Abundance conjecture and canonical bundle formula

–アバンダンス予想と標準因子公式–

権業善範

Contents

Preface iii

Acknowledgments v

1 Preliminaries 1
- 1.1 Divisors, singularities of pairs, and asymptotic base loci . . . 1
- 1.2 Kodaira dimesnsions and numerical Kodaira dimesions . . . 4
- 1.3 Log minimal model program with scaling 6
- 1.4 Divisorial Zariski decomposition 8

2 Minimal model theory of numerical Kodaira dimension zero 11
- 2.1 Introduction 11
- 2.2 Existence of minimal model in the case $\nu = 0$ 12
- 2.3 Abundance theorem in the case $\nu = 0$ 13
  - 2.3.1 Klt pairs with nef log canonial divisor 14
  - 2.3.2 Lc pairs with nef log canonial divisor 14
  - 2.3.3 Lc pairs 16

3 Canonical bundle formulae and subadjunctions 17
- 3.1 Introduction 17
- 3.2 Main lemma 18
- 3.3 Ambro's canonical bundle formula 20
- 3.4 Subadjunction for minimal log canonical centers 23
- 3.5 Non-vanishing theorem for log canonical pairs 24
- 3.6 Subadjunction formula: local version 25

4 Reduction maps and minimal model theory 27
- 4.1 Introduction 27
- 4.2 Log smooth models 28

4.3 The existence of minimal models and abundance 29
4.4 Reduction maps and $\widetilde{\tau}(X, \Delta)$ 32
4.5 The MMP and the $(K_X + \Delta)$-trivial reduction map 36

5 Log pluricanonical representations and abundance conjecture 37
5.1 Introduction 37
5.2 Slc, sdlt, and log pluricanonical representations 39
5.3 Finiteness of log pluricanonical representations 43
5.3.1 Klt pairs 44
5.3.2 Lc pairs with big log canonical divisor 49
5.3.3 Lc pairs with semi-ample log canonical divisor 51
5.4 On abundance conjecture for log canonical pairs 55
5.4.1 Relative abundance conjecture 58
5.4.2 Miscellaneous applications 59
5.5 Non-vanishing, abundance, and minimal model conjectures 61

6 Images of log Fano and weak Fano varieties 63
6.1 Introduction 63
6.2 Kawamata's semipositivity theorem 64
6.3 Log Fano varieties 65
6.4 Fano and weak Fano manifolds 69
6.5 Comments and Questions 75
6.6 Appendix 76

7 Weak Fano varieties with log canonical singularities 79
7.1 Introdution 79
7.2 Preliminaries and Lemmas 81
7.3 On semi-ampleness for weak Fano varieties 88
7.4 On the Kleiman-Mori cone for weak Fano varieties 89
7.5 Examples 91

Preface

Throughout this thesis, we will work over $\mathbb{C}$, the complex number field, except Sections 6.5 and 6.6. We will make use of the standard notation and definitions as in [KoMo] and [KaMaMa].

In Chapter 1, we collect the basic notation and results on divisors, singularities of pairs, asymptotic base loci, the minimal model theory, and the divisorial Zariski decomposition.

In Chapter 2, we prove the existence of good log minimal models for dlt pairs of numerical log Kodaira dimension 0.

In Chapter 3, we consider a canonical bundle formula for generically finite proper surjective morphisms and obtain subadjunction formulae for minimal log canonical centers of log canonical pairs. We also treat related topics and applications.

In Chapter 4, we use reduction maps to study the minimal model program. Our main result is that the existence of a good minimal model for a klt pair (X, Δ) can be detected on the base of the $(K_X + \Delta)$-trivial reduction map. Thus we show that the main conjectures of the minimal model program can be interpreted as a natural statement on the existence of curves on X.

In Chapter 5, we prove the finiteness of log pluricanonical representations for projective log canonical pairs with semi-ample log canonical divisor. As a corollary, we obtain that the log canonical divisor of a projective semi log canonical pair is semi-ample if and only if so is the log canonical divisor of its normalization. We also treat many other applications.

In Chapter 6, we treat a smooth projective morphism between smooth complex projective varieties. If the source space is a weak Fano (or Fano) manifold, then so is the target space. Our proof is Hodge theoretic. We do not need mod p reduction arguments. We also discuss related topics and questions.

In Chapter 7, we consider semi-ampleness of the anti-log canonical divisor of any weak log Fano pair with log canonical singularities. We show semi-ampleness dose not hold in general by constructing several examples. Based on those examples, we propose sufficient conditions which seem to be the best possible and we prove semi-ampleness under such conditions.

Chapters 3, 5, and 6 are based on joint works with Osamu Fujino [FG1], [FG2], [FG3] and Chapter 4 is based on a joint work with Brian Lehmann [GL].

Acknowledgments

First of all, I wish to express my deep gratitude to my supervisor Hiromichi Takagi, for many suggestions, discussions, his warm encouragement, and support during the years of my master course and my doctoral studies. I am also indebted to Professors Caucher Birkar, Osamu Fujino, Yujiro Kawamata for their invaluable suggestions and discussions, and profound expertise. In particular, without many discussions with Professor Fujino and his kindness, this thesis would not be complete. Moreover, I would like to thank a lot of professors and friends, including: Professors Sébastien Boucksom, Fédéric Campana, Paolo Cascini, Stéphane Druel, Hajime Tsuji, Christpher Hacon, Nobuo Hara, Hajime Kaji, János Kollár, Yasunari Nagai, Noboru Nakayama, Yoichi Miyaoka, Keiji Oguiso, Mihai Păun, Vyacheslav V. Shokurov, Shunsuke Takagi, Doctors Dano Kim, Katsuhisa Furukawa, Tomoyuki Hisamoto, Daizo Ishikawa, Atsushi Ito, Tatsuya Ito, Ching-Jui Lai, Brian Lehmann, Shin-ichi Matsumura, Takuzo Okada, Shinnosuke Okawa, Akiyoshi Sannai, Taro Sano, Kiwamu Watanabe, Kazunori Yasutake for their encouragement, answering my questions, and many discussions. In particular Matsumura and Hisamoto kindly taught me the basic analytic methods. I wish to thank Brian for working with me.

In Chapter 2, I would like to thank Doctor Vladimir Lazić for informing the paper [D] to me. In Chapter 6, I would like to thank Doctor Kazunori Yasutake, who introduced the question on weak Fano manifolds in the seminar held at the Nihon University in December 2009, and Hiroshi Sato for constructing an interesting example (see Example 6.4.7). In Chapter 7, I would like to thank Doctor I. V. Karzhemanov for answering many questions. A part of Chapter 5 was done at CIRM and Institut de Mathématiques de Jussieu during my stay from December 2010 to March 2011. I also thank Professors Boucksom, Birkar, Claire Voisin, and Doctor Kamran Lamei and IMJ for their hospitality. I am partially supported by the Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists $\sharp 22 \cdot 7399$. Finally, I would also like to express my gratitude to my family for their moral support and warm encouragement.

June, 2011.

Yoshinori Gongyo

1 Preliminaries

In this chapter, we collect basic definitions and results that are used in the subsequent chapters. In particular we explain the *minimal model program with scaling* and results of Birkar–Cascini–Hacon–M[c]Kernan [BCHM] in Section 1.3.2.

1.1 Divisors, singularities of pairs, and asymptotic base loci

Notation and Definition 1.1.1. Let $\mathbb{K}$ be the real number field $\mathbb{R}$ or the rational number field $\mathbb{Q}$. We set $\mathbb{K}_{>0} = \{x \in \mathbb{K} | x > 0\}$.

Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and D a $\mathbb{Z}$-Cartier divisor on X. We set the complete linear system $|D/S| = \{E | D \sim_{\mathbb{Z},S} E \geq 0\}$ of D over S. The base locus of the linear system $|D/S|$ is denoted by $\mathrm{Bs}|D/S|$. When $S = \operatorname{Spec} \mathbb{C}$, we denote these by simply $|D|$ and $\mathrm{Bs}|D|$.

Definition 1.1.2. For a $\mathbb{K}$-Weil divisor $D = \sum_{j=1}^{r} d_j D_j$ such that D_j is a prime divisor for every j and $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\ulcorner D \urcorner = \sum_{j=1}^{r} \ulcorner d_j \urcorner D_j$ (resp. the *round-down* $\llcorner D \lrcorner = \sum_{j=1}^{r} \llcorner d_j \lrcorner D_j$), where for every real number x, $\ulcorner x \urcorner$ (resp. $\llcorner x \lrcorner$) is the integer defined by $x \leq \ulcorner x \urcorner < x+1$ (resp. $x - 1 < \llcorner x \lrcorner \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \llcorner D \lrcorner$. We define

$$D^{=1} = \sum_{d_j=1} D_j, \quad D^{\leq 1} = \sum_{d_j \leq 1} d_j D_j,$$
$$D^{<1} = \sum_{d_j<1} d_j D_j, \quad \text{and} \quad D^{>1} = \sum_{d_j>1} d_j D_j.$$

We call D a *boundary* $\mathbb{Q}$-divisor if $0 \leq d_j \leq 1$ for every j.

1.1.3 (Log resolution). Let X be a normal variety and let D be a $\mathbb{K}$-divisor on X. A *log resolution* $f : Y \to X$ means that

(i) f is a proper birational morphism,

(ii) Y is smooth, and

(iii) $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1}D$ is a simple normal crossing divisor on Y, where $\mathrm{Exc}(f)$ is the *exceptional locus* of f.

We recall the notion of singularities of pairs.

Definition 1.1.4 (Singularities of pairs). Let X be a normal variety and let Δ be a $\mathbb{K}$-divisor on X such that $K_X + \Delta$ is $\mathbb{K}$-Cartier. We call (X, Δ) is a *log pair*. Let $\varphi : Y \to X$ be a log resolution of (X, Δ). We set

$$K_Y = \varphi^*(K_X + \Delta) + \sum a_i E_i,$$

where E_i is a prime divisor on Y for every i. The pair (X, Δ) is called

(a) *sub kawamata log terminal* (*subklt*, for short) if $a_i > -1$ for all i, or

(b) *sub log canonical* (*sublc*, for short) if $a_i \geq -1$ for all i.

If Δ is effective and (X, Δ) is subklt (resp. sublc), then we simply call it *klt* (resp. *lc*).

Let (X, Δ) be an lc pair. If there is a log resolution $\varphi : Y \to X$ of (X, Δ) such that $\mathrm{Exc}(\varphi)$ is a divisor and that $a_i > -1$ for every φ-exceptional divisor E_i, then the pair (X, Δ) is called *divisorial log terminal* (*dlt*, for short). Assume that (X, Δ) is log canonical. If E is a prime divisor over X such that $a(E, X, \Delta) = -1$, then $c_X(E)$ is called a *log canonical center* (*lc center*, for short) of (X, Δ), where $c_X(E)$ is the closure of the image of E on X. For the basic properties of log canonical centers, see [F12, Section 9].

Definition 1.1.5 (Stratum). Let (X, Δ) be an lc pair. A *stratum* of (X, Δ) denotes X itself or an lc center of (X, Δ).

The following theorem was originally proved by Professor Christopher Hacon (cf. [F12, Theorem 10.4], [KoKov, Theorem 3.1]). For a simpler proof, see [F10, Section 4]:

Theorem 1.1.6 (Dlt blow-up). *Let X be a normal quasi-projective variety and Δ an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Suppose that (X, Δ) is log canonical. Then there exists a projective birational morphism $\varphi : Y \to X$ from a normal quasi-projective variety with the following properties:*

(i) Y *is* $\mathbb{Q}$*-factorial,*

(ii) $a(E, X, \Delta) = -1$ *for every* φ*-exceptional divisor* E *on* Y*, and*

(iii) for

$$\Gamma = \varphi_*^{-1}\Delta + \sum_{E:\varphi\text{-exceptional}} E,$$

it holds that (Y, Γ) *is dlt and* $K_Y + \Gamma = \varphi^*(K_X + \Delta)$*.*

The above theorem is very useful for studying log canonical singularities.

We will repeatedly use it in the subsequent chapters.

Definition and Lemma 1.1.7 (Different, cf. [Co])**.** Let (Y, Γ) be a dlt pair and S a union of some components of $\llcorner \Gamma \lrcorner$. Then there exists an effective $\mathbb{Q}$-divisor $\mathrm{Diff}_S(\Gamma)$ on S such that $(K_Y + \Gamma)|_S \sim_{\mathbb{Q}} K_S + \mathrm{Diff}_S(\Gamma)$. The effective $\mathbb{Q}$-divisor $\mathrm{Diff}_S(\Gamma)$ is called the *different* of Γ. Moreover it holds that $(S, \mathrm{Diff}_S(\Gamma))$ is sdlt.

Definition 1.1.8 (cf. [ELMNP])**.** Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and D an $\mathbb{R}$-Cartier divisor on X. We set

$$\mathbf{B}(D/S) = \bigcap_{D \sim_{S,\mathbb{R}} E \geq 0} \mathrm{Supp}\, E, \ \mathbf{B}_{\equiv}(D/S) = \bigcap_{D \equiv_S E \geq 0} \mathrm{Supp}\, E,$$

$$\mathbf{B}_{+}(D/S) = \bigcap_{\epsilon \in \mathbb{R}_{>0}} \mathbf{B}(D - \epsilon A/S), \text{and } \mathbf{B}_{-}(D/Y) = \bigcup_{\epsilon \in \mathbb{R}_{>0}} \mathbf{B}(D + \epsilon A/Y),$$

where A be a π-ample divisor. We remark that these definitions are independent of the choice of A. When $S = \mathrm{Spec}\, \mathbb{C}$, we write simply $\mathbf{B}(D)$, $\mathbf{B}_{\equiv}(D)$, $\mathbf{B}_{+}(D)$ and $\mathbf{B}_{-}(D)$.

Definition 1.1.9. Let X be a normal variety and D a $\mathbb{Q}$-Weil divisor. We define that

$$R(X, D) = \bigoplus_{m=0}^{\infty} H^0(X, \llcorner mD \lrcorner).$$

Finally, we identify that several different ways a divisor can be "exceptional" for a morphism.

Definition 1.1.10 ([Ft1], [Nak, III, 5.a], [Lai, Definition 2.9] and [Ta]). Let $f : X \to Y$ be a surjective projective morphism of normal quasi-projective varieties with connected fibers and D an effective f-vertical $\mathbb{R}$-Cartier divisor. We say that D is *f-exceptional* if

$$\operatorname{codim} f(\operatorname{Supp} D) \geq 2.$$

We say that D is *of insufficient fiber type* with respect to f if

$$\operatorname{codim} f(\operatorname{Supp} D) = 1$$

and there exist a codimension 1 irreducible component P of $f(\operatorname{Supp} D)$ and a prime divisor Γ such that $f(\Gamma) = P$ and $\Gamma \not\subset \operatorname{Supp}(D)$.

We call D *f-degenerate* if for any prime divisor P on Y there is some prime divisor $\Gamma \subset \operatorname{Supp}(f^*P)$ such that $f(\Gamma) = P$ and $\Gamma \not\subset \operatorname{Supp}(D)$. Note that the components of a degenerate divisor can be either f-exceptional or of insufficient fiber type with respect to f.

Lemma 1.1.11. *Let $f : X \to Y$ be a surjective projective morphism of normal quasi-projective varieties. Suppose that D is an effective f-vertical $\mathbb{Q}$-Cartier divisor such that $f_*\mathcal{O}_X(\llcorner kD \lrcorner)^{**} \cong \mathcal{O}_Y$ for every positive integer k. Then D is f-degenerate.*

Proof. If D were not f-degenerate, there would be an effective f-exceptional divisor E on X and an effective $\mathbb{Q}$-divisor T on Y such that $f^*T \leq D + E$. But since E is f-exceptional it is still true that $f_*\mathcal{O}_X(\llcorner k(D+E) \lrcorner)^{**} \cong \mathcal{O}_Y$, yielding a contradiction. □

1.2 Kodaira dimesnsions and numerical Kodaira dimesions

Definition 1.2.1 (Classical Iitaka dimension, cf. [Nak, II, 3.2, Definition]). Let X be a normal projective variety and D an $\mathbb{R}$-Cartier divisor on X. If $|\llcorner mD \lrcorner| \neq \emptyset$, we put a dominant rational map

$$\phi_{|\llcorner mD \lrcorner|} : X \dashrightarrow W_m,$$

with respect to the complete linear system of $\llcorner mD \lrcorner$. We define the *Classical Iitaka dimension* $\kappa(D)$ of D as the following:

$$\kappa(D) = \max\{\dim W_m\}$$

if $H^0(X, \llcorner mD \lrcorner) \neq 0$ for some positive integer m or $\kappa(D) = -\infty$ otherwise.

Lemma 1.2.2. *Let Y be a normal projective variety, $\varphi : Y \to X$ a projective birational morphism onto a normal projective variety, and let D be an $\mathbb{R}$-Cartier divisor on X. Then it holds the following:*

(1) $\kappa(\varphi^*D) = \kappa(\varphi^*D + E)$ *for any φ-exceptional effective $\mathbb{R}$-divisor E, and*

(2) $\kappa(\varphi^*D) = \kappa(D)$.

Proof. (1) and (2) follows from [Nak, II, 3.11, Lemma].

□

The following is remarked by Shokurov:

Remark 1.2.3. In general, $\kappa(D)$ may not coincide with $\kappa(D')$ if $D \sim_{\mathbb{R}} D'$. For example, let X be the $\mathbb{P}^1$, P and Q closed points in X such that $P \neq Q$ and a irrational number. Set $D = a(P - Q)$. Then $\kappa(D) = -\infty$ in spite of the fact that $D \sim_{\mathbb{R}} 0$.

However, fortunately, $\kappa(D)$ coincides with $\kappa(D')$ if D and D' are effective divisors such that $D \sim_{\mathbb{R}} D'$ ([Ch, Corollary 2.1.4]). Hence it seems reasonable that we define the following as the *Iitaka* (*Kodaira*) dimension for $\mathbb{R}$-divisors, which are introduced by Choi and Shokurov [Ch, Definition 2.2.1], [CS, Section 7].

Definition 1.2.4 (Invariant Iitaka dimension)**.** Let X be a normal projective variety and D an $\mathbb{R}$-Cartier divisor on X. We define the *invariant Iitaka dimension* $\kappa(D)$ of D as the following:

$$K(D) = \kappa(D')$$

if there exists an effective divisor D' such that $D \sim_{\mathbb{R}} D'$ or $K(D) = -\infty$ otherwise. Let (X, Δ) be a log canonical. Then we call $K(K_X + \Delta)$ the *log Kodaira dimension* of (X, Δ).

Definition 1.2.5 (Numerical Iitaka dimension)**.** Let X be a normal projective variety, D an $\mathbb{R}$-Cartier divisor and A an ample Cartier divisor on X. We set

$$\sigma(D, A) = \max\{k \in \mathbb{Z}_{\geq 0} | \limsup_{m \to \infty} m^{-k} \dim H^0(X, \llcorner mD \lrcorner + A) > 0\}$$

if $H^0(X, \llcorner mD \lrcorner + A) \neq 0$ for infinitely many $m \in \mathbb{N}$ and $\sigma(D, A) = -\infty$ otherwise. We define

$$\nu(D) = \max\{\sigma(D, A) | A \text{ is a ample divisor on } Y\}.$$

Note that this maximum will be computed by some sufficiently ample divisor A. Let (X, Δ) be a log pair. Then we call $\nu(K_X + \Delta)$ the *numerical log Kodaira dimension* of (X, Δ). If $\Delta = 0$, we simply say $\nu(K_X)$ is the *numerical Kodaira dimension* of X.

Lemma 1.2.6. *Let Y be a normal projective variety, $\varphi : Y \to X$ a projective birational morphism onto a normal projective variety, and let D be an $\mathbb{R}$-Cartier divisor on X. Then it holds the following:*

(1) $\nu(\varphi^*D) = \nu(\varphi^*D + E)$ for any φ-exceptional effective $\mathbb{R}$-divisor E,

(2) $\nu(\varphi^*D) = \nu(D)$, and

(3) $\nu(D) = \max\{k \in \mathbb{Z}_{\geq 0} | D^k \not\equiv 0\}$ when D is nef.

Proof. See [Nak, V, 2.7, Proposition]. □

Lemma 1.2.7 ([Nak, V, 2.7, Proposition, (1)]). *Let X be a projective variety and D and D' $\mathbb{R}$-Cartier divisors on X such that $D \equiv D'$. Then $\nu(D) = \nu(D')$.*

Remark 1.2.8. $\nu(D)$ is denoted as $\kappa_\sigma(D)$ in [Nak, V, § 2]. Moreover Nakayama also defined $\kappa_\sigma^-(D)$, $\kappa_\sigma^+(D)$ and $\kappa_\nu(D)$ as some numerical Iitaka dimensions. In this article we mainly treat in the case where $\nu(D) = 0$, i.e. $\kappa_\sigma(D) = 0$. Then it holds that $\kappa_\sigma^-(D) = \kappa_\sigma^+(D) = \kappa_\nu(D) = 0$ (cf. [Nak, V, 2.7, Proposition (8)]). Moreover, if a log canonical pair (X, Δ) has a *weakly log canonical model* in the sense of Shokurov, then $\nu(K_X + \Delta)$ coincides with the *numerical log Kodaira dimension* in the sense of Shokurov by Lemma 1.2.6 (cf. [Sh2, 2.4, Proposition]).

1.3 Log minimal model program with scaling

In this section, we review a *log minimal model program with scaling* and introduce works by Birkar–Cascini–Hacon–McKernan.

Lemma 1.3.1 (cf. [B1, Lemma 2.1] and [F12, Theorem 18.9]). *Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and (X, Δ) a $\mathbb{Q}$-factorial projective log canonical pair such that Δ is a $\mathbb{K}$-divisor. Let H be an effective $\mathbb{K}$-divisor such that $K_X + \Delta + H$ is π-nef and $(X, \Delta + H)$ is log canonical. Suppose that $K_X + \Delta$ is not π-nef. We put*

$$\lambda = \inf\{\alpha \in \mathbb{R}_{\geq 0} | K_X + \Delta + \alpha H \text{ is } \pi\text{-nef}\}.$$

Then $\lambda \in \mathbb{K}_{>0}$ and there exists an extremal ray $R \subseteq \overline{NE}(X/S)$ such that $(K_X + \Delta).R < 0$ and $(K_X + \Delta + \lambda H).R = 0$.

Definition 1.3.2 (Log minimal model program with scaling)**.** Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and (X, Δ) a $\mathbb{Q}$-factorial projective divisorial log terminal pair such that Δ is a $\mathbb{K}$-divisor. Let H be an effective $\mathbb{K}$-divisor such that $K_X + \Delta + H$ is π-nef and $(X, \Delta + H)$ is divisorial log terminal. We put

$$\lambda_1 = \inf\{\alpha \in \mathbb{R}_{\geq 0} | K_X + \Delta + \alpha H \text{ is } \pi\text{-nef}\}.$$

If $K_X + \Delta$ is not π-nef, then $\lambda_1 > 0$. By Lemma 1.3.1, there exists an extremal ray $R_1 \subseteq \overline{NE}(X/S)$ such that $(K_X + \Delta).R_1 < 0$ and $(K_X + \Delta + \lambda_1 H).R_1 = 0$. We consider an extremal contraction with respect to this R_1. If it is a divisorial contraction or a flipping contraction, let

$$(X, \Delta) \dashrightarrow (X_1, \Delta_1)$$

be the divisorial contraction or its flip. Since $K_{X_1} + \Delta_1 + \lambda_1 H_1$ is π-nef, we put

$$\lambda_2 = \inf\{\alpha \in \mathbb{R}_{\geq 0} | K_{X_1} + \Delta_1 + \alpha H_1 \text{ is } \pi\text{-nef}\},$$

where H_1 is the strict transform of H on X_1. Then we find an extremal ray R_2 by the same way as the above. We may repeat the process. We call this program a *log minimal model program with scaling of H over S*. When this program runs as the following:

$$(X_0, \Delta_0) = (X, \Delta) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_i) \cdots,$$

then

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots,$$

where $\lambda_i = \inf\{\alpha \in \mathbb{R}_{\geq 0} | K_{X_{i-1}} + \Delta_{i-1} + \alpha H_{i-1} \text{ is } \pi\text{-nef}\}$ and H_{i-1} is the strict transform of H on X_{i-1}.

The following theorems are slight generalizations of [BCHM, Corollary 1.4.1] and [BCHM, Corollary 1.4.2]. These seem to be well-known for the experts.

Theorem 1.3.3 (cf. [BCHM, Corollary 1.4.1])**.** *Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and (X, Δ) be a $\mathbb{Q}$-factorial projective divisorial log terminal pair such that Δ is an $\mathbb{R}$-divisor. Suppose that $\varphi : X \to Y$ is a flipping contraction of (X, Δ). Then there exists the log flip of φ.*

Proof. Since $-(K_X + \Delta)$ is φ-ample, so is $-(K_X + \Delta - \epsilon \llcorner \Delta \lrcorner)$ for a sufficiently small $\epsilon > 0$. Because $\rho(X/Y) = 1$, it holds that $K_X + \Delta \sim_{\mathbb{R},Y} c(K_X + \Delta - \epsilon \llcorner \Delta \lrcorner)$ for some positive number c. By [BCHM, Corollary 1.4.1], there exists the log flip of $(X, \Delta - \epsilon \llcorner \Delta \lrcorner)$. This log flip is also the log flip of (X, Δ) since $K_X + \Delta \sim_{\mathbb{R},Y} c(K_X + \Delta - \epsilon \llcorner \Delta \lrcorner)$. □

Theorem 1.3.4 (cf. [BCHM, Corollary 1.4.2]). *Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and (X, Δ) be a $\mathbb{Q}$-factorial projective divisorial log terminal pair such that Δ is an $\mathbb{R}$-divisor. Suppose that there exists a π-ample $\mathbb{R}$-divisor A on X such that $\Delta \geq A$. Then any log minimal model programs with scaling of H starting from (X, Δ) with scaling of H over S terminate, where H satisfies that $(X, \Delta + H)$ is divisorial log terminal and $K_X + \Delta + H$ is π-nef.*

The above theorem is proved by the same argument as the proof of [BCHM, Corollary1.4.2] because [BCHM, Theorem E] holds on the above setting.

Theorem 1.3.5 (cf. [BCHM, Corollary 1.4.2]). *Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and (X, Δ) be a $\mathbb{Q}$-factorial projective divisorial log terminal pair such that Δ is an $\mathbb{R}$-divisor. Suppose that there exists an π-ample effective $\mathbb{R}$-divisor H on X such that $(X, \Delta + H)$ is divisorial log terminal and $K_X + \Delta + H$ is π-nef. If $K_X + \Delta$ is not a π-pseudo-effective divisor then any minimal model programs starting from (X, Δ) with scaling of H over S terminate. Thus we get a Mori fiber space of (X, Δ).*

1.4 Divisorial Zariski decomposition

In this section, we introduce the *divisorial Zariski decomposition* for a pseudo-effective divisor.

Definition 1.4.1 (cf. [Nak, III, 1.13, Definition] and [Ka3]). Let $\pi : X \to S$ be a projective morphism of normal quasi-projective varieties and D an $\mathbb{R}$-Cartier divisor. We call that D is a *limit of movable* $\mathbb{R}$*-divisors* over S if $[D] \in \overline{\mathrm{Mov}}(X/S) \subseteq N^1(X/S)$ where $\overline{\mathrm{Mov}}(X/S)$ is the closure of the convex cone spanned by classes of fixed part free $\mathbb{Z}$-Cartier divisors over S. When $S = \operatorname{Spec}\mathbb{C}$, we denote simply $\overline{\mathrm{Mov}}(X)$.

Definition 1.4.2 (cf. [Nak, III, 1.6, Definition and 1.12, Definition]). Let X be a smooth projective variety and B an $\mathbb{R}$-big divisor. We define

$$\sigma_\Gamma(B) = \inf\{\mathrm{mult}_\Gamma B' | B \equiv B' \geq 0\}$$

for a prime divisor Γ. Let D be a pseudo-effective divisor. Then we define the following:

$$\sigma_\Gamma(D) = \lim_{\epsilon \to 0+} \sigma_\Gamma(D + \epsilon A)$$

for some ample divisor A. We remark that $\sigma_\Gamma(D)$ is independent of the choice of A. Moreover the above two definitions coincide for a big divisor because a function $\sigma_\Gamma(\cdot)$ on $\mathrm{Big}(X)$ is continuous where $\mathrm{Big}(X) := \{[B] \in \mathrm{N}^1(X) | B \text{ is big}\}$ (cf. [Nak, III, 1.7, Lemma]). We set

$$N(D) = \sum_{\Gamma:\text{prime divisor}} \sigma_\Gamma(D)\Gamma \text{ and } P(D) = D - N(D).$$

We remark that $N(D)$ is a finite sum. We call the decomposition $D \equiv P(D) + N(D)$ the *divisorial Zariski decomposition* of D. We say that $P(D)$ (resp. $N(D)$) is the *positive part* (resp. *negative part*) of D.

Remark that the decomposition $D \equiv P(D)+N(D)$ is called several names: the *sectional decomposition* ([Ka3]), the *σ-decomposition* ([Nak]), the *divisorial Zariski decomposition* ([Bo]), and the *numerical Zariski decomposition* ([Ka9]).

Proposition 1.4.3. *Let X be a smooth projective variety and D a pseudo-effective $\mathbb{R}$-divisor on X. Then it holds the following:*

(1) $\sigma_\Gamma(D) = \lim_{\epsilon\to 0+} \sigma_\Gamma(D + \epsilon E)$ *for a pseudo-effective divisor E, and*

(2) $\nu(D) = 0$ *if and only if* $D \equiv N(D)$.

Proof. (1) follows from [Nak, III, 1.4, Lemma]. (2) follows from [Nak, V, 2.7, Proposition (8)]. □

The basic properties of the σ-decomposition are:

Lemma 1.4.4 ([Nak, III.1.4 Lemma] and [Nak, V.1.3 Lemma])**.** *Let X be a smooth projective variety and D a pseudo-effective $\mathbb{R}$-Cartier divisor. Then*

(1) $\kappa(X, D) = \kappa(X, P_\sigma(D))$ *and*

(2) $\mathrm{Supp}(N_\sigma(D)) \subset \mathbf{B}_-(D)$.

Remark 1.4.5. By the results of [Leh2], this definition coincides with the notions of $\kappa_\nu(L)$ from [Nak, V, 2.20, Definition] and $\nu(L)$ from [BDPP, 3.6, Definition].

The numerical dimension satisfies a number of natural properties.

Lemma 1.4.6 ([Leh2, 6.7])**.** *Let X be a normal variety, D an $\mathbb{R}$-Cartier divisor on X. If X is smooth then* $\nu(D) = \nu(P_\sigma(D))$.

Degenerate divisors behave well with respect to the σ-decomposition.

Lemma 1.4.7 (cf. [Ft1, (1.9)], [Nak, III.5.7 Proposition]). *Let $f : X \to Y$ be a surjective projective morphism from a smooth quasi-projective variety to a normal quasi-projective variety and let D be an effective f-degenerate divisor. For any pseudo-effective divisor L on Y we have $D \leq N_\sigma(f^*L + D/Y)$.*

Proof. [Nak, III.5.1 Proposition] and [Nak, III.5.2 Proposition] together show that for a degenerate divisor D there is some component $\Gamma \subset \mathrm{Supp}(D)$ such that $D|_\Gamma$ is not $f|_\Gamma$-pseudo-effective. Since $P_\sigma(f^*L + D/Y)|_\Gamma$ is pseudo-effective, we see that Γ must occur in $N_\sigma(f^*L+D/Y)$ with positive coefficient.

Set D' to be the coefficient-wise minimum

$$D' = \min\{N_\sigma(f^*L + D/Y), D\}.$$

If $D' < D$, then $D - D'$ is still f-degenerate. Thus, there is some component of $D - D'$ contained in $\mathrm{Supp}(N_\sigma(f^*L + (D - D')/Y))$ with positive coefficient, a contradiction. □

Corollary 1.4.8 ([Lai, Lemma 2.10]). *Let $f : X \to Y$ be a surjective projective morphism of normal quasi-projective varieties and D an f-degenerate divisor. Suppose that L is a pseudo-effective divisor on Y. Then there are codimension 1 components of $\mathbf{B}_-(f^*L + D/Y)$.*

2

Minimal model theory of numerical Kodaira dimension zero

2.1 Introduction

The minimal model conjecture for smooth varieties is the following:

Conjecture 2.1.1 (Minimal model conjecture). *Let X be a smooth projective variety. Then there exists a minimal model or a Mori fiber space of X.*

This conjecture is true in dimension 3 and 4 by Kawamata, Kollár, Mori, Shokurov and Reid (cf. [KaMaMa], [KoMo] and [Sh3]). In the case where K_X is numerically equivalent to some effective divisor in dimension 5, this conjecture is proved by Birkar (cf. [B1]). When X is of general type or K_X is not pseudo-effective, Birkar, Cascini, Hacon and M[c]Kernan prove Conjecture 2.1.1 for arbitrary dimension ([BCHM]). Moreover if X has maximal Albanese dimension, Conjecture 2.1.1 is true by [F16]. In this chapter, among other things, we consider Conjecture 2.1.1 in the case where $\nu(K_X) = 0$ (for the definition of ν, see Definition 1.2.5):

Theorem 2.1.2. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial dlt pair such that $\nu(K_X + \Delta) = 0$. Then there exists a minimal model (X_m, Δ_m) of (X, Δ).*

Actually the author heared from Vladimir Lazić that Theorem 2.1.2 for klt pairs has been proved by by Druel (cf. [D]) after he finished this work. In this chapter, moreover, we give the generalization of his result for dlt pairs by using the sophisticated Birkar–Cascini–Hacon–M[c]Kernan's results and Druel's method. Essentially our method seems to be same as Druel's. However, by expanding this result to dlt pairs we give the different proof of the abundance theorem for log canonical pairs in the case where $\nu = 0$ from [CKP] and [Ka9]:

Theorem 2.1.3 (=Theorem 2.3.3). *Let X be a normal projective variety and Δ an effective $\mathbb{Q}$-divisor. Suppose that (X, Δ) is a log canonial pair such that $\nu(K_X + \Delta) = 0$. Then $K_X + \Delta$ is abundant, i.e. $\nu(K_X + \Delta) = \kappa(K_X + \Delta)$.*

First, Nakayama proved Theorem 2.1.3 when (X, Δ) is klt. Nakayama's proof is independent of Simpson's results [Sim]. Simpson's results are used to approach the abundance conjecture in [CPT]. Campana–Perternell–Toma prove Theorem 2.1.3 when X is smooth and $\Delta = 0$. Siu also gave an analytic proof of it (cf. [Siu]). The results of [CKP], [Ka9] and [Siu] depend on [Sim] and [Bu]. In this chapter, we show Theorem 2.1.3 by using a method different from [CPT], [CKP], [Ka9] and [Siu]. In our proof of Theorem 2.1.3, we do not need results of [Sim] and [Bu]. Our proof depends on [BCHM] and [G2].

