論文の内容の要旨

論文題目

Hybridized Discontinuous Galerkin Methods for Elliptic Problems
(楕円型問題に対するハイブリッド型不連続ガレルキン法の研究)

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近年の不連続ガレルキン法の発達はめざましく，様々な偏微分方程式に対して数値計算スキームが提案され，理論解析の研究もなされるようなった。不連続ガレルキン法では，自由に近似関数や要素形状が選べるという利点がある反面，その代償として，係数行列のサイズとバンド幅の増加が避けられない。その問題点を解決するために，個体力学の数値計算で用いられていた「ハイブリッド変位法」という手法と，不連続ガレルキン法とを組み合わせることを考えた。ハイブリッド変位法では，要素内部の未知関数 u_h と要素間境界上の未知関数 $\hat{u}_h$ の二種類を用いる。ある要素 K 上の未知関数 $u_h|_K$ はその周り ∂K 上の未知関数 $\hat{u}_h|_{\partial K}$ から決定されるように定式化する。そうすることで，u_h の方は消去され，最終的に残る未知関数は $\hat{u}_h$ だけになる。一般に，u_h より $\hat{u}_h$ の方が少ない未知数で済むので，u_h だけを用いる不連続ガレルキン法より出来上がる行列のサイズが小さくなる。特に，高次多項式を用いた場合にはその差が顕著に現れる。ハイブリッド変位法はそのような優れた点をもつ手法ではあったが，安定性が弱いという欠点があった。安定性を改善するために，ハイブリッド変位法に不連続ガレルキン法のペナルティ法を導入した手法が「ハイブリッド型不連続ガレルキン法 (以下 HDG 法)」である。ポアソン方程式や弾性体の問題については，HDG 法のスキームは既に提案されており，論文で理論的な誤差評価や数値計算例などが報告されている。最近では，B. Cockburn らを中心に，本論文の HDG 法とは異なるアプローチによる不連続ガレルキン法のハイブリッド化の研究もなされている。楕円型問題を始めとし，ストークス方程式やナビエ・ストークス方程式等についても，彼らの研究例がある。

本論文の第一部では，以下の移流拡散方程式に対する新たな HDG 法の定式化を提案する。

$$-\varepsilon\Delta u + \boldsymbol{b}\cdot\nabla u + cu = f \quad \text{in } \Omega, \tag{1a}$$

$$u = g_D \quad \text{on } \Gamma_D, \tag{1b}$$

$$\varepsilon\nabla u\cdot\boldsymbol{n} = g_N \quad \text{on } \Gamma_N. \tag{1c}$$

ここで，$\varepsilon > 0$ は拡散係数で，$\boldsymbol{b}, c, f, g_D, g_N$ は与えられた関数である。また，$\overline{\Gamma}_D \cup \overline{\Gamma}_N = \partial\Omega$, $\Gamma_N \cap \Gamma_D = \emptyset$ であるとし，ディリクレ境界は移流ベクトル $\boldsymbol{b}$ の流入境界であると仮定する。つまり，

$$\Gamma_- = \{x \in \partial\Omega : \boldsymbol{b}(x) \cdot \boldsymbol{n}(x) < 0\} \subset \Gamma_D$$

であるとする。$\boldsymbol{n}$ は $\partial\Omega$ 上の外向き単位法線ベクトルである。さらに，以下の条件も仮定する。

$$c(x) - \frac{1}{2}\mathrm{div}\boldsymbol{b}(x) \geq \exists\rho_0 \geq 0, \quad \forall x \in \Omega.$$

移流拡散方程式の数値計算を行うときには，移流項の近似に工夫を要する。その原因は，ε が $\boldsymbol{b}$ に比べて非常に小さい場合に，厳密解に境界層が生じる可能性があるためである。そのようなケースは特殊というわけではない。例えば，Ω を正方形領域とし，$\Gamma_D = \partial\Omega$, $\boldsymbol{b} = (1,0)^T$，$c = f = g_D = 0$, という場合，$x = 1$ の付近で境界層が現れる。この例について，従来の有限要素法をそのまま適用すると，厳密解には見られない不自然な振動が生じてしまうことが知られている。第一部で提案するHDGスキームでは，移流反応項を次のように離散化することで，そのような不都合を回避している。

$$\sum_{K \in \mathcal{T}_h} \Big[(\boldsymbol{b} \cdot \nabla u_h + c u_h, v_h)_K + \langle u_h - \hat{u}_h, \max(0, -\boldsymbol{b} \cdot \boldsymbol{n}) \cdot v_h - \max(\boldsymbol{b} \cdot \boldsymbol{n}, 0) \cdot \hat{v}_h \rangle_{\partial K} \Big].$$

第二項目は，強圧性が成立するように加えた項であるが，同時にある種の上流化の役割を果たしているため，移流が卓越する場合でも，安定性が保たれている。理論的には，区分 H^1 セミノルムに関して，最良オーダーの事前誤差評価を与えた。さらに，近似解と $\varepsilon = 0$ の場合の厳密解との関係を調べることによって，ε が極めて小さい値であっても，不安定性が生じないことの理論的な根拠を明らかにした。

第二部では，安定化にリフティング作用素を用いたHDG法について述べる。モデル問題としてポアソン方程式を対象とした。元々は，不連続ガレルキン法で導入されていたリフティング作用素のアイデアを，HDG法にも取り入れ，ハイブリッド法に合うように新たに定義した。リフティング作用素による安定化項をオリジナルのHDG法に付け加えることによって，任意の正のペナルティパラメータについて，安定性が成り立つスキームが得られる。リフティング作用素を用いないオリジナルのHDG法では，ペナルティパラメータが小さすぎると，不安定性を示すことがあるので，「任意の正の値」について安定性が保証されることは重要である。ただし，リフティング作用素による安定化項を加えることは，係数行列のバンド幅と条件数を増加させることになり，反復解法の収束性に悪影響を及ぼす可能性が考えられる。それを調査するため，小さなペナルティパラメータについて数値実験を実施し，反復法の収束速度を比較した。理論的な誤差評価としては，区分 H^1 セミノルム及び L^2 ノルムに関して，最良オーダーであることを示した。

Hybridized Discontinuous Galerkin Methods for Elliptic Problems

楕円型問題に対するハイブリッド型不連続ガレルキン法の研究

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Part I
Hybridized Discontinuous Galerkin Method for Convection-Diffusion-Reaction Problems

1 Introduction

The discontinuous Galerkin method (DGM) is now widely applied to various problems in science and engineering because of its flexibility for the choice of approximate functions and of element shapes. An issue of DGM is, however, the size and band-widths of the resulting matrices could be much larger than those of the standard finite element method, since the DGM is formulated in terms of the usual node values defined in each elements together with those corresponding to inter-element discontinuities. In order to surmount this difficulty, it is worth-while trying to extend the idea of DGM by combining with the hybrid displacement method (see, for example, [8], [9], [10] and [11]). Thus, we introduce new unknown functions on inter-element edges. We can then obtain a formulation that the resulting discrete system contains inter-element unknowns only and, consequently, the size of the system becomes smaller. Recently, in [12], [13] and [14], the author and his colleagues proposed and analyzed a new class of DGM, the *hybridized DGM*, that is based on the hybrid displacement approach by stabilizing their old method ([10] and [11]). In [12], we examined our idea by using a linear elasticity problem as a model problem and offered several numerical examples to confirm the validity of our formulation. After that, we carried out theoretical analysis by using the Poisson equation as a model problem. In [14], stability and convergence of symmetric and nonsymmetric interior penalty methods of hybrid type were studied. The usefulness of the lifting operator in order to ensure a better stability was studied in [14]. Furthermore, B. Cockburn and his colleagues are actively contributing to the hybridized DGM for elliptic([18], [15] and [19]), Stokes and Navier-Stokes problems ([20], [21] and [22]).

The purpose of this paper is to propose a new hybridized DGM for stationary convection-diffusion-reaction problems with mixed boundary conditions. In [17], Cockburn et al. proposed hybridization for the diffusion-convection-reaction problems. The stability of their method is achieved by choosing stabilization parameters according to the convection. They reported a lot of numerical results and confirmed the validity of their schemes. However, error analysis seems to be not undertaken. The scheme we are going to propose is close to the original DGM for convection-diffusion problem ([3] and [4]) and is based on a certain upwinding technique. As is well-known, there are a lot of methods of upwinding. Our method, however, differs form any previous methods. For example, our upwinding technique does not need information on neighboring elements, whereas most of upwinding use information upwind elements. Instead, our upwinding method is introduced in terms of neighboring edges. To be more specific, we find a hybridized approximation to convection and reaction terms in the following form

$$\sum_{K\in\mathcal{T}_h}[(\boldsymbol{b}\cdot\nabla u_h + cu_h,\ v_h)_K + \langle u_h-\hat{u}_h, \alpha v_h - \beta\hat{v}_h\rangle_{\partial K}], \tag{1.1}$$

where coefficients α and β are decided to satisfy coercivity, as it will be shown later. Moreover, our proposed scheme is stable even when ε is sufficiently small and it can be applied to the case $\varepsilon = 0$. We furthermore give stability and optimal order error estimates.

Now let us formulate our continuous problem. Let Ω be a bounded polyhedral domain in R^n. In this paper, we propose a new hybridized discontinuous Galerkin method for the convection-diffusion-reaction problems with mixed boundary conditions:

$$-\varepsilon\Delta u + \boldsymbol{b}\cdot\nabla u + cu = f \text{ in } \Omega, \tag{1.2a}$$

$$u = g_D \text{ on } \Gamma_D, \tag{1.2b}$$

$$\varepsilon\nabla u\cdot\boldsymbol{n} = g_N \text{ on } \Gamma_N, \tag{1.2c}$$

where $\varepsilon > 0$ is the diffusion coefficient and $f \in L^2(\Omega)$, $\boldsymbol{b} \in W^{1,\infty}(\Omega)^n$, $c \in L^\infty(\Omega)$, $g_D \in H^{3/2}(\Omega)$, and $g_N \in H^{1/2}(\Omega)$ are given functions. We assume $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$, and that the inflow boundary is included by Γ_D, i.e.,

$$\Gamma^- := \{x\in\partial\Omega : \boldsymbol{b}(x)\cdot\boldsymbol{n}(x) < 0\} \subset \Gamma_D,$$

where $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$. Moreover, we assume that there exists a non-negative constant ρ_0 such that

$$\rho(x) := c(x) - \frac{1}{2}\mathrm{div}\boldsymbol{b}(x) \geq \rho_0 \geq 0, \quad \forall x\in\Omega. \tag{1.3}$$

Under these assumptions, there exists a unique weak solution $u \in H^1(\Omega)$ by the Lax-Milgram theory. We shall pose further regularity on u in the error analysis. This paper is organized as follows. In Section 2, we introduce finite element spaces to describe our method, and norms and projections to use in our error analysis. Section 3 is devoted to the formulation of our hybridized method, and mathematical analysis is given in Section 4. We explain why our proposed DGM is stable even when ε is close to 0 in Section 5. In Section 6, we report some results of numerical computations. Finally, we conclude this paper in Section 7.

2 Preliminaries

2.1 Notation

Function spaces and norms Let $\mathcal{T}_h = \{K_i\}_i$ be a triangulation of Ω in the sense of [13]. Thus, each $K \in \mathcal{T}_h$ is an m-polyhedral domain, where m denotes an integer $m \geq n+1$. The boundary ∂K of $K \in \mathcal{T}_h$ is composed of m-faces. We assume that m is bounded from above independently a family of triangulations $\{\mathcal{T}_h\}_h$, and ∂K does not intersect with itself. Furthermore, we set $h = \max_{K \in \mathcal{T}_h} h_K$, where h_K denotes the diameter of K. The *skeleton* of $\mathcal{T}_h$ is defined by

$$\Gamma_h := \bigcup_{K \in \mathcal{T}_h} \partial K.$$

We use the broken Sobolev space over $\mathcal{T}_h$ defined by

$$H^k(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K \in H^k(K)\},$$

and L^2-spaces on Γ_h as follows

$$\begin{aligned} L^2_D(\Gamma_h) &= \{\hat{v} \in L^2(\Gamma_h) : \hat{v}|_{\Gamma_D} = g_D, \hat{v}|_{\Gamma_N} = 0\}, \\ L^2_0(\Gamma_h) &= \{\hat{v} \in L^2(\Gamma_h) : \hat{v}|_{\Gamma_D} = 0, \hat{v}|_{\Gamma_N} = 0\}. \end{aligned}$$

Then, we set $\boldsymbol{V} = H^2(\mathcal{T}_h) \times L^2_D(\Gamma_h)$ and $\boldsymbol{V}_0 = H^2(\mathcal{T}_h) \times L^2_0(\Gamma_h)$. The inner products are defined as follows

$$(u, v)_K = \int_K uv dx, \qquad \langle \hat{u}, \hat{v} \rangle_e = \int_e \hat{u}\hat{v} ds,$$

for an element K and an edge e, respectively. Let $\|\cdot\|_m$ and $|\cdot|_m$ be the usual Sobolev norms and seminorms. We introduce auxiliary seminorms:

$$|v|^2_{m,h} := \sum_{K \in \mathcal{T}_h} h_K^{2(m-1)} |v|^2_{m,K} \quad \text{for } v \in H^m(\mathcal{T}_h),$$

$$|\boldsymbol{v}|^2_{j,h} := \sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K} \left\| \sqrt{\frac{\eta_e}{h_e}} (v - \hat{v}) \right\|^2_{0,e} \quad \text{for } \boldsymbol{v} \in \boldsymbol{V},$$

where h_K is the diameter of K and h_e is the length of e. For error analysis, we define the *HDG norm* defined by

$$
\begin{aligned}
|||\boldsymbol{v}|||^2 &:= |||\boldsymbol{v}|||_d^2 + |||\boldsymbol{v}|||_{rc}^2, \\
|||\boldsymbol{v}|||_d^2 &:= \varepsilon \left(|v|_{1,h}^2 + |v|_{2,h}^2 + |\boldsymbol{v}|_{j,h}^2 \right), \\
|||\boldsymbol{v}|||_{rc}^2 &:= \frac{1}{2} \sum_{K \in \mathcal{T}_h} \| |\boldsymbol{b} \cdot \boldsymbol{n}_K|^{1/2} (v - \hat{v}) \|_{0,\partial K}^2 + \rho_0 \|v\|_{0,\Omega}^2,
\end{aligned}
$$

where $\boldsymbol{n}_K$ is the unit outward normal vector to ∂K and ρ_0 is the positive constant defined in (1.3).

