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# *A Category of Probability Spaces*

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**Abstract.** We introduce a category **Prob** of probability spaces whose objects are all probability spaces and whose arrows correspond to measurable functions satisfying an absolutely continuous requirement. We can consider a **Prob**-arrow as an evolving direction of information. We introduce a contravariant functor $\mathcal{E}$ from **Prob** to **Set**, the category of sets. The functor $\mathcal{E}$ provides conditional expectations along arrows in **Prob**, which are generalizations of the classical conditional expectations. For a **Prob**-arrow $f^-$, we introduce two concepts $f^-$-measurability and $f^-$-independence and investigate their interaction with conditional expectations along $f^-$. We also show that the completion of probability spaces is naturally formulated as an endofunctor of **Prob**.

## 1. Introduction

One of the most prominent examples of applying category theory to probability theory is Lawvere and Giry's approach of formulating transition probabilities in a monadic example ( [Law62], [Gir82] ). However, there are few of making categories consisting of all probability spaces due to a difficulty of finding an appropriate condition of their arrows. One of the trials is a way to adopt measure-preserving functions as arrows. With this setting, for example, Franz develops a stochastic independence theory in [Fra03]. Our approach is one of this simple-minded trials. Another recent trial of generalizing arrows is made by Motoyama and Tanaka [Mot16]. They introduce a notion of bounded arrows between probability spaces and define the category of all probability spaces and all bounded arrows between them, called **CPS**.

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We have two main results in this paper. One is an introduction of the category **Prob** of all probability spaces and null-presearving maps between them. The other one is that we show the existence of the conditional expectation functor from **Prob** to **Set**, which is a natural generalization of the classical notion of conditional expectation.

We introduce a category **Prob** of all probability spaces in order to see a possible generalization of some classical tools in probability theory including conditional expectations. Actually, [Ada14] provides a simple category for formulating conditional expectations, but its objects and arrows are so limited that we cannot use it as a foundation of categorical probability theory. We will also see that all arrows in the category **CPS** defined in [Mot16] are arrows in **Prob** as well if ignoring they have opposite directions. Therefore, $\mathbf{CPS}^{op}$ is a subcategory of **Prob**.

The original idea of the category **Prob** comes when we sought a generalization of the notion of financial risk measures that is one of the crucial tools for managing risks in the financial industry. The risk measure is a means of evaluating a future risk that is represented as a random variable, with current information by calculating its conditional expectation given the information. The reason of using the conditional expectation is that we have less information than we will have in future. The conditional expectation is a perfect tool as long as the only difference between today and future is the information we can access, that is, the changing part of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ from now to future is just the $\sigma$-field $\mathcal{F}$. However, after experiencing recent financial crises, we are suspecting that the probability measure $\mathbb{P}$ also varies through time, which created the disasters since we treated it as invariant when we calculated the risk. A trial of making the probability measure vary was the motivation of [Ada14].

In this paper, beyond that, we treat the situation when the underlying set $\Omega$ of elementary events also varies, that is, all the three components of the probability space are changing. We represent this change of entire probability spaces by an arrow between them, thinking within a category of probability spaces. So a natural requirement for the arrow is that we can extend the classical conditional expectation given the (current) $\sigma$-field to a sort-of conditional expectation along the arrow. The category **Prob** was developed so that this requirement is satisfied.

The arrows of **Prob** are maps corresponding to measurable functions satisfying an absolutely continuous requirement that is weaker than the measure-preserving requirement (Section 2). The requirement can be restated the inverse of the arrow preserves null sets. The resulting **Prob**-arrow can be seen as an evolving direction of information together with its interpretation. We will see that the requirement allows us to extend some important notions relativized by a $\sigma$-algebra in classical probability theory to notions relativized by a **Prob**-arrow $f^-$. For example, we introduce notions of a conditional expectation along $f^-$ (Section 3), a $f^-$-measurable function (Section 4) and a random variable independent of $f^-$ (Section 5). These are considered as generalizations of the classical counterparts. The existence of those natural generalizations may support a claim saying that the requirement for **Prob**-arrows is a natural one. We also see that the completion procedure of probability spaces becomes an endofunctor of **Prob** (Section 6).

The category **Prob** and functors developed in this paper convey more natural and richer structures than those introduced in [Ada14].

## 2. Category of Probability Spaces

In this paper, $\bar{X} = (X, \Sigma_X, \mathbb{P}_X)$, $\bar{Y} = (Y, \Sigma_Y, \mathbb{P}_Y)$ and $\bar{Z} = (Z, \Sigma_Z, \mathbb{P}_Z)$ are probability spaces.

Definition 2.1 [Null-preserving functions]. A measurable function $f : \bar{Y} \to \bar{X}$ is called ***null-preserving*** if $f^{-1}(A) \in \mathcal{N}_Y$ for every $A \in \mathcal{N}_X$, where $\mathcal{N}_X := \mathbb{P}_X^{-1}(0) \subset \Sigma_X$ and $\mathcal{N}_Y := \mathbb{P}_Y^{-1}(0) \subset \Sigma_Y$.

The following characterization is straightforward.

Proposition 2.1. *Let $f : \bar{Y} \to \bar{X}$ be a measurable function. Then, $f$ is null-preserving if and only if $\mathbb{P}_Y \circ f^{-1} \ll \mathbb{P}_X$, where $\mu \ll \nu$ means that $\mu$ is **absolutely continuous** with respect to $\nu$, that is, $\mu(A) = 0$ whenever $\nu(A) = 0$.*

The following diagram (that does not commute in general) might be helpful to see the situation we consider.

$$
\begin{array}{ccccccc}
Y & & \mathcal{N}_Y & \subset & \Sigma_Y & \xrightarrow{\mathbb{P}_Y} & \\
f \downarrow & & \uparrow f^{-1} & & \uparrow f^{-1} & & [0,1] \\
X & & \mathcal{N}_X & \subset & \Sigma_X & \xrightarrow{\mathbb{P}_X} &
\end{array}
$$

DEFINITION 2.2 [Bounded functions (Motoyama-Tanaka [Mot16])]. A measurable function $f : \bar{Y} \to \bar{X}$ is called ***bounded*** if there exists a positive number $M > 0$ such that $\mathbb{P}_Y(f^{-1}(A)) \leq M\mathbb{P}_X(A)$ for every $A \in \Sigma_X$.

By Proposition 2.1, the following proposition is obvious.

PROPOSITION 2.2. *Every bounded function $f : \bar{Y} \to \bar{X}$ is null-preserving.*

PROPOSITION 2.3. *Let $f$ and $g$ be two null-preserving functions as follows:*

$$\bar{Z} \xrightarrow{g} \bar{Y} \xrightarrow{f} \bar{X}$$

*Then, $f \circ g$ is also null-preserving.*

PROOF. Immediate. □

Proposition 2.3 makes the following definition well-defined.

DEFINITION 2.3 [Category **Prob**]. A category **Prob** is the category whose objects are all probability spaces and the set of arrows between them are defined by

$$\mathbf{Prob}(\bar{X},\bar{Y}) := \{f^- \mid f : \bar{Y} \to \bar{X} \text{ is a null-preserving function.}\},$$

where $f^-$ is a symbol corresponding uniquely to a function $f$.

