博士学位申請論文

Essays on Modeling, Valuation, and Hedging in Modern Financial Markets

（和訳：現代の金融市場におけるモデリング，価格評価，ヘッジ手法に関する研究）

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金融システム専攻

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序文

現在の金融市場はデリバティブ取引の飛躍的な成長に伴い，株式，為替，金利などのオプションやクレジット・デフォルト・スワップ（credit default swap：以下，CDS）の流動性が拡大すると共に，これら市場間の関連性が以前にも増して高まった．こうしたなか，学術と実務の双方で，伝統的な金融理論が所与とする簡潔で単純化された前提条件を拡張し，実際の金融市場で観測される多様な資産価格の変動や複雑な依存関係を適切に表現できる新しい枠組みが求められている．本論文では，こうした金融市場の実状に沿ったモデリングや価格評価方法，ヘッジ手法を幾つか提案することで，新しい金融理論の一端を開拓することは勿論のこと，金融実務の一助として金融市場の更なる進歩と発展に貢献することを目的としている．

0.1 本論文の構成

本論文は,「価格評価方法」,「モデリング」,「ヘッジ手法」に大別して以下の三部構成とする．

1. 確率分布のキュムラントを利用した価値評価方法
2. 株式とクレジットの金融派生商品の統合評価モデル
3. オプションを用いた新しいヘッジ手法

各部で一定の関連性があるものの，本論文ではそれぞれが独立した構成としている．まず第一部では，確率分布のキュムラントを利用した「価格評価方法」を提案する．この方法はエッジワース展開などの確率分布の漸近展開理論を基礎としており，こうした数理統計学の一般的な手法が，最近の数理ファイナンスで注目されている二次ガウシアン過程（quadratic Gaussian process）や時間変更レヴィ過程（time-changed Lévy process）などの重要かつ複雑な確率過程に適用できることを示す．ファイナンスへの具体的な応用例として，金融工学の分野で 1980 年代から重要な課題と認識されている住宅ローン担保証券（residential mortgage-backed securities：以下，RMBS）の価値評価と時間変更レヴィ過程の下での平均オプション価格に対する近似解析公式の導出を行う．第二部では，条件付き請求権の価格評価のための新しい「モデリング」として，ある参照企業の株価と信用リスクの変動を同一のモデルで統合的に扱うための枠組みを 2 つ提案する．さらには，これら 2 つのモデルを含む株式とクレジットの統合モデルにおける一般的な仮定の下で，株価のインプライド・ボラティリティとリスク中立確率測度の下での倒産確率の関係式を導出する．第三部では，オプションを利用した新しい「ヘッジ手法」を提案する．前半では，原資産価格の変動に対してジャンプや確率的なボラティリティを仮定した市場環境下で，ヨーロピアン・デリバティブをプレーン・バニラ・オプションで静的に複製する手法を提案する．後半では，原資産価格のボラティリティの不確実性を直接ヘッジする方法とデフォルト可能な条件付き請求権を株式オプションで静的にヘッジする方法を示す．

0.2 「確率分布のキュムラントを利用した価値評価方法」に関する研究

0.2.1 背景

金融商品の現在価値評価では，確率過程の汎関数である割引ペイオフの期待値を計算することになるが，確率過程やペイオフの関数形が複雑な場合には価格評価のための閉じた解が得られないことが多い．こう

したとき，モンテカルロ法や有限差分法，有限要素法といった数値計算方法が用いられるが，これらの数値解法には幾つかの問題が存在する．有限差分法などの偏微分方程式の差分近似で扱える確率過程の次元は高々2 次元であり，差分間隔の取り方によっては解が振動するといった欠点を持つ．さらにこの手法を非連続な確率過程に適用することは一般的には難しく，有限差分法の具体的なアルゴリズムが知られているのは一部のレヴィ過程だけである（例えば，Hirsa and Madan [2004]，Cariboni and Schoutens [2007] を参照）．一方，モンテカルロ法は多次元の確率過程を扱えるが，計算負荷が重く，数値解が乱数の系列に依存する．また，ブラウン運動の汎関数に対するパスの効率的な生成方法や分散減少法は多数知られているものの，レヴィ過程や時間変更レヴィ過程といった非連続な確率過程に関するモンテカルロ法の高速化技術は未だ発展途上にあるといえる．

第一部では，厳密解が得られない金融商品の価値評価に対して，確率分布の漸近展開を利用した近似解析的な評価式を導出するアプローチをとる．ここで登場するエッジワース展開やキュムラント展開，グラム・シャリエ展開などの漸近展開は確率統計学の一般的な手法であり，ファイナンスの分野でも既に利用されている技法である．例えば，Collin-Defresne and Goldstein [2002] は，多次元ガウシアン金利モデルと CIR モデルを仮定してエッジワース展開によりスワップションの近似価格公式を導出している．また，Tanaka et al. [2005] はグラム・シャリエ展開を用いてアファイン型金利モデルの下でスワップション，コンスタント・マチュリティ・スワップ（constant maturity swap：CMS），CMS オプションの近似価格公式を導いている．しかしながら，これらの先行研究はプレーン・バニラ型のオプションを比較的単純な確率過程に限定して価格公式を導出したに過ぎず，他の近似解析的手法でも代替が可能であることが知られている．

0.2.2 成果と貢献

1 章では，確率過程を共変量に持つ比例ハザード・モデル（Cox [1972]）をプリペイメント・モデルに設定した RMBS の価格評価公式を導出する．比例ハザード・モデルは機械の故障率や生物の死滅率の分析など，生存時間解析の分野で考案されたモデルであるが，RMBS の分野では，Schwartz and Trous [1989] の研究以降，同モデルが住宅ローン債務者の期限前償還行動を表現するスタンダード・モデルとなり，数多くの学術研究や実務での応用例が報告されている（例えば，Sugimura [2002]，Ciochetti et al. [2003]，Ozeki et al. [2009] 等を参照）．ところが，これまで RMBS 価格の解析的な評価方法は知られておらず，一昔前であれば，大掛かりな計算機環境を用意し，かなりの計算コストを掛けてモンテカルロ・シミュレーションで価格を算出していた．プリペイメントを比例ハザード・モデルで表現した RMBS の解析的な価格公式の導出は本研究が初めてであり，Schwartz and Trous [1989] の研究以来，20 年以上も未解決であった問題に一つの解決策を提示したことになる．また，この価格公式はリアル・タイムでのプライシングの実現のみならず，リスク指標の計算やカリブレーションにも有用であり，実務的にも優れた評価手法といえる．

実証分析では，本邦市場の RMBS の価格評価を実施する．本邦 RMBS 市場で最も流動性の高い住宅金融支援機構債券を分析対象とし，共変量にはハル・ホワイト型金利モデルを採用して満期 35 年の RMBS の価格評価を行う．その結果，RMBS は超長期債であるにもかかわらず，2 次オーダーの近似公式の適用によって実務で十分と思われる近似精度が得られることを示した．さらに，実効デュレーションや実効コンベキシティといった RMBS のリスク指標も近似公式によって安定的かつ高精度で計算できることを示す．

比例ハザード・モデルは RMBS の評価のみならず，ファイナンスの幅広い分野で利用されている．例えば，青沼・木島 [1998] は，本邦の定期預金を分析対象に比例ハザード・モデルを用いて預金者の解約行動を分析し，中途解約オプションの経済価値を推計した．信用リスクの分野では，Lane et al. [1986] や Whalen [1991]，Wheelock and Wilson [2000]，Duffie et al. [2007] などが個別企業のデフォルト確率の推定に同モデルを適用して実証分析を実施している．

こうした現状を鑑み，巻末の補論では，比例ハザード・モデルの共変量を連続過程であるガウシアン過程，アファイン過程， 2 次ガウシアン過程に加え，非連続過程であるレヴィ過程，時間変更レヴィ過程といった非常に広いクラスの確率過程に拡張し，価格評価の一般論を展開した上で，その近似評価公式を導出し，比例ハザード・モデルを適用した多くの価格評価問題に対して近似解析的評価が可能であることを示す．

2 章では，原資産価格が時間変更レヴィ過程によって変動する平均オプションの価格評価公式を導出する．平均オプションはそのペイオフが原資産の平均値に依存したオプションであり，エキゾティック・デリバティブ市場の代表的なプロダクツである一方で，単純な資産価格モデルであっても閉じた解が得られない評価問題として知られている．最近になり，確率ボラティリティ・モデルやジャンプを含むモデルで近似解析的な評価手法が提案され始めているが，時間変更レヴィ過程の下での平均オプションの解析的な価格評価公式の導出は本研究が初めてである．時間変更レヴィ過程は，従来の確率ボラティリティ・モデルやレヴィ過程を含む非常に広い確率過程のクラスであり，株式や為替などの価格変動の記述に適していることが先行研究で実証されているが，ヨーロピアン・オプションの価格評価はよく知られているものの，エキゾティック・デリバティブの価格評価問題を扱った研究は殆んど存在しないため，本研究の成果は，時間変更レヴィ過程の下でのエキゾティック・デリバティブの先駆的な研究として位置付けることができよう．

数値例では，ヘストン・モデルやバリアンス・ガンマ＋ CIR モデル，ノーマル・インバース・ガウシアン＋ CIR モデル等，計 6 つの時間変更レヴィ過程による原資産価格モデルに対して，近似公式を利用して平均オプションの価格を計算し，6 つの全てのモデルで十分な精度の計算結果が得られることを示す．

計算ファイナンスの観点から，これらの価格評価公式を得ることが出来た要因は以下になる．今回の価格評価問題では，漸近展開の対象が累積ハザード率や原資産価格の累積値といった複雑な確率変数となるが，確率分布の漸近展開に必要な任意の次数のキュムラントが“ 変動因子となる確率過程の異時点間同時分布に関する積率母関数 ”もしくはその変形として表現できることを示すことが，その後の解析の手掛かりとなる．これにより，特定の確率過程を採用した場合の分析対象が明確になる．ただし，“ 確率過程の異時点間同時分布に関する積率母関数 ”は正規過程を除いて，その陽的表現は自明ではなく，採用する確率過程のクラスに応じて個別の解析が必要になる．

金利の期間構造モデルとして用いられるアファイン過程や二次ガウシアン過程では，分析対象の積率母関数を与える方程式となる“ 後向きに定義される再帰的なリカッチ型連立常微分方程式 ”を導出する．また，資産価格などのジャンプが表現できるレヴィ過程では，異時点間レヴィ過程の並び替えを考えることで，レヴィ過程の独立増分性と定常増分性の性質を利用し，レヴィ・ヒンチンの公式に帰着させて問題を解く．時間変更レヴィ過程では，相関中立測度変換を逐次適用することで時間変更過程とレヴィ過程の 2 つの確率過程に分離させ，アファイン過程や二次ガウシアン過程，レヴィ過程の場合の解法へと帰着させる．

こうした確率解析の技法によって，非常に幅広い評価公式の導出が実現できる．特に，非連続過程であるレヴィ過程と時間変更レヴィ過程に関しては，ファイナンスの数値問題に応用できる近似解析手法は未だ希少であり，応用範囲の広い手法として計算ファイナンスの分野で新規性が高いものと考えられる．

なお，第二部「株式とクレジットの金融派生商品の統合評価モデル」の 4 章で議論する指数レヴィ型ジャンプ・トゥ・デフォルトモデルの下での価格評価公式の導出では，キュムラントを用いた特性関数の近似計算に同手法を応用する．

0.3 「株式とクレジットの金融派生商品の統合評価モデル」に関する研究

0.3.1 背景

CDS と株式オプションの急速な発展に伴い，学術と実務の双方でクレジット市場と株式市場の相互関係に着目する研究が盛んになってきている．

学術分野では，例えば，Zhang et al. [2005] や Cremers et al. [2008a] が企業価値モデルを用いた実証分析でクレジット・スプレッドと株式インプライド・ボラティリティの間には一定の関係があるとの結論を導いている．また，Cremers et al. [2008b] は，統計的手法によって株式インプライド・ボラティリティの水準のみならず，その傾きがクレジット・スプレッドを決定する重要な情報であることを示した．Carr and Wu [2010] は誘導型アプローチの枠組みで CDS スプレッドと株価オプションの動的な相互関係を調べた．さらに，Carr and Wu [2011] は，上場の株式アメリカン・オプションを用いて個別企業のデフォルト確率を推定している．

一方，実務の分野では，キャピタル・ストラクチャー・アービトラージ取引（capital structure arbitrage trading）と呼ばれるクレジットと株式の市場を跨ぐ裁定取引がヘッジファンドの代表的な投資戦略となり，クレジットと株式の中間的な金融商品である転換社債やコンティンジェント・キャピタル（contingent capital）の価格評価やリスク管理が金融機関の重要な課題となっている．

以前は，株式とクレジットの市場がある意味で分断されていたこともあり，伝統的なデリバティブ・モデリングでは株式とクレジットで別々のアプローチが取られてきた．古典的な株式デリバティブ理論では，ブラック・ショールズ・モデルを代表とする株価変動モデルに対して，株式オプションやその他の株式エキゾティック・デリバティブの評価方法が多数提案された一方で，これらの評価モデルでは参照企業のクレジットに関する金融商品を評価できないという欠点があった．逆に，ダフィ・シングルトン・モデルに代表される誘導型クレジット・モデルは，主に社債や CDS，CDO などの信用リスク商品の評価に用いられ，株式デリバティブの評価に直接用いられることはなかった．マートン・モデルなどの構造型クレジット・モデル（企業価値モデル）では，モデリングの対象が企業のバランスシートであるため，株式とクレジットの構造的な関係がモデルの本質であるにも拘らず，少なくとも数年前までは，株式デリバティブの評価に対して積極的に活用されることはなかった．

ところが，昨今のデリバティブ市場の変化に対応して，同一企業の株式とクレジットに関する条件付き請求権を一つのモデルで統合的に評価しようとする研究が現れてきた．例えば，Hull et al. [2005] では，Merton [1974] の企業価値モデルを拡張して，株式オプションを企業価値の上に書かれたコンパウンド・オプションとして評価している．Finger et al. [2002] や Stamicar and Finger [2006] はクレジット・グレイズ（CreditGrades：以下 CG）モデルと呼ばれる企業価値モデルを開発し，同一企業の CDS と株式オプションを同時に評価する枠組みを提案した．CG モデルは，構造型アプローチによる株式とクレジットの統合評価モデルとして実務家に浸透し，CG モデルを用いた実証分析やトレーディング戦略の開発（例えば，Veraart [2004]，Bystrom [2006]，Yu [2006]，Bedendo et al. [2007]，Bajlum and Larsen [2007] を参照）に加え，Sepp [2006] により，確率ボラティリティ又は二重指数分布のジャンプ・サイズを持つ複合ポアソン過程を導入した拡張モデルが提案されている．

一方，誘導型アプローチの統合評価モデルとして，ジャンプ・トゥ・デフォルト（jump-to-default：以下，JtD）モデルと呼ばれるモデリングが最近の研究で注目されている．JtD モデルは，伝統的な株価過程モデルとデフォルト強度モデルを統合した枠組みの総称であり，例えば，Takahashi et al. [2001] では，Black-Scholes 型の JtD モデルで転換社債（convertible bond）の価格評価を行い，株価変動のみならず信用力の変化が転換社債に与える影響を調べた．Linetsky [2006] では Black-Scholes 型，Carr and Linetsky [2006] では CEV(constant elasticity of variance) 型の JtD モデルで株式オプションと社債の解析評価公式を導いている．また，Andersen and Buffum [2003] や Carr and Madan [2010] では一般的な局所ボラティリティ型，Carr and Schoutens [2008] や Bayraktar and Yang [2011]，Carr and Wu [2010] では確率ボラティリティ型，Mendoza et al. [2010] では時間変更マルコフ過程を用いた JtD モデルを扱っている．

しかしながら，こうした株式とクレジットの統合モデルの研究は途に就いたばかりであり，特に伝統的な株式モデルにおいて主要かつ重要なクラスである“ レヴィ過程 ”による統合モデルの構築が未整備のままとなっている．さらには，構造型と誘導型の統合モデルに関して一般的な理論構築も未着手の研究課題である．

0.3.2 成果と貢献

3 章では，構造型アプローチの統合評価モデルとして，CG モデルにレヴィ過程を導入した拡張モデルを提案し，その枠組みの中で株式オプションと CDS の準解析的評価公式を導く．

レヴィ過程はジャンプを表現する確率過程のクラスであり，この確率過程を導入することでオリジナル・モデルの次の 3 つの欠点を克服する．1 つ目は，デフォルトの可予測性である．古典的な企業価値モデルの変動はブラウン運動で記述されているため，企業価値を観測することでデフォルト時刻が予測でき，その結果，短期のクレジット・スプレッドが実際よりも非常に低くなるという欠点を持つ．レヴィ過程を導入することで，企業価値の非連続な変動をモデル化し，デフォルト時刻の可予測性を排除することができる．2 つ目は，株式ボラティリティ・スキューの形状に関する欠点である．オリジナル・モデルはボラティリティ・スキューを表現できるが，この形状は企業の財務レバレッジ比率のみに依存しており，その他の要因は反映されない．レヴィ過程を導入することで，ボラティリティ・スキューに企業価値の突発的

な変化の影響を反映させることができる．3 つ目は，企業価値変動の表現力の問題である．オリジナル・モデルでは，企業価値がブラウン運動で変動しているため，ボラティリティだけが唯一のパラメータであるため表現力が低い．一方，レヴィ過程はブラウン運動を含む幅広い確率過程のクラスであるため，様々な確率過程を企業価値の変動因子として選択できるという利点が生まれる．

株式オプションと CDS の準解析的評価公式の導出では，レヴィ過程の特性関数に関するウィナー・ホップ分解という技法を用いて，企業価値過程とその最小値過程の同時分布に関する積率母関数を求め，この積率母関数のラプラス・フーリエ変換によって株式オプションと CDS の価格を導出する．ウィナー・ホップ分解では，ウィナー・ホップ因子と呼ばれるレヴィ過程の特性関数に対応するある量を計算することが必要となるが，一般的にこれを計算することは難しい．そこで負の方向のジャンプだけを認めたスペクトラリー・ネガティブ（spectrally negative）レヴィ過程を企業価値の変動因子に採用し，この過程の性質を利用して，ある代数方程式の根を求めることでウィナー・ホップ因子を効率的に計算する方法を採る．

数値例では，準解析解による評価公式を用いて株式インプライド・ボラティリティと CDS プレミアムを計算し，レヴィ過程によるジャンプの影響を調査する．

4 章では，誘導型アプローチの統合評価モデルとして，指数レヴィ型の JtD モデルを提案し，4 章と同様に株式オプションと CDS の準解析的な評価公式を導出する．

先行研究では，代表的な株価変動モデルにデフォルト強度過程を導入することで幾つかの JtD モデルが提案されてきたが，株価の指数レヴィモデルを拡張した JtD モデルは未だ提案されていない．そこで本章では，株価とデフォルト強度の両方がレヴィ過程で変動する新しいモデルを提案する．このモデルでは，レヴィ過程の線形結合により株価とデフォルト強度が変動し，共通因子となるレヴィ過程の重み係数により企業間や株式とクレジットの依存関係を表現することができる．また，変動因子は特性指数が既知である任意のレヴィ過程を採用できるため，非常に柔軟なモデリングが可能となる．デフォルト強度はレヴィ過程を共変量とする比例ハザード・モデルを採用して，第一部での計算技法を拡張し，応用する．

株式オプションと CDS の準解析的評価公式の導出では,「価格生成関数」と呼ぶ独自に定義した関数の性質を利用する．価格生成関数を計算するための一般公式の導出では，この関数がレヴィ過程の特性指数に関する無限級数展開として表現できることを示す．一般公式では，価格生成関数の計算に多重の繰り返し積分が必要となるが，比例ハザード・モデルのベースライン・ハザード関数が多項式と指数関数の積を基底とする関数のクラスに属するとき，この繰り返し積分の閉じた解が得られる．株式オプション価格は価格生成関数の逆フーリエ変換により与えられ，CDS プレミアムは価格生成関数の媒介変数に関する微分と時間に関する積分で得られる．また，デフォルト強度と瞬間的フォワード・クレジット・スプレッドの関係式を導く．

数値例では，共通因子にバリアンス・ガンマ過程，個別因子にブラウン運動を採用した指数レヴィ型 JtD モデルを用いて，株式ボラティリティ・スキューと CDS プレミアムの期間構造を描き，共通因子の影響による株式ボラティリティと CDS の変化を試算する．

5 章では，株式とクレジットの統合評価モデルの枠組みの下で，株式インプライド・ボラティリティとリスク中立確率の下でのデフォルト確率の関係を理論的に考察する．

過去の実証分析では，株式インプライド・ボラティリティとデフォルト確率に対して一定の関係が指摘されていたが，理論的な関係を探る研究は殆ど存在しなかった．本章では，構造型と誘導型の両方のアプローチに適合する緩やかな仮定の下で，特定のモデルに依存しない株式インプライド・ボラティリティとデフォルト確率の関係式を導く．この関係式から，リスク中立確率の下のデフォルト確率は株価インプライド・ボラティリティの傾きに関するある極限値として特徴付けられることが明らかになる．また，企業のデフォルト可能性を考慮した株式市場では，行使価格を 0 に近づけると，株式ボラティリティ・スキューは無裁定条件を満たす最大のスピードで発散しなければならないことを証明する．これらの結果は，株式とクレジットの統合評価モデルにおける基本原理と考えられる．

また，株式とクレジットの統合評価モデルの枠組みにおけるデフォルト可能な金融派生商品のヘッジに関する手法を第三部「オプションを用いた新しいヘッジ手法」の 8 章で提案する．

0.4 「オプションを用いた新しいヘッジ手法」に関する研究

0.4.1 背景

伝統的なデリバティブ理論では，デリバティブの原資産と安全資産を動的に組み替えることで自己充足的なポートフォリオを構成し，デリバティブのペイオフを複製することが価格付けの原理となる．ブラック・ショールズ理論（Black and Scholes [1973]）の発表以降，動的ポートフォリオによるデリバティブ・ペイオフの複製理論がヘッジ手段として活用されてきた．ところが，デリバティブ市場の発展によりプレーン・バニラ・オプションがコモディティ化したこともあり，最近の実務では原資産のみならずオプションもヘッジ・ツールとして利用することが一般的となっている．また，学術研究でも，オプションの組み合わせでアロー・デブリュー証券が構成できることに着目したオプション・ポートフォリオによる静的ヘッジ手法が幾つか提案されている．例えば，Bowie and Carr [1994] や Carr et al. [1998] では，プットとコールの対称性を利用したバリア・オプションとルックバック・オプションの静的ヘッジ手法を考案している．また，Derman et al. [1995] はカレンダー・スプレッドによるバリア・オプションの静的ヘッジの方法を提示した．さらには，Carr and Chou [1997, 2002] や Carr and Madan [1998] では，静的ヘッジ手法の基本定理となる「原資産に関して二階微分可能な経路依存のないペイオフは行使価格の異なる無限個のプレーン・バニラ・オプションの静的ポートフォリオで複製できる」ことを証明し，この定理を用いた各種デリバティブの複製手法を導出している．

一方で，これらの静的ヘッジ手法はブラック・ショールズ・モデルを前提にしたものが多く，確率ボラティリティ・モデルやジャンプを含むモデルへの拡張を試みる研究は存在するものの，必ずしも洗練された手法が提案されているとはいえない状況にある．しかしながら，こうした拡張の試みは非完備市場の完備化や離散取引で生じるヘッジ誤差の解消といった問題解決の糸口であり，理論と実務の両方で重要な課題である．そこで第三部では，1) ジャンプや確率ボラティリティを持つ価格過程への静的ヘッジ手法，2) デフォルト・リスクを持つ金融商品に対する静的ヘッジ手法，3) ボラティリティ変動過程が未知の環境下でのボラティリティ変動リスクのヘッジ手法，の 3 つについて既存の静的ヘッジ手法の拡張に取り組む．

0.4.2 成果と貢献

6 章では，経路依存のないヨーロピアン・デリバティブを短い満期のプレーン・バニラ・オプションの静的ポートフォリオで複製する手法を提案する．

本章の手法は，Carr and Chou [1997, 2002] や Carr and Madan [1998] が示した静的ヘッジ手法の基本公式を修正して，微分不可能なペイオフを持つデリバティブにも適用できる公式とし，さらにはガウス型求積法 (Gaussian quadrature rule）を利用することでヘッジ・ツールとなるプレーン・バニラ・オプションを効率的に配置する方法を与えるものである．この手法は Carr and Wu [2002] が提案した長期プレーン・バニラ・オプションの静的ヘッジ手法の一般的なヨーロピアン・デリバティブへの拡張とみることができ，ヘッジ対象デリバティブの価格関数が原資産価格に関するマルコフ過程で表現されている場合には，レヴィ過程などのジャンプを含む原資産価格モデルであっても適用可能な汎用性の高い方法であり，また，以降の章での基本的な道具立てとなる．

7 章では，確率ボラティリティ・モデルの下での静的ヘッジ手法を提案する．

確率ボラティリティ・モデルの下では，ヘッジ対象となるヨーロピアン・デリバティブの価格は原資産とボラティリティの 2 つの状態変数をとる関数として表現されるため，ボラティリティの分だけリスク要因の次元が増えるという問題が生じる．Fink [2003] では，プレーン・バニラ・オプションのストライク・スプレッドとカレンダー・スプレッドの組合せによって，ヘッジ対象デリバティブの原資産とボラティリティの 2 つの変動リスクをヘッジする方法を提案した．しかしながら，この手法は非常に多くのプレーン・バニラ・オプションを必要とするため，実務への適用は難しい．そこで本章では，原資産の価格過程を表現する確率ボラティリティ・モデルの確率微分方程式に対して，これと弱い解が一致する局所ボラティリティ・モデルの確率微分方程式を考えることで，デリバティブ価格関数を原資産価格に関するマルコフ過程に射影した後にヘッジ・ポートフォリオを構築する方法を考案する．数理ファイナンスの分野では，この技法をマルコフ射影（Markovian projection）と呼ぶが，従来はプライシングなどのテクニックとして

知られており（例えば，Avellaneda et al. [2002]，Henry-Labordere [2005]，Piterbarg [2006] を参照），ヘッジ手法に応用するのは本論文が初めてとなる．

実証分析では，ヘストン型の確率ボラティリティ・モデル（Heston [1993]）の下で通貨オプション市場のヒストリカル・データに対して同手法を適用し，静的ヘッジ手法の有効性を示す．

8 章では，デフォルト・リスクが存在する金融市場の枠組みの中で，デフォルト可能な条件付請求権を株式オプションで静的にヘッジする方法を与える．

この手法は，既存の静的ヘッジ手法の「株式とクレジットの金融派生商品の統合評価モデル（第二部）」への拡張であり，株式市場とクレジット市場を横断する新しいヘッジ手法である．ヘッジ対象となるデフォルト可能な条件付請求権は，株式のプレーン・バニラ・オプションやデジタル・オプションの他，社債などのクレジット商品も含まれる．

株式とクレジットの統合評価モデルの開発は近年盛んであるものの，この枠組みでの先進的なヘッジ手法の研究は未だ数少ない．例えば，Carr [2005] や Carr and Schoutens [2008] は誘導型アプローチの統合評価モデルで，デフォルト可能な条件付請求権のヘッジ戦略を提案したが，特定のモデルに依存した極めて限定的な手法である．これらの先行研究と比較すると，本章で提案するヘッジ手法は幾つかの長所を持つ．1 つ目は，ヘッジ手法が特定のモデルに依存せず，汎用的な方法である点である．2 つ目は，ヘッジ・ツールとなるプロダクツが相対的に流動性の高い短期の株式プレーン・バニラ・オプションである点である．3 つ目は，ヘッジが静的である点である．一般に，クレジットが悪化している企業の金融商品は流動性が低下する傾向にあるため，動的に売買を繰り返すことは難しく，その点で静的ヘッジは非常に有効な手段を与える．4 つ目は，金融実務で実効可能な有限個のオプションでヘッジが構成できる点である．

ヘッジ・ポートフォリオ組成のポイントは，株式プット・オプションについて，行使価格に関する極限を考えることで擬似的なクレジット・デリバティブを複製することにある．数値例では，バリアンス・ガンマ過程を導入した CG モデルと Carr and Linetsky [2006] が提案した CEV 型 JtD モデルを採用し，それぞれ割引社債をヘッジ対象の条件付請求権として静的ヘッジ・ポートフォリオを試算してその複製精度を検証する．

9 章では，ボラティリティ変動過程が未知の環境下でヨーロピアン・デリバティブのボラティリティ変動リスクをヘッジする手法を提案する．

昨今のデリバティブ市場では，バリアンス・スワップ（variance swap：以下，VS）が標準的なデリバティブとして取引されている．VS とは，予め決めた固定レートと原資産収益率の 2 次変分（quadratic variance）を交換するスワップ取引であり，ボラティリティ・トレーディングのツールとして活用されている．また，一部の実務家はボラティリティ変動リスクのヘッジ・ツールとして VS を利用しているが，その効果や理論的正当性を検証する学術研究はこれまで存在しなかった．

そこで本章では，VS によるボラティリティ変動リスクのヘッジ効果に対する限界を指摘した後に，ボラティリティ変動リスクのヘッジに適した「ポリノミアル・バリアンス・スワップ（polynomial variance swap：以下，PVS）」と呼ぶ VS を一般化したボラティリティ・デリバティブを考案する．VS では原資産価格の水準によらず一定のボラティリティ・エクスポージャーが発生するのに対して，PVS は原資産の水準に応じたボラティリティ・エクスポージャーを享受できるという長所がある．一般に，デリバティブのボラティリティ変動リスクは原資産価格の水準に依存するため，PVS のボラティリティ・エクスポージャーと巧く適合させることで適切なボラティリティ・ヘッジが実現できることになる．

ところが，PVS は現在の金融市場では取引されていないデリバティブであるため，PVS 自体を複製する方法も併せて提案する．VS と同様に，PVS もプレーン・バニラ・オプションの静的ポートフォリオと原資産の動的ポートフォリオによって，特定のモデルを仮定することなく複製可能であり，PVS に対する複製の頑健性が保証されることを示す．

数値例では，モンテカルロ・シミュレーション環境下で PVS によるデリバティブのボラティリティ変動リスクの効果を検証する．ブラック・ショールズ・モデルによるダイナミック・ヘッジや Bakshi et al. [1997] が提案したヘストン・モデルでの最小分散ヘッジとの比較により，PVS を活用したヘッジ手法はモデル・リスクが低く，頑健性の高いヘッジ効果が得られることを確認する．

Contents

Part I

確率分布のキュムラントを利用した価値評価方法

Chapter 1

Valuation of Residential Mortgage-Backed Securities with Proportional Hazard Model: Cumulant Expansion Approach to Pricing RMBS

The residential mortgage-backed security (hereafter, RMBS) market is currently a very large segment in the Japanese fixed income markets. Under this circumstance, the valuation problem of RMBS is significantly important for practitioners. This chapter develops a pricing formula for not only RMBS, but also interest only (hereafter, IO) and principal only (hereafter, PO) with the proportional hazard model.

Usually, RMBS is a pass-through security with monthly payments. Since mortgage contracts composed of RMBS allow the borrowers to prepay the principal at any time prior to maturity without any penalty, the cash flows of RMBS include their prepayment. That is, there is uncertainty in the RMBS cash flows due to prepayment. Almost all practitioners recognize that prepayment risk is the most important issue for RMBS valuation and risk management. On the other hand, in past literature, many researchers focused upon modeling and evaluating prepayment risk. In general, the approaches for modeling prepayment behavior can be classified into two categories: the structural approach and the intensity-based approach.

In the structural approach, it is assumed that the borrowers behave rationally and exercise their optimal prepayment strategy. Since this strategy can be seen as the early exercise problem of the American option, it is also well-known as the option-based approach. However, due to the existence of various reasons for prepayment, the situation of RMBS is more complicated than that of the American option. The structural approach was pioneered by Dunn and McConnell [1981a, 1981b]. Since their studies, much literature dealing with the structural approach has been published. For example; see McConnell and Singh [1994], Stanton [1995], Kariya and Kobayashi [2000], Nakamura [2001], and Nakagawa and Shouda [2004]. Recently, Longstaff [2005] and Pliska [2005, 2006] have conducted research on sequential refinancing and have considered multi-stage decision models.

Instead of much energetic research on the structural approach, it seems that practitioners in the RMBS market hesitate to employ practical applications of this approach. Several reasons exist: Evaluating RMBS using the structural approach is computationally demanding and extremely time-consuming. Apart from the computational difficulties, the outputs of the structural approach do not completely match with time-series prepayment data and market prices of RMBS. In addition, the assumption of the optimal prepayment strategy is often violated in market practice. This is because in actuality there are many irrational borrowers who do not fit the assumption of the structural approach.

In the intensity-based approach, it is assumed that timing of prepayment is a random time governed by some hazard rate processes. Thus, a mortgage prepayment event is regarded as a default in the credit risk modeling (e.g. the monograph on the intensity-based credit risk modeling by Bielecki and Rutkowski

[2002]). However, in contrast to credit risk modeling, research on the theoretical and mathematical foundations of prepayment modeling in the intensity-based framework is limited. The notable references include Pliska [2005, 2006] and Sugimura [2004].

Conversely, there are a large number of empirical studies based on the intensity-based approach. In the empirical intensity-based approach, a certain prepayment hazard rate function is statistically estimated from historical prepayment data. In particular, the proportional hazard model is frequently applied to describe prepayment behavior both academically and in practice, and its estimation methods have been established. For example, see Schwartz and Torous [1989], Aonuma and Kijima [1998], Ichijo and Moridaira [2001], and Sugimura [2002] for studies on empirical prepayment analysis by the proportional hazard model. Since the proportional hazard model is considered to be a typical prepayment model in the RMBS market, we assume that prepayment behavior follows the proportional hazard model.

It is well-known that evaluating RMBS using numerical calculations such as Monte Carlo and the finite-difference method is highly demanding. Particularly, in multi-dimensional stochastic cases, there is nothing for practical RMBS pricing methods other than Monte Carlo. On the other hand, literature on analytical RMBS valuation is very limited. Gorovoy and Linetsky [2007] derives an analytical pricing formula for RMBS by using the eigenfunction expansion method. Yamazaki [2005] proposes a closed-form pricing formula for RMBS under a simple Gaussian assumption. However, Gorovoy and Linetsky [2007] and Yamazaki [2005] intentionally choose analytically tractable prepayment models, which might possibly be unsuitable to market practice; in order to derive analytical formula.

In this chapter, assuming the proportional hazard model to describe prepayment behavior, we derive a pricing formula for RMBS by using the cumulant expansion method. Because the proportional hazard model already has empirical evidence as a prepayment model, the assumption of the proportional hazard model is more appropriate than that of Yamazaki [2005] or Gorovoy and Linetsky [2007]. The pricing formula gives very accurate approximate prices of RMBS quickly like a closed-form pricing formula. Moreover, the formula is applicable to various types of the proportional hazard models; i.e. it is able to deal with multi-dimensional stochastic environments and jumps. Through numerical examples, we also show that the formula is very useful from a practical point of view.

1.1 Models

This section develops risk-neutral valuation models for RMBS, IO, and PO. Our interest is prepayment risk, which is the risk of uncertain RMBS cash flow due to borrowers' prepayment, while for simplicity we ignore other risks of RMBS such as default risk, earthquake risk, and commingling risk.

1.1.1 Mortgage Contracts without Prepayment

First, we consider a mortgage contract with fixed-rate c and maturity T without prepayment. The borrower takes out a loan of $M(0)$ dollars at time 0, then he pays back periodically a constant amount denoted by A, where the payment times of A are $t_i = i/m,\ i = 0, 1, \cdots, mT$. It is obvious that A is given by

$$A = M(t_i) - M(t_{i+1}) + \frac{c}{m}M(t_i) = P(t_{i+1}) + I(t_{i+1}), \tag{1.1}$$

where $M(t_i)$, $P(t_i) := M(t_{i-1}) - M(t_i)$, and $I(t_i) := \frac{c}{m}M(t_{i-1})$ are remaining mortgage principal, payment amount of mortgage principal, and coupon amount at time t_i respectively. From Eq.(1.1), we obtain

$$M(0) = A\frac{1-(1+c/m)^{-mT}}{c/m}.$$

Inversely, A can be rewritten as

$$A = M(0)\frac{c/m\,(1+c/m)^{mT}}{(1+c/m)^{mT}-1}.$$

Thus, the constant payment amount A is determined by the initial principal $M(0)$, maturity T, coupon rate c and payment interval m. Moreover, remaining mortgage principal at time t_i is given by

$$M(t_i) = M(0)\frac{(1+c/m)^{mT}-(1+c/m)^{mt_i}}{(1+c/m)^{mT}-1}.$$

See, for instance, Fabozzi [2001] or Yamazaki [2005] for derivations of the above cash flow models.

1.1.2 Mortgage Contracts with Prepayment in the Intensity-based Framework

We assume frictionless markets and arbitrage-free. Let τ denote a prepayment time of a mortgage contract on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, where $\mathbb{Q}$ is an equivalent martingale measure. We denote the associated filtration of τ by $\mathbb{H} := (\mathcal{H}_t)_{t\geq 0}$, where $\mathcal{H}_t = \sigma(\mathbf{1}_{\{\tau>s\}} : s \leq t)$. Let $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ be an arbitrary filtration on $(\Omega, \mathcal{G}, \mathbb{Q})$. Furthermore, we assume an auxiliary filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ such that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$; i.e. $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for any $t \in [0, T]$. In order to model prepayment, we introduce a positive prepayment intensity (hazard rate) process $(h_t)_{t\geq 0}$ adapted to the filtration $\mathbb{F}$. We model the random time of prepayment τ as

$$\tau = \inf\left\{t \geq 0 : \int_0^t h_s ds \geq e\right\},$$

where $e \sim \mathrm{Exp}(1)$. Here we define that $H_t := \mathbf{1}_{\{\tau\leq t\}}$ is the prepayment indicator, $F_t := \mathbb{Q}(\tau \leq t \,|\, \mathcal{F}_t)$ is the conditional prepayment probability of τ, and $\Gamma_t := -\ln(1-F_t) = \int_0^t h_s ds$ is the hazard process of τ under $\mathbb{Q}$. Note that the compensated process $H_t - \Gamma_{t\wedge\tau} = H_t - \int_0^{t\wedge\tau} h_s ds$ is a $\mathbb{G}$-martingale.

Next, we consider a mortgage contract with prepayment. Let M_{t_i}, P_{t_i}, and I_{t_i} be remaining mortgage principal, payment amount of mortgage principal, and coupon amount with prepayment at time t_i respectively. Then,

$$M_{t_i} = M(t_i)\mathbf{1}_{\{\tau>t_i\}}, \quad I_{t_i} = I(t_i)\mathbf{1}_{\{\tau>t_{i-1}\}}, \quad P_{t_i} = P(t_i)\mathbf{1}_{\{\tau>t_{i-1}\}}. \tag{1.2}$$

Moreover, a prepayment amount[1] at time t_i denoted by PR_{t_i} is given by

$$PR_{t_i} = (M_{t_{i-1}} - P_{t_i})\mathbf{1}_{\{t_{i-1}<\tau\leq t_i\}} = M(t_i)\left(\mathbf{1}_{\{\tau>t_{i-1}\}} - \mathbf{1}_{\{\tau>t_i\}}\right). \tag{1.3}$$

Let CF_{t_i} be a cash flow of the mortgage contract with prepayment at time t_i. From Eq.(1.2) and (1.3), and the definition of $P(t_i)$, we obtain

$$CF_{t_i} = P_{t_i} + I_{t_i} + PR_{t_i} = \left(1+\frac{c}{m}\right)M(t_{i-1})\mathbf{1}_{\{\tau>t_{i-1}\}} - M(t_i)\mathbf{1}_{\{\tau>t_i\}}.$$

[1] We assume that a prepayment cash flow occurs at time t_i, where $t_{i-1} < \tau \leq t_i$. This is a reasonable assumption when RMBS is considered, because RMBS cash flows including prepayment occur only at time t_i $(i = 0, 1, \cdots, mT)$. Although τ should be called *prepayment decision time* in terms of this assumption, we shall call it *prepayment time* for convenience.

1.1.3 RMBS Valuation Models

We consider RMBS to be composed of a homogeneous mortgage pool. In this case, RMBS pricing can be identified with valuation of an arbitrary mortgage contract in the pool. If RMBS consists of a heterogeneous pool, it is sufficient to divide it into homogeneous pools and to evaluate each homogeneous pool.

Assumption 1.1 *(Short Rate Process) The short rate process $(r_t)_{t\geq 0}$ is adapted to the filtration $\mathbb{F}$ and a unique strong solution of the stochastic differential equation:*

$$dr_t = \mu(t, r_t)dt + \mathbf{b}^\top(t, r_t)d\mathbf{W}_t,$$

where $(\mathbf{W}_t)_{t\geq 0}$ is an $\mathcal{F}_t$-adapted d-dimensional Brownian motion under $\mathbb{Q}$, and $\mu : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ and $\mathbf{b} : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}^d$ are deterministic functions. In addition, the discount bond price process with maturity U denoted by $(B_t(U))_{t\geq 0}$ is a unique strong solution of the stochastic differential equation:

$$\frac{dB_t(U)}{B_t(U)} = r_t dt + \mathbf{g}^\top(t, U, r_t)d\mathbf{W}_t,$$

where $\mathbf{g} : \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}^d$ is a deterministic function.

Assumption 1.1 implies that the equivalent martingale measure $\mathbb{Q}$ is a spot neutral measure. Hence, below we call it a *spot neutral measure* to distinguish it from other measures such as *forward neutral measures*.

Let V_{t^*} denote a present value[2] of RMBS at time t^*. Using the standard intensity-based framework, we obtain

$$\begin{aligned} V_{t^*} &= \mathbb{E}\left[\sum_{i=i^*}^{mT} e^{-\int_{t^*}^{t_i} r_s ds} CF_{t_i} \;\Big|\; \mathcal{G}_{t^*}\right] \\ &= \sum_{i=i^*}^{mT} \left\{\left(1+\frac{c}{m}\right) M(t_{i-1})\,\mathbb{E}\left[e^{-\int_{t^*}^{t_i} r_s ds}\mathbf{1}_{\{\tau>t_{i-1}\}} \mid \mathcal{G}_{t^*}\right] - M(t_i)\,\mathbb{E}\left[e^{-\int_{t^*}^{t_i} r_s ds}\mathbf{1}_{\{\tau>t_i\}} \mid \mathcal{G}_{t^*}\right]\right\} \\ &= \sum_{i=i^*}^{mT} \left\{\left(1+\frac{c}{m}\right) M(t_{i-1})\,\mathbb{E}\left[e^{-\int_{t^*}^{t_i} r_s ds} e^{-\int_{t^*}^{t_{i-1}} h_s ds} \mid \mathcal{F}_{t^*}\right] - M(t_i)\,\mathbb{E}\left[e^{-\int_{t^*}^{t_i} (r_s+h_s) ds} \mid \mathcal{F}_{t^*}\right]\right\}, \end{aligned}$$

where $\mathbb{E}[\,\cdot\,]$ is an expectation operator under a spot neutral measure $\mathbb{Q}$ and $i^* := \inf\{i : t_i > t^*\}$. Taking a forward neutral measure $\mathbb{Q}_{t_i}$ for each cash flow CF_{t_i}, where the numéraire is a discount bond with maturity t_i, RMBS price can be described as

$$V_{t^*} = \sum_{i=i^*}^{mT} \left\{\left(1+\frac{c}{m}\right) M(t_{i-1})\, B_{t^*}(t_i)\,\mathbb{E}^{t_i}\left[e^{-\int_{t^*}^{t_{i-1}} h_s ds} \mid \mathcal{F}_{t^*}\right] - M(t_i) B_{t^*}(t_i)\,\mathbb{E}^{t_i}\left[e^{-\int_{t^*}^{t_i} h_s ds} \mid \mathcal{F}_{t^*}\right]\right\},$$

[2] Strictly speaking, V_{t^*} denotes a RMBS price per one unit against the *remaining principal*. However, in order to avoid redundant notations, we do not explicitly mention the *remaining principal*.

where $\mathbb{E}^{t_i}[\,\cdot\,]$ is an expectation operator under a forward neutral measure $\mathbb{Q}_{t_i}$. Similarly, PO and IO prices can be expressed as

$$\begin{aligned}
\mathrm{PO}_{t^*} &= \mathbb{E}\left[\sum_{i=i^*}^{mT} e^{-\int_{t^*}^{t_i} r_s ds}(P_{t_i} + PR_{t_i}) \,\Big|\, \mathcal{G}_{t^*}\right] \\
&= \sum_{i=i^*}^{mT} \left\{ M(t_{i-1})B_{t^*}(t_i)\,\mathbb{E}^{t_i}\left[e^{-\int_{t^*}^{t_{i-1}} h_s ds} \,\big|\, \mathcal{F}_{t^*}\right] - M(t_i)B_{t^*}(t_i)\,\mathbb{E}^{t_i}\left[e^{-\int_{t^*}^{t_i} h_s ds} \,\big|\, \mathcal{F}_{t^*}\right]\right\}, \\
\mathrm{IO}_{t^*} &= \mathbb{E}\left[\sum_{i=i^*}^{mT} e^{-\int_{t^*}^{t_i} r_s ds} I_{t_i} \,\Big|\, \mathcal{G}_{t^*}\right] \\
&= \frac{c}{m}\sum_{i=i^*}^{mT} M(t_{i-1})B_{t^*}(t_i)\,\mathbb{E}^{t_i}\left[e^{-\int_{t^*}^{t_{i-1}} h_s ds} \,\big|\, \mathcal{F}_{t^*}\right].
\end{aligned}$$

According to the above valuation models, we have to only evaluate

$$\mathbb{E}^{t_i}\left[e^{-\int_{t^*}^{t_{i-1}} h_s ds} \,\big|\, \mathcal{F}_{t^*}\right] \quad \text{and} \quad \mathbb{E}^{t_i}\left[e^{-\int_{t^*}^{t_i} h_s ds} \,\big|\, \mathcal{F}_{t^*}\right],$$

for RMBS, PO, and IO pricing. Further generalizing this argument, RMBS, PO, and IO valuations can be reduced to the problem of calculating the following equation:

$$\mathbb{E}^{U}\left[\exp\left\{-\int_{t^*}^{t} h_s ds\right\} \,\Big|\, \mathcal{F}_{t^*}\right], \tag{1.4}$$

where $\mathbb{E}^{U}[\,\cdot\,]$ is an expectation operator a under forward neutral measure $\mathbb{Q}_U$, and where the numéraire is a discount bond with maturity $U(\geq t)$. Note that Eq.(1.4) can be seen as a survival probability of a mortgage contract under $\mathbb{Q}_U$.

Assumption 1.2 *(Prepayment Model) The prepayment intensity h_t is described as the proportional hazard model, that is,*

$$h_t := h_0(t)\exp\left\{\mathbf{w}^\top \mathbf{X}_t\right\}, \tag{1.5}$$

where $h_0 : \mathbf{R}_+ \to \mathbf{R}_+$ called the base-line hazard function is a non-negative deterministic function with respect to time t, $(\mathbf{X}_t)_{t\geq 0}$ called covariate vector is an $\mathcal{F}_t$-adapted m-dimensional stochastic process under a spot neutral measure $\mathbb{Q}$, and $\mathbf{w}$ is a coefficient vector on $\mathbf{R}^m$.

Assumptions 2.1 and 2.2 are more general settings than in Yamazaki [2005] or in Gorovoy and Linetsky [2007], in both of which analytically tractable prepayment models and simple interest rate processes are set in order to derive closed-form pricing formulas. In particular, Assumption 1.2 is meaningful; because there is much literature documenting research on empirical analysis of prepayment behavior by the proportional hazard model. For example; see Schwartz and Torous [1989], Aonuma and Kijima [1998], Ichijo and Moridaira [2001], and Sugimura [2002].

1.2 Pricing Formula

This section provides a general formula for not only RMBS, but also IO and PO; which is the main finding of this chapter. In the previous section, it is shown that the RMBS pricing problem can be reduced to evaluating the survival probability of a mortgage contract under a forward neutral measure. Therefore, we shall concentrate on evaluating Eq.(1.4) for RMBS valuation. Since the pricing formula is given by an infinite series that is known as the cumulant expansion, it gives an approximate value of RMBS prices.

Theorem 1.3 *Under Assumptions 2.1 and 2.2, Eq.(1.4) is given by*

$$\mathbb{E}^U\left[\exp\left\{-\int_{t^*}^t h_s ds\right\} \Big| \mathcal{F}_{t^*}\right] = \exp\left\{\sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}\kappa_n\right\}, \tag{1.6}$$

where

$$\begin{aligned}
\kappa_1 &= \chi_1, \\
\kappa_2 &= \chi_2 - \chi_1^2, \\
\kappa_3 &= \chi_3 - 3\chi_1\chi_2 + 2\chi_1^3, \\
\kappa_4 &= \chi_4 - 4\chi_1\chi_3 - 3\chi_2^2 + 12\chi_1^2\chi_2 - 6\chi_1^4, \\
\kappa_5 &= \chi_5 - 5\chi_1\chi_4 - 10\chi_2\chi_3 + 20\chi_1^2\chi_3 + 30\chi_1\chi_2^2 - 60\chi_1^3\chi_2 + 24\chi_1^5, \\
& \quad \cdots\cdots\cdots,
\end{aligned}$$

and

$$\chi_n = n! \int_{t^*}^t \int_{t^*}^{s_{n-1}} \cdots \int_{t^*}^{s_2} \prod_{k=1}^n h_0(s_k)\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^n \mathbf{w}^\top \mathbf{X}_{s_k}\right\} \Big| \mathcal{F}_{t^*}\right] ds_1 ds_2 \cdots ds_n. \tag{1.7}$$

Note that κ_n and χ_n are respectively the nth cumulant and moment of $\tilde{\Gamma}_t := \Gamma_t - \Gamma_{t^*} = \int_{t^*}^t h_s ds$. Thus, it can be said that the pricing formula is an approximation around normal distribution with respect to $\tilde{\Gamma}_t$. When $\tilde{\Gamma}_t$ happens to be a Gaussian process, the formula coincides with the closed-form pricing formula in Yamazaki [2005] which gives exact prices of RMBS.

Before the proof of Theorem 1.3, the below lemma is provided.

Lemma 1.4 *Suppose that $f : \mathbf{R}_+ \to \mathbf{R}$ is an integrable function and*

$$F(x) := \int_\alpha^x f(u)du,$$

where α is an arbitrary non-negative constant. Then, for all $n \in \mathbf{N}$,

$$(F(x))^n = n! \int_\alpha^x \int_\alpha^{u_{n-1}} \cdots \int_\alpha^{u_2} f(u_n)f(u_{n-1})\cdots f(u_1)du_1 du_2 \cdots du_n. \tag{1.8}$$

Proof of Lemma 1.4: In the case of $n = 2$, Eq.(1.8) is well-known and omitted. Next, assume that Eq.(1.8) is valid in the case of $n \leq k$. Then,

$$\begin{aligned}
\frac{d}{dx}\left[\frac{1}{(k+1)!}(F(x))^{k+1}\right] &= \frac{1}{k!}(F(x))^k f(x) \\
&= f(x)\int_\alpha^x \int_\alpha^{u_{k-1}} \cdots \int_\alpha^{u_2} f(u_k)f(u_{k-1})\cdots f(u_1)du_1 du_2 \cdots du_k.
\end{aligned}$$

The second equality of the above equation is due to the assumption. Therefore,

$$\begin{aligned}
\frac{1}{(k+1)!}(F(y))^{k+1} &= \int_\alpha^y f(x)\int_\alpha^x \int_\alpha^{u_{k-1}} \cdots \int_\alpha^{u_2} f(u_k)f(u_{k-1})\cdots f(u_1)du_1 du_2 \cdots du_k dx \\
&= \int_\alpha^y \int_\alpha^{u_k} \int_\alpha^{u_{k-1}} \cdots \int_\alpha^{u_2} f(u_{k+1})f(u_k)f(u_{k-1})\cdots f(u_1)du_1 du_2 \cdots du_k du_{k+1}.
\end{aligned}$$

Here we rewrite $u_{k+1} := x$. By mathematical induction, the proof of Lemma 1.4 is completed. □

Proof of Theorem 1.3: First, we shall derive Eq.(1.6). Let $\psi_{\tilde{\Gamma}_t}(\theta)$ denote the moment generating function of $\tilde{\Gamma}_t$. Then, by cumulant expansion,

$$\log \psi_{\tilde{\Gamma}_t}(\theta) = \kappa_1\theta + \frac{1}{2}\kappa_2\theta^2 + \frac{1}{6}\kappa_3\theta^3 + \cdots + \frac{1}{n!}\kappa_n\theta^n + \cdots ,$$

where κ_n is the nth cumulant of $\tilde{\Gamma}_t$. When $\theta = -1$, Eq.(1.6) is obtained.

Next, we consider nth moment of $\tilde{\Gamma}_t$:

$$\chi_n := \mathbb{E}^U\left[\tilde{\Gamma}_t^n \mid \mathcal{F}_{t^*}\right] = \mathbb{E}^U\left[\left(\int_{t^*}^t h_s ds\right)^n \,\middle|\, \mathcal{F}_{t^*}\right]. \tag{1.9}$$

Applying Lemma 1.4 to Eq.(1.9),

$$\begin{aligned}\mathbb{E}^U\left[\left(\int_{t^*}^t h_s ds\right)^n \,\middle|\, \mathcal{F}_{t^*}\right] &= \mathbb{E}^U\left[n!\int_{t^*}^t\int_{t^*}^{s_{n-1}}\cdots\int_{t^*}^{s_2} h_{s_n}h_{s_{n-1}}\cdots h_{s_1}ds_1ds_2\cdots ds_n \,\middle|\, \mathcal{F}_{t^*}\right] \\ &= n!\int_{t^*}^t\int_{t^*}^{s_{n-1}}\cdots\int_{t^*}^{s_2}\mathbb{E}^U\left[\prod_{k=1}^n h_{s_k} \,\middle|\, \mathcal{F}_{t^*}\right]ds_1ds_2\cdots ds_n.\end{aligned}$$

By Assumption 1.2,

$$\begin{aligned}\mathbb{E}^U\left[\prod_{k=1}^n h_{s_k} \,\middle|\, \mathcal{F}_{t^*}\right] &= \mathbb{E}^U\left[\prod_{k=1}^n h_0(s_k)\exp\{\mathbf{w}^\top \mathbf{X}_{s_k}\} \,\middle|\, \mathcal{F}_{t^*}\right] \\ &= \prod_{k=1}^n h_0(s_k)\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^n \mathbf{w}^\top \mathbf{X}_{s_k}\right\} \,\middle|\, \mathcal{F}_{t^*}\right].\end{aligned}$$

Therefore, Eq.(1.7) is obtained. The proof of Theorem 1.3 is completed. □

According to Theorem 1.3, the RMBS pricing problem is eventually reduced to evaluating the following equation:

$$\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^n \mathbf{w}^\top \mathbf{X}_{s_k}\right\} \,\middle|\, \mathcal{F}_{t^*}\right]. \tag{1.10}$$

Note that, when covariate vector $\mathbf{X}_t$ of the proportional hazard model is decomposed to some independent vectors, it is sufficient to evaluate Eq.(1.10) of each independent vector in $\mathbf{X}_t$. If a closed-form expression of Eq.(1.10) is obtained; RMBS, IO, and PO can be evaluated simultaneously by an appropriate numerical method of multiple integrals in Eq.(1.7). Although numerical multiple integrals are very time-consuming in general, numerical examples in the following will show that single or double integral, which can be calculated very quickly, is enough to obtain accurate RMBS prices and their sensitivity.

Corollary 1.5 *Suppose that the covariate vector $\mathbf{X}_t := (X_t^1, X_t^2, \cdots, X_t^m)$ of the proportional hazard model (1.5) is an m-dimensional Gaussian process under a forward neutral measure $\mathbb{Q}_U$. Then,*

$$\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^n \mathbf{w}^\top \mathbf{X}_{s_k}\right\} \,\middle|\, \mathcal{F}_{t^*}\right] = \exp\left\{\mu + \frac{v}{2}\right\},$$

where

$$\begin{aligned}\mu &:= \sum_{k=1}^{n}\sum_{j=1}^{m} w_j \mathbb{E}^U\left[X^j_{s_k} | \mathcal{F}_{t^*}\right],\\ v &:= \sum_{k_1,k_2}^{n}\sum_{j_1,j_2}^{m} w_{j_1}w_{j_2}\mathrm{Cov}^U\left[X^{j_1}_{s_{k_1}}, X^{j_2}_{s_{k_2}} \mid \mathcal{F}_{t^*}\right],\end{aligned}$$

and w_j *($j = 1, 2, \cdots, m$) is the* j*th component of coefficient vector* $\mathbf{w}$.

1.3 Stochastic Proportional Hazard Models

In this section, we demonstrate that Theorem 1.3 is applicable to various types of the proportional hazard models. Presenting four examples, we shall show explicit expressions of Eq.(1.10). These examples deal with the stochastic proportional hazard model not only with Gaussian processes, but also with Lévy processes. In the following, we set current time $t^* = 0$ to evaluate RMBS without loss of generality.

Example 1.6 (Multi-Dimensional OU Process) Let $(\mathbf{X}_t)_{t\geq 0}$ be an m-dimensional OU process under a spot neutral measure $\mathbb{Q}$, that is, the covariate vector $\mathbf{X}_t = (X^1_t, X^2_t, \cdots, X^m_t)$ of the proportional hazard model (1.5) is given by:

$$dX^j_t = (\phi_j(t) - a_j X^j_t)dt + \mathbf{b}_j^\top d\mathbf{W}_t, \quad j = 1, 2, \cdots, m, \tag{1.11}$$

where $X^1_t := r_t$, $(\mathbf{W}_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion under $\mathbb{Q}$, $\phi_j(\cdot)$ is a deterministic function with respect to time t, $\mathbf{b}_j$ is $\mathbf{R}^d$-constant vector, and a_j is constant. Note that the spot rate process $(r_t)_{t\geq 0}$ is well-known as the Hull-White model (Hull and White [1990]).

Under a forward neutral measure $\mathbb{Q}_U$, SDE (1.11) is transformed into

$$dX^j_t = (\xi^U_j(t) - a_j X^j_t)dt + \mathbf{b}_j^\top d\mathbf{W}^U_t, \quad j = 1, 2, \cdots, m, \tag{1.12}$$

where

$$\xi^U_j(t) \quad := \quad \phi_j(t) - \frac{1 - e^{-a_j(U-t)}}{a_j}\mathbf{b}_1^\top \mathbf{b}_j,$$

and $(\mathbf{W}^U_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion under $\mathbb{Q}_U$. Solving Eq.(1.12), we obtain the following Gaussian processes:

$$X^j_t = X^j_0 e^{-a_j t} + \int_0^t \xi^U_j(s)e^{-a_j(t-s)}ds + \mathbf{b}_j^\top \int_0^t e^{-a_j(t-s)}d\mathbf{W}^U_s, \quad j = 1, 2, \cdots, m. \tag{1.13}$$

Since it is obvious that the covariate vector $\mathbf{X}_t$ is an m-dimensional Gaussian process under $\mathbb{Q}^U$, and

$$\mathbb{E}^U\left[X^j_t\right] = X^j_0 e^{-a_j t} + \int_0^t \xi^U_j(s)e^{-a_j(t-s)}ds,$$

and as $t_1 \geq t_2$,

$$\begin{aligned}\mathrm{Cov}^U\left[X^{j_1}_{t_1}, X^{j_2}_{t_2}\right] &= \mathbf{b}_{j_1}^\top \mathbf{b}_{j_2}\mathbb{E}^U\left[\left(\int_0^{t_1} e^{-a_{j_1}(t_1-s)}d\mathbf{W}^U_s\right)^\top \left(\int_0^{t_2} e^{-a_{j_2}(t_2-s)}d\mathbf{W}^U_s\right)\right]\\ &= \mathbf{b}_{j_1}^\top \mathbf{b}_{j_2}\int_0^{\min\{t_1,t_2\}} e^{-a_{j_1}(t_1-s)-a_{j_2}(t_2-s)}ds\\ &= \frac{\mathbf{b}_{j_1}^\top \mathbf{b}_{j_2}}{a_{j_1}+a_{j_2}}\left(e^{-a_{j_1}(t_1-t_2)} - e^{-(a_{j_1}t_1 + a_{j_2}t_2)}\right),\end{aligned}$$

the closed-form solution of Eq.(1.10) can be obtained from Corollary 1.5.

In the following, we consider a component X_t in the covariate vector $\mathbf{X}_t$ of the proportional hazard model, and we assume that X_t is independent of all other components of $\mathbf{X}_t$ and interest rate r_t. Under this assumption, we have only to calculate

$$\mathbb{E}^U\left[\exp\left\{w\sum_{k=1}^{n} X_{s_k}\right\}\right], \tag{1.14}$$

as regards $(X_t)_{t\geq 0}$, where w is a coefficient of X_t. Note that $(X_t)_{t\geq 0}$ is invariant under any forward neutral measures due to the independence assumption.

Example 1.7 (Function of a Gaussian process) Suppose that $X_t := f(Y_t)$, where $f(\cdot)$ is a deterministic function and Y_t is a Gaussian process under a spot neutral measure $\mathbb{Q}$. Obviously, Eq.(1.14) is given by

$$\mathbb{E}^U\left[\exp\left\{w\sum_{k=1}^{n} X_{s_k}\right\}\right] = \int_{\mathbf{R}^n}\exp\left\{w\sum_{k=1}^{n} f(y_k)\right\}p_{s_1,s_2,\ldots,s_n}(y_1,y_2,\cdots,y_n)dy_1dy_2\cdots dy_n, \tag{1.15}$$

where $p_{s_1,s_2,\ldots,s_n}(y_1,y_2,\cdots,y_n)$ is the density of the n-dimensional random variable $(Y_{s_1},Y_{s_2},\ldots,Y_{s_n})$ with a normal distribution. For quick numerical calculation of the improper integral on the right hand side of Eq.(1.15), we can use the Gauss-Hermite quadrature rule.

Example 1.8 (Compound Poisson Process) Let $(X_t)_{t\geq 0}$ be a compound Poisson process under a spot neutral measure $\mathbb{Q}$, that is,

$$X_t := \sum_{j=1}^{N_t} Y_j,$$

where N_t is a Poisson process with intensity λ, and $Y_j,\ j=1,2,\cdots$, are i.i.d. random variables which are independent of N_t, and have the same characteristic function $\varphi_Y(\theta)$. To derive the closed-form solution of Eq.(1.14), we first see

$$\sum_{k=1}^{n} X_{s_k} = n(X_{s_1}-X_{s_0})+\cdots+(n-k+1)(X_{s_k}-X_{s_{k-1}})+\cdots+(X_{s_n}-X_{s_{n-1}}),$$

where we set $X_{s_0}=0$ for convention. Then, Eq.(1.14) is written as

$$\begin{aligned}\mathbb{E}^U\left[\exp\left\{w\sum_{k=1}^{n} X_{s_k}\right\}\right] &= \mathbb{E}^U\left[\exp\left\{w\sum_{k=1}^{n}(n-k+1)(X_{s_k}-X_{s_{k-1}})\right\}\right]\\ &= \prod_{k=1}^{n}\mathbb{E}^U\left[e^{w(n-k+1)(X_{s_k}-X_{s_{k-1}})}\right].\\ &= \prod_{k=1}^{n}\mathbb{E}^U\left[e^{w(n-k+1)X_{s_k-s_{k-1}}}\right]. \end{aligned}\tag{1.16}$$

The last equality is shown by the stationary increments property of the compound Poisson process X_t. By the Lévy-Khinchine formula,

$$\mathbb{E}^U\left[e^{\imath\theta X_{s_k-s_{k-1}}}\right] = \exp\left\{(s_k-s_{k-1})\lambda\left(\varphi_Y(\theta)-1\right)\right\}. \tag{1.17}$$

where $\imath := \sqrt{-1}$. Substituting Eq.(1.17) for Eq.(1.16) with $\theta = -\imath w(n-k+1)$, we can obtain

$$\mathbb{E}^U\left[\exp\left\{w\sum_{k=1}^{n} X_{s_k}\right\}\right] = \prod_{k=1}^{n}\exp\left\{(s_k - s_{k-1})\lambda\left(\varphi_Y(-\imath w(n-k+1)) - 1\right)\right\}. \tag{1.18}$$

Example 1.9 (Infinite Activity Lévy Process) Let $(X_t)_{t\geq 0}$ be a pure jump infinite activity Lévy process with a Lévy measure ν under a spot neutral measure $\mathbb{Q}$. By the Lévy-Itô decomposition, X_t can be represented as a sum of a compound Poisson process and an almost sure limit of compensated compound Poisson processes:

$$X_t = \sum_{s\leq t}\Delta X_s 1_{|\Delta X_s|\geq 1} + \lim_{\varepsilon\downarrow 0} N_t^{\varepsilon},$$

where

$$N_t^{\varepsilon} = \sum_{s\leq t}\Delta X_s 1_{\varepsilon\leq|\Delta X_s|<1} - t\int_{\varepsilon\leq|x|<1} x\nu(dx).$$

Therefore, X_t can be approximated by

$$X_t^{\varepsilon} = \sum_{s\leq t}\Delta X_s 1_{|\Delta X_s|\geq 1} + N_t^{\varepsilon},$$

and the residual term is given by

$$R_t^{\varepsilon} = -N_t^{\varepsilon} + \lim_{\delta\downarrow 0} N_t^{\delta},$$

which is a pure jump Lévy process with Lévy measure $1_{|x|\leq\varepsilon}\nu(dx)$ satisfying $\mathbb{E}^U[R_t^{\varepsilon}] = 0$. In the finite variation case, the process $(X_t^{\varepsilon})_{t\geq 0}$ can be written as

$$X_t^{\varepsilon} = \sum_{s\leq t}\Delta X_s 1_{\varepsilon\leq|\Delta X_s|} + \mathbb{E}^U\left[\sum_{s\leq t}\Delta X_s 1_{|\Delta X_s|<\varepsilon}\right].$$

Since the approximation process $(X_t^{\varepsilon})_{t\geq 0}$ is composed of a compound Poisson process and a deterministic term, we can use Eq.(1.18) in Example 1.8 for deriving the closed-form solution of Eq.(1.14) with respect to X_t^{ε}. Moreover, for the sake of more accurate approximation, using

$$t\sigma^2(\varepsilon) := \mathrm{Var}^U[R_t^{\varepsilon}] = t\int_{|x|<\varepsilon} x^2\nu(x)dx,$$

X_t can be approximated by

$$\hat{X}_t^{\varepsilon} := X_t^{\varepsilon} + \sigma(\varepsilon)B_t,$$

where B_t is an independent Brownian motion. See pp.184-192 in Cont and Tankov [2004] for details. Since the infinite activity Lévy process can be approximately decomposed into a compound Poisson process and a Brownian motion, which are mutually independent; we can also obtain the closed-form solution of Eq.(1.14) with respect to $\hat{X}_t^{\varepsilon}$ by applying Corollary 1.5 and Eq.(1.18) in Example 1.8 to the Brownian motion and the compound Poisson process respectively.

Remark 1.10 *In Example 1.9, because an infinite activity Lévy process is approximated to obtain an explicit expression of Eq.(1.14), there is the approximation error in RMBS price by the pricing formula. On the other hand, even if it is obtained by Monte Carlo, the same approximation scheme is usually implemented (see Cont and Tankov [2004]). Therefore, it can be said that the same error exists in RMBS price by Monte Carlo.*

1.4 Numerical Examples

This section provides two numerical examples: the accuracy test and the sensitivity calculations. First, we specify an interest rate model and the proportional hazard model for the numerical examples. The parameters of these models are set based on the Japanese RMBS market. Second, by comparing RMBS, IO, and PO prices by the pricing formula with the exact prices, we examine the accuracy of the first and second order cumulant expansion prices. The result implies that the prices obtained by the second order cumulant expansion are sufficient in practice. Third, as an application of the pricing formula, we calculate effective duration and convexity of RMBS, IO, and PO. It is shown that the formula is very useful to compute these values accurately and quickly.

1.4.1 Model Specification

First, we specify spot rate dynamics $(r_t)_{t\geq 0}$ using the Hull-White model (Hull and White [1990]):

$$dr_t = (\phi(t) - ar_t)dt + \sigma dW_t,$$

where $a, \sigma > 0$ are constant, and $\phi(t)$ is a deterministic function with respect to time t. Next, we suppose that initial yield curves are given by the augmented Nelson-Siegel model (Björk and Christensen [1999]):

$$f(0,t) := z_1 + z_2 \exp(-at) + z_3 t \exp(-at) + z_4 \exp(-2at),$$

where $f(0,t)$ denotes an instantaneous t-forward rate at time 0, and z_k, $k = 1, 2, 3, 4$, are parameters. It is well-known that the augmented Nelson-Siegel model is consistent with the Hull-White model (see Björk and Christensen [1999]). By trivial calculations, we obtain

$$\phi(t) = az_1 + z_3 \exp(-at) - az_4 \exp(-2at) + \sigma^2 \frac{1 - \exp(-2at)}{2a},$$

and

$$r_0 = f(0,0) = z_1 + z_2 + z_4.$$

The parameters of the Hull-White model are set $a = 0.2$ and $\sigma = 0.008$, which are taken from the numerical example in Sugimura [2004]. In the accuracy test, the augmented Nelson-Siegel model is calibrated to the JGB yield curve as of May 9, 2008; i.e. $z_1 = 0.0308$, $z_2 = -0.0874$, $z_3 = 0.0049$, and $z_4 = 0.0725$. On the other hand, we suppose flat yield curves with parallel shift in the sensitivity calculation; i.e. $z_1 = r_0$ and $z_2 = z_3 = z_4 = 0$. These parameters are listed in Table 1.1 and 1.2.

Next, we specify the proportional hazard model as follows:

$$h_t = h_0(t) \exp\{wX_t\} := \lambda(1 - \exp(-\gamma t)) \exp\{w(R - r_t)\},$$

where $h_0(t) := \lambda(1 - \exp(-\gamma t))$, $X_t := R - r_t$, and λ, γ, R are constant.

In the Japanese RMBS market, a standard prepayment model called PSJ model is presented as a benchmark of prepayment rate by Japan Securities Dealers Association. PSJ model is a simple deterministic function with respect to time t. It rises in a linear fashion for 60 months to a certain level, at which point it remains constant. The base-line hazard function is calibrated to PSJ model; i.e. $\lambda = 0.0614$ and $\gamma = 0.3607$. Thus, we have set up the proportional hazard model if the interest rate does not move, then it behaves like PSJ model (see Figure 1.1).

When w is positive, if the interest rate goes down then the prepayment rate goes up and vice versa. Therefore, the parameter w can be interpreted as prepayment sensitivity with respect to interest rate shift. In the accuracy test, we set $R = r_0$ and $w = 3, 5, 10, 20, 30, 50$ to examine the impact of the prepayment sensitivity. Note that the proportional hazard model approaches a linear function with respect to r_t when w is small enough. In this case, the proportional hazard model can be approximated to a Gaussian function, which gives us a closed-form pricing formula for RMBS (see Yamazaki [2005]). This fact implies that the larger w is, the larger approximation errors are. For RMBS issued by Japan Housing Finance Agency (hereafter, JHFA), market consensus of prepayment sensitivity to interest rate is presented by Bloomberg. Figure 1.2 plots prepayment sensitivity curves in the case of the accuracy test against prepayment consensus of May 9, 2008 on the 13th RMBS issued by JHFA. On the other hand, we set $R = 0.05$ and $w = 20$ in the sensitivity calculations. The parameters of the proportional hazard model are also listed in Tables 1.1 and 1.2.

1.4.2 Accuracy Test

By the pricing formula, we compute RMBS, IO, and PO prices, which are set to the same condition as the 13th RMBS issued by JHFA; i.e. face value = 100, $m = 1/12$, $c = 0.0216$, $T = 35$. We regard the prices by Monte Carlo with 10^7 sample paths as the exact prices (hereafter, EP). In order to verify the accuracy of the pricing formula, we compare the first and second order cumulant expansion prices (hereafter, CEP1 and CEP2 respectively) with EP. Moreover, we realize very quick computation of the second order RMBS prices (hereafter, CEP2*) by a simple acceleration scheme introduced by Shibasaki and Nakamura [2001]. That is, we do not compute the monthly cash flow value for each month, but we compute a specific monthly cash flow value per year and interpolate into the interim values by using the spline function. By virtue of the acceleration scheme, the number of the cash flow valuations can be reduced from $35 \times 12 = 420$ to 35, and the computation speed can be dramatically improved. Figures 1.3-1.5 plot with the spline interpolation each cash flow value of RMBS, IO, and PO, respectively.

Tables 1.3 and 1.4 show RMBS, IO and PO prices, errors:=EP$-$CEP, and error ratios:=100$\times$errors/EP. In the case of CEP1, although the errors are small when w is small, they might not be ignored when w is large. In contrast, in the case of CEP2 and CEP2*, even when $w = 50$, a stress scenario of prepayment behavior, the error of RMBS price is only 0.007 and the errors of IO and PO prices are less than 0.05. According to market consensus of prepayment sensitivity of May 9, 2008; w is nearly 10 in the Japanese RMBS market, and the errors of CEP2 and CEP2* with $w = 10$ are 0.002 or below. As a result of the accuracy test, it can be said that CEP2 and CEP2* are substantially accurate in practice. In particular, CEP2* is very practical in terms of computation time.

1.4.3 Sensitivity Calculations

As an application of the pricing formula, we compute effective duration and convexity of RMBS, IO and PO by CEP2*. Effective duration (hereafter, ED) and convexity (hereafter, EC) of RMBS are defined as follows: Let $V(\Delta y)$ be a RMBS price with Δy-yield curve shift at time 0. Then,

$$\text{ED} := \frac{V(-\Delta y) - V(\Delta y)}{2\Delta y V(0)},$$

and

$$\text{EC} := \frac{V(-\Delta y) - 2V(0) + V(\Delta y)}{(\Delta y)^2 V(0)}.$$

Here we set $\Delta y = 0.001$ (10bp). ED and EC of IO and PO are defined in the same manner.

In the sensitivity calculations, we need higher interest rate environments than these of the current Japanese interest rate market. We set $c = 0.100$, $r_0 = 0.00 \sim 0.100$ (see Table 1.2 for other parameters). This is because negative convexity effects can not be observed clearly in the Japanese RMBS market due to the low interest rate environment. In order to demonstrate the usefulness of the pricing formula, we compare ED and EC by CEP2* with these by Monte Carlo with 10^4 sample paths, which is in merely tolerable for practical calculation time without any special techniques. In Monte Carlo, we adopt two types of sensitivity computation methods; that is, the Monte Carlo difference method with independent sample paths (hereafter, MCI) and fixed sample paths (hereafter, MCF).

Figures 1.6-1.8 plot RMBS, IO, and PO prices respectively against interest rate shift. Negative convexity effects can be seen on the left-hand side in Figures 1.6 and 1.8. In Figure 1.7 they are observed everywhere. All prices in Figures 1.6-1.8 seem to be very stable and smooth. However, the prices by CEP2* are more accurate than these by MCI and MCF. The Monte Carlo prices with 10^4 paths generate 0.023% absolute error ratio against EP in average. In contrast, the absolute error ratios of CEP2* are less than 0.003%.

Figures 1.9-1.11 plot effective duration of RMBS, IO, and PO respectively; and Figures 1.12-1.14 plot their effective convexity. Note that all effective duration curves of MCI are jagged and the effective convexity is highly unstable due to simulation error by Monte Carlo. Obviously, MCI is useless for sensitivity computations in practice. On the other hand, Figures 1.9-1.14 show that CEP2* and MCF are able to draw stable and smooth ED and EC curves. However, in terms of accuracy, the sensitivity calculations by CEP2* are more appropriate than these by MCF. As a result of the numerical example, it can be said that CEP2* is a very powerful tool for RMBS valuation.

1.5 Concluding Remarks

This chapter presents a general pricing formula for RMBS with the proportional hazard model. Since the assumption that prepayment behavior follows the proportional hazard model has empirical evidence, the formula seems to be more appropriate for practitioners than that in Yamazaki [2005] or Gorovoy and Linetsky [2007]. In addition, it is shown that the formula is applicable to the proportional hazard models not only with Gaussian processes, but also with Lévy processes. Numerical examples demonstrate that the accuracy of RMBS prices by the formula is remarkably better than sufficient, even when the second order approximation is adopted. Moreover, the formula is useful to calculate the effective durations and convexities very quickly.

Finally, our next research topic will be to establish more general pricing formulas for RMBS with other typical prepayment models such as the generalized additive model (see Jegadeesh and Ju [2000]) and the Poisson regression model (see Schwartz and Torous [1993]).

Table 1.1: RMBS Setting and Model Parameters in the Accuracy Test

RMBS	face value	coupon	amortization	maturity
	100	0.0216	monthly	35
proportional hazard model	λ	γ	w	R
	0.0614	0.3607	3∼50	0.0159
Hull-White model	r_0	a	σ	
	0.0159	0.2000	0.0080	
augmented Nelson-Siegel model	z_1	z_2	z_3	z_4
	0.0308	-0.0874	0.0049	0.0725

Table 1.1 shows RMBS setting and the model parameters for the accuracy test in the numerical examples. RMBS setting is the same as the 13th RMBS issued by JHFA. See Figure 1.1 and 1.2 for the parameters of the proportional hazard model. The augmented Nelson-Siegel model is calibrated to JGB yield curve as of May 9, 2008. The parameters of the Hull-White model are taken from Sugimura [2004].

Table 1.2: RMBS setting and Model Parameters in the Sensitivity Calculations

RMBS	face value	coupon	amortization	maturity
	100	0.100	monthly	35
proportional hazard model	λ	γ	w	R
	0.0614	0.3607	20	0.05
Hull-White model parameters	r_0	a	σ	
	0.000∼0.100	0.2000	0.0080	
augmented Nelson-Siegel model	z_1	z_2	z_3	z_4
	r_0	0.0000	0.0000	0.0000

Table 1.2 shows RMBS setting and and the model parameters for the sensitivity calculations in the numerical examples. The setting in Table 1.2 is higher interest rate environment than in Table 1.1 to generate negative convexity effect clearly. Note that the augmented Nelson-Siegel parameters imply flat yield curves.

Table 1.3: RMBS, IO and PO Prices in the Case of $w = 3, 5, 10$

$w = 3$	RMBS			IO			PO		
	price	error	error ratio	price	error	error ratio	price	error	error ratio
EP	100.855			21.997			78.858		
CEP1	100.854	0.001	0.001	21.994	0.003	0.015	78.860	-0.002	-0.003
CEP2	100.853	0.001	0.001	21.995	0.002	0.008	78.858	0.000	0.000
CEP2*	100.853	0.001	0.001	21.995	0.002	0.007	78.858	0.000	0.000

$w = 5$	RMBS			IO			PO		
	price	error	error ratio	price	error	error ratio	price	error	error ratio
EP	100.783			22.068			78.715		
CEP1	100.783	0.000	0.000	22.062	0.006	0.027	78.721	-0.006	-0.007
CEP2	100.781	0.001	0.001	22.066	0.002	0.008	78.715	0.000	0.000
CEP2*	100.781	0.001	0.001	22.066	0.002	0.007	78.715	0.000	0.000

$w = 10$	RMBS			IO			PO		
	price	error	error ratio	price	error	error ratio	price	error	error ratio
EP	100.609			22.229			78.380		
CEP1	100.613	-0.004	-0.004	22.211	0.018	0.079	78.401	-0.021	-0.027
CEP2	100.608	0.001	0.001	22.227	0.002	0.008	78.381	0.000	0.000
CEP2*	100.608	0.001	0.001	22.227	0.002	0.007	78.381	0.000	0.000

Table 1.3 shows RMBS, IP and PO prices, errors, and error ratios in the case of $w = 3, 5, 10$. We regard the prices computed by Monte Carlo with 10^7 sample paths as the exact prices denoted by EP. CEP1 and CEP2 denote the prices by the first and second order cumulant expansion respectively. CEP2* is the prices by the second order cumulant expansion with the spline interpolation. Here we define error:=EP−CEP and error ratio:= 100×error/EP.

Table 1.4: RMBS, IO and PO Prices in the Case of $w = 20, 30, 50$

$w = 20$	RMBS			IO			PO		
	price	error	error ratio	price	error	error ratio	price	error	error ratio
EP	100.292			22.478			77.813		
CEP1	100.307	-0.015	-0.015	22.418	0.060	0.266	77.889	-0.076	-0.098
CEP2	100.290	0.002	0.002	22.478	0.001	0.003	77.812	0.001	0.001
CEP2*	100.290	0.002	0.002	22.478	0.000	0.002	77.812	0.001	0.001

$w = 30$	RMBS			IO			PO		
	price	error	error ratio	price	error	error ratio	price	error	error ratio
EP	100.013			22.631			77.383		
CEP1	100.044	-0.031	-0.031	22.502	0.129	0.569	77.542	-0.159	-0.205
CEP2	100.011	0.003	0.003	22.634	-0.004	-0.016	77.376	0.006	0.008
CEP2*	100.011	0.002	0.002	22.635	-0.004	-0.018	77.376	0.006	0.008

$w = 50$	RMBS			IO			PO		
	price	error	error ratio	price	error	error ratio	price	error	error ratio
EP	99.561			22.655			76.906		
CEP1	99.623	-0.062	-0.063	22.298	0.357	1.576	77.325	-0.419	-0.545
CEP2	99.553	0.007	0.007	22.697	-0.042	-0.187	76.856	0.050	0.065
CEP2*	99.554	0.007	0.007	22.698	-0.043	-0.190	76.856	0.050	0.065

Table 1.4 shows RMBS, IO and PO prices, errors, and error ratios in the case of $w = 20, 30, 50$.

