}\right)\right)\frac{a}{\gamma}U_\theta\Bigg].
\end{aligned}
$$

Hence we obtain the following equation:

$$
\begin{aligned}
-\frac{d\mu_0}{\theta^2}U_\theta''=\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu_0}}\right)\left(U_\theta-U_\theta^3\right)-\frac{1}{a^2}\left(V_\theta-\frac{U_\theta}{\gamma}\right)\\
-\frac{1}{a^2\gamma}\left(1-\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu_0}}\right)\right)U_\theta.
\end{aligned}
$$

We set $\xi_\theta(y)=1-U_\theta(y)$. From (4.2), for $0<\delta_2<1$, there exists $R_0>0$ independent of $\theta>0$ such that

$$U_\theta(y)\geq\delta_2 \quad \text{for all } y\geq R_0.$$

We define $c_\theta(y)$ and $g_\theta(y)$ as follows:

$$
\begin{aligned}
c_\theta(y)&=\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu}}\right)U_\theta(y)(1+U_\theta(y)),\\
g_\theta(y)&=-\frac{1}{a^2}\left(V_\theta(y)-\frac{U_\theta(y)}{\gamma}\right)-\frac{1}{\gamma a^2}\left(1-\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu_0}}\right)\right)U_\theta(y).
\end{aligned}
$$

Then we can see

$$c_\theta(y)\geq\mu_0\delta_2 \quad \text{for all } \ y\geq R_0. \tag{5.15}$$

Moreover we can see

$$\|g_\theta\|_{L^\infty(\mathbb{R})}\leq\frac{C}{\gamma^{3/4}}$$

from Lemma 14 and the boundedness of $U_\theta$. The function $\xi_\theta$ satisfies

$$-\frac{d\mu_0}{\theta^2}\xi''_\theta(y) = c_\theta(y)\xi_\theta(y) + g_\theta(y).$$

We recall that from Kato's inequality [1], for all $u \in H^1_{loc}(\mathbb{R})$,

$$(|u|)'' \geq u'' \operatorname{sgn}(u)$$

holds in $H^1$ sense. Hence we have

$$\begin{aligned}\frac{d\mu_0}{\theta^2}(|\xi_\theta|)'' &\geq \frac{d\mu_0}{\theta^2}\xi''_\theta(y) \operatorname{sgn}(\xi_\theta(y))\\ &= c_\theta\,|\xi_\theta(y)| + g_\theta(y) \operatorname{sgn}(\xi_\theta(y))\\ &\geq c_\theta\,|\xi_\theta(y)| - \frac{C}{\gamma^{3/4}}.\end{aligned}$$

Noting (5.15) and $d/\theta^2 = 1 + o(1)$, we obtain

$$\begin{cases} -(|\xi_\theta(y)|)'' + \dfrac{\delta_2}{2}\,|\xi_\theta(y)| \leq \dfrac{2C}{\gamma^{3/4}}, & y \geq R_0,\\ |\xi_\theta(R_0)| \leq 1, & \\ |\xi_\theta(y)| \to 0, & y \to \infty.\end{cases}$$

On the other hand, we note that for any constant $C' > 0$,

$$u(y) = \frac{4C}{\delta_2\gamma^{3/4}} + C'e^{-y\sqrt{\delta_2/2}}$$

satisfies

$$-u''(y) + \frac{\delta_2}{2}u(y) = \frac{2C}{\gamma^{3/4}}.$$

Take $C' > 0$ large enough so that $C'e^{-\sqrt{\delta_2/2}R_0} \geq 1$. Then we have $u(R_0) \geq 1$. Put $v(y) = |\xi(y)| - u(y)$. Then $v(y)$ satisfies

$$-v''(y) + \frac{\delta_2}{2}v(y) \leq 0, \quad \text{for all } y > R_0$$

in $H^1$ sense and $v(R_0) \leq 0$, $v(y) \to -4C/(\delta_2\gamma^{3/4}) < 0$ $(y \to \infty)$. Hence, by using the weak maximum principle, we have

$$|\xi_\theta(y)| \leq u(y) = \frac{4C}{\delta_2\gamma^{3/4}} + C'e^{-y\sqrt{\delta_2/2}} \quad \text{for all } \ y > R_0.$$

Moreover, since $\xi_\theta(y) = 1 - U_\theta(y)$ is uniformly bounded in $[0, R_0]$, there exists a constant $C'' > 0$ such that

$$|\xi_\theta(y)| \le \frac{C''}{\gamma^{3/4}} + C'' e^{-y\sqrt{\delta_2/2}} \quad \text{for all } \ y > 0.$$

Thus we conclude the statement. □

Since $U_\theta \to U_0$ in $C^1_{loc}(\mathbb{R})$ as $\theta \to 0$, we can see qualitatively that the measures of $\{U_\theta(y) \ge 1\}$ and $\{U_\theta(y) \le -1\}$ tend to zero as $\theta \to 0$. The next lemma gives a quantitative estimate for the measures of $\{U_\theta(y) \ge 1\}$ and $\{U_\theta(y) \le -1\}$. Moreover, this lemma is used in the proof of Lemma 18.

Lemma 16.

*(1) Assume $\theta^2 \ll 1/\gamma \ll \theta$. Let $\psi_\theta$ be a minimizer of (1.9), $u_\theta = \hat{u} + \psi_\theta$ and $U_\theta$ be defined in (5.1). Then there exists a positive constant $C$ such that*

$$\int_0^\infty (1 - U_\theta(y))^2 \chi^\theta(y)\, dy \le \frac{C}{\theta^2\gamma^2}, \tag{5.16}$$

$$\int_{-\infty}^0 (1 + U_\theta(y))^2 \chi_\theta(y)\, dy \le \frac{C}{\theta^2\gamma^2}, \tag{5.17}$$

*where $\chi^\theta(y) = \chi_{\{U_\theta(y)\ge 1\}}(y)$ and $\chi_\theta(y) = \chi_{\{U_\theta(y)\le -1\}}(y)$.*

*(2) Assume $\theta^2 \ll 1/\gamma \ll \theta$. Let $\psi_\theta$ be a minimizer of (1.10), $u_\theta = \hat{u} + \psi_\theta$ and $U_\theta$ be defined in (5.1). Then there exists a positive constant $C$ such that*

$$\int_0^\infty (1 - U_\theta(y))^2 \chi^\theta(y)\, dy \le \frac{C}{\theta\gamma}, \tag{5.18}$$

$$\int_{-\infty}^0 (1 + U_\theta(y))^2 \chi_\theta(y)\, dy \le \frac{C}{\theta\gamma}. \tag{5.19}$$

