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Time-Variant Structural Reliability Analysis - A Computational Perspective

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Abstract: Life-cycle oriented structural design required the consideration of time effects regarding the safety and reliability. In order to account for time effects properly in the design optimization process, computationally efficient methods must be devised. The paper establishes the framework for time-variant reliability analysis from a conceptual perspective. It then discusses several mainly Monte-Carlo based computational reliability methods in detail. The application of these concepts and methods is demonstrated using several examples involving dynamic loading.

Keywords: Time-variant reliability, first passage problem, importance sampling, asymptotic sampling.

1. Introduction

The design of structures and infrastructural systems increasingly requires the consideration of life-cycle aspects. From a safety perspective, this implies that reliability analysis must take into account all possible time-dependent factors affecting the performance of these systems. This leads to so-called time-variant reliability analysis. Moreover, life-cycle considerations play an important role in the optimal design of structures and their maintenance planning (Frangopol, Kallen, and Noortwijk 2004; Macke and Higuchi 2007).

In this context it is essential to take into account the time-dependence of the statistical uncertainty of loads and resistances in the expected life-time of a structure. This is particularly important for overall life-cycle cost minimization including the impact of maintenance planning and repair strategies. A very comprehensive survey on life-cycle oriented structural optimization utilizing a probabilistic basis is given by (Frangopol and Maute 2003; Frangopol, Kallen, and Noortwijk 2004). Further studies involving optimization under random actions and maintenance include (Higuchi and Bucher 2004; Bucher and Frangopol 2007; Bucher 2009c). At a fairly theoretical level, the optimal design of structures under consideration of time-variant reliability constraints is in the focus of the work by (Kuschel and Rackwitz 2000). This is followed up on in (Streicher and Rackwitz 2004).

In the above-mentioned approaches, it is necessary to have access to statistical information on the state of the structure, mainly regarding deterioration due to corrosive effects or damage due to possibly large loads. For this purpose, it is useful to introduce the results of structural monitoring on a probabilistic basis. Some discussion of the effect of monitoring on time-variant reliability estimates can be found in (Bucher 2010; Catbas, Gokce, and Frangopol 2013).

When designing structures for a desired or required level of safety it is one of the most challenging tasks to compute small probabilities related to potential structural failures with both reasonable accuracy and efficacy. As usually the level of safety is not reached within one design cycle but rather through an iterative process, the efficacy requirement becomes especially important. In order to be able to apply methods from probability theory and statistics, it is required to express the randomness of all parameters $X_i, i = 1 \ldots n$ in terms of probability density or probability distribution functions. The simplest possible statistical characterization in terms of mean value as and standard deviations is usually not sufficient. Hence we assume that for each random variable X_i we know the probability density function $f_{X_i}(x_i)$, and that we also know the joint probability density function of all random variables $f_{X_1,\ldots X_n}(x_1, \ldots x_n)$.

The safety analysis (or its complementary, the failure analysis) tries to identify the probability that rare combinations of the basic variables lead to structural failure. If properly designed, this failure probability should be rather small (say of the order 10^{-4} per year). The actual value would have to depend on the consequences of failure, thus introducing a cost element into the design. On this basis, the target safety level can be based on minimal expected cost during the structural life time including initial cost, cost of maintenance, and cost of failure if it occurs. Typically, large failure costs lead to higher target safety levels (or smaller failure probabilities). Since the expected cost of failure can be computed as the product of failure probability and cost of failure, again there is a need to be able to compute small probabilities efficiently. A general review of structural reliability analysis methods is given in (Rackwitz 2001).

Specific aspects regarding the material and type of structure as well as possible deterioration mechanisms play an important role in the choice of appropriate methods. As an example, the consequences of time-dependent randomness on the service life of concrete bridges are discussed in (Enright and Frangopol 1998) whereas time-variant reliability profiles for steel bridges are discussed in (Czarnecki and Nowak 2008). Corrosion in a marine or otherwise aggressive environment and its effect on reliability is considered together with repair actions in (Soares and Garbatov 1999). An application of time-variant reliability analysis to cooling towers in conjunction with the finite element method is shown in (Sudret, Defaux, and Pendola 2005).

The focus of this paper is on conceptual and computational details of various simulation methods for the estimation of small probabilities from reasonably small sample sets, for both time-invariant and time-variant problem classes. Some of the findings as reported

here have been published previously by the author (Bucher, 2012).The major advantage of such Monte-Carlo based simulation techniques is that there are virtually no restrictions on the area of applicability. Essentially, these sampling techniques consist of simple repetition of the deterministic analysis with suitably modified parameters. Unfortunately, this enormous flexibility comes with an increase in computational effort. Specialized simulation techniques allow control over the effort required thus rendering advanced Monte-Carlo methods quite competitive.
Clearly, there are cases in which immediate application of Monte Carlo methods are not actually feasible. This will most likely occur whenever the structural model under consideration requires substantial computational resources for one single run. In such cases, response surface methodology may provide a convenient way for obtaining reasonable accurate answers with acceptable computational effort (Bucher and Macke 2005). The application and necessary improvements for the response surface method in the context of time-variant reliability analysis is discussed e.g. in (Gupta and Manohar 2004).

2. Time-variant reliability

2.1 Concept and definitions

When considering the reliability of structures or infrastructure systems, the performance over the life-time is significantly affected by deterioration due to corrosion or repeated overloading. Naturally, the magnitude of the deterioration depends on the time passed since the structure was built, and consequently the probability of failure will increase with increasing time. Methods for time-variant reliability attempt to address this problem in a computationally efficient manner. For the ease of presentations, in this section it will be assumed that the reliability problem can be simplified to

$$g(R,S) = R - S$$

in which R denotes the structural resistance, and S is the load effect. Both variables may either be constant or change over time. Cases with S = const. are fairly simply treatable and are therefore not considered in the following.
The remaining time-variant reliability problems may roughly be categorized as follows:

- The loading is applied repeatedly in time with randomly changing intensity $S(t)$, the resistance R is constant over time. Here the main problem lies in the determination of the possible correlation between different load applications.
- The loading is applied repeatedly in time with randomly changing intensity $S(t)$, the resistance changes over time due to load-independent deterioration such as e.g. corrosion.
- The loading is applied repeatedly in time with randomly changing intensity $S(t)$, the resistance changes over time due to load-dependent deterioration such as e.g. low/high cycle fatigue.

These cases are schematically sketched in Fig. 1. All the cases can be characterized as *first-passage problems*, i.e. failure occurs whenever the current load effect exceeds the current resistance for the first time.

