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Evaluation of the Structural Reliability under Imperfect Knowledge of Distribution Parameters

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Abstract: We investigate the evaluation of structural reliability under imperfect knowledge about the probability distributions of random variables, with emphasis on the uncertainties of the distribution parameters. When these uncertainties are considered, the failure probability becomes a random variable that is referred to as the conditional failure probability. For the sake of transparency in communicating risk, it is necessary to determine not only the mean but also the quantile of the conditional failure probability. A novel method is proposed for estimating the quantile of the conditional failure probability by using the probability distribution of the corresponding conditional reliability index, in which a point-estimate method based on bivariate dimension-reduction integration is first suggested to compute the first three moments (i.e., mean, standard deviation and skewness) of the conditional reliability index. The probability distribution of the conditional reliability index is then approximated by a three-parameter square normal distribution. The numerical study shows that the computational efficiency of the proposed method was well above that of Monte Carlo simulations without loss of accuracy and show that neglecting parameter uncertainties will lead to the structural reliability being overestimated. The developed methodology provides a complete picture of structural reliability evaluation under imperfect knowledge about probability distributions.

Keywords: structural reliability, parameter uncertainties, conditional failure probability, conditional reliability index, point-estimate method.

1. Introduction

A fundamental problem in structural reliability theory is the computation of the multifold probability integral (Shinozuka 1983)

$$P_f = \int_{G(\mathbf{X})\le 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{1}$$

where P_f is the probability of structural failure. In Eq. (1), $\mathbf{X}=[X_1, X_2, \ldots, X_n]^T$ (where T denotes matrix transposition) is an n-dimensional vector of random variables representing uncertain quantities such as loads, material properties, geometric dimensions, and boundary conditions. Furthermore, $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function (PDF) of $\mathbf{X}$, $G(\mathbf{X})$ is the limit state function or performance function, and $G(\mathbf{X}) \le 0$ is the domain of integration, which denotes the failure region of the structure.

One may regard Eq. (1) as a theoretical formulation of the structural reliability problem because the PDFs of the basic random variables (i.e., the components of $\mathbf{X}$ in Eq. (1)) are generally assumed to be known, and their distribution parameters in the PDFs are usually assumed to be certain. However, in practical engineering, one is faced with the problem of imperfect states of knowledge about such distributions. For example, the distribution parameters of the basic random variables involved in loads, environmental actions including chloride, temperature, oxygen, carbonation, moisture, and structural resistance are estimated from statistical data of limited sample size, and these distribution parameters may change as the amount of corresponding statistical data increases. All this results in uncertainties in the distribution parameters, and parameter uncertainties associated with the basic random variables in $\mathbf{X}$ lead to uncertainty in the calculated failure probability and in the associated reliability index.

In order to consider the uncertainties in the distribution parameters of a structural system, such as the mean and standard deviation of the basic random variables in $\mathbf{X}$, the distribution parameters are treated as a random vector $\mathbf{\Theta}$ in the Bayesian approach, whereby $f_{\mathbf{X}}(\mathbf{x})$ becomes a conditional distribution function $f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})$. Therefore, the conditional probability of failure becomes (Der Kiureghian 1989)

$$P_f(\mathbf{\Theta}) = \int_{G(\mathbf{X},\,\mathbf{\Theta})\le 0} f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})d\mathbf{x} \tag{2}$$

where $G(\mathbf{X},\mathbf{\Theta})$ expresses the performance function, $f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x}, \boldsymbol{\theta})$ is the joint PDF of $\mathbf{X}$ and $\mathbf{\Theta}$, and the conditional failure probability $P_f(\mathbf{\Theta})$ is a function of the distribution parameters $\mathbf{\Theta}$. Because the distribution parameters $\mathbf{\Theta}$ are uncertain, the conditional failure probability $P_f(\mathbf{\Theta})$ is also uncertain. The corresponding conditional reliability index $\beta(\mathbf{\Theta})$ is also uncertain and is given by

$$\beta(\mathbf{\Theta}) = \Phi^{-1}\left[1 - P_f(\mathbf{\Theta})\right] \tag{3}$$

where Φ^{-1} denotes the inverse of the cumulative distribution function (CDF) of a standard normal random variable. Because $P_f(\mathbf{\Theta})$ and $\beta(\mathbf{\Theta})$ are random variables, they have probability distribution functions as well as statistical moments, such as their means, standard deviations, and skewnesses.