Proof. (1) It suffices to show (5.16). Since $\bar{J}_\theta(\psi_\theta) = \bar{\sigma}(\theta, \gamma)$, we see that

$$a^3\theta\sqrt{\mu_0}\left(\int_{\mathbb{R}}\left[\frac{1}{2}\left|U_\theta'(y)\right|^2 + \frac{\mu(x_\theta + \theta y/(a\sqrt{\mu_0}))}{4\mu_0}\left(U_\theta(y)^2 - 1\right)^2\right] dy\right)$$
$$\le \bar{\sigma}(\theta, \gamma)$$

holds. Combing the above inequality with Proposition 2, we obtain

$$\int_{\mathbb{R}} \left[\frac{1}{2}\left|U_\theta'(y)\right|^2 + \frac{1}{4}\left(U_\theta(y)^2 - 1\right)^2\right] dy \le c_* + \frac{C}{\theta^2\gamma^2}. \tag{5.20}$$

We set $\bar{U}_\theta(y)$ as follows:

$$\bar{U}_\theta(y) = \begin{cases} U_\theta(y), & U_\theta(x) \in (-1,1), \\ 1, & U_\theta(x) \ge 1, \\ -1, & U_\theta(x) \le -1. \end{cases}$$

Then we have

$$\int_{\mathbb{R}} \left|\bar{U}_\theta'(y)\right|^2 dy \le \int_{\mathbb{R}} \left|U_\theta'(y)\right|^2 dy \tag{5.21}$$

and

$$\begin{aligned} &\int_{\mathbb{R}} \left(U_\theta(y) - 1\right)^2 dy \\ &= \int_{\mathbb{R}} \left(\bar{U}_\theta(y) - 1\right)^2 dy + \int_{\{U_\theta(y)\ge 1\}} \left(U_\theta(y) - 1\right)^2 dy \\ &\quad + \int_{\{U_\theta(y)\le -1\}} \left(U_\theta(y) - 1\right)^2 dy. \end{aligned} \tag{5.22}$$

Moreover, it is easy to check that

$$\int_{\mathbb{R}} \left[\frac{1}{2}\left|\bar{U}_\theta'(y)\right|^2 + \frac{1}{4}\left(\bar{U}_\theta(y)^2 - 1\right)^2\right] dy \ge c_* \tag{5.23}$$

from Lemma 1. Thus combining (5.20) – (5.23), we can see

$$\frac{1}{4}\int_{\{U_\theta(y)\ge 1\}} \left(U_\theta(y) - 1\right)^2 dy + \frac{1}{4}\int_{\{U_\theta(y)\le -1\}} \left(U_\theta(y) - 1\right)^2 dy \le \frac{C}{\theta^2\gamma^2}$$

Remarking (5.3), we find

$$\begin{aligned} \frac{1}{4}\int_{\{U_\theta(y)\ge 1\}} \left(U_\theta(y)^2 - 1\right)^2 dy &\ge \int_0^\infty \left(U_\theta(y) - 1\right)^2 \chi^\theta(y)\, dy, \\ \frac{1}{4}\int_{\{U_\theta(y)\le -1\}} \left(U_\theta(y)^2 - 1\right)^2 dy &\ge \int_{-\infty}^0 \left(U_\theta(y) + 1\right)^2 \chi_\theta(y)\, dy. \end{aligned}$$

Hence we conclude the statement.

(2) We note that

$$\int_{\mathbb{R}} \left[ \frac{1}{2} \left| U_\theta'(y) \right|^2 + \frac{1}{4} \left( U_\theta(y)^2 - 1 \right)^2 \right] dy \le c_* + \frac{C}{\theta\gamma}$$

follows from Proposition 2. By repeating the same argument, we can prove the statement. □

### 5.3. Auxiliary lemmas

In this subsection, we give some lemmas to reduce the amount of calculation for the proof of Theorem 5. The next lemma is used in the proof of in Lemma 22.

LEMMA 17. *Assume $\theta^2 \ll 1/\gamma \ll \theta$. Let $\psi_\theta$ be a minimizer of (1.9) or (1.10), $u_\theta = \hat{u} + \psi_\theta$ and $U_\theta$ be defined in (5.1). Then the following estimate holds:*

$$\frac{a\theta}{2\gamma\sqrt{\mu_0}} \int_{\mathbb{R}} \left\{ 1 - \mu \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right) \right\} \left( U_\theta(y)^2 - 1 \right) dy = o\left( \frac{1}{\theta\gamma^2} \right).$$

PROOF. From (5.13), we see

$$\begin{aligned}
&\int_0^\infty \left\{ 1 - \mu \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right) \right\} \left| U_\theta(y) - 1 \right| dy \\
&\le \int_0^\infty \left\{ 1 - \mu \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right) \right\} \frac{C}{\gamma^{3/4}} dy \\
&\quad + \int_0^\infty \left\{ 1 - \mu \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right) \right\} C e^{-\delta_1 y} dy \\
&\le \frac{C}{\gamma^{3/4}\theta} + \frac{C}{\delta_1}.
\end{aligned}$$

Similarly we can check that

$$\int_{-\infty}^0 \left\{ 1 - \mu \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right) \right\} \left| U_\theta(y) + 1 \right| dy \le \frac{C}{\gamma^{3/4}\theta} + \frac{C}{\delta_1}.$$

Thus we estimate the left hand side as follows:

$$\begin{aligned}&\frac{a\theta}{2\gamma\sqrt{\mu_0}}\int_{\mathbb{R}}\left\{1-\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu_0}}\right)\right\}(U_\theta(y)^2-1)\,dy\\&\le\frac{a\theta}{2\gamma\sqrt{\mu_0}}\left(\frac{C}{\gamma^{3/4}\theta}+\frac{C}{\delta_1}\right)\\&=\frac{C}{\gamma^{7/4}}+\frac{C\theta}{\delta_1\gamma}.\end{aligned}$$

We can easily to check $\theta/\gamma = o(1/(\theta\gamma^2))$ and $1/\gamma^{7/4} = o(1/(\theta\gamma^2))$. □

The next lemma is used in the proof of in Lemma 20.

LEMMA 18.

*(1) Assume* $\theta^2 \ll 1/\gamma \ll \theta^{4/3}$. *Let* $\psi_\theta$ *be a minimizer of (1.9),* $u_\theta = \hat{u}+\psi_\theta$ *and* $U_\theta$ *be defined in (5.1). Then the following estimates hold:*

$$\int_R^\infty\int_0^\infty \Gamma(x,y)B(x)\,|1-U_\theta(y)|\,\chi^\theta(y)\,dxdy = o(\theta\sqrt{\gamma}), \tag{5.24}$$

$$\int_{-\infty}^{-R}\int_0^\infty \Gamma(x,y)B(x)\,|1+U_\theta(y)|\,\chi_\theta(y)\,dxdy = o(\theta\sqrt{\gamma}), \tag{5.25}$$

*where* $\chi^\theta$ *and* $\chi_\theta$ *are defined in Lemma 16 and* $R$ *is the constant defined in (5.8).*

*(2) Assume* $\theta^2 \ll 1/\gamma \ll \theta^{3/2}$. *Let* $\psi_\theta$ *be a minimizer of (1.10),* $u_\theta = \hat{u}+\psi_\theta$ *and* $U_\theta$ *be defined in (5.1). Then (5.24) and (5.25) hold:*

PROOF. We prove only (5.24) since we can prove (5.25) by the almost same argument.