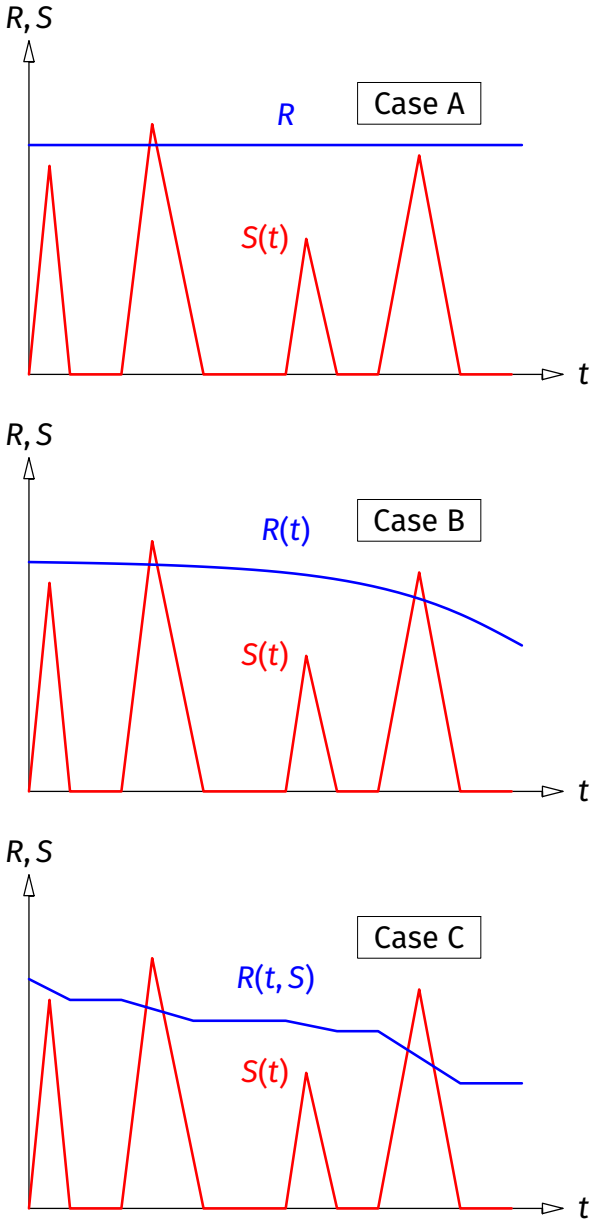

Figure 1. Classification of time-variant reliability problems

The cases as described here are further complicated if inspection and repair strategies are taken into account. Repair should basically increase the structural resistance whenever inspection reveals possible damage, either due to corrosion or excessive loading. This is not considered in the following. The complexity of Case A is mainly determined by the correlation structure of $S(t)$. If the loading process consists of a sequence of independent pulses (such as a Poisson process) then the first-passage problem can be reduced to a sequence of independent failure events whose individual probabilities may possibly be computed even in analytical form. If the load

application does not lead to immediate load effects, but is delayed and accumulated over a long time period (such as due to inertia and damping in the case of dynamic response to random excitation), then the first passage problem needs to be formulated in terms of a high-dimensional reliability problem. For Case B, the situation is fairly similar, the essential difference being the changing probability of individual failure events. Still, there is the possibility of achieving accurate results with rather small efforts. In Case C, however, the resistance and the load are physically coupled, and therefore statistically dependent. It is usually not possible to remove that dependence without crude oversimplification of the problem. Therefore, in this case a high-dimensional reliability problem arises in which all load applications had to be considered simultaneously in one high-dimensional random vector.

Common to all types as mentioned above is that the reliability problem can be expressed as a *first-passage* problem. This means that the time-dependent first passage probability

$$P_E(t) = \mathbf{Prob}\left[\inf_{\tau\in[0,t]}\{R(\tau) - S(\tau)\} \le 0\right] \qquad (1)$$

needs to be computed for all values of t between 0 and the expected service life T.

2.2. Crossing rate approach

The computation of the first passage probability as defined in Eq. 1 can be substantially simplified if the so-called expected up-crossing rate (first-passage rate) approach is applied. Simply speaking, this implies that passage events are assumed to occur independently such that the up-crossing process can be described as Poisson process. If we define the random process $Z(t) = R(t) - S(t)$ (safety margin), then the first passage probability can be written as

$$P_E(t) = \mathbf{Prob}\left[\inf_{\tau\in[0,t]} Z(\tau) \le 0\right] \qquad (2)$$

For the special case that $Z(t)$ is a stationary random process, the expected number of zero crossing of the safety margin per unit time (i.e. the zero-crossing rate) can be computed from the power spectral density function $S_{ZZ}(\omega)$ of the safety margin (Rice 1944):

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{\int_{-\infty}^{\infty}\omega^2 S_{ZZ}(\omega)\mathrm{d}\omega}{\int_{-\infty}^{\infty} S_{ZZ}(\omega)\mathrm{d}\omega}} \qquad (3)$$

For non-stationary processes as given in cases B and C above, this concept needs to be adapted. If the joint probability density function $f_{Z\dot{Z}}(z,\dot{z})$ of the safety margin Z and its time derivative $\dot{Z}$ is known, then the expected zero crossing rate of Z can alternatively be computed from (Rice 1944; Lin 1976):

$$\nu_0 = \int_0^{\infty} \dot{z}\, f_{Z\dot{Z}}(0,\dot{z})\mathrm{d}\dot{z} \qquad (4)$$

Unfortunately, the required joint density is not available readily for relevant practical situations. A reasonable simplification can be based on the assumption that Z and $\dot{Z}$ are independent, which is true for stationary Gaussian processes. In this case, the expressions of the last two equations yield identical results. For non-Gaussian situations, the probability densities need to be computed by numerical means. This can actually be achieved by re-formulating the problem in terms of a time-independent reliability analysis.

Based on the Poisson assumption and with the expected uncrossing rate, the first-passage probability may be approximated by

$$P_E(t) = \int_0^t 1 - \exp(-\nu_0\tau)\mathrm{d}\tau \qquad (4)$$

This excludes the case of instantaneous initial failure which is not related to time effects.

An excellent overview of the expected uncrossing rate approach is given by (Beck and Melchers 2004). The same authors also discuss the situation of the so-called Barrier failure dominance (BFD) which characterizes those problems where an out-crossing or overload failure is more likely to be caused by a very small realization of the barrier (resistance) than by an exceptionally large realization of the load process" (Beck and Melchers 2005). This means that in many cases the resistance variation may be significantly more important than the load variation.

From a computational perspective, it may be advantageous to treat the zero-crossing rates as mentioned above as functions of the particular value of the resistance R so that the up-crossing rates and the probabilities may be interpreted as conditional on R. Unconditional probabilities can subsequently be obtained by integrating over the probability density of R:

$$P_E(t) = \int_R P_E(t|r) f_R(r)\mathrm{d}r \qquad (5)$$

3. Simulation Methods

3.1. Definitions

Generally, failure (i.e. an undesired or unsafe state of the structure) is defined in terms of a limit state function $g(.)$, i.e. by the set $\mathcal{F} = \{\mathbf{X}: g(\mathbf{X}) \le 0\}$. Frequently, $Z = g(\mathbf{X})$ is called *safety margin*. The vector $\mathbf{X}$ contains the basic variables, i.e. all random variables relevant for the problem at hand. It is assumed that the joint probability density function $f_{\mathbf{X}}(\mathbf{x})$ is known.

From the simple problem as shown in Fig. 2 it can already be seen that the definition of the limit state function is not unique, i.e. there are several ways of expressing the failure condition

$$\begin{aligned}\mathcal{F} &= \{(F, L, M_{pl}): FL \ge M_{pl}\} \\ &= \left\{(F, L, M_{pl}): 1 - \frac{FL}{M_{pl}} \le 0\right\}\end{aligned} \qquad (6)$$

Figure 2. Failure condition for a cantilever.

The failure probability is defined as the probability of the occurrence of $\mathcal{F}$:

$$p_f = \mathbf{Prob}[\{\mathbf{X}: g(\mathbf{X}) \leq 0\}] \tag{7}$$

This quantity is *unique*, i.e. not depending on the particular choice of the limit state function. The failure probability can be written in the form of a multi-dimensional integral:

$$p_f = \mathbf{Prob}[g(X_1, \dots X_n) \leq 0] = \int_{g(\mathbf{X})\leq 0} f_{\mathbf{X}}(\mathbf{x})\mathrm{d}\mathbf{x} \tag{8}$$

The computational challenge in determining the integral of Eq. 8 lies in evaluating the limit state function $g(\mathbf{x})$, which for non-linear systems usually requires an incremental/iterative numerical approach. In this context, it is essential to realize that the limit state function $g(\mathbf{x})$ serves the sole purpose of defining the bounds of integration in Eq. (8). As an example, consider a 2-dimensional problem with standard normal random variables X_1 and X_2, and a limit state function $g(x_1, x_2) = 3 - x_1 - x_2$. In Figure 3 the integrand of Eq. 8 in the failure domain is displayed. It is clearly visible that only a very narrow region around the so-called design point $\mathbf{x}^*$ really contributes to the value of the integral, i.e., the probability of failure $P(\mathcal{F})$. This makes is difficult to locate integration points for numerical integration procedures appropriately.

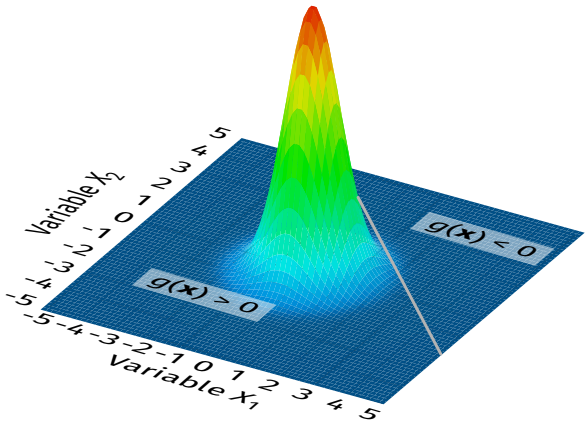

Figure 3. Integrand for calculating the probability of failure for $g(x_1, x_2) = 3 - x_1 - x_2$.

