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Structural Reliability Analysis without Exclusion of the Distribution Parameter Uncertainties

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Abstract: In conventional structural reliability evaluation, the probability distributions of the basic random variables are generally assumed to be known and their distribution parameters are usually assumed to be certain. However, since the probability distributions are estimated from statistical data of limited sample size, their distribution parameters or types may change as the amount of statistical data increases. If the parameter uncertainties are considered in structural reliability evaluation, the probability of failure and the corresponding reliability index become random variables, which are referred as the conditional failure probability and the corresponding conditional reliability index, respectively. Therefore, it is necessary to determine not only the mean but also the quantile or even the probability distribution of the conditional failure probability or conditional reliability index. Since the determination of the probability distribution of which is the focus of this study. For this purpose, the first four moments (i.e., mean, standard deviation, skewness and kurtosis) of the conditional reliability index are firstly computed by a point-estimate method based on bivariate dimension-reduction integration. The probability distribution of the conditional reliability index is then approximated by a four-parameter cubic normal distribution, in which four parameters in the probability distribution are directly defined in terms of its first four moments. Finally, an explicit formula for the quantile of the conditional failure probability is obtained by using the probability distribution of the corresponding conditional reliability index. The efficiency and accuracy of the proposed methodology for structural reliability assessment considering the uncertainties of distribution parameters are demonstrated through numerical examples, where Monte-Carlo simulations are utilized for comparison.

Keywords: structural reliability, parameter uncertainties, conditional reliability index, point-estimate method, cubic normal distribution.

1. Introduction

A fundamental problem in structural reliability theory is the computation of the multifold probability integral

$$P_f = \int_{G(\mathbf{X})\le 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{1}$$

where P_f is the probability of failure, $\mathbf{X} = [X_1, X_2, \ldots, X_n]^T$ is an n-dimensional vector of random variables representing uncertain quantities such as applied loads, material properties, geometric dimensions, and boundary conditions. $f_{\mathbf{X}}(\mathbf{x})$ represents the joint probability density function (PDF) for $\mathbf{X}$. $G(\mathbf{X})$ is the limit state function and failure occur when $G(\mathbf{X}) \le 0$.

The probability distributions of the basic random variables (i.e., the components of $\mathbf{X}$ in Eq. (1)) are generally assumed to be known and their distribution parameters are usually assumed to be certain. However, in practical application, one is faced with the problem that distribution parameters of some random variables considered in a limit state function are also uncertain. The effect of uncertainties in the distribution parameters of the basic random variables in $\mathbf{X}$ lead to uncertainty in the calculated failure probability and in the associated reliability index.

Consistent with the Bayesian notion of probability, the uncertainty distribution parameters are modeled to be a random vector $\mathbf{\Theta}$, thus $f_{\mathbf{X}}(\mathbf{x})$ becomes a conditional distribution function $f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{X}, \boldsymbol{\theta})$. The conditional failure probability is given by (Der Kiureghian 1996)

$$P_f(\mathbf{\Theta}) = \int_{G(\mathbf{X},\ \mathbf{\Theta})\le 0} f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})d\mathbf{x} \tag{2}$$

where $G(\mathbf{X}, \mathbf{\Theta})$ is the performance function, $f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x}, \boldsymbol{\theta})$ is the joint PDF of $\mathbf{X}$ and $\mathbf{\Theta}$, and the conditional failure probability $P_f(\mathbf{\Theta})$ is a function of the distribution parameters $\mathbf{\Theta}$.

It follows that, since the distribution parameters $\mathbf{\Theta}$ are uncertain, the conditional failure probability and the corresponding conditional reliability index are also uncertain. The corresponding conditional reliability index $\beta(\mathbf{\Theta})$ can be expressed as

$$\beta(\mathbf{\Theta}) = \Phi^{-1}\left[1 - P_f(\mathbf{\Theta})\right] \tag{3}$$

where Φ^{-1} denotes the inverse of the standard normal cumulative probability function. As random variables, $P_f(\mathbf{\Theta})$ and $\beta(\mathbf{\Theta})$ have probability distribution functions as well as statistical moments, such as means, standard deviations, skewnesses, and kurtosis.

