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Multifidelity Reliability Estimation

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Abstract: Multifidelity estimation combines the output of simulation models of different approximation quality and from different sources in order to obtain efficient estimators for a quantity of interest. In this contribution, possibilities to establish model hierarchies are investigated. Once a model hierarchy is established, the outputs of the models must be combined by information fusion and/or information filtering. To this end, importance sampling is extended to model hierarchies by introducing additive as well as multiplicative information fusion. The multi-fidelity reliability estimation methods are compared and critically assessed based on a simple example that highlights the main features of the methods.

Keywords: multi-fidelity, probability of failure, importance sampling, information fusion.

1. Introduction

Reliability analysis in engineering is concerned with the determination of the probability that the performance function of the system (e.g. the difference between resistance and load of a structure) becomes negative. Thus, the performance function of the system is the quantity of interest and the failure domain comprises the values of the system parameters that lead to a negative performance function. For most engineering applications, the failure probability is rather small, which precludes the application of direct Monte Carlo simulation for its evaluation.

Instead, importance sampling and importance splitting have been widely applied to the estimation of rare events (Schuëller et al. 2004). Importance sampling estimates the occurrence of rare events by generating samples from an alternative distribution and correcting for the bias by the introduction of weights. The success of this method relies on the quality of the importance sampling density, which is therefore often constructed in an adaptive way, e.g. by the cross-entropy method (Kurtz and Song 2013).

Importance splitting allows estimating small failure probabilities efficiently, even for problems that involve a high-dimensional vector of input random variables (Schuëller et al. 2004). It is based on a multiplicative decomposition of the failure probability in larger conditional probabilities that are estimated by means of Markov chain Monte Carlo simulation methods, cf. e.g. subset simulation (Au and Beck 2001).

Furthermore, the introduction of model hierarchies instead of a single model offers a great potential for increasing the efficiency of reliability analysis. In multilevel and multifidelity methods, a model hierarchy is established comprising in general a computational expensive high fidelity model and one or several less expensive low fidelity models. In multilevel methods, models are ordered by means of a discretization parameter (e.g. a mesh parameter or a time step) that is linked to the approximation error (Giles 2008). In general, this restricts the application of multilevel methods to a single mathematical model and a single discretization method; an exception is the application with a single grid but multiple discretization methods (Müller et al. 2014).

Multifidelity methods establish a model hierarchy based on the Pearson correlation parameter of the low fidelity models with respect to a high fidelity model (Peherstorfer et al. 2016). Thus, different discretization methods, different mathematical models or even experimental models might be ordered hierarchically as long as it is possible to compute the Pearson correlation between samples for the quantity of interest. The outstanding role played by the Pearson correlation in these methods stems from the fact that the information of the models is fused in an additive manner. This leads to variance reduction methods that may be interpreted as control variate methods (Gorodetsky et al. 2020).

Unlike these methods, multiplicative information fusion based on regression or Bayesian approaches has been proposed as well (Biehler 2016), but has been studied less intensively in the literature.

For reliability estimation, information filtering by adapted surrogate models is common (e.g. Chen and Quarteroni 2013, Li et al. 2011 and many others). A multilevel Monte Carlo method based on additive information fusion and information filtering has been proposed in (Elfverson et al. 2016). A multifidelity reliability estimation method with co-kriging surrogate models is due to (Sundar and Shields 2019). A multifidelity method with additive information fusion for importance sampling has been proposed in (Peherstorfer et al. 2018). A multilevel method based on additive information filtering has been applied to importance splitting in (Ullmann and Papaioannou 2015, Proppe 2020) and a multifidelity approach with additive information fusion and importance splitting can be found in (Proppe 2019). In summary, while multifidelity methods with additive information fusion have been successfully combined with both importance sampling and importance splitting, multifidelity reliability estimation methods with multiplicative information fusion have not been developed so far.