3.2. Monte Carlo simulation

The definition of the failure probability as given in Eq. 8 can be written as an expected value

$$p_f = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} I_g(x_1 \dots x_n) \cdot f_{X_1 \dots X_n}(x_1 \dots x_n) dx_1 \dots dx_n \tag{9}$$

in which $I_g(x_1 \dots x_n) = 1$ if $g(x_1 \dots x_n) \leq 0$ and $I_g(.) = 0$ else.
In order to determine p_f in principle all available statistical methods for estimation of expected values are applicable. If m independent samples $\mathbf{x}^{(k)}$ of the random vector $\mathbf{X}$ are available then the estimator

$$\bar{p}_f = \frac{1}{m}\sum_{k=1}^{m} I_g\left(\mathbf{x}^{(k)}\right) \tag{10}$$

yields a consistent and unbiased estimate for p_f.
The problem associated with this approach is this: For small values of p_f and small values of m the confidence of the estimate is very low. The variance $\sigma^2_{\bar{p}_f}$ of the estimate $\bar{p}_f$ can be determined from

$$\sigma^2_{\bar{p}_f} = \frac{p_f}{m} - \frac{p_f^2}{m} \approx \frac{p_f}{m} \rightarrow \sigma_{\bar{p}_f} = \sqrt{\frac{p_f}{m}} \tag{11}$$

It is to be noted that the required number m of simulations is independent of the dimension n of the problem! Based on the failure probability, a generalized reliability index β_G can be defined as

$$\beta_G = -\Phi^{-1}(p_f) \tag{12}$$

3.3. Importance Sampling (Weighted Simulation)

In order to reduce the standard deviation $\sigma_{\bar{p}_f}$ of the estimator to the order of magnitude of the probability of failure itself m must be in the range of $m = \frac{1}{p_f}$. For values of p_f in the range of 10^{-6} this cannot be achieved if each evaluation of the limit state function requires a complex structural analysis. Alternatively, strategies are employed which increase the "hit-rate" by artificially producing more samples in the failure domain than should occur according to the distribution functions. One way to approach this solution is the introduction of a positive weighting function $h_{\mathbf{Y}}(\mathbf{x})$ which can be interpreted as density function of a random vector $\mathbf{Y}$. Samples are taken according to $h_{\mathbf{Y}}(\mathbf{x})$.
The probability of failure is then estimated from

$$\bar{p_f} = \frac{1}{m}\sum_{k=1}^{m} \frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{Y}}(\mathbf{x})} I_g(\mathbf{x}) = \mathbf{E}\left[\frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{Y}}(\mathbf{x})} I_g(\mathbf{x})\right] \tag{13}$$

From the estimation procedure as outlined it can be seen that the variance of the estimator $\bar{p}_f$ becomes

$$\sigma^2_{\bar{p}_f} = \frac{1}{m}\mathbf{E}\left[\frac{f_{\mathbf{X}}(\mathbf{x})^2}{h_{\mathbf{Y}}(\mathbf{x})^2} I_g(\mathbf{x})\right] \tag{14}$$

A useful choice of $h_{\mathbf{Y}}(\mathbf{x})$ should be based on minimizing $\sigma^2_{\bar{p}_f}$. Ideally, the weighting function should reduce the sampling error to zero. However, this cannot be achieved in reality since such a function must have the property

$$h_{\mathbf{Y}}(\mathbf{x}) = \begin{cases} \frac{1}{p_f} f_{\mathbf{X}}(\mathbf{x}) & g(\mathbf{x}) \le 0 \\ 0 & g(\mathbf{x}) > 0 \end{cases} \tag{15}$$

This property requires the knowledge of p_f which, of course, is unknown. Special updating procedures such as adaptive sampling (Bucher 1988) can help to alleviate this problem.

For simplicity, consider a one-dimensional problem. Let X be normally distributed with

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{16}$$

Assume that the limit state function is given by $g(x) = \beta - x$. We will try to find an optimal sampling density function in the form

$$h_Y(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\bar{Y})^2}{2}\right) \tag{17}$$

In this form, $\bar{Y}$ will be chosen to minimize the variance of the estimated failure probability.

This variance can be calculated directly by evaluating the expectations given above:

$$\begin{aligned} \sigma^2_{\bar{p}_f} &= \frac{1}{m} \int_\beta^\infty \frac{f_X(x)^2}{h_Y(x)^2} h_Y(x) \mathrm{d}x \\ &= \frac{1}{m} \int_\beta^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{2x^2}{2} + \frac{(x-\bar{Y})^2}{2}\right) \mathrm{d}x \\ &= \frac{1}{m} \exp(\bar{Y}^2) \Phi[-(\beta + \bar{Y})] \end{aligned} \tag{18}$$

Differentiation with respect to $\bar{Y}$ yields

$$\begin{aligned} &\frac{\partial}{\partial \bar{Y}} \left(\sigma^2_{\bar{p}_f}\right) = 0 \\ &\rightarrow 2\bar{Y}\Phi[-(\beta+\bar{Y})] - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\beta+\bar{Y})^2}{2}\right) = 0 \end{aligned} \tag{19}$$

Using the following asymptotic (as $z \to \infty$) approximation for $\Phi(.)$ (Mill's ratio, see e.g. (Abramowitz and Stegun 1970))

$$\Phi(-z) \approx \frac{1}{z\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \tag{20}$$

an asymptotic solution to the minimization problem is given by

$$\frac{2\bar{Y}}{\beta + \bar{Y}} - 1 = 0 \rightarrow \bar{Y} = \beta \tag{21}$$

This means that centering the weighting function at the design point will yield the smallest variance for the estimated failure probability. For a value of $\beta = 3.0$ the variance is reduced by a factor of 164 as compared to plain Monte Carlo simulation which means that the computational effort to obtain a certain level of confidence is reduced substantially.

3.4. Importance Sampling at the Design Point

Based on the previous FORM analysis it may be attempted to obtain a general importance sampling concept. This can be accomplished in two steps:

1. Determine the design point $\mathbf{x}^*$ as shown in the context of the FORM-procedure.
2. Choose a weighting function (sampling density) $h_{\mathbf{Y}}(\mathbf{x})$ with the statistical moments $E[\mathbf{Y}] = \mathbf{x}^*$ and $\mathbf{C_{YY}} = \mathbf{C_{XX}}$ in the following form (multi-dimensional Gaussian distribution, cf. Fig. 4):

$$\begin{aligned} h_{\mathbf{Y}}(\mathbf{x}) &= \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det \mathbf{C_{XX}}}} \\ &\cdot \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{x}^*)^T \mathbf{C_{XX}^{-1}} (\mathbf{x}-\mathbf{x}^*)\right] \end{aligned} \tag{22}$$

3. Perform random sampling and statistical estimation according to Eq. 13.

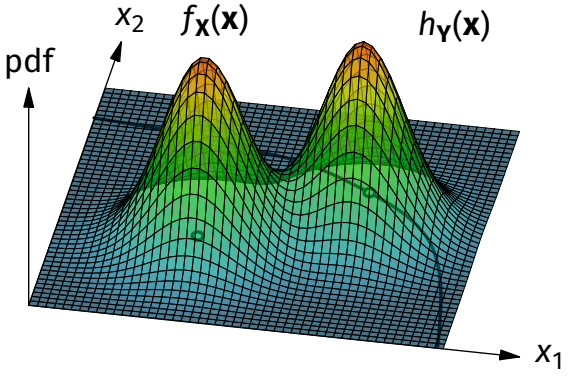

Figure 4. Original and importance sampling probability density functions.

The efficiency of this concept depends on the geometrical shape of the limit state function. In particular, limit state functions with high curvatures or almost circular shapes cannot be covered very well.

