# Construction of invariant group orderings from topological point of view

# (位相幾何の視点からの群の不変順序の構成)

Tetsuya Ito

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# Contents

# Chapter 1

# Introduction

The main object of the present thesis is the *invariant ordering* of groups, a total ordering which is compatible with the group structure. The theory of orderable groups begun around 1950s, and had been studied mainly from the group theoretical point of view.

In 1990's Patrick Dehornoy found a left-ordering of the braid group $B_n$ in his study of left-distributive operations. This left-ordering is today called the *Dehornoy ordering.* Soon after the discovery of the Dehornoy ordering, in [15], Fenn-Greene-Rourke-Rolfsen-Wiest established a geometric description of the Dehornoy ordering. Thurston also pointed out the left-orderability of the braid group rather easily follows from the work of Nielsen [43], by using the hyperbolic geometry. These observations reveal that a topological or geometric method will be useful in studying group orderings.

The discovery and the development of the theory of braid group orderings gathered attention to researchers who are working in topology and geometry, and shed a new light on the theory of group orderings, a rather classical branch of the group theory: The study of group ordering in *topological* and *geometric* prospective. Today this "topological orderable group theory" is an active research area, and is developing rapidly.

Roughly speaking, there are four types of problem in (topological) orderable group theory:

1. *Orderability Problem:*
   Is a group $G$ (mainly taken from topology or geometry, such as, the fundamental group of low-dimensional manifolds or lattice of Lie groups) left- (bi-) orderable ?

2. *Description/Construction Problem:*
   Find a detailed description or an explicit construction left- (bi-) orderings of an orderable group. More generally, construct an explicit example of left- (bi-) ordering having required additional properties.

3. *Moduli Problem:*

Study the topological space of left-orderings $\mathrm{LO}(G)$ of a left-orderable group $G$ (See Section 1.3 below), and the action of $G$ on $\mathrm{LO}(G)$. What are the topological type of $\mathrm{LO}(G)$ and $\mathrm{LO}(G)/G$ ?

4. *Application Problem:*
   Find a relationship between group orderings and other objects in topology and geometry, and use group orderings to solve problems in topology and geometry. (See authro's paper [20], [21] for results in this direction.)

In the present thesis we will mainly treat the problem (1) and (2) above. We will study constructions of group orderings by using various methods derived from (algebraic) topology. Our result is also related to the problem (3), and potentially will be used to attack the problem (4).

The main part of present thesis is divided into four chapters. Each chapter is almost logically independent, and can be read independently. In the rest of the introduction, we briefly review basic results and notions in left- or bi- ordering of groups which will be used throughout this paper, and give an overview of our results.

## 1.1 Left- and bi- ordering of groups

We begin with setting up basic notations related to group orderings.

A total ordering $<_G$ of a group $G$ is a *left-ordering* if the ordering relation $<_G$ is preserved by the left action of $G$ itself: that is, $a <_G b$ implies $ga <_G gb$ for all $a, b, g \in G$. A *right-ordering* of $G$ is defined in a similar way. $<_G$ is a *bi-ordering* if $<_G$ is both right- and left- ordering.

A group $G$ is called *left-orderable (LO)* if $G$ admits at least one left ordering. Similarly, $G$ is *bi-orderable (BO)* if $G$ has at least one bi-ordering. A pair $(G, <_G)$ consisting of the group $G$ and its left- or bi- ordering $<_G$ is called a *left-* or *bi- ordered group.*

The *positive cone* of an ordering $<_G$ is a subset

$$P(<_G) = \{g \in G \mid g >_G 1\}.$$

It is easy to see the positive cone $P = P(<_G)$ of a left-ordering $<_G$ satisfies the following two properties:

**LO1** $P \cdot P \subset P$.

**LO2** $G = P \coprod \{1\} \coprod P^{-1}$.

Moreover, if $<_G$ is a bi-ordering, then $P(<_G)$ satisfies the additional property:

**BO** $g^{-1}Pg = P$ for all $g \in G$.

Conversely, as the next lemma shows the subset of $G$ satisfying these properties determines a left- or bi- ordering.

**Lemma 1.1.1.** *A subset $P$ of $G$ is a positive cone of a left-ordering of $<_G$ if and only if $P$ has the property* **[LO1]** *and* **[LO2]**. *Similarly, $P$ is a positive cone of a bi-ordering if and only if $P$ satisfies* **[LO1]**, **[LO2]** *and* **[BO]**.

*Proof.* Let us define the relation $<_P$ by $g <_P g'$ if and only if $g^{-1}g' \in P$. Then $<_P$ is a left- (resp. bi-) ordering if $P$ has the property **[LO1]** and **[LO2]** (resp. **[LO1]**, **[LO2]** and **[BO]**). □

It is classically known that the left-orderability of groups is closely related to the 1-dimensional dynamics, as the next Theorem shows.

**Theorem 1.1.1** (Folklore: See [11] for example.). *A countable group $G$ is left-orderable if and only if $G$ faithfully acts on the real line $\mathbb{R}$ as orientation preserving homeomorphisms.*

*Proof.* Assume that $G$ faithfully acts on the real line $\mathbb{R}$ as orientation preserving homeomorphisms, hence $G$ is a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$. Fix $X = \{x_i\}_{i=1,2,\dots}$ be a countable sequence of points of $\mathbb{R}$ which is dense in $\mathbb{R}$. We define an ordering $<_X$ of $G$ by $a <_X b$ if there exists $j$ such that $a(x_i) = b(x_i)$ for all $i < j$ and $a(x_j) <_{\mathbb{R}} b(x_j)$ holds. Here $<_{\mathbb{R}}$ is a standard ordering of $\mathbb{R}$ defined by the orientation. Clearly $<_X$ is a left-ordering of $G$.

Conversely, assume that $G$ has a left-ordering $<_G$. Take an enumeration $\{g_i\}_{i\geq 0}$ of $G$, and define the map $t : G \to \mathbb{R}$ as follows .

First we define $t(g_0) = 0$. For $i \geq 1$, we define $t(g_i)$ by

$$t(g_i) = \begin{cases} \max\{t(g_0), \dots, t(g_{i-1})\} + 1 & g_i >_P \max\{g_0, \dots, g_{i-1}\} \\ \min\{t(g_0), \dots, t(g_{i-1})\} - 1 & g_i <_P \min\{g_0, \dots, g_{i-1}\} \\ (t(g_m) + t(g_M))/2 & g_m <_P g_i <_P g_M \text{ and} \\ & \quad (g_m, g_M) \cap \{g_1, \dots, g_{i-1}\} = \phi \end{cases}$$

Here $(g_m, g_M)$ is a subset $\{g \in G \mid g_m <_P g <_P g_M\}$. We define an action of $G$ on the subset $\{t(g_i)\}$ of $\mathbb{R}$ by $g \cdot t(g_i) = t(gg_i)$. By extending this action to the whole of $\mathbb{R}$, we obtain a faithful action of $G$. □

The correspondence between left-orderings and faithful actions on the real line is by no means one-to-one: as the above proof shows, a left ordering derived from dynamics depends on a choice of the sequence $X$, and a dynamics derived from left-ordering depends on a choice of an enumeration $G = \{g_i\}$. So the theory of left-orderings is not a part of one-dimensional dynamical systems, though dynamics is useful in studying orderings.

In the construction of a left-ordering from dynamics, assume that the stabilizer of a finite initial subsequence $I = \{x_1, \dots, x_k\}$ of $X$ is trivial. Then to define the ordering, we do not need to consider the rest of the sequence $\{x_{k+1}, x_{k+2}, \dots\}$. In such cases, we denote the ordering $<_X$ by $<_{\{x_1,\dots,x_k\}}$ and call it the *ordering defined by* $\{x_1, \dots, x_k\}$.

## 1.2 Basic property of orderable groups

In this section we review some basic properties of left- or bi-orderable groups. First of all, we observe that the existence of left-ordering give a restriction of possible relations in the groups. Here we present one of the typical usage of the existence of left-orderings.

**Lemma 1.2.1.** *If $G$ is left-orderable then $G$ is torsion-free.*

*Proof.* Assume that there exists $g \in G - \{1\}$ such that $g^N = 1$ $(N > 1)$. Take a left ordering $<_G$ of $G$. If necessary, by using $g^{-1}$ instead, we may assume $1 <_G g$. By the left-invariance of $<_G$, we get

$$1 <_G g <_G g^2 <_G \cdots <_G g^N = 1$$

which implies $g = 1$. This is a contradiction. □

Next we review the quotient construction of orderings. A subset $A$ of an ordered group $(G, <_G)$ is *convex* if the inequality $a <_G g <_G a'$ for $a, a' \in A$, $g \in G$ implies $g \in A$. If we would like to emphasize the ordering $<_G$, we will say that $A$ is $<_G$*-convex*. For a convex normal subgroup $H$ of $(G, <_G)$, the quotient group $G/H$ has a natural ordering $<_{G/H}$ induced by the ordering $<_G$ defined by $xH <_{G/H} yH$ if $x <_G y$ where $x, y \in G$ are arbitrary chosen coset representatives. This ordering $<_{G/H}$ is a left- (resp. bi-)ordering if $<_G$ is a left (resp, bi-)ordering. We call the ordered group $(G/H, <_{G/H})$ *the ordered quotient group.*

An element $g \in G$ is called the $<_G$*-minimal positive element* if $g$ is the $<_G$-minimal element of the positive cone $P(<_G)$. That is, the inequality $1 <_G g' \leq_G g$ implies $g = g'$. An ordering $<_G$ of $G$ is called *discrete* if there is the $<_G$-minimal positive element. Otherwise, $<_G$ is called *dense.*

Among all left-orderings there is a special class called a *Conradian ordering.* A Conradian ordering shares many properties of bi-ordering, and can be regarded as an intermediate of left-orderings and bi-orderings.

Here we simply recall the definition. For details on Conradian orderings, see [28]. A left-ordering $<_G$ is called a *Conradian ordering* if $fg^k >_G g$ holds for all $<_G$-positive $f, g \in G$ and $k \geq 2$. It is known that in the definition of Conradian orderings it is sufficient to consider the case $k = 2$. That is, $<_G$ is Conradian if and only of $fg^2 >_G g$ holds for all $<_G$-positive elements $f, g \in G$ (See [34]). Recall that a group $G$ is called *locally indicable* if every subgroup $H$ of $G$ admits a surjective homomorphism onto $\mathbb{Z}$. It was classically known that locally indicable groups are left-orderable, but converse is not true. However, for groups admitting a Conradian ordering (called Conradian-orderable) the converse is true:

**Theorem 1.2.1.** *A group $G$ is locally indicable if and only if $G$ is Conradian orderable (that is, $G$ admits a Conradian ordering).*

For a left-ordering $<_G$, the $<_G$*-Conradian soul* is the maximal (with respect to inclusions) $<_G$-convex subgroup of $G$ such that the restriction of $<_G$

is Conradian. The Conradian soul plays an important role in Navas' study of left-orderable groups [34], and is an interesting invariant of left-ordered groups.

## 1.3 Topology of the space of orderings

For a left-orderable group $G$ let $\mathrm{LO}(G)$ be the set of all left-orderings of $G$. In [44], Sikora defined the natural topology on $\mathrm{LO}(G)$.

Recall that there is an one-to-one correspondence between the set of subset of $G - \{1\}$ having the properties **[LO1]**, **[LO2]** and the set of all left-orderings $\mathrm{LO}(G)$. So $\mathrm{LO}(G)$ is regarded as a subset of the powerset $2^{G-\{1\}}$. Now let us consider the powerset topology on $2^{G-\{1\}}$, and equip the topology on $\mathrm{LO}(G)$ as the relative topology as the subspace of $2^{G-\{1\}}$. By this topology, we regard $\mathrm{LO}(G)$ as a topological space rather as a set, and call it the *space of left-orderings.*

This topology has an alternative description. For $g \in G$, let $U_g$ be a subset of $\mathrm{LO}(G)$ defined by

$$U_g = \{<_G \in \mathrm{LO}(G) \mid 1 <_G g\}.$$

The topology of $\mathrm{LO}(G)$ coincides with the topology defined so that $\{U_g\}_{g\in G}$ is an open sub-basis.

The group $G$, or more generally, the group $\mathrm{Aut}(G)$ acts on $\mathrm{LO}(G)$ as homeomorphisms from right as follow: For $\phi \in \mathrm{Aut}(G)$ and $<_G \in \mathrm{LO}(G)$, we define a left-ordering $\phi(<_G)$ by $g\phi(<_G)g'$ if and only if $\phi(g) <_G \phi(g')$.

It is known that $\mathrm{LO}(G)$ has the following properties:

1. $\mathrm{LO}(G)$ is compact and totally disconnected. Moreover, if $G$ is countable, $G$ is metrizable. ([44], [34])

2. $\mathrm{LO}(G)$ is either finite or uncountable [31].

These results suggest that as a topological space, $\mathrm{LO}(G)$ is rather similar to the Cantor set. The main difference is that $\mathrm{LO}(G)$ might be non-perfect: that is, $\mathrm{LO}(G)$ might contain isolated points.

An *isolated ordering* is a left ordering which is an isolated point in $\mathrm{LO}(G)$. Isolated orderings exist: the most trivial example is a standard ordering $<$ of $\mathbb{Z}$. Since $\mathbb{Z}$ admits only two left-orderings ($<$ and its reverse), $<$ is an isolated ordering. The following proposition provides a useful criterion to show certain left-orderings are isolated.

**Propotition 1.3.1** (Navas [34]). *If the positive cone of a left-ordering $<_G$ is a finitely generated as a semigroup, then $<_G$ is isolated.*

However, in a combinatorial group theory point of view, the ordering having finitely generated positive cone seems to be quite strange. If a group $G$ is finitely generated, then every element $g \in G$ is written as a product of a certain finite generating set $\{g_1, \ldots, g_n\}$. However, in general to express $g$ as a product of generating set, we often need to use both *positive* and *negative* generators. An existence of left-ordering whose positive cone is finitely generated implies that every element of $g$ is written as a product of either a positive generating set or a negative generating set: This is a quite restrictive condition, and we have few examples of such left-orderings. Finding an example of isolated ordering is a interesting problem.

## 1.4 Examples of group orderings

In this section we present important examples of left- or bi- orderings. These examples motivate our study of orderable groups, and provides a lot of important insights and stimulating phenomenons.

### 1.4.1 The Dehornoy ordering

Let $B_n$ be the $n$-strand braid group, defined by the presentation

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, \begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & |i-j| = 1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| > 1 \end{array} \right\rangle.$$

**Definition 1.4.1.** For $1 \leq i \leq n-1$, An $n$-braid $\beta$ is called *$i$-positive* (resp. *$i$-negative*) if $\beta$ admits a word expression which contains at least one letter $\sigma_i$ (resp. $\sigma_i^{-1}$), but contains no $\sigma_1^{\pm 1}, \ldots, \sigma_{i-1}^{\pm 1}, \sigma_i^{-1}$. (resp. $\sigma_1^{\pm 1}, \ldots, \sigma_{i-1}^{\pm 1}, \sigma_i$). $\beta$ is *$\sigma$-positive* (resp. $\sigma$-negative) if $\beta$ is $i$-positive (resp. $i$-negative) for some $i \geq 1$.

Using these notions, the Dehornoy ordering is defined as follows.

**Definition 1.4.2** (The Dehornoy ordering). The Dehornoy ordering $<_D$ is a left ordering of $B_n$ which is defined as follows: For $\alpha, \beta \in B_n$, we define $\alpha <_D \beta$ if $\alpha^{-1}\beta$ is $\sigma$-positive.

The remarkable theorem of Dehornoy says that $<_D$ defines a left-ordering.

**Theorem 1.4.1** (Dehornoy [9]). *$<_D$ is a left-ordering of the braid groups.*

The statement of this theorem is highly non-trivial: it says that every non-trivial braid is either $\sigma$-positive or $\sigma$-negative. There are many words which are neither $\sigma$-positive nor $\sigma$-negative (for example, $\sigma_1\sigma_2\sigma_1^{-1}$). Dehornoy's theorem says, we can modify such words into $\sigma$-positive or $\sigma$-negative word by using the relations of the braid groups.

Today there are many conceptually different proofs of Dehornoy's theorem. See [11]. Each approach gives a new insight in the braid groups, or more generally, (left)-orderable groups. Moreover, the Dehornoy ordering has various interesting properties and serves as a useful source to find or create various interesting phenomenon in group orderings. One remarkable feature of the Dehornoy ordering is that it can produce an isolated ordering, as next example shows.

### 1.4.2 Dubrovin-Dubrovina ordering

The Dehornoy ordering can be modified so that it produces another intersting left-ordering, called the *Dubrovin-Dubrovina ordering.*

**Definition 1.4.3.** The *Dubrovin-Dubrovina ordering* $<_{DD}$ is a left-ordering of $B_n$ defined as follows: For $\alpha, \beta \in B_n$, we define $\alpha <_{DD} \beta$ if the braid $\alpha^{-1}\beta$ is $(2i-1)$-positive or $(2i)$-negative for some $i$.

By using Dehornoy's theorem, in [12] Dubrovin-Dubrovina showed the following.

**Propotition 1.4.1** (Dubrovina-Dubrovin [12]). *The Dubrovin-Dubrovina ordering defines a left-ordering of $B_n$. Moreover, the positive cone of $<_{DD}$ is a finitely generated semi-group, generated by $\{a_1, \dots, a_{n-1}\}$ where $a_i$ is defined by*

$$a_i = (\sigma_i \sigma_{i+1} \cdots \sigma_{n-1})^{(-1)^i}.$$

By Proposition 1.3.1, this implies that the Dubrovina-Dubrovin ordering is an isolated ordering. On the other hand, the Dehornoy ordering itself is not isolated (See [34],[36]). Thus the space of the left-ordering of $B_n$, $\mathrm{LO}(B_n)$ has an interesting property: it contains both Cantor set and isolated points. The complete description of $\mathrm{LO}(B_n)$ is still unknown.

### 1.4.3 Magnus ordering of free groups

The simplest way to define a total ordering is to use the lexicographical ordering. Let $\{v_i : G \to \mathbb{R}\}_{i \in \mathcal{I}}$ be a family of maps, not necessarily a homomorphism, indexed by a well-ordered set $\mathcal{I}$. We say $\{v_i\}_{i\in\mathcal{I}}$ is a *lexicographical expression* of a total ordering $<$ of $G$ if $a < b$ is equivalent to the sequence of reals $\{v_i(b)\}_{i\in\mathcal{I}}$ is bigger than the sequence of reals $\{v_i(a)\}_{i\in\mathcal{I}}$ with respect to the lexicographical ordering of $\mathbb{R}^{\mathcal{I}}$.

Let $F_n$ be the free group of rank $n$ generated by $\{x_1, \dots, x_n\}$. $F_n$ has a standard bi-ordering called the *Magnus ordering* defined as follows.

Let $\mathbb{R}\langle\langle X_1, \dots, X_n\rangle\rangle$ be the algebra of non-commutative formal power series of the variables $\{X_1, \dots, X_n\}$. The *Magnus expansion* is an injective homomorphism $\mu : F_n \to \mathbb{R}\langle\langle X_1, \dots, X_n\rangle\rangle$ defined by $\mu(x_i) = 1 + X_i$ (See [33]).

Let $\mathcal{I}$ be the set of monomials of $\mathbb{R}\langle\langle X_1, \dots, X_n\rangle\rangle$. We define the DegLex-ordering $<_{\mathsf{DegLex}}$ on $\mathcal{I}$ as follows. For monomials $X_I = X_{i_1} \cdots X_{i_k}$ and $X_J = X_{j_1} \cdots X_{j_l}$ we define $X_I <_{\mathsf{DegLex}} X_J$ if $k < l$, or $k = l$ and the sequence of integers $(j_1, \dots, j_k)$ is bigger than $(i_1, \dots, i_k)$ with respect to the lexicographical ordering of $\mathbb{R}^k$. For a monomial $I \in \mathcal{I}$, let $\overline{v_I} : \mathbb{R}\langle\langle X_1, \dots, X_n\rangle\rangle \to \mathbb{R}$ be a map defined by $\overline{v_I}(\Sigma_{J\in\mathcal{I}}\, r_J \cdot J) = r_I$ and let $v_I = \overline{v_I} \circ \mu : F_n \to \mathbb{R}$.

**Definition 1.4.4** (Magnus ordering of free groups). The *Magnus ordering* $<_M$ is a total ordering of $F_n$ defined by a lexicographical expression $\{v_I\}_{I\in\mathcal{I}}$.

It is directly checked that the Magnus ordering $<_M$ is a bi-ordering. See [41] for details. The Magnus ordering is a fundamental piece in constructing of bi-orderings: see [4], [27].

## 1.5 Summary and overview of results

In this section we give a summary of main results of the present thesis. The precise and complete statements will be given in each Chapter.

### 1.5.1 A construction of bi-ordering via Chen's iterated integral

It was classically known that a residually torsion-free nilpotent group is bi-orderable. This was proved by using a rather classical algebraic technique, by utilizing the sequence of central extensions. In Chapter 2 we provide a new construction of bi-ordering based on Chen's iterated integral theory and the universal holonomy map. We will call the constructed bi-ordering *holonomy ordering*. Chapter 2 is based on the author's paper [22].

The main result in Chapter 2 is the following:

**Theorem 1.5.1.** *Let $G = \pi_1(M)$ be a residually torsion-free nilpotent group. Then each holonomy ordering is equivalent to bi-ordering constructed by the classical algebraic method, based on the iterated extension of bi-ordering.*

Thus, our construction provides an alternative interpretation of the classical constructions of bi-orderings. This theorem shows that bi-orderings of residually torsion-free nilpotent group can be understood in the rational homotopy theory. Based on this observation, we give various relationships between bi-orderings and topological objects, such as finite type invariants of pure braids.

### 1.5.2 Alexander polynomial criterion of bi-orderability

In [8], Clay and Rolfsen gave a necessary condition for the fundamental group of fibered 3-manifolds to be bi-orderable. Their condition uses the roots of the (classical) Alexander polynomial. In Chapter 3, we will study the possibility of extension of Clay-Rolfsen's criterion by using the *twisted* Alexander polynomial, a generalization of the Alexander polynomials. Chapter 3 is based on the author's paper [23].

