Essays on Collective Choice of Locations of Public Facilities

A Thesis Submitted for the Degree of

Doctor of Philosophy in Economics at the University of Tokyo

Kazuo Yamaguchi

October, 2011

Abstract

This thesis comprises three essays on the collective choice of locations of public facilities.

In the first essay, we investigate a model where, on a tree network, players collectively choose the location of a single public facility by noncooperative alternating-offer bargaining with the unanimity rule. We show the existence of a stationary subgame perfect equilibrium and the characterization of stationary subgame perfect equilibria. We also show that the equilibrium location converges to the Rawls location (the Rawlsian social welfare maximizer) as the discount factor tends to 1; however, it does not relate to the Weber location (the Benthamite social welfare maximizer).

In the second essay, we examine a model where, on a line network, individuals collectively choose the location of an undesirable public facility through bargaining with the unanimity rule. We show the existence of a stationary subgame perfect equilibrium and the characterization of stationary subgame perfect equilibria when the discount factor is sufficiently large. Furthermore, we show that as the discount factor tends to 1, the equilibrium location can converge to a location that is least desirable according to both the Benthamite and Rawlsian criteria.

In the third essay, we consider the outcome of majority voting in multiple undesirable facility location problems where the locations of two facilities are planned, any individual is concerned about the location of the nearest facility but not about the location of the other facility, and any individual prefers that the location of the nearest facility be as far as possible from his/her location. We show that a Condorcet winner is a subset of the set of pendant vertices and the vertices adjacent to pendant vertices on a tree network

with an odd number of individuals. Furthermore, we derive a necessary and sufficient condition for a set of locations to be a Condorcet winner on a line network with an odd number of individuals.

Acknowledgments

First of all, I would like to express my deep gratitude to supervisors Toshihiro Matsumura and Masahiro Okuno-Fujiwara for their beneficial advice and great encouragement. I also would like to thank Tomohiko Kawamori for his interesting discussion. Chapter 2 in this thesis is based on collaborative research with him. Furthermore, I am very grateful to Hiroaki Ino and Yoshihiro Ohashi for their useful comments and suggestions. Finally, I bless my father Masumi Yamaguchi and my girlfriend Akiko Terakura for their kindness. I could not complete this thesis if they did not give me moral support.

Contents

Chapter 1

Overview

1.1 Collective choice of locations of public facilities

It is important for us where public facilities are located. For example, each individual can surely enjoy beautiful scenery everyday if a park is located in the vicinity of him/her. We call a facility such that each individual prefers that the facility be located as near as possible to his/her location a *desirable facility*. Examples of desirable facilities are a park, a hospital, a station, and so on. Conversely, each individual is surely bothered by a bad odor everyday if a dump is located in the vicinity of him/her. We call a facility such that each individual prefers that the facility be located as far as possible from his/her location an *undesirable facility*, an *obnoxious facility*, or a *NIMBY*.[1] Examples of undesirable facilities are a dump that generates a bad odor, a nuclear power plant that generates a massive dose of radiation once a serious accident occurs, a military base that generates a noise pollution, and so on.

Hence, a social planner should choose socially desirable locations from the viewpoint of some criterion if it chooses the locations of public facilities. Many classical studies on the locations of public facilities have been devoted to the analysis on the optimal locations of public facilities.[2] The analysis is roughly classified into the minisum, minimax, maxisum,

[1] NIMBY is an acronym for the phrase "not in my back yard."

[2] See Tansel *et al.* (1983a, 1983b) and Hansen *et al.* (1997) for surveys on the normative analysis of

and maximin problems according to the types of facilities and the criteria.

Consider the situation where a social planner chooses the location of a desirable facility from the viewpoint of the Benthamite criterion. Then, the social planner should choose a location that minimizes the average distance from an individual's location. We call such a location a *minisum location*, a *median*, or a *Weber location* after Alfred Weber. The fundamental properties of a minisum location were derived by Hakimi (1964). Hakimi (1964) showed that the set of vertices contains a minisum location on a general network.[3]

Consider the situation where a social planner chooses the location of a desirable facility from the viewpoint of the Rawls criterion. Then, the social planner should choose a location that minimizes the maximum distance from an individual's location. We call such a location a *minimax location*, a *center*, or a *Rawls location* after John Rawls. Due to Minieka (1970), it is known that a minimax location belongs to the set of vertices and equidistant points which are not bottleneck points.[4]

Consider the situation where a social planner chooses the location of an undesirable facility from the viewpoint of the Benthamite criterion. Then, the social planner should choose a location that maximizes the average distance from an individual's location. We call such a location a *maxisum location* or an *anti-median*. Church and Garfinkel (1978) showed that the set of bottleneck points and pendant vertices contains a maxisum location on a general network. Furthermore, Zelinka (1968) showed that the set of pendant vertices contains a maxisum location on a tree network.[5]

Consider the situation where a social planner chooses the location of an undesirable facility from the viewpoint of the Rawls criterion. Then, the social planner should choose a location that maximizes the minimum distance from an individual's location. We call such a location a *maximin location* or an *anti-center*. Due to Minieka (1983), a maximin

locations of public facilities.

[3]For specialized algorithms for finding a minisum location on a tree network, see Goldman (1971).

[4]For algorithms for finding a minimax location on a general network, see Hakimi (1964), Hakimi *et al.* (1978), Kariv and Hakimi (1979), Minieka (1981), Cunninghame-Green (1984). Furthermore, see Goldman (1972) and Handler (1973) for specialized algorithms for finding a minimax location on a tree network.

[5]For an algorithm for finding a maxisum location on a tree network, see Ting (1984).

location is known to belong to the set of bottleneck points or pendant vertices on a general network.

However, since it is generally impossible to locate public facilities to meet the wishes of all individuals perfectly, it is common in a democratic society that the locations of public facilities are collectively chosen by themselves to adjust the difference of opinion. Then, which locations do individuals choose if they collectively choose the locations of public facilities? Especially, are the collectively chosen locations socially desirable from the viewpoint of some criterion? Since some intervention is needed if the collectively chosen locations are expected to be socially undesirable, the clarification of these questions is an important task for our communities.

In this thesis, we investigate which locations individuals choose if they collectively choose the locations of public facilities. Especially, we confine our attention to the important cases where individuals collectively choose the locations of public facilities through the majority rule or the unanimity rule.

1.2 Outcome of majority rule in facility location problems

In our communities, the majority rule is often used as a decision rule in collective decision making.

The majority rule has been investigated since a long time ago in voting theory. In voting theory, an alternative that is unbeatable through pairwise voting is employed as the collectively chosen alternative. Especially, an alternative that is unbeatable through pairwise majority voting, which is employed as the collectively chosen alternative through the majority rule, is called a *Condorcet winner*. Black (1948) showed that the peak of median voter is a Condorcet winner for a problem where the choice set is single-dimensional and the voters have single-peaked preferences over the choice set, which is well known as the median voter theorem.[6,7] Furthermore, the properties of a Condorcet winner in some facility location problems are already known.

Consider the situation where individuals collectively choose the location of a desirable facility through the majority rule. Hansen and Thisse (1981) showed that the set of Condorcet winners equals the set of minisum locations on a tree network. Labbe (1985) showed that the set of Condorcet winners equals either the empty set or the set of minisum locations on a cactus network. Furthermore, Hansen and Labbé (1988) derived an algorithm for finding a Condorcet winner on a general network. Unfortunately, a Condorcet winner is not necessarily a minisum location on a general network.[8] However, Hansen and Thisse (1981) showed that the ratio of the average distance from an individual's location to a Condorcet winner to the average distance from an individual's location to a minisum location is bounded above on a general network. Furthermore, a Condorcet winner is not necessarily a minimax location, even on a line network. However, Hansen and Thisse (1981) showed that the ratio of the maximum distance from an individual's location to a Condorcet winner to the maximum distance from an individual's location to a minimax location is bounded above on a general network.[9]

Consider the situation where individuals collectively choose the location of an undesirable facility through the majority rule. Labbé (1990) showed that a Condorcet winner is a pendant vertex or a bottleneck location on a general network with an odd number of individuals. Furthermore, Labbé (1990) revealed that the set of Condorcet winners equals the set of the pendant vertices that satisfy some condition on a tree network with an odd number of individuals. Unfortunately, a Condorcet winner is not necessarily a

[6]Note that in the situation where individuals located along a street choose the location of a desirable facility somewhere on the street, the choice set is single-dimensional and the individuals have single-peaked preferences on the choice set.

[7]Unfortunately, a Condorcet winner does not often exist, especially for a problem where the choice set is multidimensional. See Plott (1967), Davis *et al.* (1972), Rubinstein (1979), Schofield (1983), Cox (1984), Le Breton (1987), and McKelvey and Schofield (1987).

[8]Bandelt (1985) characterized the networks on which the set of Condorcet winners equals the set of minisum locations.

[9]Since the existence of a Condorcet winner is not assured, some studies employ other voting solutions weaker than a Condorcet winner. For example, see Bandelt and Labbé (1986) for a Simpson solution, Campos Rodríguez and Moreno Pérez (2000) for a tolerant Condorcet solution, and Campos Rodríguez and Moreno Pérez (2003) for their mixture.

maxisum location, even on a line network. However, Labbé (1990) showed that the ratio of the average distance from an individual's location to a maxisum location to the average distance from an individual's location to a Condorcet winner is bounded above on a general network.

Consider the situation where individuals collectively choose the locations of multiple desirable facilities through the majority rule, where any individual is concerned about the location of the nearest facility but not about the locations of the other facilities. Barberà and Beviá (2006) showed that a Condorcet winner is efficient, internally consistent, and Nash stable on a line network. Hajduková (2010) derived the additional necessary conditions and a sufficient condition for a location set to be a Condorcet winner on a line network. Furthermore, Campos Rodríguez and Moreno Pérez (2008) derived an algorithm for finding a Condorcet winner on a general network. Unfortunately, a Condorcet winner is not necessarily a minisum location set, even on a line network. Furthermore, the ratio of the average distance from an individual's location to the nearest location in a Condorcet winner to the average distance from an individual's location to the nearest location in a minisum location set is not bounded above, even on a line network.

Following these studies, Chapter 4 considers the outcome of majority rule in multiple undesirable facility location problems. We consider a model where the locations of multiple facilities are planned, any individual is concerned about the location of the nearest facility but not about the locations of the other facilities, and any individual prefers that the location of the nearest facility be as far as possible from his/her location. In these problems, a Condorcet winner is a set of locations that is unbeatable through pairwise majority voting. We assume that the locations of two facilities are planned. We show that a Condorcet winner is a subset of the set of pendant vertices and the vertices adjacent to pendant vertices on a tree network with an odd number of individuals. Furthermore, we derive a necessary and sufficient condition for a set of locations to be a Condorcet winner on a line network with an odd number of individuals. In these problems, we show that the ratio of the average distance from an individual's location to the nearest location in an antimedian to the average distance from an individual's location to the nearest location in a Condorcet winner is not bounded above, even on a line network. This chapter is based on Yamaguchi (2011b).

1.3 Outcome of unanimity rule in facility location problems

In our communities, the unanimity rule is also often used as a decision rule in collective decision making.

Unfortunately, it is well known that the situations that voting theory can predict are very limited. Especially, voting theory can not predict anything about the outcome of unanimity rule because there does not exist an alternative that is unbeatable in pairwise unanimity voting unless the bliss points of all individuals coincide. However, it is also well known that bargaining theory can predict much more situations than voting theory. Bargaining is an activity such that two or more individuals voluntarily continue to negotiate until some conclusion is reached, which is practiced in many situations such as a commercial transaction, reconciliation, and so on. In bargaining theory, bargaining is formulated as a game which comprises infinite rounds, where a round consists of the following steps: (i) a player is selected as a proposer according to a predetermined proposer selection protocol;[10] (ii) the proposer proposes an alternative; (iii) players announce acceptance or rejection to the proposal; (iv) the game ends with the proposal implemented if the announcements satisfy the requirement of a predetermined decision rule, and the game proceeds to the next round otherwise. Then, the equilibrium chosen alternative in this game is employed as the collectively chosen alternative. Many studies investigated bargaining on the split of the pie where the proposer proposes players' shares such that

[10] Especially, the random-proposer protocol and the rejecter-proposer protocol are often considered. In the random-proposer protocol, a player is selected as a proposer with some probability. In the rejecter-proposer protocol, a predetermined player is selected as a proposer in the first round, and the rejector in the previous round is selected as a proposer otherwise.

the sum is less than or equal to the pie. Rubinstein (1982) characterized the unique subgame perfect equilibrium in bargaining on this problem through the unanimity rule. Baron and Ferejohn (1989) considered bargaining on this problem through the majority rule. They showed that not only the uniqueness of subgame perfect equilibria but also the uniqueness of stationary subgame perfect equilibria are not generally assured. However, Eraslan (2002) showed that the uniqueness of the stationary subgame perfect equilibrium payoffs is assured in bargaining on this problem through the q-majority rule.[11] Some studies also investigated bargaining on a problem where the choice set is single-dimensional and individuals have single-peaked preferences over the choice set. Cho and Duggan (2003), Cardona and Ponsatí (2007), and Herings and Predtetchinski (2010) showed that the uniqueness of stationary subgame perfect equilibria is assured in bargaining on this problem for some situations. Predtetchinski (2011) derived the asymptotic properties of stationary subgame perfect equilibria in bargaining on this problem through a general decision rule. Furthermore, Banks and Duggan (2000) showed the existence of a stationary subgame perfect equilibrium with no delay in bargaining on a general problem through a general decision rule. The problem they considered includes the split of the pie, the choice of location of a desirable facility on a multidimensional space, and so on as a special case. Furthermore, the decision rule they considered includes the unanimity rule, the majority rule, and so on as a special case. Hence, bargaining theory could generate useful predictions even to the situations to which voting theory can not generate useful predictions. By applying bargaining theory, Chapters 2 and 3 consider the outcome of unanimity rule in single facility location problems.

Chapter 2 considers the outcome of unanimity rule in single desirable facility location problems. We consider a model where, on a tree network, individuals collectively choose the location of a desirable public facility through bargaining with the unanimity rule and employ the equilibrium location as a solution. We show that the equilibrium location relates to a minimax location: If the discount factor tends to 1, the equilibrium location

[11] Some authors have investigated decision rules other than the simple quota rules. See Winter (1996) for the decision rule with veto and Snyder *et al.* (2005) for the decision rule with weighted voting.

converges to a minimax location. However, we also show that the equilibrium location does not relate to a minisum location: It does not correspond to a minisum location, and the ratio of the average distance from an individual's location to the equilibrium location to that to a minisum location is not bounded above. This result implies that the collective choice of the location of a desirable public facility through bargaining with the unanimity rule results in the best outcome according to the Rawlsian criterion. This chapter is based on Kawamori and Yamaguchi (2010).

Chapter 3 considers the outcome of unanimity rule in single undesirable facility location problems. We consider a model where, on a line network, individuals collectively choose the location of an undesirable public facility through bargaining with the unanimity rule and employ the equilibrium location as a solution. We consider only the case where each individual obtains a non-negative net benefit regardless of wherever the facility is located. We show the existence of a stationary subgame perfect equilibrium and the characterization of stationary subgame perfect equilibria when the discount factor is sufficiently large. Furthermore, we show that as the discount factor tends to 1, the equilibrium location can converge to a location that minimizes both the average and minimum distance from an individual's location. This result implies that the collective choice of the location of an undesirable public facility through bargaining with the unanimity rule can result in the worst outcome according to both the Benthamite and Rawlsian criteria. This chapter is based on Yamaguchi (2011a).

References

Bandelt, H. -J. (1985). Networks with Condorcet solutions. *European Journal of Operational Research*, 20, 314–326.

Bandelt, H. -J., Labbé, M. (1986). How bad can a voting location be. *Social Choice and Welfare*, 3, 125–145.

Banks, J. S., Duggan, J. (2000). A bargaining model of collective choice. *American*

Political Science Review, 94, 73–88.

Barberà, S., Beviá, C. (2006). Locating public facilities by majority: stability, consistency and group formation. *Games and Economic Behavior*, 56, 185-200.

Baron, D., Ferejohn, J. (1989). Bargaining in legislatures. *American Political Science Review*, 83, 1181–1206.

Black, D. (1948). On the rationale of group decision making. *Journal of Political Economy*, 56, 23–34.

Campos Rodríguez, C. M., Moreno Pérez, J. A. (2000). The tolerant Condorcet points. *Studies in Locational Analysis*, 14, 173–190.

Campos Rodríguez, C. M., Moreno Pérez, J. A. (2003). Relaxation of the Condorcet and Simpson conditions in voting location. *European Journal of Operational Research*, 145, 673–683.

