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Application of Normal Inverse Gaussian Distribution in Reliability Analysis

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Abstract: Constructing a probability density function (PDF) with the available statistical moments as constraints is the main problem in many engineering problems. In this paper, the approximate PDF is reconstructed by using the normal inverse Gaussian (NIG) distribution based on the known first four moments, in which the parameters of the NIG are available very easily and precisely according to the closed-form formulas. Besides, the effectiveness of the distribution is demonstrated through an application to two reliability problems.

Keywords: moment method, normal inverse Gaussian distribution, probabilistic information approximation.

1. Introduction

Uncertainties widely exist in practical engineering problems, which should be appropriately quantified for the reliability and safety of a product(Guo and Du 2007). In general, the probability density function (PDF) or cumulative distribution function (CDF) can be considered as a complete description of the information of a random variable. However, it is difficult to determine the PDF or CDF for the random variable in the engineering applications, and only a finite number of sample data or statistical moments can be obtained. Thus, the problem of constructing a PDF or CDF from a limited number of moments arises in a diverse field of statistical applications. It is known that the statistical moments include a large amount of probabilistic information and can be easily obtained. Thus, these have motivated various probability estimation methods, among which the moment method is the most significant. The moment method is first employed to compute statistical moments, and then the probability estimation method is used to approximate the PDF.

This paper is interesting in the probability estimation method. The probability estimation method can be divided into three categories. Firstly, an approximate function between the random variable and a commonly used variable (usually standard normal variable) is obtained based on the finite probability information of the random variable, such as polynomial normal transform (Headrick 2002), in which the cubic normal method (Fleishman 1978; Zhao and Lu 2007) is the most frequent-used. Secondly, an approximate PDF or CDF of the random variable is derived according to a given rule, e.g. maximum entropy method (Sobczyk and Trcebicki 1999; Li and Zhang 2011), saddlepoint approximation method (Huang and Zhang 2012), etc. Lastly, the random variable is assumed to be subjected to a specific flexible distribution, e.g. Pearson system (Zhao and Ono 2001), Johnson distribution (Hong and Lind 1996), the shifted generalized lognormal distribution (Low 2013), etc. Obviously, the precision of moment methods based on different approximate probability information is different. Further, the difficulty in determining the approximate probability information from the known first few moments. The above methods involve the solutions for nonlinear equations and approximate coefficients, which may affect the accuracy of the probability estimation method. Therefore, a friendly and easy-to-implement approximate PDF with high accuracy is helpful for improving the performance of the moment method.

This paper is organized as follows. Section 2 briefly presents the NIG distribution. In section 3, some examples are investigated to verify the proposed method. In section 4, some conclusions are summarized.

2. NIG approximation of PDF

The NIG distribution is a variance-mean mixture of a Gaussian distribution with an inverse Gaussian(Hanssen and Oigard 2001), which is a powerful tool to improve the calculations of tail probabilities when the information set is restricted to the first four moments (Eriksson et al.2005). If random variable Z is assumed to follow the NIG distribution, its PDF is

$$f_Z(z) = \frac{a\delta}{\pi} \frac{\exp\left[\delta\sqrt{a^2-b^2}+b(z-v)\right]}{\sqrt{(z-v)^2+\delta^2}} K_1\left[a\cdot\sqrt{(z-v)^2+\delta^2}\right] \quad (1)$$

where $K_1(\cdot)$ denotes the modified Bessel function of the third kind with index 1, and a, b, v, δ are parameters satisfying the boundary conditions $a \geqslant |b| \geqslant 0$, $v \in R$ and $\delta > 0$, and determined analytically and exactly by(Eriksson et al.2009)

$$\begin{cases} a = \dfrac{3\sqrt{\lambda}}{\sigma_Z(\lambda-1)} \cdot \dfrac{1}{|\gamma_{Z,3}|} \\ b = \dfrac{3}{\sigma_Z(\lambda-1)} \cdot \dfrac{1}{\gamma_{Z,3}} \\ v = \mu_Z - \dfrac{3\sigma_Z}{\lambda} \cdot \dfrac{1}{\gamma_{Z,3}} \\ \delta = \dfrac{3\sigma_Z\sqrt{\lambda-1}}{\lambda} \cdot \dfrac{1}{|\gamma_{Z,3}|} \end{cases} \quad (2)$$

where

$$\lambda = \frac{3(\alpha_{4Z}-3)}{\alpha_{3Z}^2} - 4 \quad (3)$$

here μ_Z, σ_Z, α_{3Z}, and α_{4Z} are the mean, standard deviation, skewness and kurtosis of Z.