For vector $\mathbf{X}$ of the random variables in Eq. (2), whose joint PDF includes uncertain parameters $\mathbf{\Theta}$, the overall probability of failure is then defined as the expectation of the conditional failure probability $P_f(\mathbf{\Theta})$

over the outcome space of the uncertain parameters **Θ**, which can be formulated as

$$P_F = \int_{G(\mathbf{X},\,\mathbf{\Theta})\le 0} f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})d\mathbf{x}d\boldsymbol{\theta} \tag{4}$$

In most cases, Eq. (4) cannot be solved because of the difficulty in determining the explicit expression of the performance function $G(\mathbf{X}, \mathbf{\Theta})$ and the joint PDF $f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x}, \boldsymbol{\theta})$. This is because **Θ** represents the distribution parameters of **X**, but **X** is a function of **Θ**. However, the conditional failure probability of the structural system for given distribution parameter values $\mathbf{\Theta} = \boldsymbol{\theta}$ can be evaluated readily using state-of-the-art techniques such as the first- and second-order reliability methods, moment methods, and simulation methods (Ang and Tang 1984; Zhao and Ono 2001; Choi et al. 2007). Therefore, the overall probability of failure incorporating the uncertainties of the distribution parameters can be formulated generally as (Der Kiureghian 1996)

$$P_F = \int_{\boldsymbol{\theta}} P_f(\mathbf{\Theta}) f_{\Theta}(\boldsymbol{\theta})d\boldsymbol{\theta} \tag{5}$$

where $P_f(\boldsymbol{\theta})$ is the conditional probability of failure for a given $\mathbf{\Theta} = \boldsymbol{\theta}$ (which can be evaluated from state-of-the-art techniques), and $f_{\mathbf{\Theta}}(\boldsymbol{\theta})$ is the joint PDF of **Θ**.

An advanced first-order second method, which developed from the first-order reliability method by introducing an auxiliary variable, for solving Eq. (5) has been proposed by Zhao and Jiang (1992), in which the effect of distribution parameter uncertainties on the overall probability of failure was discussed. An efficient analysis procedure was proposed by Hong (1996) to evaluate the overall probability of failure by using the point-estimate method to discretize the uncertain distribution parameters; the overall probability of failure was then obtained by weighting the conditional probability of failure at each discrete point. Later, Der Kiureghian (2008) derived a simple approximate formula by using the first-order approximation method to compute the mean of the conditional reliability index. It should be noted that the aforementioned studies were focused mainly on evaluating the overall probability of failure, which is defined essentially as the mean of the conditional failure probability $P_f(\mathbf{\Theta})$ when the parameter uncertainties of basic random variables **X** are considered.

For the sake of transparency in communicating risk, it is necessary to determine not only the mean value but also the quantile or even the probability distribution of the conditional failure probability $P_f(\mathbf{\Theta})$, or the corresponding conditional reliability index $\beta(\mathbf{\Theta})$. Der Kiureghian (2009) obtained the probability distributions of the conditional reliability index $\beta(\mathbf{\Theta})$ and the corresponding conditional probability of failure for cases in which the explicit PDF of $\beta(\mathbf{\Theta})$ could be determined easily. However, in general, the explicit PDF of the conditional reliability index $\beta(\mathbf{\Theta})$ cannot be obtained in engineering practice. It is in this regard that Ang and De Leon (2005) utilized Monte Carlo simulation (MCS) to obtain both the mean and quantile of the conditional failure probability $P_f(\mathbf{\Theta})$. However, this is time-consuming for large-scale structures because very many samples are required.

The objective of the present paper is therefore to develop an efficient method for evaluating the quantile or even the distribution of the conditional failure probability $P_f(\mathbf{\Theta})$, or the corresponding conditional reliability index $\beta(\mathbf{\Theta})$. This paper is organized as follows. In Section 2, a point-estimate method based on univariate dimension reduction integration is used to approximate the mean of the conditional failure probability. In Section 3, a novel method is proposed for estimating the quantile of the conditional failure probability by using the probability distribution of the corresponding conditional reliability index. In the same section, a point-estimate method based on bivariate dimension-reduction integration is first suggested to compute the first three moments (i.e., mean, standard deviation and skewness) of the conditional reliability index. The probability distribution of the conditional reliability index is then approximated by a three-parameter square normal distribution (Zhao et al. 2001), in which three parameters in the probability distribution are directly defined in terms of its first three moments. In Section 4, to demonstrate the accuracy and efficiency of the proposed methodology for evaluating structural reliability under imperfect knowledge about the probability distributions, we present a numerical example of conditional failure probability with implicit expression, and we conduct MCS for comparison. Finally, we summarize the main conclusions of the present paper in Section 5.

