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Series System Reliability Estimation of Randomly Excited Uncertain Linear Structures by Cross Entropy-based Importance Sampling

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Abstract: This paper presents a simulation-based method to assess the reliability of linear structures with parameter uncertainties subject to Gaussian process excitation. The focus is on estimating failure probabilities of series systems, where structural failure is defined as a union of multiple first-passage failure events. The proposed approach estimates the system failure probability conditional on a given value of the uncertain structural parameters by importance sampling. To determine the failure probability involving parameter uncertainties, an integration of the conditional probability over the space of uncertain parameters is required. The novel contribution lies in the development of an adaptive importance sampling strategy based on the cross entropy method to efficiently perform this integration. A numerical example demonstrates the performance of the proposed method.

Keywords: First-passage probability, series systems, linear structures, importance sampling, cross entropy.

1. Introduction

Reliability assessment of dynamically excited structures is a challenging problem because of uncertainties in the external load and structural properties. In structural dynamics, failure is usually defined in terms of the first passage of a structural response over a specified threshold. This paper focuses on estimating failure probabilities of series systems, where the structure failure event is a union of multiple failure modes, each corresponding to a different first-passage failure. Monte Carlo simulation (MCS) methods can be applied to solve this class of problems. The main challenge in applying MCS lies in controlling the sampling variance of the failure probability estimator; the aim is to obtain probability estimates of acceptable accuracy with a small number of dynamic model runs.

For the particular case where the structural behavior is linear and the applied dynamic load is modeled as a Gaussian random process, efficient sampling methods have been developed that take advantage of the linearity of the structural response with respect to the Gaussian loading. Jensen and Valdebenito (2010) and Valdebenito et al. (2014) propose importance sampling (IS) techniques based on this approach. The application of line sampling in this context has been explored by Pradlwarter and Schueller (2010). These methods require system specific information to facilitate reliability estimation. In (Jensen and Valdebenito 2010, Pradlwarter and Schueller 2010), a pseudo-design point with respect to the uncertain structural parameters has to be identified. These approaches can be effective when there is a unique design point contributing to the failure probability. The IS method in (Valdebenito et al. 2014) makes use of a surrogate model for the probability of failure as a function of the uncertain structural parameters. The performance of the method thus relies on the proper choice of the surrogate model, which is not always a straightforward task.

The present contribution develops an adaptive importance sampling method to estimate the series system reliability of uncertain linear structures subject to Gaussian loading. The proposed approach is based on the cross entropy (CE) method. The CE method performs a pre-sampling step to determine the parameters of a near-optimal IS density through minimizing the Kullback-Leibler (KL) divergence between the theoretically optimal IS density and a chosen parametric family of densities. Unlike the existing studies discussed earlier, the proposed method can be used as a black-box algorithm, as it does not require problem-specific adjustments. It is therefore more robust and generally applicable to any linear dynamical system.

2. Problem Formulation

2.1 Dynamical System

An n degrees-of-freedom linear structure, discretized from a continuum system, e.g., by the finite element method, is governed by the equation

$$\mathbf{M}(\boldsymbol{\Theta})\ddot{\boldsymbol{X}}(t) + \mathbf{C}(\boldsymbol{\Theta})\dot{\boldsymbol{X}}(t) + \mathbf{K}(\boldsymbol{\Theta})\boldsymbol{X}(t) = \boldsymbol{D}f(t), \quad (1)$$

where $\boldsymbol{X}$, $\dot{\boldsymbol{X}}$ and $\ddot{\boldsymbol{X}}$ are the $n \times 1$ displacement, velocity and acceleration vectors; $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices of dimension $n \times n$; $\boldsymbol{\Theta}$ is an $n_{\boldsymbol{\theta}} \times 1$ vector of basic random variables that models the uncertain structural parameters; f is the random dynamic excitation acting on the structure over a time span $t \in [0, T]$; and $\boldsymbol{D}$ is an $n \times 1$ vector that couples the external excitation with the degrees of freedom of the structure. We assume that the components of $\boldsymbol{\Theta}$ are independent and identically distributed (i.i.d.) standard normal random variables. Structural parameters that follow a non-Gaussian distribution or are mutually dependent can be generated by an iso-probabilistic transformation of the basic standard normal random variables (Hohenbichler and Rackwitz 1981).

Let $\{h_i; \mathrm{i} = 1, \dots, m\}$ be m critical structural responses that are of interest, e.g., displacements, accelerations. For a particular value of the uncertain parameters specified by $\boldsymbol{\Theta} = \boldsymbol{\theta}$, the relationship between the input excitation and the i-th output response of interest is linear and is represented by a convolution integral as

$$h_i(t,\boldsymbol{\theta}) = \int_0^t K_i(t-\tau;\boldsymbol{\theta}) f(\tau) d\tau. \quad (2)$$

In the above equation $K_i(t;\boldsymbol{\theta})$ is the impulse response function for the response h_i at time t due to a unit impulse applied at time 0, where zero initial conditions have been assumed without loss of generality.