We note that for $0 < s < t$, the following inequality holds:

$$0 < e^{-s} - e^{-t} \le e^{-s}(t-s).$$

Thus for $x > 0$ and $y > 0$, we can calculate as follows:

$$\begin{aligned}&0<\Gamma(x,y)\\&\quad\le\exp\left\{-\sqrt{1-\frac{1}{d\gamma^2}}\cdot\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\,|x-y|\right\}\\&\qquad-\exp\left\{-\sqrt{1-\frac{1}{d\gamma^2}}\cdot\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\,|x+y|\right\}\end{aligned}$$

$$\leq \frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\big|\,|x+y|-|x-y|\,\big|\exp\left\{-\sqrt{1-\frac{1}{d\gamma^2}}\cdot\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\,|x-y|\right\}$$
$$\leq C\theta\sqrt{\gamma}\,|x|\exp\left\{-\sqrt{1-\frac{1}{d\gamma^2}}\cdot\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\,|x-y|\right\}.$$

We note that we have used (5.7) for the above calculation. Now we set

$$K(x,y)=\exp\left\{-\sqrt{1-\frac{1}{d\gamma^2}}\cdot\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\,|x-y|\right\}\chi_{[0,\infty)}(x)\chi_{[R,\infty)}(y).$$

Then by (5.7) we can easily check that

$$\int_{\mathbb{R}}K(x,y)\,dx\leq\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}}\cdot\frac{1}{\sqrt{1-1/(d\gamma^2)}}\int_{\mathbb{R}}e^{-|z|}\,dz\leq\frac{C}{\theta\sqrt{\gamma}}.$$

We can also check that

$$\int_{\mathbb{R}}K(x,y)\,dy\leq\frac{C}{\theta\sqrt{\gamma}}.$$

From the Schur lemma [9], we can see

$$\int_{\mathbb{R}}\int_{\mathbb{R}}K(x,y)\,|x|\,B(x)\,|1-U_\theta(y)|\,\chi^\theta(y)\,dxdy$$
$$\leq\frac{C}{\theta\sqrt{\gamma}}\big\|\,|x|\,B(x)\big\|_{L^2(\mathbb{R})}\cdot\Big\||1-U_\theta(y)|\,\chi^\theta(y)\Big\|_{L^2([0,\infty))}.$$

From (5.16), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}K(x,y)\,|x|\,B(x)\,|1-U_\theta(y)|\,\chi^\theta(y)\,dxdy\leq\frac{C}{\theta\sqrt{\gamma}}\cdot\frac{1}{\theta\gamma}=\frac{C}{\theta^2\gamma^{3/2}}.$$

Note that $1/(\theta^2\gamma^{3/2})=o(1)$ by the assumption $1/\gamma=o(\theta^{4/3})$. Thus we obtain

$$\int_R^\infty\int_0^\infty\Gamma(x,y)B(x)\,|1-U_\theta(y)|\,\chi^\theta(y)\,dxdy$$
$$\leq C\theta\sqrt{\gamma}\int_{\mathbb{R}}\int_{\mathbb{R}}K(x,y)\,|x|\,B(x)\,|1-U_\theta(y)|\,\chi^\theta(y)\,dxdy$$
$$\leq C\theta\sqrt{\gamma}\cdot\frac{1}{\theta^2\gamma^{3/2}}=o\left(\theta\sqrt{\gamma}\right).$$

Thus we have proved (5.24).

(2) We note that

$$\left\| (U_\theta - 1)\, \chi^\theta \right\|_{L^2([0,\infty))} < \frac{C}{\theta^{1/2}\gamma^{1/2}}$$

follows from (5.18). Then we can check that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} K(x,y)\, |x|\, B(x)\, |1 - U_\theta(y)|\, \chi^\theta(y)\, dxdy \le \frac{C}{\theta\sqrt{\gamma}} \cdot \frac{1}{\theta^{1/2}\gamma^{1/2}} = \frac{C}{\theta^{3/2}\gamma}$$

by repeating the same argument. Thus we obtain (5.24). □

Lemmas 19 and 20 are used in Lemma 24

LEMMA 19. *Assume $\theta^2 \ll 1/\gamma \ll \theta$. Let $\psi_\theta$ be a minimizer of (1.9) or (1.10), $u_\theta = \hat{u} + \psi_\theta$ and $U_\theta$ be defined in (5.1). Moreover, let $P_1$ be*

$$P_1 = -\frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \int_{-R}^{R} \int_0^\infty \Gamma(x,y) B(x) \mu \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right) (B(y) - B_\theta(y))\, dxdy,$$

*where $R > 0$ is the constant defined in (5.8). Then $P_1 = o(1/(\theta\gamma^2))$ holds.*

PROOF. From (5.12), we can see

$$|P_1| \le \frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \cdot \frac{4\theta\sqrt{\gamma}}{a\sqrt{\mu_0}} \int_0^\infty |x|\, B(x)\, dx \cdot \int_{-R}^{R} (B(y) - B_\theta(y))\, dy.$$

We easily see

$$\int_{-R}^{R} (B(y) - B_\theta(y))\, dy = o(1)$$

from $C^1_{loc}$ convergence and (5.10). Thus we conclude $|P_1| = o(1/(\theta\gamma^2))$. □

LEMMA 20. *Define $P_2$ and $P_3$ as follows:*

$$\begin{aligned} P_2 = &-\frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \int_R^\infty \int_0^\infty \Gamma(x,y) B(x) \mu \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right) \\ &\qquad \times (B(y) - B_\theta(y))\, dxdy, \\ P_3 = &-\frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \int_{-\infty}^{-R} \int_0^\infty \Gamma(x,y) B(x) \mu \left( x_\theta + \frac{\theta y}{a\sqrt{\mu_0}} \right) \\ &\qquad \times (B(y) - B_\theta(y))\, dxdy, \end{aligned}$$

*where $R > 0$ is the constant defined in (5.8).*

*(1) Assume $\theta^2 \ll 1/\gamma \ll \theta^{4/3}$. Let $\psi_\theta$ be a minimizer of (1.9), $u_\theta = \hat{u} + \psi_\theta$ and $U_\theta$ be defined in (5.1). Then $P_2 = o(1/(\theta\gamma^2))$ and $P_3 = o(1/(\theta\gamma^2))$ hold.*

*(2) Assume $\theta^2 \ll 1/\gamma \ll \theta^{3/2}$. Let $\psi_\theta$ be a minimizer of (1.10), $u_\theta = \hat{u} + \psi_\theta$ and $U_\theta$ be defined in (5.1). Then $P_2 = o(1/(\theta\gamma^2))$ and $P_3 = o(1/(\theta\gamma^2))$ hold.*