2.2 Existence of minimal model in the case $\nu = 0$

Theorem 2.2.1 (cf. [D, Corollaire 3.4]). *Let X be a $\mathbb{Q}$-factorial projective variety and Δ an effective $\mathbb{R}$-divisor such that (X, Δ) is divisorial log terminal. Suppose that $\nu(K_X + \Delta) = 0$. Then any log minimal model programs starting from (X, Δ) with scaling of H terminate, where H satisfies that $H \geq A$ for some effective $\mathbb{Q}$-ample divisor A and $(X, \Delta + H)$ is divisorial log terminal and $K_X + \Delta + H$ is nef.*

Proof. Let $(X, \Delta) \dashrightarrow (X_1, \Delta_1)$ be a divisorial contraction or a log flip. Remark that it holds that

$$\nu(K_X + \Delta) = \nu(K_{X_1} + \Delta_1)$$

from Lemma 1.2.6 (1) and the negativity lemma. Now we run a log minimal model program

$$(X_i, \Delta_i) \dashrightarrow (X_{i+1}, \Delta_{i+1})$$

starting from $(X_0, \Delta_0) = (X, \Delta)$ with scaling of H. Assume by contradiction that this program does not terminate. Let $\{\lambda_i\}$ be as in Definition 1.3.2. We set

$$\lambda = \lim_{i \to \infty} \lambda_i.$$

If $\lambda \neq 0$, the sequence is composed by $(K_X + \Delta + \frac{1}{2}\lambda H)$-log minimal model program. Thus the sequence terminates by Theorem 1.3.4. Therefore we see that $\lambda = 0$. Now there exists j such that $(X_i, \Delta_i) \dashrightarrow (X_{i+1}, \Delta_{i+1})$ is a log flip for any $i \geq j$. Replace (X, Δ) by (X_j, Δ_j), we lose the fact that A is ample. Then we see the following:

Claim 2.2.2. *$K_X + \Delta$ is a limit of movable $\mathbb{R}$-divisors.*

Proof of Claim 2.2.2. See [B2, Step 2 of the proof of Theorem 1.5] or [F10, Theorem 2.3]. □

Let $\varphi : Y \to X$ be a log resolution of (X, Δ). We consider the divisorial Zariski decomposition

$$\varphi^*(K_X + \Delta) = P(\varphi^*(K_X + \Delta)) + N(\varphi^*(K_X + \Delta))$$

(Definition 1.4.2). Since

$$\nu(\varphi^*(K_X + \Delta)) = \nu(K_X + \Delta) = 0,$$

we see $P(\varphi^*(K_X + \Delta)) \equiv 0$ by Proposition 1.4.3 (2). Moreover we see the following claim:

Claim 2.2.3. *$N(\varphi^*(K_X + \Delta))$ is a φ-exceptional divisor.*

Proof of Claim 2.2.3. Let G be an ample divisor on X and ϵ a sufficiently small positive number. By Proposition 1.4.3 (1), it holds that

$$\operatorname{Supp} N(\varphi^*(K_X + \Delta)) \subseteq \operatorname{Supp} N(\varphi^*(K_X + \Delta + \epsilon G)).$$

If it holds that $\varphi_*(N(\varphi^*(K_X + \Delta))) \neq 0$, we see that $\mathbf{B}_{\equiv}(K_X + \Delta + \epsilon G)$ has codimension 1 components. This is a contradiction to Claim 2.2.2. Thus $N(\varphi^*(K_X + \Delta))$ is a φ-exceptional divisor. □

Hence $K_X + \Delta \equiv 0$, in particular, $K_X + \Delta$ is nef. This is a contradiction to the assumption. □

Corollary 2.2.4. *Let X be a $\mathbb{Q}$-factorial projective variety and Δ an effective $\mathbb{R}$-divisor such that (X, Δ) is divisorial log terminal. Suppose that $\nu(K_X + \Delta) = 0$. Then there exists a log minimal model of (X, Δ).*

Remark 2.2.5. These results are on the absolute setting. It may be difficult to extends these to the relative settings. See [F10], recent preprints [B3], [HX].

2.3 Abundance theorem in the case $\nu = 0$

In this section, we prove the abundance theorem in the case where $\nu = 0$ for an $\mathbb{R}$-divisor:

2.3.1 Klt pairs with nef log canonial divisor

In this subsection, we introduce the abundance theorem for klt pairs with nef log canonial divisor of numerical Kodaira dimension zero.

Ambro and Nakayama prove the abundance theorem for klt pairs whose log canonical divisors are numerically trivial, i.e.,

Theorem 2.3.1 (cf. [Am5, Theorem 4.2], [Nak, V, 4.9. Corollary]). *Let (X, Δ) be a projective klt pair such that Δ is a $\mathbb{Q}$-divisor. Suppose that $K_X + \Delta \equiv 0$. Then $K_X + \Delta \sim_{\mathbb{Q}} 0$.*

Remark that Nakayama proved the abundance conjecture when $K_X + \Delta$ is pseudo-effective without MMP. In Subsection 2.3.3, we also generalize the result for pseudo-effective log canonical divisors with MMP.

2.3.2 Lc pairs with nef log canonial divisor

Theorem 2.3.2 (cf. [FG2, Theorem 3.1], [G2, Theorem 1.2]). *Let (X, Δ) be a projective log canonical pair such that Δ is a $\mathbb{K}$-divisor. Suppose that $K_X + \Delta \equiv 0$. Then $K_X + \Delta \sim_{\mathbb{K}} 0$.*

Proof. By taking a dlt blow-up (Theorem 1.1.6), we also may assume that (X, Δ) is a $\mathbb{Q}$-factorial dlt pair. Now, we assume that $\mathbb{K} = \mathbb{R}$. Let $\sum_i B_i$ be the irreducible decomposition of $\operatorname{Supp} \Delta$. We put $V = \bigoplus_i \mathbb{R} B_i$. Then it is well known that

$$\mathcal{L} = \{B \in V \mid (X, B) \text{ is log canonical}\}$$

is a rational polytope in V. We can also check that

$$\mathcal{N} = \{B \in \mathcal{L} \mid K_X + B \text{ is nef}\}$$

is a rational polytope and $\Delta \in \mathcal{N}$ (cf. [B2, Proposition 3.2] and [Sh2, 6.2 First Main theorem]). We note that $\mathcal{N}$ is known as Shokurov's polytope. Therefore, we can write

$$K_X + \Delta = \sum_{i=1}^{k} r_i (K_X + \Delta_i)$$

such that

(i) Δ_i is an effective $\mathbb{Q}$-divisor such that $\Delta_i \in \mathcal{N}$ for every i,

(ii) (X, Δ_i) is log canonical for every i, and

(iii) $0 < r_i < 1, r_i \in \mathbb{R}$ for every i, and $\sum_{i=1}^{k} r_i = 1$.

Since $K_X + \Delta$ is numerically trivial and $K_X + \Delta_i$ is nef for every i, $K_X + \Delta_i$ is numerically trivial for every i.Thus we may assume that $\mathbb{K} = \mathbb{Q}$. Moreover, by Theorem 2.3.1, we may assume that $\llcorner \Delta \lrcorner \neq 0$. We set

$$S = \epsilon \llcorner \Delta \lrcorner \text{ and } \Gamma = \Delta - S$$

for some sufficiently small positive number ϵ. Then (X, Γ) is klt, $K_X + \Gamma \equiv -S$ is not pseudo-effective. By Theorem 1.3.5, there exist a composition of $(K_X + \Gamma)$-log flips and $(K_X + \Gamma)$-divisorial contractions

$$\psi : X \dashrightarrow X',$$

and a Mori fiber space

$$f' : X' \to Y'$$

for (X, Γ). It holds that $K_{X'} + \Delta' \equiv 0$, where Δ' is the strict transform of Δ on X'. By the negativity lemma, it suffices to show that $K_{X'} + \Delta' \sim_{\mathbb{Q}} 0$. We put $S' = \psi_* S$ and $\Gamma' = \psi_* \Gamma$. Since $(S'.C) > 0$ for any f'-contracting curve C, we conclude that $S' \neq 0$ and the support of S' dominates Y'. Since $K_{X'} + \Delta' \equiv 0$ and f' is a $(K_{X'} + \Gamma')$-extremal contraction, there exists a $\mathbb{Q}$-Cartier divisor D' on Y' such that $K_{X'} + \Delta' \sim_{\mathbb{Q}} f'^* D'$ and $D' \equiv 0$ (cf. [KaMaMa, Lemma 3-2-5]). We remark that (X', Δ') is not necessarily dlt, but it is a $\mathbb{Q}$-factorial log canonical pair. Hence we can take a dlt blow-up

$$\varphi : (X'', \Delta'') \to (X', \Delta')$$

of (X', Δ'). Since the support of S' dominates Y', there exists an lc center C'' of (X'', Δ'') such that C'' dominates Y'. Then we see that

$$K_{C''} + \Delta''_{C''} \sim_{\mathbb{Q}} (f''_{C''})^* D',$$

where $(K_{X''} + \Delta'')|_{C''} = K_{C''} + \Delta''_{C''}$ and $f''_{C''} = f' \circ \varphi|_{C''}$. From induction on the dimension of X, it holds that $K_{C''} + \Delta''_{C''} \sim_{\mathbb{Q}} 0$. In particular, we conclude that $D' \sim_{\mathbb{Q}} 0$. Thus we see that

$$K_{X'} + \Delta' \sim_{\mathbb{Q}} 0.$$

We finish the proof of Theorem 2.3.2. $\square$

2.3.3 Lc pairs

Theorem 2.3.3. *Let X be a normal projective variety and Δ an effective $\mathbb{R}$-divisor. Suppose that (X, Δ) is a log canonical pair such that $\nu(K_X + \Delta) = 0$. Then $\nu(K_X + \Delta) = K(K_X + \Delta)$. Moreover, if Δ is a $\mathbb{Q}$-divisor, then $\nu(K_X + \Delta) = \kappa(K_X + \Delta) = K(K_X + \Delta)$.*

Proof. By taking a dlt blow-up (Theorem 1.1.6), we may assume that (X, Δ) is a $\mathbb{Q}$-factorial dlt pair. By Corollary 2.2.4, there exists a log minimal model (X_m, Δ_m) of (X, Δ). From Lemma 1.2.6 (3), it holds that $K_{X_m} + \Delta_m \equiv 0$. By Theorem 2.3.2, it holds that $K(K_{X_m} + \Delta_m) = 0$. Lemma 1.2.2 implies that $K(K_X + \Delta) = 0$. If Δ is a $\mathbb{Q}$-divisor, then there exists an effective $\mathbb{Q}$-divisor E such that $K_X + \Delta \sim_{\mathbb{Q}} E$ by Corollary 2.2.4 and Theorem 2.3.2. Thus we see that $\kappa(K_X + \Delta) = K(K_X + \Delta)$. We finish the proof of Theorem 2.3.3. □

Corollary 2.3.4. *Let $\pi : X \to S$ be a projective surjective morphism of normal quasi-projective varieties, and let (X, Δ) be a projective log canonical pair such that Δ is an effective $\mathbb{K}$-divisor. Suppose that $\nu(K_F + \Delta_F) = 0$ for a general fiber F, where $K_F + \Delta_F = (K_X + \Delta)|_F$. Then there exists an effective $\mathbb{K}$-divisor D such that $K_X + \Delta \sim_{\mathbb{K},\pi} D$.*

Proof. This follows from Theorem 2.3.3 and [BCHM, Lemma 3.2.1]. □

3

Canonical bundle formulae and subadjunctions

3.1 Introduction

The following lemma is one of the main results of this chapter, which is missing in the literature. It is a canonical bundle formula for generically finite proper surjective morphisms.

Lemma 3.1.1 (Main Lemma). *Let X and Y be normal varieties and let $f : X \to Y$ be a generically finite proper surjective morphism. Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Suppose that there exists an effective $\mathbb{K}$-divisor Δ on X such that (X, Δ) is log canonical and that $K_X + \Delta \sim_{\mathbb{K},f} 0$. Then there exists an effective $\mathbb{K}$-divisor Γ on Y such that (Y, Γ) is log canonical and that*

$$K_X + \Delta \sim_{\mathbb{K}} f^*(K_Y + \Gamma).$$

Moreover, if (X, Δ) is kawamata log terminal, then we can choose Γ such that (Y, Γ) is kawamata log terminal.

As an application of Lemma 3.1.1, we prove a subadjunction formula for minimal lc centers. It is a generalization of Kawamata's subadjunction formula (cf. [Ka8, Theorem 1]). For a local version, see Theorem 3.6.2 below.

Theorem 3.1.2 (Subadjunction formula for minimal lc centers). *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal projective variety and let D be an effective $\mathbb{K}$-divisor on X such that (X, D) is log canonical. Let W be a minimal log canonical center with respect to (X, D). Then there exists an effective $\mathbb{K}$-divisor D_W on W such that*

$$(K_X + D)|_W \sim_{\mathbb{K}} K_W + D_W$$

and that the pair (W, D_W) is kawamata log terminal. In particular, W has only rational singularities.

We summarize the contents of this chapter. Section 3.2 is devoted to the proof of Lemma 3.1.1. In Section 3.3, we discuss Ambro's canonical bundle formula for projective kawamata log terminal pairs with a generalization for $\mathbb{R}$-divisors (cf. Theorem 3.3.1). It is one of the key ingredients of the proof of Theorem 3.1.2. Although Theorem 3.3.1 is sufficient for applications in the subsequent sections, we treat slight generalizations of Ambro's canonical bundle formula for projective log canonical pairs. In Section 3.4, we prove a subadjunction formula for minimal log canonical centers (cf. Theorem 3.1.2), which is a generalization of Kawamata's subadjunction formula (cf. [Ka8, Theorem 1]). In Section 3.5, we give a quick proof of the non-vanishing theorem for log canonical pairs as an application of Theorem 3.1.2, which is the main theorem of [F11]. In Section 3.6, we prove a local version of our subadjunction formula for minimal log canonical centers (cf. Theorem 3.6.2). It is useful for local studies of singularities of pairs. This local version does not directly follow from the global version: Theorem 3.1.2. It is because we do not know how to compactify log canonical pairs.

3.2 Main lemma

In this section, we prove Lemma 3.1.1.

Proof of Lemma 3.1.1. Let

$$f : X \xrightarrow{g} Z \xrightarrow{h} Y$$

be the Stein factorization. By replacing (X, Δ) with $(Z, g_*\Delta)$, we can assume that $f : X \to Y$ is finite. Let D be a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor on Y such that $K_X + \Delta \sim_{\mathbb{K}} f^*D$. We consider the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\nu} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\mu]{} & Y, \end{array}$$

where

(i) μ is a resolution of singularities of Y,

(ii) there exists an open set $U \subseteq Y$ such that μ is isomorphic over U and f is étale over U. Moreover, $\mu^{-1}(Y - U)$ has a simple normal crossing support and $Y - U$ contains $\mathrm{Supp} f_* \Delta$, and

(iii) X' is the normalization of the irreducible component of $X \times_Y Y'$ which dominates Y'. In particular, f' is finite.

Let $\Omega = \sum_i \delta_i D_i$ be a $\mathbb{K}$-divisor on X' such that

$$K_{X'} + \Omega = \nu^*(K_X + \Delta).$$

We consider the ramification formula:

$$K_{X'} = f'^* K_{Y'} + R,$$

where $R = \sum_i (r_i - 1) D_i$ is an effective $\mathbb{Z}$-divisor such that r_i is the ramification index of D_i for every i. Note that it suffices to show the above formula outside codimension two closed subsets of X'. Then it holds that

$$(\mu \circ f')^* D \sim_{\mathbb{K}} f'^* K_{Y'} + R + \Omega.$$

By pushing forward the above formula by f', we see

$$\deg f' \cdot \mu^* D \sim_{\mathbb{K}} \deg f' \cdot K_{Y'} + f'_*(R + \Omega).$$

We set

$$\Gamma := \frac{1}{\deg f'} \mu_* f'_*(R + \Omega)$$

on Y. Then Γ is effective since

$$\mu_* f'_*(R + \Omega) = f_* \nu_*(R + \Omega) = f_*(\nu_* R + \Delta).$$

Let $Y' \setminus \mu^{-1} U = \bigcup_j E_j$ be the irreducible decomposition, where $\sum_j E_j$ is a simple normal crossing divisor. We set

$$I_j := \{i | f'(D_i) = E_j\}.$$

The coefficient of E_j in $f'_*(R + \Omega)$ is

$$\frac{\sum_{i \in I_j} (r_i + \delta_i - 1) \deg(f'|_{D_i})}{\deg f'}.$$

Since $\delta_i \leq 1$, it holds that

$$\sum_{i \in I_j} (r_i + \delta_i - 1) \deg(f'|_{D_i}) \leq \sum_{i \in I_j} r_i \deg(f'|_{D_i}) = \deg f'.$$

Thus (Y, Γ) is log canonical since $K_{Y'} + f'_*(R + \Omega) = \mu^*(K_Y + \Gamma)$. Moreover, if (X, Δ) is kawamata log terminal, then $\delta_i < 1$. Hence (Y, Γ) is kawamata log terminal. □

3.3 Ambro's canonical bundle formula

Theorem 3.3.1 is Ambro's canonical bundle formula for projective klt pairs (cf. [Am5, Theorem 4.1]) with a generalization for $\mathbb{R}$-divisors. We need it for the proof of our subadjunction formula: Theorem 3.1.2.

Theorem 3.3.1 (Ambro's canonical bundle formula for projective klt pairs). *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let (X, B) be a projective kawamata log terminal pair and let $f : X \to Y$ be a projective surjective morphism onto a normal projective variety Y with connected fibers. Assume that*

$$K_X + B \sim_{\mathbb{K},f} 0.$$

Then there exists an effective $\mathbb{K}$-divisor B_Y on Y such that (Y, B_Y) is klt and

$$K_X + B \sim_{\mathbb{K}} f^*(K_Y + B_Y).$$

Proof. If $\mathbb{K} = \mathbb{Q}$, then the statement is nothing but [Am5, Theorem 4.1]. From now on, we assume that $\mathbb{K} = \mathbb{R}$. Let $\sum_i B_i$ be the irreducible decomposition of Supp$B$. We put $V = \bigoplus_i \mathbb{R}B_i$. Then it is well known that

$$\mathcal{L} = \{\Delta \in V \,|\, (X, \Delta) \text{ is log canonical}\}$$

is a rational polytope in V. We can also check that

$$\mathcal{N} = \{\Delta \in \mathcal{L} \,|\, K_X + \Delta \text{ is } f\text{-nef}\}$$

is a rational polytope (cf. [B2, Proposition 3.2. (3)]) and $B \in \mathcal{N}$. We note that $\mathcal{N}$ is known as Shokurov's polytope. We also note that the proof of [B2, Proposition 3.2. (3)] works for our setting without any changes by [F12, Theorem 18.2]. Therefore, we can write

$$K_X + B = \sum_{i=1}^{k} r_i(K_X + \Delta_i)$$

such that

(i) $\Delta_i \in \mathcal{N}$ is an effective $\mathbb{Q}$-divisor on X for every i,

(ii) (X, Δ_i) is klt for every i, and

(iii) $0 < r_i < 1$, $r_i \in \mathbb{R}$ for every i, and $\sum_{i=1}^{k} r_i = 1$.

Since $K_X + B$ is numerically f-trivial and $K_X + \Delta_i$ is f-nef for every i, $K_X + \Delta_i$ is numerically f-trivial for every i. Thus,

$$\kappa(X_\eta, (K_X + \Delta_i)_\eta) = \nu(X_\eta, (K_X + \Delta_i)_\eta) = 0$$

for every i, where η is the generic point of Y, by Nakayama (cf. [Nak, Chapter V 2.9. Corollary]). See also [Am5, Theorem 4.2]. Therefore, $K_X + \Delta_i \sim_{\mathbb{Q},f} 0$ for every i by [F7, Theorem 1.1]. By the case when $\mathbb{K} = \mathbb{Q}$, we can find an effective $\mathbb{Q}$-divisor Θ_i on Y such that (Y, Θ_i) is klt and

$$K_X + \Delta_i \sim_{\mathbb{Q}} f^*(K_Y + \Theta_i)$$

for every i. By putting $B_Y = \sum_{i=1}^k r_i \Theta_i$, we obtain

$$K_X + B \sim_{\mathbb{R}} f^*(K_Y + B_Y),$$

and (Y, B_Y) is klt. □

Corollary 3.3.2 is a direct consequence of Theorem 3.3.1.

Corollary 3.3.2. *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let (X, B) be a log canonical pair and let $f : X \to Y$ be a projective surjective morphism between normal projective varieties. Assume that*

$$K_X + B \sim_{\mathbb{K},f} 0$$

and that every lc center of (X, B) is dominant onto Y. Then we can find an effective $\mathbb{K}$-divisor B_Y on Y such that (Y, B_Y) is kawamata log terminal and that

$$K_X + B \sim_{\mathbb{K}} f^*(K_Y + B_Y).$$

Proof. By taking a dlt blow-up (cf. Theorem 1.1.6), we can assume that (X, B) is dlt. By replacing (X, B) with its minimal lc center and taking the Stein factorization, we can assume that (X, B) is klt and that f has connected fibers (cf. Lemma 3.1.1). Therefore, we can take a desired B_Y by Theorem 3.3.1. □

From now on, we treat Ambro's canonical bundle formula for projective log canonical pairs. We note that Theorem 3.3.1 is sufficient for applications in subsequent sections.

3.3.3 (Observation). Let (X, B) be a log canonical pair and let $f : X \to Y$ be a projective surjective morphism between normal projective varieties with connected fibers. Assume that $K_X + B \sim_{\mathbb{Q},f} 0$ and that (X, B) is kawamata log terminal over the generic point of Y. We can write

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + M_Y + \Delta_Y)$$

where M_Y is the *moduli* $\mathbb{Q}$*-divisor* and Δ_Y is the *discriminant* $\mathbb{Q}$*-divisor*. For details, see, for example, [Am4]. It is conjectured that we can construct a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\nu} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\mu]{} & Y, \end{array}$$

with the following properties.

(i) ν and μ are projective birational.

(ii) X' is normal and $K_{X'} + B_{X'} = \nu^*(K_X + B)$.

(iii) $K_{X'} + B_{X'} \sim_{\mathbb{Q}} f'^*(K_{Y'} + M_{Y'} + \Delta_{Y'})$ such that Y' is smooth, the moduli $\mathbb{Q}$-divisor $M_{Y'}$ is semi-ample, and the discriminant $\mathbb{Q}$-divisor $\Delta_{Y'}$ has a simple normal crossing support.

In the above properties, the non-trivial part is the semi-ampleness of $M_{Y'}$. We know that we can construct the desired commutative diagrams of $f' : X' \to Y'$ and $f : X \to Y$ when

(1) $\dim X - \dim Y = 1$ (cf. [Ka7, Theorem 5] and so on),

(2) $\dim Y = 1$ (cf. [Am4, Theorem 0.1] and [Am5, Theorem 3.3]),

(3) general fibers of f are Abelian varieties or smooth surfaces with $\kappa = 0$ (cf. [F5, Theorem 1.2, Theorem 6.3]),

and so on. We take a general member $D \in |mM_{Y'}|$ of the free linear system $|mM_{Y'}|$ where m is a sufficiently large and divisible integer. We put

$$K_Y + B_Y = \mu_*(K_{Y'} + \frac{1}{m}D + \Delta_{Y'}).$$

Then it is easy to see that

$$\mu^*(K_Y + B_Y) = K_{Y'} + \frac{1}{m}D + \Delta_{Y'},$$

(Y, B_Y) is log canonical, and

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y).$$

By the above observation, we have Ambro's canonical bundle formula for projective log canonical pairs under some special assumptions.

Theorem 3.3.4. *Let (X, B) be a projective log canonical pair and let $f : X \to Y$ be a projective surjective morphism onto a normal projective variety Y such that $K_X + B \sim_{\mathbb{Q},f} 0$. Assume that* $\dim Y \leq 1$ *or* $\dim X - \dim Y \leq 1$. *Then there exists an effective $\mathbb{Q}$-divisor B_Y on Y such that (Y, B_Y) is log canonical and*

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y).$$

Proof. By taking a dlt blow-up (cf. Theorem 1.1.6), we can assume that (X, B) is dlt. If necessary, by replacing (X, B) with a suitable lc center of (X, B) and by taking the Stein factorization (cf. Lemma 3.1.1), we can assume that $f : X \to Y$ has connected fibers and that (X, B) is kawamata log terminal over the generic point of Y. We note that we can assume that $\dim Y = 1$ or $\dim X - \dim Y = 1$. By the arguments in 3.3.3, we can find an effective $\mathbb{Q}$-divisor B_Y on Y such that (Y, B_Y) is log canonical and that $K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y)$. □

3.4 Subadjunction for minimal log canonical centers

The following theorem is a generalization of Kawamata's subadjunction formula (cf. [Ka8, Theorem 1]). Theorem 3.4.1 is new even for threefolds. It is an answer to Kawamata's question (cf. [Ka7, Question 1.8]).

Theorem 3.4.1 (Subadjunction formula for minimal lc centers). *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal projective variety and let D be an effective $\mathbb{K}$-divisor on X such that (X, D) is log canonical. Let W be a minimal log canonical center with respect to (X, D). Then there exists an effective $\mathbb{K}$-divisor D_W on W such that*

$$(K_X + D)|_W \sim_{\mathbb{K}} K_W + D_W$$

and that the pair (W, D_W) is kawamata log terminal. In particular, W has only rational singularities.

Remark 3.4.2. In [Ka8, Theorem 1], Kawamata proved

$$(K_X + D + \varepsilon H)|_W \sim_{\mathbb{Q}} K_W + D_W,$$

where H is an ample Cartier divisor on X and ε is a positive rational number, under the extra assumption that D is a $\mathbb{Q}$-divisor and there exists an effective $\mathbb{Q}$-divisor D^o such that $D^o < D$ and that (X, D^o) is kawamata log terminal. Therefore, Kawamata's theorem claims nothing when $D = 0$.

Proof of Theorem 3.4.1. By taking a dlt blow-up (cf. Theorem 1.1.6), we can take a projective birational morphism $f : Y \to X$ from a normal projective variety Y with the following properties.

(i) $K_Y + D_Y = f^*(K_X + D)$.

(ii) (Y, D_Y) is a $\mathbb{Q}$-factorial dlt pair.

We can take a minimal lc center Z of (Y, D_Y) such that $f(Z) = W$. We note that $K_Z + D_Z = (K_Y + D_Y)|_Z$ is klt since Z is a minimal lc center of the dlt pair (Y, D_Y). Let

$$f : Z \xrightarrow{g} V \xrightarrow{h} W$$

be the Stein factorization of $f : Z \to W$. By the construction, we can write

$$K_Z + D_Z \sim_{\mathbb{K}} f^*A$$

where A is a $\mathbb{K}$-divisor on W such that $A \sim_{\mathbb{K}} (K_X + D)|_W$. We note that W is normal (cf. [F11, Theorem 2.4 (4)]). Since (Z, D_Z) is klt, we can take an effective $\mathbb{K}$-divisor D_V on V such that

$$K_V + D_V \sim_{\mathbb{K}} h^*A$$

and that (V, D_V) is klt by Theorem 3.3.1. By Lemma 3.1.1, we can find an effective $\mathbb{K}$-divisor D_W on W such that

$$K_W + D_W \sim_{\mathbb{K}} A \sim_{\mathbb{K}} (K_X + D)|_W$$

and that (W, D_W) is klt. □

3.5 Non-vanishing theorem for log canonical pairs

The following theorem is the main result of [F11]. It is almost equivalent to the base point free theorem for log canonical pairs. For details, see [F11].

Theorem 3.5.1 (Non-vanishing theorem). *Let X be a normal projective variety and let B be an effective $\mathbb{Q}$-divisor on X such that (X, B) is log canonical. Let L be a nef Cartier divisor on X. Assume that $aL - (K_X + B)$ is ample for some $a > 0$. Then the base locus of the linear system $|mL|$ contains no lc centers of (X, B) for every $m \gg 0$, that is, there is a positive integer m_0 such that* $\mathrm{Bs}|mL|$ *contains no lc centers of (X, B) for every $m \geq m_0$.*

Here, we give a quick proof of Theorem 3.5.1 by using Theorem 3.4.1.

Proof. Let W be any minimal lc center of the pair (X, B). It is sufficient to prove that W is not contained in $\mathrm{Bs}|mL|$ for $m \gg 0$. By Theorem 3.4.1, we can find an effective $\mathbb{Q}$-divisor B_W on W such that (W, B_W) is klt and $K_W + D_W \sim_{\mathbb{Q}} (K_X + B)|_W$. Therefore, $aL|_W - (K_W + B_W) \sim_{\mathbb{Q}} (aL - (K_X + B))|_W$ is ample. By the Kawamata–Shokurov base point free theorem, $|mL|_W|$ is free for $m \gg 0$. By [F11, Theorem 2.2],

$$H^0(X, \mathcal{O}_X(mL)) \to H^0(W, \mathcal{O}_W(mL))$$

is surjective for $m \geq a$. Therefore, W is not contained in $\mathrm{Bs}|mL|$ for $m \gg 0$. □

Remark 3.5.2. The above proof of Theorem 3.5.1 is shorter than the original proof in [F11]. However, the proof of Theorem 3.4.1 depends on very deep results such as the existence of dlt blow-ups. The proof of Theorem 3.5.1 in [F11] only depends on various well-prepared vanishing theorems and standard techniques.

3.6 Subadjunction formula: local version

In this section, we give a local version of our subadjunction formula for minimal log canonical centers. Theorem 3.6.1 is a local version of Ambro's canonical bundle formula for kawamata log terminal pairs: Theorem 3.3.1. It is essentially [F1, Theorem 1.2].

Theorem 3.6.1. *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let (X, B) be a kawamata log terminal pair and let $f : X \to Y$ be a proper surjective morphism onto a normal* affine *variety Y with connected fibers. Assume that*

$$K_X + B \sim_{\mathbb{K}, f} 0.$$

Then there exists an effective $\mathbb{K}$-divisor B_Y on Y such that (Y, B_Y) is klt and

$$K_X + B \sim_{\mathbb{K}} f^*(K_Y + B_Y).$$

We just explain how to modify the proof of [F1, Theorem 1.2].

Comments on the proof. First, we assume that $\mathbb{K} = \mathbb{Q}$. In this case, the proof of [F1, Theorem 1.2] works with some minor modifications. We note that M in the proof of [F1, Theorem 1.2] is μ-nef. We also note that we can assume $H = 0$ in [F1, Theorem 1.2] since Y is affine. Next, we assume that $\mathbb{K} = \mathbb{R}$. In this case, we can reduce the problem to the case when $\mathbb{K} = \mathbb{R}$ as in the proof of Theorem 3.3.1. So, we obtain the desired formula. □

By Theorem 3.6.1, we can obtain a local version of Theorem 3.4.1. The proof of Theorem 3.4.1 works without any modifications.

Theorem 3.6.2 (local version). *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal* affine *variety and let D be an effective $\mathbb{K}$-divisor on X such that (X, D) is log canonical. Let W be a minimal log canonical center with respect to (X, D). Then there exists an effective $\mathbb{K}$-divisor D_W on W such that*

$$(K_X + D)|_W \sim_{\mathbb{K}} K_W + D_W$$

and that the pair (W, D_W) is kawamata log terminal. In particular, W has only rational singularities.

Theorem 3.6.2 does not directly follow from Theorem 3.4.1. If we use a *log canonical closure* (in recent preprint [HX]) we can extend Theorem 3.6.2 to any quasi projective situations.

4

Reduction maps and minimal model theory

4.1 Introduction

Suppose that X is a smooth projective variety over $\mathbb{C}$. The minimal model program predicts that there is a birational model of X that satisfies particularly nice properties. More precisely, if X has non-negative Kodaira dimension then X should admit a good minimal model: a birational model X' with mild singularities such that some multiple of $K_{X'}$ is base point free. In this paper we use reduction maps to study the existence of good minimal models for pairs (X, Δ).