The aim of this contribution is twofold: first, new measures to establish model hierarchies are introduced and compared. These measures are either based on the dependence structure or on the information content of the quantity of interest and it is shown that dependence-based measures and information-based measures may lead to different model hierarchies. There are even cases in which one of the two classes of measures cannot be utilized, while the other one still yields a reasonable model hierarchy. Second, once a model hierarchy has been established, information fusion and information filtering has to be combined with importance sampling and importance splitting in order to yield new and efficient reliability estimation methods that are based on model hierarchies and not on a single model anymore. To this end, a new adaptive importance sampling strategy is formulated based on additive information fusion. Moreover, multiplicative information fusion is combined with importance sampling. The reduction of the approximation error and of the estimator variance of the newly introduced and of already established methods is compared by means of examples.

The paper is organized as follows: in the next section, methods to establish a model hierarchy are discussed. Following this, information fusion and information filtering for the combination of the model output are briefly introduced. After this, importance sampling is combined with additive and multiplicative information fusion. The proposed algorithms are tested on a simple example. Finally, conclusions are drawn.

2. Model Hierarchies

Additive information fusion follows the idea of variance reduction by means of control variates and therefore, the Pearson correlation between the model outputs leads to a hierarchical structure of the low fidelity models in a natural way.

Multiplicative information fusion is based on the concept of statistical dependence. In this case, copula-based measures of association are more adequate. They depend solely on the copula and not on the marginal distributions and can be further classified into measures of concordance between the variables (such as, e.g. Spearman's ρ and Kendall's τ) or measures of dependence, such as Schweizer-Wolff's σ (Nelsen 2007).

Besides Pearson correlation and copula-based measures, the difference between the distributions of output quantities is often important for reliability estimation and can be quantified by information-theoretic measures. They can be based on distribution functions (such as the Kolmogorov-Smirnov distance) or on probability density functions of the variables (such as the Kullback-Leibler divergence (Deza and Deza 2016)).

Copula-based and information-theoretic measures focus on different aspects in order to establish a relationship between output variables. Therefore, model hierarchies based on these measures might differ. The following example illustrates this situation.

Example 1: Suppose that the performance function of the high fidelity model follows a normal distribution with mean value 1 and standard deviation 2. Consider a family of low fidelity models that lead to normally distributed values for the performance function with mean value 1 and standard deviation $\sigma \in [1.5{:}\,1.9]$. Moreover, assume that the dependence structure between the low and the high fidelity model can be described by a Frank copula with parameter $\alpha = 20 \cdot (2 - \sigma)$. For the high and low fidelity models, the failure probability shall be given by $1 - \varphi(1/\sigma)$, where $\varphi(.)$ denotes the standard normal distribution function.

Figure 1 displays the development of the Pearson correlation coefficient, of Spearman's ρ and Kendall's τ as a function of σ. As can be seen, the correlation decreases with increasing σ, i.e. when the distribution of the low-fidelity approaches the high-fidelity distribution. On the other hand, cf. Figure 2, the Kullback-Leibler divergence decreases with increasing σ. The curve corresponds well to the approximation of the failure probability by the low-fidelity model. In summary, the Kullback-Leibler divergence would predict the correct order of the low-fidelity models with respect to accuracy, while correlation and copula-based measures would predict the reverse order.

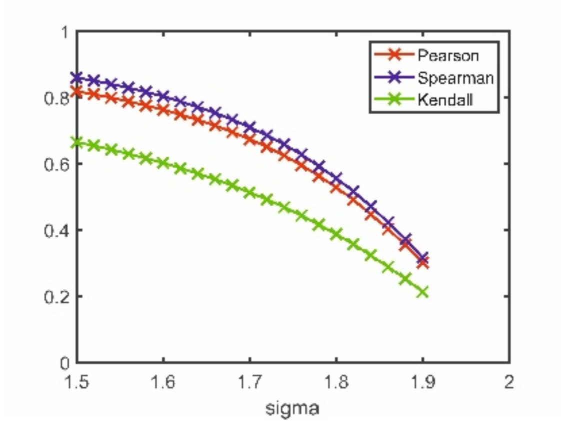

Figure 1. Development of Pearson correlation coefficient, of Spearman's ρ and Kendall's τ with σ.