For vector X of the random variables in Eq. (2), whose joint PDF includes uncertain parameters $\mathbf{\Theta}$, the overall probability of failure, denoted P_F, is then defined as the expectation of the conditional failure probability $P_f(\mathbf{\Theta})$ over the outcome space of the uncertain parameters $\mathbf{\Theta}$, which can be formulated as

$$P_F = \int_{G(\mathbf{X},\ \mathbf{\Theta})\le 0} f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})d\mathbf{x}d\boldsymbol{\theta} \tag{4}$$

In most circumstances, the integral in Eq. (4) cannot be evaluated because of the difficulty in determining the explicit expression of the performance function $G(\mathbf{X}, \mathbf{\Theta})$ and the joint PDF $f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\mathbf{\theta})$. This is because $\mathbf{\Theta}$ represents the distribution parameters of $\mathbf{X}$, but $\mathbf{X}$ is a function of $\mathbf{\Theta}$. However, the conditional failure probability of the structural system for given distribution parameter values $\mathbf{\Theta} = \mathbf{\theta}$ can be evaluated readily using state-of-the-art techniques such as the first- and second-order reliability methods, moment methods and simulation methods (Choi et al. 2007; Ang and Tang 1984; Zhao and Ono 2001). Therefore, the overall probability of failure incorporating the uncertainties of the distribution parameters can be formulated generally as

$$P_F = \int_{\mathbf{\theta}} P_f(\mathbf{\Theta}) f_{\Theta}(\mathbf{\theta}) d\mathbf{\theta} \tag{5}$$

where $P_f(\mathbf{\Theta})$ is the conditional probability of failure for a given $\mathbf{\Theta} = \mathbf{\theta}$, and $f_{\mathbf{\Theta}}(\mathbf{\theta})$ is the joint PDF of $\mathbf{\Theta}$.

In the past several decades, many researchers focused on the problems of the distribution parameters uncertainties and various approximation methods have been developed for the determination the probability of failure considering the uncertainties of distribution parameters.

To evaluate the overall probability of failure, Hong (1996) proposed an efficient analysis procedure by using the point-estimate method to obtain the overall probability of failure. Later, Der Kiureghian (2008) derived a simple approximate formula by using the first-order approximation method to compute the mean of the conditional reliability index, and then the overall probability of failure was obtained.

However, for the sake of transparency in communicating risk, it is necessary to determine not only the overall probability of failure but also the quantile or even the probability distribution of the conditional failure probability or conditional reliability index. For this purpose, Der Kiureghian (2009) obtained the probability distributions of the conditional reliability index and the corresponding conditional probability of failure for cases in which the explicit PDF of the conditional probability could be determined easily. However, in general, the explicit PDF of the conditional reliability index cannot be obtained in engineering practice. It is in this regard that Ang and De Leon (2005) utilized Monte Carlo simulation (MCS) to obtain both the mean and quantile of the conditional failure probability. However, it is time-consuming for large-scale structures because many samples are required. Recently, Zhao et al. (2018) approximate the probability distribution of the conditional reliability index by using a three-parameter square normal distribution with explicit expression. However, this distribution uses only the first three moments (i.e., mean, standard deviation, and skewness) of the conditional reliability index to approximate its probability distribution, so this distribution is not flexible enough to reflect the kurtosis of the conditional reliability index. The kurtosis as well as the mean value, standard deviation, and skewness of the conditional reliability index are essential to determine its probability distribution (Zhao and Lu 2008), and has impact on conducting the accurate analysis of the structural reliability. Therefore, a new method with good flexibility, accuracy, wide range of applications for structural reliability analysis under the condition of the probability distribution parameter uncertainties of fundamental random variables is required.

In the present paper, an efficient method for evaluating the quantile or even the distribution of the conditional failure probability or conditional reliability index by utilizing a four-parameter cubic normal distribution (Zhao and Lu 2008) with high robustness for a wide range of applications under the condition of uncertainty in probability distribution parameters of fundamental random variables is proposed.

2. Review Point-Estimate Method for Evaluating the Overall Probability of Failure

It is obvious that the right-hand side of Eq. (5) represents the mean of the conditional failure probability $E[P_f(\mathbf{\Theta})]$. Rewriting Eq. (5) in standard normal space

$$P_F = E[P_f(\mathbf{\Theta})] = \int_{\mathbf{u}} P_f[T^{-1}(\mathbf{u})]\phi(\mathbf{u})d\mathbf{u} \tag{6}$$

where $T^{-1}(\mathbf{u})$ denotes the inverse Rosenblatt transformation (Rackwitz and Fiessler 1978) and $\phi(\mathbf{u})$ denotes the PDF of standard normal variables.