It is also interesting to note that the concept of importance sampling can very well be extended for application in the context of dynamic problems (first passage failure) as shown below.

Example: Linear limit state function

Consider a reliability problem described by two independent standardized Gaussian random variables X_1 and X_2. Let failure be defined by the limit state function

$$g(X_1, X_2) = 3 - X_1 - X_2 \tag{23}$$

The design point for this problem is located at $\mathbf{x}^* = [1.5, 1.5]^T$. A crude Monte Carlo run results in $p_f = 0.015$, an importance sampling run with 200 samples results in $p_f = 0.0175$.

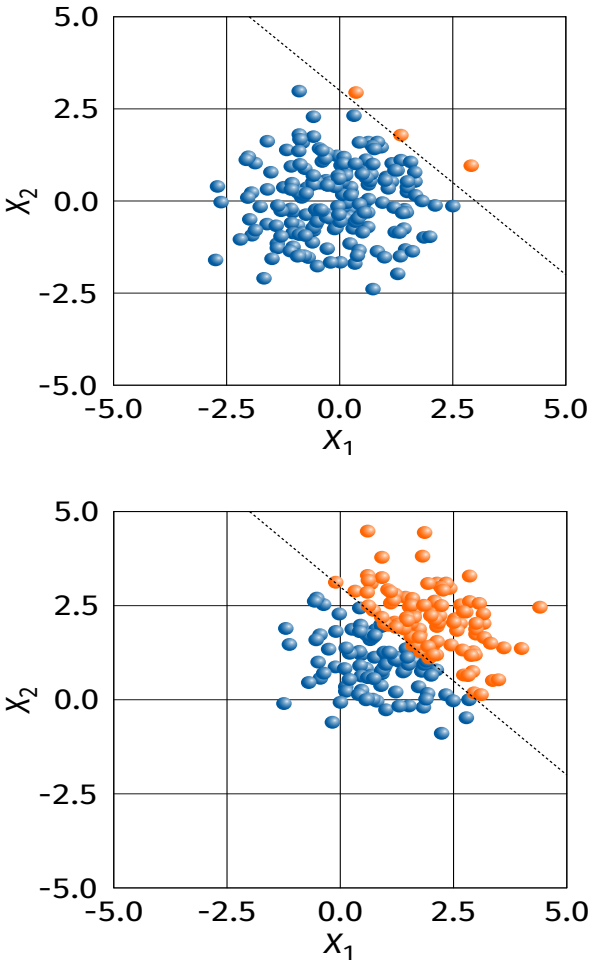

Figure 5: Plain Monte Carlo (top) and Importance Sampling (bottom).

The corresponding 200 sample points are shown in Fig. 5 for crude Monte Carlo as well as for importance sampling. The exact result for this problem is $p_f = \Phi(-1.5\sqrt{2}) = 0.01695$

3.5. Asymptotic Sampling

3.5.1 General concept

The following approach has been presented in (Bucher 2009a, 2009b). It relies on the asymptotic behavior of the failure probability in n-dimensional i.i.d Gaussian space as the reliability index β tends to infinity (see (Breitung 1984)). This can equivalently be expressed by the limit that the standard deviation σ of the variables and hence the failure probability P_F approaches zero. Breitung also states that the *generalized reliability index* β_G as defined in Eq. 12 is asymptotically equal to the linearized reliability index β as both tend to infinity.
Consider a (possibly highly nonlinear) limit state function $g(\mathbf{X})$ in which $g < 0$ denotes failure. Let σ be the standard deviation of the i.i.d. Gaussian variables $X_k, k = 1 \ldots n$. We are going to determine the functional dependence of the generalized safety index β on the standard deviation σ by using an appropriate sampling technique. This is aided by some analytical considerations in which we study the case of a linear limit state function. Let

$$g(\mathbf{X}) = -\sum_{i=1}^{n} \frac{u_i}{s_i} + 1 \tag{24}$$

as used previously in the section on FORM. If $\mathbf{X}$ is a vector of standardized Gaussian variables, then we have

$$\frac{1}{\beta^2} = \sum_{i=1}^{n} \frac{1}{s_i^2} \tag{25}$$

If we now change the standard deviation from unity to a value of $1/f$ for all variables x_i and keep the limit state function as it was, then we can relate this to standard variables $U_i = fX_i$ such that

$$g(\mathbf{U}) = -\sum_{i=1}^{n} \frac{U_i}{f s_i} + 1 \tag{26}$$

and the reliability index is given by

$$\frac{1}{\beta^2} = \sum_{i=1}^{n} \frac{1}{f^2 s_i^2} \tag{27}$$

The geometric relations are shown in Fig. 7.

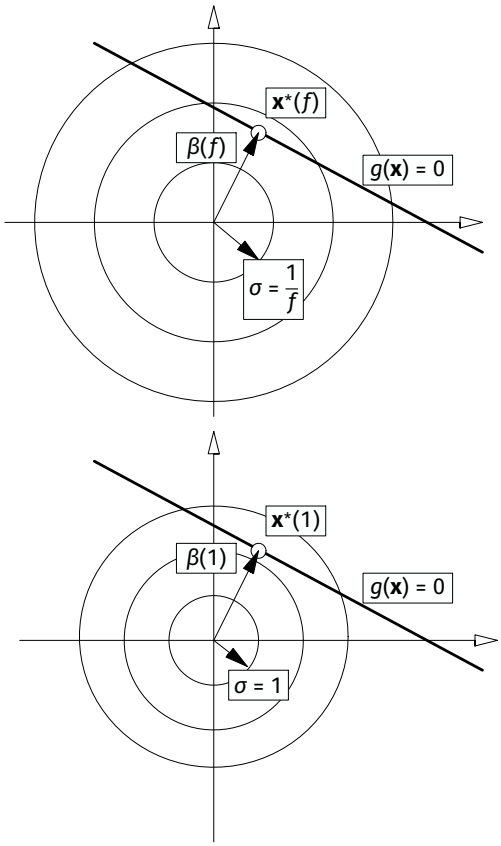

Figure 6. Linear limit state under scaling of standard deviation.

Upon comparing this result to the case of unit standard deviations we observe

$$\beta(f) = f \cdot \beta(1) \tag{28}$$

in which $\beta(1)$ is the safety index evaluated for $f = 1$. This means that in order to obtain a good estimate for $\beta(1)$, we can compute the safety index for a smaller value of the scale f using Monte Carlo simulation and then simply extrapolate by multiplying the obtained result with f. Further details are given in (Bucher 2009a, 2009b). While this result is exact for linear limit state functions, it holds only asypmtotically as $\beta \to \infty$ for general nonlinear cases as shown by (Breitung 1984). Therefore, the concept of asymptotic sampling utilizes the asymptotic behavior of the safety index β by applying an extrapolation technique. Here the (assumed) functional dependence for β is chosen as

$$\beta = A \cdot f + \frac{B}{f} \tag{29}$$

This choice is motivated in order to ensure asymptotically linear behavior as $f \to \infty$ (which is equivalent to $\sigma \to 0$). As the coefficients A and B are conveniently determined from a least-squares fit using Monte Carlo estimates of β for different values of f (typically for values of $f < 1$) as support points. For this fitting process, Eq. 29 is rewritten in terms of a scaled safety index as

$$\frac{\beta}{f} = A + \frac{B}{f^2} \tag{30}$$

This is illustrated qualitatively in Fig. 7.

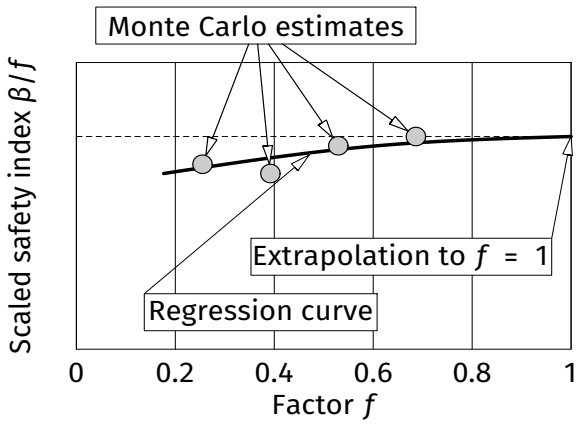

Figure 7. Basic concept of Asymptotic Sampling.

