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T. Takada, I. Yoshida & T. Itoi (editors)

MCMC ENHANCED FORM ESTIMATES AND SUBSET SIMULATION

K. Breitung[1]

[1] TU Munich, Munich, Germany: Email: breitu@aol.com

Abstract: The calculation of failure probabilities in high dimensional spaces is an important problem in structural reliability. FORM/SORM concepts are based on using the Laplace method for the pdf of the failure domain in its modes. A mathematical justification for this was given in Breitung (1984, 1994). With increasing dimensions it was found that the quality of SORM decreases considerably here. The obvious solution would have been to modify the SORM approximations. However instead of this, a new approach, subset simulation (SS) was championed by many researchers. By those it is generally maintained that this method does not suffer from the deficiencies of SORM and can solve high-dimensional reliability problems for very small probabilities easily. But in Breitung (2018, 2019a, 2019b) in numerous examples the shortcomings of SS were revealed. In Breitung (2019b) it was finally demonstrated that SS is in fact a camouflaged Monte Carlo copy of asymptotic SORM. The points computed by SS converge towards the design points as seen for example in the diagrams in Papaioannou (2016). With MCMC one can calculate integrals over F with the pdf $\varphi_n(\mathbf{x})$, but not the normalizing constant $\mathrm{P}(F)$. However, a little trick helps here. Comparing the failure domain F with another domain F having a known probability content $\mathrm{P}(F^*)$; not $\mathrm{P}(F)$ has to be estimated, but the quotient of these two probabilities. A possible choice for this is $F_L = \{\mathbf{x}; g(\mathbf{x}) < 0\}$ given by the linearized LSF gL(x), so $P(F_L) = \Phi(-|\mathbf{x}^*|)$ with $\mathbf{x}^*$ the design point. Running two MCMC's, one on F and one on F_L, by comparing them it is possible to obtain an estimate for the failure probability $\mathrm{P}(F)$ which improves the FORM/SORM estimate. Improving FORM/SORM by MCMC adds the advantages of analytic methods to the flexibility of the Monte Carlo approach. This algorithm works also for non-smooth limit state surfaces. So the costs and deficiencies of the sequential approach of SS style methods are avoided.

Keywords: FORM, SORM, asymptotic analysis, Bennett's acceptance ratio, subset simulation.

1. Introduction

A key problem of failure probability estimation methods is that many of these are focused only on producing some numbers. The whole question at issue should be seen in a larger context. Why numbers are needed and what for? Is the reliability problem for which the probability is computed already in a fixed form and the result of the computation is only some additional information? Or will these results be used to reconfigure the model, i.e. change its parameters? All these questions influence the meaning of results. The numbers are not only numbers, they have a significance which one can understand only from the environment of the model and the involved intentions of the people making the computations.

2. Structuralism and Inverse Problems

Now it will be attempted to look at this hodgepodge of questions and problems from a philosophical viewpoint. In most cases there is given a mathematical/mechanical model. There are the mechanical relations between the elements of the system and there is a set of probability distributions describing the uncertainties for the parameters.

As said in Feyerabend (1993), new theories normally never include the whole content of the older, some parts of the research field explained by those they cannot handle. In structural reliability it is the same. Here it will be outlined later where the newer concepts miss some important points covered by dust-covered approaches as FORM/SORM. And that then by using these stone age tools as starting point one can construct methods which can cope with those used today.

Structuralism is a philosophical viewpoint who sees the main goal to study the relations between elements of structures. Further to find similarities between structures, in mathematics in this case one is especially interested in isomorphisms and projections. More about this can be found in Rickart (1995). If

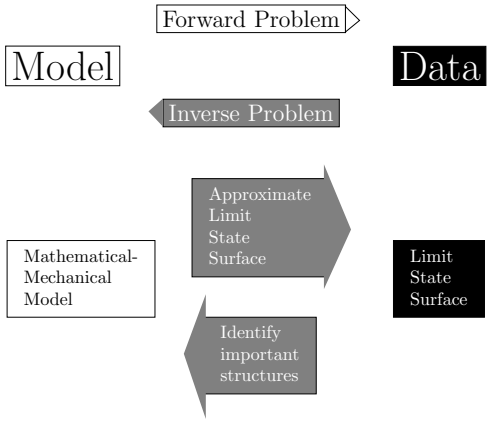

Figure 1: Forward and inverse problems: General and in structural reliability

one takes a structuralist view on structural reliability, what structures can one detect here? This now leads immediately to the concept of inverse problems.