Practically, the integral in Eq. (6) cannot be evaluated analytically because of the high dimensionality and the complicated integration. In order to avoid this problem, the point-estimate method (Zhao and Ono 2000a) is used to solve the mean of $P_f(\mathbf{\Theta})$, which is one of the moments of function $P_f(\mathbf{\Theta})$. Using the standard point estimate, the mean of $P_f(\mathbf{\Theta})$ (i.e., P_F) is estimated as

$$P_F = E[P_f(\mathbf{\Theta})] = \sum \prod_{i=1}^{n} P_{ci} \left\{ P_f \left[T^{-1}\left(u_{c1}, \ldots, u_{ci}, \ldots, u_{cn}\right) \right] \right\} \tag{7}$$

where n is the dimension of random vector $\mathbf{\Theta}$; c is a distinct combination of n items from group [1, 2, …, m]; m is the number of estimating points, ci is the ith item of c; u_{ci} is the cith estimating point; and P_{ci} is the weight corresponding to u_{ci}.

As all distinct combinations have to be considered, m^n times of function calls for computing $P_f(\mathbf{\Theta})$ are required. The computations involved in Eq. (7), therefore, can be massive when n is large. In order to avoid this problem, it is necessary to adopt dimension-reduction inte-gration. Since only the first-order moment (i.e., the mean of $P_f(\mathbf{\Theta})$) is considered, the univariate dimension-reduction method (Xu and Rahman 2004) is used here. The function $P_f(\mathbf{\Theta})$ may then be approximated by $P_f^*(\mathbf{\Theta})$ as follows

$$P_f(\mathbf{\Theta}) \cong P_f^*(\mathbf{\Theta}) = \sum_{i=1}^{n} P_{fi} - (n-1) P_f(\mathbf{\mu}) \tag{8}$$

where

$$P_{fi} = P_f(\mathbf{\Theta}_i) = P_f[T^{-1}(\mathbf{U}_i)] \tag{9}$$

and $\mathbf{\mu}$ represents the vector in which all the random variables take their mean values; $\mathbf{\Theta}_i = [\mu_1, \ldots, \mu_{i-1}, \theta_i, \mu_{i+1}, \ldots, \mu_n]^T$; $\mathbf{U}_i = [u_{\mu 1}, \ldots, u_{\mu i-1}, u_i, u_{\mu i+1}, \ldots, u_{\mu n}]^T$, where $u_{\mu k}$, $k = 1, \ldots, n$ except i is the kth value of u_μ, which is the vector in u-space corresponding to $\mathbf{\mu}$.

Since P_{fi} is a function of only one standard normal random variable u_i for specific $P_f{}^*(\mathbf{\Theta})$, for independent random variables $\mathbf{\Theta}$, P_{fi} can be expressed simply as

$$\begin{aligned} P_{fi} &= P_f\left(\mu_1,\dots,\ \mu_{i-1},\ \theta_i, \mu_{i+1},\ \dots,\ \mu_n\right) \\ &= P_f\left[\mu_1,\ \dots,\ \mu_{i-1},\ T^{-1}(u_i), \mu_{i+1},\ \dots,\ \mu_n\right] \end{aligned} \tag{10}$$

Observe that u_i (i = 1, …, n) are independent and P_{fi} is a function of only u_i; therefore, P_{fi}, i = 1, …, n are also independent. Hence, the mean of $P_f{}^*(\mathbf{\Theta})$, i.e., the mean of the conditional failure probability, can be written as

$$P_F = E[P_f(\mathbf{\Theta})] \cong E[P_f{}^*(\mathbf{\Theta})] = \sum_{i=1}^{n}\mu_{P_{fi}} - (n-1)P_f(\boldsymbol{\mu}) \tag{11}$$

where μ_{Pfi} is the mean value of P_{fi} and can be point estimated from

$$\mu_{P_{fi}} = E\left(P_{fi}\right) = E\left\{P_f\left[T^{-1}(\mathbf{U}_i)\right]\right\} = \sum_{k=1}^{m}P_k P_f\left[T^{-1}(u_{ik})\right] \tag{12}$$

where u_{i1}, u_{i2}, …, u_{im} are the estimating points of random variable u_i, and P_1, P_2, …, P_m are the corresponding weights.