In practical applications, the dynamic response of the structure is calculated at discrete time steps by numerical integration. Let $\{t_1, \dots, t_{n_T}\}$ be a set of discrete time points at a uniform time spacing $\Delta t = T/(n_T - 1)$ over the duration $[0, T]$. The excitation in discrete-time is taken to be a band-limited Gaussian white noise:

$$f(t_k) = \sqrt{2\pi S/\Delta t}\, \Xi_k, \quad (3)$$

where S is the spectral intensity, and $\{\Xi_k; k = 1, \dots, n_T\}$ are i.i.d. standard Gaussian random variables which are collectively represented by the n_T-dimensional vector $\boldsymbol{\Xi} = \{\Xi_1; \dots; \Xi_{n_T}\}$. The i-th dynamic response of the structure at time t_k is then written as

$$h_i(t_k, \boldsymbol{\theta}, \boldsymbol{\Xi}) = \sum_{s=1}^{k} K_i(t_k - t_s; \boldsymbol{\theta}) \sqrt{2\pi S \Delta t}\, \Xi_s. \quad (4)$$

2.2 Series system reliability

In reliability estimation of dynamical systems, an important problem is the computation of the first-passage probability, which is the probability that any one of the m output responses $\{h_i, i = 1, \dots, m\}$ exceeds a corresponding threshold level h_i^* within the time duration T. The system level failure event F is therefore expressed as

$$F = \bigcup_{i=1}^{m} F_i, \quad (5)$$

where

$$F_i = \{\boldsymbol{\theta} \in \mathrm{R}^{n_{\boldsymbol{\theta}}}, \boldsymbol{\xi} \in \mathrm{R}^{n_T} : \max_{k=1,\dots,n_T} |h_i(t_k, \boldsymbol{\theta}, \boldsymbol{\xi})| \geq h_i^*\} \quad (6)$$

denotes first-passage failure with respect to the i-th response measure. The probability of occurrence of F is defined by means of the multi-dimensional integral

$$P_F = \int_{\boldsymbol{\theta} \in \mathrm{R}^{n_{\boldsymbol{\theta}}}} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}, \quad (7)$$

where

$$P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) = \int_{\boldsymbol{\xi} \in \mathrm{R}^{n_T}} \mathrm{I}\{(\boldsymbol{\theta}, \boldsymbol{\xi}) \in F\}\, p_{\boldsymbol{\Xi}}(\boldsymbol{\xi}) d\boldsymbol{\xi} \quad (8)$$

is the system failure probability conditional on the particular realization $\boldsymbol{\Theta} = \boldsymbol{\theta}$ of the uncertain structural parameters. In the above equations, $p_{\boldsymbol{\Xi}}(\boldsymbol{\xi})$ and $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ denote the joint probability density functions (PDF) of $\boldsymbol{\Xi}$ and $\boldsymbol{\Theta}$, respectively, and $\mathrm{I}\{(\boldsymbol{\theta}, \boldsymbol{\xi}) \in F\}$ is the indicator function for the failure event which takes the value 1 if $(\boldsymbol{\theta}, \boldsymbol{\xi}) \in F$ and is 0 otherwise.

In principle, one can evaluate both $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ and P_F by crude MCS. However, when these probabilities are small, which is typically the case in engineering problems, crude MCS requires a large number of samples to estimate these quantities with sufficient accuracy. In the subsequent sections, we formulate an alternative strategy based on IS to efficiently estimate the failure probability of the series system. In Sec. 3 we discuss a procedure to evaluate the conditional failure probability $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$. The proposed approach for computing the unconditional failure probability by efficiently solving Eq. 7 is subsequently described in Sec. 4.

3. Probability of failure conditional on $\boldsymbol{\Theta} = \boldsymbol{\theta}$

In this section, we discuss the evaluation of the probability of failure conditional on a given value of the uncertain structural parameters $\boldsymbol{\Theta} = \boldsymbol{\theta}$. Let the event $F_{i,k}(\boldsymbol{\theta})$ denote exceedance of the threshold level h_i^* by the i-th structural response at time $t = t_k$. Occurrence of any one of the instantaneous failure events $\{F_{i,k}(\boldsymbol{\theta}); k = 1, \dots, n_T, i = 1, \dots, m\}$ leads to failure of the structure. Therefore, the system level failure event conditional on $\boldsymbol{\Theta} = \boldsymbol{\theta}$ is expressed as

$$F(\boldsymbol{\theta}) = \bigcup_{i=1}^{m} \bigcup_{k=1}^{n_T} F_{i,k}(\boldsymbol{\theta}), \quad (9)$$

where $F_{i,k}(\boldsymbol{\theta}) = \{\boldsymbol{\xi} \in \mathrm{R}^{n_T} : |h_i(t_k, \boldsymbol{\theta}, \boldsymbol{\xi})| \geq h_i^*\}$.

