論文の内容の要旨

論文題目: How to estimate Seshadri constants (セシャドリ定数を評価する方法)

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1 はじめに

代数多様体などはすべて複素数体上で考える. Seshadri 定数とは Demailly [Dem] により定義された以下のような不変量である.

定義 1.1. 対 (X, L) を偏極代数多様体, $p \in X$ を閉点とする. この時, L の p における *Seshadri* 定数 $\varepsilon(X, L; p)$ を

$$\varepsilon(X, L; p) = \inf_C \frac{C.L}{\mathrm{mult}_p(C)}$$

と定義する. ここで C は p を通る X 上のすべての曲線を動く.

定義 1.2. 対 (X, L) を偏極代数多様体とする. *Seshadri* 定数はある種の下半連続性をもつので, 非常に一般な点で L の *Seshadri* 定数は一定になる. 従って, 非常に一般な点における L の *Seshadri* 定数 $\varepsilon(X, L; 1)$ を

$$\varepsilon(X, L; 1) := \varepsilon(X, L; p)$$

と定義できる. ここで $p \in X$ は非常に一般な点とする.

同様にして多重点の場合も以下のように定義できる. 重み $\overline{m} = (m_1, \ldots, m_r) \in \{x \in \mathbb{R} \mid x > 0\}^r$ に対し, $\varepsilon(X, L; \overline{m})$ を

$$\varepsilon(X, L; \overline{m}) = \inf_C \frac{C.L}{\sum_{i=1}^r m_i \, \mathrm{mult}_{p_i}(C)}$$

と定義する. ここで $p_1, \ldots, p_r \in X$ は非常に一般な点であり, C は少なくとも 1 つの p_i を通る X 上のすべての曲線を動く.

Seshadri 定数は様々な興味深い性質を持ち, また他の概念とも関わっていることが分かっている (cf. [La, Chapter 5]). しかしながら与えられた偏極代数多様体に対し, 具体的に Seshadri 定数を計算することは非常に難しい. 特に高次元の場合には知られている計算例が極めて少ない状況である.

定義からわかるように Seshadri 定数の上からの評価は比較的得られやすい．つまり $C.L$ が小さく $\mathrm{mult}_p(C)$ が大きい曲線 C を見つけてやれば良い．一方下からの評価については有効な方法は余り知られていない．本論文では任意次元における Seshadri 定数の（特に下からの）評価方法を調べた．

2 Cayley 多面体の代数幾何的特徴付け

まず Cayley 多面体と r-平面を以下のように定義する．

定義 2.1. 自然数 r に対し，整多面体 $P \subset \mathbb{R}^n$ が長さ $r+1$ の *Cayley* 多面体であるとは，ある全射群準同型 $\mathbb{Z}^n \to \mathbb{Z}^r$ から誘導される線形写像 $\mathbb{R}^n \to \mathbb{R}^r$ による P の像が r 次元ユニモジュラー単体になることとする．

定義 2.2. 自然数 r に対し，$(\mathbb{P}^r, \mathcal{O}(1))$ と同型な偏極代数多様体を r-平面と呼ぶ．偏極代数多様体 (X, L) が r-平面で覆われるとは一般の点 $p \in X$ に対し，p を含む部分多様体 $Z \subset X$ で $(Z, L|_Z)$ が r-平面であるものが存在することとする．$r = 1$ の時は，直線で覆われると呼ぶこととする．

第 2 章では，次を証明した．

定理 2.3 (=Theorem 2.1.1)**.** *$P \subset \mathbb{R}^n$ を次元 n の整多面体とする．この時 P が長さ $r+1$ の Cayley 多面体であることと P に対応する偏極トーリック多様体 (X_P, L_P) が r-平面で覆われることは同値である．*

この定理を用いて，$\varepsilon(X_P, L_P; 1_P) = 1$ となる整多面体 P の具体的な記述を得た；

定理 2.4 (=Theorem 2.1.2)**.** *$P \subset \mathbb{R}^n$ を次元 n の整多面体とする．この時以下は同値である；*

i) P は長さ 2 の *Cayley* 多面体である，
ii) (X_P, L_P) は直線で覆われる，
iii) $\varepsilon(X_P, L_P; p) = 1$ が任意の $p \in X_P$ で成り立つ，
iv) $\varepsilon(X_P, L_P; 1_P) = 1$ がトーラスの単位元 $1_P \in (\mathbb{C}^\times)^n \subset X_P$ で成り立つ．

3 トーリック退化を用いた Seshadri 定数の評価

第 3 章ではまず n 次元整多面体 $P \subset \mathbb{R}^n$ とその面 σ に対し，不変量 $s_1(P;\sigma), s_2(P;\sigma)$ を定義した．この不変量を用いて以下の定理を示した；

定理 3.1 (=Theorem 3.1.1)**.** *$P \subset \mathbb{R}^n$ を次元 n の整多面体，σ を P の面とする．この時，*

$$s_1(P;\sigma) \le \varepsilon(X_P, L_P; p) \le s_2(P;\sigma)$$

が任意の $p \in O_\sigma$ について成り立つ．ここで $O_\sigma \subset X_P$ は σ に対応する軌道である．

ここで注意すべきことは，$s_1(P;\sigma), s_2(P;\sigma)$ が $\varepsilon(X_P, L_P; p)$ に比べ計算しやすいということである．さらにしばしば $s_1(P;\sigma) = s_2(P;\sigma)$ が成立するので，その場合は $\varepsilon(X_P, L_P; p)$ が計算できる．

定義 1.2 で述べたように Seshadri 定数は下半連続性を持つ．従って，トーリック退化を考えることで様々な場合の Seshadri 定数の下界を求める事ができる．

定理 3.2 (=Theorem 3.1.3)**.** *$X_d^n \subset \mathbb{P}^{n+1}$ を次数 d の非常に一般な超曲面とする．この時*

$$\lfloor \sqrt[n]{d/(m_1^n + \cdots + m_r^n)} \rfloor \le \varepsilon(X_d^n, \mathcal{O}(1); \overline{m}) \le \sqrt[n]{d/(m_1^n + \cdots + m_r^n)}$$

が任意の $\overline{m} = (m_1, \ldots, m_r) \in (\mathbb{N} \setminus 0)^r$ に対して成り立つ．

特に

$$\lfloor \sqrt[n]{d} \rfloor \leq \varepsilon(X_d^n, \mathcal{O}(1); 1) \leq \sqrt[n]{d}$$

が成り立つ.

定理 3.3 (=Theorem 3.1.5). *Picard 数が 1 の滑らかな Fano 3 次元代数多様体の各族に対し, (そのような族は 17 個ある),* $\varepsilon(X, -K_X; 1)$ は論文中の表 3.1 の様になる. ここで X は族の中の非常に一般の元とする.

4 Seshadri 定数と Okounkov 体

第 4 章ではまず, 代数多様体 X 上の双有理 (ある種の正値性を保証する条件) な次数付き線形系 $W_\bullet$ と重み $\overline{m}$ に対し非常に一般の点における重み $\overline{m}$ の Seshadri 定数 $\varepsilon(W_\bullet; \overline{m})$ を定義した. 非常に一般の点での Seshadri 定数は巨大な因子については Nakamaye [Na] により定義されており, $\varepsilon(W_\bullet; \overline{m})$ はその自然な一般化である.

$\Delta \subset \mathbb{R}^n$ を n 次元凸集合とすると, $(\mathbb{C}^\times)^n$ 上の双有理な次数付き線形系 $W_{\Delta,\bullet} := \{V_{k\Delta}\}_k$ が定まる. ここで $V_{k\Delta} = \bigoplus_{u \in k\Delta \cap \mathbb{Z}^n} \mathbb{C}x^u \subset \mathbb{C}[\mathbb{Z}^n]$ である. 従って Δ の不変量 $s(\Delta; \overline{m}) := \varepsilon(W_{\Delta,\bullet}; \overline{m})$ を重み $\overline{m}$ に対して定義することができる.

一方, n 次元代数多様体 X 上の双有理な次数付き線形系 $W_\bullet$, X のある滑らかな点での局所座標系 $z = (z_1, \ldots, z_n)$, 及び $\mathbb{N}^n$ 上の単項式順序 $>$ を決めると, Okounkov 体 $\Delta_{z,>}(W_\bullet)$ と呼ばれる $\mathbb{R}^n$ 内の n 次元閉凸集合が定義できる. これは Lazarsfeld-Mustaţă [LM] と Kaveh-Khovanskii [KK] により独立に導入された. Okounkov 体はトーリック多様体におけるモーメント多面体のある種の一般化であり, $W_\bullet$ の様々な漸近的性質を含んでいる.

第 4 章では Seshadri 定数と Okounkov 体の以下のような関係性を証明した;

定理 4.1 (=Theorem 4.1.1). $W_\bullet$ を n 次元代数多様体 X 上の双有理な次数付き線形系とする. X のある滑らかな点での局所座標系 $z = (z_1, \ldots, z_n)$ と $\mathbb{N}^n$ 上の単項式順序 $>$ を固定する.

この時 $\varepsilon(W_\bullet; \overline{m}) \geq s(\Delta_{z,>}(W_\bullet); \overline{m})$ が任意の $r \in \mathbb{N} \setminus 0$ と $\overline{m} \in \{x \in \mathbb{R} \,|\, x > 0\}^r$ について成り立つ.

この定理により Okounkov 体が Seshadri 定数の情報を持っていることがわかる.

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論文題目: How to estimate Seshadri constants
(セシャドリ定数を評価する方法)

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Contents

Preface

Seshadri constants, which are invariants measuring the local positivity of line bundles, were defined by Demailly [Dem] about twenty years ago. It turns out that Seshadri constants have some interesting properties and relate to other topics, e.g., jet separations of adjoint bundles, Ross-Thomas' slope stabilities of polarized varieties, Gromov width, and so on. But unfortunately, it is very difficult to compute or estimate Seshadri constants in general, especially in higher dimensions. The purpose of this thesis is to investigate how to estimate Seshadri constants in any dimension.

In Chapter 1, we denote some notations, conventions, and definitions about Seshadri constants and toric varieies.

In Chapter 2, we verify an algebro-geometric characterization of Cayley polytopes by seeing lines, planes, etc. on the corresponding toric varieties. From this characterization, we clarify when the Seshadri constant on a polarized toric variety is one.

In Chapter 3, we give lower and upper bounds of Seshadri constants on toric varieties at first. By using this lower bounds and toric degenerations, we obtain some new computations or estimations of Seshadri constants on non-toric varieties. In particular, we investigate Seshadri constants on hypersurfaces on projective spaces, and Fano 3-folds with Picard number one in detail.

In Chapter 4, we consider a relation between Seshadri constants and Okounkov bodies. Firstly, we define Seshadri constants for graded linear series. Secondly, we show the main theorem of this chapter, which states that Okounkov bodies give lower bounds of Seshadri constants. We also investigate Seshadri constants on toric varieties from a viewpoint different from that of Chapter 3.

Chapters 2, 3, and 4 are based on [It1], [It2], and [It3] respectively. Throughout this thesis, we will consider varieties or schemes over the complex number field $\mathbb{C}$.

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1 Preliminaries

1.1 Notations and conventions

We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ the set of all natural numbers, integers, rational numbers, real numbers, and complex numbers respectively. In this thesis, $\mathbb{N}$ contains 0. Set $\mathbb{R}_{>0} = \{x \in \mathbb{R} \,|\, x > 0\}, \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \,|\, x \geq 0\}$, and $\mathbb{C}^\times = \mathbb{C} \setminus 0$. For $x \in \mathbb{R}$, $\lfloor x \rfloor, \lceil x \rceil \in \mathbb{Z}$ are the round down and the round up of x respectively. We denote by $e_1, \dots, e_n$ the standard basis of $\mathbb{Z}^n$ or $\mathbb{R}^n$.

Unless otherwise stated, M stands for a free abelian group of rank $n \in \mathbb{N}$ in this thesis. We define $M_K = M \otimes_{\mathbb{Z}} K$ for any field K. For a subset $S \subset M_{\mathbb{R}}$ and $t \in \mathbb{R}_{\geq 0}$, $tS := \{tu \mid u \in S\}$. For another subset $S' \subset \mathbb{R}^n$, $S + S' = \{u + u' \,|\, u \in S, u' \in S'\}$ means the Minkowski sum of S and S'. For $u \in M_{\mathbb{R}}$, $S + u := \{u' + u \mid u' \in S\}$ is the parallel translation of S by u. We denote the convex hull of S by $\mathrm{conv}(S)$. We write $\Sigma(S)$ for the closed convex cone S spans.

For a convex set $\Delta \subset M_{\mathbb{R}}$, we denote by $\mathrm{vol}_M(\Delta)$ or $\mathrm{vol}(\Delta)$ the Euclidean volume of Δ under an identification of $M \subset M_{\mathbb{R}}$ with $\mathbb{Z}^n \subset \mathbb{R}^n$. Of course, $\mathrm{vol}(\Delta)$ does not depend on the identification. When $n = 1$, we write it $|\Delta|_M$ or $|\Delta|$, and call it the length of Δ. The dimension of Δ is the dimension of the affine space spanned by Δ.

A subset $P \subset M_{\mathbb{R}}$ is called a polytope if it is the convex hull of a finite set in $M_{\mathbb{R}}$. A polytope P is integral (resp. rational) if all vertices are in M (resp. $M_{\mathbb{Q}}$).

For free abelian groups M and M' of rank n and r, an $\mathbb{R}$-linear map $\pi : M_{\mathbb{R}} \to M'_{\mathbb{R}}$ is called a lattice projection if π is induced from a surjective group homomorphism $M \to M'$.

For a subset S in a topological space, we denote by $\bar{S}$ and S° the closure and the interior of S respectively.

For a variety X, we say a property holds at a general point of X if it holds for all points in a non-empty Zariski open subset. A property holds at a very general point of X if it holds for all points in the intersection of countably many non-empty Zariski open subset.

Throughout this thesis, a divisor means a Cartier divisor. Hence a $\mathbb{Q}$-,$\mathbb{R}$-divisor means a $\mathbb{Q}$-,$\mathbb{R}$-Cartier $\mathbb{Q}$-,$\mathbb{R}$-Weil divisor respectively. We use the words "divisor", "line bundle", and "invertible sheaf" interchangeably. Thus we sometimes denote $L^{\otimes k}$ by kL for a line bundle L and $k \in \mathbb{Z}$. For divisors D and D', the inequality $D \geq D'$ means $D - D'$ is effective.

We call a pair (X, L) a ($\mathbb{Q}$-)polarized variety if X is a projective variety and L is an ample ($\mathbb{Q}$-)line bundle on X. The normalization of a $\mathbb{Q}$-polarized variety (X, L) is $(X^{nor}, \pi^* L)$, where $\pi : X^{nor} \to X$ is the normalization of X.

1.2 Seshadri constants

Demailly [Dem] defined an interesting invariant, Seshadri constant, which measures the local positivity of a line bundle on a projective variety:

Definition 1.2.1. Let L be a nef line bundle on a projective variety X, and take a (possibly singular) closed point $p \in X$. We define the Seshadri constant of L at p to be

$$\varepsilon(X, L; p) = \varepsilon(L; p) := \inf_C \left\{ \frac{C.L}{\operatorname{mult}_p(C)} \right\},$$

where C moves all reduced and irreducible curves on X passing through p, and $\operatorname{mult}_p(C)$ is the multiplicity of C at p.

Remark 1.2.2. It is easily shown that $\varepsilon(L; p) = \max\{\, t \geq 0 \mid \mu^* L - tE \text{ is nef} \,\}$, where $\mu : \widetilde{X} \to X$ is the blowing up at p and $E = \mu^{-1}(p)$ is the exceptional divisor (cf. [La2, Chapter 5]). Hence there is an inequality $\varepsilon(L; p) \leq \sqrt[n]{L^n / \operatorname{mult}_p(X)}$ for any point $p \in X$, where n is the dimension of X.

For a subvariety Y of X, $\varepsilon(X, L; p) \leq \varepsilon(Y, L|_Y; p)$ holds for any $p \in Y \subset X$ by the definition of Seshadri constants. We will use this later repeatedly.

In flat families, ampleness is an open condition in the base. Thus the map $p \mapsto \varepsilon(X, L; p)$ from the set of smooth closed points in X to $\mathbb{R}$ has some lower-semicontinuity. Hence $\varepsilon(X, L; p)$ does not depend on the choice of p if p is very general (cf. [La2, Example 5.1.11]). Thus we can define the following:

Definition 1.2.3. Let L be a nef line bundle on a projective variety X. The Seshadri constant $\varepsilon(X, L; 1)$ of L at a very general point is defined to be

$$\varepsilon(X, L; 1) := \varepsilon(X, L; p)$$

for a very general point $p \in X$.

The definition of Seshadri constants can be generalized to multi-points cases easily (cf. [La2, Definition 5.4.1], [BDH+, Definition 1.9]):

Definition 1.2.4. Let L be a nef line bundle on a projective variety X. For $r \in \mathbb{N} \setminus 0$, $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{R}^r_{>0}$, and r points $p_1, \ldots, p_r \in X$, the Seshadri constant $\varepsilon(X, L; m_1 p_1, \ldots, m_r p_r)$ of L at $p_1, \ldots, p_r$ with wight $\overline{m}$ is

$$\varepsilon(X, L; m_1 p_1, \ldots, m_r p_r) := \inf_C \left\{ \frac{C.L}{\Sigma_{i=1}^r m_i \operatorname{mult}_{p_i}(C)} \right\},$$

where C moves all reduced and irreducible curves on X passing through at least one of $p_1, \ldots, p_r$. In the same way as Remark 1.2.2, it holds that

$$\varepsilon(X, L; m_1 p_1, \ldots, m_r p_r) = \max\{\, t \geq 0 \mid \mu^* L - t \Sigma_{i=1}^r m_i E_i \text{ is nef} \,\},$$

where $\mu : \widetilde{X} \to X$ is the blowing up at $p_1, \ldots, p_r$ and $E_i = \mu^{-1}(p_i)$ is the exceptional divisor over p_i.

As Definition 1.2.3, we define the Seshadri constant $\varepsilon(X, L; \overline{m})$ of L at very general points with weight $\overline{m}$ as follows:

$$\varepsilon(X, L; \overline{m}) = \varepsilon(X, L; m_1, \ldots, m_r) := \varepsilon(X, L; m_1 p_1, \ldots, m_r p_r)$$

for very general points $p_1, \ldots, p_r \in X$.

Seshadri constants sometimes have interesting geometric consequences. For example, lower bounds of Seshadri constants induce jet separations of adjoint linear series [Dem] and lower bounds of Gromov width (an invariant in symplectic geometry) [MP]. Upper bounds sometimes give fibrations or foliations [Na1], [Na2], [HW]. Seshadri constants are used to define the Ross-Thomas' slope stabilities for polarized varieties [RT].

But unfortunately it is not easy to compute or estimate Seshadri constants in general. Many authors study about surfaces, but estimations in higher dimensional cases are very few. In higher dimensional cases, the following are known:

In [EKL], Ein, Küchle, and Lazarsfeld show that $\varepsilon(X, L; 1) \geq 1/\dim X$ holds for any polarized variety (X, L). By [La1] and [Bau], lower bounds of Seshadri constants are obtained for abelian varieties. In [Di] or [BDH+], Seshadri constants on toric varieties at torus invariants points are computed. Somewhat surprisingly, we do not know how to compute the Seshadri constant on a polarized toric variety at a not necessarily torus invariant point in general.

1.3 Toric varieties

In this section, we prepare notations about toric varieties used in this thesis. We refer the reader to [Fu] for a further treatment.

As stated in Notations and conventions, M is a free abelian group of rank n.

Definition 1.3.1. Let $\Gamma \subset \mathbb{N} \times M$ be a finitely generated subsemigroup such that $\Gamma \cap (\{0\} \times M) = \{0\}$ and Γ generates $\mathbb{Z} \times M$ as a group. We define a not necessarily normal $\mathbb{Q}$-polarized toric variety $(X(\Gamma), L(\Gamma))$ as follows:

$$(X(\Gamma), L(\Gamma)) := (\operatorname{Proj} \mathbb{C}[\Gamma], \mathcal{O}_{\operatorname{Proj} \mathbb{C}[\Gamma]}(1)).$$

Note that the torus $T_M := \operatorname{Spec} \mathbb{C}[M]$ naturally acts on $(X(\Gamma), L(\Gamma))$, and T_M is embedded in $X(\Gamma)$ as the maximal orbit. If we fix an isomorphism $M \cong \mathbb{Z}^n$, the action $T_M \times X(\Gamma) \to X(\Gamma)$ is the extension of the group structure

$$(\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^n : (a, b) \mapsto (a_1 b_1, \ldots, a_n b_n),$$

where $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in (\mathbb{C}^\times)^n \cong T_M$.

The moment polytope $\Delta(\Gamma)$ of $(X(\Gamma), L(\Gamma))$ is defined to be

$$\Delta(\Gamma) := \Sigma(\Gamma) \cap (\{1\} \times M_\mathbb{R}) \subset \{1\} \times M_\mathbb{R},$$

which can be regarded as a rational polytope in $M_\mathbb{R}$ naturally.

The above $X(\Gamma)$ is not necessarily normal. From rational polytopes, we can obtain normal toric varieties:

Definition 1.3.2. For a rational polytope $P \subset M_\mathbb{R}$ of dimension n, we define the normal $\mathbb{Q}$-polarized toric variety (X_P, L_P) as

$$(X_P, L_P) := (X(\Gamma_P), L(\Gamma_P)),$$

where $\Gamma_P := \Sigma(\{1\} \times P) \cap (\mathbb{N} \times M)$. We write the maximal orbit of X_P as O_P, and denote by $1_P \in T_M = O_P$ the identity of the torus. For a face σ of P, there is a natural closed embedding $X_\sigma \hookrightarrow X_P$. Hence we can regard X_σ as a closed subvariety of X_P, and O_σ is considered as a T_M-orbit in X_P.

Assume P is integral. Then $L_P = \mathcal{O}_{\mathrm{Proj}[\Gamma_P]}(1)$ is an invertible sheaf, i.e., (X_P, L_P) is a polarized variety. By definition, a lattice point u in $P \cap \mathbb{Z}^n$ corresponds to the global section x^u in $H^0(X_P, L_P)$. It is well known that such global sections form a basis of $H^0(X_P, L_P)$ and the linear system $|L_P|$ is base point free. We denote by ϕ_P the morphism $X_P \to \mathbb{P}^N$ defined by $|L_P|$, where $N = \#(P \cap \mathbb{Z}^n) - 1$. Note that ϕ_P is a finite morphism onto the image.

Remark 1.3.3. For any Γ as in Definition 1.3.1, the normalization of $(X(\Gamma), L(\Gamma))$ is $\mu : (X_{\Delta(\Gamma)}, L_{\Delta(\Gamma)}) \to (X(\Gamma), L(\Gamma))$ induced by $\Gamma \hookrightarrow \Sigma(\Gamma) \cap (\mathbb{N} \times M)$ (cf. [Ei, Exercise 4.22]).

Remark 1.3.4. For any integral polytope $P \subset M_{\mathbb{R}}$ of dimension n and any face $\sigma \prec P$, $\varepsilon(X_P, L_P; p)$ is constant for $p \in O_\sigma$ because of the torus action. In particular, $\varepsilon(X_P, L_P; 1)$ (in the sense of Definition 1.2.3) coincides with $\varepsilon(X_P, L_P; 1_P)$.

2

Algebro-geometric characterization of Cayley polytopes

2.1 Introduction

Let $P_0, \ldots, P_r$ be integral polytopes in $\mathbb{R}^s$. The Cayley sum $P_0 * \cdots * P_r$ is defined to be the convex hull of $(P_0 \times 0) \cup (P_1 \times e_1) \cup \ldots \cup (P_r \times e_r)$ in $\mathbb{R}^s \times \mathbb{R}^r$ for the standard basis $e_1, \ldots, e_r$ of $\mathbb{R}^r$.

An integral polytope $P \subset \mathbb{R}^n$ is said to be a Cayley polytope of length $r+1$, if there exists an affine isomorphism $\mathbb{Z}^n \cong \mathbb{Z}^{n-r} \times \mathbb{Z}^r$ identifying P with the Cayley sum $P_0 * \cdots * P_r$ for some integral polytopes $P_0, \ldots, P_r$ in $\mathbb{R}^{n-r}$. In other words, P is a Cayley polytope of length $r+1$ if and only if P is mapped onto a unimodular r-simplex by a lattice projection $\mathbb{R}^n \to \mathbb{R}^r$.

Cayley polytopes are related to discriminants, resultants, and dual defects. See for instance [DDP], [DN], [GKZ].

On the other hand, a polarized toric variety (X_P, L_P) is defined for any integral polytope $P \subset \mathbb{R}^n$ of dimension n.

In this chapter, we give an algebro-geometric characterization of Cayley polytopes:

Theorem 2.1.1. *Let $P \subset \mathbb{R}^n$ be an integral polytope of dimension n. Then P is a Cayley polytope of length $r+1$ if and only if (X_P, L_P) is covered by r-planes.*

See Definition 2.2.4 for the definition of "covered by r-planes". An important point to note here is that we do not need any assumption on the singularities of X_P nor the lattice spanned by $P \cap \mathbb{Z}^n$. As a corollary of this theorem, we obtain a sufficient condition such that an integral polytope P is a Cayley polytope by using dual defects.

We investigate the case $r = 1$ a little more. For a polarized variety (X, L), we can define the Seshadri constant $\varepsilon(X, L; 1)$ of (X, L) at a very general point as Definition 1.2.3. In the following theorem, we characterize Cayley polytopes of length 2 by using Seshadri constants:

Theorem 2.1.2. *Let $P \subset \mathbb{R}^n$ be an integral polytope of dimension n. Then the following are equivalent:*

i) P is a Cayley polytope of length 2,

ii) (X_P, L_P) is covered by lines,

iii) $\varepsilon(X_P, L_P; p) = 1$ for any $p \in X_P$,

iv) $\varepsilon(X_P, L_P; 1_P) = 1$ for the identity of the torus $1_P \in (\mathbb{C}^\times)^n \subset X_P$.

In general, it is very difficult to compute Seshadri constants. Theorem 2.1.2 gives an explicit description for which integral polytope P the Seshadri constant $\varepsilon(X_P, L_P; 1)$ is one.

This chapter is organized as follows: In Section 2, we make some preliminaries. In Section 3, we give the proof of Theorem 2.1.1. In Section 4, we state a relation of Cayley polytopes and dual defects. In Section 5, we prove Theorem 2.1.2.

2.2 Preliminaries

2.2.1 Cayley polytopes and r-planes

Firstly, we define Cayley polytopes and lattice width.

Definition 2.2.1. Let P be an integral polytope in $\mathbb{R}^n$ and r a positive integer. We say P is a Cayley polytope of length $r+1$ if there exists a lattice projection onto a unimodular r-simplex. A unimodular r-simplex is an integral polytope in $\mathbb{R}^r$ which is identified with $\mathrm{conv}(0, e_1, \ldots, e_r)$ by a $\mathbb{Z}$-affine translation.

Definition 2.2.2. Let P be an integral polytope in $\mathbb{R}^n$. The lattice width of P is the minimum of $\max_{u\in P}\langle u, v\rangle - \min_{u\in P}\langle u, v\rangle$ over all non-zero integer linear forms v.

Remark 2.2.3. See [BN] or [DHNP] for other definitions of Cayley polytopes. Note that P has lattice width one if and only if P is an n-dimensional Cayley polytope of length 2.

We define r-planes as follows:

Definition 2.2.4. Let r be a positive integer. A polarized variety (X, L) is called an r-plane if it is isomorphic to $(\mathbb{P}^r, \mathcal{O}(1))$ as a polarized variety. Sometimes we say X is an r-plane if the polarization L is clear (e.g. a subvariety in a polarized variety).

Let (X, L) be a polarized variety. We say that (X, L) is covered by r-planes, if for any general point $p \in X$ there exists an r-plane $Z \subset X$ containing p. When $r = 1$, we say that (X, L) is covered by lines.

2.2.2 Lemma about toric varieties

We will use the following lemma repeatedly in the subsequent sections. This is well known, but we prove it for the convenience of the reader:

Lemma 2.2.5. *Let $P \subset \mathbb{R}^n$ be an integral polytope of dimension n and $\pi : \mathbb{R}^n \to \mathbb{R}^r$ a lattice projection. Then there is a birational finite morphism onto the image $\iota : X_{\pi(P)} \to X_P$ such that $\iota^*(L_P) = L_{\pi(P)}$.*

Proof. Consider the following diagram:

$$\begin{array}{ccc} \Sigma(\{1\}\times P) & \hookrightarrow & \mathbb{R}\times\mathbb{R}^n \\ \downarrow & \circlearrowleft & \downarrow{\scriptstyle id_{\mathbb{R}}\times\pi} \\ \Sigma(\{1\}\times \pi(P)) & \hookrightarrow & \mathbb{R}\times\mathbb{R}^r. \end{array}$$

By intersecting with $\mathbb{N}\times\mathbb{Z}^n$ or $\mathbb{N}\times\mathbb{Z}^r$, we have

$$\begin{array}{ccc} \Gamma_P = \Sigma(\{1\}\times P)\cap(\mathbb{N}\times\mathbb{Z}^n) & \hookrightarrow & \mathbb{N}\times\mathbb{Z}^n \\ \downarrow & \circlearrowleft & \downarrow{\scriptstyle id_{\mathbb{N}}\times\pi|_{\mathbb{Z}^n}} \\ \Gamma_{\pi(P)} = \Sigma(\{1\}\times \pi(P))\cap(\mathbb{N}\times\mathbb{Z}^r) & \hookrightarrow & \mathbb{N}\times\mathbb{Z}^r. \end{array}$$

Set $\Gamma' = (id_{\mathbb{N}} \times \pi|_{\mathbb{Z}^r})(\Gamma_P)$. Then the above diagram induces

$$\begin{array}{ccc} X_P = \operatorname{Proj} \mathbb{C}[\Gamma_P] & \hookleftarrow & (\mathbb{C}^\times)^n \\ \uparrow & \circlearrowleft & \uparrow \\ \operatorname{Proj} \mathbb{C}[\Gamma'] & \hookleftarrow & (\mathbb{C}^\times)^r. \end{array}$$

Note that Γ' generates $\mathbb{Z} \times \mathbb{Z}^r$ as a group and $\Sigma(\Gamma') = \Sigma(\{1\} \times \pi(P))$ because P is n-dimensional. Thus there exists the normalization morphism

$$\iota : X_{\pi(P)} \to \operatorname{Proj} \mathbb{C}[\Gamma'] \ (\hookrightarrow X_P).$$

By the construction of ι, it is clear that $\iota^*(L_P) = L_{\pi(P)}$. □

2.3 Proof of Theorem 2.1.1

This section is devoted to the proof of Theorem 2.1.1=Theorem 2.3.1:

Theorem 2.3.1. *Let $P \subset \mathbb{R}^n$ be an integral polytope of dimension n. Then P is a Cayley polytope of length $r+1$ if and only if (X_P, L_P) is covered by r-planes.*

Proof. If P is a Cayley polytope of length $r+1$, then there exists a birational finite morphism onto the image $\iota : \mathbb{P}^r \to X_P$ by Lemma 2.2.5. Furthermore, there exists a finite morphism $\phi = \phi_P : X_P \to \mathbb{P}^N$. We denote by $Z \subset X_P$ the image of $\mathbb{P}^r$ by the morphism ι. Set $Z' = \phi \circ \iota(\mathbb{P}^r) = \phi(Z)$ in $\mathbb{P}^N$. Since $(\phi \circ \iota)^* \mathcal{O}_{\mathbb{P}^N}(1) = \iota^* L_P = \mathcal{O}_{\mathbb{P}^r}(1)$, it holds that

$$1 = \mathcal{O}_{\mathbb{P}^r}(1)^r = \deg(\phi \circ \iota) \cdot \mathcal{O}_{Z'}(1)^r,$$

where $\deg(\phi \circ \iota)$ is the degree of the finite morphism $\phi \circ \iota : \mathbb{P}^r \to Z'$. Both of $\deg(\phi \circ \iota)$ and $\mathcal{O}_{Z'}(1)^r$ are positive integers. Hence we have $\deg(\phi \circ \iota) = \mathcal{O}_{Z'}(1)^r = 1$. Thus $Z' \subset \mathbb{P}^N$ is an r-plane and $\phi \circ \iota$ is birational. Since Z' is smooth and $\phi \circ \iota$ is a birational finite morphism, $\phi \circ \iota : \mathbb{P}^r \to Z'$ is an isomorphism. Hence $\iota : (\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \to (Z, L_P|_Z)$ is also an isomorphism, i.e., Z is an r-plane in X_P. Since $Z \cap O_P$ is not empty, (X_P, L_P) is covered by r-planes by the torus action.