Campos Rodríguez, C. M., Moreno Pérez, J. A. (2008). Multiple voting location problems. *European Journal of Operational Research*, 191, 437–453.

Cardona, D., Ponsatí, C. (2007). Bargaining one-dimensional social choices. *Journal of Economic Theory*, 137, 627–651.

Cho, S., Duggan, J. (2003). Uniqueness of stationary equilibria in a one-dimensional model of bargaining. *Journal of Economic Theory*, 113, 118–130.

Church, R. L., Garfinkel, R. S. (1978). Locating an obnoxious facility on a network. *Transportation Science*, 12, 107–118.

Cox, G. (1984). Non-collegial simple games and the nowhere denseness of the set of preference profiles having a core. *Social Choice and Welfare*, 1, 159–164.

Cunninghame-Green, R. A. (1984). The absolute center of a graph. *Discrete Applied Mathematics*, 7, 275–283.

Davis, O., DeGroot, M., Hinich, M. (1972). Social preference orderings and majority rule. *Econometrica*, 40, 147–157.

Eraslan, H. (2002). Uniqueness of stationary equilibrium payoffs in the Baron-Ferejohn model. *Journal of Economic Theory*, 103, 11–30.

Goldman, A. J. (1971). Optimal center location in simple networks. *Transportation Science*, 5, 212–221.

Goldman, A. J. (1972). Minimax location of a facility in a network. *Transportation Science*, 6, 407–418.

Hajduková, J. (2010). Condorcet winner configurations of linear networks. *Optimization*, 59, 461–475.

Hakimi, S. L. (1964). Optimum locations of switching centers and the absolute centers and medians of a graph. *Operations Research*, 12, 450–459.

Hakimi, S. L., Schmeichel, E. F., Pierce, J. G. (1978). On p-centers in networks. *Transportation Science*, 12, 1–15.

Handler, G. Y. (1973). Minimax location of a facility in an undirect tree graph. *Transportation Science*, 7, 287–293.

Hansen, P., Labbé, M. (1988). Algorithms for voting and competitive location on a network. *Transportation Science*, 22, 278–288.

Hansen, P., Labbé, M., Peeters, D., Thisse, J. -F. (1997). Facility location analysis. In R. Arnott (Eds.), *Regional and Urban Economics Part 1* (pp. 441–507). London: Routledge.

Hansen, P., Thisse, J. F. (1981). Outcomes of voting and planning: Condorcet, Weber and Rawls locations. *Journal of Public Economics*, 16, 1–15.

Herings, P. J. J., Predtetchinski, A. (2010). One-dimensional bargaining with Markov recognition probabilities. *Journal of Economic Theory*, 145, 189–215.

Kariv, O., Hakimi, S. L. (1979). An algorithmic approach to network location problems, I: the p-centers. *SIAM Journal on Applied Mathematics*, 37, 513–538.

Kawamori, T., Yamaguchi, K. (2010). Outcomes of bargaining and planning in single facility location problems. *Mathematical Social Sciences*, 59, 38–45.

Labbe, M. (1985). Outcomes of voting and planning in single facility location problems. *European Journal of Operational Research*, 20, 299–313.

Labbé, M. (1990). Location of an obnoxious facility on a network: a voting approach. *Networks*, 20, 197–207.

Le Breton, M. (1987). On the core of voting games. *Social Choice and Welfare*, 4, 295–305.

McKelvey, R., Schofield, N. (1987). Generalized symmetry conditions at a core point. *Econometrica*, 55, 923–934.

Minieka, E. (1970). The m-center problem. *SIAM Review*, 12, 138–139.

Minieka, E. (1981). A polynomial time algorithm for finding the absolute center of a network. *Networks*, 11, 351–355.

Minieka, E. (1983). Anti-centers and anti-medians of a network. *Networks*, 13, 359–365.

Plott, C. (1967). A notion of equilibrium and its possibility under majority rule. *American Economic Review*, 57, 787–806.

Predtetchinski, A. (2011). One-dimensional bargaining. *Games and Economic Behavior*, 72, 526–543.

Rubinstein, A. (1979). A note on the nowhere denseness of societies having an equilibrium under majority rule. *Econometrica*, 47, 511–514.

Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica*, 50, 97–110.

Schofield, N. (1983). Generic instability of majority rule. *Review of Economic Studies*, 50, 695–705.

Snyder, J. M., Ting, M. M., Ansolabehere, S. (2005). Legislative bargaining under weighted voting. *American Economic Review*, 95, 981–1004.

Tansel, B. C., Francis, R. L., Lowe, T. J. (1983a). Location on networks: a survey. part I: the p-center and p-median problems. *Management Science*, 29, 482–497.

Tansel, B. C., Francis, R. L., Lowe, T. J. (1983b). Location on networks: a survey. part II: exploiting tree network structure. *Management Science*, 29, 498–511.

Ting, S. S. (1984). A linear-time algorithm for maxisum facility location on tree networks. *Transportation Science*, 18, 76–84.

Winter, E. (1996). Voting and vetoing. *American Political Science Review*, 90, 813–823.

Yamaguchi, K. (2011a). Location of an undesirable facility on a network: a bargaining approach. *Mathematical Social Sciences*, 62, 104–108.

Yamaguchi, K. (2011b). Outcome of majority voting in multiple undesirable facility location problems. Mimeo.

Zelinka, B. (1968). Medians and peripherians of trees. *Archivum Mathematicum*, 4, 87–95.

Chapter 2

Outcomes of bargaining and planning in single facility location problems

Abstract: In this chapter, we investigate a model where, on a tree network, players collectively choose the location of a single public facility by noncooperative alternating-offer bargaining with the unanimity rule. We show the existence of a stationary subgame perfect equilibrium and the characterization of stationary subgame perfect equilibria. We also show that the equilibrium location converges to the Rawls location (the Rawlsian social welfare maximizer) as the discount factor tends to 1; however, it does not relate to the Weber location (the Benthamite social welfare maximizer).

Keywords: Tree network; Location of public facility; Unanimity rule; Bargaining; Rawls location

JEL Classification: D72; C78; R53; H41

Based article: Kawamori, T., Yamaguchi, K. (2010). Outcomes of bargaining and planning in single facility location problems. *Mathematical Social Sciences*, 59, 38–45.

2.1 Introduction

Where is a public facility located on a network? Since it incurs a cost to travel to the facility, a social planner and individuals on the network are interested in the choice of the location. If the social planner chooses the location, the chosen location is a socially desirable one such as a *Weber location*, at which the average distance from an individual's location to the facility is minimized, or a *Rawls location*, at which the maximum distance from an individual's location to the facility is minimized. Instead, if the individuals collectively choose the location, what location do they choose? Is the chosen location socially desirable?

Many papers have been devoted to answering these questions in the case where individuals choose the location by the majority rule, which is one of the most usual decision rules in collective decision making, by employing a *Condorcet location*, which is unbeatable by pairwise majority voting, as a solution under the majority rule. Hansen and Thisse (1981) showed that on a tree network, a Condorcet location exists, and Condorcet locations coincide with Weber locations.[1] On a cactus network, the existence of a Condorcet location is no longer assured. However, Labbe (1985) showed that if a Condorcet location exists, Condorcet locations coincide with Weber locations. On a general network, not only the existence of a Condorcet location but also the coincidence of Condorcet locations and Weber locations are no longer assured. However, Hansen and Thisse (1981) showed that if a Condorcet location exists, the ratio of the average distance from an individual's location to a Condorcet location to that to a Weber location is bounded above.[2,3]

With this result under the majority rule in mind, this chapter is devoted to answering

[1]Bandelt (1985) characterized the networks on which a Condorcet location exists and those on which Condorcet locations coincide with Weber locations.

[2]Hansen and Thisse (1981) also showed that if a Condorcet location exists, the ratio of the maximum distance from an individual's location to a Condorcet location to that to a Rawls location is bounded above.

[3]Since the existence of a Condorcet location is not assured, other voting solutions weaker than a Condorcet location have been proposed. For example, see Bandelt and Labbé (1986) for a Simpson location, Campos Rodríguez and Moreno Pérez (2000) for a tolerant Condorcet location, and Campos Rodríguez and Moreno Pérez (2003) for their mixture.

the questions in the case where individuals choose the location by the unanimity rule, which is another of the most usual decision rules in collective decision making. Since, generically, there does not exist a location that is unbeatable by pairwise unanimity voting, we consider an alternative approach. We consider a model where, on a tree network, individuals collectively choose the location by alternating-offer bargaining, which was formulated by Rubinstein (1982), and employ the equilibrium location as a solution under the unanimity rule.[4,5] We show that the equilibrium location relates to a Rawls location: If the discount factor tends to 1, the equilibrium location converges to a Rawls location. However, we also show that the equilibrium location does not relate to a Weber location: It does not correspond to a Weber location, and the ratio of the average distance from an individual's location to the equilibrium location to that to a Weber location is not bounded above.

The remainder of this chapter is organized as follows. Section 2.2 describes the model, Section 2.3 characterizes the equilibria, and Section 2.4 concludes the chapter. We provide proof of each proposition in the Appendix.

2.2 Model

In this section, we describe a noncooperative bargaining model.

Let (V, E) be a *geometric graph*: For some metric space X, V is a finite subset of X, and E is a finite set of some continuous injections from $[0, 1]$ to X such that for each $e \in E$, $e(\{0, 1\}) \subset V$ and $e([0, 1] \setminus \{0, 1\}) \subset X \setminus V$, and for each $(e, e') \in E^2$ with $e \neq e'$, $e((0, 1)) \cap e'((0, 1)) = \emptyset$. Let $N := \left(\bigcup_{e \in E} e([0, 1])\right) \cup \left(\bigcup_{v \in V} \{v\}\right)$. We call N a *network*, and $n \in N$, a *location*. Further, we call $v \in V$ a *vertex*. For each $(n, n') \in N^2$, we call

[4] Cardona and Ponsatí (2007) considered a related model where, in a one-dimensional interval, individuals collectively choose an alternative by bargaining with the q-majority rule. The preferences that we consider are a particular case of those of them, but the space that we consider is more general than that of them. In the present chapter, we show that a unique equilibrium location exists. Cardona and Ponsatí (2007) also showed that under natural conditions, a unique equilibrium exists.

[5] The space and preferences that we consider are only a specific case of those of Banks and Duggan (2000). However, due to this specification, we can provide the explicit characterization of equilibria that they did not.

$R \in 2^N$ a *route* between n and n' if (i) $R \ni n$, $R \ni n'$, and R is connected and (ii) there does not exist $S \subsetneq R$ such that $S \ni n$, $S \ni n'$, and S is connected. We assume that there is a unique route between two locations of the network, i.e., the network is a *tree*. For each $(n, n') \in N^2$, let $R(n, n')$ be the route between n and n'. Let d be a map from N^2 to $\mathbb{R}_+$ such that for each $(n, n') \in N^2$, $d(n, n')$ denotes the length of $R(n, n')$. Then, d denotes a metric on N.

Let G be an extensive form game as follows. Let I be a nonempty finite set of players of G. Players collectively choose the location of a public facility on the network by bargaining. The bargaining consists of infinitely many rounds. In a round, a player proposes a location, and each player sequentially announces acceptance or rejection of the proposal according to a predetermined order until either all players accept the proposal or one player rejects it. If all players accept it, the bargaining ends with the proposed location implemented. Otherwise, the bargaining continues to the next round. In the first round, the proposer is a predetermined player. Otherwise, the proposer is the rejector in the previous round. Each player $i \in I$ locates at a vertex $v_i \in V$ of the network. For each $i \in I$, let d_i be a map from N to $\mathbb{R}_+$ such that for each $n \in N$, $d_i(n) = d(n, v_i)$. Then, $d_i(n)$ denotes player i's distance from location n. Each player incurs the cost to travel to the public facility by her distance. Each player obtains the benefit from the public facility by a common value $\bar{u}$. For each $i \in I$, let u_i be a map from N to $\mathbb{R}$ such that for each $n \in N$, $u_i(n) = \bar{u} - d_i(n)$. Then, $u_i(n)$ denotes player i's net benefit from the public facility located at n. We assume that for each $i \in I$ and $n \in N$, $u_i(n) > 0$, i.e., each player obtains a positive net benefit, regardless of wherever the public facility is located. Each player discounts her net benefit by a common discount factor $\delta \in [0, 1)$ as the bargaining continues. Summing up, if the bargaining ends with location n in the t-th round, each player $i \in I$ obtains payoff $\delta^{t-1} u_i(n)$. Otherwise, each player obtains payoff 0.

We consider pure strategies. Our equilibrium concept is a stationary subgame perfect equilibrium (SSPE), i.e., a subgame perfect equilibrium (SPE) in which each player takes the same actions in all rounds.

2.3 Results

In this section, we investigate the equilibria. Let $n^* \in N$ be a location such that for some $(i, i') \in \arg\max_{(\iota, \iota') \in I^2} d(v_\iota, v_{\iota'})$, $n^* \in R(v_i, v_{i'})$ and $d_i(n^*) = d_{i'}(n^*)$. Note that under our assumption that the network is a tree, such a location uniquely exists. Then, for each $(i, i') \in \arg\max_{(\iota, \iota') \in I^2} d(v_\iota, v_{\iota'})$, $n^* \in R(v_i, v_{i'})$ and $d_i(n^*) = d_{i'}(n^*)$. Let $l := \max_{(i, i') \in I^2} d(v_i, v_{i'})$: l is the maximum distance between two players' locations. Let $r := \frac{1-\delta}{1+\delta}\left(\bar{u} - \frac{l}{2}\right)$, and $A^* := \{n \in N \mid d(n, n^*) \leq r\}$. For each $i \in I$, let n_i^* be a minimizer of $d_i(n)$ with respect to $n \in A^*$. Note that under our assumption that the network is a tree, such a location uniquely exists.

Proposition 2.1 shows the existence of an SSPE, and Proposition 2.2 shows the characterization of SSPEs.

Proposition 2.1. *There exists an SSPE in G.*

Proposition 2.2. *In each SSPE in G, for each $i \in I$, player i proposes n_i^* at each proposing node, and her proposal n_i^* is accepted by all players.*

Remark. There exists a unique equilibrium location, which is the proposal of the proposer in the first round. On the other hand, there may exist multiple Condorcet locations.[6] For example, if the number of players is even, half of the players locate at a vertex, and the other half of the players locate at a distinct vertex, then all locations between the two vertices are Condorcet locations.

Figure 2.1 graphically characterizes the equilibrium location in each subgame. The equilibrium location in each subgame is in the r-closed ball of n^*. Moreover, since A^* is decreasing in δ and $\bigcap_{\delta \in [0,1)} A^* = \{n^*\}$, the equilibrium location in each subgame converges to n^* as δ tends to 1.

[6] A location $n \in N$ is a *Condorcet location* if for each $n' \in N$, $|\{i \in I \mid d_i(n') < d_i(n)\}| \leq \frac{|I|}{2}$.

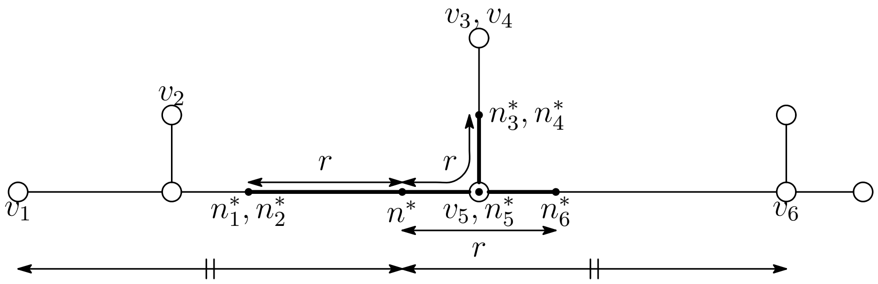


Figure 2.1: Equilibrium location in each subgame: The distance between player 1's and player 6's locations is the largest between any two players' locations. The middle location between their locations is n^*. The bold area represents A^*. The proposal of each player is the location that is the nearest to her location in A^*.

We now consider the evaluation of the equilibria from the welfare viewpoint. In equilibrium, in each subgame, the agreement is immediately achieved, i.e., no delay occurs. Hence, the important point is whether the equilibrium location is desirable or not.