Figure 1.1: PSJ model and Base-Line Hazard Function

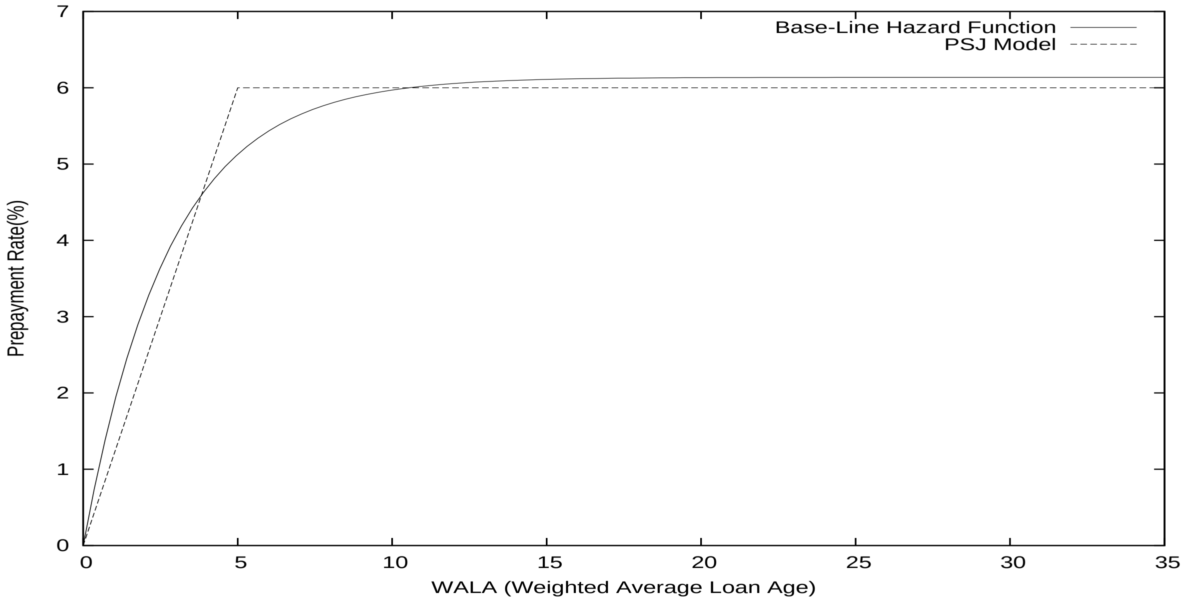


Figure 1.1 plots PSJ model and the base-line hazard function $h_0(t) := 0.0614(1 - \exp(-0.3607t))$. The base-line hazard function is calibrated to PSJ model. See *Prepayment Standard Japan Model Guide Book* published by Japan Securities Dealers Association [2006] for the definition of PSJ model.

Figure 1.2: Prepayment Sensitivity to Interest Rate Shift

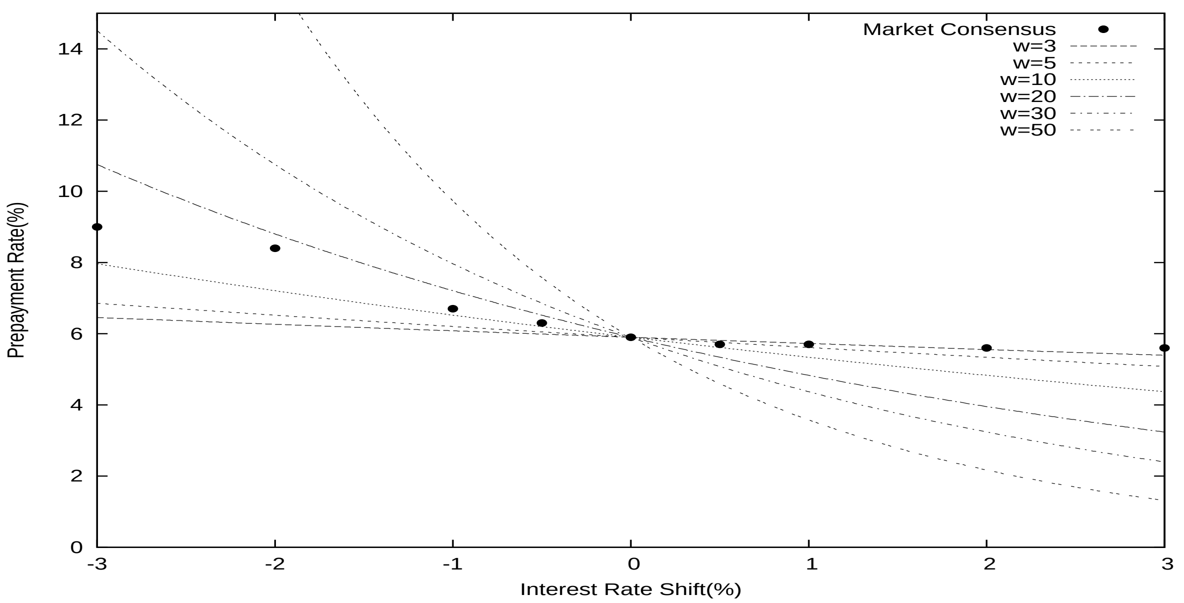


Figure 1.2 plots prepayment rates with respect to interest rate shift. The dots denote the prepayment rates of market consensus on the 13th RMBS issued by JHFA as of May 9, 2008, whose data are downloaded from Bloomberg. The lines are the exponential curve $h_0 e^{w\Delta r}$.

Figure 1.3: RMBS Cash Flow Values with Interpolation

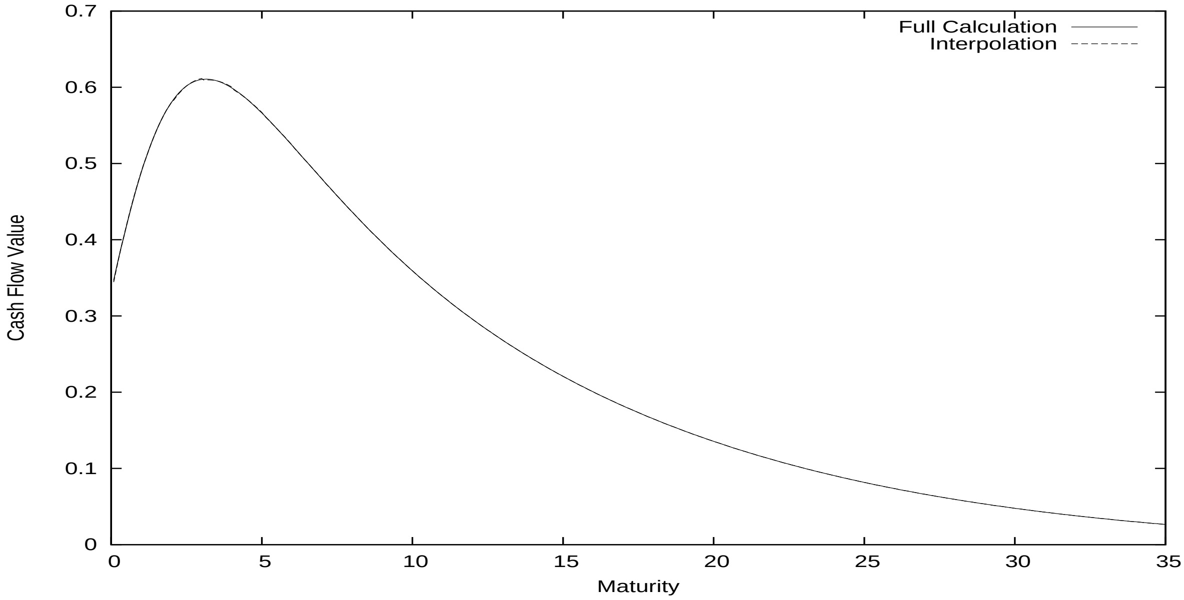


Figure 1.3 plots RMBS cash flow values at each maturity. The solid line denotes RMBS cash flow values without interpolation, in which we compute the monthly cash flow value for each month; i.e. $12 \times 35 = 420$ cash flows, by the second order cumulant expansion. The dash line denotes RMBS cash flow values with interpolation, in which we compute a specific monthly cash flow value per year; i.e. only 35 cash flows, by the same order expansion, and interpolate into the interim cash flow values by the spline function.

Figure 1.4: IO Cash Flow Values with Interpolation

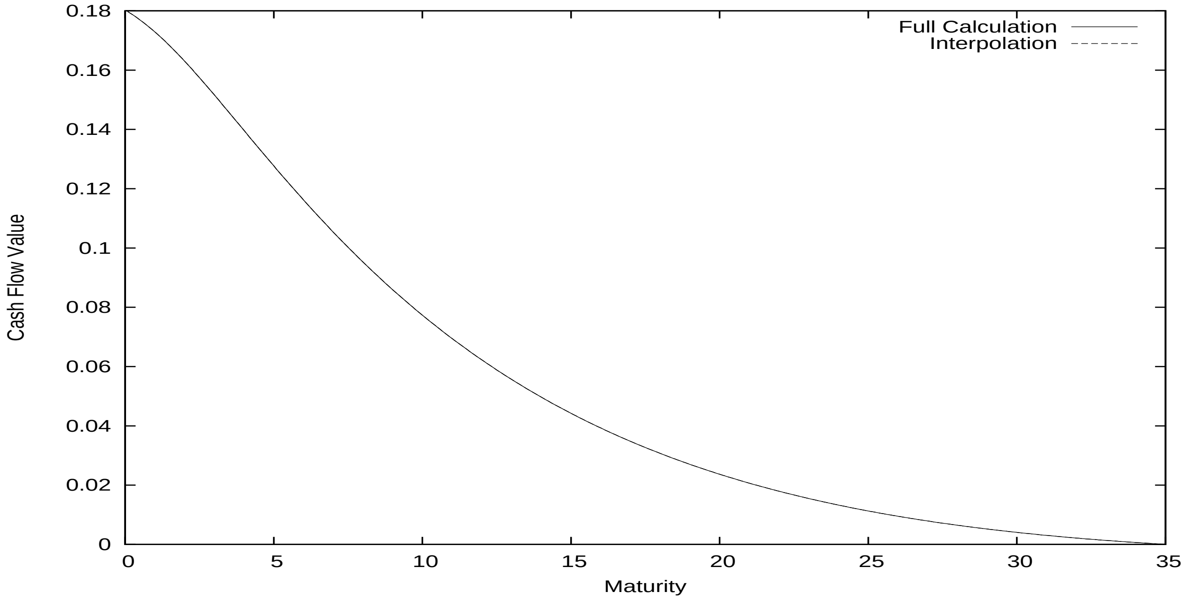


Figure 1.4 plots IO cash flow values at each maturity. The solid line denotes IO cash flow values without interpolation and the dash line denotes IO cash flow values with interpolation. These calculation methods are the same manners as in Figure 1.3.

Figure 1.5: PO Cash Flow Values with Interpolation

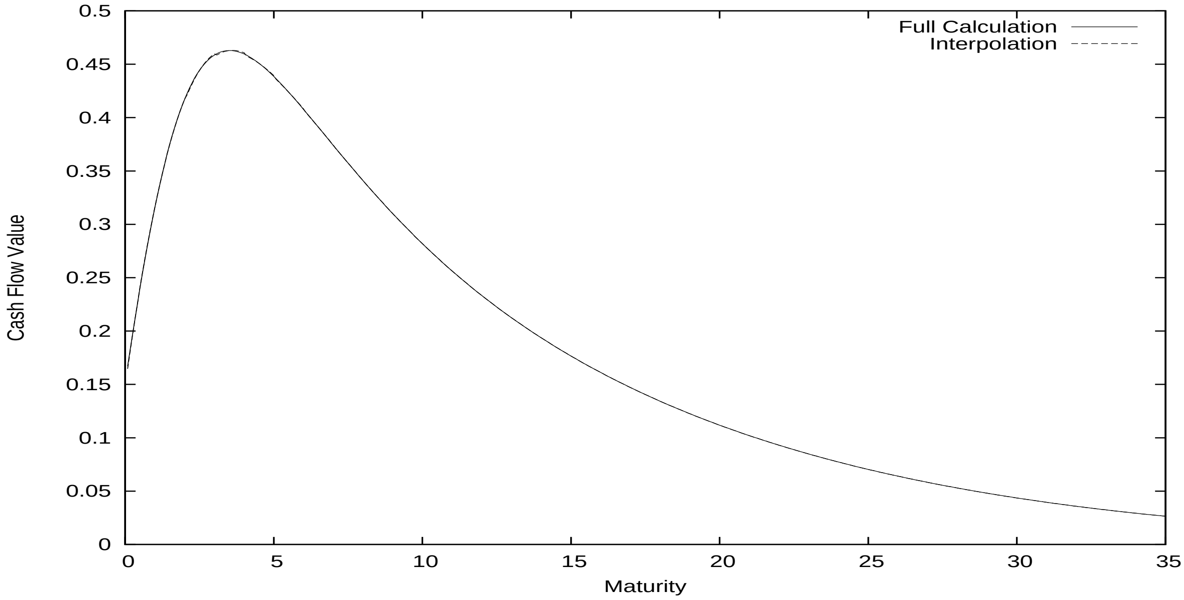


Figure 1.5 plots PO cash flow values at each maturity. The solid line denotes PO cash flow values without interpolation and the dash line denotes PO cash flow values with interpolation. These calculation methods are the same manners as in Figure 1.3.

Figure 1.6: RMBS Price

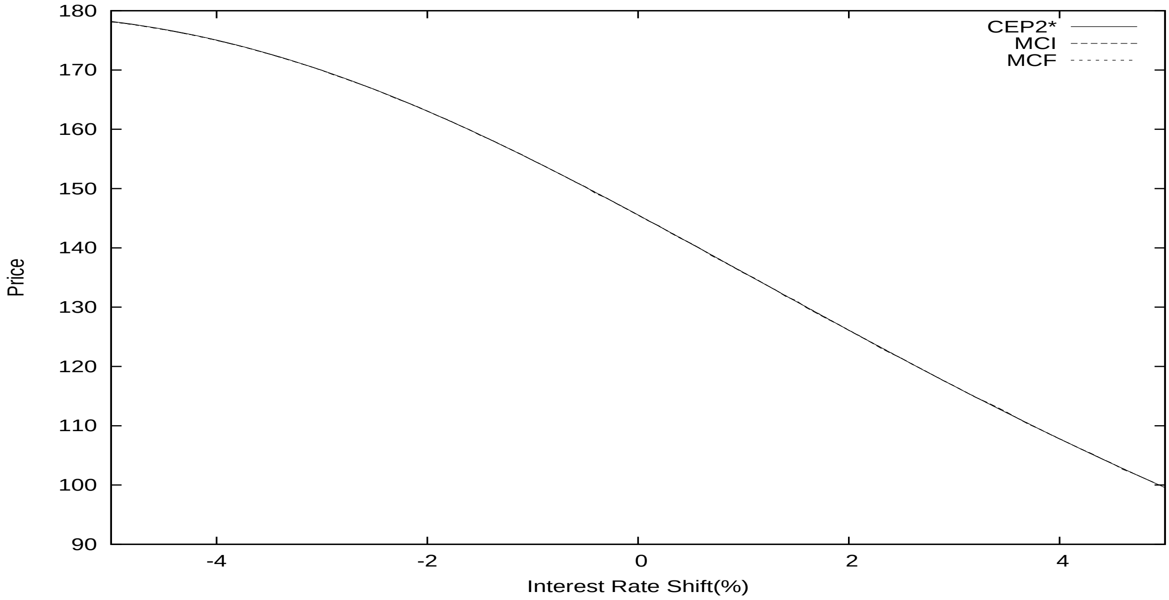


Figure 1.6 plots RMBS prices. The solid line denotes RMBS prices by the second order cumulant expansion with interpolation denoted by CEP2*. The dash line denotes RMBS prices by Monte Carlo with independent 10^4 sample paths denoted by MCI. The dotted line denotes RMBS prices by Monte Carlo with fixed 10^4 sample paths denoted by MCF.

Figure 1.7: IO Price

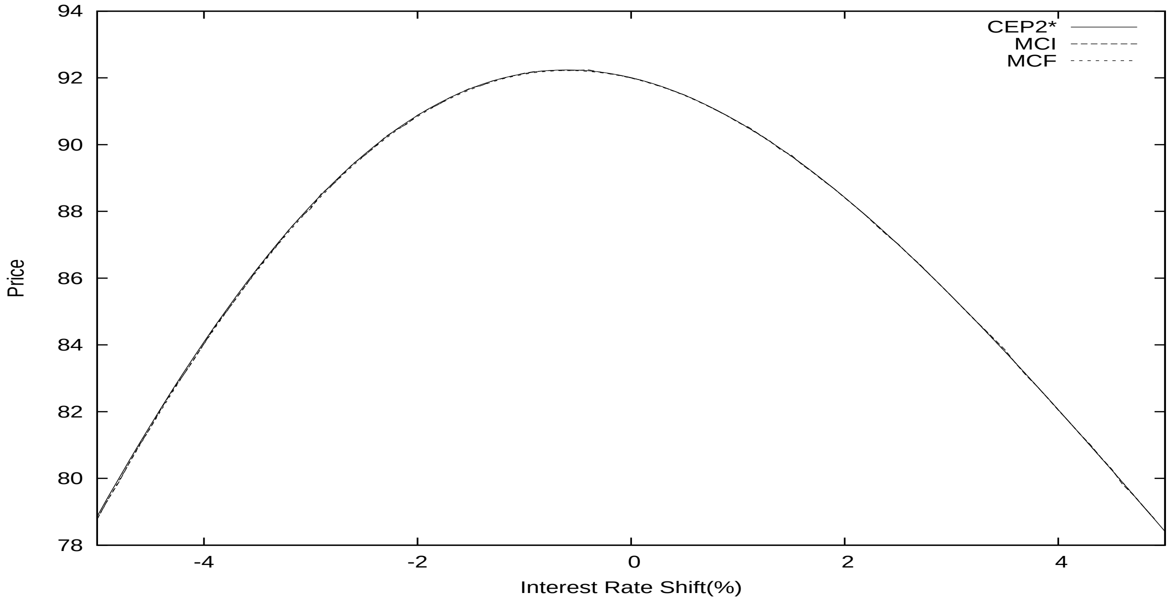


Figure 1.7 plots IO prices. The solid line denotes IO prices by CEP2*. The dash line denotes IO prices by MCI. The dotted line denotes IO prices by MCF.

Figure 1.8: PO Price

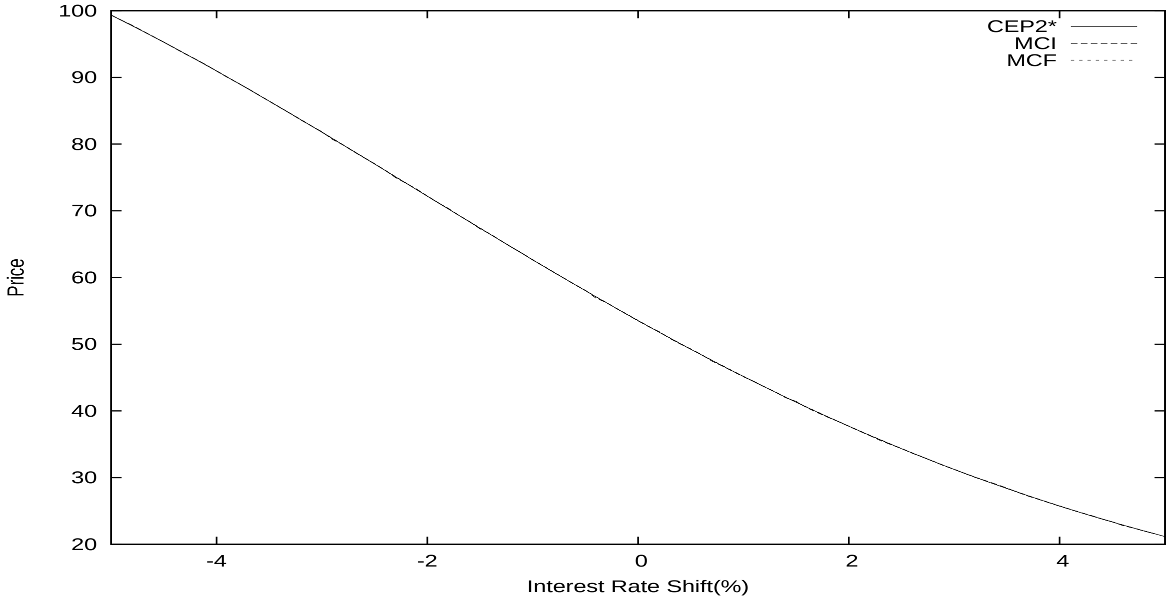


Figure 1.8 plots PO prices. The solid line denotes PO prices by CEP2*. The dash line denotes PO prices by MCI. The dotted line denotes PO prices by MCF.

Figure 1.9: Effective Duration of RMBS

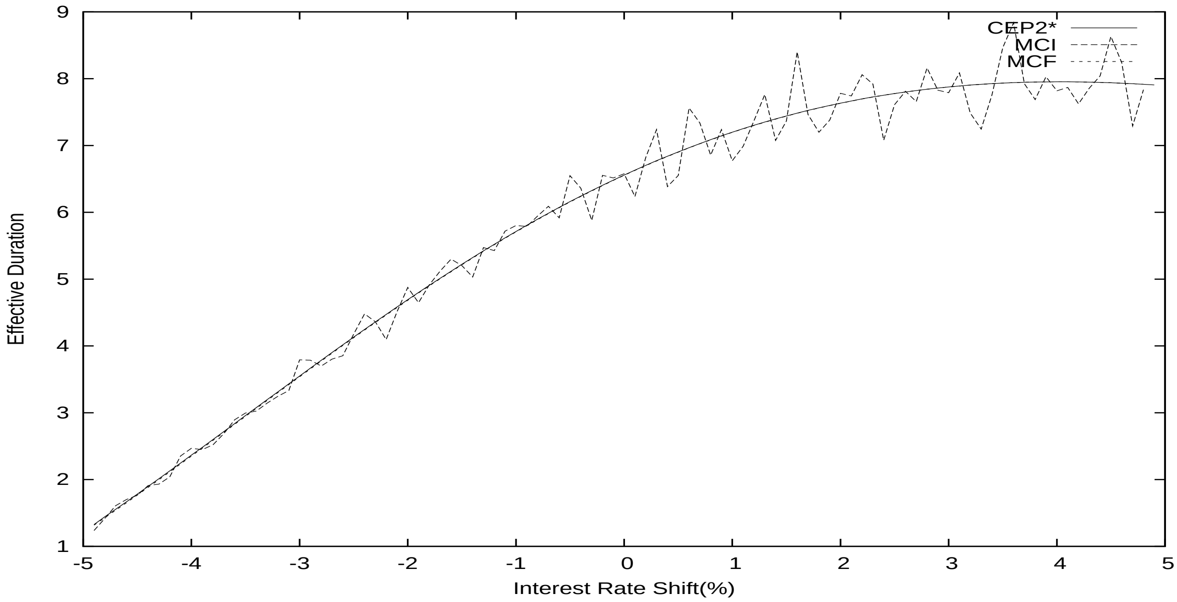


Figure 1.9 plots effective duration of RMBS. The effective duration is defined as

$$\mathrm{ED} := \frac{V(-\Delta y) - V(\Delta y)}{2\Delta y V(0)}.$$

The solid line denotes the effective duration of RMBS by CEP2*. The dash line denotes the effective duration of RMBS by MCI. The dotted line denotes the effective duration of RMBS by by MCF.

Figure 1.10: Effective Duration of IO

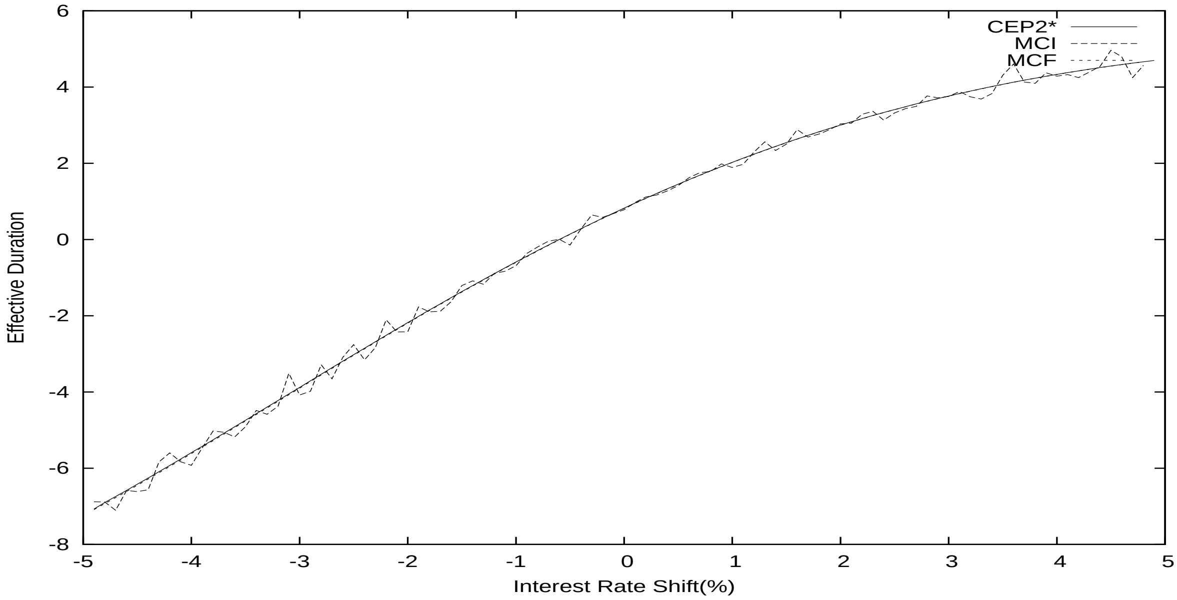


Figure 1.10 plots effective duration of IO. The solid line denotes the effective duration of IO by CEP2*. The dash line denotes the effective duration of IO by MCI. The dotted line denotes the effective duration of IO by MCF. The definition of IO effective duration is the same as of RMBS.

Figure 1.11: Effective Duration of PO

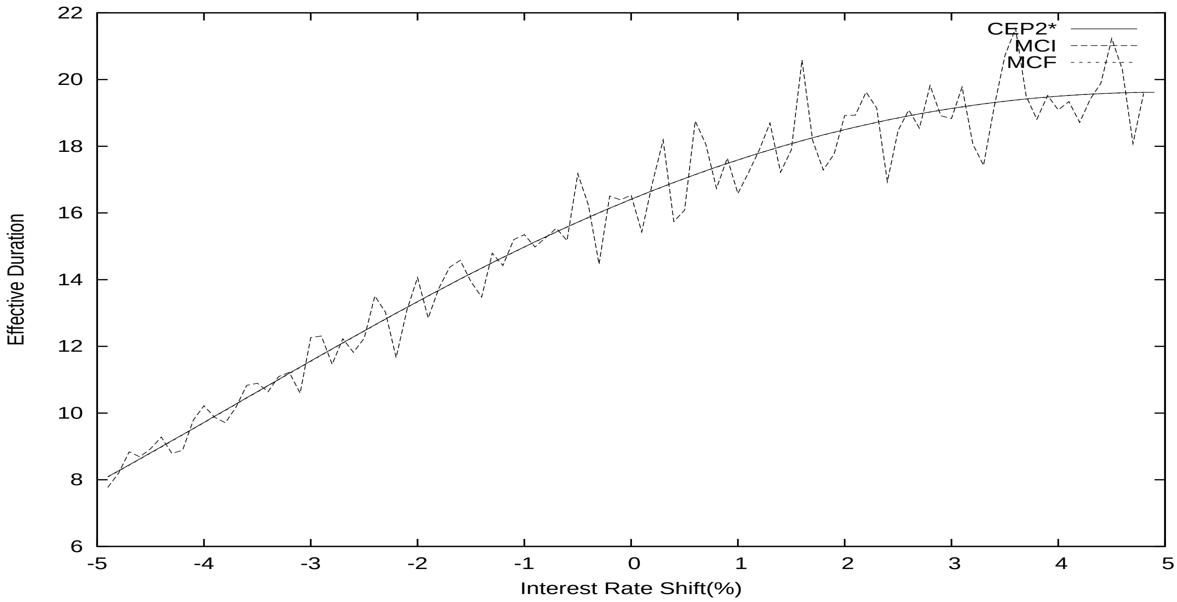


Figure 1.11 plots effective duration of PO. The solid line denotes the effective duration of PO by CEP2*. The dash line denotes the effective duration of PO by MCI. The dotted line denotes the effective duration of PO by MCF. The definition of PO effective duration is the same as of RMBS.

Figure 1.12: Effective Convexity of RMBS

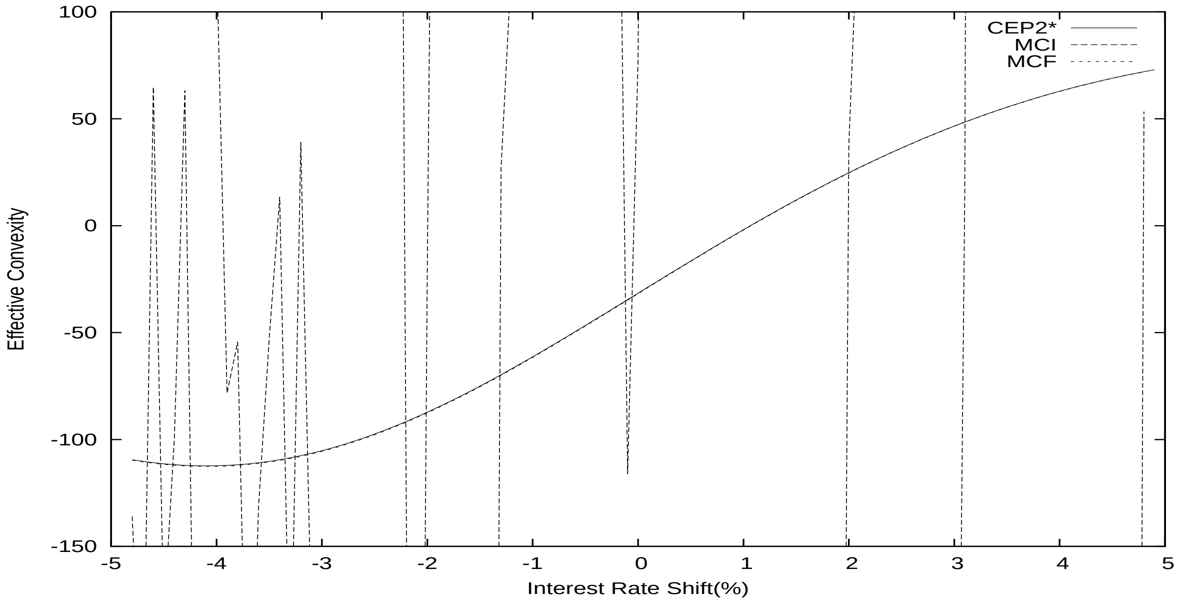


Figure 1.12 plots effective convexity of RMBS. The effective convexity is defined as

$$\mathrm{EC} := \frac{V(-\Delta y) - 2V(0) + V(\Delta y)}{(\Delta y)^2 V(0)}.$$

The solid line denotes the effective convexity of RMBS by CEP2*. The dash line denotes the effective convexity of RMBS by MCI. The dotted line denotes the effective convexity of RMBS by MCF.

Figure 1.13: Effective Convexity of IO

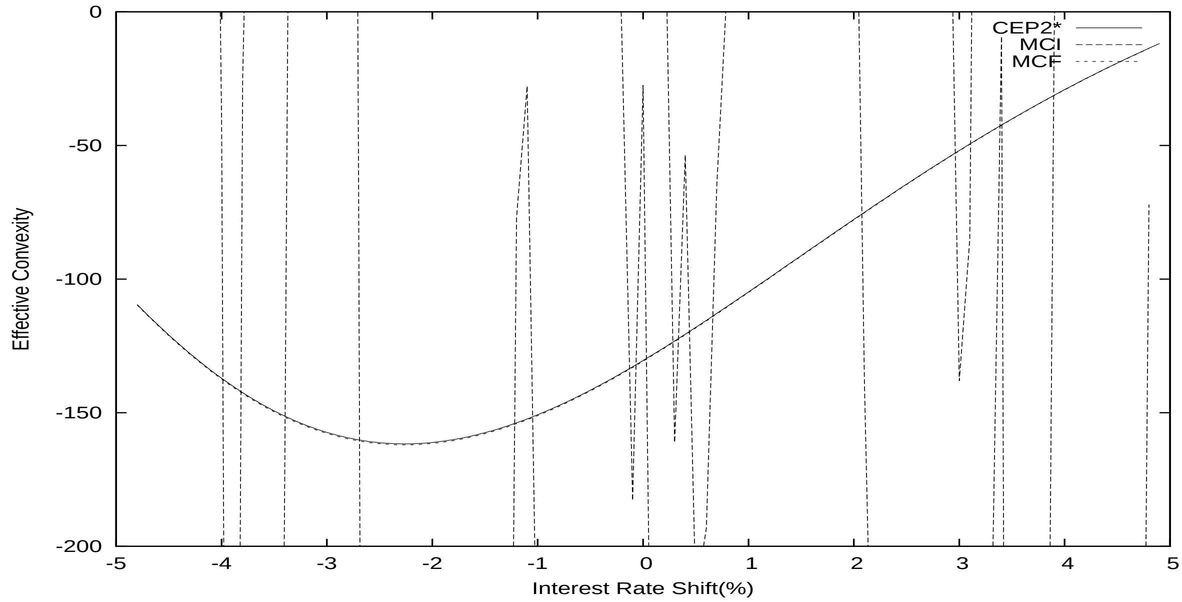


Figure 1.13 plots effective convexity of IO. The solid line denotes the effective convexity of IO by CEP2*. The dash line denotes the effective convexity of IO by MCI. The dotted line denotes the effective convexity of IO by MCF. The definition of IO effective convexity is the same as of RMBS.

Figure 1.14: Effective Convexity of PO

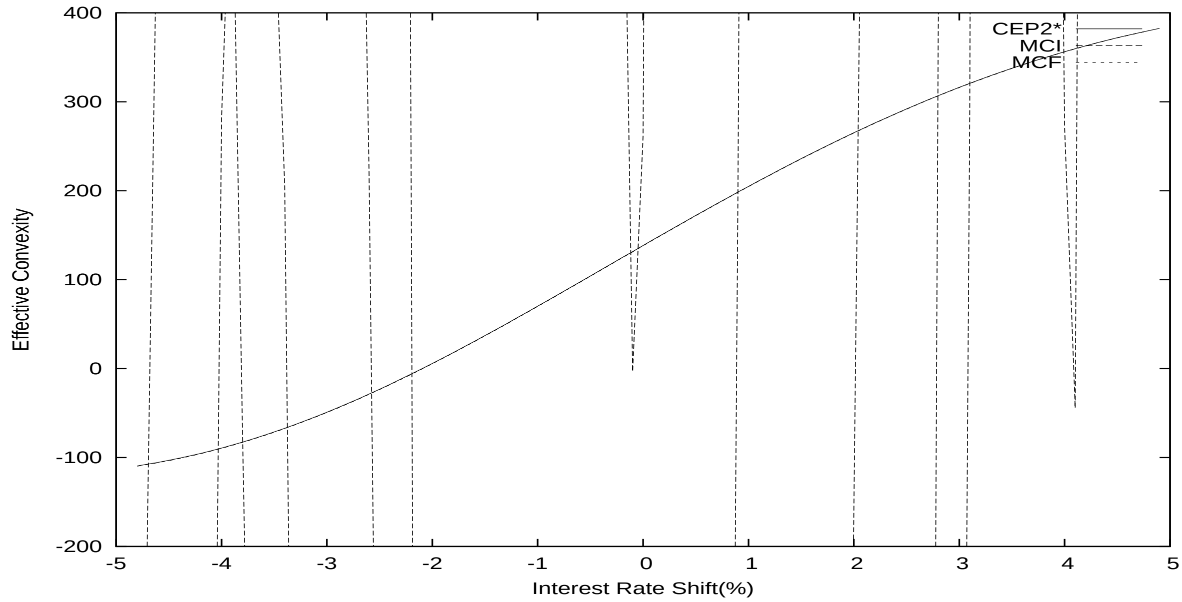


Figure 1.14 plots effective convexity of PO. The solid line denotes the effective convexity of PO by CEP2*. The dash line denotes the effective convexity of PO by MCI. The dotted line denotes the effective convexity of PO by MCF. The definition of PO effective convexity is the same as of RMBS.

Chapter 2

Pricing Average Options under Time-Changed Lévy Processes

The payoff of an average option depends on the arithmetic average value of the underlying asset price over a given time period. This chapter presents an analytic formula for pricing average options when the underlying asset price is driven by time-changed Lévy processes. We demonstrate that our formula gives an accurate approximation of the average option prices with fast computation.

Recently, it is a necessary and an important task to evaluate exotic derivatives based on calibration to liquid plain-vanilla option prices. In order to capture time series of underlying asset prices and reproduce implied volatilities in plain-vanilla option market, Carr et al. [2003], and Carr and Wu [2004] propose time-changed Lévy processes that are Lévy processes with stochastic time change as a driving factor of the underlying asset prices. Time-change Lévy processes are attractive from a practical point of view because they provide a flexible framework for generating jumps, capturing stochastic volatility as the random time change, and introducing the leverage effect. It is well-known that the class of time-changed Lévy processes has a wide variety of stochastic processes including pure Lévy processes and traditional stochastic volatility models. In addition, the processes are tractable to compute prices of European derivatives such as plain-vanilla and digital options. There have also been applications of these processes to volatility derivatives; for example, see Itkin and Carr [2010], and Carr et al. [2011]. However, in existing literature, analytic formulas for pricing path-dependent exotic options written on the underlying asset driven by time-changed Lévy processes were rarely obtained. Since it is never a trivial task to apply finite different/element methods to valuation of the exotic options on the processes, pricing these options should usually rely on Monte Carlo simulation. Straightforward application of Monte Carlo simulation is significantly time-consuming or/and produces only inaccurate estimates. Therefore, there is surely a strong need to develop some sophisticated technique of exotic derivative pricing to satisfy practical requirements. Alternatively, if we can obtain an analytic formula giving the accurate and fast-computing approximation prices, it becomes very useful.

Several approaches have been attempted to obtain pricing formulas for average options under the Black-Scholes model; for instance, see Levy [1992], Geman and Yor [1993], Curran [1994], Rogers and Shi [1995], Vecer [2001], Linetsky [2004]. As for analytic pricing methods of average options with stochastic volatility models and exponential Lévy models, there have been some works. As an example of stochastic volatility environment, Fouque and Han [2003] have derived asymptotic solutions to arithmetic average options by using a perturbation technique of a partial differential equation under a certain class of fast mean-reverting stochastic volatility models. Wong and Cheung [2004] extended the pricing method in Fouque and Han [2003] to geometric average options. Shiraya and Takahashi [2011] presented an asymptotic expansion formula based on Malliavin calculus for pricing average options on commodities under the Heston and an extended λ-SABR stochastic volatility models. On the other hand, Albrecher and Predota [2002, 2004], and Albrecher [2004] developed a moment matching approach to evaluate discretely monitored average options under exponential Lévy models. Albrecher et al. [2003], and Albrecher and Schoutens [2005] derived upper bounds of average option prices with discrete monitoring on Lévy processes by constructing supper static replication made up of plain-vanilla option portfolio. Fusai and Meucci [2008] provided a recursive algorithm as well as a control variate technique to price average options monitored at discrete times under the general assumption that the underlying evolves according to a Lévy process. However, to the best of our knowledge, this chapter is the first one that puts forward an analytic formula for valuation of both continuously and discretely monitored average options under time-changed Lévy processes.

Our pricing formula is based on the Gram-Charlier expansion, which is regarded as the generalized Edgeworth expansion around the Gaussian distribution and gives the approximations of the density function of an arbitrary random variable. There have already been some applications of the Gram-Charlier or Edgeworth expansion to derivative pricing. For example, Jarrow and Rudd [1982] developed an approximate pricing formula of equity options by using the generalized Edgeworth expansion. Assuming the multi-dimensional Gaussian model and CIR model of interest rate and using the Edgeworth expansion, Collin-Defresne and Goldstein [2002] provided a swaption pricing formula. Tanaka et al. [2005] proposed an approximation method of swaptions, constant maturity swaps, and options on constant maturity swaps under the affine interest rate models by applying the Gram-Charlier expansion. However, the key of our formula is not to apply the Gram-Charlier expansion, but to derive an explicit algorithm to compute the moments of the normalized average asset price under time-changed Lévy processes. The algorithm is represented as a recursive conditional expectation and this computational procedure has the same tractability as the calculation of plain-vanilla option prices under time-changed Lévy processes. In addition, it is demonstrated that in some cases the closed-form of the conditional expectation can be acquired.

2.1 Setup

2.1.1 Lévy Processes for Asset Price Dynamics

We start with a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ carrying a one-dimensional Lévy process $(Y_t)_{t\geq 0}$ with the associated filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$. A stochastic process $(Y_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{Q})$ with values in $\mathbf{R}$ such that $Y_0 = 0$ is called a Lévy process if it possesses the following properties: (1) Y_t is adapted to $\mathcal{F}_t$. (2) The sample paths of $(Y_t)_{t\geq 0}$ are right continuous with left limits. (3) $Y_u - Y_t$ is independent of $\mathcal{F}_t$ for $0 \leq t < u$. (4) $Y_u - Y_t$ has the same distribution as X_{u-t} for $0 \leq t < u$. Moreover, we assume frictionless markets and absence of arbitrage opportunities, and take an equivalent martingale measure $\mathbb{Q}$ as given.

Many studies in past financial literature have modeled dynamics of an underlying asset price $(S_t)_{t\geq 0}$ under $\mathbb{Q}$ as

$$S_t = S_0 e^{(r-q+\xi)t+Y_t}, \quad t \geq 0, \tag{2.1}$$

where r and q denote the instantaneous risk-free interest rate and dividend yield (instantaneous foreign interest rate), respectively, which are assumed to be constant over time for simplicity; and ξ is some constant such that it makes $e^{\xi t+Y_t}$ a $\mathbb{F}$-martingale. This modeling is well-known as *the exponential Lévy model* and the parameter ξ is called *convexity correction* in the context of the exponential Lévy model.

When analytically treating with the model in Eq.(2.1), the characteristic function of the distribution of Y_t plays various important roles. The Lévy-Khintchine formula provided by the following proposition gives a general representation for the characteristic function of any Lévy processes. The proof of the proposition can be found on pp.35-45 in Sato [1999].

Proposition 2.1 *(Lévy-Khintchine formula) Let $(Y_t)_{t\geq 0}$ be a Lévy process on $\mathbf{R}$. The characteristic function of the distribution of Y_t has the form*

$$\phi_{Y_t}(\theta) := \mathbb{E}\left[e^{i\theta Y_t}\right] = e^{-t\psi_Y(\theta)}, \quad t \geq 0, \tag{2.2}$$

where the function $\psi_Y(\theta)$, $\theta \in \mathbf{R}$ called the characteristic exponent is given by

$$\psi_Y(\theta) = -i\mu\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{\infty}(1 - e^{i\theta y} + i\theta x\mathbf{1}_{|y|\geq 1})\Pi(dy). \tag{2.3}$$

Here $\sigma \geq 0$ and $\mu \in \mathbf{R}$ are constant, and Π is a positive Radon measure on $\mathbf{R} \setminus \{0\}$ verifying

$$\int_{-\infty}^{\infty}(1 \wedge y^2)\Pi(dy) < \infty.$$

The parameter σ^2 is called *the Gaussian coefficient* and the measure Π is called *the Lévy measure*. The triplet (μ, σ^2, Π) is referred to as *the Lévy characteristics* of $(Y_t)_{t\geq 0}$. Intuitively, μ describes the constant drift of the process and the Gaussian coefficient σ^2 denotes constant variance of the continuous component of the process. The Lévy measure expresses the jump structure of the jump component of the process. If $\Pi = 0$ the Lévy process is identified with Gaussian process, and if $\sigma = 0$ the process becomes a pure jump process without the diffusion component. It is obvious from the Lévy-Khintchine formula that the convexity correction must be $\xi = \psi_Y(-i)$ so as to make $e^{\xi t + Y_t}$ a martingale.

One of the classes of Lévy processes is *finite-activity jump processes* that exhibit a finite number of jumps within any finite time interval. The examples of finite-activity jump processes are compound Poisson jump processes with normally distributed jump size (Merton [1976]), double-exponentially distributed jump size (Kou [2002]), and one-sided exponentially distributed jump size (Eraker [2001] and Eraker et al. [2003]). Another important class of Lévy processes is infinite-activity jump processes that generate an infinite number of jumps within any finite time interval. Examples in this class include the normal inverse Gaussian (NIG) process (Barndorff-Nielsen [1998]), the variance gamma (VG) process (Madan and Milne [1991] and Madan et al. [1998]), the finite moment log-stable (LS) process (Carr and Wu [2003]), the Meixner process (Schoutens [2002]), and the CGMY process (Carr et al. [2002]). Their Lévy measure and characteristic exponents are listed in Table 2.1. See Cont and Tankov [2004], and Boyarchenko and Levendorskiĭ [2002] for more details of Lévy processes in finance.

2.1.2 Time-Changed Lévy Processes for Asset Price Dynamics

Time-changed Lévy processes are proposed by Carr et al. [2003], and Carr and Wu [2004] as a driving factor of asset price dynamics in order to introduce the concept of stochastic volatility into Lévy processes. Let $t \to \mathbf{T}_t,\ t \geq 0$ be an increasing right-continuous process with left limits such that for each fixed t the random variable $(\mathbf{T}_t)_{t\geq 0}$ is a *stopping time* with respect to $(\mathcal{F}_t)_{t\geq 0}$. Suppose that $\mathbf{T}_t$ is finite $\mathbb{Q}$-a.s. for all $t \in [0, \infty)$ and $\mathbf{T}_t \to \infty$ as $t \to \infty$. Then the family of the stopping times $\{\mathbf{T}_t\}$ defines a *random time change*. Without loss of generality, we can normalize the random time change so that $\mathbb{E}[\mathbf{T}_t] = t$. With this normalization, the family of the stopping times becomes an unbiased reflection of calendar time. Time-changed Lévy process is a stochastic process $(X_t)_{t\geq 0}$ defined as

$$X_t = Y_{\mathbf{T}_t}, \quad t \geq 0, \tag{2.4}$$

where $(Y_t)_{t\geq 0}$ is a one-dimensional Lévy process. Obviously, by specifying different Lévy processes for Y_t and different stochastic time change for $\mathbf{T}_t$, we can generate various types of discontinuous stochastic processes from this step. In time-changed Lévy processes, the process $(Y_t)_{t\geq 0}$ is called *background Lévy process.*

We assume that the random time is characterized as follows:

$$\mathbf{T}_t = \int_0^t V_s ds,$$

where V_t called *instantaneous activity rate* is a one-dimensional continuous $\mathbb{F}$-adapted process on $(\Omega, \mathcal{F}, \mathbb{Q})$. Intuitively, one can regard t as calender time and $\mathbf{T}_t$ as business time at calender time t. A more active business day, on which the corresponding active rate becomes higher, generates higher volatility in the economy. This randomness in business active induces the randomness in volatility. The instantaneous activity rate needs to be non-negative in order to ensure that $\mathbf{T}_t$ is a non-decreasing process.

According to the existing literature, we model asset price dynamics $(S_t)_{t\geq 0}$ under $\mathbb{Q}$ by a time-changed Lévy process X_t defined by Eq.(2.4):

$$S_t = S_0 e^{(r-q)t+\xi_t+X_t}, \quad t \geq 0, \tag{2.5}$$

where $(\xi_t)_{t\geq 0}$ is some process such that it makes $e^{\xi_t+X_t}$ a $\mathbb{F}$-martingale. It is easy to check that the process $(\xi_t)_{t\geq 0}$ must be

$$\xi_t = \psi_Y(-i)\mathbf{T}_t.$$

2.2 Pricing Average Options

This section provides a fundamental pricing formula of average options in the case that the asset price follows Eq.(2.5). The formula is based on the Gram-Charlier expansion and the key for using the formula is how to obtain the moments/cumulants of the average process of the normalized asset price, which will be shown in the next section.

Consider the value of both continuously and discretely monitored average call options on a given asset S_t with strike K and maturity T. The terminal payoff of the continuously monitored call option (hereafter, CM) is given by

$$\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+,$$

and the terminal payoff of the discretely monitored call option (hereafter, DM) is given by

$$\left(\frac{1}{L}\sum_{l=1}^{L} S_{t_l} - K\right)^+,$$

where $0 < t_1 < \cdots < t_L$ are monitoring dates. Then, the value of the average call option at initial time denoted by $C(T, K)$ can be written as

$$C(T,K) = \begin{cases} \mathbb{E}\left[e^{-rT}\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] & \text{in the case of CM,} \\ \mathbb{E}\left[e^{-rT}\left(\frac{1}{L}\sum_{l=1}^{L} S_{t_l} - K\right)^+\right] & \text{in the case of DM,} \end{cases}$$

where $\mathbb{E}[\,\cdot\,]$ is the expectation operator under $\mathbb{Q}$.

Before presenting the pricing formula of the average call option, the following technical lemma is provided.

Lemma 2.2 *Let $g : \mathbf{R}_+ \to \mathbf{R}$ be an integrable function and*

$$G(x) := \int_0^x g(u)du.$$

Then for all $n \in \mathbf{N}$,

$$G(x)^n = n! \int_0^x \int_0^{u_n} \cdots \int_0^{u_2} g(u_n)g(u_{n-1})\cdots g(u_1)du_1du_2\cdots du_n. \tag{2.6}$$

Theorem 2.3 *Let*

$$\begin{cases} \Gamma := \displaystyle\int_0^T \frac{S_t}{S_0}dt \quad \text{and} \quad A := T \quad \text{in the case of CM} \\ \Gamma := \displaystyle\sum_{l=1}^{L} \frac{S_{t_l}}{S_0} \quad \text{and} \quad A := L \qquad \text{in the case of DM} \end{cases}$$

Suppose that the cumulative asset price Γ has a density function and has cumulants c_n, $n \geq 1$, all of which are finite. Then we have

$$\begin{aligned} C(T,K) &= \frac{e^{-rT}S_0}{A}\Bigg\{(c_1 - \tilde{K})N\left(\frac{c_1 - \tilde{K}}{\sqrt{c_2}}\right) + \sqrt{c_2}n\left(\frac{c_1 - \tilde{K}}{\sqrt{c_2}}\right) \\ &\quad + \sum_{k=3}^{\infty} \sqrt{c_2}(-1)^k q_k H_{k-2}\left(\frac{c_1 - \tilde{K}}{\sqrt{c_2}}\right) n\left(\frac{c_1 - \tilde{K}}{\sqrt{c_2}}\right)\Bigg\}, \end{aligned} \tag{2.7}$$

where $\tilde{K} := \frac{AK}{S_0}$, and $n(x)$ and $N(x)$ denote the standard normal density function and the standard normal distribution function, respectively, i.e.,

$$n(x) := \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \quad \text{and} \quad N(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy.$$

Here, $H_k(x) = (-1)^k n(x)^{-1}\frac{d^k}{dx^k}n(x)$ with $H_0(x) = 1$ is the Chebyshev-Hermite polynomial,

$$\begin{aligned} c_1 &= m_1, \\ c_2 &= m_2 - m_1^2, \\ c_3 &= m_3 - 3m_1m_2 + 2m_1^3, \\ c_4 &= m_4 - 4m_1m_3 - 3m_2^2 + 12m_1^2m_2 - 6m_1^4, \\ c_5 &= m_5 - 5m_1m_4 - 10m_2m_3 + 20m_1^2m_3 + 30m_1m_2^2 - 60m_1^3m_2 + 24m_1^5, \\ &\quad \cdots\cdots\cdots . \end{aligned}$$

where m_n is the n-th moment of Γ, i.e., $m_n = \mathbb{E}[\Gamma^n]$, and q_k is defined as

$$q_k = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k = 1, 2, \\ \displaystyle\sum_{m=1}^{[k/3]} \sum_{k_1+\cdots+k_m=k, k_i \geq 3} \frac{c_{k_1}\cdots c_{k_m}}{m!k_1!\cdots k_m!}\left(\frac{1}{\sqrt{c_2}}\right)^k & \text{if } k \geq 3. \end{cases}$$

Moreover, the n-th moment m_n is given by

$$m_n = \begin{cases} n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} e^{(r-q)\sum_{k=1}^n t_k} \\ \qquad \times \mathbb{E}\left[e^{\sum_{k=1}^n \left\{ \psi_Y(-i)\mathbf{T}_{t_k} + X_{t_k} \right\}} \right] dt_1 dt_2 \cdots dt_n & \textit{in the case of CM} \\ \\ \sum_{1 \le l_1 \le \cdots \le l_n \le L} e^{(r-q)\sum_{k=1}^n t_{l_k}} \mathbb{E}\left[e^{\sum_{k=1}^n \left\{ \psi_Y(-i)\mathbf{T}_{t_{l_k}} + X_{t_{l_k}} \right\}} \right] & \textit{in the case of DM} \end{cases} \tag{2.8}$$

Proof of Theorem 2.3: First of all, we derive Eq.(2.7). Using the Gram-Charlier expansion provided at the end of this chapter, the density function of the random variable Γ denoted by f can be written as

$$f(x) = \sum_{k=0}^{\infty} \frac{q_k}{\sqrt{c_2}} H_k\left(\frac{x-c_1}{\sqrt{c_2}}\right) n\left(\frac{x-c_1}{\sqrt{c_2}}\right).$$

Therefore, we have

$$\begin{aligned} \mathbb{E}\left[\Gamma \mathbf{1}_{\{\Gamma > \tilde{K}\}}\right] &= \int_{\tilde{K}}^{\infty} x f(x) dx \\ &= \sum_{k=0}^{\infty} \int_{\tilde{K}}^{\infty} x \frac{q_k}{\sqrt{c_2}} H_k\left(\frac{x-c_1}{\sqrt{c_2}}\right) n\left(\frac{x-c_1}{\sqrt{c_2}}\right) dx \\ &= c_1 N\left(\frac{c_1-\tilde{K}}{\sqrt{c_2}}\right) + \sqrt{c_2}\, n\left(\frac{c_1-\tilde{K}}{\sqrt{c_2}}\right) \\ &\quad + \sum_{k=3}^{\infty} (-1)^k q_k \left\{ \sqrt{c_2} H_{k-2}\left(\frac{c_1-\tilde{K}}{\sqrt{c_2}}\right) - \tilde{K} H_{k-1}\left(\frac{c_1-\tilde{K}}{\sqrt{c_2}}\right) \right\} n\left(\frac{c_1-\tilde{K}}{\sqrt{c_2}}\right), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\left[\mathbf{1}_{\{\Gamma > \tilde{K}\}}\right] &= \int_{\tilde{K}}^{\infty} f(x) dx \\ &= \sum_{k=0}^{\infty} \int_{\tilde{K}}^{\infty} \frac{q_k}{\sqrt{c_2}} H_k\left(\frac{x-c_1}{\sqrt{c_2}}\right) n\left(\frac{x-c_1}{\sqrt{c_2}}\right) dx \\ &= N\left(\frac{c_1-\tilde{K}}{\sqrt{c_2}}\right) - \sum_{k=3}^{\infty} (-1)^k q_k H_{k-1}\left(\frac{c_1-\tilde{K}}{\sqrt{c_2}}\right) n\left(\frac{c_1-\tilde{K}}{\sqrt{c_2}}\right). \end{aligned}$$

Since

$$C(T,K) = \frac{e^{-rT} S_0}{T} \mathbb{E}\left[(\Gamma - \tilde{K})^+\right] = \frac{e^{-rT} S_0}{T} \left\{ \mathbb{E}\left[\Gamma \mathbf{1}_{\{\Gamma > \tilde{K}\}}\right] - \tilde{K} \mathbb{E}\left[\mathbf{1}_{\{\Gamma > \tilde{K}\}}\right] \right\},$$

we have Eq.(2.7).

In the case of CM, consider the n-th moment of Γ; i.e.,

$$m_n = \mathbb{E}\left[\Gamma^n\right] = \mathbb{E}\left[\left(\int_0^T \frac{S_t}{S_0} dt\right)^n\right]. \tag{2.9}$$

Applying Lemma 2.2 to Eq.(2.9), we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T \frac{S_t}{S_0}dt\right)^n\right] &= \mathbb{E}\left[n!\int_0^T\int_0^{t_n}\cdots\int_0^{t_2}\frac{S_{t_n}}{S_0}\frac{S_{t_{n-1}}}{S_0}\cdots\frac{S_{t_1}}{S_0}dt_1dt_2\cdots dt_n\right] \\
&= n!\int_0^T\int_0^{t_n}\cdots\int_0^{t_2}\mathbb{E}\left[\prod_{k=1}^n\frac{S_{t_k}}{S_0}\right]dt_1dt_2\cdots dt_n,
\end{aligned}$$

Because the asset price S_t follows Eq.(2.5), we have

$$\begin{aligned}
\mathbb{E}\left[\prod_{k=1}^n\frac{S_{t_k}}{S_0}\right] &= \mathbb{E}\left[\prod_{k=1}^n\exp\left\{(r-q)t_k+\psi_Y(-i)\mathbf{T}_{t_k}+X_{t_k}\right\}\right] \\
&= e^{(r-q)\sum_{k=1}^n t_k}\,\mathbb{E}\left[e^{\sum_{k=1}^n\left\{\psi_Y(-i)\mathbf{T}_{t_k}+X_{t_k}\right\}}\right].
\end{aligned}$$

In the case of DM,

$$\begin{aligned}
m_n &= \mathbb{E}\left[\Gamma^n\right]=\mathbb{E}\left[\left(\sum_{l=1}^L\frac{S_{t_l}}{S_0}\right)^n\right] \\
&= \sum_{1\le l_1\le\cdots\le l_n\le L} e^{(r-q)\sum_{k=1}^n t_{l_k}}\,\mathbb{E}\left[e^{\sum_{k=1}^n\left\{\psi_Y(-i)\mathbf{T}_{t_{l_k}}+X_{t_{l_k}}\right\}}\right].
\end{aligned}$$

Therefore, Eq.(2.8) is obtained. The proof of Theorem 2.3 is completed. □

The coefficients q_k in Theorem 2.3 are easily expressed by the given cumulants as follows

$$\begin{aligned}
&q_0=1,\quad q_1=q_2=0,\quad q_3=\frac{c_3}{3!c_2^{3/2}},\quad q_4=\frac{c_4}{4!c_2^2}, \\
&q_5=\frac{c_5}{5!c_2^{5/2}},\quad q_6=\frac{c_6+10c_3^2}{6!c_2^3},\quad q_7=\frac{c_7+35c_3c_4}{7!c_2^{7/2}}.
\end{aligned}$$

According to Theorem 2.3, to evaluate the average call options under time-changed Lévy processes we have to compute the following equation:

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^n\left(\psi_Y(-i)\mathbf{T}_{t_k}+X_{t_k}\right)\right\}\right]. \tag{2.10}$$

If an analytic expression of Eq.(2.10) can be obtained, the price of the continuously monitored average option can be approximately computed by Theorem 2.3 with a suitable numerical procedure of the iterated integrals in Eq.(2.8). It will be shown that under Lévy processes, which are included in the class of time-changed Lévy processes, the closed-form expression of the iterated integral can be obtained as well. In the case of discretely monitored average options, the computation is easier than that of continuously monitored average options. The following two sections provide the analytic treatment of Eq.(2.10).

2.3 General Analysis

In this section we present analytic treatments of Eq.(2.10). For time-changed Lévy processes, Carr and Wu [2004] show that the generalized Fourier transform can be converted into the Laplace transform of the time change under a certain complex-valued measure. That is, the time-changed process $X_t = Y_{\mathbf{T}_t}$ has the characteristic function

$$\phi_{X_t}(\theta) = \mathbb{E}\left[e^{i\theta Y_{\mathbf{T}_t}}\right] = \mathbb{E}^{\theta}\left[e^{-\mathbf{T}_t \psi_Y(\theta)}\right], \tag{2.11}$$

where $\mathbb{E}^{\theta}[\,\cdot\,]$ denotes the expectation operator under a new complex-valued measure $\mathbb{Q}(\theta)$. The measure $\mathbb{Q}(\theta)$ is absolutely continuous with respect to the risk-neutral measure $\mathbb{Q}$ and is defined by a complex-valued exponential martingale

$$\mathbb{M}_T(\theta) := \frac{d\mathbb{Q}(\theta)}{d\mathbb{Q}}\Big|_T = \exp\left\{i\theta X_T + \mathbf{T}_T \psi_Y(\theta)\right\}, \tag{2.12}$$

where $\mathbb{M}_T$ is the Radon-Nikodym derivative of the new measure $\mathbb{Q}(\theta)$ with respect to the risk-neutral measure $\mathbb{Q}$ up to time T. Furthermore, optimal stopping theorem ensures that

$$\mathbb{M}_t(\theta) = \mathbb{E}\left[\mathbb{M}_T(\theta) \mid \mathcal{F}_t\right] = \exp\left\{i\theta X_t + \mathbf{T}_t \psi_Y(\theta)\right\}$$

is a $\mathbb{Q}$-martingale and that an arbitrary random variable Z_T on $(\Omega, \mathcal{F}, \mathbb{Q})$ satisfies

$$\mathbb{E}^{\theta}\left[Z_T \mid \mathcal{F}_t\right] = \mathbb{E}\left[\frac{\mathbb{M}_T(\theta)}{\mathbb{M}_t(\theta)} Z_T \mid \mathcal{F}_t\right],$$

for all $\mathcal{F}_t$. As we will show, since the parameter θ always takes imaginary numbers in the equations presented below, $\mathbb{Q}(\theta)$ becomes not complex-valued but real-valued measure in our case. If the background Lévy process $(Y_t)_{t\geq 0}$ is independent of the random time $(\mathbf{T}_t)_{t\geq 0}$, the characteristic function of the distribution of $X_t = Y_{\mathbf{T}_t}$ in Eq.(2.11) can be simply rewritten as

$$\phi_{X_t}(\theta) = \mathbb{E}\left[e^{-\mathbf{T}_t \psi_Y(\theta)}\right],$$

and thus changing the new measure $\mathbb{Q}(\theta)$ is unnecessary.

The next theorem gives us the general analytic treatment of Eq.(2.10).

Theorem 2.4 *Define the backward recurrence relation*

$$I_{k-1} = \mathbb{E}^{-i(n-k+1)}\left[\exp\left\{-\lambda_k \int_{t_{k-1}}^{t_k} V_s ds\right\} I_k \mid \mathcal{F}_{t_{k-1}}\right], \tag{2.13}$$

for $k = 1, \ldots, n$, *where* $I_n = I_{n-1} = 1$ *and* $\lambda_k := \psi_Y(-i(n-k+1)) - (n-k+1)\psi_Y(-i)$. *Then, we have*

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} (\psi_Y(-i)\mathbf{T}_{t_k} + X_{t_k})\right\}\right] = I_0. \tag{2.14}$$

Proof of Theorem 2.4: Note that

$$\sum_{k=1}^{n} (\psi_Y(-i)\mathbf{T}_{t_k} + X_{t_k}) = \sum_{k=1}^{n} (n-k+1)\left[\psi_Y(-i)(\mathbf{T}_{t_k} - \mathbf{T}_{t_{k-1}}) + (X_{t_k} - X_{t_{k-1}})\right]. \tag{2.15}$$

Defining

$$A_k = \psi_Y(-i)(\mathbf{T}_{t_k} - \mathbf{T}_{t_{k-1}}) + (X_{t_k} - X_{t_{k-1}}),$$

Eq.(2.10) can be written as

$$\begin{aligned}\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)\mathbf{T}_{t_k}+X_{t_k})\right\}\right] &= \mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(n-k+1)A_k\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{\sum_{k=1}^{n-1}(n-k+1)A_k\right\}\mathbb{E}\left[e^{A_n}\mid\mathcal{F}_{t_{n-1}}\right]\right].\end{aligned}$$

In the second equality of the above equation we use the law of iterated expectations. Since we have

$$\mathbb{E}\left[e^{A_n}\mid\mathcal{F}_{t_{n-1}}\right]=\mathbb{E}\left[\frac{\mathbb{M}_{t_n}(-i)}{\mathbb{M}_{t_{n-1}}(-i)}\mid\mathcal{F}_{t_{n-1}}\right]=1=I_{n-1},$$

Eq.(2.10) is reduced to

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)\mathbf{T}_{t_k}+X_{t_k})\right\}\right]=\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n-2}(n-k+1)A_k\right\}\mathbb{E}\left[e^{2A_{n-1}}I_{n-1}\mid\mathcal{F}_{t_{n-2}}\right]\right].$$

To complete the proof of Theorem 2.4, it is sufficient to verify

$$I_{k-1}=\mathbb{E}\left[e^{(n-k+1)A_k}I_k\mid\mathcal{F}_{t_{k-1}}\right],$$

for $1\le k\le n-1$. Here, we have

$$\begin{aligned}&\mathbb{E}\left[e^{(n-k+1)A_k}I_k\mid\mathcal{F}_{t_{k-1}}\right]\\ &=\mathbb{E}\Big[\exp\Big\{(n-k+1)(X_{t_k}-X_{t_{k-1}})+(\mathbf{T}_{t_k}-\mathbf{T}_{t_{k-1}})\psi_Y(-i(n-k+1))\\ &\quad-(\mathbf{T}_{t_k}-\mathbf{T}_{t_{k-1}})\psi_Y(-i(n-k+1))+(n-k+1)(\mathbf{T}_{t_k}-\mathbf{T}_{t_{k-1}})\psi_Y(-i)\Big\}I_k\mid\mathcal{F}_{t_{k-1}}\Big]\\ &=\mathbb{E}\Bigg[\frac{\mathbb{M}_{t_k}(-i(n-k+1))}{\mathbb{M}_{t_{k-1}}(-i(n-k+1))}\\ &\quad\times\exp\left\{[(n-k+1)\psi_Y(-i)-\psi_Y(-i(n-k+1))](\mathbf{T}_{t_k}-\mathbf{T}_{t_{k-1}})\right\}I_k\mid\mathcal{F}_{t_{k-1}}\Bigg]\\ &=\mathbb{E}^{-i(n-k+1)}\left[\exp\left\{-\lambda_k\int_{t_{k-1}}^{t_k}V_s ds\right\}I_k\mid\mathcal{F}_{t_{k-1}}\right]=I_{k-1}.\end{aligned}$$

□

Corollary 2.5 *Suppose that the background Lévy process $(Y_t)_{t\ge0}$ of a time-changed Lévy process $X_t=Y_{\mathbf{T}_t}$ is independent of its random time $(\mathbf{T}_t)_{t\ge0}$. Then, we have*

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)\mathbf{T}_{t_k}+X_{t_k})\right\}\right]=\mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n-1}\lambda_k\int_{t_{k-1}}^{t_k}V_s ds\right\}\right]. \tag{2.16}$$

Proof of Corollary 2.5: Although we can utilize Theorem 2.4 to prove Corollary 2.5, we directly derive Eq.(2.16) here. It holds

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)\mathbf{T}_{t_k}+X_{t_k})\right\}\right]\\
&=\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(n-k+1)\left\{\psi_Y(-i)(\mathbf{T}_{t_k}-\mathbf{T}_{t_{k-1}})+(Y_{\mathbf{T}_{t_k}}-Y_{\mathbf{T}_{t_{k-1}}})\right\}\right\}\right]\\
&=\mathbb{E}\Bigg[\exp\left\{\sum_{k=1}^{n}(n-k+1)\psi_Y(-i)(\mathbf{T}_{t_k}-\mathbf{T}_{t_{k-1}})\right\}\\
&\quad\times\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(n-k+1)(Y_{u_k}-Y_{u_{k-1}})\right\}\mid\mathbf{T}_{t_k}=u_k,\,k=1,\dots,n\right]\Bigg]\\
&=\mathbb{E}\Bigg[\exp\left\{\sum_{k=1}^{n}(n-k+1)\psi_Y(-i)(\mathbf{T}_{t_k}-\mathbf{T}_{t_{k-1}})\right\}\\
&\quad\times\exp\left\{-\sum_{k=1}^{n}(\mathbf{T}_{t_k}-\mathbf{T}_{t_{k-1}})\psi_Y(-i(n-k+1))\right\}\Bigg]\\
&=\mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n-1}\lambda_k\int_{t_{k-1}}^{t_k}V_s ds\right\}\right].
\end{aligned}$$

In the second equality of the above equation we use the law of iterated expectation and the independent assumption between $(X_t)_{t\geq 0}$ and $(\mathbf{T}_t)_{t\geq 0}$, and then in the third equality we apply the Lévy-Khinchine formula. □

Corollary 2.6 *Suppose that $(X_t)_{t\geq 0}$ follows a Lévy process, i.e. the random time $\mathbf{T}_t = t$ for all $t \geq 0$. Then, we have*

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)t_k+X_{t_k})\right\}\right]=\exp\left\{-\sum_{k=1}^{n-1}\lambda_k(t_k-t_{k-1})\right\}. \tag{2.17}$$

Proof of Corollary 2.6: In the case that X_t follows a Lévy process, we have in Corollary 2.5

$$\int_{t_{k-1}}^{t_k}V_s ds = t_k - t_{k-1}.$$

□

It is worthwhile noting that, in the case of Lévy processes, the iterated integral in Eq.(2.8) can be expressed as a closed form because Eq.(2.17) is merely an exponential function with respect to t_k. That is, an arbitrary moment of Eq.(2.8) can be obtained without any numerical methods of the iterated integrals and then the high order approximation of the average option pricing formula in Theorem 2.3 is easily implementable.

2.4 Activity Rate Processes

This section shows the explicit representations of Eq.(2.10) when adopting some processes as the random time of time-changed Lévy processes. In the first two subsections, we assume the market in the absence of leverage effect, i.e. the background Lévy process is independent of the activity rate of time-changed Lévy processes. Firstly, the affine processes is set up to be the activity rate of time-changed Lévy processes. It is well-known that the CIR model (Cox et al. [1985]) belonging to the class of the affine processes is very popular among practitioners to model the random time. Secondly, we adopt the quadratic Gaussian processes as the activity rate. Note that in the presence of leverage effect the analysis highly depends on the choice of the background Lévy process because of changing the measure Eq.(2.12). In the third subsection, we consider Heston's stochastic volatility model (Heston [1993]) as an example of existing leverage effect. The Heston model is well-known as the simplest model involved in the class of time-changed Lévy processes in the presence of leverage effect, but the most approved asset pricing model in practice.

2.4.1 Affine Processes for Time Change

Let $(\mathbf{Z}_t)_{t\geq 0}$ be a m-dimensional Markov process that starts at $\mathbf{z}_0$ and satisfies the following SDE:

$$d\mathbf{Z}_t = \mu(\mathbf{Z}_t)dt + \sigma(\mathbf{Z}_t)d\mathbf{W}_t, \tag{2.18}$$

where $(\mathbf{W}_t)_{t\geq 0}$ is a m-dimensional Brownian motion under $\mathbb{Q}$. It is assumed that the $m\times 1$ vector $\mu(\mathbf{Z}_t)$ and $m \times m$ matrix $\sigma(\mathbf{Z}_t)$ satisfy some technical condition such that the SDE (2.18) has a unique strong solution.

The affine process is defined as the SDE (2.18) having

$$\mu(\mathbf{z}) = K_0 + K_1\mathbf{z}, \quad K_0 \in \mathbf{R}^m, K_1 \in \mathbf{R}^{m\times m}, \tag{2.19}$$

$$[\sigma(\mathbf{z})\sigma(\mathbf{z})^\top]_{ij} = (H_0)_{ij} + (H_1)_{ij}^\top\mathbf{z}, \quad H_0 \in \mathbf{R}^{m\times m}, H_1 \in \mathbf{R}^{m\times m\times m}. \tag{2.20}$$

The following lemma is developed by the original work of Duffie and Kan [1996] for the affine term structure models of interest rates, and its extension to compound Poisson-type jumps is due to Duffie at al. [2000]. This chapter, however, does not deal with any jumps of the random time for simplicity.

Lemma 2.7 *Let $(\mathbf{Z}_t)_{t\geq 0}$ be an m-dimensional affine process under $\mathbb{Q}$, and $V_t = \rho_0 + \rho_1^\top\mathbf{Z}_t$, $\rho_0 \in \mathbf{R}$, $\rho_1 \in \mathbf{R}^m$. Define for any $\theta \in \mathbf{R}^m$*

$$\Phi(\theta, \mathbf{Z}_t, t, T) = \mathbb{E}\left[\exp\left\{-\int_t^T V_s ds\right\} e^{\theta^\top \mathbf{Z}_T} \mid \mathcal{F}_t\right].$$

Then, it satisfies

$$\Phi(\theta, \mathbf{z}, t, T) = e^{\alpha_T(t)+\beta_T(t)^\top \mathbf{z}}, \tag{2.21}$$

where $\alpha_T : \mathbf{R}_+ \to \mathbf{R}$ and $\beta_T : \mathbf{R}_+ \to \mathbf{R}^m$ satisfy the following ODEs

$$\frac{d}{dt}\beta_T(t) = \rho_1 - K_1^\top\beta_T(t) - \frac{1}{2}\beta_T(t)^\top H_1\beta_T(t), \tag{2.22}$$

$$\frac{d}{dt}\alpha_T(t) = \rho_0 - K_0^\top\beta_T(t) - \frac{1}{2}\beta_T(t)^\top H_0\beta_T(t), \tag{2.23}$$

with boundary conditions $\alpha_T(T) = 0$ and $\beta_T(T) = \theta$.

Let us set the affine process as the instantaneous activity rate of a time-changed Lévy process.

Proposition 2.8 *Suppose that $X_t = Y_{\mathbf{T}_t}$ follows a time-changed Lévy process under $\mathbb{Q}$ with an activity rate processes $(V_t)_{t\geq 0}$ such that*

$$V_t := \rho_0 + \rho_1^\top \mathbf{Z}_t \geq 0, \ \text{for all } t \geq 0, \quad \rho_0 \in \mathbf{R}, \ \rho_1 \in \mathbf{R}^d,$$

where $(\mathbf{Z}_t)_{t\geq 0}$ is a d-dimensional affine process defined in Eq.(2.19) and (2.20). Moreover, assume that the background Lévy process $(Y_t)_{t\geq 0}$ is independent of the activity rate process $(V_t)_{t\geq 0}$. Then, it satisfies

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)\mathbf{T}_{t_k} + X_{t_k})\right\}\right] = \exp\left\{\sum_{k=1}^{n-1}\alpha_{t_k}(t_{k-1}) + \beta_{t_1}(0)^\top \mathbf{z}\right\}, \tag{2.24}$$

where $\mathbf{z} := \mathbf{Z}_0$, $t_0 := 0$, and $\alpha_{t_k} : [t_{k-1}, t_k] \to \mathbf{R}$ and $\beta_{t_k} : [t_{k-1}, t_k] \to \mathbf{R}^d$ are recursively defined by the following ODEs:

$$\frac{d}{dt}\beta_{t_k}(t) \ = \ \lambda_k \rho_1 - K_1^\top \beta_{t_k}(t) - \frac{1}{2}\beta_{t_k}(t)^\top H_1 \beta_{t_k}(t), \tag{2.25}$$

$$\frac{d}{dt}\alpha_{t_k}(t) \ = \ \lambda_k \rho_0 - K_0^\top \beta_{t_k}(t) - \frac{1}{2}\beta_{t_k}(t)^\top H_0 \beta_{t_k}(t), \tag{2.26}$$

with boundary conditions $\alpha_{t_k}(t_k) = 0$ for $k = 1, \ldots, n-1$, and $\beta_{t_{n-1}}(t_{n-1}) = 0$ and $\beta_{t_k}(t_k) = \beta_{t_{k+1}}(t_k)$ for $k = 1, \ldots, n-2$.

Proof of Proposition 2.8: Using Corollary 2.5 and the law of iterated expectations, we have

$$\begin{aligned}\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)\mathbf{T}_{t_k} + X_{t_k})\right\}\right] &= \mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n-1}\lambda_k\int_{t_{k-1}}^{t_k} V_s ds\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n-2}\lambda_k\int_{t_{k-1}}^{t_k} V_s ds\right\} H_{n-2}\right],\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}H_{n-2} &:= \mathbb{E}\left[\exp\left\{-\lambda_{n-1}\int_{t_{n-2}}^{t_{n-1}} V_s ds\right\} \mid \mathcal{F}_{t_{n-2}}\right] \\ &= \exp\left\{\alpha_{t_{n-1}}(t_{n-2}) + \beta_{t_{n-1}}(t_{n-2})^\top \mathbf{Z}_{t_{n-2}}\right\}.\end{aligned} \tag{2.28}$$

Here, the second equality of Eq.(2.28) is obtained from Lemma 2.7, and $\alpha_{t_{n-1}}(t)$ and $\beta_{t_{n-1}}(t)$ are deterministic functions satisfying the ODEs (2.25) and (2.26) with boundary conditions $\alpha_{t_{n-1}}(t_{n-1}) = 0$ and $\beta_{t_{n-1}}(t_{n-1}) = 0$.

Next, substituting Eq.(2.28) into Eq.(2.27), we obtain

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)\mathbf{T}_{t_k} + X_{t_k})\right\}\right] = e^{\alpha_{t_{n-1}}(t_{n-2})}\,\mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n-3}\lambda_k\int_{t_{k-1}}^{t_k} V_s ds\right\} H_{n-3}\right],$$

where

$$\begin{aligned}H_{n-3} &:= \mathbb{E}\left[\exp\left\{-\lambda_{n-2}\int_{t_{n-3}}^{t_{n-2}} V_s ds\right\} e^{\beta_{t_{n-1}}(t_{n-2})^\top \mathbf{Z}_{t_{n-2}}} \mid \mathcal{F}_{t_{n-3}}\right] \\ &= \exp\left\{\alpha_{t_{n-2}}(t_{n-3}) + \beta_{t_{n-2}}(t_{n-3})^\top \mathbf{Z}_{t_{n-3}}\right\}.\end{aligned}$$

The second equality of the above equation is due to Lemma 2.7, and $\alpha_{t_{n-2}}(t)$ and $\beta_{t_{n-2}}(t)$ satisfy the ODEs (2.25) and (2.26) with boundary conditions $\alpha_{t_{n-2}}(t_{n-2}) = 0$ and $\beta_{t_{n-2}}(t_{n-2}) = \beta_{t_{n-1}}(t_{n-2})$.

Repeating this procedure, we obtain

$$\begin{aligned} \mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)\mathbf{T}_{t_k} + X_{t_k})\right\}\right] &= \exp\left\{\sum_{k=1}^{n-2}\alpha_{t_k}(t_{k-1})\right\} H_0 \\ &= \exp\left\{\sum_{k=1}^{n-1}\alpha_{t_k}(t_{k-1}) + \beta_{t_1}(0)^\top \mathbf{Z}_0\right\}. \end{aligned}$$

□

2.4.2 Quadratic Gaussian Processes for Time Change

Let $(\mathbf{Z}_t)_{t\geq 0}$ be a m-dimensional OU process, i.e.,

$$d\mathbf{Z}_t = -(\mathbf{b}_Z + K\mathbf{Z}_t)dt + d\mathbf{W}_t, \tag{2.29}$$

where $\mathbf{b}_Z$ is a vector on $\mathbf{R}^m$, K is a matrix on $\mathbf{R}^{m\times m}$, and $(\mathbf{W}_t)_{t\geq 0}$ is a m-dimensional Brownian motion under $\mathbb{Q}$. The quadratic Gaussian process is a one-dimensional process defined as the following form:

$$\mathbf{Z}_t^\top A\mathbf{Z}_t + \mathbf{b}^\top \mathbf{Z}_t + c. \tag{2.30}$$

Here, A is a $m \times m$ matrix, $\mathbf{b}$ is a m-dimensional vector, and c is a scalar.

The following lemma is developed by Leippold and Wu [2002] for asset pricing under the quadratic Gaussian class. The proof of this lemma can be found in Appendix C of Leippold and Wu [2002].

Lemma 2.9 *Let $(U_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ be quadratic Gaussian processes under $\mathbb{Q}$ such that*

$$\begin{aligned} U_t &= \mathbf{Z}_t^\top A_U \mathbf{Z}_t + \mathbf{b}_U^\top \mathbf{Z}_t + c_U, \\ V_t &= \mathbf{Z}_t^\top A_V \mathbf{Z}_t + \mathbf{b}_V^\top \mathbf{Z}_t + c_V, \end{aligned}$$

for any $t \geq 0$, where $A_U, A_V \in \mathbf{R}^{m\times m}$, $\mathbf{b}_U, \mathbf{b}_V \in \mathbf{R}^m$, and $c_U, c_V \in \mathbf{R}$. Define

$$\Psi(\mathbf{Z}_t, t, T) = \mathbb{E}\left[\exp\left\{-\int_t^T V_s ds\right\} e^{-U_T} \mid \mathcal{F}_t\right].$$

Then, it satisfies

$$\Psi(\mathbf{z}, t, T) = \exp\left\{-\mathbf{z}^\top A_T(t)\mathbf{z} - \mathbf{b}_T(t)^\top \mathbf{z} - c_T(t)\right\}, \tag{2.31}$$

where $A_T : \mathbf{R}_+ \to \mathbf{R}^{m\times m}$, $\mathbf{b}_T : \mathbf{R}_+ \to \mathbf{R}^m$, and $c_T : \mathbf{R}_+ \to \mathbf{R}$ satisfy the following ODEs

$$\frac{d}{dt}A_T(t) = -A_V + A_T(t)K + K^\top A_T(t) + 2A_T(t)^2, \tag{2.32}$$

$$\frac{d}{dt}\mathbf{b}_T(t) = -\mathbf{b}_V + 2A_T(t)\mathbf{b}_Z + K^\top \mathbf{b}_T(t) + 2A_T(t)\mathbf{b}_T(t), \tag{2.33}$$

$$\frac{d}{dt}c_T(t) = -c_V + \mathbf{b}_T(t)^\top \mathbf{b}_Z - \operatorname{tr} A_T(t) + \frac{1}{2}\mathbf{b}_T(t)^\top \mathbf{b}_T(t), \tag{2.34}$$

with boundary conditions $A_T(T) = A_U$, $\mathbf{b}_T(T) = \mathbf{b}_U$, and $c_T(T) = c_U$.

Let us assume that the instantaneous activity rate of a time-changed Lévy process follows a quadratic Gaussian process.