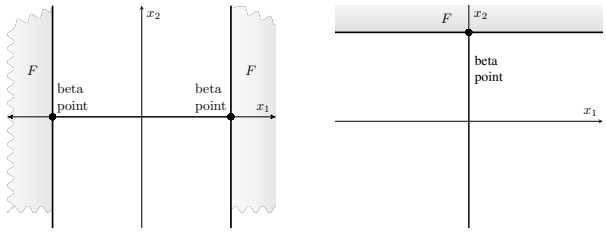

Figure 2: Two different structures

In applied mathematics one can distinguish between forward problems and inverse problems. In a forward problem there is a given model and the goal is to determine additional properties of it. Whereas in inverse problems, from given data one wants to make some inference about the causes which produced these data. The relations between forward and inverse problems in general and in special for strucural reliability are illustrated in Fig.1. For failure probability calculations all important information is in the limit state surface. For its structure one can make inferences about how the parameters of given model influence the occurence of failures.

In structural reliability in many cases one wants not only a failure probability, but also information what is causing the failure of the structure. Monte Carlo methods spit out only numbers. Consider now a simple example as in Fig. 2. Given is a system consisting of two elements X_1 and X_2, all is in the standard normal space. If these two different systems are studied with a MC approach where the LSF is treated as a black box, no basic difference will be seen in the results; a probability estimate is produced with an error bound. The geometry of the problems is ignored. This is not satisfying from the structuralist view.

In structural reliability the asymptotic SORM concept gives a structuralist view by providing estimates as functions of the underlying parameters (Breitung (1994)). Here this approach will be enhanced to give more precise probability estimates and also structural information.

3. Design points in structural reliability

In none of these mathematical fields the fact that the concept of critical points is not a panacea and that there are problems where it does not work has led to the deplorable consequences as in structural reliability. Here in the last decades there was and still is a campaign to discredit this approach as useless. Certainly it is no vice to try to show that a specific scientific method is not efficient. The weak point of these papers is that they do not show a way how to replace the information which design points give about the structure of the reliability problem under consideration. In the contrary they intend to destroy the whole geometric structure which ahs benn built around the FORM/SORm concepts and declare that it is all Monte Carlo, only numbers no structure.

The most cited publications which seem to show that design points are of no particular use in structural reliability are Katafygiotis and Zuev (2007) and Valdebenito et al. (2010). Unfortunately in these papers the asymptotic analysis concepts, the basis of FORM/SORM, are ignored, not even mentioned. The conclusions about design points in these publications are questionable.

The main example in Katafygiotis and Zuev (2007) is a parabolic region in a space with dimension $N = 1000$, defined by

$$g(\mathbf{x}) = x_1 + a\sum_{i=2}^{N} x_i^2 - b \tag{1}$$

with parameter values $a = 0.025,\ b = 20.27$. Now, the authors define as failure domain F the interior of the parabola, i.e. $F = \{g(\mathbf{x}) < 0\}$. Then they continue to say that the point $\boldsymbol{\theta}^* = (-b, 0, \ldots, 0)$ is the design point of this failure domain F. This contradicts the definition of a design point. For a failure domain F a design point $\boldsymbol{\theta}^* \in F$ is a point with $|\boldsymbol{\theta}^*| = \min_F |\boldsymbol{\theta}|$, i.e. a point in the domain (usually on its boundary) which has minimal distance to the origin. The point with them minimal distance origin is for the interior of this parabolic region the origin itself. Therefore all arguments brought forward in this example are not only moot, but simply wrong.