We evaluate the equilibrium location based on the Rawlsian and Benthamite criteria. A location $n \in N$ is a *Rawls location* if for each $n' \in N$, $\max_{i \in I} d_i(n) \leq \max_{i \in I} d_i(n')$. That is, a Rawls location is a Rawlsian social welfare maximizer. Note that under our assumption that the network is a tree, n^* is the unique Rawls location. Hence, the equilibrium location in each subgame is in the r-closed ball of the Rawls location. Moreover, the equilibrium location in each subgame converges to the Rawls location as δ tends to 1. Therefore, the equilibrium location in each subgame is almost desirable from the Rawlsian criterion. Instead, a location $n \in N$ is a *Weber location* if for each $n' \in N$, $\sum_{i \in I} d_i(n) \leq \sum_{i \in I} d_i(n')$. That is, a Weber location is a Benthamite social welfare maximizer. Generically, the equilibrium location does not relate to Weber locations. For example, if the number of players is greater than or equal to three, one player locates at a vertex v, and the other players locate at a distinct vertex v', then n^* is the middle location between v and v', but the unique Weber location is v'. Hence, the equilibrium location in each subgame does not correspond to the Weber location both for sufficiently large δ and in the limit as δ tends to 1. Moreover, $\frac{\sum_{i \in I} d_i(n^*)}{\min_{n \in N} \sum_{i \in I} d_i(n)} = \frac{|I|}{2}$ diverges to infinity as the number of players tends to infinity. To sum up, in unanimity bargaining, the equilibrium location in each subgame corresponds to a Rawls location but not necessarily a Weber location, and $\frac{\sum_{i\in I} d_i(n^*)}{\min_{n\in N}\sum_{i\in I} d_i(n)}$ is *not* bounded above. On the other hand, as shown by Hansen and Thisse (1981), in majority voting, a Condorcet location corresponds with a Weber location but not necessarily a Rawls location, and $\frac{\max_{i\in I} d_i\left(n^C\right)}{\min_{n\in N}\max_{i\in I} d_i(n)}$ is bounded above,[7] where n^C is a Condorcet location.

2.4 Conclusion

In this chapter, we investigated a model where players collectively decide the location of a public facility on a tree network by bargaining with the unanimity rule. We proved the existence of an SSPE and characterized SSPEs. We also showed that the equilibrium location of the public facility converges to the Rawls location as the discount factor tends to 1, but it does not relate to the Weber location.

This chapter assumed that the network is a tree network, and the decision rule is the unanimity rule. The result may depend on the network or the decision rule. It remains for future research to investigate bargaining on other networks or with other decision rules.

Acknowledgments

I am very grateful to coauthor Tomohiko Kawamori. I am also grateful to supervisor Masahiro Okuno-Fujiwara and an anonymous referee of Mathematical Social Sciences for their useful comments.

Appendix

Let $\precsim$ be the total order on I such that for each $(i, i') \in I^2$ with $i \neq i'$, $i \precsim i'$ if and only if i responds earlier than i'. Let $\prec$ be the binary relation on I such that $i \prec i'$ if and only

[7]This boundedness also holds on a general network.

if $i' \precsim i$ does not hold.

Proof of Proposition 2.1 For each $i \in I$, let $A'_i := \{n \in N \mid u_i(n) \geq \delta u_i(n^*_i)\}$ and $A_i := \left(\bigcap_{\iota \succsim i} A'_\iota\right)$. Let σ be the strategy profile such that each player i proposes n^*_i at each proposing node, and each player i accepts n if and only if $n \in A_i$ at each responding node following proposal $n \in N$. It suffices to show that σ is an SSPE in G.

For each $i \in I$, since for each $i' \in I$,

$$\begin{aligned}
u_{i'}(n^*_i) &= \bar{u} - d_{i'}(n^*_i) && \text{By definition} \\
&\geq \bar{u} - (d_{i'}(n^*) + r) && \text{By } d_{i'}(n^*_i) \leq d_{i'}(n^*) + r \\
&= \delta(\bar{u} + r) + (1-\delta)\frac{l}{2} - d_{i'}(n^*) && \text{By the definition of } r \\
&\geq \delta\{\bar{u} - (d_{i'}(n^*) - r)\} && \text{By } d_{i'}(n^*) \leq \frac{l}{2} \\
&\geq \delta(\bar{u} - \max\{d_{i'}(n^*) - r, 0\}) && \\
&= \delta(\bar{u} - d_{i'}(n^*_{i'})) && \text{By } d_{i'}(n^*_{i'}) = \max\{d_{i'}(n^*) - r, 0\} \\
&= \delta u_{i'}(n^*_{i'}) && \text{By definition,}
\end{aligned}$$

in σ, at each round with player i's proposal, n^*_i is accepted by each player. Therefore, in the subgame beginning with player i's proposal, player i's payoff by σ is $u_i(n^*_i)$.

Consider each proposing node of each player i. Let $n \in \{n' \in N \mid u_i(n') > u_i(n^*_i)\}$. Then, $d_i(n^*_i) > d_i(n)$. Thus, by the definition of n^*_i, $d(n, n^*) > r$. Thus, there exists $i' \in I$ such that $d_{i'}(n) > r + \frac{l}{2}$. Hence, $u_{i'}(n) = \bar{u} - d_{i'}(n) < \bar{u} - \left(\frac{l}{2} + r\right)$. Note that by the definition of r, $\bar{u} - \left(\frac{l}{2} + r\right) = \delta\left\{\bar{u} - \left(\frac{l}{2} - r\right)\right\}$. Then, $u_{i'}(n) < \delta\left\{\bar{u} - \left(\frac{l}{2} - r\right)\right\}$. Note that by $d_i(n^*_i) > d_i(n)$, $n^*_i \neq v_i$; thus, $v_i \notin A^*$; hence, $r < d(v_i, n^*) = d_i(n^*) \leq \frac{l}{2}$ and that $d_{i'}(n^*) \leq \frac{l}{2}$. Then, $u_{i'}(n) < \delta\{\bar{u} - (\max\{d_{i'}(n^*), r\} - r)\} = \delta(\bar{u} - \max\{d_{i'}(n^*) - r, 0\})$. Note that $d_{i'}(n^*_{i'}) = \max\{d_{i'}(n^*) - r, 0\}$. Then, $u_{i'}(n) < \delta(\bar{u} - d_{i'}(n^*_{i'})) = \delta u_{i'}(n^*_{i'})$. Therefore, $n \notin A_{i'}$. Hence, by a one-stage deviation of proposing n such that $u_i(n) > u_i(n^*_i)$, player i obtains $\delta u_i(n^*_{i'}) \leq \max_{n \in A^*} u_i(n) = u_i(n^*_i)$ for some $i' \in I$. By a one-stage deviation of proposing n such that $u_i(n) \leq u_i(n^*_i)$, player i obtains $u_i(n) \leq u_i(n^*_i)$

if n is accepted by each player, and $\delta u_i\left(n_{i'}^*\right) \leq \max_{n \in A^*} u_i\left(n\right) = u_i\left(n_i^*\right)$ for some $i' \in I$ if n is rejected by some player. Therefore, player i's proposal n_i^* is optimal.

Consider each responding node of each player i following each proposal n. By rejecting n, she obtains $\delta u_i\left(n_i^*\right)$. By accepting n, she obtains (i) $u_i\left(n\right) \geq \delta u_i\left(n_i^*\right)$ if $n \in A_i$, (ii) $u_i\left(n\right) \leq \delta u_i\left(n_i^*\right)$ if $n \in \left(\bigcap_{\iota \succ i} A'_\iota\right) \setminus A'_i$, and (iii) $\delta u_i\left(n_{i'}^*\right) \leq \delta \max_{n \in A^*} u_i\left(n\right) = \delta u_i\left(n_i^*\right)$ for some $i' \in I$ if $n \notin \bigcap_{\iota \succ i} A'_\iota$. Therefore, player i's response is optimal.

Hence, σ is an SPE in G. Obviously, σ is stationary. Q.E.D.

Proof of Proposition 2.2 Let σ be an SSPE in G. For each $i \in I$, let n_i be player i's proposal in σ. For each $i \in I$, let π_i be player i's payoff by σ at her proposing node. For each $i \in I$, let A_i be the set of locations of N that player i accepts in σ. For each $i \in I$, let $A_i^O := \{n \in N \mid u_i\left(n\right) > \delta \pi_i\}$ and $A_i^C := \{n \in N \mid u_i\left(n\right) \geq \delta \pi_i\}$. Let $A := \bigcap_{i \in I} A_i$, $A^O := \bigcap_{i \in I} A_i^O$, and $A^C := \bigcap_{i \in I} A_i^C$.

Lemma 2.1. $A^O \subset A \subset A^C$.

Proof. It suffices to show that for each $i \in I$, $\bigcap_{\iota \succsim i} A_\iota^O \subset \bigcap_{\iota \succsim i} A_\iota \subset \bigcap_{\iota \succsim i} A_\iota^C$. We show this by induction. (i) Let i be the maximum of $(I, \precsim)$. Let $n \in N$. Consider a response of player i to proposal n. If she accepts it, she obtains $u_i\left(n\right)$. If she rejects it, she obtains $\delta \pi_i$. Since σ is an SPE in G, $n \in A_i$ if $u_i\left(n\right) > \delta \pi_i$, and $n \notin A_i$ if $u_i\left(n\right) < \delta \pi_i$. Then, $A_i^O \subset A_i \subset A_i^C$, i.e., $\bigcap_{\iota \succsim i} A_\iota^O \subset \bigcap_{\iota \succsim i} A_\iota \subset \bigcap_{\iota \succsim i} A_\iota^C$. (ii) Let $i, i' \in I$, and suppose that i is the successor of i'. Suppose that $\bigcap_{\iota \succsim i} A_\iota^O \subset \bigcap_{\iota \succsim i} A_\iota \subset \bigcap_{\iota \succsim i} A_\iota^C$. Let $n \in \bigcap_{\iota \succsim i'} A_\iota^O$. Consider a response of player i' to proposal n. If she rejects it, she obtains $\delta \pi_{i'}$. Since $n \in \bigcap_{\iota \succsim i'} A_\iota^O \subset \bigcap_{\iota \succsim i} A_\iota^O \subset \bigcap_{\iota \succsim i} A_\iota$, if she accepts it, she obtains $u_{i'}\left(n\right)$. Note that since $n \in \bigcap_{\iota \succsim i'} A_\iota^O$, $u_{i'}\left(n\right) > \delta \pi_{i'}$. Then, since σ is an SPE in G, $n \in A_{i'}$. Note that $n \in \bigcap_{\iota \succsim i} A_\iota$. Then, $n \in \bigcap_{\iota \succsim i'} A_\iota$. Let $n \in \bigcap_{\iota \succsim i'} A_\iota$. Consider a response of player i' to proposal n. If she rejects it, she obtains $\delta \pi_{i'}$. Since $n \in \bigcap_{\iota \succsim i'} A_\iota \subset \bigcap_{\iota \succsim i} A_\iota$, if she accepts it, she obtains $u_{i'}\left(n\right)$. Note that since $n \in \bigcap_{\iota \succsim i'} A_\iota \subset A_{i'}$, she accepts n in σ. Then, since σ is an SPE in G, $u_{i'}\left(n\right) \geq \delta \pi_{i'}$, i.e., $n \in A_{i'}^C$. Note that $n \in \bigcap_{\iota \succsim i'} A_\iota \subset \bigcap_{\iota \succsim i} A_\iota \subset \bigcap_{\iota \succsim i} A_\iota^C$. Then, $n \in \bigcap_{\iota \succsim i'} A_\iota^C$. Q.E.D.

Lemma 2.2. *For each $i \in I$, $n_i \in A$.*

Proof. Suppose that for each $i \in I$, $n_i \notin A$. Then, for each $i \in I$, $\pi_i = 0$. Thus, for each $i \in I$, $u_{i'}(n_i) > 0 = \delta \pi_{i'}$ for each $i' \in I$, and thus, $n_i \in A^O$. Therefore, by Lemma 2.1, for each $i \in I$, $n_i \in A$, which is a contradiction. Hence, for some $i' \in I$, $n_{i'} \in A$. Suppose that for some $i \in I$, $n_i \notin A$. If $\pi_i = 0$, player i can improve her payoff from 0 to $u_i(n_{i'}) > 0$ by proposing $n_{i'}$, which is a contradiction. If $\pi_i > 0$, $\pi_i = \delta^t u_i(n)$ for some $n \in A$ and $t \in \mathbb{N}$, and thus, player i can improve her payoff from $\delta^t u_i(n)$ to $u_i(n)$ by proposing n, which is a contradiction. Therefore, for each $i \in I$, $n_i \in A$. Q.E.D.

Lemma 2.3. *For each $i \in I$, $\pi_i = u_i(n_i)$ and $n_i \in \arg\max_{n \in A} u_i(n)$.*

Proof. This lemma follows Lemma 2.2. Q.E.D.

Lemma 2.4. *For each $(n, n') \in A^2$, $R(n, n') \subset A$.*

Proof. Let $n'' \in R(n, n') \setminus \{n, n'\}$ and $i \in I$. Then, since $n'' \in R(n, n') \setminus \{n, n'\}$, $u_i(n'') > \min\{u_i(n), u_i(n')\}$. Note that since $n, n' \in A \subset A^C$ by Lemma 2.1, $\min\{u_i(n), u_i(n')\} \geq \delta\pi_i$. Then, $u_i(n'') > \delta\pi_i$. Thus, $n'' \in \bigcap_{i \in I} A_i^O = A^O$. Therefore, by Lemma 2.1, $n'' \in A$. Q.E.D.

Lemma 2.5. *For each $i \in I$, there exists $(i', i'') \in I^2$ such that $n_i \in R(v_{i'}, v_{i''})$.*

Proof. Suppose that for some $i \in I$, for each $(i', i'') \in I^2$, $n_i \notin R(v_{i'}, v_{i''})$. Then, since $n_i \in A \subset A^C$ by Lemmas 2.1 and 2.2, there exists $n \in A^O$ such that $u_i(n) > u_i(n_i)$. Thus, by Lemma 2.1, there exists $n \in A$ such that $u_i(n) > u_i(n_i)$. This contradicts Lemma 2.3. Q.E.D.

Lemma 2.6. *For each $i \in I$, for each $n \in A$, $R(v_i, n_i) \cap R(n_i, n) = \{n_i\}$.*

Proof. By Lemma 2.3, $R(v_i, n_i) \cap A = \{n_i\}$. Therefore, since $R(n_i, n) \subset A$ by Lemma 2.4, $R(v_i, n_i) \cap R(n_i, n) = \{n_i\}$. Q.E.D.

Lemma 2.7. *For each $i \in I$, if $n_i \in A^O$, then $n_i = v_i$.*

Proof. Suppose that $n_i \neq v_i$. Then, since $n_i \in A^O$, by Lemma 2.1, there exists $n \in A$ such that $u_i(n) > u_i(n_i)$. This contradicts Lemma 2.3. Q.E.D.

Lemma 2.8. *Suppose that for some $(i, i') \in I^2$, $n_i = n_{i'}$ and $u_i(n_i) > u_{i'}(n_{i'})$. Then, $A \subset A_i^O$.*

Proof. Suppose that for some $n \in A$, $n \notin A_i^O$. Then,

$$\begin{aligned}
u_{i'}(n) &= \bar{u} - d_{i'}(n) && \text{By definition} \\
&= \bar{u} - d_{i'}(n_{i'}) - d(n_{i'}, n) && \text{By Lemma 2.6} \\
&= \bar{u} - d_i(n_i) - d(n_i, n) + d_i(n_i) - d_{i'}(n_{i'}) && \text{By } n_i = n_{i'} \\
&= u_i(n) + d_i(n_i) - d_{i'}(n_{i'}) && \text{By Lemma 2.6} \\
&\leq \delta u_i(n_i) + d_i(n_i) - d_{i'}(n_{i'}) && \text{By } n \notin A_i^O \text{ and Lemma 2.3} \\
&= \delta(\bar{u} - d_i(n_i)) + d_i(n_i) - d_{i'}(n_{i'}) && \text{By definition} \\
&< \delta(\bar{u} - d_{i'}(n_{i'})) && \begin{array}{c}\text{By } u_i(n_i) > u_{i'}(n_{i'}) \\ \iff d_i(n_i) < d_{i'}(n_{i'})\end{array} \\
&= \delta u_{i'}(n_{i'}) && \text{By definition} \\
&= \delta \pi_{i'} && \text{By Lemma 2.3.}
\end{aligned}$$

Thus, $n \notin A^C$, and thus, $n \notin A$ by Lemma 2.1. This is a contradiction. Q.E.D.