We estimate the probability of occurrence of $F(\boldsymbol{\theta})$ by IS using the IS density proposed in (Au and Beck 2001). This IS density is expressed as

$$h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta}) = \sum_{i=1}^{m} \sum_{k=1}^{n_T} w_{i,k}(\boldsymbol{\theta}) p_{\boldsymbol{\Xi}}\left(\boldsymbol{\xi} \middle| F_{i,k}(\boldsymbol{\theta})\right), \quad (10)$$

where $p_{\boldsymbol{\Xi}}\left(\boldsymbol{\xi} \middle| F_{i,k}(\boldsymbol{\theta})\right)$ is the PDF of $p_{\boldsymbol{\Xi}}(\boldsymbol{\xi})$ truncated on the failure domain $F_{i,k}(\boldsymbol{\theta})$, and $\{w_{i,k}(\boldsymbol{\theta}); k = 1, \dots, n_T, i = 1, \dots, m\}$ are normalized weights associated with the instantaneous failure events. The weight $w_{i,k}(\boldsymbol{\theta})$ is defined as

$$w_{i,k}(\boldsymbol{\theta}) = \Pr\left[F_{i,k}(\boldsymbol{\theta})\right] \Big/ \sum_{r=1}^{m} \sum_{s=1}^{n_T} \Pr\left[F_{r,s}(\boldsymbol{\theta})\right], \quad (11)$$

where $\Pr\left[F_{i,k}(\boldsymbol{\theta})\right]$ is the probability content of the domain of $F_{i,k}(\boldsymbol{\theta})$. The probability of $F_{i,k}(\boldsymbol{\theta})$ is given by

$$\Pr\left[F_{i,k}(\boldsymbol{\theta})\right] = 2\Phi\left[-\beta_{i,k}(\boldsymbol{\theta})\right], \quad (12)$$

where $\beta_{i,k}(\boldsymbol{\theta}) = h_i^* \Big/ \sqrt{\sum_{s=1}^{k} K_i^2(t_k - t_s; \boldsymbol{\theta}) 2\pi S \Delta t}$ is the Euclidean norm of the design point associated with the domain $F_{i,k}^{+}(\boldsymbol{\theta}) = \{\boldsymbol{\xi} \in \mathrm{R}^{n_T} : h_i(t_k, \boldsymbol{\theta}, \boldsymbol{\xi}) \geq h_i^*\}$.

Using the IS density described in Eqs. 10-12, the following modified expression for the conditional probability of failure $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is obtained:

$$\begin{aligned} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) &= \int_{\boldsymbol{\xi} \in \mathrm{R}^{n_T}} \mathrm{I}\{(\boldsymbol{\theta}, \boldsymbol{\xi}) \in F\} \frac{p_{\boldsymbol{\Xi}}(\boldsymbol{\xi})}{h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta})} h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta}) d\boldsymbol{\xi} \\ &= \int_{\boldsymbol{\xi} \in \mathrm{R}^{n_T}} \frac{\tilde{P}(\boldsymbol{\theta})}{\sum_{i=1}^{m} \sum_{k=1}^{n_T} \mathrm{I}\{(\boldsymbol{\theta}, \boldsymbol{\xi}) \in F_{i,k}(\boldsymbol{\theta})\}} h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta}) d\boldsymbol{\xi} \end{aligned} \quad (13)$$

where $\tilde{P}(\boldsymbol{\theta}) = \sum_{r=1}^{m} \sum_{s=1}^{n_T} \Pr\left[F_{r,s}(\boldsymbol{\theta})\right]$. The above integral is estimated by Monte Carlo integration by generating random samples of $\boldsymbol{\Xi}$ from $h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta})$ using the algorithm proposed in (Au and Beck 2001).

4. Probability of failure considering uncertainty in the structural parameters

Evaluation of the probability of failure of the uncertain linear structure requires integration of the conditional failure probability $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ over the whole domain of $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. This leads to the integral in Eq. 7. We evaluate this

integral by IS. Let $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ be the IS density for $\boldsymbol{\Theta}$. The integral in Eq. 7 is written in modified form

$$P_F = \int_{\boldsymbol{\theta}\in \mathrm{R}^{n_{\boldsymbol{\theta}}}} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) W(\boldsymbol{\theta}) h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}, \qquad (14)$$

where $W(\boldsymbol{\theta}) = p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})/h_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is the importance weight function. The coefficient of variation (CV) of the IS estimator of P_F based on Eq. 14 depends on the choice of $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. The optimal IS density that leads to an estimator with zero variance is given by

$$h^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) = \frac{1}{P_F} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta}) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}). \qquad (15)$$

In this study we propose an IS density $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ that is a close approximation of $h^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ and is constructed by the CE method.