We first interpret the existence of good minimal models in terms of the numerical dimension of [Nak] and [BDPP]. Using results of [Lai], we show that for a kawamata log terminal pair (X, Δ) the existence of a good minimal model is equivalent to the abundance of $K_X + \Delta$. Thus we will focus on the following conjecture:

Conjecture 4.1.1. *Let (X, Δ) be a kawamata log terminal pair. Then $K_X + \Delta$ is abundant, that is, $\kappa(K_X + \Delta) = \nu(K_X + \Delta)$.*

Our main goal is to show that the abundance of $K_X + \Delta$ can be detected on the base of certain morphisms:

Theorem 4.1.2 (=Corollary 4.3.4)**.** *Let (X, Δ) be a kawamata log terminal pair. Suppose that $f : X \to Z$ is a morphism with connected fibers to a variety Z such that $\nu((K_X + \Delta)|_F) = 0$ for a general fiber F of f. Then there exist a smooth birational model Z' of Z and a kawamata log terminal pair $(Z', \Delta_{Z'})$ such that $K_X + \Delta$ is abundant if and only if $K_{Z'} + \Delta_{Z'}$ is abundant.*

In order to apply Theorem 4.1.2 in practice, the key question is whether one can find a map such that the numerical dimension of $(K_X+\Delta)|_F$ vanishes for a general fiber F. The $(K_X + \Delta)$-trivial reduction map constructed in [Leh1] satisfies precisely this property. We will consider a birational version of this map better suited for the study of adjoint divisors; we then define $\widetilde{\tau}(X, \Delta)$ to be the dimension of the image of this birational version. Thus we obtain the following:

Theorem 4.1.3 (Corollary of Theorem 4.5.1)**.** *Let* (X, Δ) *be a kawamata log terminal pair. If* $0 \leq \widetilde{\tau}(K_X + \Delta) \leq 3$ *then* (X, Δ) *has a good minimal model.*

The $(K_X + \Delta)$-trivial reduction map is constructed by quotienting by curves with $(K_X+\Delta)\cdot C = 0$. Thus, another way to approach the problem is to focus on the properties of curves on X. Recall that an irreducible curve C is said to be movable if it is a member of a family of curves dominating X. Conjecture 4.1.1 yields the following prediction:

Conjecture 4.1.4. *Let* (X, Δ) *be a kawamata log terminal pair. Suppose that* $(K_X + \Delta).C > 0$ *for every movable curve* C *on* X*. Then* $K_X + \Delta$ *is big.*

Our final goal is to show that the two conjectures are equivalent:

Theorem 4.1.5 (=Corollary 4.5.2)**.** *Conjecture 4.1.4 holds up to dimension n iff Conjecture 4.1.1 holds up to dimension n.*

The use of reduction maps to study the minimal model program was initiated by [Am3]. Our work relies on Ambro's techniques. Note that our main theorem generalizes [Am3] and [Fk4]. Related topics have been considered in [BDPP]. The method of using fibrations to study the abundance conjecture seems to appear first in [Ka2]. Finally, similar ideas have appeared independently in the recent preprint [Siu].

We summarize the contents of this chapter. In Section 4.2, we introduce the definitions of some log smooth models for the proof of main results. In Section 4.3, we prove Theorem 4.1.2 and show the equivalence of the abundance conjecture and the existence of good minimal models. In Section 4.4, we introduce the D-trivial reduction map of [Leh1] and define $\widetilde{\tau}(X, \Delta)$ for a log pair (X, Δ). In Section 4.5, we show Theorem 4.1.3 and Theorem 4.1.5.

4.2 Log smooth models

In this section, we introduce the definition of some log smooth models for the proof of main results.

Definition 4.2.1. Let (X, Δ) be a kawamata log terminal pair and $\varphi : W \to X$ a log resolution of (X, Δ). Choose Δ_W so that

$$K_W + \Delta_W = \phi^*(K_X + \Delta) + E$$

where Δ_W and E are effective $\mathbb{Q}$-divisors that have no common component. We call (W, Δ_W) a log smooth model of (X, Δ). Note that a minimal model of (W, Δ_W) may not be a minimal model of (X, Δ). To correct this deficiency, define

$$F = \sum_{F_i:\varphi\text{-exceptional}} F_i \text{ and } \Delta_W^\epsilon = \Delta_W + \epsilon F$$

for a positive number ϵ. We call (W, Δ_W^ϵ) an *ϵ-log smooth model.*

Remark 4.2.2. Note that our definition of a log smooth model differs from that of Birkar and Shokurov (cf. [B1] and [B2]).

4.3 The existence of minimal models and abundance

In this section, we show that the abundance conjecture is equivalent to the existence of good minimal models. We also prove Theorem 4.1.2, the main technical tool for the inductive arguments of Section 4.5.

Lemma 4.3.1 ([Nak, V.4.2 Corollary]). *Let (X, Δ) be a kawamata log terminal pair with $\kappa(K_X + \Delta) \geq 0$. Then the following are equivalent:*

1. $\kappa(K_X + \Delta) = \nu(K_X + \Delta)$.
2. *Let $\mu : X' \to X$ be a birational morphism and $f : X' \to Z'$ a morphism resolving the Iitaka fibration for $K_X + \Delta$. Then*

   $$\nu(\mu^*(K_X + \Delta)|_F) = 0$$

   for a general fiber F of f.

If either of these equivalent conditions hold, we say that L is abundant.

The following theorem is known to experts; for example, see [DHP, Remark 2.6]. The main ideas of the proof are from [Lai].

Theorem 4.3.2 (cf. [DHP]). *Let (X, Δ) be a klt pair. Then $K_X + \Delta$ is abundant if and only if (X, Δ) has a good minimal model.*

Proof. First suppose that (X, Δ) has a good minimal model (X', Δ'). Let Y be a common resolution of X and X' (with morphisms f and g respectively) and write

$$f^*(K_X + \Delta) = g^*(K_{X'} + \Delta') + E$$

where E is an effective exceptional $\mathbb{Q}$-divisor. Thus

$$P_\sigma(f^*(K_X + \Delta)) = P_\sigma(g^*(K_{X'} + \Delta'))$$

and since the latter divisor is semi-ample, the abundance of $K_X + \Delta$ follows.

Conversely, suppose that $K_X + \Delta$ is abundant. Let $f : (X, \Delta) \dashrightarrow Z$ be the Iitaka fibration of $K_X + \Delta$. Choose an ϵ-log smooth model $\varphi : (W, \Delta_W^\epsilon) \to X$ with sufficiently small $\epsilon > 0$ so that f is resolved on W. By [BCHM, Lemma 3.6.10] we can find a minimal model for (X, Δ) by constructing a minimal model of (W, Δ_W^ϵ) . Moreover we see that $f \circ \varphi$ is also the Iitaka fibration of $K_W + \Delta_W^\epsilon$ and $\nu(K_W + \Delta_W^\epsilon) = \nu(K_X + \Delta)$. Replacing (X, Δ) by (W, Δ_W^ϵ), we may suppose that the Iitaka fibration f is a morphism on X.

By [Nak, V.4.2 Corollary], $\nu(K_F + \Delta_F) = 0$ where F is a general fiber of f and $K_F + \Delta_F = (K_X + \Delta)|_F$. Therefore (F, Δ_F) has a good minimal model by [D, Corollaire 3.4].

The arguments of [Lai, Theorem 4.4] for (X, Δ) now show that (X, Δ) has a good minimal model. □

The following lemma is key for proving our main results. It is a consequence of Ambro's work on LC-trivial fibrations (cf. [Am5]).

Lemma 4.3.3. *Let (X, Δ) be a projective klt pair. Suppose that $f : X \to Z$ is a projective morphism with connected fibers to a smooth projective variety Z such that $\nu((K_X + \Delta)|_F) = 0$ for a general fiber. Then there exists a log resolution $\mu : X' \to X$ of (X, Δ), a klt pair $(Z', \Delta_{Z'})$, and a projective morphism $f' : X' \to Z'$ birationally equivalent to f such that*

$$P_\sigma(\mu^*(K_X + \Delta)) \sim_{\mathbb{Q}} P_\sigma(f'^*(K_{Z'} + \Delta_{Z'})).$$

We may assume that Z' is any sufficiently high birational model of Z.

Proof. By [Nak, V.4.9 Corollary] $\kappa((K_X + \Delta)|_F) = 0$. By [Am5, Theorem 3.3] and [FM, 4.4], there exist a log resolution $\mu : X' \to X$ of (X, Δ), a morphism $f' : X' \to Z'$, an effective divisor $\Delta_{Z'}$ on Z', and a (not necessarily effective) $\mathbb{Q}$-divisor $B = B^+ - B^-$ that satisfy:

(1) (X', Δ') is a log smooth model and $(Z', \Delta_{Z'})$ is klt,

(2) $K_{X'} + \Delta' \sim_{\mathbb{Q}} f'^*(K_{Z'} + \Delta_{Z'}) + B$,

(3) there exist positive integers m_1 and m_2 such that

$$H^0(X', mm_1(K_{X'} + \Delta')) = H^0(Z', mm_2(K_{Z'} + \Delta_{Z'})),$$

for any positive integers m,

(4) B^- is f'-exceptional and

(5) $f_*O_X(\llcorner lB^+ \lrcorner) = O_Y$ for every sufficiently divisible integer l.

Moreover f' is the resolution of a flattening by the Fujino–Mori construction (cf. [FM, 4.4]). Thus B^- is μ-exceptional so that

$$P_\sigma(\mu^*(K_X + \Delta)) = P_\sigma(K_{X'} + \Delta' + B^-).$$

We now turn our attention to B^+. Note that $\nu((K_{X'} + \Delta')|_{F'}) = 0$ for a general fiber F' of f' (since the same is true on the general fiber of f). In particular $P_\sigma(K_{X'} + \Delta')|_{F'} \equiv 0$. Thus

$$B^+|_{F'} \leq N_\sigma((K_{X'} + \Delta')|_{F'}) = N_\sigma(K_{X'} + \Delta')|_{F'}.$$

This implies that $B^+_h \leq N_\sigma(K_{X'} + \Delta')$, where B^+_h is the horizontal part of B^+. Therefore

$$\begin{aligned} P_\sigma(K_{X'} + \Delta' + B^-) &\sim_{\mathbb{Q}} P_\sigma(f'^*(K_{Z'} + \Delta_{Z'}) + B^+_v + B^+_h) \\ &= P_\sigma(f'^*(K_{Z'} + \Delta_{Z'}) + B^+_v) \\ &= P_\sigma(f'^*(K_{Z'} + \Delta_{Z'})) \end{aligned}$$

where the last step follows from the fact that B^+_v is f'-degenerate by Lemma 1.1.11.

□

Corollary 4.3.4. *Let (X, Δ) be a kawamata log terminal pair. Suppose that $f : X \to Z$ is a projective morphism with connected fibers to a normal projective variety Z such that $\nu((K_X + \Delta)|_F) = 0$ for a general fiber F of f. Then there exists a higher birational model Z' of Z and a kawamata log terminal pair $(Z', \Delta_{Z'})$ such that $K_X + \Delta$ is abundant if and only if $K_{Z'} + \Delta_{Z'}$ is abundant.*

Proof. This follows from Lemma 4.3.3 and the fact that the numerical dimension is invariant under pull-back and under passing to the positive part $P_\sigma(L)$. □

4.4 Reduction maps and $\widetilde{\tau}(X, \Delta)$

The results of the previous section are most useful when combined with the theory of numerical reduction maps. We will focus on the D-trivial reduction map as defined in [Leh1]:

Theorem 4.4.1 ([Leh1, Theorem 1.1]). *Let X be a normal variety and D a pseudo-effective $\mathbb{R}$-Cartier divisor on X. Then there exist a projective birational morphism $\varphi : W \to X$ and a surjective projective morphism $f : W \to Y$ with connected fibers such that*

(0) W is smooth,

(1) $\nu(\varphi^* D|_F) = 0$ for a general fiber F of f,

(2) if $w \in W$ is very general and C is an irreducible curve through w with $\dim f(C) = 1$, then $\varphi^* L.C > 0$, and

(3) If there exist a projective birational morphism $\varphi' : W' \to X$ and a dominant projective morphism $f' : W' \to Y'$ with connected fibers satisfying condition (2), then f' factors birationally through f.

We call the composition $f \circ \phi^{-1} : X \dashrightarrow Y$ the D-trivial reduction map. Note that it is only unique up to birational equivalence.

Remark 4.4.2. The D-trivial reduction map is different from the pseudo-effective reduction map (cf. [E2] and [Leh1]), the partial nef reduction map (cf. [BDPP]), and Tsuji's numerically trivial fibration with minimal singular metrics (cf. [Ts] and [E1]).

Definition 4.4.3. Let X be a normal variety and D a pseudo-effective $\mathbb{R}$-Cartier divisor on X. If $f : X \dashrightarrow Y$ denotes the D-trivial reduction map, we define

$$\tau(D) := \dim Y.$$

The following properties follow immediately from the definition.

Lemma 4.4.4. *Let X be a normal projective variety and D a pseudo-effective $\mathbb{R}$-Cartier divisor on X. Then*

(1) $\nu(D) = 0$ if $\tau(D) = 0$,

(2) if D' is a pseudo-effective $\mathbb{R}$-Cartier divisor on X such that $D' \geq D$, then $\tau(D') \geq \tau(D)$, and

(3) $\tau(f^*D) = \tau(D)$ *for every surjective morphism* $f : Y \to X$ *from a normal variety.*

Since the canonical divisor is not a birational invariant, we need to introduce a slight variant of this construction that accounts for every ϵ-log smooth model.

Remark 4.4.5. Let (X, Δ) be a kamawata log terminal pair with $K_X + \Delta$ pseudo-effective. Suppose that $\phi : W \to X$ is a log resolution of (X, Δ). Then the value of $\tau(K_W + \Delta_W^\epsilon)$ for the ϵ-log smooth model (W, Δ_W^ϵ) is independent of the choice of $\epsilon > 0$, since if C is a movable curve with $(K_W + \Delta_W^\epsilon).C = 0$ then $E.C = 0$ for any ϕ-exceptional divisor E.

Definition 4.4.6. Let (X, Δ) be a projective kawamata log terminal pair such that $K_X + \Delta$ is pseudo-effective. We define

$$\begin{aligned}\widetilde{\tau}(X, \Delta) &= \max\{\tau(K_W + \Delta_W^\epsilon) | (W, \Delta_W^\epsilon) \text{ is an } \epsilon\text{-log smooth model} \\ &\qquad \text{of } (X, \Delta) \text{ with } 0 < \epsilon \ll 1\}.\end{aligned}$$

Lemma 4.4.7. *Let* (X, Δ) *be a kawamata log terminal pair such that* $K_X + \Delta$ *is pseudo-effective. Then there exists an* ϵ*-log smooth model* $\varphi : (W, \Delta_W^\epsilon) \to (X, \Delta)$ *such that the* $(K_W + \Delta_W^\epsilon)$*-trivial reduction map is a morphism on* W *whose image has dimension* $\widetilde{\tau}(X, \Delta)$ *for a sufficiently small positive number* ϵ.

Proof. We may certainly assume that $\tau(K_W + \Delta_W^\epsilon) = \widetilde{\tau}(X, \Delta)$. Suppose W' resolves the $(K_W + \Delta_W^\epsilon)$-trivial reduction map. By the maximality of the definition, the $(K_{W'} + \Delta_{W'}^{\epsilon'})$-trivial reduction map is birationally equivalent to the $(K_W + \Delta_W^\epsilon)$-trivial reduction map for any sufficiently small $\epsilon' > 0$. Thus it can be realized as the (resolved) morphism on W'. □

Remark 4.4.8. If D is a nef divisor, the D-trivial reduction map is birationally equivalent to the nef reduction map of D (see [BCEK+]). Thus $n(D) = \tau(D)$. Moreover, for a klt pair (X, Δ) such that $K_X + \Delta$ is nef, $\tau(K_X + \Delta) = \widetilde{\tau}(X, \Delta)$ since the nef reduction map is almost holomorphic.

Next, we prove that $\widetilde{\tau}(X, \Delta)$ is preserved under flips and divisorial contractions. Although we do not need this property to prove our main results, we include it for completeness.

Definition 4.4.9. Let X be an n-dimensional normal projective variety and $T \subset \mathrm{Chow}(X)$ an irreducible and compact normal covering family of 1-cycles in the sense of Camapana (cf. [C]). Let D be a $\mathbb{R}$-Cartier divisor on X. A covering family $\{C_t\}_{t \in T}$ is *D-trivial* if $D.C_t = 0$ for all $t \in T$. A covering family $\{C_t\}_{t \in T}$ is *1-connected* if for general x and $y \in X$ there is $t \in T$ such that C_t is an irreducible curve containing x and y.

Proposition 4.4.10 (cf. [Leh1, Proposition 4.8]). *Let X be a normal variety and D an $\mathbb{R}$-divisor on X. Suppose that there exists a D-trivial 1-connected covering family $\{C_t\}_{t\in T}$. Then $\nu(D) = 0$.*

Proof. For any birational map $\varphi : W \to X$, the strict transforms of the curves C_t are still 1-connecting. Thus, the generic quotient (in the sense of [Leh1, Construction 3.2]) of X by the family $\{C_t\}_{t\in T}$ contracts X to a point. Thus $\nu(D) = \nu(f^*D) = 0$ by [Leh1, Theorem 1.1]. □

Proposition 4.4.11. *Let (X, Δ) be a kawamata log terminal pair. Then $\nu(K_X+\Delta) = 0$ if and only if there exists a $(K_X + \Delta)$-trivial 1-connected covering family $\{C_t\}_{t\in T}$ such that $C_t \cap \mathbf{B}_-(K_X + \Delta) = \emptyset$ for general $t \in T$.*

Proof. The reverse implication follows from Proposition 4.4.10. Now assume that $\nu(K_X + \Delta) = 0$. By [D, Corollaire 3.4], we get a good minimal model (Y, Γ) of (X, Δ) with $K_Y + \Gamma \sim_{\mathbb{Q}} 0$. Take the following log resolutions:

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed] & & Y. \end{tikzcd}$$

Set

$$p^*(K_X + \Delta) = q^*(K_Y + \Gamma) + E,$$

where E is an effective q-exceptional divisor. Now, since $K_Y + \Gamma \sim_{\mathbb{Q}} 0$, it holds that

$$p^*(K_X + \Delta) \sim_{\mathbb{Q}} E.$$

Because $\operatorname{codim} q(\operatorname{Supp} E) \geq 2$, there exists a complete intersection irreducible curve C with respect to very ample divisors $H_1, \ldots, H_{n-1}$ containing two general points x, y such that $C \cap q(\operatorname{Supp} E) = \emptyset$. Let $\bar{C}$ be the strict transform of C on X. Then

$$(K_X + \Delta).\bar{C} = 0.$$

Since $p(\operatorname{Supp} E) = \mathbf{B}_-(K_X + \Delta)$, the desired family can be constructed by deforming C. □

Proposition 4.4.12. *Let (X, Δ) be a klt pair such that $K_X + \Delta$ is pseudo-effective. Suppose that*

$$\varphi : (X, \Delta) \dashrightarrow (X', \Delta')$$

is a flip or a divisorial contraction. Then $\widetilde{\tau}(X, \Delta) = \widetilde{\tau}(X', \Delta')$.

Proof. Consider a log resolution

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed] & & X'. \end{tikzcd}$$

Write

$$K_W + \Delta_W^{\epsilon} = p^*(K_X + \Delta) + G$$

for the ϵ-log smooth structure induced by (X, Δ) and

$$K_W + \Delta_W'^{\epsilon} = q^*(K_{X'} + \Delta') + G',$$

for the ϵ-log smooth structure induced by (X', Δ') for a sufficiently small positive number ϵ. (Note that these structures might differ, if for example ϕ is centered in a locus along which the discrepancy is negative.) Using Lemma 4.4.7, we may find a log resolution W so that

(1) the $(K_W + \Delta_W^{\epsilon})$-trivial reduction map is a morphism $f : W \to Y$ with $\dim Y = \widetilde{\tau}(X, \Delta)$, and

(2) the $(K_W + \Delta_W'^{\epsilon})$-trivial reduction map is a morphism $f' : W \to Y'$ with $\dim Y' = \widetilde{\tau}(X', \Delta')$ and Y' is smooth.

Since ϕ is a $(K_X+\Delta)$-negative contraction, there is some effective q-exceptional divisor E' such that $K_W + \Delta_W^{\epsilon} = K_W + \Delta_W'^{\epsilon} + E'$. From Lemma 4.4.4 (2), it holds that $\widetilde{\tau}(X, \Delta) \geq \widetilde{\tau}(X', \Delta')$. Note that as E' is q-exceptional and $(W, \Delta_W'^{\epsilon})$ is an ϵ-log smooth model with $\epsilon > 0$, we have $\tau E' \leq N_\sigma(K_W + \Delta_W'^{\epsilon})$ for some $\tau > 0$.

Every movable curve C with $(K_W+\Delta_W^{\epsilon}).C = 0$ also satisfies $(K_W+\Delta_W'^{\epsilon}).C = 0$. Conversely, by Proposition 4.4.11 a very general fiber F' of f' admits a 1-connecting covering family of $K_W + \Delta_W'^{\epsilon}$-trivial curves $\{C_t\}_{t\in T}$ such that $C_t \cap \mathbf{B}_-((K_W + \Delta_W'^{\epsilon})|_{F'}) = \emptyset$ for general $t \in T$.

Since $E'|_{F'}$ is effective and $\nu((K_W + \Delta_W'^{\epsilon})|_{F'}) = 0$, we know that $\tau E'|_{F'} \leq N_\sigma((K_W + \Delta_W'^{\epsilon})|_{F'})$. Thus $E'.C_t = 0$ for general t since C_t avoids $\mathbf{B}_-((K_W + \Delta_W'^{\epsilon})|_{F'})$. So

$$\begin{aligned}(K_W + \Delta_W^{\epsilon}).C_t &= (K_W + \Delta_W'^{\epsilon} + E').C_t \\ &= 0.\end{aligned}$$

By the universal property of the D-trivial reduction map, f and f' are birationally equivalent. □

Corollary 4.4.13. *Let (X, Δ) be a klt pair such that $K_X + \Delta$ is pseudo-effective. Suppose that (X, Δ) has a good minimal model. Then*

$$\widetilde{\tau}(X, \Delta) = \kappa(X, \Delta).$$

4.5 The MMP and the $(K_X + \Delta)$-trivial reduction map

In this section we use our main technical result Theorem 4.3.3 to analyze the main conjectures of the minimal model program inductively.

Theorem 4.5.1. *Assume that the existence of good minimal models for klt pairs in dimension d. Let (X, Δ) be a kawamata log terminal pair such that $K_X + \Delta$ is pseudo-effective and $\widetilde{\tau}(X, \Delta) = d$. Then there exists a good log minimal model of (X, Δ).*

Proof. Using Lemma 4.4.7, we can find a birational morphism $\varphi : W \to X$ from an ϵ-log smooth model (W, Δ_W^ϵ) of (X, Δ) for a sufficiently small positive number ϵ and a projective morphism $f : W \to Y$ with connected fibers such that

(i) $\nu((K_W + \Delta_W^\epsilon)|_F) = 0$ for the general fiber F of f and

(ii) $\dim Y = \widetilde{\tau}(X, \Delta)$.

Theorem 4.3.2 and Corollary 4.3.4 imply that (W, Δ_W^ϵ) has a good minimal model. (X, Δ) then has a good minimal model by [BCHM, Lemma 3.6.10]. □

Corollary 4.5.2. *Conjecture 4.1.4 holds up to dimension n if and only Conjecture 4.1.1 holds up to dimension n.*

Proof. Assume that Conjecture 4.1.4 holds up to dimension n. By induction on dimension, we may assume that Conjecture 4.1.1 holds up to dimension $n-1$. Let (X, Δ) be a kawamata log terminal pair of dimension n. If $\widetilde{\tau}(X, \Delta) < \dim X$ then $K_X + \Delta$ is abundant by Theorem 4.5.1. If $\widetilde{\tau}(X, \Delta) = \dim X$ then the abundance of $(K_X + \Delta)$ follows by assumption from Conjecture 4.1.4.

Conversely, assume that Conjecture 4.1.1 holds up to dimension n. By Theorem 4.3.2 we obtain the existence of good minimal models up to dimension n. Let (X, Δ) be a kawamata log terminal pair of dimension $k \leq n$ such that $\widetilde{\tau}(X, \Delta) = k$. By Corollary 4.4.13 X is covered by irreducible curves C such that $(K_X + \Delta).C = 0$ unless $\kappa(K_X + \Delta) = n$. □

Remark 4.5.3. It seems likely that one could formulate a stronger version of Theorem 4.5.2 using the pseudo-effective reduction map for $K_X + \Delta$ (cf. [E2] and [Leh1]). The difficulty is that the pseudo-effective reduction map only satisfies the weaker condition $\nu(P_\sigma(K_X + \Delta)|_F) = 0$ on a general fiber F, so it is unclear how to use the inductive hypothesis to relate F with X.

5 Log pluricanonical representations and abundance conjecture

5.1 Introduction

The following theorem is one of the main results of this chapter (cf. Theorem 5.3.13). It is a solution of the conjecture raised in [F2] (see [F2, Conjecture 3.2]). For the definition of the *log pluricanonical representation* ρ_m, see Definitions 5.2.6 and 5.2.9 below.

Theorem 5.1.1 (cf. [F2, Section 3], [G2, Theorem B]). *Let* (X, Δ) *be a projective log canonical pair. Suppose that* $m(K_X + \Delta)$ *is Cartier and that* $K_X + \Delta$ *is semi-ample. Then* $\rho_m(\mathrm{Bir}(X, \Delta))$ *is a finite group.*

In the framework of [F2], Theorem 5.1.1 will play important roles in the study of Conjecture 5.1.2 (see [Ft2], [AFKM], [Ka4], [KeMaMc], [F2], [F13], [G2], and so on).

Conjecture 5.1.2 ((Log) abundance conjecture). *Let* (X, Δ) *be a projective semi log canonical pair such that* Δ *is a* $\mathbb{Q}$*-divisor. Suppose that* $K_X + \Delta$ *is nef. Then* $K_X + \Delta$ *is semi-ample.*

Theorem 5.1.1 was settled for surfaces in [F2, Section 3] and for the case where $K_X + \Delta \sim_{\mathbb{Q}} 0$ by [G2, Theorem B]. In this paper, to carry out the proof of Theorem 5.1.1, we introduce the notion of *$\widetilde{B}$-birational maps* and *$\widetilde{B}$-birational representations* for sub kawamata log terminal pairs, which is new and is indispensable for generalizing the arguments in [F2, Section 3] for higher dimensional log canonical pairs. For the details, see Section 5.3.

By Theorem 5.1.1, we obtain a key result.

Theorem 5.1.3 (cf. Proposition 5.4.3). *Let (X, Δ) be a projective semi log canonical pair. Let $\nu : X^\nu \to X$ be the normalization. Assume that $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$ is semi-ample. Then $K_X + \Delta$ is semi-ample.*

By Theorem 5.1.3, Conjecture 5.1.2 is reduced to the problem for log canonical pairs.

Let X be a smooth projective n-fold. By our experience on the low-dimensional abundance conjecture, we think that we need the abundance theorem for projective semi log canonical pairs in dimension $\leq n-1$ in order to prove the abundance conjecture for X. Therefore, Theorem 5.1.3 seems to be an important step for the inductive approach to the abundance conjecture. The general strategy for proving the abundance conjecture is explained in the introduction of [F2]. Theorem 5.1.3 is a complete solution of Step (v) in [F2, 0. Introduction].

As applications of Theorem 5.1.3 and [F9, Theorem 1.1], we have the following useful theorems.

Theorem 5.1.4 (cf. Theorem 5.4.2). *Let (X, Δ) be a projective log canonical pair. Assume that $K_X + \Delta$ is nef and log abundant. Then $K_X + \Delta$ is semi-ample.*

It is a generalization of the well-known theorem for kawamata log terminal pairs (see, for example, [F7, Corollary 2.5]). Theorem 5.1.5 may be easier to understand than Theorem 5.1.4.

Theorem 5.1.5 (cf. Theorem 5.4.6). *Let (X, Δ) be an n-dimensional projective log canonical pair. Assume that the abundance conjecture holds for projective divisorial log terminal pairs in dimension $\leq n-1$. Then $K_X + \Delta$ is semi-ample if and only if $K_X + \Delta$ is nef and abundant.*

We have many other applications. In this introduction, we explain only two of them. The first one is an answer to Professor János Kollár's question. For a more general result, see Corollary 5.4.11.

Theorem 5.1.6 (cf. Theorem 5.4.9). *Let $f : X \to Y$ be a projective morphism between projective varieties. Let (X, Δ) be a log canonical pair such that $K_X + \Delta$ is numerically trivial over Y. Then $K_X + \Delta \sim_{\mathbb{Q}, Y} 0$.*

The second one is a generalization of [Fk5, Theorem 0.1] and [CKP, Corollary 3]. It also contains Theorem 5.1.4. For a further generalization, see Remark 5.4.19.

Theorem 5.1.7 (cf. Theorem 5.4.18). *Let (X, Δ) be a projective log canonical pair and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that D is nef and log abundant with respect to (X, Δ). Assume that $K_X + \Delta \equiv D$. Then $K_X + \Delta$ is semi-ample.*

The reader can find many applications and generalizations in Section 5.4.

We summarize the contents of this chapter. In Section 5.2, we collects some basic notations and results. Section 5.3 is the main part of this paper. In this section, we prove Theorem 5.1.1. We divide the proof into the three steps: sub kawamata log terminal pairs in 5.3.1, log canonical pairs with big log canonical divisor in 5.3.2, and log canonical pairs with semi-ample log canonical divisor in 5.3.3. Section 5.4 contains various applications of Theorem 5.1.1. They are related to the abundance conjecture: Conjecture 5.1.2. For example, we give an affirmative answer to Professor János Kollár's question (cf. Theorem 5.1.6). In the subsection 5.4.2, we generalize the main theorem in [Fk5] (cf. [CKP, Corollary 3]), and so on. In Section 5.5, we discuss the relationship among the various conjectures in the minimal model program

5.2 Slc, sdlt, and log pluricanonical representations

Definition 5.2.1 (Slc and sdlt)**.** Let X be a reduced S_2 scheme. We assume that it is pure n-dimensional and normal crossing in codimension one. Let Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We assume that $\Delta = \sum_i a_i\Delta_i$ where $a_i \in \mathbb{Q}$ and Δ_i is an irreducible codimension one closed subvariety of X such that $\mathcal{O}_{X,\Delta_i}$ is a DVR for every i. Let $X = \cup_i X_i$ be the irreducible decomposition and let $\nu : X^\nu := \amalg_i X_i^\nu \to X = \cup_i X_i$ be the normalization. A $\mathbb{Q}$-divisor Θ on X^ν is defined by $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$ and a $\mathbb{Q}$-divisor Θ_i on X_i^ν by $\Theta_i := \Theta|_{X_i^\nu}$. We say that (X, Δ) is a *semi log canonical n-fold* (an *slc n-fold,* for short) if (X^ν, Θ) is lc. We say that (X, Δ) is a *semi divisorial log terminal n-fold* (an *sdlt n-fold*, for short) if X_i is normal, that is, X_i^ν is isomorphic to X_i, and (X^ν, Θ) is dlt.

We recall a very important example of slc pairs.

Example 5.2.2. Let (X, Δ) be a $\mathbb{Q}$-factorial lc pair. We put $S = \llcorner\Delta\lrcorner$. Assume that $(X, \Delta - \varepsilon S)$ is klt for some $0 < \varepsilon \ll 1$. Then (S, Δ_S) is slc where $K_S + \Delta_S = (K_X + \Delta)|_S$.

Remark 5.2.3. Let (X, Δ) be a dlt pair. We put $S = \llcorner\Delta\lrcorner$. Then it is well known that (S, Δ_S) is sdlt where $K_S + \Delta_S = (K_X + \Delta)|_S$.

The following theorem was originally proved by Professor Christopher Hacon (cf. [F12, Theorem 10.4], [KoKov, Theorem 3.1]). For a simpler proof, see [F10, Section 4].

5.2.4 (Log pluricanonical representations). Nakamura–Ueno ([NakUe]) and Deligne proved the following theorem (see [U, Theorem 14.10]).

Theorem 5.2.5 (Finiteness of pluricanonical representations). *Let X be a compact complex Moishezon manifold. Then the image of the group homomorphism*

$$\rho_m : \mathrm{Bim}(X) \to \mathrm{Aut}_{\mathbb{C}}(H^0(X, mK_X))$$

is finite, where $\mathrm{Bim}(X)$ *is the group of bimeromorphic maps from* X *to itself.*

For considering the logarithmic version of Theorem 5.2.5, we need the notion of *B-birational maps* and *B-pluricanonical representations*.

Definition 5.2.6 ([F2, Definition 3.1]). Let (X, Δ) (resp. (Y, Γ)) be a pair such that X (resp. Y) is a normal scheme with a $\mathbb{Q}$-divisor Δ (resp. Γ) such that $K_X + \Delta$ (resp. $K_Y + \Gamma$) is $\mathbb{Q}$-Cartier. We say that a proper birational map $f : (X, \Delta) \dashrightarrow (Y, \Gamma)$ is *B-birational* if there exists a common resolution

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "f"'] & & Y \end{tikzcd}$$

such that

$$\alpha^*(K_X + \Delta) = \beta^*(K_Y + \Gamma).$$

This means that it holds that $E = F$ when we put $K_W = \alpha^*(K_X + \Delta) + E$ and $K_W = \beta^*(K_Y + \Gamma) + F$.

Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. Then we define

$$f^*D := \alpha_*\beta^*D.$$

It is easy to see that f^*D is independent of the common resolution $\alpha : W \to X$ and $\beta : W \to Y$.

Finally, we put

$$\mathrm{Bir}(X, \Delta) = \{\sigma \,|\, \sigma : (X, \Delta) \dashrightarrow (X, \Delta) \text{ is } B\text{-birational}\}.$$

It is obvious that $\mathrm{Bir}(X, \Delta)$ has a natural group structure.

Remark 5.2.7. In Definition 5.2.6, let $\psi : X' \to X$ be a proper birational morphism from a normal scheme X' such that $K_{X'} + \Delta' = \psi^*(K_X + \Delta)$. Then we can easily check that $\mathrm{Bir}(X, \Delta) \simeq \mathrm{Bir}(X', \Delta')$ by $g \mapsto \psi^{-1} \circ g \circ \psi$ for $g \in \mathrm{Bir}(X, \Delta)$.

We give a basic example of B-birational maps.

Example 5.2.8 (Quadratic transformation). Let $X = \mathbb{P}^2$ and let Δ be the union of three general lines on $\mathbb{P}^2$. Let $\alpha : W \to X$ be the blow-up at the three intersection points of Δ and let $\beta : W \to X$ be the blow-down of the strict transform of Δ on W. Then we obtain the *quadratic transformation* φ.

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\varphi"'] & & X \end{tikzcd}$$

For the details, see [H, Chapter V Example 4.2.3]. In this situation, it is easy to see that

$$\alpha^*(K_X + \Delta) = K_W + \Theta = \beta^*(K_X + \Delta).$$

Therefore, φ is a B-birational map of the pair (X, Δ).

Definition 5.2.9 ([F2, Definition 3.2]). Let X be a pure n-dimensional normal scheme and let Δ be a $\mathbb{Q}$-divisor, and let m be a nonnegative integer such that $m(K_X + \Delta)$ is Cartier. A B-birational map $\sigma \in \mathrm{Bir}(X, \Delta)$ defines a linear automorphism of $H^0(X, m(K_X+\Delta))$. Thus we get the group homomorphism

$$\rho_m : \mathrm{Bir}(X, \Delta) \to \mathrm{Aut}_{\mathbb{C}}(H^0(X, m(K_X + \Delta))).$$

The homomorphism ρ_m is called a *B-pluricanonical representation* or *log pluricanonical representation* for (X, Δ). We sometimes simply denote $\rho_m(g)$ by g^* for $g \in \mathrm{Bir}(X, \Delta)$ if there is no danger of confusion.

Let X be a pure n-dimensional normal scheme and $g : X \dashrightarrow X$ a proper birational (or bimeromorphic) map. Set $X = \coprod_{i=1}^k X_i$. The map g defines $\sigma \in \mathcal{S}_k$ such that $g|_{X_i} : X_i \dashrightarrow X_{\sigma(i)}$, where $\mathcal{S}_k$ is the symmetric group of degree k. Hence $g^{k!}$ induces $g^{k!}|_{X_i} : X_i \dashrightarrow X_i$. By Burnside's theorem ([CR, (36.1) Theorem]), we remark the following:

Remark 5.2.10. For proving the finiteness of log pluricanonical representation, we can check that it suffices to show it under the assumption that X is connected. Moreover, Theorem 5.2.5 for a pure dimensional disjoint union of some compact Moishezon complex manifolds holds.

In the subsection 5.3.1, we will consider *$\widetilde{B}$-pluricanonical representations* for subklt pairs (cf. Definition 5.3.1). In some sense, they are generalizations of Definitions 5.2.6 and 5.2.9. We need them for our proof of Theorem 5.1.1.

Remark 5.2.11. Let (X, Δ) be a projective dlt pair. We note that $g \in \mathrm{Bir}(X, \Delta)$ does not necessarily induce a birational map $g|_T : T \dashrightarrow T$, where $T = \llcorner \Delta \lrcorner$ (see Example 5.2.8). However, $g \in \mathrm{Bir}(X, \Delta)$ induces an automorphism

$$g^* : H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))) \xrightarrow{\sim} H^0(T, \mathcal{O}_T(m(K_T + \Delta_T)))$$

where $(K_X + \Delta)|_T = K_T + \Delta_T$ and m is a nonnegative integer such that $m(K_X + \Delta)$ is Cartier (see the proof of [F2, Lemma 4.9]). More precisely, let

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "g"'] & & X \end{tikzcd}$$

be a common log resolution such that

$$\alpha^*(K_X + \Delta) = K_W + \Theta = \beta^*(K_X + \Delta).$$

Then we can easily see that

$$\alpha_* \mathcal{O}_S \simeq \mathcal{O}_T \simeq \beta_* \mathcal{O}_S,$$

where $S = \Theta^{=1}$, by the Kawamata–Viehweg vanishing theorem. Thus we obtain an automorphism

$$\begin{aligned} g^* : H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))) &\xrightarrow{\beta^*} H^0(S, \mathcal{O}_S(m(K_S + \Theta_S))) \\ &\xrightarrow{\alpha^{*-1}} H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))) \end{aligned}$$

where $(K_W + \Theta)|_S = K_S + \Theta_S$.

Let us recall an important lemma on B-birational maps, which will be used in the proof of the main theorem (cf. Theorem 5.3.13).

Lemma 5.2.12. *Let $f : (X, \Delta) \to (X', \Delta')$ be a B-birational map between projective dlt pairs. Let S be an lc center of (X, Δ) such that $K_S + \Delta_S = (K_X + \Delta)|_S$. We take a suitable common log resolution as in* Definition 5.2.6.

$$\begin{tikzcd} & (W, \Gamma) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ (X, \Delta) \arrow[rr, dashed, "f"'] & & (X', \Delta') \end{tikzcd}$$

Then we can find an lc center V of (X, Δ) contained in S with $K_V + \Delta_V = (K_X + \Delta)|_V$, an lc center T of (W, Γ) with $K_T + \Gamma_T = (K_X + \Delta)|_T$, and an lc center V' of (X', Δ') with $K_{V'} + \Delta'_{V'} = (K_{X'} + \Delta')|_{V'}$ such that the following conditions hold.

(a) $\alpha|_T$ *and* $\beta|_T$ *are B-birational morphisms.*

$$\begin{tikzcd} & (T, \Gamma_T) \arrow[dl, "\alpha|_T"'] \arrow[dr, "\beta|_T"] & \\ (V, \Delta_V) & & (V', \Delta'_{V'}) \end{tikzcd}$$

Therefore, $(\beta|_T) \circ (\alpha|_T)^{-1} : (V, \Delta_V) \dashrightarrow (V', \Delta'|_{V'})$ *is a B-birational map.*

(b) $H^0(S, m(K_S + \Delta_S)) \simeq H^0(V, m(K_V + \Delta_V))$ *by the natural restriction map where* m *is a nonnegative integer such that* $m(K_X + \Delta)$ *is Cartier.*

Proof. See Claim (A_n) and Claim (B_n) in the proof of [F2, Lemma 4.9]. □

We close this section with a remark on the minimal model program with scaling. For the details, see Section 1.3.2.

5.2.13 (Minimal model program with ample scaling). Let $f : X \to Z$ be a projective morphism between quasi-projective varieties and let (X, B) be a $\mathbb{Q}$-factorial dlt pair. Let H be an effective f-ample $\mathbb{Q}$-divisor on X such that $(X, B + H)$ is lc and that $K_X + B + H$ is f-nef. Under these assumptions, we can run the minimal model program on $K_X + B$ with scaling of H over Z. We call it *the minimal model program with ample scaling.*

Assume that $K_X + B$ is not pseudo-effective over Z. We note that the above minimal model program always terminates at a Mori fiber space structure over Z. By this observation, the results in [F2, Section 2] hold in every dimension. Therefore, we will freely use the results in [F2, Section 2] for *any* dimensional varieties.

From now on, we assume that $K_X + B$ is pseudo-effective and $\dim X = n$. We further assume that the weak non-vanishing conjecture (cf. Conjecture 5.5.1) for projective $\mathbb{Q}$-factorial dlt pairs holds in dimension $\leq n$. Then the minimal model program on $K_X + B$ with scaling of H over Z terminates with a minimal model of (X, B) over Z by [B2, Theorems 1.4, 1.5].

5.3 Finiteness of log pluricanonical representations

In this section, we give a proof of Theorem 5.1.1. We divide the proof into the three steps: subklt pairs in 5.3.1, lc pairs with big log canonical divisor in 5.3.2, and lc pairs with semi-ample log canonical divisor in 5.3.3.

5.3.1 Klt pairs

In this subsection, we prove Theorem 5.1.1 for klt pairs. More precisely, we prove Theorem 5.1.1 for *$\widetilde{B}$-pluricanonical representations* for projective subklt pairs without assuming the semi-ampleness of log canonical divisors. This formulation is indispensable for the proof of Theorem 5.1.1 for lc pairs.

First, let us introduce the notion of *$\widetilde{B}$-pluricanonical representations for subklt pairs*.

Definition 5.3.1 ($\widetilde{B}$-pluricanonical representations for subklt pairs)**.** Let (X, Δ) be an n-dimensional projective subklt pair such that X is smooth and that Δ has a simple normal crossing support. We write $\Delta = \Delta^+ - \Delta^-$ where Δ^+ and Δ^- are effective and have no common irreducible components. Let m be a positive integer such that $m(K_X + \Delta)$ is Cartier. In this subsection, we always see

$$\omega \in H^0(X, m(K_X + \Delta))$$

as a meromorphic m-ple n-form on X which vanishes along $m\Delta^-$ and has poles at most $m\Delta^+$. By $\mathrm{Bir}(X)$, we mean the group of all the birational mappings of X onto itself. It has a natural group structure induced by the composition of birational maps. We define

$$\widetilde{\mathrm{Bir}}_m(X, \Delta) = \left\{ g \in \mathrm{Bir}(X) \,\middle|\, \begin{array}{l} g^*\omega \in H^0(X, m(K_X + \Delta)) \text{ for} \\ \text{every } \omega \in H^0(X, m(K_X + \Delta)) \end{array} \right\}.$$

Then it is easy to see that $\widetilde{\mathrm{Bir}}_m(X, \Delta)$ is a subgroup of $\mathrm{Bir}(X)$. An element $g \in \widetilde{\mathrm{Bir}}_m(X, \Delta)$ is called a *$\widetilde{B}$-birational map* of (X, Δ). By the definition of $\widetilde{\mathrm{Bir}}_m(X, \Delta)$, we get the group homomorphism

$$\widetilde{\rho}_m : \widetilde{\mathrm{Bir}}_m(X, \Delta) \to \mathrm{Aut}_{\mathbb{C}}(H^0(X, m(K_X + \Delta))).$$

The homomorphism $\widetilde{\rho}_m$ is called the *$\widetilde{B}$-pluricanonical representation* of $\widetilde{\mathrm{Bir}}_m(X, \Delta)$. We sometimes simply denote $\widetilde{\rho}_m(g)$ by g^* for $g \in \widetilde{\mathrm{Bir}}_m(X, \Delta)$ if there is no danger of confusion. There exists a natural inclusion $\mathrm{Bir}(X, \Delta) \subset \widetilde{\mathrm{Bir}}_m(X, \Delta)$ by the definitions.

Next, let us recall the notion of *$L^{2/m}$-integrable m-ple n-forms*.

Definition 5.3.2. Let X be an n-dimensional connected complex manifold and let ω be a meromorphic m-ple n-form. Let $\{U_\alpha\}$ be an open covering of X with holomorphic coordinates

$$(z^1_\alpha, z^2_\alpha, \cdots, z^n_\alpha).$$

We can write

$$\omega|_{U_\alpha} = \varphi_\alpha (dz_\alpha^1 \wedge \cdots \wedge dz_\alpha^n)^m,$$

where φ_α is a meromorphic function on U_α. We give $(\omega \wedge \bar{\omega})^{1/m}$ by

$$(\omega \wedge \bar{\omega})^{1/m}|_{U_\alpha} = \left(\frac{\sqrt{-1}}{2\pi}\right)^n |\varphi_\alpha|^{2/m} dz_\alpha^1 \wedge d\bar{z}_\alpha^1 \cdots \wedge dz_\alpha^n \wedge d\bar{z}_\alpha^n.$$

We say that a meromorphic m-ple n-form ω is *$L^{2/m}$-integrable* if

$$\int_X (\omega \wedge \bar{\omega})^{1/m} < \infty.$$

We can easily check the following two lemmas.

Lemma 5.3.3. *Let X be a compact connected complex manifold and let D be a reduced normal crossing divisor on X. Set $U = X \setminus D$. If ω is an L^2-integrable meromorphic n-form such that $\omega|_U$ is holomorphic, then ω is a holomorphic n-form.*

Proof. See, for example, [Sak, Theorem 2.1] or [Ka1, Proposition 16]. □

Lemma 5.3.4 (cf. [G2, Lemma 4.8]). *Let (X, Δ) be a projective subklt pair such that X is smooth and Δ has a simple normal crossing support. Let m be a positive integer such that $m\Delta$ is Cartier and let $\omega \in H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$ be a meromorphic m-ple n-form. Then ω is $L^{2/m}$-integrable.*

Proof. Since (X, Δ) is subklt, we may write $\Delta = \sum_i a_i \Delta_i$, where Δ_i is a prime divisor and $a_i < 1$. We see that $1 - 1/m > a_i$ and ma_i is an integer for any i. Thus ω is a meromorphic m-ple n-form with at most $(m-1)$-ple pole along Δ_i for all i. By [Sak, Theorem 2.1] and holomorphicity of $\omega|_U$, $\int_X (\omega \wedge \bar{\omega})^{1/m} = \int_U (\omega|_U \wedge \bar{\omega}|_U)^{1/m} < \infty$, where $U = X \setminus \operatorname{Supp} \Delta$. □

By Lemma 5.3.4, we obtained the following result.

Proposition 5.3.5 ([G2, Proposition 4.9]). *Let (X, Δ) be an n-dimensional projective subklt pair such that X is smooth, connected, and Δ has a simple normal crossing support. Let $g \in \widetilde{\mathrm{Bir}}_m(X, \Delta)$ be a $\widetilde{B}$-birational map where m is a positive integer such that $m\Delta$ is Cartier, and let*

$$\omega \in H^0(X, m(K_X + \Delta))$$

be a nonzero meromorphic m-ple n-form on X. Suppose that $g^*\omega = \lambda\omega$ for some $\lambda \in \mathbb{C}$. Then there exists a positive integer $N_{m,\omega}$ such that $\lambda^{N_{m,\omega}} = 1$ and $N_{m,\omega}$ does not depend on g.

Proof. We consider the projective space bundle

$$\pi : M := \mathbb{P}_X(\mathcal{O}_X(-K_X) \bigoplus \mathcal{O}_X) \to X.$$

Set $\Delta = \Delta^+ - \Delta^-$, where Δ^+ and Δ^- are effective, and have no common components. Let $\{U_\alpha\}$ be coordinate neighborhoods of X with holomorphic coordinates $(z_\alpha^1, z_\alpha^2, \cdots, z_\alpha^n)$. Since $\omega \in H^0(X, m(K_X + \Delta))$, we can write ω locally as

$$\omega|_{U_\alpha} = \frac{\varphi_\alpha}{\delta_\alpha}(dz_\alpha^1 \wedge \cdots \wedge dz_\alpha^n)^m,$$

where φ_α and δ_α are holomorphic with no common factors, and $\frac{\varphi_\alpha}{\delta_\alpha}$ has poles at most $m\Delta^+$. We may assume that $\{U_\alpha\}$ gives a local trivialization of M, i.e. $M|_{U_\alpha} := \pi^{-1}U_\alpha \simeq U_\alpha \times \mathbb{P}^1$. We set a coordinate $(z_\alpha^1, z_\alpha^2, \cdots, z_\alpha^n, \xi_\alpha^1 : \xi_\alpha^2)$ of $U_\alpha \times \mathbb{P}^1$ with homogeneous coordinates $(\xi_\alpha^1 : \xi_\alpha^2)$ of $\mathbb{P}^1$. Note that

$$\frac{\xi_\alpha^1}{\xi_\alpha^2} = k_{\alpha\beta}\frac{\xi_\beta^1}{\xi_\beta^2} \text{ in } M|_{U_\alpha \bigcap U_\beta},$$

where $k_{\alpha\beta} = \det(\partial z_\beta^i / \partial z_\alpha^j)_{1 \le i,j \le n}$. Set

$$Y_{U_\alpha} = \{(\xi_\alpha^1)^m \delta_\alpha - (\xi_\alpha^2)^m \varphi_\alpha = 0\} \subset U_\alpha \times \mathbb{P}^1.$$

We can patch $\{Y_{U_\alpha}\}$ easily and denote the patching by Y. Note that Y may have singularities and be reducible. Let $\pi_1 : M' \to M$ be a log resolution of $(M, Y \cup \pi^{-1}(\mathrm{Supp}\Delta))$ such that Y' is smooth, where Y' is the strict transform of Y on M'. We set $F' = \pi \circ \pi_1$ and $f' = F'|_{Y'}$. Remark that Y' may be disconnected and a general fiber of f' is m points. Define a meromorphic n-form on M by

$$\Theta|_{M|_{U_\alpha}} = \frac{\xi_\alpha^1}{\xi_\alpha^2} dz_\alpha^1 \wedge \cdots \wedge dz_\alpha^n.$$

We put $\theta' = \pi_1^*\Theta|_{Y'}$. By the definition,

$$(\theta')^m = f'^*\omega.$$

Since $\int_X (\omega \wedge \bar{\omega})^{1/m} < \infty$ by Lemma 5.3.4, it holds that $\int_{Y'} \theta' \wedge \bar{\theta}' < \infty$. Hence θ' is L^2-integrable. Since $f'^{-1}(\mathrm{Supp}\Delta)$ is simple normal crossings, θ' is a holomorphic n-form on Y' by Lemma 5.3.3.

We take a $\nu \in \mathbb{R}$ such that $\nu^m = \lambda$. We define a birational map $\bar{g}_\nu : M \dashrightarrow M$ by

$$\bar{g}_\nu : (z_\alpha^1, z_\alpha^2, \cdots, z_\alpha^n, \xi_\alpha^1 : \xi_\alpha^2) \to (g(z_\alpha^1, z_\alpha^2, \cdots, z_\alpha^n), \nu(\det(\partial g / \partial z_\alpha))^{-1}\xi_\alpha^1 : \xi_\alpha^2)$$

on U_α. Then $\bar{g}_\nu$ induces a birational map $h' : Y' \dashrightarrow Y'$. It satisfies that

$$\begin{tikzcd} Y' \arrow[r, dashed, "h'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\circlearrowleft"] & Y' \arrow[d, "f'"] \\ X \arrow[r, dashed, "g"] & X. \end{tikzcd}$$

Thus we see

$$h'^*(\theta')^m = h'^*f'^*\omega = f'^*g^*\omega = \lambda f'^*\omega = \lambda(\theta')^m.$$

Because Theorem 5.2.5 holds for pure dimensional possibly disconnected projective manifolds (Remark 5.2.10), there exists a positive integer $N_{m,\omega}$ such that $\lambda^{N_{m,\omega}} = 1$ and $N_{m,\omega}$ does not depend on g. We finish the proof of Proposition 5.3.5.

□

Remark 5.3.6. By the proof of [G2, Proposition 4.9] and [U, Theorem 14.10], we know that $\varphi(N_{m,\omega}) \leq b_n(Y')$, where $b_n(Y')$ is the n-th Betti number of Y' which is in the proof of [G2, Proposition 4.9] and φ is the Euler function.

Proposition 5.3.7 (cf. [U, Proposition 14.7]). *Let (X, Δ) be a projective subklt pair such that X is smooth, connected, and Δ has a simple normal crossing support, and let*

$$\widetilde{\rho}_m : \widetilde{\mathrm{Bir}}_m(X, \Delta) \to \mathrm{Aut}_{\mathbb{C}}(H^0(X, m(K_X + \Delta)))$$

be the $\widetilde{B}$-pluricanonical representation of $\widetilde{\mathrm{Bir}}_m(X, \Delta)$ where m is a positive integer such that $m\Delta$ is Cartier. Then $\widetilde{\rho}_m(g)$ is semi-simple for every $g \in \widetilde{\mathrm{Bir}}_m(X, \Delta)$.

Proof. If $\widetilde{\rho}_m(g)$ is not semi-simple, there exist two linearly independent elements φ_1, $\varphi_2 \in H^0(X, m(K_X + \Delta))$ and nonzero $\alpha \in \mathbb{C}$ such that

$$g^*\varphi_1 = \alpha\varphi_1 + \varphi_2,\ g^*\varphi_2 = \alpha\varphi_2$$

by considering Jordan's decomposition of g^*. Here, we denote $\widetilde{\rho}_m(g)$ by g^* for simplicity. By Proposition 5.3.5, we see that α is a root of unity. Let l be a positive integer. Then we have

$$(g^l)^*\varphi_1 = \alpha^l\varphi_1 + l\alpha^{l-1}\varphi_2.$$

Since g is a birational map, we have

$$\int_X (\varphi_1 \wedge \bar{\varphi}_1)^{1/m} = \int_X ((g^l)^*\varphi_1 \wedge (g^l)^*\bar{\varphi}_1)^{1/m}.$$

On the other hand, we have

$$\lim_{l\to\infty}\int_X ((g^l)^*\varphi_1 \wedge (g^l)^*\bar{\varphi}_1)^{1/m} = \infty.$$

For details, see the proof of [U, Proposition 14.7]. However, we know $\int_X (\varphi_1 \wedge \bar{\varphi}_1)^{1/m} < \infty$ by Lemma 5.3.4. This is a contradiction. □

Proposition 5.3.8. *The number $N_{m,\omega}$ in* Proposition 5.3.5 *is uniformly bounded for every $\omega \in H^0(X, m(K_X + \Delta))$. Therefore, we can take a positive integer N_m such that N_m is divisible by $N_{m,\omega}$ for every ω.*

Proof. We consider the projective space bundle

$$\pi : M := \mathbb{P}_X(\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X) \to X$$

and

$$\begin{aligned} V &:= M \times \mathbb{P}(H^0(X, \mathcal{O}_X(m(K_X + \Delta)))) \\ &\to X \times \mathbb{P}(H^0(X, \mathcal{O}_X(m(K_X + \Delta)))). \end{aligned}$$

We fix a basis $\{\omega_0, \omega_1, \dots, \omega_N\}$ of $H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$. By using this basis, we can identify $\mathbb{P}(H^0(X, \mathcal{O}_X(m(K_X + \Delta))))$ with $\mathbb{P}^N$. We write the coordinate of $\mathbb{P}^N$ as $(a_0 : \cdots : a_N)$ under this identification. Set $\Delta = \Delta^+ - \Delta^-$, where Δ^+ and Δ^- are effective and have no common irreducible components. Let $\{U_\alpha\}$ be coordinate neighborhoods of X with holomorphic coordinates $(z^1_\alpha, z^2_\alpha, \cdots, z^n_\alpha)$. For any i, we can write ω_i locally as

$$\omega_i|_{U_\alpha} = \frac{\varphi_{i,\alpha}}{\delta_{i,\alpha}}(dz^1_\alpha \wedge \cdots \wedge dz^n_\alpha)^m,$$

where $\varphi_{i,\alpha}$ and $\delta_{i,\alpha}$ are holomorphic with no common factors, and $\frac{\varphi_{i,\alpha}}{\delta_{i,\alpha}}$ has poles at most $m\Delta^+$. We may assume that $\{U_\alpha\}$ gives a local trivialization of M, i.e. $M|_{U_\alpha} := \pi^{-1}U_\alpha \simeq U_\alpha \times \mathbb{P}^1$. We set a coordinate $(z^1_\alpha, z^2_\alpha, \cdots, z^n_\alpha, \xi^0_\alpha : \xi^1_\alpha)$ of $U_\alpha \times \mathbb{P}^1$ with the homogeneous coordinate $(\xi^0_\alpha : \xi^1_\alpha)$ of $\mathbb{P}^1$. Note that

$$\frac{\xi^0_\alpha}{\xi^1_\alpha} = k_{\alpha\beta}\frac{\xi^0_\beta}{\xi^1_\beta} \text{ in } M|_{U_\alpha \cap U_\beta},$$

where $k_{\alpha\beta} = \det(\partial z^i_\beta / \partial z^j_\alpha)_{1\le i,j\le n}$. Set

$$Y_{U_\alpha} = \{(\xi^0_\alpha)^m \prod_{i=0}^{N} \delta_{i,\alpha} - (\xi^1_\alpha)^m \sum_{i=0}^{N} \hat{\delta}_{i,\alpha} a_i \varphi_{i,\alpha} = 0\} \subset U_\alpha \times \mathbb{P}^1 \times \mathbb{P}^N,$$

where $\hat{\delta}_{i,\alpha} = \delta_{0,\alpha} \cdots \delta_{i-1,\alpha} \cdot \delta_{i+1,\alpha} \cdots \delta_{N,\alpha}$. By easy calculations, we see that $\{Y_{U_\alpha}\}$ can be patched and we obtain Y. We note that Y may have singularities and be reducible. The induced projection $f : Y \to \mathbb{P}^N$ is surjective and equidimensional. Let $q : Y \to X$ be the natural projection. By the same arguments as in the proof of [U, Theorem 14.10], we have a suitable stratification $\mathbb{P}^N = \amalg_i S_i$, where S_i is smooth and locally closed in $\mathbb{P}^N$ for every i, such that $(f^{-1}(S_i)^\nu, q^*\Delta|_{f^{-1}(S_i)^\nu}) \to S_i$ has a simultaneous log resolution for every i, where $f^{-1}(S_i)^\nu$ is the normalization of $f^{-1}(S_i)$. Therefore, there is a positive constant b such that for every $p \in \mathbb{P}^N$ we have a resolution $\mu_p : \widetilde{Y}_p \to Y_p := f^{-1}(p)$ with the properties that $b_n(\widetilde{Y}_p) \le b$ and that $\mu_p^*(q^*\Delta|_{Y_p})$ has a simple normal crossing support. Thus, by Remark 5.3.6, we obtain Proposition 5.3.8. □

Now we have the main theorem of this subsection. We will use it in the following subsections.

Theorem 5.3.9. *Let (X, Δ) be a projective subklt pair such that X is smooth, Δ has a simple normal crossing support, and $m(K_X + \Delta)$ is Cartier where m is a positive integer. Then $\widetilde{\rho}_m(\widetilde{\mathrm{Bir}}_m(X, \Delta))$ is a finite group.*

Proof. By Proposition 5.3.7, we see that $\widetilde{\rho}_m(g)$ is diagonalizable. Moreover, Proposition 5.3.8 implies that the order of $\widetilde{\rho}_m(g)$ is bounded by a positive constant N_m which is independent of g. Thus $\widetilde{\rho}_m(\widetilde{\mathrm{Bir}}_m(X, \Delta))$ is a finite group by Burnside's theorem. □

As a corollary, we obtain Theorem 5.1.1 for klt pairs without assuming the semi-ampleness of log canonical divisors.

Corollary 5.3.10. *Let (X, Δ) be a projective klt pair such that $m(K_X + \Delta)$ is Cartier where m is a positive integer. Then $\rho_m(\mathrm{Bir}(X, \Delta))$ is a finite group.*

Proof. Let $f : Y \to X$ be a log resolution of (X, Δ) such that $K_Y + \Delta_Y = f^*(K_X + \Delta)$. Since

$$\rho_m(\mathrm{Bir}(Y, \Delta_Y)) \subset \widetilde{\rho}_m(\widetilde{\mathrm{Bir}}_m(Y, \Delta_Y)),$$

$\rho_m(\mathrm{Bir}(Y, \Delta_Y))$ is a finite group by Theorem 5.3.9. Therefore, we obtain that $\rho_m(\mathrm{Bir}(X, \Delta)) \simeq \rho_m(\mathrm{Bir}(Y, \Delta_Y))$ is a finite group. □

5.3.2 Lc pairs with big log canonical divisor

In this subsection, we prove the following theorem. The proof is essentially the same as that of Case 1 in [F2, Theorem 3.5].

Theorem 5.3.11. *Let (X, Δ) be a projective sublc pair such that $K_X + \Delta$ is big. Let m be a positive integer such that $m(K_X + \Delta)$ is Cartier. Then $\rho_m(\mathrm{Bir}(X, \Delta))$ is a finite group.*

Before we start the proof of Theorem 5.3.11, we give a remark.

Remark 5.3.12. By Theorem 5.3.11, when $K_X + \Delta$ is big, Theorem 5.1.1, the main theorem of this paper, holds true without assuming that $K_X + \Delta$ is semi-ample. Therefore, we state Theorem 5.3.11 separately for some future usage. In Case 2 in the proof of Theorem 5.3.13, which is nothing but Theorem 5.1.1, we will use the arguments in the proof of Theorem 5.3.11.

Proof. By taking a log resolution, we can assume that X is smooth and Δ has a simple normal crossing support. By Theorem 5.3.9, we can also assume that $\Delta^{=1} \neq 0$. Since $K_X + \Delta$ is big, for a sufficiently large and divisible positive integer m', we obtain an effective Cartier divisor $D_{m'}$ such that

$$m'(K_X + \Delta) \sim_{\mathbb{Z}} \Delta^{=1} + D_{m'}$$

by Kodaira's lemma. It is easy to see that $\mathrm{Supp}\, g^*\Delta^{=1} = \mathrm{Supp}\, \Delta^{=1}$ for every $g \in \mathrm{Bir}(X, \Delta)$. This implies that $g^*\Delta^{=1} \geq \Delta^{=1}$. Thus, we have a natural inclusion

$$\mathrm{Bir}(X, \Delta) \subset \widetilde{\mathrm{Bir}}_{m'}\left(X, \Delta - \frac{1}{m'}\Delta^{=1}\right).$$

We consider the $\widetilde{B}$-birational representation

$$\widetilde{\rho}_{m'} : \widetilde{\mathrm{Bir}}_{m'}\left(X, \Delta - \frac{1}{m'}\Delta^{=1}\right) \to \mathrm{Aut}_{\mathbb{C}} H^0(X, m'(K_X + \Delta) - \Delta^{=1}).$$

Then, by Theorem 5.3.9,

$$\widetilde{\rho}_{m'}\left(\widetilde{\mathrm{Bir}}_{m'}\left(X, \Delta - \frac{1}{m'}\Delta^{=1}\right)\right)$$

is a finite group. Therefore, $\widetilde{\rho}_{m'}(\mathrm{Bir}(X, \Delta))$ is also a finite group. We put $a = |\widetilde{\rho}_{m'}(\mathrm{Bir}(X, \Delta))| < \infty$. In this situation, we can find a $\mathrm{Bir}(X, \Delta)$-invariant non-zero section $s \in H^0(X, a(m'(K_X + \Delta) - \Delta^{=1}))$. By using s, we have a natural inclusion

$$H^0(X, m(K_X + \Delta)) \subseteq H^0(X, (m + m'a)(K_X + \Delta) - a\Delta^{=1}). \qquad (\spadesuit)$$

By the construction, $\mathrm{Bir}(X, \Delta)$ acts on the both vector spaces compatibly. We consider the $\widetilde{B}$-pluricanonical representation

$$\widetilde{\rho}_{m+m'a} : \widetilde{\mathrm{Bir}}_{m+m'a}\left(X, \Delta - \frac{a}{m + m'a}\Delta^{=1}\right)$$
$$\to \mathrm{Aut}_{\mathbb{C}} H^0(X, (m + m'a)(K_X + \Delta) - a\Delta^{=1}).$$

Since

$$\left(X, \Delta - \frac{a}{m+m'a}\Delta^{=1}\right)$$

is subklt, we have that

$$\widetilde{\rho}_{m+m'a}\left(\widetilde{\mathrm{Bir}}_{m+m'a}\left(X, \Delta - \frac{a}{m+m'a}\Delta^{=1}\right)\right)$$

is a finite group by Theorem 5.3.9. Therefore, $\widetilde{\rho}_{m+m'a}(\mathrm{Bir}(X,\Delta))$ is also a finite group. Thus, we obtain that $\rho_m(\mathrm{Bir}(X,\Delta))$ is a finite group by the $\mathrm{Bir}(X,\Delta)$-equivariant embedding (♠). □

5.3.3 Lc pairs with semi-ample log canonical divisor

Theorem 5.3.13 is one of the main results of this paper (see Theorem 5.1.1). We will treat many applications of Theorem 5.3.13 in Section 5.4.

Theorem 5.3.13. *Let (X,Δ) be an n-dimensional projective lc pair such that $K_X+\Delta$ is semi-ample. Let m be a positive integer such that $m(K_X+\Delta)$ is Cartier. Then $\rho_m(\mathrm{Bir}(X,\Delta))$ is a finite group.*

Proof. We show the statement by the induction on n. By taking a dlt blow-up (cf. Theorem 1.1.6), we may assume that (X,Δ) is a $\mathbb{Q}$-factorial dlt pair. Let $f: X \to Y$ be a projective surjective morphism associated to $k(K_X+\Delta)$ for a sufficiently large and divisible positive integer k. By Corollary 5.3.10, we may assume that $\llcorner \Delta \lrcorner \neq 0$.

