The Seventh Asian-Pacific Symposium on Structural Reliability and Its Applications (APSSRA2020)
October 4–7 2020, Tokyo, Japan
T. Takada, I. Yoshida & T. Itoi (editors)

Bayesian Learning of Vector Autoregressive Models for System Identification

Jia-Hua Yang[1,*] and Heung-Fai Lam[2]

[1,*]*Corresponding author. Assistant Professor, Department of Disaster Mitigation for Structures, College of Civil Engineering, Tongji University, 1239 Siping Road, Shanghai, China. Email: javayang@tongji.edu.cn*
[2]*Associate Professor, Department of Architecture and Civil Engineering, City University of Hong Kong, Tat Chee Avenue, Hong Kong, China. Email: paullam@cityu.edu.hk*

Abstract: Finite element models (FEMs) have been widely used for modeling and analyzing structures. However, they are usually very complicated for full-scale structures, so it is inefficient to use FEMs when instant analyses and predictions of structural responses are required in practice. Considering that autoregressive (AR) models have a simple mathematical form, this paper uses a vector AR (VAR) model to model measured responses. In the meantime, theoretical investigations show that the equation of motion of a structure can be written as a VAR model, so structural dynamic properties can be captured by a VAR model. Because a large amount of data is available nowadays, identification of a VAR model is treated as a Bayesian learning problem, where the posterior PDF of uncertain VAR parameters is learned from measured data. With the novel formulations, the most probable values (MPVs) and uncertainties of VAR parameters can be computed efficiently. The new formulations also enable modal parameters and their uncertainties to be quickly extracted from the identified VAR model. The proposed method tackles the algebraically involved derivation to provide a mathematically manageable algorithm, which paves the way to understand the uncertainties in complex dynamic systems, and offers the opportunity for rigorous risk analysis and decision making.

Keywords: Bayesian learning, vector autoregressive model, system identification, uncertainty.

1. Introduction

Due to the development of sensing technology, a dense network of sensors is instrumented on large-scale buildings, bridges and tunnels, so a huge amount of data is taken. This provides us much information to understand practical structural behaviors under operational conditions. In the meantime, we also face the challenge of effectively extracting this information from a huge amount of data. The problem comes down to system identification, i.e., identifying a mathematical model of a structural system using measured data. The first issue is to construct a mathematical model for a structural system. Using FEMs is a usual and reasonable choice. However, detailed modeling by FEMs requires large computational power. When multiple evaluations of FEMs are conducted for large-scale structures (e.g., optimization involved in identification, or updating models using continuously available new data), it is time consuming, and thus FEMs may not be suitable for system identification using much data in practice. Another issue is that uncertainties always exist due to modeling errors and measurement noise. We need a theoretical basis to rigorously extract information from measured data and quantify uncertainties in a way that is consistent with modeling assumptions.

To address the first issue, instead of using a FEM to model a structure by focusing on physical details, we use the VAR model to describe measured data directly (Pi and Mickleborough 1989, Yang and Lam 2019). The advantage of the VAR model is that it has a simple mathematical form, and its evaluation can be efficient. Moreover, the modal parameters such as natural frequencies, damping ratios and mode shapes can be extracted from the identified VAR model. These dynamic properties are useful for reliability analysis, response prediction and damage detection.

The Bayesian framework (Beck and Katafygiotis 1998, Beck 2010) is potential to address the second issue. By treating system identification as a Bayesian inference problem, the posterior PDF of the VAR model parameters is learned from measured data. We consider that the model parameters are uncertain due to incomplete information, e.g., unmodeled system characteristics by the model, rather than treat the model parameters as “random” with inherent randomness. The MPVs and posterior uncertainties of the VAR parameters can be obtained in closed form. The MPVs of modal parameters are then calculated based on the MPVs of the VAR parameters. It is shown that using the first-order Taylor series expansion, the posterior uncertainties of modal parameters can also be obtained analytically. The proposed Bayesian approach is efficient in identifying high-dimensional systems and at the same time quantifying the associated uncertainties.

2. Posterior PDF of The VAR Parameters

The VAR model is constructed by writing the current-step response in terms of the responses in several previous time steps:

$$\mathbf{x}_\mathrm{i} = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \cdots \quad \mathbf{B}_\mathrm{n}] \begin{bmatrix} -\hat{\mathbf{x}}_{\mathrm{i}-1} \\ -\hat{\mathbf{x}}_{\mathrm{i}-2} \\ \vdots \\ -\hat{\mathbf{x}}_{\mathrm{i}-\mathrm{n}} \end{bmatrix} + \mathbf{e}_\mathrm{i} \qquad (1)$$