The main result in Chapter 3 is the following:

**Theorem 1.5.2.** *The twisted Alexander polynomial for a finite dimensional representation with finite image cannot be used to strengthen Clay-Rolfsen's result.*

The above statement is a bit vague: more precise statement will be given in Chapter 3. This is surprising since in almost all known criterions using the Alexander polynomial can be strengthened by using twisted Alexander polynomials. In the course of proof, we will also observe an interesting property of ordered quotient of bi-orderable groups.

### 1.5.3 Dehornoy-like ordering

In Chapter 4, motivated by the Dehornoy ordering of the braid group, we introduce a class of left-orderings called a *Dehornoy-like ordering.* Chapter 4 is based on the author's preprint [24].

In the first part of Chapter 4 we establish fundamental properties of Dehornoy-like ordering. We will show that under the condition which we call *Property F*, a Dehornoy-like ordering produces an isolated ordering and vice versa.

**Theorem 1.5.3.** *Let $\mathcal{S}$ be an ordered finite generating set of $G$ having Property F and $\mathcal{A}$ be the twisted generating set of $\mathcal{S}$. Then $\mathcal{S}$ defines a Dehornoy-like ordering of $G$ if and only if $\mathcal{A}$ defines an isolated left ordering of $G$.*

In the second part of Chapter 4, we construct new examples of Dehornoy-like orderings. Our examples are the group of the form $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$, the amalgamated free product of two cyclic groups.

**Theorem 1.5.4.** *Let $G_{m,n} = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} = \langle x, y \mid x^m = y^n \rangle$. Take a generating set $\mathcal{S} = \{s_1 = xyx^{-m+1}, s_2 = x^{m-1}y^{-1}\}$ and $\mathcal{A} = \{a = x, b = yx^{-m+1}\}$.*

1. *$\mathcal{S}$ defines a Dehornoy-like ordering $<_D$.*
2. *$\mathcal{A}$ defines an isolated left ordering $<_A$.*

We will also study more deitailed properties of the Dehornoy-like ordering of $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$, and show that they provide another interesting example of left-ordering: a left-ordering having no non-trivial convex subgroups.

### 1.5.4 Isolated ordering

As we have already mentioned, it is an interesting problem to find a method to construct isolated orderings. In Chapter 5, we will show the following theorem which provides a lot of new examples of isolated orderings.

**Theorem 1.5.5.** *Let $G$ and $H$ be finitely generated groups, $z_G$ be a non-trivial central element of $G$, and $z_H$ be a non-trivial element of $H$.*

*Let $\mathcal{G} = \{g_1, \ldots, g_m\}$ be a finite generating set of $G$ which defines an isolated left ordering $<_G$ of $G$ such that*

$$1 <_G g_1 <_G \cdots <_G g_m <_G z_G$$

*holds. Similarly, let $\mathcal{H} = \{h_1, \ldots, h_n\}$ be a finite generating set of $H$ which defines an isolated left ordering $<_H$ of $H$ such that*

$$1 <_H h_1 <_H \cdots <_H h_n <_H z_H$$

*holds. Assume that the left-ordering $<_H$ is $z_H$-right invariant.*

*Let $X = G *_{\mathbb{Z}} H = G *_{\langle z_G = z_H \rangle} H$ be an amalgamated free product of $G$ and $H$. For $i = 1, \ldots, m$, let $x_i = g_i z_H^{-1} h_1$. Then the generating set $\{x_1, \ldots, x_m, h_1, \ldots, h_n\}$ of $X$ defines an isolated left ordering $<_X$ of $X$ which only depends on the isolated orderings $<_G, <_H$ and $z_G, z_H$.*

Chapter 5 is based on the author's paper [25]. More precise properties of the constructed isolated ordering $<_X$ will be also studied. By using this theorem, we will construct an isolated ordering with trivial center, or group having a lot of non-conjugate isolated orderings.

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# Chapter 2

# Bi-orderings via iterated integrals

In this section we give a new point of view of a construction of bi-orderings based on Chen's iterated integral theory.

## 2.1 Classical construction of bi-orderings for a residually torsion-free nilpotent group

A group $G$ is called *residually torsion-free nilpotent* if $G_\infty = \bigcap_{i \geq 1} G_i = \{1\}$, where $G_k$ is the $k$-th dimension subgroup of $G$ defined by $G_k = \{g \in G \,|\, g - 1 \in J^k\}$. Here $J$ denotes the augmentation ideal of the group ring $\mathbb{Z}G$. The free group is a typical example of residually torsion-free nilpotent groups. It was classically known that a residually torsion-free nilpotent group is bi-orderable.

First of all, we review the classical construction of bi-orderings based on the following well-known lemma (see [32], for example).

**Lemma 2.1.1.** *Let $H, K$ be bi-orderable groups and $1 \to H \to G \xrightarrow{p} K \to 1$ be a group extension. For a bi-ordering $<_K$ of $K$ and a bi-ordering $<_H$ of $H$, define an ordering $<_G$ of $G$ by $g <_G g'$ if $p(g) <_K p(g')$ or, $p(g) = p(g')$ and $1 <_H g^{-1}g'$. If $<_H$ is invariant under the action of $G$, that is, if $h <_H h'$ implies $ghg^{-1} <_H gh'g^{-1}$ for all $g \in G$ and $h, h' \in H$, then $<_G$ is a bi-ordering of $G$.*

Using Lemma 2.1.1, we construct a bi-ordering of a residually torsion-free nilpotent group $G$ as follows. We inductively construct a bi-ordering $<_k$ of $G/G_k$ for each $k > 0$. To begin with, $G/G_1 = \{1\}$ so let $<_1$ be the trivial bi-ordering.

To define $<_{k+1}$, first we choose a bi-ordering $<'_k$ of the torsion-free abelian group $G_k/G_{k+1}$. Observe that there is a central extension

$$1 \to G_k/G_{k+1} \to G/G_{k+1} \to G/G_k \to 1.$$

From this central extension and bi-orderings $<_k$ and $<'_k$, we construct an ordering $<_{k+1}$ of $G/G_{k+1}$ as in Lemma 2.1.1. Since the extension is central, the assumption of Lemma 2.1.1 is automatically satisfied so $<_{k+1}$ is a bi-ordering of $G/G_{k+1}$.

Since $G$ is torsion-free nilpotent, the sequence of bi-orderings $<_k$ defines a bi-ordering $<_G$ of $G$. We call this bi-ordering $<_G$ an *iterated extension ordering* of $G$ derived from $\{<'_k\}$.

## 2.2 Chen's iterated integral and holonomy representation

In this section we briefly review Chen's iterated integral theory. For details, see [6]. We restrict our attention to smooth manifolds, although there is a generalization of Chen's iterated integral theories for simplicial complexes due to Hain [19], and our methods can also be applied for such general iterated integrals.

Let $M$ be a connected smooth manifold with a base point. We denote the de Rham DGA of $M$ by $A^*_{DR}(M)$ and the based loop space of $M$ by $\Omega M$. Let $\Delta_q$ be the standard $q$-dimensional simplex

$$\Delta_q = \{(t_1, \ldots, t_q) \in \mathbb{R}^q \mid 0 \leq t_1 \leq \cdots \leq t_q \leq 1\}.$$

and $ev : \Omega M \times \Delta_q \to M^q$ be the evaluation map

$$ev(\gamma, (t_1, \ldots, t_q)) = (\gamma(t_1), \ldots, \gamma(t_q)).$$

Formally, the iterated integral map $\int$ is defined as the composition

$$\int : A^*_{DR}(M)^{\otimes k} \xrightarrow{\times} A^*_{DR}(M^k) \xrightarrow{ev^*} A^*_{DR}(\Delta_k \times \Omega M) \xrightarrow{\int_{\Delta_k}} A^*_{DR}(\Omega M)$$

where $\times$ is a cross-product and $\int_{\Delta_k}$ is an integration along fiber.

We mainly use iterated integrals of 1-forms that is described as follows. Let $\omega_1, \ldots, \omega_m$ be 1-forms of $M$, and $\gamma : [0,1] \to M$ be a piecewise smooth path. Let us denote the pull-back of $\omega_i$ by $\gamma$ by $\gamma^*\omega_i = \alpha_i(t)dt$. Then the iterated integral $\int_\gamma \omega_1 \cdots \omega_q$ along $\gamma$ is explicitly written as the multiple integral

$$\int_\gamma \omega_1 \cdots \omega_q = \int_{0 \leq t_1 \leq \cdots \leq t_q \leq 1} \alpha_1(t_1)\alpha_2(t_2) \cdots \alpha_q(t_q) dt_1 dt_2 \cdots dt_q.$$

From now on, we always assume that the real homology group of $M$ is finite dimensional. Let $\{X_1, \ldots, X_m\}$ be a homogeneous, ordered basis of $H_*(M;\mathbb{R})$ chosen so that $\{X_1, \ldots, X_k\}$ forms a basis of $H_1(M;\mathbb{R})$.

Let $J$ be the augmentation ideal of $TH$, the tensor algebra of $H = H_*(M;\mathbb{R})$ and $\widehat{TH} = \varprojlim TH/J^N$ be the nilpotent completion of $TH$. Then $\widehat{TH}$ is identified with the algebra of non-commutative formal power series $\mathbb{R}\langle\langle X_1, \ldots, X_m\rangle\rangle$.

Let $\widehat{J}$ be the nilpotent completion of $J$, which is an ideal of $\widehat{TH}$ that consists of formal power series with zero constant term. We define a grading of $\widehat{TH}$ by the formula $\deg X_1 \cdots X_k = p_1 + \cdots + p_k - k$ where $p_i$ denotes the degree of $X_i$, and define an involution $\varepsilon : A^*_{DR}(M) \to A^*_{DR}(M)$ by $\varepsilon(\omega) = (-1)^{\deg \omega}\omega$.

Let $\delta$ be a degree 1 derivation of $\widehat{TH}$ and $\overline{\omega} = \sum_{i_1,\ldots,i_p} \omega_{i_1\cdots i_p} X_{i_1} \cdots X_{i_p}$ be an element of $A^*_{DR}(M) \otimes \widehat{TH}$. We call a pair $(\overline{\omega}, \delta)$ is a *formal homology connection* if the following three conditions hold.

1. $\delta X_i \in \widehat{J}^2$.
2. $\deg \omega_{i_1\cdots i_p} = \deg X_{i_1} \cdots X_{i_p}$.
3. $d\overline{\omega} + \delta\overline{\omega} = \varepsilon(\overline{\omega}) \wedge \overline{\omega}$.

Such a pair $(\overline{\omega}, \delta)$ always exists and we can choose $\overline{\omega}$ so that the coefficient $\omega_i$ represents the cohomology class which is the dual of $X_i \in H_*(M;\mathbb{R})$. There are many choices of formal homology connections, but in some cases there is a canonical choice. For example, if $M$ is a compact Riemannian manifold, one can find a canonical formal homology connection by using the de Rham-Hodge decomposition of $A^*_{DR}(M)$.

Let $\mathcal{R}$ be the degree 0 part of $\widehat{TH}$, identified with $\widehat{TH_1} = \mathbb{R}\langle\langle X_1, \ldots, X_k \rangle\rangle$, the nilpotent completion of the tensor algebra of $H_1 = H_1(M;\mathbb{R})$. Let $\omega$ be the degree 0 part of the formal homology connection $\overline{\omega}$ and $\mathcal{N}$ be the degree 0 part of $\delta(\widehat{TH})$. Let us denote the quotient algebra $\mathcal{R}/\mathcal{N}$ (resp. $\mathcal{R}/(\mathcal{N} + \widehat{J}^k)$) by $R$ (resp. $R_k$). $R$ is nothing but the 0-th homology group of the complex $(\widehat{TH}, \delta)$. The (Chen's) *holonomy representation* is a homomorphism $\Theta : \pi_1(M) \to R$ defined by

$$\Theta([\gamma]) = 1 + \sum_{i=1}^{\infty} \int_\gamma \underbrace{\omega\omega\cdots\omega}_{i}$$

and the *order k holonomy representation* is a homomorphism $\Theta_k : \pi_1(M) \to R_k$ defined by

$$\Theta_k([\gamma]) = 1 + \sum_{i=1}^{k} \int_\gamma \underbrace{\omega\omega\cdots\omega}_{i}$$

Chen's results are summarized as follows.

**Theorem 2.2.1** (Chen [6]). *Let $\Theta$, $\Theta_k$ be as above. Then,*

1. *$Ker\Theta = \pi_1(M)_\infty$ and $Ker\Theta_k = \pi_1(M)_k$.*
2. *$\Theta$ induces an isomorphism of Hopf algebras $\Theta : \widehat{\mathbb{R}\pi_M} \to R$, where $\widehat{\mathbb{R}\pi_M}$ is the nilpotent completion of the group ring $\mathbb{R}\pi_1(M)$.*
3. *$\Theta_k$ induces an isomorphism of Hopf algebras $\Theta_k : \mathbb{R}\pi_1(M)/J^k \to R_k$.*

Thus, if $\pi_1(M)$ is residually torsion-free nilpotent, then the universal holonomy map is injective.

**Remark 2.2.1.** If $M$ is simply connected (more generally, if $M$ is a nilpotent space), a formal homology connection computes the homology group of the based loop space $\Omega M$. More precisely, there exists a natural isomorphism of Hopf algebras

$$H_*(\Omega M) \cong H_*(\widehat{TH}, \delta).$$

**Remark 2.2.2.** To define the holonomy map, we do not need the whole formal homology connection $(\overline{\omega}, \delta)$. We only need the degree 0 and 1 parts. The degree $\leq 1$ part of a formal homology connection is computed by $A^{\leq 2}_{DR}(M)$, the degree $\leq 2$ part of the de Rham DGA, so to define the holonomy map it is sufficient to assume that $H_{\leq 2}(M;\mathbb{R})$ is finite dimensional.

We say $M$ has a *quadratic formal homology connection* if the ideal $\mathcal{N}$ is quadratic. That is, the ideal $\mathcal{N}$ is generated by elements of $H_1(M) \otimes H_1(M)$. Let $\cup^* : H_2(M) \to H_1(M) \otimes H_1(M)$ be the dual of the cup product. It is known that the quadratic part of the ideal $\mathcal{N}$ is generated by the image of $\cup^*$. Thus, if $M$ has a quadratic formal homology connection, then the ideal $\mathcal{N}$ is determined by the cup products.

The condition that $M$ has a quadratic formal homology connection is equivalent to the condition that $M$ is *formal* in the sense of Sullivan's rational homotopy theory [46]. That is, if we regard the de Rham cohomology $H^*_{DR}(M)$ as a DGA having zero differential, then there is a sequence of DGAs and DGA morphisms

$$A^*_{DR}(M) \to A_1 \leftarrow A_2 \to \cdots \leftarrow A_n \to H^*_{DR}(M)$$

such that each map induces an isomorphism on cohomology groups.

# 2.3 Holonomy construction of bi-orderings

In this section we give a construction of bi-invariant orderings by using Chen's holonomy map.

## 2.3.1 Definition of holonomy orderings

Let $M$ be a connected smooth manifold and $G$ be its fundamental group. We always assume that $H_{\leq 2}(M;\mathbb{R})$ is finite dimensional so that the holonomy map is defined. We use the notations in section 2.2.

Let $R = \mathcal{F}^0 R \supset \mathcal{F}^1 R \supset \cdots$ be the decreasing filtration of $R$ induced by the powers of the ideal $\widehat{J}$. This filtration is multiplicative: $\mathcal{F}^k R \cdot \mathcal{F}^l R \subset \mathcal{F}^{k+l} R$ holds. Let us choose a subspace $R^i$ of $R$ so that $\mathcal{F}^k = \bigoplus_{i \geq k} R^i$ holds. Then this filtration defines the grading $R = \bigoplus_{i=0}^{\infty} R^i$.

Let $\mathcal{B} = \{v_i\}_{i \in \mathcal{I}}$ be an ordered, homogeneous basis of $R$ such that the degree is non-decreasing. That is, $\deg v_i \leq \deg v_j$ if $i < j$. Let $\{v_i^* : R \to \mathbb{R}\}_{i \in \mathcal{I}}$ be the dual basis of $R$. We denote by $<_{\mathcal{B}}$ the total ordering of the vector space $R$ defined by the lexicographical expression $v_i^*$. That is, for $a, b \in R$, we define $a <_{\mathcal{B}} b$ if the sequence of reals $\{v_i^*(b)\}_{i \in \mathcal{I}}$ are bigger than $\{v_i^*(a)\}_{i \in \mathcal{I}}$ with respect to the lexicographical ordering of $\mathbb{R}^{\mathcal{I}}$.

The *holonomy ordering* is a total ordering $<$ of $G/G_\infty$ defined by $a < b$ if $\Theta(a) <_{\mathcal{B}} \Theta(b)$. In other words, the holonomy ordering is an ordering defined by the lexicographical expression $\{v_i^* \circ \Theta\}_{i\in\mathcal{I}}$.

**Theorem 2.3.1.** *The holonomy ordering is a bi-ordering of $G/G_\infty$.*

*Proof.* First observe that from the definition of Chen's holonomy map $\Theta$, the image of $\Theta$ lies in $1 + \mathcal{F}^1(R)$. Now assume that $a < b$ for $a, b \in G$. Then we write their images as

$$\Theta(a) = 1 + A_{<i} + A_i + A_{>i},\ \Theta(b) = 1 + A_{<i} + B_i + B_{>i}$$

where $A_{<i}$ (resp. $A_i$, $A_{>i}$) is degree $< i$ (resp. $i$, $> i$) part. Since $a < b$, $A_i <_{\mathcal{B}} B_i$ hold. We show the left-invariance of the holonomy ordering $<$. The proof of right-invariance is similar. For $g \in G$, let us write $\Theta(g) = 1 + G_{<i} + G_i + G_{>i}$, where $G_{<i}$ (resp. $G_i, G_{>i}$) represents the degree $< i$ (resp. $i$, $i >$) part. Since $R_i \cdot R_j \subset \mathcal{F}^{i+j}R$, the degree $< i$ part of $\Theta(ga)$ is the same as the degree $< i$ part of $\Theta(g) \cdot (1 + A_{<_i})$. Similarly, the degree $i$ part of $\Theta(ga)$ is given by $A_i + G_i + [(1 + A_{<i})(1 + G_{<i})]_i$, where $[(1 + A_{<i})(1 + G_{<i})]_i$ represents the degree $i$ part of $(1 + A_{<i})(1 + G_{<i})$.

On the other hand, the degree $< i$ part of $\Theta(gb)$ is the same as the degree $< i$ part of $\Theta(g) \cdot (1 + A\ _{<i})$, and the degree $i$ part of $\Theta(gb)$ is given by $B_i + G_i + [(1 + A_{<i})(1 + G_{<i})]_i$. Thus, $\Theta(gb) - \Theta(ga) = B_i - A_i +$ (higher parts), so we conclude $ga < gb$. □

This provides a geometric proof of bi-orderability of residually torsion-free nilpotent groups.

**Corollary 2.3.1.** *If a group $G$ is residually torsion-free nilpotent, then $G$ is bi-orderable.*

The map $v_i^* \circ \Theta : G \to \mathbb{R}$ is written as the iterated integral $[\gamma] \mapsto \int_\gamma \omega_1 \cdots \omega_k$ of some 1-forms $\omega_1, \ldots, \omega_k$. Thus, the lexicographical expression of the holonomy ordering is given as the iterated integrals.

### 2.3.2 Comparison with classical constructions

Now we show that the classical construction of bi-ordering is equivalent to the holonomy construction.

**Theorem 2.3.2.** *Let $G = \pi_1(M)$ be a residually torsion-free nilpotent group. Then each holonomy ordering is an iterated extension ordering and conversely, each iterated extension ordering is a holonomy ordering. Thus, the holonomy ordering construction is equivalent to the iterated extension ordering construction.*

*Proof.* We use notations in Section 2.2 and 2.3.1. Let $\{v_i\}$ be an ordered, homogeneous, degree non-decreasing basis of $R$ which defines a holonomy ordering $<_H$. By definition of $R_i$, the universal holonomy map provides an isomorphism

$(G_k/G_{k+1})\otimes\mathbb{R} \cong R_i$. So we can regard $G_k/G_{k+1}$ as an integer lattice of $R_i$. Let $<'_k$ be the restriction of the ordering $<_H$ to $G_k/G_{k+1}$. Then $<'_k$ is a bi-ordering, and the holonomy ordering $<_H$ is nothing but the iterated extension ordering derived from this sequence of orderings $\{<'_k\}$.

Conversely, let $<_G$ be an iterated extension ordering of $G$ derived from $\{<'_k\}$. Then the ordering $<'_k$ is naturally extended as a bi-ordering $\widetilde{<'_k}$ of $(G_k/G_{k+1})\otimes\mathbb{R} = R_i$. Let us take an ordered basis $\{v_{(k),i}\}_{i=1,2,\dots}$ of $R_k$ so that the dual basis $\{v^*_{(k),i}\}$ gives a lexicographical expression of the ordering $\widetilde{<'_k}$. Now by correcting the basis $\{v_{(k),i}\}$ of $R_k$ for all $k$, we obtain an ordered, homogeneous, degree non-decreasing basis $\{v_i\}$ of $R$. The holonomy ordering defined by the basis $\{v_i\}$ is identical with $<_G$. □

We show that the holonomy ordering construction is indeed an extension of the Magnus ordering of the free group $F_n$. The following proposition is directly obtained from Theorem 2.3.2 since the Magnus ordering is an iterated extension ordering. However, here we present a different and direct proof which is based on the fact that Magnus expansion is equivalent to Chen's holonomy map.