Let us prove the converse. Assume that (X_P, L_P) is covered by r-planes. We may assume that $0 \in \mathbb{Z}^n$ is a vertex of P. Thus throughout this proof, set $P \cap \mathbb{Z}^n = \{u_0, \ldots, u_N\}$ and $u_0 = 0$.

Since (X_P, L_P) is covered by r-planes, there exists an r-plane $Z \subset X_P$ containing $1_P = (1, \ldots, 1) \in (\mathbb{C}^\times)^n = O_P \subset X_P$. Let $Z' \subset \mathbb{P}^N$ be the image of Z by $\phi = \phi_P : X_P \to \mathbb{P}^N$. Then it is easy to see that $\phi : (Z, L_P|_Z) \to (Z', \mathcal{O}(1))$ is an isomorphism and $Z' \subset \mathbb{P}^N$ is an r-plane by the argument similar to that of the "only if" part of this proof. Note that Z' contains $1_N = (1, \ldots, 1) \in (\mathbb{C}^\times)^N \subset \mathbb{P}^N$ because $\phi(1_P) = 1_N$.

Our idea of the proof is the following:
Step 1. *By using the torus action, we degenerate Z to another r-plane $\widetilde{Z} \subset X_P$ containing 1_P such that the embedding $\mathbb{P}^r \cong \phi(\widetilde{Z}) \hookrightarrow \mathbb{P}^N$ is a toric morphism, i.e., a morphism induced by a lattice projection $\mathbb{R}^N \to \mathbb{R}^r$ as Lemma 2.2.5.*
Step 2. *By using the above lattice projection, we define another lattice projection $\mathbb{R}^n \to \mathbb{R}^r$ which maps P onto an r-simplex.*

To clarify the idea of the proof, we first show the case $r = 1$. When $r = 1$, we write C, l instead of Z, Z'.

Step 1. By definition, l is a line on $\mathbb{P}^N$ containing 1_N. Thus we can write

$$l \cap \mathbb{C}^N = 1_N + \mathbb{C}a$$

for some vector $a = (a_1, \ldots, a_N) \in \mathbb{C}^N \setminus 0$, where $\mathbb{C}^N \subset \mathbb{P}^N$ is the open set defined by $T_0 \neq 0$ for the homogeneous coordinates $T_0, \ldots, T_N$. For any point p in O_P, there exist automorphisms $p^{-1} \cdot : X_P \to X_P$ and $\phi(p)^{-1} \cdot : \mathbb{P}^N \to \mathbb{P}^N$ by torus actions. Hence any point p in $C \cap O_P$ induces an isomorphism

$$\phi|_{p^{-1} \cdot C} : p^{-1} \cdot C \to \phi(p)^{-1} \cdot l.$$

Since $1_P = p^{-1} \cdot p$ is contained in $p^{-1} \cdot C$, the line $\phi(p)^{-1} \cdot l$ contains 1_N. From this, we have

$$(\phi(p)^{-1} \cdot l) \cap \mathbb{C}^N = 1_N + \mathbb{C}\ \phi(p)^{-1} \cdot a.$$

Let us denote $\phi(p) = 1_N + t_p a \in l \cap (\mathbb{C}^\times)^N$ for $t_p \in \mathbb{C}$. Moving $p \in C \cap O_P$ so that $|t_p| \to +\infty$ and taking limits, we have a morphism

$$\phi|_{\widetilde{C}} : \widetilde{C} \to \tilde{l},$$

where $\widetilde{C} \subset X_P$ and $\tilde{l} \subset \mathbb{P}^N$ are the limits of $p^{-1} \cdot C$ and $\phi(p)^{-1} \cdot l$ in the Hilbert schemes respectively. A limit of lines is also a line, hence $\phi : (\widetilde{C}, L_P|_{\widetilde{C}}) \to (\tilde{l}, \mathcal{O}(1)) = (\mathbb{P}^1, \mathcal{O}(1))$ is also isomorphic. Since $p^{-1} \cdot C$ contains 1_P for any p, so does $\widetilde{C}$.

When $|t_p| \to +\infty$,

$$\phi(p)^{-1} \cdot t_p a = ((1 + t_p a_1)^{-1} t_p a_1, \ldots, (1 + t_p a_N)^{-1} t_p a_N)$$

converges to $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_N) \in \mathbb{C}^N$, where

$$\tilde{a}_j = \begin{cases} 1 & \text{if } a_j \neq 0 \\ 0 & \text{if } a_j = 0\ . \end{cases}$$

Since $\tilde{l} \cap \mathbb{C}^N$ is the limit of $1_N + \mathbb{C}\ \phi(p)^{-1} \cdot a = 1_N + \mathbb{C}\ \phi(p)^{-1} \cdot t_p a$, we can write

$$\tilde{l} \cap \mathbb{C}^N = 1_N + \mathbb{C}\tilde{a}.$$

From this description, $\mathbb{C}^\times = \tilde{l} \cap (\mathbb{C}^\times)^N \hookrightarrow (\mathbb{C}^\times)^N \subset \mathbb{P}^N$ can be written as

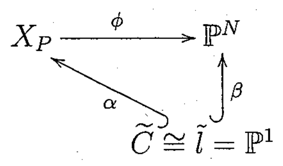
$$t \mapsto (t^{\tilde{a}_1}, \ldots, t^{\tilde{a}_N})$$

for $t \in \mathbb{C}^\times$. Therefore $\mathbb{P}^1 = \tilde{l} \hookrightarrow \mathbb{P}^N$ is the toric morphism defined by the lattice projection

$$\mathbb{R}^N \to \mathbb{R} : e_j \mapsto \tilde{a}_j$$

and $\operatorname{conv}(0, e_1, \ldots, e_N) \subset \mathbb{R}^N$ as Lemma 2.2.5.

Step 2. Restricting the diagram



to the maximal orbits, we have

$$\begin{array}{ccc} (\mathbb{C}^\times)^n & \xrightarrow{\phi} & (\mathbb{C}^\times)^N \\ & \nwarrow^{\alpha} & \uparrow \beta \\ & & \mathbb{C}^\times. \end{array}$$

Considering the coordinate rings, we have

$$\begin{array}{ccc} \mathbb{C}[\mathbb{Z}^n] & \xleftarrow{\phi^*} & \mathbb{C}[\mathbb{Z}^N] \\ & \searrow^{f} & \downarrow g \\ & & \mathbb{C}[\mathbb{Z}] \end{array}$$

By the definition of ϕ and the assumption $u_0 = 0$, the ring homomorphism ϕ^* is induced by the following group homomorphism

$$\pi : \mathbb{Z}^N \to \mathbb{Z}^n; e_j \mapsto u_j$$

for $1 \leq j \leq N$, i.e.,

$$\phi^*(x^{u'}) = x^{\pi(u')}$$

holds for each $u' \in \mathbb{Z}^N$. On a while, g is induced by the surjective group homomorphism

$$\mu : \mathbb{Z}^N \to \mathbb{Z}; e_j \mapsto \tilde{a}_j$$

from Step 1. By using π and μ, we can define a group homomorphism $\pi' : \mathbb{Z}^n \to \mathbb{Z}$ which induces the ring homomorphism f as follows:

For any $u \in \mathbb{Z}^n$, there exists a positive integer m such that mu is contained in $\pi(\mathbb{Z}^N)$. Thus we can take $u' \in \mathbb{Z}^N$ such that $mu = \pi(u')$. By the commutativity of the above diagram, we have $f(x^{mu}) = g(x^{u'}) = x^{\mu(u')}$. Since $f(x^{mu}) = f(x^u)^m$ holds, $\mu(u')/m$ must be contained in $\mathbb{Z}$ and $f(x^u) = c \cdot x^{\mu(u')/m}$ for some $c \in \mathbb{C}^\times$. Furthermore $1 \in \mathbb{C}^\times$ is mapped to $1_P \in X_P$ by α, hence c must be 1. Thus we can define $\pi' : \mathbb{Z}^n \to \mathbb{Z}$ by

$$\pi'(u) = \frac{\mu(u')}{m}.$$

It is easy to show that π' is well defined and a group homomorphism which induces f.

Since $\alpha : \mathbb{P}^1 \to X_P$ is a closed embedding, f is surjective. This means that π' is also surjective. Furthermore $\mu = \pi' \circ \pi$ holds by the definition of π', thus we have $\pi'(u_j) = \pi' \circ \pi(e_j) = \mu(e_j) = \tilde{a}_j \in \{0, 1\}$ for each $j \in \{1, \dots, N\}$. Since P is the convex hull of $0, u_1, \dots, u_N$, the lattice projection induced by π' maps P onto the closed interval $[0, 1]$ in $\mathbb{R}$. This means that P is a Cayley polytope of length 2.

When $r > 1$, the idea is same. Let $Z \subset X_P$ be an r-plane containing 1_P and $Z' = \phi(Z) \subset \mathbb{P}^N$. Since Z' is an r-plane in $\mathbb{P}^N$ containing 1_N, we can write

$$Z' \cap \mathbb{C}^N = 1_N + V,$$

where V is an r-dimensional linear subspace of $\mathbb{C}^N$. Then we can choose a basis $a_1, \dots, a_r \in \mathbb{C}^N$ of V such that $j_1 > j_2 > \dots > j_r \cdots (*)$ holds, where $a_i = (a_{i1}, \dots, a_{iN})$ and $j_i = \min\{j \mid a_{ij} \neq 0\}$. In other words, the matrix ${}^t({}^ta_1, \dots, {}^ta_r)$ is the following type:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} 0 & * & \dots & & & & & \\ 0 & 0 & 0 & * & \dots & & & \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 0 & * & \dots \end{pmatrix}.$$

Similarly to the case $r = 1$, any point $p \in O_P$ induces an isomorphism

$$\phi|_{p^{-1}\cdot Z} : p^{-1} \cdot Z \to \phi(p)^{-1} \cdot Z'.$$

Choose $p \in Z \cap O_P$ satisfying $\phi(p) = 1_N + t_p a_r$ for some $t_p \in \mathbb{C}$ and take limits of $p^{-1} \cdot Z$ and $\phi(p)^{-1} \cdot Z'$ by moving p so that $|t_p| \to +\infty$. We denote the limits by $Z^{(1)} \subset X_P$ and $Z'^{(1)} \subset \mathbb{P}^N$ respectively, both of which are r-planes.

Then

$$\begin{aligned}(\phi(p)^{-1} \cdot Z') \cap \mathbb{C}^N &= 1_N + \phi(p)^{-1} \cdot V \\ &= 1_N + \bigoplus_{i=1}^{r} \mathbb{C}\ \phi(p)^{-1} \cdot a_i\end{aligned}$$

holds because $1_N = \phi(p)^{-1} \cdot \phi(p)$ is contained in $\phi(p)^{-1} \cdot Z'$. For $t_p \neq 0$, it holds that

$$\begin{aligned}\mathbb{C}\ \phi(p)^{-1} \cdot a_i &= \mathbb{C}((1 + t_p a_{r1})^{-1} a_{i1}, \ldots, (1 + t_p a_{rN})^{-1} a_{iN}) \\ &= \mathbb{C}((1 + t_p a_{r1})^{-1} t_p a_{i1}, \ldots, (1 + t_p a_{rN})^{-1} t_p a_{iN}).\end{aligned}$$

When $|t_p| \to +\infty$,

$$\phi(p)^{-1} \cdot a_i = ((1 + t_p a_{r1})^{-1} a_{i1}, \ldots, (1 + t_p a_{rN})^{-1} a_{iN})$$

converges to $a_i^{(1)} = (a_{ij}^{(1)})_j$ for $i \neq r$, where

$$a_{ij}^{(1)} = \begin{cases} a_{ij} & \text{if } a_{rj} = 0 \\ 0 & \text{if } a_{rj} \neq 0\,. \end{cases}$$

Note that $a_i^{(1)}$ is not 0 since $a_{ij_i} \neq 0$ and $a_{rj_i} = 0$ for any $i \neq r$ by $(*)$.

On the other hand,

$$\phi(p)^{-1} \cdot t_p a_r = ((1 + t_p a_{r1})^{-1} t_p a_{r1}, \ldots, (1 + t_p a_{rN})^{-1} t_p a_{rN})$$

converges to $a_r^{(1)} = (a_{rj}^{(1)})_j$, where

$$a_{rj}^{(1)} = \begin{cases} 0 & \text{if } a_{rj} = 0 \\ 1 & \text{if } a_{rj} \neq 0\,. \end{cases}$$

It is easy to see that $a_1^{(1)}, \ldots, a_N^{(1)}$ are linearly independent by the condition $(*)$, thus we have

$$Z'^{(1)} \cap \mathbb{C}^N = 1_N + \bigoplus_{i=1}^{r} \mathbb{C} a_i^{(1)}.$$

Now $Z^{(1)}$ is an r-plane containing 1_P and $\phi|_{Z^{(1)}} : Z^{(1)} \to Z'^{(1)}$ is also isomorphic. We choose $p \in Z^{(1)} \cap O_P$ satisfying $\phi(p) = 1_N + t_p a_{r-1}^{(1)}$ for some t_p, and take limits of $p^{-1} \cdot Z^{(1)}$ and $\phi(p)^{-1} \cdot Z'^{(1)}$ by moving p so that $|t_p| \to +\infty$. We denote the limits by $Z^{(2)}$ and $Z'^{(2)}$ respectively. Note that $a_1^{(1)}, \ldots, a_N^{(1)}$ also satisfy the condition $(*)$. In fact $\min\{j \mid a_{ij} \neq 0\} = \min\{j \mid a_{ij}^{(1)} \neq 0\}$ holds by definition. By similar arguments, we have

$$Z'^{(2)} \cap \mathbb{C}^N = 1_N + \bigoplus_{i=1}^{r} \mathbb{C} a_i^{(2)},$$

where $a_i^{(2)} = (a_{ij}^{(2)})_j$ and

$$a_{ij}^{(2)} = \begin{cases} a_{ij}^{(1)} & \text{if } a_{r-1,j}^{(1)} = 0 \\ 0 & \text{if } a_{r-1,j}^{(1)} \neq 0 \end{cases}$$

for $i \neq r-1$, and

$$a_{r-1,j}^{(2)} = \begin{cases} 0 & \text{if } a_{r-1,j}^{(1)} = 0 \\ 1 & \text{if } a_{r-1,j}^{(1)} \neq 0\,. \end{cases}$$

Note that $a_{rj}^{(1)} = 0$ if $a_{r-1,j}^{(1)} \neq 0$, thus $a_r^{(2)} = a_r^{(1)}$ holds.

By repeating these operations r times, we obtain $Z^{(r)} \subset X_P$ and $Z'^{(r)} \subset \mathbb{P}^N$ containing 1_P and 1_N respectively, and linearly independent vectors $a_1^{(r)}, \ldots, a_r^{(r)} \in \mathbb{C}^N$ such that

i) $\phi|_{Z^{(r)}} : (Z^{(r)}, L_P|_{Z^{(r)}}) \to (Z'^{(r)}, \mathcal{O}(1))$ is an isomorphism,

ii) $Z'^{(r)} \cap \mathbb{C}^N = 1_N + \bigoplus_{i=1}^r \mathbb{C}a_i^{(r)}$,

iii) $a_{ij}^{(r)} = 0$ or 1. For each j, there are at most one i such that $a_{ij}^{(r)} = 1$.

Note that $Z^{(r)}$ and $Z'^{(r)}$ are r-planes by i) and ii). By iii), we can define $i_j \in \{0, 1, \ldots, r\}$ for $j = 1, \ldots, N$ as follows:
If $\{i \,|\, a_{ij}^{(r)} = 1\}$ is not empty, we set $i_j = i$ satisfying $a_{ij}^{(r)} = 1$. Otherwise we set $i_j = 0$.

As in the case $r = 1$, we have the following diagram

$$\begin{array}{ccc} \mathbb{C}[\mathbb{Z}^n] & \xleftarrow{\phi^*} & \mathbb{C}[\mathbb{Z}^N] \\ & \searrow^{f} & \downarrow g \\ & & \mathbb{C}[\mathbb{Z}^r] \end{array}$$

induced from

$$\begin{array}{ccc} X_P & \xrightarrow{\phi} & \mathbb{P}^N \\ & \nwarrow^{\alpha} & \uparrow \beta \\ & & Z^{(r)} \cong Z'^{(r)} = \mathbb{P}^r. \end{array}$$

From the construction of $Z'^{(r)}$ and e_{i_j}, it is easy to see that g is induced by the group homomorphism

$$\mu : \mathbb{Z}^N \to \mathbb{Z}^r : e_j \mapsto e_{i_j},$$

where we consider e_0 as $0 \in \mathbb{Z}^r$. Similar to the case $r = 1$, we can define a surjective group homomorphism $\pi' : \mathbb{Z}^n \to \mathbb{Z}^r$ such that f is induced by π' and $\mu = \pi' \circ \pi$. The surjectivity of π' follows from the that of f. It is easily shown that the lattice projection induced from π' maps P onto $(0, e_1, \ldots, e_r) \in \mathbb{R}^r$ because $\pi'(u_j) = e_{i_j}$ holds for each j. Thus P is a Cayley polytope of length $r + 1$. $\square$

2.4 Dual defects

Cayley polytopes are often studied with related to dual defects.

Definition 2.4.1. Let $X \subset \mathbb{P}^N$ be a projective variety. The dual variety X^* of X is the closure of all points $H \in (\mathbb{P}^N)^\vee$ such that as a hyperplane H contains the tangent space $T_{X,p}$ for some smooth point $p \in X$, where $(\mathbb{P}^N)^\vee$ is the dual projective space. A variety X in $\mathbb{P}^N$ is said to be dual defective if the dimension of X^* is less than $N - 1$. The dual defect of X is the natural number $N - 1 - \dim X^*$.

As an easy corollary of Theorem 2.1.1, we obtain a sufficient condition such that P is a Cayley polytope by using dual defects. This is a generalization of a result proved in [CC] or [Es], which is the case $r = 1$ of the following:

Corollary 2.4.2. *Let P be an integral polytope of dimension n in $\mathbb{R}^n$. Assume that the lattice points in P span the lattice $\mathbb{Z}^n$, and the dual defect of the image $\phi_P(X_P) \subset \mathbb{P}^N$ is a positive integer r. Then P is a Cayley polytope of length $r+1$. In particular, P has lattice width one.*

Proof. It is well known that if a projective variety $X \subset \mathbb{P}^N$ has dual defect r, then X is covered by r-planes (cf. [Te, Theorem 1.18] for example). Thus if the dual defect of $\phi_P(X_P)$ is r, then $\phi_P(X_P)$ is covered by r-planes. The assumption that $P \cap \mathbb{Z}$ spans the lattice $\mathbb{Z}^n$ means that $\phi_P : X_P \to \phi_P(X_P)$ is birational. Since $\phi_P(X_P)$ is covered by r-planes and $\phi_P : X_P \to \phi_P(X_P)$ is a birational finite morphism, (X_P, L_P) is also covered by r-planes. Therefore P is a Cayley polytope of length $r + 1$ by Theorem 2.1.1. □

Remark 2.4.3. (1) In Corollary 2.4.2, the assumption that $P \cap \mathbb{Z}$ spans $\mathbb{Z}^n$ is necessary. For example, let $P \subset \mathbb{R}^3$ be the convex hull of $(0,0,0), (1,1,0), (1,0,1)$ and $(0,1,1)$. Then the image $\phi_P(X_P) \subset \mathbb{P}^3$ is $\mathbb{P}^3$, hence the dual defect of $\phi_P(X_P)$ is 3. But P is not a Cayley polytope of length 4.
(2) The converse of Corollary 2.4.2 does not hold. For example, let $P \subset \mathbb{R}^2$ be the convex hull of $(0,0), (1,0), (0,1), (1,1)$. Then P is a Cayley polytope of length 2, but $\phi_P(X_P) = X_P \subset \mathbb{P}^3$ is a smooth quadric surface, which is not dual defective.
(3) There exists an explicit description of the dual defectivity for (X_P, L_P) if X_P is smooth [DN]. But in singular cases, dual defectivities of toric varieties are not so well known.

2.5 Lattice width one and Seshadri constants

In this section, we characterize integral polytopes with lattice width one by using Seshadri constants.

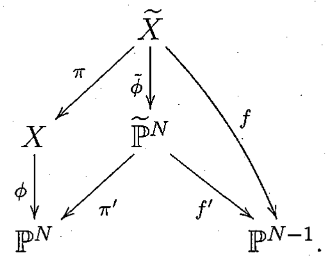
As stated in Chapter 1, it is very difficult to compute Seshadri constants in general. However, if $|L|$ is base point free, we know whether $\varepsilon(X, L; 1) = 1$ or not by considering lines on (X, L):

Proposition 2.5.1. *Let (X, L) be a polarized variety and assume that the linear system $|L|$ is base point free. Then $\varepsilon(X, L; 1) = 1$ if and only if (X, L) is covered by lines.*

Proof. "If" part is easy. In fact if (X, L) is covered by lines, then clearly $\varepsilon(X, L; 1) \leq 1$ holds by the definition of Seshadri constants. On the other hand, it is well known that $\varepsilon(X, L; 1) \geq 1$ holds for any base point free L [La2, Example 5.1.18]. Hence $\varepsilon(X, L; 1) = 1$ holds.

Thus it is enough to show the "only if" part. Assume that $\varepsilon(X, L; 1) = 1$ holds. Let $\phi : X \to \mathbb{P}^N$ be the morphism defined by $|L|$ and d the degree of the finite morphism $\phi : X \to \phi(X)$. Fix a very general point $p \in X$ and set $q = \phi(p) \in \mathbb{P}^N$. Then $\phi^{-1}(q) = \{p_1, \ldots, p_d\}$ is a set of d points in X, where $p = p_1$.

We consider the following diagram:



In the above diagram, $\pi : \widetilde{X} \to X$ and $\pi' : \widetilde{\mathbb{P}}^N \to \mathbb{P}^N$ are the blowing ups along $\{p_1, \ldots, p_d\}$ and q respectively. Then ϕ, π, and π' induce a finite morphism $\tilde{\phi} : \widetilde{X} \to \widetilde{\mathbb{P}}^N$. Let $E_1, \ldots, E_d$ and E be the exceptional divisors over $\{p_1, \ldots, p_d\}$ and q respectively. Since $\pi'^*\mathcal{O}_{\mathbb{P}^N}(1) - E$ is base point free, this induces a morphism $f' : \widetilde{\mathbb{P}}^N \to \mathbb{P}^{N-1}$. Set $f = f' \circ \tilde{\phi} : \widetilde{X} \to \mathbb{P}^{N-1}$. Note that $f^*\mathcal{O}_{\mathbb{P}^{N-1}}(1) = \pi^*L - \Sigma_{i=1}^d E_i$ and f is nothing but the morphism induced by $|\pi^*L - \Sigma_{i=1}^d E_i|$.

By the assumption that $\varepsilon(X, L; 1) = 1$ and p is very general, $\pi^*L - E_1$ is nef but not ample. Thus there exists a subvariety $\widetilde{Z}$ in $\widetilde{X}$ such that $\widetilde{Z}.(\pi^*L - E_1)^{\dim \widetilde{Z}} = 0$ (see [La2, Proposition 5.1.9]). Furthermore, $\widetilde{Z}.(\pi^*L - \Sigma_{i=1}^d E_i)^{\dim \widetilde{Z}} \geq 0$ by the freeness of $\pi^*L - \Sigma_{i=1}^d E_i$. Hence

$$\begin{aligned} 0 &\leq \widetilde{Z}.(\pi^*L - \Sigma_{i=1}^d E_i)^{\dim \widetilde{Z}} \\ &= Z.L^{\dim Z} - \Sigma_{i=1}^d \operatorname{mult}_{p_i}(Z) \\ &\leq Z.L^{\dim Z} - \operatorname{mult}_{p_1}(Z) \\ &= \widetilde{Z}.(\pi^*L - E_1)^{\dim \widetilde{Z}} = 0, \end{aligned}$$

where $Z = \pi(\widetilde{Z})$ is the image of $\widetilde{Z}$ by π and $\operatorname{mult}_{p_i}(Z)$ is the multiplicity of Z at p_i. Thus we obtain

$$0 = \widetilde{Z}.(\pi^*L - \Sigma_{i=1}^d E_i)^{\dim \widetilde{Z}} = \widetilde{Z}.(\pi^*L - E_1)^{\dim \widetilde{Z}}$$

and $\operatorname{mult}_{p_i}(Z) = 0$ for $i \neq 1$. This means $p_i \notin Z$, or equivalently $\widetilde{Z} \cap E_i = \emptyset$ for $i \neq 1$. The equality $\widetilde{Z}.(\pi^*L - \Sigma_{i=1}^d E_i)^{\dim \widetilde{Z}} = 0$ implies $\dim f(\widetilde{Z}) < \dim \widetilde{Z}$. Thus there exists a curve $\widetilde{C} \subset \widetilde{Z}$ such that $f(\widetilde{C})$ is a point. Set $C = \pi(\widetilde{C})$ be the image of $\widetilde{C}$ in X. Since the morphism $\tilde{\phi}$ is finite, $\tilde{\phi}(\widetilde{C})$ is a curve on $\widetilde{\mathbb{P}}^N$ which is contracted by f', that is, $\phi(C)(= \pi'(\tilde{\phi}(\widetilde{C})))$ is a line on $\mathbb{P}^N$ containing q. Note that ϕ is étale onto the image at p, and $\phi(C)$ is smooth at $q = \phi(p)$, hence C is also smooth at p. Since $f(\widetilde{C})$ is a point, $\widetilde{C}.(\pi^*L - \Sigma_{i=1}^d E_i) = 0$ holds and $\widetilde{C} \cap E_i \subset \widetilde{Z} \cap E_i = \emptyset$ for $i \neq 1$. Thus $0 = \widetilde{C}.(\pi^*L - E_1) = C.L - \operatorname{mult}_p(C)$ holds. From this, we have

$$1 = \operatorname{mult}_p(C) = C.L = \phi_*(C).\mathcal{O}_{\mathbb{P}^N}(1) = \deg(\phi|_C : C \to \phi(C)).$$

Thus $\phi|_C : C \to \phi(C) \cong \mathbb{P}^1$ is an isomorphism and $C.L = 1$, i.e., C is a line on X containing p. Therefore (X, L) is covered by lines. □

Remark 2.5.2. The assumption that $|L|$ is base point free is necessary in Proposition 2.5.1. For example, let (S, L) be a non-rational polarized smooth surface such that $L^2 = 1$, e.g., S is a Godeaux surface and L is the canonical divisor K_S. Then $\varepsilon(S, L; 1) = 1$ holds by [EL] and the assumption $L^2 = 1$. But S is not covered by lines because S is non-rational.

By Theorem 2.1.1 and Proposition 2.5.1 , we obtain the following:

Corollary 2.5.3 (=Theorem 2.1.2). *Let $P \subset \mathbb{R}^n$ be an integral polytope of dimension n. Then the following statements are equivalent:*

i) P has lattice width one,

ii) (X_P, L_P) is covered by lines,

iii) $\varepsilon(X_P, L_P; p) = 1$ for any $p \in X_P$,

iv) $\varepsilon(X_P, L_P; 1_P) = 1$ for the identity of the torus $1_P \in (\mathbb{C}^\times)^n \subset X_P$.

Proof. i) $\Leftrightarrow$ ii) follows from Theorem 2.1.1 and Remark 2.2.3. If (X_P, L_P) is covered by lines, there exists a line on X containing p for any $p \in X_P$ since any degeneration of lines is also a line. Thus $\varepsilon(X_P, L_P; p) \leq 1$ holds. The inverse inequality $\varepsilon(X_P, L_P; p) \geq 1$ holds since $|L_P|$ is base point free. Hence ii) $\Rightarrow$ iii) holds. iii) $\Rightarrow$ iv) is clear. iv) $\Leftrightarrow$ ii) follows from Proposition 2.5.1 since $|L_P|$ is base point free. □

As stated in Introduction, this corollary tells us for which P the Seshadri constant $\varepsilon(X_P, L_P; 1)$ is one. We note that Nakamaye [Na1] gives an explicit description for which polarized abelian variety (A, L) the Seshadri constant $\varepsilon(A, L; 1)$ is one.

3

Seshadri constants via toric degenerations

3.1 Introduction

In this chapter, we study explicit estimations of Seshadri constants. First, we give lower and upper bounds of the Seshadri constant on a toric variety at any point. Next, we obtain some new estimations of Seshadri constants on non-toric varieties by using toric degenerations.

Let M be a free abelian group of rank n and set $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. For an integral polytope $P \subset M_{\mathbb{R}}$ of dimension n and a face σ of P, we will define positive real numbers $s_1(P;\sigma), s_2(P;\sigma)$ and show:

Theorem 3.1.1 (=Theorem 3.2.14). *Let P be an integral polytope of dimension n in $M_{\mathbb{R}}$, and σ a face of P. Then,*

$$s_1(P;\sigma) \leq \varepsilon(X_P, L_P; p) \leq s_2(P;\sigma)$$

holds for any $p \in O_\sigma$.

An important point is that $s_1(P;\sigma)$ and $s_2(P;\sigma)$ are computed or estimated more easily than $\varepsilon(X_P, L_P; p)$. Besides, $s_1(P;\sigma) = s_2(P;\sigma)$ often holds, thus we can compute $\varepsilon(X_P, L_P; p)$ explicitly in those cases.

Next we study non-toric cases. Since Seshadri constants have some lower semicontinuities, degenerations are useful to get lower bounds of Seshadri constants. From Theorem 3.1.1, we obtain the following theorem:

Theorem 3.1.2 (special case of Corollary 3.3.4). *Let $f : \mathcal{X} \to T$ be a flat projective morphism over a smooth variety T with reduced and irreducible general fibers. Let $\mathcal{L}$ be an f-ample line bundle on $\mathcal{X}$ and $0 \in T$. Set $X_t = f^{-1}(t), L_t = \mathcal{L}|_{X_t}$ for $t \in T$. If the normalization of the central fiber (X_0, L_0) is isomorphic to the polarized toric variety (X_P, L_P) for an integral polytope $P \subset M_{\mathbb{R}}$, then*

$$\varepsilon(X_t, L_t; 1) \geq s_1(P;P)$$

holds for very general $t \in T$.

Roughly speaking, this theorem states that we can obtain a lower bound of the Seshadri constant of (X, L) at a very general point if (X, L) degenerates to a polarized toric variety.

By using Corollary 3.3.4, we obtain explicit estimations of Seshadri constants on hypersurfaces and Fano 3-folds with Picard number 1:

Theorem 3.1.3 (=Theorem 3.4.4). *Let X_d^n be a very general hypersurface of degree d in $\mathbb{P}^{n+1}$. Then it holds that*

$$\lfloor \sqrt[n]{d/(m_1^n + \cdots + m_r^n)} \rfloor \leq \varepsilon(X_d^n, \mathcal{O}(1); \overline{m}) \leq \sqrt[n]{d/(m_1^n + \cdots + m_r^n)}$$

for any $\overline{m} = (m_1, \ldots, m_r) \in (\mathbb{N} \setminus 0)^r$.

