Engineering analysis with imprecise probabilities: a state-of-the-art review on P-boxes

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Abstract: The consideration of imprecise probability in engineering analysis to account for missing, vague or imcomplete data in the description of model uncertainties is a currently fast growing field of research. Especially probability-boxes (p-boxes) are of interest in an engineering context since they offer a mathematically simple description of the deep uncertainty, as well as allow for an intuitive visualisation. In essence, p-boxes are defined via lower and upper bounds on cumulative distribution functions of a variable quantity whose exact probability distribution is unknown. However, the propagation of p-boxes on quantities of interest towards bounds on probabilistic measures is numerically still very demanding, and hence is subject of intensive research. In order to maintain an overview on the available methods, this paper gives a state gives a state-of-the art review for the propagation of p-boxes with a special focus on structural reliability analysis. Specifically, methods to decouple the so-called 'double loop', as well as surrogate modelling methodologies are discussed in detail.

Keywords: imprecise probability, p-boxes, literature review, reliability analysis, surrogate modelling

1. Introduction

Let $\mathcal{M}$ represent a function that maps a set of n_x input parameters $x \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$, with $\mathcal{X}$ a set of feasible input parameters (e.g., non-negative Young's moduli or contact stiffness values), to a set of n_y output parameters $y \in \mathbb{R}^{n_y}$ via following relationship:

$$y = \mathcal{M}(x), \tag{1}$$

where $\mathcal{M}$ may represent numerical model that provides a discretized approximation of the continuum physics that describe the modelling problem at hand. These models give an unparalleled insight into the response of the structure under consideration to a set of predefined loading conditions, and hence, allow for a largely virtualized design optimization workflow. However, despite the highly detailed numerical predictions that can be obtained, these results often do not achieve a satisfactory level of agreement with 'reality', i.e., the actual physical behaviour of the structure in the effective operational environment. This discrepancy is caused by epistemic (= lack of knowledge) and aleatory (= caused by variation) uncertainty in the model. In recent years, several highly performing methods based on stochastic analysis (Schuëller 2011), Fuzzy set theory and Interval analysis (Faes and Moens 2019b) have been introduced in literature to account for these type of uncertainties in the model parameters x. Also several authors compared the applicability of several of these techniques in several applications, for instance in the context of Geotechnical engineering (Beer et al. 2013) or inverse uncertainty quantification for stochastic dynamics (Broggi et al. 2018; Faes, Broggi, et al. 2019).

A random (i.e., aleatory uncertain) property of a model is usually modelled by assigning a probability P to a set of possible values the variable parameters, denoted by $X = (X_1, \ldots, X_{n_x})$, can assume. This is modelled as a joint cumulative distribution function (CDF) $F_X(x) = P(X_1 \leq x_1, \ldots, X_{n_x} \leq x_{n_x})$ for $x \in \mathbb{R}^{n_x}$." Moreover, its derivative, the joint probability density function (PDF) f_X is also used. Usually, given f_X, an analyst is interested in computing the reliability of the structure they are designing, or rather, its complement: the probability of failure $p_f = P(g(X) \leq 0)$, with $g : \mathbb{R}^{n_x} \to \mathbb{R}$ the so-called performance function that indicates whether the structure failed or not (g could be interpreted as the numerical model $\mathcal{M}$ here). In case a crisp density function f_X is known, such calculation is performed by solving following integral equation:

$$p_f = \int_{\mathbb{R}^{n_x}} I_g(x) f_X(x) \, dx, \tag{2}$$

where I_g is the indicator function which is 1 in case $g(x) \leq 0$, $x \in \mathbb{R}^{n_x}$, and 0 otherwise. This equation can readily be solved by asymptotic approximations (Breitung 1989) or advanced simulation methods such as Subset Simulation (Au and Beck 2001), Directional Importance Sampling (Misraji et al. 2020) or the Probability density evolution method (Li and Chen 2006) in case of stochastic dynamics.