Finite element spaces Let U_h and $\hat{U}_h$ be finite dimensional spaces of $H^2(\mathcal{T}_h)$ and of $L_D^2(\Gamma_h)$, respectively. Then we set $\boldsymbol{V}_h := U_h \times \hat{U}_h$, which is included by $\boldsymbol{V}$. Similarly, we define $\boldsymbol{V}_{0h} := U_h \times \hat{U}_{0h} \subset \boldsymbol{V}_0$. In this paper, we assume (H1) $\nabla v_h \in [U_h]^n$ for all $v_h \in U_h$ and that (H2) U_h includes the piecewise constant functions $\mathcal{P}^0(\mathcal{T}_h)$. For example, we can use polynomials of degree k as U_h or $\hat{U}_h$.

Projections Let P_h denote the L^2-projection from $H^2(\mathcal{T}_h)$ onto U_h, and let $\hat{P}_h$ denote the L^2-projection from $L_D^2(\Gamma_h)$ onto $\hat{U}_h$. We define $\boldsymbol{P}_h : \boldsymbol{V} \to \boldsymbol{V}_h$ by $\boldsymbol{P}_h \boldsymbol{v} := \{P_h v, \hat{P}_h \hat{v}\}$. We introduce the L^2-projection $\boldsymbol{P}_h^0 : W^{1,\infty}(\Omega) \to \mathcal{P}^0(\mathcal{T}_h)^n$. We also use the projection $\tilde{\cdot}$ defined by $\tilde{\boldsymbol{v}} := \{v, \hat{v}|_{\Gamma_h \setminus \partial\Omega} + \hat{P}_h v|_{\partial\Omega}\}$, which affects only on $\partial\Omega$. In this paper we assume the approximate properties (H3): for all $v \in H^{k+1}(K)$, we have

$$
|v - P_h v|_{i,K} \le C h^{k+1-i} |v|_{k+1,K} \quad \text{for } i = 0, 1, \tag{2.1}
$$

$$
\|v - \hat{P}_h(v|_e)\|_{0,e} \le C h^{k+1/2} |v|_{k+1,K}. \tag{2.2}
$$

Remark Throughout this paper, a boldface lowercase letters except $\boldsymbol{b}$ and $\boldsymbol{n}$ denotes a function of $\boldsymbol{V}$, i.e., $\boldsymbol{v}$ indicates $\{v, \hat{v}\} \in \boldsymbol{V}$. Moreover, the symbol C denotes a generic constant.

2.2 Inequalities

Theorem 2.1. Let $K \in \mathcal{T}_h$ and e be an edge of K.

1. (Trace inequality) There exists a constant C independent of K and e such that

$$
\|v\|_{0,e} \le C h_e^{-1/2} \left(\|v\|_{0,K}^2 + h_K^2 |v|_{1,K}^2 \right)^{1/2} \qquad \forall v \in H^1(K). \tag{2.3}
$$

2. (Inverse inequality) There exists a constant C independent of K such that

$$|v_h|_{1,K} \le C h_K^{-1} \|v_h\|_{0,K} \qquad \forall v_h \in U_h. \tag{2.4}$$

Proof. See [5, p.745]. □

Lemma 2.2. Assume (H3). Let $v \in H^{k+1}$ and $\boldsymbol{v} = \{v, v|_{\Gamma_h}\}$. Then we have

$$\begin{aligned} |||\boldsymbol{v} - \boldsymbol{P}_h\boldsymbol{v}|||_d &\le C\varepsilon^{1/2} h^k |v|_{k+1}, \\ |||\boldsymbol{v} - \boldsymbol{P}_h\boldsymbol{v}|||_{rc} &\le C h^{k+1/2} |v|_{k+1}. \end{aligned}$$

Proof. This follows immediately from the definitions. □

3 A new hybridized DGM

We are able to state a new hybridized DGM, which we propose in this paper. We first state our formulation: Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$B_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = (f, v_h)_\Omega + \langle g_N, v_h \rangle_{\Gamma_N} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.1}$$

where

$$B_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := B_h^d(\boldsymbol{u}_h, \boldsymbol{v}_h) + B_h^{rc}(\boldsymbol{u}_h, \boldsymbol{v}_h), \tag{3.2}$$

$$\begin{aligned} B_h^d(\boldsymbol{u}_h, \boldsymbol{v}_h) = \varepsilon \sum_{K \in \mathcal{T}_h} \Bigg[&(\nabla u_h, \nabla v_h)_K - \left\langle \frac{\partial u_h}{\partial n}, v_h - \hat{v}_h \right\rangle_{\partial K} \\ &- \left\langle \frac{\partial v_h}{\partial n}, u_h - \hat{u}_h \right\rangle_{\partial K} + \sum_{e \subset \partial K} \frac{\eta_e}{h_e} \langle u_h - \hat{u}_h, v_h - \hat{v}_h \rangle_e \Bigg], \end{aligned} \tag{3.3}$$

$$\begin{aligned} B_h^{rc}(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{K \in \mathcal{T}_h} \Bigg[&(\boldsymbol{b} \cdot \nabla u_h + c u_h, v_h)_K \\ &+ \langle u_h - \hat{u}_h, [\boldsymbol{b} \cdot \boldsymbol{n}]_- v_h - [\boldsymbol{b} \cdot \boldsymbol{n}]_+ \hat{v}_h \rangle_{\partial K} \Bigg], \end{aligned} \tag{3.4}$$

$$(f, v_h)_\Omega = \int_\Omega f v_h dx, \tag{3.5}$$

$$\langle g_N, v_h \rangle_{\Gamma_N} = \int_{\Gamma_N} g_N v_h ds. \tag{3.6}$$

Here η_e is a penalty parameter with $\eta_e \geq \eta_{min} > 0$, h_e is the length of an edge e, and the functions $[\,\cdot\,]_+$ and $[\,\cdot\,]_-$ are defined by

$$[x]_+ = \max(0, x), \quad [x]_- = \max(0, -x). \tag{3.7}$$

Note that $[x]_+ + [x]_- = |x|$ and $[x]_+ - [x]_- = x$.

Before proceeding to the analysis of (3.1), we state the derivation of it. Multiplying the both sides of (1.2) by a test function $v_h \in \boldsymbol{V}_h$ and integrating the both sides over Ω, we have, by integration by parts,

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \left[(\nabla u_h, \nabla v_h)_K - \left\langle \frac{\partial u_h}{\partial n}, v_h \right\rangle_{\partial K} + (\boldsymbol{b}\cdot\nabla u_h + c u_h, v_h)_K \right] \\ = (f, v_h)_\Omega + \langle g_N, v_h\rangle_{\Gamma_N} \qquad \boldsymbol{v}_h \in \boldsymbol{V}_h.\end{aligned} \tag{3.8}$$

We denote the diffusion part and convection part in (3.8) by D and C, respectively, i.e.,

$$D(u_h, v_h) := \sum_{K\in\mathcal{T}_h} \left[(\nabla u_h, \nabla v_h)_K - \left\langle \frac{\partial u_h}{\partial n}, v_h \right\rangle_{\partial K} \right], \tag{3.9}$$

$$C(u_h, v_h) := \sum_{K\in\mathcal{T}_h} (\boldsymbol{b}\cdot\nabla u_h + c u_h, v_h)_K. \tag{3.10}$$

We first derive our formulation of the diffusion part. From the continuity of the flux, we have

$$\sum_{K\in\mathcal{T}_h} \left\langle \frac{\partial u_h}{\partial n}, \hat{v}_h \right\rangle_{\partial K} = 0. \tag{3.11}$$

Adding (3.11) to (3.9) yields

$$D(u_h, v_h) = \sum_{K\in\mathcal{T}_h} \left[(\nabla u_h, \nabla v_h)_K - \left\langle \frac{\partial u_h}{\partial n}, v_h - \hat{v}_h \right\rangle_{\partial K} \right]. \tag{3.12}$$

Symmetrizing (3.12) and adding a penalty term

$$\sum_{K\in\mathcal{T}_h} \sum_{e\subset\partial K} \left\langle \frac{\eta_e}{h_e}(u_h - \hat{u}_h), v_h - \hat{v}_h \right\rangle_{\partial K}, \tag{3.13}$$

we obtain (3.3).

Next, we derive the formulation of the convection part. Let α and β be coefficients to be determined later, and consider the following form:

$$C_h(u_h, v_h) := \sum_{K\in\mathcal{T}_h} \left[(\boldsymbol{b}\cdot\nabla u_h + cu_h, v_h)_K + \langle u_h - \hat{u}_h, \alpha v_h - \beta\hat{v}_h\rangle_{\partial K}\right]. \tag{3.14}$$

The coefficients α and β are chosen so that

$$\begin{aligned} C_h(v_h, v_h) \quad &= \sum_{K\in\mathcal{T}_h} \Big[(\rho v_h, v_h)_K + \langle((\boldsymbol{b}\cdot\boldsymbol{n})/2 + \alpha)\, v_h, v_h\rangle_{\partial K} \\ &\qquad - \langle(\alpha+\beta)v_h, \hat{v}_h\rangle_{\partial K} + \langle\beta\hat{v}_h, \hat{v}_h\rangle_{\partial K}\Big] \geq |||\boldsymbol{v}|||_{rc} \end{aligned} \tag{3.15}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$. We can find the following sufficient conditions

$$\alpha + \beta = 2((\boldsymbol{b}\cdot\boldsymbol{n})/2 + \alpha) = |\boldsymbol{b}\cdot\boldsymbol{n}|,$$

from which it follows that

$$\alpha = [\boldsymbol{b}\cdot\boldsymbol{n}]_-, \quad \beta = [\boldsymbol{b}\cdot\boldsymbol{n}]_+.$$

Thus we obtain our formulation (3.1).

4 Error analysis

In this section, we shall establish an error estimates for (3.1).

Lemma 4.1. The following hold.

1. (Boundedness) There exists a constant $C_b^d > 0$ such that

$$|B_h^d(\boldsymbol{w}, \boldsymbol{v})| \leq C_b^d\, |||\boldsymbol{w}|||_d\, |||\boldsymbol{v}|||_d \quad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{V}. \tag{4.1}$$

2. (Coercivity) There exists a constant $C_c^d > 0$ such that

$$B_h^d(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq C_c^d\, |||\boldsymbol{v}_h|||_d^2 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.2}$$

Proof. We first prove the boundedness. Applying the Schwarz inequality for each term of (3.3), we have

$$
\begin{aligned}
|B_h^d(\boldsymbol{w},\boldsymbol{v})| \;\le\; & \varepsilon \sum_{K\in\mathcal{T}_h} \Bigg[\|\nabla w\|_{0,K}\|\nabla v\|_{0,K} \\
& + \sum_{e\subset\partial K} \left(\left\|\frac{\partial w}{\partial n}\right\|_{0,e} \|v-\hat{v}\|_{0,e} + \left\|\frac{\partial v}{\partial n}\right\|_{0,e} \|w-\hat{w}\|_{0,e} \right. \\
& \left. + \left\|\sqrt{\frac{\eta_e}{h_e}}(w-\hat{w})\right\|_{0,e} \left\|\sqrt{\frac{\eta_e}{h_e}}(v-\hat{v})\right\|_{0,e} \right) \Bigg].
\end{aligned} \tag{4.3}
$$

By the trace theorem, we have

$$
\left\|\frac{\partial w}{\partial n}\right\|_{0,e} \le C h_e^{-1/2}(|w|_{1,K}^2 + h_K^2|w|_{2,K}^2)^{1/2}. \tag{4.4}
$$

From (4.3), (4.4), and the Cauchy-Schwarz inequality, it follows that

$$
|B_h^d(\boldsymbol{w},\boldsymbol{v})| \le \max(1 + C\eta_{min}^{-1/2}, 2)\, |||\boldsymbol{w}|||_d \, |||\boldsymbol{v}|||_d \,.
$$

Next, we prove the coercivity. By definition,

$$
B_h^d(\boldsymbol{v}_h,\boldsymbol{v}_h) \;\ge\; |v_h|_{1,h}^2 - 2\sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K} \left\|\frac{\partial v_h}{\partial n}\right\|_{0,e} \|v_h-\hat{v}_h\|_{0,e} + |\boldsymbol{v}_h|_{j,h}^2 \tag{4.5}
$$

By the trace theorem, the inverse inequality and the Young inequality, we have for $\delta \in (0,1)$,

$$
\begin{aligned}
2\left\|\frac{\partial v_h}{\partial n}\right\|_{0,e} \|v_h-\hat{v}_h\|_{0,e} \;\le\; & \frac{2C}{h_e}|v_h|_{1,K}\|v_h-\hat{v}_h\|_{0,e} \\
\le\; & \frac{C}{\delta\eta_e}|v_h|_{1,K}^2 + \delta\left\|\sqrt{\frac{\eta_e}{h_e}}(v_h-\hat{v}_h)\right\|_{0,e}^2 \qquad \forall v_h \in U_h.
\end{aligned} \tag{4.6}
$$

From (4.5) and (4.6), we obtain

$$
B_h^d(\boldsymbol{v}_h,\boldsymbol{v}_h) \ge \left(1 - \frac{C}{\delta\eta_{min}}\right)|v_h|_{1,h}^2 + (1-\delta)\,|\boldsymbol{v}_h|_{j,h}^2, \tag{4.7}
$$

If $\eta_{min} > 4C$, then we can take $\delta = \sqrt{C/\eta_{min}} < 1/2$, which implies that

$$
1 - \frac{C}{\delta\eta_{min}} > 1/2, \quad 1-\delta > 1/2.
$$

Hence we have

$$B_h^d(\boldsymbol{v}_h, \boldsymbol{v}_h) \;\geq\; \frac{1}{2}(|v_h|_{1,h}^2 + |\boldsymbol{v}_h|_{j,h}^2). =: \frac{1}{2}\,|||\boldsymbol{v}_h|||_{d,h}^2\,.$$

Since the norms $|||\cdot|||_d$ and $|||\cdot|||_{d,h}$ are equivalent each other over $\boldsymbol{V}_h$, we obtain the coercivity (4.2) . □

Lemma 4.2. The following hold.