In particular, it holds that

$$\lfloor \sqrt[n]{d} \rfloor \leq \varepsilon(X_d^n, \mathcal{O}(1); 1) \leq \sqrt[n]{d}.$$

Remark 3.1.4. Note that Theorem 3.1.3 does not hold for $\overline{m} \in \mathbb{R}_{>0}^n$ in general.

Theorem 3.1.5 (=Theorem 3.4.6). *For each family of smooth Fano 3-folds with Picard number* 1 *(note that there are 17 such families), $\varepsilon(X, -K_X; 1)$ is as in Table 3.1, where X is a very general member in the family.*

No.	Index	$(-K_X)^3$	Description	$\varepsilon(X, -K_X; 1)$
1	1	2	smooth hypersurface of degree 6 in $\mathbb{P}(1,1,1,1,3)$ (double cover of $\mathbb{P}^3$ ramified over smooth sextic)	6/5
2	1	4	the general element of the family is quartic	4/3
3	1	6	V_6, smooth complete intersection of quadric and cubic	3/2
4	1	8	V_8, smooth complete intersection of three quadrics	2
5	1	10	the general element is V_{10}, a section of $G(2,5)$ by 2 hyperplanes in Plücker embedding and quadric	2
6	1	12	variety V_{12}	2
7	1	14	variety V_{14}, a section of $G(2,6)$ by 5 hyperplanes in Plücker embedding	2
8	1	16	variety V_{16}	2
9	1	18	variety V_{18}	2
10	1	22	variety V_{22}	2
11	2	$8 \cdot 1$	smooth hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$ (double cover of the cone over the Veronese surface branched in a smooth cubic)	2
12	2	$8 \cdot 2$	smooth hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,2)$ (double cover of $\mathbb{P}^3$ ramified over smooth quartic)	2
13	2	$8 \cdot 3$	smooth cubic	2
14	2	$8 \cdot 4$	smooth intersection of two quadrics	2
15	2	$8 \cdot 5$	variety V_5, a section of $G(2,5)$ by 3 hyperplanes in Plücker embedding	2
16	3	$27 \cdot 2$	smooth quadric	3
17	4	$64 \cdot 1$	$\mathbb{P}^3$	4

Table 3.1: Seshadri constants on Fano 3-folds with $\rho = 1$

This chapter is organized as follows: In Section 2, we examine Seshadri constants on toric varieties and show Theorem 3.1.1. We also compute some examples. In Section 3, we prove Theorem 3.1.2. In Section 4, we verify Theorems 3.1.3 and Theorem 3.1.5.

3.2 Seshadri constants on toric varieties

In this section, we investigate Seshadri constants on toric varieties and prove Theorem 3.1.1.

3.2.1 At a point in the maximal orbit

In this subsection, we estimate $\varepsilon(X_P, L_P; 1_P)$ for an integral polytope P. The following lemma is a paraphrase of Lemma 2.2.5:

Lemma 3.2.1. *Let $\pi : M_\mathbb{R} \to M'_\mathbb{R}$ be a lattice projection with* $\operatorname{rank} M = n, \operatorname{rank} M' = r$, *$P \subset M_\mathbb{R}$ an integral polytope of dimension n.*

Then the closure $(\overline{T_{M'}}, L_P|_{\overline{T_{M'}}})$ of $T_{M'}$ in X_P is a not necessarily normal polarized toric variety whose moment polytope is $\pi(P) \subset M'_\mathbb{R}$, where $T_{M'} \hookrightarrow T_M = O_P \subset X_P$ is induced by the surjection $\pi|_M : M \to M'$.

Proof. See Lemma 2.2.5. □

Let $\pi : M_\mathbb{R} \to M'_\mathbb{R}, P$, and $P' = \pi(P)$ be as in Lemma 3.2.1, and set

$$P(u') = \pi^{-1}(u') \cap P$$

for $u' \in P' \cap M'_\mathbb{Q}$. An splitting $M \cong \ker \pi|_M \oplus M'$ of $0 \to \ker \pi|_M \to M \overset{\pi|_M}{\to} M' \to 0$ induces the identification of $\pi^{-1}(u')$ with $\ker \pi$, hence we can consider $P(u')$ as a rational polytope in $\ker \pi = (\ker \pi|_M)_\mathbb{R}$. Assume that the dimension of $P(u')$ is $n - r$. Then $P(u')$ defines the polarized toric variety $(X_{P(u')}, L_{P(u')})$, and the isomorphic class of $(X_{P(u')}, L_{P(u')})$ does not depend on the choice of $M \cong \ker \pi|_M \oplus M'$.

Lemma 3.2.2. *Let $\pi : M_\mathbb{R} \to M'_\mathbb{R}, P$, and $P' = \pi(P)$ be as in Lemma 3.2.1, and take $u' \in P' \cap M'_\mathbb{Q}$ such that* $\dim P(u') = n - r$. *Then, there exists a generically surjective rational map $\varphi : X_P \dashrightarrow X_{P(u')}$ such that for any resolution $\mu : Y \to X_P$ of the indeterminacy of φ, the following hold:*

(i) $\mu^* L_P - f^* L_{P(u')}$ is $\mathbb{Q}$-effective, where $f = \varphi \circ \mu$,

(ii) $\mu(f^{-1}(1_{P(u')})) \cap O_P = T_{M'}$ holds for $1_{P(u')} \in O_{P(u')} \subset X_{P(u')}$.

$$\begin{array}{ccc} Y & \xrightarrow{\ \mu\ } & X_P \\ & \searrow^{f} & \downarrow{\scriptstyle \varphi} \\ & & X_{P(u')} \end{array}$$

Proof. By considering kP for sufficiently large and divisible $k \in \mathbb{N}$, we may assume u' is contained in M' and $P(u')$ is an integral polytope. Furthermore, by considering $P - u$ for $u \in (\pi|_M)^{-1}(u')$, we may assume $u' = 0 \in M'$. Hence $P(u')$ is an integral polytope in $\pi^{-1}(0) = (\ker \pi|_M)_\mathbb{R}$.

There is a commutative diagram

$$\begin{array}{ccc}
\Sigma(\{1\} \times P) & \hookrightarrow & \mathbb{R} \times M_{\mathbb{R}} \\
\uparrow & \circlearrowleft & \uparrow \\
\Sigma(\{1\} \times P(u')) & \hookrightarrow & \mathbb{R} \times \ker \pi.
\end{array}$$

By intersecting with $\mathbb{N} \times M$ or $\mathbb{N} \times \ker \pi|_M$, we have

$$\begin{array}{ccc}
\Gamma_P = \Sigma(\{1\} \times P) \cap (\mathbb{N} \times M) & \hookrightarrow & \mathbb{N} \times M \\
\psi \uparrow & \circlearrowleft & \uparrow \\
\Gamma_{P(u')} = \Sigma(\{1\} \times P(u')) \cap (\mathbb{N} \times \ker \pi|_M) & \hookrightarrow & \mathbb{N} \times \ker \pi|_M.
\end{array}$$

This diagram induces

$$\begin{array}{ccc}
X_P = \operatorname{Proj} \mathbb{C}[\Gamma_P] & \hookleftarrow & O_P = T_M \\
\downarrow \varphi & \circlearrowleft & \downarrow \varphi|_{T_M} \\
X_{P(u')} = \operatorname{Proj} \mathbb{C}[\Gamma_{P(u')}] & \hookleftarrow & O_{P(u')} = T_{\ker \pi|_M}.
\end{array}$$

Then φ is generically surjective because $\varphi|_{T_M}$ is surjective. We show this φ satisfies (i) and (ii) in the statement of this lemma.

Clearly $\varphi|_{T_M}^{-1}(1_{P(u')}) = T_{M'}$, hence (ii) holds.

Let $\mu : Y \to X_P$ be a resolution of indeterminacy of φ. Then $X_P, X_{P(u')}$ are normal and μ, f have connected fibers. Thus

$$\bigoplus_{k \in \mathbb{N}} H^0(Y, kf^*L_{P(u')}) = \bigoplus_{k \in \mathbb{N}} H^0(X_{P(u')}, kL_{P(u')}) = \Gamma_{P(u')},$$

$$\bigoplus_{k \in \mathbb{N}} H^0(Y, k\mu^*L_P) = \bigoplus_{k \in \mathbb{N}} H^0(X_P, kL_P) = \Gamma_P.$$

Therefore an injection $f^*L_{P(u')} \hookrightarrow \mu^*L_P$ is induced from the injection

$$\bigoplus_{k \in \mathbb{N}} H^0(Y, kf^*L_{P(u')}) = \Gamma_{P(u')} \overset{\psi}{\hookrightarrow} \Gamma_P = \bigoplus_{k \in \mathbb{N}} H^0(Y, k\mu^*L_P).$$

Hence (i) holds. $\square$

We need one more lemma, which states that lower and upper bounds of Seshadri constants are obtained from surjective morphisms.

Lemma 3.2.3. *Let $f : Y \to Z$ be a surjective morphism between projective varieties. Assume that L, L' are nef and big $\mathbb{Q}$-divisors on Y, Z respectively such that $L - f^*L'$ is $\mathbb{Q}$-effective. Set $\mathbf{B}(L - f^*L') = Bs(|k(L - f^*L')|)$ for sufficiently large and divisible $k \in \mathbb{N}$ (which is called the stable base locus of $L - f^*L'$, and does not depend on k. See [La2, Remark 2.1.24]). Then*

$$\min\{\min_{1 \le i \le r} \varepsilon(Y_i, L|_{Y_i}; y), \varepsilon(Z, L'; f(y))\} \le \varepsilon(Y, L; y) \le \min_{1 \le i \le r} \varepsilon(Y_i, L|_{Y_i}; y)$$

holds for $y \notin \mathbf{B}(L - f^*L')$, where $Y_1, \ldots, Y_r$ are all the irreducible components of $f^{-1}(f(y))$ containing y with the reduced structures.

Proof. We may assume L and L' are ample. In fact, for nef and big L, L', choose ample divisors A, A' on Y, Z such that $y \notin \mathbf{B}(A - f^*A')$, and consider $L + \delta A, L' + \delta A' (\delta > 0)$ instead of L, L'. Then we can show this lemma from ample cases by $\delta \to 0$.

The second inequality is clear by the definition of Seshadri constants, thus it is enough to show the first one. For the sake of simplicity, we set $z = f(y)$.

Fix a curve $C \subset Y$ containing y.

It suffices to show $\min\{\min_i \varepsilon(Y_i, L|_{Y_i}; y), \varepsilon(Z, L'; f(y))\} \leq \dfrac{C.L}{\mathrm{mult}_y(C)}$. $\cdots (*)$

Case 1. $C \subset f^{-1}(z)$.

Since $C \subset Y_i$ for some i, $\dfrac{C.L}{\mathrm{mult}_y(C)} = \dfrac{C.L|_{Y_i}}{\mathrm{mult}_y(C)} \geq \varepsilon(Y_i, L|_{Y_i}; y)$ holds.

Case 2. $C \not\subset f^{-1}(z)$.

Set $C' = f(C)$ with the reduced structure and fix a rational number $0 < t < \varepsilon(Z, L'; z)$. Then for any sufficiently large and divisible $k \in \mathbb{N}$, there exists $D' \in |kL' \otimes \mathfrak{m}_z^{kt}|$ such that $C' \not\subset \mathrm{Supp}\, D'$ by the ampleness of L' and [La2, Lemma 5.4.24]. Clearly $f^*D' \in |kf^*L' \otimes \mathfrak{m}_y^{kt}|$ and $C \not\subset \mathrm{Supp}\, f^*D'$, hence

$$\begin{aligned} k(C.L) &= kC.(L - f^*L' + f^*L') \\ &= kC.(L - f^*L') + C.f^*D' \\ &\geq C.f^*D' \\ &\geq kt \cdot \mathrm{mult}_y(C). \end{aligned}$$

Note $C.(L - f^*L') \geq 0$ holds by the assumption $y \notin \mathbf{B}(L - f^*L')$. Therefore $\dfrac{C.L}{\mathrm{mult}_y(C)} \geq t$ holds and we have $\dfrac{C.L}{\mathrm{mult}_y(C)} \geq \varepsilon(Z, L'; z)$ by $t \to \varepsilon(Z, L'; z)$.

Thus for any curve $C \subset Y$ containing y, $(*)$ holds. $\square$

By Lemmas 3.2.1, 3.2.2, and 3.2.3, we obtain the following proposition, which is useful to estimate $\varepsilon(X_P, L_P; 1_P)$:

Proposition 3.2.4. *Let $\pi : M_{\mathbb{R}} \to M'_{\mathbb{R}}$ be a lattice projection for free abelian groups M and M' of rank n and r. Let $P \subset M_{\mathbb{R}}$ be an n-dimensional integral polytope, and set $P' = \pi(P)$. Fix $u' \in P' \cap M'_{\mathbb{Q}}$ and assume* $\dim P(u') = n - r$. *Then it holds that*

$$\min\{\varepsilon(X_{P'}, L_{P'}; 1_{P'}), \varepsilon(X_{P(u')}, L_{P(u')}; 1_{P(u')})\} \leq \varepsilon(X_P, L_P; 1_P) \;\leq\; \varepsilon(X_{P'}, L_{P'}; 1_{P'}).$$

Proof. Let $\varphi : X_P \dashrightarrow X_{P(u')}$ be the rational map defined in Lemma 3.2.2. For a toric resolution $\mu : Y \to X_P$ of the indeterminacy of φ, the stable base locus $\mathbf{B}(\mu^*L_P - f^*L_{P(u')})$ is contained in $Y \setminus O$, where O is the maximal orbit of Y. By applying Lemma 3.2.3 to $f : Y \to X_{P(u')}, \mu^*L_P, L_{P(u')}$ and the identity 1_Y of the torus $O \subset Y$, we have

$$\begin{aligned} &\min\{\varepsilon(Y_1, (\mu^*L_P)|_{Y_1}; 1_Y), \varepsilon(X_{P(u')}, L_{P(u')}; 1_{P(u')})\} \\ &\qquad \leq \; \varepsilon(Y, \mu^*L_P; 1_Y) \;\leq\; \varepsilon(Y_1, (\mu^*L_P)|_{Y_1}; 1_Y), \end{aligned}$$

where Y_1 is the irreducible component of $f^{-1}(1_{P(u')})$ containing 1_Y. Since $\mu : Y \to X_P$ and $f|_{Y_1} : Y_1 \to \overline{T_{M'}} (\subset X_P)$ are birational and isomorphic around 1_Y from the proof of Lemma 3.2.2, it holds that $\varepsilon(Y, \mu^*L_P; 1_Y) = \varepsilon(X_P, L_P; 1_P)$, $\varepsilon(Y_1, (\mu^*L_P)|_{Y_1}; 1_Y) = \varepsilon(\overline{T_{M'}}, L_P|_{\overline{T_{M'}}}; 1_P)$. The normalization of $(\overline{T_{M'}}, L_P|_{\overline{T_{M'}}})$ is $(X_{P'}, L_{P'})$ by Lemma 3.2.1. Thus we have

$$\varepsilon(Y_1, (\mu^*L_P)|_{Y_1}; 1_Y) = \varepsilon(\overline{T_{M'}}, L_P|_{\overline{T_{M'}}}; 1_P) = \varepsilon(X_{P'}, L_{P'}; 1_{P'}).$$

From these equalities, this proposition follows. $\square$

In view of Proposition 3.2.4, we define invariants $s_1(P)$ and $s_2(P)$ for a rational polytope $P \subset \mathbb{R}^n$ as follows:

Definition 3.2.5. Let P be a rational polytope in $M_{\mathbb{R}}$. We define $s_1(P) = s_1^M(P) \in \mathbb{R}_{\geq 0}$ for which $s_1(P+u) = s_1(P)$ holds for any $u \in M_{\mathbb{Q}}$ by induction of n as follows:

When $n = 1$, we define $s_1(P) = |P|_M$, the length of P. Note that $M \subset M_{\mathbb{R}}$ is identified with $\mathbb{Z} \subset \mathbb{R}$. Clearly $s_1(P+u) = s_1(P)$ holds for any $u \in M_{\mathbb{Q}}$.

Assume such $s_1(P)$ is defined in the case of rank $n-1$, and set

$$\Phi = \{\, \pi : M_{\mathbb{R}} \to (\mathbb{Z})_{\mathbb{R}} = \mathbb{R} \,|\, \pi \text{ is a lattice projection} \,\}.$$

Fix $\pi \in \Phi$ and choose a splitting $M \cong \ker \pi|_M \oplus \mathbb{Z}$ of $0 \to \ker \pi|_M \to M \stackrel{\pi|_M}{\to} \mathbb{Z} \to 0$. Then for $u' \in \mathbb{Q}$,

$$\pi^{-1}(u') \cap P$$

can be regarded as a rational polytope in $\ker \pi = (\ker \pi|_M)_{\mathbb{R}}$ naturally. Thus we can define $s_1^{\ker \pi|_M}(\pi^{-1}(u') \cap P) \in \mathbb{R}_{\geq 0}$ by the induction hypothesis. Another choice of the splitting only causes a parallel translation of $\pi^{-1}(u') \cap P$ in $\ker \pi$, hence $s_1^{\ker \pi|_M}(\pi^{-1}(u') \cap P)$ does not depend on the splitting by the induction hypothesis. We define

$$s_1(P) = s_1^M(P) := \sup_{\pi \in \Phi} \min\{\, |\pi(P)|_{\mathbb{Z}}, \sup_{u' \in \mathbb{Q}} s_1^{\ker \pi|_M}(\pi^{-1}(u') \cap P) \,\}.$$

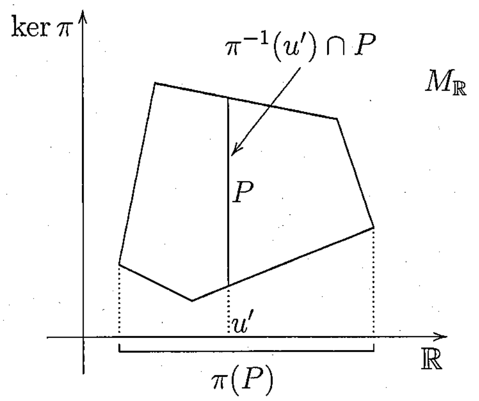
Clearly, $s_1(P+u) = s_1(P)$ holds for any $u \in M_{\mathbb{Q}}$.

The definition of $s_2(P)$ is more simple. For a rational polytope $P \subset M_{\mathbb{R}}$, $s_2(P) \in \mathbb{R}_{\geq 0}$ is defined to be

$$s_2(P) = \inf_{\pi \in \Phi} |\pi(P)|_{\mathbb{Z}}.$$

By definition, $s_2(P+u) = s_2(P)$ holds for any P and $u \in M_{\mathbb{Q}}$.

We define $s_1(\{0\}) = s_2(\{0\}) = +\infty, s_1(\emptyset) = s_2(\emptyset) = 0$ if $n = 0$.



Remark 3.2.6. Let V be a finite dimensional $\mathbb{R}$-vector space, and take two lattices M_1, M_2 of V, thus $(M_1)_{\mathbb{R}} = (M_2)_{\mathbb{R}} = V$. In general $s_i^{M_1}(P) \neq s_i^{M_2}(P)$ for $P \subset V$ and $i = 1, 2$. Hence we have to notice which lattice we consider about when we deal with $s_1(\cdot), s_2(\cdot)$.

By Proposition 3.2.4, we can show that $s_1(P)$ and $s_2(P)$ give a lower bound and a upper bound of $\varepsilon(X_P, L_P; 1_P)$ respectively:

Proposition 3.2.7. *Let $P \subset M_{\mathbb{R}}$ be a rational polytope of dimension n. Then for any $p \in O_P \subset X_P$, it holds that*

$$s_1(P) \leq \varepsilon(X_P, L_P; p) \leq s_2(P).$$

Proof. By the torus action, we may assume $p = 1_P$. We show this proposition by induction of n.

If $n = 1$, then $\varepsilon(X_P, L_P; 1_P) = \deg(L_P) = |P|$. By definitions $s_1(P) = s_2(P) = |P|$, thus the inequalities in the proposition follow.

We assume this proposition holds if the rank of M is $n-1$, and show the case of rank n.

We use the notations in Definition 3.2.5. Fix $\pi \in \Phi$, i.e., $\pi : M_{\mathbb{R}} \to \mathbb{R} = (\mathbb{Z})_{\mathbb{R}}$ is a lattice projection. We can apply Proposition 3.2.4 to P and $u' \in \pi(P) \cap \mathbb{Q}$ such that $\dim(\pi^{-1}(u') \cap P) = n-1$. Then we obtain inequalities

$$\begin{aligned}\min\{\varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)}), \varepsilon(X_{\pi^{-1}(u')\cap P}, L_{\pi^{-1}(u')\cap P}; 1_{\pi^{-1}(u')\cap P})\} \\ \leq \ \varepsilon(X_P, L_P; 1_P) \ \leq \ \varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)}).\end{aligned}$$

Note that Proposition 3.2.4 can be applied to rational polytopes. Now $\varepsilon(X_{\pi(P)}, L_{\pi(P)}; 1_{\pi(P)}) = |\pi(P)|$, and by the induction hypothesis we have

$$s_1(\pi^{-1}(u') \cap P) \leq \varepsilon(X_{\pi^{-1}(u')\cap P}, L_{\pi^{-1}(u')\cap P}; 1_{\pi^{-1}(u')\cap P}).$$

Thus these inequalities induce

$$\min\{|\pi(P)|, s_1(\pi^{-1}(u') \cap P)\} \leq \varepsilon(X_P, L_P; 1_P) \leq |\pi(P)|.$$

Note that this also holds if $\dim(\pi^{-1}(u') \cap P) < n-1$ since $s_1(\pi^{-1}(u') \cap P) = 0$ for such $u' \in \mathbb{Q}$. (This can be shown easily by the definition of s_1.) Moving u', we have

$$\min\{|\pi(P)|, \sup_{u' \in \mathbb{Q}} s_1(\pi^{-1}(u') \cap P)\} \leq \varepsilon(X_P, L_P; 1_P) \leq |\pi(P)|.$$

By moving π, we obtain $s_1(P) \leq \varepsilon(X_P, L_P; 1_P) \leq s_2(P)$. □

Remark 3.2.8. (1) Note that $s_2(P)$ is nothing but the lattice width of P defined in Definition 2.2.2. Theorem 2.1.2 says that $\varepsilon(X_P, L_P; 1_P) = 1$ if and only if $s_2(P) = 1$ for any integral polytope $P \subset M_{\mathbb{R}}$ of dimension n. But in general, $\varepsilon(X_P, L_P; 1_P) \neq s_2(P)$. See Example 3.2.9 (3).
(2) If $|\pi(P)| \leq s_1(\pi^{-1}(u') \cap P)$ holds for some $\pi \in \Phi$ and $u' \in \mathbb{Q}$, we have $\varepsilon(X_P, L_P; 1_P) = |\pi(P)| = s_1(P) = s_2(P)$ by Proposition 3.2.7.
(3) The upper bound $s_2(P)$ can be a little improved. In fact,

$$\varepsilon(X_P, L_P; 1_P) \leq \inf_{\pi : M_{\mathbb{R}} \to M'_{\mathbb{R}}} \sqrt[\operatorname{rank} M']{(\operatorname{rank} M')! \operatorname{vol}_{M'_{\mathbb{R}}}(\pi(P))}$$

holds, where $\pi : M_{\mathbb{R}} \to M'_{\mathbb{R}}$ moves all lattice projections from $M_{\mathbb{R}}$. This is shown from Proposition 3.2.4 and Remark 1.2.2 immediately.

3.2.2 At a point in the maximal orbit, Examples

By using Propositions 3.2.4, 3.2.7, we estimate $\varepsilon(X_P, L_P; 1_P)$ for some P.

Example 3.2.9. (1) Set $P_n = \mathrm{conv}(0, e_1, \ldots, e_n)$ for the standard basis $e_1, \ldots, e_n$ of $\mathbb{Z}^n$. We apply Proposition 3.2.4 to the n-th projection $\pi : \mathbb{R}^n \to \mathbb{R}$ and $u' = 0 \in \mathbb{R}$. Since $P' = \pi(P_n) = [0,1] \subset \mathbb{R}$, we have $\varepsilon(X_{P'}, L_{P'}; 1_{P'}) = |P'| = 1$. On the other hand, $P(u') = P_{n-1} = \mathrm{conv}(0, e_1, \ldots, e_{n-1}) \subset \mathbb{R}^{n-1}$. By Proposition 3.2.4, it holds that

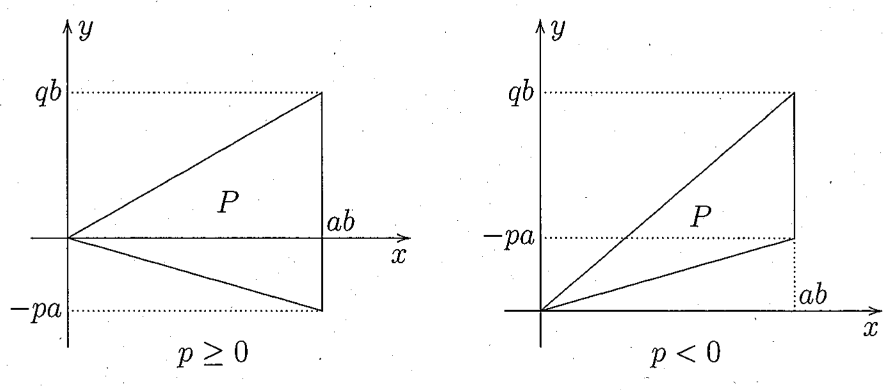
$$\min\{1, \varepsilon(X_{P_{n-1}}, L_{P_{n-1}}; 1_{P_{n-1}})\} \leq \varepsilon(X_{P_n}, L_{P_n}; 1_{P_n}) \leq 1.$$

Since $\varepsilon(X_{P_1}, L_{P_1}; 1_{P_1}) = |P_1| = 1$, we have $\varepsilon(X_{P_n}, L_{P_n}; 1_{P_n}) = 1$ for any n inductively. Note that $(X_{P_n}, L_{P_n}) = (\mathbb{P}^n, \mathcal{O}(1))$.
(2) Set $P = \mathrm{conv}((0,0),(a,0),(0,b),(a,b)) \subset \mathbb{R}^2$ for $a \leq b$ in $\mathbb{N}\backslash 0$. We apply Proposition 3.2.4 to the first projection $\pi : \mathbb{R}^2 \to \mathbb{R}$ and $u' = 0 \in \mathbb{R}$. Then $\varepsilon(X_P, L_P; 1_P) = \varepsilon(X_{P'}, L_{P'}; 1_{P'}) = a$ by Remark 3.2.8 (2) because $|P'| = a \leq b = s_1(\pi^{-1}(0) \cap P)$. Note that $(X_P, L_P) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a,b))$.
(3) Set $P = \mathrm{conv}((1,0),(0,1),(-1,-1)) \subset \mathbb{R}^2$. Since P is a triangle, it holds $|\pi^{-1}(u') \cap P| \cdot \pi(P) = 2 \cdot \mathrm{vol}(P) = 3$ for any lattice projection $\pi : \mathbb{R}^2 \to \mathbb{R}$, where $\pi^{-1}(u') \cap P$ is the longest fiber of $P \to \pi(P)$. Thus $s_1(P) = \min\{s_2(P), 3/s_2(P)\}$ holds. It is easy to see $s_2(P) = 2$, hence we have $s_1(P) = 3/s_2(P) = 3/2$. Thus $3/2 \leq \varepsilon(X_P, L_P; 1_P) \leq 2$ holds by Proposition 3.2.7. Note that X_P is the singular cubic surface in $\mathbb{P}^3 = \mathrm{Proj}\, \mathbb{C}[T_0, T_1, T_2, T_3]$ defined by $T_0^3 = T_1T_2T_3$ and $L_P = \mathcal{O}(1)$. For any (integral and not necessarily smooth) cubic surface $S \subset \mathbb{P}^3$ and a general point $p \in S$, the plane in $\mathbb{P}^3$ tangent to S at p induces a singular curve $C \sim \mathcal{O}_S(1)$. Thus $\varepsilon(S, \mathcal{O}(1); p) \leq 3/2$ holds. Hence we have $s_1(P) = 3/2 = \varepsilon(X_P, L_P; 1_P) < s_2(P)$ in this case.
(4) It is well known that there are five toric Del Pezzo surfaces. For an integral polytope $P \subset \mathbb{R}^2$ such that X_P is a Del pezzo surface and $L_P = -K_{X_P}$, we can easily find a projection π and $u' \in \mathbb{Q}$ as in Remark 3.2.8 (2) and compute $\varepsilon(X_P, L_P; 1_P)$. As a consequence, we have

$$\varepsilon(X_P, L_P; 1_P) = \begin{cases} 3 & \text{if } X_P = \mathbb{P}^2 \\ 2 & \text{otherwise} \end{cases}$$

for such P.

In the above examples, Seshadri constants can be computed without using Propositions 3.2.4, 3.2.7. The following examples are new computations of Seshadri constants on toric varieties.
(5) We consider a weighted projective space $\mathbb{P}(a,b,c)$ with $c = \max\{a,b,c\}$. We may assume any two of a,b,c are coprime. Since a and b are coprime, we can denote $c = pa + qb$ for integers p, q such that $0 < q < a$. Let $P \subset \mathbb{R}^2$ be the convex hull of $(0,0), (ab, qb)$ and $(ab, -pa)$.



It is easy to see that $(X_P, L_P) = (\mathbb{P}(a,b,c), \mathcal{O}(abc))$. Since P is a triangle, we have $s_1(P) = \min\{s_2(P), abc/s_2(P)\} \leq \varepsilon(X_P, L_P; 1_P) \leq s_2(P)$ as (3). In other words, it holds that

$$\min\{s_2(P)/abc, 1/s_2(P)\} \leq \varepsilon(\mathbb{P}(a,b,c), \mathcal{O}(1); 1) \leq s_2(P)/abc.$$

Since $s_2(P)$ can be computed by finite calculations for any given a, b, c (or more generally, any given integral polytope in $\mathbb{R}^n$), we obtain an explicit estimation. If $s_2(P) \leq \sqrt{abc}$, it holds $\varepsilon(\mathbb{P}(a,b,c), \mathcal{O}(1); 1) = s_2(P)/abc$. For example,

(i) When $p \geq 0$, we consider the first or second projections $\mathbb{R}^2 \to \mathbb{R}$ as π. Then we have $|\pi(P)| = \min\{ab, c\} \leq \sqrt{abc}$. Thus it holds

$$\varepsilon(\mathbb{P}(a,b,c), \mathcal{O}(1); 1) = \min\{ab, c\}/abc = \min\{1/c, 1/ab\}$$

by Remark 3.2.8 (2). For instance, $p \geq 0$ holds if $a = 1$, 2, or $ab \leq c$.

(ii) When $p < 0$, we have $|p_2(P)| = qb$ for the second projection $p_2 : \mathbb{R}^2 \to \mathbb{R}$. Thus, if $qb \leq \sqrt{abc}$, i.e., $q^2 b \leq ac$, it holds that $\varepsilon(\mathbb{P}(a,b,c), \mathcal{O}(1); 1) = qb/abc = q/ac$. For instance,

$$\varepsilon(\mathbb{P}(3,5,7), \mathcal{O}(1); 1) = 2/21$$

holds since $7 = -1 \cdot 3 + 2 \cdot 5$.