**Propotition 2.3.1.** *The Magnus ordering $<_M$ of the free group is a holonomy ordering.*

*Proof.* Let $D_n$ be the $n$-punctured disc and $\{x_1,\dots,x_n\}$ be a generator of $\pi_1(D_n) = F_n$. Take 1-forms $\omega_1,\dots,\omega_n$ of $D_n$ so that their representing cohomology classes $\{[\omega_i]\}$ are dual to $\{x_i\}$, and they satisfy the equation $\omega_i\wedge\omega_j = 0$. Then, the formal homology connection is taken as $(\omega = \sum_{i=1}^{n}\omega_i X_i, 0)$ and the holonomy representation defines an injective homomorphism

$$\Theta : F_n \to \mathbb{R}\langle\langle X_1,\dots,X_n\rangle\rangle.$$

Let us denote $\Theta(x_i) = 1 + X_i + X_i^{\geq 2}$, where $X_i^{\geq 2}$ is the degree $\geq 2$ part.

Let $\mathcal{B}$ be DegLex-ordered monomial basis of $R = \langle\langle X_1,\dots,X_m\rangle\rangle$, and $<$ be the holonomy ordering defined by $(\omega,0)$ and $\mathcal{B}$. Let $\alpha$ be an automorphism of $\mathbb{R}\langle\langle X_1,\dots,X_n\rangle\rangle$ defined by $\alpha(X_i) = X_i + X_i^{\geq 2}$. Then $\Theta = \alpha\circ\mu$ holds. For each element $x \in \mathbb{R}\langle\langle X_1,\dots,X_n\rangle\rangle$, $\alpha$ preserves the lowest degree part of $x$. Hence if $a <_M b$ then $a < b$ holds, so these two orderings are identical. □

### 2.3.3 Application: bi-ordering of pure braid groups and finite type invariants

In this section we describe a relationship between holonomy orderings and finite type invariants of pure braids. This relation is based on the work of Kohno [29], which relates iterated integral and finite type invariants of braids. For details of pure braid groups and their finite type invariants, see [29].

For $1 \leq i < j \leq n$, let $H_{i,j} = \ker(z_i - z_j)$ be a hyperplane of $\mathbb{C}^n$. The hyperplane arrangement $\mathcal{A}_n = \{H_{i,j} \mid 1 \leq i < j \leq n\}$ is called the *braid arrangement*. We denote by $M_{\mathcal{A}}$ the complement of the arrangement $\mathbb{C}^n -$

$\bigcup_{H\in\mathcal{A}} H$. The fundamental group of $M_{\mathcal{A}}$ is the pure braid group $P_n$, which is torsion-free nilpotent.

First let us review the definition of finite type invariants. A singular pure braid is a pure braid having transversal double points. We denote the set of singular pure braids having $k$ singular points by $S^kP_n$ and let $SP_n = \bigcup_{k\geq 0} S^kP_n$. Each map $v : P_n \to \mathbb{R}$ can be extended to the map $\widetilde{v} : SP_n \to \mathbb{R}$ by defining $\widetilde{v}(\times) = v(\diagup\!\!\!\!\diagdown) - v(\diagdown\!\!\!\!\diagup)$. A map $v$ is called a *finite type invariant* of order $k$ if $\widetilde{v}(\beta) = 0$ for all $\beta \in S^{\geq k}P_n$. Let $V_k(P_n)$ be the set of order $k$ finite type invariants and $V(P_n) = \bigcup_{k>0} V_k(P_n)$ be the set of all finite type invariants. Then $V_k(P_n)$ and $V(P_n)$ are $\mathbb{R}$-vector spaces.

Finite type invariants and holonomy orderings are related as follows.

**Propotition 2.3.2.** *Let $\mathcal{B} = \{v_i\}_{i\in\mathcal{I}}$ be an ordered basis of $V(P_n)$ such that the order of $v_i$ are non-decreasing. Let us define a total ordering $<_{\mathcal{B}}$ of $P_n$ by $\alpha <_{\mathcal{B}} \beta$ if $\{v_i(\beta)\}_{i\in\mathcal{I}}$ is bigger than $\{v_i(\alpha)\}_{i\in\mathcal{I}}$ with respect to the lexicographical ordering. Then $<_{\mathcal{B}}$ is a holonomy ordering, hence bi-invariant total ordering. Thus, the sequence of finite type invariants $\{v_i\}_{i\in\mathcal{I}}$ gives a lexicographical expression of a holonomy ordering of $P_n$.*

*Proof.* Let $\Theta : P_n \to R$ be a holonomy map. There is a natural isomorphism $V_k(P_n) \cong \mathrm{Hom}(\mathbb{R}P_n/J^{k+1}, \mathbb{R})$, so $R^* = \mathrm{Hom}_{\mathbb{R}}(R,\mathbb{R}) \cong V(P_n)$ [29]. Therefore a degree non-decreasing dual basis of the algebra $R$ corresponds to an order non-decreasing basis of $V(P_n)$. Thus the basis $\mathcal{B}$ provides a lexicographical expression of a holonomy ordering of $P_n$. $\square$

**Remark 2.3.1.** The holonomy map $\Theta : P_n \to R$ is known as the universal finite type invariant of the pure braid groups (the universal representation of the quantum group representations of pure braid groups), which is a prototype of the Kontsevich invariant of knots. By using the Drinfel'd associator, we can construct the universal holonomy map over the rationals in more explicit form (See [30] for details). In particular, the conclusion of Proposition 2.3.2 holds for $\mathbb{Q}$-valued finite type invariants.

## 2.4 Properties and generalization

In this section we study structures and properties of the holonomy orderings, based on the relationship between Chen's theory and rational homotopy theory.

### 2.4.1 General case

First of all, we study properties of holonomy orderings for general spaces. The following proposition is a refinement of famous Stallings' Theorem [45].

**Propotition 2.4.1.** *If $H_2(M;\mathbb{R}) = 0$, then $\pi_1(M)$ contains the rank $b_1(M)$ free group $F$ such that the restriction of a holonomy ordering $<$ to $F$ is the Magnus ordering, where $b_1(M)$ is the 1st Betti number of $M$.*

*Proof.* Let $\{X_1, \dots, X_m\}$ be a basis of $H_1(M;\mathbb{R})$ and $\{g_1, \dots, g_m\}$ be elements of $\pi_1(M)$ such that $h(g_i) = X_i$, where $h : \pi_1(M) \to H_1(M;\mathbb{R})$ is the Hurewicz homomorphism. Let $\mathcal{B}_{\mathsf{DegLex}}$ be an ordered basis of $R = \langle\langle X_1, \dots, X_m\rangle\rangle$ which consists of the monomials of $R$ and ordered by the DegLex-ordering. Take 1-forms $\{\omega_1, \dots, \omega_m\}$ on $M$ so that their representing cohomology classes are dual to $\{X_1, \dots, X_m\}$ and take a formal homology connection $\omega = \sum_{i=1}^{m} \omega_i + \cdots$. Since $H_2(M;\mathbb{R}) = 0$, the ideal $\mathcal{N}$ must be trivial. Let $\Theta : \pi_1(M) \to \mathcal{R} = \mathbb{R}\langle\langle X_1, \dots, X_m\rangle\rangle$ be the holonomy map. We denote the holonomy ordering with respect to the basis $\mathcal{B}_{\mathsf{DegLex}}$ by $<_M$. Let $F$ be a subgroup of $G$ generated by $\{g_1, \dots, g_m\}$. Then $\Theta(g_i) = 1 + X_i +$ (Higher term), so as in the proof of Proposition 2.3.1, we conclude that $F$ is the rank $m$ free groups, and the restriction of $<_M$ to $F$ is the Magnus ordering. □

One of the benefits to consider the holonomy orderings is that we can associate the properties of groups with the properties of manifolds (topological space having the given group as its fundamental group).

**Theorem 2.4.1.** *Let $N$ be a smooth manifold which is a retract of $M$ and $\iota : N \hookrightarrow M$ be the inclusion. Then for each holonomy ordering $<_N$ of $\pi_1(N)$, there exists a holonomy ordering $<_M$ of $\pi_1(M)$ such that the restriction of $<_M$ to $\iota_*(\pi_1(N))$ is $<_N$. Thus, every holonomy ordering of $N$ can be extended as a holonomy ordering of $M$.*

*Proof.* Let $(\omega_N, \delta_N)$ be a formal homology connection of $N$ and $\mathcal{B}_N$ be an ordered, degree non-decreasing basis of $R_N$ which defines the holonomy ordering $<_N$. Let $r : M \to N$ be a retraction. Since $N$ is a retract of $M$, the de Rham DGA of $M$ is written as a direct sum $A^*_{DR}(M) = r^*A^*_{DR}(N) \oplus A'$ for some DGA $A'$. Thus, we can construct a formal homology connection $(\omega_M, \delta_M)$ so that it is an extension of $(\omega_N, \delta_N)$. Hence, the non-commutative algebra $R_M$ is also written as a direct sum $R_M = R_N \oplus R'$ for some non-commutative algebra $R'$. Let $\mathcal{B}_M$ be an ordered, degree non-decreasing basis of $R_M$ obtained by adding vectors to $\iota_*(\mathcal{B}_N)$. Then the holonomy ordering defined by $(\omega_M, \delta_M)$ and $\mathcal{B}_M$ is an extension of the holonomy ordering of $<_N$. □

### 2.4.2 Formal space case

In the case of formal space, the structure of holonomy ordering is determined by the cohomology algebra, in particular the cup product of the 1st cohomology groups.

**Theorem 2.4.2.** *Assume that $M$ and $N$ are formal and that there exists a continuous map $\iota : N \to M$ which satisfies the following conditions.*

1. *$\iota_* : H_{\leq 2}(N;\mathbb{R}) \to H_{\leq 2}(M;\mathbb{R})$ is injection.*
2. *$\iota_*$ induces a decomposition $H_{\leq 2}(M;\mathbb{R}) = \iota_* H_{\leq 2}(N;\mathbb{R}) \oplus A$ which preserves the dual of the cup product. That is, $\cup^*(\iota_* H_2(N;\mathbb{R})) \subset \iota_* H_1(N;\mathbb{R}) \otimes \iota_* H_1(N;\mathbb{R})$ and $\cup^*(A_2) \subset A_1 \otimes A_1$ hold.*

*Then $\iota_* : \pi_1(N) \to \pi_1(M)$ is an injection, and each holonomy ordering $<_N$ can be extended as a holonomy ordering $<_M$ of $\pi_1(M)$.*

*Proof.* First observe that $\iota$ induces an injection $\widehat{T\iota} : \widehat{TH_1(N)} \to \widehat{TH_1(M)}$. Moreover, since $M$ and $N$ are formal, $M$ and $N$ have quadratic formal homology connections. Fix such a quadratic formal homology connection, and let $\Theta_N : \pi_1(N) \to \widehat{TH_1(N)}/\mathcal{N}_N = R_N$ and $\Theta_M : \pi_1(M) \to \widehat{TH_1(N)}/\mathcal{N}_M = R_M$ be holonomy maps.

Now the ideal $\mathcal{N}_N$ and $\mathcal{N}_M$ are generated by the image of the dual of the cup products. Thus from the assumption we conclude that $\mathcal{N}_M = \iota_*\mathcal{N}_N \widehat{+} \mathfrak{a}$ holds, where $\mathfrak{a}$ is an ideal generated by $\cup^*(A_2)$. Hence we conclude that $\widehat{T\iota}$ induces an injection $\widehat{T\iota_*} : R_N \hookrightarrow R_M$, thus $\iota_* : \pi_1(N) \to \pi_1(M)$ is also an injection.

Let $\mathcal{B}_N$ be the ordered degree non-decreasing basis of $R_N$, which defines a holonomy ordering $<_N$ of $\pi_1(N)$. By adding vectors to $\iota_*\mathcal{B}_N$, we form an ordered degree non-decreasing basis $\mathcal{B}_M$ of $R_M$. Then the holonomy ordering $<_M$ defined by the basis $\mathcal{B}_M$ provides an extension of the holonomy ordering $<_N$. □

**Example 2.4.1.** Assume that $M$ is a formal space and the cup product $\cup : H^1(M;\mathbb{R}) \otimes H^1(M;\mathbb{R}) \to H^2(M;\mathbb{R})$ is the zero map. Then, by Theorem 2.4.2 (take $N$ as the $b_1(M)$-punctured disc), we conclude that $\pi_1(M)$ contains the rank $b_1(M)$ free group $F$, and every holonomy ordering of $F$ can be extended as a holonomy ordering of $M$.

### 2.4.3 Generalizations of holonomy ordering and examples

We close the paper by providing a generalized construction of the holonomy ordering construction which can be seen as a special case of group extension construction described in Lemma 2.1.1.

We call an ordered basis $\mathcal{B} = \{v_i\}_{i\in\mathcal{I}}$ of $R_M = \widehat{\mathbb{R}\pi_1 M}$ is *bi-ordering basis* if $\{v_i^* \circ \Theta_M\}_{i\in\mathcal{I}}$ is a lexicographical expression of a bi-ordering $<$ of $\pi_1(M)/\pi_1(M)_\infty$. Theorem 2.3.1 shows that if $\mathcal{B}$ is degree-non-decreasing homogeneous basis, then $\mathcal{B}$ is bi-ordering basis.

Now we construct a bi-ordering basis which is not degree non-decreasing by using quasi-nilpotent fibration. Let $B$ and $F$ be connected manifolds having the residually torsion-free nilpotent fundamental groups, and assume that $\pi_2(B) = 1$. A fibration $F \to E \to B$ is called a *quasi-nilpotent fibration* if $\pi_1(B)$ acts on $H_1(F;\mathbb{R})$ nilpotently.

We can construct a bi-ordering basis of $R_E = \widehat{\mathbb{R}\pi_1(E)}$ as follows. First of all, let us take an arbitrary bi-ordering basis $\{v_j^B\}_{j\in\mathcal{J}}$ of $R_B = \widehat{\mathbb{R}\pi_1(B)}$. Since $\pi_1(B)$ acts on $H_1(F;\mathbb{R})$ nilpotently, this fibration induces an exact sequence of the nilpotent completions of the fundamental groups [5].

$$1 \to R_F \to R_E \to R_B \to 1.$$

Since $\pi_1(B)$ acts on $H_1(F;\mathbb{R})$ nilpotently, there is a basis $\{X_1,\ldots,X_m\}$ of $H_1(F;\mathbb{R})$ such that $g(X_i) = X_i + \sum_{i'>i}^{m} a_{i'}X_{i'}$ holds for all $g \in \pi_1(B)$. Let us choose an ordered, degree non-decreasing basis $\{v_i^F\}_{i\in\mathcal{I}}$ of $R_F = \widehat{\mathbb{R}\pi_1(F)}$ as the subset of the DegLex-ordered monomials of $\mathbb{R}\langle\langle X_1,\ldots,X_m\rangle\rangle$.

Now we are ready to give a bi-ordering basis of $R_E$. Let us take an ordered basis $\{v_j^B v_i^F\}_{(j,i)\in\mathcal{J}\times\mathcal{I}}$ of $R_E$. Here the ordering of $\mathcal{J}\times\mathcal{I}$ is the lexicographical ordering defined by $(j,i) > (j',i')$ if and only if $j > j'$, or $j = j'$ and $i' > i$. From the choice of the basis $\{v_i^F\}$, $g(v_i^F) = v_i^F + \sum_{i'>i} c_{i'} v_{i'}^F$ holds for all $g \in \pi_1(B)$, and $v_i^F \cdot v_j^B = v_j^B v_i^F + \sum_{i'>i} v_j^B v_{i'}^F$. This implies that the basis $\{v_j^B v_i^F\}_{(j,i)\in\mathcal{J}\times\mathcal{I}}$ is a bi-ordering basis.

Thus, we can construct a bi-ordering basis and bi-ordering using a sequence of quasi-nilpotent fibration. We call such a bi-ordering $<$ *generalized holonomy orderings*. In algebraic point of view, this is a construction of bi-ordering of $\pi_1(E)$ from bi-orderings of $\pi_1(F)$ and $\pi_1(B)$ and is a special case of a group extension construction described in Lemma 2.1.1.

**Example 2.4.2** (The fundamental groups of fiber-type arrangements [27]). Recall that a hyperplane arrangement $\mathcal{A} = \{H_i\}$ of $\mathbb{C}^n$ is called a *fiber-type arrangement* if its complement $M_{\mathcal{A}} = \mathbb{C}^n - \bigcup_{H\in\mathcal{A}} H$ has a tower of fibrations

$$M_{\mathcal{A}} = M_n \overset{p_n}{\to} M_{n-1} \overset{p_{n-1}}{\to} \cdots \overset{p_2}{\to} M_1 = \mathbb{C} - \{0\}$$

where each fiber $F_k$ of $p_k$ is homeomorphic to $\mathbb{C} - \{\text{finite points}\}$. See [14] for the precise definition. The braid arrangement is a typical example of fiber-type arrangement. It is known that for each fibration $p_k$, the action of $\pi_1(M_{k-1})$ on $H_1(F_k)$ is trivial [14]. In particular, each fibration is quasi-nilpotent.

Thus, by using the tower of quasi-nilpotent fibrations we can construct a bi-ordering basis of $R_{M_{\mathcal{A}}} = \widehat{\mathbb{R}\pi_1(M_{\mathcal{A}})}$, and bi-ordering of $\pi_1(M_{\mathcal{A}})$.

On the other hand, in [27] Kim-Rolfsen constructed a bi-ordering of $\pi_1(M_{\mathcal{A}})$ by using the tower of quasi-nilpotent fibrations and the group extension construction (Lemma 2.1.1). Thus Kim-Rolfsen's bi-ordering can be seen as a generalized holonomy ordering.

# Chapter 3

# Bi-orderability of fibered 3-manifold groups

In this chapter we study the bi-orderability problem for fibered 3-manifold groups, by using the twisted Alexander polynomials.

Our starting point is the result of Perron-Rolfsen and Clay-Rolfsen.

**Theorem 3.0.3** (Alexander polynomial criterion for the bi-orderability). *Let $M$ be a fibered 3-manifold and $\phi : \pi_1(M) \to \mathbb{Z}$ be a surjective homomorphism induced by a fibration map $M \to S^1$.*

1. *([37], [38]) If* all *roots of the Alexander polynomial $\Delta_M^\phi(t)$ are positive real, then $\pi_1(M)$ is bi-orderable.*

2. *([8]) If $\pi_1(M)$ is bi-orderable, then the Alexander polynomial $\Delta_M^\phi(t)$ has* at least one *positive real root.*

These results suggest that the Alexander polynomial is a useful tool to study the bi-orderability of fibered 3-manifold groups.

For a finite dimensional linear representation of the fundamental group, a generalization of the Alexander polynomial called the *twisted Alexander polynomial* is defined. We will use finite-dimensional representations over $\mathbb{Q}$ having finite image, which we simply call a *finite representation.*

Many results on 3-manifolds using the classical Alexander polynomial are generalized by using the twisted Alexander polynomials for finite representation. In most cases generalized arguments are stronger than the original one. For example, the twisted Alexander polynomials for finite representations give a necessary and sufficient condition for 3-manifolds to be fibered [17]. Moreover, the twisted Alexander polynomials are rather easily calculated, hence they provides practical a method to show for given 3-manifolds to have or not to have certain properties. Thus it is natural to try to generalize Clay-Rolfsen's result for the twisted Alexander polynomials.

However, unfortunately the main result in this section is negative:
*The twisted Alexander polynomial for finite representations cannot be used to strengthen Clay-Rolfsen's obstruction.* See Theorem 3.2.1 and Corollary 3.2.1 for precise statement.

## 3.1 The maximal ordered abelian quotient

In this section we study the *maximal ordered abelian quotient* of bi-ordered groups, which is an analog of the maximal abelian quotient which takes into account of the bi-ordering structure. First of all we introduce the *convex commutator subgroup*, which is a refinement of the commutator subgroup in the category of bi-ordered groups.

**Definition 3.1.1.** The *convex commutator subgroup* of a bi-ordered group $(G, <_G)$ is a convex subgroup $C_G$ defined as the intersection of all convex subgroups of $(G, <_G)$ which contain the commutator subgroup $[G, G]$.

Since the intersection of convex subgroups is a convex subgroup, $C_G$ is the minimal convex subgroup of $(G, <_G)$ which contains the commutator subgroup $[G, G]$.

**Lemma 3.1.1.** *Let $(G, <_G)$, $(H, <_H)$ be bi-ordered groups and $\theta : (G, <_G) \to (H, <_H)$ be an order-preserving homomorphism.*

1. $\theta(C_G) \subset C_H$.
2. *$C_G$ is a normal subgroup of $G$.*

*Proof.* Let $x \in C_G$. Then there is $c, c' \in [G, G]$ such that $c <_G x <_G c'$. Since $\theta$ preserves bi-orderings, $\theta(c) <_H \theta(x) <_H \theta(c')$. $\theta(c), \theta(c') \in [H, H]$, so we conclude $\theta(x) \in C_H$. To show (ii), observe that every inner automorphism of $G$ preserves the bi-ordering $<_G$. Hence by (i), $C_G$ is preserved by all inner automorphisms, so $C_G$ is normal. □

Now we are ready to define the maximal ordered abelian quotient.

**Definition 3.1.2.** The *maximal ordered abelian quotient group* of a bi-ordered group $(G, <_G)$ is an ordered quotient group

$$A(G, <_G) = (G/C_G, <_{G/C_G}).$$

$A(G, <_G)$ plays a role similar to the maximal abelian quotient $G/[G, G]$ in the category of bi-ordered groups.