2. Point-estimate Method for Evaluating the Mean of the Conditional Failure Probability

We note that the right-hand side of Eq. (5) represents the mean of the conditional failure probability $E[P_f(\mathbf{\Theta})]$. Therefore, the overall probability of failure incorporating the uncertainties of the distribution parameters is essentially the problem of estimating the mean of the conditional failure probability $P_f(\mathbf{\Theta})$. Rewriting Eq. (5) in standard normal space, we obtain

$$P_F = E[P_f(\mathbf{\Theta})] = \int_{\mathbf{u}} P_f[T^{-1}(\mathbf{u})]\phi(\mathbf{u})d\mathbf{u} \tag{6}$$

where $T^{-1}(\mathbf{u})$ denotes the inverse Rosenblatt transformation and $\phi(\mathbf{u})$ denotes the PDF of each standard normal variables.

Equation (6) gives the mean of the conditional failure probability $P_f(\mathbf{\Theta})$, which is a function of the random vector **Θ**. In practice, the integral in Eq. (6) cannot be evaluated analytically because of its high dimensionality and the complicated integration required. To avoid this problem, we use the point-estimate method (Zhao and Ono 2000a) to solve Eq. (6), i.e., we evaluate the mean of $P_f(\mathbf{\Theta})$, which is one of the moments of function $P_f(\mathbf{\Theta})$.

Using the standard point estimate, the mean of $P_f(\mathbf{\Theta})$ (i.e., P_F) is estimated as

$$P_F = E[P_f(\mathbf{\Theta})] = \sum\prod_{i=1}^{n} P_{ci}\left\{P_f\left[T^{-1}\left(u_{c1},\ \ldots,\ u_{ci},\ \ldots,\ u_{cn}\right)\right]\right\} \tag{7}$$

where n is the dimension of random vector **Θ**, c is a distinct combination of n items from group [1, 2, …, m], m is the number of estimating points, ci is the ith item of c, u_{ci} is the cith estimating point, and P_{ci} is the weight corresponding to u_{ci}.

Because all distinct combinations have to be considered, m^n function calls are required to compute $P_f(\mathbf{\Theta})$. Therefore, the computations involved in Eq. (7) can be massive if n is large. To avoid this problem, we need to adopt dimension-reduction integration (Xu and Rahaman 2004). Because only the first-order moment (i.e., the mean of $P_f(\mathbf{\Theta})$) is considered, the univariate dimension-reduction method (Rahaman and Xu 2004) is used here. The function $P_f(\mathbf{\Theta})$ may then be approximated by $P_f{}^*(\mathbf{\Theta})$ as follows

$$P_f(\mathbf{\Theta}) \cong P_f{}^*(\mathbf{\Theta}) = \sum_{i=1}^{n} P_{f\,i} - (n-1)P_f(\boldsymbol{\mu}) \tag{8}$$

where

$$P_{f\,i} = P_f(\mathbf{\Theta}_i) = P_f[T^{-1}(\mathbf{U}_i)] \tag{9}$$

and $\boldsymbol{\mu}$ represents the vector in which all the random variables take their mean values. In addition, $P_f(\boldsymbol{\mu})$ is a constant because it is the function of the mean of each random variable. Furthermore, we have $\mathbf{\Theta}_i = [\mu_1, \ldots, \mu_{i-1}, \theta_i, \mu_{i+1}, \ldots, \mu_n]^T$; $\mathbf{U}_i = [u_{\mu 1}, \ldots, u_{\mu i-1}, u_i, u_{\mu i+1}, \ldots, u_{\mu n}]^T$, where $u_{\mu k}$, $k = 1, \ldots, n$ except i is the kth value of u_μ, which is the vector in u-space corresponding to $\boldsymbol{\mu}$. Finally, P_{fi} is a function of only u_i for specific $P_f{}^*(\mathbf{\Theta})$. For independent random variables $\mathbf{\Theta}$, P_{fi} can be expressed simply as

$$\begin{aligned} P_{fi} &= P_f\left(\mu_1, \ldots, \mu_{i-1}, \theta_i, \mu_{i+1}, \ldots, \mu_n\right) \\ &= P_f\left[\mu_1, \ldots, \mu_{i-1}, T^{-1}(u_i), \mu_{i+1}, \ldots, \mu_n\right] \end{aligned} \tag{10}$$