In the second paper the quality of SORM approximations is studied. Now, asymptotic analysis shows that for small failure probabilities the probability mass will be asymptotically in the relative neighborhood of the design points. From this result one can derive using the Laplace method for multivariate integrals the SORM approximations. It is shown in the paper that for increasing dimensions they become worse and worse. The authors conclude: In consequence, probability estimates generated using FORM and SORM are inaccurate in high dimensions. The only meaningful conclusion one can draw from the decreasing quality of the SORM approximations that better approximations will be needed in high dimensional spaces. These results say nothing at all about the role and meaning of design points. Here there is an example of a logical fallacy; asymptotic analysis concepts, i.e. design points, are used to derive the SORM approximation. Now, these approximations are not good for large dimensions; therefore the authors conclude that design points are irrelevant for problems in large dimensions. This is wrong logic, nothing else.

4. Subset simulation (SS), Sequential Importance Sampling and Variants

Before the improved SORM concept is described a short account of the now popular MC methods and their deficiencies will be given. The subset simulation concept is a variant of Monte Carlo methods; here it is tried to avoid the large number of data points needed in the usual Monte Carlo. This is done by using an iterative procedure. The algorithm can be seen as a sort of a stochastic optimization procedure combined with MC integration.

While importance sampling methods try to improve the efficiency of Monte Carlo by identifying regions with high probability content and moving more date points there, SS starts from an enlarged failure domain whose beta points are nearer to the origin and moves step by step towards the original failure domain. These intermediate regions are defined in the form $F_i = \{g(u) < a_i\}$ where the a_i's are positive and $a_i \to 0$. The basic idea (Au and Beck (2001), Au and Wang (2014)) is to write the failure probability $\mathrm{P}(F)$ as a product of conditional probabilities

$$\mathrm{P}(F_n) = \mathrm{P}(F_1|F_0) \cdot \mathrm{P}(F_2|F_1) \ldots \mathrm{P}(F_n|F_{n-1}) = \prod_{k=0}^{n-1} \mathrm{P}(F_{k+1}|F_k)$$

Here $\mathbf{R}^n = F_0 \supset F_1 \supset F_2 \supset \ldots \supset F_n = F$. Since the suitably chosen conditional probabilities are relatively large compared with the failure probability $\mathrm{P}(F)$ which has to be estimated, such an access to the problem has the advantage that these conditional probabilities can be estimated more efficiently with much smaller sample sizes.

The proponents of SS claim that this is a MCMC algorithm. One important point in MCMC methods is that the chains have to run quite long to visit the whole integration domain. SS users claim that the MCMC results are valid fro SS too, since here many short Markov chains are run. However, it is not correct to assume that instead of some very long chains one can replace them with very many short chains. In general this works only if the starting points of the short chains have a stationary distributions over the integration domain. However this is what should be obtained by the MCMC algorithm. This problem is a little bit tricky, it is a sort of catch 22. The justification

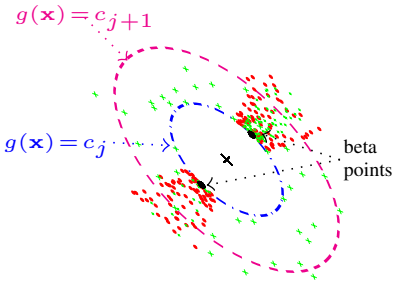

Figure 3: The distribution of points for stationary MCMC (crosses) and for SS (dots)

that SS methods do this is usually the claim that the seeds of the SS sequence in the next domain F_{j+1} have already a stationary distribution in this domain. But this in the subset community often repeated assertion is wrong. This has been shown in Breitung (2018), based on earlier work by Botev and Kroese (2012). From this follows that the data points of the SS method have no stationary distribution in these failure domains F_i, i.e. their PDF is not equal to the standard normal density constrained to F_i. Therefore then the data of the algorithm cluster around the seed points since the chain length is very short, in general it is chosen as ten. Since the seeds are near the beta points, most points are their neighborhood. In Fig. 3 this is shown.

A further clear disadvantage of the method is that with decreasing failure probability the corresponding error variances for the estimator are increasing. The problems in SS methods are:

1. The calculated data sets can converge to local minima and not to the beta points (see Breitung (2019b)),
2. The variance of the estimators increases with decreasing failure probabilities,
3. The information about the location of the beta points obtained from the convergence of the data points is ignored (see Breitung (2019a)).