However, in most real-life applications, we only have partial information about these probabilities, leading also to the presence of epistemic uncertainty. In these cases, it is imperative to expand classical stochastic methods to explicitly take this additional uncertainty into account following a rigorous framework. Such framework is given by imprecise probability theory (Augustin et al. 2014). This class of methods, rather than assuming a specific probability measure, incorporates credal sets of probability measures to fully represent all sources of uncertainty. However, the application of the general framework of imprecise probability theory requires complex mathematical descriptions and methods. Therefore, simplified imprecise probability models are often preferable for simpler a utilization and representation. A popular representative hereof are probability-boxes (p-boxes) which provide lower and upper bounds on the CDF, see Section 2. In particular, they are useful for engineering analysis with CDFs like reliability analysis , see Eq. (2). Still, computing with p-boxes typically suffer from large computational expenses due to the aleatoric and epistemic type of uncertainty that have to be combined into the same analysis. This paper aims at giving an overview of a selection of recently introduced approaches for the propagation of p-boxes. However, by no means it is an all-encompassing work of the quickly expanding literature on imprecise probabilistic methods.

2. Probability boxes
2.1. Theoretical background
In this subsection, we consider the case $n_x = 1$ for notational simplicity. This is furthermore warranted since most engineering literature on the subject, as will be clear from Section 3, either considers the univariate case of $n_x = 1$, or when $n_x > 1$ full independence among all X_i, $i = 1, \ldots, n_x$, where $F_X(x) = F_{X_1}(x_1) \cdots F_{X_{n_x}}(x_{n_x})$, $x \in \mathbb{R}^{n_x}$ holds. As mentioned above, a p-box can be described by a lower CDF $\underline{F}_X \in \mathbb{F}$ and an upper CDF $\overline{F}_X \in \mathbb{F}$, where $\mathbb{F}$ expresses the the set of all CDFs on $\mathbb{R}$. They are collected as a pair $\left[\underline{F}_X, \overline{F}_X\right]$ which yields a set of possible CDFs via $\underline{F}_X(x) \leq F_X(x) \leq \overline{F}_X(x)$, $x \in \mathbb{R}$. This corresponds to defining a lower probability $\underline{P}$ and upper probability $\overline{P}$ on events $\{X \leq x\} = (-\infty, x]$, i.e., $\underline{P}(X \leq x) = \underline{F}_X(x)$ and $\overline{P}(X \leq x) = \overline{F}_X(x)$ for $x \in \mathbb{R}$, which define a credal set of probability measures. Via the p-box framework, the epistemic uncertainty that comes for example from incomplete data on $F_X(x)$ is accounted for by assigning an interval $\left[\underline{F}_X(x), \overline{F}_X(x)\right]$ for each value of $x \in \mathbb{R}$, see (Ferson et al. 2003). In case sufficient high quality information over the entire range of possible values for x is available to the analyst, $\left[\underline{F}_X(x), \overline{F}_X(x)\right]$ will be a tight interval, and the p-box will be close to a crisp distribution. On the other hand, when the information is less informative, the bounds may become wider to acknowledge weaker confidence in the results. In case no further assumptions are made concerning concerning the set of possible CDFs, this type of p-box is also denoted a *distribution-free p-box*. Clearly, this is the most general type of p-box, which allows for the most flexibility in the modelling of parameters subjected to aleatory and epistemic uncertainty. Indeed, it can be shown that crisp (deterministic) values, intervals and crisp probability distributions are all special cases of the distribution-free p-box (Kreinovich and Ferson 2004).