1. There exists a constant $C_b^{rc} > 0$ such that for all $\boldsymbol{v} \in \boldsymbol{V}$, $\boldsymbol{w}_h \in \boldsymbol{V}_h$,

$$|B_h^{rc}(\tilde{\boldsymbol{v}} - \boldsymbol{P}_h\boldsymbol{v}, \boldsymbol{w}_h)| \leq C_b^{rc} |||\tilde{\boldsymbol{v}} - \boldsymbol{P}_h\boldsymbol{v}|||_{rc} |||\boldsymbol{w}_h|||_{rc}$$

2. (Coercivity) There exists a positive constant $C_c^{rc} > 0$ such that

$$B_h^{rc}(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq C_c^{rc} |||\boldsymbol{v}_h|||_{rc}^2 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_{0h}.$$

Proof. For the proof of (1), we first show the following equality:

$$\begin{aligned} B_h^{rc}(\boldsymbol{v}_h, \boldsymbol{w}_h) &= \sum_{K\in\mathcal{T}_h} \Big(-(v_h, \boldsymbol{b}\cdot\nabla w_h)_K + ((c - \mathrm{div}\boldsymbol{b})v_h, w_h)_K \\ &\qquad + \langle [\boldsymbol{b}\cdot\boldsymbol{n}]_+ v_n - [\boldsymbol{b}\cdot\boldsymbol{n}]_- \hat{v}_h, w_h - \hat{w}_h\rangle_{\partial K} \Big). \end{aligned} \tag{4.8}$$

By Green's formula,

$$\begin{aligned} B_h^{rc}(\boldsymbol{v}_h, \boldsymbol{w}_h) &= \sum_{K\in\mathcal{T}_h} (v_h, -\boldsymbol{b}\cdot\nabla w_h)_K + ((c - \mathrm{div}\boldsymbol{b})v_h, w_h)_K \\ &\qquad + \langle(\boldsymbol{b}\cdot\boldsymbol{n})v_h, w_h\rangle_{\partial K} + \langle v_n - \hat{v}_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_- w_h - [\boldsymbol{b}\cdot\boldsymbol{n}]_+ \hat{w}_h\rangle_{\partial K} \\ &=: \sum_{K\in\mathcal{T}_h} (I_K + II_K + III_{\partial K}). \end{aligned}$$

Rewrite $III_{\partial K}$ as follows:

$$\begin{aligned} III_{\partial K} &= \langle([\boldsymbol{b}\cdot\boldsymbol{n}]_+ - [\boldsymbol{b}\cdot\boldsymbol{n}]_-)v_h, w_h\rangle_{\partial K} + \langle v_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_- w_h\rangle_{\partial K} \\ &\quad - \langle \hat{v}_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_- w_h\rangle_{\partial K} - \langle v_h - \hat{v}_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_+ \hat{w}_h\rangle_{\partial K} \\ &= \langle([\boldsymbol{b}\cdot\boldsymbol{n}]_+ v_h - [\boldsymbol{b}\cdot\boldsymbol{n}]_- \hat{v}_h, w_h\rangle_{\partial K} - \langle v_h - \hat{v}_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_+ \hat{w}_h\rangle_{\partial K}. \end{aligned}$$

Since

$$\sum_{K\in\mathcal{T}_h} \langle \hat{v}_h, ([\boldsymbol{b}\cdot\boldsymbol{n}]_+ - [\boldsymbol{b}\cdot\boldsymbol{n}]_-)\hat{w}_h\rangle_{\partial K} = 0,$$

we have

$$\sum_{K\in\mathcal{T}_h} III_{\partial K} = \sum_{K\in\mathcal{T}_h} \langle [\boldsymbol{b}\cdot\boldsymbol{n}]_+ v_h - [\boldsymbol{b}\cdot\boldsymbol{n}]_- \hat{v}_h, w_h - \hat{w}_h\rangle_{\partial K}.$$

Thus we obtain (4.8). Next, we will estimate I_K. Let us denote $\boldsymbol{\eta} = \{\eta, \hat{\eta}\} := \tilde{\boldsymbol{u}} - \boldsymbol{P}_h \boldsymbol{u}$, then

$$\begin{aligned} I_K &= (\boldsymbol{\eta}, -\boldsymbol{b} \cdot \nabla w_h)_K \\ &= (\boldsymbol{\eta}, (\boldsymbol{P}_h^0 \boldsymbol{b} - \boldsymbol{b}) \cdot \nabla w_h)_K - (\boldsymbol{\eta}, (\boldsymbol{P}_h^0 \boldsymbol{b}) \cdot \nabla w_h)_K. \end{aligned}$$

By the property of the projection $\boldsymbol{P}_h^0$ and $\boldsymbol{P}_h \boldsymbol{u} = \boldsymbol{P}_h \tilde{\boldsymbol{u}}$, we have

$$(\boldsymbol{\eta}, (\boldsymbol{P}_h^0 \boldsymbol{b}) \cdot \nabla w_h)_K = 0.$$

Using the inverse inequality, we see that

$$|I_K| = |(\boldsymbol{\eta}, (\boldsymbol{P}_h^0 \boldsymbol{b} - \boldsymbol{b}) \cdot \nabla w_h)_K| \leq C |\boldsymbol{b}|_{1,\infty} \|\eta\|_{0,K} \|w_h\|_{0,K}. \tag{4.9}$$

By using the Schwarz inequality, we have

$$|II_K| \leq C(|c|_{0,\infty} + |\boldsymbol{b}|_{1,K}) \|v_h\|_{0,K} \|w_h\|_{0,K}, \tag{4.10}$$

and

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} |III_{\partial K}| &\leq \sum_{K \in \mathcal{T}_h} \langle |\boldsymbol{b} \cdot \boldsymbol{n}| (v_h - \hat{v}_h), w_h - \hat{w}_h \rangle_{\partial K} \\ &\leq \sum_{K \in \mathcal{T}_h} \| |\boldsymbol{b} \cdot \boldsymbol{n}|^{1/2} (v_h - \hat{v}_h) \|_{0,\partial K} \cdot \| |\boldsymbol{b} \cdot \boldsymbol{n}|^{1/2} (v_h - \hat{v}_h) \|_{0,\partial K} \end{aligned} \tag{4.11}$$

From (4.9), (4.10), and (4.11), we conclude that (4.8) holds.

We now turn to the proof of (2). By Green's formula, we have

$$\begin{aligned} B_h^{rc}(\boldsymbol{v}_h, \boldsymbol{v}_h) &= \sum_{K \in \mathcal{T}_h} \Big(\int_K (c - \mathrm{div}\boldsymbol{b}/2) v_h^2 dx + \frac{1}{2} \int_{\partial K} (\boldsymbol{b} \cdot \boldsymbol{n}) v_h^2 dx \\ &\qquad + \langle v_h - \hat{v}_h, [\boldsymbol{b} \cdot \boldsymbol{n}]_- v_h - [\boldsymbol{b} \cdot \boldsymbol{n}]_+ \hat{v}_h \rangle_{\partial K} \Big) \\ &=: \sum_{K \in \mathcal{T}_h} (I_K + II_{\partial K} + III_{\partial K}). \end{aligned}$$

By the assumption (1.3), we have

$$I_K \geq \rho_0 \|v_h\|_{0,K}^2.$$

Since $\boldsymbol{b}\cdot\boldsymbol{n} = [\boldsymbol{b}\cdot\boldsymbol{n}]_+ - [\boldsymbol{b}\cdot\boldsymbol{n}]_-$, we have

$$\begin{aligned} II_{\partial K} + III_{\partial K} &= \frac{1}{2}\langle v_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_- v_h\rangle_{\partial K} - \langle \hat{v}_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_- v_h\rangle_{\partial K} \\ &\quad + \frac{1}{2}\langle v_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_+ v_h\rangle_{\partial K} - \langle v_h, [\boldsymbol{b}\cdot\boldsymbol{n}]_+ \hat{v}_h\rangle_{\partial K} \\ &= \frac{1}{2}\langle([\boldsymbol{b}\cdot\boldsymbol{n}]_- + [\boldsymbol{b}\cdot\boldsymbol{n}]_-)(v_h - \hat{v}_h), v_h - \hat{v}_h\rangle_{\partial K} \\ &\quad + \frac{1}{2}\langle([\boldsymbol{b}\cdot\boldsymbol{n}]_+ - [\boldsymbol{b}\cdot\boldsymbol{n}]_-)\hat{v}_h, \hat{v}_h\rangle_{\partial K} \\ &= \frac{1}{2}\langle|\boldsymbol{b}\cdot\boldsymbol{n}|(v_h - \hat{v}_h), v_h - \hat{v}_h\rangle_{\partial K} + \frac{1}{2}\langle(\boldsymbol{b}\cdot\boldsymbol{n})\hat{v}_h, \hat{v}_h\rangle_{\partial K}. \end{aligned} \tag{4.12}$$

Since

$$\sum_{K\in\mathcal{T}_h} \langle(\boldsymbol{b}\cdot\boldsymbol{n})\hat{v}_h, \hat{v}_h\rangle_{\partial K} = 0,$$

summing (4.12) over all elements $K \in \mathcal{T}_h$ gives us

$$\sum_{K\in\mathcal{T}_h} (II_{\partial K} + III_{\partial K}) = \frac{1}{2}\sum_{K\in\mathcal{T}_h} \||\boldsymbol{b}\cdot\boldsymbol{n}|^{1/2}(v_h - \hat{v}_h)\|^2_{0,\partial K}.$$

Thus we obtain the coercivity (4.8). □

From Lemma 4.1 and Lemma 4.2, we get the following lemma

Lemma 4.3. We have the following three properties.

1. (Galerkin orthogonality) Let u be the exact solution of (1.2), and let $\boldsymbol{u} = \{u, u|_{\Gamma_h}\}$. Let $\boldsymbol{u}_h$ be the approximate solution by (3.1). Then we have

$$B_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

2. There exists a constant C_b independent of h and ε such that

$$|B_h(\boldsymbol{v} - \boldsymbol{P}_h\boldsymbol{v}, \boldsymbol{w}_h)| \leq C_b \,|||\boldsymbol{v} - \boldsymbol{P}_h\boldsymbol{v}|||\, |||\boldsymbol{w}_h||| \qquad \boldsymbol{v} \in \boldsymbol{V}, \boldsymbol{w}_h \in \boldsymbol{V}_h.$$

3. (Coercivity) There exists a constant C_c independent of h and ε such that

$$B_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq C_c \,|||\boldsymbol{v}_h|||^2 \qquad \boldsymbol{v}_h \in \boldsymbol{V}_{0h}. \tag{4.13}$$

Theorem 4.4. Let u be the exact solution of (1.2), and let $\boldsymbol{u} = \{u, u|_{\Gamma_h}\}$. Let $\boldsymbol{u}_h$ be the approximate solution by (3.1). Recall that we are assuming (H1), (H2), and (H3). If $u \in H^{k+1}(\Omega)$ then we have the following error estimate:

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq C(\varepsilon^{1/2} + h^{1/2})h^k |u|_{k+1}, \tag{4.14}$$

where C denotes a positive constant independent of h and ε.