One major advantage of this approach is its independence of the dimensionality. The accuracy is governed only by the relation between the number of samples and the probability of failure as well as the particular geometry of the limit state surface $g(\mathbf{U}) = 0$.

In this context it is essential to use a sampling method which provides very stable results. One obvious choice is Latin Hypercube Sampling (LHS) (Imam and Conover 1982; Florian 1992). Alternatively, pseudo-random sequences with low discrepancy (Niederreiter 1992; Sobol and Asotsky 2003) can be utilized. In particular, randomized Sobol sequences using an algorithm as discussed in (Bratley and Fox 1988) and (Hong and Hickernell 2003) are very useful (Bucher 2009a).

It also should be noted that a conceptually similar approach on the basis of expected upcrossing rates is applied in (Naess and Gaidai 2008; Naess, Gaidai, and Batsevych 2010).

As an example, consider a high-dimensional linear problem. This example serves as a test case to demonstrate the independence of the dimensionality. The limit state function is

$$g(\mathbf{X}) = 5\sqrt{n} - \sum_{k=1}^{n} X_k \tag{31}$$

in which n is the number of random variables. All random variables are i.i.d. standard Gaussian. The problem has a safety index of $\beta = 5$ or $P_F = 3 \cdot 10^{-7}$, independent of n. Table 1 shows the mean values and standard deviations of the safety index as computed from asymptotic sampling (20 repetitions with 1000 Monte Carlo samples each) for different dimension n. Hence dimension does not affect accuracy. Further examples are given in (Bucher 2009b).

Table 1. Statistics of estimated safety index for high-dimensional linear problem.

n	$\bar{\beta}$	σ_β
10	4.95	0.26
100	4.94	0.22
1000	4.95	0.24
10000	4.94	0.22
100000	5.00	0.23

4. First passage problem in structural dynamics

4.1 Application of importance sampling

In dynamic structural analysis, it is usually necessary to apply numerical methods based on a time discretization of the loads and responses. For a linear multi-degree-of-freedom system the equations of motion can be written in matrix-vector form as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F}(t) \tag{32}$$

together with appropriate initial conditions for $\mathbf{x}$ and $\dot{\mathbf{x}}$. Here, the vectors $\mathbf{X}$ and $\dot{\mathbf{X}}$ have the dimension n, the symmetric and non-negative matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ have the size $n \times n$, and $\mathbf{F}(t)$ is an n-dimensional vector valued random process. We assume that at least the second order statistics of $\mathbf{F}$ are known. For the sake of notational simplicity, the subsequent derivations will be limited to the case where the vector $\mathbf{F}$ can be represented by one scalar excitation $F(t)$ only. Again, this is an assumption most frequently made in the earthquake analysis of structures

$$\mathbf{F} = \mathbf{P}F(t) \tag{33}$$

The process $F(t)$ has a given auto-covariance function $R_{FF}(t_1, t_2)$

$$R_{FF}(t_1, t_2) = \mathbf{E}[F(t_1)F(t_2)] \tag{34}$$

The discretization of the loading process $F(t)$ in time by means of discrete random variables $F_k; k = 1 \dots n$ at time values t_ℓ typically leads to a large number n of random variables. The response $\mathbf{x}(t)$ is then automatically discretized as well, i.e. in terms of its values $\mathbf{x}_\ell; \ell = 1, m$ at time points t_ℓ. In the following, we assume $n = m = N$.

For quiescent initial conditions, the response of the system (32) at any time t can be calculated from Duhamel's integral

$$\mathbf{x}(t) = \int_0^t \mathbf{h}\,(t-\tau)\mathbf{P}F(\tau)d\tau \tag{35}$$

in which $\mathbf{h}(u)$ is the impulse response function matrix of the system. One particular component x_i of the vector $\mathbf{x}$ can be calculated from

$$x_i(t) = \sum_{k=1}^{n} \int_0^t h_{ik}\,(t-\tau)P_k F(\tau)d\tau \tag{36}$$

From this equation, linear combinations $y(t)$ of the $x_i(t)$ (such as required e.g. for the analysis of internal forces) can be calculated quite easily

$$y(t) = \sum_{i=1}^{M} c_i\, x_i(t) \tag{37}$$

The first passage problem can then be formulated as the probability that the largest of the response variables becomes larger than a predefined threshold level ξ

$$P_E = \mathbf{Prob}\left[\max_{\ell=1,\dots m} x_{i,\ell} > \xi\right] \tag{38}$$

in which it is to be noted that $x_{i,\ell}$ depends on all load variables F_k with a time $t_k \le t_\ell$, but not those with $t_k > t_\ell$ (principle of causality).

Utilizing a time-discrete representation (frequently called lumped-impulse procedure), the excitation $F(t)$ can be represented by a random pulse train (Lin 1976):

$$F(t) = \sum_{j=1}^{N} F\,(\tau_j)\Delta t\delta(t-\tau_j) \tag{39}$$

in which Δt is the time step and $\delta(.)$ denotes Dirac's Delta function. Consequently, the response $x_i(t)$ for $t \in [0, T_E]$ can be written as (cf. Eq. 35)

$$x_i(t) = \sum_{k=1}^{n}\sum_{j=1}^{N} h_{ik}\,(t-\tau_j)P_k F(\tau_j)\Delta t \tag{40}$$

Here, an equidistant spacing with N subdivisions of the interval $[0, T_E]$ is assumed. Obviously, Eq. 40 represents the response x_i as a linear combination of N Gaussian random variables $F(\tau_j); \quad j = 1 \dots N$. Based on this observation, it is clear that the probability of reaching or of exceeding the threshold ξ at time t can be directly and accurately calculated by applying the First-Order-Reliability-Method (FORM, (Hasofer and Lind 1974)). Thus, the limit state condition at time t_r becomes

$$\begin{aligned}\xi \quad &= y(t_r) = \sum_{i=1}^{M} c_i \sum_{k=1}^{N}\sum_{j=1}^{N} h_{ik}\,(t_r-\tau_j)P_k F(\tau_j)\Delta t \\ &= \sum_{j=1}^{N} A_{rj}\, f_j; \quad f_j = F(\tau_j)\end{aligned} \tag{41}$$

The most probable combination of the f_j's leading to the limit state is then readily calculated by applying FORM. From the auto-covariance function as defined in Eq. 34 a discrete covariance matrix for the sequence $f_j; j = 1 \dots N$ can be obtained

$$\mathbf{C_{ff}} = E[\mathbf{f}\mathbf{f}^T] \tag{42}$$

which can be Cholesky-decomposed into

$$\mathbf{C_{ff}} = \mathbf{L}\mathbf{L}^T \tag{43}$$

Here, $\mathbf{L}$ is a lower triangular matrix and $\mathbf{f}$ is a vector containing the sequence $f_j; j = 1 \dots N$. The transformation

$$\mathbf{f} = \mathbf{L}\mathbf{u}; \quad \mathbf{u} = \mathbf{L}^{-1}\mathbf{f} \tag{44}$$

yields a representation of Eq. 41 in terms of uncorrelated standardized Gaussian variables u_s

$$\xi = y(t_r) = \sum_{j=1}^{N}\sum_{s=1}^{N} A_{rj}\, L_{js} u_s = \sum_{s=1}^{N} b_{rs}\, u_s \tag{45}$$

The safety index β_r is easily found from (see e.g. (Madsen, Krenk, and Lind 1986))

$$\frac{1}{\beta_r^2} = \sum_{s=1}^{N}\left(\frac{b_{rs}}{\xi}\right)^2 \tag{46}$$

The design point $\mathbf{u_r}^*$ (the most likely combination of uncorrelated variables leading to failure at time t_r) is then calculated from

$$u_{rs}^* = \frac{b_{rs}}{\xi}\beta_r^2; \quad s = 1 \dots N \tag{47}$$

Due to the linearity of the limit state function, the probability P_{t_r} that the response reaches or exceeds the threshold at time t_r is given by

$$P_{t_r} = \Phi(-\beta_r) \tag{48}$$

It should be emphasized that (Li and DerKiureghian 1995) obtained similar results for the mean outcrossing rate assuming filtered white noise as input. A numerical study performed by (Vijalapura, Conte, and Meghella 2000) gave an analogous result for a nonlinear SDOF-system.

