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Efficient Reliability-based Design Updating Method for Piles in Spatially Variable Soils

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Abstract: Although various uncertainties in geotechnical engineering can be incorporated into geotechnical reliability-based design (RBD) in a straightforward manner using full probabilistic design approach, it is nontrivial to obtain feasible designs for different design scenarios when the spatial variability of soil parameters is considered in geotechnical design. This paper develops an efficient RBD updating approach for piles in spatially variable soils, which uses sample reweighting technique and equivalent variance technique. The proposed approach updates feasible designs and design points for different design scenarios based on a single run of direct MCS, avoiding repeatedly performing direct MCS for each design scenario. A drilled shaft design example is used to illustrate the proposed approach.

Keywords: Reliability-based design, Design updating, Spatial variability, Sample reweighting, Design point.

1. Introduction

There are various uncertainties (e.g., uncertainties in geotechnical properties and loads, etc.) in geotechnical engineering (e.g., Baecher and Christian 2003) affecting decision making in geotechnical design process. It is necessary to incorporate these uncertainties into geotechnical design to determine a rational design. Reliability-based design (RBD) approaches can deal, rationally, with these uncertainties. Several semi-probabilistic RBD codes and/or methodologies have been developed in geotechnical engineering in the past few decades, such as load and resistance factor design (LRFD) (Paikowsky 2004; Fenton et al. 2016) and the multiple resistance factor design (MRFD) (Phoon et al. 2003a, 2003b). These semi-probabilistic RBD codes and/or methodologies use load and resistance factors (or partial factors) to implicitly consider uncertainties in geotechnical engineering. However, these factors must be calibrated given some assumptions and simplifications (Wang et al. 2011) before used in practice. This renders difficulties in implementing semi-probabilistic RBD approach in geotechnical practice because of a wide range of design scenarios due to site-specific nature and relatively large variability of geotechnical materials.

Among various geotechnical-related uncertainties, it has been demonstrated in literature that the spatial variability of soil properties significantly affects the performance of geotechnical structures (Phoon and Kulhawy 1999; Fenton and Griffiths 2002; Li et al. 2014; Xiao et al. 2017). However, it is very difficult to directly incorporate the effects of spatial variability into the design process using semi-probabilistic RBD codes and/or methodologies. The problem can be tackled with relative ease using the full probabilistic RBD approaches. Under a full probabilistic design framework, the statistics and probability distributions of loads and geotechnical parameters, which are used to evaluate the failure probability P_f of each possible design in design domain, can be changed according to the design scenarios (Wang et al. 2011; Wang and Cao 2013; Phoon et al. 2016). The full probabilistic design approach identifies feasible designs as those with P_f less than the target probability of failure P_T prescribed in the design scenario.

Monte Carlo simulation (MCS) is a robust method to evaluate the failure probability in full probabilistic design. Several MCS-based full probabilistic design approaches have been developed for geotechnical engineering (Wang 2011; Wang et al. 2011; Wang and Cao 2013; Li et al. 2016; Gao et al. 2019). Feasible designs can be obtained for a given design scenario using these approaches. For different design scenarios, feasible designs might be different. Hence, repeated runs of full probabilistic RBD are often required to obtain the respective feasible designs for each design scenario. This requires extensive computational efforts and is not a trivial task obviously. To avoid repeatedly implementing MCS-based reliability evaluations, sample reweighting technique is used to calculate the failure probabilities of possible designs for different design scenarios (Cao et al. 2019). However, it remains a challenging task to use the sample reweighting technique when random field theory is employed to explicitly model the spatial variability of soil parameters because sample reweighting technique is not applicable to high dimension problems.

This paper presents an efficient RBD updating approach for piles in spatially variable soils, and only a single run of direct MCS is required to obtain respective feasible designs and design points of all the design scenarios. The updated feasible designs are obtained using sample reweighting technique, in which only the samples that produced by direct MCS are used repeatedly but without re-calculating the responses of possible designs for different design scenarios, so that the computational efficiency is significantly improved. The spatial variability of soil parameters is modeled using the equivalent variance technique to reduce the dimension of uncertainty parameters. The design point of different design scenarios is updated according to the samples of direct MCS without repeatedly performing simulations for different design scenarios which provides the information of the failure domain and parameter sensitivity (Low 2017). The paper starts with descriptions of the proposed approach, and then followed by an illustration of the proposed approach using a drilled shaft design example.