Proof. By using the three properties in Lemma 4.3, we deduce that

$$\begin{aligned} C_c \,|||\boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}|||^2 &\leq B_h(\boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}) \\ &= B_h(\boldsymbol{u} - \boldsymbol{P}_h\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}) \\ &= B_h(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}) + B_h(\tilde{\boldsymbol{u}} - \boldsymbol{P}_h\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}) \\ &\leq C_b(|||\boldsymbol{u} - \tilde{\boldsymbol{u}}||| + |||\tilde{\boldsymbol{u}} - \boldsymbol{P}_h\boldsymbol{u}|||)\, |||\boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}|||\,. \end{aligned}$$

Hence we have

$$|||\boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}||| \leq C(|||\boldsymbol{u} - \tilde{\boldsymbol{u}}||| + |||\boldsymbol{u} - \boldsymbol{P}_h\boldsymbol{u}|||).$$

By the triangle inequality and Lemma 2.2, we have

$$\begin{aligned} |||\boldsymbol{u} - \boldsymbol{u}_h||| &\leq |||\boldsymbol{u}_h - \boldsymbol{P}_h\boldsymbol{u}||| + |||\boldsymbol{u} - \boldsymbol{P}_h\boldsymbol{u}||| \\ &\leq (C+1)\,|||\boldsymbol{u} - \boldsymbol{P}_h\boldsymbol{u}||| + C\,|||\boldsymbol{u} - \tilde{\boldsymbol{u}}||| \\ &\leq C(\varepsilon^{1/2} + h^{1/2})h^k |u|_{k+1}. \end{aligned}$$

Thus, the proof is completed. □

5 The relation between u_h and the solution of the reduced problem

Let u_0 be the solution of the reduced problem of (1.2) :

$$\boldsymbol{b} \cdot \nabla u_0 + c u_0 = f \text{ in } \Omega, \tag{5.1a}$$

$$u_0 = g_D \text{ on } \Gamma_D. \tag{5.1b}$$

Here we assume that $\Gamma_D = \Gamma_-$ and $g_N \equiv 0$, and suppose that the unique existence of a solution $u_0 \in H^2(\Omega)$ to (5.1). Let $\boldsymbol{u}_0 = \{u_0, u_0|_{\Gamma_h}\}$. The aim of this section is to show the approximate solution $\boldsymbol{u}_h$ is also close to $\boldsymbol{u}_0$ when ε is very small. This suggests that our hybridized DG method (3.1) is stable even when ε is sufficiently small.

Theorem 5.1. Let $\boldsymbol{u}_h$ be the approximate solution defined by (3.1), and let $\tilde{\boldsymbol{u}}_0$ be defined as above. Then we have the following inequality:

$$|||\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_0||| \le C\left(|||\tilde{\boldsymbol{u}}_0|||_d + |||\tilde{\boldsymbol{u}}_0 - \boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||\right), \tag{5.2}$$

where C is a constant independent of ε and h.

Proof. By the consistency of $B_h^{rc}(\cdot,\cdot)$, we have

$$B_h^{rc}(\boldsymbol{u}_0, \boldsymbol{v}_h) = (f, v_h), \tag{5.3}$$

from which it follows that

$$B_h(\tilde{\boldsymbol{u}}_0, \boldsymbol{v}_h) = (f, v_h) + B_h^d(\boldsymbol{u}_0, \boldsymbol{v}_h) + B_h(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h). \tag{5.4}$$

Subtracting (5.4) from (3.1) gives us

$$B_h(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_h, \boldsymbol{v}_h) = B_h^d(\boldsymbol{u}_h^0, \boldsymbol{v}_h) + B_h(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{5.5}$$

Here we claim that

$$B_h(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h) = B_h^d(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h). \tag{5.6}$$

In fact, we have

$$\begin{aligned} B_h(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h) &= B_h^d(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h) + B_h^{rc}(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h) \\ &= B_h^d(\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0, \boldsymbol{v}_h) \\ &\quad + \left\langle \hat{P}_h(u_0|_{\partial\Omega}) - u_0|_{\partial\Omega}, [\boldsymbol{b}\cdot\boldsymbol{n}]_- v_h - [\boldsymbol{b}\cdot\boldsymbol{n}]_+ \hat{v}_h \right\rangle_{\partial\Omega\setminus\Gamma_-} \end{aligned}$$

Since $[\boldsymbol{b}\cdot\boldsymbol{n}]_-$ and $\hat{v}_h$ vanish on $\partial\Omega\setminus\Gamma_-$, we have (5.6). Thus (5.5) becomes

$$\begin{aligned} B_h(\tilde{\boldsymbol{u}}_0-\boldsymbol{u}_h,\boldsymbol{v}_h) &= B_h^d(\boldsymbol{u}_0,\boldsymbol{v}_h)+B_h^d(\tilde{\boldsymbol{u}}_0-\boldsymbol{u}_0,\boldsymbol{v}_h) \\ &= B_h^d(\tilde{\boldsymbol{u}}_h^0,\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h. \end{aligned} \tag{5.7}$$

Choosing $\boldsymbol{v}_h=\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0\in\boldsymbol{V}_{0h}$ in (4.13), we have

$$\begin{aligned} C_c\,|||\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||^2 &\le B_h(\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0,\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0) \\ &= B_h(\boldsymbol{u}_h-\tilde{\boldsymbol{u}}_0,\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0)+B_h(\tilde{\boldsymbol{u}}_0-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0,\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0) \\ &\le |B_h^d(\tilde{\boldsymbol{u}}_0,\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0)|+|B_h(\tilde{\boldsymbol{u}}_0-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0,\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0)| \\ &\le C_b^d\,|||\tilde{\boldsymbol{u}}_0|||_d\,|||\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||_d+C_b\,|||\tilde{\boldsymbol{u}}_0-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||\,|||\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0||| \end{aligned}$$

Then we have

$$C_c\,|||\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||\le C_b^d\,|||\tilde{\boldsymbol{u}}_0|||_d+C_b\,|||\tilde{\boldsymbol{u}}_0-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||\,.$$

By the triangle inequality,

$$\begin{aligned} |||\boldsymbol{u}_h-\tilde{\boldsymbol{u}}_0||| &\le |||\boldsymbol{u}_h-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||+|||\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0-\tilde{\boldsymbol{u}}_0||| \\ &\le \frac{C_b^d}{C_c}\,|||\tilde{\boldsymbol{u}}_0|||_d+\left(1+\frac{C_b}{C_c}\right)|||\tilde{\boldsymbol{u}}_0-\boldsymbol{P}_h\tilde{\boldsymbol{u}}_0|||\,, \end{aligned}$$

where C_b^d, C_c, and C_b are independent of ε. Thus we obtain the inequality (5.2). □

6 Numerical results

6.1 Convection-dominated case

We consider the case that the diffusion coefficient is very small, $\varepsilon = 10^{-9}$, so that the exact solution has a boundary layer. Let Ω be the unit square domain, $\boldsymbol{b} = (1,1)^T$, and $c \equiv 0$. The example problem is as follows:

$$-\varepsilon\Delta u + \boldsymbol{b}\cdot\nabla u = f \text{ in } \Omega, \tag{6.1a}$$
$$u = 0 \text{ on } \Gamma_D = \partial\Omega, \tag{6.1b}$$

where f is given so that the exact solution is

$$u(x,y) = \sin(\pi x/2)\sin(\pi y/2)\left(1 - e^{(x-1)/\varepsilon}\right)\left(1 - e^{(y-1)/\varepsilon}\right).$$

This solution has a boundary layer near $x = 1$ or $y = 1$. The meshes we use are the rectangular meshes with the length of $h = 1/N$. We computed the approximate solutions for $h = 1/10, 1/20, 1/40$, and $1/80$ with linear elements. In Figure 3, we display the graph of the approximate solution for $h = 1/10$. We can see that no oscillation appears unlike the classical finite element method. Figure 1 shows that the convergence diagram in the L^2 norm and $H^1(\mathcal{T}_h)$ seminorm on $\Omega_{0.9} := (0, 0.9)^2$. Here we restrict the domain to $\Omega_{0.9}$ in order to remove the boundary layers. We observe that the convergence rates of the L^2-error and the $H^1(\mathcal{T}_h)$-error are optimal,i.e., h^2 and h, respectively. We also computed for $\varepsilon = 10^{-1}$ to compare with the convection-dominated case, see Figure 2. In this case, it can be observed that the convergence rates of the error on the entire Ω are h^2 and h in the L^2-norm and $H^1(\mathcal{T}_h)$-seminorm, respectively.

6.2 Rotating flow problem

Next, we consider the example where $\boldsymbol{b}$ is not constant. Let Ω be the unit square domain with a slit, i.e., $\Omega = (0,1)^2 \setminus \{(1/2, y) : 0 \le y \le 1/2\}$. We consider the same equation (6.1) for different coefficients: $\varepsilon = 10^{-9}$, $\boldsymbol{b} = (1/2 - y, x - 1/2)^T$, and $f \equiv 0$. The non-homogeneous Dirichlet boundary condition, $g_D(x,y) = \sin^2(2\pi y)$, is imposed on the inflow-side slit, and $g_D = 0$ otherwise, see Figure 4. We used the same meshes and finite element spaces as the previous example. In Figure 5, we display the graphs of the approximate solution u_h and $\hat{u}_h$ with $h = 1/20$. Figure 6 shows the cross section of u_h at $x = 1/2$, which confirms us that our method works well and is stable.

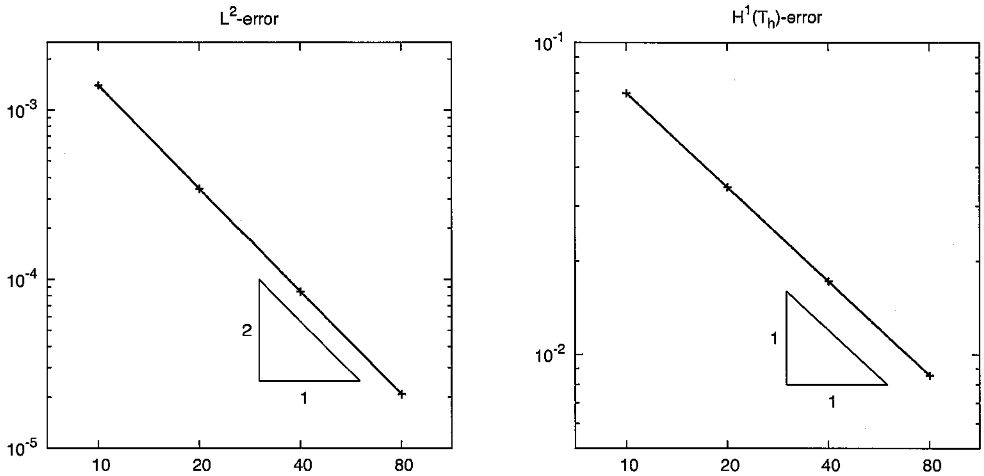


Figure. 1: L^2-error (left) and $H^1(\mathcal{T}_h)$-error(right) on $\Omega_{0.9}$ for $\varepsilon = 10^{-9}$.

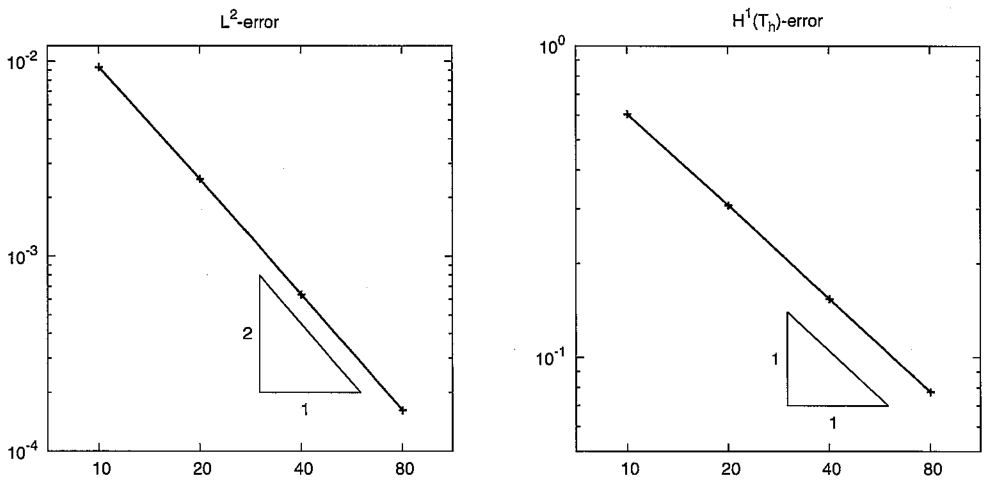


Figure. 2: L^2-error(left) and $H^1(\mathcal{T}_h)$-error(right) on Ω for $\varepsilon = 10^{-1}$.

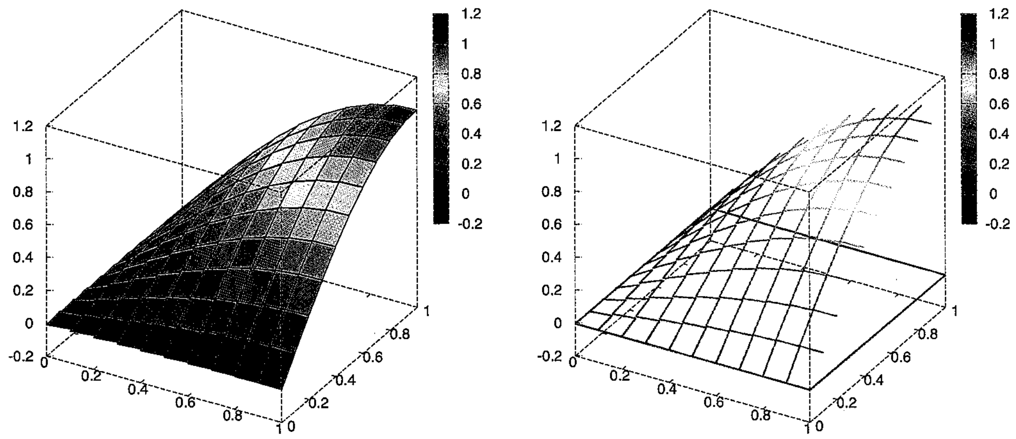


Figure. 3: Approximate solutions u_h(left) and $\hat{u}_h$(right) for $h = 1/10$ and $\varepsilon = 10^{-9}$.