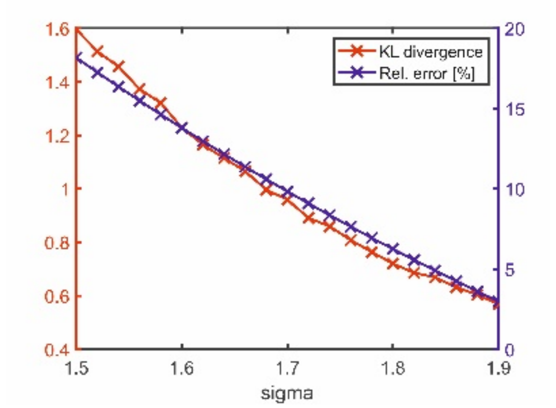

Figure 2. Development of Kullback-Leibler divergence and relative error with σ.

3. Information Fusion and Information Filtering

Once a model hierarchy has been established, the next step is to combine the information from the models. In principle, this can be done in two different ways. Either the model hierarchy is applied in parallel and an appropriate model of the model hierarchy is selected (this will be called information filtering in the following, according to (Peherstorfer et al. 2016)) - or the model hierarchy is applied in series and the information of each model is fused such that the overall computational effort is less than that of using solely the high fidelity model (this is called information fusion in the following, according to (Peherstorfer et al. 2016)). Information filtering and information fusion are not necessarily competing techniques, but can be combined.

3.1 Information fusion

Concerning information fusion, a first method that combines a high and a low fidelity model in a multiplicative manner has been presented as "global-local approximation" in (Haftka 1991). Both additive and multiplicative combinations of output quantities obtained from a high- and a low-fidelity model have been investigated for the analysis of a crack in a stiffened composite panel in (Vitali et al. 2002). The generalization of these ideas to model hierarchies leads to information fusion based on telescoping sums (Peherstorfer et al. 2016) and information fusion based on telescoping products (Biehler 2016), resp. The former method associates costs to the evaluation of the different models and determines an optimal number of simulation runs in order to obtain the same total error as with the high fidelity model. For reliability estimation, this method has been combined with information filtering in (Elfverson et al. 2016). The possibility to apply telescoping products of conditional probabilities for the output variables has been mentioned in (Biehler 2016), although the investigations are limited to the case of two models. This method does not optimize costs. Another difference between the two methods is that the use of telescoping sums requires the same random variables for the input of the models, while the use of telescoping products makes no restrictions in this regard.

Reliability estimation deals with the evaluation of the failure probability

$$P_F = \int_F p(\boldsymbol{\theta}) \mathrm{d}\,\boldsymbol{\theta}, \qquad (1)$$

where $F = \{\boldsymbol{\theta} \in \mathbb{R}^n | g(\boldsymbol{\theta}) < 0\}$ denotes the failure region and $p(\boldsymbol{\theta})$ the joint probability density function of the random vector $\boldsymbol{\Theta}$. In general, the performance function $g(\boldsymbol{\theta})$ is not known exactly, but is computed by numerical approximation.

In the following, an ordered family of approximated performance functions $g_l(\boldsymbol{\theta})$, l=0, ..., L is considered, where the index l refers to the performance function of the lth model of the model hierarchy. The index l=0 denotes the model with lowest fidelity and l=L the high-fidelity model.

3.1.1 Additive information fusion

For additive information fusion, denote by Q_l a quantity of interest related to the lth model. In the context of reliability estimation, the quantity of interest could be the indicator function $I_{g_l<0}(\boldsymbol{\theta})$, i.e. $I_{g_l<0}(\boldsymbol{\theta}) = 1$, if $g_l(\boldsymbol{\theta}) < 0$ and $I_{g_l<0}(\boldsymbol{\theta}) = 0$ elsewhere. An estimator for such a quantity of interest that combines the information from the model hierarchy in an additive manner is obtained from the telescoping sum (Heinrich 2001, Giles 2008)

$$E[Q_L] = E[Q_0] + \sum_{l=1}^{L} E[Q_l - Q_{l-1}]. \qquad (2)$$

The aim is to compute each of the estimates on the right-hand side of this equation individually by Monte Carlo simulation. A reduction of the overall computational effort can be expected from the fact that the variance of the differences decreases to zero with increasing index l and thus, for a given coefficient of variation, estimates of the contributions from highly accurate performance function will require less samples. Moreover, the fact that there is a nested sequence of approximations might be beneficial in a similar manner as for multigrid methods.