where $\mathbf{x}_i \in R^{N_d}$ is the response at the i-th time step; $\mathrm{N_d}$ is the number of the measured degrees of freedom (DOFs); the responses with ∧ represent measured responses; $\{\mathbf{B}_\mathrm{i} \in \mathrm{R}^{\mathrm{N_d}\times\mathrm{N_d}}: \mathrm{i} = 1,2,\cdots,\mathrm{n}\}$ are the parameter matrices; $\mathbf{e}_\mathrm{i} \in \mathrm{R}^{\mathrm{N_d}}$ is the model error vector. Eq. *(1)* can be written compactly for all the time steps:

$$\mathbf{X} = \mathbf{U}\hat{\mathbf{A}} + \mathbf{E} \qquad (2)$$

where $\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_{N_t}]$; $\mathbf{U} = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \cdots \quad \mathbf{B}_n]$; $\hat{\mathbf{A}} = [\hat{\mathbf{a}}_1 \quad \hat{\mathbf{a}}_2 \quad \cdots \quad \hat{\mathbf{a}}_{N_t}]$; $\hat{\mathbf{a}}_i = [-\hat{\mathbf{x}}_{i-1}^T \quad -\hat{\mathbf{x}}_{i-2}^T \quad \cdots \quad -\hat{\mathbf{x}}_{i-n}^T]^T$; $\mathbf{E} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_{N_t}]$.

To derive the posterior PDF of the VAR parameters, it is convenient to transform Eq. (2) into a vector form:

$$\mathbf{d} = \left(\hat{\mathbf{A}}^T \otimes \mathbf{I}_{N_d \times N_d}\right)\mathbf{u} + \mathbf{g} \tag{3}$$

where

$$\mathbf{d} = \text{vec}(\mathbf{X}) \tag{4}$$

$$\mathbf{u} = \text{vec}(\mathbf{U}) \tag{5}$$

$$\mathbf{g} = \text{vec}(\mathbf{E}) \tag{6}$$

where vec is the vectorization operator that stack the columns of a matrix on top of each other; $\otimes$ denotes the Kronecker product. According to Bayes' theorem, the posterior PDF of $\mathbf{u}$ conditional on measured data $\mathbf{D}$ is expressed as

$$p(\mathbf{u}|\mathbf{D}) = \frac{p(\mathbf{u})p(\mathbf{D}|\mathbf{u})}{p(\mathbf{D})} \tag{7}$$

where $p(\mathbf{u})$ is the prior PDF; $p(\mathbf{D}|\mathbf{u})$ is the likelihood function; $p(\mathbf{D})$ is the normalizing constant such that the integration of the posterior PDF over the parameter space equals unity. To further derive Eq. (7), a stochastic embedding is done for the VAR model by assuming that $\mathbf{e}_i$ follows a Gaussian distribution:

$$\mathbf{e}_i \sim \mathcal{N}\left(0, \sigma^2 \mathbf{I}_{N_d \times N_d}\right) \tag{8}$$

where σ^2 is the variance of the model error at each time step. Moreover, we assume that the prior PDF is a uniform PDF, so the posterior PDF is proportional to the likelihood function. The detailed derivation for obtaining the MPVs of the uncertain model parameters and their posterior uncertainties can be found in Yang and Lam 2019. Only a brief summarization is provided here. The MPVs of $\mathbf{u}$ and σ^2 are obtained by maximizing the posterior PDF. Putting back the elements of the MPV of $\mathbf{u}$ to the MPV of the VAR model matrices $\hat{\mathbf{U}}$ gives

$$\hat{\mathbf{U}} = \sum_i \hat{\mathbf{x}}_i \hat{\mathbf{a}}_i^T \left(\sum_i \hat{\mathbf{a}}_i \hat{\mathbf{a}}_i^T\right)^{-1} \tag{9}$$

The MPV of σ^2 is

$$\hat{\sigma}^2 = \frac{\left(\text{vec}\left(\mathbf{X} - \hat{\mathbf{U}}\hat{\mathbf{A}}\right)\right)^T \text{vec}\left(\mathbf{X} - \hat{\mathbf{U}}\hat{\mathbf{A}}\right)}{N_d N_t} \tag{10}$$

By using the Laplace's method for asymptotic expansion at the MPVs, the posterior uncertainties of $\hat{\mathbf{U}}$ and $\hat{\sigma}^2$ can be obtained by taking the second derivative of the natural logarithm of the likelihood function.