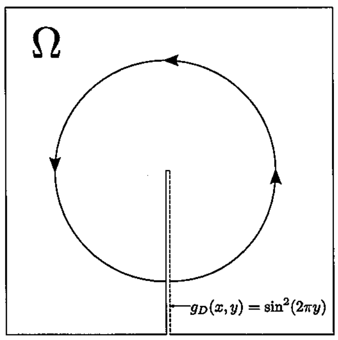


Figure. 4: Rotating flow problem.

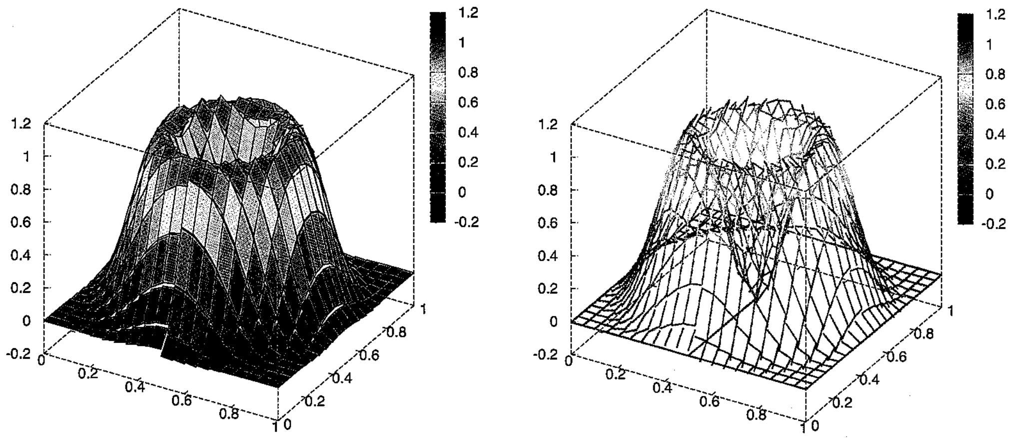


Figure. 5: Approximate solutions u_h(left) and $\hat{u}_h$(right) of the rotating flow problem for $h = 1/20$.

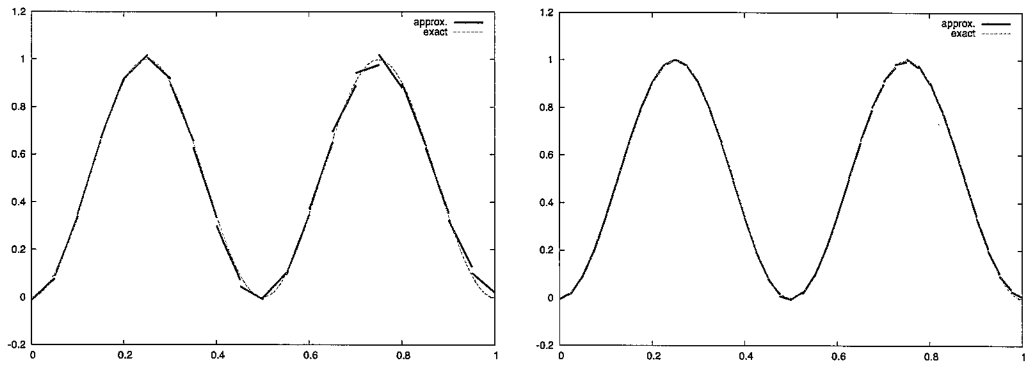


Figure. 6: Approximate solution u_h at $x = 1/2$ of the rotating flow problem for $h = 1/20$ (left) and $h = 1/40$ (right).

7 Conclusions

We have presented a new hybridized scheme for the convection-diffusion-reaction problems. In our formulation, a unwinding term is added to stabilize the convection-reaction part. As a result, our scheme is stable even when $\varepsilon \downarrow 0$. Indeed, numerical results show that no oscillation appears in our approximate solutions. We have proved the error estimates of optimal order in the HDG norm, and discussed the relation between our approximate solution and the solution of the reduced problem.

Acknowledgement

I thank Professors B. Cockburn, F. Kikuchi and N. Saito who encouraged me through valuable discussions. This work is supported by JST, CREST.

Part II

Hybridized Discontinuous Galerkin Method with Lifting Operator

1 Introduction

The discontinuous Galerkin(DG) methods is one of the active research fields of numerical analysis in the last decade. They allow us to use discontinuous approximate functions across the element boundaries and have the robustness to variation of element geometry. That is, we can utilize many kind of polynomials as approximate functions on elements and many kind of polyhedral domains as elements simultaneously. Consequently, DG method fits adaptive computations, so that mathematical analysis as well as actual applications has been developed for various problems. For more details, we refer to [2, 5, 6]. However, the size and band-widths of the resulting matrices can be much larger than those of the conventional FEM, which is a disadvantage from the viewpoint of computational cost. To surmount this obstacle, recently new class of DG method, which is called hybridized DG methods, is proposed and analyzed by B. Cockburn and his colleagues; for example, see [15]. Thus, we introduce new unknown function $\hat{U}_h$ on inter-element edges and characterize it as the weak solution of a target PDE. We then obtain the discrete system for $\hat{U}_h$ and the size of the system becomes smaller. On the other hand, it should be kept in mind that DG method has another origin. Some class of nonconforming and hybrid FEM's, which are called hybrid displacement method, use discontinuous functions as approximate field functions; see for example [8, 9]. In [10] and [11], F. Kikuchi and Y. Ando developed a variant of the hybrid displacement one, and applied it to plate problems. Their approach enables one to use conventional element matrices and vectors. It, however, suffered from numerical instability and was not fully successful. Recently, the author and his colleagues proposed a new DG method that is based on the hybrid displacement approach by stabilizing their old method and applied it to linear elasticity problems in [12]. A key point of our method is to introduce penalty terms in order to ensure the stability. We, then, carried out theoretical analysis by using the 2D Poisson equation as a model problem, and gave some concrete finite element models with numerical results and observations in [13]. However, an issue still remains. The stability is guaranteed only when the penalty parameters are taken from a certain interval, and we know only the existence of such an interval and do not know concrete information about it.

The purpose of this paper is to propose a new hybridized DG method that is stable for arbitrary penalty parameters. Our strategy is to introduce the lifting operator and define the penalty term in terms of the lifting operator. In order to state our idea as clearly as possible, we consider the Poisson equation with homogeneous Dirichlet condition:

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a convex polygonal domain and $f \in L^2(\Omega)$.

This paper is composed of six sections. In Section 2, we introduce the triangulation and finite element spaces, and then describe the lifting operator. Section 3 is devoted to the formulation of our proposed hybridized DG method, and mathematical analysis including error estimates is given in Section 4. In Section 5, we report some results of numerical computations and confirm our theoretical results. Finally, we conclude this paper in Section 6.

2 Preliminaries

2.1 Notation

Let $\Omega \subset \mathbb{R}^n$, for an integer $n \geq 2$, be a convex polygonal domain. We introduce a triangulation $\mathcal{T}_h = \{K\}$ of Ω in the sense [13], where $h = \max_{K \in \mathcal{T}_h} h_K$ and h_K stands for the diameter of K. That is each $K \in \mathcal{T}_h$ is an m-polygonal domain, where m is an integer and can differ with K. We assume that m is bounded from above independently of a family of triangulations $\{\mathcal{T}_h\}_h$, and ∂K does not intersect with itself. Let $\mathcal{E}_h = \{e \subset \partial K : K \in \mathcal{T}_h\}$ be the set of all edges of elements, and let $\Gamma_h = \bigcup_{K \in \mathcal{T}_h} \partial K$. We define the so-called broken Sobolev space for $k \geq 0$,

$$H^k(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K \in H^k(K) \quad \forall K \in \mathcal{T}_h\}.$$

Let $L^2_0(\Gamma_h) = \{\hat{v} \in L^2(\Gamma_h) : \hat{v}|_{\partial\Omega} = 0\ \}$. We introduce the inner products

$$(u, v)_K = \int_K uv dx \quad \text{for } K \in \mathcal{T}_h,$$
$$\langle \hat{u}, \hat{v} \rangle_e = \int_e \hat{u}\hat{v} ds \quad \text{for } e \in \mathcal{E}_h.$$

The usual m-th order Sobolev seminorm and norm on K are denoted by $|u|_{m,K}$ and $||u||_{m,K}$, respectively. We use finite element spaces:

$$U_h \subset H^2(\mathcal{T}_h), \quad \hat{U}_h \subset L^2_0(\Gamma_h),$$

and we set $\boldsymbol{V}_h = U_h \times \hat{U}_h$ and $\boldsymbol{V} = H^2(\mathcal{T}_h) \times L_0^2(\Gamma_h)$. We assume that

$$\nabla U_h(K) = \{\nabla v_h : v_h \in U_h(K)\} \subset U_h(K)^n$$

for all $K \in \mathcal{T}_h$.

2.2 Lifting operators

We state the definition of the lifting operator which plays a crucial role in our formulation and analysis. To this end, we fix $K \in \mathcal{T}_h$ and $e \subset \partial K$ for the time being, and set

$$U_h(K) = \{w_h|_K : w_h \in U_h\},$$
$$\hat{U}_h(e) = \{\hat{w}_h|_e : \hat{w}_h \in \hat{U}_h\}.$$

Then, for any $\hat{v} \in L^2(e)$, there exists a unique $\boldsymbol{u}_h \in U_h(K)^n$ such that

$$(\boldsymbol{u}_h, \boldsymbol{w}_h)_K = \langle \hat{v}, \boldsymbol{w}_h \cdot \boldsymbol{n}_K \rangle_{\partial K} \quad \forall \boldsymbol{w}_h \in U_h(K)^n, \tag{2.1}$$

where $\boldsymbol{n}_K$ is the unit outward normal vector to ∂K. The lifting operator $\boldsymbol{R}_h : L^2(\partial K) \to U_h(K)^n$ is defined as $\boldsymbol{R}_h(\hat{v}) = \boldsymbol{u}_h$. Thus,

$$(\boldsymbol{R}_h(\hat{v}), \boldsymbol{w}_h)_K = \langle \hat{v}, \boldsymbol{w}_h \cdot \boldsymbol{n}_K \rangle_{\partial K} \quad \forall \boldsymbol{w}_h \in U_h(K)^n. \tag{2.2}$$

Here we prove the following proposition to use in our analysis.

Proposition 2.1. Let $K \in \mathcal{T}_h$, then we have

$$\|\boldsymbol{R}_h(\hat{v})\|_{0,K} \le C \sum_{e \subset \partial K} h_e^{-1/2} \|\hat{v}\|_{0,e} \quad \forall \hat{v} \in L^2(\partial K). \tag{2.3}$$

Proof. In (2.2), taking $\boldsymbol{w}_h = \boldsymbol{R}_h(\hat{v})$ yields

$$\begin{aligned} \|\boldsymbol{R}_h(\hat{v})\|_{0,K}^2 &= (\boldsymbol{R}_h(\hat{v}), \boldsymbol{R}_h(\hat{v}))_K \\ &= \langle \hat{v}, \boldsymbol{R}_h(\hat{v}) \cdot \boldsymbol{n}_K \rangle_{\partial K} \\ &\le \sum_{e \subset \partial K} \|\hat{v}\|_{0,e} \|\boldsymbol{R}_h(\hat{v})\|_{0,e}. \end{aligned} \tag{2.4}$$

By the trace theorem, there exists C such that

$$\|\boldsymbol{R}_h(\hat{v})\|_{0,e} \le C h_e^{-1/2} \|\boldsymbol{R}_h(\hat{v})\|_{0,K}. \tag{2.5}$$

The constant C depends on $U_h(K)$ and $\hat{U}_h(\partial K)$. Combining (2.4) with (2.5), we obtain (2.3). □

3 New hybridized DG scheme

This section is devoted to the presentation of our proposed hybridized DG method. Before doing so, we convert the Poisson problem (1.1) into a suitable weak form (3.4). A key idea is to introduce unknown functions on inter-element edges. First, multiplying both the sides of (1.1) by a test function $v \in U_h$ and integrating over each $K \in \mathcal{T}_h$, we have by the integration by parts

$$\sum_{K \in \mathcal{T}_h} [(\nabla u, \nabla v)_K - \langle \boldsymbol{n}_K \cdot \nabla u, v \rangle_{\partial K}] = (f, v) \tag{3.1}$$

From the continuity of the flux, we have

$$\sum_{K \in \mathcal{T}_h} \langle \boldsymbol{n}_K \cdot \nabla u, \hat{v} \rangle = 0 \quad \forall \hat{v} \in L_0^2(\Gamma_h). \tag{3.2}$$

This, together with (3.1), implies

$$\sum_{K \in \mathcal{T}_h} [(\nabla u, \nabla v)_K - \langle \boldsymbol{n}_K \cdot \nabla u, v - \hat{v} \rangle_{\partial K}] = (f, v) \tag{3.3}$$

Here we set, for $\boldsymbol{u} = (u, \hat{u})$ and $\boldsymbol{v} = (v, \hat{v}) \in V$,

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) &= \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_K, \\ b_h(\boldsymbol{u}, \boldsymbol{v}) &= -\sum_{K \in \mathcal{T}_h} \langle \boldsymbol{n}_K \cdot \nabla u, v - \hat{v} \rangle_{\partial K}. \end{aligned}$$

Then, (3.3) is rewritten as

$$a_h(\boldsymbol{u}, \boldsymbol{v}) + b_h(\boldsymbol{u}, \boldsymbol{v}) = (f, v). \tag{3.4}$$

Now we can state our hybridized DG method: find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$B_h^L(\boldsymbol{u}_h, \boldsymbol{v}_h) = (f, v_h) \qquad \forall \boldsymbol{v}_h = (v_h, \hat{v}_h) \in \boldsymbol{V}_h, \tag{3.5}$$

where

$$\begin{aligned} B_h^L(\boldsymbol{u}_h, \boldsymbol{v}_h) := a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, \boldsymbol{u}_h) \\ + l_h(\boldsymbol{v}_h, \boldsymbol{u}_h) + j_h(\boldsymbol{u}_h, \boldsymbol{v}_h). \end{aligned} \tag{3.6}$$

Here, the third term $b_h(\boldsymbol{v}_h, \boldsymbol{u}_h)$ of B_h^L is added in order to symmetrize the scheme and the penalty terms $l_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$ and $j_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$ are defined by

$$l_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{K\in\mathcal{T}_h} (\boldsymbol{R}_h(u_h - \hat{u}_h), \boldsymbol{R}_h(v_h - \hat{v}_h))_K,$$

$$j_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{K\in\mathcal{T}_h} \sum_{e\subset\partial K} \int_e \eta_e h_e^{-1}(u_h - \hat{u}_h)(v_h - \hat{v}_h)ds,$$

where η_e is the penalty parameter on an edge e with $\eta_e \geq \eta_{min} > 0$ and h_e is the length of e.