Observe that u_i ($i = 1, \ldots, n$) are independent and P_{fi} is a function of only u_i; therefore, P_{fi}, $i = 1, \ldots, n$ are also independent. Hence, the mean of $P_f{}^*(\mathbf{\Theta})$, i.e., the mean of the conditional failure probability, can be expressed as

$$P_F = E[P_f(\mathbf{\Theta})] \cong E[P_f{}^*(\mathbf{\Theta})] = \sum_{i=1}^{n} \mu_{P_{fi}} - (n-1)P_f(\boldsymbol{\mu}) \tag{11}$$

where μ_{Pfi} is the mean value of P_{fi} and can be point-estimated from

$$\mu_{P_{fi}} = E\left(P_{fi}\right) = E\left\{P_f\left[T^{-1}(\mathbf{U}_i)\right]\right\} = \sum_{k=1}^{m} P_k P_f\left[T^{-1}(u_{ik})\right] \tag{12}$$

where $u_{i1}, u_{i2}, \ldots, u_{im}$ are the estimating points of random variable u_i, and $P_1, P_2, \ldots, P_m$ are the corresponding weights.

The estimating points u_{ik} and their corresponding weights P_k can be readily obtained as

$$u_{ik} = \sqrt{2}x_k\ ,\ P_k = \frac{w_k}{\sqrt{\pi}} \tag{13}$$

where x_k and w_k are the abscissas and weights, respectively, for Hermite integration with the weight function $\exp(-x^2)$ that can be found in Abramowitz and Stegum (1972).

3. A Novel Method for the Evaluation of Quantile of the Conditional Failure Probability

In order to quantitatively estimate the uncertainty in the failure probability $P_f(\mathbf{\Theta})$ induced by the parameter uncertainties, it is often necessary to obtain the quantile of $P_f(\mathbf{\Theta})$. For this purpose, the distributions of $P_f(\mathbf{\Theta})$ need to be determined. Because $P_f(\mathbf{\Theta})$ is a monotonic function of $\beta(\mathbf{\Theta})$, the same values of the quantile of $P_f(\mathbf{\Theta})$ or $\beta(\mathbf{\Theta})$ can be obtained using the distribution of $P_f(\mathbf{\Theta})$ or $\beta(\mathbf{\Theta})$. Because the variability of $\beta(\mathbf{\Theta})$ is much smaller than that of $P_f(\mathbf{\Theta})$, in this paper we will approximate the distribution of $\beta(\mathbf{\Theta})$ rather than that of $P_f(\mathbf{\Theta})$.

3.1 First three moments of the conditional reliability index

Using the standard point estimate, the first three moments of conditional reliability index $\beta(\mathbf{\Theta})$ can be estimated as

$$\mu_\beta = \sum\prod_{i=1}^{n} P_{ci}\left\{\beta\left[T^{-1}\left(u_{c1}, \ldots, u_{ci}, \ldots, u_{cn}\right)\right]\right\} \tag{14}$$

$$\sigma_\beta^2 = \sum\prod_{i=1}^{n} P_{ci}\left\{\beta\left[T^{-1}\left(u_{c1}, \ldots, u_{ci}, \ldots, u_{cn}\right) - \mu_\beta\right]\right\}^2 \tag{15}$$

$$\alpha_{3\beta}\sigma_\beta^3 = \sum\prod_{i=1}^{n} P_{ci}\left\{\beta\left[T^{-1}\left(u_{c1}, \ldots, u_{ci}, \ldots, u_{cn}\right) - \mu_\beta\right]\right\}^3 \tag{16}$$

where μ_β, σ_β, and $\alpha_{3\beta}$ are the first three moments, i.e., the mean, standard deviation, and skewness of $\beta(\mathbf{\Theta})$, respectively.

Like the computation of Eq. (7), the computation involved in Eqs. (14)-(16) requires m^n function calls to determine the conditional reliability index $\beta(\mathbf{\Theta})$; hence, the computation becomes excessive if n is large. To avoid this problem, we again adopt dimension-reduction integration. Because the first three moments of $\beta_f(\mathbf{\Theta})$ are considered, bivariate dimension-reduction is used here.