Case 1. $\llcorner \Delta^h \lrcorner \neq 0$, where Δ^h is the horizontal part of Δ with respect to f.

In this case, we put $T = \llcorner \Delta \lrcorner$. Since $m(K_X+\Delta) \sim_{\mathbb{Q},Y} 0$, we see that

$$H^0(X, \mathcal{O}_X(m(K_X+\Delta)-T)) = 0.$$

Thus the restricted map

$$H^0(X, \mathcal{O}_X(m(K_X+\Delta))) \to H^0(T, \mathcal{O}_T(m(K_T+\Delta_T)))$$

is injective, where $K_T+\Delta_T = (K_X+\Delta)|_T$. Let (V_i, Δ_{V_i}) be the disjoint union of all the i-dimensional lc centers of (X,Δ) for $0 \le i \le n-1$. We note that $\rho_m(\mathrm{Bir}(V_i,\Delta_{V_i}))$ is a finite group for every i by the induction on dimension. We put $k_i = |\rho_m(\mathrm{Bir}(V_i,\Delta_{V_i}))| < \infty$ for $0 \le i \le n-1$. Let l be the least common multiple of k_i for $0 \le i \le n-1$. Let $T = \cup_j T_j$ be the irreducible decomposition. By repeatedly using Lemma 5.2.12, for every T_j, we can find lc centers S_j^i of (X, Δ)

$$\begin{array}{ccccccccc} X & \overset{g}{\dashrightarrow} & X & \overset{g}{\dashrightarrow} & X & \overset{g}{\dashrightarrow} & \cdots & \overset{g}{\dashrightarrow} & X & \dashrightarrow \\ \cup & & \cup & & \cup & & & & \cup & \\ S_j^0 & & S_j^1 & & S_j^2 & & & & S_j^k & \end{array}$$

such that $S_j^0 \subset T_j$, $S_j^i \dashrightarrow S_j^{i+1}$ is a B-birational map for every i, and

$$H^0(T_j, m(K_{T_j} + \Delta_{T_j})) \simeq H^0(S_j^0, m(K_{S_j^0} + \Delta_{S_j^0}))$$

by the natural restriction map, where $K_{T_j} + \Delta_{T_j} = (K_X + \Delta)|_{T_j}$ and $K_{S_j^0} + \Delta_{S_j^0} = (K_X + \Delta)|_{S_j^0}$. Since there are only finitely many lc centers of (X, Δ), we can find $p_j < q_j$ such that $S_j^{p_j} = S_j^{q_j}$ and that $S_j^{p_j} \neq S_j^r$ for $r = p_j + 1, \cdots, q_j - 1$. Therefore, g induces a B-birational map

$$\widetilde{g} : \coprod_{p_j \le r \le q_j - 1} S_j^r \dashrightarrow \coprod_{p_j \le r \le q_j - 1} S_j^r$$

for every j. Thus, we have an embedding

$$H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))) \subset \bigoplus_j H^0(S_j^{p_j}, m(K_{S_j^{p_j}} + \Delta_{S_j^{p_j}})),$$

where $K_{S_j^{p_j}} + \Delta_{S_j^{p_j}} = (K_X + \Delta)|_{S_j^{p_j}}$ for every j. First, by the following commutative diagram (cf. Remark 5.2.11)

$$\begin{tikzcd} 0 \arrow[r] & H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))) \arrow[r] \arrow[d, "(g^*)^l"'] & \bigoplus_j H^0(S_j^{p_j}, m(K_{S_j^{p_j}} + \Delta_{S_j^{p_j}}))) \arrow[d, "(\widetilde{g}^*)^l = \mathrm{id}"] \\ 0 \arrow[r] & H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))) \arrow[r] & \bigoplus_j H^0(S_j^{p_j}, m(K_{S_j^{p_j}} + \Delta_{S_j^{p_j}}))), \end{tikzcd}$$

we obtain $(g^*)^l = \mathrm{id}$ on $H^0(T, m(K_T + \Delta_T))$. Next, by the following commutative diagram (cf. Remark 5.2.11)

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \arrow[r] \arrow[d, "(g^*)^l"'] & H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))) \arrow[d, "(g^*)^l = \mathrm{id}"] \\ 0 \arrow[r] & H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \arrow[r] & H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))), \end{tikzcd}$$

we have that $(g^*)^l = \mathrm{id}$ on $H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$. Thus we obtain that $\rho_m(\mathrm{Bir}(X, \Delta))$ is a finite group by Burnside's theorem (cf. [U, Theorem 14.9]).

Case 2. $\llcorner \Delta^h \lrcorner = 0$.

We can construct the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\psi"'] & Y \end{tikzcd}$$

with the following properties:

(a) $\varphi : X' \to X$ is a log resolution of (X, Δ).

(b) $\psi : Y' \to Y$ is a resolution of Y.

(c) there is a simple normal crossing divisor Σ on Y' such that f' is smooth and $\operatorname{Supp} \varphi_*^{-1}\Delta \cup \operatorname{Exc}(\varphi)$ is relatively normal crossing over $Y' \setminus \Sigma$.

(d) $\operatorname{Supp} f'^*\Sigma$ and $\operatorname{Supp} f'^*\Sigma \cup \operatorname{Exc}(\varphi) \cup \operatorname{Supp} \varphi_*^{-1}\Delta$ are simple normal crossing divisors on X'.

Then we have

$$K_{X'} + \Delta_{X'} = f'^*(K_{Y'} + \Delta_{Y'} + M),$$

where $K_{X'} + \Delta_{X'} = \varphi^*(K_X + \Delta)$, $\Delta_{Y'}$ is the discriminant divisor and M is the moduli part of $f' : (X', \Delta_{X'}) \to Y'$. Note that

$$\Delta_{Y'} = \sum (1 - c_Q)Q,$$

where Q runs through all the prime divisors on Y' and

$$c_Q = \sup\{t \in \mathbb{Q} \,|\, K_{X'} + \Delta_{X'} + tf'^*Q \text{ is sublc over the generic point of } Q\}.$$

We can further assume that $\operatorname{Supp} \Delta_{X'}^{=1} \subset \operatorname{Supp} f'^*\Delta_{Y'}^{=1}$ by taking more blow-ups. We can check that every $g \in \operatorname{Bir}(X', \Delta_{X'}) = \operatorname{Bir}(X, \Delta)$ induces $g_{Y'} \in \operatorname{Bir}(Y', \Delta_{Y'})$ which satisfies the following commutative diagram (see [Am4, Theorem 0.2] for the subklt case, and [Ko2, Proposition 8.4.9 (3)] for the sublc case).

$$\begin{tikzcd} X' \arrow[r, dashed, "g"] \arrow[d, "f'"'] \arrow[dr, phantom, "\circlearrowleft"] & X' \arrow[d, "f'"] \\ Y' \arrow[r, dashed, "g_{Y'}"'] & Y' \end{tikzcd}$$

Therefore, we have $\operatorname{Supp} g_{Y'}^*\Delta_{Y'}^{=1} = \operatorname{Supp} \Delta_{Y'}^{=1}$. This implies that

$$g_{Y'}^*\Delta_{Y'}^{=1} \geq \Delta_{Y'}^{=1}.$$

Thus there is an effective Cartier divisor E_g on X' such that

$$g^*f'^*\Delta_{Y'}^{=1} + E_g \geq f'^*\Delta_{Y'}^{=1}$$

and that the codimension of $f'(E_g)$ in Y' is ≥ 2. We note the definitions of g^* and $g_{Y'}^*$ (cf. Definition 5.2.6). Therefore, $g \in \operatorname{Bir}(X', \Delta_{X'})$ induces an automorphism g^* of $H^0(X', m'(K_{X'} + \Delta_{X'}) - f'^*\Delta_{Y'}^{=1})$ where m' is a sufficiently large and divisible positive integer m'. It is because

$$\begin{aligned}
&H^0(X', m'(K_{X'} + \Delta_{X'}) - g^*f'^*\Delta_{Y'}^{=1}) \\
&\subset H^0(X', m'(K_{X'} + \Delta_{X'}) - f'^*\Delta_{Y'}^{=1} + E_g) \\
&\simeq H^0(X', m'(K_{X'} + \Delta_{X'}) - f'^*\Delta_{Y'}^{=1}).
\end{aligned}$$

Here, we used the facts that $m'(K_{X'} + \Delta_{X'}) = f'^*(m'(K_{Y'} + \Delta_{Y'} + M))$ and that $f'_*\mathcal{O}_{X'}(E_g) \simeq \mathcal{O}_{Y'}$. Thus we have a natural inclusion

$$\operatorname{Bir}(X', \Delta_{X'}) \subset \widetilde{\operatorname{Bir}}_{m'}\left(X', \Delta_{X'} - \frac{1}{m'}f'^*\Delta_{Y'}^{=1}\right).$$

Since $K_{Y'} + \Delta_{Y'} + M$ is (nef and) big, for a sufficiently large and divisible positive integer m', we obtain an effective Cartier divisor $D_{m'}$ such that

$$m'(K_{Y'} + \Delta_{Y'} + M) \sim_{\mathbb{Z}} \Delta_{Y'}^{=1} + D_{m'}.$$

This means that

$$H^0(X', m'(K_{X'} + \Delta_{X'}) - f'^*\Delta_{Y'}^{=1}) \neq 0.$$

By considering the natural inclusion

$$\operatorname{Bir}(X', \Delta_{X'}) \subset \widetilde{\operatorname{Bir}}_{m'}\left(X', \Delta_{X'} - \frac{1}{m'}f'^*\Delta_{Y'}^{=1}\right),$$

we can use the same arguments as in the proof of Theorem 5.3.11. Thus we obtain the finiteness of B-pluricanonical representations. □

Remark 5.3.14. Although we did not explicitly state it, in Theorem 5.3.9, we do not have to assume that X is connected. Similarly, we can prove Theorems 5.3.11 and 5.3.13 without assuming that X is connected. For the details, see Remark 5.2.10.

We close this section with comments on [F2, Section 3] and [G2, Theorem B]. In [F2, Section 3], we proved Theorem 5.3.13 for surfaces. There, we do not need the notion of $\widetilde{B}$-birational maps. It is mainly because Y' in Case 2 in the proof of Theorem 5.3.13 is a curve if (X, Δ) is not klt and $K_X + \Delta$ is not big. Thus, $g_{Y'}$ is an automorphism of Y'. In [G2, Theorem B], we proved Theorem 5.3.13 under the assumption that $K_X + \Delta \sim_{\mathbb{Q}} 0$. In that case, Case 1 in the proof of Theorem 5.3.13 is sufficient. Therefore, we do not need the notion of $\widetilde{B}$-birational maps in [G2].

5.4 On abundance conjecture for log canonical pairs

In this section, we treat various applications of Theorem 5.1.1 on the abundance conjecture for (semi) lc pairs (cf. Conjecture 5.1.2).

Let us introduce the notion of *nef and log abundant* $\mathbb{Q}$*-divisors*.

Definition 5.4.1 (Nef and log abundant divisors). Let (X, Δ) be a sublc pair. A closed subvariety W of X is called an *lc center* if there exist a resolution $f : Y \to X$ and a divisor E on Y such that $a(E, X, \Delta) = -1$ and $f(E) = W$. A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on X is called *nef and log abundant with respect to* (X, Δ) if and only if D is nef and abundant, and $\nu_W^* D|_W$ is nef and abundant for every lc center W of the pair (X, Δ), where $\nu_W : W^\nu \to W$ is the normalization. Let $\pi : X \to S$ be a proper morphism onto a variety S. Then D is π*-nef and* π*-log abundant with respect to* (X, Δ) if and only if D is π-nef and π-abundant and $(\nu_W^* D|_W)|_{W_\eta^\nu}$ is abundant, where W_η^ν is the generic fiber of $W^\nu \to \pi(W)$. We sometimes simply say that D is nef and log abundant over S.

The following theorem is one of the main theorems of this section (cf. [F3, Theorem 0.1], [F15, Theorem 4.4]). For a relative version of Theorem 5.4.2, see Theorem 5.4.12 below.

Theorem 5.4.2. *Let (X, Δ) be a projective lc pair. Assume that $K_X + \Delta$ is nef and log abundant. Then $K_X + \Delta$ is semi-ample.*

Proof. By replacing (X, Δ) with its dlt blow-up (cf. Theorem 1.1.6), we can assume that (X, Δ) is dlt and that $K_X + \Delta$ is nef and log abundant. We put $S = \llcorner \Delta \lrcorner$. Then (S, Δ_S), where $K_S + \Delta_S = (K_X + \Delta)|_S$, is an sdlt $(n-1)$-fold and $K_S + \Delta_S$ is semi-ample by the induction on dimension and Proposition 5.4.3 below. By applying Fukuda's theorem (cf. [F9, Theorem 1.1]), we obtain that $K_X + \Delta$ is semi-ample. $\square$

We note that Proposition 5.4.3 is a key result in this paper. It heavily depends on Theorem 5.1.1.

Proposition 5.4.3. *Let (X, Δ) be a projective slc pair. Let $\nu : X^\nu \to X$ be the normalization. Assume that $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$ is semi-ample. Then $K_X + \Delta$ is semi-ample.*

Proof. The arguments in [F2, Section 4] work by Theorem 5.1.1. As we pointed out in 5.2.13, we can freely use the results in [F2, Section 2]. The finiteness of B-pluricanonical representations, which was only proved in dimension ≤ 2 in [F2, Section 3], is now Theorem 5.1.1. Therefore, the results in [F2, Section 4] hold in any dimension. □

By combining Proposition 5.4.3 with Theorem 5.4.2, we obtain an obvious corollary (see also Corollary 5.4.13, Theorem 5.4.18, and Remark 5.4.19).

Corollary 5.4.4. *Let (X, Δ) be a projective slc pair and let $\nu : X^\nu \to X$ be the normalization. If $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$ is nef and log abundant, then $K_X + \Delta$ is semi-ample.*

We give one more corollary of Proposition 5.4.3.

Corollary 5.4.5. *Let (X, Δ) be a projective slc pair such that $K_X + \Delta$ is nef. Let $\nu : X^\nu \to X$ be the normalization. Assume that X^ν is a toric variety. Then $K_X + \Delta$ is semi-ample.*

Proof. It is well known that every nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a projective toric variety is semi-ample. Therefore, this corollary is obvious by Proposition 5.4.3. □

Theorem 5.4.6. *Let (X, Δ) be a projective n-dimensional lc pair. Assume that the abundance conjecture holds for projective dlt pairs in dimension $\leq n - 1$. Then $K_X + \Delta$ is semi-ample if and only if $K_X + \Delta$ is nef and abundant.*

Proof. It is obvious that $K_X + \Delta$ is nef and abundant if $K_X + \Delta$ is semi-ample. So, we show that $K_X + \Delta$ is semi-ample under the assumption that $K_X + \Delta$ is nef and abundant. By taking a dlt blow-up (cf. Theorem 1.1.6), we can assume that (X, Δ) is dlt. By the assumption, it is easy to see that $K_X + \Delta$ is nef and log abundant. Therefore, by Theorem 5.4.2, we obtain that $K_X + \Delta$ is semi-ample. □

The following theorem is an easy consequence of the arguments in [KeMaMc, Section 7] and Proposition 5.4.3 by the induction on dimension. We will treat related topics in Section 5.5 more systematically.

Theorem 5.4.7. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial dlt n-fold such that $K_X + \Delta$ is nef. Assume that the abundance conjecture for projective $\mathbb{Q}$-factorial klt pairs in dimension $\leq n$. We further assume that the minimal model program with ample scaling terminates for projective $\mathbb{Q}$-factorial klt pairs in dimension $\leq n$. Then $K_X + \Delta$ is semi-ample.*

Proof. This follows from the arguments in [KeMaMc, Section 7] by using the minimal model program with ample scaling with the aid of Proposition 5.4.3. Let H be a general effective sufficiently ample Cartier divisor on X. We run the minimal model program on $K_X + \Delta - \varepsilon \llcorner \Delta \lrcorner$ with scaling of H. We note that $K_X + \Delta$ is numerically trivial on the extremal ray in each step of the above minimal model program if ε is sufficiently small by [B2, Proposition 3.2]. We also note that, by the induction on dimension, $(K_X + \Delta)|_{\llcorner \Delta \lrcorner}$ is semi-ample. For the details, see [KeMaMc, Section 7]. $\square$

Remark 5.4.8. In the proof of Theorem 5.4.7, the abundance theorem and the termination of the minimal model program with ample scaling for projective $\mathbb{Q}$-factorial klt pairs in dimension $\leq n - 1$ are sufficient if $K_X + \Delta - \varepsilon \llcorner \Delta \lrcorner$ is not pseudo-effective for every $0 < \varepsilon \ll 1$ by [BCHM] (cf. 5.2.13).

The next theorem is an answer to Professor János Kollár's question for *projective* varieties. He was mainly interested in the case where f is birational.

Theorem 5.4.9. *Let $f : X \to Y$ be a projective morphism between projective varieties. Let (X, Δ) be an lc pair such that $K_X + \Delta$ is numerically trivial over Y. Then $K_X + \Delta \sim_{\mathbb{Q}, Y} 0$.*

Proof. By replacing (X, Δ) with its dlt blow-up (cf. Theorem 1.1.6), we can assume that (X, Δ) is a $\mathbb{Q}$-factorial dlt pair. Let $S = \llcorner \Delta \lrcorner = \cup S_i$ be the irreducible decomposition. If $S = 0$, then $K_X + \Delta \sim_{\mathbb{Q}, Y} 0$ by Kawamata's theorem (see [F7, Theorem 1.1]). It is because $(K_X + \Delta)|_{X_\eta} \sim_{\mathbb{Q}} 0$, where X_η is the generic fiber of f, by Nakayama's abundance theorem for klt pairs with numerical trivial log canonical divisor (cf. [Nak, Chapter V. 4.9. Corollary]). By the induction on dimension, we can assume that $(K_X + \Delta)|_{S_i} \sim_{\mathbb{Q}, Y} 0$ for every i. Let H be a general effective sufficiently ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y such that $\llcorner H \lrcorner = 0$. Then $(X, \Delta + f^*H)$ is dlt, $(K_X + \Delta + f^*H)|_{S_i}$ is semi-ample for every i. By Proposition 5.4.3, $(K_X + \Delta + f^*H)|_S$ is semi-ample. By applying [F9, Theorem 1.1], we obtain that $K_X + \Delta + f^*H$ is f-semi-ample. We note that $(K_X + \Delta + f^*H)|_{X_\eta} \sim_{\mathbb{Q}} 0$ (see, for example, [G2, Theorem 1.2]). Therefore, $K_X + \Delta$ is f-semi-ample. This means that $K_X + \Delta \sim_{\mathbb{Q}, Y} 0$. $\square$

Remark 5.4.10. In Theorem 5.4.9, if Δ is an $\mathbb{R}$-divisor, then we obtain $K_X + \Delta \sim_{\mathbb{R}, Y} 0$ by the same arguments as in Theorem 2.3.2.

As a corollary, we obtain a relative version of the main theorem of [G2].

Corollary 5.4.11 (cf. [G2, Theorem 1.2]). *Let $f : X \to Y$ be a projective morphism from a projective slc pair (X, Δ) to a (not necessarily irreducible) projective variety Y. Assume that $K_X + \Delta$ is numerically trivial over Y. Then there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that $K_X + \Delta \sim_{\mathbb{Q}} f^*D$.*

Proof. Let $\nu : X^\nu \to X$ be the normalization such that $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$. By Theorem 5.4.9, $K_{X^\nu} + \Theta \sim_{\mathbb{Q},Y} 0$. Let H be a general sufficiently ample $\mathbb{Q}$-divisor on Y such that $K_{X^\nu} + \Theta + \nu^* f^* H$ is semi-ample and that $(X, \Delta + f^*H)$ is slc. By Proposition 5.4.3, $K_X + \Delta + f^*H$ is semi-ample. In particular, $K_X + \Delta + f^*H$ is f-semi-ample. Then we can find a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that $K_X + \Delta \sim_{\mathbb{Q}} f^*D$. □

By the same arguments as in the proof of Theorem 5.4.9 (resp. Corollary 5.4.11), we obtain the following theorem (resp. corollary), which is a relative version of Theorem 5.4.2 (resp. Corollary 5.4.4).

Theorem 5.4.12. *Let $f : X \to Y$ be a projective morphism between projective varieties. Let (X, Δ) be an lc pair such that $K_X + \Delta$ is f-nef and f-log abundant. Then $K_X + \Delta$ is f-semi-ample.*

Corollary 5.4.13. *Let $f : X \to Y$ be a projective morphism from a projective slc pair (X, Δ) to a (not necessarily irreducible) projective variety Y. Let $\nu : X^\nu \to X$ be the normalization such that $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$. Assume that $K_{X^\nu} + \Theta$ is nef and log abundant over Y. Then $K_X + \Delta$ is f-semi-ample.*

5.4.1 Relative abundance conjecture

In this subsection, we make some remarks on the relative abundance conjecture.

Let us recall the minimal model conjecture.

Conjecture 5.4.14 (Minimal model conjecture). *Let $f : X \to Y$ be a projective morphism between quasi-projective varieties and let (X, B) be an lc pair. If $K_X + B$ is pseudo-effective over Y, then it has a minimal model over Y.*

Conjecture 5.4.14 is very useful for the *relative* abundance conjecture by Lemma 5.4.15 below.

Lemma 5.4.15. *Assume that* Conjecture 5.4.14 *holds. Let $f : X \to Y$ be a projective morphism between quasi-projective varieties such that (X, B) is lc and that $K_X + B$ is f-nef. Let $\overline{f} : \overline{X} \to \overline{Y}$ be any projective completion of $f : X \to Y$.*

Then we can construct a projective morphism $g : V \to \overline{Y}$ from a normal projective variety V and an effective $\mathbb{Q}$-divisor B_V on V such that (V, B_V) is a $\mathbb{Q}$-factorial dlt pair, $K_V + B_V$ is g-nef, $(V, B_V)|_{g^{-1}(Y)}$ is a minimal model of (X, B) over Y, and no lc center of (V, B_V) is contained in $g^{-1}(\overline{Y} \setminus Y)$.

In particular, if $\alpha : W \to X$, $\beta : W \to g^{-1}(Y)$ is a common resolution of X and $g^{-1}(Y)$, then $\alpha^*(K_X + B) = \beta^*((K_V + B_V)|_{g^{-1}(Y)})$. Therefore, $K_X + B$ is semi-ample over Y if and only if so is $K_V + B_V$.

Proof. Let $h : Z \to \overline{X}$ be a resolution such that $\operatorname{Supp} h_*^{-1}B \cup \operatorname{Exc}(h) \cup h^{-1}(\overline{X} \setminus X)$ is a simple normal crossing divisor. We take a minimal model (V, B_V) of $(Z, h_*^{-1}B + \sum E)$, where E runs through all the h-exceptional prime divisors on Z with $h(E) \not\subset \overline{X} \setminus X$, over Y. Then it is easy to see that (V, B_V) has the desired properties. □

We close this subsection with a remark on the relative abundance conjecture.

Remark 5.4.16 (Relative setting)**.** We assume that Conjecture 5.4.14 holds. Then, by Lemma 5.4.15, we can prove Theorems 5.4.9 and 5.4.12 for any projective morphisms between (not necessarily quasi-projective) algebraic varieties. We can also formulate and prove the relative version of Theorem 5.4.6 by Lemma 5.4.15 (cf. the proof of Theorem 5.4.9). We do not know how to prove Corollary 5.4.11 and Corollary 5.4.13 for projective morphisms between arbitrary algebraic varieties even when Conjecture 5.4.14 holds. We think that there are no reasonable minimal model theories for *reducible* varieties.

5.4.2 Miscellaneous applications

In this subsection, we collect some miscellaneous applications related to the base point free theorem and the abundance conjecture.

The following theorem is the log canonical version of Fukuda's result.

Theorem 5.4.17 (cf. [Fk5, Theorem 0.1])**.** *Let (X, Δ) be a projective lc pair. Assume that $K_X + \Delta$ is numerically equivalent to some semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D. Then $K_X + \Delta$ is semi-ample.*

Proof. By taking a dlt blow-up (cf. Theorem 1.1.6), we can assume that (X, Δ) is dlt. By the induction on dimension and Proposition 5.4.3, we have that $(K_X + \Delta)|_{\llcorner \Delta \lrcorner}$ is semi-ample. By [F9, Theorem 1.1], we can prove the semi-ampleness of $K_X + \Delta$. For the details, see the proof of [G2, Theorem 6.3]. □

By using the deep result in [CKP], we have a slight generalization of Theorem 5.4.17 and [CKP, Corollary 3]. It is also a generalization of Theorem 5.4.2.

Theorem 5.4.18 (cf. [CKP, Corollary 3]). *Let (X, Δ) be a projective lc pair and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that D is nef and log abundant with respect to (X, Δ). Assume that $K_X + \Delta \equiv D$. Then $K_X + \Delta$ is semi-ample.*

Proof. By replacing (X, Δ) with its dlt blow-up (cf. Theorem 1.1.6), we can assume that (X, Δ) is dlt. Let $f : Y \to X$ be a log resolution. We put $K_Y + \Delta_Y = f^*(K_X + \Delta) + F$ with $\Delta_Y = f_*^{-1}\Delta + \sum E$ where E runs through all the f-exceptional prime divisors on Y. We note that F is effective and f-exceptional. By [CKP, Corollary 1],

$$\kappa(X, K_X + \Delta) = \kappa(Y, K_Y + \Delta_Y) \geq \kappa(Y, f^*D + F) = \kappa(X, D).$$

By the assumption, $\kappa(X, D) = \nu(X, D) = \nu(X, K_X + \Delta)$. On the other hand, $\nu(X, K_X + \Delta) \geq \kappa(X, K_X + \Delta)$ always holds. Therefore, $\kappa(X, K_X + \Delta) = \nu(X, K_X + \Delta)$, that is, $K_X + \Delta$ is nef and abundant. By applying the above argument to every lc center of (X, Δ), we obtain that $K_X + \Delta$ is nef and log abundant. Thus, by Theorem 5.4.2, we obtain that $K_X + \Delta$ is semi-ample. $\square$

Remark 5.4.19. By the proof of Theorem 5.4.18, we see that we can weaken the assumption as follows. Let (X, Δ) be a projective lc pair. Assume that $K_X + \Delta$ is numerically equivalent to a nef and abundant $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and that $\nu_W^*((K_X + \Delta)|_W)$ is numerically equivalent to a nef and abundant $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor for every lc center W of (X, Δ), where $\nu_W : W^\nu \to W$ is the normalization of W. Then $K_X + \Delta$ is semi-ample.

Theorem 5.4.20 is a generalization of Theorem 7.1.5. The proof is the same as Theorem 7.1.5 once we adopt [F9, Theorem 1.1].

Theorem 5.4.20 (cf. [G2, Theorems 6.4, 6.5]). *Let (X, Δ) be a projective lc pair such that $-(K_X + \Delta)$ (resp. $K_X + \Delta$) is nef and abundant. Assume that* $\dim \mathrm{Nklt}(X, \Delta) \leq 1$ *where* $\mathrm{Nklt}(X, \Delta)$ *is the non-klt locus of the pair (X, Δ). Then $-(K_X + \Delta)$ (resp. $K_X + \Delta$) is semi-ample.*

Proof. Let T be the non-klt locus of (X, Δ). By the same argument as in the proof of Theorem 7.3.1, we can check that $-(K_X + \Delta)|_T$ (resp. $(K_X + \Delta)|_T$) is semi-ample. Therefore, $-(K_X + \Delta)$ (resp. $K_X + \Delta$) is semi-ample by [F9, Theorem 1.1]. $\square$

Similarly, we can prove Theorem 5.4.21.

Theorem 5.4.21. *Let (X, Δ) be a projective lc pair. Assume that $-(K_X + \Delta)$ is nef and abundant and that $(K_X + \Delta)|_W \equiv 0$ for every lc center W of (X, Δ). Then $-(K_X + \Delta)$ is semi-ample.*

Proof. By taking a dlt blow-up (cf. Theorem 1.1.6), we can assume that (X, Δ) is dlt. By [G2, Theorem 1.2] (cf. Corollary 5.4.11), $(K_X + \Delta)|_{\llcorner \Delta \lrcorner}$ is semi-ample. Therefore, $K_X + \Delta$ is semi-ample by [F9, Theorem 1.1]. □

5.5 Non-vanishing, abundance, and minimal model conjectures

In this final section, we discuss the relationship among various conjectures in the minimal model program.

First, let us recall the weak non-vanishing conjecture for projective lc pairs (cf. [B2, Conjecture 1.3]).

Conjecture 5.5.1 (Weak non-vanishing conjecture). *Let (X, Δ) be a projective lc pair such that Δ is an $\mathbb{R}$-divisor. Assume that $K_X + \Delta$ is pseudo-effective. Then there exists an effective $\mathbb{R}$-divisor D on X such that $K_X + \Delta \equiv D$.*

Conjecture 5.5.1 is known to be one of the most important problems in the minimal model theory (cf. [B2]).

Remark 5.5.2. By [CKP, Theorem 1], $K_X + \Delta \equiv D \geq 0$ in Conjecture 5.5.1 means that there is an effective $\mathbb{R}$-divisor D' such that $K_X + \Delta \sim_{\mathbb{R}} D'$.

By Remark 5.5.2 and Lemma 5.5.3 below, Conjecture 5.5.1 in dimension $\leq n$ is equivalent to Conjecture 1.3 of [B2] in dimension $\leq n$ with the aid of dlt blow-ups (cf. Theorem 1.1.6).

Lemma 5.5.3. *Assume that* Conjecture 5.5.1 *holds in dimension $\leq n$. Let $f : X \to Z$ be a projective morphism between quasi-projective varieties with* $\dim X = n$. *Let (X, Δ) be an lc pair such that $K_X + \Delta$ is pseudo-effective over Z. Then there exists an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M on X such that $K_X + \Delta \sim_{\mathbb{R}, Z} M$.*

Proof. Apply Conjecture 5.5.1 and Remark 5.5.2 to the generic fiber of f. Then, by [BCHM, Lemma 3.2.1], we obtain M with the required properties. □

Before we discuss the main result of this section, we give a remark on Birkar's paper [B2].

Remark 5.5.4 (Absolute versus relative). Let $f : X \to Z$ be a projective morphism between *projective* varieties. Let (X, B) be a $\mathbb{Q}$-factorial dlt pair and let $(X, B+C)$ be an lc pair such that $C \geq 0$ and that $K_X + B + C$ is nef over Z. Let H be a very ample Cartier divisor on Z. Let D be a general member of $|2(2\dim X + 1)H|$. In this situation, $(X, B + \frac{1}{2}f^*D)$ is dlt, $(X, B + \frac{1}{2}f^*D + C)$ is lc, and $K_X + B + \frac{1}{2}f^*D + C$ is nef by Kawamata's bound on the length of extremal rays. The minimal model program on $K_X + B + \frac{1}{2}f^*D$ with scaling of C is the minimal model program on $K_X + B$ *over* Z with scaling of C. By this observation, the arguments in [B2] work without appealing relative settings if the considered varieties are *projective*.

The following theorem is the main theorem of this section.

Theorem 5.5.5. *The abundance theorem for projective klt pairs in dimension* $\leq n$ *and* Conjecture 5.5.1 *for projective* $\mathbb{Q}$*-factorial dlt pairs in dimension* $\leq n$ *imply the abundance theorem for projective lc pairs in dimension* $\leq n$.

Proof. Let (X, Δ) be an n-dimensional projective lc pair such that $K_X + \Delta$ is nef. As we explained in 5.2.13, by [B2, Theorems 1.4, 1.5], the minimal model program with ample scaling terminates for projective $\mathbb{Q}$-factorial klt pairs in dimension $\leq n$. Moreover, we can assume that (X, Δ) is a projective $\mathbb{Q}$-factorial dlt pair by taking a dlt blow-up (cf. Theorem 1.1.6). Thus, by Theorem 5.4.7, we obtain the desired result. □

The final result is on a generalized abundance conjecture formulated by Nakayama's numerical Kodaira dimension κ_σ. For the details of κ_σ, see [Nak] (see also [Leh2]).

Corollary 5.5.6 (Generalized abundance conjecture). *Assume that the abundance conjecture for projective klt pairs in dimension* $\leq n$ *and* Conjecture 5.5.1 *for* $\mathbb{Q}$*-factorial dlt pairs in dimension* $\leq n$. *Let* (X, Δ) *be an* n*-dimensional projective lc pair. Then* $\kappa(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta)$.

Proof. We can assume that (X, Δ) is a $\mathbb{Q}$-factorial projective dlt pair by replacing it with its dlt blow-up (cf. Theorem 1.1.6). Let H be a general effective sufficiently ample Cartier divisor on X. We can run the minimal model program with scaling of H by 5.2.13. Then we obtain a good minimal model by Theorem 5.5.5 if $K_X + \Delta$ is pseudo-effective. When $K_X + \Delta$ is not pseudo-effective, we have a Mori fiber space structure. In each step of the minimal model program, κ and κ_σ are preserved. So, we obtain $\kappa(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta)$. □

6 Images of log Fano and weak Fano varieties

6.1 Introduction

Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties defined over $\mathbb{C}$. The following theorem is one of the main results of this chapter.