When more information about the shape of CDFs, such as an admissible distribution family, symmetry, or about bounds on one or more statistical moments of F_X is available to the analyst, the p-box can also be described by a quintuple $\left(\overline{F}_X, \underline{F}_X, \mu_X^I, \sigma_X^I, \mathcal{F}\right)$. Here, e.g. the confidence interval of the mean value $\mu_x^I \subseteq \mathbb{IR}$, the confidence interval $\sigma_x^I \subseteq \mathbb{IR}$ of the standard deviation, and the family of admissible CDFs $\mathcal{F} \subseteq \mathbb{F}$ can be specified, with $\mathbb{IR}$ the set of real-valued intervals. The p-box description can further be simplified if the type of CDF is known, as in this case, the epistemic uncertainty is only present in the CDF's hyper-parameters. This is also referred to as a *parametric p-box*. Note that also a distribution-free p-box can also be represented as a quintuple, noted $\left(\overline{F}_X, \underline{F}_X, [-\infty, +\infty], [0, \infty], \mathcal{F}\right)$. Finally, it should be noted that a p-box is in fact a computationally efficient description of a credal set (Hall 2006), that has furthermore workable algorithms for standard mathematical functions (Ferson et al. 2003). This framework was recently extended to account for imprecision in stochastic processes by explicitly accounting for additional epistemic uncertainty in the process' autocorrelation structure (Dannert et al. 2018; Faes and Moens 2019a).

2.2. Bounds on the failure probability
In the case where X is represented as a p-box, a direct calculation of p_f, as introduced in Eq. (2), is no longer possible since a set of PDFs that are consistent with the definition of the p-box has to be considered. Indeed, the consideration of a set of f_X causes the failure probability to become set-valued. In most engineering applications, the analyst is concerned with the bounds on p_f, being the lower bound $\underline{p}_f$ and upper bound $\overline{p}_f$, which can be obtained by solving the following optimization problems:

$$\underline{p}_f = \min_{f_X} \int_{\mathbb{R}^{n_x}} I_g(x) f_X(x) \, dx \tag{3}$$

to obtain the lower bound and:

$$\overline{p}_f = \max_{f_X} \int_{\mathbb{R}^{n_x}} I_g(x) f_X(x) \, dx \tag{4}$$

to obtain the upper bound. Note that these optimization problems are not optimization problems in a strict sense, since the optimization has to be carried out over the set of all possible f_X consistent with the definition of the p-box. Indeed, since the optimization has to be carried out over all possible f_X, the solution space is in general inherently infinite. In certain cases, tighter bounds on p_f can be obtained by means of linear programming, without having to construct the probability bounds of the input random variables (Wang et al. 2018).

A first approach to tackle this problem is to *slice* the p-box in order to transform the above problem into the propagation of a large number of intervals, each having a corresponding probability mass. The result of these interval propagations can then be reassembled into a p-box description of $Y = g(X)$. The propagation of intervals is a well-understood problem in the context of uncertainty propagation (Faes and Moens 2019b). However, following this approach the required number of evaluations of Eq. (1) scales exponentially with n_x (H. Zhang, Mullen, et al. 2010). This led to the development of methods such as interval Monte Carlo simulation (H. Zhang, Mullen, et al. 2010) or interval Quasi Monte Carlo simulation (H. Zhang, Dai, et al. 2013), which manage to break this exponential scaling, which bound p_f using following formulations:

$$\underline{p}_f = \frac{1}{n} \sum_{k=1}^{N} I_g(\overline{g}(r_k) \leq 0), \tag{5}$$

$$\overline{p}_f = \frac{1}{n} \sum_{k=1}^{N} I_g(\underline{g}(r_k) \leq 0) \tag{6}$$

with $\overline{g}(r_k)$ and $\underline{g}(r_k)$ defined as:

$$\overline{g}_j(r_k) = \max\{g(x) \mid \overline{F}_X^{-1}(r_k) \leq x \leq \underline{F}_X^{-1}(r_k)\}, \tag{7}$$

$$\underline{g}_j(r_k) = \min\{g(x) \mid \overline{F}_X^{-1}(r_k) \leq x \leq \underline{F}_X^{-1}(r_k)\}. \tag{8}$$