Proposition 2.10 *Suppose that $X_t := Y_{\mathbf{T}_t}$ follows a time-changed Lévy process under $\mathbb{Q}$ with an activity rate $(V_t)_{t\geq 0}$ such that*

$$V_t := \mathbf{Z}_t^\top A_V \mathbf{Z}_t + \mathbf{b}_V^\top \mathbf{Z}_t + c_V \geq 0, \ \text{for all} \ t \geq 0, \quad A_V \in \mathbf{R}^{d\times d}, \ \mathbf{b}_V \in \mathbf{R}^d, \ c_V \in \mathbf{R},$$

where $(\mathbf{Z}_t)_{t\geq 0}$ is a d-dimensional OU process defined in Eq.(2.29). Moreover, assume that the background Lévy process $(Y_t)_{t\geq 0}$ is independent of the activity rate process $(V_t)_{t\geq 0}$. Then, it satisfies

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)\mathbf{T}_{t_k} + X_{t_k})\right\}\right] = \exp\left\{-\mathbf{z}^\top A_{t_1}(0)\mathbf{z} - \mathbf{b}_{t_1}(0)^\top \mathbf{z} - c_{t_1}(0)\right\}, \tag{2.35}$$

where $\mathbf{z} := \mathbf{Z}_0$, and $A_{t_k} : [t_{k-1}, t_k] \to \mathbf{R}^{m\times m}$, $\mathbf{b}_{t_k} : [t_{k-1}, t_k] \to \mathbf{R}^m$, and $c_{t_k} : [t_{k-1}, t_k] \to \mathbf{R}$ are recursively defined by the following ODEs:

$$\frac{d}{dt}A_{t_k}(t) = -\lambda_k A_V + A_{t_k}(t)K + K^\top A_{t_k}(t) + 2A_{t_k}(t)^2, \tag{2.36}$$

$$\frac{d}{dt}\mathbf{b}_{t_k}(t) = -\lambda_k \mathbf{b}_V + 2A_{t_k}(t)\mathbf{b}_Z + K^\top \mathbf{b}_{t_k}(t) + 2A_{t_k}(t)\mathbf{b}_{t_k}(t), \tag{2.37}$$

$$\frac{d}{dt}c_{t_k}(t) = -\lambda_k c_V + \mathbf{b}_{t_k}(t)^\top \mathbf{b}_Z - \operatorname{tr} A_{t_k}(t) + \frac{1}{2}\mathbf{b}_{t_k}(t)^\top \mathbf{b}_{t_k}(t), \tag{2.38}$$

with boundary conditions

$$A_{t_k}(t_k) = A_{t_{k+1}}(t_k), \tag{2.39}$$

$$\mathbf{b}_{t_k}(t_k) = \mathbf{b}_{t_{k+1}}(t_k), \tag{2.40}$$

$$c_{t_k}(t_k) = c_{t_{k+1}}(t_k), \tag{2.41}$$

for $k = 1, \ldots, n-2$, and $A_{t_{n-1}}(t_{n-1}) = \mathbf{b}_{t_{n-1}}(t_{n-1}) = c_{t_{n-1}}(t_{n-1}) = 0$.

Proof of Proposition 2.10: Using Corollary 2.5 and the law of iterated expectations, we have

$$\begin{aligned}\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)\mathbf{T}_{t_k} + X_{t_k})\right\}\right] &= \mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n-1}\lambda_k\int_{t_{k-1}}^{t_k}V_s ds\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n-2}\lambda_k\int_{t_{k-1}}^{t_k}V_s ds\right\}J_{n-2}\right],\end{aligned} \tag{2.42}$$

where

$$\begin{aligned}J_{n-2} &:= \mathbb{E}\left[\exp\left\{-\lambda_{n-1}\int_{t_{n-2}}^{t_{n-1}}V_s ds\right\} \mid \mathcal{F}_{t_{n-2}}\right] \\ &= \exp\left\{-\mathbf{Z}_{t_{n-2}}^\top A_{t_{n-1}}(t_{n-2})\mathbf{Z}_{t_{n-2}} - \mathbf{b}_{t_{n-1}}(t_{n-2})^\top \mathbf{Z}_{t_{n-2}} - c_{t_{n-1}}(t_{n-2})\right\}.\end{aligned} \tag{2.43}$$

Here, the second equality of Eq.(2.43) is obtained from Lemma 2.9, and $A_{t_{n-1}}(t)$, $\mathbf{b}_{t_{n-1}}(t)$, and $c_{t_{n-1}}(t)$ are deterministic functions satisfying the ODEs (2.36)-(2.38) with boundary conditions $A_{t_{n-1}}(t_{n-1}) = \mathbf{b}_{t_{n-1}}(t_{n-1}) = c_{t_{n-1}}(t_{n-1}) = 0$.

Next, substituting Eq.(2.43) into Eq.(2.42), we obtain

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)\mathbf{T}_{t_k}+X_{t_k})\right\}\right]=\mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n-3}\lambda_k\int_{t_{k-1}}^{t_k}V_s ds\right\}J_{n-3}\right],$$

where

$$\begin{aligned} J_{n-3} &:= \mathbb{E}\left[\exp\left\{-\lambda_{n-2}\int_{t_{n-3}}^{t_{n-2}}V_s ds\right\}\right. \\ &\qquad \times \left.\exp\left\{-\mathbf{Z}_{t_{n-2}}^\top A_{t_{n-1}}(t_{n-2})\mathbf{Z}_{t_{n-2}}-\mathbf{b}_{t_{n-1}}(t_{n-2})^\top\mathbf{Z}_{t_{n-2}}-c_{t_{n-1}}(t_{n-2})\right\} \mid \mathcal{F}_{t_{n-3}}\right] \\ &= \exp\left\{-\mathbf{Z}_{t_{n-3}}^\top A_{t_{n-2}}(t_{n-3})\mathbf{Z}_{t_{n-3}}-\mathbf{b}_{t_{n-2}}(t_{n-3})^\top\mathbf{Z}_{t_{n-3}}-c_{t_{n-2}}(t_{n-3})\right\}. \end{aligned}$$

The second equality of the above equation is due to Lemma 2.9, and $A_{t_{n-2}}(t)$, $\mathbf{b}_{t_{n-2}}(t)$, and $c_{t_{n-2}}(t)$ satisfy the ODEs (2.36)-(2.38) with boundary conditions (2.39)-(2.41).

Repeating this procedure, we obtain

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n}(\psi_Y(-i)\mathbf{T}_{t_k}+X_{t_k})\right\}\right]=J_0=\exp\left\{-\mathbf{Z}_0^\top A_{t_1}(0)\mathbf{Z}_0-\mathbf{b}_{t_1}(0)^\top\mathbf{Z}_0-c_{t_1}(0)\right\}.$$

□

2.4.3 Heston Model as an Example of Leverage Effect

In this subsection we assume that the asset price follows the Heston model (Heston [1993]). This case is an example of leverage effect under time-changed Lévy processes. The Heston model under $\mathbb{Q}$ is specified as follows:

$$Y_t = \sigma W_t^1, \tag{2.44}$$

$$dV_t = a(1-V_t)dt + c\sqrt{V_t}(\rho dW_t^1+\sqrt{1-\rho^2}dW_t^2), \tag{2.45}$$

where $\mathbf{W}=(W^1,W^2)$ is a 2-dimensional Brownian motion, and $\sigma,a,c>0$ and $\rho\in(-1,1)$ are the constant model parameters. The leverage effect can be accommodated by negatively correlating Y_t and V_t, i.e., $\rho<0$.

The new measure $\mathbb{Q}(\theta)$ is defined by the following exponential martingale:

$$\mathbb{M}_t(\theta)=\exp\left\{i\theta\sigma\int_0^t\sqrt{V_s}dW_s^1+\frac{1}{2}\theta^2\sigma^2\int_0^t V_s ds\right\}.$$

Girsanov's theorem implies that under $\mathbb{Q}(\theta)$, the activity rate process V_t follows the SDE:

$$dV_t=(a-[a-i\theta\sigma c\rho]V_t)dt+c\sqrt{V_t}(\rho dW_t^\theta+\sqrt{1-\rho^2}dW_t^2), \tag{2.46}$$

where $W_t^\theta := W_t^1 - i\theta\sigma\int_0^t\sqrt{V_s}ds$ is a Brownian motion under $\mathbb{Q}(\theta)$. Since Eq.(2.46) belongs to the class of affine processes, by repeatedly applying Lemma 2.7 to Eq.(2.13), Eq.(2.14) can be expressed as

$$I_0=\exp\left\{\sum_{k=1}^{n-1}\alpha_{t_k}(t_{k-1})+\beta_{t_1}(0)V_0\right\},$$

where $\alpha_{t_k} : [t_{k-1}, t_k] \to \mathbf{R}$ and $\beta_{t_k} : [t_{k-1}, t_k] \to \mathbf{R}$ satisfy the following ODEs:

$$\frac{d}{dt}\beta_{t_k}(t) = \lambda_k + [a - (n-k+1)\sigma c\rho]\beta_{t_k}(t) - \frac{1}{2}c^2\beta_{t_k}(t)^2, \tag{2.47}$$

$$\frac{d}{dt}\alpha_{t_k}(t) = -a\beta_{t_k}(t), \tag{2.48}$$

with boundary conditions $\alpha_{t_k}(t_k) = 0$ for $k = 1, \ldots, n-1$, and $\beta_{t_{n-1}}(t_{n-1}) = 0$ and $\beta_{t_k}(t_k) = \beta_{t_{k+1}}(t_k)$ for $k = 1, \ldots, n-2$. Note that in this case

$$\lambda_k = -\frac{1}{2}\sigma^2(n-k)(n-k+1). \tag{2.49}$$

Solving the ODE (2.47) and (2.48), we obtain

$$\beta_{t_k}(t) = \frac{2\chi'_{t_k}(t)}{c^2\chi_{t_k}(t)} \quad \text{and} \quad \alpha_{t_k}(t) = -\frac{2a}{c^2}\ln|\chi_{t_k}(t)|, \quad t_{k-1} \le t \le t_k,$$

where

$$\begin{aligned}
\chi_{t_k}(t) &:= D_k e^{-\frac{1}{2}B_k(t_k-t)}\left\{B_k \sinh(\gamma_k(t_k-t)) + 2\gamma_k\cosh(\gamma_k(t_k-t))\right\} \\
&\quad - E_k e^{-(\frac{1}{2}B_k-\gamma_k)(t_k-t)}, \\
\chi'_{t_k}(t) &:= F_k e^{-\frac{1}{2}B_k(t_k-t)}\sinh(\gamma_k(t_k-t)) - G_k e^{-(\frac{1}{2}B_k-\gamma_k)(t_k-t)}.
\end{aligned}$$

Here, we define

$$\begin{aligned}
B_k &= a - (n-k+1)\sigma c\rho, \\
\gamma_k &= \frac{1}{2}\sqrt{B_k^2 + 2c^2\lambda_k}, \\
D_k &= -\frac{1}{2c^2\lambda_k\gamma_k}\left(\frac{1}{2}B_k + \gamma_k\right)\left(B_k - 2\gamma_k - c^2\beta_{t_{k+1}}(t_k)\right), \\
E_k &= -\frac{c^2\beta_{t_{k+1}}(t_k)}{B_k - 2\gamma_k}, \\
F_k &= \frac{1}{2\gamma_k}\left(\frac{1}{2}B_k + \gamma_k\right)\left(B_k - 2\gamma_k - c^2\beta_{t_{k+1}}(t_k)\right), \\
G_k &= -\frac{1}{2}c^2\beta_{t_{k+1}}(t_k).
\end{aligned}$$

2.5 Numerical Examples

In this section, we provide numerical examples in which we set the Brownian motion (BM), variance gamma (VG), normal inverse Gaussian (NIG) as the background Lévy processes and the CIR process as the activity rate process of time change. VG is a finite variation process with infinite but relatively low activity of small jumps, and NIG is an infinite variation process with stable-like behavior of small jumps. Combining these processes, we can generate six types of dynamics of underlying asset prices whose driving factors belong to the class of Lévy processes with/without stochastic time change; that is, the Black-Scholes model (BS), VG, NIG, Heston model (HS), VG-CIR, and NIG-CIR. Following the result of Section 5.3, leverage effect is considered in the case of the Heston model.

The Lévy measures and characteristic exponents of the background Lévy processes are exhibited in Table 2.1. The CIR process as the activity rate of the time-changed Lévy processes is given by Eq.(2.45).

The model parameters are listed in Table 2.2. Of course, when the driving factors of the asset prices are BM, VG, or NIG, only the parameters of the background Lévy processes are effective. Setting the underlying asset price at initial time $S_0 = 100$, the risk-free interest rate $r = 0.01$, and dividend yield $q = 0.02$, we compute prices of *continuously monitored average call options* with maturity $T = 1$ and strike $K \in [80, 120]$ by using Theorem 2.3. Because the pricing formula (2.7) in Theorem 2.3 includes the infinite series, replacing it with the finite sum, we use the following approximation formula:

$$C(T,K) \approx \frac{e^{-rT}S_0}{T}\Bigg\{(c_1-\tilde{K})N\left(\frac{c_1-\tilde{K}}{\sqrt{c_2}}\right)+\sqrt{c_2}n\left(\frac{c_1-\tilde{K}}{\sqrt{c_2}}\right) + \sum_{k=3}^{L}\sqrt{c_2}(-1)^k q_k H_{k-2}\left(\frac{c_1-\tilde{K}}{\sqrt{c_2}}\right)n\left(\frac{c_1-\tilde{K}}{\sqrt{c_2}}\right)\Bigg\}. \quad (2.50)$$

In the numerical examples we calculate the prices in the case of $L = 2, \ldots, 7$, where $L = 2$ means that we compute only the first and second terms in the bracket on the right hand side of Eq.(2.50). We apply the 7 points Gauss-Legendre quadrature rule to numerical calculation of the iterated integral in Eq.(2.8). Moreover, for fast computation, we approximate

$$q_6 \approx \frac{10c_3^2}{6!c_2^3} \quad \text{and} \quad q_7 \approx \frac{35c_3c_4}{7!c_2^{7/2}},$$

because the high order cumulant may be negligible. In order to verify the accuracy of our formula, we compare the approximation prices with estimated prices by Monte Carlo simulations with 1,000 time steps and 1 million sample paths as benchmark prices.

Figures 2.1-2.12 display the benchmark prices of the average options and the differences between the benchmark prices and the approximate prices. When adopting BS, VG, and NIG, the differences are within 0.02 even when $L = 3$. This is because the average asset prices have relatively low kurtosis. In contrast, in the case of HS, VG-CIR, and NIG-CIR, high order approximations are necessary to obtain accurate values due to higher kurtosis generated by stochastic time change. However, the level of accuracy with $L = 6$ or 7 is substantially sufficient. As a result of the numerical examples, it can be said that satisfactorily accurate prices of average options are obtained by Eq.(2.50).

2.6 Concluding Remarks

We provide a pricing formula of average call options when the underlying asset price is driven by time-changed Lévy processes. The key of the pricing formula based on the Gram-Charlier expansion is to find a computation scheme of the moments of the normalized average price of the underlying asset. We show an analytic treatment of the moments; in particular, an explicit algorithm for calculating the moments are derived when the activity rate processes of the time-changed Lévy processes are either affine processes or quadratic Gaussian processes. Furthermore, numerical examples demonstrate that our formula can give accurate approximations of average call option prices under the Heston, VG-CIR, and NIG-CIR models.

It is worthwhile noting that the class of underlying asset prices driven by time-changed Lévy processes includes a variety of stochastic volatility models and all of the exponential Lévy models. Our formula is more widely applicable to asset price processes for evaluating average options than the analytic pricing methods proposed in existing literature. Therefore, it can be said that the formula is very useful and efficient from a practical point of view.

Gram-Charlier expansion This supplementation shows the Gram-Charlier expansion of an arbitrary density function. The derivation of the expansion follows the proof of Proposition 1 in Tanaka et al. [2005].

Assume that a random variable X has the density function f and has cumulants $c_n, n \geq 1$, all of which are finite and known. From the definition of cumulants, the characteristic function of X, ϕ_X, is given by

$$\ln \phi_X(\theta) = \sum_{n=1}^{\infty} \frac{c_k}{n!}(i\theta)^n.$$

Consider the normal random variable Y with mean c_1 and variance c_2, and let ϕ_Y denote the characteristic function of the distribution of Y. Then it holds

$$\ln \frac{\phi_X(\theta)}{\phi_Y(\theta)} = \sum_{n=3}^{\infty} \frac{c_n}{n!}(i\theta)^n.$$

By using the Taylor expansion, the above equation can be rewritten as

$$\begin{aligned}
\phi_X(\theta) &= \exp\left\{\sum_{n=3}^{\infty} \frac{c_n}{n!}(i\theta)^n\right\}\phi_Y(\theta) \\
&= \left[1 + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\sum_{n=3}^{\infty} \frac{c_n}{n!}(i\theta)^n\right)^k\right]\phi_Y(\theta) \\
&= \left[1 + \frac{c_3}{3!}(i\theta)^3 + \frac{c_4}{4!}(i\theta)^4 + \frac{c_5}{5!}(i\theta)^5 + \frac{c_6 + 10c_3^2}{6!}(i\theta)^6 + \frac{c_7 + 35c_3c_4}{7!}(i\theta)^7 + \cdots\right]\phi_Y(\theta) \\
&= \sum_{k=0}^{\infty} C_k (i\theta)^k \phi_Y(\theta),
\end{aligned}$$

where $C_0 = 1,\ C_1 = C_2 = 0$, and for $k \geq 3$

$$C_k = \sum_{m=1}^{[k/3]} \sum_{k_1+\cdots+k_m=k, k_i \geq 3} \frac{c_{k_1}\cdots c_{k_m}}{m! k_1! \cdots k_m!}.$$

Applying the inverse Fourier transform to ϕ_X, we have

$$f(x) = \sum_{k=0}^{\infty} C_k \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta x}(i\theta)^k \phi_X(\theta)d\theta = \sum_{k=0}^{\infty} \frac{(-1)^k C_k}{\sqrt{c_2}} \frac{d^k}{dx^k} n\left(\frac{x - c_1}{\sqrt{c_2}}\right).$$

Using the relationship between the Chebyshev-Hermite polynomial and the standard normal density function, we obtain

$$\frac{d^k}{dx^k} n\left(\frac{x - c_1}{\sqrt{c_2}}\right) = (-1)^k \left(\frac{1}{\sqrt{c_2}}\right)^k H_k\left(\frac{x - c_1}{\sqrt{c_2}}\right) n\left(\frac{x - c_1}{\sqrt{c_2}}\right).$$

Therefore, the following equation holds.

$$f(x) = \sum_{k=0}^{\infty} \frac{q_k}{\sqrt{c_2}} H_k\left(\frac{x - c_1}{\sqrt{c_2}}\right) n\left(\frac{x - c_1}{\sqrt{c_2}}\right),$$

where

$$q_k = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k = 1, 2, \\ \displaystyle\sum_{m=1}^{[k/3]} \sum_{k_1+\cdots+k_m=k, k_i \geq 3} \frac{c_{k_1}\cdots c_{k_m}}{m! k_1! \cdots k_m!}\left(\frac{1}{\sqrt{c_2}}\right)^k & \text{if } k \geq 3. \end{cases}$$

Table 2.1: Lévy measures and their corresponding characteristic exponents

Jump component	Lévy measure $\Pi(dx)/dx$	Characteristic exponent $\psi_X(\theta)$
Pure continuous Lévy component		
$\mu t + \sigma W_t$	$-$	$-i\mu\theta + \frac{1}{2}\sigma^2\theta^2$
Finite-activity jump Lévy component		
Merton [1976]	$\lambda \frac{1}{\sqrt{2\pi\eta^2}} \exp\left\{-\frac{(x-\kappa)^2}{2\eta^2}\right\}$	$\lambda\left(1-\exp\left\{i\theta\kappa - \frac{1}{2}\eta^2\theta^2\right\}\right)$
Kou [2002]	$\lambda \frac{1}{2\eta} \exp\left\{-\frac{\lvert x-\kappa\rvert}{\eta}\right\}$	$\lambda\left(1-e^{i\theta\kappa}\frac{1-\eta^2}{1+\theta^2\eta^2}\right)$
Eraker [2001]	$\lambda \frac{1}{\eta} \exp\left\{-\frac{x}{\eta}\right\}$	$\lambda\left(1-\frac{1}{1-i\theta\eta}\right)$
Infinite-activity jump Lévy component		
VG	$\frac{1}{\kappa\lvert x\rvert} e^{Ax - B\lvert x\rvert}$ $\left(A := \frac{\mu}{\sigma^2}, \quad B := \frac{\sqrt{\mu^2 + 2\sigma^2/\kappa}}{\sigma^2}\right)$	$\frac{1}{\kappa}\ln\left(1+\frac{1}{2}\kappa\sigma^2\theta^2 - i\mu\kappa\theta\right)$
NIG	$\frac{C}{\lvert x\rvert} e^{Ax} K_1(B\lvert x\rvert)$ $\left(A := \frac{\mu}{\sigma^2}, \quad B := \frac{\sqrt{\mu^2+\sigma^2/\kappa}}{\sigma^2}, \quad C := \frac{\sqrt{\mu^2+\sigma^2/\kappa}}{2\pi\sigma\sqrt{\kappa}}\right)$	$\frac{1}{\kappa}\sqrt{1+\sigma^2\kappa\theta^2 - 2i\mu\kappa\theta} - \frac{1}{\kappa}$
LS	$c\lvert x\rvert^{-\alpha-1}, \quad x<0$	$-c\Gamma(-\alpha)(i\theta)^\alpha$
Meixner	$\delta \frac{\exp(\frac{b}{a}x)}{x\sinh(\frac{\pi x}{a})}$	$2\delta\ln\left(\frac{\cos(b/2)}{\cosh(\frac{a\theta - ib}{2})}\right)$
CGMY	$\begin{cases} C\frac{\exp\{-G\lvert x\rvert\}}{\lvert x\rvert^{Y+1}}, & x<0 \\ C\frac{\exp\{-M\lvert x\rvert\}}{\lvert x\rvert^{Y+1}}, & x>0 \end{cases}$	$C\Gamma(-Y)\left[M^Y - (M-i\theta)^Y + G - (G+i\theta)^Y\right]$

Table 2.2: Parameters of Time-Changed Lévy Processes

		HS (BS)	VG-CIR (VG)	NIG-CIR (NIG)
background	σ	0.10	0.10	0.10
Lévy processes	μ	0.00	-0.50	-0.50
	κ	–	0.01	0.01
CIR process	V_0	1.00	1.00	1.00
	a	1.00	1.00	1.00
	c	1.00	1.00	1.00
	ρ	-0.70	–	–

Figure 2.1: Average Call Option Prices under BS

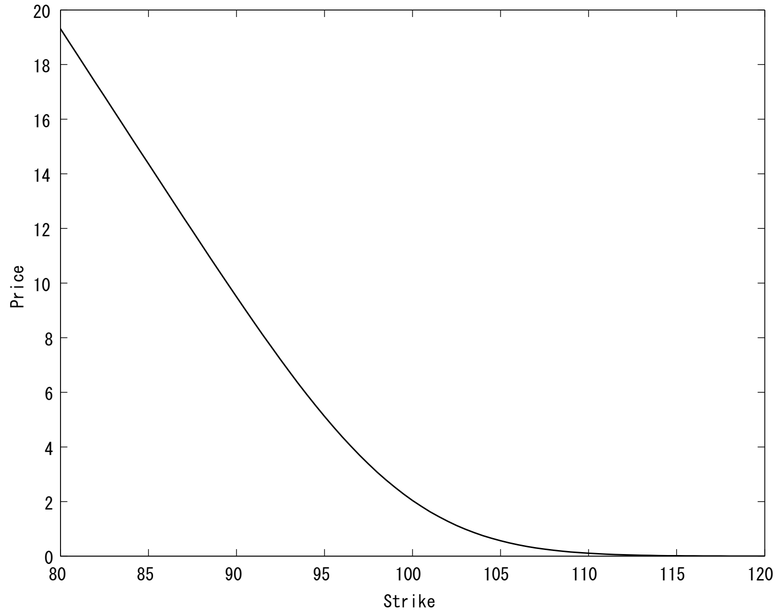


Figure 2.2: Differences BS Prices between MC and Our Formula

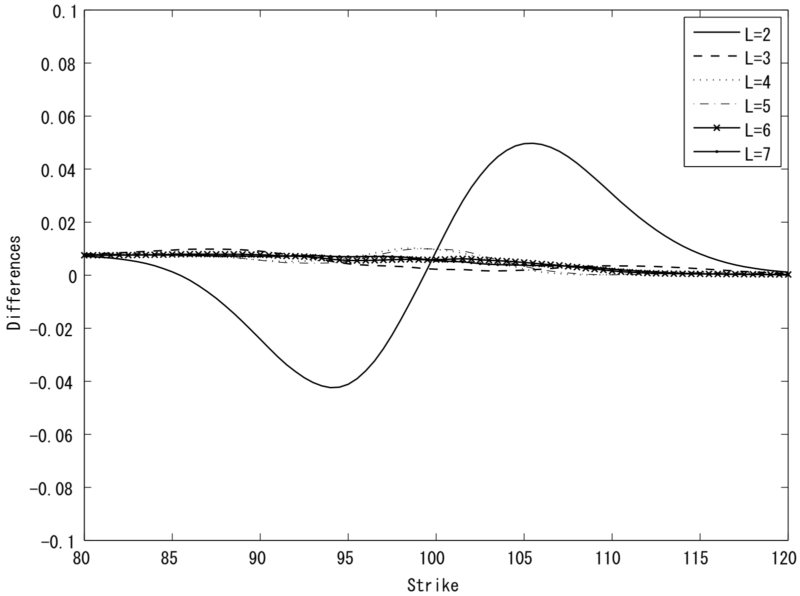


Figure 2.3: Average Call Option Prices under VG

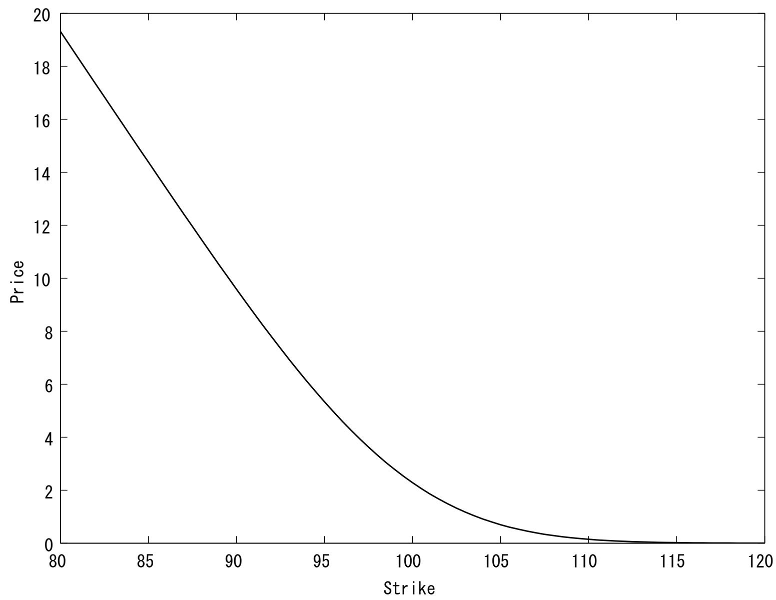


Figure 2.4: Differences VG Prices between MC and Our Formula

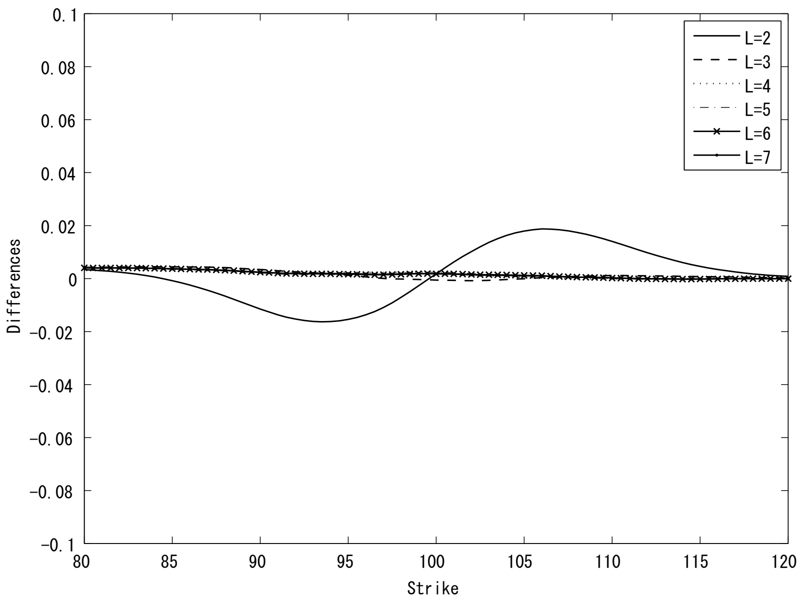


Figure 2.5: Average Call Option Prices under NIG

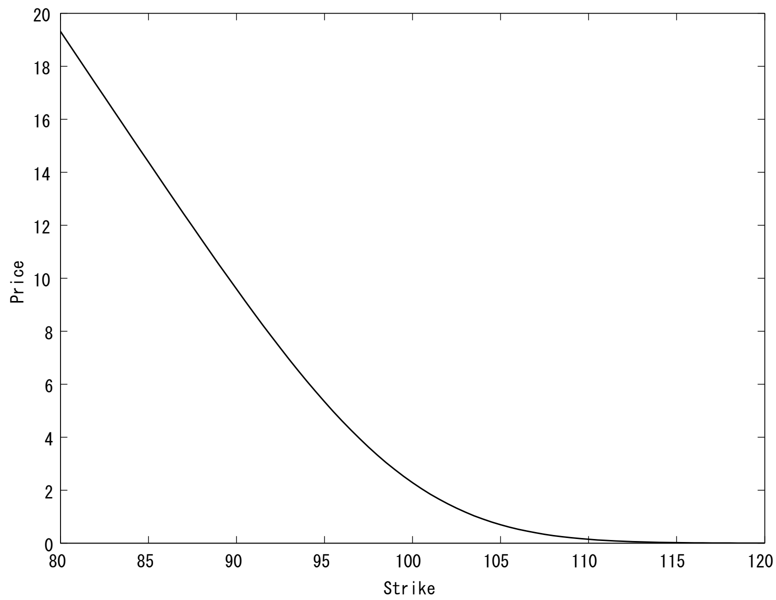


Figure 2.6: Differences NIG Prices between MC and Our Formula

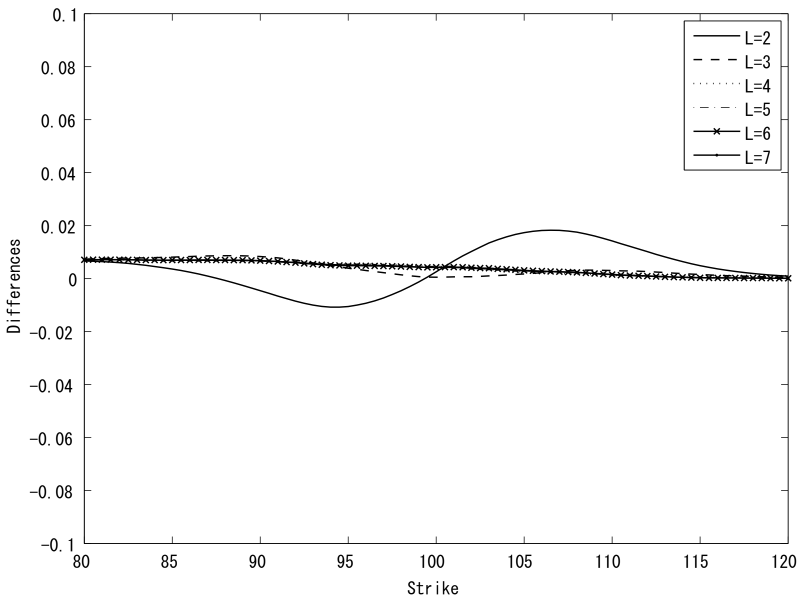


Figure 2.7: Average Call Option Prices under HS

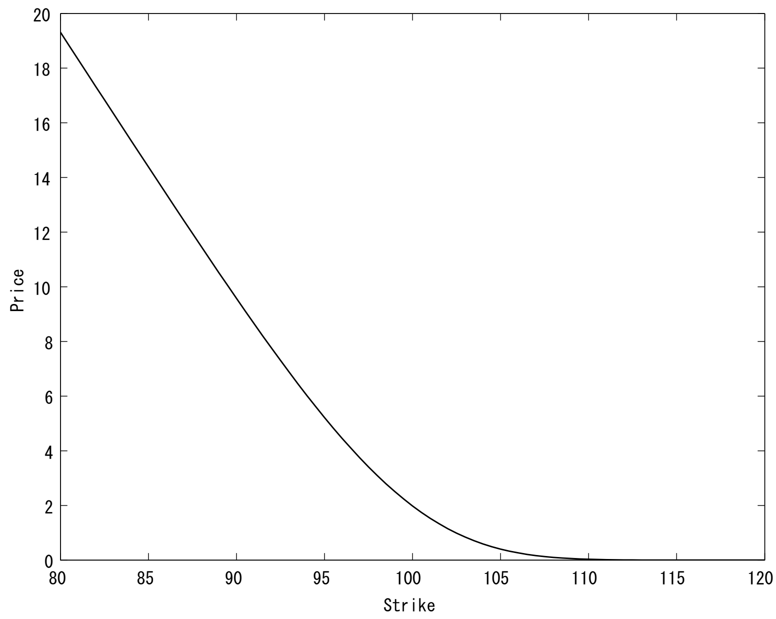


Figure 2.8: Differences HS Prices between MC and Our Formula

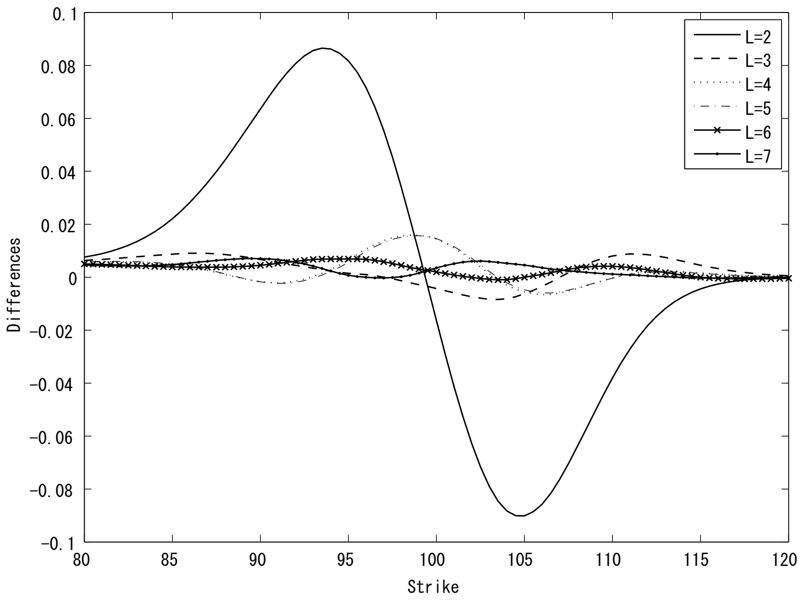


Figure 2.9: Average Call Option Prices under VG-CIR

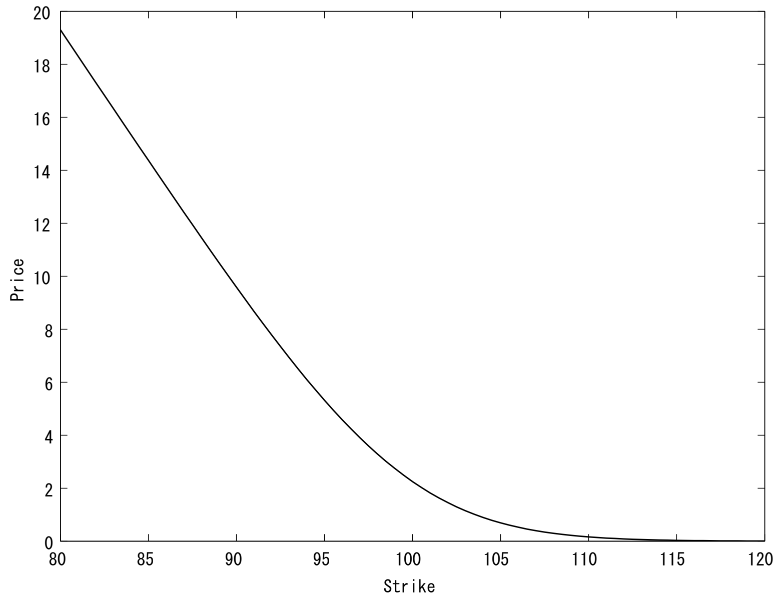


Figure 2.10: Differences VG-CIR Prices between MC and Our Formula

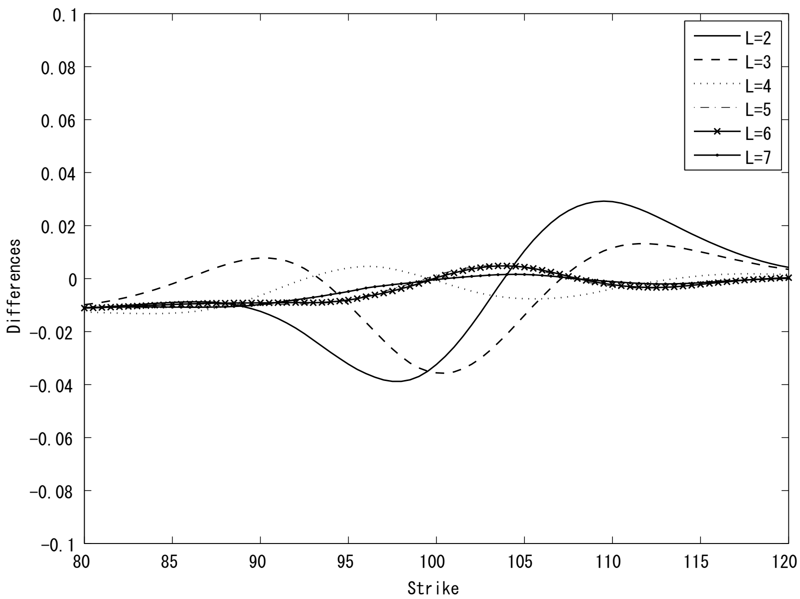


Figure 2.11: Average Call Option Prices under NIG-CIR

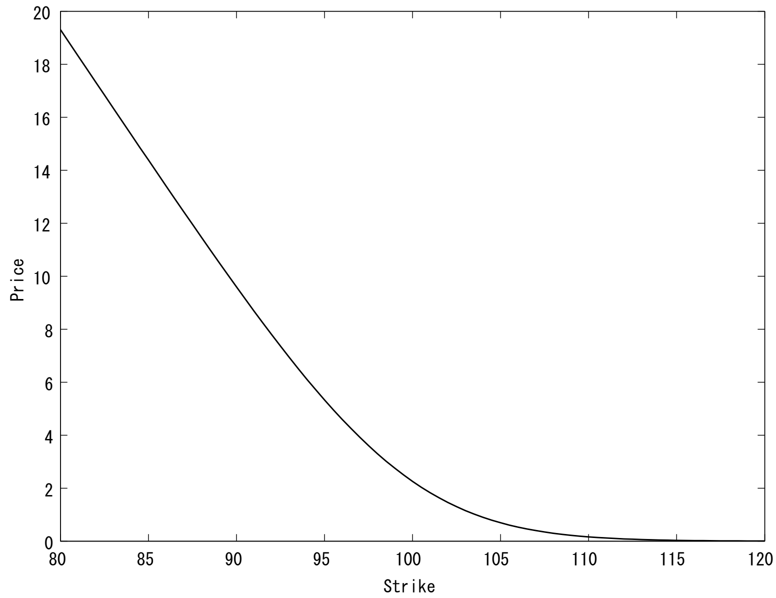
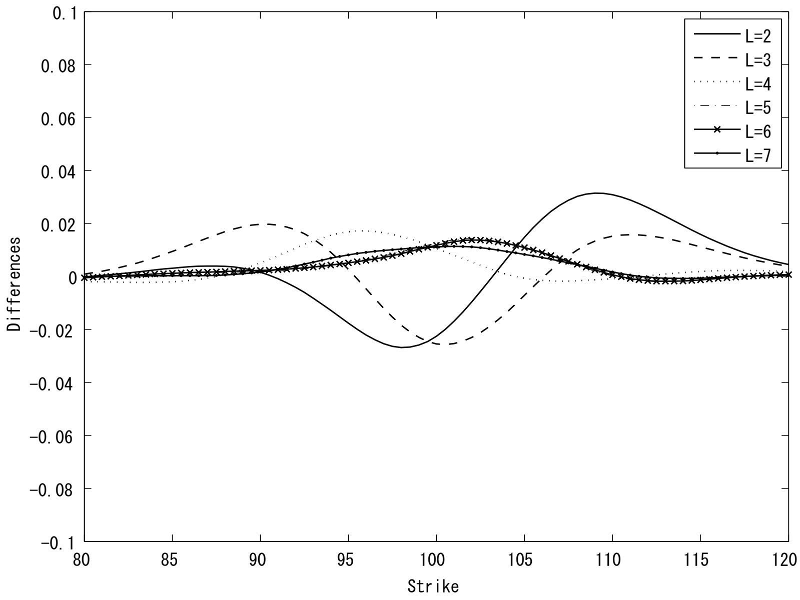


Figure 2.12: Differences NIG-CIR Prices between MC and Our Formula



Part II

株式とクレジットの金融派生商品の統合評価モデル

Chapter 3

An Extension of CreditGrades Model Approach with Lévy Processes

This chapter proposes an extended CreditGrades model for pricing equity options and CDSs simultaneously. The original CreditGrades model presented by Finger et al. [2002], and Stamicar and Finger [2006] is one of the most approved approaches to link between credit and equity markets. Lévy processes are then introduced into the original model in order to describe non-continuous dynamics of reference firm's asset value. In this setting, quasi closed-form formulae for pricing equity options and for calculating survival probabilities of the firm are derived, and we focus on investigating jump effects of the firm's value on short term credit spread and equity volatility skew.

With remarkable development of derivatives such as CDSs and equity options, linkage between credit and equity markets is one of the leading issues among practitioners. For example, capital structure arbitrage, convertible bond arbitrage, and credit relative value trading have become preferred strategies among hedge funds. Moreover, many asset managers and banks measure firm-specific credit risk in fixed-income portfolios by careful examination of the equity market, in particular, the equity option market. By incorporating the interaction between credit and equity risk, sophisticated trading strategies and risk management can be implemented.

In Merton's seminal paper [1974], a classical firm value model is introduced in order to deal with the credit risk of a specific entity, in which the company defaults if the asset value becomes less than its debt payment at maturity. Black and Cox [1976], and Leland and Toft [1996] extended Merton's model to take into account the possibility that default may happen prior to the maturity date. Besides these, there are many extensions of Merton's model; for example, stochastic interest rates (Longstaff and Schwartz [1995]), stochastic default barriers (Finger et al. [2002]), jumps in the dynamics of the firm's asset value (Zhou [1997, 2001]). These credit risk modelings are known as *structural approach*, in contrast to *intensity-based approach.*

The CreditGrades model also belongs to the class of structural approach. Although Merton's original model provides no connection to the equity option market, the CreditGrades model explicitly connects credit risk with the equity option market. There have been various empirical investigations using the model of linkage between credit and equity markets since the model was presented. For example, Veraart [2004] examined default probabilities of some commercial banks and compared the CreditGrades model with the KMV model; Bystrom [2006] investigated the predictive ability of the CreditGrades model by using empirically observed CDS spreads of iTraxx indices covering Europe. Yu [2006], Bedendo et al. [2007], and Bajlum [2007] studied capital structure arbitrage trading using the CreditGrades model.

One of the problems in the structural model approach is so-called predictability of default, which is discussed in detail in Bielecki and Rutkowski [2002], Lando [2004], and Elizalde [2005]. That is, since most of the structural models assume a continuous diffusion process for dynamics of the firm's asset value and complete information about the firm's value and the default barrier, the distance from the current firm's value to the default barrier completely inform us about the nearness of default. As a result, if the current value of the firm is remote from the barrier, both the default probability and the credit spread in short-term are close to zero; because the process of the firm's value requires time to reach the barrier. This phenomenon contradicts empirical data in credit markets.

Another issue in the original CreditGrades model is that the implied volatility skew on equity options highly depends on the leverage ratio of the firm's financial structure. In the original CreditGrades model, because the equity process of the firm follows a shifted log-normal distribution, the equity volatility becomes the local volatility function, which depends on only the current stock price. Thus, although the CreditGrades model naturally introduces the volatility skew, it is not able to reflect unpredictable credit events into the implied volatility.

One of the approaches to overcome these problems is to include jumps in the firm's asset value process. For example, Zhou [1997, 2001] introduced an extended Merton model with jump risk, and using this model, he examined jump impacts of the firm's value. Although there is much literature researching structural approach with jumps (e.g., Lipton [2002a], Rogers and Hilberink [2002], Cariboni and Schoutens [2007], Madan and Schoutens [2008], and Jönsson and Schoutens[2008]), these studies focused upon credit risk of individual firms rather than equity derivative pricing. In contrast, Sepp [2006] proposed an extended CreditGrades model with jumps to price both equity options and CDSs simultaneously. However, his jump CreditGrades model dealt with only a double-exponential jump-diffusion process. In this chapter we present the framework of an extended CreditGrades model with general Lévy processes. In a sense, this chapter provides a generalization of Sepp's CreditGrades model.

Lévy processes are well-known as an appropriate class of stochastic processes with jumps in order to express various underlying asset dynamics and to price many derivative products. The processes are studied by numerous financial researchers such as Merton [1976], Barndorff-Nielsen [1998], Madan, et al. [1998], Boyarchenko and Levendorskiǐ [2002], Carr, et al. [2002], Kou [2002, 2003], Eraker, et al. [2003], Nguyen-Ngoc [2003], Sepp and Skachkov [2003], Asmussen et al. [2005], Jeannin and Pistorius [2010].

3.1 Models

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T^*}, \mathbb{Q})$ be a filtered probability space, where T^* is some time horizon, and $\mathbb{Q}$ is a risk neutral probability measure. We consider a certain reference firm and use the following notations: S_t and V_t denote the firm's equity price per share and its asset value per share, respectively. For simplicity, it is assumed that the firm's total debt per share; denoted by B, is a strictly positive constant value.

Suppose that the asset value V_t is driven by a suitable stochastic process under the risk neutral measure $\mathbb{Q}$ and the firm defaults when the asset value falls below the barrier B. Thus, the time τ of the default on time interval $(0, T]$ is defined as

$$\tau = \inf\{t \in (0, T] : V_t \leq B\}, \tag{3.1}$$

with τ being an $\mathbb{F}$-stopping time. In this chapter, the firm's debt B is identified with the default barrier for simplicity.

Next, we propose a new model introducing a Lévy process into the original CreditGrades model. In the sequel it is called the *Lévy CreditGrades model*. Thus, under the Lévy CreditGrades model approach, the firm's asset value follows

$$V_t = V_0 e^{X_t}, \tag{3.2}$$

where $V_0 := S_0 + B$ is the initial asset value, $X := (X_t)_{t\geq 0}$ is a one-dimensional Lévy process on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T^*}, \mathbb{Q})$; i.e. X_t is adapted to $\mathcal{F}_t$, the sample paths of X are right continuous with left limits, and $X_u - X_t$ is independent of $\mathcal{F}_t$ and has the same distribution as X_{u-t} for

$0 \le t < u$. In addition, assume that the Lévy process X is exponential martingale under the risk neutral measure $\mathbb{Q}$. Note that the default time τ is not a predictable stopping time, but a *totally inaccessible stopping time* because of discontinuity property of Lévy processes (see chapter III.2. in Protter [2003]). This fact is very important for credit risk modeling since the default time is considered to be one of unpredictable events in financial markets.

We define the dynamics of the equity price as

$$S_t = \begin{cases} (V_t - B)e^{\int_0^t (r_s - d_s)ds} & \text{if} \quad t < \tau, \\ 0 & \text{otherwise}, \end{cases} \tag{3.3}$$

where r_t is a deterministic risk-free interest rate, and d_t is a deterministic dividend yield of the firm's equity. By virtue of the specification (3.3), we can estimate parameters of the asset value process V_t, which is not directly observable, by linking the asset value to the equity process and using available market data such as implied volatilities on the equity option market.

In order to analyze the Lévy CreditGrades models, we apply the characteristic function approach, which is based on the characteristic function of a Lévy process X_t. The characteristic function Ψ_{X_t} of the distribution of the random variable X_t can be represented in the following form:

$$\Psi_{X_t}(\theta) := \mathbb{E}\left[e^{i\theta X_t}\right] = \exp\left\{-t\psi_X(\theta)\right\}, \tag{3.4}$$

where $\mathbb{E}[\,\cdot\,]$ is the expectation operator under the risk neutral measure $\mathbb{Q}$. The function ψ_X is called the characteristic exponent of X. The following proposition gives us the explicit representation of the characteristic exponent. The proof can be found on pp.37-45 of Sato [1999].

Proposition 3.1 *(Lévy-Khintchine formula) Let $X = (X_t)_{t \ge 0}$ be a Lévy process on $\mathbf{R}$. Then its characteristic exponent ψ_X is given by*

$$\psi_X(\theta) = -i\gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{+\infty} \left(1 - e^{i\theta x} + i\theta x \mathbf{1}_{|x| \le 1}\right) \Pi(dx), \tag{3.5}$$

where $\sigma \ge 0$ and $\gamma \in \mathbf{R}$ are constants, and Π is a measure on $\mathbf{R} \setminus \{0\}$ satisfying

$$\int_{-\infty}^{+\infty} \left(1 \wedge x^2\right) \Pi(dx) < +\infty. \tag{3.6}$$

The parameter σ^2 is called the Gaussian coefficient and the measure Π is called the Lévy measure. The triplet (γ, σ^2, Π) is referred to as the *Lévy characteristics* of X. Intuitively, γ describes the constant drift of the process and the Gaussian coefficient σ^2 describes constant variance of the continuous component of the process. The Lévy measure Π describes the jump structure of the jump component of the process. If $\Pi = 0$, the Lévy process is Gaussian, and if $\sigma^2 = 0$, the process is a jump process without the diffusion component.

If the Lévy CreditGrades model has the following Gaussian process, we call it the *standard model*:

$$X_t = \sigma W_t - \frac{1}{2}\sigma^2 t, \tag{3.7}$$

where W_t is a one-dimensional standard Brownian motion under the risk neutral measure $\mathbb{Q}$, and σ is asset volatility. Here, the term $-\frac{1}{2}\sigma^2 t$ in the equation (3.7) is convexity correction to make the asset value V_t a martingale. In the case of the standard model, since the equity price process (3.3) is a shifted log-normal process, the equity volatility σ_t^S becomes the following local volatility function:

$$\sigma_t^S = \sigma \frac{S_t + B}{S_t}. \tag{3.8}$$

Therefore, the standard model can describe the implied equity volatility skew naturally. However, because it strongly depends on the leverage ratio of the firm's debt; the standard model of different firms describes exactly the same volatility skew shapes when the firms have the same debt-equity ratio and asset volatility, and the volatility skew cannot reflect rare credit events in the future, which might damage the firm's value. Furthermore, the standard model cannot describe higher shorter credit spreads of CDS; because the process (3.7) is continuous. The idea of making higher shorter spreads is to set a stochastic default barrier with some distribution (e.g., see Finger et al. [2002]). However it is difficult to choose the appropriate distribution of the stochastic default barrier, because the stochastic behavior of the barrier is usually unobservable.

On the other hand, by introducing a Lévy process into the model, our models can describe the jump risk of the firm's asset value and the default event becomes unpredictable without a stochastic default barrier. Therefore, not only can the Lévy CreditGrades models generate higher shorter credit spreads of CDS; but also they can draw the volatility skew, including speculation of rare event risks.

3.2 Pricing Equity Options and CDSs

In this section, how to price equity options and CDSs of a certain firm under the Lévy CreditGrades model are considered; also the formulae of the equity option prices and the survival probabilities are derived. These formulae, which are quasi closed-form solutions, are the main contributions.

3.2.1 Fundamental formulae of Equity Options and CDSs

We first provide the fundamental formula of equity option pricing in the general CreditGrades model approach. In this approach, equity options are evaluated as down and out options, which vanish after the firm's asset value falls below the default barrier. Therefore the payoff function of the call option with strike K and maturity T is defined by

$$(S_T - K)^+ \mathbf{1}_{\{\tau > T\}}, \tag{3.9}$$

where τ is the default time. Then, the call option price at the initial time is given by

$$C = \mathbb{E}\left[e^{-\int_0^T r_t dt} (S_T - K)^+ \mathbf{1}_{\{\tau > T\}}\right]. \tag{3.10}$$

Next, we provide the fundamental formula of CDS par premiums, and it is shown that survival probabilities of the firm denoted by $\mathbb{Q}(\tau > t)$ allow us to price CDS par premium. Indeed, the fixed leg of a CDS can be represented as follows:

$$\begin{aligned} \text{Fixed Leg} &= \mathbb{E}\left[\int_0^T e^{-\int_0^t r_u du} cN \mathbf{1}_{\{\tau > t\}} dt\right] \\ &= cN \int_0^T e^{-\int_0^t r_u du} \mathbb{Q}(\tau > t)\, dt, \end{aligned} \tag{3.11}$$

where c is the premium of the CDS contract with maturity T, and N is the notional amount of the contract. On the other hand, the floating leg can be represented as follows:

$$
\begin{aligned}
\text{Floating Leg} &= \mathbb{E}\left[\int_0^T e^{-\int_0^t r_u du}(1-R)N\mathbf{1}_{\{\tau\in dt\}}dt\right] \\
&= (1-R)N \\
&\times \left(1 - e^{-\int_0^T r_t dt}\mathbb{Q}(\tau > T) - \int_0^T r_t e^{-\int_0^t r_u du}\mathbb{Q}(\tau > t)dt\right),
\end{aligned} \tag{3.12}
$$

where R is a constant recovery rate of the firm. The CDS par premium is chosen to equate the fixed leg and the floating leg, thus it can be calculated by

$$
c = (1-R)\frac{1 - e^{-\int_0^T r_t dt}\mathbb{Q}(\tau > T) - \displaystyle\int_0^T r_t e^{-\int_0^t r_u du}\mathbb{Q}(\tau > t)dt}{\displaystyle\int_0^T e^{-\int_0^t r_u du}\mathbb{Q}\left(\tau > t\right)dt}. \tag{3.13}
$$

Note that if the survival probabilities of the firm under the risk neutral measure $\mathbb{Q}$ is obtained, the CDS par premiums can be calculated. Hence, it is sufficient for pricing the premiums to know how to calculate the survival probabilities.

3.2.2 Equity Option Prices and Survival Probabilities under the Standard Model

In this subsection, we provide the formulae of equity option prices and survival probabilities under the standard model. Although these formulae, which are derived by Finger et al. [2002], Stamicar and Finger [2006], and Sepp [2006], are simple; they play important roles for the Lévy CreditGrades model. In following subsection, we apply the formulae for robust calculation of both equity option prices and survival probabilities under our model.

The call option of the standard model is evaluated as a down and out call option with a zero knock-out barrier under a shifted log-normal equity process. Thus, the call price with an asset volatility σ at the initial time, which is denoted by C^σ, is given by

$$
\begin{aligned}
C^\sigma =& C_{\mathrm{BS}}(T, S_0 + B, K + B, \bar{r}, \bar{d}) \\
&- \frac{S_0 + B}{B} C_{\mathrm{BS}}(T, \frac{B^2}{S_0 + B}, K + B, \bar{r}, \bar{d}), \\
\text{where} \quad & \bar{r} = \frac{1}{T}\int_0^T r_t dt, \quad \bar{d} = \frac{1}{T}\int_0^T d_t dt,
\end{aligned} \tag{3.14}
$$

$C_{\mathrm{BS}}(T, S, K, r, d)$ is the Black-Scholes price of a call option with maturity T, strike K, constant interest rate r, and constant dividend rate d on underlying price S with volatility σ. The formula (3.14) is quoted from Sepp [2006].

The survival probability of the standard model with an asset volatility σ, which is denoted by $\mathbb{Q}(\tau > T; \sigma)$, can be calculated by the following formula:

$$\mathbb{Q}(\tau > T;\sigma) = \mathcal{N}\left(\frac{\log\left(\frac{S_0+B}{B}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - \frac{S_0+B}{B}\mathcal{N}\left(\frac{\log\left(\frac{B}{S_0+B}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right), \tag{3.15}$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution function of standard normal distribution. The formula (3.15) is quoted from Finger et al. [2002].

3.2.3 Equity Option Prices and Survival Probabilities under the Lévy CreditGrades Model

Before we derive the formulae of the equity option prices and survival probabilities under the Lévy CreditGrades models, we introduce the Wiener-Hopf factors of a Lévy process, which are beneficial in evaluating the Fourier transforms of quantities related to maximum and minimum of a Lévy process. First, the maximum and minimum processes associated with a Lévy process X_t are defined as

$$M_t := \max_{0\le s\le t} X_s, \qquad N_t := \min_{0\le s\le t} X_s \tag{3.16}$$

In the following proposition, the Wiener-Hopf factors associated with the process X_t are introduced. The proof of Proposition 3.2 can be found on p.334 of Sato [1999].

Proposition 3.2 *Let $q > 0$. There exist a unique pair of characteristic functions $\Phi^+_{q,X}(\theta)$ and $\Phi^-_{q,X}(\theta)$ of infinitely divisible distributions having zero drift and supported on $[0,\infty)$ and $(-\infty,0]$ respectively such that*

$$\frac{q}{q+\psi_X(\theta)} = \Phi^+_{q,X}(\theta)\Phi^-_{q,X}(\theta), \quad \theta \in \mathbf{R}. \tag{3.17}$$

These functions have the following representations.

$$\Phi^+_{q,X}(\theta) := \exp\left\{\int_0^{+\infty} t^{-1}e^{-qt}dt \int_0^{+\infty} \left(e^{i\theta x} - 1\right) dF_{X_t}(x)\right\}, \tag{3.18}$$

$$\Phi^-_{q,X}(\theta) := \exp\left\{\int_0^{+\infty} t^{-1}e^{-qt}dt \int_{-\infty}^{0} \left(e^{i\theta x} - 1\right) dF_{X_t}(x)\right\}, \tag{3.19}$$

where $F_{X_t}(\cdot)$ is the distribution function of a random variable X_t.

The function $\Phi^+_{q,X}(\theta)$ and $\Phi^-_{q,X}(\theta)$ are called the *Wiener-Hopf factors*. The function $\Phi^+_{q,X}(\theta)$ can be continuously extended to a bounded analytic function without zeros on the upper half plane and $\Phi^-_{q,X}(\theta)$ can be similarly extended to the lower half plane.

The following theorem shows the equity option pricing formula under the Lévy CreditGrades model. The Wiener-Hopf factors play crucial roles for deriving the pricing formulae. The proof of Theorem 3.3 is provided at the end of this chapter.

Theorem 3.3 *(Option pricing formula under the Lévy CreditGrades model) Let X_t be a Lévy process driving the CreditGrades model, and $\alpha, \beta > 0, \gamma \in \mathbf{R}$ and $\sigma > 0$ be some real values. Then the equity option price C with strike K and maturity T is given by the following representation:*

$$C = e^{-\int_0^T d_t dt}(S_0 + B)f(T,k,b) + C^\sigma, \tag{3.20}$$

where

$$
\begin{aligned}
f(T,k,b) &:= \frac{e^{-(\alpha k+\beta b-\gamma T)}}{(2\pi)^3}\iiint_{\mathbf{R}^3} e^{-i(uk+vb-\omega T)}\kappa(\gamma+i\omega,u,v)dudvd\omega,\\
\kappa(q,u,v) &:= \frac{1}{q(iu+\alpha)(iv+\beta)(iu+\alpha+1)}\\
&\quad\times\Big\{\Phi^{+}_{q,X}(u-i[\alpha+1])\Phi^{-}_{q,X}(u+v-i[\alpha+\beta+1])\\
&\qquad -\Phi^{+}_{q,Y}(u-i[\alpha+1])\Phi^{-}_{q,Y}(u+v-i[\alpha+\beta+1])\Big\},\\
k &:= \log\left(\frac{Be^{\int_0^T(r_t-d_t)dt}+K}{(S_0+B)e^{\int_0^T(r_t-d_t)dt}}\right),\\
b &:= \log\left(\frac{B}{S_0+B}\right),
\end{aligned}
\tag{3.21}
$$

$\Phi^{\pm}_{q,X}(\cdot)$ *and* $\Phi^{\pm}_{q,Y}(\cdot)$ *denote the Wiener-Hopf factors of the Lévy process* X_t *and a Gaussian process* $Y_t := \sigma W_t - \frac{1}{2}\sigma^2 t$ *respectively.*

Next, we derive the survival probability formula under the Lévy CreditGrades model. In the following theorem, the Wiener-Hopf factors again play crucial roles for calculating survival probabilities. The proof of Theorem 3.4 is provided at the end of this chapter.

Theorem 3.4 *(Survival probability formula under the Lévy CreditGrades models) Let* X_t *be a Lévy process driving a CreditGrades model, and* $\alpha > 0, \gamma \in \mathbf{R}$ *and* $\sigma > 0$ *be some real values. Then the survival probability* $\mathbb{Q}(\tau > t)$ *is given by the following representation:*

$$
\mathbb{Q}(\tau > t) = g(t,b) + \mathbb{Q}(\tau > t;\sigma), \tag{3.22}
$$

where

$$
\begin{aligned}
g(t,b) &:= \frac{e^{-(\alpha b-\gamma t)}}{(2\pi)^2}\iint_{\mathbf{R}^2} e^{-i(ub-\omega t)}\xi(\gamma+i\omega,u)dud\omega,\\
\xi(q,u) &:= \frac{\Phi^{-}_{q,X}(u-i\alpha)-\Phi^{-}_{q,Y}(u-i\alpha)}{q(iu+\alpha)},\\
b &:= \log\left(\frac{B}{S_0+B}\right),
\end{aligned}
\tag{3.23}
$$

$\Phi^{-}_{q,X}(\cdot)$ *and* $\Phi^{-}_{q,Y}(\cdot)$ *denote the Wiener-Hopf factors of the Lévy process* X_t *and a Gaussian process* $Y_t := \sigma W_t - \frac{1}{2}\sigma^2 t$ *respectively.*

By using the parameter α and β in Theorem 3.3 and 3.4, singularity on the integrands in the Fourier inversion can be avoided, since the numerical computation method, such as the *fast Fourier transform method*, evaluates the integrands at $u = 0$ and $v = 0$. The parameter γ is used for changing the inverse Laplace transform, which has some difficult problems in the numerical computation, into the Fourier transform. In our numerical examples, we set $\alpha = \beta = \gamma = 1$. Moreover, in Theorem 3.3, by considering the difference between option prices of the Lévy CreditGrades model and the standard model, the Fourier inversion converges quickly at infinity. The same technique is applied for Theorem

3.3. See pp.361-363 in Cont and Tankov [2003] for details. Numerical algorithms employed to compute the equation (3.21) and (3.23) can be found at the end of this chapter.

Note that in order to compute option prices and survival probabilities, we need to derive the explicit expression of the Wiener-Hopf factors of the Lévy process X_t driving the model. However, in general, it is difficult to find the explicit expression of the factors. Therefore, in the following section, a certain tractable class of Lévy processes for numerical examples are introduced.

3.3 Model Specifications

In this section, we provide three tractable examples of the Lévy CreditGrades model. According to Theorem 3.3 and 3.4, in order to calculate equity option prices and CDS premiums, we have to obtain the Wiener-Hopf factors of the Lévy process X_t driving the model. However, in general, it is difficult to find explicit forms of the factors and we must evaluate them numerically. Computations using the equation (3.18) and (3.19) are not very efficient, because these equations involve the probability density function of X_t which is usually not available in closed form.

Boyarchenko and Levendorskiĭ [2002] provides a more efficient expression which is valid for tempered stable, normal inverse Gaussian and several other Lévy processes:

$$\Phi^{+}_{q,X}(\theta) = \exp\left\{\frac{\theta}{2\pi i}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-}\frac{\log\left(q+\psi_X(\xi)\right)}{\xi\left(\theta-\xi\right)}d\xi\right\}, \tag{3.24}$$

with some $\omega_- < 0$ such that $\Phi^{+}_{q,X}(\theta)$ is analytic in the half plane $\Re\,\theta > \omega_-$. Similarly,

$$\Phi^{-}_{q,X}(\theta) = \exp\left\{-\frac{\theta}{2\pi i}\int_{-\infty+i\omega_+}^{+\infty+i\omega_+}\frac{\log\left(q+\psi_X(\xi)\right)}{\xi\left(\theta-\xi\right)}d\xi\right\}, \tag{3.25}$$

with some $\omega_+ > 0$ such that $\Phi^{-}_{q,X}(\theta)$ is analytic in the half plane $\Re\,\theta < \omega_+$. Note that the above integral must be computed for all values of θ and q to obtain a equity option price and a survival probability, and yet these computations are still time-consuming.

In our model specifications, we apply a certain class of Lévy processes called the *spectrally negative Lévy processes*. The spectrally negative Lévy processes have only negative jumps, i.e. the Lévy measure Π of X_t satisfies that $\Pi\left((0,+\infty)\right) = 0$. Using spectrally negative Lévy processes, we can express negative jumps of the firm's asset value with causes of some credit events, e.g. accounting practices, a scandal concerning the executives, detection of defective products, and other adverse occurrences. In order to price credit derivatives, it seems to be sufficient for firm's asset modeling to be described by only negative jumps with a diffusion component. In fact, Lipton [2002], Madan and Schoutens [2008], and Jönsson and Schoutens [2008] introduce spectrally negative Lévy processes to firm's asset value modeling. Madan and Schoutens [2008] demonstrates that negative pure jump processes such as CMY, Gamma, and inverse Gaussian process are able to calibrate historical CDS curves, and Jönsson and Schoutens [2008] shows that these processes are reasonable CDS spread generators to price vanilla and exotic options on single name CDSs. On the other hand, to price equity derivatives, it also seems to be satisfactory to apply the spectrally negative Lévy processes. There is a great deal of empirical evidence that stock volatility is negatively related to stock price. As a consequence, in the equity option markets it has been observed that equity implied volatility is decreasing in the option's strike price; it is commonly referred to as the *implied volatility skew.* Negative jumps of stock dynamics generate such an implied volatility skew. For these reasons, we support the spectrally negative Lévy process as the driving factor of the Lévy CreditGrades model.

If X_t is a spectrally negative Lévy process, the Wiener-Hopf factors are given by

$$\begin{aligned}\Phi^+_{q,X}(\theta) &= \frac{\eta_q}{\eta_q - i\theta},\\ \Phi^-_{q,X}(\theta) &= \frac{q\left(\eta_q - i\theta\right)}{\eta_q\left(q + \psi_X(\theta)\right)},\end{aligned} \tag{3.26}$$

where η_q is the unique positive real root of $q + \psi_X(-i\eta_q)$. See pp.346-348 of Sato [1999] for details.

In the case of the standard model with a Gaussian process $Y_t := \sigma W_t - \frac{1}{2}\sigma^2 t$, its characteristic exponent is $\psi_Y(\theta) = \frac{1}{2}\sigma^2\left(\theta^2 + i\theta\right)$. Since the process belongs to the class of spectrally negative Lévy processes, the Wiener-Hopf factors $\Phi^+_{q,Y}(\theta)$ and $\Phi^-_{q,Y}(\theta)$ can be obtained as the following expressions:

$$\begin{aligned}\Phi^+_{q,Y}(\theta) &= \frac{\eta_+}{\eta_+ - i\theta}, \quad \text{where} \quad \eta_+ = +\frac{1}{2} + \frac{1}{\sigma}\sqrt{\frac{\sigma^2}{4} + 2q},\\ \Phi^-_{q,Y}(\theta) &= \frac{\eta_-}{\eta_- + i\theta}, \quad \text{where} \quad \eta_- = -\frac{1}{2} + \frac{1}{\sigma}\sqrt{\frac{\sigma^2}{4} + 2q}.\end{aligned} \tag{3.27}$$

Note that the equations (3.27) can be used for calculating (3.21) in Theorem 3.3 and (3.23) in Theorem 3.4.

3.3.1 Exponential Jump (EJCG)

First, we consider a jump-diffusion process driving the Lévy CreditGrades model as follows:

$$X^E_t = \mu t + \sigma W_t - \sum_{j=1}^{N_t} Y_j, \tag{3.28}$$

where $\sigma > 0$, N_t and W_t denote Poisson process with intensity λ and Brownian motion respectively under the risk-neutral measure $\mathbb{Q}$, and the sequence of jump sizes $(Y_j)_{j\in\mathbf{N}}$ are i.i.d. random variables according to exponential distribution with parameter a. We refer to this model as the *Exponential Jump CreditGrades model* (EJCG). Note that the process X^E_t is a spectrally negative Lévy process and if the jump intensity $\lambda = 0$, this model is equivalent to the standard model.

We derive the characteristic exponent of X^E_t. Let $Z^E_t := \sigma W_t - \sum_{j=1}^{N_t} Y_j$, which is the random part of X^E_t. By Lévy-Khintchine formula, the characteristic exponent of Z^E_t is given by

$$\begin{aligned}\psi_{Z^E}(\theta) &= \frac{1}{2}\sigma^2\theta^2 + \lambda\left(1 - \Psi_Y(-\theta)\right)\\ &= \frac{1}{2}\sigma^2\theta^2 + \lambda\left(1 - \frac{a}{a + i\theta}\right),\end{aligned} \tag{3.29}$$

where $\Psi_Y(\theta) := a/(a - i\theta)$ is the characteristic function of the jump size Y_j. Because the process X^E_t satisfies exponential martingale under the risk-neutral measure $\mathbb{Q}$; its drift μ, which is called *convexity correction*, must be

$$\mu = \psi_{Z^E}(-i) = -\frac{1}{2}\sigma^2 + \frac{\lambda}{a+1}. \tag{3.30}$$

Therefore, the characteristic exponent of the process (3.28) is given by

$$\psi_{X^E}(\theta) = -i\mu\theta + \frac{1}{2}\sigma^2\theta^2 + \lambda\left(1 - \frac{a}{a + i\theta}\right). \tag{3.31}$$

Then, in order to obtain the Wiener-Hopf factors of X_t^E, the equation $q + \psi_{X^E}(-i\eta_q) = 0$ is solved with respect to η_q. This equation can be rewritten as

$$\sigma^2\eta_q^3 + (a\sigma^2 + 2\mu)\eta_q^2 + 2(a\mu - \lambda - q)\eta_q - 2aq = 0. \tag{3.32}$$

Since the equation (3.32) is a third degree polynomial equation, we can apply Cardano formula to solve it, i.e. the unique positive real solution η_q is given by

$$\begin{aligned}\eta_q = &-\frac{a_2}{3a_1} + \sqrt[3]{-\frac{a_2^3}{27a_1^3} + \frac{a_2a_3}{6a_1^2} - \frac{a_4}{2a_1} + \frac{1}{6}\sqrt{\frac{D}{3}}} \\ &+ \sqrt[3]{-\frac{a_2^3}{27a_1^3} + \frac{a_2a_3}{6a_1^2} - \frac{a_4}{2a_1} - \frac{1}{6}\sqrt{\frac{D}{3}}}, \\ \text{where} \quad & a_1 = \sigma^2, \quad a_2 = a\sigma^2 + 2\mu, \quad a_3 = 2(a\mu - \lambda - q), \quad a_4 = -2aq, \\ & D = 4\left(-\frac{a_2^2}{3a_1^2} + \frac{a_3}{a_1}\right)^3 + 27\left(\frac{2a_2^3}{27a_1^3} - \frac{a_2a_3}{3a_1^2} + \frac{a_4}{a_1}\right)^2.\end{aligned} \tag{3.33}$$

Substituting (3.33) for (3.26), we obtain the Wiener-Hopf factors of the process X_t^E for all q.

3.3.2 Gamma Jump (GJCG)

Second, we consider the following process driving the Lévy CreditGrades model:

$$X_t^G = \mu t + \sigma W_t - G_t, \tag{3.34}$$

where G_t is the Gamma process under $\mathbb{Q}$. We refer to this model as the *Gamma Jump CreditGrades model* (GJCG). The Gamma process is defined as the pure jump process which starts at zero, and has stationary and independent Gamma distributed increments. The Lévy density of the Gamma process is given by

$$\Pi^G(x) = \frac{\lambda e^{-ax}}{x}\mathbf{1}_{x>0}, \tag{3.35}$$

where λ and a are parameters of the Gamma process. X_t^G is obviously a spectrally negative Lévy process, and when $\lambda = 0$ this model is equivalent to the standard model.

Let $Z_t^G := \sigma W_t - G_t$. By Lévy-Khintchine formula, the characteristic exponent of Z_t^G is given by

$$\psi_{Z^G}(\theta) = \frac{1}{2}\sigma^2\theta^2 + \lambda\log\left(1 + \frac{i\theta}{a}\right), \tag{3.36}$$

and convexity correction μ must be

$$\mu = \psi_{Z^G}(-i) = -\frac{1}{2}\sigma^2 + \lambda\log\left(1 + \frac{1}{a}\right). \tag{3.37}$$

Therefore, the characteristic exponent of the process (3.34) is given by

$$\psi_{X^G}(\theta) = -i\mu\theta + \frac{1}{2}\sigma^2\theta^2 + \lambda\log\left(1 + \frac{i\theta}{a}\right). \tag{3.38}$$

For the sake of the Wiener-Hopf factors, we only have to obtain the unique positive real root of $q + \psi_{X^G}(-i\eta_q)$ numerically.

3.3.3 Inverse Gaussian Jump (IGJCG)

Third, we consider the following process driving the Lévy CreditGrades model:

$$X_t^I = \mu t + \sigma W_t - I_t, \tag{3.39}$$

where I_t is the Inverse Gaussian process under $\mathbb{Q}$. We refer to this model as the *Inverse Gaussian Jump CreditGrades model* (IGJCG). The Inverse Gaussian process is defined as the pure jump process which starts at zero, and has stationary and independent Inverse Gaussian distributed increments. The Lévy density of the Inverse Gaussian process is given by

$$\Pi^I(x) = \frac{\lambda \exp\left(-\frac{1}{2}a^2 x\right)}{\sqrt{2\pi}x^{3/2}} \mathbf{1}_{x>0}, \tag{3.40}$$

where λ and a are parameters of the Inverse Gaussian process. X_t^I is also a spectrally negative Lévy process, and when $\lambda = 0$ this model is equivalent to the standard model.

Let $Z_t^I := \sigma W_t - I_t$. By Lévy-Khintchine formula, the characteristic exponent of Z_t^I is given by

$$\psi_{Z^I}(\theta) = \frac{1}{2}\sigma^2\theta^2 + \lambda\left(\sqrt{a^2 + 2i\theta} - a\right), \tag{3.41}$$

and convexity correction μ must be

$$\mu = \psi_{Z^I}(-i) = -\frac{1}{2}\sigma^2 + \lambda\left(\sqrt{a^2 + 2} - a\right). \tag{3.42}$$

Therefore, the characteristic exponent of the process (3.39) is given by

$$\psi_{X^I}(\theta) = -i\mu\theta + \frac{1}{2}\sigma^2\theta^2 + \lambda\left(\sqrt{a^2 + 2i\theta} - a\right). \tag{3.43}$$

For the sake of the Wiener-Hopf factors, we only have to obtain the unique positive real root of $q + \psi_{X^I}(-i\eta_q)$ numerically.

Remark 3.5 *Similarly, other single sided jump processes such as the compound Poisson process with Gamma distribution and the CMY process (Madan and Schoutens [2008]) can be applied as jump components of spectrally negative Lévy process. Note that their Wiener-Hopf factors are obtained in the same manner as the above discussion.*

3.4 Numerical Examples

In this section, we show the numerical examples of equity option prices and CDS par premiums computed by EJCG, GJCG, or IGJCG with arbitrary parameters. Note that each model has only three parameters, i.e. σ, λ and a. We set $\sigma = 0.2$ and $\lambda = 0.00$, 0.25, 0.50, or 1.00 for all models, and $a = 10$, 8, and 4 for EJCG, GJCG, and IGJCG respectively. In addition, it is necessary to set an initial stock price S_0, a constant total debt B, deterministic interest rate r_t, and deterministic dividend yield d_t. Let us suppose that $S_0 = 100$, $B = 100$, and $r_t = d_t = 0$ for all $t \geq 0$.

First, we compute equity option prices with different strikes and maturities. Figure 3.1-3.6 plot implied equity volatilities with 3-month and 6-month maturities, which are generated by EJCG, GJCG, and IGCG. As for the numerical results of the option pricing, we can say that even if two firms have the same debt-equity ratio, different volatility skew curves of the firms can be drawn by using different types of the Lévy CreditGrades models with suitable parameters. Hence, the Lévy CreditGrades models have more extensive ability of drawing implied volatility skews than the original models. Furthermore, the volatility skews with longer maturity tend to be flatten by the central limit theorem on the Lévy process. This phenomenon consists with observed implied volatilities in equity option markets in general.

Next, we compute CDS par premiums with the recovery rate $R = 0.40$. Table 3.1 shows CDS par premium for each maturity by EJCG, GJCG, and IGCG. Moreover Table 3.2 shows the decay ratios of CDS premiums for each period by number of years. That is, the ratio of n to $(n-1)$ year is defined as follows:

$$\text{decay ratio} := \frac{n\text{-year CDS par premium} - (n-1)\text{-year CDS par premium}}{n\text{-year CDS par premium}} (\%)$$

Note that the existence of jump risk generates higher short-term spreads, which is in accordance with empirical observation. On the other hand, when the standard model (in the case of $\lambda = 0.00$) is used, short-term spreads decrease very rapidly and seem to converge to zero under 1-year maturity. As with the numerical results of CDS pricing, we find that the Lévy CreditGrades models are able to generate higher short-term spreads without a stochastic default barrier.

3.5 Concluding Remarks

In this chapter, we propose an extended CreditGrades model called the Lévy CreditGrades model, which is driven by Lévy process. Our main contribution is to introduce Lévy process into the original CreditGrades model and to derive the pricing formulae of both equity options and CDSs. In addition, we provide three tractable examples of the Lévy CreditGrades model using spectrally negative Lévy processes and present concrete calculation procedures in these modelings. Moreover, numerical examples show that our models have the extensive representation of equity option and CDS pricing.

Although the standard assumption that the firm's asset value dynamics follows a geometric Brownian motion is simple, this generates unrealistic low value for short-term CDS spreads and fits poorly to the implied volatility surface from equity options. Our extension overcomes this drawback by applying Lévy process which describes more realistic firm's value dynamics.

Finally, our next research topic will be to examine historical time-series data of equity options and CDS curves by the Lévy CreditGrades model, and to compare it with other firm value models such as the extended Merton model developed by Hull et al. [2005].

Mathematical Tools Mathematical tools for the proof of Theorem 3.3 and 3.4 are introduced below. The first two lemmas show that the Wiener-Hopf factors can be used for computing quantities related to the maximum and minimum of a Lévy process. The proof of the lemmas can be found on p.341 of Sato [1999].

Lemma 3.6 *(Wiener-Hopf factorization for a maximum process) The Laplace transform in t of the joint characteristic function of $(M_t, X_t - M_t)$ is given by*

$$q \int_0^{+\infty} e^{-qt} \mathbb{E}\left[e^{ixM_t + iy(X_t - M_t)}\right] dt = \Phi^+_{q,X}(x) \Phi^-_{q,X}(y), \tag{3.44}$$

for any $q > 0$ and $x, y \in \mathbf{R}$.

Lemma 3.7 *(Wiener-Hopf factorization for a minimum process) The Laplace transform in t of the joint characteristic function of $(N_t, X_t - N_t)$ is given by*

$$q \int_0^{+\infty} e^{-qt} \mathbb{E}\left[e^{ixN_t + iy(X_t - N_t)}\right] dt = \Phi^+_{q,X}(y) \Phi^-_{q,X}(x), \tag{3.45}$$

for any $q > 0$ and $x, y \in \mathbf{R}$.

By the following lemma, we change the inverse Laplace transform into the Fourier transform.

Lemma 3.8 *Let γ be a constant number and $f(t)$ be the inverse Laplace transform of $\bar{f}(\kappa)$:*

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\kappa t} \bar{f}(\kappa) d\kappa. \tag{3.46}$$

Then $f(t)$ is expressed as the following Fourier transform.

$$f(t) = \frac{e^{\gamma t}}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \bar{f}(\gamma + i\omega) d\omega. \tag{3.47}$$

Proof of Lemma 3.8: By changing the parameter κ into $\kappa = \gamma + i\omega$, then substituting this parameter for (3.46), the expression (3.47) can be obtained. □

In the equation (3.46), the parameter γ is a vertical contour in the complex plane chosen so that all singularities of $\bar{f}(\kappa)$ are to the left of it.

Proof of Theorem 3.3 The payoff function (3.9) can be expressed as follows:

$$\begin{aligned}(S_T - K)^+ \mathbf{1}_{\{\tau > T\}} &= \left((V_T - B) e^{\int_0^T (r_s - d_s) ds} - K \right)^+ \mathbf{1}_{\left\{\min_{0 \le s \le T} V_s > B\right\}} \\ &= \left(\tilde{V}_0 e^{X_T} - \tilde{K} \right)^+ \mathbf{1}_{\left\{\min_{0 \le s \le T} X_s > b\right\}},\end{aligned} \tag{3.48}$$

where $\tilde{V}_0 := V_0 e^{\int_0^T (r_s - d_s) ds}$, $\tilde{K} := B e^{\int_0^T (r_s - d_s) ds} + K$ and $b := \log(B/(S_0 + B))$. Thus the call price of the Lévy CreditGrades model with a Lévy process X_t is given by

$$\begin{aligned}C &= \mathbb{E}\left[e^{-\int_0^T r_t dt} (S_T - K)^+ \mathbf{1}_{\{\tau > T\}} \right] \\ &= \mathbb{E}\left[e^{-\int_0^T r_t dt} \left(\tilde{V}_0 e^{X_T} - \tilde{K} \right)^+ \mathbf{1}_{\left\{\min_{0 \le s \le T} X_s > b\right\}} \right] \\ &= e^{-\int_0^T r_t dt} \tilde{V}_0 \, \mathbb{E}\left[\left(e^{X_T} - e^k \right)^+ \mathbf{1}_{\left\{N_T^X > b\right\}} \right],\end{aligned} \tag{3.49}$$

where $k := \log(\tilde{K}/\tilde{V}_0)$ and $N_T^X := \min_{0 \le s \le T} X_s$. Similarly, the call price of the standard model with a Gaussian process $Y_t := \sigma W_t - \frac{1}{2}\sigma^2 t$ is given by

$$C^\sigma = e^{-\int_0^T r_t dt} \tilde{V}_0 \, \mathbb{E}\left[\left(e^{Y_T} - e^k \right)^+ \mathbf{1}_{\left\{N_T^Y > b\right\}} \right], \tag{3.50}$$

where $N_T^Y := \min_{0 \le s \le T} Y_s$.