(iii) If $a = 3, b = 4, c = 5$, we have $s_2(P) = 8$. In this case, $s_2(P) = 8 > 2\sqrt{15} = \sqrt{abc}$. Thus we have only the estimation

$$1/8 \leq \varepsilon(\mathbb{P}(3,4,5), \mathcal{O}(1); 1) \leq 2/15.$$

(6) There are 18 smooth toric Fano 3-folds (cf. [Bat], [WW]). As (4), we can easily compute $\varepsilon(X_P, L_P; 1_P)$ if X_P is a smooth toric Fano 3-fold and $L_P = -K_{X_P}$. For such P, we can show

$$\varepsilon(X_P, L_P; 1_P) = \begin{cases} 4 & \text{if } X_P = \mathbb{P}^3 \\ 3 & \text{if } X_P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \\ 2 & \text{otherwise.} \end{cases}$$

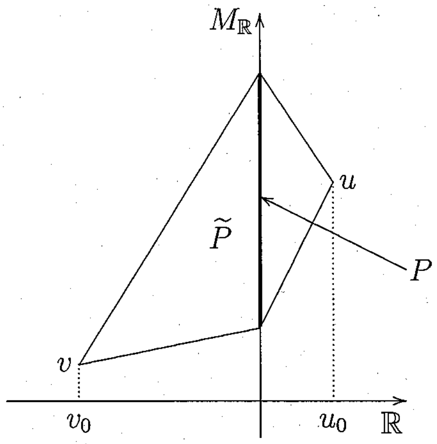
(7) We give examples of a polarized variety (X, L) satisfying

$$\varepsilon(X, L; 1) = \sqrt[n]{L^n} \in \mathbb{N} \cdots (*)$$

for $n = \dim X$. We construct such examples by induction of n as follows:

When $n = 1$, (X_P, L_P) satisfies the condition $(*)$ for any integral polytope P in $M_{\mathbb{R}} \cong \mathbb{R}$. Note $(*)$ always holds if X is a curve.

Let $P \subset M_{\mathbb{R}}$ be an integral polytope such that (X_P, L_P) satisfies $(*)$. Choose $u, v \in \mathbb{Z} \times M$ such that $\overline{uv} \cap (\{0\} \times P) \neq \emptyset$ and $|u_0 - v_0| = \sqrt[n]{L_P^n} =: m \in \mathbb{N}$, where $\overline{uv}$ is the segment in $(\mathbb{Z} \times M)_{\mathbb{R}}$ whose end points are u and v, and u_0, v_0 are $\mathbb{Z}$-components of $u, v \in \mathbb{Z} \times M$ respectively. Set $\widetilde{P} = \mathrm{conv}(u, v, \{0\} \times P)$ in $(\mathbb{Z} \times M)_{\mathbb{R}}$. By applying Proposition 3.2.4 to the first projection $\pi : \mathbb{R} \times M_{\mathbb{R}} \to \mathbb{R}$ and $0 \in \mathbb{R}$, we have $\varepsilon(X_{\widetilde{P}}, L_{\widetilde{P}}; 1_{\widetilde{P}}) = m$. Since $L_{\widetilde{P}}^{n+1} = (n+1)! \operatorname{vol}(\widetilde{P}) = m^{n+1}$, $(X_{\widetilde{P}}, L_{\widetilde{P}})$ is an $n+1$-dimensional example satisfying $(*)$.



(8) Set $P = \mathrm{conv}(e_1, \ldots, e_n, -\Sigma_{i=1}^n a_i e_i) \subset \mathbb{R}^n$ for rational numbers $a_1, \ldots, a_n \geq 0$. Then we have

$$\varepsilon(X_P, L_P; 1_P) \geq \min_{1 \leq i \leq n} \frac{a_i + \cdots + a_n + 1}{a_{i+1} + \cdots + a_n + 1}. \cdots (**)$$

We show this by induction of n. When $n = 1$, $\varepsilon(X_P, L_P; 1_P) = |P| = a_1 + 1 = \min_{1 \leq i \leq n} \frac{a_i + \cdots + a_n + 1}{a_{i+1} + \cdots + a_n + 1}$. Thus $(**)$ holds.

Assume $(**)$ holds for $n-1$. We apply Proposition 3.2.4 to the n-th projection $\pi : \mathbb{R}^n \to \mathbb{R}$, P, and $0 \in \pi(P) \cap \mathbb{Q}$. Then $P(0) = \pi^{-1}(0) \cap P = \mathrm{conv}(e_1, \ldots, e_{n-1}, -1/(a_n + 1)\Sigma_{i=1}^{n-1} a_i e_i)$ and $P' = \pi(P) = [-a_n, 1] \subset \mathbb{R}$. By induction hypothesis,

$$\begin{aligned} \varepsilon(X_{P(0)}, L_{P(0)}; 1_{P(0)}) &\geq \min_{1 \leq i \leq n-1} \frac{a_i/(a_n + 1) + \cdots + a_{n-1}/(a_n + 1) + 1}{a_{i+1}/(a_n + 1) + \cdots + a_{n-1}/(a_n + 1) + 1} \\ &= \min_{1 \leq i \leq n-1} \frac{a_i + \cdots + a_n + 1}{a_{i+1} + \cdots + a_n + 1} \end{aligned}$$

holds. By Proposition 3.2.4, it follows that

$$\begin{aligned} \varepsilon(X_P, L_P; 1_P) &\geq \min\{\varepsilon(X_{P'}, L_{P'}; 1_{P'}), \varepsilon(X_{P(0)}, L_{P(0)}; 1_{P(0)})\} \\ &\geq \min\{a_n + 1, \min_{1 \leq i \leq n-1} \frac{a_i + \cdots + a_n + 1}{a_{i+1} + \cdots + a_n + 1}\} \\ &= \min_{1 \leq i \leq n} \frac{a_i + \cdots + a_n + 1}{a_{i+1} + \cdots + a_n + 1}. \end{aligned}$$

We will use this lower bound in Section 3.4.

3.2.3 At a point in any orbit

Next, we consider the Seshadri constant on a toric variety at a point not necessarily contained in the maximal orbit.

Definition 3.2.10. Let P be an integral polytope of dimension n in $M_\mathbb{R}$, and v a vertex of P. We define

$$s(P; v) = \min\{\, |\tau|_{M_\tau} \mid v \prec \tau \prec P, \dim \tau = 1\} \in \mathbb{N} \setminus 0,$$

where $M_\tau = \mathbb{R}(\tau - \tau) \cap M$ and we consider τ as a subset in $(M_\tau)_\mathbb{R} = \mathbb{R}(\tau - \tau)$ by a parallel translation. If $M = \{0\}$, we set $s(P; v) = +\infty$ for $P = v = \{0\}$.

Let σ be a face of P (we denote this by $\sigma \prec P$). Let $\pi : M_{\mathbb{R}} \to M_{\mathbb{R}}/\mathbb{R}(\sigma - \sigma)$ be the natural projection and set $M' = \pi(M), P' = \pi(P)$, and $v' = \pi(\sigma)$. Note that P' is an integral polytope in $M'_{\mathbb{R}} = M_{\mathbb{R}}/\mathbb{R}(\sigma - \sigma)$ and v' is a vertex of P'. Then $s_1(P;\sigma), s_2(P;\sigma) \in \mathbb{R}_{>0}$ are defined to be

$$s_1(P;\sigma) = \min\{s_1^{M_\sigma}(\sigma), s(P';v')\}, \ \ s_2(P;\sigma) = \min\{s_2^{M_\sigma}(\sigma), s(P';v')\},$$

where $M_\sigma = \mathbb{R}(\sigma - \sigma) \cap M$ and we regard σ as an integral polytope in $\mathbb{R}(\sigma - \sigma) = (M_\sigma)_{\mathbb{R}}$.

Note that $s_1(P;P) = s_1(P), s_2(P;P) = s_2(P)$, and $s_1(P;v) = s_2(P;v) = s(P;v)$ holds for any vertex v.

Proposition 3.2.11. *Let σ be a face of an n-dimensional integral polytope P in $M_{\mathbb{R}}$. Set $\pi : M_{\mathbb{R}} \to M_{\mathbb{R}}/\mathbb{R}(\sigma - \sigma), P' = \pi(P), v' = \pi(\sigma)$ as in Definition 3.2.10. Then,*

$$\varepsilon(X_P, L_P; p) = \min\{\varepsilon(X_\sigma, L_\sigma; 1_\sigma), s(P';v')\}$$

holds for any $p \in O_\sigma$.

Proof. We use notations in Definition 3.2.10. We may assume $0 \in \sigma$, thus $v' = 0$ in $M' = \pi(M)$.

Firstly, we show $\varepsilon(X_P, L_P; p) \leq \min\{\varepsilon(X_\sigma, L_\sigma; p), s(P';v')\}$. Note that $\varepsilon(X_\sigma, L_\sigma; p) = \varepsilon(X_\sigma, L_\sigma; 1_\sigma)$ by the torus action. Since $L_P|_{X_\sigma} = L_\sigma$, the inequality

$$\varepsilon(X_P, L_P; p) \leq \varepsilon(X_\sigma, L_\sigma; p) \cdots (*)$$

is clear. By the definition of π, there is a natural 1 to 1 correspondence between $\Xi = \{\tau \mid \sigma \prec \tau \prec P, \dim \tau = \dim \sigma + 1\}$ and $\Xi' = \{\tau' \mid v' \prec \tau' \prec P', \dim \tau' = 1\}$ by corresponding $\tau \in \Xi$ to $\pi(\tau) \in \Xi'$. Fix $\tau' \in \Xi'$ and let $\tau \in \Xi$ be the corresponding face of P. Then by Proposition 3.2.7, $\varepsilon(X_\tau, L_\tau; q) \leq s_2(\tau) \leq |\tau'|$ holds for $q \in O_\tau$. Since $\mathrm{codim}(X_\sigma, X_\tau) = 1$ and X_τ is normal, X_τ is smooth at p. Therefore by the lower semicontinuity of Seshadri constants (see [La2, Example 5.1.11]), it holds that $\varepsilon(X_P, L_P; p) \leq \varepsilon(X_\tau, L_\tau; p) \leq \varepsilon(X_\tau, L_\tau; q) \leq |\tau'|$. Hence by definition of $s(P';v')$,

$$\varepsilon(X_P, L_P; p) \leq \min_{\tau' \in \Xi'} |\tau'| = s(P';v'). \cdots (**)$$

From $(*)$ and $(**)$, we have $\varepsilon(X_P, L_P; p) \leq \min\{\varepsilon(X_\sigma, L_\sigma; p), s(P';v')\}$.

Next we show the opposite inequality. Let C be a curve on X_P containing p. It is enough to show

$$C.L_P \geq \mathrm{mult}_p(C) \cdot \min\{\varepsilon(X_\sigma, L_\sigma; p), s(P';v')\}. \cdots (***)$$

Case 1. $C \subset X_\sigma$.

In this case, $C.L_P \geq \mathrm{mult}_p(C) \cdot \varepsilon(X_\sigma, L_\sigma; p)$ is clear by the definition of Seshadri constants, thus $(***)$ holds.

Case 2. $C \not\subset X_\sigma$.

We use the following claim:

Claim 3.2.12. *In this case, there exist $\tau' \in \Xi'$ and an effective divisor $D \in |L_P \otimes \mathfrak{m}_p^{|\tau'|}|$ on X_P such that $C \not\subset \mathrm{Supp}\, D$.*

If there exists such a divisor D, it holds that

$$\begin{aligned} C.L_P = C.D &\geq \operatorname{mult}_p(C) \cdot |\tau'| \\ &\geq \operatorname{mult}_p(C) \cdot s(P'; v') \\ &\geq \operatorname{mult}_p(C) \cdot \min\{\varepsilon(X_\sigma, L_\sigma; p), s(P'; v')\}. \end{aligned}$$

Thus the proof is completed by showing this claim.

Proof of Claim 3.2.12. For $\tau' \in \Xi'$, let $v'_{\tau'}$ be the vertex of τ' different from v'. If $\tau \in \Xi$ corresponds to τ', there exists a vertex v_τ of τ such that $\pi(v_\tau) = v'_{\tau'}$. Many vertices of τ may satisfy this condition, but we choose one of them. Let $x^{v_\tau} \in H^0(X_P, L_P)$ be the section corresponding to v_τ, and $D_{v_\tau} \in |L_P|$ the corresponding effective divisor on X_P. Since $\operatorname{Supp} D_{v_\tau} = \bigcup_{v_\tau \notin \rho \prec P} X_\rho$, we have

$$\bigcap_{\tau \in \Xi} \operatorname{Supp} D_{v_\tau} = \bigcup_{v_\tau \notin \rho \text{ for}^\forall \tau \in \Xi} X_\rho.$$

By the choices of v_τ, σ does not contain any v_τ. If $\rho \succ \sigma$ and $\rho \neq \sigma$, then ρ contains some $\tau \in \Xi$, hence $v_\tau \in \tau \subset \rho$. Consequently, it holds that

$$X_\sigma \subset \bigcap_{\tau \in \Xi} \operatorname{Supp} D_{v_\tau} \subset X_\sigma \cup \bigcup_{\sigma \not\prec \rho \prec P} X_\rho.$$

Since $\bigcup_{\sigma \not\prec \rho \prec P} X_\rho$ is a closed set not containing p, $\bigcap_{\tau \in \Xi} \operatorname{Supp} D_{v_\tau}$ coincides with X_σ around p. Now C contains p and is not contained in X_σ by assumption, thus C is not contained in $\bigcap_{\tau \in \Xi} \operatorname{Supp} D_{v_\tau}$. Hence we can choose $\tau_0 \in \Xi$ such that $\operatorname{Supp} D_{v_{\tau_0}}$ does not contain C. Let $\tau'_0 \in \Xi'$ be the corresponding face, and set $e' = |\tau'_0|^{-1} v'_{\tau'_0} \in M'$. (Note that we assume $v' = 0$, and τ'_0 is the convex hull of v' and $v'_{\tau'_0}$.) Then e' is the generator of $\mathbb{R}(\tau'_0 - \tau'_0) \cap M' = \mathbb{R}\tau'_0 \cap M' \cong \mathbb{Z}$ contained in τ'_0. Fix $e \in M \cap \pi^{-1}(e')$. Since $v'_{\tau'_0} = |\tau'_0| e'$, $u := v_{\tau_0} - |\tau'_0| e$ is contained in $\pi^{-1}(0) \cap M = \mathbb{R}(\sigma - \sigma) \cap M$. This means $x^{v_{\tau_0}} = x^u \cdot (x^e)^{|\tau'_0|}$ is contained in $H^0(X_P, L_P \otimes \mathfrak{m}_p^{|\tau'|})$, hence this τ'_0 and $D_{v_{\tau_0}}$ satisfies the condition in the claim. ☐

☐

Remark 3.2.13. For a vertex v of P, we have $\varepsilon(X_P, L_P; p) = s(P; v)$ for the torus invariant point $p = O_v$ by Proposition 3.2.11. When X_P is smooth, this is Corollary 4.2.2 in [BDH+].

The invariant $s(P'; v')$ in Proposition 3.2.11 is easily computed. Thus, it is enough to see $\varepsilon(X_\sigma, L_\sigma; 1_\sigma)$ to compute $\varepsilon(X_P, L_P; p)$ for $p \in O_\sigma$. But we can use Proposition 3.2.7 to estimate $\varepsilon(X_\sigma, L_\sigma; 1_\sigma)$. Therefore we obtain the following theorem:

Theorem 3.2.14 (=Theorem 3.1.1). *Let $P \subset M_\mathbb{R}$ be an integral polytope, σ a face of P, and $p \in O_\sigma$. Then, it holds that*

$$s_1(P; \sigma) \leq \varepsilon(X_P, L_P; p) \leq s_2(P; \sigma).$$

Proof. This is easily shown from Propositions 3.2.7, 3.2.11, and the definitions of $s_1(P; \sigma)$ and $s_2(P; \sigma)$. ☐

Remark 3.2.15. Since $s_1(\sigma) = s_2(\sigma)$ for $\sigma \subset (M_\sigma)_\mathbb{R}$ when $\operatorname{rank} M_\sigma = 0$ or 1, $s_1(P; \sigma) = \varepsilon(X_P, L_P; p) = s_2(P; \sigma)$ holds if $\dim \sigma = 0$ or 1.

3.2.4 At a point in any orbit, Examples

Example 3.2.16. Unless otherwise stated, π, P' and v' are as in Proposition 3.2.11.
(1) Let P_n be as in Example 3.2.9 (1). We apply Proposition 3.2.11 to P_n and any face $\sigma \prec P_n$ of codimension r. Then the image of P by π is $P' = P_r$, thus $s(P', v') = 1$. Since σ is identified with P_{n-r} by some integral affine translation, $\varepsilon(X_\sigma, L_\sigma; 1_\sigma) = 1$ holds by Example 3.2.9 (1). Hence we have $\varepsilon(X_{P_n}, L_{P_n}; p) = 1$ for any $\sigma \prec P$ and any $p \in O_\sigma$ by Proposition 3.2.11.
(2) Let P be as in Example 3.2.9 (2). Then we have $\varepsilon(X_P, L_P; p) = a$ for any $p \in X_P$ by Proposition 3.2.11.
(3) Let P be as in Example 3.2.9 (3). Then for any 1-dimensional face σ of P, $s(P', v') = |P'| = 3$ and $\varepsilon(X_\sigma, L_\sigma; 1_\sigma) = 1$. Thus $\varepsilon(X_P, L_P; p) = \min\{1, 3\} = 1$ for $p \in O_\sigma$. For any vertex v of P, $\varepsilon(X_P, L_P; p) = s(P; v) = 1$ by Remark 3.2.13. Thus we have $\varepsilon(X_P, L_P; p) = 1$ for $p \in X_P \setminus O_P$.
(4) For an integral polytope $P \subset \mathbb{R}^2$ such that X_P is a Del pezzo surface and $L_P = -K_{X_P}$, we can easily compute $\varepsilon(X_P, L_P; p)$ for any p by Propositions 3.2.4 and 3.2.11. As a consequence, we know $\varepsilon(X_P, L_P; p) \in \{1, 2, 3\}$ for such P and any $p \in X_P$.
(5) As (4), we can easily compute $\varepsilon(X_P, L_P; p)$ if X_P is a smooth toric Fano 3-fold and $L_P = -K_{X_P}$. As a consequence, we know $\varepsilon(X_P, L_P; p) \in \{1, 2, 3, 4\}$ for such P and any $p \in X_P$.
(6) Let P be as in Example 3.2.9 (7), and $\sigma \prec P$ a 1-dimensional face. Then it is easy to see $s_1(P; \sigma) = s_2(P; \sigma) = \min\{|\sigma|, abc/|\sigma|\}$. Thus we have

$$\varepsilon(\mathbb{P}(a, b, c), \mathcal{O}(1); p) = (abc)^{-1} \min\{|\sigma|, abc/|\sigma|\} = \min\{|\sigma|/abc, 1/|\sigma|\}$$

for $p \in O_\sigma$. For example, $\varepsilon(\mathbb{P}(a, b, c), \mathcal{O}(1); p) = \min\{b/abc, 1/b\} = 1/ac$ for $p \in O_\sigma$ if σ is the convex hull of $(0, 0)$ and (ab, qb). Note that $|\sigma|$ is not the Euclidean length of σ in $\mathbb{R}^2$, i.e., $|\sigma|$ is not $b\sqrt{a^2 + q^2}$ but b.

When $\sigma \prec P$ is a vertex, we can easily compute $s(P; \sigma)$. For example,

$$\varepsilon(\mathbb{P}(a, b, c), \mathcal{O}(1); p) = s(P; \sigma) = \min\{1/bc, 1/ac\}$$

holds if $\sigma = (0, 0)$ and $p = O_\sigma$.

3.3 Seshadri constants and toric degenerations

In the above section, we study the Seshadri constants on toric varieties. In this section, we investigate non-toric cases by using toric degenerations.

Definition 3.3.1. Let L be a nef $\mathbb{R}$-divisor on a projective variety X and $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{R}^r_{>0}$ for $r > 0$. We say $L(\overline{m})$ or $L(m_1, \ldots, m_r)$ is nef (resp. ample) if so is

$$\mu^* L - \sum_{i=1}^{r} m_i E_i,$$

where $p_1, \ldots, p_r$ are very general r points on X, $\mu : \widetilde{X} \to X$ is the blowing up at $p_1, \ldots, p_r$ and E_i is the exceptional divisor over p_i. In other words, $L(\overline{m})$ is nef if and only if $\varepsilon(X, L; \overline{m}) \geq 1$. We sometimes denote $\mu^* L - \sum_{i=1}^{r} m_i E_i$ by $L(\overline{m})$ for very general p_i.

Remark 3.3.2. To show the nefness of $L(\overline{m})$, it is enough to show $\mu^* L - \sum_{i=1}^{r} m_i E_i$ is nef for *one* choice of $p_1, \ldots, p_r$. This follows from the openness of the ampleness condition as in [Bi, Lemma.6.1.A].

By using degenerations, we can show the nefness (resp. ampleness) of a divisor from the nefness (resp. ampleness) of other divisors. The following theorem is a straightforward generalization of Theorem 2.A in [Bi]:

Theorem 3.3.3. *Let $f : \mathcal{X} \to T$ be a flat projective morphism over a smooth variety T with reduced and irreducible general fibers, and $\mathcal{L}$ an f-nef (resp. f-ample) divisor on $\mathcal{X}$. Let $X_t = f^{-1}(t)$ be the scheme theoretic fiber of f, $L_t = \mathcal{L}|_{X_t}$ for $t \in T$. Assume that Y_i $(1 \leq i \leq r)$ are irreducible components of the central fiber X_0 $(0 \in T)$ with the reduced structures (other components may exist). We assume the following:*

(i) X_0 is reduced at the generic point of Y_i for any i,

(ii) There exist $k_i \in \mathbb{N}$ and $\overline{m}^{(i)} = (m_1^{(i)}, \ldots, m_{k_i}^{(i)}) \in \mathbb{R}_{>0}^{k_i}$ for $1 \leq i \leq r$ such that $\mathcal{L}|_{Y_i}(\overline{m}^{(i)})$ is nef (resp. ample) for any i.

Then $L_t(\overline{m}^{(1)}, \ldots, \overline{m}^{(r)})$ is nef (resp. ample) for very general $t \in T$.

Proof. Fix very general points $p_1^{(i)}, \ldots, p_{k_i}^{(i)}$ in Y_i for $i = 1, \ldots, r$.

Firstly, we assume that there exist sections of f, $\{\sigma_j^{(i)}\}$ for $1 \leq i \leq r$, $1 \leq j \leq k_i$ satisfying $\sigma_j^{(i)}(0) = p_j^{(i)}$. By shrinking T if necessary, we may assume $\sigma_j^{(i)}(T) \cap \sigma_{j'}^{(i')}(T) = \emptyset$ for $(i,j) \neq (i',j')$. Let $\mu : \mathcal{X}' \to \mathcal{X}$ be the blowing up along $\bigcup_{i,j} \sigma_j^{(i)}(T)$, $\mathcal{E}_j^{(i)}$ the exceptional divisor over $\sigma_j^{(i)}(T)$, and set $\mathcal{L}' = \mu^*\mathcal{L} - \Sigma_{i,j} m_j^{(i)} \mathcal{E}_j^{(i)}$. Then for very general t,

$$\mu_t : (f \circ \mu)^{-1}(t) \to f^{-1}(t) = X_t$$

is the blowing up along $\Sigma_{i=1}^r k_i$ smooth points $\{\sigma_j^{(i)}(t)\}$, and it holds that

$$\mathcal{L}'|_{(f\circ\mu)^{-1}(t)} = \mu_t^* L_t - \Sigma_{i,j} m_j^{(i)} E_{j,t}^{(i)},$$

where $E_{j,t}^{(i)}$ is the exceptional divisor over $\sigma_j^{(i)}(t)$. By the assumption ii) and the choice of $p_j^{(i)}$, the restriction of $\mathcal{L}'$ on the fiber of $f \circ \mu : \mathcal{X}' \to T$ over 0 is nef (resp. ample). Hence $\mathcal{L}'|_{(f\circ\mu)^{-1}(t)} = \mu_t^* L_t - \Sigma_{i,j} m_j^{(i)} E_{j,t}^{(i)}$ is also nef (resp. ample) for very general $t \in T$. Thus $L_t(\overline{m}^{(1)}, \ldots, \overline{m}^{(r)})$ is nef (resp. ample).

In general there may not exist such sections, but we can make sections by a base change as follows.

From the assumption i) and by cutting by sufficiently ample divisors on $\mathcal{X}$, there exists a subvariety $U \subset \mathcal{X}$ such that U contains all $p_j^{(i)}$ and the restriction $f|_U : U \to T$ is étale at $p_j^{(i)}$ for any i, j. Set $U^{(k)} = \underbrace{U \times_T \ldots \times_T U}_{k}$ for $k \in \mathbb{N}$. Then the natural morphism $\alpha : U^{(\Sigma k_i)} = U^{(k_1)} \times_T \ldots \times_T U^{(k_r)} \to T$ is étale at $\tilde{p} = (p^{(i)})_i \in U^{(k_1)} \times_T \ldots \times_T U^{(k_r)}$, where $p^{(i)} = (p_j^{(i)})_j \in U^{(k_i)}$. Thus there is an open neighborhood $V \subset U^{(k_1)} \times_T \ldots \times_T U^{(k_r)}$ of $\tilde{p}$ such that $\alpha|_V : V \to T$ is étale. Then by base change we have a diagram

$$\begin{array}{ccc} \mathcal{X} \times_T V & \xrightarrow{\beta} & \mathcal{X} \\ \downarrow g & \circlearrowleft & \downarrow f \\ V & \xrightarrow{\alpha|_V} & T. \end{array}$$

Since $\alpha|_V$ is étale, $g^{-1}(v) \cong f^{-1}(\alpha(v)) = X_{\alpha(v)}$ for $v \in V$, and g and $\beta^*\mathcal{L}$ satisfies the conditions i) and ii) for the central fiber $g^{-1}(\tilde{p}) = X_0$ and Y_i. (Note $\alpha(\tilde{p}) = 0 \in T$.) The morphism

$$V \hookrightarrow U^{(k_1)} \times_T \ldots \times_T U^{(k_r)} \stackrel{\pi_i}{\to} U^{(k_i)} \stackrel{\varpi_j}{\to} U \hookrightarrow \mathcal{X}$$

induces a section $\sigma_j^{(i)}$ of g, where π_i and ϖ_j are the i-th and j-th projections respectively. Since $\sigma_j^{(i)}(\tilde{p}) = p_j^{(i)} \in X_0 = g^{-1}(\tilde{p})$ by the definition of $\sigma_j^{(i)}$, we can use the first part of this proof. Thus, $(\beta^*\mathcal{L})|_{g^{-1}(v)}(\overline{m}^{(1)}, \ldots, \overline{m}^{(r)})$ is nef (resp. ample) for very general $v \in V$. If we identify $X_{\alpha(v)}$ with $g^{-1}(v)$, $L_{\alpha(v)}(\overline{m}^{(1)}, \ldots, \overline{m}^{(r)})$ is identified with $(\beta^*\mathcal{L})|_{g^{-1}(v)}(\overline{m}^{(1)}, \ldots, \overline{m}^{(r)})$. Since $\alpha|_V$ is étale, particularly generically surjective, $L_t(\overline{m}^{(1)}, \ldots, \overline{m}^{(r)})$ is nef (resp. ample) for very general $t \in T$. □

Corollary 3.3.4. *Let $f : \mathcal{X} \to T, \mathcal{L}, X_0$ and Y_i be as in Theorem 3.3.3 satisfying condition (i). Moreover assume that there exists an integral polytope P_i such that the normalization of $(Y_i, \mathcal{L}|_{Y_i})$ is isomorphic to (X_{P_i}, L_{P_i}) as a polarized variety for $1 \le i \le r$. Then $L_t(\varepsilon_1, \ldots, \varepsilon_r)$ is nef for very general $t \in T$, where $\varepsilon_i = \varepsilon(X_{P_i}, L_{P_i}; 1_{P_i})$.*

In particular, $L_t(s_1(P_1), \ldots, s_1(P_r))$ is nef for very general $t \in T$.

Proof. Since the normalization is isomorphic over a non-empty open set in Y_i, it holds that $\varepsilon(X_{P_i}, L_{P_i}; 1) = \varepsilon(Y_i, \mathcal{L}|_{Y_i}; 1)$. Applying Theorem 3.3.3 to $k_i = 1, m_1^{(i)} = \varepsilon_i$, the nefness of $L_t(\varepsilon_1, \ldots, \varepsilon_r)$ follows for very general $t \in T$. The last statement is clear from $\varepsilon_i \ge s_1(P_i)$. □

3.4 Examples in non-toric cases

Theorem 3.3.3 and Corollary 3.3.4 tell us a strategy for obtaining lower bounds of (multi-point) Seshadri constants at very general points:

Finding degenerations to (unions of) polarized varieties whose Seshadri constants are more computable, such as toric varieties.

Toric degenerations are studied very well, thus we know many such degenerations. Furthermore the assumption that the normalizations are toric in Corollary 3.3.4 is weaker than usual toric degenerations, which assume the irreducible components are normal toric themselves. Therefore we can find more such degenerations. Of course, we do not know when such degenerations exist in general. The obtained lower bounds may not be good even if such degenerations exist. But if we can find good degenerations, we sometimes get good lower bounds as we will see in the rest of this chapter.

In this section, we estimate Seshadri constants on some non-toric varieties by using Theorem 3.3.3 and Corollary 3.3.4.

3.4.1 Hypersurfaces and complete intersections in projective spaces

In this subsection, we study Seshadri constants on hypersurfaces or complete intersections in projective spaces. For positive integers $d_1, \ldots, d_k$ and n, we denote by $X^n_{d_1,\ldots,d_k}$ a very general complete intersection of hypersurfaces of degrees $d_1, \ldots, d_k$ in $\mathbb{P}^{n+k}$.

Firstly, we estimate $\varepsilon(X, \mathcal{O}(1); 1)$ for a very general complete intersection X.

Proposition 3.4.1. *Let $d_1, \ldots, d_k$ and n be positive integers. Suppose that there exist a positive integer c and natural numbers $l_1, \ldots, l_k$ such that $\Sigma_{j=1}^k l_j = n$ and $d_j \ge c^{l_j}$ hold for any $1 \le j \le k$. Then $\varepsilon(X^n_{d_1,\ldots,d_k}, \mathcal{O}(1); 1) \ge c$ holds.*

In particular, $\varepsilon(X^n_d, \mathcal{O}(1); 1) \ge \lfloor \sqrt[n]{d} \rfloor$ holds for any $d \in \mathbb{N} \setminus 0$.

Proof. We prove this proposition by 3 steps.

Step 1. Firstly, we find a not necessarily normal toric variety which is a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_k$ in $\mathbb{P}^{n+k}$. Let $d_j^{(i)}$ be natural numbers for $1 \leq i \leq n, 1 \leq j \leq k$ such that $1 + \Sigma_{i=1}^{n} d_j^{(i)} = d_j$ holds for any j. We consider the following homogeneous polynomials

$$\begin{aligned} T_0^{d_1} &= T_1^{d_1^{(1)}} T_2^{d_1^{(2)}} \cdots T_n^{d_1^{(n)}} T_{n+1} \\ T_{n+1}^{d_2} &= T_1^{d_2^{(1)}} T_2^{d_2^{(2)}} \cdots T_n^{d_2^{(n)}} T_{n+2} \\ &\vdots \\ T_{n+k-1}^{d_k} &= T_1^{d_k^{(1)}} T_2^{d_k^{(2)}} \cdots T_n^{d_k^{(n)}} T_{n+k}, \end{aligned}$$

where $T_0, \ldots, T_{n+k}$ are the homogeneous coordinates of $\mathbb{P}^{n+k}$. It is not hard to see that the intersection X of the hypersurfaces defined by these polynomials is reduced, irreducible, and n-dimensional, i.e., a complete intersection variety in $\mathbb{P}^{n+k}$.