Theorem 6.1.1 (cf. Theorem 6.4.6). *If X is a weak Fano manifold, that is, $-K_X$ is nef and big, then so is Y.*

Our proof of Theorem 6.1.1 is Hodge theoretic. We do not need mod p reduction arguments. More precisely, we obtain Theorem 6.1.1 as an application of Kawamata's positivity theorem (cf. [Ka8]). By the same method, we can recover the well-known result on Fano manifolds.

Theorem 6.1.2 (cf. Theorem 6.4.8). *If X is a Fano manifold, that is, $-K_X$ is ample, then so is Y.*

Our proof of Theorem 6.1.2 is completely different from the original one by Kollár, Miyaoka, and Mori in [KoMiMo]. It is the first proof which does not use mod p reduction arguments. We raise a conjecture on the semi-ampleness of anti-canonical divisors.

Conjecture 6.1.3. *If $-K_X$ is semi-ample, then so is $-K_Y$.*

We reduce Conjecture 6.1.3 to another conjecture on canonical bundle formulas and give affirmative answers to Conjecture 6.1.3 in some special cases (cf. Remark 6.4.3 and Theorem 6.4.5). In this chapter, we obtain the following theorem, which is a key result for the proof of Theorem 6.1.1 and Theorem 6.1.2.

Theorem 6.1.4 (cf. Theorem 6.4.2). *If $-K_X$ is semi-ample, then $-K_Y$ is nef.*

We note that the proof of Theorem 6.1.4 is also an application of Kawamata's positivity theorem. We note that it is the first time that Theorem 6.1.4 is proved without mod p reduction arguments. The reader will recognize that Kawamata's positivity theorem is very powerful. We can find related topics in [Z] and [Dbook, Section 3.6]. Note that both of them depend on mod p reduction arguments.

We summarize the contents of this chapter. Section 6.2 is a preliminary section. We recall Kawamata's positivity theorem (cf. Theorem 6.2.2) here. In Section 6.3, we treat log Fano varieties with only kawamata log terminal singularities. The result obtained in this section will be used in Section 6.4. In Section 6.3, we also treat images of log Fano varieties by generically finite surjective morphisms as an application of Lemma 3.1.1. Theorem 6.3.7 is an answer to the question raised by Professor Karl Schwede (cf. [ScSm, Remark 6.5]). In Section 6.4, we prove Theorem 6.1.1, Theorem 6.1.2, and some related theorems. In Section 6.5, we give some comments and questions on related topics. In the final section: Section 6.6, which is an appendix, we give a mod p reduction approach to Theorem 6.1.1.

6.2 Kawamata's semipositivity theorem

Definition 6.2.1 (Relative normal crossing divisors). Let $f : X \to Y$ be a smooth surjective morphism between smooth varieties with connected fibers and $D = \sum_i D_i$ a reduced divisor on X such that $D^h = D$, where D_i is a prime divisor for every i. We say that D is *relatively normal crossing* if D satisfies the condition that for each closed point $x \in X$, there exits an analytic open neighborhood U and $u_1, \ldots, u_k \in \mathcal{O}_{X,x}$ inducing a regular system of parameter on $f^{-1}f(x)$ at x, where $k = \dim f^{-1}f(x)$, such that $D \cap U = \{u_1 \cdots u_l = 0\}$ for some l with $0 \le l \le k$.

Let us recall Kawamata's positivity theorem in [Ka8]. It is the main ingredient of this chapter.

Theorem 6.2.2 (Kawamata's positivity theorem). *Let $f : X \to Y$ be a surjective morphism of smooth projective varieties with connected fibers. Let $P = \sum_j P_j$ and $Q = \sum_l Q_l$ be simple normal crossing divisors on X and Y, respectively, such that $f^{-1}(Q) \subseteq P$ and f is smooth over $Y \setminus Q$. Let $D = \sum_j d_j P_j$ be a $\mathbb{Q}$-divisor (d_j's may be negative or zero), which satisfies the following conditions:*

(1) $f : \mathrm{Supp} D^h \to Y$ is relatively normal crossing over $Y \setminus Q$ and $f(\mathrm{Supp} D^v) \subseteq Q$,

(2) $d_j < 1$ *unless* $\mathrm{codim}_Y f(P_j) \geq 2$,

(3) $\dim_{\mathbb{C}(\eta)} f_*\mathcal{O}(\ulcorner -D \urcorner) \otimes_{\mathcal{O}_Y} \mathbb{C}(\eta) = 1$, *where* η *is the generic point of* Y, *and*

(4) $K_X + D \sim_{\mathbb{Q}} f^*(K_Y + L)$ *for some* $\mathbb{Q}$*-divisor* L *on* Y.

Let

$$\begin{aligned} f^*(Q_l) &= \sum_j w_{lj} P_j, \text{ where } w_{lj} > 0, \\ \bar{d}_j &= \frac{d_j + w_{lj} - 1}{w_{lj}} \text{ if } f(P_j) = Q_l, \\ \delta_l &= \max\{\bar{d}_j | f(P_j) = Q_l\}, \\ \Delta_0 &= \sum \delta_l Q_l, \text{ and} \\ M &= L - \Delta_0. \end{aligned}$$

Then M *is nef. We sometimes call* M (*resp.* Δ_0) *the* moduli part (*resp.* discriminant part).

Remark 6.2.3. In Theorem 6.2.2, we note that δ_l can be characterized as follows. If we put

$$c_l = \sup\{t \in \mathbb{Q} \,|\, K_X + D + tf^*Q_l \text{ is lc over the generic point of } Q_l\},$$

then $\delta_l = 1 - c_l$.

6.3 Log Fano varieties

The proof of the following theorem is essentially the same as [F1, Theorem 1.2]. We will use similar arguments in Section 6.4.

Theorem 6.3.1. *Let* $f : X \to Y$ *be a proper surjective morphism between normal projective varieties with connected fibers. Let* Δ *be an effective* $\mathbb{Q}$*-divisor on* X *such that* (X, Δ) *is klt. Assume that* $-(K_X + \Delta + \varepsilon f^*H)$ *is semi-ample, where* ε *is a positive rational number and* H *is an ample Cartier divisor on* Y. *Then we can find an effective* $\mathbb{Q}$*-divisor* Δ_Y *on* Y *such that* (Y, Δ_Y) *is klt and* $-(K_Y + \Delta_Y)$ *is ample. In particular, if* K_Y *is* $\mathbb{Q}$*-Cartier, then* $-K_Y$ *is big.*

Proof. By replacing H with mH and ε with $\frac{\varepsilon}{m}$ for some sufficiently large positive integer m, we can assume that H is very ample and $\varepsilon < 1$. By replacing H with a general member of $|H|$, we can further assume that

$(X, \Delta + \varepsilon f^*H)$ is klt. Let A be a general member of a free linear system $|-m(K_X + \Delta + \varepsilon f^*H)|$ such that $(X, \Delta + \varepsilon f^*H + \frac{1}{m}A)$ is klt and

$$K_X + \Delta + \varepsilon f^*H + \frac{1}{m}A \sim_{\mathbb{Q}} 0.$$

We put $\Gamma = \Delta + \varepsilon f^*H + \frac{1}{m}A$. Then we consider the following commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "\nu"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\mu"'] & Y, \end{tikzcd}$$

where

(i) X' and Y' are smooth projective varieties,

(ii) ν and μ are projective birational morphisms,

(iii) we put $L = -K_{Y'}$ and define a $\mathbb{Q}$-divisor D on X' as follows:

$$K_{X'} + D = \nu^*(K_X + \Gamma),$$

and

(iv) there are simple normal crossing divisors P on X' and Q on Y' which satisfy the conditions (1) of Theorem 6.2.2 and there exists a set of sufficiently small non-negative rational numbers $\{s_l\}$ such that $\mu^*H - \sum_l s_l Q_l$ is ample.

We see that $f' : X' \to Y'$, D, and L satisfy the conditions (1), (2), and (4) in Theorem 6.2.2. Now we check the condition (3) in Theorem 6.2.2. We put $h = f \circ \nu$.

Claim 6.3.2. $\mathcal{O}_Y = h_*\mathcal{O}_{X'}(\ulcorner -D \urcorner)$

Proof of Claim 6.3.2. Since (X, Γ) is klt, we see that $\ulcorner -D \urcorner$ is effective and ν-exceptional. Thus it holds that $\nu_*\mathcal{O}_{X'}(\ulcorner -D \urcorner) \simeq \mathcal{O}_X$. Since $f_*\mathcal{O}_X = \mathcal{O}_Y$, we have $\mathcal{O}_Y = h_*\mathcal{O}_{X'}(\ulcorner -D \urcorner)$. □

By Claim 6.3.2, we see that $f' : X' \to Y'$ and D satisfy the condition (3) in Theorem 6.2.2 since μ is birational. If we take $\mathbb{Q}$-divisors Δ_0 and M on Y' as in Theorem 6.2.2, then

$$K_{X'} + D \sim_{\mathbb{Q}} f'^*(K_{Y'} + M + \Delta_0)$$

and M is nef. We have the following claim about Δ_0.

Claim 6.3.3. $\Delta_0^+ \geq \varepsilon \mu^* H$.

Proof of Claim 6.3.3. Since H is general, $h^* H$ is reduced. We set $h^* H = \sum_j P_{k_j}$. Note that the coefficient of P_{k_j} in D is ε for every j by the generality of H and A. By the definition of $\bar{d}_{k_j}$, it holds that

$$\bar{d}_{k_j} = d_{k_j} = \varepsilon.$$

Thus we have $\Delta_0^+ \geq \varepsilon \mu^* H$. □

We decompose $\varepsilon = \varepsilon' + \varepsilon''$ such that ε' and ε'' are positive rational numbers. Since M is nef, $M + \varepsilon'(\mu^* H - \sum_l s_l Q_l)$ is ample. Hence, there exists an effective $\mathbb{Q}$-divisor B such that $M + \varepsilon'(\mu^* H - \sum_l s_l Q_l) \sim_{\mathbb{Q}} B$, $(Y', B + \varepsilon' \sum_l s_l Q_l + \Delta_0^+ + \varepsilon'' \mu^* H)$ is klt, and $\mathrm{Supp}(B + \varepsilon' \sum_l s_l Q_l + \Delta_0^+ + \varepsilon'' \mu^* H - \Delta_0^-)$ is simple normal crossing. If ε' is a sufficiently small positive rational number, then we see that

$$\mathrm{Supp}(B + \varepsilon' \sum_l s_l Q_l + \Delta_0^+ + \varepsilon'' \mu^* H - \Delta_0^-)^- = \mathrm{Supp}\, \Delta_0^-.$$

We set

$$\Delta_0' = \Delta_0^+ - \varepsilon \mu^* H \text{ and } \Omega' = B + \varepsilon' \sum_l s_l Q_l + \Delta_0' + \varepsilon'' \mu^* H - \Delta_0^-.$$

It holds that

$$K_{Y'} + \Omega' \sim_{\mathbb{Q}} K_{Y'} + L \sim_{\mathbb{Q}} 0.$$

By the following claim, $\mu_* \Omega'$ is effective.

Claim 6.3.4 (cf. Claim (B) in [F1]). $\mu_* \Delta_0^- = 0$.

Proof of Claim 6.3.4. Let $\Delta_0^- = -\sum_k \delta_{l_k} Q_{l_k}$, where $\delta_{l_k} < 0$. If there exists k and j such that $\ulcorner -d_j \urcorner < w_{l_k j}$, it holds that $-d_j + 1 \leq w_{l_k j}$ since $w_{l_k j}$ is an integer. Then we obtain $\delta_{l_k} \geq 0$. Thus, it holds that $\ulcorner -d_j \urcorner \geq w_{l_k j}$ for all k and j. Therefore we have $\ulcorner -D \urcorner \geq f'^* Q_{l_k}$. Since $\mathcal{O}_{Y'} = f'_* \mathcal{O}_{X'}$, we see that $f'_* \mathcal{O}_{X'}(\ulcorner -D \urcorner) \supseteq \mathcal{O}_{Y'}(Q_{l_k})$. By Claim 6.3.2, $\mu_* Q_{l_k} = 0$. We finish the proof of Claim 6.3.4. □

We put $\Omega = \mu_* \Omega'$. Then we see that Ω is effective by Claim 6.3.4,

$$K_{Y'} + \Omega' = \mu^*(K_Y + \Omega),\ K_Y + \Omega \sim_{\mathbb{Q}} 0, \text{ and } \Omega \geq \varepsilon'' H.$$

Thus (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y) \sim_{\mathbb{Q}} \varepsilon'' H$ is ample if we put $\Delta_Y = \Omega - \varepsilon'' H \geq 0$. We finish the proof of Theorem 6.3.1. □

Remark 6.3.5. Let (X, B) be a projective klt pair. Then $-(K_X + B)$ is semi-ample if and only if $-(K_X + B)$ is nef and abundant by [F7, Theorem 1.1].

The following corollary is obvious by Theorem 6.3.1.

Corollary 6.3.6 (cf. [PrSh, Theorem 2.9]). *Let $f : X \to Y$ be a proper surjective morphism between normal projective varieties with connected fibers. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is klt and $-(K_X + \Delta)$ is ample. Then there is an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is ample.*

Moreover, we give an easy application of Lemma 3.1.1. Theorem 6.3.7 is an answer to the question raised by Karl Schwede (cf. [ScSm, Remark 6.5]).

Theorem 6.3.7. *Let (X, Δ) be a projective klt pair such that $-(K_X + \Delta)$ is ample. Let $f : X \to Y$ be a generically finite surjective morphism to a normal projective variety Y. Then we can find an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is ample.*

Proof. Without loss of generality, we can assume that Δ is a $\mathbb{Q}$-divisor by perturbing the coefficients of Δ. Let H be a general very ample Cartier divisor on Y and let ε be a sufficiently small positive rational number. Then $K_X + \Delta + \varepsilon f^*H$ is anti-ample and $(X, \Delta + \varepsilon f^*H)$ is klt. We can take an effective $\mathbb{Q}$-divisor Θ on X such that $m\Theta$ is a general member of the free linear system $|-m(K_X + \Delta + \varepsilon f^*H)|$ where m is a sufficiently large and divisible integer. Then

$$K_X + \Delta + \varepsilon f^*H + \Theta \sim_{\mathbb{Q}} 0.$$

Let δ be a positive rational number such that $0 < \delta < \varepsilon$. Then

$$K_X + \Delta + (\varepsilon - \delta) f^*H + \Theta \sim_{\mathbb{Q}} f^*(-\delta H).$$

By Lemma 3.1.1, we can find an effective $\mathbb{Q}$-divisor Δ_Y on Y such that

$$K_Y + \Delta_Y \sim_{\mathbb{Q}} -\delta H$$

and that (Y, Δ_Y) is klt. We note that

$$-(K_Y + \Delta_Y) \sim_{\mathbb{Q}} \delta H$$

is ample. □

By combining Theorem 6.3.7 with Theorem 6.3.1, we can easily obtain the following corollary.

Corollary 6.3.8. *Let (X, Δ) be a projective klt pair such that $-(K_X + \Delta)$ is ample. Let $f : X \to Y$ be a projective surjective morphism onto a normal projective variety Y. Then we can find an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is ample.*

We close this section with an easy corollary of Theorem 6.3.1.

Corollary 6.3.9. *Let (X, Δ) be a projective klt pair such that $-(K_X + \Delta)$ is semi-ample. Let n be a positive integer such that $n(K_X + \Delta)$ is Cartier. Then there is an effective $\mathbb{Q}$-divisor Δ_Y on*

$$Y = \mathrm{Proj} \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(-mn(K_X + \Delta)))$$

such that (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is ample.

Proof. By definition, Y is a normal projective variety and there is a projective surjective morphism $f : X \to Y$ with connected fibers such that $-(K_X + \Delta) \sim_{\mathbb{Q}} f^*H$, where H is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. Then we can apply Theorem 6.3.1. □

6.4 Fano and weak Fano manifolds

In this section, we apply Kawamata's positivity theorem to smooth projective morphisms between smooth projective varieties.

We note that the statement of the following theorem is weaker than [Dbook, Corollary 3.15 (a)]. However, the proof of Theorem 6.4.2 has potential for further generalizations. We describe it in details.

First, we give a remark on the Stein factorization. We will use Lemma 6.4.1 in this section See also Remark 6.5.3 below.

Lemma 6.4.1 (Stein factorization). *Let $f : X \to Y$ be a smooth projective morphism between smooth varieties. Let*

$$f : X \xrightarrow{h} Z \xrightarrow{g} Y$$

be the Stein factorization. Then $g : Z \to Y$ is étale. Therefore, $h : X \to Z$ is a smooth projective morphism between smooth varieties with connected fibers.

Proof. By assumption, $R^i f_* \mathcal{O}_X$ is locally free and

$$R^i f_* \mathcal{O}_X \otimes \mathbb{C}(y) \simeq H^i(X_y, \mathcal{O}_{X_y})$$

for every i and any $y \in Y$. By definition, $Z = \mathrm{Spec}_Y f_*\mathcal{O}_X$. Since $g_*\mathcal{O}_Z \simeq f_*\mathcal{O}_X$ is locally free, g is flat. By construction,

$$Z_y = \mathrm{Spec} H^0(X_y, \mathcal{O}_{X_y})$$

consists of n copies of $\mathrm{Spec}\mathbb{C}$ for any $y \in Y$, where n is the rank of $f_*\mathcal{O}_X$. Therefore, g is unramified. This implies that g is étale. Thus, Z is a smooth variety and $h : X \to Z$ is a smooth morphism with connected fibers. □

Theorem 6.4.2 (cf. [Dbook, Corollary 3.15 (a)]). *Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties with connected fibers. If $-K_X$ is semi-ample, then $-K_Y$ is nef.*

Proof. Let C be an integral curve on Y. Let A be a general member of the free linear system $|-mK_X|$. Then there is a non-empty Zariski open set U of Y such that $C \cap U \neq \emptyset$ and that A is smooth over U. By construction, $K_X + \frac{1}{m}A \sim_{\mathbb{Q}} 0$. Let $\mu : Y' \to Y$ be a resolution such that μ is an isomorphism over U and $\mu^{-1}(Y \setminus U)$ is a simple normal crossing divisor on Y'. We consider the following commutative diagram.

$$\begin{tikzcd}
\widetilde{X} = X \times_Y Y' \arrow[r, "\varphi"] \arrow[d, "\widetilde{f}"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "\mu"] & Y
\end{tikzcd}$$

We note that $\widetilde{f} : \widetilde{X} \to Y'$ is smooth. We write $K_{Y'} = \mu^* K_Y + E$. Then $\mathrm{Supp} E = \mathrm{Exc}(\mu)$, where $\mathrm{Exc}(\mu)$ is the exceptional locus of μ, and E is effective. We put

$$K_{\widetilde{X}} + \widetilde{D} = \varphi^*(K_X + \frac{1}{m}A) \sim_{\mathbb{Q}} 0.$$

Then

$$\widetilde{D} = -\widetilde{f}^* E + \varphi^* \frac{1}{m} A.$$

Note that $K_{\widetilde{X}} = \varphi^* K_X + \widetilde{f}^* E$. We put $U' = \mu^{-1}(U)$. Then $\mu : U' \to U$ is an isomorphism. Let $\psi : X' \to \widetilde{X}$ be a resolution such that ψ is an isomorphism over $\widetilde{f}^{-1}(U')$ and that $\mathrm{Supp} A' \cup \mathrm{Supp} f'^{-1}(Y' \setminus U')$ is a simple normal crossing divisor, where A' is the strict transform of A on X' and $f' = \widetilde{f} \circ \psi : X' \to Y'$. We define

$$K_{X'} + D = \psi^*(K_{\widetilde{X}} + \widetilde{D}) \sim_{\mathbb{Q}} 0.$$

We can write

$$K_{X'} + D = f'^*(K_{Y'} + \Delta_0 + M)$$

as in Kawamata's positivity theorem (see Theorem 6.2.2). We put $E = \sum_i e_iE_i$, where E_i is a prime divisor for every i and $E_i \neq E_j$ for $i \neq j$. The coefficient of E_i in Δ_0 is $1 - c_i$, where

$$c_i = \sup\{t \in \mathbb{Q} \,|\, K_{X'} + D + tf'^*E_i \text{ is lc over the generic point of } E_i\}.$$

By construction,

$$c_i = \sup\{t \in \mathbb{Q} \,|\, K_{\widetilde{X}} + \widetilde{D} + t\widetilde{f}^*E_i \text{ is lc over the generic point of } E_i\}.$$

Since

$$\widetilde{D} = -\widetilde{f}^*E + \varphi^*\frac{1}{m}A,$$

and $\varphi^*\frac{1}{m}A$ is effective, we can write $c_i = e_i + a_i$ for some $a_i \in \mathbb{Q}$ with $a_i \leq 1$. Thus, we have $1 - c_i = 1 - e_i - a_i$. Therefore, the coefficient of E_i in $E + \Delta_0$ is

$$e_i + 1 - e_i - a_i = 1 - a_i \geq 0.$$

So, we can see that $E + \Delta_0$ is effective. Since $K_{Y'} + \Delta_0 + M \sim_{\mathbb{Q}} 0$ and $K_{Y'} = \mu^*K_Y + E$, we have

$$-\mu^*K_Y = -K_{Y'} + E \sim_{\mathbb{Q}} E + \Delta_0 + M.$$

Let C' be the strict transform of C on Y'. Then

$$\begin{aligned} C \cdot (-K_Y) &= C' \cdot (-\mu^*K_Y) \\ &= C' \cdot (E + \Delta_0 + M) \geq 0. \end{aligned}$$

It is because M is nef and $\operatorname{Supp}(E + \Delta_0) \subset Y' \setminus U'$. Therefore, $-K_Y$ is nef. □

We give a very important remark on Theorem 6.4.2.

Remark 6.4.3 (Semi-ampleness of $-K_Y$). We use the same notation as in Theorem 6.4.2 and its proof. It is conjectured that the *moduli part* M is semi-ample (see, for example [Am4, 0. Introduction]). Some very special cases of this conjecture were treated in [F5] before [Am4]. Unfortunately, the results in [F5] are useless for our purposes here. If this semi-ampleness conjecture is solved, then we will obtain that $-K_Y$ is semi-ample.

Let $y \in Y$ be an arbitrary point. We can choose A such that $y \in U$. Since

$$-\mu^*K_Y \sim_{\mathbb{Q}} M + E + \Delta_0,$$

$E + \Delta_0$ is effective, and $\operatorname{Supp}(E + \Delta_0) \subset Y' \setminus U'$, we can find a positive integer m and an effective Cartier divisor D on Y such that $-mK_Y \sim D$ and that $y \notin \operatorname{Supp} D$. It implies that $-K_Y$ is semi-ample.

By [Ka7], M is semi-ample if $\dim Y = \dim X - 1$. Therefore, $-K_Y$ is semi-ample when $\dim Y = \dim X - 1$.

In [Am5, Theorem 3.3], Ambro proved that M is nef and abundant. So, if Y is a surface, then we can check that $-K_Y$ is semi-ample as follows. If $\nu(Y', M) = \kappa(Y', M) = 0$ or 1, then M is semi-ample. Therefore, we can apply the same argument as above. If $\nu(Y', M) = \kappa(Y', M) = 2$, then M is big. Since

$$-\mu^* K_Y \sim_{\mathbb{Q}} M + E + \Delta_0$$

and $E + \Delta_0$ is effective, $-\mu^* K_Y$ is big. Therefore, $-K_Y$ is nef and big. In this case, $-K_Y$ is semi-ample by the Kawamata–Shokurov base point free theorem. Anyway, for an arbitrary point $y \in Y$, we can always find a positive integer m and an effective Cartier divisor D on Y such that $-mK_Y \sim D$ and that $y \notin \operatorname{Supp} D$. It means that $-K_Y$ is semi-ample.

In the end, in Theorem 6.4.2, $-K_Y$ is semi-ample if $\dim Y \leq 2$. By combining the above results, we know that $-K_Y$ is semi-ample when $\dim X \leq 4$. We conjecture that $-K_Y$ is semi-ample if $-K_X$ is semi-ample without any assumptions on dimensions.

Remark 6.4.4. In Remark 6.4.3, we used Ambro's results in [Am4] and [Am5]. When we investigate the moduli part M on Y by the theory of variations of Hodge structures, we note the following construction. Let $\pi : V \to X$ be a cyclic cover associated to $m(K_X + \frac{1}{m}A) \sim 0$. In this case, π is a finite cyclic cover which is ramified only along $\operatorname{Supp} A$. Since $\operatorname{Supp} A$ is relatively normal crossing over U, we can construct a simultaneous resolution $f \circ \pi : V \to Y$ and make the union of the exceptional locus and the inverse image of $\operatorname{Supp} A$ a simple normal crossing divisor and relatively normal crossing over U by the canonical desingularization theorem. Therefore, the moduli part M on X behaves well under pull-backs. It is a very important remark.

The semi-ampleness of $-K_Y$ is not so obvious even when $-K_X \sim_{\mathbb{Q}} 0$. The proof of the following theorem depends on some deep results on the theory of variations of Hodge structures (cf. [Am5] and [F7]).

Theorem 6.4.5. *Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties. Assume that $-K_X \sim_{\mathbb{Q}} 0$. Then $-K_Y$ is semi-ample.*

Proof. By the Stein factorization (cf. Lemma 6.4.1), we can assume that f has connected fibers. In this case, we can write

$$K_X \sim_{\mathbb{Q}} f^*(K_Y + M),$$

where M is the moduli part. By [Am5, Theorem 3.3], we know that M is nef and abundant. Therefore, $-K_Y$ is nef and abundant. This implies that $-K_Y$ is semi-ample by [F7, Theorem 1.1]. $\square$

The following theorem is one of the main results of this chapter. We note that it was proved by Yasutake in a special case where $f: X \to Y$ is a $\mathbb{P}^n$-bundle (cf. [Y]).

Theorem 6.4.6 (Weak Fano manifolds). *Let $f: X \to Y$ be a smooth projective morphism between smooth projective varieties. If X is a weak Fano manifold, then so is Y.*

Proof. By taking the Stein factorization, we can assume that f has connected fibers (cf. Lemma 6.4.1). By Theorem 6.4.2, $-K_Y$ is nef since $-K_X$ is semi-ample by the Kawamata–Shokurov base point free theorem. By Kodaira's lemma, we can find an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is klt and that $-(K_X + \Delta)$ is ample. By Theorem 6.3.1, we can find an effective $\mathbb{Q}$-divisor Δ_Y such that $-(K_Y + \Delta_Y)$ is ample. Therefore, $-K_Y$ is big. So, $-K_Y$ is nef and big. This means that Y is a weak Fano manifold. $\square$

The following example is due to Hiroshi Sato.

Example 6.4.7 (Sato). Let Σ be the fan in $\mathbb{R}^3$ whose rays are generated by

$$\begin{aligned} &x_1 = (1,0,1), \quad x_2 = (0,1,0), \quad x_3 = (-1,3,0), \quad x_4 = (0,-1,0), \\ &y_1 = (0,0,1), \quad y_2 = (0,0,-1), \end{aligned}$$

and their maximal cones are

$$\begin{aligned} &\langle x_1, x_2, y_1\rangle, \langle x_1, x_2, y_2\rangle, \langle x_2, x_3, y_1\rangle, \langle x_2, x_3, y_2\rangle, \\ &\langle x_3, x_4, y_1\rangle, \langle x_3, x_4, y_2\rangle, \langle x_4, x_1, y_1\rangle, \langle x_4, x_1, y_2\rangle. \end{aligned}$$

Let Δ be the fan obtained from Σ by successive star subdivisions along the rays spanned by

$$z_1 = x_2 + y_1 = (0,1,1)$$

and

$$z_2 = x_2 + z_1 = 2x_2 + y_1 = (0,2,1).$$

We see that $V = X(\Sigma)$, the toric threefold corresponding to the fan Σ with respect to the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$, is a $\mathbb{P}^1$-bundle over $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3))$. We note that the $\mathbb{P}^1$-bundle structure $V \to Y$ is induced by the projection $\mathbb{Z}^3 \to \mathbb{Z}^2 : (x,y,z) \mapsto (x,y)$. The toric variety $X = X(\Delta)$ corresponding to the fan Δ was obtained by successive blow-ups from V. We can check that X is a three-dimensional toric weak Fano manifold and that the induced morphism $f: X \to Y$ is a flat morphism onto Y since every fiber of f is one-dimensional. It is easy to see that $-K_Y$ is big but not nef.

Therefore, if f is only flat, then $-K_Y$ is not always nef even when X is a weak Fano manifold.

Let us give a new proof of the well-known theorem by Kollár, Miyaoka, and Mori (cf. [KoMiMo]). We note that Y is not always Fano if f is only flat. There exists an example in [W].

Theorem 6.4.8 (cf. [KoMiMo, Corollary 2.9]). *Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties. If X is a Fano manifold, then so is Y.*

Proof. By taking the Stein factorization, we can assume that f has connected fibers (cf. Lemma 6.4.1). By Theorem 6.4.6, $-K_Y$ is nef and big. Therefore, $-K_Y$ is semi-ample by the Kawamata–Shokurov base point free theorem. Thus, it is sufficient to see that $C \cdot (-K_Y) > 0$ for every integral curve C on Y. Let C be an integral curve C on Y. We take a general very ample divisor H on Y. Let ε be a small positive rational number. Then $K_X + \varepsilon f^*H$ is anti-ample. Let A be a general member of the free linear system $|-m(K_X + \varepsilon f^*H)|$. We can assume that there is a non-empty Zariski open set U of Y such that H is smooth on U, $\mathrm{Supp}(A + f^*H)$ is simple normal crossing on $f^{-1}(U)$, $\mathrm{Supp}A$ is smooth over U, and $C \cap H \cap U \neq \emptyset$. Apply the same arguments as in the proof of Theorem 6.4.2 to

$$K_X + \varepsilon f^*H + \frac{1}{m}A \sim_{\mathbb{Q}} 0.$$

Then we obtain a projective birational morphism $\mu : Y' \to Y$ from a smooth projective variety Y' such that μ is an isomorphism over U and $\mathbb{Q}$-divisors Δ_0 and M on Y' as before. By construction, Δ_0 contains $\varepsilon H'$, where H' is the strict transform of H on Y' (cf. the proof of Theorem 6.3.1). Therefore, we have

$$C \cdot (-K_Y) = C' \cdot (E + \Delta_0 + M) > 0$$

as in the proof of Theorem 6.4.2. Thus, $-K_Y$ is ample. □

We can prove the following theorem by the same arguments. It is a generalization of Theorem 6.4.8.

Theorem 6.4.9. *Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties. Let H be an ample Cartier divisor on Y. Assume that $-(K_X + \varepsilon f^*H)$ is semi-ample for some positive rational number ε. Then $-K_Y$ is ample, that is, Y is a Fano manifold.*

Proof. By Lemma 6.4.1, we can assume that f has connected fibers. By Theorem 6.3.1, we see that $-K_Y$ is big. By the proof of Theorem 6.4.8, we can see that $C \cdot (-K_Y) > 0$ for every integral curve C on Y. By the Kawamata–Shokurov base point free theorem, $-K_Y$ is semi-ample. Thus, $-K_Y$ is ample. □

6.5 Comments and Questions

In this section, we will work over an algebraically closed field k of arbitrary characteristic. We denote the characteristic of k by $\mathrm{char} k$.

6.5.1. Let $f : X \to Y$ be a smooth projective morphism between smooth projective varieties defined over k.

(A) If $-K_X$ is ample, that is, X is Fano, then so is $-K_Y$.

It was obtained by Kollár, Miyaoka, and Mori in [KoMiMo]. Their proof is an application of the deformation theory of morphisms from curves invented by Mori. It needs mod p reduction arguments even when $\mathrm{char} k = 0$. In the case $\mathrm{char} k = 0$, we gave a Hodge theoretic proof without using mod p reduction arguments in Theorem 6.4.8.

(N) If $-K_X$ is nef, then so is $-K_Y$.

This result can be proved by the same method as in [KoMiMo] (cf. [Miy], [Z], and [Dbook, Corollary 3.15 (a)]). In the case $\mathrm{char} k = 0$, we do not know whether we can prove it without mod p reduction arguments or not.

(NB) If $-K_X$ is nef and big, that is, X is weak Fano, then so is $-K_Y$ when $\mathrm{char} k = 0$.

It was proved in Theorem 6.4.6. We do not know whether this statement holds true or not in the case $\mathrm{char} k > 0$. See also Section 6.6: Appendix.

(SA) If $-K_X$ is semi-ample, is $-K_Y$ semi-ample?

We have only some partial answers to this question. For details, see Remark 6.4.3 and Theorem 6.4.5. In the case $\mathrm{char} k = 0$, we note that $-K$ is semi-ample if and only if $-K$ is nef and abundant (see Remark 6.3.5).

(B) If $-K_X$ is big, is $-K_Y$ big?

The following example gives a negative answer to this question.

Example 6.5.2. Let $E \subset \mathbb{P}^2$ be a smooth cubic curve. We consider $f : X = \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(1)) \to E = Y$. Then, we see that $-K_X$ is big. However, $-K_Y$ is not big since E is a smooth elliptic curve.

Anyway, it seems to be difficult to construct nontrivial examples. It is because the smoothness of f is a very strong condition.

We close this section with a remark on Lemma 6.4.1. It may be indispensable when $k \neq \mathbb{C}$.