Lemma 2.9. *Suppose that* $\max_{i \in I} d_i(n_i) = 0$. *Then, for each $i \in I$, $n_i = n_i^*$.*

Proof. Let $(i, i') \in \arg\max_{(\iota, \iota')} d(v_\iota, v_{\iota'})$. By Lemma 2.2, $n_i \in A$ and $n_{i'} \in A$. Hence, by Lemmas 2.1 and 2.3, $u_{i'}(n_i) \geq \delta \pi_{i'} = \delta u_{i'}(n_{i'})$. Note that by the supposition of this lemma, $n_i = v_i$ and $n_{i'} = v_{i'}$. Then, $u_{i'}(v_i) \geq \delta u_{i'}(v_{i'})$. Thus, $(1 - \delta)\bar{u} \geq d_{i'}(v_i) = d(v_i, v_{i'}) = l$. Let $i \in I$. Suppose that $v_i \notin A^*$. Then, $\frac{l}{2} \geq d(v_i, n^*) > r = \frac{1-\delta}{1+\delta}\left(\bar{u} - \frac{l}{2}\right)$. Thus, $(1 - \delta)\bar{u} < l$, which is a contradiction. Hence, $v_i \in A^*$. Therefore, $n_i^* = v_i$. Note that by the supposition of this lemma, $n_i = v_i$. Then, $n_i = n_i^*$. Q.E.D.

Lemma 2.10. *Suppose that* $\max_{i \in I} d_i(n_i) > 0$. *Then, for each $i \in I$, $n_i = n_i^*$.*

Proof. Let $i^0 \in \arg\max_{i \in I} d_i(n_i)$. Then, by Lemma 2.7, $n_{i^0} \notin A^O$. Thus, for some $i^1 \in I$, $u_{i^1}(n_{i^0}) \leq \delta \pi_{i^1}$. Note that by Lemmas 2.1 and 2.2, $n_{i^0} \in A \subset A^C$, and thus, $u_{i^1}(n_{i^0}) \geq \delta \pi_{i^1}$. Then, $u_{i^1}(n_{i^0}) = \delta \pi_{i^1}$. Thus, by Lemma 2.3,

$$u_{i^1}(n_{i^0}) = \delta u_{i^1}(n_{i^1}). \tag{2.1}$$

Thus,

$$\begin{aligned}
u_{i^0}(n_{i^1}) &= u_{i^1}(n_{i^0}) - d_{i^0}(n_{i^0}) + d_{i^1}(n_{i^1}) && \text{By Lemmas 2.2 and 2.6} \\
&= \delta u_{i^1}(n_{i^1}) - d_{i^0}(n_{i^0}) + d_{i^1}(n_{i^1}) && \text{By (2.1)} \\
&= \delta u_{i^0}(n_{i^0}) + (1-\delta)\left(d_{i^1}(n_{i^1}) - d_{i^0}(n_{i^0})\right) && \text{By definition} \\
&\leq \delta u_{i^0}(n_{i^0}) && \text{By } i^0 \in \arg\max_{i \in I} d_i(n_i).
\end{aligned}$$

Note that by Lemmas 2.1, 2.2, and 2.3, $u_{i^0}(n_{i^1}) \geq \delta u_{i^0}(n_{i^0})$. Then,

$$u_{i^0}(n_{i^1}) = \delta u_{i^0}(n_{i^0}). \tag{2.2}$$

Then, by (2.1), (2.2), and Lemma 2.6,

$$d_{i^0}(n_{i^0}) = d_{i^1}(n_{i^1}) = \bar{u} - \frac{d(n_{i^0}, n_{i^1})}{1-\delta}. \tag{2.3}$$

Therefore, since $i^0 \in \arg\max_{i \in I} d_i(n_i)$, $i^1 \in \arg\max_{i \in I} d_i(n_i)$. Note that $n_{i^0} \neq n_{i^1}$.

Let $n \in R(n_{i^0}, n_{i^1})$ be the unique point such that $d(n, n_{i^0}) = d(n, n_{i^1})$. Let $n' \in A$. Suppose that $d(n', n) > \frac{d(n_{i^0}, n_{i^1})}{2}$. Then, there exists $(i', i'') \in \{i^0, i^1\}^2$ such that $d(n', n_{i'}) > d(n_{i''}, n_{i'})$. Note that by $n' \in A$ and Lemmas 2.2 and 2.6, $d_{i'}(n') = d(n', n_{i'}) + d_{i'}(n_{i'})$ and $d_{i'}(n_{i''}) = d(n_{i''}, n_{i'}) + d_{i'}(n_{i'})$. Then, $d_{i'}(n') > d_{i'}(n_{i''})$, and thus, $u_{i'}(n') < u_{i'}(n_{i''})$. Thus, by (2.1) and (2.2), $u_{i'}(n') < \delta u_{i'}(n_{i'})$. Therefore, by Lemma 2.3, $u_{i'}(n') < \delta \pi_{i'}$. Thus, $n' \notin A^C$. Hence, by Lemma 2.1, $n' \notin A$, which is a contradiction. Therefore, for each $n' \in A$, $d(n', n) \leq \frac{d(n_{i^0}, n_{i^1})}{2}$. Hence, for each $(n', n'') \in A^2$, $d(n', n'') \leq$

$d(n', n) + d(n'', n) \leq d(n_{i^0}, n_{i^1})$. Therefore, $(n_{i^0}, n_{i^1}) \in \arg\max_{(n,n') \in A^2} d(n, n')$.

Summing up, for each $(i, i') \in I^2$,

$$
\begin{aligned}
d(v_i, v_{i'}) &= d_i(n_i) + d(n_i, n_{i'}) + d_{i'}(n_{i'}) && \text{By Lemma 2.6} \\
&\leq d_{i^0}(n_{i^0}) + d(n_{i^0}, n_{i^1}) + d_{i^1}(n_{i^1}) && \begin{array}{c}\text{By } i^0, i^1 \in \arg\max_{i \in I} d_i(n_i) \\ \text{and } (n_{i^0}, n_{i^1}) \in \arg\max_{(n,n') \in A^2} d(n, n')\end{array} \\
&= d(v_{i^0}, v_{i^1}) && \text{By Lemma 2.6.}
\end{aligned}
$$

Therefore, $d(v_{i^0}, v_{i^1}) = \max_{(i,i') \in I^2} d(v_i, v_{i'}) = l$, $n^* \in R(v_{i^0}, v_{i^1})$, and $d_{i^0}(n^*) = d_{i^1}(n^*) = \frac{l}{2}$. By (2.3) and Lemma 2.6, $n^* \in R(n_{i^0}, n_{i^1})$ and $d(n_{i^0}, n^*) = d(n_{i^1}, n^*) = \frac{1-\delta}{1+\delta}\left(\bar{u} - \frac{l}{2}\right) = r$.

Let $n \in \{n' \in N \mid d(n', n^*) < r\}$. Let $i \in I$. Then, $u_i(n) > \min\{u_i(n_{i^0}), u_i(n_{i^1})\}$. Note that since $n_{i^0}, n_{i^1} \in A \subset A^C$ by Lemmas 2.1 and 2.2, $\min\{u_i(n_{i^0}), u_i(n_{i^1})\} \geq \delta\pi_i$. Then, $u_i(n) > \delta\pi_i$. Thus, by Lemma 2.1, $n \in A$. Therefore, $\{n' \in N \mid d(n', n^*) < r\} \subset A$. Let $n \in \{n' \in N \mid d(n', n^*) > r\}$. Then, for some $(i, i') \in \{i^0, i^1\}^2$ such that $i \neq i'$, $u_i(n) < u_i(n_{i'})$. Note that by (2.1) and (2.2), $u_i(n_{i'}) = \delta u_i(n_i)$. Then, $u_i(n) < \delta u_i(n_i)$. Thus, by Lemmas 2.1 and 2.3, $n \notin A$. Therefore, $A \subset \{n' \in N \mid d(n', n^*) \leq r\}$.

Suppose that for some $i \in I$, $n_i \neq n_i^*$. Let $n \in R(n_i, n_i^*) \setminus \{n_i, n_i^*\}$. Then, $d(n, n^*) < \max\{d(n_i, n^*), d(n_i^*, n^*)\}$. Note that since $n_i, n_i^* \in \{n' \in N \mid d(n', n^*) \leq r\}$ by Lemma 2.2 and the definition of n_i^*, $\max\{d(n_i, n^*), d(n_i^*, n^*)\} \leq r$. Then, $d(n, n^*) < r$. Thus, $n \in \{n' \in N \mid d(n', n^*) < r\}$. Therefore, $R(n_i, n_i^*) \setminus \{n_i, n_i^*\} \subset \{n' \in N \mid d(n', n^*) < r\}$ and $R(n_i, n_i^*) \subset \{n' \in N \mid d(n', n^*) \leq r\}$. By the definition of n_i^*, it follows that $R(v_i, n_i^*) \cap \{n' \in N \mid d(n', n^*) \leq r\} = \{n_i^*\}$. Therefore, since $R(n_i^*, n_i) \subset \{n' \in N \mid d(n', n^*) \leq r\}$, $R(v_i, n_i^*) \cap R(n_i^*, n_i) = \{n_i^*\}$. Hence, for each $n \in R(n_i, n_i^*) \setminus \{n_i, n_i^*\}$, $u_i(n) > u_i(n_i)$. This contradicts Lemma 2.3. Q.E.D.

The conclusion of Proposition 2.2 follows Lemmas 2.2, 2.9, and 2.10. Q.E.D.

References

Bandelt, H. -J. (1985). Networks with Condorcet solutions. *European Journal of Operational Research*, 20, 314–326.

Bandelt, H. -J., Labbé, M. (1986). How bad can a voting location be. *Social Choice and Welfare*, 3, 125–145.

Banks, J. S., Duggan, J. (2000). A bargaining model of collective choice. *American Political Science Review*, 94, 73–88.

Campos Rodríguez, C. M., Moreno Pérez, J. A. (2000). The tolerant Condorcet points. *Studies in Locational Analysis*, 14, 173–190.

Campos Rodríguez, C. M., Moreno Pérez, J. A. (2003). Relaxation of the Condorcet and Simpson conditions in voting location. *European Journal of Operational Research*, 145, 673–683.

Cardona, D., Ponsatí, C. (2007). Bargaining one-dimensional social choices. *Journal of Economic Theory*, 137, 627–651.

Hansen, P., Thisse, J. F. (1981). Outcomes of voting and planning: Condorcet, Weber and Rawls locations. *Journal of Public Economics*, 16, 1–15.

Labbe, M. (1985). Outcomes of voting and planning in single facility location problems. *European Journal of Operational Research*, 20, 299–313.

Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica*, 50, 97–110.

Chapter 3

Location of an undesirable facility on a network: a bargaining approach

Abstract: We examine a model where, on a line network, individuals collectively choose the location of an undesirable public facility through bargaining with the unanimity rule. We show the existence of a stationary subgame perfect equilibrium and the characterization of stationary subgame perfect equilibria when the discount factor is sufficiently large. Furthermore, we show that as the discount factor tends to 1, the equilibrium location can converge to a location that is least desirable according to both the Benthamite and Rawlsian criteria.

Keywords: Location of undesirable public facility; Network; Bargaining with the unanimity rule; Benthamite criterion; Rawlsian criterion

JEL Classification: D72; C78; R53; H41

Based article: Yamaguchi, K. (2011). Location of an undesirable facility on a network: a bargaining approach. *Mathematical Social Sciences*, 62, 104–108.

3.1 Introduction

The choice of the location of a public facility is an important problem from the perspective of welfare. For a desirable public facility—a public facility such that each individual prefers that the facility be built as near as possible to his/her location—such as a park, it is socially desirable to choose a *minisum location*, which minimizes the average distance from an individual's location, according to the Benthamite criterion, or a *minimax location*, which minimizes the maximum distance from an individual's location, according to the Rawlsian criterion. For an undesirable public facility—a public facility such that each individual prefers that the facility be built as far as possible from his/her location—such as a nuclear power plant, it is socially desirable to choose a *maxisum location*, which maximizes the average distance from an individual's location, according to the Benthamite criterion, or a *maximin location*, which maximizes the minimum distance from an individual's location, according to the Rawlsian criterion. Then, if individuals collectively choose the location of a public facility, is the chosen location socially desirable?

Hansen and Thisse (1981) considered a model where individuals collectively choose the location of a desirable public facility through majority voting by employing a *Condorcet location*—a location such that no other location is closer to a strict majority of the individuals—as a solution. Hansen and Thisse (1981) showed that on a tree network, a Condorcet location exists, and a Condorcet location is a minisum location. Furthermore, Hansen and Thisse (1981) showed that on a general network, if a Condorcet location exists, the ratio of the average distance from an individual's location to a Condorcet location to that to a minisum location is bounded above. Hansen and Thisse (1981) also showed that on a general network, if a Condorcet location exists, the ratio of the maximum distance from an individual's location to a Condorcet location to that to a minimax location is bounded above. Labbé (1990) considered a model where individuals collectively choose the location of an undesirable public facility through majority voting by employing an *anti-Condorcet location*—a location such that no other location is farther from a strict majority of the individuals—as a solution. Labbé (1990) showed that on a general network, if an anti-Condorcet location exists, the ratio of the average distance from an individual's location to a maxisum location to that to an anti-Condorcet location is bounded above. Kawamori and Yamaguchi (2010) considered a model where individuals collectively choose the location of a desirable public facility through bargaining with the unanimity rule by employing the equilibrium location in the alternating-offer bargaining formulated by Rubinstein (1982) as a solution.[1] Kawamori and Yamaguchi (2010) showed that on a tree network, as the discount factor tends to 1, the equilibrium location converges to a minimax location. These results imply that the collective choice of the location of a desirable public facility through majority voting, that of an undesirable public facility through majority voting, and that of a desirable public facility through bargaining with the unanimity rule result in a favorable outcome according to at least either the Benthamite or Rawlsian criterion.

With these results in mind, we consider a model where, on a line network, individuals collectively choose the location of an undesirable public facility through bargaining with the unanimity rule by employing the equilibrium location in the alternating-offer bargaining as a solution. We consider only the case where each individual obtains a non-negative net benefit regardless of wherever the facility is located.[2] We show the existence of a stationary subgame perfect equilibrium and the characterization of stationary subgame perfect equilibria when the discount factor is sufficiently large. Furthermore, we show that as the discount factor tends to 1, the equilibrium location can converge to a location that minimizes both the average and minimum distance from an individual's location. This result implies that the collective choice of the location of an undesirable public facility through bargaining with the unanimity rule can result in the worst outcome according

[1] Herings and Predtetchinski (2010) also considered a similar model.

[2] In reality, when an undesirable public facility is built in a certain area, while each individual in the area suffers a loss such as a bad odor, noise, and the threat of a serious accident, he/she can often obtain a benefit such as the public favorable treatment of taxes and subsidies, a job offer from the facility, and the enjoyment of beneficial services that the facility provides. The case that we consider is just the case where the benefit that each individual obtains is greater than or equal to the loss he/she suffers regardless of wherever the facility is located.

to both the Benthamite and Rawlsian criteria.

Hence, if individuals collectively choose the location of a public facility, the chosen location may or may not be socially desirable according to the type of public facility and the rule of collective choice. Thus, a need for some intervention from the perspective of welfare may arise.

The remainder of this chapter is organized as follows. Section 3.2 describes the model, Section 3.3 characterizes the equilibria, and Section 3.4 concludes the chapter. We provide the proof of propositions in the Appendix.

3.2 The model

In this section, we describe a model analogous to that of Kawamori and Yamaguchi (2010) where, on a line network, individuals collectively choose the location of an undesirable public facility through bargaining with the unanimity rule.

Suppose that $N := [0, 1]$. We call N a *network*, and $n \in N$ a *location*. Let d be a map from N^2 to $\mathbb{R}_+$ such that for each $(n, n') \in N^2$, $d\,(n, n') = |n - n'|$. Then, d denotes a metric on N.

Let G be an extensive form game as follows. Let $I := \{1, \cdots, p\}$ be the set of players of G, where $p \geq 2$. The players collectively choose the location of an undesirable public facility on the network through bargaining. The bargaining comprises infinite rounds. In a round, a player proposes a location, and each player sequentially announces his/her acceptance or rejection of the proposal according to a predetermined order until either all players accept the proposal or one player rejects it. Let $\precsim$ be the total order on I such that for each $(i, i') \in I^2$ with $i \neq i'$, $i \precsim i'$ if and only if i responds earlier than i'. Let $\prec$ be the binary relation on I such that $i \prec i'$ if and only if $i' \precsim i$ does not hold. If all players accept it, the bargaining ends with the acceptance of the proposed location. Otherwise, the bargaining proceeds to the next round. In the first round, the proposer is a predetermined player. Otherwise, the proposer is the rejector in the previous round.