Set P be the image of $\mathrm{conv}(0, e_1, \ldots, e_{n+k}) \subset \mathbb{R}^{n+k}$ by the lattice projection

$$\pi : \mathbb{R}^{n+k} = (\mathbb{Z}^{n+k})_{\mathbb{R}} \to (\mathbb{Z}^{n+k}/M)_{\mathbb{R}},$$

where M is the subgroup of $\mathbb{Z}^{n+k}$ spanned by $\Sigma_{i=1}^{n} d_1^{(i)} e_i + e_{n+1}$ and $\Sigma_{i=1}^{n} d_j^{(i)} e_i - d_j e_{n+j-1} + e_{n+j}$ for $2 \leq j \leq k$. Note that $\Sigma_{i=1}^{n} d_1^{(i)} e_i + e_{n+1}$ comes from $T_0^{d_1} = T_1^{d_1^{(1)}} T_2^{d_1^{(2)}} \cdots T_n^{d_1^{(n)}} T_{n+1}$ for example. By Lemma 3.2.1, $(X, \mathcal{O}(1))$ is a not necessarily normal toric variety whose normalization is (X_P, L_P). Since $X^n_{d_1,\ldots,d_k}$ degenerates to X, we have $\varepsilon(X^n_{d_1,\ldots,d_k}, \mathcal{O}(1); 1) \geq \varepsilon(X_P, L_P; 1)$ by Corollary 3.3.4. Thus it suffices to show $\varepsilon(X_P, L_P; 1) \geq c$ for a suitable choice of $d_j^{(i)}$.

Step 2. Secondly, we estimate $\varepsilon(X_P, L_P; 1)$ by $d_j^{(i)}$. We denote $\pi(e_l)$ by $[e_l]$ for $1 \leq l \leq n+k$. Since the coefficient of e_{n+1} in $\Sigma_{i=1}^{n} d_1^{(i)} e_i + e_{n+1}$ and that of e_{n+j} in $\Sigma_{i=1}^{n} d_j^{(i)} e_i - d_j e_{n+j-1} + e_{n+j}$ are 1 for $2 \leq j \leq k$, we can take $[e_1], \ldots, [e_n]$ as a basis of $\mathbb{Z}^{n+k}/M$. It is easy to see that P is the convex hull of $[e_1], \ldots, [e_n]$, and $[e_{n+k}]$. Since $[e_{n+1}] = -\Sigma_{i=1}^{n} d_1^{(i)} [e_i]$ and $[e_{n+j}] = -\Sigma_{i=1}^{n} d_j^{(i)} [e_i] + d_j [e_{n+j-1}]$, we can show $[e_{n+k}] = -\Sigma_{i=1}^{n} a^{(i)} e_i$ for

$$\begin{aligned} a^{(i)} &= \Sigma_{j=1}^{k} d_j^{(i)} d_{j+1} \cdots d_k \\ &= d_1^{(i)} d_2 \cdots d_k + \cdots + d_{k-1}^{(i)} d_k + d_k^{(i)}. \end{aligned}$$

By Example 3.2.9 (8), we have

$$\varepsilon(X_P, L_P; 1) \geq \min_{1 \leq i \leq n} b^{(i)}/b^{(i+1)},$$

where $b^{(i)} = a^{(i)} + a^{(i+1)} + \cdots + a^{(n)} + 1$ for $1 \leq i \leq n$ and $b^{(n+1)} = 1$.

Step 3. Note that $X^n_{d_1,\ldots,d_k} = \bigcap_j X^{n+k-1}_{d_j}$ degenerates to $\bigcap_j (X^{n+k-1}_{c^{l_j}} \cup X^{n+k-1}_{d_j - c^{l_j}})$. Since $X^{n+k-1}_{c^{l_j}}$ and $X^{n+k-1}_{d_j - c^{l_j}}$ are very general, $X^n_{c^{l_1},\ldots,c^{l_k}} = \bigcap_j X^{n+k-1}_{c^{l_j}}$ is an irreducible component of $\bigcap_j (X^{n+k-1}_{c^{l_j}} \cup X^{n+k-1}_{d_j - c^{l_j}})$. By applying Theorem 3.3.3 to this degeneration, we have

$$\varepsilon(X^n_{d_1,\ldots,d_k}, \mathcal{O}(1); 1) \geq \varepsilon(X^n_{c^{l_1},\ldots,c^{l_k}}, \mathcal{O}(1); 1).$$

Thus it is enough to show this proposition for $d_j = c^{l_j}$.

Let us define $d_j^{(i)}$ for $d_j = c^{l_j}$ such that $b^{(i)}/b^{(i+1)} = c$ for any $1 \leq i \leq n$. Set $d_j^{(i)}$ as follows:

$$d_j^{(i)} = \begin{cases} (c-1)c^{h_j - i} & \text{if } h_{j-1} < i \leq h_j \\ 0 & \text{otherwise,} \end{cases}$$

where $h_j = l_1 + \cdots + l_j$ for $1 \leq j \leq k$ and $h_0 = 0$. Note $h_k = \Sigma_j l_j = n$ and $\Sigma_i d_j^{(i)} = c^{l_j} - 1 = d_j - 1$. For each i, we define j_i to be the unique j satisfying $h_{j-1} < i \leq h_j$. Then we have

$$a^{(i)} = d_{j_i+1} \cdots d_k (c-1) c^{h_{j_i} - i} = (c-1)c^{n-i}.$$

Since $b^{(i)} = a^{(i)} + b^{(i+1)}$, we have $b^{(i)} = c^{n+1-i}$. Thus $b^{(i)}/b^{(i+1)} = c^{n+1-i}/c^{n-i} = c$, which proves this proposition. □

Example 3.4.2. If we choose $d_j^{(i)}$ carefully, we may obtain a better estimation than that of Proposition 3.4.1. We use notations as in the proof of Proposition 3.4.1.
(1) Let $2 \leq d_1 \leq \ldots \leq d_k$ be positive integers such that $\Sigma_j d_j \leq n+k$. Then $X^n_{d_1,\ldots,d_k}$ is a Fano n-fold such that $-K_{X^n_{d_1,\ldots,d_k}} = \mathcal{O}(n+k+1-\Sigma_j d_j)$. If $\Sigma_j d_j < n+k$, it is known that $X^n_{d_1,\ldots,d_k}$ is covered by lines (cf. [Deb, Proposition 2.13]). Hence we have $\varepsilon(X^n_{d_1,\ldots,d_k}, \mathcal{O}(1); 1) = 1$.

Now assume $\Sigma_j d_j = n+k$. Then $X^n_{d_1,\ldots,d_k}$ is a Fano n-fold such that $-K_{X^n_{d_1,\ldots,d_k}} = \mathcal{O}(1)$. We can show $\varepsilon(X^n_{d_1,\ldots,d_k}, \mathcal{O}(1); 1) = d_k/(d_k - 1)$ as follows:

We define $d_j^{(i)}$ by

$$d_j^{(i)} = \begin{cases} 1 & \text{if } h'_{j-1} < i \leq h'_j \\ 0 & \text{otherwise,} \end{cases}$$

where $h'_j = (d_1 - 1) + \cdots + (d_j - 1)$. Note $h'_k = \Sigma_{j=1}^k (d_j - 1) = \Sigma_j d_j - k = n$. Then we have $a^{(i)} = d_{j+1} \cdots d_k$ and $b^{(i)} = d_{j+1} \cdots d_k (h'_j + 2 - i)$ for $h'_{j-1} < i \leq h'_j$. By Steps 1 and 2 in the proof of Theorem 3.4.1, it holds that

$$\varepsilon(X^n_{d_1,\ldots,d_k}, \mathcal{O}(1); 1) \geq \min_{1 \leq i \leq n} \frac{b^{(i)}}{b^{(i+1)}} = \min_{1 \leq j \leq k} \frac{d_j}{d_j - 1} = \frac{d_k}{d_k - 1}.$$

Next, we show $\varepsilon(X^n_{d_1,\ldots,d_k}, \mathcal{O}(1); 1) \leq d_k/(d_k - 1)$ by finding a curve $C \subset X^n_{d_1,\ldots,d_k}$ such that $C.\mathcal{O}(1)/\operatorname{mult}_p(C) = d_k/(d_k - 1)$.

Let $F_1, \ldots, F_k$ be homogeneous polynomials in $\mathbb{C}[T_0, \ldots, T_{n+k}]$ of degrees $d_1, \ldots, d_k$ respectively such that $X^n_{d_1,\ldots,d_k} = (F_1 = \cdots = F_k = 0)$. We may assume $p = [1 : 0 : \ldots : 0] \in \mathbb{P}^{n+k}$. Then there exist homogeneous polynomials $F_j^i \in \mathbb{C}[T_1, \ldots, T_{n+k}]$ such that $\deg F_j^i = i$ and $F_j = \sum_{i=1}^{d_j} T_0^{d_j - i} F_j^i$. Let D_j and $D_j^i \subset \mathbb{P}^{n+k}$ be the hypersurfaces defined by F_j and F_j^i respectively. Then

$$\begin{aligned} D_j \cap \bigcap_{i=1}^{d_j - 1} D_j^i &= (F_j = F_j^1 = \cdots = F_j^{d_j - 1} = 0) \\ &= (F_j^1 = \cdots = F_j^{d_j - 1} = F_j^{d_j} = 0) \\ &= \bigcap_{i=1}^{d_j} D_j^i \end{aligned}$$

and

$$\begin{aligned} D_k \cap \bigcap_{i=1}^{d_k-2} D_k^i &= (F_k = F_k^1 = \cdots = F_k^{d_k-2} = 0) \\ &= (F_k^1 = \cdots = F_k^{d_k-2} = T_0 F_k^{d_k-1} + F_k^{d_k} = 0) \\ &= \bigcap_{i=1}^{d_k-2} D_k^i \cap (T_0 F_k^{d_k-1} + F_k^{d_k} = 0). \end{aligned}$$

Note that all F_j^i are general since $X^n_{d_1,\ldots,d_k}$ and p are general. Hence

$$C := \bigcap_{j=1}^{k-1} \bigcap_{i=1}^{d_j} D_j^i \cap \left(\bigcap_{i=1}^{d_k-2} D_k^i \cap (T_0 F_k^{d_k-1} + F_k^{d_k} = 0) \right)$$

is a curve on $X^n_{d_1,\ldots,d_k} = \cap_{j=1}^k D_j$ containing p. By definition,

$$\deg C = \prod_{j=1}^{k-1} d_j! \cdot (d_k - 2)! \cdot d_k$$

and

$$\operatorname{mult}_p(C) = \prod_{j=1}^{k-1} d_j! \cdot (d_k - 1)!.$$

Thus we have $\varepsilon(X^n_{d_1,\ldots,d_k}, \mathcal{O}(1); 1) = \varepsilon(X^n_{d_1,\ldots,d_k}, \mathcal{O}(1); p) \leq \deg C / \operatorname{mult}_p(C) = d_k/(d_k - 1)$. Therefore $\varepsilon(X^n_{d_1,\ldots,d_k}, \mathcal{O}(1); 1) = d_k/(d_k - 1)$ holds.

For example, we have

$$\begin{aligned} \varepsilon(X^3_4, \mathcal{O}(1); 1) &= 4/3, \\ \varepsilon(X^3_{2,3}, \mathcal{O}(1); 1) &= 3/2, \\ \varepsilon(X^3_{2,2,2}, \mathcal{O}(1); 1) &= 2 \end{aligned}$$

when $n = 3$.
(2) When $k = 1$, we denote $d = d_1, d^{(i)} = d_1^{(i)}$ for simplicity. Then, $a^{(i)} = d^{(i)}$ for any i. Thus we have

$$\varepsilon(X^n_d, \mathcal{O}(1); 1) \geq \min_{1 \leq i \leq n} \frac{d^{(i)} + \cdots + d^{(n)} + 1}{d^{(i+1)} + \cdots + d^{(n)} + 1}.$$

In other words,

$$\varepsilon(X^n_d, \mathcal{O}(1); 1) \geq \min \left\{ \frac{c_n}{1}, \frac{c_{n-1}}{c_n}, \ldots, \frac{c_2}{c_3}, \frac{c_1}{c_2} \right\}$$

holds for any increase sequence of positive integers $1 \leq c_n \leq c_{n-1} \leq \cdots \leq c_1 = d$.

When $n = 2$, set $c_1 = d, c_2 = \lceil \sqrt{d} \rceil$. Then $\varepsilon(X^2_d, \mathcal{O}(1); 1) \geq \min\{\lceil \sqrt{d} \rceil, d/\lceil \sqrt{d} \rceil\} = d/\lceil \sqrt{d} \rceil$ holds. From this and Proposition 3.4.1, we have

$$\varepsilon(X^2_d, \mathcal{O}(1); 1) \geq \max\{\lfloor \sqrt{d} \rfloor, d/\lceil \sqrt{d} \rceil\}.$$

When $d \geq 4$, $\varepsilon(X^2_d, \mathcal{O}(1); 1) \geq \lfloor \sqrt{d} \rfloor$ follows also from Proposition 1 in [St] since $\operatorname{Pic} X = \mathbb{Z}\mathcal{O}_X(1)$. But $d/\lceil \sqrt{d} \rceil$ is a new estimation. For example, $\varepsilon(X^2_7, \mathcal{O}(1); 1) \geq 7/3$ holds.

Next, we study multi-point cases. The following proposition looks like Theorem 2.A in [Bi] somehow:

Proposition 3.4.3. *Let $d_1, \dots, d_k, a, b$, and n be positive integers for $k \in \mathbb{N}$. We denote by L_a, L_b, and L_{a+b} the invertible sheaf $\mathcal{O}(1)$ on $X^n_{d_1,\dots,d_k,a}, X^n_{d_1,\dots,d_k,b}$, and $X^n_{d_1,\dots,d_k,a+b}$ respectively.*

If $L_a(\overline{m}_1)$ and $L_b(\overline{m}_2)$ are nef (resp. ample) for $\overline{m}_1 \in \mathbb{R}^{r_1}_{>0}, \overline{m}_2 \in \mathbb{R}^{r_2}_{>0}$, then $L_{a+b}(\overline{m}_1, \overline{m}_2)$ is also nef (resp. ample).

Proof. A very general hypersurface $X^{n+k}_{a+b} \subset \mathbb{P}^{n+k+1}$ of degree $a+b$ degenerates to the union $X^{n+k}_a \cup X^{n+k}_b$ of hypersurfaces of degrees a and b. Thus $X^n_{d_1,\dots,d_k,a+b} = X^{n+1}_{d_1,\dots,d_k} \cap X^{n+k}_{a+b}$ degenerates to $X^{n+1}_{d_1,\dots,d_k} \cap (X^{n+k}_a \cup X^{n+k}_b) = X^n_{d_1,\dots,d_k,a} \cup X^n_{d_1,\dots,d_k,b}$. Applying Theorem 3.3.3 to this degeneration, the proposition follows. □

As a corollary of Proposition 3.4.1 and Theorem 3.3.3 or Proposition 3.4.3, we obtain estimations of multi-point Seshadri constants on hypersurfaces in projective spaces:

Theorem 3.4.4 (=Theorem 3.1.3). *Let X^n_d be a very general hypersurface of degree d in $\mathbb{P}^{n+1}$. Then it holds that*

$$\lfloor \sqrt[n]{d/(m_1^n + \cdots + m_r^n)} \rfloor \leq \varepsilon(X^n_d, \mathcal{O}(1); \overline{m}) \leq \sqrt[n]{d/(m_1^n + \cdots + m_r^n)}$$

for any $\overline{m} = (m_1, \dots, m_r) \in (\mathbb{N} \setminus 0)^r$.

Proof. The second inequality is clear since $\mathcal{O}_{X^n_d}(1)^n = d$. Thus it remains to prove the first inequality. For simplicity, set $c = \lfloor \sqrt[n]{d/(m_1^n + \cdots + m_r^n)} \rfloor$. Let $d_1, \dots, d_r$ be natural numbers such that $d = d_1 + \ldots + d_r$ and $d_i \geq (cm_i)^n$. We can choose such d_i because $d \geq \Sigma_i (cm_i)^n$. By Proposition 3.4.1, $\varepsilon(X^n_{d_i}, \mathcal{O}(1); 1) \geq cm_i$ holds. Note that cm_i is an integer. Since X^n_d degenerates to $\bigcup_{i=1}^r X^n_{d_i}$, we can apply Theorem 3.3.3 and we know $L_d(cm_1, \dots, cm_r)$ is nef, where L_d is the invertible sheaf $\mathcal{O}(1)$ on X^n_d. Hence $\varepsilon(X^n_d, \mathcal{O}(1); \overline{m}) \geq c$ holds. □

3.4.2 Fano 3-folds with Picard number 1

In this subsection, we estimate Seshadri constants on a smooth Fano 3-fold X with Picard number 1, i.e., X is a smooth projective variety of dimension 3 such that $-K_X$ is ample and $\operatorname{Pic} X \cong \mathbb{Z}$. The index of X is the positive integer r such that $-K_X = rH$, where $H \in \operatorname{Pic} X$ is the ample generator.

Toric degenerations of Fano 3-folds are studied by many authors. Small toric degenerations of Fano 3-folds are treated by [Gal], and [CI] investigated complete intersection cases in (weighted) projective spaces and homogeneous spaces. In [ILP], Ilten, Lewis, and Przyjalkowski studied remaining cases of Fano 3-folds with Picard number 1. They showed that every smooth Fano 3-fold of Picard number 1 has a toric degeneration and gave an explicit description of the moment polytope of the central fiber. Most of the degenerations in [ILP] give good lower bounds of Seshadri constants.

Example 3.4.5. Let $X \subset \mathbb{P}(1,1,1,1,3)$ be a very general hypersurface of degree 6. By [ILP, First Main Theorem], $(X, \mathcal{O}(1))$ degenerates to (X_P, L_P) (as a $\mathbb{Q}$-polarized variety) for $P := \operatorname{conv}(e_1, e_2, e_3, -1/3(e_1+e_2+e_3)) \subset \mathbb{R}^3$. It is easy to see $s_1(P) \geq 6/5$. Thus we have $\varepsilon(X, \mathcal{O}(1); 1) \geq 6/5$ by Corollary 3.3.4.

We can show $\varepsilon(X, \mathcal{O}(1); 1) \leq 6/5$ by similar arguments as Example 3.4.2 (1), but we give a little more geometrical proof here.

Fix a very general point $p \in X$. Define $p' \in X$ by $\{p, p'\} := \varphi^{-1}(\varphi(p))$, where $\varphi : X \to \mathbb{P}^3$ is the double cover defined by $|\mathcal{O}_X(1)|$. Since $\dim H^0(X, \mathcal{O}(3)) = 21$ and $\dim \mathcal{O}_X/\mathfrak{m}_p^4 = 20$, there exists $S \in |\mathcal{O}_X(3) \otimes \mathfrak{m}_p^4|$. Then $\operatorname{mult}_p(S) = 4$ because X and p are very general. It is not hard to see that S does not contain p'. Let $\pi : \widetilde{X} \to X$ be the blowing up at $\{p, p'\}$, and set E, E' be the exceptional divisors over p and p' respectively. Let $\widetilde{S} \subset \widetilde{X}$ be the strict transform of S, and set $\psi = \varphi|_{\widetilde{S}} : \widetilde{S} \to S$ and $F = E|_{\widetilde{S}}$. Then $F^2 = -\operatorname{mult}_p(S) = -4$. Since $\varphi^*\mathcal{O}_X(1) - E - E'$ is base point free, so is $(\varphi^*\mathcal{O}_X(1) - E - E')|_{\widetilde{S}} = \psi^*\mathcal{O}_S(1) - F$. Let $f : \widetilde{S} \to \mathbb{P}^2$ be the morphism defined by $\varphi^*\mathcal{O}_S(1) - F$. By $\varepsilon(S, \mathcal{O}(1); p) \geq \varepsilon(X, \mathcal{O}(1); p) > 1$, we know $\varphi^*\mathcal{O}_S(1) - F$ is ample. Thus f is finite morphism. Since $f_*F.\mathcal{O}_{\mathbb{P}^2}(1) = F.f^*\mathcal{O}_{\mathbb{P}^2}(1) = F.(\varphi^*\mathcal{O}_S(1) - F) = 4$, we have $f_*F \sim \mathcal{O}_{\mathbb{P}^2}(4)$. Thus $D := f^*f_*F - F$ is an effective divisor and $D \sim \psi^*\mathcal{O}_S(4) - 5F$. Hence $\psi^*\mathcal{O}_S(1) - 6/5F$ is not ample because $D.(\psi^*\mathcal{O}_S(1) - 6/5F) = (\psi^*\mathcal{O}_S(4) - 5F).(\psi^*\mathcal{O}_S(1) - 6/5F) = 0$. Thus $\varepsilon(X, \mathcal{O}(1); 1) = \varepsilon(X, \mathcal{O}(1); p) \leq \varepsilon(S, \mathcal{O}(1); p) \leq 6/5$ holds and we have $\varepsilon(X, \mathcal{O}(1), 1) = 6/5$.

It is known that there are 17 families of smooth Fano 3-folds with Picard number 1. For each case, we can compute the Seshadri constant as follows:

Theorem 3.4.6 (=Theorem 3.1.5). *For each family of smooth Fano 3-folds with Picard number* 1*,* $\varepsilon(X, -K_X; 1)$ *is as in Table 3.1, where* X *is a very general member in the family.*

Proof. For No.1 - 4 in Table 3.1, $\varepsilon(X, -K_X; 1)$ is computed in Examples 3.4.2 (1) and 3.4.5. (In fact, degenerations in [ILP] for No.2 - 4 give same lower bounds as Examples 3.4.2 (1) though some of their degenerations are different from those of Examples 3.4.2 (1).)

For No.5 - 17, we can show the following:

$$\varepsilon(X, -K_X; 1) \geq \begin{cases} 2 & \text{for No.5 - 15} \\ 3 & \text{for No.16} \\ 4 & \text{for No.17.} \end{cases} \qquad \cdots (*)$$

Except No.11, these lower bounds are obtained by applying Corollary 3.3.4 to the degenerations in [ILP, First Main Theorem].

In No.11 case, the moment polytope of the central fiber of the degeneration in [ILP] is $P' = \operatorname{conv}(e_3, 2e_1 - e_3, e_2 - e_3, -2/3e_1 - 2/3e_2 - e_3)$. By the 2nd projection, we have $s_1(P') \leq s_2(P') \leq 5/3$. Thus $s_1(P')$ is not so large. Instead of this degeneration, we consider the following degeneration, whose construction is essentially same as Proposition 3.4.1.

Let T_0, T_1, T_2, T_3, T_4 be weighted homogeneous coordinates on $\mathbb{P}(1,1,1,2,3)$ with $\deg T_0 = \deg T_1 = \deg T_2 = 1, \deg T_3 = 2, \deg T_4 = 3$. Then $X_0 := (T_4^2 = T_1^2 T_2^2 T_3) \subset \mathbb{P}(1,1,1,2,3)$ is a non-normal toric variety whose moment polytope is $P = \operatorname{conv}(0, e_1, e_2, -e_1 - e_2 + e_3)$. A very general hypersurface X in $\mathbb{P}(1,1,1,2,3)$ of degree 6 degenerates to X_0. Since $-K_X = \mathcal{O}_X(2)$, $(X, -K_X)$ degenerates to $(X_0, \mathcal{O}_{X_0}(2))$. Thus $\varepsilon(X, -K_X; 1) \geq \varepsilon(X_0, \mathcal{O}_{X_0}(2); 1) = \varepsilon(X_{2P}, L_{2P}; 1) \geq s_1(2P) = 2$.

Next, we think about the upper bounds. For No.5 - 10, it is known that X is covered by conics, i.e., for any general $p \in X$, there exists a smooth rational curve C containing p such that $C.(-K_X) = 2$ (cf. [IP, Chapter 4]). Thus $\varepsilon(X, -K_X; 1) \leq 2$ in these cases. For No.11 - 15, $-K_X = 2H$ holds for the ample generator H. Assume that $\varepsilon(X, -K_X; 1) > 2$, i.e., $-K_{\widetilde{X}} = \mu^*(-K_X) - 2E = 2(\mu^*H - E)$ is ample for the blowing up $\mu : \widetilde{X} \to X$ at a very general point $p \in X$ and $E = \mu^{-1}(p)$. Then $\widetilde{X}$ is a Fano 3-fold of index 2, i.e., a Del Pezzo 3-fold, and the Picard number is 2. By the classification of Del Pezzo manifolds (cf. [IP, §12.1]), $(-K_{\widetilde{X}})^3 = 8(\pi^*H - E)^3$ must be $8 \cdot 6$ or $8 \cdot 7$, which contradicts $H^3 \leq 5$. Thus $\varepsilon(X, -K_X; 1) \leq 2$ holds for No.11 - 15. For No.16 and 17, X is covered by lines since X is a smooth quadric or $\mathbb{P}^3$. Hence $\varepsilon(X, -K_X; 1) = r\varepsilon(X, H; 1) \leq r$ holds for the index r and the ample generator H for No.16 and 17.

Thus the inequalities in $(*)$ are in fact equalities, and the proof is completed. □

3.4.3 Other examples

To apply Corollary 3.3.4, we have to find toric degenerations. We give some examples which degenerate to (unions of) toric varieties.

Example 3.4.7. Let G be a connected reductive group. Alexeev and Brion [AB] proved that any polarized spherical G-variety (X, L) admits a flat degeneration to a polarized toric variety over $\mathbb{A}^1$ and gave an explicit description of the moment polytope of the central fiber. Note that this degeneration is trivial over $\mathbb{A}^1 \setminus \{0\}$. Hence we can get a lower bound of $\varepsilon(X, L; 1)$ by applying Corollary 3.3.4 to this degeneration.

Example 3.4.8. For an n-dimensional polarized variety (X, L) and a flag $Y_\bullet$ of subvarieties of X, that is, a chain $X = Y_0 \supset Y_1 \supset \cdots \supset Y_n$, where Y_i is a subvariety of codimension i in X which is nonsingular at the point Y_n, we can define the Okounkov body $\Delta_{Y_\bullet}(L) \subset \mathbb{R}^d$ (see [LM] or [KK]). Roughly, we defines a graded semigroup $\Gamma \subset \mathbb{N} \times \mathbb{N}^n$ from (X, L) and $Y_\bullet$, and $\Delta(L) = \Delta_{Y_\bullet}(L)$ is defined to be the intersection of $\{1\} \times \mathbb{R}^n$ with the closure of the convex hull of Γ in $\mathbb{R} \times \mathbb{R}^n$. Note that $\Delta(L)$ is nothing but the moment polytope $\Delta(\Gamma)$ if Γ is finitely generated (cf. Definition 1.3.1). Anderson [An] showed that if Γ is finitely generated, (X, L) admits a flat degeneration to the not necessarily normal polarized toric variety $(X(\Gamma), L(\Gamma))$ over $\mathbb{A}^1$ which is trivial over $\mathbb{A}^1 \setminus \{0\}$. Thus $\varepsilon(X, L; 1) \geq \varepsilon(X_{\Delta(L)}, L_{\Delta(L)}; 1_{\Delta(L)})$ holds by Corollary 3.3.4 in this case. In Chapter 4, we will define $\varepsilon(X_\Delta, L_\Delta; 1_\Delta)$ for any closed convex set $\Delta \subset \mathbb{R}^n$ suitably, and prove that $\varepsilon(X, L; 1) \geq \varepsilon(X_{\Delta(L)}, L_{\Delta(L)}; 1_{\Delta(L)})$ holds without the assumption that Γ is finitely generated. See Theorem 4.1.1.

Example 3.4.9. (cf. [BBC+, 3.10]) Let P be an integral polytope of dimension n in $M_\mathbb{R}$. A polytope decomposition $\mathcal{P}$ of P is a finite subset of $\{\sigma \mid \sigma \text{ is a polytope in } M_\mathbb{R}\}$ such that

(i) $P = \bigcup_{\sigma \in \mathcal{P}} \sigma$,

(ii) if $\sigma \in \mathcal{P}$ and τ is a face of σ, then $\tau \in \mathcal{P}$,

(iii) if $\sigma, \sigma' \in \mathcal{P}$, then $\sigma \cap \sigma'$ is either a common face of σ, σ' or empty.

We say $\mathcal{P}$ is integral (resp. rational) if all $\sigma \in \mathcal{P}$ are integral (resp. rational) polytopes. For example, a rational affine function $f : M_\mathbb{R} \to \mathbb{R}$ defines a rational polytope decomposition $\mathcal{P}_f$ of P by $\mathcal{P}_f = \{\sigma \cap f^{-1}([0, +\infty)), \sigma \cap f^{-1}(0), \sigma \cap f^{-1}((-\infty, 0])\}_{\sigma \prec P}$.

Let $\mathcal{P}$ be an integral polytope decomposition of P. If there exists a function $\varphi : P \to \mathbb{R}$ such that

(a) φ is piecewise affine and strictly convex with respect to $\mathcal{P}$,

(b) φ takes integral values at all $u \in P \cap M$,

then one can construct an $n+1$ dimensional toric variety $\mathcal{X}$, an ample line bundle $\mathcal{L}$ on $\mathcal{X}$ and a projective toric morphism $f : \mathcal{X} \to \mathbb{A}^1$ such that $X_0 = \bigcup_{i=1}^r X_{P_i}, \mathcal{L}|_{X_{P_i}} = L_{P_i}$ and $X_t = X_P, L_t = L_P$ for any $t \in \mathbb{A}^1 \setminus \{0\}$, where $\mathcal{P}^{[n]} = \{\sigma \in \mathcal{P} \mid \dim \sigma = n\} = \{P_1, \ldots, P_r\}$. See [GS] for example. Thus in this case $L_P(s_1(P_1), \ldots, s_1(P_r))$ is nef by Corollary 3.3.4. Such φ exists at least for the decomposition $k\mathcal{P}_f = \{k\sigma\}_{\sigma \in \mathcal{P}_f}$ of kP defined by a rational affine function f if $k \in \mathbb{N}$ is sufficiently large and divisible. See also Proposition 4.5.11.

For example, Theorem0.6 in [Ec], which states $\varepsilon(\mathbb{P}^2, \mathcal{O}(1); \overbrace{1, \ldots, 1}^{10}) \geq 4/13$, follows from this argument by using his decomposition of $\mathrm{conv}(0, e_1, e_2) \subset \mathbb{R}^2$ in his paper. In fact, the author defined $s_1(P)$ inspired by Theorem2.2 in [Ec].

4

Seshadri constants and Okounkov bodies

4.1 Introduction

In this chapter, we extend the notion of Seshadri constants slightly. Let $W_\bullet = \{W_k\}_k$ be a birational graded linear series associated to a line bundle L on a variety X (see Definition 4.2.3 for the definition of "birational"), $r \in \mathbb{N} \setminus 0$, and $\overline{m} \in \mathbb{R}^r_{>0}$. We define the Seshadri constant $\varepsilon(X, W_\bullet; \overline{m})$ of $W_\bullet$ at very general points with weight $\overline{m}$. We will show in Lemma 4.2.11 that this is a generalization of $\varepsilon(X, L; \overline{m})$ defined for a polarized variety (X, L) in Definition 1.2.4.

Let $W_\bullet$ be a graded linear series associated to a line bundle L on an n-dimensional variety X. Fix a local coordinate system $z = (z_1, \ldots, z_n)$ on X at a smooth point and a monomial order $>$ on $\mathbb{N}^n$. If $W_\bullet$ is birational, we can construct an n-dimensional closed convex set $\Delta(W_\bullet) = \Delta_{z,>}(W_\bullet)$ in $\mathbb{R}^n$, which is called the Okounkov body of $W_\bullet$ with respect to z and $>$. Okounkov bodies were introduced and studied by Lazarsfeld and Mustaţă [LM] and independently by Kaveh and Khovanskii [KK], based on the work of Okounkov [Ok1], [Ok2]. The Okounkov body $\Delta(W_\bullet)$ seems to have rich information of $W_\bullet$. For example the volume of $W_\bullet$ is $n!$ times the Euclidean volume of $\Delta(W_\bullet)$.