4 Error estimates

In this section, we give a mathematical analysis of our hybridized DG method. To this end, we introduce

$$|||\boldsymbol{v}|||^2 = \sum_{K\in\mathcal{T}_h} \|\nabla v - \boldsymbol{R}_h(v - \hat{v})\|_{0,K}^2 + |\boldsymbol{v}|_{j,h}^2,$$

$$|||\boldsymbol{v}|||_h^2 = \sum_{K\in\mathcal{T}_h} |v|_{1,K}^2 + |\boldsymbol{v}|_{j,h}^2,$$

where

$$|\boldsymbol{v}|_{j,h}^2 = \sum_{K\in\mathcal{T}_h} \sum_{e\subset\partial K} \frac{\eta_e}{h_e}\|v - \hat{v}\|_{0,e}^2.$$

Proposition 4.1. There exist constants C_1 and C_2 such that

$$C_1\, |||\boldsymbol{v}|||_h \leq |||\boldsymbol{v}||| \leq C_2\, |||\boldsymbol{v}|||_h \qquad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{4.1}$$

Proof. By Proposition 2.1, we have for all $\boldsymbol{v} \in \boldsymbol{V}$,

$$|||\boldsymbol{v}|||^2 \leq 2 \sum_{K\in\mathcal{T}_h} (|v|_{1,K}^2 + \|\boldsymbol{R}_h(v - \hat{v})\|_{0,K}^2) + |\boldsymbol{v}|_{j,h}^2 \leq C\, |||\boldsymbol{v}|||_h^2.$$

Next, we prove the other inequality of (4.1). For any $\varepsilon > 0$, we have

$$\begin{aligned} |||\boldsymbol{v}|||^2 &\geq \sum_{K\in\mathcal{T}_h} \left((1-\varepsilon)|v|_{1,K}^2 + (1-\varepsilon^{-1})\|\boldsymbol{R}_h(v - \hat{v})\|_{0,K}^2\right) + |\boldsymbol{v}|_{j,h}^2 \\ &\geq (1-\varepsilon) \sum_{K\in\mathcal{T}_h} |v|_{1,K}^2 + \left(1 + \frac{C}{\eta_{min}}(1-\varepsilon^{-1})\right) |\boldsymbol{v}|_{j,h}^2. \\ &\geq \min\left\{1-\epsilon, 1 + \frac{C}{\eta_{min}}(1-\varepsilon^{-1})\right\} |||\boldsymbol{v}|||_h^2. \end{aligned}$$

We denote $\alpha = C/\eta_{min}$. Taking $\varepsilon = (1+2\alpha)/2(1+\alpha)$ gives us

$$|||\boldsymbol{v}|||^2 \geq \frac{1}{1+\alpha} |||\boldsymbol{v}|||_h^2 .$$

□

Theorem 4.2. The bilinear form B_h^L satisfies the following three properties.

(Consistency) Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be the exact solution. For $\boldsymbol{u} = (u, u|_{\Gamma_h})$, we have

$$B_h^L(\boldsymbol{u}, \boldsymbol{v}) = (f, v) \quad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

(Boundedness)

$$|B_h^L(\boldsymbol{v}, \boldsymbol{w})| \leq |||\boldsymbol{v}|||\,|||\boldsymbol{w}||| \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}.$$

(Coercivity)

$$B_h^L(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq |||\boldsymbol{v}_h|||^2 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Furthermore, the scheme (3.6) admits a unique solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$ for any $f \in L^2(\Omega)$ and $\{\eta_e\}_e$.

Proof. The consistency is trivial since $u - u|_{\Gamma_h} = 0$ on Γ_h. The coercivity is a direct consequence of the expression

$$b_h(\boldsymbol{v}_h, \boldsymbol{w}) = -\sum_K (\nabla v_h, \boldsymbol{R}_h(w - \hat{w}))_K.$$

Combining this with the Schwarz inequality, we immediately deduce the boundedness. Finally, the coercivity implies the uniqueness of (3.6) and, hence, the system of linear equations (3.6) admits a unique solution. □

As results of those three properties, we obtain the following a priori error estimates in terms of $|||\cdot|||$.

Theorem 4.3. Let $\boldsymbol{u} = (u, u|_{\Gamma_h}) \in \boldsymbol{V}$ with the exact solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of the Poisson problem (1.1). Suppose that $\{\mathcal{T}_h\}_h$ satisfies

$$\tau \leq h_e/h_K \quad \forall K \in \mathcal{T}_h,\ \forall e \subset \partial K \tag{4.2}$$

with some positive constant τ. Let $\boldsymbol{u}_h \in \boldsymbol{V}_h$ be the solution of our HDG scheme (3.6) for any $\{\eta_e\}_e$ with $\eta_e > 0$. Then, we have the error estimates

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_h \leq C \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} |||\boldsymbol{u} - \boldsymbol{v}_h|||_h . \tag{4.3}$$

Proof. By Theorem 4.2, we have for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$\begin{aligned} |||\boldsymbol{u}_h - \boldsymbol{v}_h|||^2 &\leq B_h^L(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{u}_h - \boldsymbol{u}_h) && \text{(Coercivity)} \\ &= B_h^L(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{v}_h) && \text{(Consistency)} \\ &\leq |||\boldsymbol{u} - \boldsymbol{u}_h|||\ |||\boldsymbol{u}_h - \boldsymbol{v}_h|||, && \text{(Boundedness)} \end{aligned}$$

which implies that

$$|||\boldsymbol{u}_h - \boldsymbol{v}_h||| \leq |||\boldsymbol{u} - \boldsymbol{v}_h||| \quad \forall \boldsymbol{v}_h \in \boldsymbol{V_h}. \tag{4.4}$$

By the triangle inequality and (4.4), we have

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq |||\boldsymbol{u} - \boldsymbol{v}_h||| + |||\boldsymbol{u}_h - \boldsymbol{v}_h||| \leq 2\, |||\boldsymbol{u} - \boldsymbol{v}_h|||$$

Since the norms $|||\cdot|||$ and $|||\cdot|||_h$ are equivalent by Proposition 4.1, we obtain (4.3). □

As is stated in [13], we assume that the following approximate properties: for $v \in H^{k+1}(K)$ there exist positive constants $C_1 = C_1(k,s)$ and $C_2 = C_2(k,s)$ such that

$$\inf_{v_h \in U_h} |v - v_h|_{s,K} \leq C_1 h_K^{k+1-s} |v|_{k+1,K}, \tag{4.5}$$

$$\inf_{\hat{v}_h \in \hat{U}_h} |v - \hat{v}_h|_{s,e} \leq C_2 h_K^{k+\frac{1}{2}-s} |v|_{k+1,K}. \tag{4.6}$$

Then we have the error estimates in Theorem 4.3 are actually of optimal order.

Theorem 4.4. Under the assumptions in Theorem 4.3 and the approximate properties (4.5) and (4.6), we have, if $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$,

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_h \leq Ch^k |u|_{k+1,\Omega}, \tag{4.7}$$

$$\|u - u_h\|_{0,\Omega} \leq Ch^{k+1} |u|_{k+1,\Omega}. \tag{4.8}$$

Proof. From (4.5) and (4.6), we can easily see that

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{V_h}} |||\boldsymbol{u} - \boldsymbol{v}_h|||_h \leq Ch^k |u|_{k+1,\Omega}. \tag{4.9}$$

Combining this with Theorem 4.3 yields (4.7). Next, we prove (4.8). Here we define $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ as the solution of the adjoint problem

$$\begin{cases} -\Delta\psi = u - u_h & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.10}$$

Let $\boldsymbol{\psi} = (\psi, \psi|_{\Gamma_h})$. Then, since B_h^L is symmetric, we have

$$B_h^L(\boldsymbol{v}, \boldsymbol{\psi}) = (u - u_h, v) \quad \forall \boldsymbol{v} = (v, \hat{v}) \in \boldsymbol{V}. \tag{4.11}$$

In particular, taking $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_h$, we have for any $\boldsymbol{\psi}_h \in \boldsymbol{V_h}$,

$$\begin{aligned}
\|u - u_h\|_{0,\Omega}^2 &\le B_h^L(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\psi}) \\
&= B_h^L(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\psi} - \boldsymbol{\psi}_h) \\
&\le |||\boldsymbol{u} - \boldsymbol{u}_h||| \, |||\boldsymbol{\psi} - \boldsymbol{\psi}_h||| \\
&\le C \, |||\boldsymbol{u} - \boldsymbol{u}_h||| \, |||\boldsymbol{\psi} - \boldsymbol{\psi}_h|||_h \, .
\end{aligned}$$

From (4.5) and (4.6), it follows that

$$|||\boldsymbol{\psi} - \boldsymbol{\psi}_h|||_h \le Ch|\psi|_{2,\Omega}. \tag{4.12}$$

By the regularity of the adjoint problem, we have

$$|\psi|_{2,\Omega} \le C\|u - u_h\|_{0,\Omega}. \tag{4.13}$$

Thus we obtain (4.8). □

Remark. In contrast to our previous results of [13], error estimates in Theorem 4.3 are valid for any positive parameters η_e. This is one of the advantages of our hybridized DG method.

5 Numerical results

We now present the numerical results of our method for the following problem:

$$\begin{cases} -\Delta u = 2\pi^2 \sin(\pi x)\sin(\pi y) \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a unit square. We use uniform rectangular meshes and P_k-elements ($k = 1, 2, 3$). We computed the approximate solutions with various mesh size $h = 1/5$, $1/10$, $1/20$, and $1/40$, see Figure 7 and Figure 8. Here we take $\eta_e = 1$ as the penalty parameters for each $e \in \mathcal{E}_h$. From them, we see that the H^1 and L^2 convergence rates of the approximate solutions are h^k and h^{k+1}, respectively. Figure 9 and Figure 10 show the approximate solutions u_h and $\hat{u}_h$ with P_1- and $\mathcal{P}_2$-elements, respectively.

Next, we provide the results for small penalty parameters in order to verify the stability of our scheme. For comparison purposes, we also show that of the symmetric scheme [13]. The mesh size h and tolerance of termination criterion are fixed to certain values. We use the BiCGSTAB method with diagonal preconditioning to solve the linear systems. Figure 11 and Figure 12 show the numbers of iterations and $H^1(\mathcal{T}_h)$-errors. We see that our scheme is successful for all positive parameters, while the symmetric scheme fails when η_e is not large enough. Moreover, we see that the number of iterations of our scheme is about the same as the symmetric one for sufficiently large parameters.

6 Conclusions

We have presented a new hybridized DG method by using the lifting operator and examined the stability for arbitrary penalty parameters. Convergence results of optimal order have been proved and confirmed by numerical experiments. As a model problem, we have considered only the Dirichlet boundary value problem for the Poisson equation. We are interested in application to other problems, for example, Neumann boundary value problem, Stokes system, and time-dependent problems. They are left here as future study.

7 Acknowledgement

I thank Professor Fumio Kikuchi who brought my attention to the present subject and encouraged me through valuable discussions. This work is supported by Grants-in-Aid for Scientific Research, JSPS and by Global COE Program (The Research and Training Center for New Development in Mathematics, The University of Tokyo), MEXT, Japan.