In the published examples the problem under 1) never shows up; since all examples have a well-behaved LSF which is a linear or homogeneous function. However if one has a complicated black-box algorithm which produces the values of the LSF, how can one verify that the LSF is well-behaved?

The examples the author studied in Breitung (2019b) show that already in simple cases all these methods can lead to wrong results. The trust that the SS-community sets into theirs methods is psychologically understandable. However, is it justified? The counterexamples show that there cannot be a general proof of convergence for these approaches. Maybe one can give a proof for special cases of well-behaved LSF's. This would not be sufficient for the claim that the SS approach works also for complex high-dimensional cases.

The proponents of SS refrain from a precise mathematical explanation what the algorithm is doing and why it is working. One problem which makes it difficult to understand the interior machinery is certainly that it is a twisted up combination of stochastic minimization and integration. Everybody who looks at the diagrams of SS calculations can see that the point clouds converge towards the beta points (Papaioannou et al. (2015), Cui and Ghosn (2019)) as explained earlier. However, in the vast SS literature the concept of beta points is practically never mentioned and is more or less subject to a *damnatio memoriae*. Only in analyzing the algorithm in context with FORM/SORM concepts and asymptotic analysis methods one is able to understand what is going on in the algorithm.

The gist of the arguments for SS is that no information about the structure of the problem is necessary. The user starts the algorithm and automatically a failure probability estimate pops up as result. This may be very nice for the people applying the method, but doing it this way any structural information is lost completely. And there is the clear danger that due to a lack of understanding the structure wrong results are not recognized.

In Monahan (2011), p. 394, it is written about the indiscriminate application of MCMC algorithms:

For MCMC, an extremely naive user can generate a lot of output without even understanding the problem. The lack of discipline of learning about the problem that other methods require can lead to unfounded optimism and confidence in the results.

The important information about the location of the beta points is thrown away. This seems to follow from the ideology of SS which claims to be basically different from the FORM/SORM approach, so any mention of these points is considered almost as an anathema.

The philosophical view here seems to be that the goal of structural reliability methods for failure probability calculation is only to produce numbers, i.e. probabilities. This contradicts the opinion of the author that failure probability calculations should be seen more under a structuralistic point of view. The efficiency of the SS/SiS approach has been illustrated by a deluge of examples. However all these examples have the same deficiency, the structure of the graph of the LSF's. These are always of a simple structure, homogeneous functions or similar types.

The bold conclusion made by the proponents of these concepts that one can conclude from the good results obtained for simple examples that they work also for more complex structured cases is a fallacy. Examples are an important tool in mathematics if they are used in the right way. Especially counterexamples help to understand the limitations of mathematical results and they show the way to possible generalizations. In no way one can give a general proof that a method works by examples only.

5. Bennett's Acceptance Ratio

But there are other methods for better failure probability estimations improving FORM/SORM. The method outlined here for calculating normalizing constants will be applied in the next section to approximate failure probabilities.

To calculate the normalizing constant many methods have been proposed. One of these is Bennett's acceptance ratio abbreviated BAR. This was proposed in Bennett (1976). Originally it was used for calculating free energy differences. This approach seems to be well suited for failure probability calculations. Here only the most simple form will be described. A number of ramifications is possible taking this as starting point. The exposition here is more or less a paraphrase of section 2 in Meng and Wong (1996).

Given are two densities $p_i(\mathbf{x})$, $i = 1,2$ with respect to the Lebesgue measure in the n-dimensional space. The support of p_i is denoted by D_i. These densities are known only up to a normalizing constant, i.e. $p_i(\mathbf{x}) = \frac{q_i(\mathbf{x})}{K_i}$. Further it is assumed that K_1 is known. The algorithm gives a method to estimate the unknown K_2 comparing it to the known K_1.