**Lemma 3.1.2.** *Let $(G, <_G)$, $(H, <_H)$ be bi-ordered groups and $\theta : (G, <_G) \to (H, <_H)$ be an order-preserving homomorphism.*

(i) *$\theta$ induces an order-preserving homomorphism $\theta_*^A : A(G, <_G) \to A(H, <_H)$.*

*(ii) Let $p_G : H_1(G;\mathbb{Z}) = G/[G,G] \to G/C_G$ be the natural projection, and let $V_G$ be the kernel of $p_G \otimes id_Q : H_1(G;\mathbb{Q}) \to A(G,<_G)\otimes\mathbb{Q}$. Then $\theta$ induces a $\mathbb{Q}$-linear map $\theta_*^V : V_G \to V_H$, and we have a commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & V_G & \longrightarrow & H_1(G;\mathbb{Q}) & \longrightarrow & A(G,<_G)\otimes\mathbb{Q} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \theta_*^V} & & \downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \theta_*^A} & & \\ 1 & \longrightarrow & V_H & \longrightarrow & H_1(H;\mathbb{Q}) & \longrightarrow & A(H,<_H)\otimes\mathbb{Q} & \longrightarrow & 1 \end{array}$$

*(iii) If $G$ is finitely generated, then $A(G,<_G)$ is non-trivial.*

*Proof.* (i) and (ii) are direct consequences of Lemma 3.1.1. (iii) follows from [8, Lemma 2.2], which asserts that for a finitely generated $G$ there exists the maximal proper convex subgroup $C$ of $(G,<_G)$, and that $G/C$ is abelian. Since $C \supset C_G$, this implies $A(G,<_G)$ is non-trivial. □

Recall the commutator $[a,b] = a^{-1}b^{-1}ab$ satisfies the commutator identities

$$[a,bc] = [a,c][a,b][[a,b],c], \quad [ab,c] = [a,c][[a,c],b][b,c].$$

**Lemma 3.1.3.** *Let $(G,<_G)$ be a bi-ordered group and $a,b \in G$.*

1. *$[a,b] <_G b$ if $b >_G 1$ and $[a,b] >_G b$ if $b <_G 1$.*
2. *$[a,b] >_G a^{-1}$ if $a >_G 1$ and $[a,b] <_G a^{-1}$ if $a <_G 1$.*
3. *If $[a,b] >_G 1$, then $[a^n,b^m] >_G [a,b]$ holds for $m,n > 1$.*

*Proof.* Proof of (i) and (ii) are routine. To show (iii), first we show $[a,b^m] >_G [a,b]$ by induction on $m$. By (ii), we have $[[a,b],b^{m-1}] >_G [a,b]^{-1}$. From inductive hypothesis $[a,b^{m-1}] >_G [a,b]$ holds. Thus by commutator identity

$$[a,b^m] = [a,b^{m-1}][a,b][[a,b],b^{m-1}] >_G [a,b][a,b][a,b]^{-1} = [a,b].$$

Similarly, by induction on $n$ we get an inequality $[a^n,b] >_G [a,b]$. These two inequalities give the desired inequality. □

Now we show the main result in this section, which is interseting in its own right.

**Theorem 3.1.1.** *Let $(G,<_G)$ be a bi-ordered group and $H$ be a finite index subgroup of $G$. Let $<_H$ be a bi-ordering of $H$ defined by the restriction of the ordering $<_G$ to $H$. Then the natural inclusion map of bi-ordered groups $i : (H,<_H) \hookrightarrow (G,<_G)$ induces an isomorphism of $\mathbb{Q}$-vector space*

$$i_* : A(H,<_H)\otimes\mathbb{Q} \xrightarrow{\cong} A(G,<_G)\otimes\mathbb{Q}.$$

Theorem 3.1.1 shows that the behavior of the rank of the maximal ordered abelian quotient is different from the behavior of the rank of the usual maximal abelian quotient. To illustrate this point more precisely, we consider finite index subgroups of a free group. Let $F = F_n$ be the free group of rank $n(> 1)$, and $<_F$ be a bi-ordering of $F$. Consider a proper, finite index subgroup $H$ of $F$, and let $<_H$ be the restriction of the bi-ordering $<_F$ to $H$. Then $H$ is a free group of rank $[F : H](n-1)+1$, hence for the rank of the maximal abelian quotient, we have a strict inequality

$$\mathrm{rank}_{\mathbb{Q}}(H/[H,H]) \otimes \mathbb{Q} = [F : H](n-1)+1 > n = \mathrm{rank}_{\mathbb{Q}}(F/[F,F]) \otimes \mathbb{Q}.$$

On the other hand, by Theorem 3.1.1, for the rank of the maximal ordered abelian quotient, we always have an equality

$$\mathrm{rank}_{\mathbb{Q}} A(H, <_H) \otimes \mathbb{Q} = A(F, <_F) \otimes \mathbb{Q}.$$

Thus, the difference between the rank of the maximal abelian quotient and that of the maximal ordered abelian quotient can be arbitrary large.

*Proof of Theorem 3.1.1.* Let $C$ be the intersection of all convex subgroups of $(G, <_G)$ which contains $[H, H]$. Observe that $<_G$-convex hull of $C_H$, that is, the subset $C'$ of $G$ defined by

$$C' = \{g \in G \mid \exists h, h' \in C_H,\ h <_G g <_G h'\}$$

is a convex subgroup of $(G, <_G)$. Thus, $C' = C$ and $C_H = C \cap H$. Since $[G : H]$ is finite, there is a finite integer $N > 0$ such that $g^N \in H$ for all $g \in G$. Thus by Lemma 3.1.3 (iii), for any commutator $[g, g']$ of elements in $G$, we can find a commutator $[h, h']$ of elements in $H$ that satisfies the inequality

$$[h, h']^{-1} <_G [g, g'] <_G [h, h'].$$

This shows that any convex subgroup of $G$ containing $[H, H]$ must contain $[G, G]$ as well. Therefore $C = C_G$, hence we get $C_H = C_G \cap H$.

Next observe that $HC_G$ is a finite index subgroup of $G$, hence $HC_G/C_G$ is a finite index subgroup of $G/C_G$. Thus, as $\mathbb{Q}$-vector spaces we have an isomorphism $(HC_G/C_G) \otimes \mathbb{Q} \cong (G/C_G) \otimes \mathbb{Q}$. Therefore we get a sequence of isomorphisms of $\mathbb{Q}$-vector spaces

$$(H/C_H) \otimes \mathbb{Q} \cong (H/C_G \cap H) \otimes \mathbb{Q} \cong (HC_G/C_G) \otimes \mathbb{Q} \cong (G/C_G) \otimes \mathbb{Q}.$$

□

## 3.2 The twisted Alexander polynomials criterion

In this section we show that the twisted Alexander polynomials only provides the same obstruction of the bi-orderability as the classical Alexander polynomial.

### 3.2.1 Twisted Alexander polynomial

We review the definition and basic properties of the twisted Alexander polynomials. For details, see [18].

Let $\phi : \pi_1(M) \to \mathbb{Z} = \langle t \rangle$ be a non-trivial homomorphism and $V_\alpha$ be a finite dimensional left $\mathbb{Q}\pi_1(M)$-module defined by the presentation $\alpha : \pi_1(M) \to \mathrm{GL}(V_\alpha)$. Recall we say $\alpha$ is a *finite representation* if the image of $\alpha$ is finite. The classical Alexander polynomial is obtained as the twisted Alexander polynomial for the trivial representation $\varepsilon : \pi_1(M) \to \mathrm{GL}(\mathbb{Q}) = \mathbb{Q}$.

Let $\alpha \otimes \phi : \pi_1(M) \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}])$ be the product representation given by $[\alpha \otimes \phi](g) : v \otimes t^i \mapsto [\alpha(g)](v) \otimes t^{i+\phi(g)}$. Then $V_{\alpha\otimes\phi}$ is a left $\mathbb{Q}\pi_1(M)$-module and the action of $\mathbb{Q}\pi_1(M)$ commutes with the right action of $\mathbb{Q}[t, t^{-1}]$.

The *i-th twisted Alexander module* is the $\mathbb{Q}[t, t^{-1}]$-module defined by the $i$-th twisted coefficient homology group

$$H_i(M; V_{\alpha\otimes\phi}) = H_i(C_*(\widetilde{M}) \otimes_{\mathbb{Q}\pi_1(M)} V_{\alpha\otimes\phi})$$

where $C_*(\widetilde{M})$ is the singular chain complex of the universal cover $\widetilde{M}$ of $M$, viewed as a right $\mathbb{Q}\pi_1(M)$-module.

Since $H_i(M; V_{\alpha\otimes\phi})$ is a finitely generated $\mathbb{Q}[t, t^{-1}]$-module, there exist Laurant polynomials $p_j(t) \in \mathbb{Q}[t, t^{-1}]$ and an isomorphism as $\mathbb{Q}[t, t^{-1}]$-module

$$H_i(M; V_{\alpha\otimes\phi}) \cong \mathbb{Q}[t, t^{-1}]^k \oplus \bigoplus_{j=1}^{m} \mathbb{Q}[t, t^{-1}]/(p_j(t)).$$

The elements $p_j(t)$ are well-defined up to multiplication by a unit of $\mathbb{Q}[t, t^{-1}]$ if we add the condition that $p_j(t)$ divides $p_{j+1}(t)$ for each $j$. The *i-th twisted Alexander polynomial* $\Delta_{M,i}^{\alpha\otimes\phi}$ is a Laurant polynomial defined by

$$\Delta_{M,i}^{\alpha\otimes\phi}(t) = \begin{cases} \prod_{j=1}^{m} p_j(t) & k = 0 \\ 0 & k \neq 0. \end{cases}$$

They are well-defined up to multiplication by a unit of $\mathbb{Q}[t, t^{-1}]$. In particular, the (non-zero) roots of the twisted Alexander polynomial are well-defined. For 3-manifolds, we only use the 1st twisted Alexander polynomial and simply denote the 1st twisted Alexander polynomial by $\Delta_M^{\alpha\otimes\phi}$.

We use the following properties of the twisted Alexander polynomials.

**Lemma 3.2.1.** *Let $d \cdot \phi : \pi_1(M) \to \mathbb{Z} = \langle t \rangle$ be the homomorphism defined by $d \cdot \phi(g) = t^{\phi(g)\cdot d}$. Then we have an equality*

$$\Delta_M^{\alpha\otimes d\cdot\phi}(t) = \Delta_M^{\alpha\otimes\phi}(t^d).$$

**Lemma 3.2.2.** *Let $\alpha : \pi_1(M) \to GL(V)$, $\beta : \pi_1(M) \to GL(W)$ be finite dimensional representations, and $\alpha \oplus \beta : \pi_1(M) \to GL(V \oplus W)$ be its direct sum. Then we have an equality*

$$\Delta_M^{(\alpha\oplus\beta)\otimes\phi}(t) = \Delta_M^{\alpha\otimes\phi}(t) \cdot \Delta_M^{\beta\otimes\phi}(t).$$

Now we prove (a special case of) Shapiro's lemma for the twisted Alexander polynomials. Let $G$ be a finite group and $p : \widetilde{M_G} \to M$ be a $G$-covering of $M$ corresponding to a surjective homomorphism $f : \pi_1(M) \to G$. For a non-trivial homomorphism $\phi : \pi_1(M) \to \mathbb{Z}$, let $p^*\phi : \pi_1(\widetilde{M_G}) \to \mathbb{Z}$ be its pull-back. Let $\alpha = \iota \circ f : \pi_1(M) \to \mathrm{GL}(\mathbb{Q}G)$ be a finite representation where $\iota : G \to \mathrm{GL}(\mathbb{Q}G)$ be the regular representation of $G$. We call such a finite representation a *regular finite representation.*

**Lemma 3.2.3** ((Shapiro's Lemma for twisted Alexander polynomials [18])). *Let $\alpha : \pi_1(M) \to GL(\mathbb{Q}G)$ be a regular finite representation. Then there is an equality*

$$\Delta_M^{\alpha\otimes\phi}(t) = \Delta_{\widetilde{M_G}}^{p^*\phi}(t)$$

*where $\Delta_{\widetilde{M_G}}^{p^*\phi}$ is the classical Alexander polynomial of $\widetilde{M_G}$ with respect to $p^*\phi$.*

*Proof.* Let $\widetilde{M}$ be the common universal covering of $M$ and $\widetilde{M_G}$. We have the isomorphisms of the chain complexes

$$\begin{aligned} C_*(\widetilde{M}) \otimes_{\mathbb{Q}\pi_1(\widetilde{M_G})} V_{\alpha\otimes p^*\phi} &= C_*(\widetilde{M}) \otimes_{\mathbb{Q}\pi_1(M)} (\mathbb{Q}\pi_1(M) \otimes_{\mathbb{Q}\pi_1(\widetilde{M_G})} V_{p_*\alpha\otimes\phi}) \\ &= C_*(\widetilde{M}) \otimes_{\mathbb{Q}\pi_1(M)} V_{p_*\alpha\otimes\phi}. \end{aligned}$$

Hence we obtain isomorphisms of the twisted Alexander modules and the desired equality of the twisted Alexander polynomials. □

### 3.2.2 Getting stronger criterion from classical Alexander polynomials

Assume that we have a statement of the form *"If a 3-manifold $M$ has a property $X$, then its Alexander polynomial has a property $Y$"*. Then we extend the argument for the twisted Alexander polynomials for finite representation according to the following strategy, as in [16].

1. Consider the twisted Alexander polynomials for regular finite representations. By Shapiro's lemma, these twisted Alexander polynomials are the classical Alexander polynomial of finite coverings.

2. Verify that the property $X$ of 3-manifolds we are considering is preserved by taking finite coverings. (For example, the property that the fundamental group is bi-orderable is preserved by taking any coverings.)

3. By the classical Alexander polynomial argument, the twisted Alexander polynomial for a regular finite representation has the property $Y$.

4. Consider the irreducible decomposition of the regular representation and the corresponding factorization of the twisted Alexander polynomials (See Remark 3.2.1 below).

5. Study how the property $Y$ behaves under the factorization and obtain the property of twisted Alexander polynomials. For example, if each factor of the polynomial also has the property $Y$, then we conclude that every twisted Alexander polynomial for finite representation has the property $Y$.

**Remark 3.2.1.** By Maschke's theorem, representations of a finite group $G$ over $\mathbb{Q}$ are completely reducible and each irreducible representation appears as an irreducible summand of the regular representation. Thus by Lemma 3.2.2, we can obtain all twisted Alexander polynomials for finite representations from the twisted Alexander polynomials for regular finite representations.

### 3.2.3 Review of Clay-Rolfsen's argument

Now we review Clay-Rolfsen's argument. Throughout the rest of this paper, we always assume that 3-manifold $M$ is fibered, and a homomorphism $\phi : \pi_1(M) \to \mathbb{Z}$ is derived from a fibration map $M \to S^1$. We also denote the monodromy map by $\theta : \Sigma \to \Sigma$, and put $F = \pi_1(\Sigma)$.

The starting point of Clay-Rolfsen's argument is the following well-known fact of an HNN-extension of bi-orderable groups.

**Lemma 3.2.4.** *Let $H$ be a bi-orderable group and $G$ be an HNN-extension of $H$ by the automorphism $\phi : H \to H$. Then $G$ is bi-orderable if and only if there exists a bi-ordering of $H$ which is preserved by $\phi$.*

By Lemma 3.2.4, if $\pi_1(M)$ is bi-orderable, there exists a bi-ordering $<_F$ of $F$ which is invariant under the monodromy $\theta_* : F \to F$. So $\theta$ induces an order-preserving map $\theta_*^A : A(F, <_F) \to A(F, <_F)$ by Lemma 3.1.1. Let $\chi_A(t)$ be the characteristic polynomial of $\theta_*^A \otimes id_{\mathbb{Q}}$. The key lemma shown in [8] is the following.

**Lemma 3.2.5.** *Let $(A, <_A)$ be a bi-ordered abelian group of finite rank and $\theta : A \to A$ be an order-preserving automorphism. Then the $\mathbb{Q}$-linear map $\theta \otimes id_{\mathbb{Q}} : A \otimes \mathbb{Q} \to A \otimes \mathbb{Q}$ has at least one positive real eigenvalue.*

Thus, $\chi_A(t)$ has at least one positive real root. Recall that the classical Alexander polynomial $\Delta_M^{\phi}(t)$ is equal to the characteristic polynomial of $\theta_* : H_1(F;\mathbb{Q}) \to H_1(F;\mathbb{Q})$. By Lemma 3.1.2, $\Delta_M^{\phi}(t)$ divides $\chi_A(t)$. Therefore we conclude that $\Delta_M^{\phi}(t)$ has at least one positive real root.

Thus, the important part of the Alexander polynomial which contains information about bi-ordering (Clay-Rolfsen's obstruction) is not the Alexander polynomial itself, but its factor $\chi_A(t)$, the characteristic polynomial of the monodromy map $\theta_*^A \otimes id_{\mathbb{Q}}$ induced on the maximal ordered abelian quotient. We say this factor of the Alexander polynomial the *essential factor*.

### 3.2.4 The failure of twisted Alexander polynomial argument

Now we are ready to show that we cannot get better obstruction by using the twisted Alexander polynomials for finite representations. According to the strategy described in Section 3.2.2, we expect the following generalization of Clay-Rolfsen's obstruction.

**Expected generalization 1.** *Let $\rho : \pi_1(M) \to GL(V)$ be a finite representation. If $\pi_1(M)$ is bi-orderable, then the twisted Alexander polynomial $\Delta_M^{\rho\otimes\phi}(t)$ has at least one positive real root.*

**Expected generalization 2.** *Let $\rho : \pi_1(M) \to GL(\mathbb{Q}G)$ be a regular finite presentation. If $\pi_1(M)$ is bi-orderable, then the twisted Alexander polynomial $\Delta_M^{\rho\otimes\phi}(t)$ has more positive real roots than the classical Alexander polynomial.*

However these expected generalizations are false as the following simple example shows.

**Example 3.2.1** (The simplest counter example). Let $K$ be the figure-eight knot. $K$ is fibered, and its Alexander polynomial $\Delta_K(t) = t^2 - 3t + 1$ has two positive real roots. Thus, $\pi_1(S^3 - K)$ is bi-orderable by Perron-Rolfsen's criterion [37]. Let $a_2 : \pi_1(S^3 - K) \to \mathbb{Z}_2$ be the mod 2 abelianization map.

First let us consider the alternating representation $alt : \pi_1(S^3 - K) \to \mathrm{GL}(\mathbb{Q})$, defined by $alt(g)(1) = a_2(g) \cdot 1$. Then its twisted Alexander polynomial $\Delta_K^{alt\otimes\phi}$ is equal to $(t^2 + 3t + 1)$, hence it has no positive real root. Next let us consider the regular finite representation $\alpha = \pi_1(S^3 - K) \to \mathrm{GL}(\mathbb{Q}\mathbb{Z}_2)$. Then $\Delta_K^{\alpha\otimes\phi} = (t^2+3t+1)(t^2-3t+1)$, which has exactly the same positive real roots as the classical Alexander polynomial.

Now we show that these "expected generalizations" are impossible. First of all, we clarify the essential factor of the twisted Alexander polynomials which contains the information of bi-orderability.

Let $M$ be a fibered 3-manifold whose fundamental group $\pi_1(M)$ has a bi-ordering $<_M$. Let $\alpha : \pi_1(M) \to G \to \mathrm{GL}(\mathbb{Q}G)$ be a regular finite representation.

Let $p : \widetilde{M} \to M$ be the corresponding $G$-cover and $\widetilde{\phi} : \pi_1(\widetilde{M}) \to \mathbb{Z}$ be the surjective homomorphism induced by the induced fibration $\widetilde{M} \to S^1$. Then there exists an integer $d$ such that $p^*\phi = d \cdot \widetilde{\phi}$. Let $\Sigma$, $\widetilde{\Sigma}$ be the fiber of $M$, $\widetilde{M}$ and put $F = \pi_1(\Sigma)$, $\widetilde{F} = \pi_1(\widetilde{\Sigma})$ respectively. $\widetilde{\Sigma}$ is a regular finite covering of $\Sigma$, so $\widetilde{F}$ is normal subgroup of $F$ having finite index in $F$. We regard all of $F$, $\widetilde{F}$ and $\pi_1(\widetilde{M})$ as subgroups of $\pi_1(M)$. Let $<_F, <_{\widetilde{F}}$ be the restrictions of the bi-ordering $<_M$ to $F, \widetilde{F}$ respectively. By Lemma 3.2.5, the orderings $<_F$, $<_{\widetilde{F}}$ are preserved by the monodromy map $\theta$, $\widetilde{\theta}$ respectively. By definition of $d$, the monodromy $\widetilde{\theta}$ is a lift of $\theta^d$.

By Lemma 3.2.3, the twisted Alexander polynomial associated to $\alpha$ is given as the classical Alexander polynomial

$$\Delta_M^{\alpha\otimes\phi}(t) = \Delta_{\widetilde{M}}^{p^*\phi}(t).$$

On the other hand, since $p^*\phi = d \cdot \widetilde{\phi}$, by Lemma 3.2.1 we get an equality

$$\Delta_M^{\alpha\otimes\phi}(t) = \Delta_{\widetilde{M}}^{p^*\phi}(t) = \Delta_{\widetilde{M}}^{\widetilde{\phi}}(t^d).$$

For fibered 3-manifolds, the classical Alexander polynomial is equal to the characteristic polynomial of the monodromy. So finally we get a description of the twisted Alexander polynomial as the characteristic polynomial of the monodromy $\widetilde{\theta}$,

$$\Delta_M^{\alpha\otimes\phi}(t) = \Delta_{\widetilde{M}}^{\widetilde{\phi}}(t^d) = \det(t^d I - \widetilde{\theta}_*)$$

By Clay-Rolfsen's argument reviewed in Section 3.3, the factor of $\Delta_M^{\alpha\otimes\phi}(t)$ which leads Clay-Rolfsen's obstruction is $\det(t^d - (\widetilde{\theta}_*^A \otimes id_{\mathbb{Q}}))$, the factor derivied from the linear map

$$\widetilde{\theta}_*^A \otimes id_{\mathbb{Q}} : A(\widetilde{F}; <_{\widetilde{F}}) \otimes \mathbb{Q} \to A(\widetilde{F}; <_{\widetilde{F}}) \otimes \mathbb{Q}.$$

We denote this by $\chi_{\widetilde{A}}(t)$ and call it the *essential factor* of the twisted Alexander polynomial.

Now we show that the essential factors for the classical and the twisted Alexander polynomial are essentially the same.