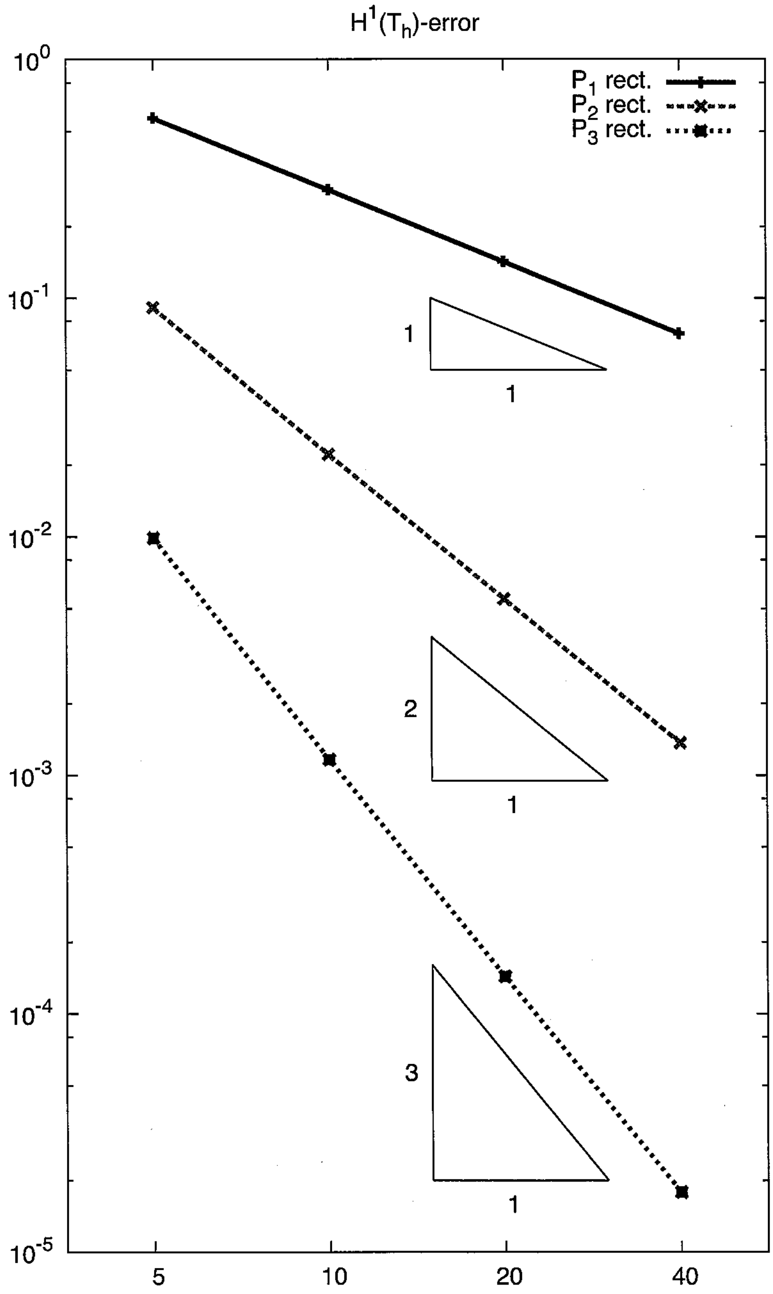


Figure. 7: Convergence diagram in the $H^1(\mathcal{T}_h)$-norm.

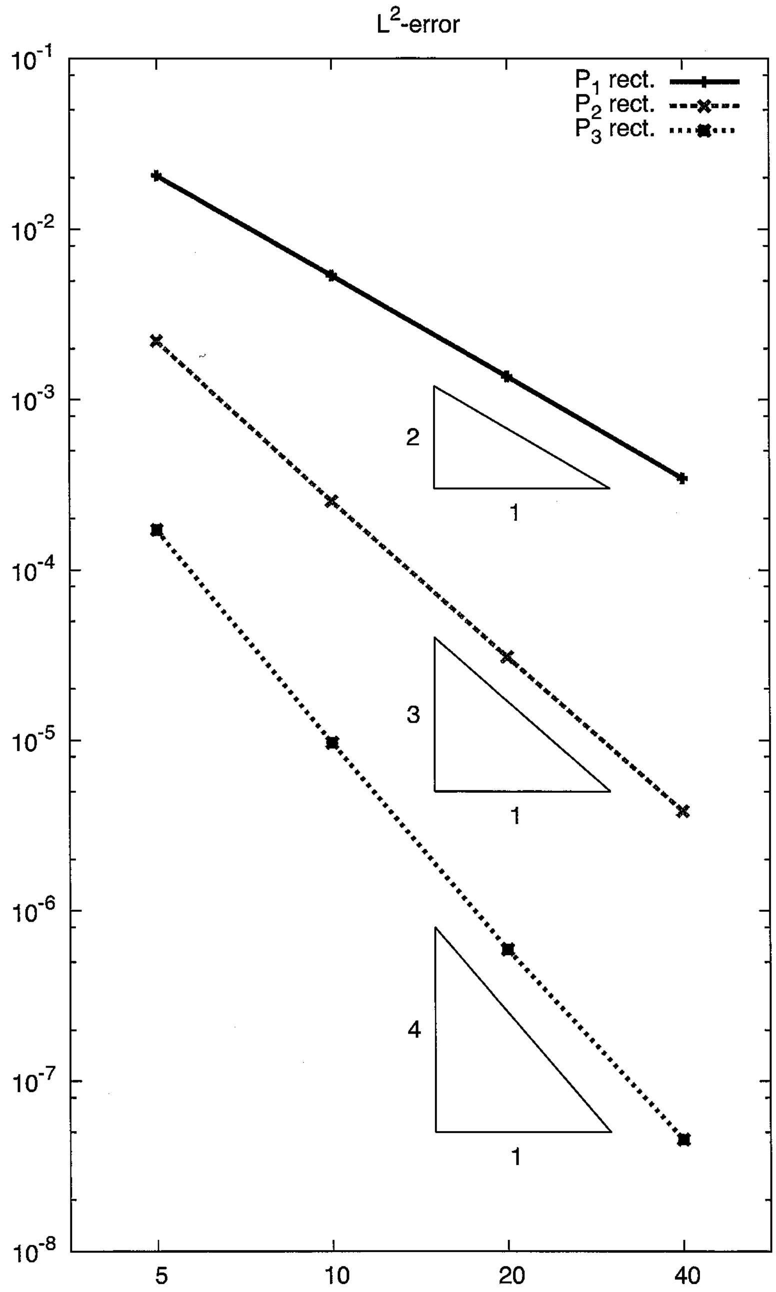


Figure. 8: Convergence diagram in the L^2-norm.

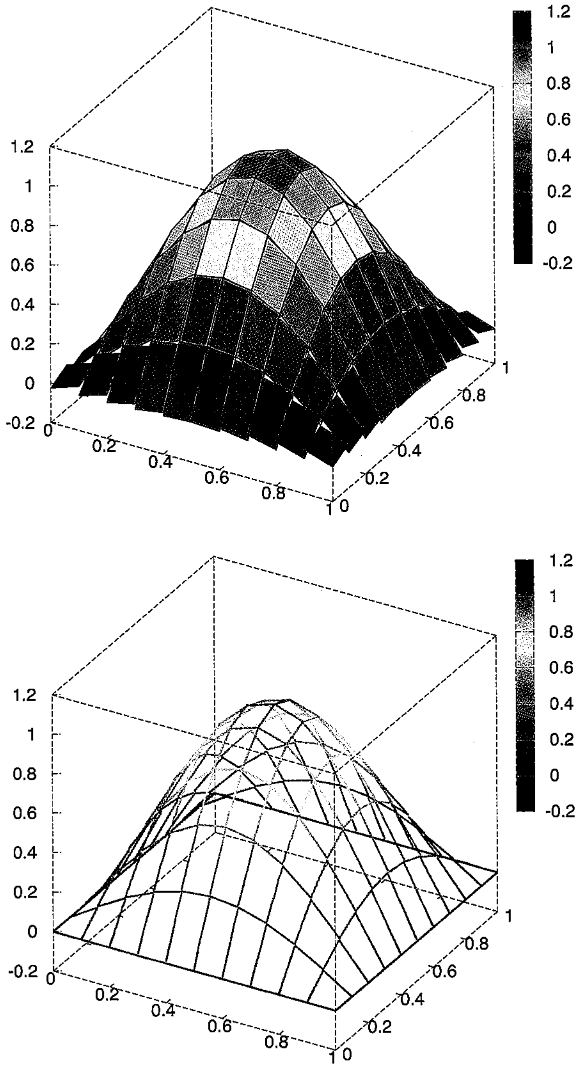


Figure. 9: Approximate solutions u_h(top) and $\hat{u}_h$ (bottom) with P_1-elements and $h = 1/10$.

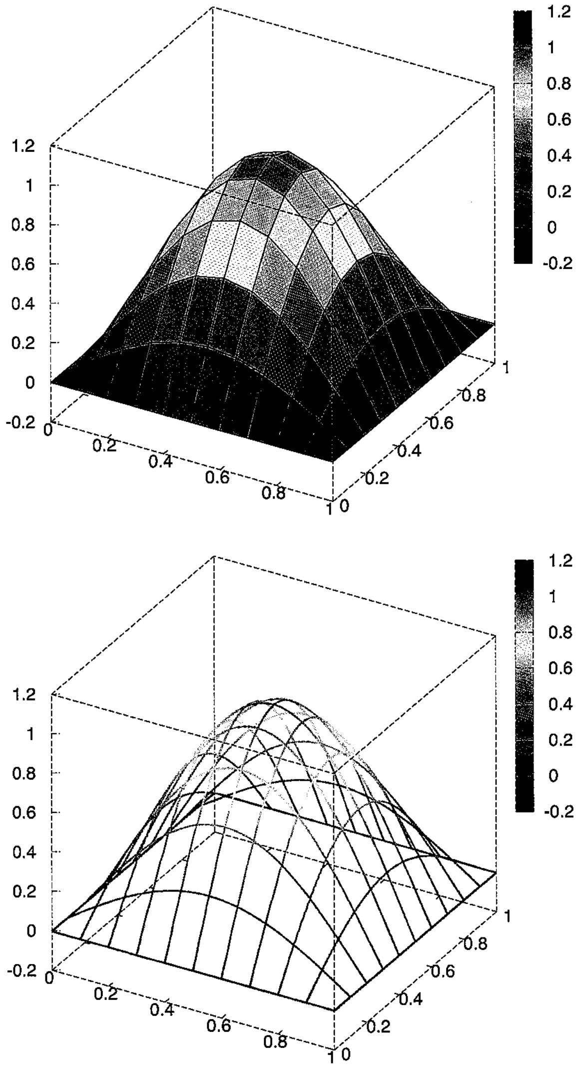


Figure. 10: Approximate solutions u_h (top) and $\hat{u}_h$ (bottom) with P_2-elements and $h = 1/10$.

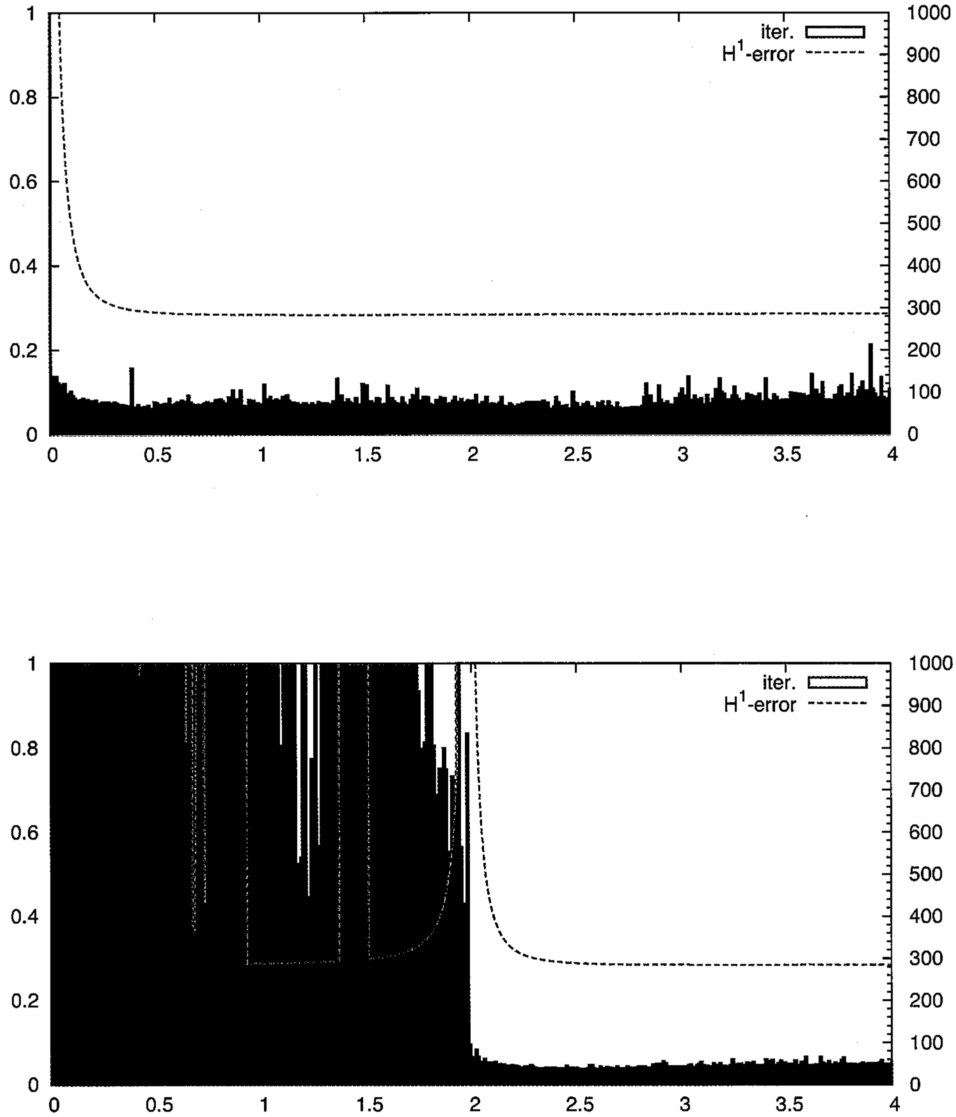
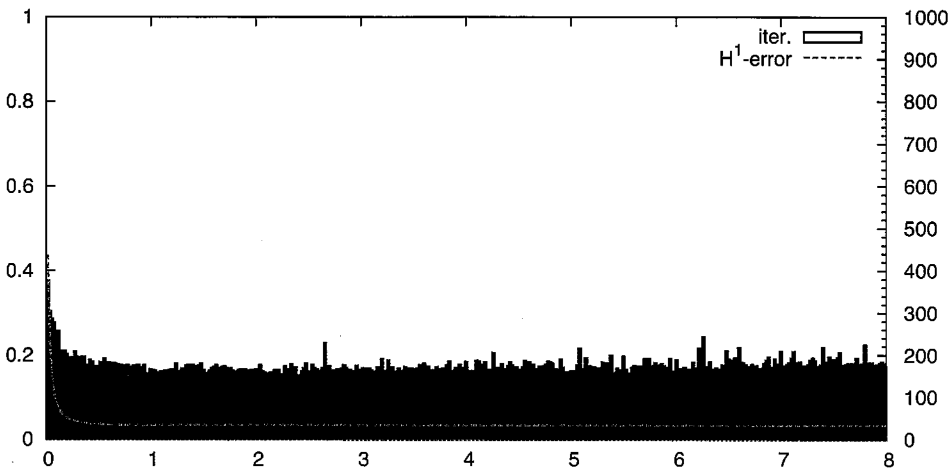


Figure. 11: The number of iterations and $H^1(\mathcal{T}_h)$-errors with P_1-elements and $h = 1/10$ by our scheme (top) and the symmetric one(bottom). The x-axis denotes the penalty parameter η_e.