We write $Id_X$ for an identity *measurable* function from $\bar{X}$ to $\bar{X}$, while writing $id_X$ for an identity function from $X$ to $X$. Therefore, the identity arrow of a **Prob**-object $\bar{X}$ is $Id_X^-$.

Motoyama and Tanaka [Mot16] introduce the category consisting of all probability spaces and all bounded arrows between them, and call it **CPS**.

By Proposition 2.2, $\mathbf{CPS}^{op}$ is a subcategory of **Prob**. So the category **CPS** is, in a sense, not large enough for developing the theory of financial risk measures as we mentioned in Section 1.

An arrow $f^-$ in **Prob** can be considered to represent an evolving direction of information with a way of its interpretation. The information is evolving along $f^{-1}$ but with a restriction to its accompanying probability measure.

In order to see it more concretely, let us consider the case where $f$ is an identity function on $X$, that is, consider a **Prob**-arrow $id_X^- : (X, \Sigma_1, \mathbb{P}_1) \to (X, \Sigma_2, \mathbb{P}_2)$. Then, we have $\Sigma_1 \subset \Sigma_2$ and $\mathbb{P}_2 = \mathbb{P}_2 \circ id_X^{-1} \ll \mathbb{P}_1$. This means that the information is growing while the support of the probability measure is decreasing. The latter makes sense if we think of the following situation: someone believed that some event among many other events may happen, but now she has changed her mind to believe that the event will never occur, and so she can concentrate on other possible events.

Actually, [Ada14] treats this special situation. It introduces, for a given measurable space $(\Omega, \mathcal{G})$, a category $\chi(\Omega, \mathcal{G})$ whose objects are all the pairs of the form $(\mathcal{F}_\mathcal{U}, \mathbb{P}_\mathcal{U})$ where $\mathcal{F}_\mathcal{U}$ is a sub-$\sigma$-field of $\mathcal{G}$ and $\mathbb{P}_\mathcal{U}$ is a probability measure on $\mathcal{G}$. And it has a unique arrow from $(\mathcal{F}_\mathcal{V}, \mathbb{P}_\mathcal{V})$ to $(\mathcal{F}_\mathcal{U}, \mathbb{P}_\mathcal{U})$ only when $\mathcal{F}_\mathcal{V} \subset \mathcal{F}_\mathcal{U}$ and $\mathbb{P}_\mathcal{U} \ll \mathbb{P}_\mathcal{V}$. Note that there is a natural embedding $\iota$ of the category $\chi(\Omega, \mathcal{G})$ into **Prob**.

$$\chi(\Omega, \mathcal{G}) \xrightarrow{\quad \iota \quad} \mathbf{Prob}$$

$$\begin{array}{ccc}
(\mathcal{F}_\mathcal{V}, \mathbb{P}_\mathcal{V}) & \xrightarrow{\quad \iota \quad} & (\Omega, \mathcal{F}_\mathcal{V}, \mathbb{P}_\mathcal{V} \mid \mathcal{F}_\mathcal{V}) \\
{\scriptstyle *}\big\downarrow & & \big\downarrow{\scriptstyle \iota(*) := id_\Omega^-} \\
(\mathcal{F}_\mathcal{U}, \mathbb{P}_\mathcal{U}) & \xrightarrow{\quad \iota \quad} & (\Omega, \mathcal{F}_\mathcal{U}, \mathbb{P}_\mathcal{U} \mid \mathcal{F}_\mathcal{U})
\end{array}$$

Proposition 2.4. *A probability space* $\nvdash := (\{*\}, \{\{*\}, \emptyset\}, \mathbb{P}_\nvdash)$, *where* $\mathbb{P}_\nvdash(\{*\}) := 1$ *and* $\mathbb{P}_\nvdash(\emptyset) := 0$, *is an initial object of the category* **Prob**. *Actually, for a probability space* $\bar{X}$, $!_X^- : \nvdash \to \bar{X}$ *is a unique arrow in* **Prob**, *where* $!_X : X \to \{*\}$ *is a function such as* $!_X(x) = *$ *for all* $x \in X$.

Proof. First, we show the uniqueness of $!_X^-$. But it is a straightforward consequence from the fact that there exists only one arrow $!_X$ from $X$ to $\{*\}$. Next, we prove that $!_X^-$ is a **Prob**-arrow. Obviously $!_X$ is measurable, so all we need to show is that $!_X^{-1}$ is null-preserving. Since $\emptyset$ is the only null set of $\not\vdash$ and any inverse image of $\emptyset$ is also $\emptyset$, we conclude $!_X^{-1}$ is null-preserving. □

In the following discussions, we fix the state space to be the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for simplicity. $\mathcal{L}^\infty(\bar{X})$ is a vector space consisting of $\mathbb{R}$-valued random variables $v$ such that $\mathbb{P}_X\text{-}\operatorname{ess\,sup}_{x\in X}|v(x)| < \infty$, while $\mathcal{L}^1(\bar{X})$ is a vector space consisting of $\mathbb{R}$-valued random variables $v$ such that $\int_X |v|\, d\mathbb{P}_X$ has a finite value. For two random variables $u_1$ and $u_2$, we write $u_1 \sim_{\mathbb{P}_X} u_2$ or $u_1 = u_2$ $\mathbb{P}_X$-a.s. when $\mathbb{P}_X(u_1 \neq u_2) = 0$. $L^\infty(\bar{X})$ and $L^1(\bar{X})$ are quotient spaces $\mathcal{L}^\infty(\bar{X})/\sim_{\mathbb{P}_X}$ and $\mathcal{L}^1(\bar{X})/\sim_{\mathbb{P}_X}$, respectively.

PROPOSITION 2.5. *Let $u_1$ and $u_2$ be two elements of $\mathcal{L}^\infty(\bar{X})$, and $f^-$ be an arrow in* $\mathbf{Prob}(\bar{X}, \bar{Y})$. *Then, $u_1 \sim_{\mathbb{P}_X} u_2$ implies $u_1 \circ f \sim_{\mathbb{P}_Y} u_2 \circ f$.*

PROOF. Assume that $u_1 \sim_{\mathbb{P}_X} u_2$. Then, $\mathbb{P}_X(u_1 \neq u_2) = 0$. Hence, we have $\mathbb{P}_Y\big(f^{-1}\{u_1 \neq u_2\}\big) = (\mathbb{P}_Y \circ f^{-1})(u_1 \neq u_2) = 0$ since $\mathbb{P}_Y \circ f^{-1} \ll \mathbb{P}_X$. Therefore $\mathbb{P}_Y(u_1 \circ f \neq u_2 \circ f) = 0$ since $\{u_1 \circ f \neq u_2 \circ f\} \subset f^{-1}\{u_1 \neq u_2\}$, which means $u_1 \circ f \sim_{\mathbb{P}_Y} u_2 \circ f$. □

Proposition 2.5 makes the following definition well-defined.