4.1 IS density for $\boldsymbol{\Theta}$ based on the cross entropy method

The CE method is an adaptive approach that determines an IS density through minimizing the KL divergence between the theoretically optimal IS density $h^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ and a chosen parametric family of distributions. Let $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\boldsymbol{\nu})$ be a family of parametric densities, where $\boldsymbol{\nu} \in \mathcal{V}$ is the parameter vector. The KL divergence between $h^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ and $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\boldsymbol{\nu})$ is defined as (Rubinstein and Kroese 2017)

$$D\big(h^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta}), h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\boldsymbol{\nu})\big) = \mathrm{E}_{h^*_{\boldsymbol{\Theta}}}\left[\ln\left(\frac{h^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\boldsymbol{\nu})}\right)\right]. \qquad (16)$$

The basic idea in the CE method is to determine the parameter vector $\boldsymbol{\nu} \in \mathcal{V}$ through minimizing the KL divergence of Eq. 16, i.e. through solving

$$\boldsymbol{\nu} = \underset{\boldsymbol{a}\in\mathcal{V}}{\operatorname{argmin}}\, D\big(h^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta}), h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\boldsymbol{a})\big). \qquad (17)$$

Upon substituting the expression of $h^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ in Eq. 15, the optimization problem in Eq. 17 becomes equivalent to

$$\boldsymbol{\nu} = \underset{\boldsymbol{a}\in\mathcal{V}}{\operatorname{argmax}}\, \mathrm{E}_{p_{\boldsymbol{\Theta}}}\big[P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})\ln\big(h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\boldsymbol{a})\big)\big]. \qquad (18)$$

The above optimization can be solved by approximating the expectation in Eq. 18 using a set of samples from $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. To obtain a good sample approximation, a considerable number of these samples should come from the high probability region of $h^*_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$, i.e. the region where the value of $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is large. Ensuring this could require a large number of samples to be drawn from $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$, which makes the direct solution of Eq. 18 computationally impractical.

In order to circumvent this problem, we solve the CE optimization in Eq. 18 using a multi-level approach. We introduce a sequence of intermediate target densities $\{h^k_{\boldsymbol{\Theta}}(\boldsymbol{\theta}); k = 1, \dots, L\}$ that gradually approach the optimal IS density. These intermediate densities are defined as

$$h^k_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) = \frac{1}{C_k} P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})^{\gamma_k} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}), \qquad (19)$$

where $0 < \gamma_1 < \cdots < \gamma_L = 1$ and C_k is the normalizing constant of $h^k_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. The idea is to solve the CE optimization sequentially for each of the intermediate target densities, which leads to a sequence of parameter vectors $\{\boldsymbol{\nu}^k, k = 1, \dots, L\}$. The final parameter vector $\boldsymbol{\nu}^L$ should approximate well the solution of Eq. 18.

To determine $\boldsymbol{\nu}^k$ we minimize the KL divergence between $h^k_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ and $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\boldsymbol{\nu})$, which results in the following optimization problem

$$\boldsymbol{\nu}^k = \underset{\boldsymbol{a}\in\mathcal{V}}{\operatorname{argmax}}\, \mathrm{E}_{p_{\boldsymbol{\Theta}}}\big[P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})^{\gamma_k}\ln\big(h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\boldsymbol{a})\big)\big]. \qquad (20)$$

The expectation in Eq. 20 is approximated by IS using a set of samples drawn from $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\hat{\boldsymbol{\nu}}^{k-1})$, $\hat{\boldsymbol{\nu}}^{k-1}$ being the solution of the problem in the previous step. The sample counter-part of the CE optimization problem in Eq. 20 is given by

$$\hat{\boldsymbol{\nu}}^k = \underset{\boldsymbol{a}\in\mathcal{V}}{\operatorname{argmax}} \sum_{i=1}^{N} W_k(\boldsymbol{\theta}^i;\hat{\boldsymbol{\nu}}^{k-1})\ln\Big(h_{\boldsymbol{\Theta}}(\boldsymbol{\theta}^i;\boldsymbol{a})\Big). \qquad (21)$$

Here $W_k(\boldsymbol{\theta};\hat{\boldsymbol{\nu}}^{k-1}) = P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})^{\gamma_k} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})/h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\hat{\boldsymbol{\nu}}^{k-1})$ and $\{\boldsymbol{\theta}^i, i = 1, \dots, N\}$ are i.i.d. samples drawn from $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\hat{\boldsymbol{\nu}}^{k-1})$. A default choice of $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\hat{\boldsymbol{\nu}}^0)$ is the nominal density $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$.