Proof. (1) We shall prove only $P_2 = o(1/(\theta\gamma^2))$.
We note that from the definition of $B(x)$, (5.9) and (5.11),

$$B(x) > 0, \quad x > 0,$$

$$B(y) - B_\theta(y) = \begin{cases} > 0, & U_0(y) < U_\theta(y) \text{ and } |y| \geq R, \\ < 0, & U_0(y) > U_\theta(y) \text{ and } |y| \geq R, \end{cases}$$

and

$$\Gamma(x, y) > 0, \quad x > 0, \ y > 0.$$

hold. This implies that we may assume

$$U_\theta(y) > U_0(y) \tag{5.26}$$

for the lower estimate of $P_2$. Now we set $P_4$ and $P_5$ as follows:

$$\begin{aligned} P_4 = \frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \int_R^\infty \int_0^\infty \Gamma(x,y) B(x) \mu\left(x_\theta + \frac{\theta y}{a\sqrt{\mu_0}}\right) \\ \times (B(y) - B_\theta(y))\, \chi_{\{0 \leq U_\theta(y) \leq 1\}}(y)\, dxdy, \\ P_5 = \frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \int_R^\infty \int_0^\infty \Gamma(x,y) B(x) \mu\left(x_\theta + \frac{\theta y}{a\sqrt{\mu_0}}\right) \\ \times (B(y) - B_\theta(y))\, \chi^\theta(y)\, dxdy. \end{aligned}$$

We shall estimates $P_4$ and $P_5$. From (5.10) and (5.12), we estimate $P_4$ as follows:

$$\begin{aligned} P_4 \leq \frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \cdot \frac{4\theta\sqrt{\gamma}}{a\sqrt{\mu_0}} \int_0^\infty |x|\, B(x) dx \\ \cdot \int_R^\infty C\, |U_0(y) - U_\theta(y)|\, \chi_{\{0 \leq U_\theta(y) \leq 1\}}(y)\, dy. \end{aligned}$$

Moreover, we can see that if $0 \leq U_\theta(y) \leq 1$, then

$$|U_0(y) - U_\theta(y)| = 1 - U_0(y) - (1 - U_\theta(y)) \leq 2(1 - U_0(y))$$

holds from (5.26). Hence we have

$$\int_R^\infty |U_0(y) - U_\theta(y)| \, \chi_{\{0 \leq U_\theta(y) \leq 1\}}(y) \, dy = o(1)$$

by the dominated convergence theorem. It follows that

$$P_4 = o\left(\frac{1}{\theta\gamma^2}\right).$$

We next calculate $P_5$ as follows:

$$\begin{aligned}
P_5 \leq & \frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \int_R^\infty \int_0^\infty \Gamma(x,y)B(x)\left(B(y) - B_\theta(y)\right)\chi^\theta(y)\,dxdy \\
\leq & \frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \int_R^\infty \int_0^\infty \Gamma(x,y)B(x)C\,|U_0(y) - U_\theta(y)|\,\chi^\theta(y)\,dxdy \\
\leq & C\frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \int_R^\infty \int_0^\infty \Gamma(x,y)B(x)\,|1 - U_\theta(y)|\,\chi^\theta(y)\,dxdy \\
& + C\frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \int_R^\infty \int_0^\infty \Gamma(x,y)B(x)\,|1 - U_0(y)|\,\chi^\theta(y)\,dxdy.
\end{aligned}$$

We note we used the relations (5.10) and $|U_0(y) - U_\theta(y)| \leq |1 - U_\theta(y)| + |1 - U_0(y)|$ for the above calculation. For the first term, we can see

$$\begin{aligned}
\frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \int_R^\infty \int_0^\infty \Gamma(x,y)B(x)\,|1 - U_\theta(y)|\,\chi^\theta(y)\,dxdy &= \frac{a^4\sqrt{\gamma}\theta^2}{d^2\gamma^3} \cdot o(\theta\sqrt{\gamma}) \\
&= o\left(\frac{1}{\theta\gamma^2}\right)
\end{aligned}$$

from (1) of Lemma 18. For the second term, we can readily see that

$$\begin{aligned}
& \int_R^\infty \int_0^\infty \Gamma(x,y)B(x)\,|1 - U_0(y)|\,\chi^\theta(y)\,dxdy \\
& < C\frac{4\theta\sqrt{\gamma}}{a\sqrt{\mu_0}} \int_0^\infty |x|\,B(x)\,dx \cdot \int_R^\infty (1 - U_0(y))\chi^\theta(y)\,dy \\
& = O(\theta\sqrt{\gamma}) \cdot o(1) = o(\theta\sqrt{\gamma})
\end{aligned}$$

from (5.12) and the dominated convergence theorem. Thus we obtain $P_5 = o(1/(\theta\gamma^2))$. Since $P_2 = -P_4 - P_5$ holds from (5.3), we conclude the statement of (1). By repeating the same argument with (2) of Lemma 18, we can prove (2). □

### 5.4. Proof of Theorem 5

In this section, we derive the lower estimate. To show this, we calculate each term of $J_\theta(\psi_\theta)$, where $\psi_\theta$ is a minimizer of (1.10). For reader's convenience, we recall $J_\theta^{(i)}(\psi)$ $(i = 1, 2, \ldots, 5)$:

$$J_\theta^{(1)}(\psi) = \int_{\mathbb{R}} \frac{\theta^2}{2} \left|u'\right|^2 dx, \tag{3.5}$$

$$J_\theta^{(2)}(\psi) = \int_{\mathbb{R}} \frac{\mu(x)}{4}(u^2 - a^2)^2 \, dx, \tag{3.6}$$

$$J_\theta^{(3)}(\psi) = \int_{\mathbb{R}} \frac{1}{2} \left(v' - \frac{u'}{\gamma}\right)^2 dx, \tag{3.7}$$

$$J_\theta^{(4)}(\psi) = \int_{\mathbb{R}} \frac{\gamma}{2} \left(v - \frac{u}{\gamma}\right)^2 dx, \tag{3.8}$$

$$J_\theta^{(5)}(\psi) = \int_{\mathbb{R}} \frac{1 - \mu(x)}{2\gamma} u^2 \, dx. \tag{3.9}$$

We begin with $J_\theta^{(1)}(\psi_\theta) + J_\theta^{(2)}(\psi_\theta)$. This lemma can be proved as in Lemma 2.

Lemma 21. *Assume $\theta^2 \ll 1/\gamma \ll \theta$. Let $\psi_\theta$ be a minimizer of (1.9) or (1.10). Then the following inequality holds:*

$$J_\theta^{(1)}(\psi_\theta) + J_\theta^{(2)}(\psi_\theta) \geq a^3 \sqrt{\mu_0} c_* \theta.$$

Next, we estimate $J^{(5)}(\psi_\theta)$.

