The Seventh Asian-Pacific Symposium on Structural Reliability and Its Applications (APSSRA2020)
October 4–7 2020, Tokyo, Japan
T. Takada, I. Yoshida & T. Itoi (editors)

Directional Importance Sampling for Reliability Assessment of Linear Structures subject to Dynamic Gaussian Loading

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Abstract: This paper presents an approach for calculating the first excursion probability of linear structural systems subject to stochastic Gaussian loading. The probability estimation is carried out by Directional Sampling in combination with Importance Sampling. The approach fully exploits the linearity of the structural system with respect to the loading. In this way, it is possible to estimate small failure probabilities (within the range of 10^{-3}) with high precision and high efficiency (a few hundred simulations). A numerical example illustrates the application of the sampling technique.

Keywords: First excursion probability, Linear structure, Gaussian load, Directional importance sampling.

1. Introduction

The uncertainty associated with time-dependent loading acting over structural systems can be described by means of stochastic processes (Soong & Grigoriu, 1993). In such case, the response of the structural system becomes uncertain as well. A means to quantify the uncertainty associated with the response consists of calculating the first excursion probability (Au & Beck, 2001), which measures the chances that the response exceeds a prescribed threshold level within the duration of the stochastic load. Calculation of first excursion probabilities remains as one of the most challenging problems in stochastic dynamics. Usually this probability is assessed by means of simulation, as closed form solutions are not known for cases of practical interest (Schuëller et al., 2004). However, applying simulation may not be straightforward, as the computation of the dynamic response for different samples of the input loading can be numerically demanding. In view of this challenge, this contribution explores the application of an approach for the estimation of first excursion probabilities of linear structural systems subject to dynamic Gaussian load. The approach is based on directional importance sampling (Ditlevsen et al., 1988). The main idea behind such sampling technique consists of sampling a direction in the standard normal space using a specially designed importance sampling density function. Afterwards, this sampled direction is explored radially, taking advantage of the linearity of the structural response. The application of directional importance sampling is illustrated by means of an application example involving a large-scale finite element model. The results obtained suggest that small failure probabilities (in the order of 10^{-3} or less) can be estimated with high accuracy and a low number of realizations of the structural response.

2. Problem Formulation

Consider a linear elastic structural system subjected to a stochastic Gaussian loading, whose equation of motion is:

$$\boldsymbol{M}\,\ddot{x}(t,\boldsymbol{z}) + \boldsymbol{C}\dot{\boldsymbol{x}}(t,\boldsymbol{z}) + \boldsymbol{K}\boldsymbol{x}(t,\boldsymbol{z}) = \boldsymbol{g}p(t,\boldsymbol{z}) \qquad (1)$$

where t denotes time; p is the Gaussian loading of duration T, which depends on time t and $\boldsymbol{z}$, the latter being a realization of a standard Gaussian multivariate probability distribution with density function $f_{\boldsymbol{Z}}(\boldsymbol{z})$; $\boldsymbol{x}$ is the vector of displacements; $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ represent the mass, damping and stiffness matrices; and $\boldsymbol{g}$ is a vector that couples the Gaussian loading with the corresponding degrees-of-freedom of the structure. Due to the uncertain nature in the loading, the displacement vector becomes also uncertain.

A possible means for quantifying the uncertainty associated with the structural response is calculating the first excursion probability, which measures the chances that the structural response η (which depends linearly on $\boldsymbol{x}$) exceeds a prescribed threshold level b within the duration of the stochastic loading. Formally, the first excursion probability p_F is expressed by means of the following integral:

$$p_F = \int_{\boldsymbol{z}\in F} f_{\boldsymbol{Z}}(\boldsymbol{z})d\boldsymbol{z} \qquad (2)$$

where F is the set of all values of $\boldsymbol{z}$ for which failure occurs, that is $F = \{\boldsymbol{z}: \eta(t,\boldsymbol{z}) \geq b \wedge t \in [0,T]\}$. In practical problems, the dimensionality of the above integral is quite high, in the order of hundreds or thousands. Furthermore, the response η is not known analytically, but is the result of a structural analysis conducted applying, e.g. the finite element method. These two issues favor the application of Monte Carlo simulation for evaluating the integral associated with the first excursion probability. Nonetheless, Monte Carlo simulation may demand performing repeated structural analyses for different realizations of the stochastic loading, which can become prohibitive when dealing with involved structural models. A possible means for overcoming such issue consists of designing an advanced simulation scheme that exploits specific features of the problem at hand, as described in the sequence.