Let $h(\mathbf{x})$ be an arbitrary function defined on $D_1 \cap D_2$ the intersection of the domains such that

$$0 < \int_{D_1 \cap D_2} |h(\mathbf{x}) p_1(\mathbf{x}) p_2(\mathbf{x}) d\mathbf{x}|\, d\mathbf{x} < \infty. \tag{2}$$

Such a function exists iff $\int_{D_1 \cap D_2} p_1(\mathbf{x}) p_2(\mathbf{x}) d\mathbf{x} > 0$. This quantity is measuring the overlap between the support of the both PDF's. For such a function $h(\mathbf{x})$ one can write the identity:

$$\frac{\int_{D_1 \cap D_2} h(\mathbf{x}) p_1(\mathbf{x}) p_2(\mathbf{x}) d\mathbf{x}}{\int_{D_1 \cap D_2} h(\mathbf{x}) p_1(\mathbf{x}) p_2(\mathbf{x}) d\mathbf{x}} = 1 \tag{3}$$

Replacing in the nominator and the denominator the functions $p_2(\mathbf{x})$ and $p_1(\mathbf{x})$ by their unnormalized densities yields

$$\begin{aligned} & \frac{\int_{D_1 \cap D_2} h(\mathbf{x}) \left[\frac{q_2(\mathbf{x})}{K_2}\right] p_1(\mathbf{x}) d\mathbf{x}}{\int_{D_1 \cap D_2} h(\mathbf{x}) \left[\frac{q_1(\mathbf{x})}{K_1}\right] p_2(\mathbf{x}) d\mathbf{x}} \\ = \quad & \frac{K_1}{K_2} \times \frac{\int_{D_1 \cap D_2} h(\mathbf{x}) q_2(\mathbf{x}) p_1(\mathbf{x}) d\mathbf{x}}{\int_{D_1 \cap D_2} h(\mathbf{x}) q_1(\mathbf{x}) p_2(\mathbf{x}) d\mathbf{x}} = 1 \,. \end{aligned} \tag{4}$$

Using the Rule of Three, one gets:

$$\begin{aligned} K_2 &= K_1 \times \frac{\int_{D_1 \cap D_2} h(\mathbf{x}) q_2(\mathbf{x}) p_1(\mathbf{x}) d\mathbf{x}}{\int_{D_1 \cap D_2} h(\mathbf{x}) q_1(\mathbf{x}) p_2(\mathbf{x}) d\mathbf{x}} \\ &= K_1 \times \underbrace{\frac{I\!E_1(h(\mathbf{x}) q_2(\mathbf{x}))}{I\!E_2(h(\mathbf{x}) q_1(\mathbf{x}))}}_{= r} \end{aligned} \tag{5}$$

Here $I\!E_i(.)$ denotes the expected value with respect to the probability measure with PDF $p_i(\mathbf{x})$. Since the unnormalized densities q_i are zero outside of D_i, one can replace the integral over $D_1 \cap D_2$ by the integral over D_i in the nominator and denominator, respectively. If one knows the quantities on the right side or can estimate them, one gets an estimator for the constant K_2. Important here is that the function $h(\mathbf{x})$ can be chosen freely as long as Eq. 2 is satisfied. So it can be adjusted to decrease the variance of the estimation.

The expected values in the fraction can be estimated using MCMC methods. Assume that one has two MCMC chains, one producing points $\mathbf{x}_i$ with the PDF p_1 as target distribution and the other points $\mathbf{y}_j$ with target distribution PDF p_2, each with run length n. Then an estimator of r is given by:

$$\widehat{r} = \frac{\sum_{i=1}^{n} h(\mathbf{x}_i) q_2(\mathbf{x}_i)}{\sum_{j=1}^{n} h(\mathbf{y}_j) q_1(\mathbf{y}_j)} \tag{6}$$

6. Failure Probability Calculation Using Bennett's Acceptance Ratio

Bennett's acceptance ratio can be used for calculating failure probabilities. Here the difference between both integrals is only the integration domain. The integrand is always the PDF, i.e. in the standard normal case $\varphi_n(\mathbf{x})$. The idea is to compare a failure domain with unknown probability with another one whose probability content is known. Obvious choices for domains with known contents are domains defined in the FORM/SORM algorithms.