2. Expanded RBD for a Given Design Scenario

For a given design scenario Y_i, the expanded RBD approach (Wang 2011; Wang et al. 2011) based on direct MCS is adopted to calculate failure probabilities of different possible designs in this study. In the context of expanded RBD, the design parameters d (e.g., the diameter and length of pile) of geotechnical structures are artificially considered as independent discrete random variables $d^{(t)}$, t = 1, 2, …, n_d, with a uniform probability mass function $P(d^{(t)})$. Let $F^{(t)}$ denotes the failure event of $d^{(t)}$. Then, the process of RBD is viewed as a process of evaluating conditional failure probabilities $P(F^{(t)}|d^{(t)}, Y_i)$ of all the possible designs given Y_i and identifying feasible designs. The failure probability, $P(F^{(t)}|d^{(t)}, Y_i)$, of $d^{(t)}$ can be calculated using the Bayes' theorem as follow (Ang and Tang 2007; Wang 2011; Wang et al. 2011):

$$P(F^{(t)} \mid d^{(t)}, Y_i) = \frac{P(d^{(t)} \mid F, Y_i) P(F \mid Y_i)}{P(d^{(t)})} \tag{1}$$

where $P(d^{(t)}|F, Y_i)$ is the conditional probability of $d^{(t)}$ given Y_i and failure; $P(F|Y_i)$ is failure probability given Y_i. Based on the statistical information of uncertainty parameters X (e.g., materials parameters and loads parameters) obtained from the site-specific design scenario Y_i and the possible ranges of design parameters, the random samples of X and d can be generated by MCS to calculate the performance functions of different possible designs. Then, $P(F^{(t)}|d^{(t)}, Y_i)$ can be expressed as (Wang et al. 2011):

$$P(F^{(t)} \mid d^{(t)}, Y_i) \approx \frac{(n_f^{(t)} / n_f)(n_f / n)}{1 / n_d} = \frac{n_f^{(t)}}{n} n_d \tag{2}$$

where n and n_f are the numbers of direct MCS samples and failure samples, respectively; $n_f^{(t)}$ is the number of failure samples of $d^{(t)}$; n_d is the number of possible discrete values of d.

In the process of expanded RBD, the failure samples of all the possible designs will be identified, and their corresponding values of the joint probability density function (PDF) of X are calculated. The design point of each design can be approximately determined as the failure sample with the maximum joint PDF value (Gao et al. 2019). However, the accuracy of results of the design point depends on the number of failure samples. It may require significant computational costs to generate failure samples of possible designs, especially for the feasible designs of interest that have relatively small failure probabilities. For different design scenarios, repeated simulations of the above process are required for each design scenario to obtain respective feasible designs and design points, which is often a difficult task. The next section presents an efficient method to update the feasible designs and design points for different design scenarios based on direct MCS samples that have been generated in the expanded RBD for the given design scenario.

3. RBD Updating for Different Design Scenarios

3.1 RBD updating

As the design scenario varies from Y_i to Y_j, the $P(F^{(t)}|d^{(t)}, Y_j)$ shall be calculated accordingly. In this study, a sample reweighting technique is employed to calculate $P(F^{(t)}|d^{(t)}, Y_j)$ instead of repeating direct MCS for the new design scenario. Let $f(X|Y_i)$ and $f(X|Y_j)$ denote the joint PDF of X given Y_i and Y_j, respectively. $P(F^{(t)}|d^{(t)}, Y_j)$ can be re-written as (Baecher and Christian 2003; Fonseca et al. 2007; Cao et al. 2019):

$$\begin{aligned} P(F^{(t)} \big| d^{(t)}, Y_j) &= \int I(F^{(t)} \mid X, d^{(t)}) f(X \mid Y_j) dX \\ &= \int I(F^{(t)} \mid X, d^{(t)}) \frac{f(X \mid Y_j)}{f(X \mid Y_i)} f(X \mid Y_i) dX \end{aligned} \tag{3}$$