3. Directional Importance Sampling

As the structural systems considered in this contribution exhibit a linear behavior, it is straightforward to demonstrate that its response is linear with respect to $\boldsymbol{z}$ at each time instant of analysis (Au & Beck, 2001; Der

Kiureghian, 2000). Hence, the failure domain is bounded by a series of hyperplanes, as represented schematically in Figure 1.

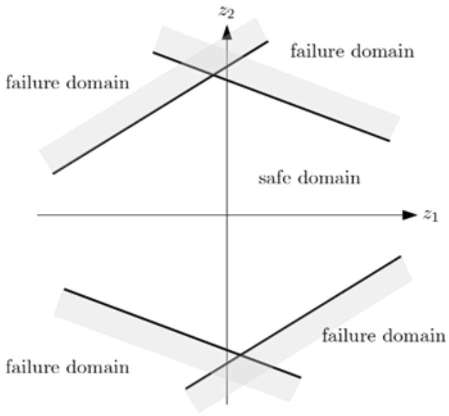

Figure 1. Schematic representation of failure domain.

Considering the geometry of the failure domain, it is possible to formulate the probability integral within the framework of Directional Importance Sampling, that is:

$$p_F = \int_{\boldsymbol{u}\in\Omega_U} \int_{r^*(\boldsymbol{u})}^{\infty} f_R(r) \frac{f_U(\boldsymbol{u})}{f_U^{IS}(\boldsymbol{u})} f_U^{IS}(\boldsymbol{u}) dr d\boldsymbol{u} \tag{3}$$

where $\boldsymbol{u}$ is a vector of unit Euclidean norm that points in the direction of $\boldsymbol{z}$, with associated probability distribution $f_U(\boldsymbol{u})$; Ω_U is the set of all points belonging to the unit hypersphere; r is the Euclidean norm of $\boldsymbol{z}$, with associated probability distribution $f_R(r)$; $r^*(\boldsymbol{u})$ is the value of r that fulfills the equation $b = \eta(r\boldsymbol{u})$; and where $f_U^{IS}(\boldsymbol{u})$ is the importance sampling density function associated with the direction vector. The latter importance sampling density function is equal to a weighted summation of the probability density function associated with $\boldsymbol{u}$ conditioned on the occurrence of the failure event at each discrete time instant. That is:

$$f_U^{IS}(\boldsymbol{u}) = \sum_{k=1}^{n_T} \omega_k f_U(\boldsymbol{u}|F_k) \tag{4}$$

Where ω_k is a real number that acts as a weight (Au & Beck, 2001); n_T is the number of discrete time instants of analysis; and F_k denotes occurrence of the elementary failure event at discrete time t_k, that is $F_k = \{\boldsymbol{z}: \eta(t_k, \boldsymbol{z}) \geq b\}$. It can be shown that explicit expressions associated with $f_U^{IS}(\boldsymbol{u})$ can be deduced by applying Bayes' theorem, as discussed in (Misraji et al., 2020), leading to:

$$f_U^{IS}(\boldsymbol{u}) = \frac{f_U(\boldsymbol{u})}{\hat{P}_F} \sum_{k=1}^{n_T} \left(1 - F_{\chi^2_{n_T}}(d_k(\boldsymbol{u})^2)\right) \tag{5}$$

where $\hat{P}_F$ denotes the summation of the probabilities of occurrence of all elementary failure events (that is, $\hat{P}_F = \sum_{k=1}^{n_T} P[F_k]$); $d_k(\boldsymbol{u}) = b/|\eta(t_k, \boldsymbol{u})|$; and $F_{\chi^2_{n_T}}(\cdot)$ is the Chi-squared distribution of n_T degrees-of-freedom. It should be noted that $\hat{P}_F$ can be calculated in closed form, as discussed in detail (Au & Beck, 2001; Der Kiureghian, 2000).

Eq. (3) is estimated by generating N random samples $\boldsymbol{u}^{(j)}, j = 1, \ldots, N$ that follow the importance sampling density function $f_U^{IS}(\boldsymbol{u})$, resulting in the following estimator for the failure probability (Misraji et al., 2020):

$$p_F \approx \tilde{p}_F = \frac{\hat{P}_F}{N} \sum_{j=1}^{N} \frac{1 - F_{\chi^2_{n_T}}(d_{min}(\boldsymbol{u})^2)}{\sum_{k=1}^{n_T} \left(1 - F_{\chi^2_{n_T}}(d_k(\boldsymbol{u})^2)\right)} \tag{6}$$

where $d_{min}(\boldsymbol{u}) = \min(d_1(\boldsymbol{u}), \ldots, d_{n_T}(\boldsymbol{u}))$.