3. The MPVs and Posterior Uncertainties of Modal Parameters

The detailed derivation of the analytical expressions for extracting the MPVs and posterior uncertainties of modal parameters from the MPVs and posterior uncertainties of the VAR parameters has been developed in Yang and Lam 2019. A brief introduction is given here. It can be shown that a structural dynamic system is equivalent to a VAR model (Pi and Mickleborough 1989, Yang and Lam 2019), and the modal parameters can be extracted from the VAR matrices. To do this, Eq. *(1)* is used to construct a first-order VAR model as follows

$$\mathbf{y}_i = \mathbf{F}\mathbf{y}_{i-1} + \mathbf{h}_i \tag{11}$$

where

$$\mathbf{y}_i = [\mathbf{x}_{i-n+1}^T \quad \mathbf{x}_{i-n+2}^T \quad \cdots \quad \mathbf{x}_i^T]^T \tag{12}$$

$$\mathbf{h}_i = [\mathbf{0}^T \quad \mathbf{0}^T \quad \cdots \quad \mathbf{e}_i^T]^T \tag{13}$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \\ -\mathbf{B}_n & -\mathbf{B}_{n-1} & \cdots & -\mathbf{B}_2 & -\mathbf{B}_1 \end{bmatrix} \tag{14}$$

By solving the eigenvalue problem for $\mathbf{F}$, the eigenvalues will give the natural frequencies and damping ratios of the corresponding structural dynamic system, and the first N_d components of the eigenvectors will give the mode shapes.

The posterior uncertainties of modal parameters are obtained by propagating the posterior uncertainties of the VAR parameters. This is done by considering that modal parameters are the function of the VAR model parameters. Specifically, we need three functions to obtain natural frequencies and damping ratios from $\mathbf{F}$. The first function is about solving the eigenvalue problem of $\mathbf{F}$ to get the discrete-time eigenvalues. The second function is about transforming the discrete-time eigenvalues to the continuous-time eigenvalues. The third function is about getting natural frequencies and damping ratios from the continuous-time eigenvalues. By applying the first-order Taylor series expansion for each function at the MPVs, the covariance matrix of one parameter transfers to another by pre- and post-multiplying the first derivative of the function to the covariance matrix. That is, let α_m denote the m-th discrete-time eigenvalue, β_m the m-th continuous-time eigenvalue, $\boldsymbol{\eta}_m$ contain the natural frequency and damping ratio of the m-th mode, and $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{G}_3$ the three functions, respectively, the covariance matrix of $\mathbf{F}$, $\mathbf{C}_F$ and the covariance matrix of the m-th mode natural frequency and damping ratio, $\mathbf{C}_\eta$ has the following relationship

$$\mathbf{C}_{\eta} = \frac{\partial \mathbf{G}_3}{\partial \beta_m} \frac{\partial \mathbf{G}_2}{\partial \alpha_m} \frac{\partial \mathbf{G}_1}{\partial \mathrm{vec}(\mathbf{F})} \mathbf{C}_{\mathrm{F}} \left(\frac{\partial \mathbf{G}_1}{\partial \mathrm{vec}(\mathbf{F})}\right)^{\mathrm{T}} \left(\frac{\partial \mathbf{G}_2}{\partial \alpha_m}\right)^{\mathrm{T}} \left(\frac{\partial \mathbf{G}_3}{\partial \beta_m}\right)^{\mathrm{T}} \tag{15}$$

The analytical expression for the covariance matrix has been developed (Yang and Lam 2019), so the posterior uncertainties of modal parameters can be quantified efficiently in practice.

4. Case Study

Simulated data of a six-story shear building was used to validate the proposed method. The same story mass 10^6 kg was chosen for all the stories. The same inter-story stiffness 4×10^9 N/m was chosen for all the stories. To simulate the measured data for system identification, a Gaussian excitation with a constant power spectral density (PSD) $19.6\ \mathrm{N}/\sqrt{\mathrm{Hz}}$ was applied at each story. The acceleration data were simulated for 300 s with sampling frequency 256 Hz. A Gaussian noise with PSD $0.1 \times 10^{-6}\ \mathrm{g}/\sqrt{\mathrm{Hz}}$ was added to the data.

The acceleration data were used in the VAR model for system identification. Following the Bayesian framework, the MPVs of the VAR matrices were calculated by Eq. (9). Using the MPVs of the VAR matrices, the model-predicted acceleration data were obtained and compared against the measured ones in Figure 1. It can be seen that the matching between the measured and predicted accelerations is good. The posterior uncertainties of the MPVs of the VAR model can also be calculated using analytical formulations. Due to the limited space, they are not shown here.

The MPVs of the modal parameters of the shear building was then calculated by constructing the first-order VAR model (Eq. (11)) using the MPVs of the original VAR model, and solving the eigenvalue problem for $\mathbf{F}$ (Eq. (14)). The MPVs of the natural frequencies, damping ratios and mode shapes of the six modes are summarized in Figure 2. It can be seen that using the VAR model with a simple linear form, the vibration patterns of a structural dynamic system can be efficiently obtained. The posterior uncertainties (standard deviations) of the modal parameters were also calculated using the derived analytical formulations (see Table 1 and Table 2). These uncertainties are helpful for fast robust reliability analysis, risk analysis and decision making in practical applications.