The above results are exploited in order to construct a useful importance sampling scheme. In complete analogy to the importance sampling method for static problems, the above design point excitations can be used as "importance sampling mean" excitations. This mean excitation is simply added to the random excitation process as simulated in the usual way. As there are N possible locations for "design points it becomes necessary to weigh these points appropriately. It is suggested to use the values of $\Phi(-\beta_k)$ as weights (Stix 1983; Macke and Bucher 2003), so that the multi-modal importance sampling density $h_{\mathbf{U}}(\mathbf{u})$ in standard normal

space becomes

$$h_{\mathbf{U}}(\mathbf{u}) = \frac{1}{\sum_{r=1}^{N}\Phi(-\beta_r)}\sum_{r=1}^{N}\frac{\Phi(-\beta_r)}{(2\pi)^{\frac{N}{2}}} \cdot \exp\left(-\frac{1}{2}\sum_{s=1}^{N}(u_{rs}^{*}-u_s)^2\right) \quad (49)$$

The ratio of original density $f_{\mathbf{U}}(\mathbf{u})$ to importance sampling density $h_{\mathbf{U}}(\mathbf{u})$ for a sample function f_k^m as obtained from the uncorrelated variables u_k^m needs to be calculated based on Eq. 49. This implies that there will be considerable interaction between the N design points as well as the N limit state conditions.
The extension to non-linear problems is fairly straightforward. Details can be found in (Macke and Bucher 2003; Bucher 2009b).

4.1.1 Linear SDOF-System under stationary white noise excitation

This example mainly serves the purpose of interpretation of the above-mentioned design point excitation. It is a SDOF-system with a natural frequency $\omega_0 = 1$ and a damping ratio of 5% (Naess and Skaug 2000). A time duration $T_E = 20\pi$ is considered. This duration is divided into $N = 200$ time steps, so that $\Delta t = \pi/10$. Let $f(t)$ be a stationary white noise with a two-sided power spectral density $S_0 = 0.1/\pi$. Its auto-correlation function is given by

$$R_{ff}(t_1, t_2) = 2\pi S_0 \delta(t_1 - t_2) \quad (50)$$

A discrete representation is given in terms of uncorrelated random variables f_j with zero mean and variances $\sigma_{f_j}^2 = 2\pi S_0 \Delta t$. Following the above derivations, the design point excitation is given by

$$r_s = h(t - \tau_s)\sqrt{\frac{\Delta t}{2\pi S_0}}\frac{\beta^2}{\xi} \quad (51)$$

This is a sequence which is basically a time-reversed impulse response function. For $t = 100\Delta t$ this is shown in Fig. 8.

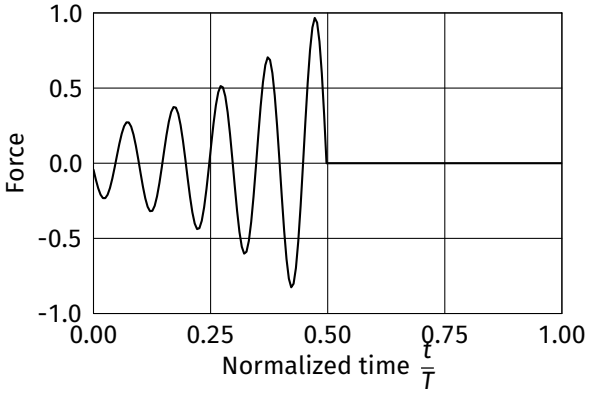

Figure 8. Design point excitation for threshold crossing at time $t = 10\pi$ (white excitation).

The corresponding sample trajectory of $x(t)$ reaches the threshold ξ exactly at time t which is shown in Fig. 9.

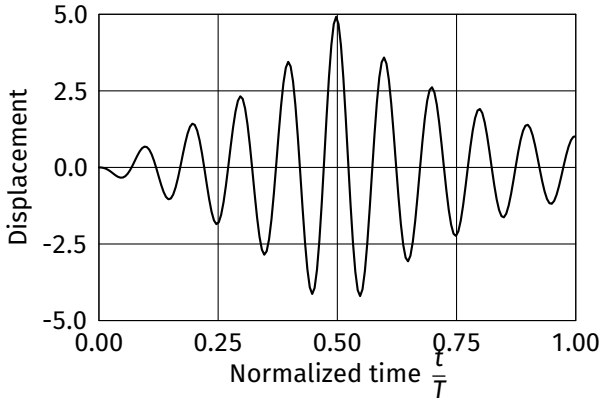

Figure 9. Response to design point excitation for threshold crossing at time $t = 10\pi$ (white excitation).

Upon inspection of Eq. 46 it can easily be seen that in transition to continuous time for white noise excitation we obtain

$$\beta^2 = \frac{\xi^2}{\int_0^t h^2(t-\tau)d\tau} = \frac{\xi^2}{\sigma_x^2(t)} \quad \rightarrow \beta = \frac{\xi}{\sigma_x} \quad (52)$$

The first passage probabilities are evaluated based on the above outlined importance sampling scheme. The threshold level ξ is varied from 0 to $10\sigma_x$. For a sample size of 500 the resulting first passage probabilities are shown in Fig. 10. This figure also shows as a dotted line comparative results from an approximation based on the upcrossing rate (Corotis, Vanmarcke, and Cornell 1972). It can be seen that down to the very low probability level of 10^{-15} the coefficient of variation of the estimated probabilities remains at approximately 10%. The standard estimation error is shown as dashed line in Fig. 10.

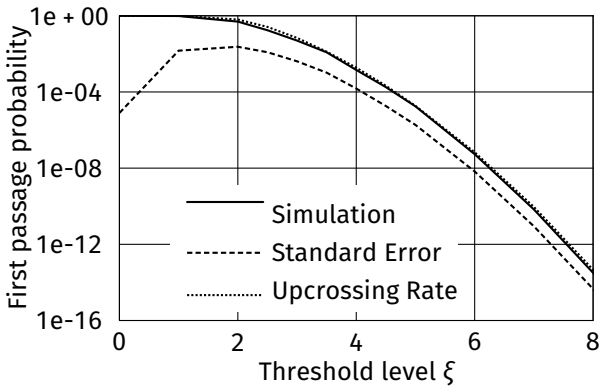

Figure 10. First Passage Probability vs. Threshold Level for White Noise Input (500 Samples).

4.1.2 Linear SDOF-System under non-white excitation

For non-white excitations the picture becomes somewhat more complicated. As an example (Bucher 2009b), consider a stationary excitation with an exponential

autocorrelation function (Orenstein-Uhlenbeck process)

$$R_{ff}(t_1, t_2) = \sigma_f^2 \exp\left(-\frac{|t_1 - t_2|}{\tau_c}\right) \qquad (53)$$

with a correlation time $\tau_c = 2$ and a variance $\sigma_f^2 = 0.02$. This corresponds to a power spectral density $S_{ff}(0) = \pi/0.1$ as in the previous example. In this case, the design point excitation extends into the future as shown in Fig. 11 which is a consequence of the temporal correlation. The corresponding response trajectory is given in Fig. 12. Again, a time interval $\Delta t = \pi/10$ is chosen for time discretization.

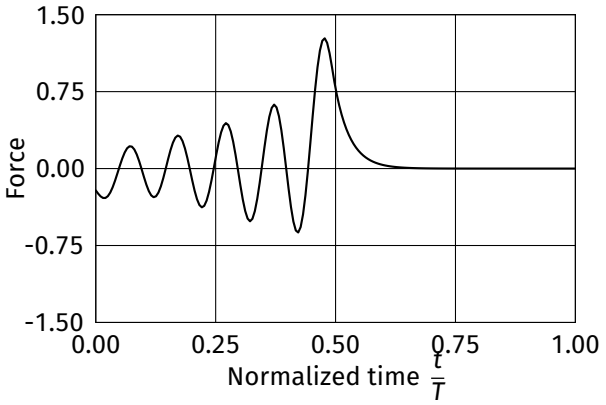

Figure 11. Design point excitation for threshold crossing at time $t = 10\pi$ (non-white excitation).