The function $\beta(\mathbf{\Theta})$ can then be approximated by $\beta^*(\mathbf{\Theta})$ as follows

$$\begin{aligned} \beta(\mathbf{\Theta}) &\cong \beta^*(\mathbf{\Theta}) = \beta^*\left[T^{-1}(\mathbf{U})\right] \\ &= \sum_{i<j} \beta_{i,j} - (n-2)\sum_{i=1}^{n} \beta_i + \frac{(n-1)(n-2)}{2}\beta_0 \end{aligned} \tag{17}$$

where

$$\beta_{i,j} = \beta_{i,j}\left[\mu_1, \ldots, T^{-1}(u_i), \ldots, T^{-1}(u_j), \ldots, \mu_n\right] \tag{18}$$

$$\beta_i = \beta_i\left[\mu_1, \ldots, T^{-1}(u_i), \ldots, \mu_n\right] \tag{19}$$

$$\beta_0 = \beta\left(\mu_1, \ldots, \mu_i, \ldots, \mu_n\right) \tag{20}$$

where $\beta_{i,j}$ is a two-dimensional function, $i, j = 1, 2, \ldots, n$ and $i < j$. Furthermore, β_i is a one-dimensional function and β_0 is a constant.

Hence, using the inverse Rosenblatt transformation (Rackwitz and Fiessler 1978), the kth raw moments of $\beta(\mathbf{\Theta})$, $\mu_{k\beta}$, can be formulated approximately as

$$\begin{aligned} \mu_{k\beta} &= E\left\{\left[\beta(\mathbf{\Theta})\right]^k\right\} \cong E\left\{\left[\beta^*(\mathbf{\Theta})\right]^k\right\} = E\left\{\left\{\beta^*\left[T^{-1}(\mathbf{U})\right]\right\}^k\right\} \\ &\cong \sum_{i<j} \mu_{\beta_{i,j}}^k - (n-2)\sum_{i=1}^{n} \mu_{\beta_i}^k + \frac{(n-1)(n-2)}{2}\beta_0^k \end{aligned} \tag{21}$$

where

$$\beta_0^k = \left[\beta\left(\mu_1,\ ...,\ \mu_i,\ ...,\ \mu_n\right)\right]^k \tag{22}$$

$$\mu_{\beta_i}^k = \int_{-\infty}^{\infty}\left\{\beta_i\left[\mu_1,...,T^{-1}\left(u_i\right),\ ...,\mu_n\right]\right\}^k \phi(u_i)du_i \tag{23}$$

$$\mu_{\beta_{i,j}}^k = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\beta_{i,j}\left[\mu_1,...,T^{-1}(u_i),\ ...,T^{-1}(u_j),...,\mu_n\right]\right\}^k \phi(u_i)\phi(u_j)du_i du_j \tag{24}$$

Using the point-estimate method (Zhao and Ono 2000a), the one-dimensional integral in Eq. (23) can be estimated as follows

$$\mu_{\beta_i}^k = \sum_{r=1}^{m} P_r\left\{\beta_i\left[\mu_1,\ ...,\ T^{-1}\left(u_r\right),\ ...,\mu_n\right]\right\}^k \tag{25}$$

Similarly, the two-dimensional integral in Eq. (24) can be estimated as

$$\mu_{\beta_{i,j}}^k = \sum_{r_1=1}^{m}\sum_{r_2=1}^{m} P_{r_1}P_{r_2}\left\{\beta_{i,j}\left[\mu_1,\ ...,T^{-1}\left(u_{r_1}\right),\ ...,T^{-1}\left(u_{r_2}\right),\ ...,\mu_n\right]\right\}^k \tag{26}$$

The estimating points and the corresponding weights can be found in the work of Abramowitz and Stegun (1972).

Finally, the mean, standard deviation, and skewness of the conditional reliability index $\beta(\mathbf{\Theta})$ can be estimated, respectively, as follows

$$\mu_\beta = \mu_{1\beta} \tag{27}$$

$$\sigma_\beta = \sqrt{\mu_{2\beta} - \mu_{1\beta}^2} \tag{28}$$

$$\alpha_{3\beta} = (\mu_{3\beta} - 3\mu_{2\beta}\mu_{1\beta} + 2\mu_{1\beta}^3)/\sigma_\beta^3 \tag{29}$$

3.2 Probability distribution of the conditional reliability index

After the first three moments of the conditional reliability index $\beta(\mathbf{\Theta})$ are obtained, the probability distribution of $\beta(\mathbf{\Theta})$ can be approximated by using a three-parameter probability distribution, in which the three parameters in the probability distribution are directly defined in terms of its first three moments. Here the square normal distribution (Zhao et al. 2001) based on the third-moment standardization function (Zhao and Ono 2000b) is used, and the PDF of the conditional reliability index $\beta(\mathbf{\Theta})$ can then be expressed as