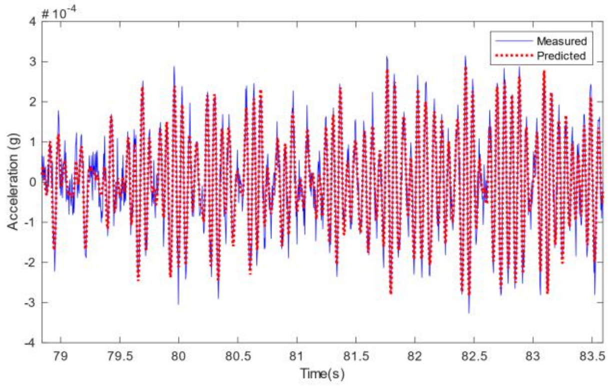

Figure 1. Comparison of the measured and predicted accelerations.

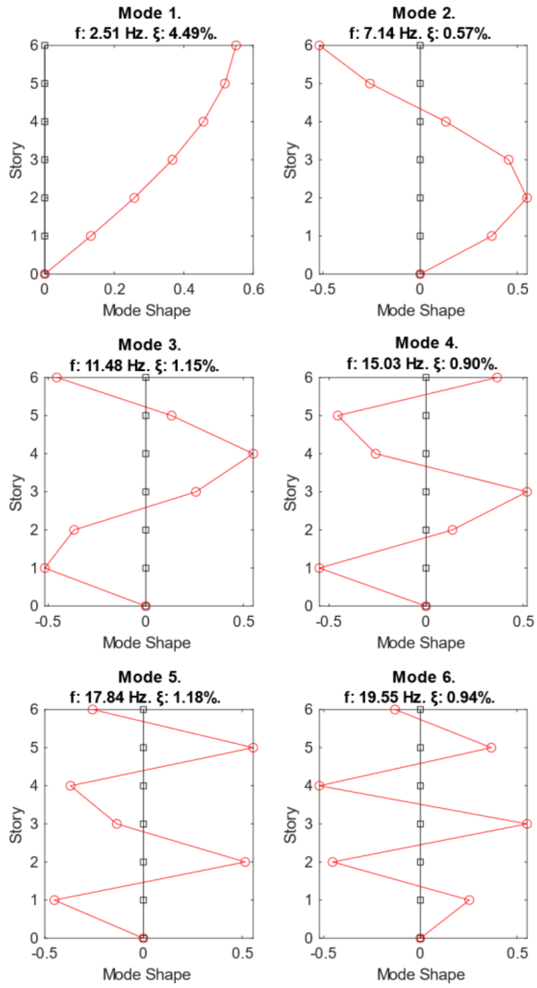

Figure 2. Identified modal parameters.

Table 1. Posterior uncertainties (standard deviations) of natural frequencies and damping ratios.

	f (Hz)	ξ
Mode 1	0.09	0.021
Mode 2	0.08	0.011
Mode 3	0.20	0.017
Mode 4	0.21	0.014
Mode 5	0.22	0.012

Table 2. Posterior uncertainties (standard deviations) of mode shapes ($\times 10^{-5}$).

Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
0.47	0.22	2.81	2.83	4.00	0.36
0.60	0.33	1.75	2.48	4.28	0.48
0.52	0.26	2.63	3.57	4.80	0.29
0.31	0.22	4.15	1.84	4.87	0.36
0.21	0.33	1.71	2.30	4.50	0.48
0.79	0.62	2.38	2.41	1.78	0.13

5. Conclusions

This paper introduces a recently developed Bayesian system identification method based on the VAR model. It has been shown that the parameters of the system that accurately predicts system responses can be easily obtained, and the “patterns” of data (in this case modal parameters) can also be obtained. The posterior uncertainties of the uncertain parameters can be efficiently calculated due to the analytical formulations. The simulated case study of a six-story shear building validates the good performance of the proposed method. The proposed method is potential to be applied for system identification, robust reliability analysis and risk analysis of large-scale structures.

Acknowledgement

The first author is funded by National Natural Science Foundation of China (Grant No.: 51808400) and Shanghai Sailing Program (Grant No. 18YF1424500. The generous support is greatly acknowledged.

References

Beck, J.L. 2010. Bayesian system identification based on probability logic. *Structural Control and Health Monitoring*, 17(7): 825-847.

Beck, J.L. and Katafygiotis, L.S. 1998. Updating models and their uncertainties. I: Bayesian statistical framework. *Journal of Engineering Mechanics*, 124(4): 455-461.

Pi, Y.L and Mickleborough, N.C. 1989 Modal identification of vibrating structures using ARMA model. *Journal of Engineering Mechanics*, 115(10): 2232-2250.

Yang, J.H. and Lam, H.F. 2019. An innovative Bayesian system identification method using autoregressive model. *Mechanical Systems and Signal Processing*, 133: 106289.