The parameters r_k, $j = 1, \ldots, N$ are realisations of a sample of N independent and identically distributed (i.i.d.) random variables according to a multivariate standard uniform distribution. As is clear from these equations, still require a

high number of model evaluations to estimate of the bounds on p_f with sufficiently small variance, especially since an optimization problem (Eq. (7)) has to be solved for each $\boldsymbol{r}_k$. Also line sampling methods to deal with imprecise probabilities have been introduced recently (de Angelis et al. 2015). In the case of parametric p-boxes, Eq.(3) and Eq.(4) can be solved directly since the set of all possible f_X is readily parametrized by a set of hyper-parameters. In this case, for realisation of the hyper-parameters of f_X, a reliability problem is solved. However, even in the simplest case where the p-box describes a set of possible f_X by means of interval-valued statistical moments, such calculation can be prohibitively demanding from a numerical standpoint. On one hand, the calculation of the failure probability for a fixed value of the parameters associated with the stochastic process is quite costly. On the other hand, solving the associated optimization problems in this simple case is far from trivial, as it constitutes a double loop problem, where the inner loop comprises probability calculation, leading to possibly non-smooth behaviour of the objective function due to the inherent variance on the estimator of p_f. Hence, apart from considering near-trivial simulation models, such the propagation of p-box valued parameters towards the bounds on the probability of failure on a structure is computationally intractable.

2.3. A short note on fuzzy probabilities

An extension to the p-box is provided by fuzzy probabilities, which allows for considering a fuzzy set of probability models, each having their own level of plausibility according to the available information (Beer et al. 2013). According to this framework, the fuzzy membership function serves as an instrument to combine various plausible intervals $[\underline{F}_X^\alpha(x), \overline{F}_X^\alpha(x)], \alpha \in [0,1]$ for $\boldsymbol{x} \in \mathbb{R}^{n_x}$ in a single scheme, and hence, allows for assessing the sensitivity of the bounds of p_f. Indeed, sensitivities of p_f are found by considering the rate of change of the bounds on the interval with respect to the size of the input intervals represented in the fuzzy numbers. Furthermore, the methods discussed further in the paper, which are developed for p-boxes, can always be applied to fuzzy probabilities in an α-cut sense.

3. Propagation methods for p-boxes

It is clear that, in general, the propagation of a p-box (parametrized or distribution-free) is very demanding from a computational point of view due to the double loop of propagation schemes that have to be considered. Therefore, two paths are recently pursued to determine the bounds on p_f: approaches to decouple the propagation efficiency and surrogate modelling techniques. The state-of-the-art in these methods is reviewed in the following sections.

3.1. Improving the propagation efficiency for p-boxes

A first approach to improve the numerical efficiency to propagate p-boxes is based on the importance sampling framework. The core idea of this class of methods is to propagate a single, well-chosen realisation $\hat{f}_X$ of a parametrized p-box (where $\hat{f}_X$ is optimal with respect to a predefined measure), and reweigh the obtained samples of $\boldsymbol{y}$ to infer bounds on p_f.

A first such method is Extended Monte Carlo simulation, as introduced by (Wei, Song, et al. 2019), which is applica-

ble to the propagation of parametrized p-boxes subjected to epistemic uncertainty in their first two moments. As a first step, the hyper-parameters $\boldsymbol{\theta}$ of the p-box, which account for μ_x and σ_x in the quintuple description, are represented by a subjective probability model $f_\Theta(\boldsymbol{\theta}) = \prod_{i=1}^{n_\theta} f_{\Theta_i}(\theta_i)$. Then, a local estimation for p_f, being $\hat{p}_f$, is derived as:

$$\hat{p}_f(\boldsymbol{\theta}) = \frac{1}{N} \sum_{k=1}^{N} I_g(\boldsymbol{x}_k) \frac{f_X(\boldsymbol{x}_k \mid \boldsymbol{\theta})}{f_X(\boldsymbol{x}_k \mid \boldsymbol{\theta}^*)} \tag{9}$$

which is an unbiased estimator, but highly affected by the selection of $\boldsymbol{\theta}^*$. 'Local' in this context denotes that the estimator is derived for a fixed value of $\boldsymbol{\theta}$ inside its support. This fixed value, $\boldsymbol{\theta}^*$, should be selected such that it minimizes the variance on the estimator $\hat{p}_f(\boldsymbol{\theta})$ (Wei, Lu, et al. 2014), similarly to conventional Importance Sampling, as:

$$\boldsymbol{\theta}^* = \arg\min \int_{\mathbb{R}^{n_\theta}} T(\boldsymbol{\theta}, \boldsymbol{\theta}^*) f_\Theta(\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{\theta} \tag{10}$$

with $T(\boldsymbol{\theta}, \boldsymbol{\theta}^*) = V\left[I_g(X) f_X(X \mid \boldsymbol{\theta}) / f_X(X \mid \boldsymbol{\theta}^*)\right]$ and V the variance operator with respect to $f_X(\cdot \mid \boldsymbol{\theta}^*)$. The global version of this approach is based on realizations $(\boldsymbol{x}_k, \boldsymbol{\theta}_k)$, $k = 1, \ldots, N$ of a joint sample distributed according to a joint PDF $f_{X,\Theta}$. The estimator $\hat{p}_f$ is in this case expressed as:

$$\hat{p}_f(\boldsymbol{\theta}) = \frac{1}{N} \sum_{k=1}^{N} I_g(\boldsymbol{x}_k) \frac{f_X(\boldsymbol{x}_k \mid \boldsymbol{\theta})}{f_X(\boldsymbol{x}_k \mid \boldsymbol{\theta}_k)} \tag{11}$$

where $\boldsymbol{x}_k$ and $\boldsymbol{\theta}_k$ are generated by applying the correct inverse probabilistic transform to the corresponding variables of a multivariate standard uniform distribution. The global estimator gives a better estimation of p_f over the entire support of $\boldsymbol{\theta}$, at the cost of lower accuracy around $\boldsymbol{\theta}^*$ and a higher computational cost, since in this case, also convergence in terms of the effect of $\boldsymbol{\theta}$ has to be ensured. An alternative optimal sampling density to propagate parametrized p-boxes following an reweighted sampling scheme was proposed by (J. Zhang and Shields 2018, 2019). Following the approach of (J. Zhang and Shields 2018, 2019), the optimal density should obtained by minimizing the total expected squared Hellinger distance between $f_X(\cdot \mid \boldsymbol{\theta})$ and the optimal sampling density $f_X(\cdot \mid \boldsymbol{\theta}^*)$ under an isoperimetric constraint that ensures that the derived optimal sampling density is a valid density function. The main difference with optimal sampling density presented in Eq. (10) is that this approach is not aimed at minimizing the variance, but rather that the sampling density is as close as possible to the target density.

Another pathway to improve the propagation efficiency for p-boxes is to decouple the double loop, as presented in Eq. (3) and (4). In (Faes, Valdebenito, et al. 2020), such a highly efficient numerical scheme to propagate imprecise stochastic loads, modelled as a distributional p-boxes with imprecisely defined hyper-parameters $\boldsymbol{\theta}$ through linear models $\mathcal{M}$, was introduced. In case an affine formulation of the imprecise random variables in terms of their hyper-parameters is possible, the propagation of the imprecise stochastic load can be performed in a two-step procedure. First, those values of the epistemic parameters that yield an extremum for p_f are determined by maximizing/minimizing

the operator norm of the product of the linear mapping provided by the numerical model $\mathcal{M}$ with a basis $\boldsymbol{B}$ that represents the autocorrelation of the load on the model:

$$\boldsymbol{\theta}^{\underline{*}} = \underset{\boldsymbol{\theta} \in \boldsymbol{\theta}^I}{\operatorname{argmin}} \|\boldsymbol{A}(\boldsymbol{\theta})\| \qquad (12)$$

with $\boldsymbol{A} = \mathcal{M}\boldsymbol{B}$, where $\boldsymbol{B}$ can for instance be determined following the well-known Karhunen-Loève expansion. The specific calculation of $\|\boldsymbol{A}(\boldsymbol{\theta})\|$ is highly case dependent; kindly refer to (Faes, Valdebenito, et al. 2020) for more details. A similar optimization problem has to be solved to determine $\boldsymbol{\theta}^*$. Then, two failure probabilities, corresponding to $\boldsymbol{\theta} = \boldsymbol{\theta}^{\underline{*}}$ and $\boldsymbol{\theta} = \boldsymbol{\theta}^*$ have to be computed to determine the bounds on p_{f}. As such, the double loop is effectively replaced by two deterministic optimizations and two crisp reliability estimations.