$$f_\beta\left[\beta(\mathbf{\Theta})\right] = \frac{3\phi\left[\frac{1}{\alpha_{3\beta}}\left(\sqrt{9+\frac{1}{2}\alpha_{3\beta}^2+6\alpha_{3\beta}\frac{\beta(\mathbf{\Theta})-\mu_\beta}{\sigma_\beta}}-\sqrt{9-\frac{1}{2}\alpha_{3\beta}^2}\right)\right]}{\sigma_\beta\sqrt{9+\frac{1}{2}\alpha_{3\beta}^2+6\alpha_{3\beta}\frac{\beta(\mathbf{\Theta})-\mu_\beta}{\sigma_\beta}}} \tag{30}$$

and the CDF of $\beta(\mathbf{\Theta})$ is expressed as

$$F_\beta\left[\beta(\mathbf{\Theta})\right] = \Phi\left[\frac{1}{\alpha_{3\beta}}\left(\sqrt{9+\frac{1}{2}\alpha_{3\beta}^2+6\alpha_{3\beta}\frac{\beta(\mathbf{\Theta})-\mu_\beta}{\sigma_\beta}}-\sqrt{9-\frac{1}{2}\alpha_{3\beta}^2}\right)\right] \tag{31}$$

where $\Phi(\cdot)$ is the CDF of a standard normal random variable u.

3.3 Quantile of the conditional failure probability

Assuming that the confidence level of the conditional failure probability $P_f(\mathbf{\Theta})$ is α, the fractile of the conditional reliability index $\beta(\mathbf{\Theta})$ will be $1 - \alpha$. The quantile corresponding to the confidence level α can then be determined by the following equation

$$F_\beta\left[\beta(\alpha)\right] = \Phi\left[\frac{1}{\alpha_{3\beta}}\left(\sqrt{9+\frac{1}{2}\alpha_{3\beta}^2+6\alpha_{3\beta}\frac{\beta(\alpha)-\mu_\beta}{\sigma_\beta}}-\sqrt{9-\frac{1}{2}\alpha_{3\beta}^2}\right)\right] = 1-\alpha \tag{32}$$

Therefore, the quantile corresponding to the confidence level α is given as

$$\beta(\alpha) = \mu_\beta + \sigma_\beta\left\{-\frac{\alpha_{3\beta}}{6} + \Phi^{-1}(1-\alpha) + \frac{\alpha_{3\beta}}{6}\left[\Phi^{-1}(1-\alpha)\right]^2\right\} \tag{33}$$

Therefore, the corresponding failure probability of the confidence level $1 - \alpha$ is given as

$$P_f(1-\alpha) = \Phi\left[-\beta(\alpha)\right] \tag{34}$$

4. Application Example

This example considers an existing reinforced concrete T-beam bridge as shown in Fig. 1, which has been investigated by Wang et al. (2015). The bridge consists of a 19.5-m simply supported span and five beams spaced equally at 1.6-m intervals. The reinforcement diameter is 32 mm. The thickness of the wearing surface (asphalt) is 5.0 cm. The initial yield strength of the reinforcement is 280 MPa. Considering the randomness of the resistance and load effects, the flexural limit state function in the midspan cross-section can be written as

$$Z = G(\mathbf{X}) = K_P \cdot f_g A_g\left(h_0 - \frac{f_g A_g}{2f_c \cdot b_i}\right) - S_G - S_Q \tag{35}$$

where R is the resistance of the bar and S is the applied load. where K_P is normally distributed with a mean of 1.098 and a standard deviation of 0.078. Furthermore, f_g is the yield strength of the corroded reinforcement: $f_g = f_{g0}(1 - \eta)$, where η is the steel strength loss, and f_{g0} is the initial yield strength of the reinforcement. In addition, A_g is the area of reinforcement, f_c is the concrete strength, S_G is the effect of the dead load, S_Q is the effect of a live load, $h_0 = 1.1$ m, and $b_i = 1.6$ m.

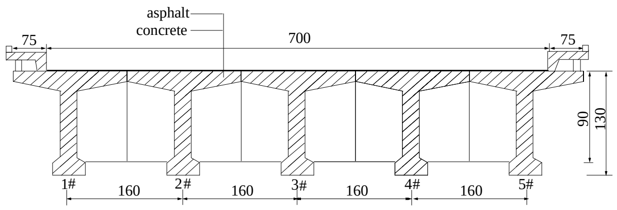

Figure 1. Cross-section of the concrete bridge (cm)

The concrete strength f_c (MPa), steel strength loss η (%), dead load effect S_G (kN·m), and the live load effect S_Q (kN·m) are assumed as random variables, the probabilistic information of which is listed in Table 1.