**Theorem 3.2.1.** *Let $M$ be a fibered 3-manifold whose fundamental group is bi-orderable, and let $\alpha : \pi_1(M) \to GL(\mathbb{Q}G)$ be a regular finite representation. Then the essential factor $\chi_{\widetilde{A}}(t)$ of the twisted Alexander polynomial $\Delta_M^{\alpha\otimes\phi}(t)$ is given by*

$$\chi_{\widetilde{A}}(t) = \det(t^d - (\theta_*^A \otimes id_{\mathbb{Q}})^d).$$

*where $d$ is an integer determined by the representation $\alpha$.*

*Proof.* Since $\widetilde{F}$ is a finite index subgroup of $F$, by Theorem 3.1.1 we get a commutative diagram

$$\begin{array}{ccc} A(\widetilde{F}; <_{\widetilde{F}}) \otimes \mathbb{Q} & \xrightarrow{\widetilde{\theta}_*^A \otimes id_{\mathbb{Q}}} & A(\widetilde{F}; <_{\widetilde{F}}) \otimes \mathbb{Q} \\ \downarrow \cong & & \cong \downarrow \\ A(F; <_F) \otimes \mathbb{Q} & \xrightarrow{(\theta_*^A \otimes id_{\mathbb{Q}})^d} & A(F; <_F) \otimes \mathbb{Q} \end{array}$$

where the vertical arrows are isomorphisms. Therefore, we obtain the desired equality

$$\chi_{\widetilde{A}}(t) = \det(t^d - (\widetilde{\theta}_*^A \otimes id_{\mathbb{Q}})) = \det(t^d - (\theta_*^A \otimes id_{\mathbb{Q}})^d).$$

□

This technical result provides the following negative result for an extension of Clay-Rolfsen's obstruction.

**Corollary 3.2.1.** *Let $M$ be a fibered 3-manifold whose fundamental group is bi-orderable. Then compared with the classical Alexander polynomial, the twisted Alexander polynomial for a finite representation contains the same or less information about bi-ordering of $\pi_1(M)$ that gives rise to Clay-Rolfsen's obstruction. In other words, the twisted Alexander polynomial for finite representations cannot be used to strengthen Clay-Rolfsen's obstruction for the bi-orderbility.*

*Proof.* Let $\beta : \pi_1(M) \xrightarrow{f} G \to \mathrm{GL}(V)$ be a finite representation, where $f$ is a surjective homomorphism. We consider the corresponding regular finite representation $\alpha : \pi_1(M) \xrightarrow{f} G \to \mathrm{GL}(\mathbb{Q}G)$.

By Lemma 3.2.2 and Remark 3.2.1, the twisted Alexander polynomial for the finite representation $\beta$, $\Delta_M^{\beta\otimes\phi}(t)$ is a factor of $\Delta_M^{\alpha\otimes\phi}(t)^k$ for some integer $k$ determined by $\beta$. Thus, as for the root of the twisted Alexander polynomials, the twisted Alexander polynomial for the regular finite representation $\alpha$ contains at least as much information as the twisted Alexander polynomial for the representation $\beta$.

However, by Theorem 3.2.1, the essential factor of the twisted Alexander polynomial $\Delta_M^{\alpha\otimes\phi}(t)$ is determined by $\alpha$ and the essential factor of the classical Alexander polynomial $\Delta_M^{\phi}(t)$. Therefore the twisted Alexander polynomial $\Delta_M^{\beta\otimes\phi}(t)$ contains at most as much information as the classical Alexander polynomial with respect to Clay-Rolfsen's obstruction. $\square$

# Chapter 4

# Dehornoy-like ordering

In this chapter we study a special class of left-orderings called *Dehornoy-like ordering*. A Dehornoy-like ordering is a left-ordering defined in a similar way to the Dehornoy ordering. We introduce a property called *Property $F$* for an ordered finite generating set $\mathcal{S}$ of a group $G$. We show that Property $F$ allows us to relate Dehornoy-like orderings and isolated orderings in a simple way. Moreover, using Property $F$ we generalize various known properties of the Dehornoy ordering to Dehornoy-like orderings.

In the latter half of this chapter we construct a new example of Dehornoy-like and isolated orderings and study their detailed properties. Our examples are generalizations of Navas' example of Dehornoy-like and isolated orderings [35]. We will also give a negative answer to the main question of [35], which asks the characterization of groups having isolated ordering whose positive cone is generated by two elements.

## 4.1 Dehornoy-like orderings

### 4.1.1 The definition of Dehornoy-like orderings

Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be an ordered finite generating set of $G$. We consider the two sub-semigroups of $G$, the *$\mathcal{S}$-word positive semigroup* and the *$\sigma(\mathcal{S})$-positive semigroup*.

A ($\mathcal{S}$-) positive word is a non-empty word on $\mathcal{S}$. We say an element $g \in G$ is ($\mathcal{S}$-) *word positive* or simply *$\mathcal{S}$-positive* if $g$ is represented by a $\mathcal{S}$-positive word. The set of all $\mathcal{S}$-word positive elements form a sub-semigroup $P_{\mathcal{S}}$ of $G$, which we call the ($\mathcal{S}$-)*word positive semigroup*. The $\mathcal{S}$-word positive semigroup is nothing but a sub-semigroup of $G$ generated by $\mathcal{S}$.

To define a Dehornoy-like ordering, we introduce slightly different notions. A word $w$ on $\mathcal{S} \cup \mathcal{S}^{-1}$ is called *$i$-positive* (or, *$i(\mathcal{S})$-positive*, if we need to indicate $\mathcal{S}$) if $w$ contains at least one $s_i$ but contains no $s_1^{\pm 1}, \ldots, s_{i-1}^{\pm 1}, s_i^{-1}$. We say an element $g \in G$ is *$i$-positive* (or, *$i(\mathcal{S})$-positive*) if $g$ is represented by an $i$-positive word. An element $g \in G$ is called *$\sigma$-positive* (*$\sigma(\mathcal{S})$-positive*) if $g$ is $i$-positive for some $1 \leq i \leq n$. The notions of *i-negative* and *σ-negative* are defined in the similar way. The set of $\sigma(\mathcal{S})$-positive elements of $G$ forms a sub-semigroup $\Sigma_{\mathcal{S}}$ of $G$. We call the semigroup $\Sigma_{\mathcal{S}}$ the *σ-positive semigroup* (or, *$\sigma(\mathcal{S})$-positive semigroup*).

**Definition 4.1.1** (Dehornoy-like ordering). A *Dehornoy-like ordering* is a left ordering $<_D$ whose positive cone is equal to the $\sigma$-positive semigroup $\Sigma_{\mathcal{S}}$ for some ordered finite generating set $\mathcal{S}$ of $G$. In this situation, we say $\mathcal{S}$ *defines* a Dehornoy-like ordering $<_D$.

As we have already mentioned, the definition of Dehornoy-like orderings is motivated from the Dehornoy ordering of the braid groups: A standard Artin generator $\mathcal{S} = \{\sigma_1, \ldots, \sigma_{n-1}\}$ of the braid group $B_n$ defines the Dehornoy-ordering $<_D$.

To study Dehornoy-like orderings we introduce the following two properties, which are also motivated from the theory of the Dehornoy ordering.

**Definition 4.1.2.** Let $\mathcal{S}$ be an ordered finite generating set of $G$.

1. We say $\mathcal{S}$ has *Property A* (*the Acyclic property*) if no $\sigma(\mathcal{S})$-positive words represent the trivial element. That is, $\Sigma_{\mathcal{S}}$ does not contains the identity element 1.

2. We say $\mathcal{S}$ has *Property C* (*the Comparison property*) if every non-trivial element of $G$ admits either $\sigma(\mathcal{S})$-positive or $\sigma(\mathcal{S})$-negative word expression.

**Propotition 4.1.1.** *Let $\mathcal{S}$ be an ordered finite generating set of a group $G$. Then $\mathcal{S}$ defines a Dehornoy-like ordering if and only if $\mathcal{S}$ has both Property A and Property C.*

*Proof.* Property $C$ implies that $G = \Sigma_{\mathcal{S}} \cup \Sigma_{\mathcal{S}}^{-1} \cup \{1\}$, and the Property $A$ implies that $\Sigma_{\mathcal{S}}$, $\Sigma_{\mathcal{S}}^{-1}$ and $\{1\}$ are disjoint. Thus, the $\sigma(\mathcal{S})$-positive monoid $\Sigma_{\mathcal{S}}$ satisfies both **LO1** and **LO2**. Converse is clear. □

Now we introduce an operation to construct new ordered finite generating sets from an ordered finite generating set which connects a Dehornoy-like ordering and an isolated ordering.

The *twisted generating set* of $\mathcal{S}$ is an ordered finite generating set $\mathcal{A} = \mathcal{A}_{\mathcal{S}} = \{a_1, \ldots, a_n\}$ where each $a_i$ is defined by

$$a_i = (s_i \cdots s_{n-1})^{(-1)^{n-i+1}}.$$

An ordered finite generating set $\mathcal{D} = \mathcal{D}_{\mathcal{S}} = \{d_1, \ldots, d_n\}$ whose twisted generating set is equal to $\mathcal{S}$ is called the *detwisted generating set* of $\mathcal{S}$. The detwisted generating set $\mathcal{D}$ is given as

$$d_i = \begin{cases} s_n^{-1} & i = n \\ s_i^{-1} s_{i+1}^{-1} & n-i: \text{ even} \\ s_i s_{i+1} & n-i: \text{ odd} \end{cases}$$

For each $1 \leq i \leq n$, let $\mathcal{S}^{(i)} = \{s_i, s_{i+1}, \ldots, s_n\}$ and $G_{\mathcal{S}}^{(i)}$ be the subgroup of $G$ generated by $\mathcal{S}^{(i)}$. Thus, $\mathcal{S}^{(i)}$ is an ordered finite generating set of $G_{\mathcal{S}}^{(i)}$. We denote the $\mathcal{S}^{(i)}$-word positive semigroup and the $\sigma(\mathcal{S}^{(i)})$-positive semigroup by $P_{\mathcal{S}}^{(i)}$, $\Sigma_{\mathcal{S}}^{(i)}$ respectively. They are naturally regarded as sub-semigroup of $G_{\mathcal{S}}^{(i)}$. By definition of the twisted generating set, $\mathcal{A}_{\mathcal{S}^{(i)}} = (\mathcal{A}_{\mathcal{S}})^{(i)}$. Thus, $G_{\mathcal{S}}^{(i)} = G_{\mathcal{A}}^{(i)}$ so we will often write $G^{(i)}$ to represent $G_{\mathcal{S}}^{(i)} = G_{\mathcal{A}}^{(i)}$.

**Remark 4.1.1.** In some literature, the notation $G^{(i)}$ is used to represent the $i$-th derived subgroup of $G$. Our $G^{(i)}$ is not related to the derived subgroups, and as its definition shows, depends on a choice of generating set $\mathcal{S}$ of $G$.

There is an obvious inclusion for $\sigma(\mathcal{S})$-positive and $\mathcal{A}$-word positive monoids.

**Lemma 4.1.1.** *Let $\mathcal{S}$ be an ordered finite generating set and $\mathcal{A} = \mathcal{A}_{\mathcal{S}}$ be the twisted generating set of $\mathcal{S}$. Then $\Sigma_{\mathcal{S}} \cup \Sigma_{\mathcal{S}}^{-1} \supset P_{\mathcal{A}} \cup P_{\mathcal{A}}^{-1}$.*

*Proof.* We show $P_{\mathcal{A}} \subset \Sigma_{\mathcal{S}} \cup \Sigma_{\mathcal{S}}^{-1}$. The proof of $P_{\mathcal{A}}^{-1} \subset \Sigma_{\mathcal{S}} \cup \Sigma_{\mathcal{S}}^{-1}$ is similar. Let $g \in P_{\mathcal{A}}$ and $w$ be an $\mathcal{A}$-positive word expression of $g$. Put

$$i = \min\{j \in \{1, 2, \ldots, n\} \mid w \text{ contains the letter } a_j\}.$$

Since $a_i = (s_i s_{i+1} \cdots s_n)^{(-1)^{n-i+1}}$, $g$ is $\sigma(\mathcal{S})$-positive if $(n-i)$ is odd and is $\sigma(\mathcal{S})$-negative if $(n-i)$ is even. □

Now we introduce a key property called *Property F* (*the Filtration property*) which allows us to generalize various properties of the Dehornoy ordering for Dehornoy-like orderings.

**Definition 4.1.3.** Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be an ordered finite generating set of $G$ and $\mathcal{A} = \{a_1, \ldots, a_n\}$ be the twisted generating set of $\mathcal{S}$. We say $\mathcal{S}$ has *Property F* (*the Filtration property*) if

**F** $a_i \cdot (P_{\mathcal{A}}^{(i+1)})^{-1} \cdot a_i^{-1} \subset P_{\mathcal{A}}^{(i)}$, $a_i^{-1} \cdot (P_{\mathcal{A}}^{(i+1)})^{-1} \cdot a_i \subset P_{\mathcal{A}}^{(i)}$

hold for all $i$.

We say a finite generating set $\mathcal{A}$ *defines* an isolated ordering if the $\mathcal{A}$-word positive semigroup is the positive cone of an isolated ordering $<_{\mathcal{A}}$. First we show a Dehornoy-like ordering and an isolated ordering are closely related if we assume Property $F$.

**Theorem 4.1.1.** *Let $\mathcal{S}$ be an ordered finite generating set of $G$ having Property $F$ and $\mathcal{A}$ be the twisted generating set of $\mathcal{S}$. Then $\mathcal{S}$ defines a Dehornoy-like ordering if and only if $\mathcal{A}$ defines an isolated left ordering.*

*Proof.* Let $n$ be the cardinal of the generating set $\mathcal{S}$. We prove theorem by induction on $n$. The case $n = 1$ is trivial. General cases follow from the following two claims.

**Claim 4.1.1.** *$\mathcal{S}$ has Property C if and only if $P_{\mathcal{A}} \cup P_{\mathcal{A}}^{-1} \cup \{1\} = G$ holds.*

By Lemma 4.1.1, $G = P_{\mathcal{A}} \cup P_{\mathcal{A}}^{-1} \cup \{1\} \subset \Sigma_{\mathcal{S}} \cup \Sigma_{\mathcal{S}}^{-1} \cup \{1\} = G$.

To show the converse assume that $\mathcal{S}$ has Property $C$. Let $g \in G$ be a non-trivial element. We assume that $g$ is $\sigma(\mathcal{S})$-positive. The case $g$ is $\sigma(\mathcal{S})$-negative is proved in a similar way.

First of all, assume that $g$ has a $k(\mathcal{S})$-positive word representative for $k > 1$. Then $g \in G^{(2)}$ and $g$ is $\sigma(\mathcal{S}^{(2)})$-positive. By inductive hypothesis, $g \in P_{\mathcal{A}}^{(2)} \cup (P_{\mathcal{A}}^{(2)})^{-1} \cup \{1\} \subset P_{\mathcal{A}} \cup P_{\mathcal{A}}^{-1} \cup \{1\}$.

Thus we assume that $g$ is $1(\mathcal{S})$-positive. We also assume that $n$ is even. The case $n$ is odd is similar. Since $s_1 = a_1 a_2$, by rewriting a $1(\mathcal{S})$-positive word representative of $g$ by using the twisted generating set $\mathcal{A}$, we write $g$ as

$$g = V_0 a_1 V_1 \cdots a_1 V_m$$

where $V_i$ is a word on $\mathcal{A}^{(2)} \cup (\mathcal{A}^{(2)})^{-1} \subset G^{(2)}$. By inductive hypothesis, we may assume that either $V_i \in P_{\mathcal{A}}^{(2)}$ or $V_i \in (P_{\mathcal{A}}^{(2)})^{-1}$.. If all $V_i$ belong to $P_{\mathcal{A}}^{(2)}$, then $g \in P_{\mathcal{A}}$. Assume that some $V_i$ belongs to $(P_{\mathcal{A}}^{(2)})^{-1}$. By Property $F$, $a_1 V_i \subset P_{\mathcal{A}} \cdot a_1$ and $V_i a_1 \subset a_1 \cdot P_{\mathcal{A}}$, so we can rewrite $g$ so that it belongs to $P_{\mathcal{A}}$.

**Claim 4.1.2.** *$\mathcal{S}$ has Property $A$ if and only if $1 \notin P_{\mathcal{A}}$.*

Assume that $1 \notin P_{\mathcal{A}}$ and let $g \in G$ be a $\sigma(\mathcal{S})$-positive element. If $g$ has a $k(\mathcal{S})$-positive word representative for $k > 1$, then $g \in G^{(2)}$ so inductive hypothesis shows $g \neq 1$. Thus we assume $g$ is $1(\mathcal{S})$-positive. Assume that $n$ is even. Then as we have seen in the proof of Claim 4.1.1, $g \in P_{\mathcal{A}}$ so we conclude $g \neq 1$. The case $n$ is odd, and the case $g$ is $\sigma(\mathcal{S})$-negative are proved in a similar way. Converse is obvious from Lemma 4.1.1. □

**Corollary 4.1.1.** *Let $G$ be a left-orderable group. If $G$ has a Dehornoy-like ordering having Property $F$, then $G$ also has an isolated left ordering.*

### 4.1.2 Property of Dehornoy-like orderings

In this section we study fundamental properties of Dehornoy-like orderings and isolated orderings derived from the Dehornoy-like orderings.

Let $\mathcal{S} = \{s_1, \ldots, s_n\} (n > 1)$ be an ordered finite generating set of a group $G$ which defines a Dehornoy-like ordering $<_D$ and $\mathcal{A}$ be the twisted generating set of $\mathcal{S}$. First of all, we observe that a Dehornoy-like ordering have the following good properties with respect to the restrictions.

**Propotition 4.1.2.** *Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be an ordered finite generating set of $G$ which defines a Dehornoy-like ordering $<_D$.*

1. *For $1 \leq i \leq n$, $\mathcal{S}^{(i)}$ defines a Dehornoy-like ordering $<_D^{(i)}$ of $G^{(i)}$. Moreover, the restriction of the Dehornoy-like ordering $<_D$ to $G^{(i)}$ is equal to the Dehornoy-like ordering $<_D^{(i)}$.*

2. *For $1 \leq i \leq n$, the subgroup $G^{(i)}$ is $<_D$-convex. In particular, $<_D$ is discrete, and the minimal $<_D$-positive element is $s_n$.*

3. *If $H$ is a $<_D$-convex subgroup of $G$, then $H = G^{(i)}$ for some $1 \leq i \leq n$.*

*Proof.* Since $\mathcal{S}$ has Property $A$, $\mathcal{S}^{(i)}$ has Property $A$. Assume that $\mathcal{S}^{(i)}$ does not have Property $C$, so there is an element $g \in G^{(i)} - \{1\}$ which is neither $\sigma(\mathcal{S}^{(i)})$-positive nor $\sigma(\mathcal{S}^{(i)})$-negative. Assume that $1 <_D g$, so $g$ is represented by a $\sigma(\mathcal{S})$-positive word $W$. The case $1 >_D g$ is similar. Since $g \in G^{(i)}$ we may find a word representative $V$ of $g$ which consists of the alphabets in $\mathcal{S}^{(i)} \cup {\mathcal{S}^{(i)}}^{-1}$. Then $V^{-1}W$ is $\sigma(\mathcal{S})$-positive word which represents the trivial element, so this contradicts the fact that $\mathcal{S}$ has Property $A$. Thus, $\mathcal{S}^{(i)}$ has Property $C$, hence $\mathcal{S}^{(i)}$ defines a Dehornoy-like ordering $<_D^{(i)}$. Now the Property $A$ and Property $C$ of $\mathcal{S}^{(i)}$ implies $\Sigma_{\mathcal{S}^{(i)}} = \Sigma_{\mathcal{S}} \cap G_{\mathcal{S}}^{(i)}$, so $<_D^{(i)}$ is equal to the restriction of $<_D$ to $G_{\mathcal{S}}^{(i)}$.

Next we show $G^{(i)}$ is $<_D$-convex. Assume that $1 <_D h <_D g$ hold for $g \in G^{(i)}$ and $h \in G$. If $h$ is $j(\mathcal{S})$-positive for $j < i$, then $g^{-1}h$ is also $j(\mathcal{S})$-positive, so $g <_D h$. This contradicts the assumption, so $h$ must be $j(\mathcal{S})$-positive for $j \geq i$. This implies $h \in G^{(i)}$, so we conclude $G^{(i)}$ is $<_D$-convex.

To show there are no $<_G$-convex subgroups other than $G^{(i)}$, it is sufficient to show if $H \supset G^{(2)}$ then $H = G^{(2)}$ or $G^{(1)} = G$. Assume that $H \neq G^{(2)}$, hence $H$ contains an element $g$ in $G - G^{(2)}$. Let us take such $g$ so that $1 <_D g$ holds. Then $g$ must be $1(\mathcal{S})$-positive, hence we may write $g = hs_1P$ where $h \in G^{(2)}$ and $P >_D 1$. Then we have

$$1 <_D hs_1 \leq_D hs_1P = g$$

Since $H$ is convex, this implies $hs_1 \in H$. Since $h \in G^{(2)} \subset H$, we conclude $s_1 \in H$ hence $H = G$. □

From now on, we will always assume that $\mathcal{S}$ has Property $F$, hence $\mathcal{A}$ defines an isolated left ordering $<_A$. First of all we observe that $<_A$ also has the same properties as we have seen in Proposition 4.1.2.

**Propotition 4.1.3.** *Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be the twisted generating set of $\mathcal{S}$ which defines an isolated left ordering $<_A$.*

1. *For $1 \leq i \leq n$, $\mathcal{A}^{(i)}$ defines an isolated ordering $<_A^{(i)}$ of $G^{(i)}$. Moreover, the restriction of the isolated ordering $<_A$ to $G^{(i)}$ is equal to the isolated ordering $<_A^{(i)}$.*

2. *For $1 \leq i \leq n$, the subgroup $G^{(i)}$ is $<_A$-convex. In particular, $<_A$ is discrete, and the minimal $<_A$-positive element is $a_n$.*

3. *If $H$ is an $<_A$-convex subgroup of $G$, then $H = G^{(i)}$ for some $1 \leq i \leq n$.*

*Proof.* The proofs of (1) and (3) are similar to the case of Dehornoy-like orderings. To show (2), assume that $1 <_A h <_A g$ hold for $g \in G^{(i)}$ and $h \in G$. If $h \neq P_{\mathcal{A}}^{(i)}$, then we may write $h$ as $h = h'a_jw$ where $h' \in P_{\mathcal{A}}^{(j+1)}$ and $w \in P_{\mathcal{A}}^{(j)} \cup \{1\}$ for $j < i$. If $g^{-1}h' \in P_{\mathcal{A}}^{(j+1)}$, then $g^{-1}h = (g^{-1}h')a_jw >_A 1$. If $g^{-1}h' \in (P_{\mathcal{A}}^{(j+1)})^{-1}$, then by Property $F$, $(g^{-1}h')a_j >_A 1$ so $g^{-1}h = [(g^{-1}h')a_j]w >_A 1$. Therefore in both cases, $g^{-1}h >_A 1$, it is a contradiction.