We concentrate on calculating the difference between the call price of the Lévy CreditGrades model and that of the standard model. To do this, we define the following function:

$$\begin{aligned}f(T, k, b) &:= \frac{C - C^\sigma}{e^{-\int_0^T r_t dt} \tilde{V}_0} \\ &= \mathbb{E}\left[\left(e^{X_T} - e^k \right)^+ \mathbf{1}_{\left\{N_T^X > b\right\}} \right] - \mathbb{E}\left[\left(e^{Y_T} - e^k \right)^+ \mathbf{1}_{\left\{N_T^Y > b\right\}} \right].\end{aligned} \tag{3.51}$$

Then we consider the double Fourier transform of the function $e^{\alpha k + \beta b} f(T, k, b)$:

$$
\begin{aligned}
&\iint_{\mathbf{R}^2} e^{iuk+ivb} e^{\alpha k+\beta b} f(T,k,b) dkdb \\
&= \iint_{\mathbf{R}^2} dkdb \iint_{\mathbf{R}^2} dxdy\, e^{(iu+\alpha)k+(iv+\beta)b} \\
&\times \Big\{ \left(e^x - e^k\right)^+ \mathbf{1}_{\{y>b\}} \rho_{X_T,N_T^X}(x,y) - \left(e^x - e^k\right)^+ \mathbf{1}_{\{y>b\}} \rho_{Y_T,N_T^Y}(x,y) \Big\} \\
&= \iint_{\mathbf{R}^2} dxdy\, \rho_{X_T,N_T^X}(x,y) \int_{-\infty}^{y} \int_{-\infty}^{x} e^{(iu+\alpha)k+(iv+\beta)b} \left(e^x - e^k\right) dkdb \\
&\quad - \iint_{\mathbf{R}^2} dxdy\, \rho_{Y_T,N_T^Y}(x,y) \int_{-\infty}^{y} \int_{-\infty}^{x} e^{(iu+\alpha)k+(iv+\beta)b} \left(e^x - e^k\right) dkdb \qquad (3.52) \\
&= \frac{1}{(iu+\alpha)(iv+\beta)(iu+\alpha+1)} \\
&\times \Big\{ \iint_{\mathbf{R}^2} e^{(iu+\alpha+1)x+(iv+\beta)y} \rho_{X_T,N_T^X}(x,y) dxdy \\
&\quad - \iint_{\mathbf{R}^2} e^{(iu+\alpha+1)x+(iv+\beta)y} \rho_{Y_T,N_T^Y}(x,y) dxdy \Big\} \\
&= \frac{\Psi_{X_T,N_T^X}(u-i\alpha-i, v-i\beta) - \Psi_{Y_T,N_T^Y}(u-i\alpha-i, v-i\beta)}{(iu+\alpha)(iv+\beta)(iu+\alpha+1)},
\end{aligned}
$$

where $\rho_{X,Z}(\cdot,\cdot)$ and $\Psi_{X,Z}(\cdot,\cdot)$ denote the joint density function and the joint characteristic function of the random vector (X,Z) respectively.

Let the function $\kappa(q,u,v)$ denote the Laplace transform in T of (3.52):

$$
\kappa(q,u,v) := \int_0^{+\infty} e^{-qT} \iint_{\mathbf{R}^2} e^{iuk+ivb} e^{\alpha k+\beta b} f(T,k,b) dkdbdT. \qquad (3.53)
$$

By using Lemma 3.7, the function $\kappa(q,u,v)$ can be expressed as

$$
\begin{aligned}
\kappa(q,u,v) &= \frac{1}{(iu+\alpha)(iv+\beta)(iu+\alpha+1)} \\
&\times \Big\{ \int_0^{+\infty} e^{-qT} \Psi_{X_T,N_T^X}(u-i\alpha-i, v-i\beta) dT \\
&\quad - \int_0^{+\infty} e^{-qT} \Psi_{Y_T,N_T^Y}(u-i\alpha-i, v-i\beta) dT \Big\} \qquad (3.54) \\
&= \frac{1}{q(iu+\alpha)(iv+\beta)(iu+\alpha+1)} \\
&\times \Big\{ \Phi_{q,X}^{+}(u-i[\alpha+1]) \Phi_{q,X}^{-}(u+v-i[\alpha+\beta+1]) \\
&\quad - \Phi_{q,Y}^{+}(u-i[\alpha+1]) \Phi_{q,Y}^{-}(u+v-i[\alpha+\beta+1]) \Big\},
\end{aligned}
$$

Thus, by inverting the double Fourier transform and the Laplace transform, the function $e^{\alpha k+\beta b} f(T,k,b)$ can be obtained:

$$
e^{\alpha k+\beta b} f(T,k,b) = \frac{1}{2\pi i} \int_{\varsigma-i\infty}^{\varsigma+i\infty} e^{qT} \frac{1}{(2\pi)^2} \iint_{\mathbf{R}^2} e^{-iuk-ivb} \kappa(q,u,v) dudvdq. \qquad (3.55)
$$

Applying Lemma 3.8 for (3.55), we complete the proof of Theorem 3.3. □

Proof of Theorem 3.4 By definition of the default time τ, the survival probability $\mathbb{Q}(\tau > t)$ can be expressed as follows:

$$\begin{aligned}\mathbb{Q}(\tau > t) &= \mathbb{Q}\left(\min_{0\le s\le t} V_s > B\right)\\ &= \mathbb{Q}\left((S_0+B)\exp\left\{\min_{0\le s\le t} X_s\right\} > B\right)\\ &= \mathbb{Q}\left(N_t^X > b\right) = \mathbb{E}\left[\mathbf{1}_{\{N_t^X > b\}}\right],\end{aligned} \tag{3.56}$$

where $b := \log(B/(S_0+B))$ and $N_t^X := \min_{0\le s\le t} X_s$. Similarly, under a standard model with a Gaussian process $Y_t := \sigma W_t - \frac{1}{2}\sigma^2 t$, its survival probability $\mathbb{Q}(\tau > t; \sigma)$ is given by

$$\mathbb{Q}(\tau > t; \sigma) = \mathbb{E}\left[\mathbf{1}_{\{N_t^Y > b\}}\right], \tag{3.57}$$

where $N_t^Y := \min_{0\le s\le t} Y_s$.

Next, the following function is defined:

$$g(b,t) := \mathbb{Q}(\tau > t) - \mathbb{Q}(\tau > t; \sigma) \tag{3.58}$$

Then we consider the Fourier transform of the function $e^{\alpha b} g(b,t)$:

$$\begin{aligned}&\int_{\mathbf{R}} e^{iub} e^{\alpha b} g(b,t) db\\ &= \int_{\mathbf{R}} e^{iub+\alpha b} \mathbb{E}\left[\mathbf{1}_{\{N_t^X > b\}} - \mathbf{1}_{\{N_t^Y > b\}}\right] db\\ &= \int_{\mathbf{R}} e^{iub+\alpha b} \int_{\mathbf{R}} \left(\mathbf{1}_{\{y > b\}} \rho_{N_t^X}(y) - \mathbf{1}_{\{y > b\}} \rho_{N_t^Y}(y)\right) dy db\\ &= \int_{\mathbf{R}} dy \rho_{N_t^X}(y) \int_{-\infty}^{y} e^{iub+\alpha b} db - \int_{\mathbf{R}} dy \rho_{N_t^Y}(y) \int_{-\infty}^{y} e^{iub+\alpha b} db\\ &= \int_{\mathbf{R}} \frac{e^{(iu+\alpha)y}}{iu+\alpha} \rho_{N_t^X}(y) dy - \int_{\mathbf{R}} \frac{e^{(iu+\alpha)y}}{iu+\alpha} \rho_{N_t^Y}(y) dy\\ &= \frac{\Psi_{N_t^X}(u-i\alpha) - \Psi_{N_t^Y}(u-i\alpha)}{iu+\alpha},\end{aligned} \tag{3.59}$$

where $\rho_Z(\cdot)$ and $\Psi_Z(\cdot)$ denote the density function and the characteristic function of the random vector Z respectively.

Let the function $\xi(q,u)$ denote the Laplace transform in t of (3.59).

$$\xi(q,u) = \int_0^{+\infty} e^{-qt} \int_{\mathbf{R}} e^{iub} e^{\alpha b} g(t,b) db dt. \tag{3.60}$$

By Lemma 3.7, the function $\xi(q,u)$ can be expressed as

$$\begin{aligned}\xi(q,u) &= \frac{1}{iu+\alpha}\\ &\quad\times \left\{\int_0^{+\infty} e^{-qt} \Psi_{N_t^X}(u-i\alpha) dt - \int_0^{+\infty} e^{-qt} \Psi_{N_t^Y}(u-i\alpha) dt\right\}\\ &= \frac{1}{q(iu+\alpha)} \left\{\Phi_{q,X}^{-}(u-i\alpha) - \Phi_{q,Y}^{-}(u-i\alpha)\right\},\end{aligned} \tag{3.61}$$

Thus, we can obtain the function $e^{\alpha b}g(t,b)$ by inverting the Fourier transform and the Laplace transform:

$$e^{\alpha b}g(t,b) = \frac{1}{2\pi i}\int_{\varsigma-i\infty}^{\varsigma+i\infty} e^{qt}\frac{1}{2\pi}\int_{\mathbf{R}} e^{-iub}\xi(q,u)dudq. \tag{3.62}$$

Applying Lemma 3.8 for (3.62), we complete the proof of Theorem 3.4. □

Numerical Implementation We apply *the double exponential formula* proposed in Ooura [2005] to compute the Fourier transforms in the equation (3.21) and (3.23). Ooura [2005] demonstrates that this formula provides significantly efficient numerical computation of the Fourier transform. Here, we briefly review the formula.

Let N_-, N_+ be two sufficiently large integers and $N = N_- + N_+ + 1$. We choose a mesh-size $h > 0$ to be sufficiently small. Then, for some $\omega_0 > 0$, the Fourier transform

$$F(\omega) = \int_0^{+\infty} f(x)e^{i\omega x}dx, \quad \omega \in (0, 2\omega_0), \tag{3.63}$$

of f is approximated by

$$\begin{aligned} F_h^{(N)}(\omega) &= \frac{2\pi i}{\omega_0}\sum_{n=-N_-}^{N_+} f\left(\frac{\pi}{\omega_0 h}\varphi(nh)\right)\sin\left(\frac{\pi}{2h}\hat{\varphi}(nh)\right)\varphi'(nh) \\ &\quad \times\exp\left(\frac{\pi i\omega}{\omega_0 h}\varphi(nh) - \frac{\pi i}{2h}\hat{\varphi}(nh)\right), \end{aligned} \tag{3.64}$$

where

$$\begin{aligned} \varphi(t) &= \frac{t}{1-\exp\left(-2t-\alpha\left(1-e^{-t}\right)-\beta\left(e^{t}-1\right)\right)}, \\ \hat{\varphi}(t) &= \varphi(t) - t, \end{aligned} \tag{3.65}$$

and

$$\begin{aligned} \alpha &= \frac{\beta}{\sqrt{1+\log(1+\pi/\omega h)/4\omega h}}, \\ \beta &= 0.25. \end{aligned} \tag{3.66}$$

See Ooura [2005] for details.

In the equation (3.21) and (3.23), the integrations of the Fourier transform are done along total real axis $(-\infty, +\infty)$. To apply the equation (3.64), we divide the integration interval into $(0, +\infty)$ and $(-\infty, 0)$ and change the latter integration interval to $(0, +\infty)$ by changing of variable.

In the equation (3.21) and (3.23), we change the inverse Laplace transform into the Fourier transform by Lemma 3.8. We can also choose another way to directly compute the inverse Laplace transform by Gaver-Stehfest algorithm proposed in Stehfest [1970]. This algorithm is straightforward. For any bounded real-valued function $f(\cdot)$ defined on $[0, \infty)$ that is continuous at t, the inverse Laplace transform $\tilde{f}$ of f is given by

$$\tilde{f}(t) := \int_0^{\infty} e^{-ts}f(s)ds = \lim_{n\to\infty}\tilde{f}_n(t), \tag{3.67}$$

where

$$\tilde{f}_n(t) = \frac{\ln 2}{t}\frac{(2n)!}{n!(n-1)!}\sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}f\left((n+k)\frac{\ln 2}{t}\right). \tag{3.68}$$

To accelerate the convergence, an n-point Richardson extrapolation can be applied. More precisely, $\tilde{f}(t)$ is approximated by $f_n^*(t)$ for sufficiently large n, where

$$f_n^*(t) = \sum_{k=1}^{n} (-1)^{n-k} \frac{k^n}{k!(n-k)!} \tilde{f}_k(t). \tag{3.69}$$

See Stehfest [1970] for details.

Table 3.1: CDS par Premiums (bp)

Model	λ	1-Year	2-Year	3-Year	4-Year	5-Year
Standard	0.00	6	59	126	185	209
	0.25	24	95	169	221	252
EJCG	0.50	45	136	210	261	293
	1.00	96	212	289	331	347
	0.25	25	92	153	190	209
GJCG	0.50	42	119	181	217	234
	1.00	79	175	236	268	278
	0.25	22	91	152	191	210
IGJCG	0.50	37	118	182	219	236
	1.00	71	172	239	273	283

Table 3.2: Decay Ratio of CDS par Premiums

Model	λ	2 to 1-Year	3 to 2-Year	4 to 3-Year	5 to 4-Year
Standard	0.00	89.97%	53.12%	31.70%	11.66%
	0.25	74.71%	43.49%	23.56%	12.27%
EJCG	0.50	67.29%	35.03%	19.60%	10.97%
	1.00	54.60%	26.88%	12.65%	4.48%
	0.25	72.61%	39.66%	19.61%	9.23%
GJCG	0.50	64.85%	34.15%	16.42%	7.19%
	1.00	54.87%	26.10%	11.69%	3.68%
	0.25	75.37%	40.44%	20.13%	9.11%
IGJCG	0.50	68.60%	35.15%	16.89%	7.02%
	1.00	58.90%	27.88%	12.30%	3.52%

Figure 3.1: Implied Volatilities on the 3-Month Options by EJCG

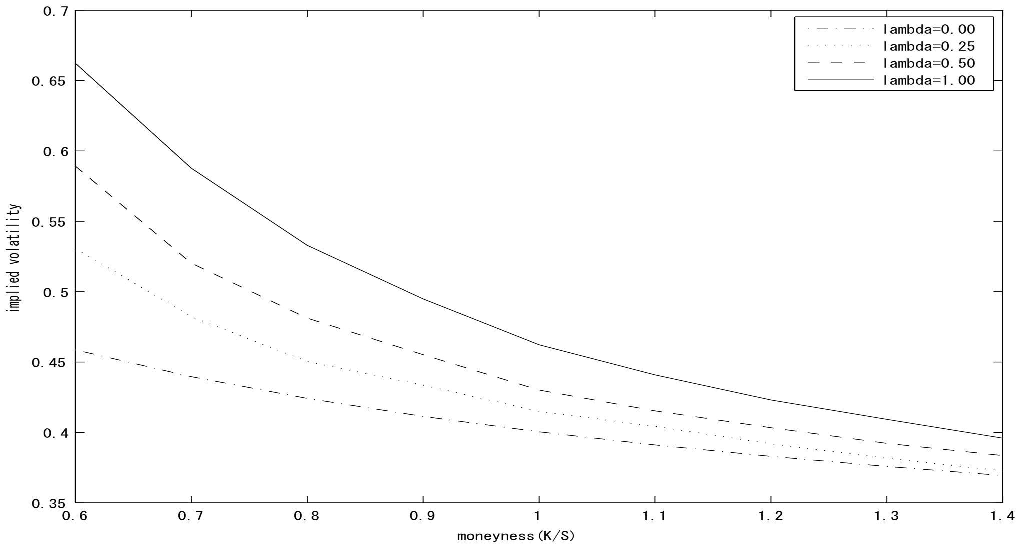


Figure 3.2: Implied Volatilities on the 6-Month Options by EJCG

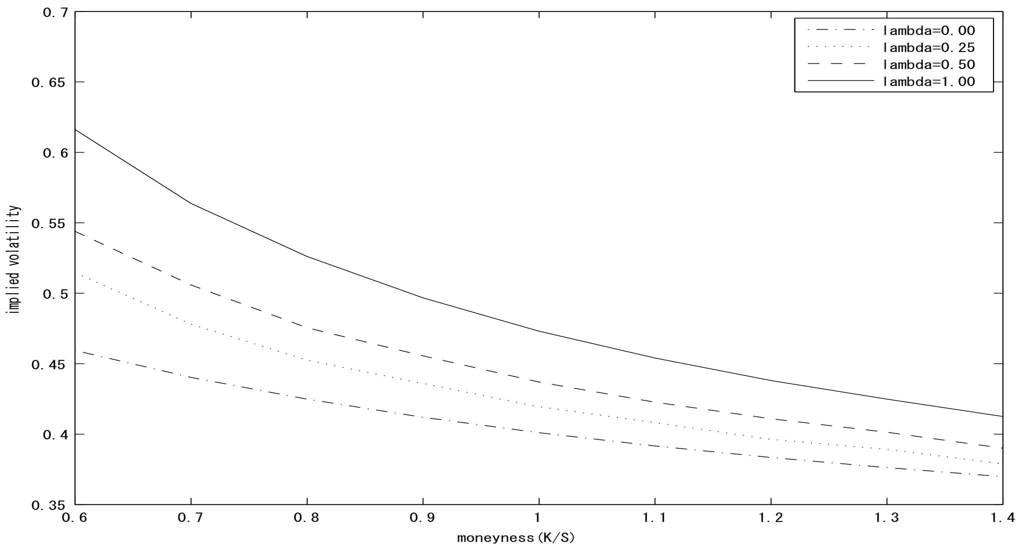


Figure 3.3: Implied Volatilities on the 3-Month Options by GJCG

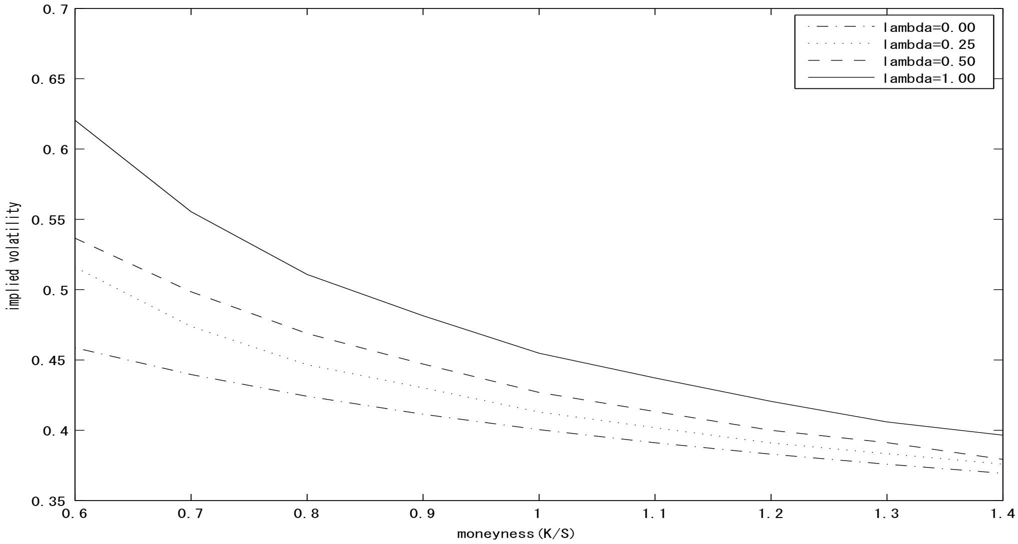


Figure 3.4: Implied Volatilities on the 6-Month Options by GJCG

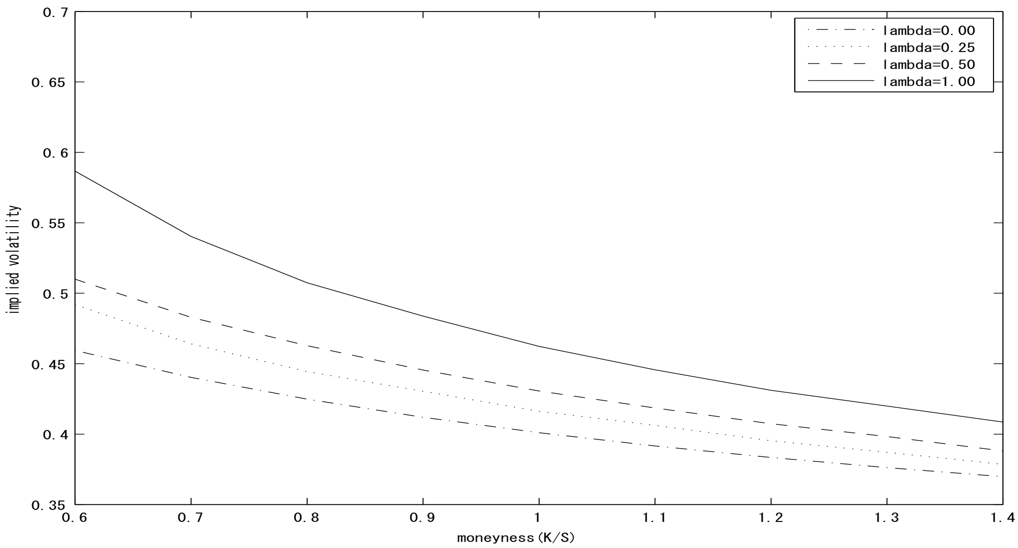


Figure 3.5: Implied Volatilities on the 3-Month Options by IGJCG

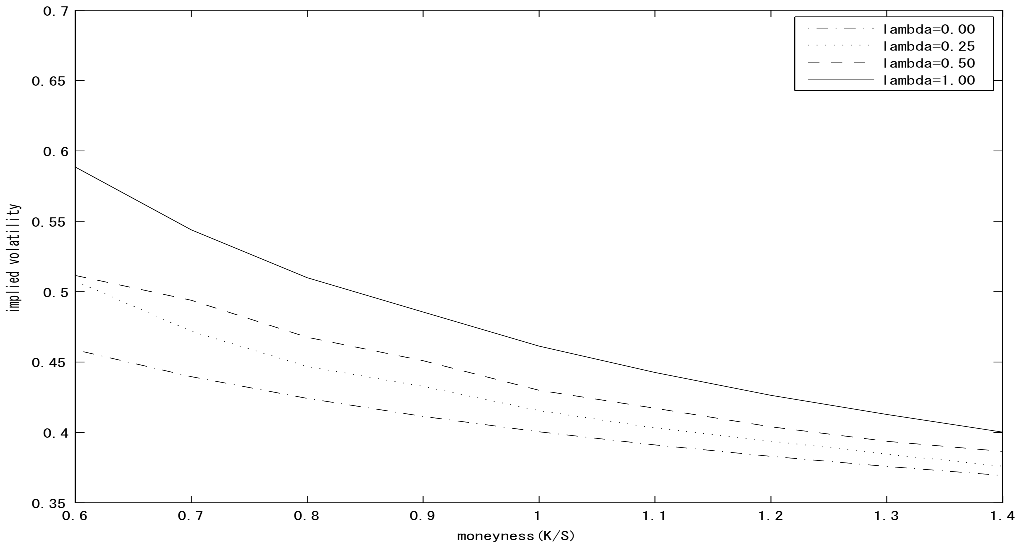
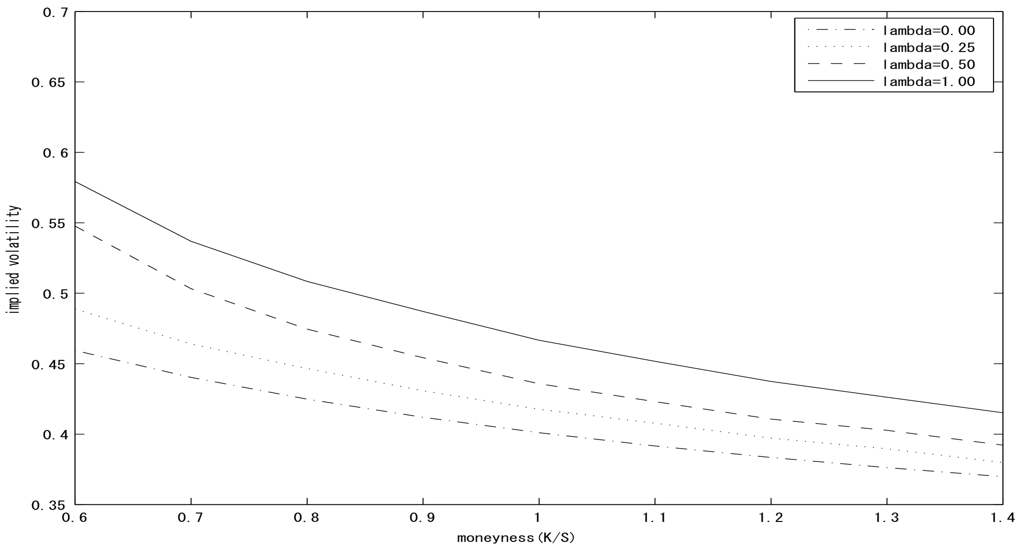


Figure 3.6: Implied Volatilities on the 6-Month Options by IGJCG



Chapter 4

Exponential Lévy Models Extended by a Jump to Default

With remarkable development of derivatives such as credit default swaps and equity options, interaction between credit and equity markets becomes one of the leading issues both academically and in practice. In fact, there is a great amount of researches investigating empirical linkage between equity and credit markets. For example, Zhang et al. [2005] and Cremers et al. [2008a] found some empirical evidences of relation between credit spread and implied volatility that are estimated by firm value models. Cremers et al. [2008b] showed by statistical analysis that implied volatility skew as well as its level contains important information for credit spread. Carr and Wu [2010] developed a joint framework for joint valuation of equity and credit derivatives and examined dynamic interaction between credit default swaps and equity options. Carr and Wu [2011] estimated default probabilities of individual firms from listed American put options. Furthermore, Ohsaki et al. [2010] demonstrated some theoretical relations between implied volatility and risk neutral default probability on a given firm under a simple assumption.

In the past few years, several hedging schemes across equity and credit markets have been proposed. Carr [2005] showed a dynamic hedging strategy by using the stock of a given firm and its call options for replication of a defaultable discount bond under an extended Black-Scholes model equipped with a constant default arrival rate. Carr and Wu [2011] provided a simple replication scheme of a virtual credit derivative named *the unit recovery claim*, which pays one dollar at default time, by using American put options. Carr and Schoutens [2008] explained how to perfectly hedge a defaultable contingent claim under Heston's stochastic volatility model with jump to default, in which not only the stock and the bond, but also the variance swap and the credit default swap are used for hedging. Ohsaki and Yamazaki [2011] developed a static hedging scheme of defaultable contingent claims. They demonstrated that any path-independent defaultable contingent claims such as not only plain vanilla, digital, and power options, but also defaultable bonds can be replicated by a static portfolio composed of a risk-free bond and plain vanilla options, all of whose maturity is shorter than that of the target defaultable contingent claims.

On the other hand, focusing on practical aspects, capital structure arbitrage, convertible bond arbitrage, and relative value trading across credit and equity markets have become preferred strategies among hedge funds. Moreover, many financial institutions measure firm-specific credit risk in corporate bond or loan portfolios by careful monitoring of equity markets, in particular, equity option market. Nowadays, the view that sophisticated trading strategies and risk management are able to be realized by incorporating the interactive relation between credit and equity risk is infiltrated into practitioners.

For further advanced researches in this field, developing more flexible and suitable unified credit-equity models is indispensable. In this chapter we propose a new dynamically consistent framework for joint valuation of equity derivatives, credit derivatives, and corporate bonds. To capture observed stock price dynamics and credit spread behaviors of a given firm, we assume that the pre-default stock price follows a geometric Lévy process, while the stock price jumps to zero when default occurs and then it remains there permanently. Furthermore, the default arrival rate is assumed to be expressed by the Cox proportional hazard model with stochastic covariates which are also driven by Lévy processes. That is, the originality of our work is to introduce Lévy processes into credit-equity hybrid modeling. Lévy processes are well-known as an appropriate class of stochastic processes with jump to express various underlying dynamics. In fact, there is a large number of financial applications of Lévy processes, and the processes themselves have been studied by numerous researchers in finance as well as in mathematics.

In our framework, we find the solution of the pricing generator for evaluating equity and credit derivatives, and then derive the quasi closed-form formulas for pricing credit default swaps and equity options by utilizing the pricing generator. We also examine the impact of randomness of Lévy processes for stochastic covariates on term structure of credit spreads. The framework proposed in this chapter can represent not only interactive dynamics of equity and credit markets, but also their dependency among individual firms. By virtue of arbitrary choice of Lévy processes, various model settings corresponding to different types of jump properties are available.

Generally speaking, the unified credit-equity modeling approaches are classified into two broad categories: *the structural approach* and *the intensity-based approach*. This classification is in accordance with traditional credit modelings (see Bielecki and Rutkowski [2002], Duffie and Singleton [2003], Schönbucher [2003], and Lando [2004] as an example of the monographs for traditional credit modelings).

An example of the structural approach of the unified modeling is Hull et al. [2005] who proposed an extension of Merton's firm value model (Merton [1974]) in which the equity option is priced as a compound option written on the firm value. The CreditGrades model introduced by Finger et al. [2002] and Stamicar and Finger [2006] is one of the most approved structural models among practitioners to jointly evaluate credit default swaps and equity plain vanilla options. In addition, several extensions of the CreditGrades model have been developed. For instance, Sepp [2002] proposed two extensions of the CreditGrades models with stochastic volatility and double exponential distributed jumps, and Ozeki et al. [2011] introduced a specified class of Lévy processes into the CreditGrades model.

On the other hand, in the intensity-based approach, such a unified modeling is well-known as *the jump to default model* pioneered by Merton [1976], who recognized the direct impact of firm's default on the stock price process and assumed that the stock price jumps to zero and stays there immutably upon the random default time with a constant default intensity. Recently, much literature dealing with the jump to default models has been published. For example, Takahashi et al. [2001], Ayache et al. [2003] and Linetsky [2006] enhanced the Black-Scholes model by equipped with a default intensity depending on the level of the stock price. Carr and Linetsky [2006] derived the closed-form formulas for pricing equity options and defaultable bonds in the jump to default extended CEV (constant elasticity of variance) model. Andersen and Buffum [2003] and Carr and Madan [2010] treated with local volatility models extended by a jump to default, while Carr and Schoutens [2008], Bayraktar and Yang [2011], Carr and Wu [2010] proposed intensity-based unified models with stochastic volatility. Moreover, Mendoza et al. [2010] introduced time-changed Markov processes into the jump to default model.

It can be said that the framework we propose belongs to the intensity-based approach. To the best of our knowledge, applying Lévy processes to the jump to default model has never been done in past literature. Although Carr and Wu [2010] added an independent jump component only to the stock dynamics, their stock price process can be essentially regarded as one of *the jump to default extended Heston's type stochastic volatility models* and their default intensity has no jump. In contrast to Carr and Wu [2010], our model can be exactly called *the jump to default extended exponential Lévy model*, because both the stock price process and the default arrival rate are fully driven by Lévy processes. In addition, arbitrary Lévy processes can be chosen as the driving factors for the model. As will be unveiled in the following, our modeling and analytical treatment are entirely different from those of Carr and Wu [2010].

4.1 Setup

We start with a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ carrying stochastic processes $(X_t^d)_{t\geq 0}$ where $d = 1, \ldots, D$, and exponential random variables with unit parameter $e^j \sim \text{Exp}(1)$ where $j = 1, \ldots, J$, that are independent of each other and all of processes $(X_t^d)_{t\geq 0}$. We denote by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by all of the processes $(X_t^d)_{t\geq 0}$. Each process $(X_t^d)_{t\geq 0}$ is assumed to be independent of other processes and follow a Lévy process on $\mathbf{R}$. A stochastic process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{Q})$ with values in $\mathbf{R}$ such that $X_0 = 0$ is called a Lévy process if it possesses the following properties: (1) X_t is adapted to $\mathcal{F}_t$. (2) The sample paths of $(X_t)_{t\geq 0}$ are right continuous with left limits. (3) $X_u - X_t$ is independent of $\mathcal{F}_t$ for $0 \leq t < u$. (4) $X_u - X_t$ has the same distribution as X_{u-t} for $0 \leq t < u$. Moreover, we assume frictionless and no arbitrage markets, and take an equivalent martingale measure $\mathbb{Q}$ as given.

4.1.1 Exponential Lévy Models in the Absence of Default Risk

Suppose $X_t := X_t^d$ as given d. In the absence of default risk, many studies in past literature have modeled dynamics of a firm's stock price $(S_t)_{t\geq 0}$ under $\mathbb{Q}$ as

$$S_t = S_0 \exp\left\{\int_0^t (r(s) - q(s))ds + \xi t + X_t\right\}, \tag{4.1}$$

where $r(\cdot)$ and $q(\cdot)$ denote the instantaneous risk-free interest rate and dividend yield, respectively, which are assumed to be deterministic functions over time; and ξ is some constant such that it makes $e^{\xi t + X_t}$ a $\mathbb{F}$-martingale. This modeling is called *the exponential Lévy model* in financial modeling and the parameter ξ is known as *convexity correction* in the context of the exponential Lévy model. Note that the stock price S_t in Eq.(4.1) is strictly positive as long as X_t is finite.

When analytically treating with the model in Eq.(4.1), the characteristic function of the distribution of X_t plays various important roles. The Lévy-Khintchine formula provided by the following proposition gives a general representation for the characteristic function of any Lévy process. The proof of the proposition can be found on pp.35-45 in Sato [1999].

Proposition 4.1 *(Lévy-Khintchine formula) Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbf{R}$. The characteristic function of the distribution of X_t has the form*

$$\phi_{X_t}(t, \theta) := \mathbb{E}\left[e^{i\theta X_t}\right] = e^{-t\psi_X(\theta)}, \quad t \geq 0, \tag{4.2}$$

where the characteristic exponent $\psi_X(\theta)$, $\theta \in \mathbf{R}$ is given by

$$\psi_X(\theta) = -i\mu\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{\infty} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{|x|\leq 1})\Pi(dx). \tag{4.3}$$

Here $\sigma \geq 0$ and $\mu \in \mathbf{R}$ are constant, and Π is a positive Radon measure on $\mathbf{R} \setminus \{0\}$ verifying

$$\int_{-\infty}^{\infty} (1 \wedge x^2)\Pi(dx) < \infty.$$

The parameter σ^2 is called *the Gaussian coefficient* and the measure Π is called *the Lévy measure*. The triplet (μ, σ^2, Π) is referred to as *the Lévy characteristics* of $(X_t)_{t\geq 0}$. Intuitively, μ describes the constant drift of the process and the Gaussian coefficient σ^2 denotes constant variance of the continuous component of the process. The Lévy measure Π expresses the jump structure of the jump component of the process. If $\Pi = 0$ the Lévy process is identified with Gaussian process, and if $\sigma^2 = 0$ the process becomes a pure jump process without the diffusion component. It is obvious from the Lévy-Khintchine formula that the convexity correction must be $\xi = \psi_X(-i)$ so as to make $e^{\xi t + X_t}$ a martingale.

One of the classes of Lévy processes is *finite-activity jump processes* that exhibit a finite number of jumps within any finite interval. The examples of finite-activity jump processes are compound Poisson jump processes with normally distributed jump size (Merton [1976]), double-exponential distributed jump size (Kou [2002]), and one-sided exponential distributed jump size (Eraker [2001], and Eraker et al. [2003]). Another class of Lévy processes is *infinite-activity jump processes* that generate an infinite number of jumps within any finite time interval. Examples in this class include the normal inverse Gaussian (NIG) process (Barndorff-Nielsen [1998]), the variance gamma (VG) process (Madan and Milne [1991], and Madan et al. [1998]), the finite moment log-stable (LS) process (Carr and Wu [2003]), the Meixner process (Schoutens [2002]), and the CGMY process (Carr et al. [2002]). Their Lévy measures and characteristic exponents are listed in Table 2.1. See Cont and Tankov [2004], and Boyarchenko and Levendorskiĭ [2002] for more details of Lévy processes in financial applications.

4.1.2 Exponential Lévy Models in the Presence of Default Risk

A main purpose of this chapter is to propose an extension of the exponential Lévy model by introducing default risk. First of all, we provide basic setup for an extended exponential Lévy model in the presence of default risk.

Suppose that there are J reference firms in markets. Let $\tau^j > 0$ be the random default time of the j-th firm, where $j \in \{1, \cdots, J\}$. Introducing a non-negative $\mathbb{F}$-progressively measurable process $(\lambda_t^j)_{t\geq 0}$, the default time τ^j is defined as

$$\tau^j = \inf\left\{t \geq 0 : \int_0^t \lambda_s^j ds \geq e^j\right\}.$$

We denote by $\mathbb{H}^j := (\mathcal{H}_t^j)_{t\geq 0}$ the associated filtration of τ^j, where $\mathcal{H}_t^j := \sigma(\mathbf{1}_{\{\tau^j > s\}} : s \leq t)$. Moreover, let $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0} = \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^J \vee \mathbb{F}$; i.e., $\mathcal{G}_t = \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^J \vee \mathcal{F}_t$ for any $t \in [0, \infty)$. Here, we also define that $H_t^j = \mathbf{1}_{\{\tau^j \leq t\}}$ is the default indicator, $F_t^j = \mathbb{Q}(\tau^j \geq t \mid \mathcal{F}_t)$ is the conditional default probability of the j-th firm, and $\Gamma_t^j = -\ln(1 - F_t^j) = \int_0^t \lambda_s^j ds$ is the hazard process of τ^j under $\mathbb{Q}$. Note that the compensated process $H_t^j - \Gamma_{t\wedge\tau^j}^j = H_t^j - \int_0^{t\wedge\tau^j} \lambda_s^j ds$ is a $\mathbb{G}$-martingale.

From now on, we assume that the stock price S_t^j for any $j = 1, \ldots, J$ is strictly positive before default while it is fixed at zero after default, i.e., for any $t \in [0, \infty)$,

$$\begin{cases} S_t^j > 0 & \text{if} \quad \tau^j > t \\ S_t^j = 0 & \text{if} \quad \tau^j \leq t, \end{cases} \tag{4.4}$$

holds for $\mathbb{Q}$-*a.e.* $\omega \in \Omega$. According to past literature, the modeling of Eq.(4.4) is named *the jump to default model*. The jump to default model is considered as a unified modeling of intensity-based credit model and traditional non-defaultable equity dynamics model in which the stock price jumps to zero at the default time and then it remains forever. In the following we propose a new type of the jump to default model that is driven by Lévy processes equipped with the Cox proportional hazard model as a default arrival rate.

Assumption 4.2 *(Jump to Default Exponential Lévy Model) For any $t < \tau^j$, the pre-default stock price dynamics is given by*

$$S_t^j = S_0^j \exp\left\{\int_0^t (r(s) - q^j(s) + \lambda_s^j) ds + \xi_\alpha^j t + \sum_{d=1}^D \alpha_d^j X_t^d\right\}, \tag{4.5}$$

where $q^j(\cdot)$ denotes the dividend yield of the j-th firm, $\alpha_1^j, \ldots, \alpha_D^j$ are coefficients of the stochastic covariates $X_t^1, \ldots, X_t^D$ for the j-th firm, and the parameter ξ_α^j given by

$$\xi_\alpha^j = \sum_{d=1}^{D} \psi_{X^d}(-i\alpha_d^j),$$

is the convexity correction of the model so as to make $e^{\xi_\alpha^j t + \sum_{d=1}^{D} \alpha_d^j X_t^d}$ a $\mathbb{F}$-martingale.

The incorporation of the default intensity $(\lambda_t^j)_{t\geq 0}$ in Eq.(4.5) compensates for the possibility of default so that the forward price of the stock remains a $\mathbb{F}$-martingale under $\mathbb{Q}$.

Next, we assume that each default intensity process is modeled as the Cox proportional hazard model (Cox [1972]) with the stochastic covariates $X_t^1, \ldots, X_t^D$.

Assumption 4.3 *(Cox Proportional Hazard Model) For $j = 1, \ldots, J$, the default intensity process $(\lambda_t^j)_{t\geq 0}$ of the j-th reference firm is given by*

$$\lambda_t^j = \bar{\lambda}^j(t) \exp\left\{ \sum_{d=1}^{D} \beta_d^j X_t^d \right\}, \quad \text{for } t \geq 0, \tag{4.6}$$

where $\bar{\lambda}^j : \mathbf{R}_+ \to \mathbf{R}_+$ is a deterministic and non-negative function with respect to time t, and $\beta_1^j, \ldots, \beta_D^j$ are coefficients of the stochastic covariates $X_t^1, \ldots, X_t^D$ for the j-th firm.

The function $\bar{\lambda}^j(t)$ called *baseline hazard function* describes arbitrary term structures of the default intensity. By virtue of introducing a Lévy process into the model, the dynamics of the default intensity as well as the stock price can have various types of jump properties. Comparing with other jump to default models, Eq.(4.6) seems to be more suitable for modeling of default intensities, because the Cox proportional hazard model is one of the most approved models in survival analysis and has been used to estimate probabilities of default in past literature; e.g., see Lane et al. [1986], Whalen [1991], Wheelock and Wilson [2000], Duffie and Singleton [2003], and Duffie et al. [2007].

Hereafter, we call the model satisfying Assumption 4.2 and 4.3 *the jump to default exponential Lévy model* (JDELM for short). Since JDELM is able to employ any Lévy processes, this modeling is capable of matching observed stock price dynamics and credit spread behaviors simultaneously. In addition, not only the interaction between credit and equity dynamics of a given firm, but also its dependency among individual firms can be represented in JDELM framework.

4.2 Pricing Generator

In the following the index j denoting individual firms is omitted for simplicity of notations. Before providing pricing formulas for CDSs and equity call options written on a reference firm under JDELM, let us consider the following function:

$$\Phi(t,\theta) := \mathbb{E}\left[\exp\left\{ -\int_0^t \lambda_s ds \right\} e^{i\theta \ln(S_t/S_0)} \right] = \exp\left\{ i\theta \int_0^t (r(s) - q(s))ds + i\theta \xi_\alpha t \right\} \Psi(t,\theta),$$

for $t \in [0, \infty)$ and $\theta \in \mathcal{D} \subset \mathbf{C}$, where $\mathcal{D}$ denotes the subset of the complex plane under which the expectation is well-defined, and

$$\Psi(t,\theta) := \mathbb{E}\left[\exp\left\{ (i\theta - 1) \int_0^t \lambda_s ds + i\theta \sum_{d=1}^{D} \alpha_d X_t^d \right\} \right].$$

The next section will show that the pricing formulas of both CDSs and equity options under JDELM can be represented as quasi closed-form solutions by using function $\Phi(t,\theta)$. In this sense the function $\Phi(t,\theta)$ is called *the pricing generator* of JDELM.

Firstly, the following technical lemma is provided.

Lemma 4.4 *Let $g:\mathbf{R}_+\to\mathbf{R}$ be an integrable function and*

$$G(x) := \int_0^x g(u)du,$$

Then for all $n\in\mathbf{N}$,

$$G(x)^n = n!\int_0^x\int_0^{u_n}\cdots\int_0^{u_2} g(u_n)g(u_{n-1})\cdots g(u_1)du_1du_2\cdots du_n. \tag{4.7}$$

Next, we provide the following lemma that plays a key role to obtain our main result.

Lemma 4.5 *Suppose that $(X_t)_{t\geq 0}$ follows a Lévy process on $\mathbf{R}$. For any $\beta\in\mathbf{R}$, $\theta\in\mathcal{D}$, and $0\leq t_1\leq\cdots\leq t_n\leq t$, it holds*

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^n \beta X_{t_k} + i\theta X_t\right\}\right] = \phi_{X_t}(t,\theta)\exp\left\{\sum_{k=1}^n a_{n,k}(\theta)t_k\right\}, \tag{4.8}$$

where

$$a_{n,k}(\theta) := \psi_X(\theta - i[n-k]\beta) - \psi_X(\theta - i[n-k+1]\beta), \quad \textit{for } k=1,\ldots,n.$$

Proof of Lemma 4.5: Rearranging the sum in the left hand side of Eq.(4.8), it can be rewritten as

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^n \beta X_{t_k} + i\theta X_t\right\}\right] = \mathbb{E}\left[\exp\left\{\sum_{k=1}^{n+1}(i\theta + [n-k+1]\beta)(X_{t_k} - X_{t_{k-1}})\right\}\right],$$

where we set $t_{n+1} := t$. By the definition of Lévy processes and the Lévy-Khintchine formula, we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{\sum_{k=1}^n \beta X_{t_k} + i\theta X_t\right\}\right] &= \prod_{k=1}^{n+1}\mathbb{E}\left[e^{(i\theta+[n-k+1]\beta)(X_{t_k}-X_{t_{k-1}})}\right] \\
&= \prod_{k=1}^{n+1}\mathbb{E}\left[e^{(i\theta+[n-k+1]\beta)X_{t_k-t_{k-1}}}\right] \\
&= \exp\left\{-\sum_{k=1}^{n+1}(t_k - t_{k-1})\psi_X(\theta - i[n-k+1]\beta)\right\}.
\end{aligned}$$

Since

$$\sum_{k=1}^{n+1}(t_k - t_{k-1})\psi_X(\theta - i[n-k+1]\beta) = t\psi_X(\theta) - \sum_{k=1}^n a_{n,k}(\theta)t_k,$$

we obtain Eq.(4.8). □

The following theorem which gives the explicit expression of the pricing generator $\Phi(t,\theta)$ is the main result of this chapter.

Theorem 4.6 *Under Assumption 4.2 and 4.3, for any* $t \in [0,\infty)$ *and* $\theta \in \mathcal{D}$*, it holds*

$$\Phi(t,\theta) = \exp\left\{ i\theta \int_0^t (r(s)-q(s))ds + i\theta\xi_\alpha t \right\} \left(\prod_{d=1}^D \phi_{X_t^d}(t,\alpha_d\,\theta) \right) \sum_{n=0}^\infty (i\theta-1)^n J_n(t,\theta),$$

where

$$J_n(t,\theta) := \begin{cases} 1 & \textit{for } n=0 \\ \displaystyle\int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \prod_{k=1}^n \bar{\lambda}(t_k) e^{A_{n,k}(\theta)t_k} dt_1 dt_2 \cdots dt_n & \textit{for } n \in \mathbf{N}, \end{cases} \tag{4.9}$$

and

$$\begin{aligned} A_{n,k}(\theta) &:= \sum_{d=1}^D a_{n,k}(\alpha_d\,\theta) \\ &= \sum_{d=1}^D \left\{ \psi_{X^d}(\alpha_d\,\theta - i[n-k]\beta_d) - \psi_{X^d}(\alpha_d\,\theta - i[n-k+1]\beta_d) \right\}. \end{aligned}$$

Proof of Theorem 4.6: By Lemma 4.4, we have

$$\begin{aligned} \Psi(t,\theta) &= \mathbb{E}\left[\left\{ \sum_{n=0}^\infty \frac{(i\theta-1)^n}{n!} \left(\int_0^t \lambda_s ds \right)^n \right\} \exp\left\{ i\theta \sum_{d=1}^D \alpha_d X_t^d \right\} \right] \\ &= \mathbb{E}\left[\left\{ 1 + \sum_{n=1}^\infty (i\theta-1)^n \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \lambda_{t_n}\lambda_{t_{n-1}} \cdots \lambda_{t_1} dt_1 dt_2 \cdots dt_n \right\} \exp\left\{ i\theta \sum_{d=1}^D \alpha_d X_t^d \right\} \right] \\ &= \prod_{d=1}^D \phi_{X_t^d}(t,\alpha_d\,\theta) \\ &\quad + \sum_{n=1}^\infty (i\theta-1)^n \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \left(\prod_{k=1}^n \bar{\lambda}(t_k) \right) \left(\prod_{d=1}^D \mathbb{E}\left[e^{\sum_{k=1}^n \beta_d X_{t_k} + i\theta\alpha_d X_t^d} \right] \right) dt_1 dt_2 \cdots dt_n. \end{aligned}$$

In the last equality, we use the assumptions that the default intensity is given by the Cox proportional hazard model in Eq.(4.6) and X_t^d, $d = 1, \cdots, D$ are independent of each other.

Applying Lemma 4.5 to the expectations in the left hand side of the above equation and using the independence assumption of X_t^d, $d = 1, \cdots, D$, we obtain Eq.(4.9). □

Note that, in the case that $\lambda_t = 0$ for all $t \geq 0$, $D = 1$, and $\alpha_1 = 1$, Eq.(4.9) is reduced to the well-known pricing generator of non-defaultable exponential Lévy models for calculating European equity derivatives. Furthermore, setting $\theta = 0$ in Eq.(4.9) we can compute survival probabilities of a given firm. The following corollary is very useful to price credit products.

Corollary 4.7 *Under Assumption 4.2 and 4.3, for any* $t \in [0,\infty)$*, it holds*

$$\begin{aligned} \frac{\partial}{\partial t}\Phi(t,0) &= -\mathbb{E}\left[\lambda_t \exp\left\{ -\int_0^t \lambda_s ds \right\} \right] \\ &= -\bar{\lambda}(t) \left(\prod_{d=1}^D \phi_{X_t^d}(t,-i\beta_d) \right) \sum_{n=0}^\infty (-1)^n K_n(t). \end{aligned} \tag{4.10}$$

where

$$K_n(t) := \begin{cases} 1 & \textit{for } n = 0 \\ \displaystyle\int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \prod_{k=1}^{n} \bar{\lambda}(t_k) e^{A_{n+1,k}(0)t_k} dt_1 dt_2 \cdots dt_n & \textit{for } n \in \mathbf{N}. \end{cases} \tag{4.11}$$

Proof of Corollary 4.7: Since $A_{n,n}(0) = -\sum_{d=1}^{D} \psi_{X^d}(-i\beta_d)$, we have

$$\frac{\partial}{\partial t} J_n(t,0) = \bar{\lambda}(t) e^{A_{n,n}(0)t} K_{n-1}(t) = \bar{\lambda}(t) \left(\prod_{d=1}^{D} \phi_{X_t^d}(t, -i\beta_d) \right) K_{n-1}(t),$$

for all $n \in \mathbf{N}$. Therefore, we obtain Eq.(4.10) from Theorem 4.6 and the above equation. □

It might not be feasible to compute high order values of the iterated integrals in Eq.(4.9) and (4.11) by any quadrature methods. However, it is worthwhile noting that the closed-form solutions of $J_n(t,\theta)$ and $K_n(t)$ can be obtained when the baseline hazard function $\bar{\lambda}(t)$ is given by an arbitrary linear combination of $t^m e^{c_m t}$, where $m = 0, 1, \ldots$ and $c_m \in \mathbf{R}$; i.e.,

$$\bar{\lambda}(t) = \sum_{m=0}^{M} b_m t^m e^{c_m t}, \quad b_m \in \mathbf{R}. \tag{4.12}$$

In this case, applying the well-known formula

$$\int t^m e^{at} dt = \frac{e^{at}}{a} \sum_{\gamma=0}^{m} (-1)^\gamma \frac{m!\, t^{m-\gamma}}{(m-\gamma)!\, a^\gamma}, \quad \text{for } a \in \mathbf{R} \setminus \{0\}, \ m = 0, 1, \ldots,$$

to the iterated integral repeatedly, we obtain the closed-form solutions of $J_n(t,\theta)$ and $K_n(t)$. The class of forms in Eq.(4.12) seems to be sufficient to describe any term structure of the default intensity because the class has much flexibility to match any curves. For instance, the Nelson-Siegel model and its extensions are strong candidates of $\bar{\lambda}(t)$ belonging to the class. The relation of the baseline hazard function, expected default intensity, and credit spread will be given in Section 4.4 analytically and shown in Section 4.5 numerically.

4.3 Joint Valuation of CDSs and Equity Options

This section provides the pricing formulas of credit default swaps and equity call options as typical examples of credit and equity products under JDELM. These formulas are represented by utilizing the pricing generator $\Phi(t,\theta)$ which is defined in the previous section. It will be also known that other contingent claims such as defaultable bonds, equity put and digital options can be easily priced in the similar manner. In the following, we assume $\tau > t \geq 0$ for the sake of simplicity of notations.

4.3.1 Pricing Credit Default Swaps

The most actively traded credit product in the over-the-counter market is credit default swaps (CDSs). The protection buyer pays a fixed premium to the protection seller periodically over time. If a credit event occurs on the reference firm, the buyer stops the premium payments and the seller pays a certain value depending on the loss of the credit event. The CDS spread (par premium) is determined such that the initial value of the premium leg is equal to that of the protection leg.

Consider a CDS contract with maturity T at time t, which has unit notional amount and continuous payments of the premium for simplicity. Let $s(t,T)$ denote the fixed premium rate of the CDS. Then, the time-t value of the premium leg can be written as

$$\begin{aligned}\text{Premium Leg} &= \mathbb{E}\left[\int_t^T s(t,T)e^{-\int_t^u r(s)ds}\mathbf{1}_{\{\tau>u\}}du \mid \mathcal{G}_t\right] \\ &= s(t,T)\int_t^T B(t,u)\mathbb{E}\left[e^{-\int_t^u \lambda_s ds} \mid \mathcal{F}_t\right]du,\end{aligned}$$

where $B(t,u)$ denotes the time-t price of a risk-free discount bond with maturity u. Furthermore, assuming constant recovery rate $\delta \in [0,1]$ upon the reference firm for simplicity, the time-t value of the protection leg can be written as

$$\begin{aligned}\text{Protection Leg} &= \mathbb{E}\left[(1-\delta)e^{-\int_t^\tau r(s)ds}\mathbf{1}_{\{\tau\le T\}} \mid \mathcal{G}_t\right] \\ &= (1-\delta)\int_t^T B(t,u)\mathbb{E}\left[\lambda_u e^{-\int_t^u \lambda_s ds} \mid \mathcal{F}_t\right]du.\end{aligned}$$

By equating the time-t value of the two legs, the par premium $s(t,T)$ can be obtained as

$$s(t,T) = (1-\delta)\frac{\int_t^T B(t,u)\mathbb{E}\left[\lambda_u e^{-\int_t^u \lambda_s ds} \mid \mathcal{F}_t\right]du}{\int_t^T B(t,u)\mathbb{E}\left[e^{-\int_t^u \lambda_s ds} \mid \mathcal{F}_t\right]du},$$

which can be regarded as a weighted average of the expected default loss.

Using the pricing generator $\Phi(t,\theta)$, we can immediately solve for the par premium of the CDS contract as follows:

Proposition 4.8 *Under Assumption 4.2 and 4.3, the par premium of CDS with maturity T at the initial time, $s(0,T)$, is given by*

$$s(0,T) = (\delta-1)\frac{\int_0^T B(0,u)\frac{\partial}{\partial u}\Phi(u,0)du}{\int_0^T B(0,u)\Phi(u,0)du},$$

where $\Phi(u,0)$ and $\frac{\partial}{\partial u}\Phi(u,0)$ are given by Theorem 4.6 and Corollary 4.7, respectively.

It is tangible that other simple credit products such as defaultable bonds can be priced by using the functions $\Phi(u,0)$ and $\frac{\partial}{\partial u}\Phi(u,0)$. Here, deriving formulas for such products is omitted because of trivial.

4.3.2 Pricing Equity Call Options

Next, consider the time-t value of a European call option on a given firm's stock S_t with strike K and maturity T. The terminal payoff of the option is $(S_T-K)^+$ if the firm has not defaulted by the maturity, and is zero otherwise. The value of the call option denoted by $c(t,T,K)$ can be written as

$$\begin{aligned}c(t,T,K) &= \mathbb{E}\left[e^{-\int_t^T r(s)ds}(S_T-K)^+\mathbf{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right] \\ &= B(t,T)\mathbb{E}\left[e^{-\int_t^T \lambda_s ds}(S_T-K)^+ \mid \mathcal{F}_t\right].\end{aligned}$$

The following formula giving equity call option prices is a kind of generalization of the pricing technique with the Fourier inversion pioneered by Carr and Madan [1999]. The main difference between the formula we develop in the following proposition and that developed by Carr and Madan [1999] is that the former is needed to consider the joint distribution of the stock price and survival probability of a given firm while the latter is sufficient to simply treat with only the distribution of the stock price.

Proposition 4.9 *Under Assumption 4.2 and 4.3, the European call price with strike K and maturity T at the initial time, $c(0,T,K)$, is given by*

$$c(0,T,K) = \frac{S_0 B(0,T) e^{-\alpha k}}{\pi} \int_0^\infty e^{-ik\theta} \frac{\Phi(T, \theta - [\alpha+1]i)}{(i\theta+\alpha)(1+i\theta+\alpha)} d\theta, \tag{4.13}$$

for any $\alpha > 0$. Here, $k := \ln(K/S_0)$ and $\Phi(T, \theta - [\alpha+1]i)$ is given by Theorem 4.6.

Proof of Proposition 4.9: Let

$$\Upsilon(k) := \frac{c(0,T,K)e^{\alpha k}}{S_0 B(0,T)} = e^{\alpha k} \mathbb{E}\left[\exp\left\{ -\int_0^T \lambda_s ds \right\} \left(e^{Z_T} - e^k \right)^+ \right],$$

where $Z_T := \ln(S_T/S_0)$. Then, consider the Fourier transform of $\Upsilon(k)$;

$$\tilde{\Upsilon}(\theta) := \int_{-\infty}^{\infty} e^{i\theta k} \Upsilon(k) dk.$$

Note that by the definition of $\Upsilon(k)$ it can be rewritten as

$$\Upsilon(k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\alpha k} y \left(e^z - e^k \right) \mathbf{1}_{\{z \geq k\}} dF(z,y).$$

Here, $F(z,y)$ denotes the joint distribution function of (Z_T, Y_T), where $Y_T := e^{-\int_0^T \lambda_s ds}$. Therefore, we have

$$\begin{aligned}
\tilde{\Upsilon}(\theta) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dF(z,y)\, y \int_{-\infty}^{\infty} dk e^{i\theta k} e^{\alpha k} \left(e^z - e^k \right) \mathbf{1}_{\{z \geq k\}} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dF(z,y)\, y \, \frac{e^{(1+i\theta+\alpha)z}}{(i\theta+\alpha)(1+i\theta+\alpha)} \\
&= \frac{1}{(i\theta+\alpha)(1+i\theta+\alpha)} \mathbb{E}\left[\exp\left\{ -\int_0^T \lambda_s ds \right\} e^{(1+i\theta+\alpha)Z_T} \right] \\
&= \frac{1}{(i\theta+\alpha)(1+i\theta+\alpha)} \Phi(T, \theta - [1+\alpha]i).
\end{aligned}$$

By the Fourier inversion of $\tilde{\Upsilon}(\theta)$, we obtain Eq.(4.13). □

The parameter α in Proposition 4.9 is used to avoid singularity on the integrand in the Fourier inversion of Eq.(4.13) because the numerical computation method, such as the fast Fourier transform method, evaluates the integrand at $\theta = 0$.

On the other hand, the time-t value of a put option denoted by $p(t,T,K)$ can be written as

$$\begin{aligned}
p(t,T,K) &= \mathbb{E}\left[e^{-\int_t^T r(s)ds} (K - S_T)^+ \mathbf{1}_{\{\tau > T\}} \mid \mathcal{G}_t \right] + \mathbb{E}\left[e^{-\int_t^T r(s)ds} K \mathbf{1}_{\{\tau \leq T\}} \mid \mathcal{G}_t \right] \\
&= B(t,T) \mathbb{E}\left[e^{-\int_t^T \lambda_s ds} (K - S_T)^+ \mid \mathcal{F}_t \right] + B(t,T) K \mathbb{E}\left[1 - e^{-\int_t^T \lambda_s ds} \mid \mathcal{F}_t \right].
\end{aligned}$$

Therefore, the pricing formula of the put option can be obtained in the similar derivation as the call option formula or by the put-call parity. Similarly to the above procedure, other European equity derivatives such as digital and power options are able to be priced and all of these formulas will be represented by the pricing generator $\Phi(t,\theta)$.

4.4 Term Structure of Credit Spread

In the setting of JDELM, any shapes of term structure of default intensity are able to be described by virtue of the baseline hazard function $\bar{\lambda}(t)$ of the Cox proportional hazard model. However, when calibrating to observed term structure of credit spread, we need to consider *convexity adjustment* against the expected value of the default intensity. In the following we examine relation of the baseline hazard function, expected default intensity and credit spread, and demonstrate how to determine the term structure of credit spreads under JDELM.

If covariate vector $(X_t^1, \ldots, X_t^d)$ is invariant over time, it is obvious that $\lambda_t = \bar{\lambda}(t)$ for all $t \geq 0$. In the special case, the term structure of the default intensity can be identified with the baseline hazard function $\bar{\lambda}(t)$ itself. However, it is necessary under JDELM to consider the randomness of the stochastic covariates.

Next, let us consider the expectation value of a stochastic default intensity λ_t. We can immediately obtain

$$\mathbb{E}[\lambda_t] = \bar{\lambda}(t) \left(\prod_{d=1}^{D} \phi_{X_t^d}(t, -i\beta_d) \right), \tag{4.14}$$

for any $t \geq 0$. That is, because of the effect of the stochastic covariates, the term structure of the expected default intensity is adjusted against the baseline hazard function $\bar{\lambda}(t)$ by $\prod_{d=1}^{D} \phi_{X_t^d}(t, -i\beta_d)$. Obviously, this adjustment depends on properties of Lévy processes adopted for the stochastic covariates of the Cox proportional hazard model. Unfortunately, although Eq.(4.14) is quite simple and can be easily computed, the expected default intensity is not directly observable in credit markets.

In order to examine relationship between observed credit markets and JDELM, we try to analytically consider credit spread of defaultable bond. Let $D(t,T)$ denote the time-t price of defaultable discount bond issued by a given firm with maturity T and zero recovery. Then, it can be written as

$$D(t,T) = \mathbb{E}\left[e^{-\int_t^T r(s)ds} \mathbf{1}_{\{\tau > T\}} \mid \mathcal{G}_t \right] = B(t,T)\mathbb{E}\left[e^{-\int_t^T \lambda_s ds} \mid \mathcal{F}_t \right], \tag{4.15}$$

for $0 \leq t \leq T$. The time-t start instantaneous forward credit spread observed at initial time denoted by $sp(t)$ is defined as

$$sp(t) = -\frac{\partial}{\partial t} \ln D(0,t) - r(t) = -\frac{\partial}{\partial t} \ln \mathbb{Q}(\tau > t).$$

Using the pricing generator, we have

$$sp(t) = -\frac{\partial}{\partial t} \ln \Phi(t,0) = -\frac{\frac{\partial}{\partial t}\Phi(t,0)}{\Phi(t,0)} = \bar{\lambda}(t) \left(\prod_{d=1}^{D} \phi_{X_t^d}(t, -i\beta_d) \right) C(t), \tag{4.16}$$

where

$$C(t) := \sum_{n=0}^{\infty} (-1)^n K_n(t) \Bigg/ \sum_{n=0}^{\infty} (-1)^n J_n(0,t).$$

Therefore, $C(t)$ can be interpreted as the convexity adjustment of the credit spread against the expected default intensity; i.e., $sp(t) = C(t)\mathbb{E}[\lambda_t]$ for any $t \geq 0$. Clearly, the convexity adjustment coefficient $C(t)$ also depends on Lévy processes as the stochastic covariates adopted to the Cox proportional hazard model. The numerical impact of the convexity adjustment will be shown in the next section.

4.5 Numerical Examples

This section gives numerical examples of term structure of CDS spreads, and implied volatility skews under JDELM with the variance gamma process and the Brownian motion. In addition, we show the impact of the convexity adjustment on term structure of forward credit spreads given the analytical explanation in the previous section.

Suppose that $D = 2$, and X_t^1 and X_t^2 follow the variance gamma (hereafter VG) process and the standard Brownian motion, respectively. VG process is an infinite-activity jump process with the Lévy measure

$$\Pi(dx) = \left(\frac{e^{-\xi_p x}}{\nu x} \mathbf{1}_{\{x>0\}} + \frac{e^{-\xi_n |x|}}{\nu |x|} \mathbf{1}_{\{x<0\}} \right) dx,$$

where

$$\xi_p = \sqrt{\frac{\eta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu}} - \frac{\eta}{\sigma^2} \quad \text{and} \quad \xi_n = \sqrt{\frac{\eta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu}} + \frac{\eta}{\sigma^2},$$

which is another representation of the Lévy measure for VG process in Table 2.1. Here, σ, ν, and η are parameters of VG process. It is well-known that VG process over unit time interval has the statistics listed in Table 4.1 and the process approaches the Gaussian process with drift η and volatility σ as $\nu \to 0$.

Table 4.1: Statistics of VG process

	VG process X_1^1
mean	η
variance	$\sigma^2 + \nu\eta^2$
skewness	$\eta\nu(3\sigma^2 + 2\nu\eta^2)/(\sigma^2 + \nu\eta^2)^{3/2}$
kurtosis	$3(1 + 2\nu - \nu\sigma^4[\sigma^2 + \nu\eta^2]^{-2})$

In the numerical examples, we assume that the initial stock price $S_0 = 50$, the covariate coefficients $\alpha_1 = 1$, $\alpha_2 = 0$, and $\beta_1 = -0.5$, $\beta_2 = 0.05$ which mean that the standard Brownian motion X_t^2 is an idiosyncratic credit risk factor, while VG process X_t^1 is a common factor for the stock price dynamics and the credit spread behaviors. For simplicity, the interest rate and the dividend yield are assumed to be constant ($r = 0.02$, $q = 0$). As a base model, the parameters of VG process and the baseline hazard function are assumed to be $\sigma = 0.2$, $\nu = 0.5$, $\eta = -0.01$ and $\bar{\lambda}(t) = \bar{\lambda} = 0.02$ for all $t \geq 0$, respectively.

Figure 4.1 plots the term structure of CDS spreads in the case of $\sigma = 0.1$, 0.2, and 0.3, where the recovery rate is set $\delta = 0.4$. The lager parameter implies that more downside risk of the stock price appears and the probability of default is increased. As expected, we can observe in Figure 4.1 that increasing σ increases the CDS spreads. We stress that although any observed term structure of credit spreads can be matched by an appropriate choice of the baseline hazard function, the baseline hazard function is assumed to be constant for simplicity in the numerical example. That is, the flexibility of the baseline hazard function realizes perfect calibration to observed term structure of CDS spreads or corporate bond yields.

Figure 4.2 and 4.3 plot implied volatilities of equity options maturing $T = 0.25$ and $T = 0.5$ against the strike prices in the case of $\bar{\lambda} = 0.01$, 0.02, and 0.03. The implied volatilities are obtained by first computing the call option pricing formula of Proposition 4.9 and then implying out the Black-Scholes implied volatilities. For comparison, implied volatility smiles for the standard exponential VG model that is the case of $\bar{\lambda} = 0$ are also plotted. Increasing the baseline hazard function increases the probabilities of default. This implies that possibility of jump to default of the firm's stock is increased. As expected, all of the skews plotted by the JDELM are above the corresponding skews plotted by the standard exponential VG model, and the larger baseline hazard function exhibits steeper implied volatility skews. Furthermore, the JDELM model skews as well as the standard exponential VG model skews tend to be gradually decreased for longer maturities, which is in accordance with empirical observations. Therefore, it can be said that JDELM has preferable ability not only capturing the skew for single maturity, but also tracing the pattern of decreasing skew steepness as maturity increases.

Finally, Figure 4.4 plots expected default intensities and instantaneous forward credit spreads. For comparison, the baseline hazard function is also potted. In this example the baseline hazard function is reset $\bar{\lambda} = 0.05$ instead of $\bar{\lambda} = 0.02$, because larger baseline hazard function causes larger convexity adjustment. We can observe the level of the convexity adjustment $C(t)$ as the difference between the expected default intensities and the forward credit spreads in Figure 4.4. The impact of the convexity adjustment is much larger for the longer maturities, while it is negligible for the shorter maturities.

4.6 Concluding Remarks

We propose a new dynamically consistent model named *the jump to default exponential Lévy model* (JDELM) for joint valuation of equity derivatives and credit products written on the same reference firm. In the framework, Lévy processes are adopted to the stochastic factors of both the stock price and the default arrival rate. That is, the pre-default stock price is assumed to follow an extended exponential Lévy model, while the default intensity is modeled by the Cox proportional hazard model with stochastic covariates driven by Lévy processes. Incorporating the Cox proportional hazard model is preferable because it is well-known as one of the most approved models in survival analysis and is also popular in credit modeling. Moreover, not only dynamic interaction between the stock price and term structure of credit spread, but also its dependency among individual firms can be represented. Under JDELM, we derive the pricing formulas for equity call options and credit default swaps, and analytically examine the impact of the convexity adjustment on term structure of credit spreads. It is demonstrated that these pricing formulas can be expressed by the pricing generator defined in Section 4.2, and we find the general solution of the pricing generator in Theorem 4.6 and Corollary 4.7. In the numerical examples, setting the variance gamma process and the standard Brownian motion into JDELM, we compute the term structure of CDS spreads and equity implied volatility smiles, and observe the impact of the convexity adjustment on the credit spread.

Finally, the next research topic will be to implement empirical analysis by using JDELM and to compare it with other jump to default models such as the jump to default extended CEV model developed by Carr and Linetsky [2006]. Although it might be difficult for us to do the analysis solely because suitable market data are not available in our current circumstance, we hope that our work will spur further research by academics and practitioners into the development of a credit-equity unified framework for pricing, trading, and risk management of credit and equity derivatives.

Figure 4.1: Term Structure of CDS Spreads

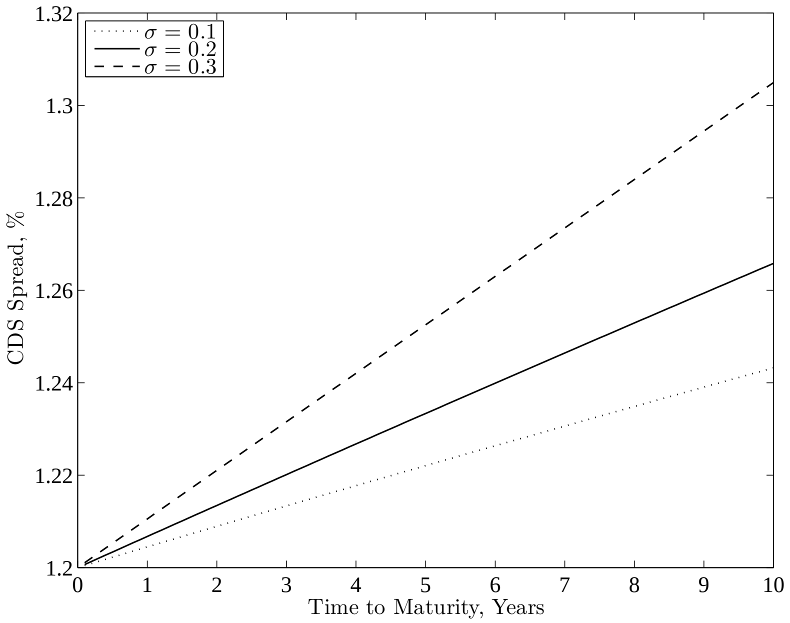


Figure 4.2: Implied Volatility Skews with Maturity $T = 0.25$

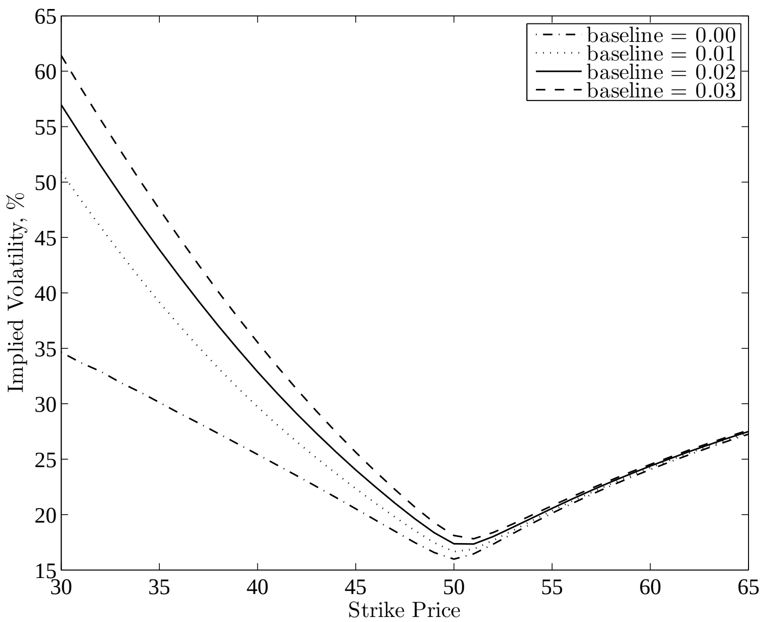


Figure 4.3: Implied Volatility Skews with Maturity $T = 0.5$

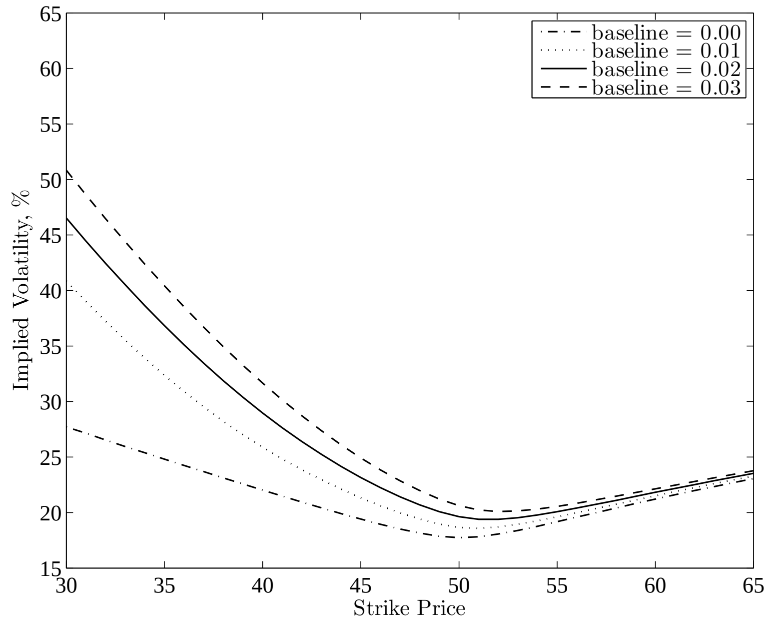
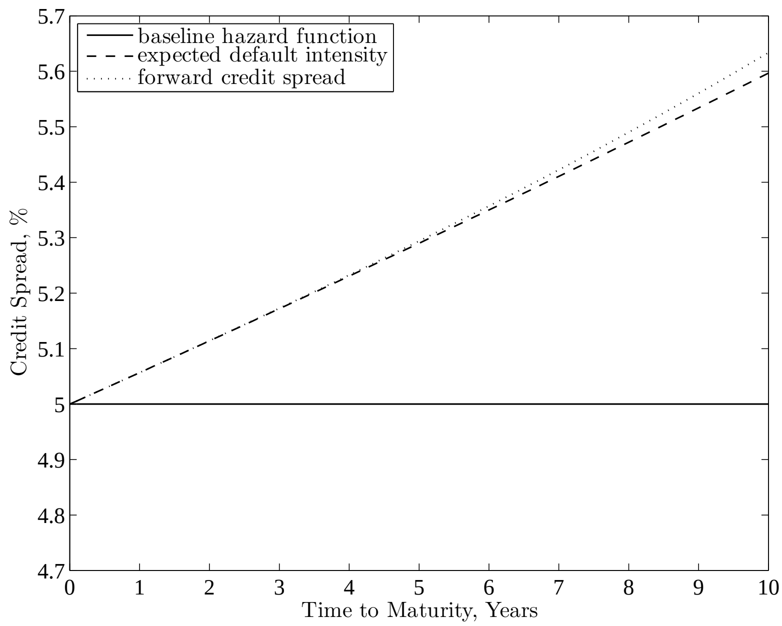


Figure 4.4: Term Structure of Credit Spreads



Chapter 5

A Note on the Black-Scholes Implied Volatility with Default Risk

It seems to be natural to expect that some connections must exist between market prices of equity and bond issued by a company. Actually, there is a great amount of literature investigating empirical linkage between equity options and credit risk. For examples, Zhang et al. [2005] and Cremers et al. [2008a] found some empirical evidences of relations between credit spread and implied volatility using firm value models. Cremers et al. [2008b] showed by statistical analysis that implied volatility skew as well as its level contains important information for credit spread. Carr and Wu [2010] developed a joint framework for credit and equity, and examined dynamic interactions between CDS and equity options. Moreover, Carr and Wu [2011] estimated default probabilities of individual firms from listed American put options.

In contrast to these empirical studies, there are not so many theoretical studies on the linkages between equity options and credit risk. In this chapter we seek a theoretical aspect of relations of the Black-Scholes implied volatility to the default probability based on a general framework that the stock price becomes zero after default occurs. The *jump to default* modeling introduced by Merton [1976], for instance, belongs to this category. In this modeling the stock price of a firm jumps to zero when default occurs. After his epoch-making work, many papers concerning the jump to default models have been published, e.g., see Madan and Unal [1998], Takahashi et al. [2001], Ayache et al. [2003], Carr and Linetsky [2006], Carr and Wu [2007, 2010, 2011], Linetsky [2006], Bielecki et al. [2007], Bayraktar and Yang [2008], Becherer and Ward [2008], Carr and Schoutens [2008], Hurd and Yi [2008], Kovalov and Linetsky [2008], Mendoza et al. [2010], and Papageorgiou and Sircar [2008]. On the other hand, structural models like the *CreditGrades* model introduced by Finger et al. [2002], and Stamicar and Finger [2006], are also within the framework. In the approach originally proposed by Black and Cox [1976], default occurs at any time as soon as the firm's asset value falls below a given default barrier and the stock price is defined as an excess value of the firm's asset over the barrier.

In this chapter, setting a general framework embracing almost all of them, we derive some formulas for relations between the implied volatility and the default probability. As a result, it is shown that the default probability restricts the divergence speed of the Black-Scholes implied volatility at extremely small strike. Therefore, it can be said that the results in this chapter are fundamental principles for unified credit-equity models. Furthermore, through a numerical test, we show whether our model-free formula that can derive survival probability from only the information of implied volatility theoretically is applicable or not in practice.

5.1 Setup

We consider a reference company which has default risk, and assume arbitrage-free and frictionless markets. Let τ denote the random default time of the company on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We denote by $\mathcal{H} := (\mathcal{H}_t)_{t\geq 0}$ the associated filtration of τ, where $\mathcal{H}_t := \sigma(\mathbf{1}_{\{\tau > s\}} : s \leq t)$, for $0 \leq t \leq T < +\infty$. Let $\mathcal{G} := (\mathcal{G}_t)_{t\geq 0}$ be an arbitrary filtration on $(\Omega, \mathcal{G}, \mathbb{P})$. Furthermore, we suppose an auxiliary filtration $\mathcal{F} := (\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{G} = \mathcal{H} \vee \mathcal{F}$; i.e. $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for any $t \in [0, T]$.

We define the pre-default stock dynamics as a $\mathcal{G}$-adapted process $(S_t)_{t\geq 0}$. It is assumed that $(S_t)_{t\geq 0}$ follows a *strictly positive* stochastic process, and the stock price is zero after default, i.e., for any $t \in [0, T]$,

$$\begin{cases} S_t > 0 & \text{if} \quad \tau > t \\ S_t = 0 & \text{if} \quad \tau \leq t, \end{cases} \tag{5.1}$$

holds for $\mathbb{P}$-$a.e$ $\omega \in \Omega$. Furthermore let us assume that the risk-free interest rate r and the dividend yield q are constant for simplicity.

The assumption (5.1) is very acceptable to many model settings. For instance the jump to default models, which are intensity-based approach, satisfy the above assumption. Introducing a non-negative $\mathcal{G}$-progressively measurable process $(\lambda_t)_{t\geq 0}$ the default time τ is defined as

$$\tau = \inf\left\{ t \geq 0 : \int_0^t \lambda_s ds \geq \eta \right\},$$

where $\eta \sim \text{Exp}(1)$; a random variable with unit exponential law under an equivalent measure $\mathbb{Q}$, see Bielecki and Rutkowski [2002] for details. The stock process is given by a stochastic process X_t that is an exponential $\mathcal{G}$-martingale and $e^{X_t} > 0$ a.s. for all $t \in [0, T]$, under $\mathbb{Q}$ as

$$S_t = \begin{cases} S_0 \exp\left\{ (r-q)t + \int_0^t \lambda_s ds + X_t \right\} & \text{if} \quad \tau > t \\ 0 & \text{if} \quad \tau \leq t. \end{cases}$$

It is worth noting that we may employ not only continuous processes driven by Brownian motion with stochastic volatility and/or local volatility, but also jump processes such as Lévy process and time-changed Lévy process (see Carr and Wu [2004] for instance) as X_t. We can also adopt various processes as the default intensity λ_t.