The reliability analysis for the performance function as expressed by Eq. (35) can be readily evaluated using state-of-the-art techniques. Here, the well-known first-order reliability method (FORM) (Hasofer and Lind

1974) is utilized, and the reliability index is readily obtained as 5.341, with the corresponding probability of failure as 4.614×10^{-8}.

Table 1. Probabilistic information about the random variables

Variable	Distribution	Mean	Standard deviation
f_c (MPa)	Normal	20	1.45
η (%)	Normal	22	2.04
S_G (kN·m)	Normal	613.18	26.45
S_Q (kN·m)	Gumbel	535.42	84.08

In this example, the distribution parameters (i.e., mean and standard deviation) of the four random variables, i.e., μ_{fc}, μ_η, μ_{SG}, μ_{SQ}, σ_{fc}, σ_η, σ_{SG}, and σ_{SQ} are assumed to be random variables, and their probabilistic information is listed in Table 2. Estimating the mean value and quantile of the conditional failure probability is described below.

Table 2. Probabilistic information for distribution parameters

Variable	Distribution	Mean	Standard deviation
μ_{fc}	Normal	21.34	1.72
μ_η	Normal	22.08	2.56
μ_{SG}	Normal	597.03	28.36
μ_{SQ}	Normal	610	7.14
σ_{fc}	Lognormal	1.51	0.072
σ_η	Lognormal	2.67	0.12
σ_{SG}	Normal	25.32	2.16
σ_{SQ}	Normal	96.23	4.68

According to Eq. (5), the overall failure probability can be formulated as

$$P_F = \int_{G(\mathbf{X},\,\mathbf{\Theta})\le 0} f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})d\mathbf{x}d\boldsymbol{\theta} = \int_{\boldsymbol{\theta}} P_f(\mathbf{\Theta}) f_{\Theta}(\boldsymbol{\theta})d\boldsymbol{\theta} \quad (36)$$

According to Eq. (8), the conditional failure probability $P_f(\mathbf{\Theta})$ can be approximated as

$$P_f(\mathbf{\Theta}) \cong P_f^*(\mathbf{\Theta}) = \sum_{i=1}^{8} P_{fi} - 7P_f(\boldsymbol{\mu}) \quad (37)$$

Although explicit expressions for P_{fi} and $P_f(\boldsymbol{\mu})$ are not available, they can be easily estimated by using FORM. Because $P_f(\boldsymbol{\mu})$ is a function of the means of all eight random variables, we replace the original mean and standard deviation of the four random variables in Table 1 by means of these parameters as given in Table 2, whereby $P_f(\boldsymbol{\mu})$ can then be easily obtained as 5.919×10^{-7} by using FORM.

Therefore, according to Eq. (11), the overall probability of failure, i.e., the mean of the conditional failure probability, is readily estimated as

$$P_F = E[P_f(\mathbf{\Theta})] \cong \sum_{i=1}^{8} \mu_{P_{fi}} - 7P_f(\boldsymbol{\mu}) = 1.001\times 10^{-6} \quad (38)$$

Using the MCS with 1,000,000 samples, the overall probability of failure, i.e., the mean of the conditional failure probability, is obtained as 1.052×10^{-6}. One can see that the result obtained by using the proposed method is almost the same as that of MCS. The mean of the conditional failure probability when considering the parameter uncertainties (1.001×10^{-6}) is larger than the failure probability without considering the parameter uncertainties (4.614×10^{-8}).

According to Eq. (17), the conditional reliability index $\beta(\mathbf{\Theta})$ can be approximated as

$$\beta(\mathbf{\Theta}) \cong \beta^*(\mathbf{\Theta}) = \beta^*\left[T^{-1}(\mathbf{U})\right] = \sum_{i<j}\beta_{i,j} - 6\sum_{i=1}^{8}\beta_i + 21\beta_0 \quad (39)$$

Using the proposed point-estimate method based on bivariate dimension-reduction integration, i.e., Eqs. (21)–(29), in which the estimation of the reliability indices for determining $\mu_{\beta_i}^k$ and $\mu_{\beta_{i,j}}^k$ in Eqs. (25) and (26) is evaluated from Eq. (35) using FORM, the first three moments of $\beta(\mathbf{\Theta})$ are easily obtained as, $\mu_\beta = 4.856$, $\sigma_\beta = 0.218$, and $\alpha_{3\beta} = -0.023$, respectively. Substituting the obtained first three moments of $\beta(\mathbf{\Theta})$ into Eq. (30), the PDF of $\beta(\mathbf{\Theta})$ is expressed as

$$f_\beta\left[\beta(\mathbf{\Theta})\right] = \frac{13.76\phi\left[-43.48\left(\sqrt{9-0.63\left[\beta(\mathbf{\Theta})-4.86\right]}-3\right)\right]}{\sqrt{9-0.63\left[\beta(\mathbf{\Theta})-4.86\right]}} \quad (40)$$

The histogram of the conditional reliability index $\beta(\mathbf{\Theta})$ obtained by using the MCS with 1,000,000 samples is shown in Fig. 2, together with the PDF curve (denoted as the thick solid line) obtained from Eq. (40). It can be seen from Fig. 2 that the histogram of the conditional reliability index $\beta(\mathbf{\Theta})$ is well behaved and can be approximated well by the PDF of the square normal distribution determined by using its first three moments.