The estimating points u_{ik} and their corresponding weights P_k can be readily obtained as

$$u_{ik} = \sqrt{2}x_k\ ,\ P_k = \frac{w_k}{\sqrt{\pi}} \tag{13}$$

where x_k and w_k are the abscissas and weights for Hermite integration with the weight function exp($-x^2$) that can be found in Abramowitz and Stegum (1972).

Specially, for a seven-point estimate (m = 7) in standard normal space (Zhao and Ono 2000), we have the following

$$u_{i1} = -3.7504397,\ P_1 = 5.48269\times10^{-4} \tag{14}$$

$$u_{i2} = -2.3667594,\ P_2 = 3.07571\times10^{-2} \tag{15}$$

$$u_{i3} = -1.1544054,\ P_3 = 0.2401233 \tag{16}$$

$$u_{i4} = 0,\quad P_4 = 0.4571427 \tag{17}$$

$$u_{i5} = 1.1544054,\quad P_5 = 0.2401233 \tag{18}$$

$$u_{i6} = 2.3667594, P_6 = 3.07571\times10^{-2} \tag{19}$$

$$u_{i7} = 3.7504397, P_7 = 5.48269\times10^{-4} \tag{20}$$

3. Methods of Moment for the Evaluation of Quantile of the Conditional Failure Probability

In order to quantitatively estimate the uncertainty in the failure probability induced by the distribution parameter uncertainties, it is often necessary to obtain the quantile of the conditional failure probability. For this purpose, the distributions of the conditional failure probability need to be determined. Since conditional failure probability is a monotonic function of the related reliability index, the percentile values of conditional failure probability or related reliability index can be obtained utilizing the distribution of conditional failure probability or related reliability index. Since the variability of conditional reliability index is much smaller than that of conditional failure probability, the distribution of conditional reliability index, rather than that of conditional failure probability, is approximated in this study.

3.1 First four moments of the conditional reliability index

Using the standard point estimate, the first four moments of the conditional reliability index $\beta(\mathbf{\Theta})$, can be estimated as

$$\mu_\beta = \sum\prod_{i=1}^{n}P_{ci}\left\{\beta\left[T^{-1}\left(u_{c1},\ \dots,\ u_{ci},\ \dots,\ u_{cn}\right)\right]\right\} \tag{21}$$

$$\sigma_\beta^2 = \sum\prod_{i=1}^{n}P_{ci}\left\{\beta\left[T^{-1}\left(u_{c1},\ \dots,\ u_{ci},\ \dots,\ u_{cn}\right)-\mu_\beta\right]\right\}^2 \tag{22}$$

$$\alpha_{3\beta}\sigma_\beta^3 = \sum\prod_{i=1}^{n}P_{ci}\left\{\beta\left[T^{-1}\left(u_{c1},\ \dots,\ u_{ci},\ \dots,\ u_{cn}\right)-\mu_\beta\right]\right\}^3 \tag{23}$$

$$\alpha_{4\beta}\sigma_\beta^4 = \sum\prod_{i=1}^{n}P_{ci}\left\{\beta\left[T^{-1}\left(u_{c1},\ \dots,\ u_{ci},\ \dots,\ u_{cn}\right)-\mu_\beta\right]\right\}^4 \tag{24}$$

where μ_β, σ_β, $\alpha_{3\beta}$, and $\alpha_{4\beta}$ are the first four moments, i.e., the mean, standard deviation, skewness, and kurtosis of $\beta(\mathbf{\Theta})$, respectively.

Similar to the calculation of Eq. (7), the calculation involved in Eqs. (21)-(24) requires m^n times of function calls to determine the conditional reliability index $\beta(\mathbf{\Theta})$. Therefore, the computation becomes excessive when n is large. In order to avoid this problem, dimension-reduction integration method will be adopted again. Since the first four moments of β_f ($\mathbf{\Theta}$) are considered, bivariate dimension-reduction (Xu and Rahman 2004) is used here. The function β ($\mathbf{\Theta}$) can then be approximated by $\beta^*(\mathbf{\Theta})$ as follows