In the context of direct MCS, Eq. (3) can be written as:

$$P(F^{(t)} \big| d^{(t)}, Y_j) \approx \frac{1}{n^{(t)}} \sum_{k=1}^{n^{(t)}} I(F^{(t)} \mid X_{i,k}^{(t)}, d^{(t)}) \frac{f(X_{i,k}^{(t)} \mid Y_j)}{f(X_{i,k}^{(t)} \mid Y_i)} \tag{4}$$

where $X_{i,k}^{(t)}, k = 1, 2, \ldots, n^{(t)}$ are the samples of X generated from $f(X|Y_i)$ during the expanded RBD given Y_i; $I(F^{(t)} \mid X_{i,k}^{(t)}, d^{(t)})$ is an indicator function which is equal to 1 when $F^{(t)}$ occurs and is equal to 0 when $F^{(t)}$ does not occur. The ratio of $f(X|Y_j)$ over $f(X|Y_i)$ are calculated at $X_{i,k}^{(t)}$ to adjust the weighting of the samples $X_{i,k}^{(t)}, k = 1, 2, \ldots, n^{(t)}$. Since $I(F^{(t)} \mid X_{i,k}^{(t)}, d^{(t)})$ in Eq. (4) has been calculated in the expanded RBD for Y_i, only the ratio of $f(X|Y_j)$ over $f(X|Y_i)$ at failure samples of X generated from $f(X|Y_i)$ is required, which avoids generating samples of X from $f(X|Y_j)$ to evaluate the corresponding performance functions and indicator functions. After the failure samples of X generated from $f(X|Y_i)$ are identified from the expanded RBD, $P(F^{(t)}|d^{(t)}, Y_j)$ can be updated without repeating direct MCS for the expanded RBD given the new design scenario, leading to significant computational saving. Then, feasible designs can be updated according to the updated failure probabilities of possible designs.

Note that failure samples of each design generated from $f(X|Y_i)$ by expanded RBD are the key to update $P(F^{(t)}|d^{(t)}, Y_j)$ of the new design scenario. As shown in Eq.(4), it requires that these failure samples shall cover the failure region corresponding to the new design scenario. However, for high dimension problems, it is a challenging task to cover the failure region of each design under the new design scenario with the failure samples generated from $f(X|Y_i)$. Hence, it is difficult to update failure probabilities for the new design scenario using the sample reweighting technique when the spatial variability of soil parameters is modeled using the random field theory, because random field modeling is usually discretized into a number of random variables, leading to a high dimension problem.

3.2 Incorporating spatial variability into RBD updating

To model the spatial variability of soil parameters, the equivalent variance technique (EVT) is adopted in this study. Consider, for example, a drilled shaft in soil layer considering spatial variability of soil parameters Z. Fig. 1 shows the influence zones for evaluating the drilled shaft capacity. In the EVT, spatial variability of Z is modeled by Z_{side} and Z_{tip}, which represent the spatial averages of Z over their respective influence zones (Vanmarcke 1977; Wang and Cao 2013). As a result, the dimension of uncertainty parameters is reduced, making it feasible to update failure probabilities using the sample reweighting technique for the new design scenario when spatial variability of soil parameters is considered. For the sake of conciseness, details of the EVT for modeling spatial variability of soil parameters are not presented herein, and are referred to Wang and Cao (2013).

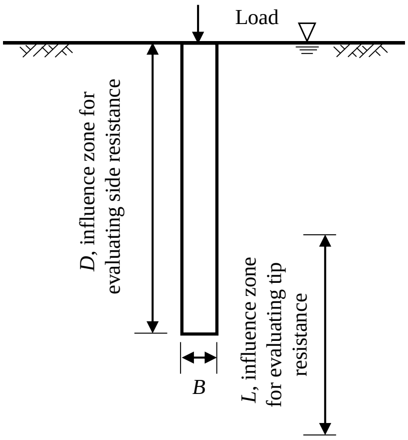


Figure 1. Illustration of influence zones for evaluating the drilled shaft capacity.

3.3 Design point updating

As mentioned in the previous subsection, feasible designs can be updated when the design scenario varies from Y_i to Y_j. The design point corresponding to the target failure probability can also be updated according to the updated design. If failure samples generated by the direct MCS for different designs cover their respective failure regions for the new design scenario, it is trivial to make use of these failure samples to update the design point instead of repeated direct MCS runs for the new design scenario. Consider, for example, a drilled shaft considering spatial variability of soil parameters Z. After the failure samples of Z_{side} and Z_{tip} generated from $f(X|Y_i)$ are identified from expanded RBD, it only requires to re-calculate the joint PDF of Z_{side} and Z_{tip} at these failure samples for the new design scenario. Then, the design point of Z_{side} and Z_{tip} can be updated as the failure sample with the maximum joint PDF value given the new design scenario.