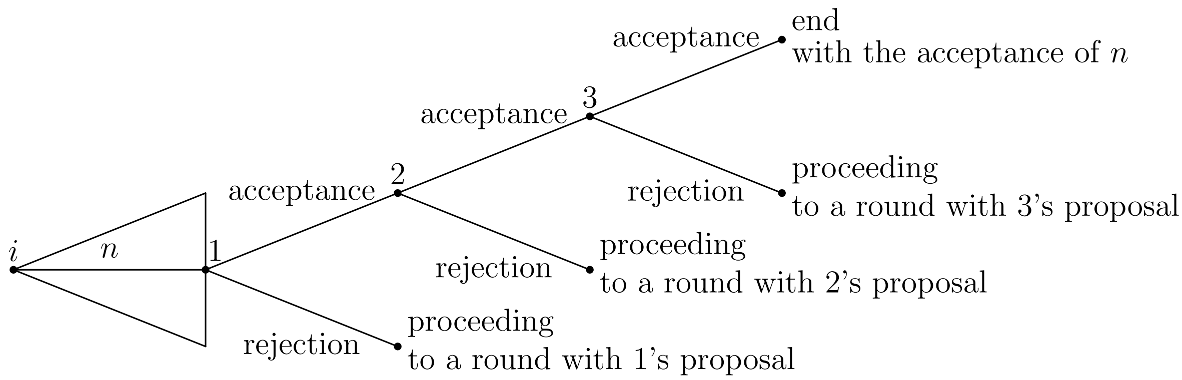


Figure 3.1: A round with i's proposal in the case where $I = \{1, 2, 3\}$ and $1 \prec 2 \prec 3$.

Figure 3.1 shows a round with player i's proposal in the case where $I = \{1, 2, 3\}$ and $1 \prec 2 \prec 3$. Each player $i \in I$ is located at a location $v_i \in N$ of the network. We assume that $0 = v_1 \leq \cdots \leq v_p = 1$. Suppose that $V := \{v_i | i \in I\}$. For each $i \in I$, let d_i be a map from N to $\mathbb{R}_+$ such that for each $n \in N$, $d_i(n) = d(n, v_i)$. Then, $d_i(n)$ denotes player i's distance from location n. We assume that each player $i \in I$ obtains payoff $\delta^{t-1} d_i(n)$ if the bargaining ends with location n in the tth round and payoff 0 otherwise, where $\delta \in [0, 1)$ denotes a common discount factor.[3]

We consider pure strategies in this chapter. A strategy of a player prescribes what location he/she proposes at his/her proposing node and how he/she responds at his/her responding node. Our equilibrium concept is a stationary subgame perfect equilibrium (SSPE), that is, a subgame perfect equilibrium (SPE) in which each player takes the same actions in all rounds.

3.3 Results

In this section, we investigate the equilibria.

[3]Note that each individual prefers that the facility be built as far as possible from his/her location. Furthermore, note that each individual obtains a non-negative net benefit regardless of where the facility is located. In this case, the bargaining ends, that is, the facility is built somewhere. However, we can also consider the case where an individual can obtain a negative net benefit. In this case, it is possible that the bargaining does not end, that is, the facility is not built anywhere.

Suppose that

$$\mathfrak{I} := \Bigg\{ \left\{i^0, i^1\right\} \subset I | v_{i^0} \neq v_{i^1}, v_i = \frac{v_{i^0} + v_{i^1}}{2} \text{ for each } i \in I \\ \text{such that } \min\left\{v_{i^0}, v_{i^1}\right\} < v_i < \max\left\{v_{i^0}, v_{i^1}\right\} \Bigg\}.$$

Then, $\mathfrak{I}$ denotes the family of sets of two players such that their locations are different, and players located between their locations are located on just the middle location between their locations.

For each $\{i^0, i^1\} \in \mathfrak{I}$ such that $0 < \min\{v_{i^0}, v_{i^1}\}$, let

$$\delta_L\left(\left\{i^0, i^1\right\}\right) := \frac{\sqrt{\left(\max\left\{v_{i^0}, v_{i^1}\right\} - \min\left\{v_{i^0}, v_{i^1}\right\}\right)^2 + \left(\min\left\{v_{i^0}, v_{i^1}\right\}\right)^2}}{\min\left\{v_{i^0}, v_{i^1}\right\}} \\ - \frac{\max\left\{v_{i^0}, v_{i^1}\right\} - \min\left\{v_{i^0}, v_{i^1}\right\}}{\min\left\{v_{i^0}, v_{i^1}\right\}}.$$

For each $\{i^0, i^1\} \in \mathfrak{I}$ such that $\max\{v_{i^0}, v_{i^1}\} < 1$, let

$$\delta_R\left(\left\{i^0, i^1\right\}\right) := \frac{\sqrt{\left(\max\left\{v_{i^0}, v_{i^1}\right\} - \min\left\{v_{i^0}, v_{i^1}\right\}\right)^2 + \left(1 - \max\left\{v_{i^0}, v_{i^1}\right\}\right)^2}}{1 - \max\left\{v_{i^0}, v_{i^1}\right\}} \\ - \frac{\max\left\{v_{i^0}, v_{i^1}\right\} - \min\left\{v_{i^0}, v_{i^1}\right\}}{1 - \max\left\{v_{i^0}, v_{i^1}\right\}}.$$

For each $\{i^0, i^1\} \in \mathfrak{I}$, suppose that

$$\delta\left(\left\{i^0, i^1\right\}\right) := \begin{cases} 0 & \text{if } 0 = \min\left\{v_{i^0}, v_{i^1}\right\} < \max\left\{v_{i^0}, v_{i^1}\right\} = 1 \\ \delta_R\left(\left\{i^0, i^1\right\}\right) & \text{if } 0 = \min\left\{v_{i^0}, v_{i^1}\right\} < \max\left\{v_{i^0}, v_{i^1}\right\} < 1 \\ \delta_L\left(\left\{i^0, i^1\right\}\right) & \text{if } 0 < \min\left\{v_{i^0}, v_{i^1}\right\} < \max\left\{v_{i^0}, v_{i^1}\right\} = 1 \\ \max\{\delta_L\left(\left\{i^0, i^1\right\}\right), \\ \quad \delta_R\left(\left\{i^0, i^1\right\}\right)\} & \text{if } 0 < \min\left\{v_{i^0}, v_{i^1}\right\} < \max\left\{v_{i^0}, v_{i^1}\right\} < 1. \end{cases}$$

Suppose that $\hat{\delta} = \max_{\{i^0, i^1\} \in \mathfrak{I}} \delta\left(\left\{i^0, i^1\right\}\right)$. Note that $\hat{\delta} \in [0, 1)$.

Suppose that $M := \left\{n \in N | n = \frac{v_i + v_{i'}}{2} \text{ for some } i, i' \in I \text{ such that } v_i \neq v_{i'}\right\}$. Suppose that $\ell := \min_{m \in M} \min_{i \in \{i \in I | v_i \neq m\}} d_i(m)$. Suppose that $\tilde{\delta} := \frac{1-2\ell}{2\ell+1}$. Note that $\tilde{\delta} \in [0,1)$.

For each $\{i^0, i^1\} \subset I$, suppose that

$$C\left(\left\{i^0, i^1\right\}\right) := \left\{n \in N | d\left(n, \frac{v_{i^0} + v_{i^1}}{2}\right) = \frac{1-\delta}{1+\delta}\frac{d\left(v_{i^0}, v_{i^1}\right)}{2}\right\} = \left\{\frac{v_{i^0} + \delta v_{i^1}}{1+\delta}, \frac{v_{i^1} + \delta v_{i^0}}{1+\delta}\right\}.$$

Then, $C\left(\{i^0, i^1\}\right)$ denotes the set of locations such that the distance from the middle location between player i^0's and player i^1's locations is $\frac{1-\delta}{1+\delta}\frac{d\left(v_{i^0}, v_{i^1}\right)}{2}$. Note that $\lim_{\delta \to 1} \frac{v_{i^0} + \delta v_{i^1}}{1+\delta} = \frac{v_{i^0} + v_{i^1}}{2}$ and $\lim_{\delta \to 1} \frac{v_{i^1} + \delta v_{i^0}}{1+\delta} = \frac{v_{i^0} + v_{i^1}}{2}$. Furthermore, for each $\{i^0, i^1\} \in \mathfrak{I}$, let $\Sigma\left(\{i^0, i^1\}\right)$ be the set of strategy profiles such that for each $i \in I$, player i proposes $n_i \in \arg\max_{n \in C(\{i^0, i^1\})} d_i(n)$ at each proposing node, and his/her proposal n_i is accepted by all players. We call $\sigma \in \Sigma\left(\{i^0, i^1\}\right)$ a strategy profile of type $\{i^0, i^1\}$. Note that as δ tends to 1, the chosen location in each subgame converges to $\frac{v_{i^0} + v_{i^1}}{2}$ in each strategy profile of type $\{i^0, i^1\}$.

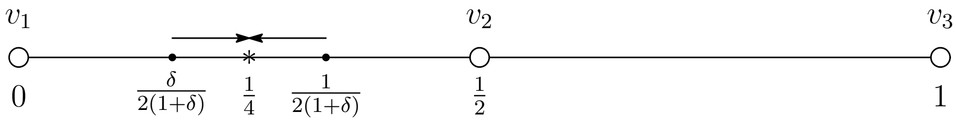
Propositions 3.1 and 3.2 show the characterization of SSPEs.

Proposition 3.1. *Suppose that $\delta \in \left(\hat{\delta}, 1\right)$. Then, for each $\{i^0, i^1\} \in \mathfrak{I}$, there exists an SSPE σ in G such that $\sigma \in \Sigma\left(\{i^0, i^1\}\right)$.*

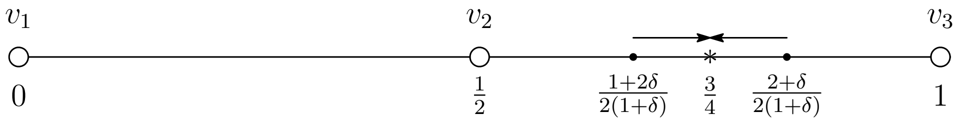
Proposition 3.2. *Suppose that $\delta \in \left[\tilde{\delta}, 1\right)$. Then, for each SSPE σ in G, for some $\{i^0, i^1\} \in \mathfrak{I}$, $\sigma \in \Sigma\left(\{i^0, i^1\}\right)$.*

Note that the equilibrium location can converge to $n \in N$ as δ tends to 1 if and only if $n = \frac{v_{i^0} + v_{i^1}}{2}$ for some $i^0, i^1 \in I$ such that $\{i^0, i^1\} \in \mathfrak{I}$. For example, consider the case where $I = \{1, 2, 3\}$, $v_1 = 0$, $v_2 = \frac{1}{2}$, and $v_3 = 1$. Figure 3.2 graphically characterizes the equilibrium location in each subgame. If δ is sufficiently large, each equilibrium is a strategy profile of type $\{1, 2\}$, $\{2, 3\}$, or $\{1, 3\}$. As δ tends to 1, the equilibrium location in each subgame converges to $\frac{1}{4}$ in each equilibrium of type $\{1, 2\}$, $\frac{3}{4}$ in each equilibrium of type $\{2, 3\}$, and $\frac{1}{2}$ in each equilibrium of type $\{1, 3\}$.

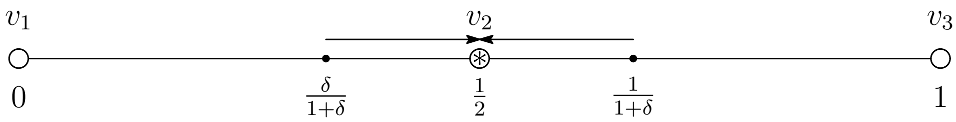
We now evaluate the equilibria from the perspective of welfare. In each equilibrium, in each subgame, the agreement is immediately achieved, that is, no delay occurs. Hence, the



(i) $\sigma \in \Sigma(\{1,2\})$



(ii) $\sigma \in \Sigma(\{2,3\})$



(iii) $\sigma \in \Sigma(\{1,3\})$

Figure 3.2: Equilibrium location in each subgame in the case where $I = \{1,2,3\}$, $v_1 = 0$, $v_2 = \frac{1}{2}$, and $v_3 = 1$: the proposal of each player is a location that is the farthest from his/her location in the locations represented by dots.

important point is whether the equilibrium location is desirable or not. We evaluate the equilibrium location according to the Benthamite and Rawlsian criteria. For example, consider the case where $I = \{1,2,3\}$, $v_1 = 0$, $v_2 = \frac{1}{2}$, and $v_3 = 1$ again. Note that $\arg\max_{n\in N} \sum_{i\in I} d_i(n) = \{0,1\}$ and $\arg\min_{n\in N} \sum_{i\in I} d_i(n) = \left\{\frac{1}{2}\right\}$. That is, according to the Benthamite criterion, the most desirable location is 0 or 1 and the least desirable location is $\frac{1}{2}$. Note that $\arg\max_{n\in N} \min_{i\in I} d_i(n) = \left\{\frac{1}{4}, \frac{3}{4}\right\}$ and $\arg\min_{n\in N} \min_{i\in I} d_i(n) = \left\{0, \frac{1}{2}, 1\right\}$. That is, according to the Rawlsian criterion, the most desirable location is $\frac{1}{4}$ or $\frac{3}{4}$ and the least desirable location is 0, $\frac{1}{2}$, or 1. Hence, as δ tends to 1, the equilibrium location in each subgame converges to a location that is most desirable according to the Rawlsian criterion in each equilibrium of type $\{1,2\}$ or $\{2,3\}$. On the other hand, as δ tends to 1, the equilibrium location in each subgame converges to a location that is least desirable according to both the Benthamite and Rawlsian criteria in each equilibrium of type $\{1,3\}$.[4]

[4] In general, note that a location is least desirable according to the Rawlsian criterion if and only if the location is an individual's location. Hence, the equilibrium location can converge to a location that

As shown by Hansen and Thisse (1981), Labbé (1990), and Kawamori and Yamaguchi (2010), the collective choice of the location of a desirable public facility through majority voting, that of an undesirable public facility through majority voting, and that of a desirable public facility through bargaining with the unanimity rule result in a favorable outcome according to at least either the Benthamite or Rawlsian criterion. On the other hand, as shown above, the collective choice of the location of an undesirable public facility through bargaining with the unanimity rule can result in the worst outcome according to both the Benthamite and Rawlsian criteria. Hence, if individuals collectively choose the location of a public facility, the chosen location may or may not be socially desirable according to the type of public facility and the rule of collective choice. Thus, a need for some intervention from the perspective of welfare may arise.

3.4 Conclusion

We examined a model where, on a line network, individuals collectively choose the location of an undesirable public facility through bargaining with the unanimity rule. We showed the existence of a stationary subgame perfect equilibrium and the characterization of stationary subgame perfect equilibria when the discount factor is sufficiently large. Furthermore, we showed that as the discount factor tends to 1, the equilibrium location can converge to a location that is least desirable according to both the Benthamite and Rawlsian criteria.

We examined a model where the network is a line network, the protocol of the selection of proposers is the rejector-proposer protocol, and the decision rule is the unanimity rule. The examination of a model with other networks such as a tree network, other protocols for the selection of proposers such as the random-proposer protocol, and other decision rules such as the majority rule is left for future research.

is least desirable according to the Rawlsian criterion as δ tends to 1 if and only if there exists $i^0, i^1, i^2 \in I$ such that $\left\{i^0, i^1\right\} \in \mathfrak{I}$ and $v_{i^2} = \frac{v_{i^0}+v_{i^1}}{2}$.

Acknowledgments

The author is grateful to Tomohiko Kawamori, an anonymous associate editor, and two anonymous referees of Mathematical Social Sciences for their useful comments.

Appendix

Proof of Proposition 3.1 Suppose that $\{i^0, i^1\} \in \mathfrak{I}$. Without loss of generality, suppose that $v_{i^0} < v_{i^1}$. For each $i \in I$, suppose that $n_i \in \arg\max_{n \in C(\{i^0,i^1\})} d_i(n)$. Then, note that for each $i \in I$,

$$n_i \in \begin{cases} \{n_{i^1}\} & \text{if } v_i < \frac{v_{i^0}+v_{i^1}}{2} \\ \{n_{i^1}, n_{i^0}\} & \text{if } v_i = \frac{v_{i^0}+v_{i^1}}{2} \\ \{n_{i^0}\} & \text{if } v_i > \frac{v_{i^0}+v_{i^1}}{2}. \end{cases}$$

For each $i \in I$, suppose that $\bar{A}_i := \{n \in N \mid d_i(n) \geq \delta d_i(n_i)\}$. For each $i \in I$, suppose that $A_i := \left(\bigcap_{\iota \succsim i} \bar{A}_\iota\right)$. Then, note that under our assumption that $\delta \in \left(\hat{\delta}, 1\right)$,

$$\begin{aligned} \cap_{i\in I} A_i &= \cap_{i \in I} \bar{A}_i \\ &= \begin{cases} \left[n_{i^1}, n_{i^0}\right] & \begin{array}{l}\text{if there does not exist } i \in I \\ \text{such that } v_i = \frac{v_{i^0}+v_{i^1}}{2}\end{array} \\ \left[n_{i^1}, \frac{(1+\delta)n_{i^1}+(1-\delta)n_{i^0}}{2}\right] \cup \left[\frac{(1-\delta)n_{i^1}+(1+\delta)n_{i^0}}{2}, n_{i^0}\right] & \text{otherwise.} \end{cases} \end{aligned}$$

Let σ be the strategy profile such that each player i proposes n_i at each proposing node, and each player i accepts n if and only if $n \in A_i$ at each responding node following proposal $n \in N$.