At each intermediate sampling step, we select the parameter γ_k adaptively such that the sample CV $\hat{\delta}_{W_k}$ of the weights $\{W_k(\boldsymbol{\theta}^i;\hat{\boldsymbol{\nu}}^{k-1}), i = 1, \dots, N\}$ is equal to a target value δ_{target}. Hence, at each iteration one solves the optimization problem

$$\gamma_k = \underset{\gamma\in(\gamma_{k-1},1)}{\operatorname{argmin}} \big(\hat{\delta}_{W_k}(\gamma) - \delta_{target}\big)^2. \qquad (22)$$

We note that Eq. 22 is equivalent to requiring that the number of effective samples available to fit the parametric density at each sampling iteration is equal to a target value (Latz et al. 2018). The choice of the value of δ_{target} is discussed in (Papaioannou et al. 2016, 2019). In the present study we set δ_{target} to 1.5. The adaptive procedure is stopped when the value of γ_k determined based on Eq. 22 is equal to 1. After convergence, the final parameter vector $\hat{\boldsymbol{\nu}}^L$ is determined by solving Eq. 21 with $\gamma_L = 1$. The sampling density $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\hat{\boldsymbol{\nu}}^L)$ is taken as the IS density for estimating the probability of failure.

Finally, we note that determination of the IS density for $\boldsymbol{\Theta}$ based on the above described procedure requires repeated evaluations of the conditional probability $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$. One could evaluate $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ by IS according to Eq. 10. However, to ensure smooth convergence of the CE method, the sampling variance of the estimate of $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ should be small. This requires a sufficient number of samples to be used in the IS estimator for $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$, which, in turn, increases the number of evaluations of the dynamical system. In order to reduce the computational effort, we employ an analytical approximation of $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ during CE optimization, instead of estimating it by IS. This approach is based on the extended Poisson approximation described in (Song and Der Kiureghian 2006). The IS estimator for $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$ described in Section 3 is only applied once the final IS density for $\boldsymbol{\Theta}$ is obtained.

4.2 IS estimator for the probability of failure

The probability of failure is estimated based on Eq. 14 by substituting $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\hat{\boldsymbol{\nu}}^L)$ for $h_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ and the expression in Eq. 13 for $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$, respectively. This leads to the following IS estimator for P_F:

$$\hat{P}_F = \frac{1}{N_R}\sum_{i=1}^{N_R} \tilde{P}(\boldsymbol{\theta}^i)\left(\frac{1}{\sum_{j=1}^{m}\sum_{k=1}^{n_T} \mathrm{I}\{(\boldsymbol{\theta}^i,\boldsymbol{\xi}^i)\in F_{j,k}(\boldsymbol{\theta}^i)\}}\right) W(\boldsymbol{\theta}^i), \qquad (23)$$

where $W(\boldsymbol{\theta}) = p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})/h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\hat{\boldsymbol{\nu}}^L)$ and $\{(\boldsymbol{\theta}^i, \boldsymbol{\xi}^i), i = 1, \dots, N_R\}$ are i.i.d. samples drawn from the joint density $h_{\boldsymbol{\Theta},\boldsymbol{\Xi}}(\boldsymbol{\theta},\boldsymbol{\xi}) = h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta})h_{\boldsymbol{\Theta}}(\boldsymbol{\theta};\hat{\boldsymbol{\nu}}^L)$. Here $h_{\boldsymbol{\Xi}}(\boldsymbol{\xi}|\boldsymbol{\theta})$ is the IS density of $\boldsymbol{\Xi}$ conditional on $\boldsymbol{\Theta} = \boldsymbol{\theta}$ defined in Eq. 10 and $\tilde{P}(\boldsymbol{\theta}) = \sum_{j=1}^{m}\sum_{k=1}^{n_T} \Pr\big[F_{j,k}(\boldsymbol{\theta})\big]$.

5. Numerical Illustration

This section illustrates the performance of the proposed CE-based IS method by application to a numerical example that is a modified version of an example given in (Valdebenito et al. 2014). It consists of a two-story linear shear frame that is excited by a stochastic ground acceleration. The dynamic response of the structure is governed by Eq. 1 with $\boldsymbol{X}(t) = \{X_1(t) \quad X_2(t)\}^{\mathrm{T}}$, where $X_i(t)$ denotes the relative displacement of the i-th floor with respect to the support. The mass and stiffness matrices, respectively, are given by

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \text{ and } \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix},$$

where m is the mass of each floor and k_i is the lateral stiffness of the i-th floor, and the damping matrix $\mathbf{C}$ corresponds to a classical damping of 4% for the two modes. It is noted that $\boldsymbol{D} = -\{m \quad m\}^{\mathbf{T}}$. Each floor possesses a mass of $m = 30\mathrm{Mg}$. The stiffness parameters $\{k_i; i = 1,2\}$ are modeled as independent uniform random variables with marginal distribution $k_i \sim \mathrm{U}[12,28]\ \mathrm{MN/m}$. The base acceleration $f(t)$ is modeled as a stationary Gaussian white noise of duration $T = 15\mathrm{s}$ and spectral intensity $S = 10^{-4}\ \mathrm{m^2/s^3}$. The stochastic excitation is discretized at time intervals of $\Delta t = 0.01\mathrm{s}$, i.e. $n_T = 1501$. Two response measures are considered: $h_1 =$ relative displacement of the first floor with respect to the support, and $h_2 =$ relative displacement between the first and the second floor. The objective is to estimate the probability that any one of these response measures exceeds a specified threshold h_i^* over the duration of the random excitation.