3.2. Surrogate modelling for p-boxes

A second approach to enable the computation of $\underline{p}_f$ and $\overline{p}_f$ with realistic engineering models is to replace $\mathcal{M}$ by a computationally more efficient surrogate model $\hat{\mathcal{M}}(\cdot \mid \boldsymbol{a})$. This surrogate $\hat{\mathcal{M}}$, parametrized by a vector $\boldsymbol{a} \in \mathbb{R}^{n_a}$, is usually *trained* by means of an a set of a priori generated training data $\{(\boldsymbol{x}_i, \boldsymbol{y}_i) \mid i = 1, ..., N\}$ via a supervised learning approach as to minimize the discrepancy between $\hat{\boldsymbol{y}}_i = \hat{\mathcal{M}}(\boldsymbol{x}_i \mid \boldsymbol{a})$ and $\boldsymbol{y}_i$, according to a predefined measure (e.g., in an L_2 sense). Examples of such maps to represent $\hat{\mathcal{M}}$ that have been used in the context of propagating p-boxes include Gaussian process models (J. Zhang, M. Xiao, et al. 2019) (also known as Kriging), polynomial response surface models (Sofi et al. 2020) or techniques based on Taylor series expansions (Gao et al. 2011). Also adaptive schemes based on Kriging have been introduced in literature (Schöbi and Sudret 2017a) that are applicable to both parametric and distribution-free p-boxes. In this section, three classes of methods are explained in detail that are highly promising from an engineering point of view due to their 'blackbox' nature (i.e., they require no interaction with the inner operations of $\mathcal{M}$), theoretical implications and numerical efficiency.

3.2.1. Polynomial Chaos Expansions & Kriging models
Polynomial chaos expansion (PCE) and Kriging are two widely used surrogate modelling techniques that approximate $\mathcal{M}$ via a intricate regression schemes. A sparse PCE surrogate model is given by:

$$\hat{\mathcal{M}}(\boldsymbol{x} \mid \boldsymbol{a}) = \sum_{\alpha \in \mathcal{A}} a_\alpha \boldsymbol{\phi}_\alpha(\boldsymbol{x}), \qquad (13)$$

where $\boldsymbol{\phi}_\alpha$ are multivariate orthonormal polynomials and $\mathcal{A} \subset \mathbb{N}^{n_x}$ is a finite set of multi-indices that is obtained by sparse decomposition. In (Schöbi and Sudret 2017b), distribution-free p-boxes are propagated in a two-level approach in which first $\mathcal{M}$, and second $\underline{\mathcal{M}}$ and $\overline{\mathcal{M}}$ (in the sense of Eq (7) and (8)) are substituted using sparse PCE. The training set is generated for an auxiliary input vector X and least angle regression (LARS) is used for training. In case of parametric p-boxes, it is proposed in (Liu et al. 2020) to model the sparse PCE coefficients a_α as quatratic

polynomial functions of the hyper-parameters $\boldsymbol{\theta}$ of the p-box and using a double-loop sampling for the propagation.

Whereas PCE methods focus on the global behaviour of $\mathcal{M}$ and are therefore suitable for a general propagation of p-boxes, Kriging methods focus on a local behaviour of $\mathcal{M}$ and are therefore often preferred for reliability analysis, where $\{g = 0\}$ is crucial. Using Kriging, a surrogate $\hat{g}$ for the limit-state function is considered to be a realization of a Gaussian process. It is:

$$\hat{g}(\boldsymbol{x} \mid \boldsymbol{a}) = \boldsymbol{\beta}_{\boldsymbol{a}}^{\mathrm{T}}\boldsymbol{\psi}(\boldsymbol{x}) + Z_{\boldsymbol{a}}(\boldsymbol{x}, \omega), \qquad (14)$$