Note that for each $i \in I$, $n_i \in \cap_{i \in I} A_i$. Therefore, in the subgame beginning with player i's proposal, player i's payoff by σ is $d_i(n_i)$.

Consider each proposing node of each player i. For each $n \in \{n' \in N | d_i(n') > d_i(n_i)\}$, there exists $i' \in I$ such that $n \notin A_{i'}$. Hence, by a one-stage deviation of proposing n such that $d_i(n) > d_i(n_i)$, player i obtains $\delta d_i(n_{i'}) \leq d_i(n_i)$ for some $i' \in I$. By a one-stage deviation of proposing n such that $d_i(n) \leq d_i(n_i)$, player i obtains $d_i(n) \leq d_i(n_i)$ if n is accepted by each player, and $\delta d_i(n_{i'}) \leq d_i(n_i)$ for some $i' \in I$ if n is rejected by some player. Therefore, player i's proposal n_i is optimal.

Consider each responding node of each player i following each proposal n. By rejecting n, player i obtains $\delta d_i(n_i)$. By accepting n, player i obtains (i) $d_i(n) \geq \delta d_i(n_i)$ if $n \in A_i$, (ii) $d_i(n) \leq \delta d_i(n_i)$ if $n \in \left(\bigcap_{\iota \succ i} \bar{A}_\iota\right) \setminus \bar{A}_i$, and (iii) $\delta d_i(n_{i'}) \leq \delta d_i(n_i)$ for some $i' \in I$ if $n \notin \bigcap_{\iota \succ i} \bar{A}_\iota$. Therefore, player i's response is optimal.

Hence, σ is an SPE in G. Obviously, σ is stationary and $\sigma \in \Sigma(\{i^0, i^1\})$. Q.E.D.

Proof of Proposition 3.2 Let σ be an SSPE in G. For each $i \in I$, let n_i be player i's proposal in σ. For each $i \in I$, let π_i be player i's payoff by σ at his/her proposing node. For each $i \in I$, let A_i be the set of locations of N that player i accepts in σ. For each $i \in I$, suppose that $A_i^O := \{n \in N \mid d_i(n) > \delta\pi_i\}$ and $A_i^C := \{n \in N \mid d_i(n) \geq \delta\pi_i\}$. Suppose that $A := \bigcap_{i \in I} A_i$, $A^O := \bigcap_{i \in I} A_i^O$, and $A^C := \bigcap_{i \in I} A_i^C$.

Lemma 3.1. $A^O \subset A \subset A^C$.

Proof. It suffices to show that for each $i \in I$, $\bigcap_{\iota \succsim i} A_\iota^O \subset \bigcap_{\iota \succsim i} A_\iota \subset \bigcap_{\iota \succsim i} A_\iota^C$. We show this by induction. (i) Let i be the maximum of $(I, \precsim)$. Suppose that $n \in N$. Consider a response of player i to proposal n. If the player accepts it, he/she obtains $d_i(n)$. If the player rejects it, he/she obtains $\delta\pi_i$. Since σ is an SPE in G, $n \in A_i$ if $d_i(n) > \delta\pi_i$, and $n \notin A_i$ if $d_i(n) < \delta\pi_i$. Then, $A_i^O \subset A_i \subset A_i^C$, that is, $\bigcap_{\iota \succsim i} A_\iota^O \subset \bigcap_{\iota \succsim i} A_\iota \subset \bigcap_{\iota \succsim i} A_\iota^C$. (ii) Suppose that $i, i' \in I$, and suppose that i is the successor of i'. Suppose that $\bigcap_{\iota \succsim i} A_\iota^O \subset \bigcap_{\iota \succsim i} A_\iota \subset \bigcap_{\iota \succsim i} A_\iota^C$. Suppose that $n \in \bigcap_{\iota \succsim i'} A_\iota^O$. Consider a response of player i' to proposal n. If the player rejects it, he/she obtains $\delta\pi_{i'}$. Since $n \in \bigcap_{\iota \succsim i'} A_\iota^O \subset \bigcap_{\iota \succsim i} A_\iota^O \subset \bigcap_{\iota \succsim i} A_\iota$, if the player accepts it, he/she obtains $d_{i'}(n)$. Note that since $n \in \bigcap_{\iota \succsim i'} A_\iota^O$, $d_{i'}(n) > \delta\pi_{i'}$. Then, since σ is an SPE in G, $n \in A_{i'}$. Note that $n \in \bigcap_{\iota \succsim i} A_\iota$. Then, $n \in \bigcap_{\iota \succsim i'} A_\iota$. Suppose that $n \in \bigcap_{\iota \succsim i'} A_\iota$. Consider a response of player i' to proposal n. If the player rejects it, he/she obtains $\delta\pi_{i'}$. Since $n \in \bigcap_{\iota \succsim i'} A_\iota \subset \bigcap_{\iota \succsim i} A_\iota$, if the player accepts it, he/she obtains $d_{i'}(n)$. Note that since $n \in \bigcap_{\iota \succsim i'} A_\iota \subset A_{i'}$, the player accepts n in σ. Then, since σ is an SPE in G, $d_{i'}(n) \geq \delta\pi_{i'}$, that is, $n \in A_{i'}^C$. Note that $n \in \bigcap_{\iota \succsim i'} A_\iota \subset \bigcap_{\iota \succsim i} A_\iota \subset \bigcap_{\iota \succsim i} A_\iota^C$. Then, $n \in \bigcap_{\iota \succsim i'} A_\iota^C$. Q.E.D.

Lemma 3.2. *A is neither the empty set nor a singleton.*

Proof. Suppose that $A = \emptyset$. Then, for each $i \in I$, $\pi_i = 0$. Thus, for each $n \in N \setminus V$, $d_i(n) > 0 = \delta\pi_i$ for each $i \in I$; thus, $n \in A^O$. Therefore, by Lemma 3.1, for each $n \in N \setminus V$, $n \in A$, which is a contradiction.

Suppose that for some $n \in N$, $A = \{n\}$. Suppose that $v_i = n$. Then, $\pi_i = 0$. Suppose that $v_i \neq n$. Consider a proposal of player i. If the player proposes n, he/she obtains $d_i(n)$. If the player proposes $n' \neq n$, he/she obtains (i) $\delta^t d_i(n) < d_i(n)$ for some $t \in \mathbb{N}$ if $n_{i'} = n$ for some $i' \in I$, and (ii) $0 < d_i(n)$ if $n_{i'} \neq n$ for each $i' \in I$. Thus, since σ is an SPE in G, $n_i = n$; thus, $\pi_i = d_i(n)$. Suppose that $\epsilon := \min_{i \in \{i' \in I | v_{i'} \neq n\}} d_i(n)$. Suppose that $n' \in \{n'' \in N | d(n'', n) < (1-\delta)\epsilon\} \setminus \{n\}$. Then, for each $i \in I$ such that $v_i = n$, $d_i(n') > 0 = \delta\pi_i$; thus, $n' \in A_i^O$. For each $i \in I$ such that $v_i \neq n$,

$$\begin{aligned}
d_i(n') &\geq d_i(n) - d(n', n) && \text{By triangle inequality} \\
&> d_i(n) - (1-\delta)\epsilon && \text{By } n' \in \{n'' \in N | d(n'', n) < (1-\delta)\epsilon\} \setminus \{n\} \\
&\geq d_i(n) - (1-\delta) d_i(n) && \text{By the definition of } \epsilon \\
&= \delta d_i(n) \\
&= \delta\pi_i;
\end{aligned}$$

thus, $n' \in A_i^O$. Hence, $n' \in A^O$. Therefore, $\{n'' \in N | d(n'', n) < (1-\delta)\epsilon\} \setminus \{n\} \subset A^O$; thus, by Lemma 3.1, $\{n'' \in N | d(n'', n) < (1-\delta)\epsilon\} \setminus \{n\} \subset A$, which is a contradiction. Q.E.D.

Lemma 3.3. *For each $i \in I$, $n_i \in A$.*

Proof. Suppose that for some $i \in I$, $n_i \notin A$. Note that by Lemma 3.2, $A \setminus \{v_i\} \neq \emptyset$. If

$\pi_i = 0$, player i can improve his/her payoff from 0 to $d_i(n) > 0$ by proposing $n \in A \setminus \{v_i\}$, which is a contradiction. If $\pi_i > 0$, $\pi_i = \delta^t d_i(n)$ for some $n \in A \setminus \{v_i\}$ and $t \in \mathbb{N}$; thus, player i can improve his/her payoff from $\delta^t d_i(n)$ to $d_i(n)$ by proposing n, which is a contradiction. Q.E.D.

Lemma 3.4. *For each $i \in I$, $\pi_i = d_i(n_i)$ and $n_i \in \arg\max_{n \in A} d_i(n)$.*

Proof. This lemma follows Lemma 3.3. Q.E.D.

Lemma 3.5. *For each $i \in I$, $n_i \notin A^O$.*

Proof. Suppose that for some $i \in I$, $n_i \in A^O$. Then, by Lemma 3.1, there exists $n \in A$ such that $d_i(n) > d_i(n_i)$. This contradicts Lemma 3.4. Q.E.D.

Lemma 3.6. *For each $i, i' \in I$, $n_i \neq v_{i'}$.*

Proof. Suppose that for some $i, i' \in I$, $n_i = v_{i'}$. Then, by Lemmas 3.1 and 3.4, $d_{i'}(n_{i'}) = 0$. This contradicts Lemmas 3.2 and 3.4. Q.E.D.

Lemma 3.7. $|\{n_i | i \in I\}| = 2$.

Proof. Suppose that $|\{n_i | i \in I\}| = 1$. Then, by Lemma 3.6, for each $i \in I$, $d_{i'}(n_i) > \delta d_{i'}(n_{i'})$ for each $i' \in I$; thus, $n_i \in A^O$. This contradicts Lemma 3.5.

Suppose that $|\{n_i | i \in I\}| > 2$. Consider a proposal of player $i \in I$ such that $\min_{i \in I} n_i < n_i < \max_{i \in I} n_i$. Note that by Lemma 3.3, for each $i \in I$, $n_i \in A$. If $v_i \leq n_i$, player i can improve his/her payoff from $d_i(n_i)$ to $d_i(\max_{i \in I} n_i) > d_i(n_i)$ by proposing $\max_{i \in I} n_i$, which is a contradiction. If $v_i > n_i$, player i can improve his/her payoff from $d_i(n_i)$ to $d_i(\min_{i \in I} n_i) > d_i(n_i)$ by proposing $\min_{i \in I} n_i$, which is a contradiction. Q.E.D.

Lemma 3.8. *There exists $\{i^0, i^1\} \subset I$ such that (i) $v_{i^0} \neq v_{i^1}$ and (ii) for each $i \in I$, $n_i \in \arg\max_{n \in C(\{i^0, i^1\})} d_i(n)$.*

Proof. Note that by Lemma 3.6, there exists $i \in I$ such that $v_i < \min_{i \in I} n_i$. Suppose that there does not exist $i \in I$ such that $v_i < \min_{i \in I} n_i$ and $d_i(\min_{i \in I} n_i) = \delta d_i(n_i)$. Then, by Lemmas 3.1 and 3.4, there exists $n \in A$ such that $n < \min_{i \in I} n_i$. Note that by Lemma 3.6, there exists $i \in I$ such that $\max_{i \in I} n_i < v_i$. Note that for each $i \in I$ such that $\max_{i \in I} n_i < v_i$, $n_i = \min_{i \in I} n_i$. Hence, there exists $i \in I$ and $n \in A$ such that $d_i(n) > d_i(n_i)$. This contradicts Lemma 3.4. Hence, there exists $i \in I$ such that $v_i < \min_{i \in I} n_i$ and $d_i(\min_{i \in I} n_i) = \delta d_i(n_i)$. Note that for each $i \in I$ such that $v_i < \min_{i \in I} n_i$, $n_i = \max_{i \in I} n_i$. Therefore, there exists $i^0 \in I$ such that $v_{i^0} < \min_{i \in I} n_i$, $n_{i^0} = \max_{i \in I} n_i$, and $d_{i^0}(\min_{i \in I} n_i) = \delta d_{i^0}(n_{i^0})$. Similarly, we can prove that there exists $i^1 \in I$ such that $\max_{i \in I} n_i < v_{i^1}$, $n_{i^1} = \min_{i \in I} n_i$, and $d_i(\max_{i \in I} n_i) = \delta d_{i^1}(n_{i^1})$. Therefore, there exists $i^0, i^1 \in I$ such that $v_{i^0} < n_{i^1} < n_{i^0} < v_{i^1}$, $d_{i^0}(n_{i^1}) = \delta d_{i^0}(n_{i^0})$, and $d_{i^1}(n_{i^0}) = \delta d_{i^1}(n_{i^1})$. Note that $n_{i^0} = \frac{v_{i^1} + \delta v_{i^0}}{1+\delta}$ and $n_{i^1} = \frac{v_{i^0} + \delta v_{i^1}}{1+\delta}$. Furthermore, note that by Lemmas 3.3 and 3.7, $n_i \in \arg\max_{n \in \{n^0, n^1\}} d_i(n)$ for each $i \in I$. Q.E.D.

Lemma 3.9. *Suppose that* $\delta \in \left[\tilde{\delta}, 1\right)$. *Then, there exists* $\{i^0, i^1\} \subset I$ *such that (i)* $v_{i^0} \neq v_{i^1}$, *(ii)* $n_i \in \arg\max_{n \in C(\{i^0, i^1\})} d_i(n)$ *for each* $i \in I$, *and (iii)* $v_i = \frac{v_{i^0} + v_{i^1}}{2}$ *for each* $i \in I$ *such that* $\min\{v_{i^0}, v_{i^1}\} < v_i < \max\{v_{i^0}, v_{i^1}\}$.

Proof. By Lemma 3.8, there exists $\{i^0, i^1\} \subset I$ such that (i) $v_{i^0} \neq v_{i^1}$ and (ii) $n_i \in \arg\max_{n \in C(\{i^0, i^1\})} d_i(n)$ for each $i \in I$. Without loss of generality, suppose that $v_{i^0} < v_{i^1}$. Suppose that there exists $i \in I$ such that $v_{i^0} < v_i < \frac{v_{i^0} + v_{i^1}}{2}$. Then, note that under our assumption that $\delta \in \left[\tilde{\delta}, 1\right)$, $d_i\left(\frac{v_{i^0} + v_{i^1}}{2}\right) \geq \ell = \frac{1}{2}\frac{1-\tilde{\delta}}{1+\tilde{\delta}} \geq \frac{1}{2}\frac{1-\delta}{1+\delta} \geq \frac{d\left(v_{i^0}, v_{i^1}\right)}{2}\frac{1-\delta}{1+\delta}$; thus, $v_{i^0} < v_i \leq n_{i_1}$. Note that $d_i(n_{i^1}) < d_{i^0}(n_{i^1}) = \frac{\delta}{1-\delta} d(n_{i^1}, n_{i^0})$. Hence, $d_i(n_{i^1}) < \delta(d_i(n_{i^1}) + d(n_{i^1}, n_{i^0})) = \delta d_i(n_{i^0}) = \delta d_i(n_i)$; thus, by Lemmas 3.1 and 3.4, $n_{i^1} \notin A$. This contradicts Lemma 3.3. Therefore, there does not exist $i \in I$ such that $v_{i^0} < v_i < \frac{v_{i^0} + v_{i^1}}{2}$. Similarly, we can prove that there does not exist $i \in I$ such that $\frac{v_{i^0} + v_{i^1}}{2} < v_i < v_{i^1}$. Q.E.D.

The conclusion of Proposition 3.2 follows Lemmas 3.3 and 3.9. Q.E.D.