It is important to note that for the estimation of $E[Q_l - Q_{l-1}]$ both Q_l and Q_{l-1} are evaluated for the same samples. If the dimension of the random vector depends on the selected model of the model hierarchy, it is necessary to generate the samples for the model with higher fidelity and to obtain the corresponding samples for the model with lower fidelity by coarse-graining.

3.1.2 Multiplicative information fusion

For multiplicative information fusion (Biehler 2016), one has with

$$\begin{aligned} p(\boldsymbol{\theta}_L) &= \int \dots \int p(\boldsymbol{\theta}_L, \dots, \boldsymbol{\theta}_l, \dots, \boldsymbol{\theta}_0)\, \mathrm{d}\,\boldsymbol{\theta}_L \dots \mathrm{d}\boldsymbol{\theta}_l \dots \mathrm{d}\boldsymbol{\theta}_0 \\ &= \int p(\boldsymbol{\theta}_L|\boldsymbol{\theta}_{L-1}) \dots \int p(\boldsymbol{\theta}_l|\boldsymbol{\theta}_{l-1}) \dots \\ &\quad \int p(\boldsymbol{\theta}_1|\boldsymbol{\theta}_0) p(\boldsymbol{\theta}_0)\, \mathrm{d}\boldsymbol{\theta}_L \dots \mathrm{d}\boldsymbol{\theta}_l \dots \mathrm{d}\boldsymbol{\theta}_0 \end{aligned} \qquad (3)$$

an expression for the high-fidelity probability density function that is computed from the low-fidelity contributions and can then be utilized to estimate the quantity of interest of the high-fidelity model.

3.2 Information Filtering

For reliability estimation, information filtering has been described in (Elfverson et al. 2016), where a selective refinement strategy has been applied, such that realizations far away from the limit state $g(\boldsymbol{\theta})$=0 are solved by a lower accuracy than those close to the limit state, which further reduces the computational effort.

3.3 Combination of Information Fusion and Information Filtering

It is obvious how information filtering and information fusion can be combined for reliability estimation. To this end, starting from a model hierarchy, the combination of the model outputs in parallel by information filtering leads to a new model hierarchy whose output is then combined in series by information fusion. It is noted that the methods for information fusion and filtering described above can also be applied to other problems, such as sensitivity analysis or Bayesian inference.

4. Information Fusion and Importance Sampling

The aim of importance sampling is to reduce the variance of Monte Carlo simulation by sampling from an alternative density, the importance sampling density. To this end, the importance sampling density $p_{IS}(\boldsymbol{\theta})$ is introduced into eq. (1):

$$P_F = \int_F \frac{p(\boldsymbol{\theta})}{p_{IS}(\boldsymbol{\theta})} p_{IS}(\boldsymbol{\theta}) \mathrm{d}\,\boldsymbol{\theta}. \qquad (4)$$

The importance sampling estimate is then computed from the weighted average

$$\hat{P}_{F,IS} = \frac{1}{N} \sum_{i=1}^{N} I_{g<0}(\boldsymbol{\theta}^i) \frac{p(\boldsymbol{\theta}^i)}{p_{IS}(\boldsymbol{\theta}^i)}, \qquad (5)$$

where the N samples $\boldsymbol{\theta}^i$, i=1, ..., N, are drawn from $p_{IS}(\boldsymbol{\theta})$. The optimal importance sampling density is given by

$$p_{IS}(\boldsymbol{\theta}) = \frac{I_{g<0}(\boldsymbol{\theta})}{p(\boldsymbol{\theta})} P_F \quad (6)$$

which however requires already the knowledge of P_F and is thus infeasible.