DEFINITION 2.4 [Functor **L**]. A functor $\mathbf{L} : \mathbf{Prob} \to \mathbf{Set}$ is defined by:

$$
\begin{array}{ccccccccc}
X & \bar{X} & \overset{\mathbf{L}}{\longmapsto} & \mathbf{L}\bar{X} & := & L^\infty(\bar{X}) & \ni & [u]_{\sim_{\mathbb{P}_X}} \\
{\scriptstyle f}\uparrow & {\scriptstyle f^-}\downarrow & & \downarrow{\scriptstyle \mathbf{L}f^-} & & & & \downarrow{\scriptstyle \mathbf{L}f^-} \\
Y & \bar{Y} & \overset{\mathbf{L}}{\longmapsto} & \mathbf{L}\bar{Y} & := & L^\infty(\bar{Y}) & \ni & [u \circ f]_{\sim_{\mathbb{P}_Y}}
\end{array}
$$

*Example* 2.6. Let $\omega$ be the category whose objects are all integers starting with 0 and for each pair of integers $s$ and $t$ with $s \leq t$ there is a unique arrow $*_{s,t} : s \to t$. That is, $\omega$ is the category corresponding to the integer set $\mathbb{N}$ with the usual total order. For a real number $p \in (0, 1)$, we define a functor $\mathcal{B} := \mathcal{B}^p : \omega \to \mathbf{Prob}$ in the following way.

For an object $t$ of $\omega$, $\mathcal{B}t$ is a probability space $\bar{X}_t := (X_t, \Sigma_t, \mathbb{P}_t)$ whose components are defined as follows:

1. $X_t := \{0,1\}^t$, the set of all binary numbers of $t$ digits,
2. $\Sigma_t := 2^{X_t}$,
3. for $a \in X_t$, $\mathbb{P}_t : \Sigma_t \to [0,1]$ is the probability measure defined by $\mathbb{P}_t(\{a\}) := p^{\#a}(1-p)^{t-\#a}$ where $\#a$ is the number of occurrences of 1 in $a$.

For an integer $t$, $F(*_{t,t+1}) := f_t$ is defined by $f_t(i_0 i_1 \dots i_t i_{t+1}) := i_0 i_1 \dots i_t$ where $i_k$ is 0 or 1. For $s < t$, We write $f_{s,t}$ for $F(*_{s,t}) = f_s \circ f_{s+1} \circ \cdots \circ f_{t-1}$.

Since $\Sigma_t$ is a powerset of $X_t$, any function from $X_t$ is measurable. Moreover by the definition of $\mathbb{P}_t$, only null set in $\Sigma_t$ is $\emptyset$. Therefore any function between $X_s$ and $X_t$ is null-preserving. Hence, $f_{s,t}$ is a **Prob**-arrow. Thus, the functor $\mathcal{B}$ is well-defined.

The functor $\mathcal{B}$ represents a filtration over the classical binomial model, for example developed in [Shr04]. So we can think $\mathcal{B}$ a sort of ***generalized filtration***.

One of the biggest difference between the classical and **Prob** versions of binomial models is that the classical version requires the terminal time horizon $T$ for determining the underlying set $\Omega := \{0,1\}^T$ while our version does not require it since the time variant probability spaces can evolve without any limit. That is, our version allows unknown future elementary events, which, we believe, shows a big philosophical difference from the Kolmogorov world.

## 3. Conditional Expectation Functor

DEFINITION 3.1. Let us consider a **Prob** arrow $f^- : \bar{X} \to \bar{Y}$. Take $v \in \mathcal{L}^1(\bar{Y})$ and put

$$v^*(B) := \int_B v \, d\mathbb{P}_Y$$

for $B \in \Sigma_Y$. Then $v^* \circ f^{-1}$ is absolutely continuous w.r.t. $\mathbb{P}_X$, since $f^{-1}$ maps $\mathbb{P}_X$-null sets to $\mathbb{P}_Y$-null sets and

$$v^* \circ f^{-1}(A) = \int_{f^{-1}(A)} v \, d\mathbb{P}_Y \ (A \in \Sigma_X).$$

So, thanks to Radon-Nikodym theorem, we have the unique (up to $\mathbb{P}_X$-a.s.) element $E^{f^-}(v)$ of $\mathcal{L}^1(\bar{X})$ such that

$$\int_A E^{f^-}(v) \, d\mathbb{P}_X = \int_{f^{-1}(A)} v \, d\mathbb{P}_Y \tag{3.1}$$

for all $A \in \Sigma_X$. We call this element $E^{f^-}(v)$ ***the conditional expectation of*** $v$ ***along*** $f^-$.

PROPOSITION 3.1. *For* $u \in \mathcal{L}^1(\bar{X})$, $E^{Id_X^-}(u) \sim_{\mathbb{P}_X} u$.

PROOF. For every $A \in \Sigma_X$, $\int_A E^{Id_X^-}(u) \, d\mathbb{P}_X = \int_{Id_X^{-1}(A)} u \, d\mathbb{P}_X = \int_A u \, d\mathbb{P}_X$. □

PROPOSITION 3.2. *Let* $f^-$ *and* $g^-$ *be arrows in* **Prob** *like:*

$$\bar{X} \xrightarrow{f^-} \bar{Y} \xrightarrow{g^-} \bar{Z} \, .$$

1. *For* $v_1, v_2 \in \mathcal{L}^1(\bar{Y})$, $v_1 \sim_{\mathbb{P}_Y} v_2$ *implies* $E^{f^-}(v_1) \sim_{\mathbb{P}_X} E^{f^-}(v_2)$.
2. *For* $w \in \mathcal{L}^1(\bar{Z})$, $E^{f^-}(E^{g^-}(w)) \sim_{\mathbb{P}_X} E^{g^- \circ f^-}(w)$.

PROOF.

1. Assume that $v_1 \sim_{\mathbb{P}_Y} v_2$. Then, it is obvious that $v_1^* = v_2^*$ as functions. The result comes from the uniqueness (up to $\mathbb{P}_X$-null sets) of conditional expectations.
2. It is sufficient to show that for every $A \in \Sigma_X$

$$\int_A E^{f^-}(E^{g^-}(w)) \, d\mathbb{P}_X = \int_{(f \circ g)^{-1}(A)} w \, d\mathbb{P}_Z.$$

However, we can get this immediately by applying (3.1) twice. □

Proposition 3.1 and Proposition 3.2 make the following definition well-defined.

Definition 3.2 [Functor $\mathcal{E}$]. A functor $\mathcal{E} : \mathbf{Prob}^{op} \to \mathbf{Set}$ is defined by:

$$
\begin{array}{ccccccccc}
X & & \bar{X} & \overset{\mathcal{E}}{\longmapsto} & \mathcal{E}\bar{X} & := & L^1(\bar{X}) & \ni & [E^{f^-}(v)]_{\sim_{\mathbb{P}_X}} \\
{\scriptstyle f}\uparrow & & {\scriptstyle f^-}\downarrow & & \uparrow{\scriptstyle \mathcal{E}f^-} & & & & \uparrow{\scriptstyle \mathcal{E}f^-} \\
Y & & \bar{Y} & \overset{\mathcal{E}}{\longmapsto} & \mathcal{E}\bar{Y} & := & L^1(\bar{Y}) & \ni & [v]_{\sim_{\mathbb{P}_Y}}
\end{array}
$$

We call $\mathcal{E}$ a ***conditional expectation functor***.