where the first term, consisting of coefficients $\boldsymbol{\beta}_{\boldsymbol{a}}$ and regression functions $\boldsymbol{\psi}$, is the mean value of the process, and the second term is a zero-mean, stationary Gaussian process, characterized by a variance and an auto-correlation function depending on $\boldsymbol{a}$. Similar to above, a two-level approach in which first g, and second $\underline{g}$ and $\overline{g}$ are substituted is considered for distribution-free p-boxes in (Schöbi and Sudret 2017a). Here, adaptive Kriging Monte Carlo simulation (AK-MCS) is used for an accurate estimation of the failure probabilities and random slicing is used to obtain $\underline{p}_{\mathrm{f}}$ and $\overline{p}_{\mathrm{f}}$, see Eq. (5) and (6). Also in (Schöbi and Sudret 2017a), a failure probability $p_{\mathrm{f}}(\boldsymbol{\theta})$ which depends on the hyper-parameters $\boldsymbol{\theta}$ is estimated via AK-MCS and efficient global optimization (EGO) for parametric p-boxes. A similar, but more detailed, Kriging-based procedure for parametric p-boxes is also described in (N.-C. Xiao et al. 2020).

3.2.2. Sobol-Hoeffding decomposition based methods
The Extended Monte Carlo framework, as introduced in Section 3.1 allows for propagating parametrized p-boxes by a single probabilistic simulation and a reweighting step. Nonetheless, still a considerable number of evaluations of g are required, which might impede practical applications. Therefore, in (Wei, Song, et al. 2019), both the local and global Extended Monte Carlo methods were integrated with a Sobol-Hoeffding decomposition (also known as HDMR) of $\mathcal{M}$ as a surrogate modelling strategy. Following a HDMR deceomposition, p_{f} can be represented as:

$$p_{\mathrm{f}}(\boldsymbol{\theta}) = p_{\mathrm{f},0} + \sum_{i=1}^{n_\theta} p_{\mathrm{f},i}(\theta_i) + \sum_{1 \le i < j \le d} p_{\mathrm{f},ij}([\theta_i, \theta_j]) + \dots$$
$$+ p_{\mathrm{f},12\dots n_\theta}(\boldsymbol{\theta}) \qquad (15)$$

Specifically, in (Wei, Song, et al. 2019), it is proposed to apply a cut-HDMR strategy in combination with the local Extended Monte Carlo Method, allowing for a rigorous estimation of the variances of the estimators, as well as an estimation of the sensitivity of the parameters in $\boldsymbol{\theta}$. Similarly, it is proposed to perform a Random Slicing HDMR decomposition in combination with the Global Method. For the details concerning the implementation of these techniques, as well as the corresponding proofs, the reader is referred to (Wei, Song, et al. 2019). These methods were recently also extended to be applied in combination with Line Sampling in (Song et al. 2020).

An alternative application of the Sobol-Hoeffding decomposition in the context of propagating imprecise probabilities through numerical models is given by (Fina et al. 2020). This

paper applies a fuzzy probabilistic approach in the study of designing cylindrical shells under geometric imperfections, which are modelled as a random field. Specifically, imprecision in the auto-correlation structure of the random field is accounted for by means of fuzzy arithmetic, and the S-H decomposition is applied to speed up the corresponding α-level optimization.

3.2.3. Interval predictor models

An interval predictor model (IPM), as introduced in (Crespo et al. 2016), is a type surrogate model that approximates $\mathcal{M}$ by means of an interval-valued map $\hat{\mathcal{M}}_I(\cdot, \boldsymbol{\theta}) : \mathbb{R}^{n_x} \to \mathbb{IR}$. This map can be constructed with a minimal number of assumptions on the mapping provided by $\mathcal{M}$. Specifically, $\hat{\mathcal{M}}_I(\boldsymbol{x}, \boldsymbol{\theta})$ given by:

$$\hat{\mathcal{M}}_I(\boldsymbol{x}, \boldsymbol{\theta}) = \left\{ y = \boldsymbol{\theta}^{\mathrm{T}} \boldsymbol{\phi}(\boldsymbol{x}) \mid \boldsymbol{\theta} \in \boldsymbol{\theta}^I \right\} \tag{16}$$