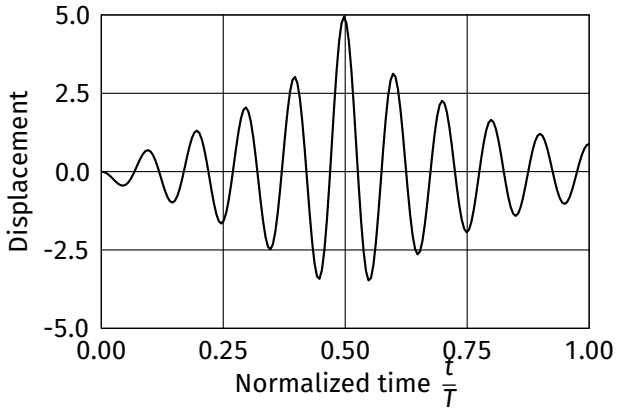

Figure 12. Response to design point excitation for threshold crossing at time $t = 10\pi$ (non-white excitation).

From a sample size of 500 the first passage probabilities as shown in Fig. 13 are obtained. Again the c.o.v. remains in the range of 10%, even for probabilities as low as 10^{-15}.

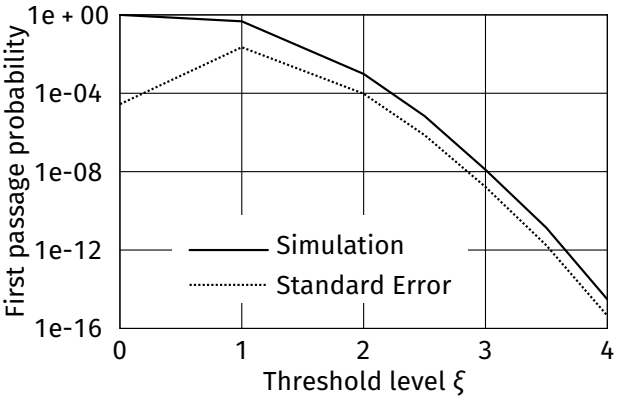

Figure 13. First Passage Probability vs. Threshold Level for White Noise Input (500 Samples).

4.2 Application of asymptotic sampling

4.2.1 Linear SDOF oscillator

Again, the equation of motion is given in the form of

$$m\ddot{x} + c\dot{x} + kx = w(t) \qquad (54)$$

$w(t)$ is a stationary white noise with intensity D_0 which is represented by a sequence of i.i.d. zero mean normal variables F_k spaced uniformly at time interval δt

$$\sigma_F^2 = \frac{D_0}{\Delta t} \qquad (55)$$

Numerical values are k = 1 N/m, m = 1 kg, c = 0.1 kg/s, $D_0 = \frac{\pi}{50}$ m^2/s^3, Δt = 0.15 s. The total time considered is T = 40 s. We carry out asymptotic sampling using 5 runs with 1000 simulations each. The threshold values are varied from 1 to 5.

The resulting probability distribution function of the peak absolute response of SDOF system for different sampling magnifications f (1000 samples each) is shown in Fig. 14. Fig. 15 shows the resulting first passage probabilities for different threshold values.

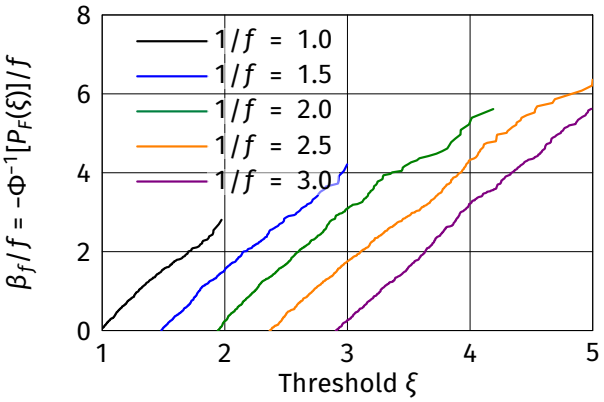

Figure 14. CDF of absolute response vs. threshold level for different scaling factors f (5x1000 Samples).

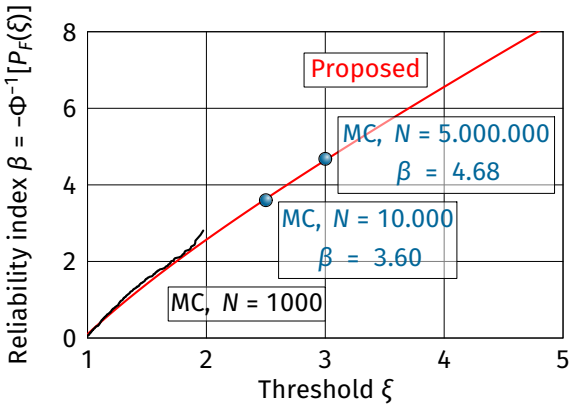

Figure 15. First Passage Probability of SDOF systems under white noise.

4.2.2 Nonlinear SDOF system

Here we consider a Duffing oscillator described by the equation of motion

$$m\ddot{x} + c\dot{x} + k(x + \gamma x^3) = w(t) \qquad (56)$$

The numerical values chosen for this example are: m = 1 kg, c = 0.63 kg/s, k = 5 N/m and γ = 10^4 1/m^2 The excitation $w(t)$ is (discrete) white noise with a total duration T = 15 s. Its spectral density is S_0 = 0.017 m^2/s^3. The time step chosen is Δt = 0.01 s. Hence the excitation process is discretized into 1500 random variables.
The following procedure was carried out:

- Asymptotic Sampling is based on five runs with each 1000 simulations.
- Scaling factor f was varied from f = 1 to f = 1/2 in five steps.
- Failure occurs if displacement at time T exceeds the threshold ξ
- Threshold values from 0.01 to 0.06 m are considered.

Two sample time histories one for f = 1 and one for f = 1/2, are shown in Fig. 16. It can be observed that failure may occur long before the end of the time interval, Also, the frequency characteristics are changed by the larger excitation intensity at f =1/2 (frequency of oscillation increases),
The results from the proposed asymptotic sampling method based on 5 runs with each 1000 samples (cf. Fig. 17, (Valdebenito, Pradlwarter, and Schuëller 2010)) match the full Monte Carlo results very well up to a threshold level of 0.04 m. At the highest threshold level of 0.05 m, however, the accuracy deteriorates.
Results indicate a bias at high threshold levels leading to over-estimation of the exceedance probability. For a threshold value ξ = 0.05, the reference value as obtained from Monte Carlo simulation is $0.7{\cdot}10^{-5}$ (β = 4.3, from ≈ 10.000.000 samples). Asymptotic Sampling (with 5.000 samples) yields a result of $1.3{\cdot}10^{-5}$ (β = 4.2). The suspected reason for the deviation is that the failure domain may not be simply connected (e.g. it contains several ”islands” of failure). This is a situation in which the asymptotic property underlying the method cannot be exploited.

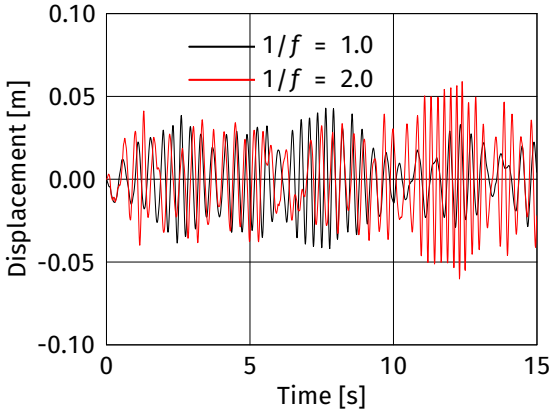

Figure 16. Sample time histories for different scaling factors f.