$$\begin{aligned} \beta(\mathbf{\Theta}) &\cong \beta^*(\mathbf{\Theta}) = \beta^*\left[T^{-1}(\mathbf{U})\right] \\ &= \sum_{i<j}\beta_{i,j} - (n-2)\sum_{i=1}^{n}\beta_i + \frac{(n-1)(n-2)}{2}\beta_0 \end{aligned} \tag{25}$$

where

$$\beta_{i,j} = \beta_{i,j}\left[\mu_1,\dots,T^{-1}(u_i),\dots,T^{-1}(u_j),\dots,\mu_n\right] \tag{26}$$

$$\beta_i = \beta_i\left[\mu_1,\dots,T^{-1}(u_i),\dots,\mu_n\right] \tag{27}$$

$$\beta_0 = \beta\left(\mu_1,\dots,\mu_i,\dots,\mu_n\right) \tag{28}$$

where $\beta_{i,j}$ is a two-dimensional function; i, j = 1, 2, …, n and i < j; β_i is a one-dimensional function; and β_0 is a constant.

Hence, using the inverse Rosenblatt transformation (Rackwitz and Fiessler 1978), the kth raw moments of $\beta(\mathbf{\Theta})$, $\mu_{k\beta}$, can be formulated approximately as

$$\begin{aligned} \mu_{k\beta} &= E\left\{\left[\beta(\mathbf{\Theta})\right]^k\right\} \cong E\left\{\left[\beta^*(\mathbf{\Theta})\right]^k\right\} = E\left\{\left\{\beta^*\left[T^{-1}(\mathbf{U})\right]\right\}^k\right\} \\ &\cong \sum_{i<j}\mu_{\beta_{i,j}}^k - (n-2)\sum_{i=1}^{n}\mu_{\beta_i}^k + \frac{(n-1)(n-2)}{2}\beta_0^k \end{aligned} \tag{29}$$

where

$$\beta_0^k = \left[\beta\left(\mu_1,\ ...,\ \mu_i,\ ...,\ \mu_n\right)\right]^k \tag{30}$$

$$\mu_{\beta_i}^k = \int_{-\infty}^{\infty}\left\{\beta_i\left[\mu_1,...,T^{-1}\left(u_i\right),\ ...,\mu_n\right]\right\}^k \phi(u_i)du_i \tag{31}$$

$$\mu_{\beta_{i,j}}^k = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\beta_{i,j}\left[\mu_1,...,T^{-1}(u_i),\ ...,T^{-1}(u_j),...,\mu_n\right]\right\}^k \phi(u_i)\phi(u_j)du_idu_j \tag{32}$$

Using the point-estimate method (Zhao and Ono 2000), the one-dimensional integral in Eq. (31) can be estimated as follow equation

$$\mu_{\beta_i}^k = \sum_{r=1}^{m} P_r\left\{\beta_i\left[\mu_1,\ ...,\ T^{-1}\left(u_r\right),\ ...,\mu_n\right]\right\}^k \tag{33}$$

Similarly, the two-dimensional integral in Eq. (32) can be estimated as

$$\mu_{\beta_{i,j}}^k = \sum_{r_1=1}^{m}\sum_{r_2=1}^{m} P_{r_1}P_{r_2}\left\{\beta_{i,j}\left[\mu_1,\ ...,T^{-1}\left(u_{r_1}\right),\ ...,T^{-1}\left(u_{r_2}\right),\ ...,\mu_n\right]\right\}^k \tag{34}$$

The estimating points and the corresponding weights can be found in the work of Abramowitz and Stegun (1972). For a seven-point estimate (m = 7) in standard normal space, these are given by Eqs. (14)-(20).

Finally, the mean, standard deviation, skewness, and kurtosis of the conditional reliability index $\beta(\Theta)$ can be estimated, respectively, as follows

$$\mu_\beta = \mu_{1\beta} \tag{35}$$

$$\sigma_\beta = \sqrt{\mu_{2\beta} - \mu_{1\beta}^2} \tag{36}$$

$$\alpha_{3\beta} = (\mu_{3\beta} - 3\mu_{2\beta}\mu_{1\beta} + 2\mu_{1\beta}^3)/\sigma_\beta^3 \tag{37}$$

$$\alpha_{4\beta} = \left(\mu_{4\beta} - 4\mu_{3\beta}\mu_{1\beta} + 6\mu_{2\beta}\mu_{1\beta}^2 - 3\mu_{1\beta}^4\right)\Big/\sigma_\beta^4 \tag{38}$$

3.2 Probability distribution of the conditional reliability index

Since the first four moments of the conditional reliability index $\beta(\Theta)$ are obtained, the probability distribution of $\beta(\Theta)$ can be approximated by using a four-parameter probability distribution, in which the four parameters in the probability distribution are directly defined in terms of its first four moments.