with $\boldsymbol{\phi}$ a basis (e.g., polynomial or trigoniometric), $\boldsymbol{\theta}$ the fitting parameters of the IPM and $\boldsymbol{\theta}^I = [\underline{\boldsymbol{\theta}}, \overline{\boldsymbol{\theta}}]$ an $n_{\boldsymbol{\theta}}$-dimensional hyper-rectangular set. An optimal IPM is constructed by minimizing $E\left[(\overline{\boldsymbol{\theta}} - \underline{\boldsymbol{\theta}})|\boldsymbol{\phi}(\boldsymbol{x})|\right]$. Scenario Optimization (Campi et al. 2018) can be used to judge the generalization properties of the IPM. In case the corresponding optimization problem is convex, the reliability R of the IPM (i.e., the probability that a future unobserved data point will be contained in the IPM) is bounded by:

$$P(R \geq 1 - \epsilon) > 1 - \beta, \tag{17}$$

where ϵ and β are the confidence and reliability parameters, which for our hyper-rectangular model can be obtained from

$$\beta \geq \binom{k + n_\theta - 1}{k} \sum_{i=0}^{k+n_\theta-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i}, \tag{18}$$

where k is the number of data points discarded by some algorithm and ϵ can be chosen as a very small number (e.g., $\epsilon = 1 \cdot 10^{-06}$). An approach to apply IPMs in the context of propagating parametrized p-boxes is introduced by (Sadeghi et al. 2020). They show that IPMs can be used as surrogate model to speed up the calculation of Eq.(3) and Eq.(4), including a strategy to intelligently construct the set $\{(\boldsymbol{x}_i, \boldsymbol{y}_i) \mid i = 1, ..., N\}$. Furthermore, they show that the IPM can also be used as a surrogate model for g, which in its turn can be used in combination with importance sampling to determine $[\underline{p}_{\mathrm{f}}, \overline{p}_{\mathrm{f}}]$. The main advantages of these techniques are that (1) they are completely black-box as they don't require any assumption on $\mathcal{M}$ and (2) that under the mild assumption of convexity of the training guaranteed reliability bounds on the accuracy are obtained based on the rigorous framework of Scenario Optimization, which was recently extended to non-convex optimization problems too (Campi et al. 2018). Unfortunately, active learning of this type of surrogate models is not feasible, since this violates the required assumptions on independence between the training samples (Faes, Sadeghi, et al. 2019).

4. Conclusions

The development of highly efficient approaches to perform engineering computations with imprecise probabilities, represented as p-boxes, is a quickly expanding field of research.

The main challenge in this context is to overcome the required double loop propagation framework to estimate the bounds on probabilistic measures of the structure under consideration (such as, e.g., the probability of failure). Apart from near-trivial numerical simulation models, such double loop calculations are computationally intractable without resorting to high-performance computing facilities.

This problem is currently being tackled from two sides: (1) by improving the propogation efficiency of p-boxes aimed at breaking the double loop and (2) developing efficient surrogate models for the numerical models to be used in the double loop. Concerning the former set of solutions, highly efficient propagation schemes have been introduced in recent years. However, these methods are either limited in terms of the admissible descriptions of the uncertainty, or the non-linearity of the underlying numerical model. Future developments in these areas should concentrate on expanding the scope of applicability of these techniques. Concerning the latter, surrogate models usually only require some smoothness constraints on the underlying numerical model, which allows for a greater flexibility. Nonetheless, the accuracy of the calculation of the bounds on the probabilistic measures is limited to the accuracy of the underlying surrogate model. Furthermore, also the training of these surrogate models can entail a non-negligible numerical cost, which is commonly mitigated by resorting to active learning.

As such, to conclude, the last 5 years brought many highly performing approaches to compute with imprecise probabilities in general, and p-boxes in specific. The main challenge at this point appears to translate this set of highly performing methods to industrial applications involving multiphysical and/or million degree-of-freedom numerical models.

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