□

To deduce more precise properties, we observe the following simple lemma.

**Lemma 4.1.2.** *If $a_{n-1}a_na_{n-1} \neq a_n$, then*

$$a_{n-1}a_n^{-1}a_{n-1}^{-1}, a_{n-1}^{-1}a_na_{n-1} \in P_A^{(n-1)} - P_A^{(n)}.$$

*Proof.* To make notations simple, we put $p = a_{n-1}$ and $q = a_n$.

By Property $F$, $pq^{-1}p^{-1}, p^{-1}q^{-1}p \in P_A^{(n-1)}$. We show $pq^{-1}p^{-1} \neq P_A^{(n)}$. The proof of $p^{-1}q^{-1}p \neq P_A^{(n)}$ is similar. Assume that $pq^{-1}p^{-1} = q^k$ for $k > 0$. Since we have assumed that $qpq \neq q$, $k > 1$. Then

$$q = p^{-1}(pq^{-1}p^{-1})^{-1}p = p^{-1}q^{-k}p = (p^{-1}q^{-1}p)^k,$$

so we have $1 <_A p^{-1}q^{-1}p <_A q$. This contradicts Proposition 4.1.3 (2), the fact that $q = a_n$ is the minimal $<_A$-positive element. □

Next we show that in most cases the Dehornoy-like ordering $<_D$ is not isolated in $LO(G)$, so it makes a contrast to the isolated ordering $<_A$. For $g \in G$ and a left ordering $<$ of $G$ whose positive cone is $P$, we define $<_g = < \cdot g$ as the left ordering defined by the positive cone $P \cdot g$. Thus, $x <_g x'$ if and only if $xg < x'g$. This defines a right action of $G$ on LO$(G)$. Two left orderings are said to be *conjugate* if they belong to the same $G$-orbit.

**Theorem 4.1.2.** *If $a_{n-1}a_na_{n-1} \neq a_n$, then the Dehornoy-like ordering $<_D$ is an accumulation point of the set of its conjugates $\{<_D \cdot g\}_{g\in G}$. Thus, $<_D$ is not isolated in $LO(G)$, and the $\sigma(\mathcal{S})$-positive monoid is not finitely generated.*

*Proof.* Our argument is a generalization of Navas-Wiest's criterion [36]. As in the proof of Lemma 4.1.2, we put $p = a_{n-1}$ and $q = a_n$ to make notation simple.

We construct a sequence of left orderings $\{<_n\}$ so that $\{<_n\}$ non-trivially converge to $<_D$ and that each $<_n$ is conjugate to $<_D$. Here the word non-trivially means that $<_n \neq <_D$ for sufficiently large $n > 0$.

Let $<_n = <_D \cdot (q^np)$. Thus, $1 <_n g$ if and only if $1 <_D (q^np)^{-1}g(q^np)$. First we show the orderings $<_n$ converge to $<_D$ for $n \to \infty$. By definition of the topology of LO$(G)$, it is sufficient to show that for an arbitrary finite set of $<_D$-positive elements $c_1, \ldots, c_r$, $1 <_N c_i$ holds for sufficiently large $N > 0$.

If $c_i \notin G^{(n-1)}$, then $1 <_n c_i$ for all $n$. Thus, we assume $c_i \in G^{(n-1)}$.

First we consider the case $c_i \notin G^{(n)}$. Then $c_i$ is $(n-1)(\mathcal{S})$-positive, hence by using generators $\{p, q\}$, $c_i$ is written as $c_i = q^mpw$ where $w \in \Sigma_{\mathcal{S}}^{(n-1)}$ and $m \in \mathbb{Z}$. For $k > m$ by Property $F$, $p^{-1}q^{m-k}p \in P_A^{(n-1)} \subset \Sigma_{\mathcal{S}}^{(n-1)}$. So, if we take $k > m$, then

$$(q^kp)^{-1}c_i(q^kp) = (p^{-1}q^{m-k}p)wq^kp$$

is $(n-1)(\mathcal{S})$-positive. So $1 <_k c_i$ for $k > m$.

Next assume that $c_i \in G^{(n)}$, so $c_i = q^{-m}$ $(m > 0)$. Then $(q^k p)^{-1} c_i (q^k p) = p^{-1} q^{-m} p$. By Lemma 4.1.2, $p^{-1}q^{-m}p \in P_{\mathcal{A}}^{(n-1)} - P_{\mathcal{A}}^{(n)}$, so $p^{-1}q^{-m}p$ is $(n-1)(\mathcal{S})$-positive. Therefore by rewriting the $\mathcal{A}$-positive word representative of $p^{-1}q^{-m}p$ using the generator $\mathcal{S}$, we conclude $1 <_D p^{-1}q^{-m}p$. Hence $1 <_k c_i$ for all $k > 1$.

To show the convergent sequence $\{<_n\}$ is non-trivial, we observe that the minimal positive element of the ordering $P_n$ is $(q^k p)^{-1} q (q^k p) = p^{-1}qp$. From the assumption, $p^{-1}qp$ is not identical with $q$, the minimal positive element of the ordering $<_D$. Thus, $<_n$ are different from the ordering $<_D$. □

It should be mentioned that our hypothesis $a_{n-1}a_n a_{n-1} \neq a_n$ is really needed. Let us consider the Klein bottle group $G = \langle s_1, s_2 \mid s_2 s_1 s_2 = s_1 \rangle$. It is known that $\mathcal{S} = \{s_1, s_2\}$ defines a Dehornoy-like ordering $<_D$ of $G$ (See [35] or the proof of Theorem 4.2.1 in Section 3.1, which is valid for the Klein bottle group case, $(m, n) = (2, 2)$). However, since $G$ has only finitely many left orderings, $<_D$ must be isolated. Observe that for the twisted generating set $\mathcal{A} = \{a_1, a_2\}$ of $\mathcal{S}$, the Klein bottle group has the same presentation $G = \langle a_1, a_2 \mid a_2 a_1 a_2 = a_1 \rangle$.

Finally we determine the Conradian soul of $<_D$ and $<_A$.

**Theorem 4.1.3** (Conradian properties of Dehornoy-like and isolated orderings). *Let $\mathcal{S} = \{s_1, \dots, s_n\} (n > 1)$ be an ordered finite generating set of a group $G$ which defines a Dehornoy-like ordering $<_D$. Assume that $\mathcal{S}$ has Property F and let $\mathcal{A} = \{a_1, \dots, a_n\}$ be the twisted generating set of $\mathcal{S}$. Let $<_A$ be the isolated ordering defined by $\mathcal{A}$. If $a_{n-1}a_n a_{n-1} \neq a_n$, then two orderings $<_D$ and $<_A$ have the following properties.*

1. *$<_D$ is not Conradian. Thus, the $<_D$-Conradian soul is $G^{(n)}$, the infinite cyclic subgroup generated by $s_n$.*
2. *$<_A$ is not Conradian. Thus, the $<_A$-Conradian soul is $G^{(n)}$, the infinite cyclic subgroup generated by $a_n$.*

*Proof.* As in the proof of Lemma 4.1.2, we put $p = a_{n-1}$ and $q = a_n$. To prove theorem it is sufficient to show for $n > 2$, $<_D$ and $<_A$ are not Conradian, since by Proposition 4.1.2 and Proposition 4.1.3 if $H$ is a $<_D$- or $<_A$- convex subgroup, then $H = G^{(i)}$ for some $i$. First we show $<_A$ is not Conradian. By Lemma 4.1.2, every $\mathcal{A}$-word positive representative of $p^{-1}q^{-1}p$ contains at least one $p$, so we put $p^{-1}qp = Np^{-1}q^{-k}$ where $N \leq_A 1$ and $k \geq 0$. Then we obtain an inequality

$$(q^k p)^{-1} q (q^k p)^2 = (p^{-1}qp) q^k p = N \leq_A 1$$

hence $<_A$ is not Conradian. To see $<_D$ is not Conradian, we observe

$$\begin{aligned}
(pq^{k+1}pq^{k+1})^{-1}(pq^{k+2})(pq^{k+1}pq^{k+1})^2 &= q^{-(k+1)}(p^{-1}qp)q^{k+1}pq^{k+1}pq^{k+1}pq^{k+1} \\
&= q^{-(k+1)}N(p^{-1}q)pq^{k+1}pq^{k+1}pq^{k+1} \\
&= \cdots \\
&= q^{-(k+1)}N^4p^{-1}q \\
&<_D 1.
\end{aligned}$$

□

## 4.2 Isolated and Dehornoy-like ordering on $\mathbb{Z}*_{\mathbb{Z}}\mathbb{Z}$

### 4.2.1 Construction of orderings

In this section we construct Dehornoy-like and isolated left orderings of $\mathbb{Z}*_{\mathbb{Z}}\mathbb{Z}$. Let $G = \mathbb{Z}*_{\mathbb{Z}}\mathbb{Z}$ be the amalgamated free product of two infinite cyclic groups, which is presented as

$$G_{m,n} = \langle x, y \mid x^m = y^n \rangle. \quad (m \geq n)$$

by using two integers $m, n$. In this section we will always assume $(m,n) \neq (2,2)$. Let us consider an ordered generating set $\mathcal{S} = \{s_1 = xyx^{-m+1}, s_2 = x^{m-1}y^{-1}\}$ and its twisted generating set $\mathcal{A} = \{a = x, b = yx^{-m+1}\}$. Using $\mathcal{S}$ or $\mathcal{A}$, the group $G_{m,n}$ is presented as

$$\begin{aligned} G_{m,n} &= \langle s_1, s_2 \mid s_2 s_1 s_2 = ((s_1 s_2)^{m-2} s_1)^{n-1} \rangle \\ &= \langle a, b \mid (ba^{m-1})^{n-1} b = a \rangle \end{aligned}$$

respectively.

The following is the main result of this section.

**Theorem 4.2.1.** *Let $\mathcal{S}, \mathcal{A}$ be the ordered finite generating sets of $G_{m,n}$ as above.*

1. *$\mathcal{S}$ defines a Dehornoy-like ordering $<_D$.*
2. *$\mathcal{A}$ defines an isolated left ordering $<_A$.*

By the presentation of $G$, it is easy to see that $\mathcal{S}$ have Property $F$ and $bab \neq a$ if $(m,n) \neq (2,2)$. Thus from general theories developed in the previous Section, we obtain various properties of $<_D$ and $<_A$.

**Corollary 4.2.1.** *Let $<_D$ be the Dehornoy-like ordering and $<_A$ be the isolated ordering of $G = G_{m,n}$ in Theorem 4.2.1.*

1. *If $H$ is a $<_D$-convex subgroup, then $H = \{1\}$ or $\langle s_2 \rangle$ or $G$.*
2. *If $H$ is a $<_A$-convex subgroup, then $H = \{1\}$ or $\langle b \rangle$ or $G$.*
3. *The Conradian soul of $<_D$ is $G^{(2)} = \langle s_2 \rangle$.*
4. *The Conradian soul of $<_A$ is $G^{(2)} = \langle b \rangle$.*
5. *$<_D$ is an accumulation point of the set of its conjugates $\{<_D \cdot g\}_{g \in G}$. Thus, $<_D$ is not isolated in $LO(G)$ and the $\sigma(\mathcal{S})$-positive monoid is not finitely generated.*

The proof of Theorem 4.2.1 given here is a generalization of Navas' argument in [35], and mainly use the twisted generating set $\mathcal{A}$ not $\mathcal{S}$. In subsequent section we will also give an outline of alternative proof which mainly uses $\mathcal{S}$.

Before giving a proof, first we recall the structure of the group $G_{m,n}$. Let $Z_{m,n} = \mathbb{Z}_m * \mathbb{Z}_n = \langle X, Y \mid X^m = Y^n = 1\rangle$, where $\mathbb{Z}_m$ is the cyclic group of order $m$ and let $\pi : G_{m,n} \to Z_{m,n}$ be a homomorphism defined by $\pi(x) = X$, $\pi(y) = Y$. The kernel of $\pi$ is an infinite cyclic group generated by $x^m = y^n$ which is the center of $G_{m,n}$. Thus, we have a central extension

$$1 \to \mathbb{Z} \to G_{m,n} \to Z_{m,n} \to 1.$$

We describe an action of $Z_{m,n}$ on $S^1$ which is used to prove Property $A$. Let $T = T_{m,n}$ be the Bass-Serre tree for the free product $Z_{m,n} = \mathbb{Z}_m * \mathbb{Z}_n$. That is, $T_{m,n}$ is a tree whose vertices are disjoint union of cosets $\mathbb{Z}_{m,n}/\mathbb{Z}_m \coprod \mathbb{Z}_{m,n}/\mathbb{Z}_n$ and edges are $Z_{m,n}$. Here an edge $g \in Z_{m,n}$ connects two vertices $g\mathbb{Z}_m$ and $g\mathbb{Z}_n$. (See Figure 4.1 left for example the case $(m,n) = (4,3)$).

In our situation, the Bass-Serre tree $T$ is naturally regarded as a planer graph. More precisely, we regard $T$ as embedded into the hyperbolic plane $\mathbb{H}^2$. Now $X$ acts on $T_{m,n}$ as a rotation centered on $P$, and $Y$ acts on $T$ as a rotation centered on $Q$. This defines a faithful action of $Z_{m,n}$ on $T$. Let $E(T)$ be the set of the ends of $T$, which is identified with the set of infinite rays emanating from a fixed base point of $T$. The end of tree $E(T)$ is a Cantor set, and naturally regarded as a subset of the points at infinity $S^1_\infty$ of $\mathbb{H}^2$. The action of $Z_{m,n}$ induces a faithful action on $E(T)$. Moreover, the action on $E(T) \subset S^1_\infty$ extends an action on $S^1 = S^1_\infty$, since the actions of $X$ and $Y$ can be extended as isometries of $\mathbb{H}^2$. (Alternatively, one can directly observe this fact by using combinatorial description of the action on $E(T)$ given in the next section) We call this action the *standard action* of $Z_{m,n}$.

The standard action is easy to describe since $X$ and $Y$ act as rotations of the tree $T$. $X$ acts on $S^1$ so that it sends an interval $[p_i, p_{i+1}]$ to the adjacent interval $[p_{i+1}, p_{i+2}]$ (here indices are taken modulo $m$), and $Y$ acts on $S^1$ so that it sends an interval $[q_i, q_{i+1}]$ to $[q_{i+1}, q_{i+2}]$ (here indices are taken modulo $n$). See Figure 4.1 right. More detailed description of the set of ends $E(T)$ and the standard action will be given in next section.

Using the standard action, we show the Property $A$ for $\mathcal{S}$, which is equivalent to the following statement by Claim 4.1.2.

**Lemma 4.2.1.** *If $g \in P_{\mathcal{A}}$, then $g \neq 1$.*

*Proof.* Let $g \in P_{\mathcal{A}}$ and put $A = \pi(a) = X$ and $B = \pi(b) = YX^{-m+1} = YX$. If $g \in \operatorname{Ker}\pi$, that is, $g = a^{mN}$ for $N > 0$, then $g \neq 1$ is obvious so we assume $g \neq a^{mN}$. Let us put

$$\pi(g) = A^{s_k}B^{r_k}\cdots A^{s_1}B^{r_1}.$$

where $0 < s_i < m$ $(i < k)$, $0 \leq s_k < m$ and $0 < b_i$ $(i > 1)$, $0 \leq b_1$.

First observe that the dynamics of $B$, $A$ and $(BA^{m-1})^iB$ are given by the following formulas.

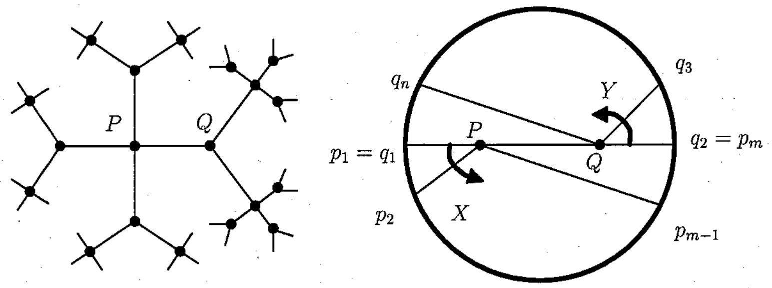


Figure 4.1: Bass-Serre Tree and action of $Z_{m,n}$ on $S^1$

$$\begin{aligned}
B[p_m, p_{m-1}] &= YX[p_m, p_{m-1}] = Y[p_1, p_m] = Y[q_1, q_2] \\
&= [q_2, q_3] \subset [p_m, p_1] \\
A^i[p_m, p_1] &\subset [p_m, p_{m-1}] \ (i \neq m-1) \\
(BA^{m-1})^i B[p_m, p_{m-1}] &= (YXX^{m-1})^i YX[p_m, p_{m-1}] = Y^{i+1}[p_1, p_m] \\
&\subset Y^{i+1}[q_1, q_2] \subset [q_{2+i}, q_{3+i}] \subset [p_m, p_1].
\end{aligned}$$

Here $[p_m, p_{m-1}]$ represents the interval $[p_m, p_1] \cup [p_1, p_2] \cup \cdots \cup [p_{m-2}, p_{m-1}]$. Since $(BA^{m-1})^{n-1}B = A$, we can assume that the above word expression does not contain a subword of the form $(BA^{m-1})^{n-1}B$. Thus, by using the above formulas repeatedly, we conclude

$$\begin{cases}
\pi(g)[p_{r_k-1}, p_{r_k}] = [p_{r_k}, p_{r_k+1}] & (s_k \neq 0, r_1 \neq 0) \\
\pi(g)[p_m, p_1] = [p_{r_k}, p_{r_k+1}] & (s_k = 0, r_1 \neq 0) \\
\pi(g)[p_1, p_2] = [p_m, p_1] & (s_k \neq 0, r_1 = 0)
\end{cases}$$

So we conclude $\pi(g) \neq 1$ if $s_k \neq 0$ or $r_1 \neq 0$. If $s_k = r_1 = 0$, we need more careful argument to treat the case the word $\pi(g)$ contains a subword of the form $(BA^{m-1})^i$. Let us write

$$\pi(g) = B^{s_k - 1}(BA^{m-1})^i \underbrace{B \cdots A^{r_1}}_{*}$$

where we take $i$ the maximal among such description of the word $\pi(g)$. That is, the prefix of the word $*$ is not $BA^{m-1}$. Then for $i \neq 0$,

$$\pi(g)[q_i, q_{i+1}] \subset B^{s_k-1}(BA^{m-1})^i[p_m, p_1] \subset B^{s_k-1}[p_m, p_1].$$

Thus if $s_k \neq 1$, then $\pi(g)[q_n, q_1] \subset [q_2, q_3]$. and if $s_k = 1$, then $\pi(g)[q_2, q_3] = [q_{2+i}, q_{3+i}]$. Thus we proved that in all cases $\pi(g)$ acts on $S^1$ non trivially, hence $g \neq 1$. □

Next we show Property $C$, which is equivalent to the following statement according to Claim 4.1.1.

**Lemma 4.2.2.** $P_{\mathcal{A}} \cup \{1\} \cup P_{\mathcal{A}}^{-1} = G$ .

*Proof.* Let $g \in G$ be a non-trivial element. Since $a^m = (ba^{m-1})^n = (a^{m-1}b)^n$ is central, we may write $g$ as

$$g = a^{mM} a^{s_k} b^{r_k} \cdots a^{s_1} b^{r_1}.$$

where $0 < s_i < m$ $(i < k)$, $0 \leq s_k < m$ and $0 < b_i$ $(i > 1)$, $0 \leq b_1$. Among such word expressions, we choose the word expression $w$ so that $k$ is minimal. If $mM + s_k \geq 0$, then $g \in P_{\mathcal{A}}$. So we assume that $mM + s_k < 0$ and we prove $g \in P_{\mathcal{A}}^{-1}$ by induction on $k$. The case $k = 1$ is a direct consequence of Property $F$.

First observe that from a relation $a^{-1}b = (a^{m-1}b)^{1-n}$, we get a relation

$$a^{-1}b^r = [(a^{m-1}b)^{2-n}b^{-1}a^{-m+2}]^{r-1}(a^{m-1}b)^{1-n}$$

for all $r > 0$. Thus, by applying this relation, $g$ is written as

$$g = X(a^{m-1}b)^{1-n} \cdot a^{s_{k-1}} b^{r_{k-1}} \cdots$$

for some $X \in P_{\mathcal{A}}^{-1}$. Unless $s_{k-1} = s_{k-2} = \cdots = s_{k-(n-1)} = m-1$ and $r_{k-1} = r_{k-2} = \cdots = r_{k-(n-1)} = 1$, by reducing this word expression, we obtain a word expression of the form

$$g = X' a^{-1} b^{r'_i} a^{s_{i-1}} \cdots$$

where $X' \in P_{\mathcal{A}}^{-1}$ and $i < k$. By inductive hypothesis, $a^{-1}b^{r'_i}a^{s_{i-1}} \cdots \in P_{\mathcal{A}}^{-1}$, hence we conclude $g \in P_{\mathcal{A}}^{-1}$.

Now assume $s_{k-1} = s_{k-2} = \cdots = s_{k-(n-1)} = m-1$ and $r_{k-1} = r_{k-2} = \cdots = r_{k-(n-1)} = 1$. Since $(a^{m-1}b)^{n-1} = b^{-1}a$, by replacing the subword $(a^{m-1}b)^{n-1}$ with $b^{-1}a$ and canceling $b^{-1}$ we obtain another word representative

$$g = a^{(m+1)M} a^{s_k} b^{r_k - 1} a^{s_{k-n}+1} b^{r_{k-n}} \cdots$$

which contradicts the assumption that we have chosen the first word representative of $g$ so that $k$ is minimal. □

These two lemmas and Theorem 4.1.1 prove Theorem 4.2.1.