As other examples, the structural models like the *CreditGrades* model also satisfy the assumption (5.1). In this model setting, the default time τ is defined as

$$\tau = \inf\{t \geq 0 : A_t \leq L\},$$

where $(A_t)_{t\geq 0}$ is the firm's asset value process that is a $\mathcal{G}$-martingale under some equivalent measure $\mathbb{Q}$, and L is the default barrier, which is assumed to be a random variable with mean $\bar{L}$. Note that we can also adopt various processes as A_t. The stock process is characterized by A_t and $\bar{L}$ as follows;

$$S_t = \begin{cases} e^{(r-q)t}(A_t - \bar{L}) & \text{if} \quad \tau > t \\ 0 & \text{if} \quad \tau \leq t. \end{cases} \tag{5.2}$$

See Finger et al. [2002], and Stamicar and Finger [2006] for details of the *CreditGrades* model, and also Sepp [2006], and Ozeki et al. [2011] for extensions with stochastic volatility and jumps.

In the following, we examine theoretical aspects of relations between the Black-Scholes implied volatility and the default probability under the assumption of Eq.(5.1) without any specification of models. In both examples shown above, we can regard $\mathbb{Q}$ as an equivalent martingale measure. So we assume the existence of such measure $\mathbb{Q}$.

5.2 Call and Put with Default Risk

We consider European call and put options with default risk. Let $C(K,T)$ and $P(K,T)$ denote call and put prices with strike K and maturity T at time $t=0$ respectively. Then, the call and the put prices under the assumption (5.1) can be written as

$$\begin{aligned} C(K,T) &= \mathbb{E}\left[e^{-rT}\left(S_T-K\right)^{+}\mathbf{1}_{\{\tau>T\}}\right], \\ P(K,T) &= \mathbb{E}\left[e^{-rT}\left(K-S_T\right)^{+}\mathbf{1}_{\{\tau>T\}}\right]+\mathbb{E}\left[e^{-rT}K\mathbf{1}_{\{\tau\le T\}}\right], \end{aligned}$$

where $\mathbb{E}[\,\cdot\,]$ is an expectation operator under $\mathbb{Q}$. The following lemma shows that default and survival probabilities can be expressed as the first derivative of the put and the call with respect to strike K at $K=0$.

Lemma 5.1 *The partial differentials $\partial P(K,T)/\partial K$ and $\partial C(K,T)/\partial K$ exist for all $K\ge 0$. Moreover, the risk-neutral default and survival probabilities of a reference company are given by*

$$\mathbb{Q}(\tau\le T)=e^{rT}\left.\frac{\partial P(K,T)}{\partial K}\right|_{K=0}, \tag{5.3}$$

and

$$\mathbb{Q}(\tau> T)=-e^{rT}\left.\frac{\partial C(K,T)}{\partial K}\right|_{K=0}, \tag{5.4}$$

respectively for any $T>0$.

Proof of Lemma 5.1 Let F be the distribution function of S_T. Then we have

$$\begin{aligned} P(K,T) &= e^{-rT}\int_{[0,+\infty)}(K-s)^{+}dF(s) \\ &= e^{-rT}\int_{[0,K]}(K-s)dF(s) \\ &= e^{-rT}K\int_{[0,K]}dF(s)-e^{-rT}\int_{[0,K]}sdF(s). \end{aligned} \tag{5.5}$$

Since

$$\int_{(-\infty,x]}dF(s)=F(x),$$

we obtain

$$\begin{aligned} \int_{[0,K]}dF(s) &= \lim_{\varepsilon\downarrow 0}\int_{(-\varepsilon,K]}dF(s)=\lim_{\varepsilon\downarrow 0}\left(\int_{(-\infty,K]}dF(s)-\int_{(-\infty,-\varepsilon]}dF(s)\right) \\ &= \lim_{\varepsilon\downarrow 0}\left(F(K)-F(-\varepsilon)\right)=F(K)=\mathbb{Q}(S_T\le K). \end{aligned} \tag{5.6}$$

On the second term in the right hand side of Eq. (5.5), we have

$$\begin{aligned} \int_{[0,K]}sdF(s) &= \int_{(0,K]}sdF(s) \\ &= KF(K)-\int_{(0,K]}F(s)ds, \end{aligned} \tag{5.7}$$

by the integration by parts formula. Substituting Eq. (5.6) and (5.7) into Eq. (5.5), we obtain

$$\begin{aligned} P(K,T) &= e^{-rT}KF(K) - e^{-rT}\left(KF(K) - \int_{(0,K]} F(s)ds\right) \\ &= e^{-rT}\int_{(0,K]} F(s)ds, \end{aligned}$$

which implies that $\partial P(K,T)/\partial K$ exists for all $K > 0$, and it satisfies

$$\frac{\partial P(K,T)}{\partial K} = e^{-rT}F(K), \tag{5.8}$$

for $K > 0$. Therefore, at $K = 0$,

$$\begin{aligned} \left.\frac{\partial P(K,T)}{\partial K}\right|_{K=0} &= \lim_{K\to 0}\frac{P(K,T)-P(0,T)}{K} = \lim_{K\to 0}\frac{P(K,T)}{K} \\ &= e^{-rT}\lim_{K\downarrow 0}\frac{1}{K}\int_{(0,K]}F(s)ds = e^{-rT}F(0) \\ &= e^{-rT}\mathbb{Q}(S_T = 0) = e^{-rT}\mathbb{Q}(\tau \le T). \end{aligned}$$

From the put-call parity, the differentiability for $C(K,T)$ and Eq. (5.4) can be proved. □

Although Lemma 5.1 is nearly identical to the result in Breeden and Litzenberger [1978], it is worthwhile connecting the equity options with the default probabilities explicitly. By virtue of Lemma 5.1, we can investigate the default probabilities of individual companies by using only information on the equity option markets.

In the end of this section, a simple example is provided in order to verify the validity of Lemma 5.1.

Example 5.2 (The Black-Scholes Model with jump to default: Merton [1976]) Let $(S_t)_{t\ge 0}$ be a unique strong solution of the stochastic differential equation:

$$\frac{dS_t}{S_t} = (r - q + \lambda)dt + \sigma dW_t, \quad t < \tau, \tag{5.9}$$

where λ and σ are constant. Merton [1976] shows that the call price is given by

$$C(K,T) = Se^{-qT}N(h_+) - Ke^{-(r+\lambda)T}N(h_-); \quad S := S_0,$$

where $N(\,\cdot\,)$ is the standard normal cumulative distribution function, and

$$h_\pm = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \lambda)T \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

Since

$$\left.\frac{\partial C(K,T)}{\partial K}\right|_{K=0} = \lim_{K\to 0}\left\{-e^{-(r+\lambda)T}N(h_-)\right\} = -e^{-(r+\lambda)T},$$

the survival probability $\mathbb{Q}(\tau > T) = e^{-\lambda T}$. Note that Eq. (5.9) is identified with the original Black-Scholes model (Black and Scholes [1973]) and $\mathbb{Q}(\tau > T) = 1$ when $\lambda = 0$, that is, the economy is the non-defaultable Black-Scholes world. □

5.3 Implied Volatility with Default Risk

Our interest is relations between the Black-Scholes implied volatility and the default probability. Before discussing them, we introduce some useful notations as follows: Let F denote T-forward price of S, that is, $F := Se^{(r-q)T}$. We define the log-moneyness as $x := \ln(K/F)$, so let $K(x) := Fe^x$ be the strike at log-moneyness x. Note that we have chosen the sign convention such that K increases as x increases. Under these notations, the market price of the call options can be written as

$$C(K(x), T) = C^{\text{BS}}(x, I(x)),$$

where

$$C^{\text{BS}}(x, \sigma) := e^{-rT}\left[FN(d_1(x)) - K(x)N(d_2(x))\right],$$

$$d_j(x) = d_j(x, \sigma) := \frac{-x}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}; \quad j = 1, 2,$$

and $I(x)$ is the Black-Scholes *implied volatility* at log-moneyness x.

5.3.1 Lee's Condition

Lee [2004] examined the implied volatility at extreme strike on a non-negative underlying random variable with arbitrary distribution under arbitrage-free condition. For the later discussion, we introduce Theorem 3.4 in Lee [2004].

Lemma 5.3 *(Theorem 3.4. in Lee [2004]) Let*

$$\beta_L := \limsup_{x \to -\infty} \frac{I^2(x)T}{|x|}.$$

Then $\beta_L \in [0, 2]$.

Lemma 5.3 is a general result for any underlying process. Of course, this statement is valid for our framework. Notice that β_L can be interpreted as the divergence speed of the implied volatility. It is obvious that if $\lim_{x \to -\infty} I(x)$ is finite, then $\beta_L = 0$ holds.

5.3.2 Default Probability and Implied Volatility

In this subsection, we examine relations between the default probabilities and the implied volatilities. From now on, we set two assumptions on the implied volatility of a reference company.

Assumption 5.1 *(i)* $I(x) > 0$ *for all* $x \in \mathbf{R}$. *(ii)* $\lim_{x \to -\infty} \partial I(x)/\partial x$ *exists.*

Note that Assumption 5.1 (ii) includes the both cases that the limit of $\partial I(x)/\partial x$ converges and diverges, however the case of oscillation is excluded. The existence of $\partial I(x)/\partial x$ for any $x \in (-\infty, \infty)$ is proved at the end of this chapter. In the following, a technical lemma is provided.

Lemma 5.4 $\lim_{x \to -\infty} \partial I(x)/\partial x = 0$.

Proof of Lemma 5.4 Suppose that

$$\alpha := \lim_{x \to -\infty} \frac{\partial I(x)}{\partial x} \neq 0.$$

Obviously, either $\alpha \in [-\infty, 0)$ or $\alpha \in (0, +\infty]$ holds. In the case of $\alpha \in [-\infty, 0)$, there exist $\tilde{x} < x^*$ and $\alpha < \tilde{\alpha} < 0$ such that

$$\frac{\partial I(x)}{\partial x} < \tilde{\alpha},$$

for any $x < \tilde{x}$. Since, for all $y < \tilde{x}$,

$$I(\tilde{x}) - I(y) = \int_y^{\tilde{x}} \frac{\partial I(x)}{\partial x} dx < \tilde{\alpha}(\tilde{x} - y),$$

we have

$$I(y) > \tilde{\alpha}(y - \tilde{x}) + I(\tilde{x}).$$

Therefore,

$$\beta_L = \limsup_{y \to -\infty} \frac{I^2(y)T}{|y|} \geq \lim_{y \to -\infty} \frac{[\tilde{\alpha}(y - \tilde{x}) + I(\tilde{x})]^2 T}{|y|} = +\infty.$$

This is a contradiction to Lemma 5.3.

On the other hand, in the case of $\alpha \in (0, +\infty]$, there exist $\bar{x} < x^*$ and $0 < \bar{\alpha} < \alpha$ such that

$$\frac{\partial I(x)}{\partial x} > \bar{\alpha},$$

for any $x < \bar{x}$. Since, for all $y < \bar{x}$,

$$I(\bar{x}) - I(y) = \int_y^{\bar{x}} \frac{\partial I(x)}{\partial x} dx > \bar{\alpha}(\bar{x} - y),$$

we have

$$I(y) < \bar{\alpha}(y - \bar{x}) + I(\bar{x}).$$

Therefore, there exists $\bar{y} < \bar{x}$ such that $I(\bar{y}) < 0$ holds. This is a contradiction to Assumption 5.1. $\square$

The following theorem shows that a simple formula gives us an explicit relation between the default probability and the implied volatility under the risk-neutral measure. Note that this fact is a general result under the assumption of Eq. (5.1).

Theorem 5.5 *The risk-neutral survival probability of a reference company is given by*

$$\mathbb{Q}(\tau > T) = \lim_{x \to -\infty} N(d_2(x)), \tag{5.10}$$

for any $T \geq 0$.

Proof of Theorem 5.5 Using the relation $\phi(d_1(x)) = \frac{K}{F}\, \phi(d_2(x))$, we obtain the following equation:

$$\begin{aligned} \frac{\partial C(K,T)}{\partial K} &= \frac{e^{-rT}}{K} \left\{ F\phi(d_1(x)) \frac{\partial d_1}{\partial x} - K\phi(d_2(x)) \frac{\partial d_2}{\partial x} - N(d_2(x)) \frac{\partial K}{\partial x} \right\} \\ &= e^{-rT} \left\{ \frac{\partial I}{\partial x} \sqrt{T} \phi(d_2(x)) - N(d_2(x)) \right\}. \end{aligned}$$

By Lemma 5.1, the survival probability can be written as

$$\mathbb{Q}(\tau > T) = \lim_{x\to-\infty}\left\{N(d_2(x)) - \frac{\partial I(x)}{\partial x}\sqrt{T}\phi(d_2(x))\right\}.$$

To complete the proof, it is sufficient to check that

$$\lim_{x\to-\infty}\frac{\partial I(x)}{\partial x}\phi(d_2(x)) = 0.$$

By Lemma 5.4, we have

$$0 \quad \leq \quad \lim_{x\to-\infty}\left|\frac{\partial I(x)}{\partial x}\phi(d_2(x))\right| \leq \frac{1}{\sqrt{2\pi}}\lim_{x\to-\infty}\left|\frac{\partial I(x)}{\partial x}\right| = 0.$$

□

According to Theorem 5.5 and the definition of $d_2(x)$, the divergence of the implied volatility at strike $K = 0$ is a necessary condition for defaultable economy. Furthermore, it can be said that the default probability is determined by the limit of $d_2(x)$.

5.3.3 Divergence Speed of Implied Volatility with Default Risk

In the previous subsection, it is shown that the default probability significantly links with the limit of the implied volatility. Our purpose in this subsection is to examine relations between the divergence speed of the implied volatility and the default probability. First, we present an insightful example.

Example 5.6 Suppose that, for all $x < x^* \leq 0$ and any $T > 0$, the implied volatility $I(x)$ is given by

$$I(x) = \frac{\beta^\alpha|x|^\alpha - \gamma(T)}{\sqrt{T}},$$

where $0 < \alpha, \beta < \infty$, and $\gamma(T)$ is a monotonic decreasing function with respect to T with $\lim_{T\to0}\gamma(T) = +\infty$ and $\lim_{T\to\infty}\gamma(T) = -\infty$. For sufficiently small x, we have

$$d_2(x) \sim \left[\frac{1}{\beta^\alpha}|x|^{1-2\alpha} - \frac{\beta^\alpha}{2}\right]|x|^\alpha + \left[\frac{|x|^{1-2\alpha}}{\beta^{2\alpha}} + \frac{1}{2}\right]\gamma(T),$$

which implies

$$\lim_{x\to-\infty} d_2(x) = \begin{cases} -\infty & (1/2 < \alpha,\ 0 < \beta) \\ & \quad \text{or} \quad (\alpha = 1/2,\ 2 < \beta), \\ +\infty & (0 < \alpha < 1/2,\ 0 < \beta) \\ & \quad \text{or} \quad (\alpha = 1/2,\ 0 < \beta < 2), \\ \gamma(T) & (\alpha = 1/2,\ \beta = 2). \end{cases} \tag{5.11}$$

Substituting Eq. (5.11) for Eq. (5.10), we obtain

$$\mathbb{Q}(\tau > T) = \begin{cases} 0 & (1/2 < \alpha,\ 0 < \beta) \\ & \quad \text{or} \quad (\alpha = 1/2,\ 2 < \beta), \\ 1 & (0 < \alpha < 1/2,\ 0 < \beta) \\ & \quad \text{or} \quad (\alpha = 1/2,\ 0 < \beta < 2), \\ N(\gamma(T)) & (\alpha = 1/2,\ \beta = 2). \end{cases}$$

In the case of $\mathbb{Q}(\tau > T) = 0$, the economy is almost surely a defaultable world and there are arbitrage opportunities; e.g. selling a T-forward contract F at $t = 0$, we can make money without any risk. Indeed, Lee's arbitrage-free condition (Lemma 5.3) is not satisfied, that is, the divergence speed of the implied volatility is greater than 2 ($\beta_L > 2$). Thus, this case should be ruled out.

In this example, $\beta_L \in [0, 2)$ holds when $\mathbb{Q}(\tau > T) = 1$; i.e. non-defaultable economy. On the other hand, $\beta_L = 2$ holds when $\mathbb{Q}(\tau > T) = N(\gamma(T))$; i.e. defaultable economy. These facts indicate that there might be a certain relation between the divergence speed of the implied volatilities and the default probabilities. $\square$

The following proposition shows that it is necessary for a defaultable economy to have the maximum speed of the implied volatility divergence. That is, the divergence speed β_L is determined uniquely in any defaultable economy.

Proposition 5.7 *If $\mathbb{Q}(\tau \leq T) > 0$ holds, then $\beta_L = 2$.*

Proof of Proposition 5.7 We suppose that $I(x)$ is bounded. In this case, there exists $M > 0$ such that $I(x) \leq M$ for all $x \in (-\infty, x^*)$. Then, we have

$$d_2(x) \geq \frac{|x|}{M\sqrt{T}} - \frac{M\sqrt{T}}{2}.$$

Consequently, $d_2(x) \to +\infty$ as $x \to -\infty$. This fact implies $\mathbb{Q}(\tau \leq T) = 0$ by Theorem 5.5.

Conversely, we suppose that $I(x) \to +\infty$ as $x \to -\infty$. To complete the proof, it is enough to show that if $\beta_L \in [0, 2)$ holds then $\lim_{x\to-\infty} d_2(x) = +\infty$. Since $\beta_L < 2$, we can choose $\delta > 0$ such that $\bar{\beta} := \beta_L + \delta < 2$. By the definition of β_L and the property of $\limsup$, we can deduce that there exists $\bar{x} < x^*$ such that

$$\frac{I^2(x)T}{|x|} < \bar{\beta} < 2,$$

for any $x < \bar{x}$. Then, for every $x < \bar{x}$,

$$d_2(x) = \left(\frac{\sqrt{|x|}}{I(x)\sqrt{T}} - \frac{I(x)\sqrt{T}}{2\sqrt{|x|}} \right) \sqrt{|x|} > \left(\frac{1}{\sqrt{\bar{\beta}}} - \frac{\sqrt{\bar{\beta}}}{2} \right) \sqrt{|x|} = \frac{2 - \bar{\beta}}{2\sqrt{\bar{\beta}}} \sqrt{|x|}.$$

This fact implies $\lim_{x\to-\infty} d_2(x) = +\infty$. $\square$

While Lemma 5.3 (Theorem 3.4. in Lee [2004]) proved that β_L must be in the interval $[0, 2]$ under arbitrage-free condition, Proposition 5.7 shows that β_L is uniquely determined to be 2 in any defaultable economy. Note that Proposition 5.7 is included by Theorem 3.4. in Lee [2004]. However, his theorem is purely considered as the moment formula for implied volatility and it just provides a mathematical expression. On the other hand, Proposition 5.7 and Example 5.6 give an interpretation to the divergence speed of implied volatility β_L in terms of economics and reveal a more concrete description of implied volatility in defaultable economy.

The inverse statement of Proposition 5.7 is not necessarily satisfied. To show this, we present a counter-example of the inverse statement.

Example 5.8 Suppose that, for sufficiently small x, the implied volatility $I(x)$ is given by

$$I(x) = \sqrt{\frac{2}{T}(|x|^{1/2} - |x|^{\gamma})},$$

where $0 < \gamma < 1/2$. In this example, $\mathbb{Q}(\tau \leq T) = 0$ holds, while $\beta_L = 2$. $\square$

5.4 Numerical Test

Theorem 5.5 is considered as a kind of limit theorem for obtaining the survival probability of a given firm. If this formula works well in real markets, we can estimate survival probabilities of individual firms from observed implied volatilities without any model specification and any information of credit markets such as CDS and/or corporate bond markets. In order to examine the practical applicability of Theorem 5.5, we implement a numerical test by the following procedure: First, a certain model is chosen in which both survival probabilities and option prices can be derived analytically. Second, we compute the implied volatilities based on the model. Third, substituting small non-zero strikes and the implied volatilities at these strikes into Eq.(5.10) in Theorem 5.5, we estimate the survival probabilities from the implied volatilities and compare the estimated survival probabilities with the analytical ones. Our interest is how implied volatility at small strike is necessary to estimate survival probabilities.

For the test, we choose two models: the CreditGrades model (CG for short) and Merton's jump to default model (MJD for short). In CG, it is assumed that the asset value process in Eq.(5.2) is given by

$$A_t = \exp\left\{ -\frac{1}{2}\sigma^2 t + \sigma W_t \right\}, \quad t \geq 0,$$

where $(W_t)_{t\geq 0}$ is the standard Brownian motion under $\mathbb{Q}$ and σ is a constant parameter; and the default barrier L is constant for simplicity. Because the closed-form formulas of the option price and the survival probability under CG is well-known (see Stamicar and Finger [2006] for instance), they are omitted. On the other hand, the model setting of MJD has been described in Example 5.2. The model parameters of CG and MJD are listed in Tables 5.1 and 5.2, respectively.

Table 5.1: Model parameters of CG

S_0	T	r	q	σ	L
100	3	0	0	0.25 or 0.40	80

Table 5.2: Model parameters of MJD

S_0	T	r	q	σ	λ
100	0.5	0	0	0.3	0.15 or 0.85

Figures 5.1 and 5.2 plot the implied volatilities that are calculated under CG and MJD, respectively. Tables 5.3–5.6 exhibit the estimated survival probabilities by Theorem 5.5 and the exact survival probabilities, which is at column of moneyness 0 (log-moneyness $-\infty$). Note that all of the estimated survival probabilities are underestimated to the exact value. Although the estimated probabilities gradually approach the exact value when moneyness is close to zero (log-moneyness is close to $-\infty$), the convergence speed is slow. Furthermore, even when moneyness is at 0.1 (log-moneyness, however, is far from $-\infty$), the estimated values are not quite close to the exact ones. This fact implies that it is difficult to estimate survival probabilities from observed implied volatilities by using Theorem 5.5 directly in real equity option markets, in which implied volatilities at extremely small strikes are unavailable in usual.

As a result of the numerical test, specifying a certain model is necessary to estimate default probability from implied volatility in practice, although a simple model-free formula exists. On the other hand, Theorem 5.5 reveals that estimating default probability from implied volatility by a certain model is equivalent to extrapolating implied volatility at strike 0 by using the model. Therefore, such estimation is highly model-dependent. In fact, in past literature, estimating default probabilities from only the information of observed implied volatilities seems to not be successful. Consequently, practitioners have to take notice of the difficulty of estimating a firm's default probability from its implied volatility.

Figure 5.1: Implied volatility under CG

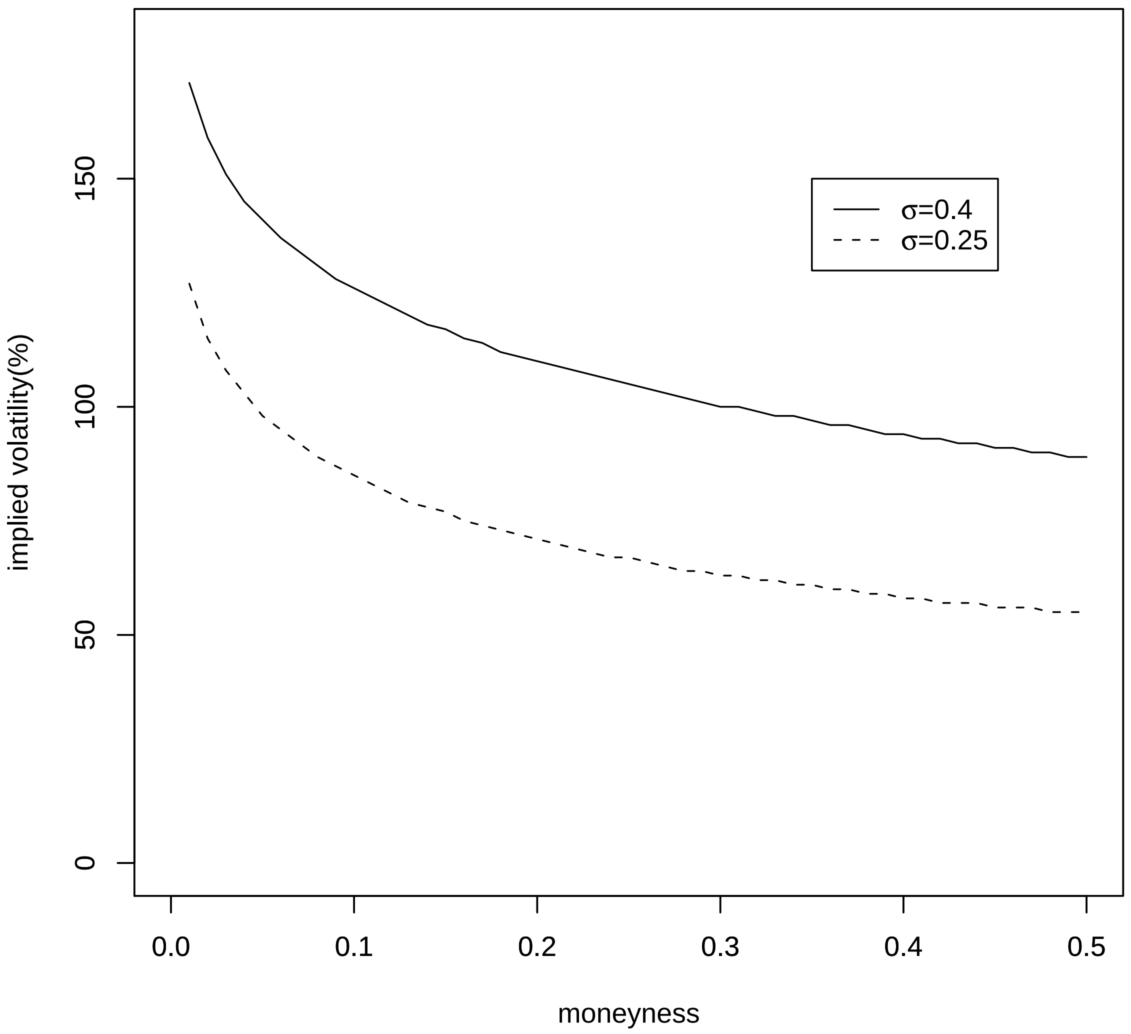


Figure 5.2: Implied volatility under MJD

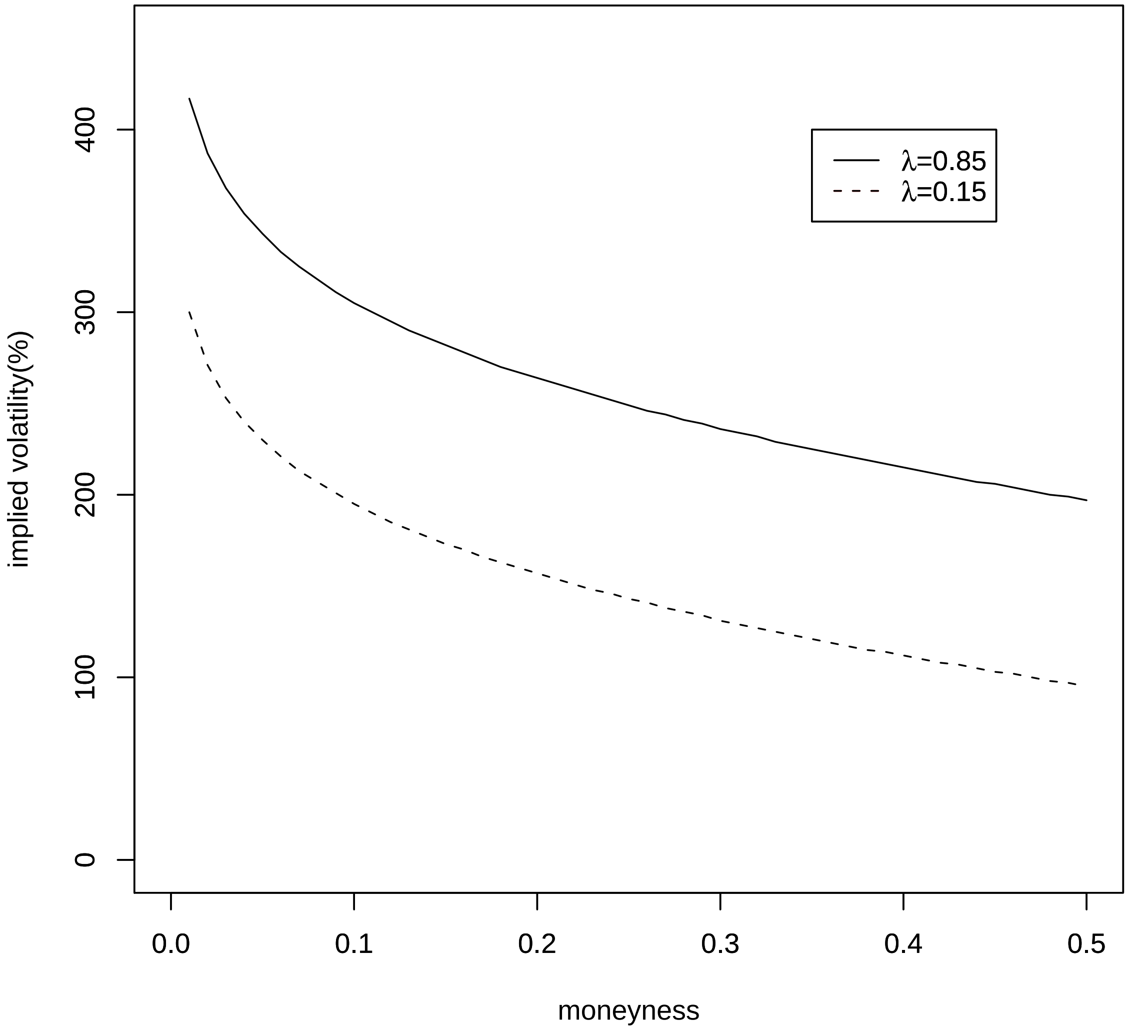


Table 5.3: Estimated survival probability under CG with $\sigma = 0.25$

Moneyness	0.5	0.4	0.3	0.2	0.1	1E-10	0
Log Moneyness	-0.693	-0.916	-1.204	-1.609	-2.303	-23.026	$-\infty$
Implied Vol (%)	54.75	58.26	63.22	70.83	84.62	329.38	∞
Survival Prob (%)	60.14	65.67	70.95	75.76	79.90	88.17	90.99

Table 5.4: Estimated survival probability under CG with $\sigma = 0.40$

Moneyness	0.5	0.4	0.3	0.2	0.1	1E-10	0
Log Moneyness	-0.693	-0.916	-1.204	-1.609	-2.303	-23.026	$-\infty$
Implied Vol (%)	88.81	93.80	100.44	109.93	125.77	378.36	∞
Survival Prob (%)	37.51	40.19	42.95	45.75	48.71	59.36	64.96

Table 5.5: Estimated survival probability under MJD with $\lambda = 0.15$

Moneyness	0.5	0.4	0.3	0.2	0.1	1E-10	0
Log Moneyness	-0.693	-0.916	-1.204	-1.609	-2.303	-23.026	$-\infty$
Implied Vol (%)	95.20	111.75	131.36	156.59	195.21	793.11	∞
Survival Prob (%)	75.59	77.77	79.72	81.59	83.59	90.35	92.77

Table 5.6: Estimated survival probability under MJD with $\lambda = 0.85$

Moneyness	0.5	0.4	0.3	0.2	0.1	1E-10	0
Log Moneyness	-0.693	-0.916	-1.204	-1.609	-2.303	-23.026	$-\infty$
Implied Vol (%)	196.94	214.94	236.30	263.74	305.48	942.31	∞
Survival Prob (%)	42.13	43.76	45.43	47.23	49.44	59.80	65.38

5.5 Concluding Remarks

In this chapter, setting a general framework under the assumption (5.1), we investigate theoretical relations between the Black-Scholes implied volatility and the default probability of a reference firm. As a result, we find that behavior of the implied volatility at extremely small strike significantly links to the default probability. Furthermore, we prove that if the reference firm has default risk, then β_L must be 2. According to Lee [2004], this is the maximum coefficient of the bound for the implied volatility under arbitrage-free conditions. These results can be considered as fundamental principles for almost all unified credit-equity models. Finally, implementing a numerical test, we show the difficulty of estimating survival probability from observed implied volatilities by Theorem 5.5, and we reveal that it is inevitable to specify a certain model in order to estimate default probability in practice. On the other hand, according to Theorem 5.5, estimating default probability from implied volatility by a specified model such as the CreditGrades model and the jump to default model is highly model-dependent. Therefore, practitioners must be very careful to choose the estimating model.

On existence of $\partial I/\partial x$ In this appendix, we show that $\partial I(x)/\partial x$ exists for any $x \in \mathbf{R}$.

Let $x = x_0 \in \mathbf{R}$. Since the map $(x,\sigma) \mapsto C^{\mathrm{BS}}(x,\sigma)$ is partial differentiable at (x_0,σ_0), where $\sigma_0 = I(x_0)$, there exist functions $\alpha(x,\sigma)$, $\beta(x,\sigma)$ at some neighborhood of (x_0,σ_0) such that α,β are continuous at (x_0,σ_0), and it satisfies

$$C^{BS}(x,\sigma) = C^{BS}(x_0,\sigma_0) + \alpha(x,\sigma)(x-x_0) + \beta(x,\sigma)(\sigma-\sigma_0), \tag{5.12}$$

$$\alpha(x_0,\sigma_0) = \frac{\partial C^{\mathrm{BS}}(x_0,\sigma_0)}{\partial x}, \quad \beta(x_0,\sigma_0) = \frac{\partial C^{\mathrm{BS}}(x_0,\sigma_0)}{\partial \sigma}. \tag{5.13}$$

Since $\beta(x_0,I(x_0)) = \beta(x_0,\sigma_0) = \partial C^{\mathrm{BS}}(x_0,\sigma_0)/\partial\sigma \neq 0$, then $\beta(x,I(x)) \neq 0$ at some neighborhood of (x_0,σ_0) due to continuity of β and I at (x_0,σ_0). Using the relation $C(K(x),T) = C^{\mathrm{BS}}(x,I(x))$ and Eq. (5.12), we have

$$\frac{I(x)-I(x_0)}{x-x_0} = \frac{1}{\beta(x,I(x))}\left\{\frac{C(K(x),T)-C(K(x_0),T)}{x-x_0} - \alpha(x,I(x))\right\}.$$

Hence we obtain

$$\begin{aligned}\lim_{x\to x_0}\frac{I(x)-I(x_0)}{x-x_0} &= \frac{1}{\beta(x_0,I(x_0))}\left\{\frac{\partial C(K(x_0),T)}{\partial x} - \alpha(x_0,I(x_0))\right\}\\ &= \frac{1}{\partial C^{BS}(x_0,I(x_0))/\partial\sigma}\left\{\frac{\partial C(K(x_0),T)}{\partial x} - \frac{\partial C^{BS}(x_0,I(x_0))}{\partial x}\right\}.\end{aligned}$$

Note that, by Lemma 5.1 and the chain rule, $\partial C(K(x),x)/\partial x$ exists for all $x \in \mathbf{R}$. This implies that $\partial I(x_0)/\partial x$ exists and is equal to the right hand side of above equation. □

Part III

オプションを利用した新しいヘッジ手法

Chapter 6

Efficient Static Replication of European Options under Exponential Lévy Models

This chapter develops a new efficient scheme for the static replication of European derivatives. Suppose the value of a target European derivative is twice differentiable in the underlying asset price; in other words, the gamma of the target derivative exists. By applying a technique similar to Carr and Chou [1997] and Carr and Madan [1998], we first show that the value of the derivative can be decomposed into a value-weighted bond, a delta-weighted forward contract and a gamma-weighted portfolio of options, all of whose maturities are shorter than the maturity of the target derivative. Based on this decomposition, a static replication can be obtained. However, theoretically an infinite number of options are needed for the replication. To overcome this problem, we introduce the Gauss-Legendre quadrature rule in order to approximate the replication based on a finite number of options. Consequently, compared with a standard static replication approach, our approach of gamma-weighted portfolio of options is more efficient; that is, a more precise hedge is derived from a smaller number of options.

To demonstrate this advantage, this chapter presents a concrete procedure for implementing our scheme by applying it to a standard plain vanilla option under exponential Lévy models. Specifically, we derive semi-analytic formulas for the price, and the delta and gamma of the target option based on modifications of the fast Fourier transform method developed by Carr and Madan [1999]. In this way, we are able to achieve a very efficient computation for constructing static replication portfolios. It should also be noted that this scheme can be applied to other European derivatives such as cash digital, asset digital and power options.

Finally, when the underlying asset price dynamics is represented by a Carr, Geman, Madan and Yor [2002] (hereafter, CGMY) type exponential Lévy model that can describe the price processes in the real world very well, numerical examples show that our scheme significantly outperforms a standard static replication model. This result demonstrates that a more accurate replication can be derived from fewer options.

For over a decade, static hedging techniques have been developed and investigated extensively for barrier type options. Bowie and Carr [1994] and Carr, Ellis and Gupta [1998] consider a static hedge method for barrier-type and lookback options by using *put call symmetry* (Carr [1994]). Derman, Ergener and Kani [1995] proposes the *calendar-spreads* method. Carr and Picron [1999] presents a method for static hedging of timing risk which is applied to pricing barrier options.

Carr and Chou [1997, 2002] shows the representation of any twice differentiable payoff function that corresponds to Lemma 6.1 in this chapter. Their paper then develops the so called *strike-spreads* method for static hedging of barrier, ratchet and lookback options under the Black-Scholes model. Andersen, Andreasen and Eliezer [2002] theoretically investigates static replication of barrier options.

Fink [2003] generalizes the method of Derman, Ergener and Kani [1995] for barrier options in an environment of stochastic volatility. More recently, Nalholm and Poulsen [2006b] proposes a new technique for static hedging of barrier options under general asset dynamics, such as a jump-diffusion process with correlated stochastic volatility. Furthermore, Nalholm and Poulsen [2006a] examines the sensitivity of dynamic and static hedging methods for barrier options to model risk.

On the other hand, Carr and Wu [2002] concentrates on an efficient replication of a plain vanilla option though their approach implies the possibility of further extensions and applications. It also applies the Gauss-Hermite quadrature rule to approximate static hedging of the option by plain vanilla options with shorter terms under the Black-Scholes and Merton [1976] jump-diffusion models. Moreover, their paper undertakes extensive simulation exercises to investigate the robustness of the method. In a certain sense, our scheme relies on and extends the methodologies developed by Carr and Wu [2002], Carr and Chou [1997, 2002] and Carr and Madan [1998, 1999].

6.1 Efficient Method for Static Replication

This section presents a general efficient method for static replication of European options. Specifically, under a single factor Markovian setting, we develop a methodology to replicate European options and their portfolios based on a *static* portfolio of shorter term plain vanilla options. *Static* portfolio implies that the weights in the portfolio remain unchanged when the price of underlying assets moves and options in the portfolio approach maturity.

Under the assumptions of a frictionless and no arbitrage market, let S_t denote the spot price of a stock, an underlying asset at time $t \in [0, T^*]$ where T^* is some arbitrarily determined time horizon. For sake of simplicity, the interest rate r and the dividend yield d are assumed to be constants. The no-arbitrage condition ensures the existence of a risk-neutral probability measure $\mathbb{Q}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ such that the instantaneous expected rate of return on every asset is equal to the instantaneous interest rate r. Furthermore, the risk-neutral process of the underlying asset price is assumed to be time-inhomogeneous Markovian. Note that all exponential Lévy models belong to this class, where an exponential Lévy model implies that stock price dynamics are driven by Lévy processes. Moreover, the analysis in this chapter concentrates on static replication of path-independent options where the final payoff of the option is solely determined by the stock price at maturity. Typical examples in this class include plain vanilla, cash digital, asset digital and power options. The following formula implies that a static portfolio of plain vanilla options allows us to replicate any European derivatives under a certain condition.

Lemma 6.1 *Suppose that the payoff function $f(S_T)$ of a European derivative with maturity T is twice differentiable. Then, for any $\kappa > 0$, it satisfies*

$$\begin{aligned} f(S_T) &= f(\kappa) + f'(\kappa)(S_T - \kappa) \\ &\quad + \int_0^{\kappa} f''(K)(K - S_T)^+ dK + \int_{\kappa}^{+\infty} f''(K)(S_T - K)^+ dK. \end{aligned} \tag{6.1}$$

Moreover, for all $t \in [0, T]$ the present value $V_t(S_t)$ of the derivative satisfies

$$\begin{aligned} V_t(S_t) &= e^{-r(T-t)} f(\kappa) + e^{-r(T-t)} f'(\kappa)\{F_t(T) - \kappa\} \\ &\quad + \int_0^{\kappa} f''(K) P_t(T, K) dK + \int_{\kappa}^{+\infty} f''(K) C_t(T, K) dK, \end{aligned} \tag{6.2}$$

where $F_t(T)$ denotes the time-t price of the forward contract with maturity T, and $P_t(T, K) := P_t(S_t; T, K)$ and $C_t(T, K) := C_t(S_t; T, K)$ represent the time-t prices of plain vanilla put and call options with spot price S_t, strike K and maturity T respectively.

Proof: See Carr and Chou [1997] or Appendix 1 in Carr and Madan [1998] for instance. □

The following proposition indicates that a European derivative can be replicated by using plain vanilla options whose maturities are shorter than the target European derivative so long as the delta and gamma for all possible values of the underlying spot price exist. Because the price function of a derivative can be regarded as a payoff function, the proof of the proposition is obvious from Lemma 6.1.

Proposition 6.2 *Let $\tau \in [0, T]$. Suppose that the time-τ price function $V_\tau(S)$ of a European derivative with maturity T is twice differentiable for all $S \geq 0$, that is, both the delta and gamma of the derivative exist at time τ. For any $\kappa > 0$, it satisfies*

$$\begin{aligned} V_\tau(S_\tau) &= V_\tau(\kappa) + \frac{\partial V_\tau}{\partial S} \mid_{S=\kappa} (S_\tau - \kappa) \\ &+ \int_0^\kappa \frac{\partial^2 V_\tau}{\partial S^2} \mid_{S=K} (K - S_\tau)^+ dK + \int_\kappa^{+\infty} \frac{\partial^2 V_\tau}{\partial S^2} \mid_{S=K} (S_\tau - K)^+ dK. \end{aligned} \tag{6.3}$$

Moreover, for all $t \in [0, \tau]$ the present value $V_t(S_t)$ of the derivative satisfies

$$\begin{aligned} V_t(S_t) &= e^{-r(\tau-t)} V_\tau(\kappa) + e^{-r(\tau-t)} \frac{\partial V_\tau}{\partial S} \mid_{S=\kappa} \{F_t(\tau) - \kappa\} \\ &+ \int_0^\kappa \frac{\partial^2 V_\tau}{\partial S^2} \mid_{S=K} P_t(\tau, K) dK + \int_\kappa^{+\infty} \frac{\partial^2 V_\tau}{\partial S^2} \mid_{S=K} C_t(\tau, K) dK. \end{aligned} \tag{6.4}$$

According to Proposition 6.2, once the replication portfolio is created, re-balancing is unnecessary until the maturity date of the options in the portfolio. This property is called *static*. Note that although nearly none of the payoff functions of the derivatives are twice differentiable, their price functions are mostly twice differentiable. Hence, Proposition 6.2 is more useful for applications. The practical implication of this proposition is that the risk embedded in a target European derivative can be hedged using a static portfolio of liquid plain vanilla options with a maturity that is shorter than the maturity of the target derivative.

Next, we present an efficient method for static replication. Proposition 6.2 shows that any derivative whose price function is twice differentiable can be completely replicated by using an infinite number of plain vanilla options. However, since an infinite number of options can not be used in practice, approximation of a static portfolio using a finite number of the options is necessary. Specifically, we apply the Gauss-Legendre quadrature rule for the approximation. The rule is a numerical computational method for an integral $\int_{-1}^{1} g(x)dx$, where $g(x) \in C^{2n}$ ($n \in \mathbf{N}$) on $[-1, 1]$. Here, C^{2n} denotes the set of $2n$-times continuously differentiable functions. For a given target function $g(x)$, the Gauss-Legendre quadrature rule provides the following formula.

$$\int_{-1}^{1} g(x)dx = \sum_{j=1}^{n} \omega_j g(x_j) + \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} g^{(2n)}(\xi), \tag{6.5}$$

for some $\xi \in [-1, 1]$, where x_j, $j = 1, 2, \cdots, n$, are roots of the nth order Legendre polynomial $L_n(x)$, $\omega_j := 2/(nL_{n-1}(x_j)L_n^{'}(x_j))$ and $g^{(2n)}$ denotes the $2n$-th derivative of g. The second term on the right hand side of equation (6.5) is the approximation error on the n-th order Gauss-Legendre quadrature rule. Note that if $g(x)$ is smooth, the error term converges to zero when $n \to \infty$. For details of the Gaussian quadrature rule, see pp. 225-230 of Sugihara and Murota [1994] for example. Application of the Gauss-Legendre quadrature rule to Proposition 6.2 provides the main result in this chapter, which can be stated as the following theorem.

Theorem 6.3 *Let $\tau \in [0,T]$, and suppose that $C_t(\tau,K) \in C^{2m}$ and $P_t(\tau,K) \in C^{2n}$ with respect to K respectively. Let $V_\tau(S) \in C^q$ be the time-τ price function of a European derivative with maturity T where $q := 2\max\{m,n\}$. Assume that there exist $S_{\min} \in [0,\kappa)$ and $S_{\max} \in (\kappa,\infty)$ such that for all $t \in [0,\tau]$*

$$\begin{aligned} &\frac{\partial^2 V_\tau}{\partial S^2}\mid_{S=y} P_t(\tau,y) = 0 \quad \text{if } y \in [0,S_{\min}], \\ &\frac{\partial^2 V_\tau}{\partial S^2}\mid_{S=z} C_t(\tau,z) = 0 \quad \text{if } z \in [S_{\max},+\infty). \end{aligned} \tag{6.6}$$

Moreover, define a static portfolio $\Lambda_t(n,m)$ as follows:

$$\begin{aligned} \Lambda_t(n,m) :=& e^{-r(\tau-t)}V_\tau(\kappa) + e^{-r(\tau-t)}\frac{\partial V_\tau}{\partial S}\mid_{S=\kappa}\{F_t(\tau)-\kappa\} \\ &+ \sum_{j=1}^{n} A_j^P P_t(\tau,K_j^P) + \sum_{l=1}^{m} A_l^C C_t(\tau,K_l^C), \end{aligned} \tag{6.7}$$

where

$$\begin{aligned} K_j^P &:= \frac{\kappa-S_{\min}}{2}x_j^n + \frac{\kappa+S_{\min}}{2}, \quad A_j^P := \omega_j^n\left(\frac{\kappa-S_{\min}}{2}\right)\frac{\partial^2 V_\tau}{\partial S^2}\mid_{S=K_j^P}, \\ K_l^C &:= \frac{S_{\max}-\kappa}{2}x_l^m + \frac{S_{\max}+\kappa}{2}, \quad A_l^C := \omega_l^m\left(\frac{S_{\max}-\kappa}{2}\right)\frac{\partial^2 V_\tau}{\partial S^2}\mid_{S=K_l^C}, \\ \omega_j^n &:= \frac{2}{nL_{n-1}(x_j^n)L_n'(x_j^n)}, \quad \omega_l^m := \frac{2}{mL_{m-1}(x_l^m)L_m'(x_l^m)}. \end{aligned} \tag{6.8}$$

Here, x_j^n, $j=1,\cdots,n$(x_l^m, $l=1,\cdots,m$) denote the roots of the n-th(m-th) order Legendre polynomial.

Then, $\Lambda_t(n,m)$ approximates $V_t(S_t)$ for all $t \in [0,\tau]$:

$$V_t(S_t) = \Lambda_t(n,m) + p_n(\xi_1) + p_m(\xi_2), \text{ for some } \xi_1,\xi_2 \in [-1,1], \tag{6.9}$$

where $p_n(\xi_1)$($p_m(\xi_2)$) is the error term of the n-th(m-th) order quadrature rule.

In particular, if $C_t(\tau,K)$, $P_t(\tau,K)$ and $V_\tau(S)$ are smooth, $\Lambda_t(n,m)$ converges to $V_t(S_t)$ for all $t \in [0,\tau]$, when $n \to +\infty$ and $m \to +\infty$.

Proof: Let us define the following integral.

$$I := \int_0^\kappa \frac{\partial^2 V_\tau}{\partial S^2}\mid_{S=K} P_t(\tau,K)dK = \int_{S_{\min}}^\kappa \frac{\partial^2 V_\tau}{\partial S^2}\mid_{S=K} P_t(\tau,K)dK \tag{6.10}$$

Changing the integral parameter K into $\frac{\kappa-S_{\min}}{2}x + \frac{\kappa+S_{\min}}{2}$, we re-write the integral (6.10) as

$$\begin{aligned} I =& \frac{\kappa-S_{\min}}{2} \\ &\times \int_{-1}^1 \frac{\partial^2 V_\tau}{\partial S^2}\mid_{S=\frac{\kappa-S_{\min}}{2}x+\frac{\kappa+S_{\min}}{2}} P_t(\tau,\frac{\kappa-S_{\min}}{2}x + \frac{\kappa+S_{\min}}{2})dx. \end{aligned} \tag{6.11}$$

Then, the Gauss-Legendre quadrature rule can be applied to the integral (6.11). That is,

$$I = \sum_{j=1}^{n} A_j^P P_t(\tau, K_j^P) + p_n(\xi_1), \quad \text{for some } \ \xi_1 \in [-1, 1], \tag{6.12}$$

where $p_n(\xi)$ denotes the error term of the n-th order quadrature rule.

Similar argument holds for:

$$J := \int_{\kappa}^{+\infty} \frac{\partial^2 V_\tau}{\partial S^2} \mid_{S=K} C_t(\tau, K) dK = \int_{\kappa}^{S_{\max}} \frac{\partial^2 V_\tau}{\partial S^2} \mid_{S=K} C_t(\tau, K) dK. \tag{6.13}$$

Further, if $C_t(\tau, K)$, $P_t(\tau, K)$ and $V_\tau(S)$ are smooth, it clearly holds that when $n \to +\infty$, $m \to +\infty$,

$$\sum_{j=1}^{n} A_j^P P_t(\tau, K_j^P) \ \longrightarrow \ \int_0^{\kappa} \frac{\partial^2 V_\tau}{\partial S^2} \mid_{S=K} P_t(\tau, K) dK, \tag{6.14}$$

and

$$\sum_{l=1}^{m} A_l^C C_t(\tau, K_l^C) \ \longrightarrow \ \int_{\kappa}^{+\infty} \frac{\partial^2 V_\tau}{\partial S^2} \mid_{S=K} C_t(\tau, K) dK. \tag{6.15}$$

□

Remark 6.4 *Although assumption (6.6) in the theorem may not hold rigorously in applications, a static portfolio $\Lambda_t(n, m)$ is very effective because the gamma of most European derivatives, such as plain vanilla, cash digital and asset digital options, approaches zero very quickly as the moneyness goes to in-the-money and out-of-the-money.*

6.2 Option Prices and Greeks under Exponential Lévy Models

This section derives the formulas for the price and the Greeks of a European plain vanilla option under exponential Lévy models. This is done because, in practical situations, efficient and accurate computation of the price and Greeks is crucial for static replication.

Suppose the stock price process is specified as $S_t = S_0 e^{(r-d)t + X_t}, t \in [0, T^*]$ under a risk-neutral measure $\mathbb{Q}$, where $(X_t)_{t \geq 0}$ is a one-dimensional stochastic process with $X_0 = 0$ and is an exponential martingale on the probability space $(\Omega, \mathcal{F}, P)$ endowed with a standard complete filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$. In particular, $(X_t)_{t \geq 0}$ is assumed to be a Lévy process with respect to the filtration $\mathbb{F}$. By the Lévy Khintchine formula (see Sato [1999] for example), the characteristic function of X_t takes the form

$$\Phi_{X_t}(\theta) := \mathbb{E}\left[e^{i\theta X_t}\right] = e^{-t\psi_X(\theta)}, \quad t \geq 0, \tag{6.16}$$

where the *characteristic exponent* $\psi_X(\theta)$, $\theta \in \mathbf{R}$, is given by

$$\psi_X(\theta) = -i\gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{+\infty} \left(1 - e^{i\theta x} + i\theta x \mathbf{1}_{|x| \leq 1}\right) \Pi(dx), \tag{6.17}$$

where $\sigma \geq 0$ and $\gamma \in \mathbf{R}$ are constants, and Π is a measure on $\mathbf{R} \setminus \{0\}$ satisfying

$$\int_{-\infty}^{+\infty} \left(1 \wedge x^2\right) \Pi(dx) < +\infty. \tag{6.18}$$

In many exponential Lévy models, the characteristic function $\Phi_{X_t}(\theta)$ can be obtained analytically. Indeed, in the case of the Black-Scholes model (i.e. $S_t = S_0 e^{(r-d)t-\frac{1}{2}\sigma^2+\sigma W_t}$, where W_t is a one-dimensional standard Brownian motion and σ is a volatility) the characteristic function $\Phi^{\sigma}_{BS_t}(\theta)$ of $BS_t := -\frac{1}{2}\sigma^2 + \sigma W_t$ is given by $\Phi^{\sigma}_{BS_t}(\theta) = \exp\left\{-\frac{\sigma^2 t}{2}(\theta^2 + i\theta)\right\}$. Other well-known examples are the characteristic functions of Merton's jump-diffusion process (Merton [1976]), Kou's jump-diffusion process (Kou [2002]), the Variance Gamma process (Mardan, Carr and Chang [1998]), the normal inverse Gaussian process (Barndorff-Nielsen [1998]), the CGMY process (Carr, Geman, Madan and Yor [2002]), the generalized hyperbolic process (Eberlein, Keller and Prause [1998]), and the finite moment log-stable process (Carr and Wu [2003]).

Carr and Madan [1999] introduces a fast Fourier transform method for option pricing. This chapter proposes to compute the time value of the option after subtracting an intrinsic value from the option price in order to avoid the oscillation of the integrand in the Fourier inversion. As a result, the option price can be obtained as the time value derived by the Fourier inversion plus the intrinsic value. On the other hand, to compute the delta and gamma of an option, we propose to subtract the Black-Scholes price with adequate volatility from the option price instead of subtracting the intrinsic value. This choice is made because the intrinsic value might not be differentiable. See also p.363 of Cont and Tankov [2004]. (Note that the Black-Scholes prices of European options are twice differentiable.)

The following proposition shows the formulas for the price, the delta and the gamma of a plain vanilla call option.

Proposition 6.5 *Let C_t denote a plain vanilla call price with strike K and maturity T at time t. Then the call price is given by*

$$C_t = \frac{Se^{-\alpha k}}{2\pi}\int_{-\infty}^{+\infty} e^{-iuk}\zeta_{T,t}(u)du + C_t^{\sigma}, \tag{6.19}$$

where

$$\begin{aligned}\zeta_{T,t}(u) :=& \frac{\exp\{[(r-d)(iu+\alpha+1)-r](T-t)\}}{(iu+\alpha)(iu+\alpha+1)} \\ &\times \left(\Phi_{X_{T-t}}(u-i\alpha-i) - \Phi^{\sigma}_{BS_{T-t}}(u-i\alpha-i)\right),\end{aligned} \tag{6.20}$$

$\alpha > 0$, $k := \ln(K/S)$ and C_t^{σ} denotes the Black-Scholes price of the plain vanilla call with some volatility $\sigma > 0$. Moreover, the delta $\frac{\partial C_t}{\partial S}$ and the gamma $\frac{\partial^2 C_t}{\partial S^2}$ are given by

$$\begin{aligned}\frac{\partial C_t}{\partial S} &= \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{+\infty}(iu+\alpha+1)e^{-iuk}\zeta_{T,t}(u)du + \frac{\partial C_t^{\sigma}}{\partial S}, \\ \frac{\partial^2 C_t}{\partial S^2} &= \frac{e^{-\alpha k}}{2\pi S}\int_{-\infty}^{+\infty}(iu+\alpha)(iu+\alpha+1)e^{-iuk}\zeta_{T,t}(u)du + \frac{\partial^2 C_t^{\sigma}}{\partial S^2}.\end{aligned} \tag{6.21}$$

Proof: See at the end of this chapter. □

6.3 Numerical Examples

This section examines the effectiveness of our replication scheme through numerical examples. First, let us specify the stock process $S_t = S_0 e^{(r-d)t+X_t}$ as the CGMY model under a risk-neutral measure

$\mathbb{Q}$, which is introduced by Carr, Geman, Madan and Yor [2002]. That is, $X_t := \omega t + Z_t^{CGMY}$ where $\omega := -\psi_{Z^{CGMY}}(-i)$. Here, ω is called a *convexity correction*, and Z_t^{CGMY} is a pure jump Lévy process whose Lévy measure Π_{CGMY} is defined by:

$$\Pi_{CGMY}(dx) := \begin{cases} C\dfrac{\exp\{-G|x|\}}{|x|^{1+Y}}dx & \text{for } x < 0 \\ \\ C\dfrac{\exp\{-M|x|\}}{|x|^{1+Y}}dx & \text{for } x > 0, \end{cases} \tag{6.22}$$

where $C > 0$, $G \geq 0$, $M \geq 0$, and $Y < 2$. The characteristic exponent of Z_t^{CGMY} is given by

$$\psi_{Z^{CGMY}}(\theta) = C\Gamma(-Y)\left[M^Y - (M - i\theta)^Y + G^Y - (G + i\theta)^Y\right], \tag{6.23}$$

where $\Gamma(\cdot)$ is the gamma function. See Carr, Geman, Madan and Yor [2002] for details.

The input parameters of the CGMY model in the numerical examples are listed in Table 6.1, where the CGMY parameters are taken from Table 2 in Carr, Geman, Madan and Yor [2002].

Table 6.1: The input parameters of the CGMY model

S_0	r	d	C	G	M	Y
100	0.00	0.00	24.79	94.45	95.79	0.2495

In order to examine effectiveness of our method, we compare it to a static replication approach that is referred to as the *standard method* in the subsequent analysis. In the standard method, the replication portfolio for a given target option consists of various plain vanilla options with different strike prices and the same maturity, which is shorter than the maturity of the target option. The replication portfolio is obtained so that the value of the portfolio is equal to that of the target option for each discretized grid of stock prices at the maturity of the portfolio. At each grid, the corresponding portfolio weights can be found by solving a system of linear equations. The general procedure of the standard method can be found in Nalholm and Poulsen[2006b], for instance. On the other hand, our replication scheme is hereafter referred to as the *efficient method*.

We compute the replication portfolios for a target plain vanilla call with strike $K = 100$ and maturity $T = 1$. Using 8, 12, or 16 plain vanilla options with maturity $\tau = 0.5$, we replicate each value of the target option for all $S_t \in [50, 150]$, $0 \leq t \leq \tau$. Figure 6.1 describes the present values of the target option with different underlying stock prices and time-to-expiries. Figures 6.2-6.4 plot the errors of the replication that are defined as the deviations of the portfolio's values from the target option's values. It is obvious that the efficient method provides more accurate approximations of the replication portfolio than the standard method in all cases. Table 6.2 shows the costs of replication that are equivalent to the portfolio values at the initial time, as well as the errors and error ratios against the corresponding option premium. In the efficient method, considerable accuracy in prices can be obtained by using only 8 options. Consequently, these numerical results show that the efficient method is effective and efficient in practice.

Remark 6.6 *A similar technique was applied to the static replication of European derivatives such as cash digital, asset digital and power options. We obtained particularly good result for these options in numerical experiments using the same CGMY model.*

6.4 Conclusion

This chapter presents a new scheme for the static replication of European derivatives under a general class of exponential Lévy models. The scheme can be applied to European derivatives including digital-type options for which dynamic hedging is sometimes difficult to implement and is therefore not very effective in practice. Our efficient method developed in a general class of the underlying price process appears to be useful and widely applicable in trading and hedging of derivatives. Moreover, numerical examples in a CGMY model confirm the validity of our scheme through comparison with a standard static replication method. Finally, our next research topic will be to establish an effective and efficient scheme for the static replication of multi-factor derivatives, such as stochastic volatility models.

Table 6.2: The cost of replication for the target call option

Exact Value	4.86176		
Number of Options	8	12	16
Efficient Method	4.86424	4.86167	4.86175
Error	-0.00248	0.00010	0.00001
Error Ratio	-0.0510%	0.0020%	0.0003%
Standard Method	5.21137	4.95555	4.89037
Error	-0.34961	-0.09379	-0.02861
Error Ratio	-7.1910%	-1.9291%	-0.5884%

Proof of Proposition 6.5 Note that

$$\begin{aligned} C_t =& \mathbb{E}\left[e^{-r(T-t)}(S_T-K)^+ \mid \mathcal{F}_t\right] \\ =& S_t e^{-r\bar{T}} \int_{-\infty}^{\infty} \left(e^{(r-d)\bar{T}+x} - e^k\right) \mathbf{1}_{\{(r-d)\bar{T}+x>k\}} \rho_{X_{\bar{T}}}(x)dx, \end{aligned} \tag{6.24}$$

where $\bar{T} := T-t$ and $\rho_{X_{\bar{T}}}(\cdot)$ is the density function of $X_{\bar{T}}$. We define the function $\bar{\zeta}_{T,t}(k)$ as

$$\begin{aligned} \bar{\zeta}_{T,t}(k) :=& \frac{e^{\alpha k}}{S_t}(C_t - C_t^{\sigma}) \\ =& e^{-r\bar{T}+\alpha k} \int_{-\infty}^{\infty} \left(\rho_{X_{\bar{T}}}(x) - \rho_{BS_{\bar{T}}}(x)\right) \\ & \times \left(e^{(r-d)\bar{T}+x} - e^k\right) \mathbf{1}_{\{(r-d)\bar{T}+x>k\}} dx. \end{aligned} \tag{6.25}$$

Figure 6.1: The present value of the target call option

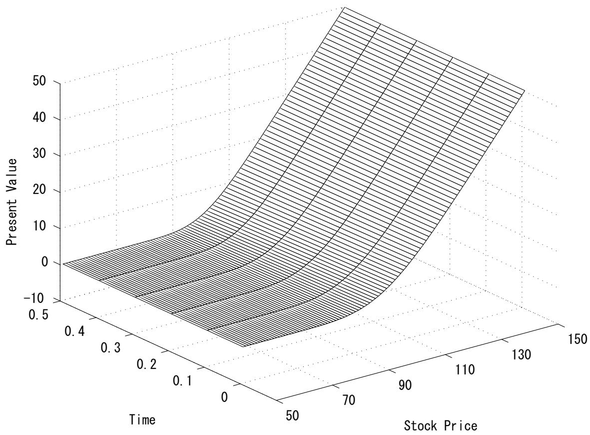


Figure 6.2: The replication error with 8 options

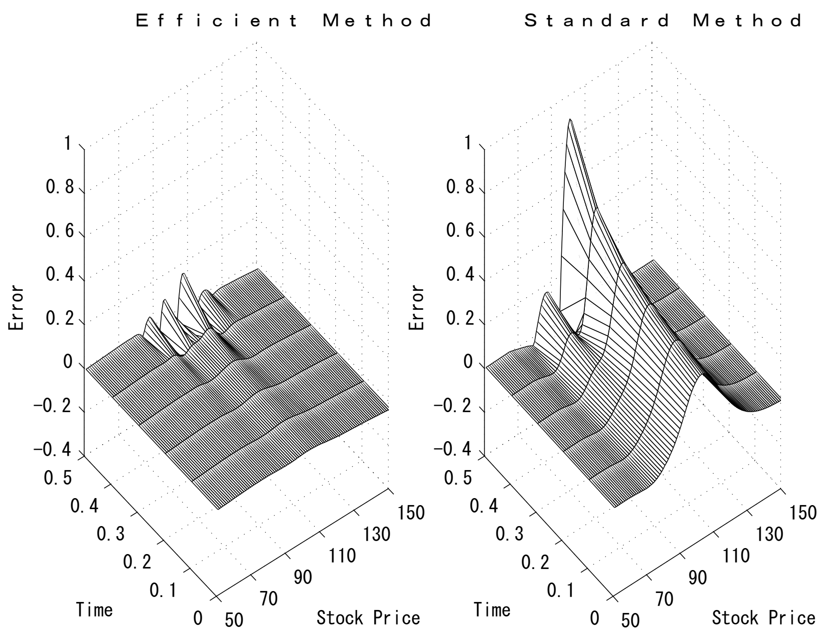


Figure 6.3: The replication error with 12 options

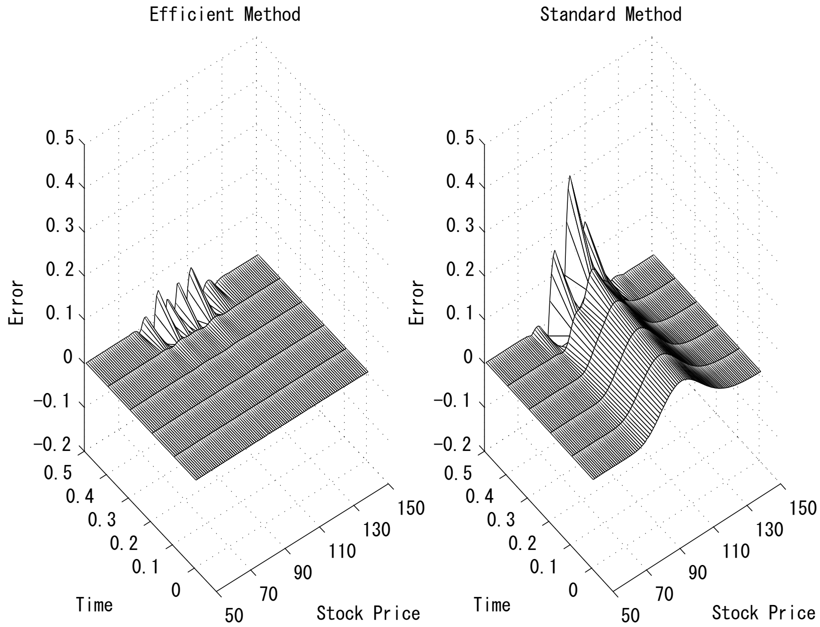
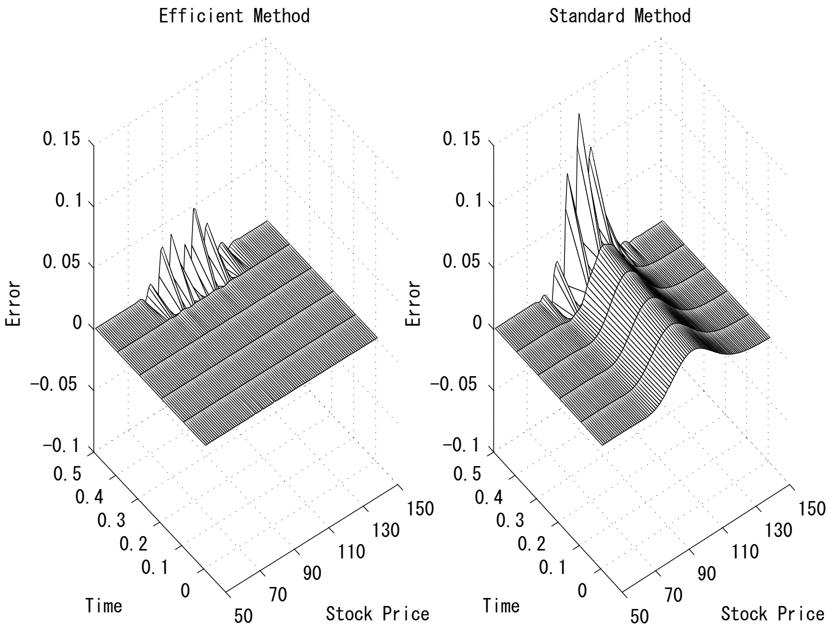


Figure 6.4: The replication error with 16 options



Let $\zeta_{T,t}(u)$ denote the Fourier transform of $\bar{\zeta}_{T,t}(k)$. Then, $\zeta_{T,t}(u)$ can be calculate as follows:

$$\begin{aligned}\zeta_{T,t}(u) =& \int_{-\infty}^{\infty} e^{iuk}\bar{\zeta}_{T,t}(k)dk \\ =& e^{-r\bar{T}} \int_{-\infty}^{\infty} \left(\rho_{X_{\bar{T}}}(x) - \rho_{BS_{\bar{T}}}(x)\right) \\ & \times \int_{-\infty}^{(r-d)\bar{T}+x} e^{(iu+\alpha)k} \left(e^{(r-d)\bar{T}+x} - e^{k}\right) dkdx \\ =& \frac{\exp\left\{[(r-d)(iu+\alpha+1)-r]\bar{T}\right\}}{(iu+\alpha)(iu+\alpha+1)} \\ & \times \int_{-\infty}^{\infty} e^{(iu+\alpha+1)x} \left(\rho_{X_{\bar{T}}}(x) - \rho_{BS_{\bar{T}}}(x)\right) dx \\ =& \frac{\exp\left\{[(r-d)(iu+\alpha+1)-r]\bar{T}\right\}}{(iu+\alpha)(iu+\alpha+1)} \\ & \times \left(\Phi_{X_{\bar{T}}}(u-i\alpha-i) - \Phi^{\sigma}_{BS_{\bar{T}}}(u-i\alpha-i)\right).\end{aligned} \tag{6.26}$$

By the Fourier inversion of $\zeta_{T,t}(u)$, the equation (6.19) is obtained. Next we define the function $\hat{\zeta}(S)$ as

$$\hat{\zeta}(S) := S \int_{-\infty}^{+\infty} e^{-(iu+\alpha)k} \zeta_{T,t}(u)du. \tag{6.27}$$

Let us differentiate $\hat{\zeta}_{T,t}(S)$ once and twice. Thus,

$$\begin{aligned}\frac{\partial\hat{\zeta}}{\partial S} =& \int_{-\infty}^{+\infty} e^{-(iu+\alpha)k}\zeta_{T,t}(u)du + S\int_{-\infty}^{+\infty} \frac{\partial k}{\partial S}\frac{\partial}{\partial k} e^{-(iu+\alpha)k}\zeta_{T,t}(u)du \\ =& \int_{-\infty}^{+\infty} (iu+\alpha+1)e^{-(iu+\alpha)k}\zeta_{T,t}(u)du, \\ \frac{\partial^2\hat{\zeta}}{\partial S^2} =& \int_{-\infty}^{+\infty} \frac{\partial k}{\partial S}\frac{\partial}{\partial k}(iu+\alpha+1)e^{-(iu+\alpha)k}\zeta_{T,t}(u)du \\ =& \frac{1}{S}\int_{-\infty}^{+\infty} (iu+\alpha)(iu+\alpha+1)e^{-(iu+\alpha)k}\zeta_{T,t}(u)du.\end{aligned} \tag{6.28}$$

Therefore the equation (6.21) is obtained. □

Chapter 7

A New Scheme for Static Hedging of European Derivatives under Stochastic Volatility Models

This chapter develops a new scheme for the static hedging of European path-independent derivatives under stochastic volatility models. When the dynamics of the underlying asset price is described by a multi-dimensional process, a one-dimensional price process that has the same distribution as the original one can be obtained by using the result proved by Gyöngy [1986](Theorem 4.6 in his paper). Piterbarg [2006] called this result *Markovian projection* in the context of financial mathematics, and noted that Dupire [1994], Derman and Kani [1998], and Savine [2001] derived essentially the same result in finance. In particular, Savine [2001] applied Tanaka's formula for the derivation.

Preceding literatures such as Avellaneda, Boyer-Olson, Busca and Friz [2002], Henry-Labordere [2005], Antonov and Misirpashaev [2009], Piterbarg [2006], and Madan, Qian and Ren [2007] used the Gyöngy's theorem mainly for pricing and calibration in some complicated multi-factor models. Due to his theorem, certain approximation formulas of European derivative prices and/or the Black-Scholes equivalent volatilities can be obtained under the models for which it is difficult to derive exact closed-form formulas.

Unlike these literatures, we propose a new application of Gyöngy's theorem in finance, that is a static hedging strategy under stochastic volatility models. Specifically, based on his theorem pricing European path-independent derivatives under stochastic volatility models is transformed to pricing those under one-factor local volatility models. Thus, we can apply an efficient method for one-dimensional price processes developed by Takahashi and Yamazaki [2009a] to forming a static hedging portfolio for a European derivative: compared with a standard static replication approach, their method of gamma-weighted portfolio of options is more efficient, that is, a more precise hedge is derived from a smaller number of options.

In particular, if the drift and diffusion terms of the one-dimensional price processes are obtained analytically, it is easy to implement this scheme. For instance, when the option price is analytically or semi-analytically obtained, the scheme is implemented through the relation between the option price and its volatility function developed by Dupire [1994]. As an example, we derive the local volatility model that corresponds to the Heston [1993]'s model.

To demonstrate how our scheme works, this chapter uses a standard plain vanilla option under the Heston [1993]'s model in a numerical example. It should also be noted that this method can be applied to other European derivatives such as cash digital, asset digital and power options. Finally, simulation exercises comparing our scheme with a dynamic hedging method, specifically *the minimum-variance hedging method* (see Bakshi,Cao and Chen [1997] for example) are used to demonstrate that our hedging scheme is effective in practice.

For over a decade, static hedging techniques have been developed and investigated extensively for barrier type options. See, for example, Derman, Ergener and Kani [1995], Carr, Ellis and Gupta [1998], Carr and Picron [1999] and Fink [2003]. Carr and Chou [1997] shows the representation of any twice differentiable payoff functions, that is the basis for Theorem 7.4 in this chapter as well as for Proposition 1 of Takahashi and Yamazaki [2009a]. Their paper then develops the so called *strike-spreads* method for static hedging of barrier under the Black-Scholes model.

More recently, Carr and Lee [2009] extends put-call symmetry(PCS) and applies it to constructing semi-static replications for barrier-type claims under general asset dynamics. For other works related with static hedging of barrier options, see their paper and references therein.

On the other hand, Carr and Wu [2002] concentrates on an efficient replication of a plain vanilla option though their approach implies the possibility of further extensions and applications. It also applies the Gauss-Hermite quadrature rule to approximate static hedging of the option by plain vanilla options with shorter terms under the Black-Scholes and Merton [1976] jump-diffusion models. Moreover, their paper undertakes extensive simulation exercises to investigate the robustness of the method. In a certain sense, this chapter extends the methodologies developed by Carr and Wu [2002], Carr and Chou [1997] and Carr and Madan [1998, 1999] to stochastic volatility models.

7.1 New scheme for static hedging of European path-independent derivatives

This section presents a new scheme for static hedging of European options. Specifically, under stochastic volatility models, we develop a methodology to hedge European path-independent derivatives and their portfolios based on a *static* portfolio of shorter term plain vanilla options. *Static* portfolio implies that the weights in the portfolio remain unchanged when the prices of underlying assets move and options in the portfolio approach maturity. This static hedging scheme is not entirely perfect, but provides much better performance than a dynamic hedging method. Robustness of our scheme will be shown in the next section.

Under the assumptions of a frictionless and no-arbitrage market, let S_t denote the spot price of a stock, an underlying asset at time $t \in [0, T^*]$ where T^* is some arbitrarily fixed time horizon. For sake of simplicity, the interest rate r and the dividend yield q are assumed to be constants. The no-arbitrage condition ensures the existence of a risk-neutral probability measure $\mathbb{Q}$ such that the instantaneous expected rate of return on every asset is equal to the instantaneous interest rate r. Furthermore, the risk-neutral process of the underlying asset price is assumed to be an Itô process under a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T^*]}, \mathbb{Q})$. In addition, the analysis in this chapter concentrates on static hedging of European path-independent options where the final payoff of the option is solely determined by the stock price at maturity. Typical examples in this class include plain vanilla, cash digital, asset digital and power options.

7.1.1 General case

Suppose that the underlying asset price S under the risk-neutral measure $\mathbb{Q}$ is evolved by a stochastic volatility model. In particular, (S, V) is a $\mathbf{R}^2_{++}$-valued process and it is the unique solution of a stochastic differential equation given $(S_0, V_0) \in \mathbf{R}^2_{++}$:

$$\begin{aligned} dS_t &= cS_t dt + \sqrt{V_t} S_t \bar{\sigma}_1 dW_t \\ dV_t &= \mu(\omega, t)dt + \sigma_2(\omega, t)\bar{\sigma}_2 dW_t, \end{aligned} \tag{7.1}$$

where $c := r - q$ is a constant and $W = (W_1, W_2)$ is a 2-dimensional Brownian motion. Here μ and σ_2 are $\mathbf{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable processes that guarantee the unique solution to the stochastic differential equation. Also, $\bar{\sigma}_i (i = 1, 2)$ are defined by $\bar{\sigma}_1 = (1, 0)$ and $\bar{\sigma}_2 = (\rho, \sqrt{1-\rho^2})(|\rho| \le 1)$ respectively.

Our subsequent analysis relies on the next result due to Gyöngy [1986].

Theorem 7.1 *(Theorem 4.6 in Gyöngy [1986])*
ξ is a $\mathbf{R}^n$-valued process and it is the unique solution of a stochastic differential equation:

$$\xi_t = x_0 + \int_0^t \beta(\omega, u)du + \int_0^t \delta(\omega, u)dW_u$$

where $x_0 \in \mathbf{R}^n$, β and δ are bounded measurable $\mathcal{F}_u$-adapted $\mathbf{R}^n$-valued and $\mathbf{R}^{n\times n}$-valued processes respectively, and W is a n-dimensional Brownian motion.
We put a condition:

$$\sum_{i,j} \alpha_{i,j} z_i z_j \geq p|z|^2$$

for every $(\omega, t) \in \Omega \times [0, \infty)$ and $z \in \mathbf{R}^n$ where $\alpha := \delta\delta^\top$ and p is a fixed positive constant. Here, $x^\top$ denotes the transpose of x.
Under the condition, the stochastic differential equation:

$$X_t = x_0 + \int_0^t b(X_u, u)du + \int_0^t \sigma(X_u, u)dW_u$$

admits a weak solution $\bar{X}_t$ which has the same one-dimensional distribution as ξ_t, where $b : \mathbf{R}^n \times [0, \infty) \to \mathbf{R}^n$ and $\sigma : \mathbf{R}^n \times [0, \infty) \to \mathbf{R}^{n\times n}$ are respectively bounded measurable functions such that

$$b(x,t) := \mathbb{E}[\beta(t)|\xi_t = x] \tag{7.2}$$
$$\sigma(x,t) := \mathbb{E}[\delta(t)\delta(t)^\top|\xi_t = x]^{\frac{1}{2}}. \tag{7.3}$$

That is, the distribution of ξ_t and $\bar{X}_t$ are the same for every $t \geq 0$.

For pricing a European path-independent derivative, only the distribution at maturity of the underlying asset price does matter. Hence, due to Gyöngy's above theorem, pricing a European path-independent derivative under the stochastic volatility model (7.1) is transformed to pricing it under a local volatility model if S is regarded as ξ in his theorem. This result is stated as the following proposition:

Proposition 7.2 *Suppose that $f_T(S)$ is the payoff at maturity T of a European path-independent derivative whose randomness depends solely on the underlying price at maturity, S_T. Suppose also that the time-0 price function $v_0(y, z)$ of the derivative under the stochastic volatility model (7.1) with $S_0 = y$ and $V_0 = z$. then, $v_0(y, z)$ is given by:*

$$v_0(y, z) = e^{-rT}\mathbb{E}[f_T(\hat{S})],$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator under the risk-neutral probability measure $\mathbb{Q}$, and $\hat{S}$ follows a local volatility model:

$$d\hat{S}_t = c\hat{S}_t dt + \sigma(\hat{S}_t, t)dW_{1t}; \ \hat{S}_0 = y. \tag{7.4}$$

Here, $\sigma(x, t)$ is defined by:

$$\sigma(x,t) := \mathbb{E}[V_t|S_t = x]^{\frac{1}{2}}. \tag{7.5}$$

Also, in stead of getting the local volatility $\sigma(x,t)$ by evaluating the right hand side of (7.5), we can sometimes obtain it easier through the following Dupire [1994]'s result:

Proposition 7.3 *(Dupire [1994]) Suppose that the underlying spot price $\hat{S}$ is evolved by a local volatility model (7.4). Let $C(t,x) := C(y;t,x)$ represent the time-0 price of a plain vanilla call option with spot price $\hat{S}_0 = y$, strike x and maturity t. Then, the local volatility $\sigma(x,t)$ is given by:*

$$\sigma^2(x,t) = 2\frac{qC(t,x) + cx\frac{\partial}{\partial x}C(t,x) + \frac{\partial}{\partial t}C(t,x)}{x^2\frac{\partial^2}{\partial x^2}C(t,x)}. \tag{7.6}$$

Note that the option prices and its derivatives appearing in the right hand side of (7.6) is equivalent to those under the stochastic volatility model (7.1). Hence, if the option prices and the derivatives under the stochastic volatility model is obtained analytically or semi-analytically, then the local volatility is easy to calculate. We will see this case using the Heston [1993]'s model in the next subsection.

Of course, when pricing derivatives under the stochastic volatility model is possible analytically, the transformation to a local volatility model does not have any advantage in terms of valuation. However, in terms of hedging, it does have advantage because the reduction of a two-factor model to a one-factor model allows us the direct application of Proposition 1 in Takahashi and Yamazaki [2009a]. The following theorem is our main result.

Theorem 7.4 *Suppose that $f_T(S)$ is the payoff at maturity T of a European path-independent derivative and that its underlying asset price is evolved by the model (7.1). Also let $\tau \in [0,T]$ and suppose that the time-τ price function $\hat{v}_\tau(\hat{S})$ of the European derivative under model (7.4) is twice differentiable for all $\hat{S} \geq 0$, that is, both the delta and gamma of the derivative exist at time τ. Here, the process $\hat{S}$ is the solution to the stochastic differential equation of (7.4). Then, it holds that for any $\kappa > 0$,*

$$\begin{aligned} v_0(y,z) &= e^{-r\tau}\hat{v}_\tau(\kappa) + e^{-r\tau}\frac{\partial \hat{v}_\tau}{\partial \hat{S}}\mid_{\hat{S}=\kappa}\{F(\tau)-\kappa\} \\ &+ \int_0^\kappa \frac{\partial^2 \hat{v}_\tau}{\partial \hat{S}^2}\mid_{\hat{S}=x} P(\tau,x)dx + \int_\kappa^{+\infty}\frac{\partial^2 \hat{v}_\tau}{\partial \hat{S}^2}\mid_{\hat{S}=x} C(\tau,x)dx, \end{aligned} \tag{7.7}$$

where $F(\tau)$ denotes the time-0 price of the forward contract with maturity τ, and $P(\tau,x)$ and $C(\tau,x)$ represent the time-0 prices of plain vanilla put and call options with spot price y, strike x and maturity τ respectively.