3. Applications to reliability analysis

In the reliability analysis, the accuracy of reliability estimation, as an important aspect in reliability analysis, measures the accuracy of pointwise CDF approximations and should also be considered as a comparison metric. Consider a threshold value z_C, and the probability can be defined as

$$R\left(f \mid z_C\right)=\int_{-\infty}^{z_C} f(z)dz \triangleq F\left(z_C\right) \tag{4}$$

In order to evaluate the accuracy of the probability, the relative error ε is defined by

$$\varepsilon=\frac{\left|R\left(\hat{f} \mid z_C\right)-R\left(f \mid z_C\right)\right|}{R\left(f \mid z_C\right)} \tag{5}$$

In this section, two examples are investigated to verify the flexibility, stability, and accuracy of the probability estimation methods in approximating the PDF of a system response with the same statistical moments. Meanwhile, comparing with other probability estimation methods, such as cubic normal method (CNTM) (Zhao and Lu 2007), approximate method based on Pearson system (AMPS) (Zhao and Ono 2001), the saddlepoint approximation method (SAM) (Huang and Zhang 2012), maximum entropy method (MEM) (Li and Zhang 2011).

Example 1

Consider a nonlinear performance function related to the ultimate bending capacity of a reinforced concrete beam, which is given by an explicit function (Zhang and Pandey 2013):

$$Z=G(\boldsymbol{X})=\left[X_1X_2X_3-\frac{X_1^2X_2^2X_4}{X_5X_6}-X_7\right] \tag{6}$$

in which the statistical information of all random variables is illustrated in Table 1.

Table 1. Statistical information of all variables in Example 1.

Variable	Distribution	Mean	S. d.[a]
X_1	Lognormal	1260 (mm2)	252
X_2	Lognormal	300 (N/mm2)	60
X_3	Lognormal	770 (mm)	154
X_4	Lognormal	0.35	0.035
X_5	Lognormal	25 (N/mm2)	5
X_6	Lognormal	200 (mm)	40
X_7	Lognormal	100 (kN·m)	20

Note [a] S. d. is the Standard deviation of random variables.

Table 2. The statistical moments for Example 1.

μ_Z	$m_{Z,2}$	$m_{Z,3}$	$m_{Z,4}$
1.7943×10^{8}	1.0398×10^{16}	9.6489×10^{23}	5.1934×10^{32}

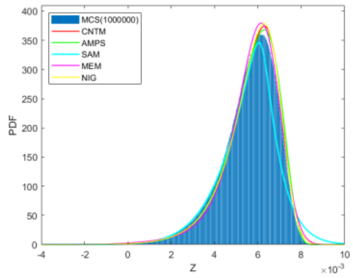

Figure 1 (a). PDF of Z.

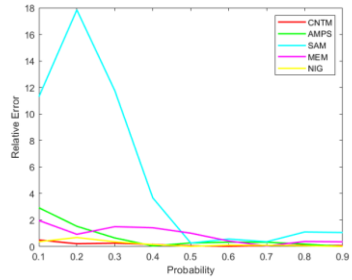

Figure 1 (b). Relative errors of F in 0.1~0.9.

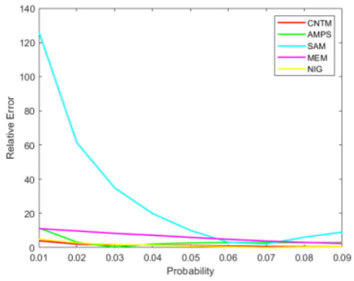

Figure 1 (c). Relative errors of F in 0.01~0.09.

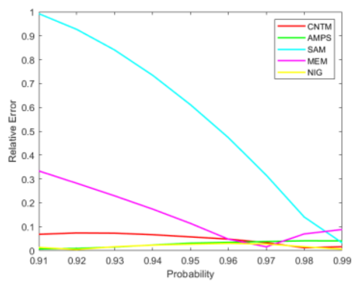

Figure 1 (d). Relative errors of F in 0.91~0.99.

The statistical moments are obtained from MCS with 106 samples, which listed in Table 2. The approximate PDF are reconstructed from different methods, which are plotted in Fig. 1(a). It is easy to find that all probability estimation methods can work well except SAM. Meanwhile, the whole probability levels ($0.1 \leqslant F \leqslant 0.9$) are shown in Fig. 1(b), where the result shows that the SAM yields undesirable accuracy and stability in terms of the overall PDF approximation. The low($F<0.1$) reliability levels and high($0.9<F$) probability levels, as shown in

Fig. 1(c) and Fig. 1(d), respectively. The MEM presents relatively large reliability errors compare with other methods. The NIG and the AMPS shows better accuracy than the CNTM in high- probability levels. The failure probabilities calculated by the CNTM, AMPS, SAM, MEM, NIG, and MCS are 0.0124, 0.011, 0.0248, 0.0133, 0.0126, and 0.0122, respectively. It is noted that the NIG provides high accuracy of failure probabilities with errors of less than 3.27%. Therefore, the accuracy and high flexibility of the NIG distribution have been validated.