Note that the functors $L$ and $\mathcal{E}$ defined in [Ada14] from the category $\chi(\Omega, \mathcal{G})$ to **Set** are representable as $\mathbf{L} \circ \iota$ and $\mathcal{E} \circ \iota$, respectively by using **L** and $\mathcal{E}$ defined in this paper. That is, **Prob** is a more general and richer category than $\chi$, while still having enough structure to define conditional expectation functor.

One may wonder why we do not use more structured category such as the category of Banach spaces instead of using **Set**. One of our hidden goals when we defined the functors **L** and $\mathcal{E}$ is to develop a model of a logical inference system based on **Prob**. In order to make it possible, we wanted to make the functor category over **Prob** be a topos. Picking **Set** as a target category is a natural consequence of this line since the functor category $\mathbf{Set}^{\mathbf{Prob}}$ becomes a topos.

The following three propositions state basic properties of our conditional expectations, which are similar to those of classical conditional expectations.

Proposition 3.3 [Linearity]. *Let $f^- : \bar{X} \to \bar{Y}$ be a* **Prob**-*arrow. Then for every pair of random variables $u, v \in \mathcal{L}^1(\bar{Y})$ and $\alpha, \beta \in \mathbf{R}$, we have*

$$E^{f^-}(\alpha u + \beta v) \sim_{\mathbb{P}_X} \alpha E^{f^-}(u) + \beta E^{f^-}(v). \tag{3.2}$$

PROOF. For all $A \in \Sigma_X$,

$$\begin{aligned}\int_A E^{f^-}(\alpha u + \beta v)\, d\mathbb{P}_X &= \int_{f^{-1}(A)} (\alpha u + \beta v)\, d\mathbb{P}_Y \\ &= \alpha \int_{f^{-1}(A)} u\, d\mathbb{P}_Y + \beta \int_{f^{-1}(A)} v\, d\mathbb{P}_Y \\ &= \alpha \int_A E^{f^-}(u)\, d\mathbb{P}_X + \beta \int_A E^{f^-}(v)\, d\mathbb{P}_X \\ &= \int_A (\alpha E^{f^-}(u) + \beta E^{f^-}(v))\, d\mathbb{P}_X. \ \square\end{aligned}$$

PROPOSITION 3.4 [Positivity]. *Let $f^- : \bar{X} \to \bar{Y}$ be a* **Prob**-*arrow. If a random variable $v \in \mathcal{L}^1(\bar{Y})$ is $\mathbb{P}_Y$-almost surely positive, i.e. $v \geq 0$ ($\mathbb{P}_Y$-a.s.), then $E^{f^-}(v) \geq 0$ ($\mathbb{P}_X$-a.s.).*

PROOF. Since $v$ is $\mathbb{P}_Y$-almost surely positive, $v^* \circ f^{-1}(A) = \int_{f^{-1}(A)} v\, d\mathbb{P}_Y$ is a measure on $(X, \Sigma_X)$ for every $A \in \Sigma_X$. Thus $E^{f^-}(v) \geq 0$ ($\mathbb{P}_X$-a.s.) because $E^{f^-}(v)$ is a Radon-Nikodym derivative $d(v^* \circ f^{-1})/d\mathbb{P}_X$. $\square$

PROPOSITION 3.5 [Monotone Convergence]. *Let $f^- : \bar{X} \to \bar{Y}$ be a* **Prob**-*arrow, $v, v_n \in \mathcal{L}^1(\bar{Y})$ be random variables for $n \in \mathbf{N}$. If $0 \leq v_n \uparrow v$ ($\mathbb{P}_Y$-a.s.), then $0 \leq E^{f^-}(v_n) \uparrow E^{f^-}(v)$ ($\mathbb{P}_X$-a.s.).*

PROOF. By Proposition 3.3 and Proposition 3.4, we have $0 \leq E^{f^-}(v_n) \leq E^{f^-}(v_{n+1})$ ($\mathbb{P}_X$-a.s.) for all $n \in \mathbf{N}$. Put $h := \limsup_n E^{f^-}(v_n)$, then obviously, $E^{f^-}(v_n) \uparrow h$ ($\mathbb{P}_X$-a.s.). So all we need to show is that for every $A \in \Sigma_X$, $\int_A E^{f^-}(v)\, d\mathbb{P}_X = \int_A h\, d\mathbb{P}_X$. But, thanks to monotone convergence theorem, we have

$$\begin{aligned}\int_A h\, d\mathbb{P}_X = \lim_{n\to\infty} \int_A E^{f^-}(v_n)\, d\mathbb{P}_X = \lim_{n\to\infty} \int_{f^{-1}(A)} v_n\, d\mathbb{P}_Y \\ = \int_{f^{-1}(A)} v\, d\mathbb{P}_Y = \int_A E^{f^-}(v)\, d\mathbb{P}_X. \ \square\end{aligned}$$

Definition 3.3 [Unconditional Expectation]. For $v \in \mathcal{L}^1(\bar{Y})$, we call $E^{!_{\bar{Y}}}(v)$ a ***unconditional expectation*** of $v$, where $!_{\bar{Y}}$ is the unique arrow $!_{\bar{Y}} : \nvdash \to \bar{Y}$ defined in Proposition 2.4.

Proposition 3.6. *Let* $!_{\bar{Y}} : \nvdash \to \bar{Y}$ *be the unique* **Prob**-*arrow and* $v \in \mathcal{L}^1(\bar{Y})$. *Then, we have*

$$E^{!_{\bar{Y}}}(v)(*) = \mathbb{E}^{\mathbb{P}_Y}[v]. \tag{3.3}$$

Proof.

$$E^{!_{\bar{Y}}}(v)(*) = \int_{\{*\}} E^{!_{\bar{Y}}}(v)\, d\mathbb{P}_{\nvdash} = \int_{!_Y^{-1}(\{*\})} v\, d\mathbb{P}_Y = \int_Y v\, d\mathbb{P}_Y = \mathbb{E}^{\mathbb{P}_Y}[v]. \ \square$$

Proposition 3.6 asserts that our unconditional expectation is a natural extension of the classical one.

## 4. $f^-$-Measurability

Definition 4.1 [$f^-$-measurability]. Let $f^- : \bar{X} \to \bar{Y}$ be a **Prob**-arrow and $v \in \mathcal{L}^\infty(\bar{Y})$. $v$ is called ***$f^-$-measurable*** if there exists $w \in \mathcal{L}^\infty(\bar{X})$ such that $v \sim_{\mathbb{P}_Y} w \circ f$.

The following proposition allows us to say that an element of $L\bar{Y}$ is $f^-$-measurable.

Proposition 4.1. *Let* $f^- : \bar{X} \to \bar{Y}$ *be a* **Prob**-*arrow and* $v_1$ *and* $v_2$ *be two elements of* $\mathcal{L}^\infty(\bar{Y})$ *satisfying* $v_1 \sim_{\mathbb{P}_Y} v_2$. *Then, if* $v_1$ *is* $f^-$-*measurable, so is* $v_2$.

Proof. Obvious. $\square$

The next proposition is well-known. For example, see Page 206 of [Wil91].