The implication of this theorem is that the risk embedded in a target European derivative can be hedged using a *static* portfolio of liquid plain vanilla options with a maturity that is shorter than the maturity of the target derivative. The equation (7.7) implies that the static portfolio consists of the following securities with maturities τ; $\frac{\partial^2 \hat{v}_\tau}{\partial \hat{S}^2}\mid_{\hat{S}=x} dx$ units of a call with strike x for each $x > \kappa$ and $\frac{\partial^2 \hat{v}_\tau}{\partial \hat{S}^2}\mid_{\hat{S}=x} dx$ units of a put with strike x for each $x < \kappa$ as well as $\frac{\partial \hat{v}_\tau}{\partial \hat{S}}\mid_{\hat{S}=\kappa}$ units of a forward contract with delivery price κ and $\hat{v}_\tau(\kappa)$ units of a zero coupon bond with face value 1. Here *static* portfolio indicates that once the hedging portfolio is created, re-balancing is unnecessary until the maturity date of the options in the portfolio.

Finally, Theorem 1 in Takahashi and Yamazaki [2009a] provides a practically efficient scheme based on the Gauss-Legendre quadrature rule for approximating the theoretical hedging portfolio given by the right hand side of (7.7). We will show the validity of our scheme in the next section through a numerical example.

Remark 7.5 *The equation (7.7) indicates that the value of the target derivative is replicated exactly by the hedging portfolio at time-0. However, after time-0 to the end of the hedging period the value may not* *be replicated for all the realization of (S_t, V_t) for $t \in (0, \tau]$; more precisely, if the realization of V_t given S_t deviates from $\mathbb{E}[V_t|S_t]$, the target derivative is not hedged perfectly. Therefore, we need to examine the performance of our hedging scheme in further detail. In fact, simulation exercises in the next section show that our static scheme provides much better performance than a dynamic hedging method.*

Remark 7.6 *When the hedging target is a plain vanilla call option under non-stochastic volatility environment, Theorem 7.4 is reduced to Theorem 1 in Carr and Wu [2002].*

7.1.2 Example: Heston Model

This subsection derives the formula for the volatility function $\sigma(x,t)$ under the Heston [1993]'s model used for a numerical example in the next section. The stochastic volatility model (7.1) becomes the following in this case:

$$\begin{aligned} dS_t &= cS_t dt + \sqrt{V_t} S_t \bar{\sigma}_1 dW_t;\ S_0 = y \\ dV_t &= \xi(\eta - V_t)dt + \theta\sqrt{V_t}\bar{\sigma}_2 dW_t;\ V_0 = z, \end{aligned} \tag{7.8}$$

where ξ, η and θ are positive constants such that $\xi\eta \geq \theta^2/2$. Also, $\bar{\sigma}_i (i = 1, 2)$ are defined by $\bar{\sigma}_1 = (1, 0)$ and $\bar{\sigma}_2 = (\rho, \sqrt{1-\rho^2})(|\rho| \leq 1)$ respectively, and W is a 2-dimensional Brownian motion. We next present expressions for the call price and its derivatives in the right hand side of (7.6) based on a slight modification of Carr and Madan [1999]'s Fourier transform method.

Carr and Madan [1999] introduces a fast Fourier transform method for option pricing. This chapter proposes to compute the time value of the option after subtracting an intrinsic value from the option price in order to avoid the oscillation of the integrand in the Fourier inversion. As a result, the option price can be obtained as the time value derived by the Fourier inversion plus the intrinsic value. On the other hand, to compute the partial derivatives of a call option with respect to strike K, we propose to subtract the Black-Scholes price with appropriate volatility from the option price instead of subtracting the intrinsic value. This choice is made because the intrinsic value of a call option is not differentiable. See also p.363 of Cont and Tankov [2004]. (Note that the Black-Scholes call price is twice differentiable with respect to strike K.)

In the Heston model, let $X_t := \ln\{S_t/S_0\} - ct$ and then $\phi_{X_t}(u)$, the characteristic function of X_t is obtained by:

$$\phi_{X_t}(u) = \exp\{A(u,t)\}B(u,t),$$

where

$$A(u,t) := \frac{\xi\eta t(\xi - i\rho\theta u)}{\theta^2} - \frac{(u^2 + iu)V_0}{\gamma\coth(\gamma t/2) + \xi - i\rho\theta u} \tag{7.9}$$

$$B(u,t) := \left\{\cosh(\gamma t/2) + \frac{\xi - i\rho\theta u}{\gamma}\sinh(\gamma t/2)\right\}^{-2\xi\eta/\theta^2} \tag{7.10}$$

$$\gamma := \sqrt{\theta^2(u^2 + iu) + (\xi - i\rho\theta u)^2},\ i = \sqrt{-1}. \tag{7.11}$$

For the case of the Black-Scholes model (i.e. $S_t^{bs} = S_0 e^{ct - \frac{1}{2}\sigma^2 t + \sigma W_{1t}}$), note that $\phi_{X_t^{bs}}(u)$, the characteristic function of $X_t^{bs} := \ln\{S_t^{bs}/S_0\} - ct$ is expressed as $\phi_{X_t^{bs}}(u) = \exp\{-\sigma^2 t(u^2 + iu)/2\}$. Then, we have the following proposition. The proof is easy and is omitted.

Proposition 7.7 *Under the Heston [1993]'s stochastic volatility model (7.8), let $C(t,x)$ the call price at time 0 with strike x and maturity t. Then, $C(t,x)$, $\frac{\partial C(t,x)}{\partial x}$, $\frac{\partial^2 C(t,x)}{\partial x^2}$ and $\frac{\partial C(t,x)}{\partial t}$ in (7.6) are given as* *follows:*

$$
\begin{aligned}
C(t,x) &= \frac{S_0 e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\zeta_t(u)du + C^{bs}(t,x) \qquad (7.12)\\
\frac{\partial C(t,x)}{\partial x} &= \frac{-e^{-(\alpha+1)k}}{2\pi}\int_{-\infty}^{\infty}(\alpha+iu)e^{-iuk}\zeta_t(u)du + \frac{\partial C^{bs}(t,x)}{\partial x}\\
\frac{\partial^2 C(t,x)}{\partial x^2} &= \frac{e^{-(\alpha+2)k}}{2\pi S_0}\int_{-\infty}^{\infty}(\alpha+iu)(\alpha+iu+1)e^{-iuk}\zeta_t(u)du\\
&+ \frac{\partial^2 C^{bs}(t,x)}{\partial x^2}\\
\frac{\partial C(t,x)}{\partial t} &= \frac{S_0 e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\frac{\partial \zeta_t(u)}{\partial t}du + \frac{\partial C^{bs}(t,x)}{\partial t}
\end{aligned}
$$

where $\alpha > 0$*,* $k := \ln\{x/S_0\}$*,* $C^{bs}(t,x)$ *denotes Black-Scholes call price at time* 0 *with strike* x *and maturity* t*, and*

$$
\zeta_t(u) := \frac{\exp\{[c(iu+\alpha+1)-r]t\}}{(iu+\alpha)(iu+\alpha+1)}\{\phi_{X_t}(u-i\alpha-i)-\phi_{X_t^{bs}}(u-i\alpha-i)\}.
$$

7.2 Numerical examples

This section shows the validity of our scheme through numerical examples under the Heston [1993]'s model. The examples are two types of simulation test, which are Monte Carlo simulation and historical simulation.

Note first that the market is incomplete under the stochastic volatility model and that the perfect hedge is not possible by dynamic trading of the underlying asset. Specifically, we implement hedging simulations comparing the performance of our scheme with that of *the minimum-variance hedging method*, a standard method of dynamic hedging in an incomplete market. In the minimum-variance hedging method, the units of the underlying asset to be held at each time t are computed as follows:[1]

$$
\frac{\partial C_t}{\partial S_t} + \frac{\rho\theta}{S_t}\frac{\partial C_t}{\partial V_t}, \qquad (7.13)
$$

where C_t denotes the time-t price of a target call option , and ρ and θ are parameters in (7.8). Here we observe that volatility risk is partially hedged through the correlation between the underlying asset's price and its instantaneous variance. Moreover, based on the equation (7.13), we re-balance the dynamic portfolio once a day in our simulations.

Let us describe briefly the procedure for implementation of our static hedging scheme. First, we transform the Heston model (7.8) into a local volatility model (7.4) by applying (7.12). Next, in order to obtain a static hedging portfolio in Theorem 1 of Takahashi and Yamazaki [2009a] that is a practical method for implementing Theorem 7.4 in this chapter, we need approximations of the price, the delta and gammas of the target option in Theorem 7.4, in other words $\hat{v}_\tau(\kappa)$, $\frac{\partial \hat{v}_\tau}{\partial \hat{S}}\mid_{\hat{S}=x}$ and $\frac{\partial^2 \hat{v}_\tau}{\partial \hat{S}^2}\mid_{\hat{S}=x}$ in (7.7) respectively. Solving the relevant partial differential equation (PDE) numerically by the Crank-Nicholson method provides those approximations.

[1]For example, see Bakshi,Cao and Chen [1997] for the detail and for a practical application of the minimum-variance hedging method.

7.2.1 Monte Carlo simulation test

In Monte Carlo simulations, we consider two cases: the first case (Case 1) is that the Heston parameters are the same under a risk-neutral measure and under the physical measure except a mean reverting rate η, while the second case (Case 2) is that volatility on the variance θ under the risk-neutral measure is higher than the one under the physical measure. Under the physical measure, we assume that S has a drift coefficient of 0.06. These parameters are taken form Carr and Lee [2007]. The initial conditions of our simulations and the Heston parameters for the first and second cases are listed in Tables 7.1 and 7.2 respectively. For both cases, the hedging period is set to be $\tau = 0.5$ while the maturity of the target option is $T = 1.0$.

Table 7.3 shows approximations of the target option's price by the values of options' portfolios used for static hedging; the target option's *true* price is given by direct application of Heston [1993]'s formula. Also these static portfolio compositions are reported in Table 7.4. Clearly, the more the number of options, the better is the approximation. A portfolio of more than eight options gives rather good approximation; the absolute values of the error and the error ratio are less than 0.002 and 0.03% respectively for the portfolio of eight options(call=4, put=4 in the table).

Next, Tables 7.5 and 7.6 provide basic statistics of Monte Carlo simulation results for Case 1 and Case 2 respectively. Moreover, Figure 7.1 shows the histograms of hedging errors. The statistics and the histograms are based on 10,000 simulated paths. All the statistics and figures shows that our static hedging scheme outperforms the dynamic hedging based on *the minimum-variance hedging method*. In particular, for Case 2, that is when the volatility on the variance under the physical measure differs from the one under the risk-neutral measure, our scheme gives more robust result than the dynamic hedging in a sense that its hedging performance is less affected by the parameter's change than the dynamic hedging's performance. Because this situation is common in practice, the result indicates that our static hedging scheme seems useful.

7.2.2 Historical simulation test

This subsection shows the historical performance of our static hedging scheme in USD/EUR currency option market. The data on USD/EUR currency options are obtained from British Bankers Association's homepage. They are daily time-series data of plain vanilla options on USD/EUR spot exchange rate from August 2001 to January 2008.

In currency option markets, option prices are provided as Black-Scholes implied volatilities and the moneyness of an option is expressed in terms of Black-Scholes delta, rather than its strike price (See Carr and Wu [2007] for the detail). Using the daily data of 25-delta call, 25-delta put and ATM with 3-month and 1-year maturities and re-calibrating the Heston model every business day, we compare the performance of the static hedging with that of the minimum-variance hedging. The target option is plain vanilla call with maturity $T = 1.0$ and ATM strike at hedging starting date. The maturity of options on a static hedging portfolio is set to be $\tau = 0.5$ and $\tau = 0.25$ for investigation of option maturity effects in our static hedging scheme. Table 7.7 shows the static portfolio compositions on 2001/08/29 as an example. To set each period of the hedging performance measurement to be one month (21 business days), we obtain 78 non-overlapping hedging experiments on the data from August 2001 to January 2008. Hedging errors in each hedging experiment are normalized by the target option price at the starting date of each month for comparison of the performance among 78 experiments.

Table 7.8 provides basic statistics of historical simulation results and Figure 7.2 shows the histograms of hedging errors in the case of $\tau = 0.5$ and $\tau = 0.25$. All the statistics and figures shows that our static hedging scheme outperforms the dynamic hedging based on the minimum-variance hedging method as in Monte Carlo simulation tests of the previous subsections. Even the static hedging with $\tau = 0.25$, which shows worse performance than the static hedging with $\tau = 0.5$, gives much more robust result than the dynamic hedging. According to the historical simulation results, our static hedging scheme seems very effective in practice.

7.3 Concluding remarks

This chapter presents a new scheme for the static hedging of European path-independent derivatives under stochastic volatility models. The scheme can be applied to European path-independent derivatives including digital-type options for which dynamic hedging is sometimes difficult to implement and is therefore not very effective in practice. Also, our efficient method can be extended to more general class of the underlying models with certain approximation methods. Moreover, a numerical example in the Heston [1993]'s stochastic volatility model confirms the validity of our scheme through comparison with a dynamic hedging method. Finally, our next research topic will be to establish an effective and efficient scheme for the static hedging of more general multi-factor derivatives, such as cross-currency derivatives with stochastic interest rates and stochastic volatilities.

Figure 7.1: Histogram of Monte Carlo hedging errors

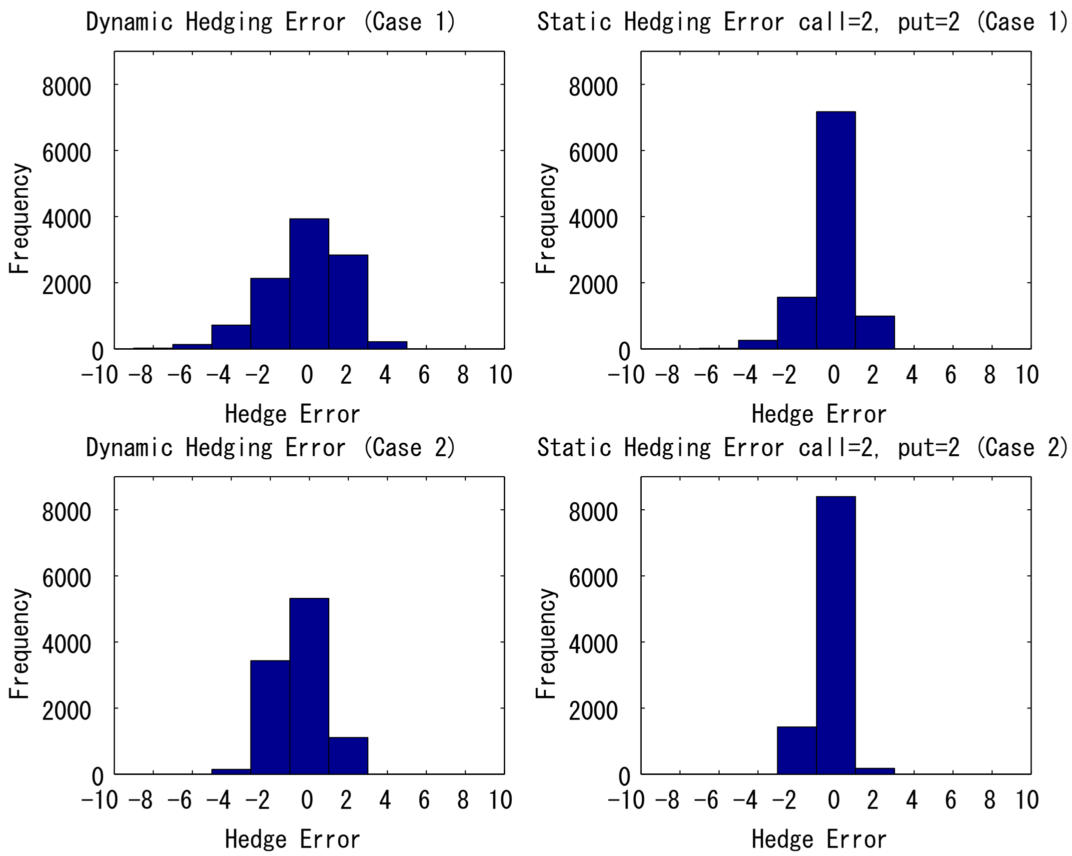


Figure 7.2: Histogram of historical hedging errors

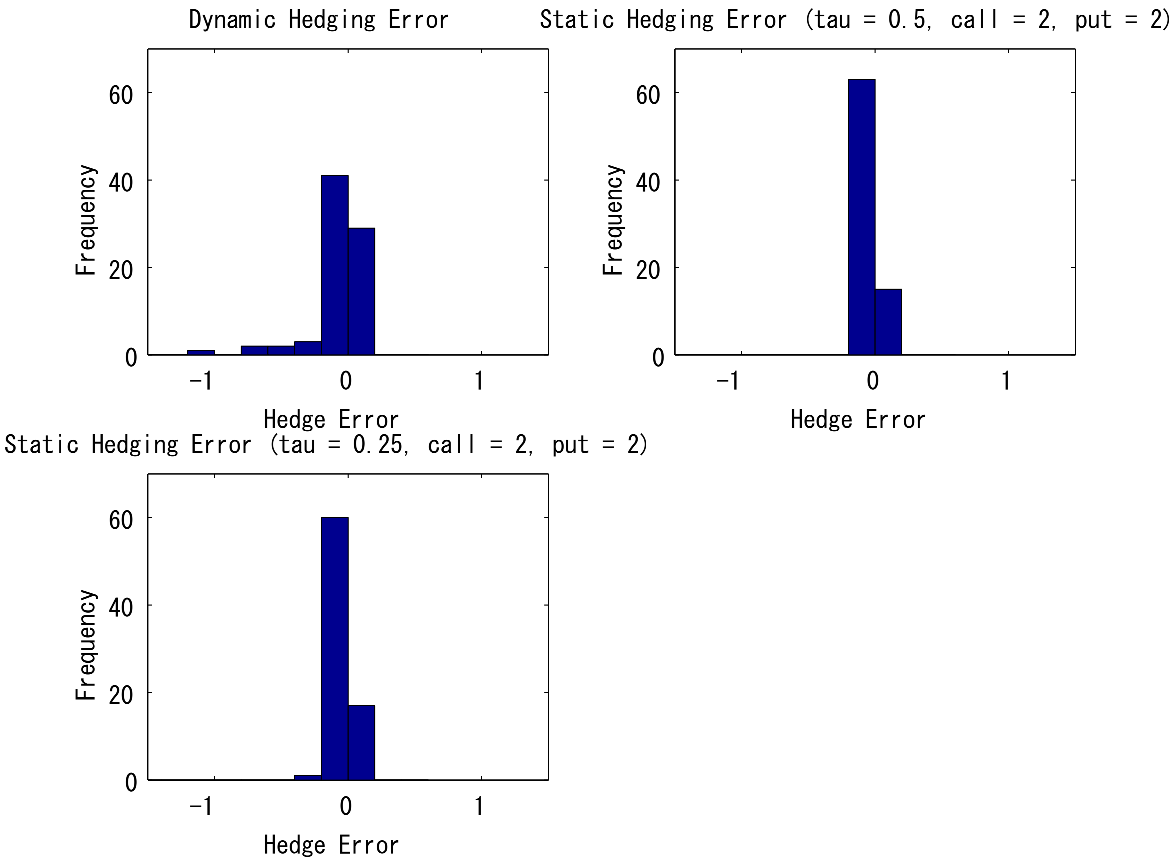


Table 7.1: Initial condition (Case 1 & Case 2)

target option	S_0	T	r	q	K	τ
call	100	1.0	0.0	0.0	100	0.5

Table 7.2: Heston parameters (Case 1 & Case 2)

parameter	V_0	ξ	η	θ	ρ
risk-neutral	0.20^2	1.15	0.20^2	0.39	-0.64
physical (Case 1)	0.20^2	1.15	0.18^2	0.39	-0.64
physical (Case 2)	0.20^2	1.15	0.18^2	0.15	-0.64

Table 7.3: Pricing (Case 1 & Case 2)

	target option	static portfolio call=2, put=2	call=4, put=4	call=8, put=8
price	7.240	7.202	7.238	7.240
error	-	-0.038	-0.002	0.001
error ratio (%)	-	-0.522	-0.026	0.007

Table 7.4: Static hedge portfolio in Monte Carlo simulation test (Case 1 & Case 2)

static option portfolio	strike / amount	No.1	No.2	No.3	No.4	No.5	No.6	No.7	No.8
call=2, put=2	call strike	106.34	123.66						
	call amount	0.507	0.055						
	put strike	68.45	91.55						
	put amount	0.043	0.424						
call=4, put=4	call strike	102.08	109.90	120.10	127.92				
	call amount	0.179	0.279	0.071	0.008				
	put strike	62.78	73.20	86.80	97.22				
	put amount	0.007	0.051	0.195	0.204				
call=8, put=8	call strike	100.60	103.05	107.12	112.25	117.75	122.88	126.95	129.40
	call amount	0.050	0.116	0.156	0.126	0.060	0.020	0.006	0.002
	put strike	60.79	64.07	69.49	76.33	83.67	90.51	95.93	99.21
	put amount	0.001	0.005	0.016	0.040	0.083	0.124	0.122	0.064

Table 7.5: Monte Carlo simulation result (Case 1)

hedge error	dynamic hedge	static hedge call=2, put=2	 call=4, put=4	 call=8, put=8
mean	−0.152	−0.045	−0.104	−0.100
standard deviation	1.939	1.203	1.165	1.161
percentile 1%	−5.555	−3.941	−3.661	−3.666
percentile 5%	−3.699	−2.448	−2.252	−2.230
percentile 10%	−2.795	−1.694	−1.585	−1.579

Table 7.6: Monte Carlo simulation result (Case 2)

hedge error	dynamic hedge	static hedge call=2, put=2	 call=4, put=4	 call=8, put=8
mean	−0.521	−0.203	−0.253	−0.252
standard deviation	1.180	0.706	0.715	0.703
percentile 1%	−3.166	−2.082	−1.956	−1.903
percentile 5%	−2.414	−1.504	−1.431	−1.397
percentile 10%	−2.069	−1.198	−1.163	−1.136

Table 7.7: Static hedge portfolio in Historical simulation test (as of 2001/08/29)

static option portfolio	strike / amount	No.1	No.2	No.3	No.4	No.5	No.6	No.7	No.8
	call strike	0.954	1.070						
$\tau = 0.5$, call=2, put=2	call amount	0.395	0.067						
	put strike	0.754	0.870						
	put amount	0.059	0.478						
	call strike	0.926	0.978	1.046	1.098				
$\tau = 0.5$, call=4, put=4	call amount	0.173	0.194	0.067	0.014				
	put strike	0.726	0.778	0.846	0.898				
	put amount	0.009	0.070	0.246	0.186				
	call strike	0.916	0.932	0.960	0.994	1.030	1.065	1.092	1.108
$\tau = 0.5$, call=8, put=8	call amount	0.053	0.106	0.117	0.087	0.049	0.023	0.010	0.003
	put strike	0.716	0.732	0.760	0.794	0.830	0.865	0.892	0.908
	put amount	0.002	0.007	0.021	0.055	0.110	0.144	0.118	0.054
	call strike	0.944	1.030						
$\tau = 0.25$, call=2, put=2	call amount	0.275	0.075						
	put strike	0.794	0.880						
	put amount	0.120	0.335						
	call strike	0.923	0.962	1.013	1.052				
$\tau = 0.25$, call=4, put=4	call amount	0.113	0.147	0.069	0.015				
	put strike	0.773	0.812	0.863	0.902				
	put amount	0.026	0.106	0.195	0.121				
	call strike	0.915	0.927	0.948	0.973	1.001	1.027	1.047	1.059
$\tau = 0.25$, call=8, put=8	call amount	0.034	0.070	0.083	0.071	0.047	0.026	0.011	0.004
	put strike	0.765	0.777	0.798	0.823	0.851	0.877	0.897	0.909
	put amount	0.006	0.019	0.040	0.070	0.097	0.103	0.078	0.035

Table 7.8: Historical simulation result

hedge error	dynamic hedge	static hedge $\tau = 0.5$ call=2, put=2	call=4, put=4	call=8, put=8	static hedge $\tau = 0.25$ call=2, put=2	call=4, put=4	call=8, put=8
mean	−0.070	−0.028	−0.026	−0.026	−0.023	−0.023	−0.023
standard deviation	0.195	0.047	0.049	0.049	0.060	0.061	0.061
max	0.188	0.090	0.100	0.100	0.139	0.145	0.145
min	−1.113	−0.186	−0.183	−0.183	−0.203	−0.213	−0.213
skewness	−3.135	−0.524	−0.436	−0.441	−0.166	−0.176	−0.178
kurtosis	14.432	4.737	4.456	4.462	4.046	4.093	4.095

Chapter 8

Static Hedging of Defaultable Contingent Claims: A Simple Hedging Scheme across Equity and Credit Markets

This chapter proposes a simple scheme for static hedging of defaultable contingent claims. We show that suitable path-independent defaultable contingent claims including plain vanilla, cash digital, asset digital, and power options, but also defaultable bonds can be replicated by a static portfolio composed of a non-defaultable bond and plain vanilla options, all of whose maturities are shorter than that of the target defaultable contingent claim. In addition, by using the Gauss-Legendre quadrature rule as an approximation technique of the static portfolio, which is proposed in Takahashi and Yamazaki [2009a], it is demonstrated that the target claim can be accurately replicated by a feasible number of options. Therefore, our static hedging scheme is able to be implemented in practice. Through numerical examples, we show that the scheme is applicable to both the structural models and the intensity-based models.

The scheme proposed in this chapter generalizes the techniques developed by Carr and Chou [1997], Carr and Madan [1998], and Takahashi and Yamazaki [2009a] to credit-equity models. Until recently, equity models and credit models have developed more or less independently of each other. Equity models mainly focused on pricing equity derivatives and describing the implied volatility smile observed in markets by introducing jumps and/or stochastic volatility into the stock price process without default risk. Conversely, credit models concentrated on modeling the default event and evaluating credit derivatives while the credit models ignored the information on the equity market (see Bielecki and Rutkowski [2002], Duffie and Singleton [2003], and Lando [2004] for instance). However, in recent literature many researchers developed unified credit-equity models in order to evaluate equity and credit derivatives simultaneously. Generally speaking, the unified modeling approaches can be classified into two categories: the structural approach and the intensity-based approach.

In the structural approach to unified modeling, for instance, Hull et al. [2005] proposed an extended Merton's firm value model to examine linkage between equity implied volatility skew and default probability on individual firms. The CreditGrades model introduced by Finger et al. [2002] and Stamicar and Finger [2006] is one of the most preferred structural models for practitioners when jointly pricing credit default swaps and equity plain vanilla options, and there are several extensions of the CreditGrades model; e.g., see Sepp [2006] and Ozeki et al. [2011]. Section 8.4 will show an application of our static hedging scheme to an extended CreditGrades model to replicate defaultable bonds.

On the other hand, in the intensity-based approach, the unified modeling is well-known as *the jump to default model* pioneered by Merton [1976]. The jump to default model is represented as a defaultable stock price process with a default intensity, in which the stock price drops to zero when the reference firm bankrupts. Recently, much literature dealing with jump to default models has been published. For example, see Takahashi et al. [2001], Linetsky [2006], Carr and Linetsky [2006], Mendoza et al. [2011], Papageorgiou and Sircar [2008], Carr and Schoutens [2008]. Section 8.5 will demonstrate that our static hedging scheme is applicable to defaultable bond replications under *the jump to default extended CEV model* proposed in Carr and Linetsky [2006].

Even though many unified credit-equity models have been studied, the issue of hedging has rarely been addressed. To the best of our knowledge, there are the following references: Carr [2005] proposed to replicate a defaultable discount bond under the jump to default extended Black-Scholes model using a stock and its call option. Carr and Wu [2011] introduced a simple replication scheme of a certain credit derivative named the *unit recovery claim*, which pays one dollar at the default time, by using American put options. Carr and Schoutens [2008] explained how to perfectly hedge under Heston's stochastic volatility model with jump to default. In their paper, not only the stock and the bond, but also the variance swap and the credit default swap are used to hedge a defaultable contingent claim.

Compared to the other approaches that have appeared in the literature, the hedging scheme proposed in this chapter seems to be more pragmatical. The reasons are as follows: First, our scheme is general. Thus it is not for a specific model, but applicable to many credit-equity hybrid models. Second, hedging instruments in our scheme are mainly plain vanilla equity options with shorter maturities than that of the target defaultable contingent claim. This implies that any relatively illiquid defaultable claim can be replicated by using liquid equity options. Third, our hedging scheme is *static*. *Static hedging* means that re-balancing the hedging portfolio is not necessary. In general, it is difficult to trade with financial products on a given firm dynamically when the firm's default probability highly increases. Therefore, dynamic hedging is not practical for a defaultable contingent claim. Fourth, although the static portfolio needs an infinite number of plain vanilla options theoretically, the Gauss-Legendre quadrature rule allows us to accurately approximate this portfolio using only a finite number of options. This fact will be shown in the numerical examples in Section 8.4 and 8.5.

8.1 Assumptions and Notation

We assume that markets are frictionless and arbitrage-free. Let $\tau \in [0, T^*]$ denote the default time of a given firm on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, where T^* is some fixed time horizon. We remark that the dynamics of τ are not explicitly specified. However, we give explicit examples in Sections 8.4 and 8.5. Let $(S_t)_{t\geq 0}$ be the stock dynamics of the firm under $\mathbb{Q}$, where $\mathbb{Q}$ is an equivalent martingale measure. We denote the associated filtrations of S_t and τ by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{H} := (\mathcal{H}_t)_{t\geq 0}$ respectively, where $\mathcal{F}_t = \sigma(S_s : s \in [0,t])$ and $\mathcal{H}_t = \sigma(\mathbf{1}_{\{\tau > s\}} : s \in [0,t])$. Finally, we set $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for $t > 0$ and $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$. Hence, in this setting the reference filtration $\mathbb{F}$ is generated by the only stock price $(S_t)_{t\geq 0}$ for simplicity.

Assumption 8.1 *The pre-default stock dynamics $(S_t)_{t\geq 0}$ follows a strictly positive stochastic process, and the stock price is zero after default; i.e.,*

$$\begin{cases} S_t > 0 & \text{if} \quad t < \tau, \\ S_t = 0 & \text{if} \quad t \geq \tau. \end{cases}$$

Assumption 8.1 is satisfied in many models. For instance, many of the jump to default models, which fall under intensity-based approach, satisfy this assumption. As other examples, the structural models like the CreditGrades model also satisfy Assumption 8.1. Concrete examples of stock price dynamics satisfying Assumption 8.1 are presented in Sections 8.4 and 8.5.

Following Carr and Linetsky [2006], we consider the prices of the following three "building blocks", which can be used to construct more complex securities:

- A European-style contingent claim with maturity T and payoff $f(S_T)$ at T given that no default has occurred by T; in the event of default, the payoff is zero.
- A recovery payment of one dollar at T if default has occurred by T.
- A recovery payment of one dollar at the default time τ, if $\tau \leq T$.

We point out that we have made minimal assumptions about the dynamics of τ and S up to this point. Hence we cannot derive the following formulas, but need to postulate them. However, in Sections 8.4 and 8.5, we present concrete models which do satisfy them.

Assumption 8.2 *The prices of the European-style contingent claim with maturity T, the recovery payment of δ^T dollars at time T, and the recovery payment of δ^τ dollars at τ satisfy:*

$$\begin{aligned}
\mathbb{E}\left[e^{-r(T-t)} f(S_T)\mathbf{1}_{\{T<\tau\}} \mid \mathcal{G}_t\right] &=: \mathbf{1}_{\{\tau>t\}} V_t^0(S_t,T), \\
\mathbb{E}\left[e^{-r(T-t)} \delta^T \mathbf{1}_{\{T\geq\tau\}} \mid \mathcal{G}_t\right] &=: \mathbf{1}_{\{\tau>t\}} V_t^1(S_t,T) + \mathbf{1}_{\{\tau\leq t\}} e^{-r(T-t)}\delta^T, \\
\mathbb{E}\left[e^{-r(\tau-t)} \delta^\tau \mathbf{1}_{\{T\geq\tau\}} \mid \mathcal{G}_t\right] &=: \mathbf{1}_{\{\tau>t\}} V_t^2(S_t,T) + \mathbf{1}_{\{\tau\leq t\}} e^{-r(\tau-t)}\delta^\tau,
\end{aligned}$$

where $V_t^0(S,T)$, $V_t^1(S,T)$, and $V_t^2(S,T)$ are Borel-measurable functions dependent on t, S, and T. All expectations are taken with respect to $\mathbb{Q}$ and r denotes the risk-free interest rate. Thus we consider three types of value functions $V_t(\tau, S_t; T)$ of defaultable contingent claims:

$$V_t(\tau,S_t;T) = \begin{cases} \mathbf{1}_{\{\tau>t\}} V_t^0(S_t,T) & \textit{(ZR)} \\ \mathbf{1}_{\{\tau>t\}}[V_t^0(S_t,T)+V_t^1(S_t,T)] + \mathbf{1}_{\{\tau\leq t\}} e^{-r(T-t)}\delta^T & \textit{(RM)} \\ \mathbf{1}_{\{\tau>t\}}[V_t^0(S_t,T)+V_t^2(S_t,T)] + \mathbf{1}_{\{\tau\leq t\}} e^{-r(\tau-t)}\delta^\tau & \textit{(RD).} \end{cases} \tag{8.1}$$

In the above equation, ZR, RM, and RD are short hand notation for "zero recovery", "recovery paid at the maturity date", and "recovery paid at the default time", respectively. Many examples, such as forward contracts with default risk, European options with default risk, and defaultable bonds, can be shown to fall into the framework presented in Assumption 8.2.

Example 8.3 (Forward contract with default risk) A defaultable equity forward price $F_t(T)$ at time t is given by

$$F_t(S_t;T) = \mathbb{E}\left[e^{-r(T-t)} S_T \mathbf{1}_{\{T<\tau\}} \mid \mathcal{G}_t\right], \tag{8.2}$$

where T is maturity. In this case, we set $f(x) = x$ under ZR in Assumption 8.2.

Example 8.4 (European options with default risk) A European call option price $C_t(K,T)$ at time t is given by

$$C_t(S_t;K,T) = \mathbb{E}\left[e^{-r(T-t)} (S_T-K)_+ \mathbf{1}_{\{T<\tau\}} \mid \mathcal{G}_t\right], \tag{8.3}$$

where K is strike price and T is maturity. In this case, we set $f(x) = (x-K)_+$ under ZR in Assumption 8.2. Similarly, a European put option price $P_t(K,T)$ at time t is given by

$$\begin{aligned}
P_t(S_t;K,T) &= \mathbb{E}\left[e^{-r(T-t)} (K-S_T)_+ \mathbf{1}_{\{T<\tau\}} \mid \mathcal{G}_t\right] \\
&\quad + \mathbb{E}\left[e^{-r(T-t)} K \mathbf{1}_{\{T\geq\tau\}} \mid \mathcal{G}_t\right].
\end{aligned} \tag{8.4}$$

In this case, we set $f(x) = (K - x)_+$ and $\delta^T = K$ under RM. Other defaultable European options such as cash digital, asset digital, and power options can be written in a similar manner.

Example 8.5 (Defaultable bonds) A defaultable discount bond price $D_t(T)$ at time t is given by

$$D_t^{\delta^T}(S_t;T) = \mathbb{E}\left[e^{-r(T-t)}\mathbf{1}_{\{T<\tau\}} \mid \mathcal{G}_t\right] + \mathbb{E}\left[e^{-r(T-t)}\delta^T\mathbf{1}_{\{T\geq\tau\}} \mid \mathcal{G}_t\right], \tag{8.5}$$

or

$$D_t^{\delta^\tau}(S_t;T) = \mathbb{E}\left[e^{-r(T-t)}\mathbf{1}_{\{T<\tau\}} \mid \mathcal{G}_t\right] + \mathbb{E}\left[e^{-r(\tau-t)}\delta^\tau\mathbf{1}_{\{T\geq\tau\}} \mid \mathcal{G}_t\right]. \tag{8.6}$$

If $\delta^T = \delta^\tau = 0$, then the bond $D_t^0(S_t;T)$ is expressed by ZR with $f(x) = 1$. Note that since a defaultable fixed-coupon bond is composed of a portfolio of defaultable discount bonds, it is sufficient to consider the case of defaultable discount bonds.

Finally, we remind the reader that since a very general model is formulated for S and τ, the equations in Assumption 8.2 could not be derived, but had to be postulated. However, concrete examples of models satisfying the assumptions are presented in Sections 8.4 and 8.5.

8.2 Static Hedging of ZR and RM Claims

In this section, we firstly study the static replication of the ZR and RM claims in Subsection 8.2.1. In Subsection 8.2.2, we show how smoothing techniques, which have been successfully applied in the context of dynamic hedging in Giles [2009] and Glasserman [2003], can also be used for static replication. Looking at Eq.(8.1), we are concerned with payoffs of the form

$$\mathbf{1}_{\{T<\tau\}}f(S_T) + \mathbf{1}_{\{T\geq\tau\}}\delta^T, \tag{8.7}$$

where $\delta^T \geq 0$ in the cases of ZR and RM. In particular, we allow for $\delta^T = 0$. We remark that for the rest of this section, we assume that the claim which is to be hedged statically is still alive, i.e., that default has not occurred by the time the static hedge is set up.

8.2.1 Static Hedging of the Payoffs

Note that from Assumption 8.1, Eq.(8.7) is equivalent to

$$f(S_T) + \mathbf{1}_{\{T\geq\tau\}}(\delta^T - f(0)).$$

This means that in order to statically hedge the contingent claims having payoffs of the form given in Eq.(8.7), we need to be able to hedge the payoffs $f(S_T)$ and $\mathbf{1}_{T\geq\tau}$. Whereas the static replication of claims of the form $f(S_T)$ was addressed in Carr and Chou [1997], and Carr and Madan [1998], the static replication of the claim $\mathbf{1}_{\{T\geq\tau\}}$ is addressed in the following lemma.

Lemma 8.6 *Under Assumption 8.1, we have*

$$\mathbf{1}_{\{T<\tau\}} = 1 - \lim_{K\downarrow 0}\frac{(K - S_T)_+}{K}. \tag{8.8}$$

Moreover, for all $t \in [0,T]$ we have

$$D_t^0(S_t;T) = B_t(T) - \lim_{K\downarrow 0}\frac{P_t(S_t;K,T)}{K}, \tag{8.9}$$

where $B_t(T) := e^{-r(T-t)}$ is the non-defaultable bond price at time t with maturity T.

Proof of Lemma 8.6 : Let $\tau > T$. Since $S_T > 0$, it holds

$$\lim_{K \downarrow 0} \frac{(K - S_T)_+}{K} = 0.$$

Conversely, let $\tau \leq T$. Since $S_T = 0$, it holds

$$\lim_{K \downarrow 0} \frac{(K - S_T)_+}{K} = 1.$$

From arbitrage-free assumption and dominated convergence theorem, Eq.(8.9) is obtained. □

Note that $1/K$ units of a European put option with strike K can be replaced with one unit of a European cash digital put option with the same strike. Lemma 8.6 shows that a defaultable discount bond can be approximately replicated by one unit of non-defaultable discount bond and $-1/\varepsilon$ units of a put option with extremely small strike ε and the *same* maturity as that of the defaultable bond. However, this replication scheme is not suitable for statically hedging defaultable bonds, because the maturities of corporate bonds are usually much longer than that of equity options in market practice. To overcome this problem, we will develop a new scheme for static hedging of defaultable contingent claims which incorporates a smoothing technique in the next subsection.

The following lemma, which stems from Carr and Chou [1997], and Carr and Madan [1998], is well-known.

Lemma 8.7 *Let $f : [0, \infty) \to \mathbb{R}$ be a twice differentiable function and continuous from the right at 0. Then, for any $\kappa > 0$, it satisfies*

$$\begin{aligned} f(S_T) &= f(\kappa) + f'(\kappa)(S_T - \kappa) \\ &\quad + \int_0^{\kappa} f''(K)(K - S_T)_+ dK + \int_{\kappa}^{\infty} f''(K)(S_T - K)_+ dK. \end{aligned} \tag{8.10}$$

Proof of Lemma 8.7 : See for instance Carr and Chou [1997] or Appendix 1 in Carr and Madan [1998]. □

We can now easily hedge claims of the form given in Eq.(8.7).

Proposition 8.8 *Let $f : [0, \infty) \to \mathbb{R}$ be a twice differentiable function and continuous from the right at 0. Then, for any $\kappa > 0$, it satisfies*

$$\begin{aligned} \mathbf{1}_{\{T<\tau\}} f(S_T) + \mathbf{1}_{\{T \geq \tau\}} \delta^T &= f(\kappa) + f'(\kappa)(S_T - \kappa) \\ &\quad + \int_0^{\kappa} f''(K)(K - S_T)_+ dK \\ &\quad + \int_{\kappa}^{\infty} f''(K)(S_T - K)_+ dK \\ &\quad + \left(\delta^T - f(0)\right) \lim_{K \downarrow 0} \frac{(K - S_T)_+}{K}. \end{aligned} \tag{8.11}$$

Moreover, for all $t \in [0, T]$ the present value $V_t(\tau, S_t; T)$ of the claim satisfies

$$\begin{aligned} V_t(\tau, S_t; T) &= f(\kappa) B_t(T) + f'(\kappa)\{C_t(S_t; \kappa, T) - P_t(S_t; \kappa, T)\} \\ &\quad + \int_0^{\kappa} f''(K) P_t(S_t; K, T) dK \\ &\quad + \int_{\kappa}^{\infty} f''(K) C_t(S_t; K, T) dK \\ &\quad + \left(\delta^T - f(0)\right) \lim_{K \downarrow 0} \frac{P_t(S_t; K, T)}{K}. \end{aligned} \tag{8.12}$$

Proof of Proposition 8.8 : The proof follows immediately from Lemma 8.6 and 8.7. □

8.2.2 Static Hedging of Smoothed Payoffs

It is well-known that payoff functions of European claims are not often twice differentiable, e.g., European calls, and hence Proposition 8.8 is not applicable. Therefore, we employ the following smoothing technique, which has been successfully used in the context of dynamic hedging, see e.g., Glassserman [2003] and Giles [2009]. We fix a date $u < T$ and consider the conditional expectation

$$\begin{aligned}
& \mathbb{E}\left[e^{-r(T-u)}\left(\mathbf{1}_{\{T<\tau\}} f(S_T) + \mathbf{1}_{\{T\geq\tau\}}\delta^T\right) \mid \mathcal{G}_u\right] \\
= & \ \mathbf{1}_{\{\tau>u\}} V_u^0(S_u, T) + \mathbf{1}_{\{\tau>u\}} V_u^1(S_u, T) + \mathbf{1}_{\{\tau\leq u\}} e^{-r(T-u)}\delta^T \\
= & \ \mathbf{1}_{\{\tau>u\}}\left[V_u^0(S_u, T) + V_u^1(S_u, T)\right] + \mathbf{1}_{\{\tau\leq u\}} e^{-r(T-u)}\delta^T,
\end{aligned}$$

which follows from Assumption 8.2. We now set $\tilde{V}_u(S_u; T) := V_u^0(S_u, T) + V_u^1(S_u, T)$ and assume that $\tilde{V}$ is twice differentiable with respect to S_u. Furthermore, we set $\tilde{\delta}^u := e^{-r(T-u)}\delta^T$. Consequently, we deal with the smoothed payoff that is payable at time $u < T$;

$$\tilde{V}_u(S_u; T) + \left(\tilde{\delta}^u - \tilde{V}_u(0; T)\right)\mathbf{1}_{\{u\geq\tau\}}.$$

We have the following theorem from Proposition 8.8, which is the main result of this chapter.

Theorem 8.9 *Let $\tilde{V}_u(S_u; T)$ be twice differentiable in S_u and continuous from the right at $S_u = 0$. Then the following holds, for any $\kappa > 0$,*

$$\begin{aligned}
& \tilde{V}_u(S_u; T) + \left(\tilde{\delta}^u - \tilde{V}_u(0; T)\right)\mathbf{1}_{\{u\geq\tau\}} \\
= & \ \tilde{V}_u(\kappa; T) + \frac{\partial \tilde{V}_u}{\partial S}\Big|_{S=\kappa}(S_u - \kappa) \\
& + \int_0^\kappa \frac{\partial^2 \tilde{V}_u}{\partial S^2}\Big|_{S=K}(K - S_u)_+ dK + \int_\kappa^\infty \frac{\partial^2 \tilde{V}_u}{\partial S^2}\Big|_{S=K}(S_u - K)_+ dK \\
& + \left(\tilde{\delta}^u - \tilde{V}_u(0, T)\right)\lim_{K\downarrow 0}\frac{(K - S_u)_+}{K}.
\end{aligned}$$

Moreover, for all $t \in [0, u]$ the present value $V_t(\tau, S_t; T)$ of the claim satisfies

$$\begin{aligned}
V_t(\tau, S_t; T) = & \ \tilde{V}_u(\kappa; T)B_t(u) + \frac{\partial \tilde{V}_u}{\partial S}\Big|_{S=\kappa}\{C_t(S_t; \kappa, u) - P_t(S_t; \kappa, u)\} \\
& + \int_0^\kappa \frac{\partial^2 \tilde{V}_u}{\partial S^2}\Big|_{S=K} P_t(S_t; K, u) dK \\
& + \int_\kappa^\infty \frac{\partial^2 \tilde{V}_u}{\partial S^2}\Big|_{S=K} C_t(S_t; K, u) dK \\
& + \left(\tilde{\delta}^u - \tilde{V}_u(0, T)\right)\lim_{K\downarrow 0}\frac{P_t(S_t; K, u)}{K}.
\end{aligned} \tag{8.13}$$

We remark that choosing u close to T, we hope that the value function

$$V_u(\tau, S_u; T) = \mathbb{E}\left[\mathbf{1}_{\{T<\tau\}} f(S_T) + \mathbf{1}_{\{T\geq\tau\}}\delta^T \mid \mathcal{G}_u\right],$$

approximates the claim $\mathbf{1}_{\{T<\tau\}}f(S_T)+\mathbf{1}_{\{T\geq\tau\}}\delta^T$ well. Note that from Lemma 8.7 it satisfies

$$\tilde{V}_u(0,T)=\tilde{V}_u(\kappa;T)-\frac{\partial\tilde{V}_u}{\partial S}\Big|_{S=\kappa}\kappa+\int_0^{\kappa}\frac{\partial^2\tilde{V}_u}{\partial S^2}\Big|_{S=K}KdK. \tag{8.14}$$

Although the left-hand side of Eq.(8.14) is used in Theorem 8.9, the right-hand expression may be convenient for computations to avoid a numerical singularity of the value function at $S=0$.

The strategy developed in Theorem 8.9 can be also called *static hedging* in the same sense as Takahashi and Yamazaki [2009a]. That is, whereas the hedging portfolio in Theorem 8.9 needs to be re-balanced at time u, no re-balancing is necessary until time u. According to Theorem 8.9, the static hedging portfolio of a defaultable contingent claim consists of the following securities with maturity u $(<T)$;

- $\tilde{V}_u(\kappa;T)$ units of a non-defaultable discount bond with face value 1
- $\frac{\partial\tilde{V}_u}{\partial S}\mid_{S=\kappa}$ units of a call option with strike κ
- $-\frac{\partial\tilde{V}_u}{\partial S}\mid_{S=\kappa}$ units of a put option with strike κ
- $\frac{\partial^2\tilde{V}_u}{\partial S^2}\mid_{S=K}dK$ units of a call option with strike K for each $K\geq\kappa$
- $\frac{\partial^2\tilde{V}_u}{\partial S^2}\mid_{S=K}dK$ units of a put option with strike K for each $K<\kappa$
- $\frac{\tilde{\delta}^u-\tilde{V}_u(0,T)}{\varepsilon}$ units of a put option with extremely small strike ε

The parameter κ, which can be set arbitrary, is called *put-call separation.* In our numerical examples in Sections 8.4 and 8.5, κ is set to be the forward price of the stock with maturity u to use out of the money puts and calls for constructing a static hedging portfolio. Note that the fifth term on the right-hand side of Eq.(8.13), which is the adjusted term for default risk, can be replaced with $\tilde{\delta}^u-\tilde{V}_u(0,T)$ units of a European cash digital put with extremely small strike ε, or $\tilde{\delta}^u-\tilde{V}_u(0,T)$ units of a non-defaultable discount bond and $-\tilde{\delta}^u+\tilde{V}_u(0,T)$ units of a defaultable discount bond with ZR (see Lemma 8.6).

The practical implication of Theorem 8.9 is that the risk, which is not only credit spread fluctuations but also loss-given-default, embedded in a target defaultable contingent claim can be hedged by a static portfolio of plain vanilla options with a maturity that is shorter than that of the target contingent claim. To check the validity of Theorem 8.9, a simple example is provided as follows:

Example 8.10 (Defaultable bond replication under the jump to default extended Black-Scholes model) Let $(S_t)_{t\geq0}$ be a unique solution of the stochastic differential equation:

$$\frac{dS_t}{S_{t-}}=(r-q+\lambda)dt+\sigma dW_t-dN_t, \tag{8.15}$$

where r and q are the risk-free interest rate and the dividend yield respectively, σ is the constant pre-default volatility, W_t is a standard Brownian motion under $\mathbb{Q}$, and N_t is a standard Poisson process with constant default intensity $\lambda\geq0$, and we assume that W_t and N_t are independent. It is obvious that $(S_t)_{t\geq0}$ satisfies Assumption 8.1.

Suppose that a defaultable discount bond with ZR $D_t^0(S_t;T)=e^{-(r+\lambda)(T-t)}$ is the target defaultable contingent claim. It is well-known that the call price is given by

$$C_t(S_t;K,T)=S_te^{-q(T-t)}N(h_+)-Ke^{-(r+\lambda)(T-t)}N(h_-), \tag{8.16}$$

where $N(\cdot)$ is the standard normal cumulative distribution function, and

$$h_{\pm} = \frac{\ln(\frac{S}{K}) + (r - q + \lambda \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{8.17}$$

First, we confirm the validity of Lemma 8.6, that is,

$$\begin{aligned}\lim_{K\downarrow 0}\frac{P_t(S_t;K,u)}{K} &= \lim_{K\downarrow 0}\frac{C_t(S_t;K,u) - C_t(S_t;0,u)}{K} + e^{-r(u-t)} \\ &= -e^{-(r+\lambda)(u-t)}N(h_+)\Big|_{K=0} + e^{-r(u-t)} \\ &= -e^{-(r+\lambda)(u-t)} + e^{-r(u-t)}.\end{aligned} \tag{8.18}$$

Here, we use the put-call parity. Next, it is clear that

$$D_u^0(\kappa;T) = D_u^0(0;T) = e^{-(r+\lambda)(T-u)} \quad \text{and} \quad \frac{\partial D_u^0}{\partial S} = \frac{\partial^2 D_u^0}{\partial S^2} = 0.$$

Thus the right-hand side of Eq.(8.13) can be written as

$$D_u^0(\kappa;T)B_t(u) - D_u^0(0;T)\lim_{K\downarrow 0}\frac{P_t(S_t;K,u)}{K} = e^{-(r+\lambda)(T-t)}. \tag{8.19}$$

Theorem 8.9 shows that any ZR and RM defaultable contingent claim whose price function is twice differentiable can be replicated by using an infinite number of plain vanilla options. However, since using an infinite number of options is not practical at all, an approximating static portfolio of a finite number of options is necessary. Takahashi and Yamazaki [2009a] proposed to apply the Gauss-Legendre quadrature rule to approximate the integral terms on the right-hand side of Eq.(8.13). The rule is a numerical method for an integral $\int_{-1}^{1} g(x)dx$, where $g(x) \in C^{2n}$ ($n \in \mathbf{N}$) on $[-1,1]$. Here, C^{2n} denotes the set of $2n$-times continuously differentiable functions. For a given target function $g(x)$, the Gauss-Legendre quadrature rule provides the following formula:

$$\int_{-1}^{1} g(x)dx = \sum_{j=1}^{n} w_j g(x_j) + \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} g^{(2n)}(\xi), \tag{8.20}$$

for some $\xi \in [-1,1]$, where $x_j, j = 1,2,\ldots,n$ are roots of the n-th order Legendre polynomial $L_n(x)$, $w_j := 2/(nL_{n-1}(x_j)L_n'(x_j))$, and $g^{(2n)}$ denotes the $2n$-th derivative of g. The second term on the right-hand side of Eq.(8.20) is the approximation error of the n-th order Gauss-Legendre quadrature rule. Note that if $g(x)$ is smooth, the error term converges to zero when $n \to \infty$ (see Sugihara and Murota [1994] for more details). In our numerical examples, we apply the Gauss-Legendre quadrature rule for efficient approximation of static hedging portfolios. See Theorem 1 in Takahashi and Yamazaki [2009a] for the details of the approximation scheme.

8.3 Static Hedging of RD Claims

To develop the static hedging scheme of the RD defaultable contingent claims, it is sufficient to replicate the conditional expectation

$$\mathbb{E}\left[e^{-r(\tau-t)}\mathbf{1}_{\{T\geq\tau\}} \mid \mathcal{G}_t\right],$$

which Carr and Wu [2011] called the *unit recovery claim*. However, the replication of the unit recovery claim by a static portfolio of equity options seems to be difficult under Assumption 8.1. Hence, following Carr and Wu [2011], in this section we introduce a notation named the *default corridor* and set the following assumption instead of Assumption 8.1.

Assumption 8.11 *The pre-default stock price S_t is bounded below by a strictly positive barrier $B > 0$, and the stock price is zero after default; i.e.,*

$$\begin{cases} S_t > B & \text{if} \quad t < \tau, \\ S_t = 0 & \text{if} \quad t \geq \tau. \end{cases}$$

Carr and Wu [2011] called the interval $(0, B]$ the default corridor, where the stock price can never enter. Note that Assumption 8.11 is stricter than Assumption 8.1.

Carr and Wu [2011] demonstrated a simple replication scheme of the unit recovery claim by using an American option having the strike within the default corridor. That is, for any $K \in (0, B]$, the time-t value of the unit recovery claim with maturity T is given by

$$\mathbb{E}\left[e^{-r(\tau-t)}\mathbf{1}_{\{T\geq\tau\}} \mid \mathcal{G}_t\right] = \frac{P_t^A(S_t; K, T)}{K},$$

where $P_t^A(S_t; K, T)$ denotes the American put price at time t with strike K and maturity T. Therefore, we can statically replicate arbitrary RD claims by using the American put option under Assumption 8.11, because any RD defaultable contingent claim can be decomposed into a ZR claim and the unit recovery claim.

The models satisfying Assumption 8.11 might not be so many, but Carr and Wu [2011] proposed such a model called the *defaultable displaced diffusion* (DDD) model. In the DDD model, the pre-default stock price follows

$$S_{t-} = e^{rt}\left\{J_{t-}\left[B + (S_0 - B)G_t\right]\right\},$$

equivalently,

$$S_{t-} = Be^{rt}J_{t-} + (S_0 - B)e^{rt}J_{t-}G_t,$$

where $S_0 > B > 0$, $G_t = \exp(\sigma W_t - \sigma^2 t/2)$, $J_t = \mathbf{1}_{\{N_t=0\}}e^{\lambda t}$, and σ and B are strictly positive constants. Here, W_t and N_t denote a standard Brownian motion and a standard Poisson process with constant default intensity λ under $\mathbb{Q}$ respectively, and we assume that W_t and N_t are independent.

Note that the DDD model unifies the displaced diffusion model (Rubinstein [1983]) and the Black-Scholes jump to default model. Before a jump to default occurs, S_{t-} is above the level Be^{rt}, but S can jump to zero if default occurs. In the DDD model, the default corridor is the interval $(0, B]$, and the time-t price of the unit recovery claim is given by

$$\mathbb{E}\left[e^{-r(\tau-t)}\mathbf{1}_{\{T\geq\tau\}} \mid \mathcal{G}_t\right] = \lambda\frac{1 - e^{-(r+\lambda)(T-t)}}{r + \lambda},$$

which is equivalent to the price of $1/K$ units of an American put option with strike $K \in (0, B]$ and maturity T.

8.4 Static Hedging under Structural Model

In this section, we provide numerical examples of static hedging under a structural model. Under the standard structural approach, the default time of a reference firm is defined as

$$\tau = \inf\{t \geq 0 : A_t \leq L\}, \tag{8.21}$$

where $(A_t)_{t\geq 0}$ denotes the firm's asset value process that is a martingale under $\mathbb{Q}$, and L is the default barrier that is modeled as a constant, a deterministic function, or an independent random variable.

One of the preferred structural models is the *CreditGrades model* introduced by Finger et. al. [2002], and Stamicar and Finger [2006]. By virtue of the following simple assumption on the stock price process S_t, it is possible to evaluate both credit and equity derivatives simultaneously:

$$S_t := \begin{cases} e^{(r-q)t}(A_t - \bar{L}) & \text{if} \quad t < \tau, \\ 0 & \text{if} \quad t \geq \tau, \end{cases} \tag{8.22}$$

where $\bar{L}$ is the mean of L. Because of its tractability the CreditGrades model has been applied to various empirical investigations of linkage between credit and equity markets; for example, see Veraart [2004], Bystrom [2006], Yu [2006], Bedendo et al. [2007], and Bajlum and Larsen [2007]. Furthermore, several extensions of the model have been proposed; see Sepp [2006] and Ozeki et al. [2011].

8.4.1 VG CreditGrades Model

We specify the firm's asset value process $(A_t)_{t\geq 0}$ as follows:

$$A_t = A_0 e^{\omega t + X_t}, \tag{8.23}$$

where $A_0 := S_0 + L$ is constant; i.e.,the default barrier L is constant, $(X_t)_{t\geq 0}$ is the Variance Gamma process (VG process), and ω called *martingale correction* is constant. The VG process is well-known as a Lévy process with infinite activity jumps and without diffusion component. The Lévy measure Π of the VG process is defined as

$$\Pi(dy) = \left(\frac{e^{-\lambda_+ y}}{\nu y} \mathbf{1}_{\{y>0\}} + \frac{e^{-\lambda_- |y|}}{\nu |y|} \mathbf{1}_{\{y<0\}} \right) dy, \tag{8.24}$$

with

$$\lambda_\pm := \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu}} \mp \frac{\theta}{\sigma^2}, \tag{8.25}$$

where σ, ν, θ are parameters. Hereafter, we will abbreviate the CreditGrades model with the VG process as the VGCG.

By the Lévy-Khinchin formula, the characteristic function of X_t is given by

$$\phi_{X_t}(u; \sigma, \nu, \theta) := \mathbb{E}\left[e^{iuX_t}\right] = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2 \nu u^2\right)^{-t/\nu}, \tag{8.26}$$

where $i := \sqrt{-1}$. Therefore, the martingale correction is determined as

$$\omega = -\frac{1}{t} \ln \phi_{X_t}(-i; \sigma, \nu, \theta) = \frac{1}{\nu} \ln \left(1 - \theta\nu - \frac{1}{2}\sigma^2 \nu\right). \tag{8.27}$$

Note that the default time of the VGCG is unpredictable because of the discontinuity property of the Lévy process, that is, τ is a *totally inaccessible stopping time*. Needless to say, perfect hedging is impossible by dynamic stock trading even if the firm is non-defaultable.

In this setting, the call price can be written as

$$\begin{aligned} C_t(S_t; K, T) &= \mathbb{E}\left[e^{-r(T-t)}(S_T - K)_+ \mathbf{1}_{\{\min_{0\leq t\leq T} A_t > L\}} \mid \mathcal{G}_t\right] \\ &= e^{-q(T-t)} c_t(A_t; k, T), \end{aligned} \tag{8.28}$$

where

$$c_t(A_t;k,T) := \mathbb{E}\left[(A_T-k)_+\mathbf{1}_{\{\min_{0\le t\le T} A_t>L\}} \mid A_t\right], \tag{8.29}$$

and $k := L+Ke^{-(r-q)(T-t)}$. Note that Eq.(8.29) can be regarded as the value of a barrier option called *down-and-out call* on the firm's asset value $(A_t)_{t\geq 0}$ with strike k, barrier L, and maturity T. To obtain the call price in Eq.(8.28), we solve the following *partial integro-differential equation* (PIDE) numerically for $v(A,t) := c_t(A;k,T)$;

$$\frac{\partial v}{\partial t}+\omega A\frac{\partial v}{\partial A}+\int_{-\infty}^{\infty}[v(A_{t-}e^y,t)-v(A_t,t)]\Pi(dy)=0, \tag{8.30}$$

with the terminal condition $v(A,T)=(A-k)_+$ and the boundary conditions

$$\begin{cases} v(A,t)=0 & \text{as} \quad A\le L, \\ v(A,t)=A-k & \text{as} \quad A\to\infty. \end{cases}$$

To derive the values $v(A,t)$ from PIDE (8.30), we adopt the finite difference method developed by Hirsa and Madan [2004], and Cariboni and Schoutens [2007].

We choose defaultable discount bonds with ZR as the target defaultable contingent claims for numerical examples later. The defaultable discount bond can be written as

$$\begin{aligned} D_t^0(S_t;T) &= \mathbb{E}\left[e^{-r(T-t)}\mathbf{1}_{\{\min_{0\le t\le T} A_t>L\}} \mid \mathcal{G}_t\right] \\ &= e^{-r(T-t)}d_t(A_t;T), \end{aligned} \tag{8.31}$$

where

$$d_t(A_t;T) := \mathbb{E}\left[\mathbf{1}_{\{\min_{0\le t\le T} A_t>L\}} \mid A_t\right]. \tag{8.32}$$

To obtain the defaultable bond price, we again solve PIDE (8.30) for $v(A,t) := d_t(A;T)$ with the terminal condition $v(A,T)=\mathbf{1}_{\{A>L\}}$ and the boundary conditions

$$\begin{cases} v(A,t)=0 & \text{as} \quad A\le L, \\ v(A,t)=1 & \text{as} \quad A\to\infty. \end{cases}$$

In order to apply the static hedging formula Eq.(8.13), it is necessary to calculate the first and second derivatives of Eq.(8.31) with respect to S_t. It is numerically straightforward due to the finite differences with respect to the defaultable bond prices.

8.4.2 Numerical Examples

The input parameters of the VGCG in the numerical examples are listed in Table 8.1. We compute

Table 8.1: The input parameters of the VGCG

S_0	r	q	σ	ν	θ	L
50	0.05	0.00	0.20	0.20	-0.20	40

the replicating portfolios of defaultable discount bonds with ZR and maturity $T=1$ and 5. Using 6, 10, 20 plain vanilla options with maturity $u=0.5$, we replicate each value of the target bonds for all

$S_t \in [0, 100]$ and $0 \le t \le u$. Tables 8.3 and 8.4 show the strikes and the number of units of the plain vanilla options composing the static replicating portfolios.

Table 8.7 shows the initial prices of the target bonds and the replicating portfolios, as well as the errors and error ratios against the corresponding bond prices. The result demonstrates that considerable accuracy in the prices can be obtained by using only 6 options. The first plots in Figures 8.1 and 8.2 display the time-t values of the target bonds maturing at $T = 1$ and $T = 5$, respectively, with different stock price and time in $[0, u]$. The remaining three plots in Figures 8.1 and 8.2 show the replication errors corresponding to the first plots. The errors are measured by the deviation of the portfolio value from the target bond value. According to these figures, larger replication errors are generated when the stock price S_t approaches zero. However, the replication errors are caused not by our static hedging scheme, but by inevitable numerical errors of the finite difference scheme used to solve PIDE (8.30). The figures show that the more options we use, the smaller the errors are, which implies the validity of our scheme.

8.5 Static Hedging under Intensity-Based Model

In this section, we show numerical examples of static hedging under an intensity-based model. In recent literature, many unified credit-equity models under the intensity-based approach, which are called *jump to default models*, have been developed. Based on our assumption, the pre-default stock price process $(S_t)_{t \ge 0}$ is modeled as follows:

$$\frac{dS_t}{S_t} = [r - q + \lambda(S_t, t)]\, dt + \sigma(S_t, t) dW_t, \tag{8.33}$$

where $\sigma(S, t)$ and $\lambda(S, t)$ are the time- and state-dependent stock volatility and the time- and state-dependent default intensity, respectively. The random time of default τ is modeled as the first time when the process $\int_0^t \lambda(S_s, s) ds$ is greater or equal to an exponential random variable $e \sim \text{Exp}(1)$ (equivalently, as the first jump time of a doubly stochastic Poisson process with intensity $\lambda(S_t, t)$):

$$\tau = \inf \left\{ t \ge 0 : \int_0^t \lambda(S_s, s) ds \ge e \right\}. \tag{8.34}$$

At the default time τ, the stock price jumps to zero, after that the stock price remains zero permanently.

The model shown in Eq.(8.33) was employed in Merton [1976], Takahashi et al. [2001], Andersen and Buffum [2003], Ayache et al. [2003], Duffie and Singleton [2003], Carr and Linetsky [2006], Linetsky [2006], etc. In particular, Eq.(8.33) seems to be the preferred model in the field of convertible bond pricing.

8.5.1 Jump to Default Extended CEV Model

For the numerical examples, we employ the *jump to default extended CEV model* (JDCEV for short) proposed in Carr and Linetsky [2006]. JDCEV specifies the instantaneous volatility as that of a constant elasticity of variance (CEV) process:

$$\sigma(S, t) = a(t) S^{\beta}, \tag{8.35}$$

where $\beta < 0$ is the volatility elasticity parameter and $a(t) > 0$ is the time-dependent volatility scale parameter. This specification is consistent with the leverage effect and the implied volatility skew. Moreover, JDCEV specifies the default intensity as an affine function of the instantaneous variance of the stock price:

$$\lambda(S, t) = b(t) + c\sigma^2(S, t) = b(t) + ca^2(t) S^{2\beta}, \tag{8.36}$$

where $b(t) \geq 0$ is a time-dependent deterministic function and $c \geq 0$ is a constant parameter. This specification is consistent with the empirical evidence linking credit spreads to volatility. The functions $a(t)$ and $b(t)$ can be determined by at-the-money implied volatilities and a term structure of credit spreads.

Carr and Linetsky [2006] provided the closed-form formula of the plain vanilla call price with strike K and maturity T as follows:

$$\begin{aligned} C_t(S_t; K, T) &= e^{-q(T-t)} S_t \Phi^+ \left(0, \frac{k^2}{t^*}; \delta_+, \frac{x^2}{t^*}\right) - e^{-r(T-t) - \int_t^T b(u)du} \\ &\quad \times K \left(\frac{x^2}{t^*}\right)^{1/(2|\beta|)} \Phi^+ \left(-\frac{1}{2|\beta|}, \frac{k^2}{t^*}; \delta_+, \frac{x^2}{t^*}\right), \end{aligned} \tag{8.37}$$

where

$$\begin{aligned} k &= \frac{1}{|\beta|} K^{|\beta|} e^{-|\beta|[(r-q)(T-t) + \int_t^T b(u)du]}, \\ t^* &= \int_t^T a^2(u) e^{-2|\beta|[(r-q)(u-t) + \int_t^u b(s)ds]} du, \\ x &= \frac{1}{|\beta|} S_t^{|\beta|}, \end{aligned} \tag{8.38}$$

and the function $\Phi^+(p, k; \delta, \alpha)$ is defined as

$$\Phi^+(p, k; \delta, \alpha) = 2^p \sum_{n=0}^{\infty} e^{-\alpha/2} \left(\frac{\alpha}{2}\right)^n \frac{\Gamma(\delta/2 + p + n, k/2)}{n! \Gamma(\delta/2 + n)}. \tag{8.39}$$

Here, $\Gamma(\alpha)$ is the standard Gamma function and $\Gamma(\alpha, x)$ is the complementary incomplete Gamma function. The plain vanilla put price is derived from the put-call parity.

We choose again defaultable discount bonds with ZR as the target defaultable contingent claims for numerical examples. Carr and Linetsky [2006] also provided the closed-form formula of the defaultable discount bond price as follows:

$$D_t^0(S_t; T) = e^{-r(T-t) - \int_t^T b(u)du} \left(\frac{x^2}{t^*}\right)^{1/(2|\beta|)} \mathcal{M}\left(-\frac{1}{2|\beta|}; \delta_+, \frac{x^2}{t^*}\right), \tag{8.40}$$

where the function $\mathcal{M}(p; \delta, \alpha)$ is defined as

$$\mathcal{M}(p; \delta, \alpha) = 2^p e^{-\alpha/2} \frac{\Gamma(\delta/2 + p)}{\Gamma(\delta/2)} {}_1F_1(\delta/2 + p, \delta/2, \alpha/2). \tag{8.41}$$

Here, the function ${}_1F_1(a, b, x)$ is the Kummer confluent hypergeometric function:

$${}_1F_1(a, b, x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, \tag{8.42}$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for any $n \geq 1$. In order to apply the static hedging formula Eq.(8.13), it is necessary to obtain the first and second derivatives of Eq.(8.40) with respect to S_t. By direct calculations, their closed-form solution can be derived easily (see at the end of this chapter for details).

Table 8.2: The input parameters of the JDCEV

S_0	r	q	$a(t)$	$b(t)$	c	β
50	0.05	0.00	20	0.02	1.00	-0.80

8.5.2 Numerical Examples

The input parameters of the JDCEV for the numerical examples are listed in Table 8.2. Here, $a(t)$ and $b(t)$ are set as time-independent parameters for simplicity. Similarly to the numerical examples in Section 8.4, we compute the replicating portfolios of the defaultable discount bonds with ZR and maturity $T = 1$ and 5 by using 6, 10, 20 plain vanilla options with maturity $u = 0.5$. Tables 8.5 and 8.6 show the strikes and the number of units of the plain vanilla options composing the static replicating portfolios.

Table 8.8 shows the initial prices of the target bonds and the replicating portfolios, as well as the errors and error ratios against the corresponding bond prices. Note that considerable accuracy in the prices can be obtained by using only 6 options. Figures 8.3 and 8.4 plot the time-t values of the target bonds with different stock price and time in $[0, u]$, and their replication errors. As a result of the numerical examples, we conclude that our static hedging scheme provides effective and feasible static replication portfolios of defaultable contingent claims.

Note that the hedging errors near the hedging expiry in Figure 8.3 are larger than those in Figure 8.4. The reason is as follows: According to Theorem 8.9, it is necessary to approximate the time-u value curve of the target defaultable bond by using a finite number of put and call payoffs which cannot describe the curve, but a line graph. Therefore, the sharper the curvature of the curve becomes, the more options in the hedging portfolio are needed to accurately trace the curve. That is, larger hedging errors are caused by sharper curvature of the value curve at hedging maturity.

8.6 Concluding Remarks

This chapter proposes a new scheme for static hedging of path-independent defaultable contingent claims. Moreover, our scheme makes it possible that long maturity illiquid defaultable contingent claims are hedged by shorter maturity plain vanilla options, which are highly liquid financial products in derivative markets. Our hedging scheme is applicable to many unified credit-equity models. Actually, Sections 8.4 and 8.5 present numerical examples of the static portfolios for hedging defaultable discount bonds under VGCG and JDCEV, which are a structural model and an intensity-based model, respectively. Moreover, as a result of the numerical examples, it is ascertained that the scheme is more pragmatical; i.e., the replicating portfolios can be composed of a feasible number of plain vanilla options. The proposed hedging scheme seems to be widely applicable, not only to hedging defaultable contingent claims, but also trading strategy to *capital structure arbitrage*, that is popular with hedge funds (see Bajlum and Larsen [2007], Bedendo et al. [2007], and Yu [2006] for instance).

Finally, our next research topic will be to extend the hedging scheme to multi-factor credit-equity hybrid models, which include stochastic volatility, stochastic interest rate, and other credit risk factors. Gyöngy's theorem (Gyöngy [1986]), which is well-known as *Markovian projection* in the context of mathematical finance (For example, see Antonov and Misirpashaev [2009], and Piterbarg [2006]), might be a key to extend our scheme to the multi-factor settings (see Takahashi and Yamazaki [2009b] for an application of Gyöngy's theorem to static hedging under stochastic volatility models in a non-defaultable economy).

Delta and Gamma of Defaultable Discount Bond under JDCEV The delta of the defaultable discount bond with ZR under JDCEV is as follows:

$$\begin{aligned}\frac{\partial D_u^0}{\partial S} &= e^{-r(T-u)-\int_u^T b(s)ds} S^{|\beta|-1} \left(\frac{x^2}{t^*}\right)^{1/(2|\beta|)} \\ &\times \left[\left(\frac{1}{|\beta|x} - \frac{x}{t^*}\right)\mathcal{M}\left(-\frac{1}{2|\beta|};\delta_+,\frac{x^2}{t^*}\right) + \frac{x}{t^*}\mathcal{M}^{(1)}\left(-\frac{1}{2|\beta|};\delta_+,\frac{x^2}{t^*}\right)\right].\end{aligned}$$

The gamma of the defaultable discount bond with ZR under JDCEV is as follows:

$$\begin{aligned}\frac{\partial^2 D_u^0}{\partial S^2} &= e^{-r(T-u)-\int_u^T b(s)ds} S^{|\beta|-2} \left(\frac{x^2}{t^*}\right)^{1/(2|\beta|)+1} \\ &\times \left[\left(\frac{|\beta|x}{t^*} - \frac{2|\beta|+1}{x}\right)\mathcal{M}\left(-\frac{1}{2|\beta|};\delta_+,\frac{x^2}{t^*}\right)\right. \\ &+ \left(\frac{|\beta|+1}{x} - \frac{2|\beta|x}{t^*}\right)\mathcal{M}^{(1)}\left(-\frac{1}{2|\beta|};\delta_+,\frac{x^2}{t^*}\right) \\ &\left.+\frac{|\beta|x}{t^*}\mathcal{M}^{(2)}\left(-\frac{1}{2|\beta|};\delta_+,\frac{x^2}{t^*}\right)\right].\end{aligned}$$

Here the function $\mathcal{M}^{(m)}(p;\delta,\alpha)$ $(m=1,2)$ is defined as

$$\mathcal{M}^{(m)}(p;\delta,\alpha) = 2^p e^{-\alpha/2}\frac{\Gamma(\delta/2+p)}{\Gamma(\delta/2)}\,{}_1F_1^{(m)}(\delta/2+p,\delta/2,\alpha/2),$$

where the function ${}_1F_1^{(m)}(a,b,x)$ is given by

$${}_1F_1^{(m)}(a,b,x) = \sum_{n=0}^{\infty}\frac{(a)_{n+m}}{(b)_{n+m}}\frac{x^n}{n!}.$$

Table 8.3: The static option portfolio for the defaultable bond replication with $T = 1$ under VGCG

options	strike/amount	No.1	No.2	No.3	No.4	No.5	No.6	No.7	No.8	No.9	No.10
	call strike	52.16	55.90	62.28	70.74	80.52	90.75	100.53	108.98	115.36	119.10
call:10	call amount	-0.10	-0.14	-0.09	-0.04	-0.02	-0.01	0.00	0.00	0.00	0.00
put:10	put strike	5.60	8.12	12.42	18.11	24.69	31.58	38.16	43.85	48.14	50.66
	put amount	-141.76	-202.59	-133.87	-57.05	-19.31	-6.07	-1.97	-0.69	-0.26	-0.08
	call strike	54.49	67.13	85.63	104.14	116.78					
call:5	call amount	-0.26	-0.11	-0.02	0.00	0.00					
put:5	put strike	7.17	15.68	28.13	40.59	49.10					
	put amount	-381.62	-158.96	-20.64	-2.44	-0.36					
	call strike	59.01	85.63	112.25							
call:3	call amount	-0.35	-0.03	0.00							
put:3	put strike	10.21	28.13	46.05							
	put amount	-512.22	-32.25	-1.29							

Note: All of option amounts in the table are 10,000 times of actual calculated amounts.

Table 8.4: The static option portfolio for the defaultable bond replication with $T = 5$ under VGCG

options	strike/amount	No.1	No.2	No.3	No.4	No.5	No.6	No.7	No.8	No.9	No.10
	call strike	52.16	55.90	62.28	70.74	80.52	90.75	100.53	108.98	115.36	119.10
call:10	call amount	-3.41	-6.42	-6.98	-5.81	-4.11	-2.63	-1.59	-0.92	-0.49	-0.19
put:10	put strike	5.60	8.12	12.42	18.11	24.69	31.58	38.16	43.85	48.14	50.66
	put amount	-12.03	-25.96	-36.83	-39.98	-35.06	-26.38	-17.89	-11.17	-6.23	-2.47
	call strike	54.49	67.13	85.63	104.14	116.78					
call:5	call amount	-10.87	-12.18	-6.32	-2.44	-0.73					
put:5	put strike	7.17	15.68	28.13	40.59	49.10					
	put amount	-41.40	-75.71	-58.80	-28.42	-9.44					
	call strike	59.01	85.63	112.25							
call:3	call amount	-20.61	-9.87	-2.04							
put:3	put strike	10.21	28.13	46.05							
	put amount	-95.54	-91.87	-25.54							

Note: All of option amounts in the table are 10,000 times of actual calculated amounts.

Table 8.5: The static option portfolio for the defaultable bond replication with $T = 1$ under JDCEV

options	strike/amount	No.1	No.2	No.3	No.4	No.5	No.6	No.7	No.8	No.9	No.10
	call strike	52.55	57.93	67.09	79.24	93.28	107.98	122.03	134.17	143.34	148.71
call:10	call amount	-60.25	-115.79	-128.09	-106.37	-74.11	-47.05	-28.65	-16.90	-9.18	-3.61
put:10	put strike	0.67	3.46	8.22	14.52	21.82	29.45	36.74	43.05	47.81	50.60
	put amount	-9.38	-55.34	-128.54	-197.97	-232.73	-223.43	-183.02	-130.85	-79.57	-33.00
	call strike	55.90	74.05	100.63	127.22	145.37					
call:5	call amount	-194.79	-223.84	-113.26	-44.26	-13.87					
put:5	put strike	2.40	11.83	25.63	39.44	48.86					
	put amount	-71.16	-328.45	-443.80	-308.95	-122.77					
	call strike	62.39	100.63	138.87							
call:3	call amount	-376.15	-176.97	-38.07							
put:3	put strike	5.78	25.63	45.49							
	put amount	-272.94	-693.44	-313.32							

Note: All of option amounts in the table are 100,000 times of actual calculated amounts.

Table 8.6: The static option portfolio for the defaultable bond replication with $T = 5$ under JDCEV

options	strike/amount	No.1	No.2	No.3	No.4	No.5	No.6	No.7	No.8	No.9	No.10
	call strike	52.55	57.93	67.09	79.24	93.28	107.98	122.03	134.17	143.34	148.71
call:10	call amount	-6.06	-13.90	-20.87	-25.85	-27.86	-26.67	-22.88	-17.45	-11.27	-4.86
put:10	put strike	0.67	3.46	8.22	14.52	21.82	29.45	36.74	43.05	47.81	50.60
	put amount	-0.28	-1.69	-4.14	-7.05	-9.63	-11.17	-11.22	-9.70	-6.86	-3.11
	call strike	55.90	74.05	100.63	127.22	145.37					
call:5	call amount	-21.86	-45.94	-52.63	-39.62	-17.64					
put:5	put strike	2.40	11.83	25.63	39.44	48.86					
	put amount	-2.16	-11.16	-20.11	-20.51	-10.94					
	call strike	62.39	100.63	138.87							
call:3	call amount	-52.39	-82.23	-43.05							
put:3	put strike	5.78	25.63	45.49							
	put amount	-8.54	-31.43	-25.08							

Note: All of option amounts in the table are 100,000 times of actual calculated amounts.