Remark 6.5.3. Lemma 6.4.1 holds true even when $k \neq \mathbb{C}$. We can check it as follows. By the proof of Lemma 6.4.1, it is sufficient to see that $f_*\mathcal{O}_X$ is locally free and $f_*\mathcal{O}_X \otimes k(y) \simeq H^0(X_y, \mathcal{O}_{X_y})$ for every closed point $y \in Y$. Without loss of generality, we can assume that Y is affine. Let us check that the natural map

$$f_*\mathcal{O}_X \otimes k(y) \to H^0(X_y, \mathcal{O}_{X_y})$$

is surjective for every $y \in Y$. We take an arbitrary closed point $y \in Y$. We can replace Y with $\mathrm{Spec}\mathcal{O}_{Y,y}$. Let m_y be the maximal ideal corresponding to $y \in Y$. We note that $f_*\mathcal{O}_X \otimes k(y) \simeq (f_*\mathcal{O}_X)^\wedge_y \otimes k(y)$, where $(f_*\mathcal{O}_X)^\wedge_y$ is the formal completion of $f_*\mathcal{O}_X$ at y. By the theorem on formal functions (cf. [H, Theorem 11.1]), we have

$$(f_*\mathcal{O}_X)^\wedge_y \simeq \varprojlim H^0(X_n, \mathcal{O}_{X_n}),$$

where $X_n = X \times_Y \mathrm{Spec}\mathcal{O}_{Y,y}/m_y^n$. Therefore, we can see that

$$(f_*\mathcal{O}_X)^\wedge_y \otimes k(y) \to H^0(X_y, \mathcal{O}_{X_y})$$

is surjective. It is because $H^0(X_{yi}, \mathcal{O}_{X_{yi}}) = k$ for every i, where $X_y = \coprod_i X_{yi}$ is the irreducible decomposition of a smooth variety X_y. By the base change theorem (cf. [H, Theorem 12.11]), we obtain the desired results.

6.6 Appendix

In this appendix, we give another proof of Theorem 6.1.1 depending on mod p reduction arguments. This proof is not related to Kawamata's positivity theorem.

First let us recall various results without proofs for the reader's convenience.

6.6.1 (Preliminary results). The following theorem was obtained by the same way as in [KoMiMo].

Theorem 6.6.2 ([Dbook, Corollary 3.15 (a)]). *Let $f : X \to Y$ be a smooth morphism of smooth projective varieties over an arbitrary algebraic closed field. If $-K_X$ is nef, then so is $-K_Y$.*

In [ScSm], Schwede and Smith established the following results on log Fano varieties and global F-regular varieties. For various definitions and details, see [ScSm] and [S]. See also [HWY] for related topics.

Theorem 6.6.3 (cf. [ScSm, Theorem 1.1]). *Let X be a normal projective variety over an F-finite field of prime characteristic. Suppose that X is globally F-regular. Then there exists an effective $\mathbb{Q}$-divisor Δ on X such that $-(K_X + \Delta)$ is ample and that (X, Δ) is klt.*

For the definition of *klt in any characteristic*, see [ScSm, Remark 4.2].

Theorem 6.6.4 (cf. [ScSm, Theorem 5.1]). *Let X be a normal projective variety defined over a filed of characteristic zero. Suppose that there exists an effective $\mathbb{Q}$-divisor Δ on X such that $-(K_X + \Delta)$ is ample and that (X, Δ) is klt. Then X has globally F-regular type.*

Theorem 6.6.5 (cf. [ScSm, Corollary 6.4]). *Let $f : X \to Y$ be a projective morphism of normal projective varieties over an F-finite field of prime characteristic. Suppose that $f_*\mathcal{O}_X = \mathcal{O}_Y$. If X is a globally F-regular variety, then so is Y.*

We can find the following lemma in [Liu, Proposition 3.7 (a)].

Lemma 6.6.6. *Let C be a smooth projective curve over a field k, let K be an extension field of k, and let D be a Cartier divisor on C. Suppose that $\pi : C_K := C \times_k K \to C$ is the natural projection. Then $\deg_k D = \deg_K \pi^* D$.*

By the above lemma, we see the following lemma.

Lemma 6.6.7. *Let X be a projective variety over a field k, let K be an extension field of k, and let D be a Cartier divisor on X. Suppose that $\pi^* D$ is nef, where $\pi : X_K := X \times_k K \to X$ is the projection. Then D is nef.*

Proof. We take a morphism $f : C \to X$ from a smooth projective curve. We consider the following commutative diagram:

$$\begin{tikzcd}
C_K \arrow[r, "\pi_C"] \arrow[d, "f_K"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "f"] \\
X_K \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d] \\
\mathrm{Spec} K \arrow[r] & \mathrm{Spec} k
\end{tikzcd}$$

where $C_K := C \times_k K$. By the assumption, $\deg_K \pi_C{}^*(f^*D) \geq 0$. Hence $\deg_k f^*D \geq 0$ by Lemma 6.6.6. Thus D is nef. $\square$

Let us start the proof of Theorem 6.1.1.

Proof of Theorem 6.1.1. First, we note that $-K_X$ is semi-ample by the Kawamata–Shokurov base point free theorem and that $-K_Y$ is nef by Theorem 6.6.2. It is sufficient to show that $(-K_Y)^{\dim Y} > 0$. By the Stein factorization, we can assume that f has connected fibers. We can take a finitely generated $\mathbb{Z}$-algebra A, a non-empty affine open set $U \subseteq \mathrm{Spec} A$, and smooth morphisms $\varphi : \mathcal{X} \to U$ and $\psi : \mathcal{Y} \to U$ such that

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "F"] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\ & U & \end{tikzcd}$$

and $F \simeq f$ over the generic point of U and that $-K_{\mathcal{X}}$ is semi-ample. We take a general closed point $\mathfrak{p} \in U$. Note that the residue field $k := \kappa(\mathfrak{p})$ of $\mathfrak{p}$ has positive characteristic p. Let $f_p : X_p \to Y_p$ be the fiber of F at $\mathfrak{p}$, and let K be an algebraic closure of k. By Theorem 6.6.4, we may assume that X_p is globally F-regular. Let $\overline{f_p} : \overline{X_p} \to \overline{Y_p}$ be the base change of f_p by $\mathrm{Spec} K$, where $\overline{X_p} := X_p \times_k K$ and $\overline{Y_p} := Y_p \times_k K$. Since $-K_{\mathcal{X}}$ is semi-ample, we see that $-K_{\overline{X_p}}$ is semi-ample. In particular, $-K_{\overline{X_p}}$ is nef. Hence, we obtain that $-K_{\overline{Y_p}}$ is nef by Theorem 6.6.2. By Lemma 6.6.7, $-K_{Y_p}$ is nef. By Theorem 6.6.5, Y_p is globally F-regular. Hence $-K_{Y_p}$ is nef and big. Thus $(-K_{Y_p})^{\dim Y} > 0$. Since ψ is flat, $(-K_Y)^{\dim Y} > 0$. Therefore, $-K_Y$ is nef and big. $\square$

7 Weak Fano varieties with log canonical singularities

7.1 Introdution

We start by some basic definitions.

Definition 7.1.1. Let X be a normal projective variety and Δ an effective $\mathbb{Q}$-Weil divisor on X. We say that (X, Δ) is a *weak log Fano pair* if $-(K_X + \Delta)$ is nef and big. If $\Delta = 0$, then we simply say that X is a *weak Fano variety*.

There are questions whether the following fundamental properties hold or not for a log canonical weak log Fano pair (X, Δ) (cf. [Sh1, 2.6. Remark-Corollary], [Pr, 11.1]):

(i) Semi-ampleness of $-(K_X + \Delta)$.

(ii) Existence of $\mathbb{Q}$-complements, i.e., existence of an effective $\mathbb{Q}$-divisor D such that $K_X + \Delta + D \sim_{\mathbb{Q}} 0$ and $(X, \Delta + D)$ is lc.

(iii) Rational polyhedrality of the Kleiman-Mori cone $\overline{NE}(X)$.

It is easy to see that (i) implies (ii). In the case where (X, Δ) is a klt pair, the above three properties hold by the Kawamata-Shokurov base point free theorem and the cone theorem (cf. [KaMaMa], [KoMo]). Shokurov proved that these three properties hold for surfaces (cf. [Sh1, 2.5. Proposition]).

Among other things, we prove the following:

Theorem 7.1.2 (=Corollaries 7.3.3 and 7.4.5). *Let X be a weak Fano 3-fold with log canonical singularities. Then $-K_X$ is semi-ample and $\overline{NE}(X)$ is a rational polyhedral cone.*

Theorem 7.1.3 (=Corollary 7.3.4 and Theorem 7.4.4). *Let X be a weak Fano variety with log canonical singularities. Suppose that any lc center of X is at most 1-dimensional. Then $-K_X$ is semi-ample and $\overline{NE}(X)$ is a rational polyhedral cone.*

On the other hand, the above three properties do not hold for d-dimensional log canonical weak *log* Fano pairs in general, where $d \geq 3$. Indeed, we give the following examples of plt weak log Fano pairs whose anti-log canonical divisors are not semi-ample in Section 7.5 (in particular, such examples of 3-dimensional weak log Fano plt pairs show the main result of [Kar1] does not hold). It is well known that there exists a $(d-1)$-dimensional smooth projective variety S such that $-K_S$ is nef and is not semi-ample (e.g. When $d = 3$, we take a very general 9-points blow up of $\mathbb{P}^2$ as S). Let X_0 be the cone over S with respect to some projectively normal embedding $S \subset \mathbb{P}^N$. We take the blow-up X of X_0 at its vertex. Let E be the exceptional divisor of the blow-up. Then the pair (X, E) is a weak log Fano plt pair such that $-(K_X + E)$ is not semi-ample. Moreover we give an example of a log canonical weak log Fano pair without $\mathbb{Q}$-complements and an example whose Kleiman-Mori cone is not polyhedral.

We now outline the proof of semi-ampleness of $-K_X$ as in Theorem 7.1.2. First, we take a birational morphism $\varphi : Y \to X$ such that $\varphi^*(K_X) = K_Y + S$, (Y, S) is dlt and S is reduced. We set $C := \varphi(S)$, which is the union of lc centers of X. By an argument in the proof of the Kawamata-Shokurov base point free theorem (Lemma 7.2.6), it is sufficient to prove that $-(K_Y + S)|_S$ is semi-ample. Moreover we have only to prove that $-K_X|_C$ is semi-ample by the formula $K_X|_C = (\varphi|_S)^*((K_Y + S)|_S)$.

It is not difficult to see semi-ampleness of the restriction of $-K_X$ on any lc center of X. The main difficulty is how to extend semi-ampleness to C from each 1-dimensional irreducible component C_i of C since the configuration of C_i's may be complicated. The key to overcome this difficulty is the abundance theorem for 2-dimensional semi-divisorial log terminal pairs ([AFKM]). We decompose $C = C' \cup C''$, where

$$\Sigma := \{i | -K_X|_{C_i} \equiv 0\},\ C' := \bigcup_{i \in \Sigma} C_i, \text{ and } C'' := \bigcup_{i \notin \Sigma} C_i.$$

Let S' be the union of the irreducible components of S whose image on X is contained in C'. We define the boundary $\mathrm{Diff}_{S'}(S)$ on S' by the formula $K_Y + S|_{S'} = K_{S'} + \mathrm{Diff}_{S'}(S)$. The pair $(S', \mathrm{Diff}_{S'}(S))$ is known to be semi-divisorial log terminal pair (sdlt, for short). Applying the abundance theorem to the pair $(S', \mathrm{Diff}_{S'}(S))$, we see that $K_{S'} + \mathrm{Diff}_{S'}(S)$ is $\mathbb{Q}$-linearly trivial, namely, there is a non-zero integer m_1 such that $-m_1(K_Y + S)|_{S'} = -m_1(K_{S'} + \mathrm{Diff}_{S'}(S)) \sim 0$. This shows that $-m_1 K_X|_{C'} \sim 0$. On the other hand, since $-K_X|_{C''}$ is ample, we can take enough sections of $H^0(C'', -m_2K_X|_{C''})$ for a sufficiently large and divisible m_2 (Lemma 7.2.11). Thus, we can find enough sections of $H^0(C, -mK_X|_C)$ for a sufficiently large and divisible m, and can conclude that $-K_X|_C$ is semi-ample.

To generalize this theorem to higher dimensional weak log Fano pairs, we need the following theorem for arbitrary dimension:

Theorem 7.1.4 (Abundance theorem in a special case, [G2]). *Let (X, Δ) be a d-dimensional projective sdlt pair whose $K_X + \Delta$ is numerically trivial. Then $K_X + \Delta$ is $\mathbb{Q}$-linearly trivial, i.e., there exists an $n \in \mathbb{N}$ such that $n(K_X + \Delta) \sim 0$.*

The abundance conjecture is one of the most famous conjecture in the minimal model program. This theorem is proved when $d \leq 3$ by the works of Fujita, Kawamata, Miyaoka, Abramovich, Fong, Kollár, M^{c}Kernan, Keel, Matsuki, and Fujino (cf. [AFKM], [F2]). Recently, this theorem for arbitrary dimension is proved in [G2].

By the same way as in the 3-dimensional case, we see the following theorem:

Theorem 7.1.5 (=Theorem 7.3.1). *Let (X, Δ) be a d-dimensional log canonical weak log Fano pair. Suppose that* $\dim \mathrm{Nklt}(X, \Delta) \leq 1$. *Then $-(K_X + \Delta)$ is semi-ample.*

We remark that by Examples 7.5.2 and 7.5.3, this condition for the dimension of lc centers is the best possible.

In Section 7.4, by the cone theorem for normal varieties by Ambro and Fujino (cf. Theorem 7.4.3), we derive the following:

Theorem 7.1.6 (=Theorem 7.4.4). *Let (X, Δ) be a d-dimensional log canonical weak log Fano pair. Suppose that* $\dim \mathrm{Nklt}(X, \Delta) \leq 1$. *Then $\overline{NE}(X)$ is a rational polyhedral cone.*

Note that rational polyhedrality of $\overline{NE}(X)$ as in Theorem 7.1.2 is a corollary of the above theorem. In Example 7.5.6, we also see that the Kleiman-Mori cone is not rational polyhedral in general when $\dim \mathrm{Nklt}(X, \Delta) \geq 2$.

This chapter is based on the minimal model theory for log canonical pairs developed by Ambro and Fujino ([Am1], [Am2], [Am6], [F12], [F14], [F15]).

7.2 Preliminaries and Lemmas

In this section, we introduce notation and some lemmas for the proof of Theorem 7.1.5 (=Theorem 7.3.1).

The following theorem is very important as a generalization of vanishing theorems (cf. [Am2, Theorem 3.1], [F14, Theorem 2.2], [F6, Theorem 2.38], [F12, Theorem 6.3]).

Theorem 7.2.1 (Torsion-freeness theorem). *Let Y be a smooth variety and B a boundary $\mathbb{R}$-divisor such that* Supp$B$ *is simple normal crossing. Let $f : Y \to X$ be a projective morphism and L a Cartier divisor on Y such that $H \sim_{\mathbb{R}} L-(K_Y+B)$ is f-semi-ample. Then every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y, B) for any non-negative integer q.*

The following theorem is proved by Fujino ([F12, Theorem 10.5]). We include the proof for the reader's convenience.

Theorem 7.2.2. *Let X be a normal quasi-projective variety and Δ an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Suppose that (X, Δ) is lc. Then there exists a projective birational morphism $\varphi : Y \to X$ from a normal quasi-projective variety with the following properties:*

(i) Y is $\mathbb{Q}$-factorial,

(ii) $a(E, X, \Delta) = -1$ for every φ-exceptional divisor E on Y,

(iii) for

$$\Gamma = \varphi_*^{-1}\Delta + \sum_{E:\varphi\text{-exceptional}} E,$$

it holds that (Y, Γ) is dlt and $K_Y + \Gamma = \varphi^*(K_X + \Delta)$, and

(iv) Let $\{C_i\}$ be any set of lc centers of (X, Δ). Let $W = \bigcup C_i$ with a reduced structure and S the union of the irreducible components of $\llcorner \Gamma \lrcorner$ which are mapped into W by φ. Then $(\varphi|_S)_* \mathcal{O}_S \simeq \mathcal{O}_W$.

Proof. Let $\pi : V \to X$ be a resolution such that

(1) $\pi^{-1}(C)$ is a simple normal crossing divisor on V for every lc center C of (X, Δ), and

(2) $\pi_*^{-1}\Delta \cup \mathrm{Exc}(\pi) \cup \pi^{-1}(\mathrm{Nklt}(X, \Delta))$ has a simple normal crossing support, where $\mathrm{Exc}(\pi)$ is the exceptional set of π.

By Hironaka's resolution theorem, we can assume that π is a composite of blow-ups with centers of codimension at least two. Then there exists an effective π-exceptional Cartier divisor B on V such that $-B$ is π-ample. We put

$$F = \sum_{\substack{a(E,X,\Delta)>-1, \\ E:\pi\text{-exceptional}}} E \text{ and } G = \sum_{a(E,X,\Delta)=-1} E.$$

Let H be a sufficiently ample Cartier divisor on X such that $-B + \pi^*H$ is ample. We choose $0 < \varepsilon \ll 1$ such that $\varepsilon G - B + \pi^*(H)$ is ample. Since $-B + \pi^*(H)$ and $\varepsilon G - B + \pi^*(H)$ are ample, we can take effective $\mathbb{Q}$-divisors H_1 and H_2 on V with small coefficients such that $G+F+\pi_*^{-1}\Delta+H_1+H_2$ has a simple normal crossing support and that $-B+\pi^*H \sim_{\mathbb{Q}} H_1$, $\varepsilon G-B+\pi^*(H) \sim_{\mathbb{Q}} H_2$. We take $0 < \nu, \mu \ll 1$ such that every divisor in F has a negative coefficient in

$$M := \Gamma_V - G - (1-\nu)F - \pi_*^{-1}\Delta^{<1} + \mu B,$$

where Γ_V is a $\mathbb{Q}$-divisor on V such that $K_V + \Gamma_V = \pi^*(K_X + \Delta)$. Now we construct a log minimal model of $(V, G + (1-\nu)F + \pi_*^{-1}\Delta^{<1} + \mu H_1)$ over X. Since

$$G + (1-\nu)F + \mu H_1 \sim_{\mathbb{Q}} (1-\varepsilon\mu)G + (1-\nu)F + \mu H_2,$$

it is sufficient to construct a log minimal model of $(V, (1-\varepsilon\mu)G + (1-\nu)F + \pi_*^{-1}\Delta^{<1} + \mu H_2)$ over X. Because $(V, (1-\varepsilon\mu)G + (1-\nu)F + \pi_*^{-1}\Delta^{<1} + \mu H_2)$ is klt, we can get a log minimal model $\varphi : Y \to X$ of $(V, (1-\varepsilon\mu)G + (1-\nu)F + \pi_*^{-1}\Delta^{<1} + \mu H_2)$ over X by [BCHM, Theorem 1.2].

We show this Y satisfies the conditions of the theorem. For any divisor D on V (appearing above), let D' denote its strict transform on Y. We see the following claim:

Claim 7.2.3. $F' = 0$.

Proof of Claim 7.2.3. By the above construction,

$$N := K_Y + G' + (1-\nu)F' + \varphi_*^{-1}\Delta^{<1} + \mu H_1'$$

is φ-nef. Then

$$-M' \sim_{\mathbb{Q},\varphi} N - (K_Y + \Gamma_Y)$$

since $(\pi^*H)' = \varphi^*H$, hence it is φ-nef. Since $\varphi_*M' = 0$, we see that M' is effective by the negativity lemma (cf. [KoMo, Lemma 3.39]). Since every divisor in F has a negative coefficient in M, F is contracted on Y. We finish the proof of Claim 7.2.3. □

From Claim 7.2.3, the discrepancy of every φ-exceptional divisor is equal to -1. We see that Y satisfies the condition (ii). By the above construction, (Y, Γ) is a $\mathbb{Q}$-factorial dlt pair since so is $(Y, G' + \varphi_*^{-1}\Delta^{<1} + \mu H_1)$. We see the condition (i). Because the support of $K_Y + \Gamma - \varphi^*(K_X + \Delta)$ coincide with F', we see the condition (iii).

Now, we show that Y and φ satisfy the condition (iv). Since we get Y by the log minimal model program over X with scaling of some effective divisor with respect to $K_V + G + (1 - \nu)F + \pi_*^{-1}\Delta^{<1} + \mu H_1$ (cf. [BCHM]), we see that the rational map $f : V \dashrightarrow Y$ is a composition of $(K_V + G + (1 - \nu)F + \pi_*^{-1}\Delta^{<1} + \mu H_1)$-negative divisorial contractions and log flips. Let Σ be an lc center of (Y, Γ). Then it is also an lc center of $(Y, \Gamma + \mu H_1')$. By the negativity lemma, $f : V \dashrightarrow Y$ is an isomorphism around the generic point of Σ. Therefore, if $\varphi(\Sigma) \subseteq W$, then $\Sigma \subseteq S$ by the conditions (1) and (2) for $\pi : V \to X$. This means that no lc centers of $(Y, \Gamma - S)$ are mapped into W by φ. Let $g : Z \to Y$ be a resolution such that

(a) $\operatorname{Supp} \Gamma_Z$ is a simple normal crossing divisor, where Γ_Z is defined by $K_Z + \Gamma_Z = g^*(K_Y + \Gamma)$, and

(b) g is an isomorphism over the generic point of any lc center of (Y, Γ).

Let S_Z be the strict transform of S on Z. We consider the following short exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_Z(\ulcorner -(\Gamma_Z^{<1})\urcorner - S_Z) \to \mathcal{O}_Z(\ulcorner -(\Gamma_Z^{<1})\urcorner) \\ \to \mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner) \to 0. \end{aligned} \tag{$*$}$$

We note that

$$\ulcorner -(\Gamma_Z^{<1})\urcorner - S_Z - (K_Z + \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z) \sim_{\mathbb{Q}} -h^*(K_X + \Delta),$$

where $h = \varphi \circ g$. Then we obtain

$$\begin{aligned} 0 \to h_*\mathcal{O}_Z(\ulcorner -(\Gamma_Z^{<1})\urcorner - S_Z) \to h_*\mathcal{O}_Z(\ulcorner -(\Gamma_Z^{<1})\urcorner) \to h_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner) \\ \xrightarrow{\delta} R^1h_*\mathcal{O}_Z(\ulcorner -(\Gamma_Z^{<1})\urcorner - S_Z) \to \cdots. \end{aligned}$$

We claim the following:

Claim 7.2.4. *δ is a zero map.*

Proof of Claim 7.2.4. Let Σ be an lc center of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$. Then Σ is some intersection of components of $\Gamma_Z^{=1} - S_Z$. By the conditions (a) and (b), $\Gamma_Z^{=1} - S_Z$ is the strict transform of $\llcorner \Gamma \lrcorner - S$. By this, the image of Σ by

g is some intersection of components of $\llcorner\Gamma\lrcorner - S$. In particular, $g(\Sigma)$ is an lc center of $(Y, \Gamma - S)$. Thus no lc centers of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$ are mapped into W by h. Assume by contradiction that δ is not zero. Then there exists a section $s \in H^0(U, h_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner))$ for some non-empty open set $U \subseteq X$ such that $\delta(s) \neq 0$. Since Supp $\delta(s) \neq \emptyset$, we can take an associated prime $x \in$ Supp $\delta(s)$. We see that $x \in W$ since $\mathrm{Supp}(h_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner))$ is contained in W. By Theorem 7.2.1, x is the generic point of the h-image of some stratum of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$. Since h is a birational morphism, x is the generic point of the h-image of some lc center of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$. Because no lc centers of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$ are mapped into W by h, it holds that $x \notin W$. But this contradicts the way of taking x.

□

Thus, we obtain

$$0 \to \mathcal{I}_W \to \mathcal{O}_X \to h_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner) \to 0,$$

where $\mathcal{I}_W$ is the defining ideal sheaf of W since $\ulcorner -(\Gamma_Z^{<1})\urcorner$ is effective and h-exceptional. This implies that $\mathcal{O}_W \simeq h_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner)$. By applying g_* to $(*)$, we obtain

$$0 \to \mathcal{I}_S \to \mathcal{O}_Y \to g_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner) \to 0,$$

where $\mathcal{I}_S$ is the defining ideal sheaf of S since $\ulcorner -(\Gamma_Z^{<1})\urcorner$ is effective and g-exceptional. We note that

$$R^1 g_*\mathcal{O}_Z(\ulcorner -(\Gamma_Z^{<1})\urcorner - S_Z) = 0$$

by Theorem 7.2.1 since g is an isomorphism at the generic point of any stratum of $(Z, \{\Gamma_Z\} + \Gamma_Z^{=1} - S_Z)$. Thus, $\mathcal{O}_W \simeq h_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner) \simeq \varphi_* g_*\mathcal{O}_{S_Z}(\ulcorner -(\Gamma_Z^{<1})\urcorner) \simeq \varphi_*\mathcal{O}_S$. We finish the proof of Theorem 7.2.2. □

The following proposition is [Fk2, Proposition 2] (for the proof, see [Fk1, Proof of Theorem 3] and [Ka5, Lemma 3]).

Proposition 7.2.5. *Let (X, Δ) be a proper dlt pair and L a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and big for some $a \in \mathbb{N}$. If $\mathrm{Bs}|mL| \cap \llcorner\Delta\lrcorner = \emptyset$ for every $m \gg 0$, then $|mL|$ is base point free for every $m \gg 0$, where $\mathrm{Bs}|mL|$ is the base locus of $|mL|$.*

By this proposition, we derive the following lemma:

Lemma 7.2.6. *Let (Y, Γ) be a $\mathbb{Q}$-factorial weak log Fano dlt pair. Suppose that $-(K_S + \Gamma_S)$ is semi-ample, where $S := \llcorner\Gamma\lrcorner$ and $\Gamma_S := \mathrm{Diff}_S(\Gamma)$. Then $-(K_Y + \Gamma)$ is semi-ample.*

Proof. We consider the exact sequence

$$0 \to O_Y(-m(K_Y + \Gamma) - S) \to O_Y(-m(K_Y + \Gamma)) \to \\ \to O_S(-m(K_Y + \Gamma)|_S) \to 0$$

for $m \gg 0$. By the Kawamata-Viehweg vanishing theorem (cf. [KaMaMa, Theorem 1-2-5.], [KoMo, Theorem 2.70]), we have

$$H^1(Y, O_Y(-m(K_Y + \Gamma) - S)) = \\ = H^1(Y, O_Y(K_Y + \Gamma - S - (m+1)(K_X + \Gamma))) = \{0\},$$

since the pair $(Y, \Gamma - S)$ is klt and $-(K_Y + \Gamma)$ is nef and big. Thus, we get the exact sequence

$$H^0(Y, O_Y(-m(K_Y + \Gamma)) \to H^0(S, O_S(-m(K_Y + \Gamma)|_S)) \to 0.$$

Therefore, we see that $\mathrm{Bs}|-m(K_Y + \Gamma)| \cap S = \emptyset$ for $m \gg 0$ since $-(K_S + \Delta_S)$ is semi-ample. Applying Proposition 7.2.5, we conclude that $-(K_Y + \Gamma)$ is semi-ample. □

We introduce the following definitions [GT, 1.1. Definition], [KoS, Definitions 7.1 and 7.2]:

Definition 7.2.7. Suppose that R is a reduced excellent ring and $R \subseteq S$ is a reduced R-algebra which is finite as an R-module. We say that the extension $i : R \hookrightarrow S$ is *subintegral* if one of the following equivalent conditions holds:

(a) $(S \bigotimes_R k(\mathfrak{p}))_{\mathrm{red}} = k(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

(b) the induced map on the spectra is bijective and i induces trivial residue field extensions.

Definition 7.2.8. Suppose that R is a reduced excellent ring. We say that R is *semi-normal* if every subintegral extension $R \hookrightarrow S$ is an isomorphism.

A scheme X is called *semi-normal at* $q \in X$ if the local ring at q is semi-normal. If X is semi-normal at every point, we say that X is *semi-normal.*

Proposition 7.2.9 ([GT, 5.3. Corollary]). *Let $(R, \mathfrak{m})$ be a local excellent ring. Then R is semi-normal if and only if $\widehat{R}$ is semi-normal, where $\widehat{R}$ is $\mathfrak{m}$-adic completion of R.*

Proposition 7.2.10 (cf. [Ko1, 7.2.2.1], [KoS, Remark 7.6]). *Let C be a pure 1-dimensional proper reduced scheme of finite type over $\mathbb{C}$, and $q \in C$ a closed point. Then C is semi-normal at q if and only if $\widehat{O}_{C,q}$ satisfies that*

(i) $\widehat{\mathcal{O}}_{C,q} \simeq \mathbb{C}[[X]]$, *or*

(ii) $\widehat{\mathcal{O}}_{C,q} \simeq \mathbb{C}[[X_1, X_2, \cdots, X_r]]/\langle X_iX_j | 1 \le i \ne j \le r\rangle$ *for some* $r \ge 2$, *i.e.,* $q \in C$ *is isomorphic to the coordinate axies in* $\mathbb{C}^r$ *at the origin as a formal germs.*

Lemma 7.2.11. *Let* $C = C_1 \cup C_2$ *be a pure* 1*-dimensional proper semi-normal reduced scheme of finite type over* $\mathbb{C}$*, where* C_1 *and* C_2 *are pure* 1*-dimensional reduced closed subschemes. Let* D *be a* $\mathbb{Q}$*-Cartier divisor on* C*. Suppose that* D_1 *is* $\mathbb{Q}$*-linearly trivial and* D_2 *is ample, where* $D_i := D|_{C_i}$*. Then* D *is semi-ample.*

Proof. Let $C_1 \cap C_2 = \{p_1, \ldots, p_r\}$. We take $m \gg 0$ which satisfies the following:

(i) $mD_1 \sim 0$,

(ii) $\mathcal{O}_{C_2}(mD_2) \otimes (\bigcap_{k \ne l} \mathfrak{m}_{p_k})$ is generated by global sections for all $l \in \{1, \ldots, r\}$, and

(iii) $\mathcal{O}_{C_2}(mD_2) \otimes (\bigcap_k \mathfrak{m}_{p_k})$ is generated by global sections,

where $\mathfrak{m}_{p_k}$ is the ideal sheaf of p_k on C_2. We choose a nowhere vanishing section $s \in H^0(C_1, mD_1)$. By (ii), we can take a section $t_l \in H^0(C_2, mD_2)$ which does not vanish at p_l but vanishes at all the p_k ($k \in \{1, \ldots, r\}, k \ne l$) for each $l \in \{1, \ldots, r\}$. By multiplying suitable nonzero constants to t_l, we may assume that $t_l|_{p_l} = s|_{p_l}$. We set $t := \sum_l t_l \in H^0(C_2, mD_2)$. Since C is semi-normal, Proposition 7.2.10 implies that $\mathcal{O}_{C_1 \cap C_2} \simeq \bigoplus_{l=1}^r \mathbb{C}(p_l)$, where $\mathbb{C}(p_l)$ is the skyscraper sheaf $\mathbb{C}$ sitting at p_l, by computations on $\widehat{\mathcal{O}}_{C,p_l}$. Thus we get the following exact sequence:

$$0 \to \mathcal{O}_C(mD) \to \mathcal{O}_{C_1}(mD_1) \oplus \mathcal{O}_{C_2}(mD_2) \to \bigoplus_{l=1}^r \mathbb{C}(p_l) \to 0,$$

where the third arrow maps (s', s'') to $((s' - s'')|_{p_1}, \ldots, (s' - s'')|_{p_r})$. Hence s and t patch together and give a section u of $H^0(C, mD)$.

Let p be any point of C. If $p \in C_1$, then u does not vanish at p. We may assume that $p \in C_2 \setminus C_1$. By (iii), we can take a section $t' \in H^0(C_2, mD_2)$ which does not vanish at p but vanishes at p_l for all $l \in \{1, \ldots, r\}$. The zero section $0 \in H^0(C_1, mC_1)$ and t' patch together and give a section u' of $H^0(C, mD)$. By construction, the section u' does not vanish at p. We finish the proof of Lemma 7.2.11. □

7.3 On semi-ampleness for weak Fano varieties

In this section, we prove Theorem 7.1.5 (=Theorem 7.3.1). As a corollary, we see that the anti-canonical divisors of weak Fano 3-folds with log canonical singularities are semi-ample. Moreover we derive semi-ampleness of the anti-canonical divisors of log canonical weak Fano varieties whose lc centers are at most 1-dimensional.