Proposition 4.2. *Let* $f^- : \bar{X} \to \bar{Y}$ *be a* **Prob**-*arrow and* $v \in \mathcal{L}^\infty(\bar{Y})$. *Then,* $v$ *is* $f^-$-*measurable if and only if* $v$ *is* $f^{-1}(\Sigma_X)/\mathcal{B}(\mathbb{R})$-*measurable.*

Proposition 4.2 says that $f^{-}$-measurability is an extension of the classical measurability.

THEOREM 4.3. *Let* $f^{-} : \bar{X} \to \bar{Y}$ *be a* **Prob**-*arrow,* $u$ *be an element of* $\mathcal{L}^1(\bar{Y})$ *and* $v$ *be a random variable in* $\mathcal{L}^\infty(\bar{Y})$*, and assume that* $v$ *is* $f^{-}$-*measurable. Then we have*

$$E^{f^{-}}(v \cdot u) \sim_{\mathbb{P}_X} w \cdot E^{f^{-}}(u), \tag{4.1}$$

*where* $w \in \mathcal{L}^\infty(\bar{X})$ *is a random variable satisfying* $v \sim_{\mathbb{P}_Y} w \circ f$*.*

PROOF. By (3.1), it is sufficient to prove that for every $A \in \Sigma_X$,

$$\int_{f^{-1}(A)} v \cdot u \, d\mathbb{P}_Y = \int_A w \cdot E^{f^{-}}(u) \, d\mathbb{P}_X. \tag{4.2}$$

But, it is obvious from the transformation theorem applying with the Jordan decomposition. □

Theorem 4.3 is a generalization of a classical formula

$$\mathbb{E}^{\mathbb{P}}[v \cdot u \mid \mathcal{G}] \sim_{\mathbb{P}} v \cdot \mathbb{E}^{\mathbb{P}}[u \mid \mathcal{G}]$$

for a $\mathcal{G}$-measurable random variable $v$.

The following theorem has some categorical taste.

THEOREM 4.4. *Let* $\mathcal{E} \boxtimes \mathbf{L}, \mathcal{E}\mathcal{P}_1 : \mathbf{Prob}^{op} \times \mathbf{Prob} \to \mathbf{Set}$ *be two parallel bifunctors defined by*

$$\mathcal{E} \boxtimes \mathbf{L} := \Box \circ (\mathcal{E} \times \mathbf{L}) \ \ \text{and} \ \ \mathcal{E}\mathcal{P}_1 := \mathcal{E} \circ \mathcal{P}_1$$

*where* $\mathcal{P}_1 : \mathbf{Prob}^{op} \times \mathbf{Prob} \to \mathbf{Prob}^{op}$ *is the projection for the first component, and* $\Box : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ *is a functor which sending an ordered pair of sets to the set product of its components.*

*Now, for each* **Prob**-*object* $\bar{X}$*, define a function* $\alpha_{\bar{X}} : L^1(\bar{X}) \times L^\infty(\bar{X}) \to L^1(\bar{X})$ *by* $\alpha_{\bar{X}}(\langle [u]_{\sim_{\mathbb{P}_X}}, [v]_{\sim_{\mathbb{P}_X}} \rangle) = [u \cdot v]_{\sim_{\mathbb{P}_X}}$*. Then the following diagram* *commutes.*

$$\mathbf{Prob}^{op} \times \mathbf{Prob} \xrightarrow{\mathcal{E} \boxtimes \mathbf{L}} \mathbf{Set} \qquad\qquad \mathbf{Set} \xleftarrow{\mathcal{E}\mathcal{P}_1} \mathbf{Prob}^{op} \times \mathbf{Prob}$$

$$\begin{array}{ccccccc}
\langle \bar{X},\, \bar{X} \rangle & & L^1(\bar{X}) \times L^\infty(\bar{X}) & \xrightarrow{\alpha_{\bar{X}}} & L^1(\bar{X}) & & \langle \bar{X},\, \bar{X} \rangle \\
{\scriptstyle \langle f^-, Id_X^- \rangle} \uparrow & & {\scriptstyle \mathcal{E}f^- \times \mathbf{L}Id_X^-} \uparrow & & \downarrow {\scriptstyle \mathcal{E}Id_X^-} & & \downarrow {\scriptstyle \langle Id_X^-, f^- \rangle} \\
\langle \bar{Y},\, \bar{X} \rangle & & L^1(\bar{Y}) \times L^\infty(\bar{X}) & & L^1(\bar{X}) & & \langle \bar{X},\, \bar{Y} \rangle \\
{\scriptstyle \langle Id_Y^-, f^- \rangle} \downarrow & & {\scriptstyle \mathcal{E}Id_Y^- \times \mathbf{L}f^-} \downarrow & & \uparrow {\scriptstyle \mathcal{E}f^-} & & \uparrow {\scriptstyle \langle f^-, Id_Y^- \rangle} \\
\langle \bar{Y},\, \bar{Y} \rangle & & L^1(\bar{Y}) \times L^\infty(\bar{Y}) & \xrightarrow{\alpha_{\bar{Y}}} & L^1(\bar{Y}) & & \langle \bar{Y},\, \bar{Y} \rangle
\end{array}$$

*In other words, $\alpha : \mathcal{E} \boxtimes \mathbf{L} \overset{\bullet\bullet}{\longrightarrow} \mathcal{E}\mathcal{P}_1$ is a dinatural transformation.*

Proof. For $\langle [v]_{\sim \mathbb{P}_Y},\, [u]_{\sim \mathbb{P}_X} \rangle \in L^1(\bar{Y}) \times L^\infty(\bar{X})$, we have

$$(\mathcal{E}Id_X^- \circ \alpha_{\bar{X}} \circ (\mathcal{E}f^- \times \mathbf{L}Id_X^-))(\langle [v]_{\sim \mathbb{P}_Y},\, [u]_{\sim \mathbb{P}_X} \rangle) = \left[ E^{f^-}(v) \cdot u \right]_{\sim \mathbb{P}_X},$$
$$(\mathcal{E}f^- \circ \alpha_{\bar{Y}} \circ (\mathcal{E}Id_Y^- \times \mathbf{L}f^-))(\langle [v]_{\sim \mathbb{P}_Y},\, [u]_{\sim \mathbb{P}_X} \rangle) = \left[ E^{f^-}(v \cdot (u \circ f)) \right]_{\sim \mathbb{P}_X}$$

since $\mathcal{E}Id_X^- = Id_{L^1(\bar{X})}$. But by Theorem 4.3, two rightmost hand sides coincide, which completes the proof. □

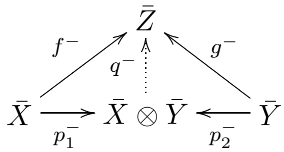
## 5. $f^-$-Independence

Definition 5.1 [Category **mpProb**]. A **Prob**-arrow $f^- : \bar{X} \to \bar{Y}$ is called ***measure-preserving*** if $\mathbb{P}_Y \circ f^{-1} = \mathbb{P}_X$. A subcategory **mpProb** of **Prob** is a category whose objects are same as those of **Prob** but arrows are limited to all measure-preserving arrows.