The failure regions for the FORM/SORM approximation functions $F_L = \{g_L(\mathbf{x}) < 0\}$ and $F_Q = \{g_Q(\mathbf{x}) < 0\}$ are approximating the original failure domain $F = \{g(\mathbf{x}) < 0\}$. Certainly these domains might be accurate approximations for the failure domain if the dimension is high and/or the shape is complex. However, in general these domains will be nearer to F than the domains $\{g(\mathbf{x}) < a_1\}$ with $a_1 > 0$ usually chosen a starting point in SS/SuS/SiS methods. Further one can assume that in the most cases the probabilities $\mathrm{P}(F)$, $\mathrm{P}(F_L)$ and $\mathrm{P}(F_Q)$ are in the same order of magnitude. In the case of SS and similar methods the first probability $\mathrm{P}(F_1)$ is usually taken as 10^{-2} which is several orders of magnitude away from the true failure probability in most examples.

Let be given two domains F_1 and F_2. For F_1 one knows the probability $\mathrm{P}(F_1)$ and $\mathrm{P}(F_2)$ has to be estimated. If one takes as F_1 either as the linearized domain or the quadratic approximation domain, one can be certain that the domains overlap, since a neighborhood of the beta point is contained in the approximating domain and the original one, i.e. Eq. 2 is satisfied.

For the failure probability estimation, one considers the two PDF's p_i and its unnormalized densities q_i

$$p_i(\mathbf{x}) = \frac{\varphi_n(\mathbf{x}) 1_{F_i}(\mathbf{x})}{\mathrm{P}(F_i)},\ q_i(\mathbf{x}) = 1_{F_i}(\mathbf{x})\varphi_n(\mathbf{x}),\ i = 1,2 \tag{7}$$

with $\varphi_n(\mathbf{x})$ the n-dimensional standard normal density. Choosing now $h(\mathbf{x}) = 1/\varphi_n(\mathbf{x})$ gives then

$$h(\mathbf{x}) q_i(\mathbf{x}) = 1_{F_i} \text{ on } F_1 \cap F_2 \tag{8}$$

Inserting this into Eq. 5 one obtains

$$\mathrm{P}(F_2) = \mathrm{P}(F_1) \times \underbrace{\frac{\mathrm{P}_1(F_1 \cap F_2)}{\mathrm{P}_2(F_1 \cap F_2)}}_{=r_p} \tag{9}$$

Here analogously one writes $\mathrm{P}_i(A) = I\!E_i(1_A)$.

The quotient on the right side one can estimate running two MCMC's, one on F_1 and the other on F_2. Denoting the points of the first chain by $\mathbf{x}_i$ and those of the second by $\mathbf{y}_j$, one obtains using Eq. 6 the following estimator for

$$\widehat{r}_p = \frac{\sum_{i=1}^n 1_{F_2}(\mathbf{x}_i)}{\sum_{j=1}^n 1_{F_1}(\mathbf{y}_j)} \tag{10}$$

or written with the number sign # as

$$\widehat{r}_p = \frac{\#\{\mathbf{x}_i; i = 1 \ldots n,\ \mathbf{x}_i \in F_2\}}{\#\{\mathbf{y}_j; j = 1 \ldots n,\ \mathbf{y}_j \in F_1\}} \tag{11}$$

So the quotient r_p can be estimated by counting the points in the Markov chains over the sets F_i, $i = 1,2$ which are in the other set.

Given a failure domain F_1 whose probability content $\mathrm{P}(F_1)$ is known, one obtains as estimate for $\mathrm{P}(F_2)$