4. Illustrative Example

In this section, the proposed approach is illustrated using a drilled shaft design example considering spatial variability of the effective stress friction angle ϕ'. Wang and Cao (2013) developed an efficient Monte Carlo simulations approach to design this example. As shown in Fig. 1, this drilled shaft example has two design parameters (i.e., the drilled shaft diameter B and depth D), and is installed in a sand layer. The unit weight γ of the sand is 20 kN/m^3. Both at-rest coefficients of horizontal soil stress K_0 of the sand and nominal operative in-situ horizontal stress coefficient ratio $(K/K_0)_n$ are 1.0. The unit weight of concrete γ_{con} and water γ_w are 24 kN/m^3 and 9.81 kN/m^3, respectively. The drilled shaft is designed to support a vertical load with a maximum allowable value F_{50} equal to 800 kN. The allowable displacement y_a in this example is 25 mm, which is adopted from Wang and Cao (2013).

4.1 Expanded RBD for a given fluctuation scale

In this study, the effective stress friction angle ϕ' of the sand is assumed to be a lognormal random variable with a mean $\mu_{\phi'} = 32^\circ$ and coefficient of variation $COV_{\phi'} = 0.17$. As mentioned above, the spatial variability of ϕ' is considered in the expanded RBD using EVT. Let ϕ'_{side} and ϕ'_{tip} represent the spatial averages of ϕ' over influence zone D and L, respectively. The spatial variability of ϕ' surrounding the drilled shaft can be approximately modelled by ϕ'_{side} and ϕ'_{tip}. A single exponential correlation function is adopted to calculate the variance reduction factor for ϕ'_{side} and ϕ'_{tip} in this study. The vertical scale of fluctuation λ is taken as 10 m in expanded RBD.

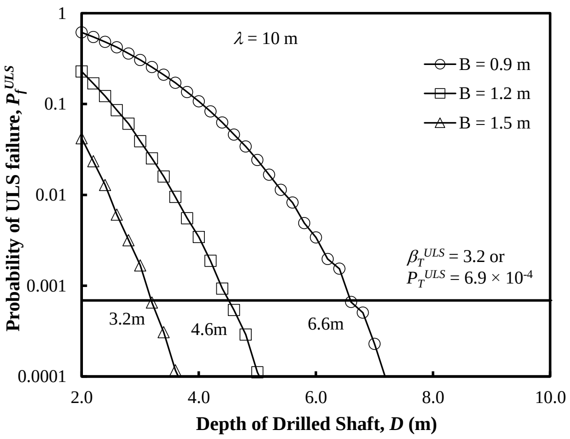
For this example, both ultimate limit state (ULS) and serviceability limit state (SLS) are considered. The factors of safety for ULS and SLS can be calculated as:

$$FS_{ULS} = (Q_{side} + Q_{tip} - W)\big/F_{50} \quad (5)$$

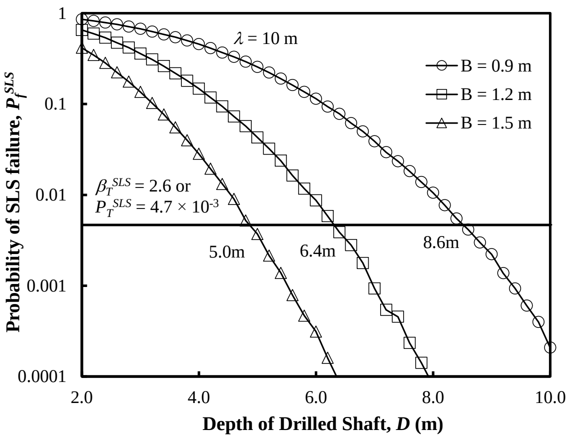
$$FS_{SLS} = 0.625a(y_a/B)^b(Q_{side} + Q_{tip} - W)\big/F_{50} \quad (6)$$

where a and b are curve fitted parameters, and are equal to 4.0 and 0.4, respectively; Q_{side} and Q_{tip} are side resistance and tip resistance, respectively; W is effective shaft weight, and can be calculated as $W = 0.25\pi B^2 D(\gamma_{con} - \gamma_W)$. For the sake of conciseness, calculation of Q_{side} and Q_{tip} are referred to Wang and Cao (2013).