Table 8.7: Initial prices of the defaultable discount bond under VGCG

	defaultable discount bond	static replication call:10, put:10	call:5, put:5	call:3, put:3
price with $T = 1$	0.948317	0.948324	0.948324	0.948360
error	-	-0.000006	-0.000007	-0.000042
error ratio (%)	-	-0.00066	-0.00074	-0.00447
price with $T = 5$	0.674995	0.679367	0.679357	0.679769
error	-	-0.004372	-0.004362	-0.004774
error ratio (%)	-	-0.64772	-0.64621	-0.70726

Table 8.8: Initial prices of the defaultable discount bond under JDCEV

	defaultable discount bond	static replication call:10, put:10	call:5, put:5	call:3, put:3
price with $T = 1$	0.501847	0.502405	0.502412	0.502391
error	-	-0.000558	-0.000564	-0.000544
error ratio (%)	-	-0.11115	-0.11246	-0.10834
price with $T = 5$	0.184798	0.184900	0.184900	0.184884
error	-	-0.000102	-0.000102	-0.000086
error ratio (%)	-	-0.05502	-0.05511	-0.04636

Figure 8.1: The static replication of the defaultable discount bond with ZR and maturity $T = 1$ under VGCG

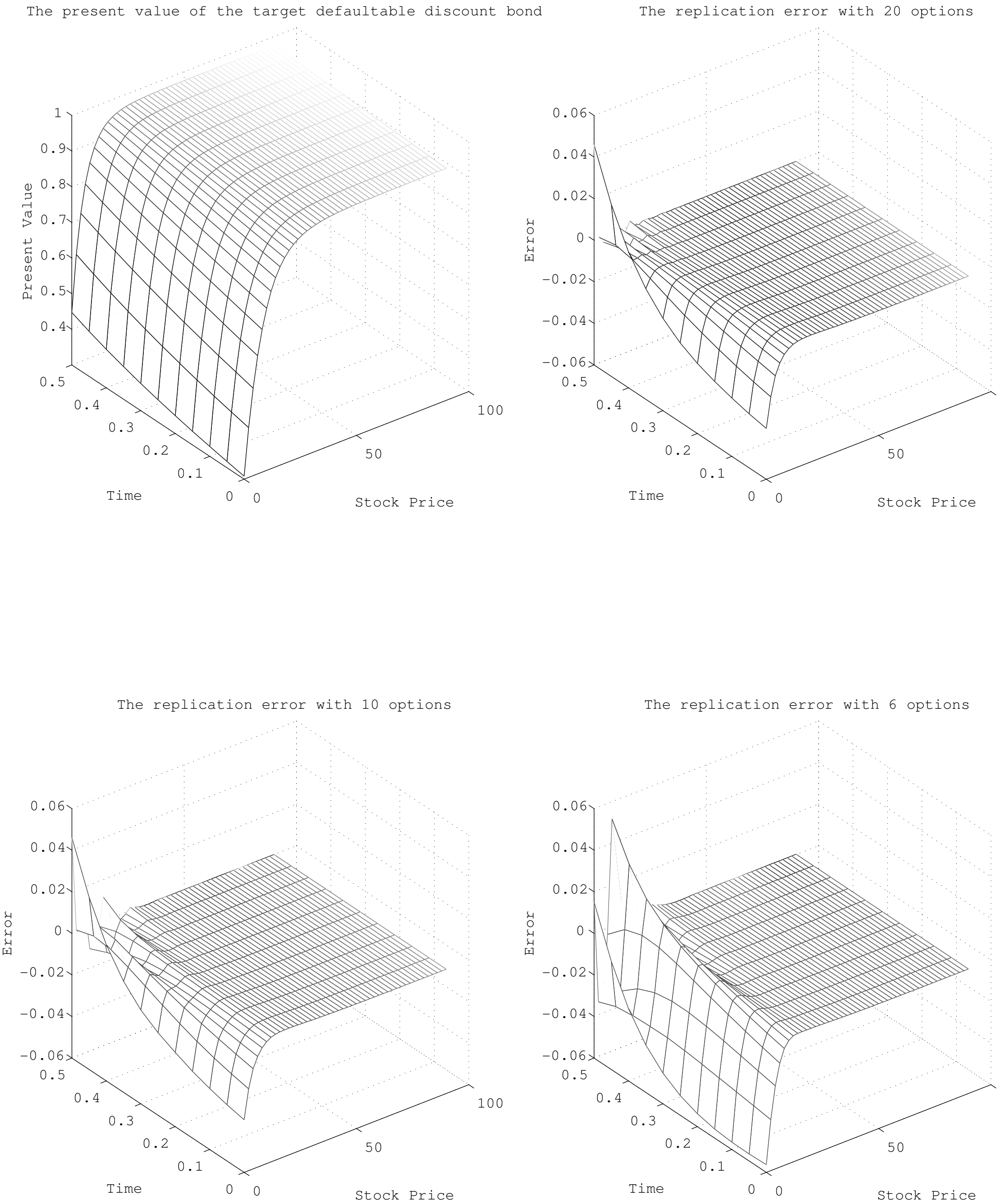


Figure 8.2: The static replication of the defaultable discount bond with ZR and maturity $T = 5$ under VGCG

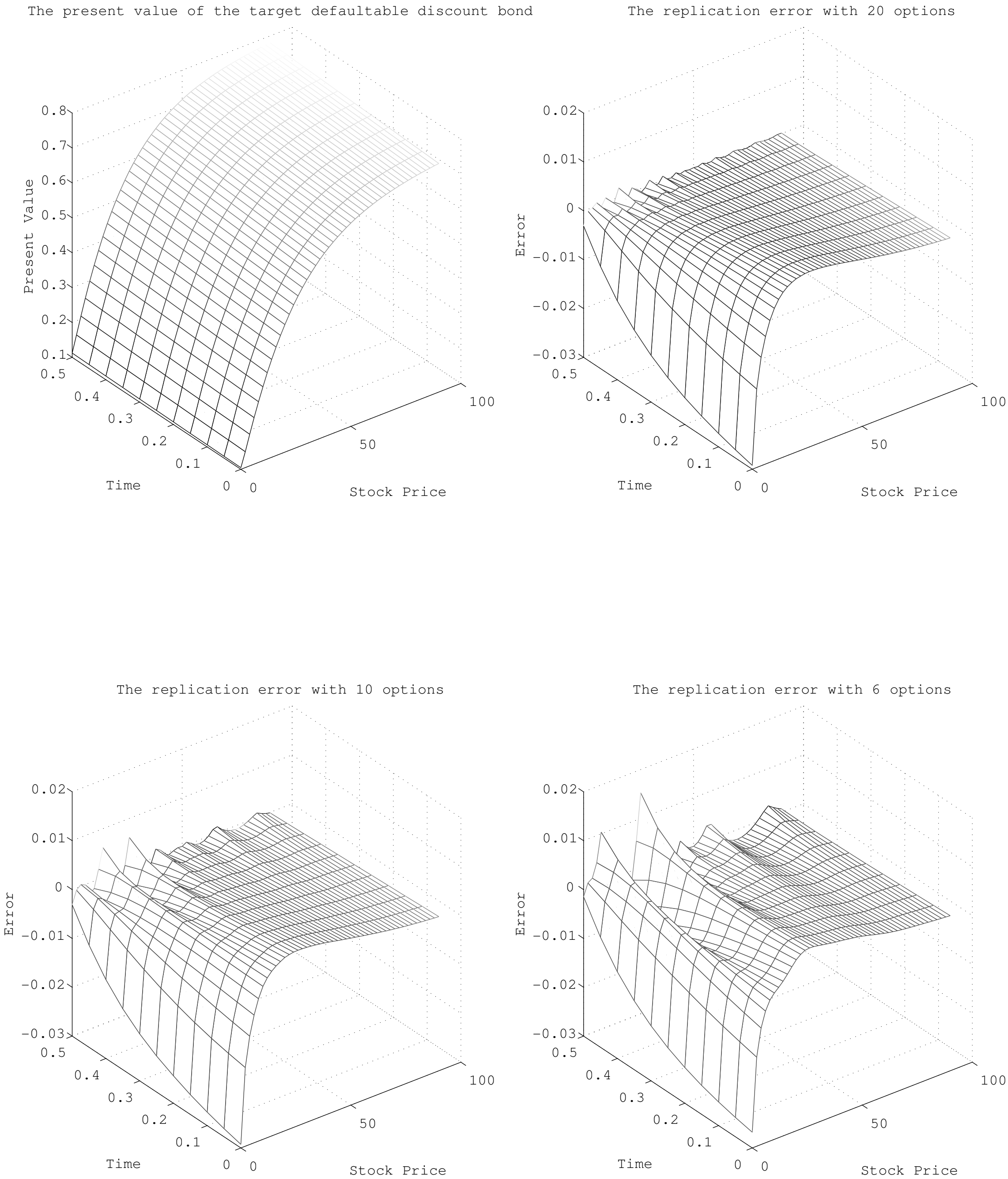


Figure 8.3: The static replication of the defaultable discount bond with ZR and maturity $T = 1$ under JDCEV

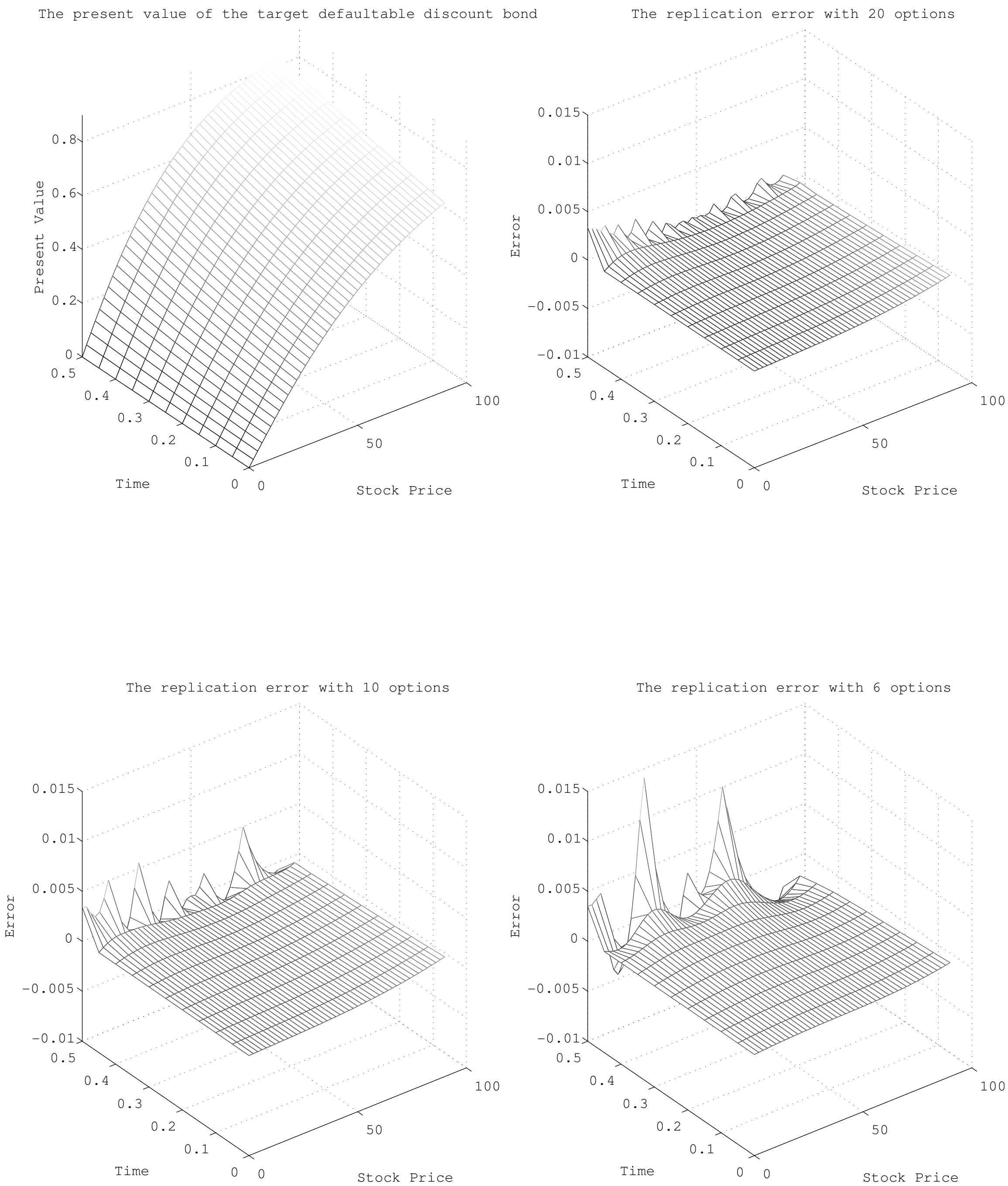
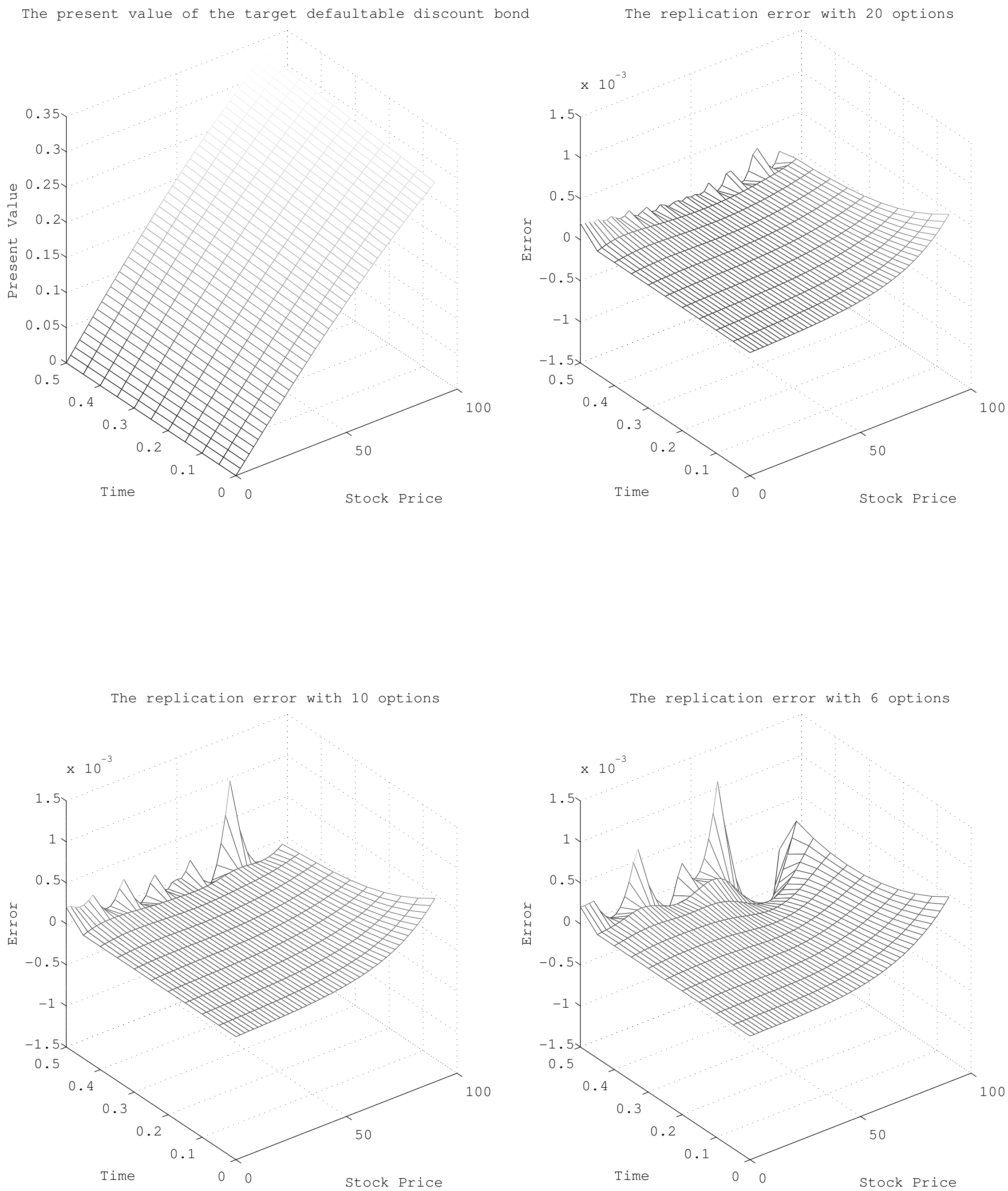


Figure 8.4: The static replication of the defaultable discount bond with ZR and maturity $T = 5$ under JDCEV



Chapter 9

Hedging European Derivatives with the Polynomial Variance Swap under Uncertain Volatility Environments

This chapter proposes a new hedging scheme of European derivatives under uncertain volatility environments, in which an exotic variance swap called the polynomial variance swap is added to the Black-Scholes delta hedging in order to hedge volatility risk. To examine robustness of our hedging scheme, we implement Monte Carlo simulation tests with three different settings of underlying processes and compare the hedging performance of our scheme with that of other standard hedging schemes.

Variance swaps, which pay the realized variance of the returns on an underlying price process and receive the fixed positive amount, have become the most approved tools for trading volatility. Moreover, with a remarkable development of volatility derivatives both in practice and academically, various types of derivatives on realized variance/volatility are proposed; for examples, corridor variance swaps (Carr and Lewis [2004]), gamma swaps (Mougeot [2005]), conditional variance swaps (Mougeot [2005]), moment swaps (Schoutens [2005]), volatility swaps (Carr and Lee [2008], and Friz and Gatheral [2005]), multi-asset stochastic local variance contracts (Carr and Laurence [2011]), and options on realized variance/volatility (Carr et al. [2005], and Carr and Lee [2007, 2010]). By virtue of much energetic research, institutions can deal with a large number of volatility-based trading strategies, and have invested in such realized variance/volatility contracts as new asset classes.

On the other hand, our interest is how robust the Black-Scholes delta hedging is under uncertain volatility environments when adding a certain variance swap. It is widely recognized that the variance swap is a useful tool for managing exposure to volatility risk. This recognition, however, is questionable. For example, volatility exposure of a call option is higher at-the-money than out/in-the-money, while the standard variance swap has invariant volatility exposure with respect to the underlying asset price. The similar problem exists with respect to time-to-maturity. In general, it can be said that there is a discrepancy between volatility risks of the hedging target derivative and the variance swap, and volatility exposure of the variance swap cannot be completely balanced with that of the derivative security. Therefore, the standard variance swap might not work effectively to hedge volatility risk. This detailed discussion is stated in the next section.

To overcome this problem, a special realized variance/volatility contract for hedging volatility exposure on European derivatives is needed, and it is necessary to consider a new hedging scheme using this contract. However, to the best of our knowledge, there is no research designing such a variance/volatility contract and proposing such a hedging strategy. Besides, in the past literature numerical experiment or empirical analysis, which examines the hedging performance in the case that the standard variance swap is used as a hedging tool of volatility exposure, is very limited.

The purpose of this chapter is as follows: First, we design a weighted variance swap called the polynomial variance swap for hedging volatility exposure on path-independent European derivatives. Second, we propose a new robust hedging scheme against exposure to volatility risk, in which the polynomial variance swap is added to the Black-Scholes delta hedging. Third, through Monte Carlo simulation tests, we confirm effectiveness of our hedging scheme. In contrast to the hedging schemes with uncertain/stochastic volatilities proposed in the past literature (e.g., Avellaneda et al. [1995], Bakshi et al. [1997], Heath et al. [2001a, 2001b], Fink [2003], and Takahashi and Yamazaki [2009b]), our scheme has a preferable property that any information about volatility process of the underlying asset is unnecessary. Therefore, it can be expected that our hedging scheme minimizes model risk and continues to be robust to the hedging performance under any uncertain volatility environments.

9.1 Hedging Uncertain Volatility Risk

In this section, we briefly review the hedging error caused by uncertain volatility risk when using the Black-Scholes delta hedging. Then, we consider applying the standard variance swap to eliminate this error. However, it will be demonstrated that the standard variance swap is not quite a suitable tool for hedging uncertain volatility risk.

9.1.1 Uncertain Volatility Risk on European Derivatives

We assume a frictionless and no-arbitrage market. Let S_t be the spot price of a certain stock, an underlying asset price at time $t \in [0, T^*]$ where T^* is some arbitrarily determined time horizon. For simplicity, both the risk-free interest rate and the dividend yield of the stock are assumed to be zero. The no-arbitrage assumption ensures the existence of a risk-neutral probability measure $\mathbb{Q}$ such that the instantaneous expected rate of return on every asset is equal to the instantaneous interest rate; i.e., it is equal to zero in our setting. Furthermore, the risk-neutral process of the underlying asset price is assumed to be an Itô process under a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T^*]}, \mathbb{Q})$. Thus the underlying asset price S under the risk-neutral measure $\mathbb{Q}$ is given by the unique solution of the following stochastic differential equation (SDE):

$$dS_t = \sigma(\omega, t) S_t dW_t, \tag{9.1}$$

where W is a Brownian motion under $\mathbb{Q}$, and σ is a $\mathbf{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable process that guarantees that unique solution to SDE (9.1).

Suppose that $f^T(S)$ is the payoff at maturity T of a path-independent European derivative whose randomness depends solely on the underlying asset price at maturity, S_T. If a trader hedges the derivative security over the period by the Black-Scholes delta hedging with a fixed volatility $\sigma_{\mathrm{H}} > 0$ called *the hedging volatility*, then it holds the following equation:

$$\begin{aligned} f^T(S_T) &= v(0, S_0; \sigma_{\mathrm{H}}, f^T) + \int_0^T \frac{\partial v}{\partial S}(t, S_t; \sigma_{\mathrm{H}}, f^T) dS_t \\ &\quad + \frac{1}{2}\int_0^T S_t^2 \frac{\partial^2 v}{\partial S^2}(t, S_t; \sigma_{\mathrm{H}}, f^T)\left[\sigma^2(\omega, t) - \sigma_{\mathrm{H}}^2\right] dt, \end{aligned} \tag{9.2}$$

where $v(t, S_t; \sigma_{\mathrm{H}}, f^T)$ denote the Black-Scholes price function with the constant volatility σ_{H} and payoff f^T at time t, and $\frac{\partial v}{\partial S}(t, S_t; \sigma_{\mathrm{H}}, f^T)$ and $\frac{\partial^2 v}{\partial S^2}(t, S_t; \sigma_{\mathrm{H}}, f^T)$ are the Black-Scholes delta and gamma, respectively. The first term on the right hand side of Eq.(9.2) is the premium of the derivative security, and the second term is the dynamic portfolio by the Black-Scholes delta with the hedging volatility σ_{H}. The third term,

$$\mathrm{HE} := \frac{1}{2}\int_0^T S_t^2 \frac{\partial^2 v}{\partial S^2}(t, S_t; \sigma_{\mathrm{H}}, f^T)\left[\sigma^2(\omega, t) - \sigma_{\mathrm{H}}^2\right] dt, \tag{9.3}$$

is the hedging error caused by uncertain volatility risk, which depends on the difference between the instantaneous volatility $\sigma(\omega, t)$ and the hedging volatility $\sigma_{\rm H}$. If $\sigma(\omega, t)$ is completely predictable for all $t \in [0, T]$, the hedging error HE can be eliminated perfectly. However, this situation is obviously unrealistic and there exists uncertainty of volatilities in a real market all the time. In this chapter, we explore that how the Black-Scholes delta hedging can be improved by using the variance swap under uncertain volatility environments.

9.1.2 Hedging Volatility Risk with Variance Swap

The hedging error HE can be decomposed into

$$\mathrm{HE} = \int_0^T g(t, S_t)\sigma^2(\omega, t)dt - \sigma_{\rm H}^2 \int_0^T g(t, S_t)dt =: A - B, \tag{9.4}$$

where

$$g(t, S_t) := \frac{1}{2} S_t^2 \frac{\partial^2 v}{\partial S^2}(t, S_t; \sigma_{\rm H}, f^T), \tag{9.5}$$

and

$$A := \int_0^T g(t, S_t)\sigma^2(\omega, t)dt, \quad B := \sigma_{\rm H}^2 \int_0^T g(t, S_t)dt. \tag{9.6}$$

In the following, $g(t, S)$ is called *the volatility risk weight* on the derivative security at time t and stock price S. In the following we consider the hedging schemes of risk A and risk B in Eq.(9.4), separately.

Hedging Risk B

Firstly, we present a hedging scheme of risk B. By regarding $g^t(S) := g(t, S)$ as a payoff function at maturity t, the volatility risk weight can be represented in the following form:

$$\begin{aligned} g(t, S_t) &= v(0, S_0; \sigma_{\rm H}, g^t) + \int_0^t \frac{\partial v}{\partial S}(u, S_u; \sigma_{\rm H}, g^t)dS_u \\ &\quad + \frac{1}{2}\int_0^t S_u^2 \frac{\partial^2 v}{\partial S^2}(u, S_u; \sigma_{\rm H}, g^t)\left[\sigma^2(\omega, u) - \sigma_{\rm H}^2\right] du, \end{aligned} \tag{9.7}$$

for $0 \le u \le t$. The third term on the right hand side of Eq.(9.7) is the hedging error of risk B; i.e., the hedging error of the hedging error HE. This error, however, is negligible because it is much smaller than HE in usual. Therefore, by time-discretization and neglecting the hedging error of risk B, it satisfies

$$B \approx \sigma_{\rm H}^2 \int_0^T v(0, S_0; \sigma_{\rm H}, g^t)dt + \sum_{n=1}^N \sigma_{\rm H}^2 \Delta t \int_0^{t_n} \frac{\partial v}{\partial S}(u, S_u; \sigma_{\rm H}, g^{t_n})dS_u, \tag{9.8}$$

where $t_n = n\Delta t$ is the re-balance timing of the dynamic hedging and $t_N = T$. Note that $\sigma_{\rm H}^2 \int_0^T v(0, S_0; \sigma_{\rm H}, g^t)dt$ is the initial cost of risk B and the second term on the right hand side of Eq.(9.8) is the dynamic portfolio for hedging risk B. The closed-form expressions of $v(u, S_u; \sigma_{\rm H}, g^t)$ and $\frac{\partial v}{\partial S}(u, S_u; \sigma_{\rm H}, g^t)$ can be found at the end of this chapter.

Hedging Risk A by Using Standard Variance Swap

Secondly, we consider a hedging scheme of risk A by using the standard variance swap. The variance swap is now actively traded in over-the-counter (OTC) on both stocks and stock indices. In this contract, one party agrees with the other to receive the realized variance of returns of a specified underlying asset over a specified future period. In return, the party pays a fixed positive amount at expiry. The fixed positive amount is agreed upon at the initial time and chosen so that the variance swap is costless to enter. That is, the variance swap in continuous-time setting has the following payoff:

$$M \left\{ \frac{1}{T_{\mathrm{VS}}} \int_0^{T_{\mathrm{VS}}} \sigma^2(\omega, t) dt - K_{\mathrm{VS}} \right\}, \tag{9.9}$$

where M, K_{VS}, and T_{VS} are a notional amount, a fixing rate, and maturity of the variance swap, respectively. Thus, in the contract (9.9) the party receives $\frac{M}{T_{\mathrm{VS}}} \int_0^{T_{\mathrm{VS}}} \sigma^2(\omega, t) dt$ while he pays MK_{VS}.

It is well-known that the floating side of the variance swap, which is the realized variance of an underlying asset price, admits model-free replication by a static position in options and dynamic trading of the underlying asset; and the fixing rate K_{VS} is determined by the initial value of the static position. See Derman et al. [1999] and Carr and Madan [1998] for details of the standard variance swap.

Consider the situation in which a trader tries to hedge risk A by using the variance swap. To do this, it is necessary to choose an appropriate notional amount of the variance swap. If he can choose the notional amount $M = g(t, S_t)T_{\mathrm{VS}}$ for all $t \in [0, T]$ and $S_t > 0$, then risk A is entirely fixed to MK_{VS}. In addition, by setting the hedging volatility $\sigma_{\mathrm{H}} = \sqrt{K_{\mathrm{VS}}}$, the initial cost for hedging risk B in Eq.(9.8) can be balanced with the fixed payment of the variance swap, MK_{VS}. By virtue of the standard variance swap, a robust hedging scheme for uncertain volatility risk is apparently able to be implemented. However, this scheme seems to be not successful because the volatility risk weight $g(t, S_t)$ highly depends on both time t and the underlying asset price S_t in general. Figure 9.1 shows over/under-hedging situations when using the variance swap as a hedging tool of risk A on a call with strike $K = 100$ and maturity $T = 1$. In order to improve this problem, we develop a new variance swap called *the polynomial variance swap* in the next section.

9.2 Polynomial Variance Swap

In this section we introduce the polynomial variance swap (PVS) for hedging uncertain volatility risk of European derivatives, instead of the standard variance swap. PVS has the following payoff:

$$\int_0^{T_{\mathrm{PVS}}} \mathbf{1}_{\{S_t \in I\}} P_M(S_t) \sigma^2(\omega, t) dt - K_{\mathrm{PVS}}, \tag{9.10}$$

where $I = [a, b]$, $0 \le a < b$ denotes a corridor interval, T_{PVS} is maturity, K_{PVS} is a fixed payment of PVS, and $P_M(x)$ is the M-th order polynomial, that is,

$$P_M(x) := a_0 + a_1 x + a_2 x^2 + \cdots + a_M x^M. \tag{9.11}$$

Here, $a_0, a_1, \cdots, a_M \in \mathbf{R}$ are coefficients of $P_M(x)$. Note that PVS is a generalization of some variations on variance swaps which several institutes have offered; e.g., the standard variance swap, the corridor variance swap, and the gamma swap. By a suitable choice of the order and the coefficients of the polynomial, PVS allows us an arbitrary allocation of the volatility risk weight with respect to the underlying asset price.

9.2.1 Replication of Polynomial Variance

Similarly to the standard variance swap, PVS admits model-free replication by a static portfolio in options and dynamic trading of the underlying asset. In this subsection the replication scheme of PVS is provided.

Note that PVS is a linear combination of power variances with a corridor:

$$\int_0^{T_{\text{PVS}}} \mathbf{1}_{\{S_t \in I\}} S_t^m \sigma^2(\omega, t) dt. \tag{9.12}$$

Hence it is sufficient to demonstrate the model-free replication scheme of each power variance.

First, in the case of $m = 0$, the power variance swap is obviously equivalent to the corridor variance swap introduced by Carr and Lewis [2004].

Proposition 9.1 *Let $I = [a, b]$ be an interval. Define $S_t^* = \max(a, \min(S_t, b))$. Then, for any $\kappa \in I$ and all $T \in [0, T^*]$, it satisfies*

$$\begin{aligned} \int_0^T \mathbf{1}_{\{S_t \in I\}} \sigma^2(\omega, t) dt &= \int_a^\kappa \frac{2}{K^2}(K - S_T)^+ dK \\ &+ \int_\kappa^b \frac{2}{K^2}(S_T - K)^+ dK \\ &- 2\left\{\ln \frac{\kappa}{S_0^*} + \frac{S_0}{\kappa} - \frac{S_0}{S_0^*}\right\} \\ &- 2\int_0^T \left(\frac{1}{\kappa} - \frac{1}{S_t^*}\right) dS_t. \end{aligned} \tag{9.13}$$

Proof: See Carr and Lewis [2004]. □

The first and second term on the right hand side of Eq.(9.13) are put and call static portfolios, respectively. Moreover, the fourth term is a dynamic portfolio of the underlying asset. Note that all of the portfolios in Eq.(9.13) are model-free.

Next, the following proposition shows the replication portfolio of power variance in the case of $m = 1$.

Proposition 9.2 *Let $I = [a, b]$ be an interval. Define $S_t^* = \max(a, \min(S_t, b))$. Then, for any $\kappa \in I$ and all $T \in [0, T^*]$, it satisfies*

$$\begin{aligned} \int_0^T \mathbf{1}_{\{S_t \in I\}} S_t \sigma^2(\omega, t) dt &= \int_a^\kappa \frac{2}{K}(K - S_T)^+ dK \\ &+ \int_\kappa^b \frac{2}{K}(S_T - K)^+ dK \\ &- 2\left\{S_0 \ln \frac{S_0^*}{\kappa} - S_0^* + \kappa\right\} \\ &- 2\int_0^T \ln \frac{S_t^*}{\kappa} dS_t. \end{aligned} \tag{9.14}$$

Proof: See at the end of this chapter. □

Note that all of the portfolios on the right hand side of Eq.(9.14), which are static option positions and a dynamic position consisting of the underlying asset, are also model-free.

Finally, the following proposition presents the replication portfolio of power variance in the case of $m \geq 2$.

Proposition 9.3 *Let $I = [a, b]$ be an interval and $m \geq 2$ be an integer. Define $S_t^* = \max(a, \min(S_t, b))$. Then, for any $\kappa \in I$ and all $T \in [0, T^*]$, it satisfies*

$$
\begin{aligned}
&\int_0^T \mathbf{1}_{\{S_t \in I\}} S_t^m \sigma^2(\omega, t) dt \\
= &\int_a^\kappa 2K^{m-2}(K - S_T)^+ dK \\
+ &\int_\kappa^b 2K^{m-2}(S_T - K)^+ dK \\
- &\frac{2}{m(m-1)} \left\{(1-m)(S_0^*)^m - mS_0 \left[\kappa^{m-1} - (S_0^*)^{m-1}\right] - (1-m)\kappa^m\right\} \\
- &\frac{2}{m-1} \int_0^T \left[(S_t^*)^{m-1} - \kappa^{m-1}\right] dS_t.
\end{aligned}
\tag{9.15}
$$

Proof: See at the end of this chapter. □

Similarly to the case of $m = 0$ and 1, all of the portfolios on the right hand side of Eq.(9.15) are model-free.

9.2.2 Strike Volatility of Polynomial Variance Swap

When giving a polynomial $P_M(x)$ and a corridor $I = [a, b]$, the fixed payment K_{PVS} of PVS can be computed easily as follows:

$$
K_{\text{PVS}} := \mathbb{E}\left[\int_0^{T_{\text{PVS}}} \mathbf{1}_{\{S_t \in I\}} P_M(S_t) \sigma^2(\omega, t) dt\right] = \sum_{m=0}^{M} a_m \beta_m, \tag{9.16}
$$

where $\mathbb{E}[\,\cdot\,]$ denotes the expectation operator under the risk-neutral measure $\mathbb{Q}$, $a_0, a_1, \cdots, a_M \in \mathbf{R}$ are coefficients of $P_M(x)$,

$$
\begin{aligned}
\beta_0 &:= \mathbb{E}\left[\int_0^{T_{\text{PVS}}} \mathbf{1}_{\{S_t \in I\}} \sigma^2(\omega, t) dt\right] \\
&= \int_a^{S_0} \frac{2}{K^2} P(K, T) dK + \int_{S_0}^b \frac{2}{K^2} C(K, T) dK,
\end{aligned}
\tag{9.17}
$$

$$
\begin{aligned}
\beta_1 &:= \mathbb{E}\left[\int_0^{T_{\text{PVS}}} \mathbf{1}_{\{S_t \in I\}} S_t \sigma^2(\omega, t) dt\right] \\
&= \int_a^{S_0} \frac{2}{K} P(K, T) dK + \int_{S_0}^b \frac{2}{K} C(K, T) dK,
\end{aligned}
\tag{9.18}
$$

and when $m \geq 2$,

$$\begin{aligned}\beta_m &:= \mathbb{E}\left[\int_0^{T_{\text{PVS}}} \mathbf{1}_{\{S_t\in I\}} S_t^m \sigma^2(\omega,t)dt\right] \\ &= \int_a^{S_0} 2K^{m-2}P(K,T)dK + \int_{S_0}^b 2K^{m-2}C(K,T)dK. \end{aligned} \tag{9.19}$$

Here, $P(K,T)$ and $C(K,T)$ represent the time-0 prices of plain vanilla put and call options with spot price S_0, strike K and maturity T, respectively.

On the other hand, the strike volatility σ_{PVS} of PVS is defined as a positive constant such that

$$\mathbb{E}\left[\int_0^{T_{\text{PVS}}} \mathbf{1}_{\{S_t\in I\}} P_M(S_t)\sigma^2(\omega,t)dt\right] = \mathbb{E}\left[\int_0^{T_{\text{PVS}}} \mathbf{1}_{\{S_t\in I\}} P_M(S_t)\sigma_{\text{PVS}}^2 dt\right]. \tag{9.20}$$

In the simulation analysis, we adopt the strike volatility σ_{PVS} as the hedging volatility of the Black-Scholes delta hedging with/without PVS. The strike volatility is computed by the following proposition:

Proposition 9.4 *Let $I = [a,b]$ be an interval. Then, the strike volatility σ_{PVS} is given by*

$$\sigma_{PVS} = \sqrt{\frac{K_{PVS}}{L_{PVS}}}, \tag{9.21}$$

where K_{PVS} is defined in Eq.(9.16), and

$$\begin{aligned} L_{PVS} &:= \int_0^{T_{PVS}} P_M(\kappa)\mathbb{E}\left[\mathbf{1}_{\{S_t\in I\}}\right]dt + \int_0^{T_{PVS}} P_M'(\kappa)\mathbb{E}\left[\mathbf{1}_{\{S_t\in I\}}(S_t-\kappa)\right]dt \\ &+ \int_0^{T_{PVS}}\int_0^{\kappa} P_M''(K)\mathbb{E}\left[\mathbf{1}_{\{S_t\in I\}}(K-S_t)^+\right]dKdt \\ &+ \int_0^{T_{PVS}}\int_{\kappa}^{\infty} P_M''(K)\mathbb{E}\left[\mathbf{1}_{\{S_t\in I\}}(S_t-K)^+\right]dKdt, \end{aligned} \tag{9.22}$$

for any $\kappa > 0$.

Proof: See at the end of this chapter. □

9.2.3 Hedging Volatility Risk with Polynomial Variance Swap

In order to improve the problem mentioned in Section 9.1.2, we apply PVS to hedge risk A. If a trader can use PVS such that $\mathbf{1}_{\{S_t\in I\}}P_M(S_t) = g(t,S_t)$ for all $t \in [0,T]$ and $S_t > 0$, then risk A is perfectly fixed to K_{PVS}. Moreover, by setting the hedging volatility $\sigma_{\text{H}} = \sigma_{\text{PVS}}$, the initial cost of risk B can be canceled out to the initial payment of PVS K_{PVS}. Similarly to the case of the standard variance swap, there exists a discrepancy between the volatility risk weight and PVS against time t and the underlying asset price S_t. However, the over/under-hedging problem of PVS is more improvable than that of the standard variance swap. To implement a suitable hedging scheme with PVS, we provide a certain procedure to set up the polynomial as follows.

Fixing a certain time τ, a corridor $I = [a, b]$ and an order M of a polynomial[1], the coefficients $a_0, a_1, \cdots, a_M$ of the polynomial $P_M(x)$ can be determined by solving the least square problem:

$$\min_{P_M \in \mathbf{P}_M} \int_a^b \{g(\tau, x) - P_M(x)\}^2 \, dx, \tag{9.23}$$

where $\mathbf{P}_M$ is the set of the M-th order polynomials. The solution of the problem (9.23) is given by

$$P_M(x) = \sum_{m=0}^{M} b_m \phi_m(x), \tag{9.24}$$

where $\phi_m(x)$ is the m-th order orthogonal polynomial such that

$$\int_a^b \phi_m(x)\phi_n(x)dx = \begin{cases} 0 & \text{if} \quad m \neq k \\ \frac{b-a}{2(m+1)} & \text{if} \quad m = k \end{cases}, \tag{9.25}$$

and

$$b_m := \frac{2(m+1)}{b-a} \int_a^b \phi_m(x) g(\tau, x) dx. \tag{9.26}$$

Therefore, the coefficients $a_0, a_1, \cdots, a_M$ of $P_M(x)$ are determined as real numbers satisfying

$$\sum_{m=0}^{M} b_m \phi_m(x) = \sum_{m=0}^{M} a_m x^m. \tag{9.27}$$

For example, Figure 9.2 plots the volatility risk weights of a call option and a polynomial of PVS. In this example, the 6-th order PVS with a corridor interval $I = [70, 140]$ is applied and the fitting point of the polynomial is set as $\tau = 0.5$, while the call option is the same contract as Figure 9.1. Looking at Figure 9.2, it can be said that although the mismatching problem cannot be completely overcome[2], PVS is much better for hedging uncertain volatility risk than the standard variance swap.

When selling a European derivative in Eq.(9.2), the hedging scheme proposed in the above discussion consists of the following positions:

- Hold $\frac{\partial v}{\partial S}(t_n, S_{t_n}; \sigma_{\text{PVS}}, f^T)$ units of the underlying asset S_{t_n} at each time t_n for the Black-Scholes delta hedging.
- Pay the fixed payment K_{PVS} and receive the realized polynomial variance,

$$\int_0^{T_{\text{PVS}}} \mathbf{1}_{\{S_t \in I\}} P_M(S_t) \sigma^2(\omega, t) dt, \tag{9.28}$$

  at maturity of PVS for hedging risk A.
- Receive the initial cost of risk B,

$$\sigma_{\text{PVS}}^2 \int_0^T v(0, S_0; \sigma_{\text{PVS}}, g^t) dt, \tag{9.29}$$

[1] Although the selection of τ, I, and M appear to be more art than science, it is not so difficult to choose them for effective hedging (see Figure 9.2).

[2] As one method to improve the mismatching problem with respect to time t, the *forward start* polynomial variance swaps can be considered as tools of time-piecewise fitting to the volatility risk weight. However, this is not pragmatical because a large number of options are needed for replicating the forward start PVSs.

and hold

$$\sigma_{\mathrm{PVS}}^2 \Delta t \sum_{t_n < t_i} \frac{\partial v}{\partial S}(t_n, S_{t_n}; \sigma_{\mathrm{PVS}}, g^{t_i}), \tag{9.30}$$

units of the underlying asset S_{t_n} at each time t_n for dynamic hedging of risk B.

9.3 Simulation Analysis

This section shows the effectiveness of the hedging scheme proposed in Section 9.2 under uncertain volatility environments through Monte Carlo simulation tests. To examine the performance of the Black-Scholes delta hedging with PVS, we compare four types of hedging strategies under different three scenarios.

9.3.1 Setup of Simulation

Let us consider the problem faced by the writer of a call option on a certain stock, whose maturity is 3-months and strike is at-the-money. The writer intends to hold this short potion until the maturity, and can hedge his market risk using various hedging schemes. For concreteness, suppose that the initial stock price is $S_0 = 100$, the strike of the target call is $K = 100$, and the option maturity is $T = 60/250$; i.e., we assume that there are 20 business days in a month and 250 business days in a year. Both the interest rate and the dividend yield are set to be zero for simplicity.

The assumed market situation in the simulation tests is as follows: All market options are priced in accordance with the Heston's stochastic volatility model (Heston [1993]) with given parameters, and can be traded without any transaction cost. That is, all traders in the option market believe that the risk-neutral dynamics of the underlying asset price take the following form:

$$\begin{aligned} dS_t &= \sqrt{V_t} S_t dW_t, \\ dV_t &= \xi(\eta - V_t)dt + \theta\sqrt{V_t} dZ_t, \end{aligned} \tag{9.31}$$

where ξ, η and θ are positive constants such that $\xi\eta \geq \theta^2/2$, and W and Z are Brownian motions with correlation ρ under a pricing measure. The writer also knows the fact that the market option prices follow the Heston model through observing the option market. The Heston parameters for the market option prices are listed in Table 9.1.

Table 9.1: Heston Parameters for Market Option Prices

V_0	ξ	η	θ	ρ
0.20^2	1.15	0.20^2	0.39	-0.64

On the other hand, the writer does not know the true generating process of the stock price. Thus, we consider the situation that the writer cannot know not only the parameters of the generating model, but also the true model itself.

9.3.2 Hedging Strategies

We employ four hedging strategies in order to compare the performance of our hedging scheme with that of standard hedging schemes. In the simulation tests, the writer carries out one of these strategies systematically to hedge his market risk on the target call until the maturity.

The first strategy is the Black-Scholes delta hedging without PVS (BS DH for short). The writer uses the Black-Scholes model for dynamic hedging of the call option:

$$dS_t = \sigma_{\mathrm{BS}} S_t dW_t, \tag{9.32}$$

where σ_{BS} is a constant volatility. At each time t, he computes the delta based on the Black-Scholes model and re-balances the dynamic hedging portfolio accordingly. To be comparison with the Black-Scholes delta hedging with PVS, the constant volatility σ_{BS} is set to be the strike volatility of PVS.

The second strategy is the minimum-variance hedging with the Heston model (HS MVH for short), which is a standard method of dynamic hedging in an incomplete market. In the minimum-variance hedging under the Heston model (9.31), the units of the underlying asset to be held at each time t are computed as follows:[3]

$$\frac{\partial C_t}{\partial S_t} + \frac{\rho\theta}{S_t}\frac{\partial C_t}{\partial V_t}, \tag{9.33}$$

where C_t denotes the time-t price of the target call option, and ρ and θ are parameters in (9.31). Note that volatility risk can be partially hedged through the correlation between the underlying asset price and its instantaneous variance if the model and its parameters are correct. Based on the equation (9.33), the writer re-balances the dynamic hedging portfolio at each time in the simulation tests. For HS MVH, he adopts the same Heston parameters as in Table 9.1.

The third strategy is the Black-Scholes delta hedging with PVS, in which the PVS maturity is the same as that of the target call; i.e., $T_{\mathrm{PVS1}} = 60/250$ (BS DH with PVS1 for short). To hedge uncertain volatility risk on the target call, the writer uses the sixth order PVS with corridor interval $I = [85, 120]$. Its polynomial is fitted with the volatility risk weight of the target call at time $\tau_{\mathrm{PVS1}} = 30/250$, and the static portfolio replicating the PVS is composed of 4 calls and 4 puts. In order to accurately approximate the static portfolio by the finite number of options, the Gauss-Legendre quadrature rule is applied. More detailed discussion for this approximation scheme can be found in Takahashi and Yamazaki [2009a]. Table 9.2 reports the static portfolio compositions. At initial time, the writer constitutes the static portfolio for the PVS. Then, he computes the Black-Scholes delta and the amount of the underlying asset for replicating the PVS at each time, and re-balances the dynamic portfolios of both BS DH and the PVS accordingly. Note that, although the writer can roughly offset the volatility risk by using the PVS, there is a discrepancy between the polynomial of the PVS1 and the volatility risk weight of the target call for all time $t \in [0, T]$. In particular, this strategy is considerably over-hedging when time t approaches the target call maturity and the stock price is apart from at-the-money.

Table 9.2: Static Portfolio for Replicating PVS1

	No.1	No.2	No.3	No.4
Call Strike	101.3886	106.6002	113.3998	118.6114
Call Amount	0.1906	0.2273	0.0560	0.0089
Put Strike	86.0415	89.9501	95.0499	98.9585
Put Amount	0.0180	0.1021	0.2341	0.1488

The fourth strategy is the Black-Scholes delta hedging with PVS, in which the PVS maturity is shorter than that of the target call (BS DH with PVS2 for short). To avoid over-hedging against the volatility risk near the target call maturity, the PVS maturity is set as $T_{\mathrm{PVS2}} = 55/250$ in this strategy. Additionally, the fitting point of its polynomial is $\tau_{\mathrm{PVS2}} = 27.5/250$. Except for the PVS maturity and the fitting point, the scheme of BS DH with PVS2 is the same as that of BS DH with PVS1. Table 9.3 shows the static portfolio compositions.

[3] For example, see Bakshi et al. [1997] for the detail and for a practical application of the minimum-variance hedging method.

Table 9.3: Static Portfolio for Replicating PVS2

	No.1	No.2	No.3	No.4
Call Strike	101.3886	106.6002	113.3998	118.6114
Call Amount	0.1837	0.2247	0.0622	0.0102
Put Strike	86.0415	89.9501	95.0499	98.9585
Put Amount	0.0206	0.1075	0.2292	0.1434

For all strategies, the writer re-balances the dynamic portfolios once a day until the target call maturity in the simulation tests. Then, we monitor the hedging error (profit and loss) of each sample path, which is defined as the difference between the final value of the total hedging portfolio and the payoff of the target call option.

9.3.3 Simulation Test

For the simulation tests, we consider three data generating processes: the Heston model with correct estimated parameters, the Heston model with misspecified parameters, and the CEV process. In each simulation, a time series of a daily underlying asset price is generated according to an Euler-Maruyama approximation of the respective data generating process. Evey simulation result is based upon 10,000 sample paths.

Simulation under Heston World

First, we generate underlying asset prices by the Heston model:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^*, \\ dV_t &= \xi(\eta^* - V_t)dt + \theta\sqrt{V_t} dZ_t^*, \end{aligned} \tag{9.34}$$

where W^* and Z^* are Brownian motions with correlation ρ under a physical measure. Note that the parameter η^*, which denotes the mean reversion level of the instantaneous variance for the data generating process, is generally smaller than η in Eq.(9.31) because of market price of volatility risk. The Heston parameters for generating underlying asset prices are listed in Table 9.4. This scenario indicates

Table 9.4: Heston Parameters for Generating Stock Prices

μ	V_0	ξ	η^*	θ	ρ
0.06	0.20^2	1.15	0.18^2	0.39	-0.64

that the option market perfectly estimates both the data generating process and its parameters. Such a situation, however, seems to be unrealistic in practice.

Table 9.5 reports the summary statistics of the Monte Carlo simulation results. Moreover, Figure 9.3 shows the histograms of hedging errors. In the Heston world with correct estimated parameters, the means of hedging errors are nearly zero for all hedging strategies. Conversely, the standard deviations are very different for each hedging strategy. Since the minimum-variance hedging can partially hedge volatility risk on the target call, the standard deviation of HS MVH is smaller than that of BS DH without PVS. Furthermore, because the PVS can directly hedge the volatility risk, the standard deviations of BS DH with PVS1 and PVS2 are reduced by half from BS DH and HS MVH. In addition, when seeing in detail, the standard deviation of BS DH with PVS2 is smaller than that of BS DH with PVS1. This is the effect of avoiding over-hedging near the target call maturity. From the viewpoint of the standard deviation of the hedging errors, it can be said that BS DH with PVS2 is the best strategy among four candidates. However, it is necessary to pay attention to high kurtosis of the hedging errors in BS DH with PVS1 and PVS2. It means that BS DH with PVS has a fat tail distribution of hedging errors. In fact, we can observe fat-tail properties from Figure 9.3. In particular, the left tail on the histograms is thick; i.e, negative skew. The high kurtosis is caused by the discrepancy between the polynomials of the PVS and the volatility risk weights of the target call with several bad scenarios for BS DH with PVS.

Table 9.5: Hedge Error under the Heston World

Hedging Scheme	BS DH	HS MVH	BS DH with PVS1	BS DH with PVS2
Mean	-0.0045	-0.0335	-0.0113	-0.0137
Std Err	0.9576	0.8685	0.4746	0.4612
Skewness	-0.7288	-0.4404	-1.8565	-1.7630
Kurtosis	5.4867	3.6180	10.8580	11.3134
Min	-7.1363	-5.6175	-4.3763	-3.9820
Max	3.2830	2.7374	1.5868	1.7767

Simulation under Heston World with Misspecified Parameters

Second, we generate underlying asset prices by the Heston model (9.34) with misspecified parameters. The misspecified Heston parameters for generating underlying asset prices are listed in Table 9.6.

Table 9.6: Misspecified Heston Parameters for Generating Stock Prices

μ	V_0	ξ	η^*	θ	ρ
0.06	0.20^2	2.00	0.25^2	0.39	-0.64

Table 9.7 reports the basic statistics of the Monte Carlo simulation results, and Figure 9.4 shows the histograms of hedging errors. Note that BS DH and HS MVH make losses on average while the means of hedging errors of BS DH with PVS1 and PVS2 can be regarded as to be nearly zero. In addition, the standard deviations of BS DH with PVS1 and PVS2 are much smaller than those of BS DH and HS MVH. That is, by using the PVS the writer can improve the mean of hedging errors as well as the standard deviation. As a result of this simulation test, it can be said that the PVS is an appropriate tool for hedging volatility risk even when the model parameters are mis-estimated, because the PVS does not depend on parameter specification at all. On the other hand, similarly to the previous case, there exists fat-tail risk of the hedging errors in BS DH with PVS.

Table 9.7: Hedge Error under the Heston World with Misspecified Parameters

Hedging Scheme	BS DH	HS MVH	BS DH with PVS1	BS DH with PVS2
Mean	-0.2800	-0.3131	-0.0609	-0.0960
Std Err	0.9247	0.8303	0.5302	0.4942
Skewness	-1.0346	-0.5237	-1.8707	-2.1345
Kurtosis	6.0194	3.7374	9.5258	11.1666
Min	-7.1393	-5.6710	-4.8121	-4.3671
Max	2.6183	2.0812	1.7729	1.4335

Simulation under CEV World

Third, we generate underlying asset prices by the CEV model:

$$dS_t = \mu S_t dt + \sigma_{\mathrm{CEV}} S_t^{\beta} dW_t^*, \tag{9.35}$$

where β and σ_{CEV} are constant parameters. The CEV parameters for generating underlying asset prices are listed in Table 9.8. This situation means a model misspecification case.

Table 9.8: CEV Parameters for Generating Stock Prices

μ	β	σ_{CEV}
0.06	0.50	2.00

Table 9.9 reports the basic statistics of the Monte Carlo simulation results, and Figure 9.5 shows the histograms of hedging errors. Similarly to the misspecified Heston world, although BS DH and HS MVH make losses on average, the means of hedging errors of BS DH with PVS1 and PVS2 are nearly equal to zero. In addition, the standard deviations of BS DH with PVS1 and PVS2 are much smaller than HS MVH. Note that the hedging performance of HS MVH is worse than BS DH without PVS because HS MVH is a fragile hedging scheme for model risk. On the other hand, by adopting BS DH with PVS the writer can improve the mean of hedging errors as well as the standard deviation. As a consequence of this simulation test, it can be also said that BS DH with PVS is a robust hedging scheme for model risk. Of course, fat-tail risk of the hedging errors exists in BS DH with PVS.

Table 9.9: Hedge Error under the CEV World

Hedging Scheme	BS DH	HS MVH	BS DH with PVS1	BS DH with PVS2
Mean	-0.1300	-0.1702	-0.0327	-0.0495
Std Err	0.4429	1.1665	0.3866	0.3254
Skewness	-0.3633	-0.0814	-1.4537	-1.5130
Kurtosis	4.7282	2.2873	6.1156	7.4525
Min	-2.5923	-3.3922	-2.6899	-2.7925
Max	1.8762	2.7595	0.9525	1.1065

9.4 Concluding Remarks

This chapter examines how robust the Black-Scholes delta hedging is against uncertain volatility risk when adding a certain variance swap. While the standard variance swap is the most approved contract to purely trade volatility of a underlying asset, it is not absolutely an appropriate tool for hedging uncertain volatility risk on derivative securities. To improve the defect of the standard variance swap, we develop the polynomial variance swap, which is a kind of exotic variance swaps and can be implemented by model-free replication. Then, a new hedging scheme applying PVS is proposed. We test the hedging performance of our scheme through Monte Carlo simulations which generate several different scenarios of the underlying price processes. As a result, it is shown that the hedging scheme proposed in this chapter is not perfect, but significantly robust rather than other standard hedging schemes such as the minimum-variance hedging. Moreover, it is found that the hedging performance of our scheme is hardly affected by model risk.

Finally, our next research topic will be to consider a robust hedging scheme of exotic derivatives such as barrier options and look-back options against uncertain volatility risk, and design a suitable volatility derivative as a hedging tool for this problem.

Price and Delta of the Volatility Risk Weight We derive the closed-form expressions of $v(0, S_0; \sigma_{\rm H}, g^t)$ and $\frac{\partial v}{\partial S}(0, S_0; \sigma_{\rm H}, g^t)$ in Eq.(9.8) when the target derivative is a plain vanilla call. In the cases of other European derivatives such as asset digital and cash digital, the closed-form expressions can be obtained by the same manner as the following discussion.

Consider the payoff function $f^T(S) = (S-K)^+$ with strike K. By Eq.(9.5) and the Black-Scholes gamma formula of the call option, the payoff function of the volatility risk weight at maturity t can be written as

$$g(t, S_t) := g^t(S_t) = \frac{1}{2} S_t \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}d_1^2}}{\sigma_{\rm H}\sqrt{T-t}}, \tag{9.36}$$

where

$$d_1 = \frac{\ln \frac{S_t}{K} + \frac{1}{2}\sigma_{\rm H}^2(T-t)}{\sigma_{\rm H}\sqrt{T-t}}. \tag{9.37}$$

Therefore, the Black-Scholes price, which is the price in the case of $\sigma(\omega, t) = \sigma_H$ in Eq.(9.1), of the derivative with payoff g^t is given by

$$\begin{aligned} v(0, S_0; \sigma_{\rm H}, g^t) &= \mathbb{E}\left[\frac{1}{2} S_t \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}d_1^2}}{\sigma_{\rm H}\sqrt{T-t}}\right] \\ &= \frac{1}{2} S_0 I(S_0), \end{aligned} \tag{9.38}$$

where

$$\begin{aligned} I(S_0) &:= \mathbb{E}\left[\frac{S_t}{S_0} \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}d_1^2}}{\sigma_{\rm H}\sqrt{T-t}}\right] \\ &= \frac{1}{\sqrt{2\pi T}\sigma_{\rm H}} \exp\left\{-\frac{1}{2}\frac{(\ln \frac{S_0}{K} + \frac{1}{2}\sigma_{\rm H}^2 T)^2}{\sigma_{\rm H}^2 T}\right\}. \end{aligned} \tag{9.39}$$

From Eq.(9.38), we have

$$\begin{aligned} \frac{\partial v}{\partial S}(0, S_0; \sigma_{\rm H}, g^t) &= \frac{1}{2}\left[I(S_0) + S_0 \frac{\partial I}{\partial S}(S_0)\right] \\ &= \left[\frac{1}{4} - \frac{\ln \frac{S_0}{K}}{2\sigma_{\rm H}^2 T}\right] I(S_0). \end{aligned} \tag{9.40}$$

Proof of Proposition 9.2 Let

$$h(x) = 2\left\{x \ln \frac{x^*}{\kappa} - x^* + \kappa\right\}. \tag{9.41}$$

Then

$$h'(x) = 2\ln\frac{x^*}{\kappa} \quad \text{and} \quad h''(x) = \frac{2}{x}\mathbf{1}_{\{x\in I\}}. \tag{9.42}$$

By Itô's formula, we obtain

$$\frac{1}{2}\int_0^T h''(S_t) S_t^2 \sigma^2(\omega, t) dt = h(S_T) - h(S_0) - \int_0^T h'(S_t) dS_t. \tag{9.43}$$

Since $h(\kappa) = h'(\kappa) = 0$, we have

$$\begin{aligned} h(S_T) &= \int_0^\kappa h''(K)(K - S_T)^+ dK + \int_\kappa^\infty h''(K)(S_T - K)^+ dK \\ &= \int_a^\kappa \frac{2}{K}(K - S_T)^+ dK + \int_\kappa^b \frac{2}{K}(S_T - K)^+ dK. \end{aligned} \tag{9.44}$$

Therefore, by substituting Eq.(9.41), (9.42), and (9.44) into Eq.(9.43), Eq.(9.14) is obtained. □

Proof of Proposition 9.3 Let

$$h(x) = \frac{2}{m(m-1)} \left\{ (1-m)(x^*)^m - mx \left[\kappa^{m-1} - (x^*)^{m-1} \right] - (1-m)\kappa^m \right\}. \tag{9.45}$$

Then

$$h'(x) = \frac{2}{m-1} \left\{ (x^*)^{m-1} - \kappa^{m-1} \right\} \quad \text{and} \quad h''(x) = 2x^{m-1} \mathbf{1}_{\{x \in I\}}. \tag{9.46}$$

Therefore, by the same discussion as Proof of Proposition 9.2, Eq.(9.15) can be obtained. □

Proof of Proposition 9.4 From the definition of the strike volatility, we have

$$\sigma^2_{\text{PVS}} = \frac{\mathbb{E}\left[\int_0^{T_{\text{PVS}}} \mathbf{1}_{\{S_t \in I\}} P_M(S_t) \sigma^2(\omega, t) dt \right]}{\mathbb{E}\left[\int_0^{T_{\text{PVS}}} \mathbf{1}_{\{S_t \in I\}} P_M(S_t) dt \right]} = \frac{K_{\text{PVS}}}{L_{\text{PVS}}}, \tag{9.47}$$

where

$$L_{\text{PVS}} = \mathbb{E}\left[\int_0^{T_{\text{PVS}}} \mathbf{1}_{\{S_t \in I\}} P_M(S_t) dt \right]. \tag{9.48}$$

Next, for any $\kappa > 0$ and all $t \in [0, T_{\text{PVS}}]$, it satisfies

$$\begin{aligned} P_M(S_t) &= P_M(\kappa) + P'_M(\kappa)(S_t - \kappa) \\ &+ \int_0^\kappa P''_M(K)(K - S_t)^+ dK + \int_\kappa^\infty P''_M(K)(S_t - K)^+ dK. \end{aligned} \tag{9.49}$$

Therefore, by substituting Eq.(9.49) into Eq.(9.48), Eq.(9.22) is obtained. □

Figure 9.1: Volatility Risk Weight of Call and Variance Swap

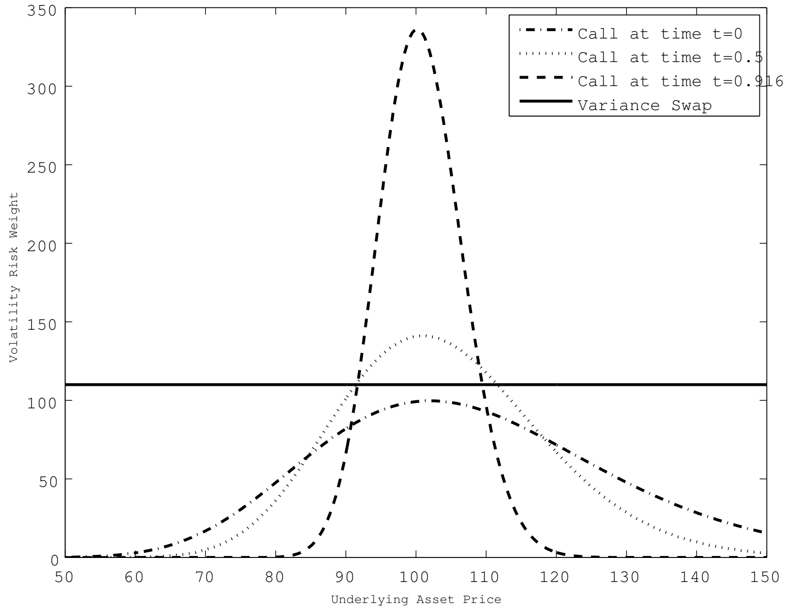


Figure 9.2: Volatility Risk Weight of Call and Polynomial Variance Swap

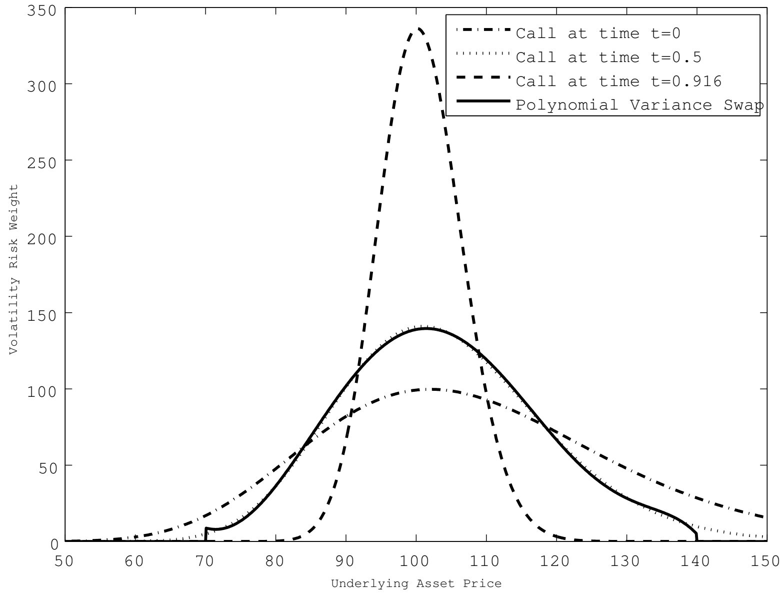


Figure 9.3: Histograms of Hedging Errors under the Heston World

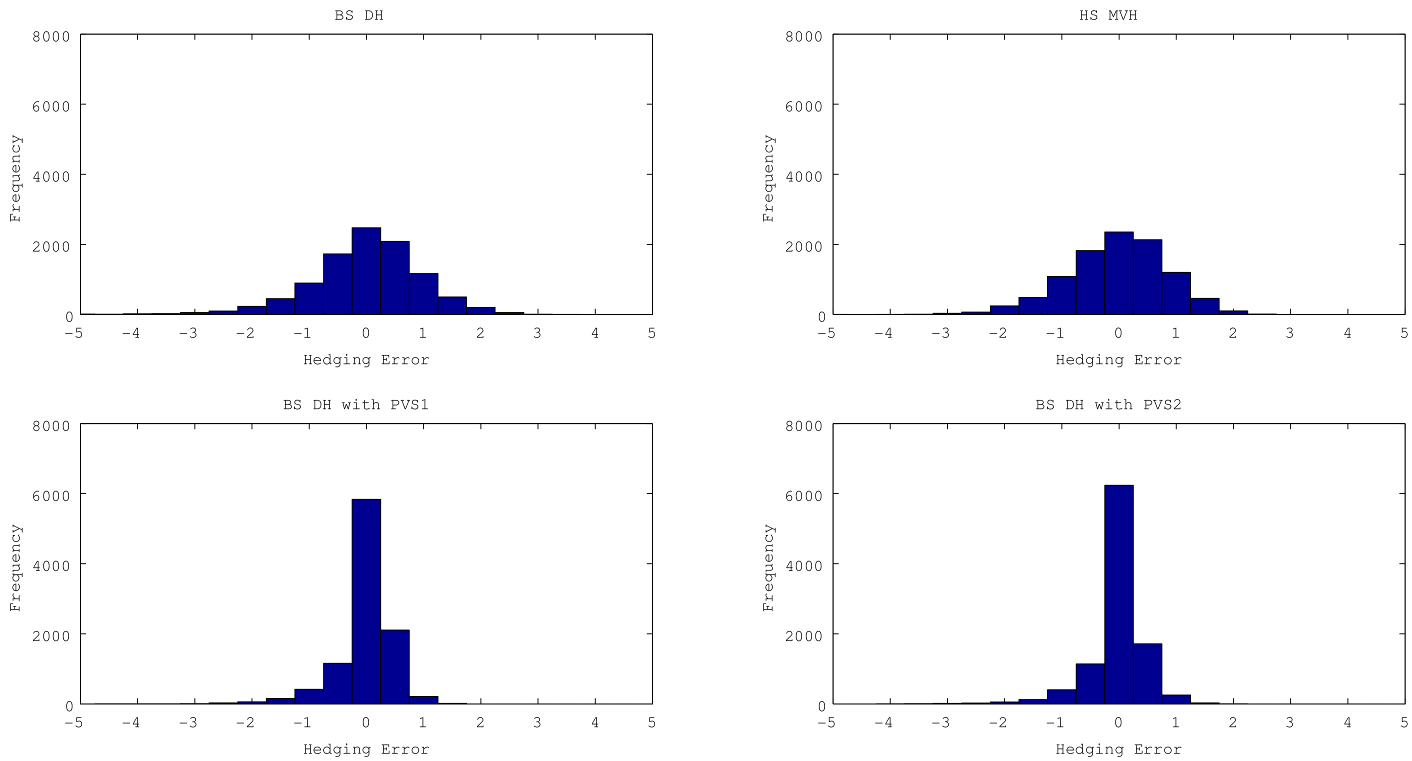


Figure 9.4: Histograms of Hedging Errors under the Misspecified Heston World

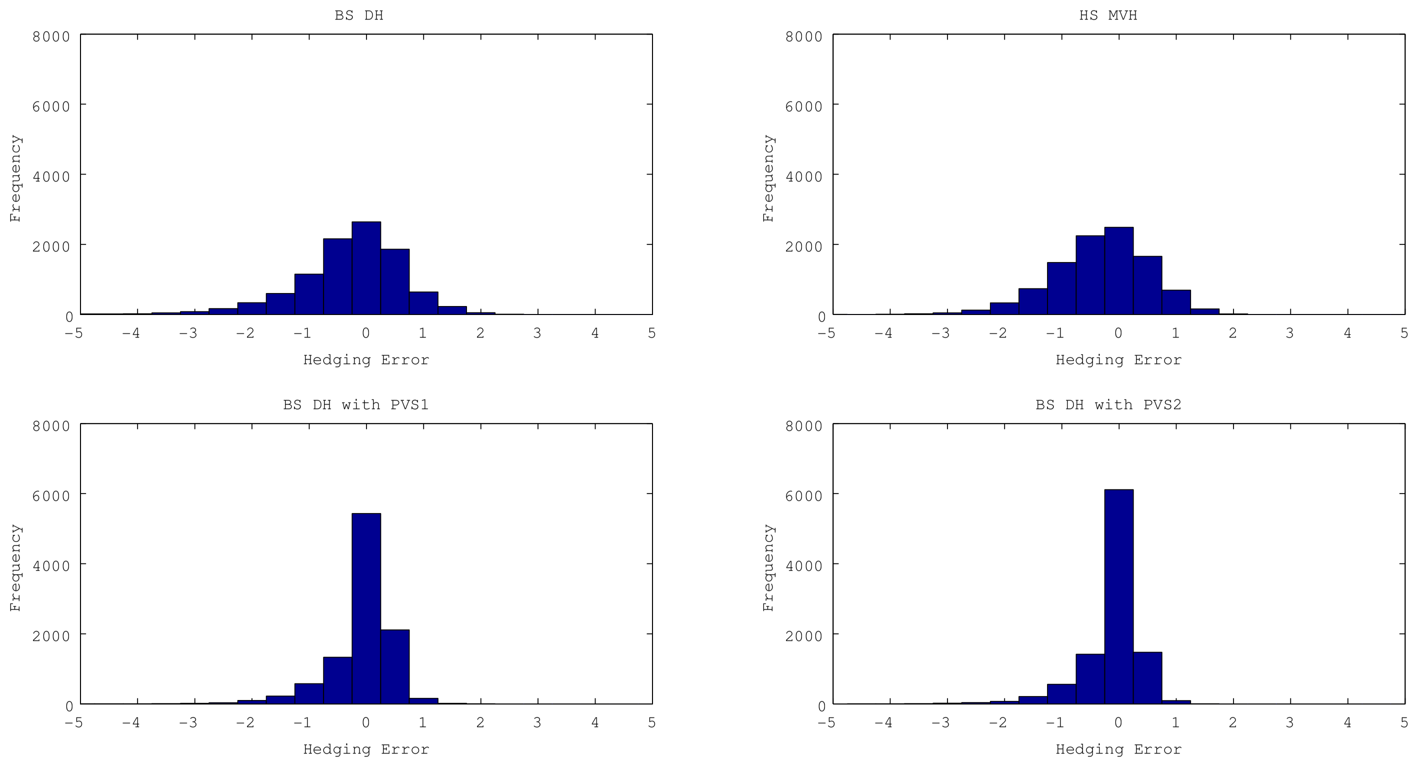
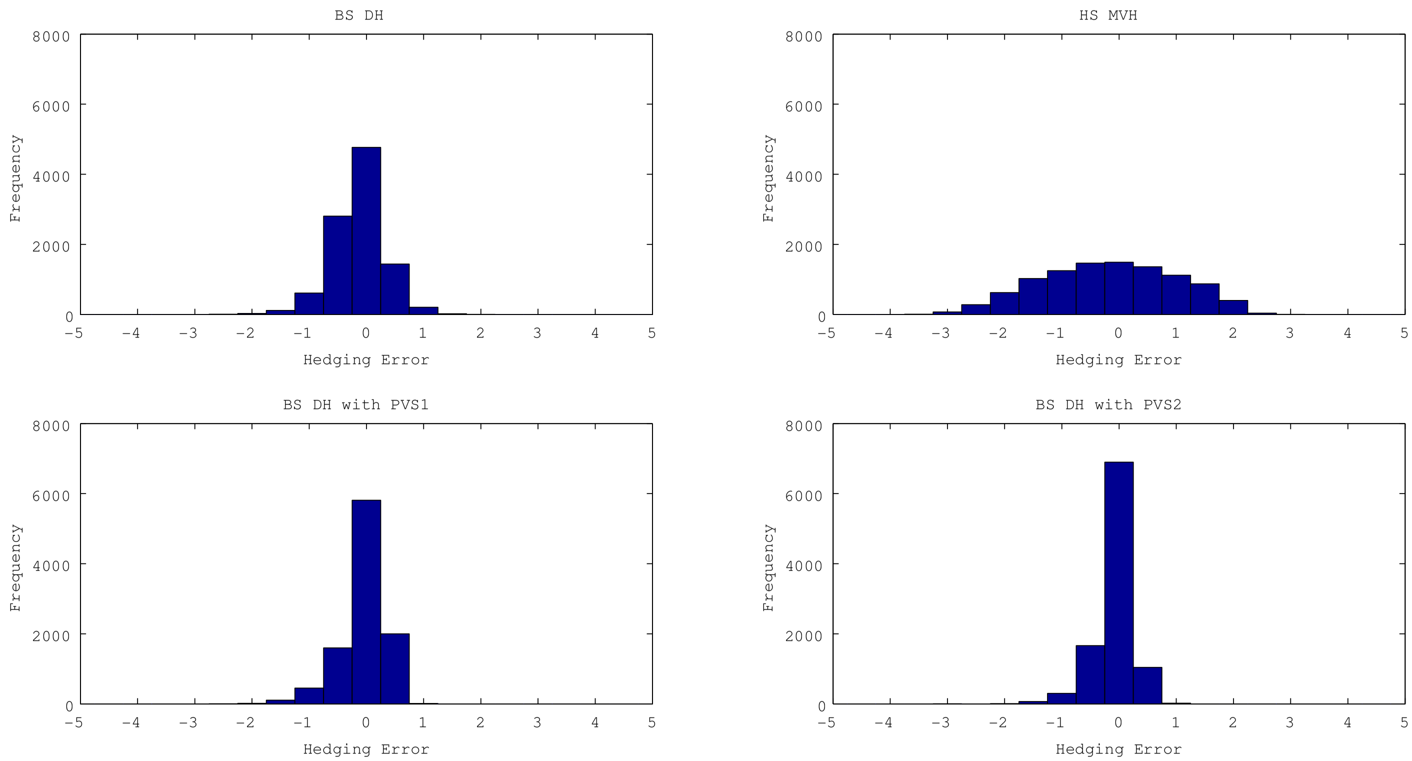


Figure 9.5: Histograms of Hedging Errors under the CEV World



Appendix A

On Valuation with Stochastic Proportional Hazard Models in Finance

The proportional hazard model proposed by Cox [1972] was originally developed for survival analysis, which is a branch of statistics and deals with death in biological organisms and failure in mechanical systems. Currently the model has become one of the most approved survival models in this field, and its estimation methods have been established. On the other hand, looking at finance, there have been many applications of the proportional hazard model to describe financial event risks such as credit risk, prepayment risk, and withdrawal risk, because the phenomenon of these financial risks are analogous with that of death or failure risk in survival analysis. Much literature documenting research on empirical analysis reported effectiveness to modeling such financial risks by the proportional hazard model. The empirical works include Aonuma and Kijima [1998], Ciochetti et al. [2003], Duffie et al. [2007], Lane et al. [1989], Quigley [1987], Schwartz and Torous [1989], Stepanova and Thomas [2002], Sugimura [2002], Whalen [1991], and Wheelock and Wilson [2000]. In this chapter we provide an analytical treatment for evaluating financial risks with the proportional hazard model in continuous time setting.

The proportional hazard model is statistically meaningful for analyzing and estimating financial event risks. In particular, the model has become a standard model among both researchers and practitioners to represent prepayment behavior of mortgage borrowers in mortgage-backed security analysis since Schwartz and Torous [1989], who pioneered the proportional hazard model into mortgage analysis and popularized the model in this field. However, while the valuation of mortgage-backed securities usually depends on some numerical method such as Monte Carlo simulations, the existing literature that analytically treats the valuation problem with the proportional hazard model is very limited. Recently, applying the cumulant expansion approach, Ozeki et al. [2009] developed an analytic pricing formula for residential mortgage-backed securities with the proportional hazard model. They demonstrated the derivation of the analytical pricing formula in the case that the proportional hazard model has Gaussian and/or compound Poisson processes as stochastic covariates. Their formula is very useful, because while the pricing problem of the mortgage-backed securities has been considered to be highly demanding, their formula shows a potential to overcome computationally hard requirement.

However, Ozeki et al, [2009] only dealt with the two processes and a discrete cash flow payment model. We significantly extend the analytical treatment of such the valuation problem in the case that the proportional hazard model has not only the two processes but also affine, quadratic Gaussian, Lévy, and time-changed Lévy processes. The family of the affine-processes and the quadratic Gaussian process is the most widely studied time-series processes in the empirical finance literature, particularly, when modeling the term structure of interest rates. Their popularity is attributable to their accommodation of stochastic volatility and correlations among the risk factors driving asset returns. (e.g., see Duffie and Kan [1996], Duffie et al. [2000], Leippold and Wu [2002, 2007].) On the other hand, a variety of models based on Lévy processes and time-changed Lévy processes has been proposed as models for asset prices having jumps and tested on empirical data. (e.g., see Barndorff-Nielsen [1998], Carr and Wu [2003, 2004], Carr et al. [2002, 2003], Cont and Tankov [2004], Huang and Wu [2004].) Therefore, the derived formulas allow us to adopt huge number of stochastic processes as covariates of the proportional hazard model. In contrast to Ozeki et al. [2009] applying the cumulant expansion approach to the mortgage-backed security valuation in the setting of a discrete time cash flow payment model, our valuation formulas are based on the Edgeworth expansion, whose merit is easily applicable to both discrete and continuous cash flow payment models.

Our fundamental formulas are widely useful to financial valuation problems other than that of mortgage-backed securities. One of examples is to evaluate withdrawal risk of time deposits and saving accounts for the asset-liability management of commercial banks. Applying the proportional hazard model to estimate withdrawing behavior of Japanese retail depositors, Aonuma and Kijima [1998] measured the present values of time deposits with withdrawal risk. Using only a one-factor Gaussian interest rate model as the stochastic covariate of the proportional hazard model, they computed the time deposit values by the trinomial tree method. However, it could be difficult to extend highly dimensional multi-factor setting when applying the lattice method as well as the finite difference method. Even if using Monte Carlo simulation, it seems to be very hard and time-consuming to obtain the accurate values when adopting jump processes and computing risk sensitivities. Our formulas are applicable to multi-dimensional stochastic environments and various types of processes including jumps, and they give very accurately approximate values quickly like a closed-form formula.

Another important application of the proportional hazard model in finance is credit risk modeling in the intensity-based approach. For instance, applying the proportional hazard model, Lane et al. [1986] estimated default probabilities of banks using time-independent covariates. Whalen [1991], and Wheelock and Wilson [2000] also used the proportional hazard model for bank default analysis. Stepanova and Thomas [2002] built credit scoring models by the proportional hazard model applied to personal loan data. More recently, Duffie et al. [2007] proposed maximum likelihood estimators of term structures of conditional probabilities of corporate default, incorporating the dynamics of firms-specific and macroeconomic covariates. And then they took the default intensities to be of the proportional hazard model, and provided an empirical implementation of this estimation method for the US-listed industrial firms.

Furthermore, a remarkable example is the Black-Karasinski model [1991]. Although the model was originally proposed for modeling interest rate dynamics that follow a log-normal distribution, it is regarded as one of the most suitable models for credit spread processes when pricing credit derivatives. For instance, Garcia et al. [2001], Chu and Kwok [2003], and Pan and Singleton [2008] introduced the Black-Karasinski model as a default intensity to price credit default swaps and credit spread options. Unfortunately, until recently, it has been well-known that the Black-Karasinski model lacks the level of analytical tractability. However, it is worthwhile noting that the Black-Karasinski model can be considered as a proportional hazard model with the one-factor stochastic covariate following the generalized Ornstein-Uhlenbeck process. Therefore, it can be said that the model is the simplest example in our setting and our formulas are applicable to more complicated modeling.

A.1 Setup

Let us start with a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ carrying a m-dimensional stochastic covariate process $(\mathbf{X}_t)_{t\geq 0}$ of a financial event risk we consider, a d-dimensional Brownian motion $(\mathbf{W}_t)_{t\geq 0}$, and an exponential random variable with unit mean $e \sim \text{Exp}(1)$. We denote by $\mathbb{F}^X := (\mathcal{F}_t^X)_{t\geq 0}$ and $\mathbb{F}^W := (\mathcal{F}_t^W)_{t\geq 0}$ the filtrations generated by $(\mathbf{X}_t)_{t\geq 0}$ and $(\mathbf{W}_t)_{t\geq 0}$, respectively. Furthermore, let $\mathbb{F} = \mathbb{F}^X \vee \mathbb{F}^W$; i.e. $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^W$ for any $t \in [0, \infty)$. All markets are assumed to be frictionless and arbitrage-free. We take an equivalent martingale measure $\mathbb{Q}$ as given and set the instantaneous risk-free interest rate $(r_t)_{t\geq 0}$ as follows.

Assumption A.1 *(Instantaneous interest rate) The instantaneous risk-free rate* $(r_t)_{t\geq 0}$ *follows an Itô process adapted to the filtration* $\mathbb{F}^W$*. That is,* $(r_t)_{t\geq 0}$ *is a unique strong solution of the SDE:*

$$dr_t = \mu_t dt + \sigma_t^\top d\mathbf{W}_t, \quad t \geq 0; \tag{A.1}$$

where $(\mu_t)_{t\geq 0}$ *and* $(\sigma_t)_{t\geq 0}$ *are* $\mathcal{F}_t^W$*-adapted processes. In addition, the discount bond price process with an arbitrary maturity* T *denoted by* $(B(t,T))_{t\geq 0}$ *is a unique strong solution of the SDE:*

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \mathbf{g}_t^\top d\mathbf{W}_t, \quad t \geq 0; \tag{A.2}$$

where $(\mathbf{g}_t)_{t\geq 0}$ *is a d-dimensional* $\mathcal{F}_t^W$*-adapted process.*

Assumption A.1 implies that the equivalent martingale measure $\mathbb{Q}$ is equal to the spot measure. Hereafter, we call the measure $\mathbb{Q}$ the "spot measure" to distinguish it from other measures such as "forward measures".

In order to model the event risk, we introduce a positive intensity (hazard rate) process $(h_t)_{t\geq 0}$ adapted to $\mathbb{F}$. The random event time τ is modeled as the first time when the hazard process $\int_0^t h_s ds$ is greater or equal to the random level $e \sim \mathrm{Exp}(1)$, i.e.,

$$\tau = \inf\left\{ t \geq 0 : \int_0^t h_s ds \geq e \right\}.$$

Therefore, it is assumed that the certain financial event we consider in this chapter occurs at time τ.

Next, we introduce an event indicator process $(H_t)_{t\geq 0}$, $H_t = \mathbf{1}_{\{t\geq\tau\}}$ which means that a one-jump process is equal to zero before the event and jumps to one at time τ. Here, $F_t := \mathbb{Q}(\tau \leq t \mid \mathcal{F}_t)$ denotes the conditional probability of τ, and $\Gamma_t := -\ln(1-F_t) = \int_0^t h_s ds$ is the hazard process of τ under $\mathbb{Q}$. We denote by $\mathbb{H} := (\mathcal{H}_t)_{t\geq 0}$ the filtration generated by $(H_t)_{t\geq 0}$. Moreover, $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ (i.e., $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for any $t \in [0,\infty)$) denotes an enlarge filtration. Thus, the compensated process $H_t - \int_0^{t\wedge\tau} h_s ds$ is a $\mathbb{G}$-martingale.

Assumption A.2 *(Cox proportional hazard model) The intensity process* $(h_t)_{t\geq 0}$ *is given by the Cox proportional hazard model (PHM for short) with a m-dimensional stochastic covariate vector* $\mathbf{X}_t$*, that is,*

$$h_t := \bar{h}(t) \exp\left\{\mathbf{w}^\top \mathbf{X}_t\right\}, \quad t \geq 0; \tag{A.3}$$

where $\bar{h} : \mathbf{R}_+ \to \mathbf{R}_+$ *called the base-line hazard function is a non-negative deterministic function with respect to time* t *and* $\mathbf{w}$ *is a coefficient vector of* $\mathbf{X}_t$ *on* $\mathbf{R}^m$*.*

A.2 Valuation of Event Risk

This section briefly reviews three examples of event risk valuations in past finance literature; i.e., credit risk, prepayment risk, and withdrawal risk. In this section, it is not necessarily assumed that the intensity process $(h_t)_{t\geq 0}$ follows PHM. Hereafter, we consider on $\{\tau > t_0\}$ for simplicity of notations, where $t_0 \geq 0$ denotes the current time.