References

Hansen, P., Thisse, J. F. (1981). Outcomes of voting and planning: Condorcet, Weber and Rawls locations. *Journal of Public Economics*, 16, 1–15.

Herings, P. J. J., Predtetchinski, A. (2010). One-dimensional bargaining with Markov recognition probabilities. *Journal of Economic Theory*, 145, 189–215.

Kawamori, T., Yamaguchi, K. (2010). Outcomes of bargaining and planning in single facility location problems. *Mathematical Social Sciences*, 59, 38–45.

Labbé, M. (1990). Location of an obnoxious facility on a network: a voting approach. *Networks*, 20, 197–207.

Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica*, 50, 97–110.

Chapter 4

Outcome of majority voting in multiple undesirable facility location problems

Abstract: We consider the outcome of majority voting in multiple undesirable facility location problems where the locations of two facilities are planned, any individual is concerned about the location of the nearest facility but not about the location of the other facility, and any individual prefers that the location of the nearest facility be as far as possible from his/her location. We show that a Condorcet winner is a subset of the set of pendant vertices and the vertices adjacent to pendant vertices on a tree network with an odd number of individuals. Furthermore, we derive a necessary and sufficient condition for a set of locations to be a Condorcet winner on a line network with an odd number of individuals.

Keywords: Locations of multiple undesirable facilities; Majority voting; Condorcet winner

JEL Classification: D72; R53; H41

Based article: Yamaguchi, K. (2011). Outcome of majority voting in multiple undesirable facility location problems. Mimeo.

4.1 Introduction

The locations of facilities have an important influence on the daily life of individuals. For example, if a park is located in the vicinity of an individual, he/she surely enjoys beautiful scenery every day. Conversely, if a dump is located in the vicinity of an individual, he/she is surely bothered by bad odor every day. Hence, individuals are very interested in the locations of facilities. Then, if individuals collectively choose the locations of facilities, which locations do they choose? Furthermore, are the chosen locations socially desirable? Many studies have been devoted to answering these questions in cases where individuals collectively choose the locations of facilities through majority voting by employing a Condorcet winner, which is unbeatable through pairwise majority voting, as a solution under majority voting.[1,2]

Initially, the studies focused on single desirable facility location problems where the location of a single facility is planned, and any individual prefers that the location of the facility be as close as possible to his/her location. In these problems, a Condorcet winner is a location that is unbeatable through pairwise majority voting. Hansen and Thisse (1981) showed that the set of Condorcet winners equals the set of medians on a tree network.[3] Labbe (1985) showed that the set of Condorcet winners equals either the empty set or the set of medians on a cactus network. Furthermore, Hansen and Labbé (1988) derived an algorithm for finding a Condorcet winner on a general network. In these problems, a Condorcet winner is not necessarily a median on a general network.[4] However, Hansen and Thisse (1981) showed that the ratio of the average distance from an

[1]Some studies have been devoted to answering those questions in cases where individuals collectively choose the location of a facility through unanimity bargaining by employing the equilibrium location in alternating-offer bargaining as a solution under unanimity bargaining. For example, see Kawamori and Yamaguchi (2010) for single desirable facility location problems.

[2]Since the existence of a Condorcet solution is not assured, other voting solutions weaker than a Condorcet solution have been also employed. For example, see Bandelt and Labbé (1986) for a Simpson solution, Campos Rodríguez and Moreno Pérez (2000) for a tolerant Condorcet solution, and Campos Rodríguez and Moreno Pérez (2003) for their mixture.

[3]A p-median is a set of p locations such that the average distance from an individual's location to the nearest location in a set of p locations is minimized, that is, a Benthamite social welfare maximizer in p-desirable facility location problems.

[4]Bandelt (1985) characterized the networks on which the set of Condorcet winners equals the set of medians.

individual's location to a Condorcet winner to the average distance from an individual's location to a median is bounded above on a general network.

Subsequently, the studies began to focus on single undesirable facility location problems where the location of a single facility is planned, and any individual prefers that the location of the facility be as far as possible from his/her location. In these problems, a Condorcet winner is a location that is unbeatable through pairwise majority voting. Labbé (1990) showed that a Condorcet winner is a pendant vertex or a bottleneck location on a general network with an odd number of individuals. Furthermore, the study revealed that the set of Condorcet winners equals the set of the pendant vertices that satisfy some condition on a tree network with an odd number of individuals. In these problems, a Condorcet winner is not necessarily an antimedian, even on a line network.[5] However, Labbé (1990) showed that the ratio of the average distance from an individual's location to an antimedian to the average distance from an individual's location to a Condorcet winner is bounded above on a general network.

Recently, the studies addressed multiple desirable facility location problems where the locations of multiple facilities are planned, any individual is concerned about the location of the nearest facility but not about the locations of the other facilities, and any individual prefers that the location of the nearest facility be as close as possible to his/her location. In these problems, a Condorcet winner is a set of locations that is unbeatable through pairwise majority voting. Barberà and Beviá (2006) showed that a Condorcet winner is efficient, internally consistent, and Nash stable on a line network. Hajduková (2010) derived the additional necessary conditions and a sufficient condition for a set of locations to be a Condorcet winner on a line network. Furthermore, Campos Rodríguez and Moreno Pérez (2008) derived an algorithm for finding a Condorcet winner on a general network. In these problems, a Condorcet winner is not necessarily a median, even on a line network. Furthermore, the ratio of the average distance from an individual's

[5] A p-antimedian is a set of p locations such that the average distance from an individual's location to the nearest location in a set of p locations is maximized, that is, a Benthamite social welfare maximizer in p-undesirable facility location problems.

location to the nearest location in a Condorcet winner to the average distance from an individual's location to the nearest location in a median is not bounded above, even on a line network.

Following these studies, this chapter is devoted to multiple undesirable facility location problems where the locations of multiple facilities are planned, any individual is concerned about the location of the nearest facility but not about the locations of the other facilities, and any individual prefers that the location of the nearest facility be as far as possible from his/her location. In these problems, a Condorcet winner is a set of locations that is unbeatable through pairwise majority voting. We assume that the locations of two facilities are planned. We show that a Condorcet winner is a subset of the set of pendant vertices and the vertices adjacent to pendant vertices on a tree network with an odd number of individuals. Furthermore, we derive a necessary and sufficient condition for a set of locations to be a Condorcet winner on a line network with an odd number of individuals. In these problems, we show that the ratio of the average distance from an individual's location to the nearest location in an antimedian to the average distance from an individual's location to the nearest location in a Condorcet winner is not bounded above, even on a line network.

The remainder of this chapter is organized as follows. Section 4.2 describes our model, and Section 4.3 presents our result.

4.2 Model

In this section, we describe our model.

Let (V, E) be a geometric graph. For some metric space X, V is a finite subset of X, and E is a finite set of some continuous injections from $[0,1]$ to X such that for any $e \in E$, $e(\{0,1\}) \subset V$ and $e([0,1] \setminus \{0,1\}) \subset X \setminus V$, and for any $(e, e') \in E^2$ with $e \neq e'$, $e((0,1)) \cap e'((0,1)) = \emptyset$. We call $v \in V$ a vertex, and for any $e \in E$, $e([0,1])$, an edge. We assume that $|V| \geq 3$. Let $N = \left(\bigcup_{e \in E} e([0,1])\right) \cup \left(\bigcup_{v \in V} \{v\}\right)$. We call N a network, and $n \in N$, a location. For any $(n, n') \in N^2$, we call $R \in 2^N$ a route between n and n' if (i) $R \ni n$, $R \ni n'$, and R is connected and (ii) there does not exist $S \subsetneq R$ such that $S \ni n$, $S \ni n'$, and S is connected. We assume that there is a unique route between two locations of the network; in other words, the network is a tree. For any $(n, n') \in N^2$, let $R(n, n')$ be the route between n and n'. Let d be a map from N^2 to $\mathbb{R}_+$ such that for any $(n, n') \in N^2$, $d(n, n')$ denotes the length of $R(n, n')$. Then, d denotes a metric on N.

Let I be a nonempty finite set of individuals. We assume that $|I|$ is odd. Individual $i \in I$ is located at a vertex $v_i \in V$ of the network. For any $i \in I$, let d_i be a map from N to $\mathbb{R}_+$ such that for any $n \in N$, $d_i(n) = d(n, v_i)$. Then, $d_i(n)$ denotes individual i's distance from location n.

These individuals collectively choose the locations of two undesirable facilities on the network through majority voting. It is permissible for the facilities to be located only on the vertices of the network, but not on the same vertex of the network. Let $\mathcal{L} = \{L \subset V \| L| = 2\}$. Then, $\{\ell, \ell'\} \in \mathcal{L}$ denotes a set of locations that is a candidate for the set of the locations of the facilities. Any individual is concerned about the location of the nearest facility but not about the location of the other facility. Any individual prefers that the location of the nearest facility be as far as possible from his/her location. For any $i \in I$, let D_i be a map from $\mathcal{L}$ to $\mathbb{R}_+$ such that for any $\{\ell, \ell'\} \in \mathcal{L}$, $D_i(\{\ell, \ell'\}) = \min\{d_i(\ell), d_i(\ell')\}$. Then, $D_i(\{\ell, \ell'\})$ denotes individual i's distance from the location of the nearest facility when $\{\ell, \ell'\}$ is the set of the locations of the facilities. We employ a Condorcet winner, which is unbeatable through pairwise majority voting, as a solution under majority voting. In our model, a Condorcet winner is a set of locations that is unbeatable through pairwise majority voting.

Definition 4.1. $C \in \mathcal{L}$ is a Condorcet winner if for any $L \in \mathcal{L}$,

$$|\{i \in I | D_i(C) < D_i(L)\}| \leq \frac{|I|}{2}.$$

Let $\mathcal{C}$ denote the set of Condorcet winners.

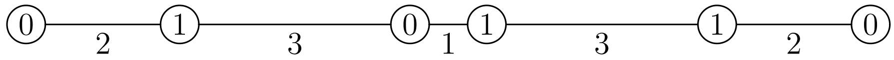


Figure 4.1: An example where a Condorcet winner does not exist: the number in a vertex denotes the number of the individuals located on the vertex; the number under an edge denotes the length of the edge

4.3 Result

In this section, we present the result.

First, we derive a necessary condition for a set of locations to be a Condorcet winner on a tree network. We refer to a vertex of degree one as a pendant vertex. Let P be the set of pendant vertices. Let Q be the set of the vertices adjacent to the pendant vertices.

Proposition 4.1. *Suppose that $C \in \mathcal{C}$. Then, $C \subset P \cup Q$.*

Proof. Suppose that $\{c, c'\} \in \mathcal{C}$. Suppose that $c \notin P \cup Q$. Let $V_1 = \{c\}$, $V_2 = \{v \in V | d(v, c') = d(v, c) + d(c, c')\} - V_1$, and $V_3 = V - V_1 - V_2$. Then, $\{V_1, V_2, V_3\}$ is a partition of V. Note that since $c \notin P \cup Q$, $|V_2| \geq 2$ and $|V_3| \geq 2$. For any $j \in \{1, 2, 3\}$, let $I_j = \{i \in I | v_i \in V_j\}$. Then, $\{I_1, I_2, I_3\}$ is a partition of I. Note that since $|I|$ is odd, either $|I_1 \cup I_2| > \frac{|I|}{2}$ or $|I_1 \cup I_3| > \frac{|I|}{2}$. Suppose that $|I_1 \cup I_2| > \frac{|I|}{2}$. Note that for any $L \in \mathcal{L}$ such that $L \subset V_3$, for any $i \in I_1 \cup I_2$, $D_i(\{c, c'\}) < D_i(L)$. Since $\{c, c'\} \in \mathcal{C}$, this is a contradiction. Suppose that $|I_1 \cup I_3| > \frac{|I|}{2}$. Note that for any $L \in \mathcal{L}$ such that $L \subset V_2$, for any $i \in I_1 \cup I_3$, $D_i(\{c, c'\}) < D_i(L)$. Since $\{c, c'\} \in \mathcal{C}$, this is a contradiction. Q.E.D.

By Proposition 4.1, it is sufficient for finding a Condorcet winner to consider only a subset of the set of pendant vertices and the vertices adjacent to pendant vertices.

Unfortunately, a Condorcet winner does not necessarily exist, even on a line network. An example is shown in Figure 4.1. As shown by Hansen and Thisse (1981) and Labbé (1990), in single facility location problems, a Condorcet winner exists on a tree network. However, as shown by Barberà and Beviá (2006) and above, in multiple facility location problems, a Condorcet winner does not necessarily exist, even on a line network. Hence, in multiple facility location problems, other voting solutions weaker than a Condorcet solution may be needed for considering the full outcome of majority voting. However, if a Condorcet winner exists on a line network, we can find it easily.

Hereafter, we assume that the network is a line. Let m be a map from N^2 to N such that for any $(n, n') \in N^2$, $m(n, n') \in R(n, n')$ and $d(n, m(n, n')) = d(n', m(n, n'))$. Then, $m(n, n')$ denotes the middle location between locations n and n'. Let p and p' denote the pendant vertices. Let q and q' denote the vertices adjacent to pendant vertices p and p' respectively. Then, by Proposition 4.1, only $\{p, q\}$, $\{q', p'\}$, $\{p, q'\}$, $\{q, p'\}$, $\{p, p'\}$, and $\{q, q'\}$ are candidates that can be a Condorcet winner. For any set of locations, we derive a necessary and sufficient condition for the set of locations to be a Condorcet winner.

Proposition 4.2. *$\{p, q\} \in \mathcal{C}$ if and only if the following conditions are met: (i) $|\{i \in I | v_i \in R(m(q, q'), p')\}| > \frac{|I|}{2}$; and (ii) $|\{i \in I | v_i \in \{p\} \cup R(m(q, p'), p')\}| > \frac{|I|}{2}$.*

Proof. (Necessity) We first prove the necessity. Suppose that $\{p, q\} \in \mathcal{C}$. (i) Suppose that $|\{i \in I | v_i \in R(m(q, q'), p')\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, it follows that $|I - \{i \in I | v_i \in R(m(q, q'), p')\}| > \frac{|I|}{2}$. Note that for any $i \in I - \{i \in I | v_i \in R(m(q, q'), p')\}$, $D_i(\{p, q\}) < D_i(\{q', p'\})$. Since $\{p, q\} \in \mathcal{C}$, this is a contradiction. (ii) Suppose that $|\{i \in I | v_i \in \{p\} \cup R(m(q, p'), p')\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, it follows that $|I - \{i \in I | v_i \in \{p\} \cup R(m(q, p'), p')\}| > \frac{|I|}{2}$. Note that $D_i(\{p, q\}) < D_i(\{p, p'\})$ for any $i \in I - \{i \in I | v_i \in \{p\} \cup R(m(q, p'), p')\}$. Since $\{p, q\} \in \mathcal{C}$, this is a contradiction.

(Sufficiency) Suppose that (i) $|\{i \in I | v_i \in R(m(q, q'), p')\}| > \frac{|I|}{2}$. Furthermore, suppose that (ii) $|\{i \in I | v_i \in \{p\} \cup R(m(q, p'), p')\}| > \frac{|I|}{2}$. Then, $|\{i \in I | v_i \in R(q, p')\}| > \frac{|I|}{2}$. Note that for any $L \in \mathcal{L}$ such that $L \cap \{p\} \neq \emptyset$ and $L \cap \{q\} = \emptyset$, for any $i \in \{i \in I | v_i \in \{p\} \cup R(m(q, p'), p')\}$, $D_i(\{p, q\}) \geq D_i(L)$. Note that for any $L \in \mathcal{L}$ such that $L \cap \{p\} = \emptyset$ and $L \cap \{q\} \neq \emptyset$, for any $i \in \{i \in I | v_i \in R(q, p')\}$, $D_i(\{p, q\}) \geq D_i(L)$. Note that for any $L \in \mathcal{L}$ such that $L \cap \{p\} = \emptyset$ and $L \cap \{q\} = \emptyset$, for any $i \in \{i \in I | v_i \in R(m(q, q'), p')\}$, $D_i(\{p, q\}) \geq D_i(L)$. Therefore, $\{p, q\} \in \mathcal{C}$. Q.E.D.

Proposition 4.3. *$\{q', p'\} \in \mathcal{C}$ if and only if the following conditions are met: (i)*

$|\{i \in I | v_i \in R(p, m(q, q'))\}| > \frac{|I|}{2}$*; and (ii)* $|\{i \in I | v_i \in R(p, m(p, q')) \cup \{p'\}\}| > \frac{|I|}{2}$.

Proof. Since the proof of this proposition is analogous to that of Proposition 4.2, we omit the proof. Q.E.D.