In this study, three possible B values of 0.9 m, 1.2 m, and 1.5 m are considered, and the possible D values vary from 2.0 to 10.0 m with an increment of 0.2 m are considered. Based on the uncertainty model and deterministic model described above, the expanded RBD based on direct MCS is performed to evaluate failure probabilities of all the possible designs. To ensure the accuracy of the failure probabilities, 20,000,000 random samples are generated during the simulation. Fig. 2 shows the variation of the failure probabilities as a function of *D* for *B*=0.9 m, 1.2 m, and 1.5 m by solid lines with circles, squares, and triangles, respectively. For a given value of *B*, the failure probability decreases as *D* increases. The target failure probabilities for ULS and SLS are taken as $P_T^{ULS} = 6.9\times10^{-4}$ and $P_T^{SLS} = 4.7\times10^{-3}$, respectively, which are the same as those taken by Wang and Cao (2013). As shown in Fig. 2, feasible designs can be determined as those below the lines of target failure probabilities. It is shown that the SLS dominates the feasible design in this example. Hence, the minimum design shaft lengths are identified as 8.6 m, 6.4 m, and 5.0 m for *B* = 0.9 m, 1.2 m, and 1.5 m, respectively.



(a) ULS



(b) SLS

Figure 2. Failure probabilities of the drilled shaft design example estimated from the expanded RBD.

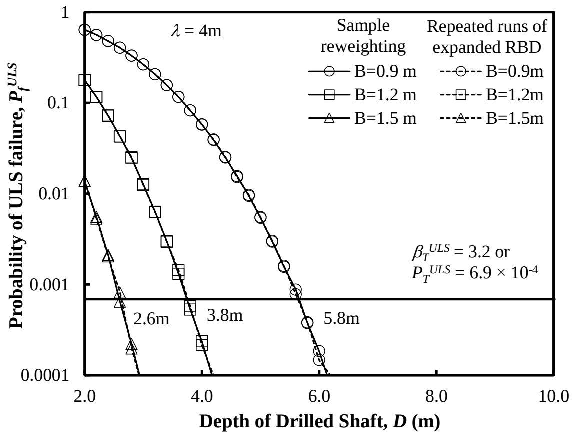
For the three minimum design above, the design points of SLS are determined according to their respective failure samples. The failure samples corresponding to these three designs are identified from the expanded RBD, the number of which are summarized in Table 1. The design points are taken as the failure samples with the maximum joint PDF value as discussed in Section 2.

Table 1. Design points obtained from the expanded RBD with direct MCS.

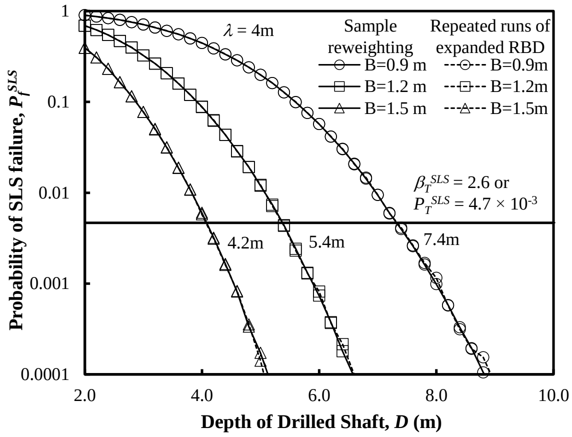
Design scheme (m)	Number of failure samples	Design points	
		ϕ_{side}^*	ϕ_{tip}^*
B=0.9, *D*=8.6	674	22.74	22.58
B=1.2, *D*=6.4	631	22.22	22.63
B=1.5, *D*=5.0	604	22.48	22.42

4.2 Updated designs and design point for different design scenarios with different scale of fluctuation λ

The spatial variability of ϕ' changes as the scale of fluctuation λ changes. As a result, the feasible designs and design points also shall be updated accordingly. In this study, a total of 9 values of scale of fluctuation vary from 1 to 9 m with an increment of 1 m are considered. These scale of fluctuation include the typical range of the vertical scale of fluctuation (e.g., 2m to 6 m) of in-situ test data on soil parameters reported by Phoon and Kulhawy (1999).



(a) ULS



(b) SLS

Figure 3. Failure probabilities of the drilled shaft design example for λ = 4m.

Based on the failure samples generated in the expanded RBD for λ = 10 m, the failure probability of each design is updated using Eq. (4) for different λ values. Fig. 3 shows the updated failure probabilities of possible designs obtained from the proposed approach for λ = 4 m that is adopted by Wang and Cao (2013). For validation, repeated runs of expanded RBD with 20,000,000 random samples are also performed for each scale of fluctuation λ to calculate the failure probabilities of possible designs. Fig. 3 also include the results from repeated runs of the expanded RBD, which are in good agreement with the results obtained from the proposed approach. Similar to design scenario λ = 10 m, the minimum design shaft lengths are identified for each scale of fluctuation. Fig. 4 shows the results obtained from the proposed approach and repeated runs of expanded RBD by solid and open symbols, respectively. These two approaches give the same results, which validates the proposed approach. Compared with repeated expanded RBD for different λ values, the proposed approach only requires to adjust the weighting of failure samples, which leads to a considerable computational saving.