Lemma 22. *Assume $\theta^2 \ll 1/\gamma \ll \theta$. Let $\psi_\theta$ be a minimizer of (1.10). Then the following estimate holds:*

$$J_\theta^{(5)}(\psi_\theta) = \frac{a^2}{2\gamma} \int_{\mathbb{R}} (1 - \mu(x)) \, dx + o\left(\frac{1}{\theta\gamma^2}\right).$$

PROOF. We transform $J_\theta^{(5)}(\psi_\theta)$ as follows:

$$J_\theta^{(5)}(\psi_\theta) = \frac{a^2}{2\gamma}\int_{\mathbb{R}} (1-\mu(x))\,dx - \frac{1}{2\gamma}\int_{\mathbb{R}} (1-\mu(x))\left(a^2 - u_\theta(x)^2\right)\,dx.$$

By changing variables, we have

$$\begin{aligned}&\left|\frac{1}{2\gamma}\int_{\mathbb{R}} (1-\mu(x))\left(a^2 - u_\theta(x)^2\right)\,dx\right|\\ &= \left|\frac{a^2}{2\gamma}\cdot\frac{\theta}{a\sqrt{\mu_0}}\int_{\mathbb{R}}\left(1-\mu\left(x_\theta + \frac{\theta y}{a\sqrt{\mu_0}}\right)\right)\left(1-U_\theta(y)^2\right)\,dx\right|.\end{aligned}$$

Then from Lemma 17, we obtain

$$\left|\frac{1}{2\gamma}\int_{\mathbb{R}} (1-\mu(x))\left(a^2 - u_\theta(x)^2\right)\,dx\right| = o\left(\frac{1}{\theta\gamma^2}\right)$$

Thus we complete the proof. □

Finally, we estimate $J_\theta^{(3)}(\psi_\theta) + J_\theta^{(4)}(\psi_\theta)$ in the Lemmas 23 – 25.

LEMMA 23. *Assume $\theta^2 \ll 1/\gamma \ll \theta$. Let $\psi_\theta$ be a minimizer of (1.9) or (1.10), $u_\theta = \hat{u} + \psi_\theta$ and $U_\theta$ be defined in (5.1). Moreover, define $\bar{H}(y)$, $\tilde{H}(y)$ as follows:*

$$\bar{H}(y) = -\frac{a^3\mu_0}{d\gamma^2} B\left(\frac{a\sqrt{\mu_0}y}{\theta\sqrt{\gamma}}\right), \tag{5.27}$$

$$\begin{aligned}\tilde{H}(y) =& \frac{a^3}{d\gamma^2}\left[\mu_0 B\left(\frac{a\sqrt{\mu_0}y}{\theta\sqrt{\gamma}}\right) - \mu\left(x_\theta + \frac{y}{\sqrt{\gamma}}\right)B_\theta\left(\frac{a\sqrt{\mu_0}y}{\theta\sqrt{\gamma}}\right)\right]\\ &+ \frac{a}{d\gamma^3}U_\theta\left(\frac{a\sqrt{\mu_0}y}{\theta\sqrt{\gamma}}\right)\left(1-\mu\left(x_\theta + \frac{y}{\sqrt{\gamma}}\right)\right),\end{aligned} \tag{5.28}$$

*where $B(x)$ and $B_\theta(x)$ are defined in (1.14) and (5.4). Then the following estimate holds:*

$$\begin{aligned}&J_\theta^{(3)}(\psi_\theta) + J_\theta^{(4)}(\psi_\theta)\\ &\geq \frac{\sqrt{\gamma}}{2}\iint_{\mathbb{R}^2} G_d(x,y)\bar{H}(x)\bar{H}(y)\,dxdy + \sqrt{\gamma}\iint_{\mathbb{R}^2} G_d(x,y)\bar{H}(x)\tilde{H}(y)\,dxdy.\end{aligned}$$

Proof. Let $\tilde{w}_\theta(x)$ be function defined as follows:

$$\tilde{w}_\theta(x) = v_\theta\left(x_\theta + \frac{x}{\sqrt{\gamma}}\right) - \frac{u_\theta\left(x_\theta + \frac{x}{\sqrt{\gamma}}\right)}{\gamma}.$$

Then we can show that $\tilde{w}_\theta$ satisfies

$$J_\theta^{(3)}(\psi_\theta) + J_\theta^{(4)}(\psi_\theta) = \frac{\sqrt{\gamma}}{2}\,\|\tilde{w}_\theta\|^2_{H^1(\mathbb{R})} \tag{5.29}$$

similarly as in Lemma 7.

Now we shall derive the equation for $\tilde{w}_\theta$. For simplicity, we write $\tilde{u}_\theta$, $\tilde{v}_\theta$ as follows:

$$\tilde{u}_\theta(x) = u_\theta\left(x_\theta + \frac{x}{\sqrt{\gamma}}\right), \quad \tilde{v}_\theta(x) = v_\theta\left(x_\theta + \frac{x}{\sqrt{\gamma}}\right).$$

Since $(u_\theta, v_\theta)$ satisfies

$$\begin{aligned} -\frac{u_\theta''(x)}{\gamma^2} &= \frac{\mu(x)}{d\gamma^2}\left(u_\theta(x) - u_\theta(x)^3\right) - \frac{v_\theta(x)}{d\gamma^2}, \\ -v_\theta''(x) &+ \gamma\left(v_\theta(x) - \frac{u_\theta(x)}{\gamma}\right) = 0, \end{aligned}$$

$(\tilde{u}_\theta, \tilde{v}_\theta)$ satisfies

$$\begin{aligned} -\frac{\tilde{u}_\theta''(x)}{\gamma} &= \frac{\mu\left(x_\theta + x/\sqrt{\gamma}\right)}{d\gamma^2}\left(\tilde{u}_\theta(x) - \tilde{u}_\theta(x)^3\right) - \frac{\tilde{v}_\theta(x)}{d\gamma^2}, \\ -\tilde{v}_\theta''(x) &+ \left(\tilde{v}_\theta(x) - \frac{\tilde{u}_\theta(x)}{\gamma}\right) = 0. \end{aligned}$$

Hence $\tilde{w}_\theta$ satisfies

$$-\tilde{w}_\theta''(x) + \left(1 - \frac{1}{d\gamma^2}\right)\tilde{w}_\theta(x) = -\frac{\mu(x_\theta + x/\sqrt{\gamma})}{d\gamma^2}\left(\tilde{u}_\theta(x) - \tilde{u}_\theta(x)^3\right) + \frac{\tilde{u}_\theta(x)}{d\gamma^3}.$$