### 4.2.2 Dynamics of the Dehornoy-like ordering of $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$

In this section we give an alternative definition of the Dehornoy-like ordering $<_D$ of $G_{m,n}$ by using the dynamics of $G_{m,n}$. Recall that $G_{m,n}$ is a central extension of $Z_{m,n}$ by $\mathbb{Z}$. By lifting the standard action of $Z_{m,n}$ on $S^1$, we obtain a faithful orientation-preserving action of $G_{m,n}$ on the real line. We call this action the *standard action* of $G_{m,n}$ and denote it by $\Theta : G_{m,n} \to \mathrm{Homeo}_+(\mathbb{R})$.

We give a detailed description of the action of $G_{m,n}$ on $\mathbb{R}$. First of all, we give a combinatorial description of the end of the tree $T = T_{m,n}$.

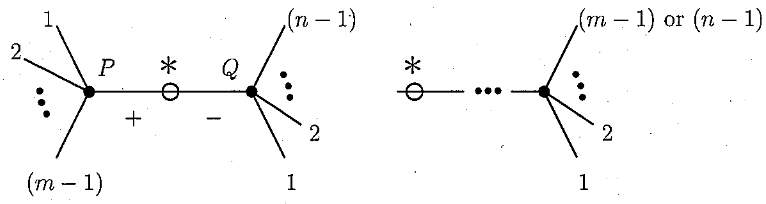


Figure 4.2: Labeling of edge

Let us take a basepoint $*$ of $T$ as the midpoint of $P$ and $Q$. For each edge of $T$, we assign a label as in Figure 4.2. Let $e$ be a point of $E(T)$, which is represented by an infinite ray $\gamma_e$ emanating from $*$. Then by reading a label on edge along the infinite path $\gamma_e$, $\gamma_e$ is encoded by an infinite sequence of integers $\pm l_1 l_2 \cdots$.

Now let $p : \mathbb{R} \to S^1$ be the universal cover, and $\widetilde{E}(T) = p^{-1}(E(T)) \subset \mathbb{R}$. Then the standard action $G_{m,n}$ preserves $\widetilde{E}(T)$. A point of $\widetilde{E}(T)$ is given as the sequence of integers $(N; \pm l_1 l_2 \cdots)$ where $N \in \mathbb{Z}$. Observe that the set of such a sequence of integers has a natural lexicographical ordering. This ordering of $\widetilde{E}(T)$ is identical with the ordering induced by the standard ordering $<_{\mathbb{R}}$ of $\mathbb{R}$, so we denote the ordering by the same symbol $<_{\mathbb{R}}$.

The action of $G_{m,n}$ on $\widetilde{E}(T)$ is easy to describe, since $X$ and $Y$ act on $T_{m,n}$ as rotations of the tree.

$$x : \begin{cases} (N; +i \cdots) & \mapsto \quad (N; +(i+1) \cdots) \quad (i \neq m-1) \\ (N; +(m-1) \cdots) & \mapsto \quad ((N+1); - \cdots) \\ (N; -i \cdots) & \mapsto \quad (N; +1i \cdots) \end{cases}$$

$$y : \begin{cases} (N; +i \cdots) & \mapsto \quad ((N+1); -1i \cdots) \\ (N; -i \cdots) & \mapsto \quad (N; -(i+1) \cdots) \quad (i \neq n-1) \\ (N; -(n-1) \cdots) & \mapsto \quad (N; + \cdots) \end{cases}$$

See Figure 4.3.

Therefore, the action of $s_1$, $s_2$ and $s_2^{-1}$ are given by the formula

$$s_1 : \begin{cases} (N; +i \cdots) & \mapsto \quad (N; +11(i+1) \cdots) \\ & \qquad (i \neq m-1) \\ (N; +(m-1)i \cdots) & \mapsto \quad (N; +1(i+1) \cdots) \\ & \qquad (i \neq n-1) \\ (N; +(m-1)(n-1)i \cdots) & \mapsto \quad (N; +(i+1) \cdots) \\ & \qquad (i \neq m-1) \\ (N; +(m-1)(n-1)(m-1) \cdots) & \mapsto \quad ((N+1); - \cdots) \\ (N; -i \cdots) & \mapsto \quad (N; +111 \cdots) \end{cases}$$

$$s_2 : \begin{cases} (N; + \cdots) & \mapsto \quad (N; +(m-1)(n-1) \cdots) \\ (N; -i \cdots) & \mapsto \quad (N; +(m-1)(i-1) \cdots) \quad (i \neq 1) \\ (N; -1i \cdots) & \mapsto \quad (N; +(i-1) \cdots) \quad (i \neq 1) \\ (N; -11 \cdots) & \mapsto \quad (N; - \cdots) \end{cases}$$

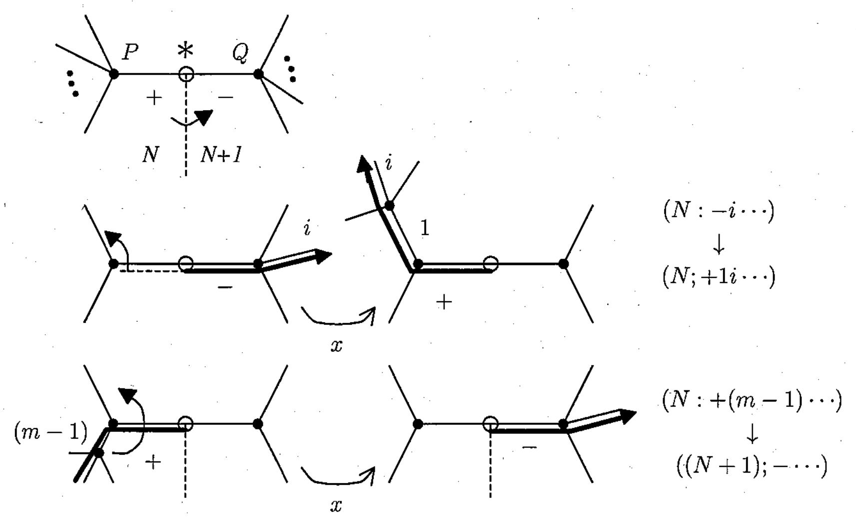


Figure 4.3: The standard action of $x$

$$s_2^{-1}: \begin{cases} (N; +i \cdots) & \mapsto (N; -1(i+1) \cdots) \quad (i \neq m-1) \\ (N; +(m-1)i \cdots) & \mapsto (N; -(i+1) \cdots) \quad (i \neq n-1) \\ (N; +(m-1)(n-1) \cdots) & \mapsto (N; + \cdots) \\ (N; - \cdots) & \mapsto (N; -11 \cdots) \end{cases}$$

Let $E = (0; -1111\cdots)$ and $F = (0; +1111\cdots)$ be the point of $\widetilde{E}(T)$ and let $<_{\{E,F\}}$ be a left-ordering of $G_{m,n}$ defined by the sequence $\{E, F\}$ and the standard action $\Theta$. The following theorem gives an alternative definition of $<_D$.

**Theorem 4.2.2.** *The left ordering $<_{\{E,F\}}$ is identical with the Dehornoy-like ordering $<_D$ defined by $\mathcal{S}$.*

*Proof.* By the formula of the action of $G_{m,n}$ on $\widetilde{E}(T_{m,n})$ given above, it is easy to see that for $1(\mathcal{S})$-positive element $g \in G_{m,n}$, $E <_{\mathbb{R}} g(E)$. Thus by Property $C$ of $\mathcal{S}$, $g(E) = E$ if and only if $g = s_2^q$ for $q \in \mathbb{Z}$. Similarly, $s_2^q(F) = (0; +(m-1)(n-1)\cdots) > F$ if $q > 0$. Thus, we conclude $1 <_D g$ then $1 <_{\{E,F\}} g$, hence two orderings are identical. □

We remark that a more direct proof is possible. That is, we can prove that $<_{\{E,F\}}$ is a Dehornoy-like ordering without using Theorem 4.2.1. In fact, the proof of Theorem 4.2.2 provides an alternative (but essentially equivalent since it uses the standard action of $G_{m,n}$) proof of the fact that $\mathcal{S}$ has Property $A$. On the other hand, using the description of the standard action given here, we can give a completely different proof of Property $C$, as we give an outline here. Let $g \in G$. If $g(E) = E$, then $g = s_2^k$ for $k \in \mathbb{Z}$. So assume $g(E) <_{\mathbb{R}} E$, so $g(E) = (N; l_1 l_2 l_3 \cdots)$ where $N \leq 0$, $l_1 \in \{+, -\}$. Let $c(g)$ be the minimal integer such that $l_{j'} = 1$ for all $j' > j$. We define the complexity of $g$ by $C(g) =$

$(|N|, c(g))$. Now we can find $1(\mathcal{S})$-positive element $p_g$ such that $C(p_g g) < C(g)$. The construction of $p_g$ is not difficult but requires complex case-by-case studies, so we omit the details. Here we compare the complexity by the lexicographical ordering of $\mathbb{Z} \times \mathbb{Z}$. Since $C(g) = (0,0)$ implies $g(E) = E$, by induction of the complexity we prove $g$ is $\sigma(\mathcal{S})$-negative.

**Remark 4.2.1.** Regard $G_{3,2} = B_3$ as the mapping class group of the 3-punctured disc $D_3$ with a hyperbolic metric and let $\widetilde{D_3} \subset \mathbb{H}^2$ be the universal cover of $D_3$. By considering the action on the set of points at infinity of $\widetilde{D_3}$, we obtain an action of $G_{3,2}$ on $\mathbb{R}$ which we call the *Nielsen-Thurston action.* The Dehornoy ordering $<_D$ of $B_3 = G_{3,2}$ is defined by the Nielsen-Thurston action. See [43]. In the case $(m,n) = (3,2)$ the standard action $\Theta$ derived from Bass-Serre tree is conjugate to the Nielsen-Thurston action hence the Dehornoy-like ordering of $G_{m,n}$ is also regarded as a generalization of the Dehornoy ordering of $B_3$, from the dynamical point of view.

We say a Dehornoy-like ordering $<_D$ defined by an ordered finite generating set $\mathcal{S}$ has *Property S* (the *Subword property*) if $g <_D wg$ holds for all $\mathcal{S}$-word positive element $w$ and for all $g \in G$. The Dehornoy ordering of the braid group $B_n$ has Property $S$ ([11]). One remarkable fact is that our Dehornoy-like ordering $<_D$ of $G_{m,n}$ does *not* have Property $S$ except for the braid group case.

**Theorem 4.2.3.** *The Dehornoy-like ordering $<_D$ of $G_{m,n}$ does not have Property $S$ unless $(m,n) = (3,2)$.*

*Proof.* We use the dynamical description of $<_D$ given in Theorem 4.2.2. If $m > 2$ and $n \neq 2$, then

$$[s_1(s_2 s_1)]E = (0; +211\cdots) <_{\mathbb{R}} (0; +(m-1)(n-1)1\cdots) = [s_2 s_1]E$$

hence $s_1(s_2 s_1) <_D (s_2 s_1)$. □

However we show that the Dehornoy-like ordering $<_D$ of $G_{m,n}$ has a slightly weaker property which can be regarded as a partial subword property.

**Theorem 4.2.4.** *Let $<_D$ be the Dehornoy-like ordering of $G_{m,n}$. Then $g <_D s_2 g$ holds for all $g \in G$.*

*Proof.* Observe that the standard action of $s_2$ on $\widetilde{E}(T)$ is monotone increasing. That is, for any $p \in \widetilde{E}(T)$, we have $p \leq_{\mathbb{R}} s_2(p)$. Thus, $g(E) \leq s_2 g(E)$. The equality holds only if $g(E) = E$, so in this case $g = s_2^k$ $(k \in \mathbb{Z})$. So in this case we also have a strict inequality $g <_D s_2 g$. □

**Remark 4.2.2.** A direct proof of Theorem 4.2.3 which does not use the dynamics is easy once we found a counter example. If $(m,n) \neq (3,2)$, then $s_2 s_1 s_2 = s_1 s_2 s_1 W$ for an $\mathcal{S}$-positive word $W$, so

$$\begin{aligned} s_1^{-1} s_2^{-1} s_1^{-1} s_2 s_1 &= s_1^{-1} s_2^{-1} s_1^{-1} (s_2 s_1 s_2) s_2^{-1} \\ &= s_1^{-1} s_2^{-1} s_1^{-1} (s_1 s_2 s_1 W) s_2^{-1} \\ &= W s_2^{-1} \end{aligned}$$

The last word is $1(\mathcal{S})$-positive hence $s_1(s_2s_1) <_D (s_2s_1)$. Using dynamics we can easily find a lot of other counter examples. The main point is that, as we can easily see, the action $s_1$ is not monotone increasing unlike the action of $s_2$. That is, there are many points $p \in \widetilde{E}(T)$ such that $s_1(p) <_{\mathbb{R}} p$.

**Remark 4.2.3.** Another good property of the standard generator $\mathcal{S} = \{\sigma_1, \sigma_2\}$ of $B_3$ is that the *$\mathcal{S}$-word positive monoid* $P_{\mathcal{S}} \cup \{1\} = B_3^+$ is a *Garside monoid* hence $B_3^+$ has various nice lattice-theoretical properties. See [3],[10] for a definition and basic facts of Garside monoid and Garside groups. In particular, the monoid $B_3^+$ is atomic. That is, if we define the partial ordering $\prec$ on $B_3^+$ by $g \prec g'$ if $g^{-1}g' \in B_3^+$, then for every $g \in G$, the length of a strict chain

$$1 \prec g_1 \prec \cdots \prec g_k = g$$

is finite. However, if $m > 3$, then the $\mathcal{S}$-word positive monoid $P_{\mathcal{S}} \cup \{1\}$ is not atomic. If $m > 3$, then $s_2s_1s_2 = s_1s_2s_1s_2W$ holds for $W \in P_{\mathcal{S}}$ so we have a chain

$$\cdots \prec s_1^2s_2s_1s_2 \prec s_1s_1s_2s_1s_2W = s_1s_2s_1s_2 \prec s_1s_2s_1s_2W = s_2s_1s_2$$

having infinite length. Thus for $m > 3$, $P_{\mathcal{S}} \cup \{1\}$ is not a Garside monoid.

On the other hand, the groups $G_{m,n}$ have a lot of Garside group structures. For example, take a generating set $\mathcal{X} = \{x, y\}$ of $G_{m,n}$ so that $G_{m,n} = \langle x, y | x^m = y^n \rangle$. Then the $\mathcal{X}$-word positive monoid $P_{\mathcal{X}} \cup \{1\}$ is a Garside monoid. Moreover, if $m$ and $n$ are coprime, that is, if $G_{m,n}$ is a torus knot group, then there are other Garside group structures due to Picantin [39]. Thus, unlike the Dehornoy ordering of $B_n$, a relationship between general Dehornoy-like orderings and Garside structures of groups seem to be weak. It is an interesting problem to find other family of left-orderings which is more related to Garside group structure.

In the remaining case $(m, n) = (3, 3)$, the author could not determine whether $P_{\mathcal{S}} \cup \{1\}$ is a Garside monoid or not.

### 4.2.3 Exotic orderings: left orderings with no non-trivial proper convex subgroups

In [7], Clay constructed left orderings of free groups which has no non-trivial proper convex subgroups by using the Dehornoy ordering of $B_3$. Such an ordering is interesting, because many known constructions of left orderings, such as a method to use group extensions, produce an ordering having proper non-trivial convex subgroups. In this section we construct such orderings by using a Dehornoy-like ordering of $G_{m,n}$. By using the dynamics, we prove a stronger result even for the 3-strand braid group case.

Let $H$ be a normal subgroup of $G = G_{m,n}$. By abuse of notations, we also use $<_D$ to represent both the Dehornoy like ordering of $G = G_{m,n}$ and its restriction to $H$. First we observe the following lemma, where the partial subword property plays an important role.

**Lemma 4.2.3.** *Let $C$ be a non-trivial $<_D$-convex subgroup of $H$ and $G^{(2)} = G^{(2)}_{m,n}$. If $G^{(2)} \cap H = \{1\}$, then $s_2^k c s_2^{-k} \in C$ for all $k \in \mathbb{Z}$ and $c \in C$.*

*Proof.* Let $c \in H$ be a $<_D$-positive element. Since $G^{(2)} \cap H = \{1\}$, $c$ must be $1(\mathcal{S})$-positive. So $s_2^k c^{-1} s_2^{-k}$ is $1(\mathcal{S})$-negative, hence $cs_2^k c^{-1} s_2^{-k} <_D c$. On the other hand, by Theorem 4.2.4, $cs_2^k c^{-1} >_D 1$. This implies that $cs_2^k c^{-1}(E) = cs_2^k c^{-1} s_2^{-k}(E) \geq_{\mathbb{R}} E$, so $cs_2^k c^{-1} s_2^{-k} \geq_D 1$. Since we assumed that $H$ is a normal subgroup, $cs_2^k c^{-1} s_2^{-k} \in H$. Since $C$ is a $<_D$-convex subgroup, $cs_2^k c^{-1} s_2^{-k} \in C$. Hence we conclude $s_2^k c s_2^{-k} \in C$. □

Now we show that in most cases, the restriction of the Dehornoy-like ordering to a normal subgroup of $G_{m,n}$ yields a left-ordering having no non-trivial proper convex subgroups.

**Theorem 4.2.5.** *Let $H$ be a normal subgroup of $G = G_{m,n}$ such that $G^{(2)} \cap H = \{1\}$. Then the restriction of the Dehornoy-like ordering $<_D$ to $H$ contains no non-trivial proper convex subgroup.*

*Proof.* Let $C$ be a non-trivial $<_D$-convex subgroup of $H$ and $c \in C$ be $<_D$-positive element. Since $c$ must be 1-positive, by Lemma 4.2.3, we may assume that $c = s_2 s_1 s_2 w$ where $w$ is a 1-positive element, by taking a power of $c$ and conjugate by $s_2$ if necessary. Similarly, we also obtain $c' \in C$ such that $c' = w' s_2 s_1 s_2$ where $w'$ is a 1-positive element. By computing the standard action of $c'c$, then we obtain

$$\begin{aligned} c'c(E) = w' s_2 s_1 s_2^2 s_1 s_2 w(E) &>_{\mathbb{R}} w' s_2 s_1 s_2^2 s_1 s_2(E) = w'(1; -11(n-1)11\cdots) \\ &>_{\mathbb{R}} (1; -1111\cdots). \end{aligned}$$

Thus, for any $h \in H$, $(c'c)^N(E) >_{\mathbb{R}} (N; -1111\cdots) >_{\mathbb{R}} h(E) >_{\mathbb{R}} E$ holds for sufficiently large $N > 0$. Since $C$ is convex and $c'c \in C$, this implies $h \in C$ so we conclude $C = H$.

□

The assumption that $G^{(2)} \cap H = \{1\}$ is necessary, since $H \cap G^{(2)}$ yields a $<_D$-convex subgroup of $H$.

**Remark 4.2.4.** Observe that the hypothesis $G^{(2)} \cap H = \{1\}$ implies that $H$ is not a finite index subgroup of $G$. Since $G = G_{m,n} = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$, [26, Corollary, Page 253] shows that if $H$ is finitely generated then $H$ is a free subgroup of $G$.

Theorem 4.2.5 provides an example of a left-ordering of the free group of *infinite* rank which does not have any non-trivial proper convex subgroups. For example, take $F = [B_3', B_3']$, where $B_3' = [B_3, B_3] \cong F_2$ be the commutator subgroup of $B_3$ that is isomorphic to the rank 2 free group.

# Chapter 5

# Construction of isolated group ordering

In this chapter we study isolated group orderings. We will concentrate our attention to *genuine* isolated left orderings: that is, we mainly consider the case $G$ admits infinitely many (uncountably) many left orderings, since the classification of groups having only finitely many left-orderings is given by Tararin. See [28]. OI it is difficult to construct genuine isolated left orderings of groups. Only two families of isolated orderings are known.

1. Dubrovina-Dubrovin ordering (See Section 1.4.2)
2. The central extension of the Hecke groups, which has a presentation of the form
$$\Gamma_n = \langle a, b \mid b = ab^n a \rangle$$
This example was found by Navas, in [35].

Now we observed in Chapter 4 that the Navas's example of isolated orderings are extended as follows.

2' The amalgamated free product of two cyclic groups, which has a presentation of the form
$$G_{m,n} = \langle a, b \mid (ba^{m-1})^{n-1} b = a \rangle$$

The main theorem of this chapter provides a new construction of isolated left orderings by means of the *partially central cyclic amalgamation.* From two groups having (not necessarily genuine) isolated orderings, we construct a new group having an isolated left ordering. In almost all cases, the constructed isolated orderings are genuine. Our construction is not related to Dehornoy-like ordering construction given in previous Chapter, and gives a lot of new isolated ordering which cannot be obtained from the deformation of Dehornoy-like orderings.

**Theorem 5.0.6** (Construction of isolated left ordering via partially central cyclic amalgamation). *Let $G$ and $H$ be finitely generated groups. Let $z_G$ be a non-trivial central element of $G$, and $z_H$ be a non-trivial element of $H$.*

*Let $\mathcal{G} = \{g_1, \ldots, g_m\}$ be a finite generating set of $G$ which defines an isolated left ordering $<_G$ of $G$. We take a numbering of elements of $\mathcal{G}$ so that $1 <_G g_1 <_G \ldots <_G g_m$ holds. Similarly, let $\mathcal{H} = \{h_1, \ldots, h_n\}$ be a finite generating set of $H$ which defines an isolated left ordering $<_H$ of $H$ such that the inequality $1 <_H h_1 <_H \ldots <_H h_n$ holds. We assume the cofinality assumption* **CF(G)**, **CF(H)**, *and the invariance assumption* **INV(H)**.