Franz defines stochastic independence in the opposite category of **mpProb** as an example of his introducing notion of stochastic independence in monoidal categories.

Definition 5.2 [Fra03]. Two **mpProb**-arrows $f^- : \bar{X} \to \bar{Z}$ and $g^- : \bar{Y} \to \bar{Z}$ are called ***independent*** if there exists an **mpProb**-arrow $q^-$ :

$\bar{X} \otimes \bar{Y} \to \bar{Z}$ such that the following diagram commutes




where $\bar{X} \otimes \bar{Y} := (X \times Y, \Sigma_X \otimes \Sigma_Y, \mathbb{P}_X \otimes \mathbb{P}_Y)$, $p_1$ and $p_2$ are projections, $\Sigma_X \otimes \Sigma_Y$ is the smallest $\sigma$-algebra of $X \times Y$ making both $p_1$ and $p_2$ measurable, and $\mathbb{P}_X \otimes \mathbb{P}_Y$ is a product measure such that $(\mathbb{P}_X \otimes \mathbb{P}_Y)(A \times B) = \mathbb{P}_X(A)\mathbb{P}_Y(B)$ for all $A \in \Sigma_X$ and $B \in \Sigma_Y$.

Franz shows that the notion of independence defined in Definition 5.2 exactly matches the classical one in the sense of the following proposition.

PROPOSITION 5.1 [Fra03]. *Two* **mpProb**-*arrows* $f^- : \bar{X} \to \bar{Z}$ *and* $g^- : \bar{Y} \to \bar{Z}$ *are independent if and only if for every pair of* $A \in \Sigma_X$ *and* $B \in \Sigma_Y$

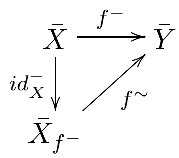
$$\mathbb{P}_Z(f^{-1}(A) \cap g^{-1}(B)) = \mathbb{P}_Z(f^{-1}(A))\mathbb{P}_Z(g^{-1}(B)). \tag{5.1}$$

Before extending the notion of independence to the category **Prob**, we need the following note.

PROPOSITION 5.2. *Let* $f^- : \bar{X} \to \bar{Y}$ *be a* **Prob**-*arrow. We define a* **Prob**-*object* $\bar{X}_{f^-}$ *by*

$$\bar{X}_{f^-} := (X, \Sigma_X, \mathbb{P}_Y \circ f^{-1}). \tag{5.2}$$

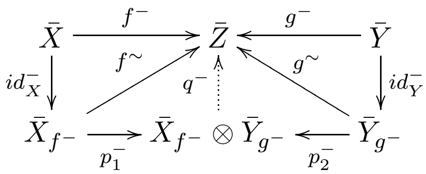
*Then, the following diagram commutes in* **Prob**




*where* $f^\sim$ *and* $id_X^-$ *are corresponding* **Prob**-*arrows of* $f : \bar{Y} \to \bar{X}_{f^-}$ *and* $id_X : \bar{X}_{f^-} \to \bar{X}$*, respectively. Moreover,* $f^\sim$ *is measure-preserving.*

PROOF. Obvious. □

DEFINITION 5.3 [Independence in **Prob**]. Two **Prob**-arrows $f^- : \bar{X} \to \bar{Z}$ and $g^- : \bar{Y} \to \bar{Z}$ are called ***independent*** if there exists a measure-preserving arrow $q^- : \bar{X}_{f^-} \otimes \bar{Y}_{g^-} \to \bar{Z}$ such that the following diagram commutes.




LEMMA 5.3. *For a measure-preserving* **Prob**-*arrow* $f^- : \bar{X} \to \bar{Y}$, $\mathcal{E}f^- \circ \mathbf{L}f^- = id_{\mathbf{L}\bar{X}}$.

PROOF. For $u \in \mathcal{L}^\infty(\bar{X})$, $(\mathbf{L}f^-)[u]_{\sim_{\mathbb{P}_X}} = [u \circ f]_{\sim_{\mathbb{P}_Y}}$ is $f^-$-measurable. Hence by Theorem 4.3,

$$\mathcal{E}f^-(\mathbf{L}f^-([u]_{\sim_{\mathbb{P}_X}})) = \left[E^{f^-}(u \circ f)\right]_{\sim_{\mathbb{P}_X}} = \left[u \cdot E^{f^-}(1_Y)\right]_{\sim_{\mathbb{P}_X}}.$$

But, since $f^-$ is measure-preserving, for all $A \in \Sigma_X$

$$\int_A E^{f^-}(1_Y)\, d\mathbb{P}_X = \int_{f^{-1}(A)} 1_Y\, d\mathbb{P}_Y = \mathbb{P}_Y(f^{-1}(A)) = \mathbb{P}_X(A) = \int_A 1_X\, d\mathbb{P}_X.$$

Therefore, $E^{f^-}(1_Y) = 1_X$, which concludes the proof. □

LEMMA 5.4. *Let* $\bar{X}$ *and* $\bar{Y}$ *be probability spaces. Then for all* $v \in \mathcal{L}^\infty(\bar{Y})$,

$$E^{p_1^-}(v \circ p_2) \sim_{\mathbb{P}_X} \mathbb{E}^{\mathbb{P}_Y}[v] 1_X \tag{5.3}$$

*where* $p_1$ *and* $p_2$ *are projections from* $X \times Y$ *to* $X$ *and* $Y$, *respectively.*

PROOF. For $A \in \Sigma_X$,

$$\begin{aligned}
&\int_A E^{p_1^-}(v \circ p_2)\, d\mathbb{P}_X \\
&\quad = \int_{p_1^{-1}(A)} v \circ p_2 \, d(\mathbb{P}_X \otimes \mathbb{P}_Y) \\
&\quad = \int_{X \times Y} (v \circ p_2) \cdot (1_A \circ p_1)\, d(\mathbb{P}_X \otimes \mathbb{P}_Y) \\
&\quad = \int_Y \int_X ((v \circ p_2)\langle x, y\rangle) \cdot ((1_A \circ p_1)\langle x, y\rangle)\, \mathbb{P}_X(dx)\, \mathbb{P}_Y(dy) \\
&\quad = \int_Y v(y)\, \mathbb{P}_Y(dy) \int_X 1_A(x)\, \mathbb{P}_X(dx) \\
&\quad = \mathbb{E}^{\mathbb{P}_Y}[v] \int_A 1_X \, d\mathbb{P}_X. \ \square
\end{aligned}$$

THEOREM 5.5. *Let $f^- : \bar{X} \to \bar{Z}$ and $g^- : \bar{Y} \to \bar{Z}$ be two independent* **Prob**-*arrows. Then, for every $v \in \mathcal{L}^\infty(\bar{Y})$, we have*

$$E^{f^-}(v \circ g) \sim_{\mathbb{P}_X} \mathbb{E}^{\mathbb{P}_Z}[v \circ g] E^{f^-}(1_Z). \tag{5.4}$$

PROOF. For the diagram in Definition 5.3, apply $\mathcal{E}$ to its left box, and apply $\mathbf{L}$ to its right box. Then, we get the following diagram.