With the relation $1 - a^2 = 1/\gamma$, we rewrite the right hand side as

$$\begin{aligned} (\text{r.h.s.}) &= -\frac{\mu(x_\theta + x/\sqrt{\gamma})}{d\gamma^2}\left(a^2\tilde{u}_\theta(x) - \tilde{u}_\theta(x)^3 + (1-a^2)\tilde{u}_\theta\right) + \frac{\tilde{u}_\theta(x)}{d\gamma^3} \\ &= -\frac{\mu(x_\theta + x/\sqrt{\gamma})}{d\gamma^2}\left(a^2\tilde{u}_\theta(x) - \tilde{u}_\theta(x)^3\right) \\ &\quad + \frac{\tilde{u}_\theta(x)}{d\gamma^3}\left(1 - \mu\left(x_\theta + \frac{x}{\sqrt{\gamma}}\right)\right). \end{aligned}$$

We note that we can see

$$a^2\tilde{u}_\theta(x) - \tilde{u}_\theta(x)^3 = a^3\left(U_\theta\left(\frac{a\sqrt{\mu_0}x}{\theta\sqrt{\gamma}}\right) - U_\theta\left(\frac{a\sqrt{\mu_0}x}{\theta\sqrt{\gamma}}\right)^3\right)$$
$$= a^3 B_\theta\left(\frac{a\sqrt{\mu_0}x}{\theta\sqrt{\gamma}}\right)$$

from the relation $\tilde{u}_\theta(x) = aU_\theta\left(a\sqrt{\mu_0}x/(\theta\sqrt{\gamma})\right)$. Thus we conclude that $\tilde{w}_\theta$ should satisfy

$$-\tilde{w}_\theta''(x) + \left(1 - \frac{1}{d\gamma^2}\right)\tilde{w}_\theta(x) = \bar{H}(x) + \tilde{H}(x). \tag{5.30}$$

It is easy to check $\bar{H},\ \tilde{H} \in L^2(\mathbb{R})$ and hence $\tilde{w}_\theta$ is represented as

$$\tilde{w}_\theta(x) = \int_{\mathbb{R}} G_d(x,y)\left(\bar{H}(y) + \tilde{H}(y)\right)dy,$$

where $G_d(x,y)$ is the Green function defined in (5.5). Moreover, multiplying (5.30) by $\tilde{w}_\theta$, we obtain

$$\int_{\mathbb{R}}\left[\left(\tilde{w}_\theta'\right)^2 + \left(1 - \frac{1}{d\gamma^2}\right)\tilde{w}_\theta^2\right]dx$$
$$= \iint_{\mathbb{R}} G_d(x,y)\left(\bar{H}(y) + \tilde{H}(y)\right)\left(\bar{H}(x) + \tilde{H}(x)\right)dydx. \tag{5.31}$$

Since we can check

$$\iint_{\mathbb{R}} G_d(x,y)\tilde{H}(x)\tilde{H}(y)\,dxdy \geq 0$$

similarly as in (2) of Lemma 4, we obtain

$$\text{(r.h.s)} \geq \iint_{\mathbb{R}^2} G_d(x,y)\bar{H}(x)\bar{H}(y)\,dydx$$
$$+ 2\iint_{\mathbb{R}^2} G_d(x,y)\bar{H}(x)\tilde{H}(y)\,dydx. \tag{5.32}$$

Combining (5.29) – (5.32), we find that

$$J_\theta^{(3)}(\psi_\theta) + J_\theta^{(4)}(\psi_\theta)$$
$$= \frac{\sqrt{\gamma}}{2}\|\tilde{w}_\theta\|_{H^1(\mathbb{R})}^2$$
$$\geq \frac{\sqrt{\gamma}}{2}\int_{\mathbb{R}}\left[\left(\tilde{w}_\theta'\right)^2 + \left(1 - \frac{1}{d\gamma^2}\right)\tilde{w}_\theta^2\right]dx$$

$$\geq \frac{\sqrt{\gamma}}{2} \iint_{\mathbb{R}^2} G_d(x,y)\bar{H}(x)\bar{H}(y)\,dydx + \sqrt{\gamma} \iint_{\mathbb{R}^2} G_d(x,y)\bar{H}(x)\tilde{H}(y)\,dydx.$$

Thus we conclude the statement. □

LEMMA 24. *Let $\psi_\theta$ is a minimizer of (1.9) or (1.10). Assume either (1) or (2):*

*(1) If $\psi_\theta$ is a minimizer of (1.9), then $\theta^2 \ll 1/\gamma \ll \theta^{4/3}$ holds.*

*(2) If $\psi_\theta$ is a minimizer of (1.10), then $\theta^2 \ll 1/\gamma \ll \theta^{3/2}$ holds.*

*Then the following estimate holds:*

$$\sqrt{\gamma} \iint_{\mathbb{R}^2} G_d(x,y)\bar{H}(x)\tilde{H}(y)\,dxdy \geq o(1/(\theta\gamma^2)).$$

PROOF. Since $\bar{H}(-x) = -\bar{H}(x)$ holds for any $x \in \mathbb{R}$ and $G_d(-x,y) = G_d(x,-y)$ holds for any $x, y \in \mathbb{R}$ , we can check that

$$\begin{aligned} &\sqrt{\gamma} \iint_{\mathbb{R}^2} G_d(x,y)\bar{H}(x)\tilde{H}(y)\,dxdy \\ &= \sqrt{\gamma} \int_{\mathbb{R}} \left[ \int_0^\infty \left(G_d(x,y) - G_d(x,-y)\right) \bar{H}(x)\,dx \right] \tilde{H}(y)\,dy \end{aligned} \tag{5.33}$$

similarly as in (1) of Lemma 4.

We now simplify (5.33). We transform $\tilde{H}(y)$ as follows:

$$\begin{aligned} \tilde{H}(y) =& \frac{a^3}{d\gamma^2} \left[ \left( \mu_0 - \mu\left(x_\theta + \frac{y}{\sqrt{\gamma}}\right)\right) B\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}} y\right) \right. \\ &\left. + \mu\left(x_\theta + \frac{y}{\sqrt{\gamma}}\right)\left( B\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}} y\right) - B_\theta\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}} y\right)\right)\right] \\ &+ \frac{a^3}{d\gamma^2} \cdot \frac{1}{a^2\gamma} U_\theta\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}} y\right)\left(1 - \mu\left(x_\theta + \frac{y}{\sqrt{\gamma}}\right)\right). \end{aligned}$$

We remark the following relations:

$$\bar{H}(x) = -\frac{a^3\mu_0}{d\gamma^2} \left\{ U_0\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}} x\right) - U_0\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}} x\right)^3 \right\} < 0 \quad \text{for all } x > 0,$$

$$\mu_0 - \mu\left(x_\theta + \frac{y}{\sqrt{\gamma}}\right) < 0 \quad \text{for all } y > 0,$$

$$G_d(x,y) - G_d(x,-y) \begin{cases} > 0 & \text{for all } \ (x,y) \in (0,\infty)\times(0,\infty), \\ < 0 & \text{for all } \ (x,y) \in (0,\infty)\times(-\infty,0), \end{cases}$$

$$B\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}}y\right) = U_0\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}}y\right) - U_0\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}}y\right)^3 \begin{cases} > 0 & \text{for all } \ y > 0, \\ < 0 & \text{for all } \ y < 0. \end{cases}$$