In the cross-entropy method, an optimal importance sampling density is computed within the family of densities $\tilde{p}(\boldsymbol{\theta}, \boldsymbol{v})$ with parameter vector $\boldsymbol{v}$ by minimizing the Kullback-Leibler divergence (i.e. the cross-entropy)

$$\int_F \mathrm{p_{IS}}(\boldsymbol{\theta}) \ln p_{IS}(\boldsymbol{\theta}, \boldsymbol{v})\ \mathrm{d}\boldsymbol{\theta} - \int_F \mathrm{p_{IS}}(\boldsymbol{\theta}) \ln \tilde{p}(\boldsymbol{\theta}, \boldsymbol{v})\ \mathrm{d}\boldsymbol{\theta} \quad (7)$$

to the optimal importance sampling density (Rubinstein and Kroese 2017). This amounts to maximizing the expression

$$\int_F \mathrm{p}(\boldsymbol{\theta}) \ln \tilde{p}(\boldsymbol{\theta}, \boldsymbol{v})\ \mathrm{d}\boldsymbol{\theta}. \quad (8)$$

The estimation of this quantity requires again an importance sampling procedure, which can be based on the same family of densities $\tilde{p}(\boldsymbol{\theta}, \boldsymbol{v})$. This leads to an iterative scheme for the determination of the optimal parameter $\boldsymbol{v}$, cf. (Kurtz and Song 2013).

Both information fusion by additive combination and multiplicative combination of the model outputs can be combined with importance sampling for reliability estimation. This is described in the following sections.

4.1 Additive information fusion

For additive combination of the model outputs, the importance sampling density for each expectation has to focus on the differences of the performance functions of neighboring models. Alternatively, the importance sampling estimators might be computed separately for each level and then the differences of the estimators are weighted, cf. (Peherstorfer et al. 2016).

Example 2: Consider the circular performance function

$$g(\theta_1, \theta_2) = r - \sqrt{\theta_1^2 + \theta_2^2} \quad (9)$$

with parameter r>0 and a standard normal distribution for the random variables Θ_1, Θ_2. The limit state function $g(\theta_1, \theta_2) = 0$ is approximated by a regular convex polygon with n+3 facets and orbiting radius r and the corresponding performance functions $g_n(\theta_1, \theta_2)$ represent the oriented Euclidean distance of a sample from the approximated limit state function.

Table 1. Relative error (e[%]) and coefficient of variation (c.o.v.), single level method.

Order	r=1		r=3		r=5	
	e	c.o.v.	e	c.o.v.	e	c.o.v.
n=1	20	1e-3	499	2e-3	21510	22e-3
n=2	5	1e-3	59	6e-3	291	17e-3
n=3	2	2e-3	23	6e-3	81	10e-3
n=4	1	2e-3	12	5e-3	39	6e-3
n=5	1	2e-3	8	5e-3	23	7e-3

Table 1 displays the convergence of the single level approximations with approximation order n. As can be seen, the relative error decreases quickly. For lower failure probabilities, a high approximation order is necessary, e.g. for r=5, an octogonal approximation is not sufficient to reduce the relative error below 10%.

For the multifidelity method with additive information fusion, the expectations for the differences of the failure probabilities are computed by an adaptive cross-entropy based importance sampling scheme that utilizes a Gaussian mixture model to represent the importance sampling density, cf. (Geyer et al. 2019). As Table 2 reveals, the relative error of the multifidelity method is similar to the single level method for r=1, lower for r=3 and n>2 and higher for r=5. The coefficient of variation of the estimator for the differences $\Delta_{n+1n} = E[g_{n+1}(\theta_1, \theta_2) - g_n(\theta_1, \theta_2)]$ is lower than that of the single level estimator only for r=5 and low approximation order (n<4). In contrast to multilevel methods, the coefficient of variation *increases* with increasing approximation order n. Thus, the multifidelity method is not efficient in this case.

Table 2. Relative error (e[%]), mean value (m) and coefficient of variation (c.o.v.) for the differences between approximation orders.