For any n-dimensional convex set $\Delta \subset \mathbb{R}^n$, we can define a birational monomial graded linear series $W_{\Delta,\bullet}$ associated to $\mathcal{O}_{(\mathbb{C}^\times)^n}$ on $(\mathbb{C}^\times)^n$. Thus we can define an invariant $s(\Delta; \overline{m}) := \varepsilon((\mathbb{C}^\times)^n, W_{\Delta,\bullet}; \overline{m})$ for $\overline{m} \in \mathbb{R}^r_{>0}$. The main theorem of this chapter states that Okounkov bodies give lower bounds of Seshadri constants:

Theorem 4.1.1 (=Theorem 4.4.8). *Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on an n-dimensional variety X. Fix a local coordinate system $z = (z_1, \ldots, z_n)$ on X at a smooth point and a monomial order $>$ on $\mathbb{N}^n$.*

Then $\varepsilon(W_\bullet; \overline{m}) \geq s(\Delta_{z,>}(W_\bullet); \overline{m}) (= \varepsilon((\mathbb{C}^\times)^n, W_{\Delta_{z,>}(W_\bullet),\bullet}; \overline{m}))$ holds for any $r \in \mathbb{N} \setminus 0$ and $\overline{m} \in \mathbb{R}^r_{>0}$.

This theorem says that the Seshadri constant of a graded linear series $W_\bullet$ is greater than or equal to that of the monomial graded linear series $W_{\Delta(W_\bullet),\bullet}$ defined by the Okounkov body.

In view of Theorem 4.1.1, it is natural to investigate $s(\Delta; \overline{m})$. It is very difficult to compute $s(\Delta; \overline{m})$ in general, but when $r = 1$ and $\overline{m} = 1$ (in this case we write $s(\Delta)$ for short), we have the following theorem:

Theorem 4.1.2 (cf. Corollary 4.5.3, Corollary 4.5.5). *For an n-dimensional bounded convex* *set* $\Delta \subset \mathbb{R}^n_{\geq 0}$*, it holds that*

$$\begin{aligned} s(\Delta) &= \sup_{k\in\mathbb{N}\setminus 0} \frac{\max\{m \in \mathbb{N} \,|\, \operatorname{rank} A_{k\Delta,m} = \binom{m+n}{n}\}}{k} \\ &= \lim_{k\in\mathbb{N}\setminus 0} \frac{\max\{m \in \mathbb{N} \,|\, \operatorname{rank} A_{k\Delta,m} = \binom{m+n}{n}\}}{k}, \end{aligned}$$

where $A_{k\Delta,m}$ *is a matrix which depends on* $m, n \in \mathbb{N}$ *and* $k\Delta$*.*

As a special case, for an integral polytope $P \subset \mathbb{R}^n_{\geq 0}$ *of dimension* n*, it holds that*

$$\begin{aligned} \varepsilon(X_P, L_P; 1) &= \sup_{k\in\mathbb{N}\setminus 0} \frac{\max\{m \in \mathbb{N} \,|\, \operatorname{rank} A_{kP,m} = \binom{m+n}{n}\}}{k} \\ &= \lim_{k\in\mathbb{N}\setminus 0} \frac{\max\{m \in \mathbb{N} \,|\, \operatorname{rank} A_{kP,m} = \binom{m+n}{n}\}}{k}. \end{aligned}$$

Since $s(\Delta)$ (or $s(P)$) does not change under parallel translations, the assumption that $\Delta \subset \mathbb{R}^n_{\geq 0}$ (or $P \subset \mathbb{R}^n_{\geq 0}$) is not essential. An important point is that $\max\{m \in \mathbb{N} \,|\, \operatorname{rank} A_{k\Delta,m} = \binom{m+n}{n}\}$ can be computed if we know the lattice points in $k\Delta$. Therefore this theorem states that $s(\Delta)$ or the Seshadri constant $\varepsilon(X_P, L_P; 1)$ is the supremum (or the limit) of computable values.

When $m > 1$, we state three methods to obtain lower bounds of $s(\Delta; \overline{m})$, though it is not enough for good estimations.

This chapter is organized as follows: In Section 2, we define Seshadri constants for graded linear series. In Section 3, we introduce invariants for convex sets. In Section 4, we give the proof of Theorem 4.1.1. In Section 5, we investigate the computations of invariants defined in Section 3.

4.2 Seshadri constants of graded linear series

In this section, we define Seshadri constants for graded linear series.

Definition 4.2.1. Let L be a line bundle on a (not necessarily projective) variety X, and W a subspace of $H^0(X, L)$. For $r \in \mathbb{N} \setminus 0$ and $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$, we say that W separates $\overline{m}$-jets at smooth r points $p_1, \ldots, p_r$ in X if the natural map

$$W \to L/(L \otimes \bigotimes_{i=1}^{r} \mathfrak{m}_{p_i}^{m_i+1}) = \bigoplus_{i=1}^{r} L/\mathfrak{m}_{p_i}^{m_i+1} L$$

is surjective, where we regard $\mathfrak{m}_{p_i}^{m_i+1} = \mathcal{O}_X$ for $m_i \leq -1$. We say W generically separates $\overline{m}$-jets if W separates $\overline{m}$-jets at general r points in X. Note that any W generically separates $\overline{m}$-jets if $m_i \leq -1$ for any i.

We define $j(W; \overline{m}) \in \mathbb{R} \cup \{+\infty\}$ for $W \subset H^0(X, L)$ and $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{R}^r_{>0}$ to be

$$j(W; \overline{m}) = \sup\{\, t \in \mathbb{R} \,|\, W \text{ generically separates } \lceil t\overline{m} \rceil\text{-jets}\},$$

where $\lceil t\overline{m} \rceil = (\lceil tm_1 \rceil, \ldots, \lceil tm_r \rceil)$. We denote it by $j(W)$ when $r = 1$ and $\overline{m} = 1 \in \mathbb{R}_{>0}$. Note that

$$j(W; \overline{m}) = \max\{\, t \in \mathbb{R} \,|\, W \text{ generically separates } \lceil t\overline{m} \rceil\text{-jets}\} \in \mathbb{R}$$

when $\dim W < +\infty$.

When $W = H^0(X, L)$, we denote $j(W, \overline{m})$ by $j(L; \overline{m})$.

Remark 4.2.2. Note that W generically separates $\overline{m}$-jets if W separates $\overline{m}$-jets at *some* smooth r points $p_1, \ldots, p_r$ in X. We use this in Sections 4.4 and 4.5.

Suppose that X is a variety of dimension n and L is a line bundle on X. Let $W_\bullet = \{W_k\}_{k \in \mathbb{N}}$ be a graded linear series associated to L, i.e., W_k is a subspace of $H^0(X, kL)$ for any $k \geq 0$ with $W_0 = \mathbb{C}$, such that $\bigoplus_{k \geq 0} W_k$ is a graded subalgebra of the section ring $\bigoplus_{k \geq 0} H^0(X, kL)$. When all W_k are finite dimensional, $W_\bullet$ is called of finite dimensional type.

Definition 4.2.3. A graded linear series $W_\bullet$ on a variety X is birational if the function field $K(X)$ of X is generated by $\{ f/g \in K(X) \mid f, g \in W_k, g \neq 0\}$ over $\mathbb{C}$ for any $k \gg 0$. When $W_\bullet$ is of finite dimensional type, this is clearly equivalent to the condition that the rational map defined by $|W_k|$ is birational onto its image for any $k \gg 0$, which is Condition (B) in [LM, Definition 2.5].

Now we define Seshadri constants for graded linear series:

Definition 4.2.4. Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on a variety X.

For $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{R}^r_{>0}$, we define the Seshadri constant of $W_\bullet$ at very general points with weight $\overline{m}$ to be

$$\varepsilon(X, W_\bullet; \overline{m}) = \varepsilon(W_\bullet; \overline{m}) := \sup_{k>0} \frac{j(W_k; \overline{m})}{k} \in \mathbb{R}_{>0} \cup \{+\infty\}.$$

Note that $j(W_k; \overline{m}) > 0$ holds for $k \gg 0$ by the birationality of $W_\bullet$.

When $W_k = H^0(X, kL)$ for any k, we denote it by $\varepsilon(X, L; \overline{m})$ or $\varepsilon(L; \overline{m})$ for short.

Remark 4.2.5. The definition of Seshadri constants by jet separations is due to Theorem 6.4 in [Dem]. See also [La2] or [BDH+]. We will show in Lemmas 4.2.11 that $\varepsilon(L; \overline{m})$ defined in Definition 4.2.4 coincides with $\varepsilon(L; \overline{m})$ defined in Definition 1.2.4. Throughout this chapter, or at least until Lemma 4.2.11, we use $\varepsilon(L; \overline{m})$ in the sense of Definition 4.2.4.

Remark 4.2.6. If subspaces W and W' in $H^0(X, L)$ satisfy the inclusion $W \subset W'$, then $j(W; \overline{m}) \leq j(W'; \overline{m})$ holds by definition. Thus $\varepsilon(W_\bullet; \overline{m}) \leq \varepsilon(W'_\bullet; \overline{m})$ holds for $W_\bullet, W'_\bullet$ associated to L if $W_k \subset W'_k$ for any k (we write $W_\bullet \subset W_\bullet'$ for such $W_\bullet, W'_\bullet$).

If $L \hookrightarrow L'$ is an injection between line bundles on X and W is a subspace of $H^0(X, L)$, then $j(W; \overline{m})$ does not change if we regard W as a subspace of $H^0(X, L')$ because we consider jets separations at general points. Hence $\varepsilon(W_\bullet; \overline{m})$ also dose not change for $W_\bullet$ which is associated to L if we consider that $W_\bullet$ is associated to L'.

Let $\pi : X' \to X$ be a birational morphism and $W_\bullet$ a graded linear series associated to L on X. Then we can consider that $W_\bullet$ is a graded linear series on X' associated to $\pi^* L$ by the natural inclusion $H^0(X, kL) \subset H^0(X', k\pi^* L)$ for any $k \in \mathbb{N}$. By the similar reason as above, $\varepsilon(X, W_\bullet; \overline{m}) = \varepsilon(X', W_\bullet; \overline{m})$ holds.

By the following lemma, we may assume that $W_\bullet$ is of finite dimensional type in many cases when we prove properties of $\varepsilon(X, W_\bullet; \overline{m})$:

Lemma 4.2.7. *Let $W_\bullet$ be a birational graded linear series. Suppose that $W_{1,\bullet} \subset W_{2,\bullet} \subset \cdots \subset W_{l,\bullet} \subset \cdots \subset W_\bullet$ is an increasing sequence of birational graded linear series in $W_\bullet$ such that $W_k = \bigcup_{l=1}^\infty W_{l,k}$ holds for all k. Then $\varepsilon(W_\bullet; \overline{m}) = \sup_l \varepsilon(W_{l,\bullet}; \overline{m}) = \lim_l \varepsilon(W_{l,\bullet}; \overline{m})$ holds.*

Proof. The existence of $\lim_l \varepsilon(W_{l,\bullet}; \overline{m})$ is clear because $\varepsilon(W_{l,\bullet}; \overline{m})$ is monotonically increasing. The inequality $\varepsilon(W_\bullet; \overline{m}) \geq \sup_l \varepsilon(W_{l,\bullet}; \overline{m}) \geq \lim_l \varepsilon(W_{l,\bullet}; \overline{m})$ is also clear. Thus it is enough to show $\varepsilon(W_\bullet; \overline{m}) \leq \lim_l \varepsilon(W_{l,\bullet}; \overline{m})$.

Fix k and a real number $t < j(W_k; \overline{m})$. By definition, W_k generically separates $\lceil t\overline{m} \rceil$-jets. Hence $W_{l,k}$ also generically separates $\lceil t\overline{m} \rceil$-jets for $l \gg 0$ by the assumption $W_k = \bigcup_l W_{l,k}$. Thus it holds that $j(W_k; \overline{m})/k = \lim j(W_{l,k}; \overline{m})/k \leq \lim_l \varepsilon(W_{l,\bullet}; \overline{m})$. By definition of $\varepsilon(W_\bullet; \overline{m})$, we have $\varepsilon(W_\bullet; \overline{m}) \leq \lim_l \varepsilon(W_{l,\bullet}; \overline{m})$. □

In Definition 4.2.4, $\varepsilon(W_\bullet; \overline{m})$ is defined by the supremum, but in fact it is the limit:

Lemma 4.2.8. *In Definition 4.2.4,* $\varepsilon(W_\bullet; \overline{m}) = \lim \dfrac{j(W_k; \overline{m})}{k}$ *holds.*

Proof. By Lemma 4.2.7, we may assume that $W_\bullet$ is of finite dimensional type.

At first we show the following claim:

Claim 4.2.9. *Let $W_\bullet$ be a birational graded linear series of finite dimensional type associated to a line bundle L on a variety X. Then $j(W_{kl}; \overline{m}) \geq l \cdot j(W_k; \overline{m})$ holds for any positive integer $k, l > 0$.*

Proof of Claim 4.2.9. For simplicity, we set $j_{k'} = j(W_{k'}; \overline{m})$ in the proof of Claim 4.2.9. We prove this claim when $r = 1$. When $r > 1$, the proof is essentially same. Thus we leave the details to the reader.

We write $m \in \mathbb{R}_{>0}$ instead of $\overline{m}$. Choose very general point $p \in X$ and set

$$W_{k',i} = W_{k'} \cap H^0(X, L^{\otimes k'} \otimes \mathfrak{m}_p^i) \subset H^0(X, L^{\otimes k'})$$

for $k', i \geq 0$. For any $j \in \mathbb{N}$, it is easy to show that,

$$W_{k'} \to L^{\otimes k'} \otimes \mathcal{O}_X / \mathfrak{m}_p^{j+1}$$

is surjective if and only if

$$W_{k',i} \to L^{\otimes k'} \otimes \mathfrak{m}_p^i / \mathfrak{m}_p^{i+1}$$

is surjective for each $i \in \{0, 1, \ldots, j\}$. Fix $i \in \{0, 1, 2, \ldots, l\lceil j_k m \rceil\}$. Then there exist integers $0 \leq i_1, \ldots, i_l \leq \lceil j_k m \rceil$ such that $i = i_1 + \ldots + i_l$. Consider the following diagram:

$$\begin{array}{ccc} W_{k,i_1} \otimes \cdots \otimes W_{k,i_l} & \xrightarrow{\alpha} & L^{\otimes k} \otimes \mathfrak{m}_p^{i_1}/\mathfrak{m}_p^{i_1+1} \otimes \cdots \otimes L^{\otimes k} \otimes \mathfrak{m}_p^{i_l}/\mathfrak{m}_p^{i_l+1} \\ \downarrow & \circlearrowleft & \downarrow \beta \\ W_{kl,i} & \longrightarrow & L^{\otimes kl} \otimes \mathfrak{m}_p^i/\mathfrak{m}_p^{i+1}. \end{array}$$

In the above diagram, α is surjective because $i_1, \ldots, i_l \leq \lceil j_k m \rceil$, and β is clearly surjective. Hence the map

$$W_{kl,i} \to L^{\otimes kl} \otimes \mathfrak{m}_p^i / \mathfrak{m}_p^{i+1}$$

is also surjective for any $i \in \{0, 1, 2, \ldots, l\lceil j_k m \rceil\}$. Thus W_{kl} generically separates $l\lceil j_k m \rceil$-jets, which means $j_{kl} \geq l\lceil j_k m \rceil / m \geq l \cdot j_k$. □

Now we return to the proof of the lemma. We only prove the case $\varepsilon(W_\bullet; \overline{m}) < +\infty$. When $\varepsilon(W_\bullet; \overline{m}) = +\infty$, the proof is similar.

By definition, it is sufficient to show

$$\liminf \frac{j(W_k; \overline{m})}{k} \geq \sup \frac{j(W_k; \overline{m})}{k}.$$

Fix $\delta > 0$ and choose k_0 such that $\dfrac{j(W_{k_0}; \overline{m})}{k_0} > \varepsilon(W_\bullet; \overline{m}) - \delta$. Let N be a sufficiently large integer and choose a general $s_{N+i} \in W_{N+i}$ for each $i = 0, 1, \ldots, k_0 - 1$.

Fix $k \geq N$. Then k is written as $k = k_0 l + N + i$ for some natural numbers $l \in \mathbb{N}$ and $0 \leq i \leq k_0 - 1$. Then there is the injection

$$W_{k_0 l} \hookrightarrow W_k : s \mapsto s \cdot s_{N+i},$$

induced by the multiplication in $\bigoplus W_j$. Therefore it holds that

$$j(W_k; \overline{m}) \geq j(W_{k_0 l}; \overline{m}) \geq l \cdot j(W_{k_0}; \overline{m})$$

by Remark 4.2.6 and Claim 4.2.9. Since $l \to +\infty$ if $k \to +\infty$,

$$\liminf_k \frac{j(W_k; \overline{m})}{k} \geq \liminf_l \frac{l \cdot j(W_{k_0}; \overline{m})}{k_0 l + N + i} \geq \frac{j(W_{k_0}; \overline{m})}{k_0} > \varepsilon(W_\bullet; \overline{m}) - \delta.$$

By $\delta \to 0$, we finish the proof of Lemma 4.2.8. □

Many properties of Seshadri constants of ample line bundles also hold for graded linear series. We use the following later:

Lemma 4.2.10. *Let $\overline{m}$ be in $\mathbb{R}^r_{>0}$ and $W_\bullet$ (resp.$W'_\bullet$) a birational graded linear series associated to a line bundles L (resp. L') on a variety X. Then the following holds:*

(1) $\varepsilon(W^{(l)}_\bullet; \overline{m}) = l \cdot \varepsilon(W_\bullet; \overline{m})$ holds for any positive integer $l \in \mathbb{N} \setminus 0$, where $W^{(l)}_\bullet$ is the graded linear series associated to $L^{\otimes l}$ defined by $W^{(l)}_k = W_{kl}$.

(2) $\varepsilon(W_\bullet; t\overline{m}) = t^{-1}\varepsilon(W_\bullet; \overline{m})$ holds for any positive real number $t > 0$.

(3) $\varepsilon(W''_\bullet; \overline{m}) \geq \varepsilon(W_\bullet; \overline{m}) + \varepsilon(W'_\bullet; \overline{m})$ holds, where $W''_\bullet$ is the graded linear series associated to $L \otimes L'$ defined by $W''_k :=$ the image of $W_k \otimes W'_k \to H^0(X, L \otimes L')$.

(4) $\varepsilon(W_\bullet; \overline{m}) \leq \sqrt[n]{\mathrm{vol}(W_\bullet)/\Sigma_{i=1}^r m_i^n}$ holds, where $\mathrm{vol}(W_\bullet) = \lim_k \dfrac{\dim W_k}{k^n/n!}$, and n is the dimension of X.

Proof. By Lemma 4.2.8, we have

$$\begin{aligned} \varepsilon(W^{(l)}_\bullet; \overline{m}) &= \lim_k \frac{j(W^{(l)}_k; \overline{m})}{k} \\ &= \lim_k \frac{j(W_{kl}; \overline{m})}{k} \\ &= l \cdot \lim_k \frac{j(W_{kl}; \overline{m})}{kl} \\ &= l \cdot \lim_k \frac{j(W_k; \overline{m})}{k} = l \cdot \varepsilon(W_\bullet; \overline{m}). \end{aligned}$$

Hence (1) is shown.

We can show (2) easily from the definition and the following:

$$\begin{aligned} j(W_k; t\overline{m}) &= \sup\{\, s \in \mathbb{R} \mid W_k \text{ generically separates } \lceil st\overline{m} \rceil\text{-jets} \,\} \\ &= t^{-1} \sup\{\, s \in \mathbb{R} \mid W_k \text{ generically separates } \lceil s\overline{m} \rceil\text{-jets} \,\} \\ &= t^{-1} j(W_k; \overline{m}). \end{aligned}$$

To prove (3), it is sufficient to show that $j(W''_k;\overline{m}) \geq j(W_k;\overline{m}) + j(W'_k;\overline{m})$ holds for any k. We can show this by the argument similar to the proof of Claim 4.2.9. We leave the details to the reader.

To show (4), we fix a positive number $0 < t < \varepsilon(W_\bullet;\overline{m})$. By Lemma 4.2.8, the equality $j(W_k;\overline{m}) > kt$ holds for any $k \gg 0$. Thus W_k separates $\lceil kt\overline{m}\rceil$-jets for any $k \gg 0$, i.e., the map

$$W_k \to \bigoplus_i L \otimes \mathcal{O}/\mathfrak{m}_{p_i}^{\lceil ktm_i\rceil+1}$$

is surjective for very general $p_1, \ldots, p_r$. This surjection induces the following inequality:

$$\begin{aligned} \dim W_k &\geq \dim \bigoplus_i L \otimes \mathcal{O}/\mathfrak{m}_{p_i}^{\lceil ktm_i\rceil+1} \\ &= \Sigma_i \binom{\lceil ktm_i\rceil + n}{n} = \Sigma_i \frac{t^n m_i^n}{n!} k^n + O(k^{n-1}). \end{aligned}$$

Thus we have $\mathrm{vol}(W_\bullet) \geq t^n \Sigma_{i=1}^r m_i^n$ by $k \to +\infty$. We finish the prove by $t \to \varepsilon(W_\bullet;\overline{m})$. □

Note that Definition 4.2.4 is a natural generalization of Definition 1.2.4 for a nef and big line bundle, i.e.,

Lemma 4.2.11. *Let X be a projective variety, L a nef and big line bundle on X, and $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{R}^r_{>0}$. Then it holds that*

$$\varepsilon(X, L;\overline{m}) = \max\{\, t \geq 0 \,|\, \mu^*L - t\,\Sigma_{i=1}^r m_i E_i \text{ is nef}\,\},$$

where $\mu : \widetilde{X} \to X$ is the blowing up along very general r points in X and E_i are the exceptional divisors.

Proof. The proof is essentially same as that of [La2, Theorem 5.1.17]. Firstly, we show $\varepsilon(X, L;\overline{m}) \leq \max\{t \geq 0 \,|\, \mu^*L - t\,\Sigma_{i=1}^r m_i E_i \text{ is nef}\}$, i.e., $\mu^*L - \varepsilon(X, L;\overline{m})\Sigma_{i=1}^r m_i E_i$ is nef. Fix a curve $C \subset X$ and let $\widetilde{C} \subset \widetilde{X}$ be the strict transform of C. It is enough to show $\widetilde{C}.(\mu^*L - \varepsilon(X, L;\overline{m})\Sigma_{i=1}^r m_i E_i) \geq 0$. For each k, the line bundle $L^{\otimes k}$ separates $\lceil j_k\overline{m}\rceil$-jets at very general points $p_1, \ldots, p_r$, where $j_k := j(kL;\overline{m})$. Hence there exists a nonzero effective divisor $D \in |L^{\otimes k} \bigotimes_i \mathfrak{m}_{p_i}^{\lceil j_k m_i\rceil}|$ such that D does not contain C. Thus we have

$$\begin{aligned} \widetilde{C}.\mu^*L &= C.L \\ &= k^{-1}C.D \\ &\geq k^{-1}\Sigma_i \,\mathrm{mult}_{p_i} C \cdot \mathrm{mult}_{p_i} D \\ &\geq k^{-1}\Sigma_i \lceil j_k m_i\rceil \,\mathrm{mult}_{p_i} C \\ &\geq k^{-1}\Sigma_i j_k m_i \,\mathrm{mult}_{p_i} C \\ &= k^{-1} j_k \Sigma_{i=1}^r m_i \widetilde{C}.E_i. \end{aligned}$$

By $k \to +\infty$, it follows that $\widetilde{C}.\mu^*L \geq \varepsilon(X, L;\overline{m})\Sigma_{i=1}^r m_i \widetilde{C}.E_i$.

We show the opposite inequality. At first, we assume L is ample. Let $p_1, \ldots, p_r$ be very general r points in X. Fix a rational number $0 < t = a/b < \max\{t \geq 0 \,|\, \mu^*L - t\,\Sigma_{i=1}^r m_i E_i \text{ is nef}\}$ with positive integers a, b. Then $b\mu^*L - a\Sigma_{i=1}^r m_i E_i$ is an ample $\mathbb{R}$-line bundle on $\widetilde{X}$. Multiplying a and b by a sufficiently large positive integer, we may assume that $b\mu^*L - \Sigma_{i=1}^r \lceil am_i\rceil E_i$ is an ample line bundle on $\widetilde{X}$. By Fujita's vanishing theorem (see [La2, Theorem 1.4.35] for instance), there is a natural number N such that

$$H^1(\widetilde{X}, l(b\mu^*L - \Sigma_{i=1}^r \lceil am_i\rceil E_i) + P) = 0$$

holds for every $l \geq N$ and every nef line bundle P. For any integer $k \geq Nb$, write $k = lb + b'$ with $0 \leq b' < b$. We can apply the above vanishing for $P = b'\mu^*L$, thus we have

$$H^1(\widetilde{X}, k\mu^*L - l\,\Sigma_{i=1}^r \lceil am_i \rceil E_i) = 0.$$

Since $H^1(\widetilde{X}, k\mu^*L - l\,\Sigma_{i=1}^r \lceil am_i \rceil E_i) = H^1(X, kL \otimes \bigotimes_i \mathfrak{m}_{p_i}^{l\lceil am_i \rceil})$, the line bundle kL separates $(l\lceil am_1 \rceil - 1, \ldots, l\lceil am_r \rceil - 1)$-jets at $p_1, \ldots, p_r$. Hence

$$\frac{j(kL; \overline{m})}{k} \geq \min_i \frac{l\lceil am_i \rceil - 1}{km_i} = \min_i \frac{l\lceil am_i \rceil - 1}{(lb + b')m_i}.$$

By $k \to +\infty$, we have

$$\varepsilon(L; \overline{m}) \geq \lim_k \min_i \frac{l\lceil am_i \rceil - 1}{(lb + b')m_i} = \frac{a}{b} = t.$$

By $t \to \max\{\, t \geq 0 \,|\, \mu^*L - t\,\Sigma_{i=1}^r m_i E_i$ is nef$\,\}$, it holds that $\varepsilon(L; \overline{m}) \geq \max\{\, t \geq 0 \,|\, \mu^*L - t\,\Sigma_{i=1}^r m_i E_i$ is nef$\,\}$.

Next we show the nef and big case. Since L is nef and big, there exists an effective divisor E on X such that $L - sE$ is ample for any $0 < s \ll 1$. Fix a rational number $0 < s \ll 1$ and take a sufficiently divisible integer $l \in \mathbb{N} \setminus 0$ such that $ls \in \mathbb{Z}$. Then $\varepsilon(l(L - sE); \overline{m}) \leq \varepsilon(lL; \overline{m}) = l \cdot \varepsilon(L; \overline{m})$ holds by Lemmas 4.2.6 and 4.2.10 (1). Since $l(L - sE)$ is ample, we have

$$\begin{aligned} \varepsilon(l(L - sE); \overline{m}) &= \max\{\, t \geq 0 \,|\, \mu^*(l(L - sE)) - t\,\Sigma_{i=1}^r m_i E_i \text{ is nef}\,\} \\ &= l \cdot \max\{\, t \geq 0 \,|\, \mu^*(L - sE) - t\,\Sigma_{i=1}^r m_i E_i \text{ is nef}\,\}. \end{aligned}$$

Hence $\varepsilon(L; \overline{m}) \geq \max\{\, t \geq 0 \,|\, \mu^*(L - sE) - t\,\Sigma_{i=1}^r m_i E_i$ is nef$\,\}$ holds. By $s \to 0$, we have $\varepsilon(L; \overline{m}) \geq \max\{\, t \geq 0 \,|\, \mu^*L - t\,\Sigma_{i=1}^r m_i E_i$ is nef$\,\}$. Thus we finish the proof. □

Note that $W_\bullet = \{H^0(X, kL)\}_k$ is birational if and only if L is big for a projective variety X. For a nef but not big line bundle L on X, we define $\varepsilon(X, L; \overline{m}) := \max\{\, t \geq 0 \,|\, \mu^*L - t\,\Sigma_{i=1}^r m_i E_i$ is nef$\,\} = 0$ according to nef and big cases.

For projective varieties, we can write Seshadri constants of graded linear series by using those of nef line bundles as follows:

Lemma 4.2.12. *Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on a projective variety X. For each $k > 0$, set*

$$M_k = \mu_k^* L - F_k,$$

where $\mu_k : X_k \to X$ is a resolution of the base ideal

$$\mathfrak{b}_k := \text{the image of } W_k \otimes L^{-k} \to \mathcal{O}_X$$

and $\mathcal{O}_{X_k}(-F_k) := \mu_k^{-1}\mathfrak{b}_k$. Set $M_k = 0$ if $W_k = 0$. Then it holds that

$$\varepsilon(X, W_\bullet; \overline{m}) = \sup_k \frac{\varepsilon(X_k, M_k; \overline{m})}{k} = \lim_k \frac{\varepsilon(X_k, M_k; \overline{m})}{k}.$$

Proof. Note that M_k is nef for any k. First, we show that $\lim_k \dfrac{\varepsilon(X_k, M_k; \overline{m})}{k}$ exists and the second equality in the above statement holds. To prove this, it suffices to show that

$$\frac{\varepsilon(X_{k_0}, M_{k_0}; \overline{m})}{k_0} \leq \liminf \frac{\varepsilon(X_k, M_k; \overline{m})}{k}$$

holds for any $k_0 > 0$. We may assume that M_{k_0} is big.

Fix a sufficiently large integer N. For each $k \geq N$, we can write $k = k_0 l + N + i$ for some $l \in \mathbb{N}$ and $0 \leq i \leq k_0 - 1$. Since $\varepsilon(X_{k'}, M_{k'}; \overline{m})$ does not depend on the resolution $\mu_{k'}$ by Remark 4.2.6 or Lemma 4.2.11, we may take a common resolution of $\mathfrak{b}_{k_0}, \mathfrak{b}_k$, and $\mathfrak{b}_{N+i}$. Then it is easy to see $M_k \geq l M_{k_0} + M_{N+i}$, in particular $M_k \geq l M_{k_0}$ holds because M_{N+i} is effective. By Remark 4.2.6 and Lemma 4.2.10 (1) or Lemma 4.2.11, it holds that

$$\varepsilon(M_k; \overline{m}) \geq \varepsilon(l M_{k_0}; \overline{m}) = l \cdot \varepsilon(M_{k_0}; \overline{m}).$$

The inequality $\dfrac{\varepsilon(X_{k_0}, M_{k_0}; \overline{m})}{k_0} \leq \liminf \dfrac{\varepsilon(X_k, M_k; \overline{m})}{k}$ follows from this immediately.

Next we show the first equality. Since $\mu_k^* |W_k| \subset |M_k|$, it holds that $\jmath(W_k; \overline{m}) \leq \jmath(M_k; \overline{m}) \leq \varepsilon(M_k; \overline{m})$ for any k. Thus we have

$$\varepsilon(X, W_\bullet; \overline{m}) = \lim \frac{\jmath(W_k; \overline{m})}{k} \leq \lim \frac{\varepsilon(M_k; \overline{m})}{k}.$$

To show the opposite inequality, we use Lemma 4.2.11. Since $W_\bullet$ is birational, the morphism $\varphi_k : X_k \dashrightarrow \mathbb{P}^{\dim |W_k|}$ defined by $\mu_k^* |W_k|$ is birational onto its image for $k \gg 0$. Denote the image of φ_k by Y_k. By Lemma 4.2.11, $\varepsilon(X_k, M_k; \overline{m}) = \varepsilon(Y_k, \mathcal{O}_{Y_k}(1); \overline{m})$ holds because φ_k is birational. Furthermore $W_k^l = H^0(Y_k, \mathcal{O}_{Y_k}(l))$ for $l \gg 0$, where W_k^l is the image of $W_k^{\otimes l} \to W_{kl}$. This implies that $\jmath(\mathcal{O}_{Y_k}(l); \overline{m}) = \jmath(W_k^l; \overline{m}) \leq \jmath(W_{kl}; \overline{m})$ for $l \gg 0$. Thus we have

$$\begin{aligned} \frac{\varepsilon(X_k, M_k; \overline{m})}{k} &= \frac{\varepsilon(Y_k, \mathcal{O}_{Y_k}(1); \overline{m})}{k} \\ &= \frac{1}{k} \lim_l \frac{\jmath(\mathcal{O}_{Y_k}(l); \overline{m})}{l} \\ &\leq \lim \frac{\jmath(W_{kl}; \overline{m})}{kl} = \varepsilon(W_\bullet; \overline{m}). \end{aligned}$$

□

4.3 Monomial graded linear series on $(\mathbb{C}^\times)^n$

Let n be a positive integer. For a subset $S \subset \mathbb{R}^n$, we define

$$V_S := \bigoplus_{u \in S \cap \mathbb{Z}^n} \mathbb{C} x^u \subset \bigoplus_{u \in \mathbb{Z}^n} \mathbb{C} x^u = H^0((\mathbb{C}^\times)^n, \mathcal{O}_{(\mathbb{C}^\times)^n}).$$

For a convex set Δ in $\mathbb{R}^n$, we define a graded linear series $W_{\Delta,\bullet}$ associated to $\mathcal{O}_{(\mathbb{C}^\times)^n}$ by

$$W_{\Delta,k} := V_{k\Delta} \subset H^0((\mathbb{C}^\times)^n, \mathcal{O}_{(\mathbb{C}^\times)^n}) = H^0((\mathbb{C}^\times)^n, \mathcal{O}_{(\mathbb{C}^\times)^n}^{\otimes k}).$$

It is easy to see that $W_{\Delta,\bullet}$ is birational if and only if $\dim \Delta = n$. We investigate the Seshadri constant of $W_{\Delta,\bullet}$ in this section.