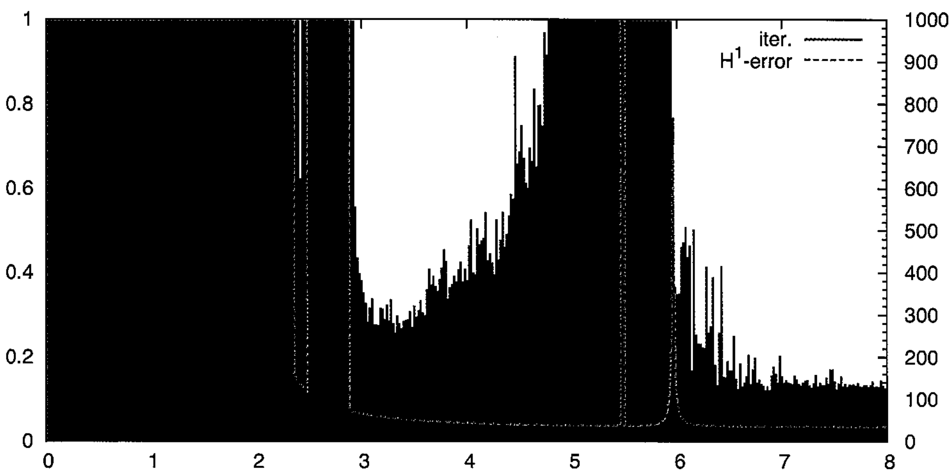


Figure. 12: The number of iterations and $H^1(\mathcal{T}_h)$-errors with P_2-elements and $h = 1/8$ by our scheme (top) and the symmetric one(bottom). The x-axis denotes the penalty parameter η_e.

Appendix

A Some remarks on the lifting operators

In this section, we consider a more general case of the lifting operators. Let K be an element of T_h. We denote by $\boldsymbol{R}_h(V_h;\hat{v})$ a lifting operator for a finite dimensional space V_h on K, which satisfies

$$(\boldsymbol{R}_h(V_h;\hat{v}), \boldsymbol{w}_h)_K = \langle \boldsymbol{n}_K \cdot \boldsymbol{w}_h, \hat{v}\rangle_{\partial K} \quad \forall \boldsymbol{w}_h \in (V_h)^n.$$

In particular, we denote $\boldsymbol{R}_h^m(\hat{v}) = \boldsymbol{R}_h(\mathcal{P}^m(K), \hat{v})$, where $\mathcal{P}^m(K)$ is polynomials of degree m on K. If $l \le m$, then

$$\|\boldsymbol{R}_h^m(\hat{v})\|_{L^2(K)^n} \ge \|\boldsymbol{R}_h^l(\hat{v})\|_{L^2(K)^n},$$

which follows from the following proposition.

Proposition A.1. If $V_h \subset W_h$, then we have

$$(\boldsymbol{R}_h(W_h;\hat{v}), \boldsymbol{R}_h(W_h;\hat{v}))_K \ge (\boldsymbol{R}_h(V_h;\hat{v}), \boldsymbol{R}_h(V_h;\hat{v}))_K \ \forall \hat{v} \in L^2(\partial K).$$

Proof. Since $\boldsymbol{R}_h(V_h;\hat{v}) \in (W_h)^n$, we have

$$\begin{aligned}(\boldsymbol{R}_h(W_h;\hat{v}), \boldsymbol{R}_h(V_h;\hat{v}))_K &= \langle \boldsymbol{n}_K \cdot \boldsymbol{R}_h(V_h;\hat{v}), \hat{v}\rangle_{\partial K} \\ &= (\boldsymbol{R}_h(V_h;\hat{v}), \boldsymbol{R}_h(V_h;\hat{v}))_K .\end{aligned}$$

Hence we have

$$\begin{aligned}0 &\le (\boldsymbol{R}_h(W_h;\hat{v}) - \boldsymbol{R}_h(V_h;\hat{v}), \boldsymbol{R}_h(W_h;\hat{v}) - \boldsymbol{R}_h(V_h;\hat{v}))_K \\ &= (\boldsymbol{R}_h(W_h;\hat{v}), \boldsymbol{R}_h(W_h;\hat{v}))_K - (\boldsymbol{R}_h(V_h;\hat{v}), \boldsymbol{R}_h(V_h;\hat{v}))_K .\end{aligned}$$ □

Next, we give a counterexample to show that the converse inequality of (2.3)

$$C \sum_{e\subset\partial K} h_e^{-1/2}\|\hat{v}\|_{0,e} \le \|\boldsymbol{R}_h(\hat{v})\|_{0,K} \quad \forall \hat{v} \in L^2(\partial K)$$

does not hold in general.

Theorem A.2. Let $K \subset \mathbb{R}^2$ be a unit square. Then, there exists $\hat{v} \in \Pi_{e\subset\partial K} P^0(e) \setminus \{0\}$ such that

$$\langle \boldsymbol{n}_K \cdot \boldsymbol{w}_h, \hat{v}\rangle_{\partial K} = 0 \quad \forall \boldsymbol{w}_h \in (\mathcal{P}^0(K))^2, \tag{A.1}$$

which implies that $\boldsymbol{R}_h^0(\hat{v}) = \boldsymbol{0}$.

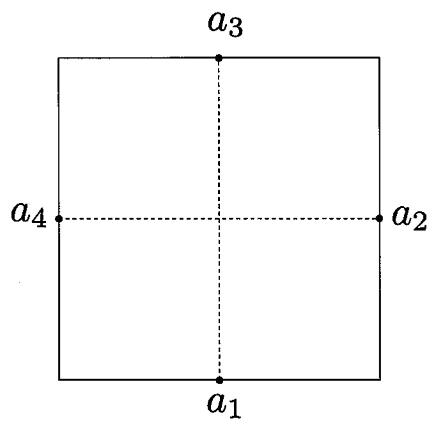


Figure. 13: Illustration of the element and midpoints $\{a_i\}_{1\le i\le 4}$.

Proof. Let $\{a_i\}_{1\le i\le 4}$ be the midpoints of ∂K, see Figure 13. Let $\boldsymbol{w}_h \equiv (w_h^1, w_h^2)^T$ and $\boldsymbol{n}_K = (n_K^1, n_K^2)^T$. Since $(\boldsymbol{n}_K \cdot \boldsymbol{w}_h \hat{v})|_e \in \mathcal{P}^0(e)$ for each $e \subset \partial K$, we have

$$\begin{aligned}\langle \boldsymbol{n}_K \cdot \boldsymbol{w}_h, \hat{v}\rangle_{\partial K} &= -w_h^2\hat{v}(a_1) + w_h^1\hat{v}(a_2) + w_h^2\hat{v}(a_3) - w_h^1\hat{v}(a_4)\\ &= w_h^1(\hat{v}(a_2) - \hat{v}(a_4)) + w_h^2(\hat{v}(a_3) - \hat{v}(a_1)).\end{aligned}$$

Hence we obtain $\hat{v}(a_2) = \hat{v}(a_4)$ and $\hat{v}(a_1) = \hat{v}(a_3)$. There exist uncountably many such functions $\hat{v}$ in $\Pi_{e\subset\partial K}P^0(e) \setminus \{0\}$. $\square$

References

[1] R. ADAMS AND J. FOURNIER, *Sobolev Spaces, 2nd edition*, Academic Press, 2003.

[2] S.C. BRENNER AND L.R. SCOTT, *The Mathematical Theory of Finite Element Methods, 3rd ed.*, Springer, 2008.

[3] W. REED AND T. HILL, *Triangular mesh methods for the neutron transport equation*, Tech. Report LA-UR-73-479, Los Alamos Scientific Laboratory, Los Alamos, NM, 1973.

[4] G. R. RICHTER, *The discontinuous Galerkin method with diffusion*, Math. Comp., 58 (1992), pp. 631–643.

[5] D. N. ARNOLD, *An interior penalty finite element method with discontinuous elements*, SIAM J. Numer. Anal., 19 (1982), pp. 742-760.

[6] D. N. ARNOLD, F. BREZZI, B. COCKBURN, AND L. D. MARINI, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., 39 (2002), pp. 1749-1779.

[7] B. AYUSO AND L.D. MARINI, *Discontinuous Galerkin methods for advection-diffusion-reaction problems*, SIAM J. Numer. Anal., 47 (2009), pp. 1391–1420.

[8] T.H.H. PIAN AND C.-C. WU, *Hybrid and Incompatible Finite Element Methods*, Chapman & Hall, 2005.

[9] P. TONG, *New displacement hybrid finite element models for solid continua*, Int. J. Num. Meth. Eng., 2 (1970), pp. 95-113.

[10] F. KIKUCHI AND Y. ANDO, *A new variational functional for the finite-element method and its application to plate and shell problems*, Nucl. Eng. Des., 21 (1972), pp. 95-113.

[11] F. KIKUCHI AND Y. ANDO, *Some finite element solutions for plate bending problems by simplified hybrid displacement method*, Nucl. Eng. Des., 23 (1972), pp. 155-178.

[12] F. KIKUCHI, K. ISHII AND I. OIKAWA, *Discontinuous Galerkin FEM of hybrid displacement type – development of polygonal elements*, Theo. & Appl. Mech. Japan, 57 (2009), pp. 395–404.

[13] I. OIKAWA AND F. KIKUCHI, *Discontinuous Galerkin FEM of hybrid type*, JSIAM Letters, vol.2 (2010), pp. 49–52.

[14] I. OIKAWA, *Hybridized discontinuous Galerkin method with lifting operator*, JSIAM Letters, vol.2 (2010), pp. 99-102.

[15] B. COCKBURN, J. GOPALAKRISHNAN, AND R. LAZAROV, *Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems,* SIAM J. Numer. Anal., 47 (2009), pp. 1319-1365.

[16] N. C. NGUYEN, J. PERAIRE, AND B. COCKBURN, *An implicit high-order hybridizable discontinuous Galerkin method for linear convection-diffusion equations*, J. Comput. Phys., 228 (2009), pp. 9941–8855.

[17] B. COCKBURN, B. DONG, J. GUZMAN, M. RESTELLI, AND R. SACCO, *A hybridizable discontinuous Galerkin method for steady-state convection-diffusion-reaction problems*, Siam J. Sci. Comput., 31 (2009), pp. 3827–3846.

[18] B. COCKBURN, J. GUZMÁN, S.-C. SOON, AND H. K. STOLARSKI, *An analysis of the embedded discontinuous Galerkin method for second-order elliptic problems*, SIAM J. Numer. Anal., 47 (2009), pp. 2686–2707.

[19] B. COCKBURN, J. GOPALAKRISHNAN, AND F.-J. SAYAS, *A projection-based error analysis of HDG methods*, Math. Comp., 79 (2010), pp. 1351–1367.

[20] B. COCKBURN, J. GOPALAKRISHNAN, N. C. NGUYEN, J. PERAIRE, AND F.-J SAYAS, *Analysis of HDG methods for Stokes flow*, Math. Comp., 80 (2011), pp. 723–760.

[21] N. C. NGUYEN, J. PERAIRE, AND B. COCKBURN, *A hybridizable discontinuous Galerkin method for Stokes flow*,Comput. Methods Appl. Mech. Engrg., 199 (2010), pp. 582–597.

[22] N. C. NGUYEN, J. PERAIRE, AND B. COCKBURN, *An implicit high-order hybridizable discontinuous Galerkin method for the incompressible Navier-Stokes equations*, J. Comput. Phys., 230 (2011), pp. 1147–1170.