Numerical experience indicates that a few hundreds of samples allow producing highly accurate estimates of the sought probability.

4. Example

The example involves a 3D finite element model of a bridge structure that involves 10068 degrees-of-freedom, which is shown in the figure below.

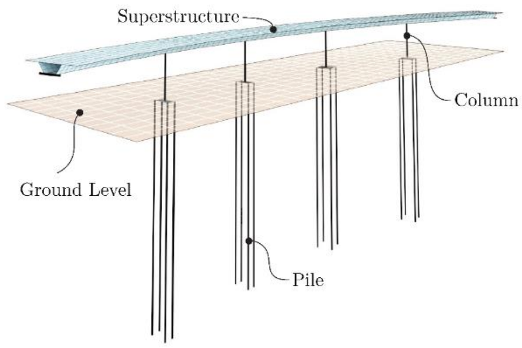

Figure 2. Bridge model, perspective view

This example has been adapted from (Jensen et al., 2015). The bridge consists of a monolithic box girder represented through shell and beam elements. The superstructure is curved and has a total length of 119 [m], divided over five spans of length 24 [m], 20 [m], 23 [m], 25 [m] and 27 [m], respectively. The bridge's substructure is modeled by means of four columns with diameter 1.6 [m] and height 8 [m]. Each of these columns is supported by four piles of 35 [m] length and diameter 0.6 [m]. The interaction between the piles and soil is modeled by means of linear springs with translational stiffness. All elements of the bridge are made of reinforced concrete with Young's modulus 20 [GPa]. Critical damping equal to 3% is considered for all mode shapes. The bridge is excited by a stochastic ground acceleration of 10 [s] duration modeled as a discrete white noise that passes through a Clough-Penzien filter. The time discretization step is equal to 0.01 [s], leading to a total of $n_T = 1001$ discrete time instants of analysis. Evidently,

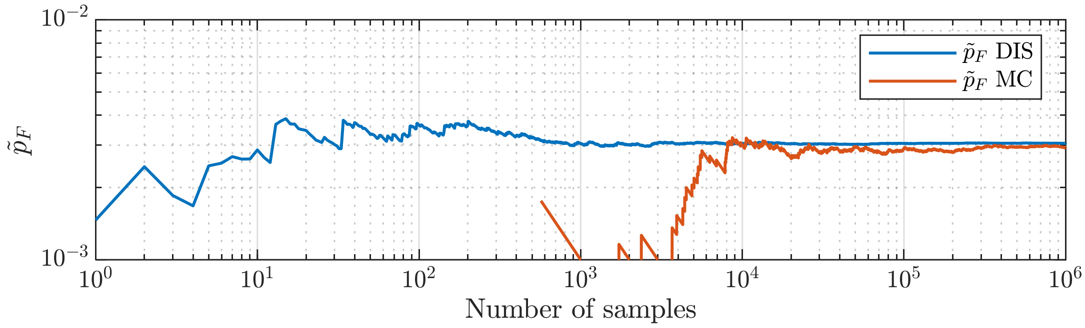

Figure 3. Evolution of failure probability estimator calculated by means of Directional Importance Sampling (DIS) and Monte Carlo Simulation (MCS)

the estimation of the first excursion probability involves a high-dimensional integral.

The failure event is defined as the maximum relative horizontal displacement between the bottom and top of each column exceeding a threshold of $b = 0.02$ [m] within the duration of the stochastic excitation. The failure probability is estimated using both Directional Importance Sampling and Monte Carlo Simulation. In order to assess the convergence properties of the estimator, a large number of samples, equal to $N = 10^6$, is considered. The results obtained are shown in Figure 3. It is readily observed that Directional Importance Sampling produces a precise estimate of the first excursion probability with as few as 100 samples. On the contrary, Monte Carlo requires about 10000 samples for producing an estimate with the same precision.

5. Conclusions

The results presented in this contribution suggest that Directional Importance Sampling allows estimating first excursion probabilities most efficiently. In fact, a reduced number of samples allows estimating the sought probability with a high level of precision.

Future research efforts aim at studying the application of Directional Importance Sampling for estimating probability sensitivity and for performing reliability-based optimization.

6. Acknowledgements

This research is partially supported by ANID (National Agency for Research and Development, Chile) under its program FONDECYT, grant number 1180271, and Universidad Tecnica Federico Santa Maria under its program PAC (Programa Asistente Cientifico 2017). This support is gratefully acknowledged by the authors.

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