Thus we have

$$\int_{\mathbb{R}}\int_0^\infty \Bigg[ \{G_d(x,y) - G_d(x,-y)\}\bar{H}(x) \times \left(\mu_0 - \mu\left(x_\theta + \frac{y}{\sqrt{\gamma}}\right)\right) B\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}}y\right)\Bigg] dxdy > 0.$$

As a consequence, we estimate (5.33) as follows:

$$\sqrt{\gamma}\iint_{\mathbb{R}^2} G_d(x,y)\bar{H}(x)\tilde{H}(y)\,dxdy \geq \frac{a^3\sqrt{\gamma}}{d\gamma^2}\int_{\mathbb{R}}\int_0^\infty \left[(G_d(x,y) - G_d(x,-y))\,\bar{H}(x)\,(Q_1(y) + Q_2(y))\right] dxdy,$$

where $Q_1(y)$ and $Q_2(y)$ are defined as follows:

$$Q_1(y) = \mu\left(x_\theta + \frac{y}{\sqrt{\gamma}}\right)\left(B\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}}y\right) - B_\theta\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}}y\right)\right),$$

$$Q_2(y) = \frac{1}{a^2\gamma}U_\theta\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}}y\right)\left(1 - \mu\left(x_\theta + \frac{y}{\sqrt{\gamma}}\right)\right).$$

Thus we find that it suffices to show the following estimate:

$$P_6 = \frac{a^3\sqrt{\gamma}}{d\gamma^2}\int_{\mathbb{R}}\int_0^\infty \left[(G_d(x,y) - G_d(x,-y))\,\bar{H}(x)Q_1(y)\right] dxdy \geq o\left(\frac{1}{\theta\gamma^2}\right), \tag{5.34}$$

$$P_7 = \frac{a^3\sqrt{\gamma}}{d\gamma^2}\int_{\mathbb{R}}\int_0^\infty \left|(G_d(x,y) - G_d(x,-y))\,\bar{H}(x)Q_2(y)\right| dxdy = o\left(\frac{1}{\theta\gamma^2}\right). \tag{5.35}$$

First, we show (5.35). By changing variables, we obtain

$$
\begin{aligned}
P_7 =& \frac{a^3\sqrt{\gamma}}{d\gamma^2}\cdot\frac{a^3\mu_0}{d\gamma^2}\cdot\frac{1}{a^2\gamma}\cdot\left(\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\right)^2 \\
&\times\int_{\mathbb{R}}\int_0^\infty \left|\left(G_d\left(\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}x,\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}y\right)-G_d\left(\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}x,-\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}y\right)\right)\right. \\
&\qquad\left.\times\left(U_0(x)-U_0(x)^3\right)\cdot U_\theta(y)\left(1-\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu}}\right)\right)\right| dxdy \\
=& \frac{a^2\theta^2\sqrt{\gamma}}{d^2\gamma^4}\cdot\int_{\mathbb{R}}\int_0^\infty |\Gamma(x,y)B(x)U_\theta(y)|\left(1-\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu}}\right)\right)dxdy,
\end{aligned}
$$

where $\Gamma(x,y)$ is defined in (5.6). Thus we can see

$$
\begin{aligned}
P_7 \leq& \frac{Ca^2\theta^2\sqrt{\gamma}}{d^2\gamma^4}\cdot\frac{4\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\int_0^\infty |x|\,B(x)\,dx\cdot\int_{\mathbb{R}}\left(1-\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu_0}}\right)\right)dy \\
\leq& \frac{C'a^2\theta^2\sqrt{\gamma}}{d^2\gamma^4}\cdot\frac{4\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\cdot\int_{\mathbb{R}}(1-\mu(X))\left(\frac{a\sqrt{\mu_0}}{\theta}dX\right) \\
=& O\left(\frac{1}{\theta^2\gamma^3}\right)=o\left(\frac{1}{\theta\gamma^2}\right)
\end{aligned}
$$

from (5.12). As a consequence, we have shown (5.35) since $P_7 \geq 0$. Next, we shall show (5.34). By changing variables, we can see that

$$
\begin{aligned}
P_6 =& -\frac{a^3\sqrt{\gamma}}{d\gamma^2}\cdot\frac{a^3\mu_0}{d\gamma^2}\cdot\left(\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\right)^2 \\
&\times\int_{\mathbb{R}}\int_0^\infty\left[\left(G_d\left(\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}x,\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}y\right)-G_d\left(\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}x,-\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}y\right)\right)\right. \\
&\qquad\left.\times B(x)\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu_0}}\right)\{B(y)-B_\theta(y)\}\right]dxdy \\
=& -\frac{a^4\theta^2\sqrt{\gamma}}{d^2\gamma^3}\int_{\mathbb{R}}\int_0^\infty \Gamma(x,y)B(x)\mu\left(x_\theta+\frac{\theta y}{a\sqrt{\mu_0}}\right)\{B(y)-B_\theta(y)\}\,dxdy.
\end{aligned}
$$

Then $P_6$ can be represented $P_6 = P_1 + P_2 + P_3$, where $P_i$ $(i = 1,2,3)$ are defined in Lemmas 19 and 20. Thus it follows $P_6 \geq o(1/(\theta\gamma^2))$ from Lemmas 19 and 20. □

Lemma 25. *Assume $\theta^2 \ll 1/\gamma \ll \theta$. Let $\psi_\theta$ be a minimizer of (1.9) or (1.10). Then the following inequality holds:*

$$\frac{\sqrt{\gamma}}{2}\iint_{\mathbb{R}^2} G_d(x,y)\bar{H}(x)\bar{H}(y)\,dxdy \geq \frac{a^3\sqrt{\mu_0}}{2\theta\gamma^2}(A+o(1)).$$

Proof. For simplicity, we write

$$P_8 = \frac{\sqrt{\gamma}}{2}\iint_{\mathbb{R}^2} G_d(x,y)\bar{H}(x)\bar{H}(y)\,dxdy.$$

From $\bar{H}(-x) = -\bar{H}(x)$ holds for all $x \in \mathbb{R}$, we see that

$$P_8 = \sqrt{\gamma}\iint_{(0,\infty)^2} \left(G_d(x,y) - G_d(x,-y)\right)\bar{H}(x)\bar{H}(y)\,dxdy.$$

From the definition of $\bar{H}(x)$, we can write $P_8$ as follows:

$$P_8 = \frac{a^6\mu_0^2\sqrt{\gamma}}{d^2\gamma^4}\iint_{(0,\infty)^2} \left(G_d(x,y) - G_d(x,-y)\right) \times B\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}}x\right) B\left(\frac{a\sqrt{\mu_0}}{\theta\sqrt{\gamma}}y\right) dxdy.$$