Example 2

This truss structure (Sun et al. 2017) is shown in Fig. 2. Ten independent input random variables are considered, namely the Young's modulus and the cross-section areas of the horizontal and the oblique bars (respectively denoted by E_1, A_1 and E_2, A_2) and the applied loads (denoted by P_i; i= 1,…,6), whose mean and standard deviation are shown in Table 3. The implicit performance function is given as

$$Z = G(\boldsymbol{X}) = v_b - |v(\boldsymbol{X})| \tag{7}$$

where v_b=0.14 m is the deterministic allowable value of displacement; $v(\boldsymbol{X})$ denotes the vertical deflection in the middle of the truss.

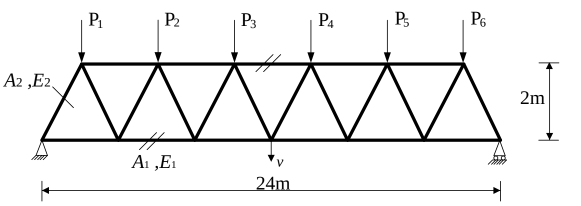

Figure 2. The truss structure.

Table 3. The statistical moments for Example 2.

Variable	Distribution	Mean	S. d.^{a}
E_1,E_2 (Pa)	Lognormal	2.0×10^{11}	2.0×10^{10}
A_1 (m^2)	Normal	2.0×10^{-3}	2.0×10^{-4}
A_2 (m^2)	Normal	1.0×10^{-3}	1.0×10^{-4}
P_1,P_6 (N)	Gumbel	5.0×10^{4}	5.0×10^{3}
P_2,P_5 (N)	Gumbel	5.5×10^{4}	5.5×10^{3}
P_3,P_4 (N)	Gumbel	6.0×10^{4}	6.0×10^{3}

Table 4. The statistical moments for Example 2.

μ_z	$m_{z,2}$	$m_{z,3}$	$m_{z,4}$
45.0329	156.5768	-1076.9297	87886.7272

The PDF and the probability errors are plotted in Fig. 3, which obtained from CNTM, AMPS, SAM, MEM, and NIG. It is noted that the NIG distribution is also accurate and robust for approximating PDF. The failure probabilities calculated by the CNTM, AMPS, SAM, MEM, NIG, and MCS are 0.0020226, 0.002041, 0.0018434, 0.0021385, 0.0020415, and 0.00209, respectively. The result of failure probabilities also demonstrates that the NIG distribution is accurate for reliability analysis.

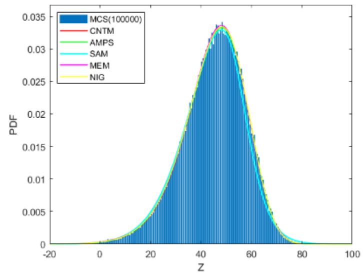

Figure 3 (a). PDF of Z.

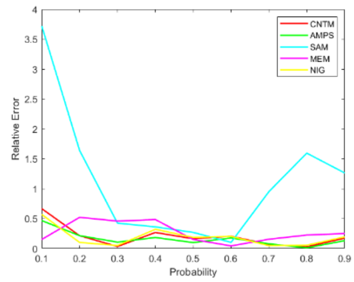

Figure 3 (b). Relative errors of F in 0.1~0.9.

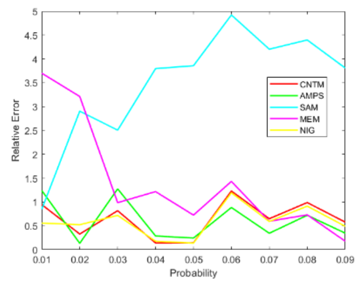

Figure 3 (c). Relative errors of F in 0.01~0.09.

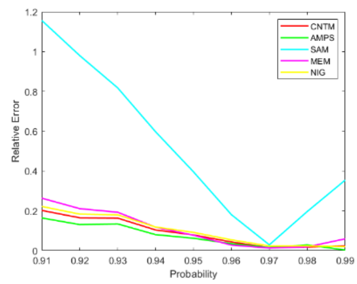

Figure 3 (d). Relative errors of F in 0.91~0.99.

3. Conclusion

A highly efficient method for probability estimation method has been presented in this paper. Two examples are presented to elucidate the efficiency and accuracy of the proposed method in comparison with other methods, and the following conclusions can be drawn:

(1) The reliability analysis moment method based on NIG approximation does not need to solve complex nonlinear equations and does not need to make cumbersome judgments,

which is easier to implement than the reference method, and the accuracy of the proposed method is higher than that of the reference method.
(2) The NIG distribution has rich flexibility in shape, covering an extensive portion of the skewness–kurtosis diagram, and the NIG distribution presents better accuracy in terms of the overall CDF quality and the tail region.

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