Proposition 4.4. $\{p, q'\} \in \mathcal{C}$ *if and only if the following conditions are met: (i) for any adjacent vertices* $v, v' \in V$ *such that* $d(p, v) < d(p, v')$ *and* $\{v, v'\} \cap \{q', p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p, v), m(v', q'))\}| > \frac{|I|}{2}$*; and (ii) for any* $v \in V$ *such that* $\{v\} \cap \{p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p, v), m(v, q')) \cup \{p'\}\}| > \frac{|I|}{2}$.

Proof. (Necessity) We first prove the necessity. Suppose that $\{p, q'\} \in \mathcal{C}$. (i) Suppose that for some adjacent vertices $v, v' \in V$ such that $d(p, v) < d(p, v')$ and $\{v, v'\} \cap \{q', p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p, v), m(v', q'))\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, $|I - \{i \in I | v_i \in R(m(p, v), m(v', q'))\}| > \frac{|I|}{2}$. Note that $D_i(\{p, q'\}) < D_i(\{v, v'\})$ for any $i \in I - \{i \in I | v_i \in R(m(p, v), m(v', q'))\}$. Since $\{p, q'\} \in \mathcal{C}$, this is a contradiction. (ii) Suppose that $|\{i \in I | v_i \in R(m(p, v), m(v, q')) \cup \{p'\}\}| \leq \frac{|I|}{2}$ for some $v \in V$ such that $\{v\} \cap \{p'\} = \emptyset$. Then, since $|I|$ is odd, $|I - \{i \in I | v_i \in R(m(p, v), m(v, q')) \cup \{p'\}\}| > \frac{|I|}{2}$. Note that for any $i \in I - \{i \in I | v_i \in R(m(p, v), m(v, q')) \cup \{p'\}\}$, $D_i(\{p, q'\}) < D_i(\{v, p'\})$. Since $\{p, q'\} \in \mathcal{C}$, this is a contradiction.

(Sufficiency) Suppose that (i) for any adjacent vertices $v, v' \in V$ such that $d(p, v) < d(p, v')$ and $\{v, v'\} \cap \{q', p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p, v), m(v', q'))\}| > \frac{|I|}{2}$, and (ii) for any $v \in V$ such that $\{v\} \cap \{p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p, v), m(v, q')) \cup \{p'\}\}| > \frac{|I|}{2}$. Then, for any vertices $v, v' \in V$ such that $d(p, v) < d(p, v')$ and $\{v, v'\} \cap \{q', p'\} = \emptyset$, $|\{i \in I | v_i \in R(m(p, v), m(v', q'))\}| > \frac{|I|}{2}$. Note that for any $\{v, v'\} \in \mathcal{L}$ such that $v' = p'$, for any $i \in \{i \in I | v_i \in R(m(p, v), m(v, q')) \cup \{p'\}\}$, $D_i(\{p, q'\}) \geq D_i(\{v, v'\})$. Note that for any $\{v, v'\} \in \mathcal{L}$ such that $v \neq p'$ and $v' = q'$, $D_i(\{p, q'\}) \geq D_i(\{v, v'\})$ for any $i \in \{i \in I | v_i \in R(m(p, v), q') \cup \{p'\}\}$. Note that for any $\{v, v'\} \in \mathcal{L}$ such that $d(p, v) < d(p, v')$ and $\{v, v'\} \cap \{q', p'\} = \emptyset$, for any $i \in \{i \in I | v_i \in R(m(p, v), m(v', q'))\}$, $D_i(\{p, q'\}) \geq D_i(\{v, v'\})$. Therefore, $\{p, q'\} \in \mathcal{C}$. Q.E.D.

Proposition 4.5. *$\{q,p'\}\in\mathcal{C}$ if and only if the following conditions are met: (i) for any adjacent vertices $v,v'\in V$ such that $d(p,v)<d(p,v')$ and $\{v,v'\}\cap\{p,q\}=\emptyset$, $|\{i\in I|v_i\in R(m(q,v),m(v',p'))\}|>\frac{|I|}{2}$; and (ii) for any $v\in V$ such that $\{v\}\cap\{p\}=\emptyset$, $|\{i\in I|v_i\in\{p\}\cup R(m(q,v),m(v,p'))\}|>\frac{|I|}{2}$.*

Proof. Since the proof of this proposition is analogous to that of Proposition 4.4, we omit the proof. Q.E.D.

Proposition 4.6. *$\{p,p'\}\in\mathcal{C}$ if and only if the following conditions are met: for any adjacent vertices $v,v'\in V$ such that $d(p,v)<d(p,v')$, $|\{i\in I|v_i\in R(m(p,v),m(v',p'))\}|>\frac{|I|}{2}$.*

Proof. (Necessity) Suppose that $\{p,p'\}\in\mathcal{C}$. Suppose that for some adjacent vertices $v,v'\in V$ such that $d(p,v)<d(p,v')$, $|\{i\in I|v_i\in R(m(p,v),m(v',p'))\}|\leq\frac{|I|}{2}$. Then, since $|I|$ is odd, $|I-\{i\in I|v_i\in R(m(p,v),m(v',p'))\}|>\frac{|I|}{2}$. Note that for any $i\in I-\{i\in I|v_i\in R(m(p,v),m(v',p'))\}$, $D_i(\{p,p'\})<D_i(\{v,v'\})$. Since $\{p,p'\}\in\mathcal{C}$, this is a contradiction.

(Sufficiency) Suppose that for any adjacent vertices $v,v'\in V$ such that $d(p,v)<d(p,v')$, $|\{i\in I|v_i\in R(m(p,v),m(v',p'))\}|>\frac{|I|}{2}$. Then, for any vertices $v,v'\in V$ such that $d(p,v)<d(p,v')$, $|\{i\in I|v_i\in R(m(p,v),m(v',p'))\}|>\frac{|I|}{2}$. Note that for any $\{v,v'\}\in\mathcal{L}$ such that $d(p,v)<d(p,v')$, for any $i\in\{i\in I|\in R(m(p,v),m(v',p'))\}$, $D_i(\{p,p'\})\geq D_i(\{v,v'\})$. Therefore, $\{p,p'\}\in\mathcal{C}$. Q.E.D.

Proposition 4.7. *$\{q,q'\}\in\mathcal{C}$ if and only if the following conditions are met: (i) $|V|=5$; (ii) $m(q,q')\in V$; (iii) $|\{i\in I|v_i\in\{p,m(q,q')\}\}|>\frac{|I|}{2}$; (iv) $|\{i\in I|v_i\in\{m(q,q'),p'\}\}|>\frac{|I|}{2}$; and (v) $|\{i\in I|v_i\in\{p,p'\}\}|>\frac{|I|}{2}$.*

Proof. (Necessity) Suppose that $\{q,q'\}\in\mathcal{C}$. (i) Suppose that $|V|=4$. Since $|I|$ is odd, either $|\{i\in I|v_i\in\{p,q\}\}|>\frac{|I|}{2}$ or $|\{i\in I|v_i\in\{q',p'\}\}|>\frac{|I|}{2}$. Suppose that $|\{i\in I|v_i\in\{p,q\}\}|>\frac{|I|}{2}$. Note that for any $i\in\{i\in I|v_i\in\{p,q\}\}$, $D_i(\{q,q'\})<D_i(\{q',p'\})$. Since $\{q,q'\}\in\mathcal{C}$, this is a contradiction. Suppose that $|\{i\in I|v_i\in\{q',p'\}\}|$

$> \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in \{q', p'\}\}$, $D_i(\{q, q'\}) < D_i(\{p, q\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. Suppose that $|V| \geq 6$. Since $|I|$ is odd, it follows that either $|\{i \in I | v_i \in R(q, q')\}| > \frac{|I|}{2}$ or $|\{i \in I | v_i \in \{p, p'\}\}| > \frac{|I|}{2}$. Suppose that $|\{i \in I | v_i \in R(q, q')\}| > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in R(q, q')\}$, $D_i(\{q, q'\}) < D_i(\{p, p'\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. Suppose that $|\{i \in I | v_i \in \{p, p'\}\}| > \frac{|I|}{2}$. Note that for any $L \in \mathcal{L}$ such that $L \cap \{p, q, q', p'\} = \emptyset$, for any $i \in \{i \in I | v_i \in \{p, p'\}\}$, $D_i(\{q, q'\}) < D_i(L)$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. (ii) Suppose that $m(q, q') \notin V$. Since $|I|$ is odd, either $|\{i \in I | v_i \in R(p, m(q, q')) \setminus \{m(q, q')\}\}| > \frac{|I|}{2}$ or $|\{i \in I | v_i \in R(m(q, q'), p') \setminus \{m(q, q')\}\}| > \frac{|I|}{2}$. We show that this is a contradiction. Suppose that $|\{i \in I | v_i \in R(p, m(q, q')) \setminus \{m(q, q')\}\}| > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in R(p, m(q, q')) \setminus \{m(q, q')\}\}$, $D_i(\{q, q'\}) < D_i(\{q', p'\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. Suppose that $|\{i \in I | v_i \in R(m(q, q'), p') \setminus \{m(q, q')\}\}| > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in R(m(q, q'), p') \setminus \{m(q, q')\}\}$, $D_i(\{q, q'\}) < D_i(\{p, q\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. (iii) Suppose that $|\{i \in I | v_i \in \{p, m(q, q')\}\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, it follows that $\{i \in I | v_i \in \{q, q', p'\}\} > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in \{q, q', p'\}\}$, $D_i(\{q, q'\}) < D_i(\{p, m(q, q')\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. (iv) Suppose that $|\{i \in I | v_i \in \{m(q, q'), p'\}\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, $\{i \in I | v_i \in \{p, q, q'\}\} > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in \{p, q, q'\}\}$, $D_i(\{q, q'\}) < D_i(\{m(q, q'), p'\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction. (v) Suppose that $|\{i \in I | v_i \in \{p, p'\}\}| \leq \frac{|I|}{2}$. Then, since $|I|$ is odd, $\{i \in I | v_i \in \{q, m(q, q'), q'\}\} > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in \{q, m(q, q'), q'\}\}$, $D_i(\{q, q'\}) < D_i(\{p, p'\})$. Since $\{q, q'\} \in \mathcal{C}$, this is a contradiction.

(Sufficiency) We prove the sufficiency. Suppose that (i) $|V| = 5$, (ii) $m(q, q') \in V$, (iii) $|\{i \in I | v_i \in \{p, m(q, q')\}\}| > \frac{|I|}{2}$, (iv) $|\{i \in I | v_i \in \{m(q, q'), p'\}\}| > \frac{|I|}{2}$, and (v) $|\{i \in I | v_i \in \{p, p'\}\}| > \frac{|I|}{2}$. Note that for any $i \in \{i \in I | v_i \in \{p, q, m(q, q')\}\}$, $D_i(\{q, q'\}) \geq D_i(\{p, q\})$. Furthermore, note that for any $i \in \{i \in I | v_i \in \{p, m(q, q')\}\}$, $D_i(\{q, q'\}) \geq D_i(\{p, m(q, q')\})$. Note that for any $i \in \{i \in I | v_i \in \{p, m(q, q'), q', p'\}\}$, $D_i(\{q, q'\}) \geq D_i(\{p, q'\})$. Note that for any $i \in \{i \in I | v_i \in \{p, p'\}\}$, $D_i(\{q, q'\}) \geq D_i(\{p, p'\})$. Note

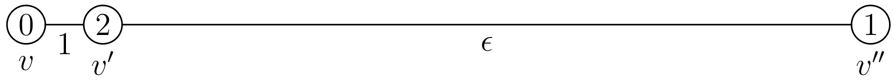


Figure 4.2: An example where the set of Condorcet winners does not intersect with the set of antimedians: ϵ denotes a number greater than 2; the number in a vertex denotes the number of the individuals located on the vertex; the number under an edge denotes the length of the edge

that for any $i \in \{i \in I | v_i \in \{p, q, m(q, q')\}\}$, $D_i(\{q, q'\}) \geq D_i(\{q, m(q, q')\})$. Note that for any $i \in \{i \in I | v_i \in \{p, q, m(q, q'), p'\}\}$, $D_i(\{q, q'\}) \geq D_i(\{q, p'\})$. Note that for any $i \in \{i \in I | v_i \in \{m\{q, q'\}, q', p'\}\}$, $D_i(\{q, q'\}) \geq D_i(\{m(q, q'), q'\})$. Note that for any $i \in \{i \in I | v_i \in \{m(q, q'), p'\}\}$, $D_i(\{q, q'\}) \geq D_i(\{m(q, q'), p'\})$. Note that for any $i \in \{i \in I | v_i \in \{m(q, q'), q', p'\}\}$, $D_i(\{q, q'\}) \geq D_i(\{q', p'\})$. Therefore, $\{q, q'\} \in \mathcal{C}$. Q.E.D.

By Propositions 4.1–4.7, if a Condorcet winner exists on a line network, we can find it easily.

Finally, we evaluate a Condorcet winner according to the Benthamite criterion. $M \in \mathcal{L}$ is an antimedian if for any $L \in \mathcal{L}$, $\sum_{i \in I} D_i(M) \geq \sum_{i \in I} D_i(L)$. That is, an antimedian is a Benthamite social welfare maximizer. Unfortunately, a Condorcet winner is not necessarily an antimedian. Furthermore, the ratio of the average distance from an individual's location to the nearest location in an antimedian to the average distance from an individual's location to the nearest location in a Condorcet winner is not bounded above. An example is shown in Figure 4.2. In this example, the unique Condorcet winner $\{v, v''\}$ is not the unique antimedian $\{v, v'\}$. Furthermore, the ratio $\frac{\sum_{i \in I} D_i(\{v, v'\})/|I|}{\sum_{i \in I} D_i(\{v, v''\})/|I|} = \frac{\epsilon}{2}$ of the average distance from an individual's location to the nearest location in the unique antimedian to the average distance from an individual's location to the nearest location in the unique Condorcet winner goes to infinity as ϵ goes to infinity. As shown by Hansen and Thisse (1981), in single desirable facility location problems, the ratio of the average distance from an individual's location to a Condorcet winner to the average distance from an individual's location to a median is bounded above on a general network, and the ratio is 1 on a tree network. Furthermore, as shown by Labbé (1990), in single undesirable facility location problems, the ratio of the average distance from an individual's location to an antimedian to the average distance from an individual's location to a Condorcet winner is bounded above on a general network. However, in multiple desirable facility location problems, the ratio of the average distance from an individual's location to the nearest location in a Condorcet winner to the average distance from an individual's location to the nearest location in a median is not bounded above, even on a line network. Furthermore, as shown above, in multiple undesirable facility location problems, the ratio of the average distance from an individual's location to the nearest location in an antimedian to the average distance from an individual's location to the nearest location in a Condorcet winner is not bounded above, even on a line network. Hence, in multiple facility location problems, mechanisms other than majority voting may be needed for implementing socially desirable outcomes from the viewpoint of the Benthamite criterion.

Acknowledgments

The author is grateful to Tomohiko Kawamori for his useful comments.

References

Bandelt, H. -J. (1985). Networks with Condorcet solutions. *European Journal of Operational Research*, 20, 314–326.

Bandelt, H. -J., Labbé, M. (1986). How bad can a voting location be. *Social Choice and Welfare*, 3, 125–145.

Barberà, S., Beviá, C. (2006). Locating public facilities by majority: stability, consistency and group formation. *Games and Economic Behavior*, 56, 185-200.

Campos Rodríguez, C. M., Moreno Pérez, J. A. (2000). The tolerant Condorcet points. *Studies in Locational Analysis*, 14, 173–190.

Campos Rodríguez, C. M., Moreno Pérez, J. A. (2003). Relaxation of the Condorcet and

Simpson conditions in voting location. *European Journal of Operational Research*, 145, 673–683.

Campos Rodríguez, C. M., Moreno Pérez, J. A. (2008). Multiple voting location problems. *European Journal of Operational Research*, 191, 437–453.

Hajduková, J. (2010). Condorcet winner configurations of linear networks. *Optimization*, 59, 461–475.

Hansen, P., Labbé, M. (1988). Algorithms for voting and competitive location on a network. *Transportation Science*, 22, 278–288.

Hansen, P., Thisse, J. F. (1981). Outcomes of voting and planning: Condorcet, Weber and Rawls locations. *Journal of Public Economics*, 16, 1–15.

Kawamori, T., Yamaguchi, K. (2010). Outcomes of bargaining and planning in single facility location problems. *Mathematical Social Sciences*, 59, 38–45.

Labbe, M. (1985). Outcomes of voting and planning in single facility location problems. *European Journal of Operational Research*, 20, 299–313.

Labbé, M. (1990). Location of an obnoxious facility on a network: a voting approach. *Networks*, 20, 197–207.