Order	r=1			r=3			r=5		
	e	m	c.o.v.	e	m	c.o.v.	e	m	c.o.v.
Δ_{21}	5	1e-1	3e-3	60	5e-2	3e-3	337	8e-4	10e-3
Δ_{32}	2	2e-2	7e-3	20	5e-3	8e-3	108	9e-6	8e-3
Δ_{43}	1	8e-3	15e-3	6	2e-3	14e-3	57	2e-6	23e-3
Δ_{54}	0	4e-3	24e-3	0	8e-4	21e-3	34	9e-7	32e-3

The reason for these results of the multifidelity method can be seen from Figures 3 and 4 that display the samples obtained from the importance sampling density for the differences Δ_{21} and Δ_{43} for r=1. As can be seen, the importance sampling density focus in both cases on the region where the approximated performance functions differ in sign; however, for Δ_{43}, these regions are so small that many samples are still outside of these regions. For increasing approximation order n, the number of regions that have to be covered by the importance sampling density increases and the regions become smaller. Thus, for higher n, it is more difficult to cover these regions very well. This leads to the increase of the coefficient of variation and also to the increase of the relative error for larger values of r.

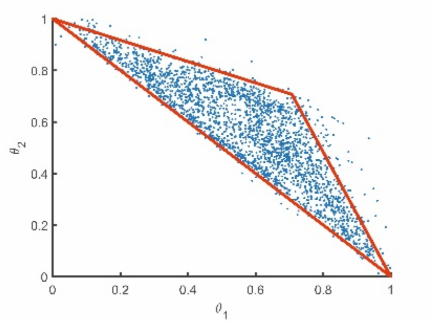

Figure 3. Samples from the importance sampling density for the difference Δ_{21}.

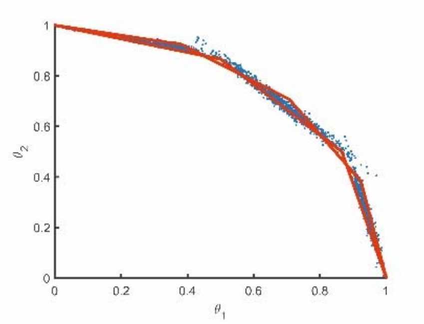

Figure 4. Samples from the importance sampling density for the difference Δ_{43}.

Next, the weighted multifidelity method proposed in (Kramer et al. 2019) is considered, where a weighted sum of importance sampling estimates is computed. This approach requires unbiased importance sampling estimates. Thus, the importance sampling densities are calibrated by means of the different approximations of the performance function $g_i(\boldsymbol{\theta})$, i=1, ..., n; however, the samples generated with the different importance sampling densities are evaluated with respect to the performance function $g_n(\boldsymbol{\theta})$ of highest fidelity. This is different from the other methods presented in this paper, where the generated samples are either evaluated by the performance function of the same or the next higher level. In order to be efficient, a reduction of the coefficient of variation for the weighted multifidelity estimator compared to the single level approach is therefore mandatory, such that in total, less samples are evaluated by the high-fidelity performance function. Table 3 indicates that the coefficient of variation of the weighted multifidelity method is indeed smaller than that of the single level approach. It decreases with increasing high-fidelity approximation order n. For n=5, the coefficient of variation of the single level approach is almost twice as large as that of the weighted multifidelity method. This is due to the fact that with increasing n the approach weights more estimators, namely a total of n, one for each level.

Table 3. Relative error (e[%]) and coefficient of variation (c.o.v.), weighted multifidelity method.

Order	r=1		r=3		r=5	
	e	c.o.v.	e	c.o.v.	e	c.o.v.
n=2	5	14e-4	59	4e-3	293	20e-3
n=3	2	10e-4	23	4e-3	81	10e-3
n=4	1	10e-4	12	3e-3	39	5e-3
n=5	1	8e-4	8	3e-3	23	5e-3

4.2 Multiplicative information fusion

For multiplicative information fusion, the importance sampling density is based on the model with lower fidelity and pairs of output quantities for neighboring models are obtained for samples generated by means of the importance sampling density. These pairs of samples are then utilized to find a functional expression by regression that relates the high fidelity model output to the low fidelity model output.