The performance of the CE method is investigated for two choices of the parametric density family: a multi-variate single Gaussian (SG) distribution and a two-component Gaussian mixture (GM) distribution. A SG density is completely specified by the mean vector and the covariance matrix. In the present examples, where the number of uncertain structural parameters is equal to 2, the bi-variate SG density comprises of 5 unknown parameters that are determined by CE optimization. For a two-component GM distribution, in addition to the parameters of the individual Gaussian densities, the normalized weight associated with each component has to be determined. This leads to a total of 11 unknown parameters for the bi-variate two-component GM model. For a SG distribution, the update rule for the unknown parameter vector $\boldsymbol{\nu}$ in each intermediate sampling step, i.e. the solution of the optimization problem in Eq. 21, is determined analytically in closed-form (Rubinstein and Kroese 2017, Kurtz and Song 2013). For the GM distribution the updated values of $\boldsymbol{\nu}$ are obtained through an expectation-maximization algorithm (Geyer et al. 2019). The IS density for $\boldsymbol{\Theta}$ obtained using the two choices of the parametric density is illustrated in Fig. 1 for thresholds $(h_1^*, h_2^*) = (0.0052\mathrm{m}, 0.0037\mathrm{m})$. The solid lines represent the contours of the optimal IS density $h_{\boldsymbol{\Theta}}^*(\boldsymbol{\theta})$, which is estimated from 10^7 direct Monte Carlo samples. The scattered points in Fig. 1(a) and 1(b) are samples of $\boldsymbol{\Theta}$ drawn from the IS density obtained using SG and GM distributions, respectively. It is seen that the samples from the GM distribution better represent the bi-modal nature of $h_{\boldsymbol{\Theta}}^*(\boldsymbol{\theta})$.

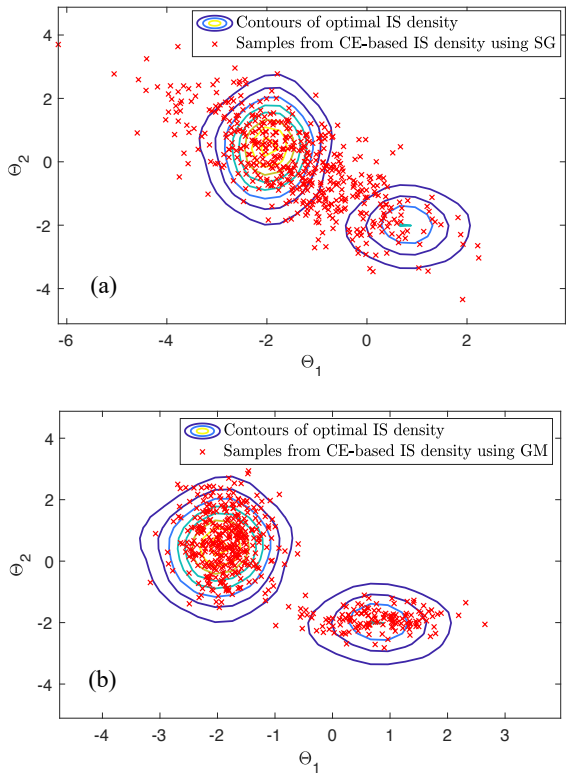

Figure 1. Comparison of IS density for $\boldsymbol{\Theta}$ from the CE method. Solid lines are contours of the optimal IS density $h_{\boldsymbol{\Theta}}^*(\boldsymbol{\theta})$. Scattered points are samples drawn from the IS density obtained using SG and two-component GM distributions.

The performance of the proposed method is assessed in terms of the sample mean and sample CV of the estimates of P_F, denoted by $\hat{P}_F$ and δ in this section, and in terms of the number of dynamical system evaluations required to implement the method. N_{CE} denotes the total number of samples needed to determine the IS density for $\boldsymbol{\Theta}$ using the CE method. N_R denotes the number of samples of $(\boldsymbol{\Theta}, \boldsymbol{\Xi})$ used to obtain a sample estimate of P_F during reliability estimation, i.e. when evaluating Eq. 23. It is noted that the dynamical system needs to be evaluated only once for every generated sample $\boldsymbol{\theta}$. Therefore, N_{CE} and N_R also indicate the number of dynamical system evaluations needed in the CE optimization step and the reliability estimation step, respectively. During CE optimization, the system is evaluated to compute the impulse response function of the critical structural responses h_i and their velocities, which are then post-processed to determine the Poisson approximation of $P_{F|\boldsymbol{\Theta}}(\boldsymbol{\theta})$. In the reliability estimation step, the impulse response functions of h_i for every sample $\boldsymbol{\theta}$ are convoluted with a sample realization of the input excitation to obtain a realization of the response time-histories. $N_T = N_{CE} + N_R$ is the total number of system evaluations. The performance measures are estimated from 50 independent simulation runs.