$$\begin{array}{ccccccc}
L^1\bar{X} & \xleftarrow{\mathcal{E}f^-} & L^1\bar{Z} & \hookleftarrow & L^\infty\bar{Z} & \xleftarrow{\mathbf{L}g^-} & L^\infty\bar{Y} \\
{\scriptstyle \mathcal{E}id_X^-}\uparrow & \swarrow{\scriptstyle \mathcal{E}f^\sim} & \downarrow{\scriptstyle \mathcal{E}q^-} & & {\scriptstyle \mathbf{L}q^-}\uparrow & \nwarrow{\scriptstyle \mathbf{L}g^\sim} & \downarrow{\scriptstyle \mathbf{L}id_Y^-} \\
L^1\bar{X}_{f^-} & \xleftarrow[\mathcal{E}p_1^-]{} & L^1(\bar{X}_{f^-} \otimes \bar{Y}_{g^-}) & \hookleftarrow & L^\infty(\bar{X}_{f^-} \otimes \bar{Y}_{g^-}) & \xleftarrow[\mathbf{L}p_2^-]{} & L^\infty\bar{Y}_{g^-}
\end{array}$$

The left and right boxes in the above diagram commute since they are images of functors $\mathcal{E}$ and $\mathbf{L}$, the center box also commutes by Lemma 5.3, and so does the whole diagram. Now for $v \in \mathcal{L}^\infty(\bar{Y})$, let us see the values at the upper-left corner of the diagram developed through two paths, which should coincide.

$$(\mathcal{E}f^- \circ \mathbf{L}g^-)[v]_{\sim_{\mathbb{P}_Y}} = [E^{f^-}(v \circ g)]_{\sim_{\mathbb{P}_X}},$$
$$(\mathcal{E}id_X^- \circ \mathcal{E}p_1^- \circ \mathbf{L}p_2^- \circ \mathbf{L}id_Y^-)[v]_{\sim_{\mathbb{P}_Y}} = [(E^{id_X^-} \circ E^{p_1^-})(v \circ p_2)]_{\sim_{\mathbb{P}_X}}.$$

Hence by Lemma 5.4,

$$E^{f^-}(v \circ g) \sim_{\mathbb{P}_X} \mathbb{E}^{\mathbb{P}_Z \circ g^{-1}}[v] E^{\bar{id_X}}(1_X) \sim_{\mathbb{P}_X} \mathbb{E}^{\mathbb{P}_Z}[v \circ g] E^{\bar{id_X}}(1_X).$$

Now for every $A \in \Sigma_X$,

$$\begin{aligned} \int_A E^{\bar{id_X}}(1_X)\, d\mathbb{P}_X &= \int_A 1_X \, d(\mathbb{P}_Z \circ f^{-1}) = (\mathbb{P}_Z \circ f^{-1})(A) \\ &= \int_{f^{-1}(A)} 1_Z \, d\mathbb{P}_Z = \int_A E^{f^-}(1_Z)\, d\mathbb{P}_X. \end{aligned}$$

Therefore, $E^{\bar{id_X}}(1_X) \sim_{\mathbb{P}_X} E^{f^-}(1_Z)$, which completes the proof. □

DEFINITION 5.4 [$f^-$-independence]. For a random variable $v \in \mathcal{L}^1(\bar{Y})$, we define a probability space $\bar{\mathbb{R}}_v$ by $\bar{\mathbb{R}}_v := (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_Y \circ v^{-1})$. Then, $v^- : \bar{\mathbb{R}}_v \to \bar{Y}$ is a **Prob**-arrow. Now for a **Prob**-arrow $f^- : \bar{X} \to \bar{Y}$, $v$ is said to be ***independent*** of $f^-$, denoted by $v \perp f^-$, if $f^-$ and $v^-$ are independent.

The following proposition allows us to say that an element of $L^1(\bar{Y})$ is independent of $f^-$.

PROPOSITION 5.6. *Let $f^- : \bar{X} \to \bar{Y}$ be a* **Prob**-*arrow and $v_1$ and $v_2$ be two elements of $\mathcal{L}^1(\bar{Y})$ satisfying $v_1 \sim_{\mathbb{P}_Y} v_2$. Then, $v_1 \perp f^-$ implies $v_2 \perp f^-$.*

PROOF. Assume that $v_1 \sim_{\mathbb{P}_Y} v_2$ and $v_1 \perp f^-$. Let $N := \{y \in Y \mid v_1(y) \neq v_2(y)\}$ and $M := Y - N$. Then, $\mathbb{P}_Y(N) = 0$.

First, we show that

$$\mathbb{P}_Y \circ v_1^{-1} = \mathbb{P}_Y \circ v_2^{-1}. \tag{5.5}$$

For every $B \in \mathcal{B}(\mathbb{R})$ and $i = 1, 2$,

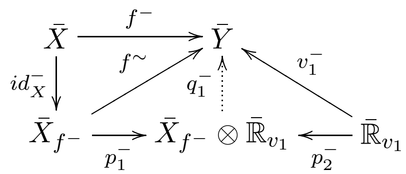
$$(\mathbb{P}_Y \circ v_i^{-1})(B) = \mathbb{P}_Y(v_i^{-1}(B) \cap N) + \mathbb{P}_Y(v_i^{-1}(B) \cap M) = \mathbb{P}_Y(v_i^{-1}(B) \cap M).$$

But,

$$y \in v_1^{-1}(B) \cap M \Leftrightarrow v_1(y) = v_2(y) \in B \Leftrightarrow y \in v_2^{-1}(B) \cap M,$$

which proves (5.5). Hence, $\bar{\mathbb{R}}_{v_1} = \bar{\mathbb{R}}_{v_2}$.

Now since $v_1 \perp f^-$, we have the following measure-preserving $q_1^-$.




Then, $q_1$ satisfies that $q_1(y) = \langle f(y), v_1(y) \rangle$ for all $y \in Y$. Similarly we define a function $q_2 : Y \to X \times R$ by $q_2(y) = \langle f(y), v_2(y) \rangle$ for all $y \in Y$. Then, all we need to show is that $q_2^-$ is a measure-preserving **Prob**-arrow, in other words,

$$\mathbb{P}_Y \circ q_2^{-1} = (\mathbb{P}_Y \circ f^{-1}) \otimes (\mathbb{P}_Y \circ v_2^{-1}). \tag{5.6}$$

However, by (5.5) and the fact that $q_1^-$ is measure-preserving, it is enough to show that

$$\mathbb{P}_Y \circ q_1^{-1} = \mathbb{P}_Y \circ q_2^{-1}. \tag{5.7}$$

For any $E \in \Sigma_X \otimes \mathcal{B}(\mathbb{R})$ and $i = 1, 2$,

$$(\mathbb{P}_Y \circ q_i^{-1})(E) = \mathbb{P}_Y(q_i^{-1}(E) \cap N) + \mathbb{P}_Y(q_i^{-1}(E) \cap M) = \mathbb{P}_Y(q_i^{-1}(E) \cap M).$$

But,

$$\begin{aligned} y \in q_1^{-1}(E) \cap M &\Leftrightarrow (\langle f(y), v_1(y) \rangle \in E \wedge v_1(y) = v_2(y)) \\ &\Leftrightarrow (\langle f(y), v_2(y) \rangle \in E \wedge v_1(y) = v_2(y)) \\ &\Leftrightarrow y \in q_2^{-1}(E) \cap M, \end{aligned}$$

which proves (5.7). □

PROPOSITION 5.7. *Let* $!_Y^- : \nvdash \to \bar{Y}$ *be a unique* **Prob**-*arrow and* $v \in \mathcal{L}^1(\bar{Y})$. *Then,* $v$ *is independent of* $!_Y^-$.