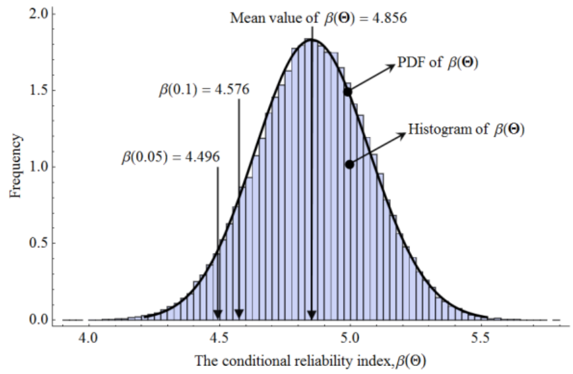

Figure 2. Histogram and PDF curve of the conditional reliability index

The histogram of the conditional failure probability $P_f(\mathbf{\Theta})$ obtained by using the MCS with 1,000,000 samples is shown in Fig. 3. It can be seen in Fig. 3 that the histogram of $P_f(\mathbf{\Theta})$ is skewed to the right and is truncated when $P_f(\mathbf{\Theta})$ tends to zero, which is difficult to approximate by well-known distributions.

According to Eq. (33), the 10% and 5% fractiles of $\beta(\mathbf{\Theta})$ can be obtained as $\beta(0.1) = 4.576$ and $\beta(0.05) = 4.496$, respectively, which are also shown in Fig. 2. Then, according to Eq. (34), the corresponding 90% and 95% confidence levels of $P_f(\mathbf{\Theta})$ are readily obtained as $P_f(0.9) = 2.370\times 10^{-6}$ and $P_f(0.95) = 3.462\times 10^{-6}$, respectively. Using MCS, the 90% and 95% confidence levels of $P_f(\mathbf{\Theta})$

are easily obtained as P_f (0.9) =2.369 × 10^{-6} and P_f (0.95) = 3.463 × 10^{-6}, which are also shown in Fig. 3.

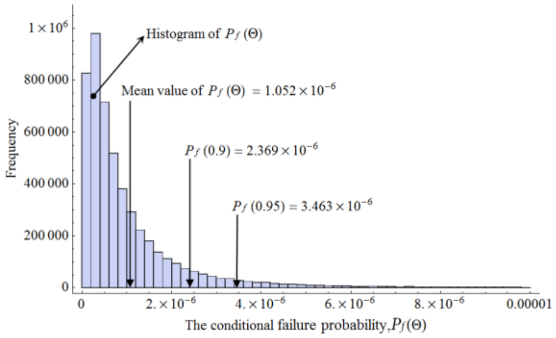

Figure 3. Histogram of the conditional failure probability

From the discussion above, it can be concluded that although it is very simple to determine the quantile of $P_f(\boldsymbol{\Theta})$ by utilizing the proposed probability distribution for the conditional reliability index, the results estimated by the proposed method are almost the same as those obtained by MCS method.

5. Conclusions

We have investigated the evaluation of structural reliability under imperfect knowledge about the probability distributions of the basic random variables, with emphasis on the uncertainties of the distribution parameters. The main contributions and conclusions are summarized as follows.

1) A point-estimate method based univariate dimension-reduction integration was used to approximate the mean of the conditional failure probability.

2) A novel method was proposed for estimating the quantile of the conditional failure probability by using the probability distribution of the corresponding conditional reliability index. In this approach, the point-estimate method based on bivariate dimension-reduction integration was first suggested for computing the first three moments of the conditional reliability index. The probability distribution of the conditional reliability index was then approximated by a three-parameter square normal distribution with explicit expression.

3) A numerical example was studied: conditional failure probability and the corresponding reliability index with implicit expression. It was found that the results obtained from the proposed method were in close agreement with those from MCS. It also showed that neglecting parameter uncertainties led to the structural reliability being overestimated.

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