A.2.1 Credit Risk Valuation

One of the most important event risk in finance is a credit risk. Therefore, suppose that τ is the random default time of a given firm in this subsection.

First, let us consider a defaultable zero coupon bond with fixed recovery paid at the maturity. In the framework of reduced-form approach, the defaultable bond price with maturity T at time $t_0 \in [0,T]$ denoted by $B^{\mathrm{m}}(t_0,T)$ can be written as

$$\begin{aligned} B^{\mathrm{m}}(t_0,T) &= \mathbb{E}\left[e^{-\int_{t_0}^T r_s ds}\mathbf{1}_{\{\tau>T\}} \mid \mathcal{G}_{t_0}\right] + \mathbb{E}\left[\delta e^{-\int_{t_0}^T r_s ds}\mathbf{1}_{\{\tau\le T\}} \mid \mathcal{G}_{t_0}\right] \\ &= \mathbb{E}\left[e^{-\int_{t_0}^T (r_s+h_s) ds} \mid \mathcal{F}_{t_0}\right] + \mathbb{E}\left[\delta e^{-\int_{t_0}^T r_s ds}\left(1-e^{-\int_{t_0}^T h_s ds}\right) \mid \mathcal{F}_{t_0}\right] \\ &= (1-\delta)B(t_0,T)\mathbb{E}^T\left[e^{-\int_{t_0}^T h_s ds} \mid \mathcal{F}_{t_0}\right] + \delta B(t_0,T), \end{aligned}$$

where $\mathbb{E}[\,\cdot\,]$ denotes the expectation under the spot measure $\mathbb{Q}$, $\mathbb{E}^T[\,\cdot\,]$ denotes the expectation under a forward measure $\mathbb{Q}_T$ in which the numéraire is a risk-free discount bond $B(t_0,T)$ expiring at time T, and $\delta \in [0,1]$ is a constant recovery rate.

Next, we consider a defaultable zero coupon bond with fixed recovery paid at the default. The defaultable bond price with maturity T at time $t_0 \in [0,T]$ denoted by $B^{\mathrm{d}}(t_0,T)$ is given by

$$\begin{aligned} B^{\mathrm{d}}(t_0,T) &= \mathbb{E}\left[e^{-\int_{t_0}^T r_s ds}\mathbf{1}_{\{\tau>T\}} \mid \mathcal{G}_{t_0}\right] + \mathbb{E}\left[\delta e^{-\int_{t_0}^\tau r_s ds}\mathbf{1}_{\{\tau\le T\}} \mid \mathcal{G}_{t_0}\right] \\ &= \mathbb{E}\left[e^{-\int_{t_0}^T (r_s+h_s) ds} \mid \mathcal{F}_{t_0}\right] + \mathbb{E}\left[\int_{t_0}^T \delta h_u e^{-\int_{t_0}^u (r_s+h_s)ds} du \mid \mathcal{F}_{t_0}\right] \\ &= B(t_0,T)\mathbb{E}^T\left[e^{-\int_{t_0}^T h_s ds} \mid \mathcal{F}_{t_0}\right] + \delta \int_{t_0}^T B(t_0,u)\mathbb{E}^u\left[h_u e^{-\int_{t_0}^u h_s ds} \mid \mathcal{F}_{t_0}\right] du. \end{aligned}$$

The next example is a credit default swap (CDS). The time-t_0 ($\in [0,T]$) value of the fixed-leg of CDS with continuous premium rate c and maturity T can be written as

$$\begin{aligned} \text{Fixed-Leg} &= \mathbb{E}\left[\int_{t_0}^T c e^{-\int_{t_0}^u r_s ds}\mathbf{1}_{\{\tau>u\}} du \mid \mathcal{G}_{t_0}\right] = \mathbb{E}\left[\int_{t_0}^T c e^{-\int_{t_0}^u (r_s+h_s)ds} du \mid \mathcal{F}_{t_0}\right] \\ &= c\int_{t_0}^T B(t_0,u)\mathbb{E}^u\left[e^{-\int_{t_0}^u h_s ds} \mid \mathcal{F}_{t_0}\right] du. \end{aligned}$$

On the other hand, the time-t_0 value of the floating-leg can be written as

$$\begin{aligned} \text{Floating-Leg} &= \mathbb{E}\left[(1-\delta)e^{-\int_{t_0}^\tau r_s ds}\mathbf{1}_{\{\tau\le T\}} \mid \mathcal{G}_{t_0}\right] = \mathbb{E}\left[\int_{t_0}^T (1-\delta)h_u e^{-\int_{t_0}^u (r_s+h_s)ds} du \mid \mathcal{F}_{t_0}\right] \\ &= (1-\delta)\int_{t_0}^T B(t_0,u)\mathbb{E}^u\left[h_u e^{-\int_{t_0}^u h_s ds} \mid \mathcal{F}_{t_0}\right] du. \end{aligned}$$

The CDS premium is chosen to equate the fixed-leg and the floating-leg, thus it can be calculated from

$$c = (1-\delta)\frac{\int_{t_0}^T B(t_0,u)\mathbb{E}^u\left[h_u e^{-\int_{t_0}^u h_s ds} \mid \mathcal{F}_{t_0}\right] du}{\int_{t_0}^T B(t_0,u)\mathbb{E}^u\left[e^{-\int_{t_0}^u h_s ds} \mid \mathcal{F}_{t_0}\right] du}.$$

A.2.2 Prepayment Risk Valuation

Typical examples of prepayment risk valuations are to price residential mortgage-backed securities and mortgage contracts. Therefore, the random time τ is regarded as prepayment timing of a mortgage contract in this subsection.

First, a discrete cash flow payment model of a mortgage contract is considered in accordance with Ozeki et al. [2009]. Let us suppose a fully amortized mortgage contract with fixed coupon rate c and maturity T in which the borrower pays a discrete stream of constant amount of cash flows including both the coupon amounts and the principal payments. The cash flow timings of the mortgage contract are $t_j = j/m$, $j = 1, \ldots, mT$. Then, the present value of the mortgage at time $t_0 \in [0, T]$ denoted by $M^{\mathrm{d}}(t_0)$ can be written as

$$\begin{aligned} M^{\mathrm{d}}(t_0) &= \sum_{j=j_0}^{mT} \left\{ \left(1+\frac{c}{m}\right) P(t_{j-1}) \mathbb{E}\left[e^{-\int_{t_0}^{t_j} r_s ds} \mathbf{1}_{\{\tau > t_{j-1}\}} \mid \mathcal{G}_{t_0} \right] - P(t_j) \mathbb{E}\left[e^{-\int_{t_0}^{t_j} r_s ds} \mathbf{1}_{\{\tau > t_j\}} \mid \mathcal{G}_{t_0} \right] \right\} \\ &= \sum_{j=j_0}^{mT} \left\{ \left(1+\frac{c}{m}\right) P(t_{j-1}) \mathbb{E}\left[e^{-\int_{t_0}^{t_j} r_s ds} e^{-\int_{t_0}^{t_{j-1}} h_s ds} \mid \mathcal{F}_{t_0} \right] - P(t_j) \mathbb{E}\left[e^{-\int_{t_0}^{t_j} (r_s + h_s) ds} \mid \mathcal{F}_{t_0} \right] \right\} \\ &= \sum_{j=j_0}^{mT} B(t_0, t_j) \left\{ \left(1+\frac{c}{m}\right) P(t_{j-1}) \mathbb{E}^{t_j}\left[e^{-\int_{t_0}^{t_{j-1}} h_s ds} \mid \mathcal{F}_{t_0} \right] - P(t_j) \mathbb{E}^{t_j}\left[e^{-\int_{t_0}^{t_j} h_s ds} \mid \mathcal{F}_{t_0} \right] \right\}, \end{aligned}$$

where $j_0 := \inf\{j : t_j > t_0\}$, and $P(t_j)$ is the remaining principal at time t_j in the absence of prepayment. Thus it is given by

$$P(t_j) = P(0) \frac{(1+c/m)^{mT} - (1+c/m)^{mt_j}}{(1+c/m)^{mT} - 1},$$

where $P(0)$ is the initial face amount of the mortgage contract.

Next, let us review a continuous cash flow payment model of the mortgage contract presented in Gorovoy and Linetsky [2007]. Consider a fully amortized mortgage contract with maturity T and continuous coupon stream at the fixed rate A including both the coupon amounts and the principal payments. We denote by c the coupon rate expressed in percent per annum. Then, the present value of the mortgage at time $t_0 \in [0, T]$ denoted by $M^{\mathrm{c}}(t_0)$ can be written as

$$\begin{aligned} M^{\mathrm{c}}(t_0) &= \mathbb{E}\left[\int_{t_0}^{T} A e^{-\int_{t_0}^{u} r_s ds} \mathbf{1}_{\{\tau > t_u\}} du \mid \mathcal{G}_{t_0} \right] + \mathbb{E}\left[e^{-\int_{t_0}^{\tau} r_s ds} P(\tau) \mathbf{1}_{\{\tau \le T\}} \mid \mathcal{G}_{t_0} \right] \\ &= \mathbb{E}\left[\int_{t_0}^{T} A e^{-\int_{t_0}^{u} (r_s + h_s) ds} du \mid \mathcal{F}_{t_0} \right] + \mathbb{E}\left[\int_{t_0}^{T} P(u) h_u e^{-\int_{t_0}^{u} (r_s + h_s) ds} du \mid \mathcal{F}_{t_0} \right] \\ &= A \int_{t_0}^{T} B(t_0, u) \mathbb{E}^{u}\left[e^{-\int_{t_0}^{u} h_s ds} \mid \mathcal{F}_{t_0} \right] du + \int_{t_0}^{T} P(u) B(t_0, u) \mathbb{E}^{u}\left[h_u e^{-\int_{t_0}^{u} h_s ds} \mid \mathcal{F}_{t_0} \right] du, \end{aligned}$$

where $P(t)$ is remaining principal at time t in the absence of prepayment. It is well known that A and $P(t)$ are given by

$$A = \frac{mP(0)}{1 - e^{-cT}}, \quad P(t) = P(0) \frac{1 - e^{-c(T-t)}}{1 - e^{-cT}},$$

where $P(0)$ is the initial face amount of the mortgage contract.

A.2.3 Withdrawal Risk Valuation

Since retail depositors can withdraw time deposits and saving accounts anytime without any penalty, it is substantially important for commercial banks to evaluate appropriate economic values of such deposits in terms of interest rate risk management. As an example, Aonuma and Kijima [1998] estimated withdrawing behaviors of Japanese retail depositors having time deposits by using PHM and evaluated withdrawal risk of the deposits as a certain American option. Hence, the random time τ is regarded as withdrawal timing of such deposits in this subsection.

First, let us consider a saving account with zero coupon rate and face amount D. This type of deposits is known as "non-maturity deposits" because they do not have predetermined maturity. The present value of the saving account at time $t_0 \in [0, T]$ denoted by $D^{\mathrm{s}}(t_0)$ can be written as

$$\begin{aligned} D^{\mathrm{s}}(t_0) &= \mathbb{E}\left[De^{-\int_{t_0}^{\tau} r_s ds} \mid \mathcal{G}_{t_0}\right] = \mathbb{E}\left[\int_{t_0}^{\infty} Dh_u e^{-\int_{t_0}^{u}(r_s+h_s)ds} du \mid \mathcal{F}_{t_0}\right] \\ &= D\int_{t_0}^{\infty} B(t_0,u)\mathbb{E}^u\left[h_u e^{-\int_{t_0}^{u} h_s ds} \mid \mathcal{F}_{t_0}\right] du. \end{aligned}$$

Next, we consider a time deposit with face amount D, fixed coupon rate c, and maturity T. The depositor receives a discrete stream of the coupon amount until either withdrawing the deposit or maturity. The coupon payment timings of the deposit are $t_j = j/m$, $j = 1, \dots, mT$. Then, the present value of the time deposit at time $t_0 \in [0, T]$ denoted by $D^{\mathrm{t}}(t_0)$ can be written as

$$\begin{aligned} D^{\mathrm{t}}(t_0) &= \sum_{j=j_0}^{mT} \mathbb{E}\left[\frac{c}{m} De^{-\int_{t_0}^{t_j} r_s ds} \mathbf{1}_{\{\tau > t_j\}} \mid \mathcal{G}_{t_0}\right] + \mathbb{E}\left[De^{-\int_{t_0}^{\tau} r_s ds}\mathbf{1}_{\{\tau \le T\}} \mid \mathcal{G}_{t_0}\right] \\ &= \sum_{j=j_0}^{mT} \mathbb{E}\left[\frac{c}{m} De^{-\int_{t_0}^{t_j}(r_s+h_s)ds} \mid \mathcal{F}_{t_0}\right] + \mathbb{E}\left[\int_{t_0}^{T} Dh_u e^{-\int_{t_0}^{u}(r_s+h_s)ds} du \mid \mathcal{F}_{t_0}\right] \\ &= \frac{c}{m} D \sum_{j=j_0}^{mT} B(t_0,t_j)\mathbb{E}^{t_j}\left[e^{-\int_{t_0}^{t_j} h_s ds} \mid \mathcal{F}_{t_0}\right] + D\int_{t_0}^{T} B(t_0,u)\mathbb{E}^u\left[h_u e^{-\int_{t_0}^{u} h_s ds} \mid \mathcal{F}_{t_0}\right] du. \end{aligned}$$

A.3 Formulas for Evaluating Event Risk with PHM

This section provides fundamental formulas for evaluating event risk with PHM. According to the discussion in the previous section, in order to evaluate various types of event risk, one has only to calculate the following two equations:

$$\mathbb{E}^U\left[\exp\left\{-\int_{t_0}^{t} h_s ds\right\} \mid \mathcal{F}_{t_0}\right] \quad \text{and} \quad \mathbb{E}^U\left[h_t \exp\left\{-\int_{t_0}^{t} h_s ds\right\} \mid \mathcal{F}_{t_0}\right], \tag{A.4}$$

which are the survival probability and the default probability, respectively, in the context of credit risk modeling. Here, $\mathbb{E}^U[\,\cdot\,]$ denotes an expectation operator under a forward measure $\mathbb{Q}_U$ in which the numéraire is a discount bond $B(t_0, U)$ expiring at time $U\,(\ge t_0)$. Before providing the valuation formulas of Eq.(A.4), let us prove the following technical lemma.

Lemma A.3 *Let $g : \mathbf{R}_+ \to \mathbf{R}$ be an integrable function and*

$$G(x) := \int_{\alpha}^{x} g(u)du,$$

where α is an arbitrary non-negative constant. Then for all $n \in \mathbf{N}$,

$$G(x)^n = n! \int_{\alpha}^{x}\int_{\alpha}^{u_n} \cdots \int_{\alpha}^{u_2} g(u_n)g(u_{n-1})\cdots g(u_1)du_1 du_2 \cdots du_n. \tag{A.5}$$

Theorem A.4 *Suppose that any moments of the cumulative hazard rate $\Gamma := \int_{t_0}^{t} h_s ds$ exist under $\mathbb{Q}_U$. Then, under Assumption A.1 and A.2 it holds*

$$\mathbb{E}^U\left[\exp\left\{-\int_{t_0}^{t} h_s ds\right\} \mid \mathcal{F}_{t_0}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_n \phi_G(i), \tag{A.6}$$

where $\phi_G(\theta)$ is the characteristic function of an arbitrary random variable G that has any moments under $\mathbb{Q}_U$, $i := \sqrt{-1}$, the coefficients C_n is given by

$$\begin{aligned}
C_0 &:= 1, \\
C_1 &:= c_1(\Gamma) - c_1(G), \\
C_2 &:= c_2(\Gamma) - c_2(G) + (c_1(\Gamma) - c_1(G))^2, \\
C_3 &:= c_3(\Gamma) - c_3(G) + 3(c_1(\Gamma) - c_1(G))(c_2(\Gamma) - c_2(G)) + (c_1(\Gamma) - c_1(G))^3, \\
C_4 &:= c_4(\Gamma) - c_4(G) + 4(c_1(\Gamma) - c_1(G))(c_3(\Gamma) - c_3(G)) + 3(c_2(\Gamma) - c_2(G))^2 \\
&\quad + 6(c_1(\Gamma) - c_1(G))^2(c_2(\Gamma) - c_2(G)) + (c_1(\Gamma) - c_1(G))^4, \\
&\quad \cdots\cdots\cdots .
\end{aligned}$$

Here, $c_n(Z)$ denotes the n-th cumulant of a random variable Z, that is,

$$\begin{aligned}
c_1(Z) &:= m_1(Z), \\
c_2(Z) &:= m_2(Z) - m_1(Z)^2, \\
c_3(Z) &:= m_3(Z) - 3m_1(Z)m_2(Z) + 2m_1(Z)^3, \\
c_4(Z) &:= m_4(Z) - 4m_1(Z)m_3(Z) - 3m_2(Z)^2 + 12m_1(Z)^2 m_2(Z) - 6m_1(Z)^4, \\
&\quad \cdots\cdots\cdots .
\end{aligned}$$

where $m_n(Z)$ is the n-th moment of Z under $\mathbb{Q}_U$. Moreover, the n-th moment $m_n(\Gamma)$ is given by

$$m_n(\Gamma) = n! \int_{t_0}^{t} \int_{t_0}^{t_n} \cdots \int_{t_0}^{t_2} \prod_{k=1}^{n} \bar{h}(t_k) \mathbb{E}^U\left[\exp\left\{\sum_{k=1}^{n} \mathbf{w}^\top \mathbf{X}_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] dt_1 dt_2 \cdots dt_n. \tag{A.7}$$

Proof of Theorem A.4: First, we shall prove Eq.(A.6). Let $\phi_Z(\theta) := \mathbb{E}^U[e^{i\theta Z}|\mathcal{F}_{t_0}]$ be the characteristic function of a random variable Z. Recall that the cumulant expansion of Z is given by

$$\ln \phi_Z(\theta) = \sum_{n=1}^{\infty} \frac{(i\theta)^n}{n!} c_n(Z).$$

By applying the cumulant expansion to the random variables Γ and G, we obtain the following equation:

$$\ln \frac{\phi_\Gamma(\theta)}{\phi_G(\theta)} = \sum_{n=1}^{\infty} \frac{(i\theta)^n}{n!} (c_n(\Gamma) - c_n(G)). \tag{A.8}$$

From Eq.(A.8), we have

$$\begin{aligned}\phi_\Gamma(\theta) &= \exp\left\{\sum_{n=1}^{\infty}\frac{(i\theta)^n}{n!}(c_n(\Gamma)-c_n(G))\right\}\phi_G(\theta)\\ &= \left\{1+\sum_{k=1}^{\infty}\frac{1}{k!}\left[\sum_{n=1}^{\infty}\frac{(i\theta)^n}{n!}(c_n(\Gamma)-c_n(G))\right]^k\right\}\phi_G(\theta)\\ &= \left\{1+(c_1(\Gamma)-c_1(G))(i\theta)+\frac{c_2(\Gamma)-c_2(G)+(c_1(\Gamma)-c_1(G))^2}{2!}(i\theta)^2+\cdots\right\}\phi_G(\theta)\\ &= \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}C_n\phi_G(\theta).\end{aligned}$$

Setting $\theta = i$ in the above equation, Eq.(A.6) can be obtained.

Next, consider n-th moment of Γ; i.e.,

$$m_n(\Gamma) := \mathbb{E}^U\left[\Gamma^n \mid \mathcal{F}_{t_0}\right] = \mathbb{E}^U\left[\left(\int_{t_0}^{t} h_s ds\right)^n \mid \mathcal{F}_{t_0}\right]. \tag{A.9}$$

Applying Lemma A.3 to Eq.(A.9), we have

$$\begin{aligned}\mathbb{E}^U\left[\left(\int_{t_0}^{t} h_s ds\right)^n \mid \mathcal{F}_{t_0}\right] &= \mathbb{E}^U\left[n!\int_{t_0}^{t}\int_{t_0}^{t_n}\cdots\int_{t_0}^{t_2} h_{t_n}h_{t_{n-1}}\cdots h_{t_1}dt_1dt_2\cdots dt_n \mid \mathcal{F}_{t_0}\right]\\ &= n!\int_{t_0}^{t}\int_{t_0}^{t_n}\cdots\int_{t_0}^{t_2}\mathbb{E}^U\left[\prod_{k=1}^{n} h_{t_k} \mid \mathcal{F}_{t_0}\right]dt_1dt_2\cdots dt_n.\end{aligned}$$

From Assumption A.2, we have

$$\begin{aligned}\mathbb{E}^U\left[\prod_{k=1}^{n} h_{t_k} \mid \mathcal{F}_{t_0}\right] &= \mathbb{E}^U\left[\prod_{k=1}^{n}\bar{h}(t_k)\exp\left\{\mathbf{w}^\top\mathbf{X}_{t_k}\right\} \mid \mathcal{F}_{t_0}\right]\\ &= \prod_{k=1}^{n}\bar{h}(t_k)\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^{n}\mathbf{w}^\top\mathbf{X}_{t_k}\right\} \mid \mathcal{F}_{t_0}\right].\end{aligned}$$

Therefore, Eq.(A.7) can be obtained. The proof of Theorem A.4 is completed. □

Theorem A.5 *Under the same condition as Theorem A.4, it holds*

$$\mathbb{E}^U\left[h_t\exp\left\{-\int_{t_0}^{t} h_s ds\right\} \mid \mathcal{F}_{t_0}\right] = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{dC_n}{dt}\phi_G(i). \tag{A.10}$$

Proof of Theorem A.5: Since

$$\begin{aligned}\mathbb{E}^U\left[h_t\exp\left\{-\int_{t_0}^{t} h_s ds\right\} \mid \mathcal{F}_{t_0}\right] &= -\mathbb{E}^U\left[\frac{d}{dt}\exp\left\{-\int_{t_0}^{t} h_s ds\right\} \mid \mathcal{F}_{t_0}\right]\\ &= -\frac{d}{dt}\mathbb{E}^U\left[\exp\left\{-\int_{t_0}^{t} h_s ds\right\} \mid \mathcal{F}_{t_0}\right],\end{aligned}$$

by applying Theorem A.4 to the right hand side of the above equation, Eq.(A.10) can be obtained. □

Note that $\frac{dC_n}{dt}$ in Eq.(A.10) can be easily calculated by using the following equation:

$$\begin{aligned}\frac{d}{dt}m_n(\Gamma) &= n!\,\bar{h}(t)\int_{t_0}^{t}\int_{t_0}^{t_{n-1}}\cdots\int_{t_0}^{t_2}\prod_{k=1}^{n-1}\bar{h}(t_k)\\ &\quad\times \mathbb{E}^U\left[\exp\left\{\sum_{k=1}^{n-1}\mathbf{w}^\top\mathbf{X}_{t_k}+\mathbf{w}^\top\mathbf{X}_t\right\}\mid\mathcal{F}_{t_0}\right]dt_1dt_2\cdots dt_{n-1}. \end{aligned}\tag{A.11}$$

As a result of the discussion in this section, the valuation problems of event risk we consider are eventually reduced to the calculation problem of the following equation:

$$\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^{n}\mathbf{w}^\top\mathbf{X}_{t_k}\right\}\mid\mathcal{F}_{t_0}\right]. \tag{A.12}$$

Note that, when covariate vector $\mathbf{X}_t$ of PHM can be decomposed into some independent vectors, it is sufficient to evaluate Eq.(A.12) in terms of each independent vector in $\mathbf{X}_t$. If an analytical expression of Eq.(A.12) is obtained, the event risk valuations with PHM can be approximately computed by a suitable numerical method of the iterated integrals in Eq.(A.7) and Eq.(A.11). In some cases, closed-form expressions of the iterated integrals can be obtained as well.

A.4 Continuous Processes as Covariates of PHM

A.4.1 Gaussian Processes

Proposition A.6 *Suppose that the covariate vector* $\mathbf{X}_t := (X_t^1, X_t^2, \ldots, X_t^m)$ *of PHM follows an* m*-dimensional Gaussian process under a forward measure* $\mathbb{Q}_U$*. Then, it satisfies*

$$\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^{n}\mathbf{w}^\top\mathbf{X}_{t_k}\right\}\mid\mathcal{F}_{t_0}\right] = \exp\left\{\mu+\frac{v}{2}\right\},$$

where

$$\begin{aligned}\mu &:= \sum_{k=1}^{n}\sum_{j=1}^{m}w_j\mathbb{E}^U\left[X_{t_k}^j\mid\mathcal{F}_{t_0}\right],\\ v &:= \sum_{k_1,k_2}^{n}\sum_{j_1,j_2}^{m}w_{j_1}w_{j_2}\mathrm{Cov}^U\left[X_{t_{k_1}}^{j_1},X_{t_{k_2}}^{j_2}\mid\mathcal{F}_{t_0}\right],\end{aligned}$$

and w_j, $j=1,2,\ldots,m$ *is the* j*-the component of the coefficient vector* $\mathbf{w}$.

Proof of Proposition A.6: This statement is trivial. □

As an example of Proposition A.6, let us consider the generalized Ornstein-Uhlenbeck (OU) process. This example is quoted from Ozeki, et al. [2009] evaluating residential mortgage-backed securities in the same setting. Suppose that $(\mathbf{X}_t)_{t\geq 0}$ is an m-dimensional generalized OU process under the spot measure $\mathbb{Q}$, i.e., the covariate vector $\mathbf{X}_t = (X_t^1, X_t^2, \ldots, X_t^m)$ of PHM is given by

$$dX_t^j = (\xi_j(t)-a_jX_t^j)dt+\mathbf{b}_j^\top d\mathbf{W}_t, \quad j=1,2,\ldots,m, \tag{A.13}$$

where $X_t^1 := r_t$, $(\mathbf{W}_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion under $\mathbb{Q}$, $\xi_j(t)$ is a deterministic function with respect to time t, $\mathbf{b}_j$ is a $\mathbf{R}^d$-constant vector, and a_j is constant. Note that the spot rate process $(r_t)_{t\geq 0}$ is well-known as the Hull-White model (Hull and White [1990]).

Under a forward measure $\mathbb{Q}_U$, the SDE (A.13) is transformed into

$$dX_t^j = (\xi_j^U(t) - a_j X_t^j)dt + \mathbf{b}_j^\top d\mathbf{W}_t^U, \quad j = 1, 2, \ldots, m, \tag{A.14}$$

where

$$\xi_j^U(t) := \xi_j(t) - \frac{1 - e^{-a_j(U-t)}}{a_j} \mathbf{b}_1^\top \mathbf{b}_j,$$

and $(\mathbf{W}_t^U)_{t \geq 0}$ is a d-dimensional standard Brownian motion under the forward measure $\mathbb{Q}_U$. The solution of Eq.(A.14) is given by

$$X_t^j = x^j e^{-a_j t} + \int_{t_0}^t \xi_j^U(s) e^{-a_j(t-s)} ds + \mathbf{b}_j^\top \int_{t_0}^t e^{-a_j(t-s)} d\mathbf{W}_s^U, \quad j = 1, 2, \ldots, m,$$

where $x^j := X_{t_0}^j \in \mathbf{R}$. Thus, we have

$$\mathbb{E}^U \left[X_t^j \mid \mathcal{F}_{t_0} \right] = x^j e^{-a_j t} + \int_{t_0}^t \xi_j^U(s) e^{-a_j(t-s)} ds,$$

and as $t_1 \geq t_2$,

$$\mathrm{Cov}^U \left[X_{t_1}^{j_1}, X_{t_2}^{j_2} \mid \mathcal{F}_{t_0} \right] = \frac{\mathbf{b}_{j_1}^\top \mathbf{b}_{j_2}}{a_{j_1} + a_{j_2}} \left[e^{-a_{j_1}(t_1 - t_2)} - e^{-a_{j_1}(t_1 - t_0) - a_{j_2}(t_2 - t_0)} \right].$$

Therefore, the explicit expression of Eq.(A.12) can be obtained from Proposition A.6 in the case that the generalized OU process is set as the covariate vector of PHM. Note that in the one-dimensional case the hazard rate h_t is reduced to the Black-Karasinski model (Black and Karasinski [1991]), which has been applied in much literature dealing with credit valuations (e.g., see Chu and Kwok [2003], Berndt et al. [2005], Pan and Singleton [2008]) and it is the simplest model in our setting.

A.4.2 Affine Processes

Let $(\mathbf{X}_t)_{t \geq 0}$ be a m-dimensional Markov process that starts at $\mathbf{x}_0$ and satisfies the following SDE:

$$d\mathbf{X}_t = \mu(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t) d\mathbf{W}_t^U, \tag{A.15}$$

where $(\mathbf{W}_t^U)_{t \geq 0}$ is a m-dimensional Brownian motion under a forward measure $\mathbb{Q}_U$. It is assumed that the $m \times 1$ vector $\mu(\mathbf{X}_t)$ and $m \times m$ matrix $\sigma(\mathbf{X}_t)$ satisfy some technical conditions such that the SDE (A.15) has a unique strong solution.

The affine process is defined as the SDE (A.15) having

$$\begin{aligned} &\mu(\mathbf{x}) = K_0 + K_1 \mathbf{x}, \quad K_0 \in \mathbf{R}^m, K_1 \in \mathbf{R}^{m \times m}, \\ &[\sigma(\mathbf{x})\sigma(\mathbf{x})^\top]_{ij} = (H_0)_{ij} + (H_1)_{ij}^\top \mathbf{x}, \quad H_0 \in \mathbf{R}^{m \times m}, H_1 \in \mathbf{R}^{m \times m \times m}. \end{aligned}$$

The following lemma is developed by the original work of Duffie and Kan [1996] for the affine term structure models of interest rates, and its extension to compound Poisson-type jumps is due to Duffie at al. [2000]. This section, however, does not deal with any jumps because the next section will focus on various types of jump processes including the compound Poisson process.

Lemma A.7 *Let $(\mathbf{X}_t)_{t\geq 0}$ be an m-dimensional affine process under $\mathbb{Q}_U$, and $V_t = \rho_0 + \rho_1^\top \mathbf{X}_t$ $\rho_0 \in \mathbf{R}, \rho_1 \in \mathbf{R}^m$. Define for any $\theta \in \mathbf{R}^m$*

$$\Phi^U(\theta, \mathbf{X}_t, t, T) = \mathbb{E}^U\left[\exp\left\{-\int_t^T V_s ds\right\} e^{\theta^\top \mathbf{X}_T} \mid \mathcal{F}_t\right].$$

Then, it satisfies

$$\Phi^U(\theta, \mathbf{x}, t, T) = e^{\alpha_T(t)+\beta_T(t)^\top \mathbf{x}}, \tag{A.16}$$

where $\alpha_T : \mathbf{R}_+ \to \mathbf{R}$ and $\beta_T : \mathbf{R}_+ \to \mathbf{R}^m$ satisfy the following ODEs

$$\frac{d}{dt}\beta_T(t) = \rho_1 - K_1^\top \beta_T(t) - \frac{1}{2}\beta_T(t)^\top H_1 \beta_T(t), \tag{A.17}$$

$$\frac{d}{dt}\alpha_T(t) = \rho_0 - K_0^\top \beta_T(t) - \frac{1}{2}\beta_T(t)^\top H_0 \beta_T(t), \tag{A.18}$$

with boundary conditions $\alpha_T(T) = 0$ and $\beta_T(T) = \theta$.

Proposition A.8 *Suppose that the covariate vector $\mathbf{X}_t := (X_t^1, X_t^2, \ldots, X_t^m)$ of PHM follows an m-dimensional affine process under a forward measure $\mathbb{Q}_U$. Then, it satisfies*

$$\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^n \mathbf{w}^\top \mathbf{X}_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] = \exp\left\{\sum_{k=1}^n \alpha_{t_{n+1-k}}(t_{n-k}) + \beta_{t_1}(t_0)^\top \mathbf{x}\right\}, \tag{A.19}$$

where $\mathbf{x} := \mathbf{X}_{t_0}$, and $\alpha_{t_k} : \mathbf{R}_+ \to \mathbf{R}$ and $\beta_{t_k} : \mathbf{R}_+ \to \mathbf{R}^m$ are recursively defined by the following ODEs:

$$\frac{d}{dt}\beta_{t_k}(t) = -K_1^\top \beta_{t_k}(t) - \frac{1}{2}\beta_{t_k}(t)^\top H_1 \beta_{t_k}(t), \tag{A.20}$$

$$\frac{d}{dt}\alpha_{t_k}(t) = -K_0^\top \beta_{t_k}(t) - \frac{1}{2}\beta_{t_k}(t)^\top H_0 \beta_{t_k}(t), \tag{A.21}$$

with boundary conditions $\alpha_{t_k}(t_k) = 0$ for $k = 1, \ldots, n$, $\beta_{t_n}(t_n) = \mathbf{w}$, and $\beta_{t_k}(t_k) = \mathbf{w} + \beta_{t_{k+1}}(t_k)$ for $k = 1, \ldots, n-1$.

Proof of Proposition A.8: Since $t_0 \leq t_1 \leq \cdots \leq t_n$, applying the law of iterated expectations and

Lemma A.7 to the left hand side of Eq.(A.19), we have

$$
\begin{aligned}
&\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^{n}\mathbf{w}^\top\mathbf{X}_{t_k}\right\} \mid \mathcal{F}_{t_0}\right]\\
&= \mathbb{E}^U\left[\exp\left\{\sum_{k=1}^{n-1}\mathbf{w}^\top\mathbf{X}_{t_k}\right\}\mathbb{E}^U\left[e^{\mathbf{w}^\top\mathbf{X}_{t_n}} \mid \mathcal{F}_{t_{n-1}}\right] \mid \mathcal{F}_{t_0}\right]\\
&= \mathbb{E}^U\left[\exp\left\{\sum_{k=1}^{n-1}\mathbf{w}^\top\mathbf{X}_{t_k}\right\}\exp\left\{\alpha_{t_n}(t_{n-1})+\beta_{t_n}(t_{n-1})^\top\mathbf{X}_{t_{n-1}}\right\} \mid \mathcal{F}_{t_0}\right]\\
&= e^{\alpha_{t_n}(t_{n-1})}\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^{n-2}\mathbf{w}^\top\mathbf{X}_{t_k}\right\}\mathbb{E}^U\left[e^{(\mathbf{w}+\beta_{t_n}(t_{n-1}))^\top\mathbf{X}_{t_{n-1}}} \mid \mathcal{F}_{t_{n-2}}\right] \mid \mathcal{F}_{t_0}\right]\\
&= e^{\alpha_{t_n}(t_{n-1})}\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^{n-2}\mathbf{w}^\top\mathbf{X}_{t_k}\right\}\exp\left\{\alpha_{t_{n-1}}(t_{n-2})+\beta_{t_{n-1}}(t_{n-2})^\top\mathbf{X}_{t_{n-2}}\right\} \mid \mathcal{F}_{t_0}\right]\\
&= \cdots\cdots\cdots\\
&= \exp\left\{\sum_{k=1}^{n}\alpha_{t_{n+1-k}}(t_{n-k})+\beta_{t_1}(t_0)^\top\mathbf{X}_{t_0}\right\}.
\end{aligned}
$$

Here, $\alpha_{t_k}(t)$ and $\beta_{t_k}(t)$ satisfy the ODEs (A.20) and (A.21) with boundary conditions $\alpha_{t_k}(t_k)=0$ for $k=1,\ldots,n$, and $\beta_{t_n}(t_n)=\mathbf{w}$ and $\beta_{t_k}(t_k)=\mathbf{w}+\beta_{t_{k+1}}(t_k)$ for $k=1,\ldots,n-1$. □

A.4.3 Quadratic Gaussian Processes

Let $(\mathbf{Z}_t)_{t\geq 0}$ be a m-dimensional OU process, i.e.,

$$d\mathbf{Z}_t = -(\mathbf{b}_Z + K\mathbf{Z}_t)dt + d\mathbf{W}_t^U, \tag{A.22}$$

where $\mathbf{b}_Z$ is a vector on $\mathbf{R}^m$, K is a matrix on $\mathbf{R}^{m\times m}$, and $(\mathbf{W}_t^U)_{t\geq 0}$ is a m-dimensional Brownian motion under a forward measure $\mathbb{Q}_U$. The quadratic Gaussian process is a one-dimensional process defined as the following form:

$$\mathbf{Z}_t^\top A\mathbf{Z}_t + \mathbf{b}^\top\mathbf{Z}_t + c. \tag{A.23}$$

Here, A is a $m\times m$ matrix, $\mathbf{b}$ is a m-dimensional vector, and c is a scalar.

The following lemma is developed by Leippold and Wu [2002] for asset pricing under the quadratic Gaussian class. The proof of this lemma can be found in Appendix C of Leippold and Wu [2002].

Lemma A.9 *Let $(Y_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ be quadratic Gaussian processes under $\mathbb{Q}_U$ such that*

$$
\begin{aligned}
Y_t &= \mathbf{Z}_t^\top A_Y\mathbf{Z}_t + \mathbf{b}_Y^\top\mathbf{Z}_t + c_Y,\\
V_t &= \mathbf{Z}_t^\top A_V\mathbf{Z}_t + \mathbf{b}_V^\top\mathbf{Z}_t + c_V,
\end{aligned}
$$

for any $t\geq 0$, where $A_Y, A_V\in\mathbf{R}^{m\times m}$, $\mathbf{b}_Y,\mathbf{b}_V\in\mathbf{R}^m$, and $c_Y, c_V\in\mathbf{R}$. Define

$$\Psi^U(\mathbf{Z}_t,t,T) = \mathbb{E}^U\left[\exp\left\{-\int_t^T V_s ds\right\}e^{-Y_T} \mid \mathcal{F}_t\right].$$

Then, it satisfies

$$\Psi^U(\mathbf{z},t,T) = \exp\left\{-\mathbf{z}^\top A_T(t)\mathbf{z} - \mathbf{b}_T(t)^\top\mathbf{z} - c_T(t)\right\}, \tag{A.24}$$

where $A_T : \mathbf{R}_+ \to \mathbf{R}^{m\times m}$, $\mathbf{b}_T : \mathbf{R}_+ \to \mathbf{R}^m$, *and* $c_T : \mathbf{R}_+ \to \mathbf{R}$ *satisfy the following ODEs*

$$\begin{aligned}
\frac{d}{dt}A_T(t) &= -A_V + A_T(t)K + K^\top A_T(t) + 2A_T(t)^2, && \text{(A.25)}\\
\frac{d}{dt}\mathbf{b}_T(t) &= -\mathbf{b}_V + 2A_T(t)\mathbf{b}_Z + K^\top\mathbf{b}_T(t) + 2A_T(t)\mathbf{b}_T(t), && \text{(A.26)}\\
\frac{d}{dt}c_T(t) &= -c_V + \mathbf{b}_T(t)^\top\mathbf{b}_Z - \operatorname{tr} A_T(t) + \frac{1}{2}\mathbf{b}_T(t)^\top\mathbf{b}_T(t), && \text{(A.27)}
\end{aligned}$$

with boundary conditions $A_T(T) = A_Y$, $\mathbf{b}_T(T) = \mathbf{b}_Y$, *and* $c_T(T) = c_Y$.

Proposition A.10 *Suppose that the covariate variable* X_t *of PHM follows a quadratic Gaussian process under a forward measure* $\mathbb{Q}_U$ *such that*

$$X_t = \mathbf{Z}_t^\top A_X\mathbf{Z}_t + \mathbf{b}_X^\top\mathbf{Z}_t + c_X, \quad t \geq 0,$$

where $A_X \in \mathbf{R}^{m\times m}$, $\mathbf{b}_X \in \mathbf{R}^m$, *and* $c_X \in \mathbf{R}$. *Then, it satisfies*

$$\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^n wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] = \exp\left\{-\mathbf{z}^\top A_{t_1}(t_0)\mathbf{z} - \mathbf{b}_{t_1}(t_0)^\top\mathbf{z} - c_{t_1}(t_0)\right\}, \tag{A.28}$$

where $\mathbf{z} := \mathbf{Z}_{t_0}$, *and* $A_{t_k} : \mathbf{R}_+ \to \mathbf{R}^{m\times m}$, $\mathbf{b}_{t_k} : \mathbf{R}_+ \to \mathbf{R}^m$, *and* $c_{t_k} : \mathbf{R}_+ \to \mathbf{R}$ *are recursively defined by the following ODEs:*

$$\begin{aligned}
\frac{d}{dt}A_{t_k}(t) &= A_{t_k}(t)K + K^\top A_{t_k}(t) + 2A_{t_k}(t)^2, && \text{(A.29)}\\
\frac{d}{dt}\mathbf{b}_{t_k}(t) &= 2A_{t_k}(t)\mathbf{b}_Z + K^\top\mathbf{b}_{t_k}(t) + 2A_{t_k}(t)\mathbf{b}_{t_k}(t), && \text{(A.30)}\\
\frac{d}{dt}c_{t_k}(t) &= \mathbf{b}_{t_k}(t)^\top\mathbf{b}_Z - \operatorname{tr} A_{t_k}(t) + \frac{1}{2}\mathbf{b}_{t_k}(t)^\top\mathbf{b}_{t_k}(t), && \text{(A.31)}
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
A_{t_k}(t_k) &= -wA_X + A_{t_{k+1}}(t_k), && \text{(A.32)}\\
\mathbf{b}_{t_k}(t_k) &= -w\mathbf{b}_X + \mathbf{b}_{t_{k+1}}(t_k), && \text{(A.33)}\\
c_{t_k}(t_k) &= -wc_X + c_{t_{k+1}}(t_k), && \text{(A.34)}
\end{aligned}$$

for $k = 1,\ldots,n-1$, *and* $A_{t_n}(t_n) = -wA_X$, $\mathbf{b}_{t_n}(t_n) = -w\mathbf{b}_X$, $c_{t_n}(t_n) = -wc_X$.

Proof of Proposition A.10: Since $t_0 \leq t_1 \leq \cdots \leq t_n$, applying the law of iterated expectations to the left hand side of Eq.(A.28), we have

$$\mathbb{E}^U\left[\exp\left\{\sum_{k=1}^n wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] = \mathbb{E}^U\left[\exp\left\{\sum_{k=1}^{n-1} wX_{t_k}\right\}\mathbb{E}^U\left[e^{wX_{t_n}} \mid \mathcal{F}_{t_{n-1}}\right] \mid \mathcal{F}_{t_0}\right]. \tag{A.35}$$

By Lemma A.9, it holds

$$\begin{aligned}
\mathbb{E}^U\left[e^{wX_{t_n}} \mid \mathcal{F}_{t_{n-1}}\right] &= \mathbb{E}^U\left[e^{\mathbf{Z}_{t_n}^\top wA_X\mathbf{Z}_{t_n} + w\mathbf{b}_X^\top\mathbf{Z}_{t_n} + wc_X} \mid \mathcal{F}_{t_{n-1}}\right]\\
&= \exp\left\{-\mathbf{Z}_{t_{n-1}}^\top A_{t_n}(t_{n-1})\mathbf{Z}_{t_{n-1}} - \mathbf{b}_{t_n}(t_{n-1})^\top\mathbf{Z}_{t_{n-1}} - c_{t_n}(t_{n-1})\right\}.
\end{aligned}$$

Here, $A_{t_n}(t)$, $\mathbf{b}_{t_n}(t)$, and $c_{t_n}(t)$ satisfy the ODEs (A.29)-(A.31) with boundary conditions $A_{t_n}(t_n) = -wA_X$, $\mathbf{b}_{t_n}(t_n) = -w\mathbf{b}_X$, $c_{t_n}(t_n) = -wc_X$.

Substituting the above equation into Eq.(A.35), we have

$$\mathbb{E}^U \left[\exp\left\{ \sum_{k=1}^{n} wX_{t_k} \right\} \mid \mathcal{F}_{t_0} \right] = \mathbb{E}^U \left[\exp\left\{ \sum_{k=1}^{n-2} wX_{t_k} \right\} G_{n-2} \mid \mathcal{F}_{t_0} \right], \tag{A.36}$$

where

$$\begin{aligned} G_{n-2} &:= \mathbb{E}^U \left[e^{\mathbf{Z}_{t_{n-1}}^\top (wA_X - A_{t_n}(t_{n-1}))\mathbf{Z}_{t_{n-1}} + (w\mathbf{b}_X - \mathbf{b}_{t_n}(t_{n-1}))^\top \mathbf{Z}_{t_n} + wc_X - c_{t_n}(t_{n-1})} \mid \mathcal{F}_{t_{n-2}} \right] \\ &= \exp\left\{ -\mathbf{Z}_{t_{n-2}}^\top A_{t_{n-1}}(t_{n-2})\mathbf{Z}_{t_{n-2}} - \mathbf{b}_{t_{n-1}}(t_{n-2})^\top \mathbf{Z}_{t_{n-2}} - c_{t_{n-1}}(t_{n-2}) \right\}. \end{aligned}$$

The second equality of the above equation is shown by Lemma A.9, and $A_{t_{n-1}}(t)$, $\mathbf{b}_{t_{n-1}}(t)$, and $c_{t_{n-1}}(t)$ satisfy the ODEs (A.29)-(A.31) with boundary conditions (A.32)-(A.34).

Repeating this procedure, we obtain

$$\mathbb{E}^U \left[\exp\left\{ \sum_{k=1}^{n} wX_{t_k} \right\} \mid \mathcal{F}_{t_0} \right] = G_0 = \exp\left\{ -\mathbf{Z}_{t_0}^\top A_{t_1}(t_0)\mathbf{Z}_{t_0} - \mathbf{b}_{t_1}(t_0)^\top \mathbf{Z}_{t_0} - c_{t_1}(t_0) \right\}.$$

□

A.5 Discontinuous Processes as Covariates of PHM

Through this section we focus on a one-dimensional process X_t in the covariance vector $\mathbf{X}_t$ of PHM, and $(X_t)_{t\geq 0}$ is assumed to be a discontinuous process and independent of both all other component of $(\mathbf{X}_t)_{t\geq 0}$ and the interest rate process $(r_t)_{t\geq 0}$. Under this assumption, we will concentrate on developing the calculation formulas of

$$\mathbb{E} \left[\exp\left\{ \sum_{k=1}^{n} wX_{t_k} \right\} \mid \mathcal{F}_{t_0} \right]. \tag{A.37}$$

Note that, by virtue of the independent assumption, Eq.(A.37) becomes an expectation value under not a forward measure $\mathbb{Q}_U$, but the spot measure $\mathbb{Q}$.

A.5.1 Lévy Processes

In this subsection, suppose that $(X_t)_{t\geq 0}$ follows a one-dimensional Lévy process; i.e., X_t is adapted to $\mathcal{F}_t$, the sample paths of $(X_t)_{t\geq 0}$ are right continuous with left limits, and $X_u - X_t$ is independent of $\mathcal{F}_t$ and has the same distribution as X_{u-t} for $0 \leq t < u$.

First, the Lévy-Khintchine formula, which gives us the explicit representation of the characteristic function of Lévy processes, is provided as the following proposition. The proof of the proposition can be found on pp.35-45 in Sato [1999].

Proposition A.11 *(Lévy-Khintchine formula) Let $(X_t)_{t\geq 0}$ be a Lévy process on* $\mathbf{R}$. *The characteristic function of the distribution of X_t has the form*

$$\phi_{X_t}(\theta) := \mathbb{E}\left[e^{i\theta X_t}\right] = e^{-t\psi_X(\theta)}, \quad t \geq 0, \tag{A.38}$$

where the characteristic exponent $\psi_X(\theta)$, $\theta \in \mathbf{R}$ is given by

$$\psi_X(\theta) = -i\mu\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{\infty}(1 - e^{i\theta x} + i\theta x \mathbf{1}_{|x|\leq 1})\Pi(dx). \tag{A.39}$$

Here $\sigma \geq 0$ and $\mu \in \mathbf{R}$ are constant, and Π is a measure on $\mathbf{R} \setminus \{0\}$ satisfying

$$\int_{-\infty}^{\infty}(1 \wedge x^2)\Pi(dx) < \infty.$$

The parameter σ^2 is called the Gaussian coefficient and the measure Π is called the Lévy measure. The triplet (μ, σ^2, Π) is referred to as the "Lévy characteristics" of $(X_t)_{t\geq 0}$. Intuitively, μ describes the constant drift of the process and the Gaussian coefficient σ^2 denotes constant variance of the continuous component of the process. The Lévy measure Π expresses the jump structure of the jump component of the process. If $\Pi = 0$, the Lévy process is Gaussian, and if $\sigma^2 = 0$, the process is a pure jump process without the diffusion component.

One of the classes of the Lévy processes is "finite-activity jump processes" that exhibit a finite number of jumps within any finite interval. The examples of finite-activity jump processes are the compound Poisson jump processes with normally distributed jump size (Merton [1976]), double-exponential distributed jump size (Kou [2002]), and one-sided exponential distributed jump size (Eraker [2001], and Eraker et al. [2003]). Another class of the Lévy process is "infinite-activity jump processes" that generate an infinite number of jumps within any finite time interval. Examples in this class include the normal inverse Gaussian (NIG) process (Barndorff-Nielsen [1998]), the variance gamma (VG) process (Madan and Milne [1991], and Madan et al. [1998]), the finite moment log-stable (LS) process (Carr and Wu [2003]), the Meixner process (Schoutens [2002]), and the CGMY process (Carr et al. [2002]). These Lévy measures and their characteristic exponents are listed in Table 2.1. See Cont and Tankov [2004], and Boyarchenko and Levendorskiĭ [2002] for financial applications and more details of Lévy processes.

Proposition A.12 *Suppose that the covariate component X_t of PHM follows a Lévy process under the spot measure $\mathbb{Q}$, and $X_{t_0} = 0$ for convention. Then, it satisfies*

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] = \exp\left\{-\sum_{k=1}^{n}(t_k - t_{k-1})\psi_X(-iw(n-k+1))\right\}, \tag{A.40}$$

where $\psi_X(\theta)$ is the characteristic exponent of X_t.

Proof of Proposition A.12: Since

$$\sum_{k=1}^{n} X_{t_k} = n(X_{t_1} - X_{t_0}) + \cdots + (n-k+1)(X_{t_k} - X_{t_{k-1}}) + \cdots + (X_{t_n} - X_{t_{n-1}}),$$

the left hand side of Eq.(A.40) can be written as

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] &= \mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} w(n-k+1)(X_{t_k} - X_{t_{k-1}})\right\} \mid \mathcal{F}_{t_0}\right] \\
&= \prod_{k=1}^{n} \mathbb{E}\left[e^{w(n-k+1)(X_{t_k} - X_{t_{k-1}})} \mid \mathcal{F}_{t_0}\right] \\
&= \prod_{k=1}^{n} \mathbb{E}\left[e^{w(n-k+1)X_{t_k - t_{k-1}}} \mid \mathcal{F}_{t_0}\right] \\
&= \exp\left\{-\sum_{k=1}^{n}(t_k - t_{k-1})\psi_X(-iw(n-k+1))\right\}.
\end{aligned}$$

The second and third equalities of the above equation are due to the independent and stationary increments property of Lévy processes, respectively. The last equality is obtained from the Lévy-Khintchine formula. □

A.5.2 Time-changed Lévy Processes

In this subsection, let us consider the "time-changed Lévy processes" proposed by Carr et al. [2003] and Carr and Wu [2004] as a covariate of PHM.

Let $t \to T_t,\ t \geq 0$ be an increasing right-continuous process with left limits such that for each fixed t the random variable $(T_t)_{t\geq 0}$ is a "stopping time" with respect to $(\mathcal{F}_t)_{t\geq 0}$. Moreover, suppose that T_t is finite $\mathbb{Q}$-a.s. for all $t \in [0,\infty)$ and $T_t \to \infty$ as $t \to \infty$. Then, the family of the stopping times $\{T_t\}$ defines a "random time change". Without loss of generality, we can normalize the random time change so that $\mathbb{E}[T_t] = t$. With this normalization, this stopping time family becomes an unbiased reflection of calendar time.

The time-changed Lévy process is a stochastic process $(X_t)_{t\geq 0}$ defined as

$$X_t = L_{T_t}, \quad \text{for } t \geq 0,$$

where $(L_t)_{t\geq 0}$ is a one-dimensional Lévy process. Evidently, by specifying different Lévy processes for L_t and different random time change for T_t, we can generate various types of discontinuous stochastic processes from this setup.

The random time can be characterized as follows:

$$T_t = \int_0^t V_s ds,$$

where V_t is called the "instantaneous activity rate". Intuitively, one can regard t as calendar time and T_t as business time at calendar time t. A more active business day, on which the corresponding active rate becomes higher, generates higher volatility in the economy. This randomness in business activity induces the randomness in volatility. The instantaneous activity rate needs to be non-negative in order to ensure that T_t is non-decreasing process. In this chapter $(T_t)_{t\geq 0}$ is assumed to be independent of $(L_t)_{t\geq 0}$ for simplicity, although the instantaneous activity rate can be correlated with the original Lévy process by the complexed-value measure change technique developed by Carr and Wu [2004].

Lemma A.13 *Suppose that the covariate component $X_t := L_{T_t}$ of PHM follows a time-changed Lévy process with an activity rate $(V_t)_{t\geq 0}$ under the spot measure $\mathbb{Q}$, and $X_{t_0} = 0$ for convention. Then, it holds*

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] = \mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n}\psi_L(-iw(n-k+1))\int_{t_{k-1}}^{t_k} V_s ds\right\} \mid \mathcal{F}_{t_0}\right],$$

where $\psi_L(\theta)$ is the characteristic exponent of L_t.

Proof of Lemma A.13: By the independence assumption between $(L_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$, it holds

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] &= \mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wL_{T_{t_k}}\right\} \mid \mathcal{F}_{t_0}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wL_{u_k}\right\} \mid T_{t_k} = u_k, \ \ k = 1, \ldots, n\right] \mid \mathcal{F}_{t_0}\right] \\
&= \mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n} (T_{t_k} - T_{t_{k-1}})\psi_L(-iw(n-k+1))\right\} \mid \mathcal{F}_{t_0}\right] \\
&= \mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n} \psi_L(-iw(n-k+1)) \int_{t_{k-1}}^{t_k} V_s ds\right\} \mid \mathcal{F}_{t_0}\right].
\end{aligned}$$

The third equality of the above equation is obtained from Proposition A.12. □

Firstly, let us set the affine process as the instantaneous activity rate of a time-changed Lévy process.

Proposition A.14 *Suppose that the covariate component $X_t := L_{T_t}$ of PHM follows a time-changed Lévy process under the spot $\mathbb{Q}$ with an activity rate $(V_t)_{t\geq 0}$ such that*

$$V_t := \rho_0 + \rho_1^\top \mathbf{Y}_t \geq 0, \ \ \textit{for all} \ \ t \geq 0, \quad \rho_0 \in \mathbf{R}, \ \ \rho_1 \in \mathbf{R}^d,$$

where $(\mathbf{Y}_t)_{t\geq 0}$ is a d-dimensional affine process with the drift term $\mu(\mathbf{Y}_t) = K_0 + K_1\mathbf{Y}_t$ and the diffusion term $[\sigma(\mathbf{Y}_t)\sigma(\mathbf{Y}_t)^\top]_{ij} = (H_0)_{ij} + (H_1)_{ij}^\top \mathbf{Y}_t$. Here, $K_0 \in \mathbf{R}^d$, $K_1 \in \mathbf{R}^{d\times d}$, $H_0 \in \mathbf{R}^{d\times d}$, and $H_1 \in \mathbf{R}^{d\times d\times d}$. Moreover, $X_{t_0} = 0$ for convention. Then, it satisfies

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] = \exp\left\{\sum_{k=1}^{n} \alpha_{t_{n+1-k}}(t_{n-k}) + \beta_{t_1}(t_0)^\top \mathbf{y}\right\}, \tag{A.41}$$

where $\mathbf{y} := \mathbf{Y}_{t_0}$, and $\alpha_{t_k} : \mathbf{R}_+ \to \mathbf{R}$ and $\beta_{t_k} : \mathbf{R}_+ \to \mathbf{R}^d$ are recursively defined by the following ODEs:

$$\frac{d}{dt}\beta_{t_k}(t) = \psi_L(-iw(n-k+1))\rho_1 - K_1^\top \beta_{t_k}(t) - \frac{1}{2}\beta_{t_k}(t)^\top H_1 \beta_{t_k}(t), \tag{A.42}$$

$$\frac{d}{dt}\alpha_{t_k}(t) = \psi_L(-iw(n-k+1))\rho_0 - K_0^\top \beta_{t_k}(t) - \frac{1}{2}\beta_{t_k}(t)^\top H_0 \beta_{t_k}(t), \tag{A.43}$$

with boundary conditions $\alpha_{t_k}(t_k) = 0$ for $k = 1, \ldots, n$, $\beta_{t_n}(t_n) = 0$, and $\beta_{t_k}(t_k) = \beta_{t_{k+1}}(t_k)$ for $k = 1, \ldots, n-1$.

Proof of Proposition A.14: Using Lemma A.13 and the law of iterated expectations, we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] &= \mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n} \lambda_k \int_{t_{k-1}}^{t_k} V_s ds\right\} \mid \mathcal{F}_{t_0}\right] \\
&= \mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n-1} \lambda_k \int_{t_{k-1}}^{t_k} V_s ds\right\} H_{n-1} \mid \mathcal{F}_{t_0}\right],
\end{aligned} \tag{A.44}$$

where $\lambda_k := \psi_L(-iw(n-k+1))$ and

$$\begin{aligned}
H_{n-1} &:= \mathbb{E}\left[\exp\left\{-\lambda_n \int_{t_{n-1}}^{t_n} V_s ds\right\} \mid \mathcal{F}_{t_{n-1}}\right] \\
&= \exp\left\{\alpha_{t_n}(t_{n-1}) + \beta_{t_n}(t_{n-1})^\top \mathbf{Y}_{t_{n-1}}\right\}.
\end{aligned} \tag{A.45}$$

Here, the second equality of Eq.(A.45) is obtained from Lemma A.7, and $\alpha_{t_n}(t)$ and $\beta_{t_n}(t)$ are deterministic functions satisfying the ODEs (A.42) and (A.43) with boundary conditions $\alpha_{t_n}(t_n) = 0$ and $\beta_{t_n}(t_n) = 0$.

Next, substituting Eq.(A.45) into Eq.(A.44), we obtain

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] = e^{\alpha_{t_n}(t_{n-1})}\,\mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n-2}\lambda_k\int_{t_{k-1}}^{t_k} V_s ds\right\} H_{n-2} \mid \mathcal{F}_{t_0}\right],$$

where

$$\begin{aligned} H_{n-2} &:= \mathbb{E}\left[\exp\left\{-\lambda_{n-1}\int_{t_{n-2}}^{t_{n-1}} V_s ds\right\} e^{\beta_{t_n}(t_{n-1})^\top \mathbf{Y}_{t_{n-1}}} \mid \mathcal{F}_{t_{n-2}}\right] \\ &= \exp\left\{\alpha_{t_{n-1}}(t_{n-2}) + \beta_{t_{n-1}}(t_{n-2})^\top \mathbf{Y}_{t_{n-2}}\right\}. \end{aligned}$$

The second equality of the above equation is due to Lemma A.7, and $\alpha_{t_{n-1}}(t)$ and $\beta_{t_{n-1}}(t)$ satisfy the ODEs (A.42) and (A.43) with boundary conditions $\alpha_{t_{n-1}}(t_{n-1}) = 0$ and $\beta_{t_{n-1}}(t_{n-1}) = \beta_{t_n}(t_{n-1})$.

Repeating this procedure, we obtain

$$\begin{aligned} \mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] &= \exp\left\{\sum_{k=1}^{n-1}\alpha_{t_{n+1-k}}(t_{n-k})\right\} H_0 \\ &= \exp\left\{\sum_{k=1}^{n}\alpha_{t_{n+1-k}}(t_{n-k}) + \beta_{t_1}(t_0)^\top \mathbf{Y}_{t_0}\right\}. \end{aligned}$$

□

Next, let us assume that the instantaneous activity rate of a time-changed Lévy process follows a quadratic Gaussian process.

Proposition A.15 *Suppose that the covariate component $X_t := L_{T_t}$ of PHM follows a time-changed Lévy process under the spot $\mathbb{Q}$ with an activity rate $(V_t)_{t\geq 0}$ such that*

$$V_t := \mathbf{Z}_t^\top A_V \mathbf{Z}_t + \mathbf{b}_V^\top \mathbf{Z}_t + c_V \geq 0, \text{ for all } t \geq 0, \quad A_V \in \mathbf{R}^{d\times d}, \ \mathbf{b}_V \in \mathbf{R}^d, \ c_V \in \mathbf{R},$$

where $(\mathbf{Z}_t)_{t\geq 0}$ is a d-dimensional OU process defined in Eq.(A.22). Moreover, $X_{t_0} = 0$ for convention. Then, it satisfies

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] = \exp\left\{-\mathbf{z}^\top A_{t_1}(t_0)\mathbf{z} - \mathbf{b}_{t_1}(t_0)^\top\mathbf{z} - c_{t_1}(t_0)\right\}, \tag{A.46}$$

where $\mathbf{z} := \mathbf{Z}_{t_0}$, and $A_{t_k} : \mathbf{R}_+ \to \mathbf{R}^{m\times m}$, $\mathbf{b}_{t_k} : \mathbf{R}_+ \to \mathbf{R}^m$, and $c_{t_k} : \mathbf{R}_+ \to \mathbf{R}$ are recursively defined by the following ODEs:

$$\begin{aligned} \frac{d}{dt}A_{t_k}(t) &= -\psi_L(-iw(n-k+1))A_V + A_{t_k}(t)K + K^\top A_{t_k}(t) + 2A_{t_k}(t)^2, && \text{(A.47)} \\ \frac{d}{dt}\mathbf{b}_{t_k}(t) &= -\psi_L(-iw(n-k+1))\mathbf{b}_V + 2A_{t_k}(t)\mathbf{b}_Z + K^\top\mathbf{b}_{t_k}(t) + 2A_{t_k}(t)\mathbf{b}_{t_k}(t), && \text{(A.48)} \\ \frac{d}{dt}c_{t_k}(t) &= -\psi_L(-iw(n-k+1))c_V + \mathbf{b}_{t_k}(t)^\top\mathbf{b}_Z - \operatorname{tr} A_{t_k}(t) + \frac{1}{2}\mathbf{b}_{t_k}(t)^\top\mathbf{b}_{t_k}(t), && \text{(A.49)} \end{aligned}$$

with boundary conditions

$$\begin{aligned} A_{t_k}(t_k) &= A_{t_{k+1}}(t_k), & \text{(A.50)}\\ \mathbf{b}_{t_k}(t_k) &= \mathbf{b}_{t_{k+1}}(t_k), & \text{(A.51)}\\ c_{t_k}(t_k) &= c_{t_{k+1}}(t_k), & \text{(A.52)} \end{aligned}$$

for $k = 1, \dots, n-1$, *and* $A_{t_n}(t_n) = \mathbf{b}_{t_n}(t_n) = c_{t_n}(t_n) = 0$.

Proof of Proposition A.15: Using Lemma A.13 and the law of iterated expectations, we have

$$\begin{aligned} \mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] &= \mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n}\lambda_k \int_{t_{k-1}}^{t_k} V_s ds\right\} \mid \mathcal{F}_{t_0}\right] \\ &= \mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n-1}\lambda_k \int_{t_{k-1}}^{t_k} V_s ds\right\} I_{n-1} \mid \mathcal{F}_{t_0}\right], \qquad \text{(A.53)} \end{aligned}$$

where $\lambda_k := \psi_L(-iw(n-k+1))$ and

$$\begin{aligned} I_{n-1} &:= \mathbb{E}\left[\exp\left\{-\lambda_n \int_{t_{n-1}}^{t_n} V_s ds\right\} \mid \mathcal{F}_{t_{n-1}}\right] \\ &= \exp\left\{-\mathbf{Z}_{t_{n-1}}^\top A_{t_n}(t_{n-1})\mathbf{Z}_{t_{n-1}} - \mathbf{b}_{t_n}(t_{n-1})^\top \mathbf{Z}_{t_{n-1}} - c_{t_n}(t_{n-1})\right\}. \qquad \text{(A.54)} \end{aligned}$$

Here, the second equality of Eq.(A.54) is obtained from Lemma A.9, and $A_{t_n}(t)$, $\mathbf{b}_{t_n}(t)$, and $c_{t_n}(t)$ are deterministic functions satisfying the ODEs (A.47)-(A.49) with boundary conditions $A_{t_n}(t_n) = \mathbf{b}_{t_n}(t_n) = c_{t_n}(t_n) = 0$.

Next, substituting Eq.(A.54) into Eq.(A.53), we obtain

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] = \mathbb{E}\left[\exp\left\{-\sum_{k=1}^{n-2}\lambda_k \int_{t_{k-1}}^{t_k} V_s ds\right\} I_{n-2} \mid \mathcal{F}_{t_0}\right],$$

where

$$\begin{aligned} I_{n-2} &:= \mathbb{E}\Bigg[\exp\left\{-\lambda_{n-1} \int_{t_{n-2}}^{t_{n-1}} V_s ds\right\} \\ &\quad \times \exp\left\{-\mathbf{Z}_{t_{n-1}}^\top A_{t_n}(t_{n-1})\mathbf{Z}_{t_{n-1}} - \mathbf{b}_{t_n}(t_{n-1})^\top \mathbf{Z}_{t_{n-1}} - c_{t_n}(t_{n-1})\right\} \mid \mathcal{F}_{t_{n-2}}\Bigg] \\ &= \exp\left\{-\mathbf{Z}_{t_{n-2}}^\top A_{t_{n-1}}(t_{n-2})\mathbf{Z}_{t_{n-2}} - \mathbf{b}_{t_{n-1}}(t_{n-2})^\top \mathbf{Z}_{t_{n-2}} - c_{t_{n-1}}(t_{n-2})\right\}. \end{aligned}$$

The second equality of the above equation is due to Lemma A.9, and $A_{t_{n-1}}(t)$, $\mathbf{b}_{t_{n-1}}(t)$, and $c_{t_{n-1}}(t)$ satisfy the ODEs (A.47)-(A.49) with boundary conditions (A.50)-(A.52).

Repeating this procedure, we obtain

$$\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} wX_{t_k}\right\} \mid \mathcal{F}_{t_0}\right] = I_0 = \exp\left\{-\mathbf{Z}_{t_0}^\top A_{t_1}(t_0)\mathbf{Z}_{t_0} - \mathbf{b}_{t_1}(t_0)^\top \mathbf{Z}_{t_0} - c_{t_0}(t_1)\right\}.$$

□

A.6 Numerical Examples

To examine the accuracy of our approximation formulas in Theorem A.4 and A.5, this section provides numerical examples with two specified processes for the covariates of PHM; Cox-Ingersoll-Ross (CIR) model (Cox et al. [1985]) and the variance gamma (VG) model. Thus, we compute the expectations

$$\mathbb{E}\left[\exp\left\{-\int_0^T h_t dt\right\}\right] \quad \text{and} \quad \mathbb{E}\left[h_T \exp\left\{-\int_0^T h_t dt\right\}\right],$$

with

$$h_t = \bar{h}(t)e^{wX_t},$$

by Theorem A.4 and A.5 when $(X_t)_{t\geq 0}$ follows CIR model or VG model. The calculation formula for CIR model that is well known as one of affine processes is presented in Proposition A.8, while the formula for VG model belonging to the class of Lévy processes is provided in Proposition A.12.

In the numerical examples, we apply the 10 points Gauss-Legendre quadrature rule to compute the iterated integration in Eq.(A.7) and (A.11), and we calculate the finite sums with $n = 4$ at most in the right hand sides of Eq.(A.6) and (A.10) as the approximation values. Thanks to the Gaussian quadrature method, these approximation values can be obtained very quickly like a closed-form solution. To verify the accuracy of our formulas, we compare the approximation values with the values computed by Monte Carlo simulation with 10^7 sample paths, which are considered to as the exact values of Eq.(A.6) and (A.10).

A.6.1 In the Case of CIR Model

CIR model follows the SDE:

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dW_t, \quad t \geq 0,$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion on $(\Omega, \mathcal{G}, \mathbb{Q})$, a, b, and $\sigma > 0$ are constant parameters, and the initial value $x := X_0$ is assumed to be strictly positive constant. In this case, solving the ODEs (A.20) and (A.21) in Proposition A.8, we have

$$\begin{aligned} \alpha_{t_k}(t) &= \frac{2a}{\sigma^2}\ln\left|\frac{(\sigma^2+\gamma_k)e^{b(t_k-t)}}{\sigma^2+\gamma_k e^{b(t_k-t)}}\right|, \\ \beta_{t_k}(t) &= \frac{2b}{\sigma^2+\gamma_k e^{b(t_k-t)}}, \end{aligned}$$

where

$$\gamma_k := \frac{2b}{w + \beta_{t_{k+1}}(t_k)} - \sigma^2.$$

We set $x = 0.02$, $a = 0.04$, $b = 2$, $\sigma = 0.2$, and $\bar{h}(t) = 0.06$ for all $t \geq 0$. Table A.1 and A.2 report the approximation values of Eq.(A.6) and (A.10), respectively, when $w = 1$. As a result of the test, we find that high order approximations are necessary to obtain more accurate values in the case of longer maturity, while low order approximations are sufficient for the valuations with shorter maturity. Table A.3 and A.4 show the results when $w = 10$. Note that this is the case having higher convexity of the hazard rate h_t with respect to the covariate X_t than that of $w = 1$. Although the accuracy in the case of $w = 10$ is worse than that of $w = 1$ due to the higher convexity, the level of the accuracy of the 4th order approximation values is substantially sufficient.

A.6.2 In the Case of VG Model

VG model is an infinite-activity jump process on $(\Omega, \mathcal{G}, \mathbb{Q})$ having the Lévy measure

$$\Pi(dx) = \left(\frac{e^{-\xi_p x}}{\kappa x} \mathbf{1}_{\{x>0\}} + \frac{e^{-\xi_n |x|}}{\kappa |x|} \mathbf{1}_{\{x<0\}} \right) dx,$$

where

$$\xi_p := \sqrt{\frac{\alpha^2}{\eta^4} + \frac{2}{\eta^2 \kappa}} - \frac{\alpha}{\eta^2} \quad \text{and} \quad \xi_n := \sqrt{\frac{\alpha^2}{\eta^4} + \frac{2}{\eta^2 \kappa}} + \frac{\alpha}{\eta^2},$$

and η, α, and κ are constant parameters. This is another representation of the Lévy measure for VG model in Table 2.1 and it is well-known that VG model approaches a Brownian motion with drift α and volatility η as $\kappa \to 0$.

We set $\eta = 0.2$, $\alpha = -0.01$, $\kappa = 0.5$, and $\bar{h}(t) = 0.06$ for all $t \geq 0$. Table A.5 and A.6 exhibit the approximation values of Eq.(A.6) and (A.10), respectively, when $w = 0.1$. And then Table A.7 and A.8 show the results when $w = 1$; i.e., this is the case of higher convexity than $w = 0.1$. It can be found from these tables that the tendency of the accuracy is almost the same as in the case of CIR model. Thus, we can obtain the accurate values by our formulas even when applying discontinuous processes to the covariates of PHM.

A.7 Concluding Remarks

We propose an analytical treatment of event risk valuation problems with the proportional hazard model. In the setting of the proportional hazard model, the derived formulas based on the Edgeworth expansion are widely useful for evaluating financial products including corporate bonds, credit derivatives, mortgage-backed securities, saving accounts, and time deposits. The formulas are applicable to various types of the proportional hazard model having not only continuous processes (Gaussian, affine, and quadratic Gaussian processes) but also discontinuous processes (Lévy and time-changed Lévy processes) as time-dependent stochastic covariates. Furthermore, our numerical examples show that the formulas can give very accurate approximations of present values to the problems.

Occasionally, in order for easy implementations we tend to choose tractable, but unrealistic models for describing hazard rates without any empirical validations. In contrast, the proportional hazard model has an ability to explain probabilities of event risks relevant to time-series data such as asset returns, macroeconomic variables, and firm's accounting information. Our contribution is to make it possible to efficiently implement the statistically meaningful model in finance.

Table A.1: Eq.(A.6) with CIR model and $w = 1$

maturity		1-year	3-year	5-year	7-year	10-year
1st order	value	0.9387833	0.8163468	0.6939103	0.5714738	0.3878191
	abs error	0.0018358	0.0158766	0.0424090	0.0799931	0.1543460
	error ratio	0.1952%	1.9077%	5.7596%	12.2789%	28.4685%
2nd order	value	0.9406572	0.8332119	0.7407574	0.6632936	0.5752054
	abs error	0.0000381	0.0009885	0.0044381	0.0118266	0.0330402
	error ratio	0.0040%	0.1188%	0.6027%	1.8154%	6.0941%
3rd order	value	0.9406189	0.8321794	0.7359773	0.6501772	0.5369658
	abs error	0.0000002	0.0000440	0.0003421	0.0012898	0.0051993
	error ratio	0.0000%	0.0053%	0.0465%	0.1980%	0.9590%
4th order	value	0.9406195	0.8322268	0.7363431	0.6515825	0.5428185
	abs error	0.0000004	0.0000034	0.0000238	0.0001155	0.0006534
	error ratio	0.0000%	0.0004%	0.0032%	0.0177%	0.1205%
exact	value	0.9406191	0.8322234	0.7363193	0.6514669	0.5421652

Table A.2: Eq.(A.10) with CIR model and $w = 1$

maturity		1-year	3-year	5-year	7-year	10-year
1st order	value	0.0612181	0.0612182	0.0612182	0.0612182	0.0612182
	abs error	0.0036349	0.0102708	0.0161416	0.0213364	0.0280277
	error ratio	6.3124%	20.1596%	35.8094%	53.4991%	84.4446%
2nd order	value	0.0574703	0.0499749	0.0424796	0.0349842	0.0237412
	abs error	0.0001130	0.0009725	0.0025970	0.0048976	0.0094494
	error ratio	0.1962%	1.9088%	5.7613%	12.2802%	28.4700%
3rd order	value	0.0575850	0.0510075	0.0453476	0.0406054	0.0352129
	abs error	0.0000017	0.0000600	0.0002710	0.0007236	0.0020223
	error ratio	0.0030%	0.1178%	0.6012%	1.8144%	6.0931%
4th order	value	0.0575827	0.0509442	0.0450550	0.0398025	0.0328719
	abs error	0.0000006	0.0000032	0.0000217	0.0000794	0.0003187
	error ratio	0.0010%	0.0063%	0.0480%	0.1990%	0.9602%
exact	value	0.0575833	0.0509474	0.0450766	0.0398818	0.0331906

Table A.3: Eq.(A.6) with CIR model and $w = 10$

maturity		1-year	3-year	5-year	7-year	10-year
1st order	value	0.9261292	0.7779852	0.6298370	0.4816889	0.2594667
	abs error	0.0026766	0.0230105	0.0609422	0.1140378	0.2176320
	error ratio	0.2882%	2.8727%	8.8222%	19.1426%	45.6157%
2nd order	value	0.9288823	0.8027826	0.6986360	0.6164375	0.5342921
	abs error	0.0000765	0.0017869	0.0078568	0.0207108	0.0571934
	error ratio	0.0082%	0.2231%	1.1374%	3.4766%	11.9877%
3rd order	value	0.9288132	0.8009236	0.6900730	0.5930055	0.4661349
	abs error	0.0000074	0.0000721	0.0007063	0.0027211	0.0109637
	error ratio	0.0008%	0.0090%	0.1022%	0.4568%	2.2980%
4th order	value	0.9288145	0.8010289	0.6908762	0.5960722	0.4788436
	abs error	0.0000087	0.0000332	0.0000969	0.0003456	0.0017449
	error ratio	0.0009%	0.0041%	0.0140%	0.0580%	0.3657%
exact	value	0.9288058	0.8009957	0.6907793	0.5957267	0.4770987

Table A.4: Eq.(A.10) with CIR model and $w = 10$

maturity		1-year	3-year	5-year	7-year	10-year
1st order	value	0.0740577	0.0740741	0.0740741	0.0740741	0.0740741
	abs error	0.0053087	0.0147869	0.0229414	0.0299806	0.0387600
	error ratio	7.7219%	24.9412%	44.8664%	67.9934%	109.7580%
2nd order	value	0.0685369	0.0575605	0.0465862	0.0356123	0.0191514
	abs error	0.0002120	0.0017266	0.0045465	0.0084812	0.0161627
	error ratio	0.3084%	2.9123%	8.8915%	19.2345%	45.7684%
3rd order	value	0.0687448	0.0594132	0.0517085	0.0456299	0.0395602
	abs error	0.0000041	0.0001261	0.0005759	0.0015364	0.0042462
	error ratio	0.0060%	0.2127%	1.1262%	3.4845%	12.0240%
4th order	value	0.0687395	0.0592736	0.0510692	0.0438846	0.0344922
	abs error	0.0000094	0.0000135	0.0000635	0.0002089	0.0008219
	error ratio	0.0137%	0.0228%	0.1241%	0.4738%	2.3274%
exact	value	0.0687490	0.0592871	0.0511327	0.0440935	0.0353141

Table A.5: Eq.(A.6) with VG model and $w = 0.1$

maturity		1-year	3-year	5-year	7-year	10-year
1st order	value	0.9400240	0.8202158	0.7005991	0.5811736	0.4023931
	abs error	0.0017634	0.0152400	0.0406865	0.0766932	0.1478586
	error ratio	0.1872%	1.8242%	5.4886%	11.6579%	26.8711%
2nd order	value	0.9418228	0.8363834	0.7454494	0.6689632	0.5811980
	abs error	0.0000354	0.0009276	0.0041639	0.0110964	0.0309463
	error ratio	0.0038%	0.1110%	0.5617%	1.6867%	5.6240%
3rd order	value	0.9417868	0.8354138	0.7409674	0.6566841	0.5454847
	abs error	0.0000005	0.0000420	0.0003182	0.0011827	0.0047671
	error ratio	0.0001%	0.0050%	0.0429%	0.1798%	0.8663%
4th order	value	0.9417874	0.8354574	0.7413035	0.6579734	0.5508417
	abs error	0.0000000	0.0000016	0.0000180	0.0001066	0.0005899
	error ratio	0.0000%	0.0002%	0.0024%	0.0162%	0.1072%
exact	value	0.9417874	0.8354558	0.7412856	0.6578668	0.5502517

Table A.6: Eq.(A.10) with VG model and $w = 0.1$

maturity		1-year	3-year	5-year	7-year	10-year
1st order	value	0.0599520	0.0598562	0.0597605	0.0596650	0.0595220
	abs error	0.0034907	0.0098538	0.0154745	0.0204349	0.0268080
	error ratio	6.1825%	19.7067%	34.9422%	52.0900%	81.9466%
2nd order	value	0.0563556	0.0490885	0.0418503	0.0346407	0.0238801
	abs error	0.0001057	0.0009138	0.0024357	0.0045893	0.0088339
	error ratio	0.1871%	1.8276%	5.5000%	11.6985%	27.0034%
3rd order	value	0.0564635	0.0500574	0.0445359	0.0398934	0.0345655
	abs error	0.0000022	0.0000551	0.0002499	0.0006633	0.0018515
	error ratio	0.0039%	0.1101%	0.5643%	1.6908%	5.6598%
4th order	value	0.0564614	0.0499993	0.0442673	0.0391577	0.0324270
	abs error	0.0000001	0.0000031	0.0000187	0.0000724	0.0002870
	error ratio	0.0001%	0.0061%	0.0423%	0.1845%	0.8772%
exact	value	0.0564613	0.0500024	0.0442860	0.0392301	0.0327140

Table A.7: Eq.(A.6) with VG model and $w = 1$

maturity		1-year	3-year	5-year	7-year	10-year
1st order	value	0.9396982	0.8172659	0.6923540	0.5649123	0.3688940
	abs error	0.0018038	0.0163037	0.0452800	0.0882653	0.1777280
	error ratio	0.1916%	1.9559%	6.1385%	13.5132%	32.5139%
2nd order	value	0.9415415	0.8346709	0.7431160	0.6694518	0.5989949
	abs error	0.0000395	0.0011013	0.0054821	0.0162742	0.0523729
	error ratio	0.0042%	0.1321%	0.7432%	2.4915%	9.5812%
3rd order	value	0.9415034	0.8335146	0.7370819	0.6507271	0.5329272
	abs error	0.0000014	0.0000550	0.0005521	0.0024505	0.0136948
	error ratio	0.0001%	0.0066%	0.0748%	0.3752%	2.5053%
4th order	value	0.9415040	0.8335753	0.7376703	0.6535968	0.5503418
	abs error	0.0000020	0.0000057	0.0000364	0.0004192	0.0037198
	error ratio	0.0002%	0.0007%	0.0049%	0.0642%	0.6805%
exact	value	0.9415020	0.8335696	0.7376339	0.6531776	0.5466220

Table A.8: Eq.(A.10) with VG model and $w = 1$

maturity		1-year	3-year	5-year	7-year	10-year
1st order	value	0.0606045	0.0618319	0.0630842	0.0643618	0.0663269
	abs error	0.0036161	0.0108792	0.0180538	0.0249389	0.0345176
	error ratio	6.3453%	21.3516%	40.0924%	63.2601%	108.5141%
2nd order	value	0.0568741	0.0498076	0.0415331	0.0318881	0.0144432
	abs error	0.0001143	0.0011451	0.0034973	0.0075347	0.0173662
	error ratio	0.2006%	2.2474%	7.7666%	19.1126%	54.5946%
3rd order	value	0.0569907	0.0510322	0.0455197	0.0410801	0.0385507
	abs error	0.0000022	0.0000795	0.0004892	0.0016573	0.0067414
	error ratio	0.0039%	0.1560%	1.0865%	4.2038%	21.1931%
4th order	value	0.0569882	0.0509445	0.0449801	0.0390910	0.0293336
	abs error	0.0000002	0.0000082	0.0000503	0.0003319	0.0024757
	error ratio	0.0004%	0.0161%	0.1117%	0.8418%	7.7830%
exact	value	0.0569884	0.0509527	0.0450304	0.0394229	0.0318093

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