Theorem 7.3.1. *Let (X, Δ) be a d-dimensional log canonical weak log Fano pair. Suppose that* $\dim \mathrm{Nklt}(X, \Delta) \leq 1$. *Then $-(K_X + \Delta)$ is semi-ample.*

Proof. By Theorem 7.2.2, we take a birational morphism $\varphi : (Y, \Gamma) \to (X, \Delta)$ as in the theorem. We set $S := \llcorner \Gamma \lrcorner$ and $C := \varphi(S)$, where we consider the reduced scheme structures on S and C. We have only to prove that $-(K_S + \Gamma_S) = -(K_Y + \Gamma)|_S$ is semi-ample from Lemma **??**. By the formula $(K_Y + \Gamma)|_S \sim_{\mathbb{Q}} (\varphi|_S)^*((K_X + \Delta)|_C)$, it suffices to show that $-(K_X + \Delta)|_C$ is semi-ample. Arguing on each connected component of C, we may assume that C is connected. Since $\dim \mathrm{Nklt}(X, \Delta) \leq 1$, it holds that $\dim C \leq 1$. When $\dim C = 0$, i.e., C is a closed point, then $-(K_X + \Delta)|_C \sim_{\mathbb{Q}} 0$, in particular, is semi-ample.
When $\dim C = 1$, C is a pure 1-dimensional semi-normal scheme by [Am1, Theorem 1.1] or [F12, Theorem 9.1]. Let $C = \bigcup_{i=1}^{r} C_i$, where C_i is an irreducible component, and let $D := -(K_X + \Delta)|_C$ and $D_i := D|_{C_i}$. We set

$$\Sigma := \{i | \, D_i \equiv 0\}, \; C' := \bigcup_{i \in \Sigma} C_i, \; C'' := \bigcup_{i \notin \Sigma} C_i.$$

Let S' be the union of irreducible components of S whose image by φ is contained in C'. We see that $K_{S'} + \Gamma_{S'} \equiv 0$, where $\Gamma_{S'} := \mathrm{Diff}_{S'}(\Gamma)$. Thus it holds that $K_{S'} + \Gamma_{S'} \sim_{\mathbb{Q}} 0$ by applying Theorem 7.1.4 to $(S', \Gamma_{S'})$. Since $(\varphi|_{S'})_* \mathcal{O}_{S'} \simeq \mathcal{O}_{C'}$ by the condition (iv) in Theorem 7.2.2, it holds that $D|_{C'} \sim_{\mathbb{Q}} 0$. We see that $D|_{C''}$ is ample since the restriction of D on any irreducible component of C'' is ample. By Lemma 7.2.11, we see that $D = -(K_X + \Delta)|_C$ is semi-ample. We finish the proof of Theorem 7.3.1. □

Corollary 7.3.2. *Let (X, Δ) be a d-dimensional log canonical weak log Fano pair. Suppose that* $\dim \mathrm{Nklt}(X, \Delta) \leq 1$. *Then $R(X, -(K_X + \Delta))$ is a finitely generated algebra over $\mathbb{C}$.*

We immediately obtain the following corollaries:

Corollary 7.3.3. *Let (X, Δ) be a 3-dimensional log canonical weak log Fano pair. Suppose that $\llcorner \Delta \lrcorner = 0$. Then $-(K_X + \Delta)$ is semi-ample and $R(X, -(K_X + \Delta))$ is a* *finitely generated algebra over $\mathbb{C}$. In particular, if X is a weak Fano 3-fold with log canonical singularities, then $-K_X$ is semi-ample and $R(X, -K_X)$ is a finitely generated algebra over $\mathbb{C}$.*

Corollary 7.3.4. *Let X is a log canonical weak Fano variety whose lc centers are at most 1-dimensional, then $-K_X$ is semi-ample and $R(X, -K_X)$ is a finitely generated algebra over $\mathbb{C}$.*

Remark 7.3.5. When $\dim \mathrm{Nklt}(X, \Delta) \geq 2$, $-(K_X + \Delta)$ is not semi-ample and $R(X, -(K_X + \Delta))$ is not a finitely generated algebra over $\mathbb{C}$, in general (Examples 7.5.2 and 7.5.3).

Remark 7.3.6. Based on Theorem 7.3.1, we expect the following statement:

> Let (X, Δ) be an lc pair and D a nef Cartier divisor. Suppose there is a positive number a such that $aD - (K_X + \Delta)$ is nef and big. If it holds that $\dim \mathrm{Nklt}(X, \Delta) \leq 1$, then D is semi-ample.

However, there is a counterexample for this statement due to Zariski (cf. [KaMaMa, Remark 3-1-2], [Z]).

This result of semi-ampleness induces that of topologies of weak Fano varieties with log canonical singularities by Hacon–M^{c}Kernan.

Corollary 7.3.7 ([HM, Corollary 1.4])**.** *Let (X, Δ) be an lc weak Fano pair. Suppose that* $\dim \mathrm{Nklt}(X, \Delta) \leq 1$*. Then the map*

$$\pi_1(\mathrm{Nklt}(X, \Delta)) \to \pi_1(X)$$

of fundamental groups is surjective. In particular, X is simply connected if $\mathrm{Nklt}(X, \Delta)$ *is simply connected.*

7.4 On the Kleiman-Mori cone for weak Fano varieties

In this section, we introduce the cone theorem for normal varieties by Ambro and Fujino and prove polyhedrality of the Kleiman-Mori cone for a log canonical weak Fano variety whose lc centers are at most 1-dimensional. We use the notion of the scheme $\mathrm{Nlc}(X, \Delta)$, whose underlying space is the set of non-log canonical singularities. For the scheme structure on $\mathrm{Nlc}(X, \Delta)$, we refer [F12, Section 7], [F8] and [FST] in detail.

Definition 7.4.1 ([F12, Definition 16.1]). Let X be a normal variety and Δ an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : X \to S$ be a projective morphism. We put

$$\overline{NE}(X/S)_{\mathrm{Nlc}(X,\Delta)} = \mathrm{Im}(\overline{NE}(\mathrm{Nlc}(X,\Delta)/S) \to \overline{NE}(X/S)).$$

Definition 7.4.2 ([F12, Definition 16.2]). An *extremal face* of $\overline{NE}(X/S)$ is a non-zero subcone $F \subset \overline{NE}(X/S)$ such that $z, z' \in F$ and $z+z' \in F$ implies that $z, z' \in F$. Equivalently, $F = \overline{NE}(X/S) \cap H^{\perp}$ for some π-nef $\mathbb{R}$-divisor H, which is called a *supporting function* of F. An *extremal ray* is a one-dimensional extremal face.

(1) An extremal face F is called $(K_X + \Delta)$*-negative* if

$$F \cap \overline{NE}(X/S)_{K_X+\Delta\geq 0} = \{0\}.$$

(2) An extremal face F is called *rational* if we can choose a π-nef $\mathbb{Q}$-divisor H as a support function of F.

(3) An extremal face F is called *relatively ample at* $\mathrm{Nlc}(X,\Delta)$ if

$$F \cap \overline{NE}(X/S)_{\mathrm{Nlc}(X,\Delta)} = \{0\}.$$

Equivalently, $H|_{\mathrm{Nlc}(X,\Delta)}$ is $\pi|_{\mathrm{Nlc}(X,\Delta)}$-ample for every supporting function H of F.

(4) An extremal face F is called *contractible at* $\mathrm{Nlc}(X,\Delta)$ if it has a rational supporting function H such that $H|_{\mathrm{Nlc}(X,\Delta)}$ is $\pi|_{\mathrm{Nlc}(X,\Delta)}$-semi-ample.

Theorem 7.4.3 (Cone theorem, cf. [F12, Theorem 16.5]). *Let X be a normal variety, Δ an effective $\mathbb{Q}$-divisor on X such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier, and $\pi : X \to S$ a projective morphism. Then we have the following properties.*

(1) $\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+\Delta\geq 0} + \overline{NE}(X/S)_{\mathrm{Nlc}(X,\Delta)} + \sum R_j$*, where R_j's are the $(K_X+\Delta)$-negative extremal rays of $\overline{NE}(X/S)$ that are rational and relatively ample at* $\mathrm{Nlc}(X,\Delta)$. *In particular, each R_j is spanned by an integral curve C_j on X such that $\pi(C_j)$ is a point.*

(2) *Let H be a π-ample $\mathbb{Q}$-divisor on X. Then there are only finitely many R_j's included in $(K_X + \Delta + H)_{<0}$. In particular, the R_j's are discrete in the half-space $(K_X + \Delta)_{<0}$.*

(3) *Let F be a $(K_X + \Delta)$-negative extremal face of $\overline{NE}(X/S)$ that is relatively ample at* $\mathrm{Nlc}(X, \Delta)$. *Then F is a rational face. In particular, F is contractible at* $\mathrm{Nlc}(X, \Delta)$.

By the above theorem, we derive the following theorem:

Theorem 7.4.4. *Let (X, Δ) be a d-dimensional log canonical weak log Fano pair. Suppose that* $\dim \mathrm{Nklt}(X, \Delta) \leq 1$. *Then $\overline{NE}(X)$ is a rational polyhedral cone.*

Proof. Since $-(K_X + \Delta)$ is nef and big, there exists an effective divisor B satisfies the following: for any sufficiently small rational positive number ε, there exists a general $\mathbb{Q}$-ample divisor A_ε such that

$$-(K_X + \Delta) \sim_{\mathbb{Q}} \varepsilon B + A_\varepsilon.$$

We fix a sufficiently small rational positive number ε and set $A := A_\varepsilon$. We also take a sufficiently small positive number δ. Thus $\mathrm{Supp}(\mathrm{Nlc}(X, \Delta + \varepsilon B + \delta A))$ is contained in the union of lc centers of (X, Δ) and $-(K_X + \Delta + \varepsilon B + \delta A)$ is ample. By applying Theorem 7.4.3 to $(X, \Delta + \varepsilon B + \delta A)$, We get

$$\overline{NE}(X) = \overline{NE}(X)_{\mathrm{Nlc}(X,\Delta+\varepsilon B+\delta A)} + \sum_{j=1}^{m} R_j \text{ for some } m.$$

Now we see that $\overline{NE}(X)_{\mathrm{Nlc}(X,\Delta+\varepsilon B+\delta A)}$ is polyhedral since $\dim \mathrm{Nlc}(X, \Delta + \varepsilon B) \leq 1$ by the assumption of $\dim \mathrm{Nklt}(X, \Delta) \leq 1$. We finish the proof of Theorem 7.4.4.

□

Corollary 7.4.5. *Let X be a weak Fano 3-fold with log canonical singularities. Then the cone $\overline{NE}(X)$ is rational polyhedral.*

Remark 7.4.6. When $\dim \mathrm{Nklt}(X, \Delta) \geq 2$, $\overline{NE}(X)$ is not polyhedral in general (Example 7.5.6).

7.5 Examples

In this section, we construct examples of log canonical weak log Fano pairs (X, Δ) such that $-(K_X + \Delta)$ is not semi-ample, (X, Δ) does not have $\mathbb{Q}$-complements, or $\overline{NE}(X)$ is not polyhedral.

Basic construction 7.5.1. Let S be a $(d-1)$-dimensional smooth projective variety such that $-K_S$ is nef and $S \subset \mathbb{P}^N$ some projectively normal embedding. Let X_0 be the cone over S and $\phi\colon X \to X_0$ the blow-up at the vertex. Then the linear projection $X_0 \dashrightarrow S$ from the vertex is decomposed as follows:

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[dr, "\pi"] & \\ X_0 & & S. \end{tikzcd}$$

This diagram is the restriction of the diagram for the projection $\mathbb{P}^{N+1} \dashrightarrow \mathbb{P}^N$:

$$\begin{tikzcd} & V := \mathbb{P}_{\mathbb{P}^N}(\mathcal{O}_{\mathbb{P}^N} \oplus \mathcal{O}_{\mathbb{P}^N}(-1)) \arrow[dl, "\phi_0"'] \arrow[dr, "\pi_0"] & \\ \mathbb{P}^{N+1} & & \mathbb{P}^N. \end{tikzcd}$$

Moreover, the ϕ_0-exceptional divisor is the tautological divisor of $\mathcal{O}_{\mathbb{P}^N} \oplus \mathcal{O}_{\mathbb{P}^N}(-1)$. Hence $X \simeq \mathbb{P}_S(\mathcal{O}_S \oplus \mathcal{O}_S(-H))$, where H is a hyperplane section on $S \subset \mathbb{P}^N$, and the ϕ-exceptional divisor E is isomorphic to S and is the tautological divisor of $\mathcal{O}_S \oplus \mathcal{O}_S(-H)$.

By the canonical bundle formula, it holds that

$$K_X = -2E + \pi^*(K_S - H),$$

thus we have

$$-(K_X + E) = \pi^*(-K_S) + \pi^*H + E$$

We see $\pi^*H + E$ is nef and big since $\mathcal{O}_X(\pi^*(H) + E) \simeq \phi^*\mathcal{O}_{X_0}(1)$ and ϕ is birational. Hence $-(K_X + E)$ is nef and big since $\pi^*(-K_S)$ is nef.

The above construction is inspired by that of Hacon and McKernan in Lazić's paper (cf. [Lazi, Theorem A.6]).

In the following examples, (X, E) is the plt weak log Fano pair given by the above construction.

Example 7.5.2. This is an example of a d-dimensional plt weak log Fano pair such that the anti-log canonical divisors are not semi-ample, where $d \geq 3$.
There exists a variety S such that $-K_S$ is nef and is not semi-ample (e.g. the surface obtained by blowing up $\mathbb{P}^2$ at very general 9 points). We see that $-(K_X + E)$ is not semi-ample since $-(K_X + E)|_E = -K_E$ is not semi-ample. In particular, $R(X, -(K_X + E))$ is not a finitely generated algebra over $\mathbb{C}$ by $-(K_X + \Delta)$ is nef and big.

Example 7.5.3. This is an example of a log canonical weak Fano variety such that the anti-canonical divisor is not semi-ample.
Let T be a k-dimensional smooth projective variety whose $-K_T$ is nef and A a $(d-k-1)$-dimensional smooth projective manifold with $K_A \sim_{\mathbb{Q}} 0$, where d and k are integers satisfying $d-1 \geq k \geq 0$. We set $S = A \times T$. Let $p_T : S \to T$ be the canonical projection. We see that $K_S = p_T^*(K_T)$. Let A_p be the fiber of p_T at a point $p \in T$, and $\varphi : X \to Y$ the birational morphism with respect to $|\phi^*(\mathcal{O}_{X_0}(1)) \otimes \pi^* p_T^* \mathcal{O}_T(H_T)|$, where H_T is some very ample divisor on T. We claim the following:

Claim 7.5.4. *It holds that:*

(i) *Y is a projective variety with log canonical singularities.*

(ii) $\mathrm{Exc}(\varphi) = E$ *and any exceptional curve of φ is contained in some A_p.*

(iii) $\varphi^* K_Y = K_X + E$.

(iv) $\varphi(E) = T$ *and* $(\varphi|_E)^* K_T = K_E$.

Proof of Claim 7.5.4. We see (ii) easily. Because $-E|_E$ is ample, E is not φ-numerical trivial. Set $\varphi^* K_Y = K_X + E + aE$ for some $a \in \mathbb{Q}$. Since $K_X + E$ is φ-numerical trivial, we see $a = 0$. Thus we obtain (iii). (i) follows from (iii). By (iii), $\varphi(E)$ is an lc center. By $(\phi^*(\mathcal{O}_{X_0}(1)) \otimes \pi^* p_T^* \mathcal{O}_T(H_T))|_E \simeq p_T^* \mathcal{O}_T(H_T)$, it holds that $\varphi|_E = p_T$. Thus (iv) follows. □

If $-K_T$ is not semi-ample, then $-K_Y$ is not semi-ample and $k \geq 2$. Thus we see that Y is a log canonical weak Fano variety with $\dim \mathrm{Nklt}(Y, 0) = k$ and $-K_Y$ is not semi-ample. In particular, $R(X, -K_X)$ is not a finitely generated algebra over $\mathbb{C}$ by $-K_X$ is nef and big (cf. [Laza, Theorem 2.3.15]).

Example 7.5.5. We construct an example of a weak log Fano plt pair without $\mathbb{Q}$-complements.
Let S be the $\mathbb{P}^1$-bundle over an elliptic curve with respect to a non-split vector bundle of degree 0 and rank 2. Then $-K_S$ is nef and S does not have $\mathbb{Q}$-complements (cf. [Sh1, 1.1. Example]). Thus (X, E) does not have $\mathbb{Q}$-complements by the adjunction formula $-(K_X + E)|_E = -K_E$.

Example 7.5.6. We construct an example of a weak log Fano plt pair whose Kleiman-Mori cone is not polyhedral. Let S be the surface obtained by blowing up $\mathbb{P}^2$ at very general 9 points. It is well known that S has infinitely many (-1)-curves $\{C_i\}$.
Then we see that the Kleiman-Mori cone $\overline{NE}(X)$ is not polyhedral. Indeed, we have the following claim:

Claim 7.5.7. $\mathbb{R}_{\geq 0}[C_i] \subseteq \overline{NE}(X)$ *is an extremal ray with* $(K_X + E).C_i = -1$. *Moreover, it holds that* $\mathbb{R}_{\geq 0}[C_i] \neq \mathbb{R}_{\geq 0}[C_j]$ $(i \neq j)$.

Proof of Claim 7.5.7. We take a semi-ample line bundle L_i on S such that L_i satisfies $L_i.C_i = 0$ and $L_i.G > 0$ for any pseudoeffective curve $[G] \in \overline{NE}(S)$ such that $[G] \notin \mathbb{R}_{\geq 0}[C_i]$. We identify E with S. Let $\mathcal{L}_i$ be a pullback of L_i by π and $\mathcal{F}_i := \phi^*(\mathcal{O}_{X_0}(1)) \otimes \mathcal{L}_i$. We show that $\mathbb{R}_{\geq 0}[C_i] \subseteq \overline{NE}(X)$ is an extremal ray. Since $(K_X + E)|_E \sim K_E$, it holds that $(K_X + E).C_i = -1$. By the cone theorem for dlt pairs, there exist finitely many $(K_X + E)$-negative extremal rays R_k such that $[C_i] - [D] \in \sum R_k$ for some $[D] \in \overline{NE}(X)_{K_X+E=0}$. It holds that $\mathcal{F}_i.D = \mathcal{F}_i.R_k = 0$ for all k since $\mathcal{F}_i.C_i = 0$ and $\mathcal{F}_i$ is a nef line bundle. We see that, if an effective 1-cycle C on X satisfies $\mathcal{F}_i.C = 0$, then $C = \alpha C_i$ for some $\alpha \geq 0$ by the construction of $\mathcal{F}_i$. Thus, any generator of R_k is equal to $\alpha_k C_i$ for some $\alpha_k \geq 0$. Hence $\mathbb{R}_{\geq 0}[C_i] \subseteq \overline{NE}(X)$ is an extremal ray. It is clear to see that $\mathbb{R}_{\geq 0}[C_i] \neq \mathbb{R}_{\geq 0}[C_j]$. Thus the claim holds. □

Bibliography

[AFKM] D. Abramovich, L.-Y. Fong, J. Kollár and J. M^{c}Kernan, Semi log canonical surfaces, *Flips and Abundance for algebraic threefolds*, Astérisque **211** (1992), 139–154.

[Am1] F. Ambro, The locus of log canonical singularities, math.AG/9806067.

[Am2] F. Ambro, Quasi-log varieties, Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220–239; translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 214–233.

[Am3] F. Ambro, Nef dimension of minimal models, Math. Ann. 330 (2004), no. 2, 309–322.

[Am4] F. Ambro, Shokurov's boundary property, J. Differential Geom. **67** (2004), no. 2, 229–255.

[Am5] F. Ambro, The moduli *b*-divisor of an lc-trivial fibration, Compos. Math. 141 (2005), no. 2, 385–403.

[Am6] F. Ambro, Basic properties of log canonical centers, math.AG/0611205.

[BCEK+] T. Bauer, F. Campana, T. Eckl, S. Kebekus, T. Peternell, S. Rams, T. Szemberg, and L. Wotzlaw, A reduction map for nef line bundles. Complex geometry, 27–36, Springer, Berlin, 2002.

[B1] C. Birkar, On existence of log minimal models, Compositio Mathematica 146 (2010), 919–928.

[B2] C. Birkar, On existence of log minimal models II, to appear in J. Reine Angew Math.

[B3] C. Birkar, Existence of log canonical flips and a special LMMP, preprint, arXiv:1104.4981.

[BCHM] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. 23 (2010), 405–468.

[Bo] S. Boucksom, Divisorial Zariski decomposition on compact complex manifolds, Ann. Sci. Ecole Norm. Sup. 37 (2004), no.4 , 45–76.

[BDPP] S. Boucksom, J-P. Demailly, M. Păun and T. Peternell, The pseudo-effective cone of a compact Kahler manifold and varieties of negative Kodaira dimension, http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/coneduality.pdf.

[Bu] N. Budur, Unitary local systems, multiplier ideals, and polynomial periodicity of Hodge numbers, Adv. Math. 221 (2009), no. 1, 217—250.

[C] F. Campana, Coréduction algébrique d'un espace analytique faiblement Kählérien compact, Invent. Math., 63 (1981), 187-223.

[CKP] F. Campana, V. Koziarz and M. Păun, Numerical character of the effectivity of adjoint line bundles, preprint, arXiv:1004.0584, to appear in Ann. Inst. Fourier.tivity of adjoint line bundles, preprint (2010).

[CPT] F. Campana, T. Peternell and M. Toma, Geometric stability of the cotangent bundle and the universal cover of a projective manifold, preprint, math/0405093, to appear in Bull. Soc. Math. France.

[Ch] S. R. Choi, The geography of log models and its applications, Ph.D thesis. Johns Hophkin University, 2008.

[CS] S. R. Choi and V. V. Shokurov, The geography of log models and its applications, arXiv:0909.0288.

[Co] A. Corti, *Adjunction of log divisors*, Flip and Abundance for algebraic threefolds, Astérisque 211 (1992), 171–182.

[CR] C. W Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1988.

[Dbook] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext. Springer-Verlag, New York, 2001.

[DHP] J-P. Demailly, C. Hacon, and M. Păun, Extension theorems, Non-vanishing and the existence of good minimal models, 2010, arXiv:1012.0493v1

[D] S. Druel, Quelques remarques sur la décomposition de Zariski divisorielle sur les variétés dont la premiére classe de Chern est nulle, Math. Z., 267, 1-2 (2011), p. 413-423.

[E1] T. Eckl, Tsuji's numerical trivial fibrations, J. Algebraic Geom. 13 (2004), no. 4, 617–639.

[E2] T. Eckl, Numerically trivial foliations, Iitaka fibrations and the numerical dimension, arXiv:math.AG/0508340

[ELMNP] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, M. Popa, Asymptotic invariants of base loci, Ann. Inst. Fourier (Grenoble) 56 (2006), no. 6, 1701–1734.

[F1] O. Fujino, Applications of Kawamata's positivity theorem, Proc. Japan Acad. Ser. A Math. Sci. **75** (1999), no. 6, 75–79.

[F2] O. Fujino, Abundance theorem for semi log canonical threefolds, Duke Math. J. **102** (2000), no. 3, 513–532.

[F3] O. Fujino, Base point free theorem of Reid-Fukuda type, J. Math. Sci. Univ. Tokyo **7** (2000), no. 1, 1–5.

[F4] O. Fujino, The indices of log canonical singularities, Amer. J. Math. **123** (2001), no. 2, 229–253.

[F5] O. Fujino, A canonical bundle formula for certain algebraic fiber spaces and its applications, Nagoya Math. J. **172** (2003), 129–171.

[F6] O. Fujino, *What is log terminal?*, Flips for 3-folds and 4-folds, Oxford Lecture Series in Mathematics and its Applications, vol.35, Oxford University Press, 2007, 49–62.

[F7] O. Fujino, On Kawamata's theorem, *Classification of Algebraic Varieties*, 305–315, EMS Ser. of Congr. Rep., Eur. Math. Soc., Zürich, 2010.

[F8] O. Fujino, Theory of non-lc ideal sheaves:basic properties, Kyoto Journal of Mathematics, Vol. **50**, No. 2 (2010), 225–245.

[F9] O. Fujino, Base point free theorems—saturation, b-divisors, and canonical bundle formula—, to appear in Algebra and Number Theory,

[F10] O. Fujino, Semi-stable minimal model program for varieties with trivial canonical divisor, Proc. Japan Acad. Ser. A Math. Sci. **87** (2011), no. 3, 25–30.

[F11] O. Fujino, Non-vanishing theorem for log canonical pairs, J. Algebraic Geom. 20 (2011), no. 4, 771–783.

[F12] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. 47 (2011), no. 3, 727–789.

[F13] O. Fujino, Minimal model theory for log surfaces, preprint, arXiv:1001.3902.

[F14] O. Fujino, Introduction to the theory of quasi-log varieties, *Classification of Algebraic Varieties*, 289–303, EMS Ser. of Congr. Rep. Eur. Math. Soc., Zürich, 2010.

[F15] O. Fujino, Introduction to the log minimal model program for log canonical pairs, preprint, arXiv:0907.1506.

[F16] O. Fujino, On maximal Albanese dimensional varieties. preprint, arXiv:0911.2851.

[FG1] O. Fujino and Y. Gongyo, On images of weak Fano manifolds, to appear in Math. Z.

[FG2] O. Fujino and Y. Gongyo, On canonical bundle formulae and subadjunctions, to appear in Michigan Math. J.

[FG3] O. Fujino and Y. Gongyo, Log pluricanonical representations and abundance conjecture, preprint (2011)

[FM] O. Fujino and S. Mori, A canonical bundle formula. J. Differential Geom. 56 (2000), no. 1, 167–188.

[FST] O. Fujino, K. Schwede and S. Takagi, Supplements to non-lc ideal sheaves, arXiv:1004.5170.

[Ft1] T. Fujita, Zariski decomposition and canonical rings of elliptic threefolds, J. Math. Soc. Japan Volume 38, Number 1 (1986), 19–37.

[Ft2] T. Fujita, Fractionally logarithmic canonical rings of algebraic surfaces, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **30** (1984), no. 3, 685–696.

[Fk1] S. Fukuda, On base point free theorem, Kodai Math. J. **19** (1996), 191–199.

[Fk2] S. Fukuda, A base point free theorem of Reid type, J. Math. Sci. Univ. Tokyo **4** (1997), 621–625.

[Fk3] S. Fukuda, On numerically effective log canonical divisors, Int. J. Math. Math. Sci. **30** (2002), no. 9, 521–531.

[Fk4] S. Fukuda, Tsuji's numerically trivial fibrations and abundance, Far East Journal of Mathematical Sciences, Volume 5, Issue 3, Pages 247–257.

[Fk5] S. Fukuda, An elementary semi-ampleness result for log canonical divisors, preprint, arXiv:1003.1388v2.

[G1] Y. Gongyo, On weak Fano varieties with log canonical singularities, to appear in J. Reine Angew. Math.

[G2] Y. Gongyo, Abundance theorem for numerically trivial log canonical divisors of semi-log canonical pairs, to appear in J. Algebraic Geom.

[G3] Y. Gongyo, On the minimal model theory for dlt pairs of numerical log Kodaira dimension zero, to appear in Math. Res. Lett.

[GL] Y. Gongyo and B. Lehmann, Reduction maps and minimal model theory, preprint, arXiv:1103.1605.

[GT] S. Greco and C. Travero, On seminormal schemes, Comp. Math. **40** (1980), no. 3, 325–365.

[HM] C. D. Hacon and J. M^{c}Kernan, On Shokurov's rational connectedness conjecture, Duke Math. J. **138** (2000), no. 1, 119–136.

[HX] C. D. Hacon and C. Xu, Existence of log canonical closures, preprint, arXiv:1105.1169.

[HWY] N. Hara, K.-i. Watanabe, K. Yoshida, Rees algebras of F-regular type, J. Algebra **247** (2002), no. 1, 191–218.

[H] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. **52**. Springer-Verlag, New York-Heidelberg, 1977.

[Kar1] I. V. Karzhemanov, Semiampleness theorem for weak log Fano varieties, Russ. Acad. Sci. Sb. Math. **197** (2006), 57–64.

[Kar2] I. V. Karzhemanov, One base point free theorem for weak log Fano threefolds, arXiv:0906.0553v6.

[Ka1] Y. Kawamata, Characterization of abelian varieties, Compositio Math. **43** (1981), no. 2, 253–276.

[Ka2] Y. Kawamata, Pluricanonical systems on minimal algebraic varieties. Invent. Math. 79 (1985), no. 3, 567—588.

[Ka3] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. Ann. of Math. 127(1988), 93-163.

[Ka4] Y. Kawamata, Abundance theorem for minimal threefolds, Invent. Math. **108** (1992), no. 2, 229–246.

[Ka5] Y. Kawamata, Log canonical models of algebraic 3-folds, Internat. J. Math. **3** (1992), 351–357.

[Ka6] Y. Kawamata, On Fujita's freeness conjecture for 3-folds and 4-folds, Math. Ann. **308** (1997), no. 3, 491–505.

[Ka7] Y. Kawamata, Subadjunction of log canonical divisors for a subvariety of codimension 2, *Birational algebraic geometry* (Baltimore, MD, 1996), 79–88, Contemp. Math., **207**, Amer. Math. Soc., Providence, RI, 1997.

[Ka8] Y. Kawamata, Subadjunction of log canonical divisors, II. Amer. J. Math. **120** (1998), no. 5, 893–899.

[Ka9] Y. Kawamata, On the abundance theorem in the case of $\nu = 0$, preprint, arXiv:1002.2682.

[KaMaMa] Y. Kawamata, K, Matsuda and K, Matsuki, *Introduction to the minimal model problem*, Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[KeMaMc] S. Keel, K. Matsuki, and J. M^{c}Kernan, Log abundance theorem for threefolds, Duke Math. J. **75** (1994), no. 1, 99–119.

[Ko1] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics.

[Ko2] J. Kollár, Kodaira's canonical bundle formula and adjunction, *Flips for 3-folds and 4-folds*, 134–162, Oxford Lecture Ser. Math. Appl., **35**, Oxford Univ. Press, Oxford, 2007.

[KoMiMo] J. Kollár, Y. Miyaoka, S. Mori, Rational connectedness and boundedness of Fano manifolds, J. Differential Geom. **36** (1992), no. 3, 765–779.

[KoKov] J. Kollár and S. J. Kovács, Log canonical singularities are Du Bois, J. Amer. Math. Soc. **23**, no. 3, 791–813.

[KoMo] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math.,134 (1998).

[KoS] S. J. Kovács and K. Schwede, Hodge theory meets the minimal model program: a survey of log canonical and Du Bois singularities, arXiv:0909.0993.

[Lai] C.-J. Lai, Varieties fibered by good minimal models, to appear in Math. Ann.

[Laza] R. Lazarsfeld, *Positivity in algebraic geometry I*, Ergeb. Math. Grenzgeb. vol. 48, Springer-Verlag, Berlin, 2004.

[Lazi] V. Lazić, Adjoint rings are finitely generated, arXiv:0905.2707.

[Leh1] B. Lehmann, On Eckl's pseudo-effective reduction map, arXiv:1103.1073v1.

[Leh2] B. Lehmann, Comparing numerical dimensions, arXiv:1103.0440v1.

[Liu] Q. Liu, *Algebraic geometry and arithmetic curves*, Translated from the French by Reinie Erné. Oxford Graduate Texts in Mathematics, **6**. Oxford Science Publications. Oxford University Press, Oxford, 2002.

[Miy] Y. Miyaoka, Relative deformations of morphisms and applications to fibre spaces, Comment. Math. Univ. St. Paul. **42** (1993), no. 1, 1–7.

[NakUe] I. Nakamura and K. Ueno, An addition formula for Kodaira dimensions of analytic fibre bundles whose fibre are Moišezon manifolds, J. Math. Soc. Japan **25** (1973), 363–371.

[Nak] N. Nakayama, *Zariski decomposition and abundance*, MSJ Memoirs, 14. Mathematical Society of Japan, Tokyo, 2004.

[Pr] Yu. G. Prokhorov, *Lectures on complements on log surfaces*, MSJ Memoirs 10 (2001).

[PrSh] Yu. G. Prokhorov, V. V. Shokurov, Towards the second main theorem on complements, J. Algebraic Geom. **18** (2009), no. 1, 151–199.

[Sak] F. Sakai, Kodaira dimensions of complements of divisors, *Complex analysis and algebraic geometry*, pp. 239–257. Iwanami Shoten, Tokyo, 1977.

[ScSm] K. Schwede, K. E. Smith, Globally F-regular and log Fano varieties, Adv. Math. **224** (2010), no. 3, 863–894.

[Sh1] V. V. Shokurov, Complements on surfaces, J. Math. Sci. **107** (2000), no. 2, 3876–3932.

[Sh2] V. V. Shokurov, 3-fold log models, Algebraic geometry, 4. J. Math. Sci. 81 (1996), no. 3, 2667–2699.

[Sh3] V. V. Shokurov, Prelimiting flips, Tr. Mat. Inst. Steklova 240 (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 82–219; translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 75–213.

[Sim] C. Simpson, Subspaces of moduli spaces of rank one local systems, Ann. Sci. École Norm. Sup. (4) 26 (1993), no. 3, 361–401.

[Siu] Y. T. Siu, Abundance conjecture, preprint, arXiv:0912.0576.

[S] K. E. Smith, Globally F-regular varieties: applications to vanishing theorems for quotients of Fano varieties, Dedicated to William Fulton on J. **48** (2000), 553–572.

[Ta] S. Takayama, On uniruled degenerations of algebraic varieties with trivial canonical divisor, Math. Z. 259 (2008), no. 3, 487–501.

[Ts] H. Tsuji, Numerically trivial fibration, math.AG/0001023.

[U] K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Lecture Notes in Math., Vol. 439, Springer, Berlin, 1975.

[W] J. A. Wiśniewski, On contractions of extremal rays of Fano manifolds, J. Reine Angew. Math. **417** (1991), 141–157.

[Y] K. Yasutake, On the classification of rank 2 almost Fano bundles on projective space, preprint (2010). arXiv:1004.2544v2

[Z] O. Zariski, The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, Ann. of Math. **76** (1962), 560–615.

[Z] Qi. Zhang, On projective manifolds with nef anticanonical bundles, J. Reine Angew. Math. **478** (1996), 57–60.