Example 3: Consider the same performance function and its approximation as in example 2 and the following bifidelity method: For the lower approximation order (n) the importance sampling density is calibrated by the same cross-entropy based importance sampling algorithm applied in example 2. After that, 100 samples (out of 50000 samples) with highest weights are identified and only for these samples, the performance function of approximation order n+1 is evaluated. A linear relation between the 100 values of the low fidelity performance function and the high fidelity performance function is then calibrated by regression. By means of this linear relation, approximations of the high fidelity performance function are computed for the remaining 49900 samples. Table 4 summarizes the relative error and the coefficient of variation obtained for the bifidelity method. It can be seen that the relative error of the bifidelity method is comparable to that of the higher approximation order n+1. Thus, the bifidelity method considerably improves the approximations of the failure probability. The coefficient of variation of the bifidelity estimator is higher than that of the single level estimator. It scales with the coefficient of determination R^2 and thus might be attributed to the regression error. For higher approximation order n and for lower values of the radius r, the coefficient of determination decreases because the differences between two successive levels are smaller in these cases. Thus, the correlation between the values of the performance functions is higher which reduces the coefficient of variation. Moreover, increasing the number of high level samples decreases the coefficient of variation.

Table 4. Relative error (e[%]), coefficient of variation (c.o.v.) and cofficient of determination (R^2) for the bifidelity method.

Order	r=1			r=3			r=5		
	e	c.o.v.	R^2	e	c.o.v.	R^2	e	c.o.v.	R^2
n=1	4	22e-3	85	42	39e-2	36	68	2.2	25
n=2	2	6e-3	99	10	11e-2	76	30	0.4	60
n=3	1	4e-3	100	7	5e-2	93	12	0.1	87
n=4	1	2e-3	100	5	2e-2	97	7	0.1	91
n=5	1	2e-3	100	5	2e-2	99	6	0.1	94

In summary, the combination of additive information fusion using telescoping sums with importance sampling requires an importance sampling density that focuses on differences between approximations of successive order; a reduction of the coefficient of variation is then difficult to achieve. Implementing additive information fusion by a weighted sum of importance sampling estimators leads to a reduction of the coefficient of variation, but requires all samples generated by the different importance sampling densities to be evaluated by the high-fidelity performance function. The combination of multiplicative information fusion with importance sampling leads to mean square errors that are comparable to those of the next higher level. A reduction of the coefficient of variation is achieved, if the regression error is small. For linear regression, this is the case if the approximations of successive order are highly correlated.

5. Conclusions

This paper compares different measures for establishing model hierarchies and combines additive and multiplicative information fusion with importance sampling in order to efficiently obtain reliability estimates.

It has been shown that dependence- and information-based measures may lead to different model hierarchies. As a rule of thumb, low or at least decreasing values of information-based measures such as the Kullback-Leibler divergence are often a prerequisite for the general applicability of a multifidelity method to reliability estimation, while values converging to 1 for dependence-based measures are often a prerequisite for the efficiency of that method. Obtaining a good importance sampling density is still a challenging task, especially in high dimensions. This is particularly the case, if the importance sampling density should focus on level differences. It therefore appears to be more advantageous to combine different importance sampling estimators by a weighted sum, even if this approach requires unbiased estimators and thus the evaluation of all samples by the high-fidelity performance function. Multiplicative information fusion might have advantages compared to additive information fusion and requires less samples to be evaluated by the high-fidelity performance function. However, multiplicative information fusion relies on establishing a precise relationship between the values of the performance function obtained with the model of lower and of higher fidelity.

The proposed methods help to balance the approximation error and the statistical error by information fusion. It is demonstrated that these methods may lead to a considerable increase in efficiency. The approaches can be extended by taking the data and model error in a Bayesian setting into account. It is also noted that the methods presented in the paper can be applied to sensitivity analysis and Bayesian inference.

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