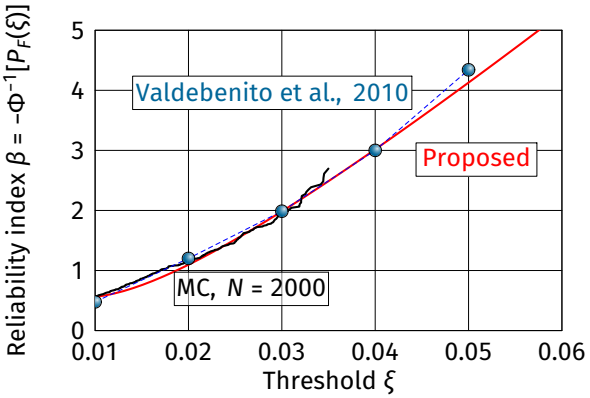

Figure 17. First passage probability for nonlinear oscillator (value at end of time interval).

4.2.3 Change failure criterion - nonlinear SDOF system

If one plans to study the effect of different failure criteria, the asymptic sampling method provides a convenient and computationally very efficient approach to do so. There is no need to re-compute sample functions of the dynamic responses but the samples simply need to be post-processed once more. In this way, we compute the reliability based on maximum absolute value of dynamic response within time interval (rather than value at end of interval). Not surprisingly this leads to substantially smaller reliability (especially at lower threshold levels). The results are shown in Fig. 18.

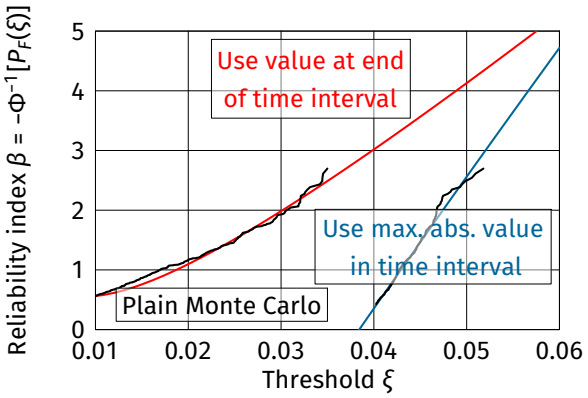

Figure 18. First passage probability for nonlinear oscillator (maximum over time).

4.2.4 Nonlinear MDOF system

Here we consider a Duffing-type oscillator subjected to filtered modulated white noise $a(t)$ governed by the following set of equations:

$$\dot{\mathbf{X}}(t) = \mathbf{A}_{EQ}\mathbf{X}(t) + \mathbf{B}_{EQ}w(t);\ a(t) = \mathbf{C}_{EQ}\mathbf{X}(t) \quad (57)$$

$$\mathbf{A}_{EQ} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\Omega_{1g}^2 & -2\zeta_{1g}\Omega_{1g} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega_{1g}^2 & 2\zeta_{1g}\Omega_{1g} & -\Omega_{2g}^2 & -2\zeta_{2g}\Omega_{2g} \end{bmatrix} \quad (58)$$

$$\mathbf{B}_{EQ} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (59)$$

$$\mathbf{C}_{EQ} = \begin{bmatrix} \Omega_{1g}^2 & 2\zeta_{1g}\Omega_{1g} & -\Omega_{2g}^2 & -2\zeta_{2g}\Omega_{2g} \end{bmatrix} \quad (60)$$

The numerical values for the parameters are Ω_{1g} =15 rad/s, ω_{2g} = 0.3 rad/s, ζ_{1g} = 0.8 and ζ_{2g} = 0.995. Here $w(t)$ is white noise with $E[w(t)w(t+\tau)] = I\delta(\tau)$ · $h(t)$, I = 0.08 m²/s³ and

$$h(t) = \begin{cases} \frac{t}{2} & ,\ 0s \le t \le 2s \\ 1 & ,\ 2s \le t \le 10s \\ \exp\left(-0.1(t-10)\right) & ,\ t \ge 10s \end{cases} \quad (61)$$

The equations of motion of the oscillator are

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\left(\mathbf{u}(t)\right)\mathbf{u}(t) = \mathbf{F}(t) = \mathbf{m}a(t) \quad (62)$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & m_{10} \end{bmatrix};\quad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{10} \end{bmatrix} \quad (63)$$

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & \dots & 0 \\ -c_2 & c_2 + c_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{10} \end{bmatrix} \quad (64)$$

$$\mathbf{K} = \begin{bmatrix} \bar{k}_1 + \bar{k}_2 & -\bar{k}_2 & \dots & 0 \\ -\bar{k}_2 & \bar{k}_2 + \bar{k}_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{k}_{10} \end{bmatrix} \quad (65)$$

$$\bar{k}_i = k_i\left(1 + \epsilon\left(\frac{u_i(t) - u_{i-1}(t)}{\delta u_{ref}}\right)^2\right);\ i = 1, \cdots, 10 \quad (66)$$

The system parameters are given as δu_{ref}=0.02 m, ϵ = 0.1, $m_1 = \cdots = m_{10}$ =10.000 kg, $k_1 = k_2 = k_3$ = 40 MN/m, $k_4 = k_5 = k_6$=36 MN/m, $k_7 = k_8 = k_9 = k_{10}$ = 32 MN/m, $c_i = 2\zeta_i\sqrt{m_i k_i}$ and ζ_i =0.04 for $i = 1, \cdots, 10$. The time duration T = 20 s, time step Δt = 0.005 s → 4000 random variables. As failure criterion we define that the relative displacement between stories 9 and 10 exceeds the predefined threshold value ξ. The range of threshold values considered is 0.01 – 0.05 m. Asymptotic sampling is carried out in 5 runs with 1000 simulations each

The resulting first passage probabilities of nonlinear MDOF system under non-stationary colored noise (as estimated from a total of 5.000 samples) are shown in Fig. 19. The reference Monte Carlo results also shown in this figure results are taken from (Schuëller and Pradlwarter 2007). In comparison, asymptotic sampling reduces computer time by a factor of about 1500. In addition, results from subset simulation taken from (Au, Ching, and Beck 2007) are shown as well. These two point results are based on about 5000 samples. Note that - in contrast to asymptotic sampling - subset simulation cannot provide the entire dependence of the first passage probability at once. Hence, in this respect, asymptotic sampling is substantially more efficient.

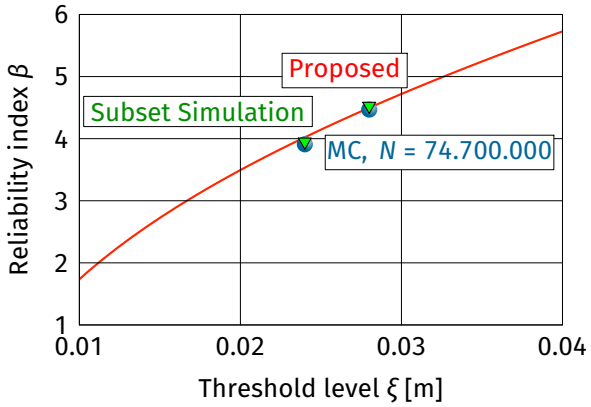

Figure 19. First passage probability for nonlinear MDOF oscillator.

5. Concluding Remarks

In the process of time-variant structural reliability analysis there is substantial need to take into account complex interactions of externally applied loads and structural resistance changes. These changes may be due to deterioration based on external corrosive agents, but also due to load-induced damage processes such as

fatigue. Furthermore, human-induced changes of the structural resistance due to maintenance and repair work have substantial influence on the life-cycle performance of the structures.

All these effects can be subsumed into a probabilistic formulation describing a first-passage problem. However, in order to solve this problem, specific computational methods need to be applied. While classical first-order reliability analysis (FORM) or, somewhat improved second-order analysis (SORM) may provide valuable first results, these approaches are in many cases not quite flexible enough to accommodate complex what-if scenarios, particularly when dealing with maintenance and repair decisions.

It turns out that Monte-Carlo based simulation methods certainly have all the flexibility required, but typically induce substantial computational effort. Efficacy of the simulation methods is therefore the primary property needed. Several highly efficient methods have been presented and the application to selected time-variant reliability problems demonstrated their use.

Future developments will need to focus on unified response surface models (see e.g. Bucher 2018) which include design variables for the life-cycle oriented optimization process simultaneously with the random variables describing the loads and the deterioration processes. Together with reliability updating based on structural health monitoring this will allow for long-term strategic decisions regarding inspection and maintenance as well as for short-term decisions for case-based repair actions.

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