Here the cubic normal distribution (Zhao and Lu 2008) based on the four moment standardization function (Zhao and Lu 2007) is used

$$\beta_s = \frac{\beta(\Theta) - \mu_\beta}{\sigma_\beta} = S_u(u) = -l_1 + k_1u + l_1u^2 + k_2u^3 \tag{39}$$

where β_s is the standardized random variable; $S_u(u)$ denotes the third polynomial of a standard normal random variable u; the coefficients l_1, k_1, and k_2 are given as

$$l_1 = \frac{\alpha_{3\beta}}{6(1+6l_2)},\ l_2 = \frac{1}{36}\left(\sqrt{6\alpha_{4\beta} - 8\alpha_{3\beta}^2 - 14} - 2\right) \tag{40a,b}$$

$$k_1 = \frac{1-3l_2}{(1+l_1^2-l_2^2)},\ k_2 = \frac{l_2}{(1+l_1^2+12l_2^2)} \tag{40c,d}$$

From Eq. (40b), $\alpha_{4\beta}$ should be limited in the range of

$$\alpha_{4\beta} \geq (7 + 4\alpha_{4\beta}^2)/3 \tag{41}$$

The CDF of the conditional reliability index $\beta(\Theta)$ corresponding to Eq. (39) can then be expressed as

$$F_\beta\left(\beta(\Theta)\right) = \Phi\left(u\right) \tag{42}$$

and the PDF of the conditional reliability index $\beta(\Theta)$ is expressed as

$$f_\beta\left(\beta(\Theta)\right) = \frac{\phi\left(u\right)}{\sigma_\beta\left(k_1 + 2l_1u + 3k_2u^2\right)} \tag{43}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the CDF and PDF of a standard normal random variable u.

3.3 Quantile of the conditional failure probability

According to the four-moment standardization function (Zhao and Lu 2007), the quantile of the conditional reliability index corresponding to the confidence level α can then be determined by the following equation

$$\beta\left(\alpha\right) = \beta_s\left(\alpha\right)\cdot\sigma_\beta + \mu_\beta \tag{44}$$

where

$$\beta_s\left(\alpha\right) = S_u(u) = -l_1 + k_1u(\alpha) + l_1u(\alpha)^2 + k_2u(\alpha)^3 \tag{45}$$

where $\beta_s(\cdot)$ is the standardized random variable related to the confidence level α; $u(\cdot)$ is the standard normal random variable corresponding to the confidence level α; the coefficients l_1, k_1, and k_2 are given by Eqs. (40a)-(40d).

Therefore, the corresponding failure probability of the confidence level $1-\alpha$ is given as

$$P_f\left(1-\alpha\right) = \Phi\left[-\beta\left(\alpha\right)\right] \tag{46}$$

4. Numerical Examples

This example considers a steel rod with a circular cross-section, which has been investigated by Lu et al. (2011). The rod fails if the axial force exceeds the yield limit of material, and the limit state function is expressed simply as

$$g(R,d,P) = \frac{\pi}{4}d^2R - P \tag{47}$$

where P represents the axial force of the rod; R represents the yield limit of material; and d represents the diameter of the round rod.

The axial force of the rod P, yield limit of material R, and diameter of the round rod d are assumed as random variables, the probabilistic information of which is listed in Table 1.

Table 1. Probabilistic information about the random variables

Variable	Distribution	Mean	Standard deviation
P (kN)	Gumbel	79.4	6.20
R (kN•cm^{-2})	Gumbel	10.5	1.00
d (cm)	Gumbel	5.8	3.00

The reliability analysis for the performance function as expressed by Eq. (47) can be readily evaluated using state-of-the-art techniques. Here, the well-known first-order reliability method (FORM) (Hasofer and Lind 1974) is utilized, and the reliability index is readily obtained as 1.723, with a corresponding failure probability of 4.242×10^{-2}. In this example, the mean of the three random variables, i.e., μ_R, μ_d, and μ_P are assumed to be random variables, and their probabilistic information are listed in Table 2. Estimating the mean value and quantile of the conditional failure probability is described below.