While implementing the CE-based IS method, the sample size N_R in the reliability estimation step is selected using two approaches which are indicated using the following nomenclature: (i) SG-NonAdap and GM-NonAdap denote the case in which N_R is taken equal to the number of samples per level for CE optimization, i.e. $N_R = N$, and (ii) SG-Adap and GM-Adap denote the case in which N_R is adapted on the fly to ensure that an estimate of the CV of the IS estimate of P_F is smaller than a specified target value δ^*.

The simulation results for $(h_1^*, h_2^*) = (0.0045\text{m}, 0.0030\text{m})$ and $(0.0052\text{m}, 0.0037\text{m})$ are reported in Tables 1 and 2, respectively. The results from SG-Adap and GM-Adap correspond to $\delta^* = 0.05$. The number of samples per level for CE optimization is taken as $N = 250$. The reference values for the probability of failure are obtained from a crude MCS using 10^7 samples. The values of N_{CE} indicate that for both pairs of response thresholds and for both choices of the parametric density, the CE method requires approximately two steps to converge to the target density. The mean estimates obtained using the SG and GM distributions with the two choices of N_R are similar but slightly biased. However, in terms of the CV of the estimates and the average computational cost, the GM distribution performs better. For similar number of system evaluations, it is seen that the CV of the estimate obtained from GM-NonAdap is smaller compared to SG-NonAdap. GM-Adap requires less number of system evaluations than SG-Adap to converge to the target CV δ^*. The superior performance of the GM distribution is due to its greater flexibility in fitting the bi-modal nature of $h_{\boldsymbol{\Theta}}^*(\boldsymbol{\theta})$.

Table 1. Results for $(h_1^*, h_2^*) = (0.0045\text{m}, 0.0030\text{m})$ from the CE-based IS method using SG and GM distributions. Reference probability of failure is 1.41×10^{-3}.

	$\hat{P}_F$	δ	N_{CE}	N_R	N_T
SG-NonAdap	1.25x10^{-3}	0.104	505	250	755
SG-Adap	1.22x10^{-3}	0.050	505	1006	1511
GM-NonAdap	1.24x10^{-3}	0.080	530	250	780
GM-Adap	1.22x10^{-3}	0.052	530	610	1140

Table 2. Results for $(h_1^*, h_2^*) = (0.0052\text{m}, 0.0037\text{m})$ from the CE-based IS method using SG and GM distributions. Reference probability of failure is 4.06×10^{-5}.

	$\hat{P}_F$	δ	N_{CE}	N_R	N_T
SG-NonAdap	3.45x10^{-5}	0.102	500	250	750
SG-Adap	3.51x10^{-5}	0.051	500	1110	1610
GM-NonAdap	3.45x10^{-5}	0.066	555	250	805
GM-Adap	3.44x10^{-5}	0.047	555	532	1087

We investigate the effect of the sample size *N*, used per level during CE optimization, on the performance of the method. For this, different values of *N* in the range 125-1000 are considered. The parametric study is conducted using the GM distribution. The sample means of the probability estimates from GM-NonAdap and GM-Adap for the different values of *N* are similar to those given in Tables 2 and 3, and hence are not reported separately. The

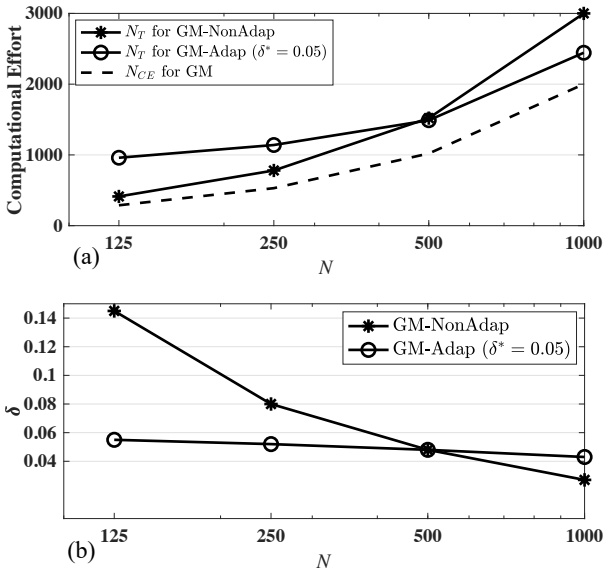

Figure 2. Variation of N_T and N_{CE} with N for $(h_1^*, h_2^*) = (0.0045\text{m}, 0.0030\text{m})$ using GM distribution

variation in the sample CV of the estimates and the computational effort is depicted in Fig. 2 and 3. Since the total effort for CE optimization is proportional to the number of samples per level, N_{CE} increases monotonically with *N*. The difference between the vertical coordinates of the dotted line and the solid lines gives N_R, the average number of dynamical system evaluations used in the reliability estimation step. With increase in *N*, the number of effective samples of $\boldsymbol{\Theta}$ available to fit the parametric distribution at each intermediate sampling step increases. This leads to better estimation of the parameters of the IS density for $\boldsymbol{\Theta}$. For GM-NonAdap, where $N_R = N$, an increase in *N* also implies an increase in the number