By changing variables and (5.6), we obtain

$$\begin{aligned} P_8 &= \frac{a^6\mu_0^2\sqrt{\gamma}}{d^2\gamma^4}\cdot\left(\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\right)^2 \iint_{(0,\infty)^2} \Gamma(x,y)B(x)B(y)\,dxdy \\ &= \frac{a^4\sqrt{\gamma}\theta^2\mu_0}{d^2\gamma^3}\iint_{(0,\infty)^2} \Gamma(x,y)B(x)B(y)\,dxdy. \end{aligned}$$

We note that

$$e^{-s} - e^{-t} \geq (t-s) - \frac{t^2}{2} \quad \text{for all } \ 0 < s < t.$$

Then we have

$$\begin{aligned} \Gamma(x,y) &\geq \frac{1}{2}\left\{\frac{\theta\sqrt{\gamma}}{a\sqrt{\mu_0}}\left(|x+y| - |x-y|\right) - \sqrt{1 - \frac{1}{d\gamma^2}\frac{\theta^2\gamma}{a^2\mu_0}}\,|x+y|^2\right\} \\ &\geq \frac{\theta\sqrt{\gamma}}{2a\sqrt{\mu_0}}\left(|x+y| - |x-y|\right) - \frac{\theta^2\gamma}{a^2\mu_0}\left(|x|^2 + |y|^2\right). \end{aligned}$$

As a consequence, we find that

$$P_8 \geq \frac{a^4\sqrt{\gamma}\theta^2\mu_0}{d^2\gamma^3}\left\{\frac{\theta\sqrt{\gamma}}{2a\sqrt{\mu_0}}A - \frac{\theta^2\gamma}{a^2\mu_0}\iint_{(0,\infty)^2}\left(|x|^2+|y|^2\right)B(x)B(y)\,dydx\right\}$$
$$\geq \frac{a^3\theta^3\sqrt{\mu_0}}{2d^2\gamma^2}A - C\frac{\gamma\sqrt{\gamma}\theta^4}{d^2\gamma^3}.$$

Noting $d = \theta^2 + o(\theta^2)$ and $1/\gamma^{3/2} = o(1/\theta\gamma^2)$, we have

$$P_8 \geq \frac{a^3\sqrt{\mu_0}}{2\theta\gamma^2}A + o\left(\frac{1}{\theta\gamma^2}\right).$$

As a consequence, we conclude the statement. □

With these lemmas, we prove Theorem 5.

Proof of Theorem 5. We can prove each statement from Lemmas 21 – 25. For the proof of (2) or (3), we only note that $J^{(3)}(\psi_\theta)+J^{(4)}(\psi_\theta)$ is estimated

$$J^{(3)}(\psi_\theta) + J^{(4)}(\psi_\theta) \geq \frac{a^3\sqrt{\mu_0}}{2\theta\gamma^2}A + o\left(\frac{1}{\theta\gamma^2}\right)$$

from Lemmas 23 – 25. □

Proof of Theorem 3. It is obvious from Proposition 3 and Theorem 5. □

*Acknowledgements*. We would like thank the referee for carefully reading our manuscript and for giving useful comments.

The second author was supported by JSPS KAKENHI Grant Number 16K05240.

## References

[1] Agmon, S., *Lectures on exponential decay of solutions of second-order elliptic equations*, Princeton Univ. Press, 1982.

[2] Bonheure, D. and L. Sanchez, Heteroclinic orbits for some classes of second and fourth order differential equations, Handbook of Differential Equations **3** (2006), 103–202.

[3] Chen, C. N. and Y. Choi, Standing pulse solutions to FitzHugh-Nagumo equations, Arch. Ration. Mech. Anal. **206** (2012), 741–777.
[4] Chen, C. N., Kung, S. Y. and Y. Morita, Planar standing wavefronts in the FitzHugh-Nagumo equations, SIAM J. Math. Anal. **46** (2014), 657–690.
[5] Chen, C. N. and K. Tanaka, A variational approach for standing waves of FitzHugh-Nagumo type systems, J. Differential Equations, **257** (2014), 109–144.
[6] Dancer, E. N. and S. Yan, A minimization problem associated with elliptic systems of FitzHugh-Nagumo type, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), 237–253.
[7] Dancer, E. N. and S. Yan, Solutions with interior and boundary peaks for the Neumann problem of an elliptic system of FitzHugh-Nagumo type, Indiana Univ. Math. J. **55** (2006), 217–258.
[8] Ei, S. and H. Matsuzawa, The motion of transition layer for a bistable reaction deffusion equation with heterogeneous environment, Discrete Contin. Dyn. Syst. **26** (2010), 901–921.
[9] Grafakos, L., *Classical Fourier Analysis*, Second edition, Springer, 2008.
[10] Kajiwara, T., A heteroclinic solution to a variational problem corresponding to FitzHugh-Nagumo type reaction-diffusion system with heterogeneity, Comm. Pure Appl. Anal. **16** (2017), 2133–2156.
[11] Kajiwara, T., The sub-supersolution method for the FitzHugh-Nagumo type reaction-diffusion system with heterogeneity, Discrete Contin. Dyn. Syst. **38** (2018), 2441–2465.
[12] Klaasen, G. A. and W. C. Troy, Stationary wave solutions of a system of reaction-diffusion equations derived from the FitzHugh-Nagumo equations, SIAM J. Appl. Math. **44** (1984), 96–110.
[13] Matsuzawa, H., Stable transition layers in a balanced bistable equation with degeneracy, Nonlinear Anal. **58** (2004), 45–67.
[14] Nakashima, K., Stable transition layers in a balanced bistable equation, Differential and Integral Equations **13** (2000), 1025–1038.
[15] Nishiura, Y., Coexistence of infinitely many stable solutions to reaction-diffusion system in the singular limit, *Dynamics Reported: Expositions in Dynamical Systems*, Vol. 3, Springer, New York, 1994.
[16] Oshita, Y., On stable nonconstant stationary solutions and mesoscopic patterns for FitzHugh-Nagumo equations in higher dimensions, J. Differential Equations **188** (2003), 110–134.
[17] Reinecke, C. and G. Sweer, A positive solution on $\mathbb{R}^N$ to a system of elliptic equations of FitzHugh-Nagumo Type, J. Differential Equations **153** (1999), 292-312.
[18] Ren, X. and J. Wei, Nucleation in the FitzHugh-Nagumo system: Interface-spike solutions, J. Differential Equations **209** (2005), 266–301.
[19] Wei, J. and M. Winter, Standing waves in the FitzHugh-Nagumo system and a problem in combinatorial geometry, Math. Z. **254** (2006), 359–383.

(Received February 20, 2018)
(Revised May 27, 2019)

Department of Mathematics and
Information Sciences
Tokyo Metropolitan University
Tokyo 192-0397, Japan
E-mail: root.twenty.kajiwara@gmail.com
kurata@tmu.ac.jp