Table 2. Probabilistic information of the distribution parameters

Variable	Distribution	Mean	Standard deviation
μ_R	Lognormal	10	0.5
μ_d	Gumbel	6	0.8
μ_P	Lognormal	80	1

Form Eq. (5), the overall failure probability can be obtained as

$$P_F = \int_{G(\mathbf{X},\,\mathbf{\Theta})\le 0} f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})d\mathbf{x}d\boldsymbol{\theta} = \int_{\boldsymbol{\theta}} P_f(\mathbf{\Theta}) f_{\mathbf{\Theta}}(\boldsymbol{\theta})d\boldsymbol{\theta} \qquad (48)$$

Based on Eq. (8), the conditional failure probability $P_f(\mathbf{\Theta})$ can be expressed as

$$P_f(\mathbf{\Theta}) \cong P_f^{\,*}(\mathbf{\Theta}) = \sum_{i=1}^{3} P_{f\,i} - 2P_f(\boldsymbol{\mu}) \qquad (49)$$

where

$$P_{f1} = P_f(\mu_R),\ P_{f2} = P_f(\mu_d),\ P_{f3} = P_f(\mu_P)$$

Since $P_f(\boldsymbol{\mu})$ is a function of the means of all three random variables, the original mean of the three random variables in Table 1 will be replaced by means of these parameters as given in Table 2, and $P_f(\boldsymbol{\mu})$ can then be easily obtained as 0.0349 by using FORM.

Using a seven-point estimate in standard normal space as shown in Eqs. (14)-(20), the estimating points of P_{f1}, i.e., $P_f(\mu_R)$ in original space, can be obtained as follows with the aid of an inverse Rosenblatt transformation

$$\mu_{R1} = 8.281,\ \mu_{R2} = 8.874,\ \mu_{R3} = 9.428,\ \mu_{R4} = 9.988$$

$$\mu_{R5} = 10.581,\ \mu_{R6} = 11.241,\ \mu_{R7} = 12.046$$

In the same way as the procedure to evaluate $P_f(\mu)$, we can use FORM to estimate the value of $P_f(\mu_{Ri})$, i = 1, …, 7. Using the point-estimate method, the mean of $P_f(\mu_R)$ or P_{f1}, μ_{Pf1}, is readily obtained as

$$\mu_{P_{f1}} = \sum_{k=1}^{7} P_k P_f(\mu_{Rk}) = 3.506\times10^{-2} \qquad (50)$$

Similarly, the means of $P_f(\mu_d)$ or P_{f2} and $P_f(\mu_P)$ or P_{f3} are obtained as $\mu_{Pf2} = 4.815\times10^{-2}$ and $\mu_{Pf3} = 3.492\times10^{-2}$, respectively.

Therefore, according to Eq. (11), the overall probability of failure, is readily estimated as

$$P_F = E[P_f(\mathbf{\Theta})] \cong \sum_{i=1}^{3} \mu_{P_{fi}} - 2P_f(\boldsymbol{\mu}) = 4.830\times10^{-2}$$

The overall probability of failure is obtained as 4.834×10^{-2} by using MCS with 1,000,000 samples.

Based on Eq. (24), the conditional reliability index $\beta(\mathbf{\Theta})$ can be approximated as

$$\beta(\mathbf{\Theta}) \cong \beta^*(\mathbf{\Theta}) = \beta^*\left[T^{-1}(\mathbf{U})\right] = \sum_{i<j} \beta_{i,j} - \sum_{i=1}^{3} \beta_i + \beta_0 \qquad (51)$$

where

$$\beta_{1,2} = \beta(\mu_R, \mu_d),\ \beta_{1,3} = \beta(\mu_R, \sigma_P),\ \beta_{2,3} = \beta(\mu_d, \sigma_P)$$

$$\beta_1 = \beta(\mu_R),\ \beta_2 = \beta(\mu_d),\ \beta_3 = \beta(\sigma_P)$$

$$\beta_0 = \Phi^{-1}\left[1 - P_f(\boldsymbol{\mu})\right] = 1.812$$

Utilizing the point-estimate method based on bivariate dimension-reduction integration, i.e., Eqs. (29)-(38), in which the estimation of the reliability indices for determining $\mu_{\beta_i}^k$ and $\mu_{\beta_{i,j}}^k$ in Eqs. (33)-(34) is evaluated from Eq. (47) using FORM, the first four moments of $\beta(\mathbf{\Theta})$ are easily obtained as, μ_β = 1.850, σ_β = 0.578, $\alpha_{3\beta}$ =2.033, and $\alpha_{4\beta}$ =12.262, respectively.