Definition 4.3.1. For a subset $S \subset \mathbb{R}^n$ and $\overline{m} \in \mathbb{R}^r_{>0}$, we define $\tilde{s}(S; \overline{m})$ to be

$$\tilde{s}(S; \overline{m}) = \jmath(V_S; \overline{m}) \in \mathbb{R} \cup \{+\infty\}.$$

For a convex set $\Delta \subset \mathbb{R}^n$, we define $s(\Delta; \overline{m})$ in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$ to be

$$s(\Delta; \overline{m}) = \begin{cases} \varepsilon(W_{\Delta,\bullet}; \overline{m}) & \text{if } \dim \Delta = n \\ 0 & \text{otherwise.} \end{cases}$$

When $r = 1$ and $\overline{m} = 1$, we denote them by $\tilde{s}(S)$ and $s(\Delta)$ instead.

Remark 4.3.2. For any Δ and $\overline{m}$, it holds that

$$s(\Delta;\overline{m}) = \sup_k \frac{\tilde{s}(k\Delta;\overline{m})}{k} = \lim_k \frac{\tilde{s}(k\Delta;\overline{m})}{k}.$$

.

When $\dim \Delta = n$, this follows from Lemma 4.2.8.

When $\dim \Delta < n$, $V_{k\Delta}$ does not generically separate 1-jets, which we will show later (see Proposition 4.5.1). Thus it holds that $\tilde{s}(k\Delta;\overline{m}) = \jmath(V_{k\Delta};\overline{m}) \leq 0$. Moreover $V_{k\Delta}$ generically separates $(\underbrace{-1,\ldots,-1}_{r})$-jets, hence $\tilde{s}(k\Delta;\overline{m}) = \jmath(V_{k\Delta};\overline{m}) \geq -\max_i\{1/m_i\}$. Therefore we have $\sup_k \tilde{s}(k\Delta;\overline{m})/k = \lim_k \tilde{s}(k\Delta;\overline{m})/k = 0 = s(\Delta;\overline{m})$.

Remark 4.3.3. The following properties are easily shown from the definitions of $\tilde{s}(S;\overline{m})$ and $s(\Delta;\overline{m})$:

(1) $\tilde{s}(S_1;\overline{m}) \leq \tilde{s}(S_2;\overline{m}), s(\Delta_1;\overline{m}) \leq s(\Delta_2;\overline{m})$ for $S_1 \subset S_2$ and $\Delta_1 \subset \Delta_2$.
(2) $\tilde{s}(S+u;\overline{m}) = \tilde{s}(S;\overline{m})$ for $u \in \mathbb{Z}^n$.
(3) $s(t\Delta;\overline{m}) = t \cdot s(\Delta;\overline{m}), s(\Delta+u;\overline{m}) = s(\Delta;\overline{m})$ for any $t > 0 \in \mathbb{Q}$ and $u \in \mathbb{Q}^n$.

Proof. (1) is clear. (2) is immediately shown by the following diagram:

$$\begin{array}{ccc} V_S & \hookrightarrow & \mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}] \\ \wr\downarrow & \circlearrowleft & \wr\downarrow \times x^u \\ V_{S+u} & \hookrightarrow & \mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]. \end{array}$$

To show (3), choose sufficiently divisible $l \in \mathbb{N} \setminus 0$ such that $lt \in \mathbb{Z}$ and $lu \in \mathbb{Z}^n$. Then

$$\begin{aligned} s(t\Delta;\overline{m}) &= \lim_k \frac{\tilde{s}(kt\Delta;\overline{m})}{k} \\ &= \lim_k \frac{\tilde{s}(klt\Delta;\overline{m})}{kl} \\ &= t \lim_k \frac{\tilde{s}(klt\Delta;\overline{m})}{klt} = t \cdot s(\Delta;\overline{m}) \end{aligned}$$

and by (2),

$$\begin{aligned} s(\Delta+u;\overline{m}) &= \lim_k \frac{\tilde{s}(k(\Delta+u);\overline{m})}{k} \\ &= \lim_k \frac{\tilde{s}(kl(\Delta+u);\overline{m})}{kl} \\ &= \lim_k \frac{\tilde{s}(kl\Delta+klu;\overline{m})}{kl} \\ &= \lim_k \frac{\tilde{s}(kl\Delta;\overline{m})}{kl} = s(\Delta;\overline{m}). \end{aligned}$$

□

Following lemmas are properties of $s(\Delta;\overline{m})$ which are used later.

Lemma 4.3.4. *Let Δ be a convex set in $\mathbb{R}^n$. Then $s(\Delta;\overline{m}) = s(\Delta^\circ;\overline{m})$.*

Proof. If $\dim \Delta < n$, then $s(\Delta; \overline{m}) = s(\Delta^\circ; \overline{m}) = 0$ by definition.

Thus we may assume $\dim \Delta = n$ and it is enough to show $s(\Delta; \overline{m}) \leq s(\Delta^\circ; \overline{m})$. Fix $u \in \Delta^\circ \cap \mathbb{Q}^n$. By the convexity of Δ, there exists the inclusion $\Delta - u \subset t(\Delta^\circ - u)$ for $t > 1$. Thus

$$s(\Delta; \overline{m}) = s(\Delta - u; \overline{m}) \leq s(t(\Delta^\circ - u); \overline{m}) = t \cdot s(\Delta^\circ; \overline{m})$$

holds for $t > 1$ in $\mathbb{Q}$ by Remark 4.3.3 (3). By $t \to 1$, this lemma is proved. □

Now we can show that the property (3) in Remark 4.3.3 holds for any $t \in \mathbb{R}_{>0}$ and $u \in \mathbb{R}^n$:

Lemma 4.3.5. *Let $\Delta \subset \mathbb{R}^n$ be a convex set, $u \in \mathbb{R}^n$, and $t \in \mathbb{R}_{>0}$. Then*

$$s(\Delta + u; \overline{m}) = s(\Delta; \overline{m}), \ s(t\Delta; \overline{m}) = t \cdot s(\Delta; \overline{m}).$$

Proof. These are all 0 if $\dim \Delta < n$, hence we may assume $\dim \Delta = n$.

Fix $u' \in \Delta^\circ \cap \mathbb{Q}^n$. As in the proof of Lemma 4.3.4, $\Delta - u' \subset (1 + t')(\Delta^\circ - u')$ holds for $t' > 0$. Translating it by $u + u'$, we have $\Delta + u \subset (1 + t')(\Delta^\circ - u') + u + u'$. Choose $u'' \in \mathbb{Q}^n$ such that $(u + u') - u'' \in t'(\Delta^\circ - u')$, then $\Delta + u \subset (1 + 2t')(\Delta^\circ - u') + u''$. Thus for $t' > 0$ in $\mathbb{Q}$,

$$s(\Delta + u; \overline{m}) \leq s((1 + 2t')(\Delta^\circ - u') + u''; \overline{m}) = (1 + 2t')s(\Delta; \overline{m})$$

holds by Remark 4.3.3 (3) and Lemma 4.3.4. Hence $s(\Delta + u; \overline{m}) \leq s(\Delta; \overline{m})$ follows by $t' \to 0$. Since the opposite inequality $s(\Delta + u; \overline{m}) \geq s(\Delta; \overline{m})$ also holds similarly, we can show that $s(\Delta + u; \overline{m}) = s(\Delta; \overline{m})$.

For $t_1, t_2 \in \mathbb{Q}$ such that $0 < t_1 \leq t \leq t_2$, there exists inclusions $t_1(\Delta - u') \subset t(\Delta - u') \subset t_2(\Delta - u')$. Thus we have

$$s(t_1(\Delta - u'); \overline{m}) \leq s(t(\Delta - u'); \overline{m}) \leq s(t_2(\Delta - u'); \overline{m}).$$

By Remark 4.3.3 (3) and the first equality of this lemma, which we have already proved, $s(t(\Delta - u'); \overline{m}) = s(t\Delta; \overline{m})$ and $s(t_i(\Delta - u'); \overline{m}) = t_i \cdot s(\Delta; \overline{m})$ hold for $i = 1, 2$. Substituting these inequalities, it holds that

$$t_1 \cdot s(\Delta; \overline{m}) \leq s(t\Delta; \overline{m}) \leq t_2 \cdot s(\Delta; \overline{m}).$$

By $t_1, t_2 \to t$, we have $s(t\Delta; \overline{m}) = t \cdot s(\Delta; \overline{m})$. □

Lemma 4.3.6. *Let $\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_i \subset \cdots$ be an increasing sequence of convex sets in $\mathbb{R}^n$ and set $\Delta = \bigcup_{i=1}^\infty \Delta_i$. Then $s(\Delta; \overline{m}) = \sup_i s(\Delta_i; \overline{m}) = \lim_i s(\Delta_i; \overline{m})$ for any $\overline{m} \in \mathbb{R}^r_{>0}$.*

Proof. This lemma follows from Lemma 4.2.7 immediately. □

Lemma 4.3.7. *Let Δ, Δ' be convex sets in $\mathbb{R}^n$. Then the following hold:*

$$s(\Delta; \overline{m}) + s(\Delta'; \overline{m}) \leq s(\Delta + \Delta'; \overline{m}), s(\Delta) \leq \sqrt[n]{n! \operatorname{vol}(\Delta)/|\overline{m}|_n}.$$

Proof. This lemma follows from $V_{k\Delta} \cdot V_{k\Delta'} \subset V_{k(\Delta+\Delta')}, \operatorname{vol}(W_{\Delta,\bullet}) = n! \operatorname{vol}(\Delta)$, and Lemma 4.2.10 (3), (4). □

4.4 Okounkov bodies and Seshadri constants

In this section, we show Theorem 4.1.1, which states that the Okounkov bodies give lower bounds of Seshadri constants.

4.4.1 Definition of Okounkov bodies

For a subsemigroup Γ in $\mathbb{N} \times \mathbb{N}^n$ and $k \in \mathbb{N}$, set

$$\begin{aligned} \Delta(\Gamma) &= \Sigma(\Gamma) \cap (\{1\} \times \mathbb{R}^n), \\ \Gamma_k &= \Gamma \cap (\{k\} \times \mathbb{N}^n). \end{aligned}$$

Recall that $\Sigma(\Gamma)$ is the closed convex cone in $\mathbb{R} \times \mathbb{R}^n$ spanned by Γ. We consider $\Delta(\Gamma)$ and Γ_k as subsets in $\mathbb{R}^n$ and $\mathbb{N}^n$ respectively in a natural way.

Definition 4.4.1. We say a semigroup Γ in $\mathbb{N} \times \mathbb{N}^n$ is birational if

i) $\Gamma_0 = \{0\} \in \mathbb{N}^n$,

ii) Γ generates $\mathbb{Z} \times \mathbb{Z}^n$ as a group.

These conditions are (2.3) and (2,5) in [LM] respectively. Note that Γ is birational if and only if so is the graded linear series $W(\Gamma)_\bullet = \{W(\Gamma)_k\}_k := \{V_{\Gamma_k}\}_k$ on $(\mathbb{C}^\times)^n$ associated to $\mathcal{O}_{(\mathbb{C}^\times)^n}$.

Remark 4.4.2. If Γ is finitely generated, i) and ii) in Definition 4.4.1 are nothing but the condition we assume to define toric variety $(X(\Gamma), L(\Gamma))$ in Definition 1.3.1. For such Γ, the above $\Delta(\Gamma)$ is the moment polytope of $(X(\Gamma), L(\Gamma))$ defined in Definition 1.3.1.

Fix a monomial order $>$ on $\mathbb{N}^n$, i.e., $>$ is a total order on $\mathbb{N}^n$ such that (i) for every $u \in \mathbb{N}^n \setminus 0$, $u > 0$ holds, and (ii) if $v > u$ and $w \in \mathbb{N}^n$, then $w + v > w + u$. In this chapter, $v > u$ does not contain the case $v = u$. Let X be a variety of dimension n and $z = (z_1, \ldots, z_n)$ a local coordinate system at a smooth point $p \in X$. (In [LM], they use "admissible flags" instead of local coordinate systems, but essentially there is no difference if $>$ is the lexicographic order. Local coordinate systems are used in [BC] for instance.) Then we define a valuation

$$\nu = \nu_{z,>} : \mathcal{O}_{X,p} \setminus \{0\} \to \mathbb{N}^n$$

as follows: for $f \in \mathcal{O}_{X,p} \setminus \{0\}$, we expand it as a formal power series

$$f = \Sigma_{u \in \mathbb{N}^n} c_u z^u$$

and define

$$\nu(f) = \min\{u \mid c_u \neq 0\},$$

where the minimum is taken with respect to the monomial order $>$.

Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on X. Then one can define the Okounkov body of $W_\bullet$ as follows:

Fix an isomorphism $L_p \cong \mathcal{O}_{X,p}$. Then this isomorphism naturally induces $L_p^{\otimes k} \cong \mathcal{O}_{X,p}$ for any $k \geq 0$ and we have the following map

$$W_k \setminus \{0\} \hookrightarrow L_p^{\otimes k} \setminus \{0\} \cong \mathcal{O}_{X,p} \setminus \{0\} \xrightarrow{\nu} \mathbb{N}^n.$$

We write $\nu(W_k) \subset \mathbb{N}^n$ for the image of $W_k \setminus \{0\}$ by this map. Note that $\nu(W_k)$ does not depend on the choice of the isomorphism $L_p \cong \mathcal{O}_{X,p}$ because ν maps any unit elements in $\mathcal{O}_{X,p}$ to $0 \in \mathbb{N}^n$. Then

$$\Gamma_{W_\bullet} = \Gamma_{W_\bullet, z, >} := \bigcup_{k \in \mathbb{N}} \{k\} \times \nu(W_k) \subset \mathbb{N} \times \mathbb{N}^n$$

is a semigroup by construction. We define the Okounkov body of $W_\bullet$ with respect to z and $>$ by

$$\Delta(W_\bullet) = \Delta_{z,>}(W_\bullet) := \Delta(\Gamma_{W_\bullet}).$$

Then $\Delta(W_\bullet)$ is an n-dimensional closed convex set in $\mathbb{R}^n$ because $\Gamma_{W_\bullet}$ is birational as we will prove in Lemma 4.4.3 later. In general, $\Delta(W_\bullet)$ is not bounded, in particular is not a convex body (= an n-dimensional compact convex set in $\mathbb{R}^n$), even if $W_\bullet$ is of finite dimensional type. But we call it Okounkov body according to custom.

In [LM, Lemma 2.12], they assume that $W_\bullet$ satisfies "condition (C)", which seems to be a little stronger condition than being birational(= Condition (B) in [LM]), to show that $\Gamma_{W_\bullet,z,>}$ generates $\mathbb{Z} \times \mathbb{Z}^n$ as a group for any z. But we can show that it is enough to assume that $W_\bullet$ is birational:

Lemma 4.4.3. *Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on a variety X. Then $\Gamma_{W_\bullet,z,>}$ is birational for any local coordinate system z at any smooth point $p \in X$ and any monomial order $>$.*

Proof. The condition i) in Definition 4.4.1 is clearly satisfied. Thus it is enough to show that $\Gamma_{W_\bullet}$ generates $\mathbb{Z} \times \mathbb{Z}^n$ as a group. Suppose that z is a local coordinate system and $>$ is a monomial order. Fix $k \gg 0$. Then the function field $K(X)$ is generated by $\{f/g \in K(X) \mid f, g \in W_k, g \neq 0\}$ over $\mathbb{C}$ because $W_\bullet$ is birational. Hence any $F \in K(X) \setminus \{0\}$ is written as $F = G/H$, where G, H are written as some polynomials over $\mathbb{C}$ of some elements in $\{f/g \in K(X) \mid f, g \in W_k, g \neq 0\}$. Therefore we can write as $F = G'/H'$ for some $G', H' \in W_{kl}$ for some $l \in \mathbb{N} \setminus 0$. Thus $\nu(F) = \nu(G') - \nu(H') \in \nu(W_{kl}) - \nu(W_{kl}) \subset \mathbb{Z}^n$ (note that ν is naturally extended to $K(X) \setminus \{0\}$). Since the valuation $\nu : K(X) \setminus \{0\} \to \mathbb{Z}^n$ is surjective, the group $\mathbb{Z}^n$ is generated by $\{\nu(W_{kl}) - \nu(W_{kl})\}_{l \in \mathbb{N}}$. Thus the subgroup $\{0\} \times \mathbb{Z}^n$ in $\mathbb{Z} \times \mathbb{Z}^n$ is generated by $\{0\} \times \{\nu(W_{kl}) - \nu(W_{kl})\}_l \subset \Gamma_{W_\bullet} - \Gamma_{W_\bullet}$.

On the other hand, $s_k \in W_k \setminus \{0\}$ and $s_{k+1} \in W_{k+1} \setminus \{0\}$ induce the element $(1, \nu(s_{k+1}) - \nu(s_k)) \in \Gamma_{W_\bullet} - \Gamma_{W_\bullet} \subset \mathbb{Z} \times \mathbb{Z}^n$. Since the group $\mathbb{Z} \times \mathbb{Z}^n$ is generated by $\{0\} \times \mathbb{Z}^n$ and $(1, \nu(s_{k+1}) - \nu(s_k))$, the semigroup $\Gamma_{W_\bullet}$ generates $\mathbb{Z} \times \mathbb{Z}^n$ as a group. □

4.4.2 Proof of Theorem 4.1.1

First, we state the idea of the proof. Let X be a projective variety and L a line bundle on X. Suppose that $V \subset H^0(X, L)$ is a subspace and set $W_\bullet = \{W_k\}$ by $W_k = V^k \subset H^0(X, kL)$. Anderson [An] proved that $(\operatorname{Proj} \bigoplus_k V^k, \mathcal{O}(1))$ degenerates to the toric variety $(\operatorname{Proj} \mathbb{C}[\Gamma_{W_\bullet}], \mathcal{O}(1))$ if $\Gamma_{W_\bullet}$ is finitely generated as a semigroup. From this result and the lower semicontinuities of Seshadri constants, Theorem 4.1.1 for $W_\bullet = \{V^k\}_k$ is easily shown if $\Gamma_{W_\bullet}$ is finitely generated. But it seems that $\Gamma_{W_\bullet}$ is seldom finitely generated, even if $W_\bullet = \{H^0(X, kL)\}_k$ for a very ample L on X (cf. [LM, Lemma 1.7]).

Thus we modify the above idea. Instead of the degeneration of the variety or the section ring $\bigoplus_k W_k$, we degenerate W_k for each k severally. This is one of the main reason why we define Seshadri constants by using jet separations in this chapter, instead of the usual definition by blowing ups as Lemmas 4.2.11 and 4.2.12.

Lemma 4.4.4. *Let $\Gamma \subset \mathbb{N} \times \mathbb{N}^n$ be a birational semigroup. Then*

$$s(\Delta(\Gamma); \overline{m}) = \sup \frac{\tilde{s}(\Gamma_k; \overline{m})}{k} = \lim \frac{\tilde{s}(\Gamma_k; \overline{m})}{k}$$

holds for any $\overline{m} \in \mathbb{R}^r_{>0}$. In other words, $\varepsilon(W_{\Delta(\Gamma),\bullet}; \overline{m}) = \varepsilon(W(\Gamma)_\bullet; \overline{m})$ holds.

Proof. By definition, it holds that

$$\varepsilon(W_{\Delta(\Gamma),\bullet};\overline{m}) = s(\Delta(\Gamma);\overline{m}) = \sup \frac{\tilde{s}(k\Delta(\Gamma);\overline{m})}{k} = \lim \frac{\tilde{s}(k\Delta(\Gamma);\overline{m})}{k}$$

and

$$\varepsilon(W(\Gamma)_{\bullet};\overline{m}) = \sup \frac{\tilde{s}(\Gamma_k;\overline{m})}{k} = \lim \frac{\tilde{s}(\Gamma_k;\overline{m})}{k}.$$

Since Γ_k is contained in $k\Delta(\Gamma)$ for any k, $\varepsilon(W_{\Delta(\Gamma),\bullet};\overline{m}) \geq \varepsilon(W(\Gamma)_{\bullet};\overline{m})$ is clear. Thus it suffices to show

$$\lim_k \frac{\tilde{s}(\Gamma_k;\overline{m})}{k} \geq \lim \frac{\tilde{s}(k\Delta(\Gamma);\overline{m})}{k}.$$

When Γ is finitely generated, there exists $\tilde{u} = (l,u) \in \Gamma \subset \mathbb{N}\times\mathbb{N}^n$ such that $(\Sigma(\Gamma)+\tilde{u}) \cap (\mathbb{N}\times\mathbb{N}^n) \subset \Gamma$ by [Kh, §3, Proposition 3] (see also [LM, Subsection 2.1]). This gives the inclusion $(k\Delta(\Gamma)\cap\mathbb{N}^n)+u \subset \Gamma_{k+l}$, hence $\tilde{s}(k\Delta(\Gamma);\overline{m}) = \tilde{s}((k\Delta(\Gamma)\cap\mathbb{N}^n)+u;\overline{m}) \leq \tilde{s}(\Gamma_{k+l};\overline{m})$ holds. Thus we have

$$\begin{aligned}\lim_k \frac{\tilde{s}(\Gamma_k;\overline{m})}{k} &= \lim_k \frac{\tilde{s}(\Gamma_{k+l};\overline{m})}{k} \\ &\geq \lim_k \frac{\tilde{s}(k\Delta(\Gamma);\overline{m})}{k}.\end{aligned}$$

In general case, we take an increasing sequence $\Gamma^1 \subset \Gamma^2 \subset \cdots \subset \Gamma$ such that each Γ^i is a finitely generated birational subsemigroup of Γ and $\bigcup_{i\geq 1}\Gamma_i = \Gamma$. Then it is easy to show $\bigcup_i \Delta(\Gamma^i)^\circ = \Delta(\Gamma)^\circ$. By Lemmas 4.3.4 and 4.3.6, we have

$$s(\Delta(\Gamma);\overline{m}) = s(\Delta(\Gamma)^\circ;\overline{m}) = \lim s(\Delta(\Gamma^i)^\circ;\overline{m}) = \lim s(\Delta(\Gamma^i);\overline{m}).$$

Since each Γ^i is finitely generated, we can apply this lemma to Γ^i. Then we have

$$s(\Delta(\Gamma^i);\overline{m}) = \lim_k \frac{\tilde{s}(\Gamma^i_k;\overline{m})}{k} \leq \lim_k \frac{\tilde{s}(\Gamma_k;\overline{m})}{k}.$$

Thus $s(\Delta(\Gamma);\overline{m}) = \lim s(\Delta(\Gamma^i);\overline{m}) \leq \lim_k \dfrac{\tilde{s}(\Gamma_k;\overline{m})}{k}$ holds. □

Broadly speaking, the geometrical meaning of Lemma 4.4.4 is that Seshadri constants of ample line bundles (on non-normal toric varieties) at very general points do not change by normalizations. In fact, $(\operatorname{Proj}\bigoplus_k W_{\Delta(\Gamma),k}, \mathcal{O}(1)) = (\operatorname{Proj}\mathbb{C}[\Sigma(\Gamma)\cap(\mathbb{N}\times\mathbb{Z}^n)], \mathcal{O}(1))$ is nothing but the normalization of the toric variety $(\operatorname{Proj}\bigoplus_k W(\Gamma)_k, \mathcal{O}(1)) = (\operatorname{Proj}\mathbb{C}[\Gamma], \mathcal{O}(1))$ when Γ is finitely generated.

To prove Proposition 4.4.7, which is the key of the proof of Theorem 4.1.1, we need one more lemma.

Lemma 4.4.5. *Let $>$ be a monomial order on $\mathbb{N}^n$ and S a finite set in $\mathbb{N}^n$. Then there exists a vector $\alpha \in (\mathbb{N}\setminus 0)^n$ satisfying the following:*

If $v > u$ for $u \in S$ and $v \in \mathbb{N}^n$, then $\alpha\cdot v > \alpha\cdot u$ holds, where $\alpha\cdot u, \alpha\cdot v$ are the usual inner products.

Proof. For each $u \in S$, set $S_u = \{v \in \mathbb{N}^n \,|\, v > u\}$. Let I_u be the ideal in the polynomial ring $\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1,\ldots,x_n]$ generated by $\{x^v \,|\, v \in S_u\}$. By Hilbert's basis theorem, I_u is generated by $x^{v_{u1}},\ldots,x^{v_{uk_u}}$ for some $k_u \in \mathbb{N}$ and $v_{u1},\ldots,v_{uk_u} \in S_u$. Therefore any $v \in S_u$ is contained in $v_{uj}+\mathbb{N}^n$ for some j.

We use the following fact (cf. [Ro, Theorem 2.5]):

Fact 4.4.6. *Let $>$ be a monomial order on $\mathbb{N}^n$. Then there exists an integer $s \geq 1$ with $1 \leq s \leq n$, s vectors $u_1, ..., u_s \in \mathbb{R}^n$ which satisfy the following:*

For $u, v \in \mathbb{N}^n$, $v > u$ if and only if $\pi(v) >_{lex} \pi(u)$, where

$$\pi : \mathbb{N}^n \to \mathbb{R}^s; u \mapsto (u_1 \cdot u, \ldots, u_s \cdot u)$$

and $>_{lex}$ is the lexicographic order on $\mathbb{R}^s$.

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Z}^n$ and consider the above $u_1, \ldots, u_s$ and π. By the definition of the lexicographic order, for any $\gamma >_{lex} \delta$ in $\mathbb{R}^s$, $\beta \cdot \gamma > \beta \cdot \delta$ holds for $\beta = (\beta_1, \ldots, \beta_s) \in \mathbb{R}^s_{>0}$ if $\beta_1 \gg \ldots \gg \beta_n$ (of course, such β depends on γ and δ). As $\gamma >_{lex} \delta$, we consider $\pi(e_i) >_{lex} \pi(0) = 0$ and $\pi(v_{uj}) >_{lex} \pi(u)$ for $1 \leq i \leq n$, $u \in S$ and $1 \leq j \leq k_u$ and choose β as above. Since S is a finite set, we can take a common $\beta \in \mathbb{R}^s_{>0}$. Since $\beta \cdot \pi(u) = (\beta_1 u_1 + \cdots + \beta_s u_s) \cdot u$,

$$\alpha' \cdot e_i > 0, \ \alpha' \cdot v_{uj} > \alpha' \cdot u \cdots (*)$$

holds for $1 \leq i \leq n, u \in S$ and $1 \leq j \leq k_u$ if we take $\alpha' \in \mathbb{Q}^n$ sufficiently near $(\beta_1 u_1 + \cdots \beta_s u_s)$. Multiplying α' by a sufficiently divisible positive integer N, and set $\alpha := N\alpha' \in \mathbb{Z}^n$. By $(*)$, it follows that $\alpha \in (\mathbb{N} \setminus 0)^n$ and $\alpha \cdot v_{uj} > \alpha \cdot u$ for $u \in S$ and j.

We show this α satisfies the condition in the statement of this lemma. Fix $u \in S$ and $v \in \mathbb{N}^n$ such that $v > u$, i.e., v is contained in S_u. Then $v \in v_{uj} + \mathbb{N}^n$ for some $1 \leq j \leq k_u$. Thus $\alpha \cdot v = \alpha \cdot v_{uj} + \alpha \cdot (v - v_{uj}) \geq \alpha \cdot v_{uj} > \alpha \cdot u$. □

Now we show the key proposition. Roughly speaking, this states that $W \subset H^0(X, L)$ generically separates jets no less than $V_{\nu(W)}$:

Proposition 4.4.7. *Let L be a line bundle on an n-dimensional variety X and set $\nu = \nu_{z,>}$ be the valuation map defined by a local coordinate system $z = (z_1, \ldots, z_n)$ at a smooth point $p \in X$ and a monomial order $>$ on $\mathbb{N}^n$.*

Then $j(W; \overline{m}) \geq \tilde{s}(\nu(W); \overline{m})$ holds for any subspace W of $H^0(X, L)$ and any $\overline{m} \in \mathbb{R}^r_{>0}$.

Proof. By considering an increasing sequence of finite dimensional subspaces in W, we may assume that W is finite dimensional.

Let $\pi : U \to \mathbb{C}^n$ be the étale morphism defined by $z_1, \ldots, z_n$ in an open neighborhood $U \subset X$ of p. By the morphism π, we can identify $\mathcal{O}^{an}_{X,p}$ with $\mathcal{O}^{an}_{\mathbb{C}^n,0} = \mathbb{C}\{x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n$ are the coordinates on $\mathbb{C}^n$ such that $\pi^* x_i = z_i$. Then we can regard W as a subspace in $\mathbb{C}\{x_1, \ldots, x_n\}$ by $W \hookrightarrow L_p \cong \mathcal{O}_{X,p} \hookrightarrow \mathbb{C}\{x_1, \ldots, x_n\}$. Note that ν is extended to $\mathcal{O}^{an}_{X,p} \setminus \{0\} = \mathbb{C}\{x_1, \ldots, x_n\} \setminus \{0\} \to \mathbb{N}^n$ naturally.

Choose and fix $f_u \in \nu^{-1}(u) \cap W$ for each $u \in \nu(W)$. Then it holds that $V = \bigoplus_{u \in \nu(W)} \mathbb{C} f_u$ because $\#\nu(W) = \dim W$ (cf. [LM] or [BC]). Since $\nu(W)$ is a finite set, there exists $\alpha \in (\mathbb{N} \setminus 0)^n$ satisfying the following by Lemma 4.4.5: if $v > u$ for $u \in \nu(W)$ and $v \in \mathbb{N}^n$, it holds that $\alpha \cdot v > \alpha \cdot u$.

The vector α induces the action $\circ$ of $\mathbb{C}^\times$ on $\mathbb{C}\{x_1, \ldots, x_n\}$ by $t \circ x^u = t^{\alpha \cdot u} x^u$ for $t \in \mathbb{C}^\times$ and $u \in \mathbb{N}^n$. For $f_u = \Sigma_v c_{uv} z^v = \Sigma_v c_{uv} x^v$ (note we identify z_i and x_i), the regular function

$$\frac{t \circ f_u}{t^{\alpha \cdot u}} = t^{-\alpha \cdot u} \Sigma_v c_{uv} t^{\alpha \cdot v} x^v = \Sigma_v c_{uv} t^{\alpha \cdot v - \alpha \cdot u} x^v$$

on a neighborhood of $\mathbb{C}^\times \times \{0\}$ is naturally extended to a regular function on a neighborhood $\mathcal{U}$ (in $\mathbb{C} \times \mathbb{C}^n$) of $\mathbb{C} \times \{0\}$. Note that $\alpha \cdot v - \alpha \cdot u \geq 0$ if $c_{uv} \neq 0$. We denote the regular function by F_u. Set $\mathcal{W} = \bigoplus_{u \in \nu(W)} \mathbb{C} F_u$.