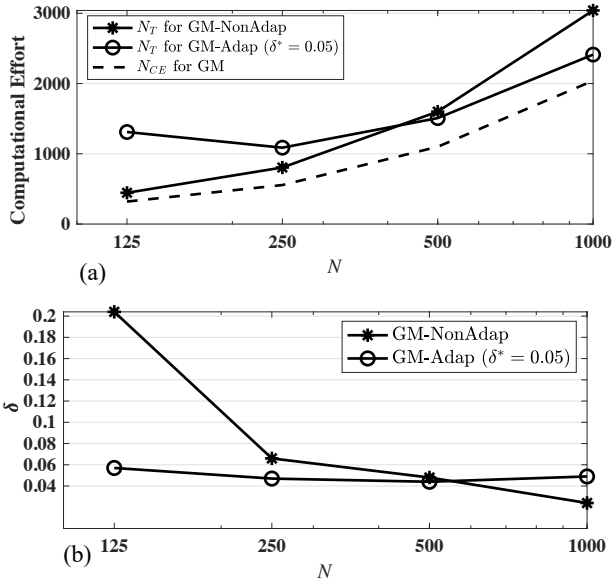

Figure 3. Variation of N_T and N_{CE} with N for $(h_1^*, h_2^*) = (0.0052\text{m}, 0.0037\text{m})$ using GM distribution

of samples of $(\boldsymbol{\Theta}, \boldsymbol{\Xi})$ used to obtain a sample estimate of P_F during reliability estimation. Due to these factors, the sample CV of the probability estimates obtained from GM-NonAdap decreases as *N* increases. In GM-Adap, it is seen that the number of samples for reliability estimation initially decreases as *N* increases, particularly for $(h_1^*, h_2^*) = (0.0052\text{m}, 0.0037\text{m})$. This is due to the sub-optimality in the IS density for $\boldsymbol{\Theta}$ obtained with a small *N*, which leads to a greater computational effort during reliability estimation necessary to meet the prescribed δ^*. As *N* increases, one obtains improved estimates of the parameter vector, and the number of samples for reliability estimation starts decreasing. Beyond $N = 500$, N_R is nearly constant, which indicates that the IS density for $\boldsymbol{\Theta}$ obtained using 500 samples per level is sufficiently optimal, and a further increase in *N* does not give any additional advantage during reliability estimation. The sample CV of the probability estimates obtained using GM-Adap remains close to the prescribed δ^* for all *N*. Finally, Figs. 2 and 3 show that the IS estimator with adaptive choice of N_R requires less number of dynamical system evaluation to meet a prescribed CV. For the case of $(h_1^*, h_2^*) = (0.0052\text{m}, 0.0037\text{m})$, it is seen that GM-NonAdap requires about 1600 system evaluations (with $N = 500$) to achieve a sample CV of 0.05, whereas GM-Adap gives the same sample CV with only 1100 system evaluations (with $N = 250$). This indicates that if the goal is to achieve a target value of the sample CV, the adaptive variant of the IS estimator is more efficient provided that the number of samples per level *N* is chosen appropriately.

6. Conclusions

The paper presents an adaptive IS method to estimate the series system reliability of uncertain linear structures subject to Gaussian loading. The main contribution lies in the development of an effective IS density for the uncertain structural parameters using the CE method. The proposed method introduces a smooth transition to the optimal IS density by choosing a sequence of intermediate target densities. The sampling density for the uncertain parameters is determined adaptively by minimizing the KL divergence between the sequence of target densities and a chosen family of parametric distributions. The IS density for the structural parameters is then combined with an efficient IS density for the input excitation to obtain the IS estimator for the probability of failure.

The CE method is implemented using the multi-variate single Gaussian distribution and the Gaussian mixture model as the parametric densities. For series systems, where the structural failure event is a union of multiple first-passage failures, the optimal IS density with respect to the uncertain structural parameters is usually multi-modal in nature. In such cases, the Gaussian mixture distribution offers more flexibility in fitting the optimal density. Results from numerical studies on a two-component series system demonstrate that the Gaussian mixture model outperforms the single Gaussian distribution, both in terms of the coefficient of variation of the failure probability estimate and the computational effort. The performance of the method in problems with a large number of uncertain structural parameters and several component failure modes is currently being investigated by the authors.

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