Substituting the obtained first four moments of $\beta(\mathbf{\Theta})$ into Eq. (43), the PDF of the conditional reliability index $\beta(\mathbf{\Theta})$ is expressed as

$$f_\beta(\beta(\mathbf{\Theta})) = \frac{\phi(u)}{0.578\left(0.662 + 0.437u + 0.265u^2\right)} \qquad (52)$$

The histogram of the conditional reliability index $\beta(\mathbf{\Theta})$ obtained by using the 1,000,000 MCS samples are shown in Figure 1 together with the PDF curve (denoted as the thick solid line) obtained from the method proposed in this paper as shown in Eq. (52)., respectively. It can be seen from Figure 1 that the histogram of the conditional reliability index $\beta(\mathbf{\Theta})$ is well behaved and can be approximated well by the PDF of the cubic normal distribution determined by using its first four moments.

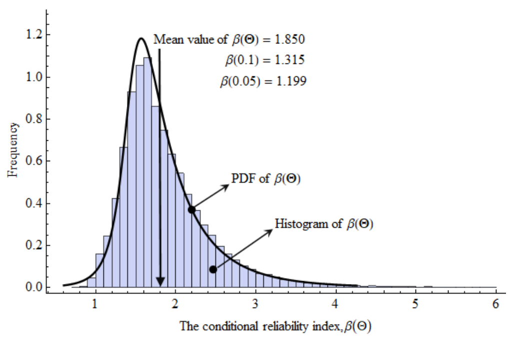

Figure 1. Histogram and PDF curve of the conditional reliability index.

The histogram of the conditional failure probability $P_f(\mathbf{\Theta})$ obtained by the MCS with 1,000,000 samples is shown in Figure 2. It can be seen in Figure 2 that the histogram of $P_f(\mathbf{\Theta})$ is skewed to the right and is truncated when $P_f(\mathbf{\Theta})$ tends to zero, as has been shown in Figure 2, which is difficult to approximate by well-known distributions.

The 90% and 95% confidence levels of $P_f(\Theta)$ are listed in Table 3, obtained from MCS, three-parameter square normal distribution, and the proposed method based on the cubic normal distribution, respectively. It also can be seen that the results obtained from the proposed method are much more accurate than the results from the three-parameter square normal distribution.

Table 3. Results of 90% and 95% confidence levels for $P_f(\Theta)$

Confidence level	MCS	Three-parameter distribution	Present
90% $P_f(\Theta)$	0.100	0.108	0.095
95% $P_f(\Theta)$	0.117	0.109	0.115

From the discussion above, it can be concluded that, the results estimated by the proposed method are almost the same as those obtained by MCS method.

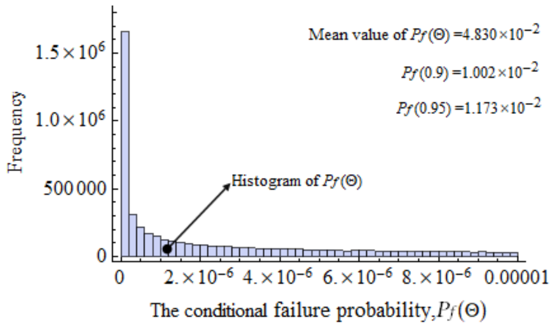

Figure 2. Histogram of the conditional failure probability.

5. Conclusions

This paper focuses on evaluating the quantile or even the distribution of the conditional failure probability or conditional reliability index by utilizing a four-parameter cubic normal distribution. It can give sufficiently accurate results and provided a complete picture of structural reliability analysis considering the parameter uncertainties. The accuracy of results obtained from the proposed method has been examined by comparisons with large sample Monte Carlo simulations (MCS).

6. Statement

This paper has been reported at the 13th International Conference on Applications of Statistics and Probability in Civil Engineering (ICASP13) held in Seoul, South Korea.

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