$$\widehat{\mathrm{P}(F_2)} = \mathrm{P}(F_1) \times \widehat{r}_p \tag{12}$$

For this procedure the acronym BM2C (Breitung's

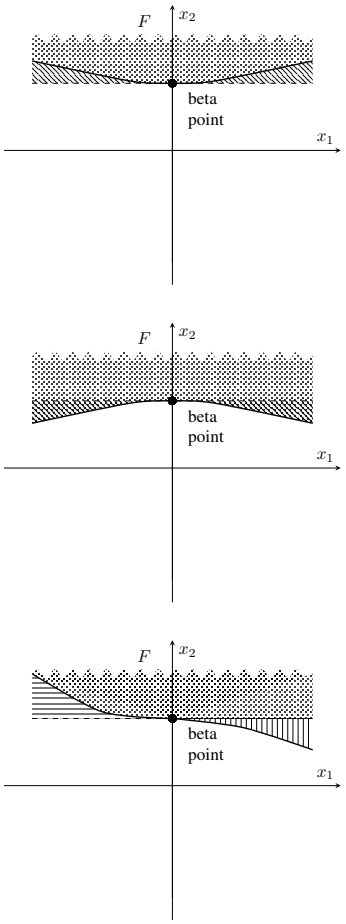

Figure 4: Bennett's acceptance ratio for failure probabilities

Münchhausen Monte Carlo) was proposed in Breitung (2020).

As starting point one can take the linear FORM approximation. The probability $P_L = \{g_L(\mathbf{x}) < 0\}$ is compared with the true failure probability $\mathrm{P}(F)$.

In Fig. 4 three cases are illustrated. As reference set with known probability the linear approximation is taken, its limit is denoted by the dashed horizontal line. In the first case $F_2 \subset F_1$, one has to estimate the

probability of the set $F_1 \setminus F_2$. In the second case $F_1 \subset F_2$, here the probability of $F_2 \setminus F_1$ has to be found. In the third case there are two sets $F_1 \setminus F_2$ (horizontal lines) and $F_2 \setminus F_1$ (vertical lines) whose joint probability has to be determined.

However in many cases the FORM approximation might be too far away and such an approach would need too many MCMC data. So a procedure starting from a SORM approximation can be better. Now on has to recognize that the asymptotic SORM approximation in Eq. 6 has a relative error which goes to zero for $\beta \to \infty$, but cannot be neglected for finite β if one wants to use it as starting point for Bennett's acceptance ratio. Since using such a probability as start would mean calculating the fraction with respect to a biased probability estimate.

But one can use as starting points other SORM approximations for which one knows the exact probability. For this one can calculate the exact probability for parabolas. For an n-dimensional problem this requires the calculation of $n-1$ one-dimensional integrals. Another possible choice – avoiding such integrations – seem to be non-central spheres as outlined in the following. For a given beta point $\mathbf{x}^*$ with $|\mathbf{x}^*| = \beta$ and main curvatures $\kappa_1, \ldots, \kappa_{n-1}$ one takes as approximating failure domain the non-central sphere:

$$F_1 = \{\mathbf{x}; |\mathbf{x} - c| \geq R\} \tag{13}$$

with $c = (R-\beta)|\mathbf{x}^*|^{-1}\mathbf{x}^*$. Here the radius R is chosen as $R = 1/(\max_{1\leq i\leq n-1} \kappa_i) - \beta$ or $R = 1/((n-1)^{-1}\sum_{i=1}^{n-1} \kappa_i) - \beta$. In the first case the curvature of the sphere is equal to the largest of the main curvatures and in the second case to the mean curvature at the beta point. (Here the sign of the curvatures are such that the curvature is positive if the surface bends towards the origin at the point Breitung (2015).

7. Examples

7.1. Approximation of a parabolic failure domain

Table 1: Results for parabolic failure domain

log of failure probability	number of points	mean of estimate	root of mean square error
-4.9935	500	-4.9985	0.12726
-4.9935	1000	-4.9896	0.065905
-4.9935	2000	-4.9929	0.059162

Given is a parabola defining the failure domain:

$$g(x_1, x_2) = \beta - x_2 - \frac{\kappa}{2} x_1^2 \tag{14}$$

As approximating failure domain the linear approximation region $F = \{\beta - x_2 < 0\}$ is taken with probability content $\Phi(-\beta)$. As parameters were taken $\beta = 4$ and $\kappa = -2$. The exact failure probability is $1.0150e-5$. For the data sets with $n = 500, 1000, 2000$ the following results were obtained. The logarithms are to the base 10.