PROOF. Obvious. □

THEOREM 5.8. *Let* $f^- : \bar{X} \to \bar{Y}$ *be a* **Prob**-*arrow and* $v \in \mathcal{L}^1(\bar{Y})$ *which is independent of* $f^-$. *Then we have,*

$$E^{f^-}(v) \sim_{\mathbb{P}_X} \mathbb{E}^{\mathbb{P}_Y}[v] E^{f^-}(1_Y). \tag{5.8}$$

PROOF. Let $\{u_n : \mathbb{R} \to \mathbb{R}\}_{n\in\mathbb{N}}$ be a sequence of functions defined by $u_n := id_{\mathbb{R}} \cdot 1_{[-n,n]}$. Then by Theorem 5.5, $E^{f^-}(u_n \circ v) \sim_{\mathbb{P}_X} \mathbb{E}^{\mathbb{P}_Y}[u_n \circ v]E^{f^-}(1_Y)$ since $u_n \in L^\infty(\bar{\mathbb{R}}_{u_n})$. On the other hand, we have

$$u_n \circ v = u_n \circ v_+ - u_n \circ v_-$$

and

$$\begin{aligned} 0 &\le u_n \circ v_+ \uparrow v_+, \\ 0 &\le u_n \circ v_- \uparrow v_- \end{aligned}$$

as $n$ goes to $\infty$, where $v_+$ ($v_-$) is the positive (negative) part of the $\mathbb{R}$-valued function $v$. So by Proposition 3.3 and Proposition 3.5, we obtain

$$\begin{aligned} E^{f^-}(v) &\sim_{\mathbb{P}_X} \lim_{n\to\infty} E^{f^-}(u_n \circ v) \\ &\sim_{\mathbb{P}_X} \lim_{n\to\infty} \left(E^{f^-}(u_n \circ v_+) - E^{f^-}(u_n \circ v_-)\right) \\ &\sim_{\mathbb{P}_X} \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_Y}[u_n \circ v_+]E^{f^-}(1_Y) - \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_Y}[u_n \circ v_-]E^{f^-}(1_Y) \\ &\sim_{\mathbb{P}_X} \mathbb{E}^{\mathbb{P}_Y}[v_+]E^{f^-}(1_Y) - \mathbb{E}^{\mathbb{P}_Y}[v_-]E^{f^-}(1_Y) \\ &\sim_{\mathbb{P}_X} \mathbb{E}^{\mathbb{P}_Y}[v]E^{f^-}(1_Y). \ \Box \end{aligned}$$

As a combination of (5.8) and (3.3), we have

$$E^{f^-}(v) \sim_{\mathbb{P}_X} E^{!_Y^-}(v)(*)E^{f^-}(1_Y), \tag{5.9}$$

which is a natural generalization of the relationship between classical conditional expectations given independent $\sigma$-fields and unconditional expectations.

## 6. Completion Functor

The following definition is taken from pages 202-203 of [Wil91].

DEFINITION 6.1 [Wil91]. Let $(X, \Sigma_X, \mathbb{P}_X)$ be a probability space.

1. $\Sigma_X^* := \{F \subset X \mid \exists A, B \in \Sigma_X, A \subset F \subset B \text{ and } \mathbb{P}_X(B - A) = 0\}$,

2. For $F \in \Sigma_X^*$, $\mathbb{P}_X^*(F)$ is defined by $\mathbb{P}_X^*(F) := \mathbb{P}_X(A) = \mathbb{P}_X(B)$, where $A, B \in \Sigma_X$ satisfies $A \subset F \subset B$ and $\mathbb{P}_X(B - A) = 0$.

Then, it is well-known that the triple $(X, \Sigma_X^*, \mathbb{P}_X^*)$ is well-defined and becomes a probability space called the ***completion*** of $(X, \Sigma_X, \mathbb{P}_X)$.

PROPOSITION 6.1. *Let* $f^- : (X, \Sigma_X, \mathbb{P}_X) \to (Y, \Sigma_Y, \mathbb{P}_Y)$ *be a* **Prob**-*arrow.*

1. *The function* $f : Y \to X$ *is* $\Sigma_Y^*/\Sigma_X^*$*-measurable.*
2. $\mathbb{P}_Y^* \circ f^{-1} \ll \mathbb{P}_X^*$.

PROOF.

1. For any $F \in \Sigma_X^*$, by Definition 6.1 there exist $A, B \in \Sigma_X$ such that $A \subset F \subset B$ and $\mathbb{P}_X(B - A) = 0$. Then, since $\mathbb{P}_Y \circ f^{-1} \ll \mathbb{P}_X$, we have $f^{-1}(A) \subset f^{-1}(F) \subset f^{-1}(B)$ and $\mathbb{P}_Y(f^{-1}(B) - f^{-1}(A)) = \mathbb{P}_Y(f^{-1}(B - A)) = 0$. Therefore, again by Definition 6.1, $f^{-1}(F) \in \Sigma_Y^*$.
2. Assume that $F \in \Sigma_X^*$ and $\mathbb{P}_X(F) = 0$. Then, it is sufficient to show that $(\mathbb{P}_Y^* \circ f^{-1})(F) = 0$. Now by Definition 6.1, there exists $B \in \Sigma_X$ such that $F \subset B$ and $\mathbb{P}_X^*(F) = \mathbb{P}_X(B) = 0$. Then by $\mathbb{P}_Y \circ f^{-1} \ll \mathbb{P}_X$, we have $(\mathbb{P}_Y^* \circ f^{-1})(F) \le (\mathbb{P}_Y^* \circ f^{-1})(B) = (\mathbb{P}_Y \circ f^{-1})(B) = 0$. □

Proposition 6.1 makes the following definition well-defined.

DEFINITION 6.2 [Functor $\mathcal{C}$]. A functor $\mathcal{C}$ : **Prob** → **Prob** is defined by:

$$\begin{array}{ccccccc}
X & & (X, \Sigma_X, \mathbb{P}_X) & \stackrel{\mathcal{C}}{\longmapsto} & \mathcal{C}(X, \Sigma_X, \mathbb{P}_X) & := & (X, \Sigma_X^*, \mathbb{P}_X^*) \\
{\scriptstyle f}\uparrow & & \downarrow{\scriptstyle f^-} & & \downarrow{\scriptstyle \mathcal{C}f^-} & & \downarrow{\scriptstyle f^-} \\
Y & & (Y, \Sigma_Y, \mathbb{P}_Y) & \stackrel{\mathcal{C}}{\longmapsto} & \mathcal{C}(Y, \Sigma_Y, \mathbb{P}_Y) & := & (Y, \Sigma_Y^*, \mathbb{P}_Y^*)
\end{array}$$

The functor $\mathcal{C}$ is called a ***completion functor***.

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