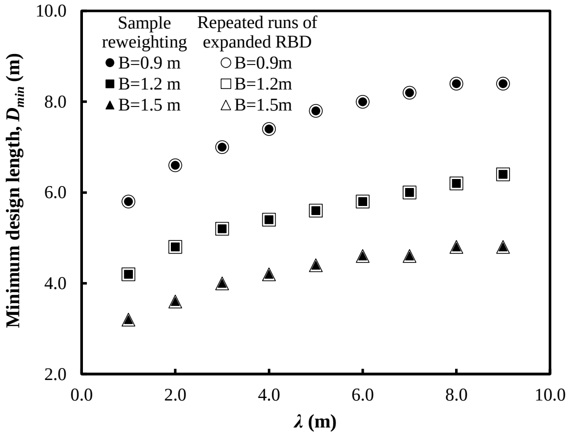
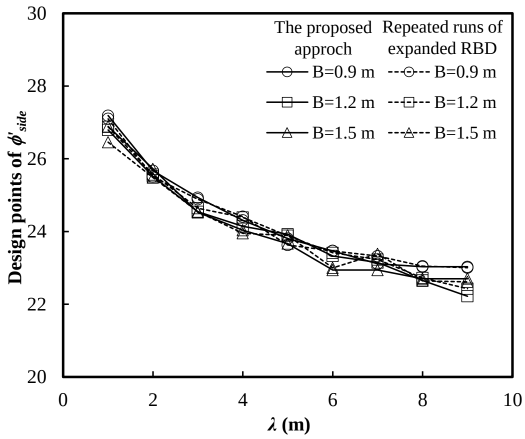


Figure 4. Comparison of minimum design lengths of the drilled shaft for different λ values obtained from the proposed approach and repeated runs of the expanded RBD.

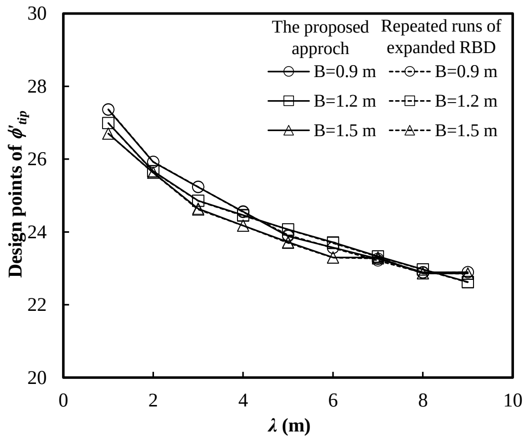
As discussed in subsection 3.3, the design points for different λ values are also updated using the failure samples generated in the expanded RBD given λ = 10 m. Fig. 5 shows the updated design points obtained from the proposed approach by solid lines. As shown in Fig. 5, the value of design point decreases as the vertical scale of fluctuation increases. For validation, repeated expanded RBD runs with direct MCS are also performed for each scale of fluctuation to calculate the design point again. The design points obtained from repeated expanded RBD runs with direct MCS are also included in Fig. 5, which agree well with those obtained from the proposed approach, which further validates the proposed approach.

5. Summary and Conclusions

This paper develops an efficient RBD updating approach for piles considering the spatial variability of soil parameters, which allows efficient updating feasible designs using direct MCS samples as the design scenario changes. A drilled shaft design example is used to illustrate the proposed approach. It was shown the feasible designs for different design scenarios can be obtained in a cost-effective manner, avoiding repeated direct MCS runs. The proposed approach allows incorporating the spatial variability of soil parameters into the efficient RBD updating by the sample reweighting technique and equivalent variance technique. The design point of different designs can also be updated according to the failure samples identified from the single run of direct MCS. This may be useful for calibrating load and resistance factors (or partial factors) for semi-probabilistic design, which will be further pursued in future study.



(a) Design point of ϕ'_{side}



(b) Design point of ϕ'_{tip}

Figure 5. Comparison of design points obtained from the proposed approach and repeated runs of the expanded RBD.

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