7.2. Sum of exponential random variables

This example is from Breitung (1984). It appears also in Papaioannou et al. (2015) as an example for SS algorithms. Unfortunately a comparison of the results is not possible, since in this paper important information about the performance of the method is missing, i.e. the number of SS steps, the bias of the estimates and the moments of the logarithm of the estimator. The coefficient of variation of the estimates given there is no good indicator of its variation, since the histogram of the data is quite skewed to the right (see Ramsey and Schafer (2012)). So it would have been more meaningful to consider the error of the estimator of the logarithm.

Given are k independent random variables $Y_1, \ldots, Y_k$, each with a standard exponential distribution, i.e. with PDF $f(x) = \exp(-x)$ for $x \geq 0$. The sum $Y = \sum_{j=1}^{k} Y_j$ has then an Erlang distribution with shape parameter n and rate 1. The LSF is given by $g(\mathbf{y}) = n + \alpha\sqrt{n} - \sum_{i=1}^{n} y_i$. Now this is transformed into the standard normal space. There is a unique beta point at :

$$\mathbf{z} = (z, \ldots, z) \text{ with } z = -\Phi^{-1}\left(\exp\left(-\frac{\alpha}{\sqrt{n}} - 1\right)\right) \tag{15}$$

In the original paper there are typos in the equations (19) and (20a,b). The corrections are: Equation (19) corrected:

$$z = -\Phi^{-1}\left(\exp\left(-\frac{\alpha}{\sqrt{n}} - 1\right)\right)$$

Equation (20a) corrected:

$$J_1 = \left\{1 - z\left[\frac{\varphi(z)}{\Phi(-z)} - z\right]\right\}^{n-1}$$

Equation (20b) corrected:

$$P(\tilde{g}(\underline{X}) < 0) \sim \Phi(-\sqrt{n}\ z) \cdot \left\{1 - z\left[\frac{\varphi(z)}{\Phi(-z)} - z\right]\right\}^{-(n-1)/2}$$

The curvature of the limit state surface at the beta point is constant

$$\kappa = \frac{z}{\sqrt{n}}\left[\frac{\varphi(z)}{\Phi(-z)} - z\right] \tag{16}$$

Now a non-central sphere with center $-(|\mathbf{z}|^{-1}\mathbf{z})(1/\kappa - \beta)$ and radius $R = 1/\kappa$ is fitted to the limit state surface at the beta point. This sphere has contact of order two with the limit state surface there (see Struik (1988), p. 23). The probability content of the domain F_S outside of the sphere $\mathrm{P}(F_S)$ is given by the complementary CDF at R^2 of the non-central χ^2 distribution with non-centrality parameter $(1/\kappa - \beta)^2$ and n degrees of freedom.

Using Eq. 11 approximations for the quotient $\mathrm{P}(F)/\mathrm{P}(F_S)$ were computed. For the domains F and F_S each 50 MCMC were run with length $n = 500, 1000, 2000$. As parameters for the LSF were taken $\alpha = 3, 4, 5$.

Table 2: Results for sum of exponential variables, $\alpha = 3,4,5$

log of failure probability	number of points	mean of estimate	root of mean square error
-2.5606	500	-2.1747	0.40348
-2.5606	1000	-2.3048	0.27882
-2.5606	2000	- 2.4299	0.16601
log of failure probability	number of points	mean of estimate	root of mean square error
-3.7930	500	-3.3157	0.49889
-3.7930	1000	-3.4680	0.34424
-3.7930	2000	-3.6086	0.21124
log of failure probability	number of points	mean of estimate	root of mean square error
-5.2273	500	-4.7933	0.44553
-5.2273	1000	-4.8846	0.35833
-5.2273	2000	-4.9397	0.30181

8. CONCLUSIONS

Here the concept of the application of the Bennett's acceptance ratio to failure probability estimation is outlined. This method uses as starting point for approximations the SORM/FORM results. Therefore it is more efficient than SS/SiS procedures in two aspects. The method starts from the regions where the PDF in the failure domain is maximal and the difference between the approximation at the start and the true failure probability is much smaller. Further with this concept one has again a meaning for the geometrical structure of the failure domain and the beta points. This approach allows a number of generalizations and variations which can be tailored to suit complex problems.

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