We prove this proposition only for $r = 1$. When $r > 1$, the proof is similar. Hence we leave the details to the reader.

By the assumption that $r = 1$, we may assume $\overline{m} = 1$. Choose a very general section σ of the first projection $\mathcal{U} \to \mathbb{C}$. Note that $\sigma(0)$ is contained in $\{0\} \times (\mathbb{C}^\times)^n \cong (\mathbb{C}^\times)^n$. (When $r > 1$, we choose very general sections $\sigma_1, \ldots, \sigma_r$ of $\mathcal{U} \to \mathbb{C}$.) Let $\mathcal{I}$ be the ideal sheaf corresponding to $\sigma(\mathbb{C})$ on $\mathcal{U} \subset \mathbb{C} \times \mathbb{C}^n$, and consider the following map between flat sheaves over $\mathbb{C}$:

$$\phi : \mathcal{W} \otimes_{\mathbb{C}} \mathbb{C}\{t\} \to \mathcal{O}_{\mathcal{U}}^{an} \to \mathcal{O}_{\mathcal{U}}^{an}/\mathcal{I}^{j+1}$$

for $j \geq 0$. We denote $W_t := \mathcal{W} \otimes \mathbb{C}\{t\}|_{\{t\} \times \mathbb{C}^n}$ and

$$\phi_t := \phi|_{\{t\} \times \mathbb{C}^n} : W_t \to \mathcal{O}_{\mathcal{U}}^{an}/\mathcal{I}^{j+1}|_{\{t\} \times \mathbb{C}^n} = \mathcal{O}_{\mathbb{C}^n}/\mathfrak{m}_{\sigma(t)}^{j+1}$$

for $t \in \mathbb{C}$. By the flatness, ϕ_t is surjective for very general t if so is ϕ_0. Thus if W_0 separates m-jets at $\sigma(0)$ for $m \in \mathbb{Z}$, then W_t also separates m-jets at $\sigma(t)$ for very general t. Since we choose a very general section, it follows that

$$j(W_t) \geq j(W_0)$$

for t in a neighborhood of 0. Note $j(W_t)$ can be defined similarly in this analytic setting. Since there is a natural identification of W_t and W for $t \in \mathbb{C}^\times$ by the action $\circ$, we have $j(W) = j(W_t)$. Note that $j(W)$ does not depend on whether we consider W as a subspace in $H^0(X, L)$ or in $\mathbb{C}\{x_1, \ldots, x_n\}$ because $\pi : U \to \mathbb{C}^n$ is étale. On the other hand, $W_0 = V_{\nu(W)} \subset \mathbb{C}[x_1, \ldots, x_n]$ since $F_u = c_{uu}x^u + t \cdot \text{higher term}, c_{uu} \neq 0$. Thus $j(W_0) = \tilde{s}(\nu(W))$ holds by the definition of $\tilde{s}(\cdot)$. From these inequalities, we have $j(W) \geq \tilde{s}(\nu(W))$. □

Now we can show the main theorem easily:

Theorem 4.4.8 (=Theorem 4.1.1). *Let $W_\bullet$ be a birational graded linear series associated to a line bundle L on an n-dimensional variety X. Fix a local coordinate system $z = (z_1, \ldots, z_n)$ on X at a smooth point and a monomial order $>$ on $\mathbb{N}^n$.*

Then $\varepsilon(W_\bullet; \overline{m}) \geq s(\Delta_{z,>}(W_\bullet); \overline{m})$ holds for any $r \in \mathbb{N} \setminus 0$ and $\overline{m} \in \mathbb{R}^r_{>0}$.

Proof. Let $\Gamma := \Gamma_{W_\bullet, z, >} \subset \mathbb{N} \times \mathbb{N}^n$ be the semigroup defined by $W_\bullet, z$ and $>$ in the definition of Okounkov bodies. Then $\Gamma_k = \nu(W_k) \subset \mathbb{N}^n$ and $\Delta(\Gamma) = \Delta_{z,>}(W_\bullet)$ by definition. By Proposition 4.4.7, we have

$$\varepsilon(W_\bullet; \overline{m}) = \lim \frac{j(W_k; \overline{m})}{k} \geq \lim \frac{\tilde{s}(\nu(W_k); \overline{m})}{k} = \lim \frac{\tilde{s}(\Gamma_k; \overline{m})}{k}.$$

Since Γ is birational by Lemma 4.4.3, it holds that

$$\lim \frac{\tilde{s}(\Gamma_k; \overline{m})}{k} = s(\Delta(\Gamma); \overline{m}) = s(\Delta_{z,>}(W_\bullet); \overline{m})$$

from Lemma 4.4.4. Thus $\varepsilon(W_\bullet; \overline{m}) \geq s(\Delta_{z,>}(W_\bullet); \overline{m})$ holds. □

Remark 4.4.9. Since $s(\Delta(W_\bullet); \overline{m}) = \varepsilon(W_{\Delta(W_\bullet),\bullet}; \overline{m})$, the above theorem says that the Seshadri constant of $W_\bullet$ is greater than or equal to that of $W_{\Delta(W_\bullet),\bullet}$. For a convex set $\Delta \subset \mathbb{R}^r_{\geq 0}$, $W_{\Delta,\bullet}$ is considered as a graded linear series on $\mathbb{C}^n$. Note that $\Delta_{z,>}(W_{\Delta,\bullet})$ is nothing but Δ itself for the standard local coordinate system $z = (x_1, \ldots, x_n)$ at $0 \in \mathbb{C}^n = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$ (cf. [LM, Proposition 6.1]). Hence Seshadri constants are minimal in monomial (or toric) cases for a fixed Okounkov body.

See [LM, Remark 5.5] for another relation between Okounkov bodies and Seshadri constants, though the relation is not written explicitly there. Note that the relation also holds for a birational graded linear series.

4.5 Computations and estimations of $s(\Delta;\overline{m}), \tilde{s}(S;\overline{m})$

4.5.1 In the case $r = 1$

It is very hard to compute $\tilde{s}(S;\overline{m})$ in general, but when $r = 1$, we can compute it by considering the rank of matrices. In this subsection, we mainly consider S or Δ which is bounded and contained in $\mathbb{R}^n_{\geq 0}$. For general S and Δ, we can reduce them to such cases by Remark 4.3.3 and Lemma 4.3.6.

For $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$, we define a natural number $\begin{bmatrix} u \\ \lambda \end{bmatrix} \in \mathbb{N}$ to be

$$\begin{bmatrix} u \\ \lambda \end{bmatrix} = \prod_{k=1}^{n} \binom{u_k}{\lambda_k},$$

where $\binom{u_k}{\lambda_k} = \dfrac{u_k(u_k-1)\cdots(u_k-\lambda_k+1)}{\lambda_k!}$ is the binomial coefficient.

Set $1_n = (1, \ldots, 1)$ in $(\mathbb{C}^\times)^n$ and let $\mathfrak{n}$ be an $\mathfrak{m}_{1_n}$-primary ideal on $(\mathbb{C}^\times)^n$. Assume that $\mathfrak{n}$ is generated by monomials of $x_1 - 1, \ldots, x_n - 1$. In particular, there exists a finite subset $\Phi_\mathfrak{n} \subset \mathbb{N}^n$ such that $\mathfrak{n}$ is the restriction of

$$\bigoplus_{\lambda \in \mathbb{N}^n \setminus \Phi_\mathfrak{n}} \mathbb{C}(x - 1_n)^\lambda \subset \mathcal{O}_{\mathbb{C}^n}$$

on $(\mathbb{C}^\times)^n$, where $(x - 1_n)^\lambda = (x_1 - 1)^{\lambda_1} \cdots (x_n - 1)^{\lambda_n}$.

For a bounded subset S in $\mathbb{R}^n_{\geq 0}$, we set a matrix $A_{S,\mathfrak{n}}$ by

$$A_{S,\mathfrak{n}} = \left(\begin{bmatrix} u \\ \lambda \end{bmatrix} \right)_{(\lambda,u) \in \Phi_\mathfrak{n} \times (S \cap \mathbb{N}^n)}.$$

When $\mathfrak{n} = \mathfrak{m}_{1_n}^{m+1}$ for $m \in \mathbb{N}$, we denote $A_{S,\mathfrak{m}_{1_n}^{m+1}}$ by $A_{S,m}$ for short.

The following proposition is a straightforward generalization of results in [Du, Proposition 13] and [BBC+, 3.10], and the proof is same. We prove it here for the convenience of the reader:

Proposition 4.5.1. *Let S be a bounded set in $\mathbb{R}^n_{\geq 0}$ and $\mathfrak{n}$ an $\mathfrak{m}_{1_n}$-primary ideal generated by monomials of $x_1 - 1, \ldots, x_n - 1$. Then the following are equivalent:*

i) the natural map $\varphi_{S,\mathfrak{n}} : V_S \to \mathcal{O}_{(\mathbb{C}^\times)^r}/\mathfrak{n} = \bigoplus_{\lambda \in \Phi_\mathfrak{n}} \mathbb{C}(x - 1_n)^\lambda : f \mapsto f + \mathfrak{n}$ is not surjective,

ii) $\operatorname{rank} A_{S,\mathfrak{n}} < \#\Phi_\mathfrak{n}$.

Furthermore, if $\#(S \cap \mathbb{N}^n) = \#\Phi_\mathfrak{n}$ then these are equivalent to

iii) there exists a nonzero element $f \in \bigoplus_{\lambda \in \Phi_\mathfrak{n}} \mathbb{R}u^\lambda \subset \mathbb{R}[u_1, \ldots, u_n]$ such that $S \cap \mathbb{N}^n$ is contained in the hypersurface in $\mathbb{R}^n$ defined by f.

Proof. Since $A_{S,\mathfrak{n}}$ is the matrix of $\varphi_{S,\mathfrak{n}}$ with respect to bases $\{x^u\}_{u \in S \cap \mathbb{N}^n}$ and $\{(x - 1_n)^\lambda\}_{\lambda \in \Phi_\mathfrak{n}}$, the equivalence of i) and ii) is clear.

We assume $\#(S \cap \mathbb{Z}^n) = \#\Phi_\mathfrak{n}$ and show the equivalence of ii) and iii). By definition, it holds that

$$\begin{bmatrix} u \\ \lambda \end{bmatrix} = \frac{1}{\lambda_1! \cdots \lambda_n!} \prod_{k=1}^{n} \prod_{l=0}^{\lambda_k - 1} (u_k - l).$$

Hence $\operatorname{rank} A_{S,\mathfrak{n}} = \operatorname{rank}(\prod_{k=1}^{n}\prod_{l=0}^{\lambda_k-1}(u_k - l))_{(\lambda,u)}$. For $\lambda \in \Phi_{\mathfrak{n}}$, any $\lambda' \in \mathbb{N}^n$ satisfying $\lambda - \lambda' \in \mathbb{N}^n$ is also contained in $\Phi_{\mathfrak{n}}$ because $\mathfrak{n}$ is an ideal. From this, the matrix $(\prod_{k=1}^{n}\prod_{l=0}^{\lambda_k-1}(u_k - l))_{(\lambda,u)}$ changes to the matrix $(u^{\lambda})_{(\lambda,u)} = (u_1{}^{\lambda_1}\cdots u_n{}^{\lambda_n})_{(\lambda,u)}$ by a suitable sequence of row operations.

Thus $A_{S,\mathfrak{n}}$ is not regular if and only if rows of $(u^{\lambda})_{(\lambda,u)}$ are linearly dependent, and this is equivalent to iii). □

Remark 4.5.2. If $\mathfrak{n} = \mathfrak{m}_{1_n}^{m+1}$ for $m \in \mathbb{N}$, then $\Phi_{\mathfrak{m}_{1_n}^{m+1}} = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 + \ldots + \lambda_n \leq m\}$. Thus in this case, iii) means that $S \cap \mathbb{N}^n$ is contained in a hypersurface of degree m in $\mathbb{R}^n$.

Corollary 4.5.3. *For a bounded set S in $\mathbb{R}^n_{\geq 0}$ and a bounded convex set Δ in $\mathbb{R}^n_{\geq 0}$, it holds that*

$$\begin{aligned}
\tilde{s}(S) &= \max\left\{m \in \mathbb{N} \,\middle|\, \operatorname{rank} A_{S,m} = \binom{m+n}{n}\right\}, \\
s(\Delta) &= \sup_{k \in \mathbb{N}\setminus 0} \frac{\max\{m \in \mathbb{N} \mid \operatorname{rank} A_{k\Delta,m} = \binom{m+n}{n}\}}{k} \\
&= \lim_{k \in \mathbb{N}\setminus 0} \frac{\max\{m \in \mathbb{N} \mid \operatorname{rank} A_{k\Delta,m} = \binom{m+n}{n}\}}{k}.
\end{aligned}$$

Proof. This corollary follows from Lemma 4.2.8, Proposition 4.5.1 and $\#\Phi_{\mathfrak{m}_{1_n}^{m+1}} = \binom{m+n}{n}$. □

Remark 4.5.4. By Corollary 4.5.3, we can compute $\tilde{s}(S)$ by finite calculations. In fact $A_{S,m}$ is a $\binom{m+n}{n} \times \#(S \cap \mathbb{N}^n)$ matrix, hence $\operatorname{rank} A_{S,m} = \binom{m+n}{n}$ only if $\binom{m+n}{n} \leq \#(S \cap \mathbb{N}^n)$. Note that $\#(S \cap \mathbb{N}^n)$ is finite by the boundedness of S.

By Corollary 4.5.3, we can describe the Seshadri constant of a polarized toric variety at a very general point as the supremum or the limit of computable numbers as follows:

Let P be an integral polytope in $\mathbb{R}^n$ of dimension n. Then $\mathbb{C}[\Gamma_P] = \bigoplus_{k \in \mathbb{N}} V_{kP}$ holds as graded $\mathbb{C}$-algebra. Thus the polarized toric variety (X_P, L_P) is also written as

$$(X_P, L_P) = (\operatorname{Proj} \bigoplus_{k \in \mathbb{N}} V_{kP}, \mathcal{O}(1)).$$

The following corollary is essentially stated in [BBC+, 3.10], at least in case of $n = 2$:

Corollary 4.5.5. *Let P be an integral polytope of dimension n contained in $\mathbb{R}^n_{\geq 0}$. Then it holds that*

$$\begin{aligned}
j(L_P) &= \max\left\{m \in \mathbb{N} \,\middle|\, \operatorname{rank} A_{P,m} = \binom{m+n}{n}\right\}, \\
\varepsilon(X_P, L_P; 1) &= \sup_{k \in \mathbb{N}\setminus 0} \frac{\max\{m \in \mathbb{N} \mid \operatorname{rank} A_{kP,m} = \binom{m+n}{n}\}}{k} \\
&= \lim_{k \in \mathbb{N}\setminus 0} \frac{\max\{m \in \mathbb{N} \mid \operatorname{rank} A_{kP,m} = \binom{m+n}{n}\}}{k}.
\end{aligned}$$

Proof. Note that there exists a natural embedding $(\mathbb{C}^\times)^n \hookrightarrow X_P$ and an identification of $W_{P,k} = V_{kP}$ and $H^0(X_P, kL_P)$. Thus it clearly holds that $j(L_P) = j(V_P) = \tilde{s}(P)$ and $\varepsilon(X_P, L_P; 1) = \varepsilon((\mathbb{C}^\times)^n, W_{P,\bullet}; 1) = s(P)$ (or more generally $j(L_P; \overline{m}) = \tilde{s}(P; \overline{m})$ and $\varepsilon(X_P, L_P; \overline{m}) = \varepsilon((\mathbb{C}^\times)^n, W_{P,\bullet}; \overline{m}) = s(P; \overline{m})$ hold for any $\overline{m} \in \mathbb{R}^r_{>0}$). Hence this corollary follows from Corollary 4.5.3. □

Remark 4.5.6. For a rational polytope $P \subset \mathbb{R}^n$ of dimension n, $s(P) = \varepsilon(X_P, L_P; 1_P)$ holds by definition. Thus $s_1(P) \leq s(P) \leq s_2(P)$ follows by Proposition 3.2.7. We defined $s_1(P)$ and $s_2(P)$ only for rational polytopes P, but it is not hard to define $s_1(\Delta)$ and $s_2(\Delta)$ for any convex set $\Delta \subset \mathbb{R}^n$. By using Lemma 4.3.6, we can easily show that $s_1(\Delta) \leq s(\Delta) \leq s_2(\Delta)$ holds for any Δ.

4.5.2 In the case $r > 1$

In the above subsection, we investigate $s(\Delta)$ or $\tilde{s}(S)$. Now we consider $s(\Delta; \overline{m})$ for general $\overline{m}$. At least there are three methods to estimate $s(\Delta; \overline{m})$ or $\tilde{s}(S; \overline{m})$ from below (though they are not enough to obtain good estimations in general). These methods may be known to specialists, at least for $S \subset \mathbb{Z}^2$ or $\Delta \subset \mathbb{R}^2$. But we state here in our settings for the convenience of the reader.

The first one uses degenerations of ideals:

Proposition 4.5.7. *Let $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. Assume that there exists a flat family $\{\mathfrak{n}_t\}_{t \in T}$ of ideals on $(\mathbb{C}^\times)^n$ over a smooth curve T such that*

i) $\mathfrak{n}_t = \mathfrak{m}_{p_{1,t}}^{m_1+1} \otimes \cdots \otimes \mathfrak{m}_{p_{r,t}}^{m_r+1}$ for general $t \in T$, where $p_{1,t}, \ldots, p_{r,t}$ are distinct r points in $(\mathbb{C}^\times)^n$,

ii) $\mathfrak{n}_0$ is an $\mathfrak{m}_{1_n}$-primary ideal generated by monomials of $x - 1_n$ for $0 \in T$.

Then V_S generically separates $\overline{m}$-jets for a bounded subset $S \subset \mathbb{R}^n_{\geq 0}$ if $\operatorname{rank} A_{S,\mathfrak{n}_0} = \dim \mathcal{O}/\mathfrak{n}_0$.

Proof. Let $\mathcal{I}$ be the ideal on $(\mathbb{C}^\times)^n \times T$ corresponding to the family $\{\mathfrak{n}_t\}_{t \in T}$ and assume $\operatorname{rank} A_{S,\mathfrak{n}_0} = \dim \mathcal{O}/\mathfrak{n}_0$.

Consider the natural map

$$\phi : V_S \otimes_{\mathbb{C}} \mathcal{O}_T \to \mathcal{O}_{(\mathbb{C}^\times)^n \times T}/\mathcal{I}.$$

By the assumption $\operatorname{rank} A_{S,\mathfrak{n}_0} = \dim \mathcal{O}/\mathfrak{n}_0$ and Proposition 4.5.1,

$$\phi_0 : V_S = V_S \otimes \mathcal{O}_T|_{(\mathbb{C}^\times)^n \times \{0\}} \to \mathcal{O}_{(\mathbb{C}^\times)^n \times T}/\mathcal{I}|_{(\mathbb{C}^\times)^n \times \{0\}} = \mathcal{O}_{(\mathbb{C}^\times)^n}/\mathfrak{n}_0$$

is surjective. Thus

$$\phi_t : V_S \otimes \mathcal{O}_T|_{(\mathbb{C}^\times)^n \times \{t\}} \to \mathcal{O}_{(\mathbb{C}^\times)^n \times T}/\mathcal{I}|_{(\mathbb{C}^\times)^n \times \{t\}} = \mathcal{O}_{(\mathbb{C}^\times)^n}/\mathfrak{n}_t$$

is also surjective for general $t \in T$ by the flatness. This means V_S separates $\overline{m}$-jets at $p_{1,t}, \ldots, p_{r,t}$ by i). In particular V_S generically separates $\overline{m}$-jets. □

Example 4.5.8. Set $T = \mathbb{C}$, $\overline{m} = (3, 1) \in \mathbb{N}^2$, and $\mathfrak{n}_t = (x-1, y-1)^4 (x-1-t, y-1)^2$ for $t \in \mathbb{C}^\times = T \setminus 0$. Then the family of ideals $\{\mathfrak{n}_t\}_{t \in \mathbb{C}^\times}$ extends over $\mathbb{C}$ by $\mathfrak{n}_0 = ((x-1)^6, (x-1)^4(y-1), (x-1)^2(y-1)^2, (x-1)(y-1)^3, (y-1)^4)$. Since the family $\{\mathfrak{n}_t\}_{t \in \mathbb{C}}$ is flat near 0, we can apply Proposition 4.5.7. In Figure 1 bellow, five $\circ$ correspond to the monomial generators of $\mathfrak{n}_0$, and thirteen $\bullet$ are all points in $\Phi_{\mathfrak{n}_0}$. Set S be thirteen $\bullet$ in Figure 2. Then $\operatorname{rank} A_{S,\mathfrak{n}_0} = 13 = \dim \mathcal{O}/\mathfrak{n}_0$. Therefore V_S generically separates $(3,1)$-jets, i.e., $\tilde{s}(S; (3,1)) \geq 1$ holds.

Note that this can be shown by the method in [Du].

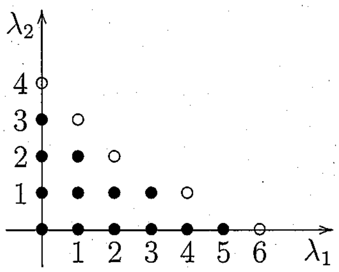


Figure 1

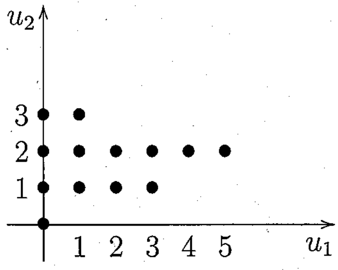


Figure 2

The second one uses finite morphisms induced by changes of lattices (cf. [Gar, Lemma 2.1]):

Proposition 4.5.9. *Let $\iota : \mathbb{Z}^n \to \mathbb{Z}^n$ be an injection between abelian groups of same rank, whose degree is $d \in \mathbb{N} \setminus 0$. For a convex set $\Delta \subset \mathbb{R}^n$, set $\Delta' = \iota_{\mathbb{R}}^{-1}(\Delta)$, where $\iota_{\mathbb{R}} = \iota \otimes id_{\mathbb{R}} : \mathbb{R}^n = \mathbb{Z}^n \otimes \mathbb{R} \to \mathbb{Z}^n \otimes \mathbb{R} = \mathbb{R}^n$. Then,*

$$s(\Delta; \underbrace{\overline{m}, \ldots, \overline{m}}_{d}) \geq s(\Delta'; \overline{m})$$

holds for any $\overline{m} = (m_1, \ldots, m_r) \in \mathbb{R}^r_{>0}$.

Proof. By Lemmas 4.3.4 and 4.3.6, we may assume that Δ is a rational polytope. Furthermore we may assume Δ, Δ' are integral polytopes by Lemma 4.3.5.

The injective morphism ι induces the quotient morphism

$$\pi : X_\Delta \to X_{\Delta'}$$

such that $\pi^* L_{\Delta'} = L_\Delta$. Note that π is a finite morphism of degree d.

We choose very general points $p_1, \ldots, p_r$ in $X_{\Delta'}$, then $\bigcup_i \pi^{-1}(p_i)$ are smooth dr-points in X_Δ. Consider the following diagram:

$$\begin{array}{ccc} \widetilde{X}_\Delta & \xrightarrow{\tilde{\pi}} & \widetilde{X}_{\Delta'} \\ \downarrow{\scriptstyle \mu} & \circlearrowleft & \downarrow{\scriptstyle \mu'} \\ X_\Delta & \xrightarrow{\pi} & X_{\Delta'}, \end{array}$$

where μ is the blowing up along $\bigcup_i \pi^{-1}(p_i)$ and μ' is the blowing up along $\{p_1, \ldots, p_r\}$. Let E_{ij} be the exceptional divisor over p_{ij}, where $\pi^{-1}(p_i) = \{p_{i1}, \ldots, p_{id}\}$, and E'_i the exceptional divisor over p_i.

By Lemma 4.2.11, $\mu'^* L_{\Delta'} - s(\Delta'; \overline{m}) \Sigma_i m_i E'_i$ is nef. Thus $\mu^* L_\Delta - s(\Delta'; \overline{m}) \Sigma_{i,j} m_i E_{ij} = \tilde{\pi}^*(\mu'^* L_{\Delta'} - s(\Delta'; \overline{m}) \Sigma_i m_i E'_i)$ is also nef. Since ampleness is an open condition in a flat family, $\mu^* L_\Delta - s(\Delta'; \overline{m}) \Sigma_{i,j} m_i E_{ij}$ is also nef if μ is the blowing up of X_Δ along very general dr-points. Thus $s(\Delta; \underbrace{\overline{m}, \ldots, \overline{m}}_{d}) \geq s(\Delta'; \overline{m})$ holds by Lemma 4.2.11. □

Example 4.5.10. Let $\Delta \subset \mathbb{R}^3$ be the convex hull of $(0,0,0), (1,1,0), (1,0,1)$, and $(0,1,1)$. Set $\iota : \mathbb{Z}^3 \to \mathbb{Z}^3$ by $\iota(e_1) = (1,1,0), \iota(e_2) = (1,0,1)$, and $\iota(e_3) = (0,1,1)$ for the standard basis e_1, e_2, e_3 of $\mathbb{Z}^3$. Then the degree of ι is 2, and $\Delta' = \iota_{\mathbb{R}}^{-1}(\Delta)$ is an integral polytope corresponding to $(\mathbb{P}^3, \mathcal{O}(1))$. Thus $s(\Delta; 1, 1) \geq s(\Delta'; 1) = 1$ holds by Proposition 4.5.9. In this case, $s(\Delta; 1, 1) = 1$ holds by Lemma 4.3.7.

The last one uses degenerations of varieties induced from polytope decomposition (cf. Example 3.4.9). The following is a convex set version of Example 3.4.9, and a simple generalization of results in [Bi], [Ec].

Proposition 4.5.11. *Let $P \subset \mathbb{R}^n$ be an integral polytope of dimension n, and $\mathcal{P}$ an integral polytope decomposition of P. We assume that $\mathcal{P}$ has a lifting function. Let Δ be a convex set in $\mathbb{R}^n$ and take $P_1, \dots, P_{r'} \in \mathcal{P}^{[n]}$ such that $\Delta_i := P_i \cap \Delta$ has the non-empty interior for each $1 \leq i \leq r'$. (There may be another $P' \in \mathcal{P}^{[n]}$ such that $(P' \cap \Delta)^\circ \neq \emptyset$.) Set $s_i \in \mathbb{N} \setminus 0$ be a positive integer and take $\overline{m}_i \in \mathbb{R}^{s_i}_{>0}$ for each $1 \leq i \leq r'$.*

Then $s(\Delta; \overline{m}_1, \dots, \overline{m}_{r'}) \geq \min_{1 \leq i \leq r'} s(\Delta_i; \overline{m}_i)$ holds.

Proof. Fix very general points $p_{i1}, \dots, p_{is_i}$ in $(\mathbb{C}^\times)^n$ for $i = 1, \dots, r'$. By the existence of a lifting function, there is a toric degeneration $f : \mathcal{X} \to \mathbb{A}^1$ as in Example 3.4.9. We consider each p_{ij} as a point in the central fiber X_0 by the inclusion $(\mathbb{C}^\times)^n \subset X_{P_i} \subset X_0$. For each $1 \leq i \leq r', 1 \leq j \leq s_i$, choose a very general section σ_{ij} of f satisfying $\sigma_{ij}(0) = p_{ij}$. Set $\overline{m}_i = (m_{i1}, \dots, m_{is_i})$.

We consider the natural map

$$\varphi : V_{\bigcup_i k\Delta_i^\circ} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^1} \hookrightarrow V_{kP} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^1} = H^0(\mathcal{X}, \mathcal{L}^{\otimes k}) \to \bigoplus_{i,j} \mathcal{L}^{\otimes k} \otimes \mathcal{O}_{\mathcal{X}} / \mathcal{I}_{ij}^{l_{ij}+1}$$

for integers $k, l_{ij} \in \mathbb{N}$, where $\mathcal{I}_{ij} \subset \mathcal{O}_{\mathcal{X}}$ is the ideal corresponding to $\sigma_{ij}(\mathbb{A}^1)$. By similar arguments in the proofs of Proposition 4.4.7 or Proposition 4.5.7, we can show that $\varphi|_{X_t} : V_{\bigcup_i k\Delta_i^\circ} \to \bigoplus_{i,j} L_P^{\otimes k} \otimes \mathcal{O}_{X_P} / \mathfrak{m}_{\sigma_{ij}(t)}^{l_{ij}+1} = \bigoplus_{i,j} \mathcal{O}_{(\mathbb{C}^\times)^n} / \mathfrak{m}_{\sigma_{ij}(t)}^{l_{ij}+1}$ is surjective for general $t \in \mathbb{A}^1$ if so is $\varphi|_{X_0}$. Note that $V_{\bigcup_i k\Delta_i^\circ} = \bigoplus_i V_{k\Delta_i^\circ}$ and $\varphi|_{X_0} = \oplus_i \psi_i$, where

$$\psi_i : V_{k\Delta_i^\circ} \to \bigoplus_j L_{P_i}^{\otimes k} \otimes \mathcal{O}_{X_{P_i}} / \mathfrak{m}_{p_{ij}}^{l_{ij}+1} = \bigoplus_j \mathcal{O}_{(\mathbb{C}^\times)^n} / \mathfrak{m}_{p_{ij}}^{l_{ij}+1}.$$

Thus $V_{\bigcup_i k\Delta_i^\circ}$ generically separates $(l_{ij})_{i,j}$-jets if $V_{k\Delta_i^\circ}$ generically separates $(l_{ij})_j$-jets for all i. Hence

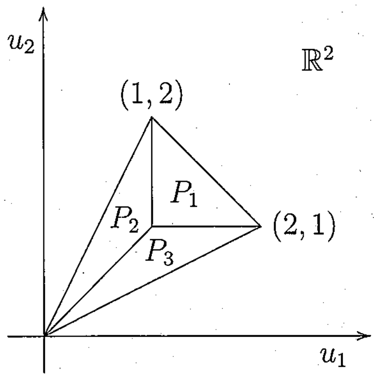
$$\begin{aligned} \tilde{s}(k\Delta; \overline{m}_1, \dots, \overline{m}_{r'}) &= j(V_{k\Delta}; \overline{m}_1, \dots, \overline{m}_{r'}) \\ &\geq j(V_{\bigcup_i k\Delta_i^\circ}; \overline{m}_1, \dots, \overline{m}_{r'}) \\ &\geq \min_i j(V_{k\Delta_i^\circ}; \overline{m}_i) = \min_i \tilde{s}(k\Delta_i^\circ; \overline{m}_i) \end{aligned}$$

holds for any k. Thus we have

$$\begin{aligned} s(\Delta; \overline{m}_1, \dots, \overline{m}_{r'}) &= \lim_k \frac{\tilde{s}(k\Delta; \overline{m}_1, \dots, \overline{m}_{r'})}{k} \\ &\geq \lim_k \frac{\min_i \tilde{s}(k\Delta_i^\circ; \overline{m}_i)}{k} \\ &= \min_i \lim_k \frac{\tilde{s}(k\Delta_i^\circ; \overline{m}_i)}{k} \\ &= \min_i s(\Delta_i^\circ; \overline{m}_i) \\ &= \min_i s(\Delta_i; \overline{m}_i). \end{aligned}$$

□

Example 4.5.12. Let $\Delta \subset \mathbb{R}^2$ be the convex hull of $(0,0),(2,1)$ and $(1,2)$. Consider the following decomposition.



It is easy to see that there exists a lifting function, hence we can apply Proposition 4.5.11 to $P = \Delta$ and this decomposition. Then we have $s(\Delta; 1,1,1) \geq \min_{1\leq i\leq 3} s(P_i; 1) = 1$. In this case $s(\Delta; 1,1,1) = 1$ holds by Lemma 4.3.7.

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