**CF(G)** *$g_i <_G z_G$ holds for all $i$.*

**CF(H)** *$h_i <_H z_H$ holds for all $i$.*

**INV(H)** *$<_H$ is $z_H$-right invariant.*

*Let $X = G *_{\mathbb{Z}} H = G *_{\langle z_G = z_H \rangle} H$ be an amalgamated free product of $G$ and $H$. For $i = 1, \ldots, m$, let $x_i = g_i z_H^{-1} h_1$. Then we have the following results:*

1. *The generating set $\{x_1, \ldots, x_m, h_1, \ldots, h_n\}$ of $X$ defines an isolated left ordering $<_X$ of $X$.*

2. *The isolated ordering $<_X$ does not depend on a choice of a generating sets $\mathcal{G}$ and $\mathcal{H}$. Thus, $<_X$ only depends on isolated orderings $<_G, <_H$ and $z_G, z_H$.*

3. *The natural inclusions $\iota_G : G \to X$ and $\iota_H : H \to X$ are order-preserving homomorphism.*

4. *$1 <_X x_1 <_X \cdots <_X x_m <_X h_1 <_X \cdots <_X h_n <_X z_H = z_G$. Moreover, $z = z_G = z_H$ is $<_X$-positive cofinal and the isolated ordering $<_X$ is $z$-right invariant.*

5. *For an isolated ordering $<$, let $r(<)$ be the minimal number of the generator of the positive cone $P(<)$. Then $r(<_X) \leq r(<_G) + r(<_H)$.*

6. *Let $Y$ be a non-trivial proper subgroup of $X$. If $Y$ is $<_X$-convex, then $Y = \langle x_1 \rangle$, the infinite cyclic group generated by $x_1$.*

We call the construction of isolated ordering described in Theorem 5.0.6 the *partially central cyclic amalgamation construction.*

As we will see in Lemma 5.1.3 in Section 5.1.1, the cofinality assumption **CF(G)** (resp. **CF(H)**) are understood as an assumption on $z_G$ and $<_G$ (resp. $z_H$ and $<_H$). Thus, Theorem 5.0.6 (2) shows that the choice of generating sets $\mathcal{G}$ and $\mathcal{H}$ is not important, though it is useful to describe and understand the isolated ordering $<_X$. Therefore the generating sets $\mathcal{G}$ and $\mathcal{H}$ play rather auxiliary roles and are not essential in our partially central cyclic amalgamation construction. This makes a contrast with the construction using Dehornoy-like orderings.

Theorem 5.0.6 (3) shows that the partially central cyclic amalgamation construction can be seen as a mixing of two isolated orderings $<_G$ and $<_H$. We remark that Theorem 5.0.6 (4) implies that we can iterate the partially central cyclic amalgamation construction. Thus, we can actually produce many isolated orderings by using the partially central cyclic amalgamation constructions.

The proof of Theorem 5.0.6 (1) given in this paper is constructive so the proof actually provide an explicit algorithm to compute the isolated ordering $<_X$. In particular, the isolated ordering $<_X$ can be determined algorithmically if we have algorithms to compute isolated orderings $<_G$ and $<_H$. See Section 5.1.7.

Unlike the partial central cyclic amalgamation (the amalgamated free product in Theorem 5.0.6), the usual free product does not preserve the property that the group has an isolated left orderings. For example, Navas showed that the free group of rank two $F_2 = \mathbb{Z} * \mathbb{Z}$ has no isolated orderings [34], whereas the infinite cyclic group $\mathbb{Z}$ has isolated orderings. Indeed, recently Rivas [40] proved that the free product of groups does not have any isolated left orderings. Similarly, the direct product of groups also does not preserve the property that the group has an isolated left orderings: $\mathbb{Z} \times \mathbb{Z}$ has no isolated orderings.

## 5.1 Construction of isolated left orderings

### 5.1.1 Cofinality and Invariance assumption

First of all we review the assumptions in the statement of Theorem 5.0.6 again, and deduce their consequences. This clarifies the role of each hypothesis in Theorem 5.0.6.

Let $G$ and $H$ be countable groups having an isolated left ordering $<_G$, $<_H$ respectively. Let $z_G \in G$ be a non-trivial central element of $G$, and let $z_H$ be a non-trivial element of $H$, which might be not central. We consider the group $X$ obtained as an amalgamated free product

$$X = G *_{\mathbb{Z}} H = G *_{\langle z_G = z_H \rangle} H,$$

which is again countable.

Let $\mathcal{G} = \{g_1, \dots, g_m\}$ be a generating set of $G$ which defines an isolated left ordering $<_G$ of $G$. We take a numbering of elements of $\mathcal{G}$ so that $1 <_G g_1 <_G \cdots <_G g_m$ holds. Similarly, let $\mathcal{H} = \{h_1, \dots, h_n\}$ be a generating set of $H$ which defines an isolated left ordering $<_H$ of $H$, and we assume that the inequality $1 <_H h_1 <_H \cdots <_H h_n$ holds.

As next lemma shows, the choice of the numbering of $\mathcal{G}$ implies that $g_1$ (resp. $h_1$) is the $<_G$-minimal (resp. $<_H$-minimal) positive element, hence $g_1$ (resp. $h_1$) is independent of a choice of the generating set $\mathcal{G}$ (resp. $\mathcal{H}$).

**Lemma 5.1.1.** *Let $\mathcal{G} = \{g_1, \dots, g_m\}$ be a generating set of a group $G$ which defines an isolated left ordering $<_G$ of $G$. Assume that $g_1$ is the $<_G$-minimal element of the set $\mathcal{G}$. Then $g_1$ is the $<_G$-minimal positive element. In particular, $<_G$ is discrete. Moreover, $<_G$ is $g_1$-right invariant.*

*Proof.* Assume $g \in G$ satisfies the inequality $1 <_G g \leq_G g_1$. Since $1 <_G g$, $g_1$ is written as a $\mathcal{G}$-positive word $g = g_{i_1} \cdots g_{i_l}$. If $i_1 \neq 1$, then $g_1^{-1} g_{i_1} >_G 1$, which is a contradiction. If $l > 1$, then $g_1^{-1} g_{i_1} >_G 1$ unless $i_1 = 1$. So we conclude $g = g_1$.

The $g_1$-right invariance of the ordering $<_G$ now follows from the fact $g_1$ is $<_G$-minimal positive element: If $a >_G b$, then $b^{-1} a g_1 >_G b^{-1} a >_G 1$. Thus, $b^{-1} a g_1 >_G g_1$ so $a g_1 >_G b g_1$. □

To obtain an isolated ordering of $X$ from $<_G$ and $<_H$, we impose the following assumptions, which we call the *cofinality assumption* and the *invariance assumption.*

**CF(G)** $g_i <_G z_G$ holds for all $i$.

**CF(H)** $h_i <_H z_H$ holds for all $i$.

**INV(H)** $<_H$ is $z_H$-right invariant.

We here remark that the invariance assumption for $<_G$ is automatically satisfied: that is, $<_G$ is $z_G$-right invariant since we have chosen $z_G$ as a central element.

First we observe the following simple lemma.

**Lemma 5.1.2.** *Let $<_H$ be a discrete left ordering of a group $H$, and let $h_1$ be the $<_H$-minimal positive element. If $<_H$ is an $h$-right invariant for $h \in H$, then $h$ commutes with $h_1$.*

*Proof.* $<_H$ is an $h$-right invariant, so $h h_1 h^{-1} >_H 1$ and $h^{-1} h_1 h >_H 1$. $h_1$ is the $<_H$-minimal positive element, $h h_1 h^{-1} \geq_H h_1$ and $h^{-1} h_1 h \geq_H h_1$. Thus, we get $h h_1 \geq_H h_1 h$ and $h_1 h \geq_H h h_1$, hence $h h_1 = h_1 h$. □

By Lemma 5.1.1 and Lemma 5.1.2, the invariance assumption [**INV(H)**] implies that $z_H$ commutes with $h_1$.

Recall that an element $g \in G$ and a left-ordering $<_G$ of $G$, $g$ is called $<_G$-*cofinal* if for all $g' \in G$, there exists integers $m$ and $M$ such that $g^m <_G g' <_G g^M$ holds. Although the cofinality assumptions [**CF(G)**] and [**CF(G)**] involves the generating sets $\mathcal{G}$ and $\mathcal{H}$, these assumptions should be regarded as assumptions on $z_G$, $z_H$ and isolated orderings $<_G$, $<_H$ as the next lemma shows.

**Lemma 5.1.3.** *Assume the invariance assumption* [**INV(H)**] *is satisfied. A generating set $\mathcal{H}$ satisfying the cofinality assumption* [**CF(H)**] *exists if and only if $z_H$ is $<_H$-positive cofinal and $H \neq \langle z_H \rangle$. Moreover, in such case we may choose a generating set $\mathcal{H}$ so that the cardinal of $\mathcal{H}$ is equal to the rank of the isolated ordering $<_H$.*

*Proof.* It is easy to see if a generating set $\mathcal{H}$ satisfies the cofinality assumption [**CF(H)**] and the invariance assumption [**INV(H)**] is satisfied, then $z_H$ is $<_H$-positive cofinal and $H \neq \langle z_H \rangle$. We show that under the invariance assumption [**INV(H)**], if $z_H$ is $<_H$-positive cofinal and $H \neq \langle z_H \rangle$, then we can choose a generating set $\mathcal{H} = \{h_1, \ldots, h_k\}$ which defines the ordering $<_H$ so that $\mathcal{H}$ satisfies [**CF(H)**] and $k = r(<_H)$.

Let us take a generating set $\mathcal{H}' = \{h'_1, \dots, h'_k\}$ of $H$ which defines the isolated ordering $<_H$ and $k = r(<_H)$. Assume that $h'_1 <_H \dots <_H h'_s \leq_H z_H <_H h'_{s+1} <_H \dots <_H h'_k$. Since $z_H$ is $<_H$-cofinal, for each $i$ there is a non-negative integer $N_i$ such that $1 <_H z_H^{-N_i} h'_i \leq z_H$. Let $h_i = z_H^{-N_i} h'_i$. By assumption, $h'_i = h_i$ if $i \leq s$. Since $H \neq \langle z_H \rangle$, $z_H >_H h'_1 = h_1$. So if necessary, by replacing $h_i$ with $h_1{}^{-1} h_i$, we can assume that $h_i \neq z_H$ for all $i$.

We show that $z_H$ is written by a $\{h'_1, \dots, h'_s\}$-positive word. Assume that $z_H = V h'_i W$, where $i > s$ and $V, W$ are $\mathcal{H}'$-positive or non-empty words. Then $z_H W^{-1} = V h'_i W W^{-1} = V h'_i >_H V z_H$, hence we get $1 \geq_H W^{-1} >_H z_H^{-1} V z_H$. However, $<_H$ is $z_H$-right invariant, $z_H^{-1} V z_H \geq_H 1$. This is a contradiction.

Therefore, the generating set $\mathcal{H} = \{h_1, \dots, h_k\}$ also defines the isolated ordering $<_H$. So $\mathcal{H}$ is a generating set which satisfies the cofinality assumption **[CF(H)]** with cardinal $k = r(<_G)$. □

Thus, under the invariance assumption **[INV(H)]**, we can always find a generating set $\mathcal{H}$ which defines $<_H$ and satisfies the cofinality assumption **[CF(H)]** if the condition on $<_H$ and $z_H$ in Lemma 5.1.3 is satisfied. Moreover, if necessary we may choose $\mathcal{H}$ so that the cardinal of $\mathcal{H}$ is equal to the rank of $<_H$. Since for $z_G$ and $<_G$ the invariance assumption is automatically satisfied, we can always find a generating set $\mathcal{G}$ which defines $<_G$ and satisfies the cofinality assumption **[CF(G)]** if $z_G$ is $<_G$-positive cofinal and $G \neq \langle z_G \rangle$.

Now we put $\Delta_H = z_H h_1^{-1}$. Since $z_H$ and $h_1$ do not depend on a choice of a generating set $\mathcal{H}$, so is $\Delta_H$. As an element of $H$, $\Delta_H$ is characterized by the following property.

**Lemma 5.1.4.** *$\Delta_H$ is the $<_H$-maximal element which is strictly smaller than $z_H$.*

*Proof.* Assume that $z_H h_1^{-1} = \Delta_H \leq_H h <_H z_H$ holds for some $h \in H$. Then $h_1^{-1} \leq_H z_H^{-1} h <_H 1$. By Lemma 5.1.1, $h_1^{-1}$ is the $<_H$-maximal element which is strictly smaller than 1, we get $z_H^{-1} h = h_1^{-1}$, hence $h = z_H h_1^{-1}$. □

Finally, we put $x_i = g_i \Delta_H^{-1} = g_i z_H^{-1} h_1$ and let $\mathcal{X} = \{x_1, \dots, x_m\}$. Then $\{\mathcal{X}, \mathcal{H}\}$ generates the group $X$. The following lemma is rather obvious, but plays an important role in the proof of Theorem 5.0.6.

**Lemma 5.1.5.** *$z_H = z_G$ commutes with all $x_i$.*

*Proof.* By Lemma 5.1.2, $z_H$ commutes with $\Delta_H = z_H h_1^{-1}$. Since $z_H = z_G$ commutes with all $g_i$, we conclude that $z_H$ commutes with all $x_i = g_i \Delta_H^{-1}$. □

### 5.1.2 Property A and Property C criterion

To prove that $\{\mathcal{X}, \mathcal{H}\}$ defines an isolated left ordering $<_X$ of $X$, we use the following criterion which is a genelarization of Property $A$ and $C$ introduced in Chapter 4.

**Definition 5.1.1.** Let $\mathcal{S} = \{s_1, \dots, s_m\}$ be a generating set of a group $G$ and let $W$ be a sub-semigroup of $(\mathcal{S} \cup \mathcal{S}^{-1})^*$.

1. We say $W$ has the *Property A (Acyclic Property)* if no words in $W$ represent the trivial element of $G$.

2. We say $W$ has the *Property C (Comparison Property)* if for each non-trivial element $g \in G$, either $g$ or $g^{-1}$ is represented by a word $w \in W$.

**Propotition 5.1.1.** *Let $W$ be a sub-semigroup of $(\mathcal{S} \cup \mathcal{S}^{-1})^*$. Let $P = \pi(W)$, where $\pi : (\mathcal{S} \cup \mathcal{S}^{-1})^* \to G$ be the natural projection. Then $P$ is equal to a positive cone of a left ordering of $G$ if and only if $W$ has Property A and C.*

*Proof.* If $W$ is a positive cone of a left ordering, then it is obvious that $W$ has Property A and C. We show the converse. Since $W$ is a sub-semigroup, $P$ is a sub-semigroup of $G$. By Property $C$, $G = P \cup \{1\} \cup P^{-1}$. Property $A$ implies $1 \notin P$, hence $G$ is decomposed as a disjoint union, $G = P \coprod \{1\} \coprod P^{-1}$. This shows that $P$ is a positive cone of a left ordering. □

**Definition 5.1.2.** The set of words $W$ in Proposition 5.1.1 is called a *language defining a left-ordering* $<_G$.

As a special case, we get a criterion for a finite generating set to define an isolated ordering, which will be used to show $\{\mathcal{X}, \mathcal{H}\}$ indeed defines an isolated ordering.

**Corollary 5.1.1.** *A finite generating set $\mathcal{G} = \{g_1, \dots, g_m\}$ of a group $G$ defines an isolated ordering of $G$ if and only if the following condition* [**Property A**] *and* [**Property C**] *hold:*

**Property A** *If $g \in G$ is represented by a $\mathcal{G}$-positive word, then $g \neq 1$.*

**Property C** *If $g \neq 1$, then $g$ is represented by either an $\mathcal{G}$-positive or an $\mathcal{G}$-negative word.*

### 5.1.3 Reduced standard factorization

Now we begin to show that $\{\mathcal{X}, \mathcal{H}\}$ indeed defines an isolated left ordering of $X$. As the first step of the proof, we introduce a notion of standard factorization.

Let $PX$ be a sub-semigroup of $X$ generated by $\mathcal{X} = \{x_1, \dots, x_m\}$. A standard factorization of $x \in X$ is a factorization of $x \in X$ of the form

$$\mathcal{F}(x) = rp_1q_1 \cdots p_lq_l$$

where $r, q_1, \cdots, q_l \in H$, $p_1, \dots, p_l \in PX$ which satisfies the conditions

1. $q_i >_H 1$ $(i \neq l)$, and $q_l \geq_H 1$.

2. $q_i \neq z_H^N$ for all $N > 0$.

For a standard factorization $\mathcal{F}(x) = rp_1q_1 \cdots p_lq_l$, we say $l$ is a *complexity* of a standard factorization $\mathcal{F}(x)$, and denote by $c(\mathcal{F})$.

A *distinguished subfactorization* of a standard factorization $\mathcal{F}(x)$ is a part of the standard factorization $\mathcal{F}(x)$ of the form

$$w = x_a q_i p_{i+1} q_{i+1} \cdots p_{i+r} q_{i+r}.$$

which satisfies the two conditions

1. $q_j = \Delta_H$ for all $j = i, i+1, \dots, i+r$.
2. $p_j \in \{x_1, \dots, x_m\}$ for all $j = i+1, \dots, i+r$.
3. $p'_i = p_i x_a^{-1} \in PX \cup \{1\}$.

Since $g_i = x_i \Delta_H$, a distinguished sub-factorization $w$ is naturally regarded as a $\mathcal{G}$-positive word. We will often denote a distinguished subfactorization $w$ by $[g]$, by using an element $g \in G$ represented by a $\mathcal{G}$-positive word $w$, as

$$\begin{aligned} x &= r p_1 q_1 \cdots p_{i-1} q_{i-1} p'_{i+1} (x_a q_i \cdots p_{i+r} q_{i+r}) p_{i+r+1} q_{i+r+1} \cdots p_l q_l \\ &= r p_1 q_1 \cdots p_{i-1} q_{i-1} p'_{i+1} [g] p_{i+r+1} q_{i+r+1} \cdots p_l q_l. \end{aligned}$$

Let us take $x_u$ so that $p'_{i+r+1} = x_u^{-1} p_{i+r+1} \in PX \cup \{1\}$. We say a distinguished subfactorization $w$ is *reducible* if $g g_u \geq_G z_G$ holds for some choice of such $x_u$. We say a distinguished subfactorization $w$ is *maximal* if there are no other distinguished subfactorization which contains $w$. That is, $q_{i-1} \neq \Delta_H$, and $q_{i+r+1} \neq \Delta_H$ if $p_{i+r+1} \in \mathcal{X} = \{x_1, \dots, x_m\}$.

Now we define the notion of a *reduced standard factorization*, which plays an important role in the proof of both Property A and Property C.

**Definition 5.1.3** (Reduced standard factorization). Let $\mathcal{F}(x) = r p_1 q_1 \cdots p_l q_l$ be a standard factorization. We say $\mathcal{F}$ is *reduced* if $q_i <_H z_H$ for all $i$ and $\mathcal{F}$ contains no reducible distinguished subfactorization.

First we show the existence of the reduced standard subfactorization. The proof of next lemma utilize the standard form of amalgamated free product, and mainly works in the generating set $\{\mathcal{G}, \mathcal{H}\}$.

**Lemma 5.1.6.** *Every element $x \in X$ admits a reduced standard subfactorization.*

*Proof.* Since $X$ is an amalgamated free product of $G$ and $H$, every $x \in X$ is written as

$$x = q_0 w_1 q_1 w_2 q_2 \cdots w_k q_k$$

where $q_i \in H$, $w_i \in G$, and $q_i \neq z_H^N$ and $w_i \neq z_G^N$ for any $N \in \mathbb{Z}$ and $i > 0$.

Since $z_G$ is $<_G$-cofinal, for each $i > 0$ there exists $N_i$ which satisfies

$$z_G^{N_i} <_G w_i <_G z_G^{N_i+1}.$$

We put $w_i^* = z_G^{-N_i} w_i$. Then $w_i^*$ satisfies the inequality

$$1 <_G w_i^* <_G z_H$$

Similarly, $z_H$ is $<_H$-cofinal, for each $i > 0$, there exists $M_i$ which satisfies the inequality

$$z_H^{M_i} \leq_H \Delta_H q_i <_H z_H^{M_i+1}.$$

Let $L_i = \sum_{j>i}(N_j + M_j)$, and put $q_i^* = z_H^{-L_i}(z_H^{-M_i}\Delta_H q_i)z_H^{L_i}$. Since $<_H$ is $z_H$-right invariant, $1 \leq_H q_i^* <_H z_H$ holds. Since we have assumed that $q_i \neq z_H^N$, $q_i^* \neq \Delta_H$. Thus, $1 \leq_H q_i^* <_H \Delta_H$.

Then we get a reduced standard factorization of $x$ as follows.

$$\begin{aligned}
x &= q_0 w_1 q_1 \cdots w_l q_l \\
&= q_0(z_G^{N_1} w_1^*) q_1 (z_G^{N_2} w_2^*) \cdots (z_G^{N_{l-1}} w_{l-1}^*) q_{l-1} (z_H^{N_l} w_l^*) q_l \\
&= (q_0 z_H^{N_1}) w_1^* (q_1 z_H^{N_2}) \cdots w_{l-1}^* (q_{l-1} z_H^{N_l}) w_l^* q_l \\
&= (q_0 z_H^{N_1}) w_1^* (q_1 z_H^{N_2}) \cdots w_{l-1}^* (q_{l-1} z_H^{N_l}) w_l^* \Delta_H^{-1} z_H^{M_l} (z_H^{-M_l} \Delta_H q_l) \\
&= (q_0 z_H^{N_1}) w_1^* (q_1 z_H^{N_2}) \cdots w_{l-1}^* \Delta_H^{-1} z_H^{N_l+M_l} (z_H^{-N_l-M_l} \Delta_H q_{l-1} z_H^{N_l+M_l}) (w_l^* \Delta_H^{-1}) q_l^* \\
&= (q_0 z_H^{N_1}) w_1^* (q_1 z_H^{N_2}) \cdots z_H^{N_l+M_l} (w_{l-1}^* \Delta_H^{-1}) q_{l-1}^* (w_l^* \Delta_H^{-1}) q_l^* \\
&= \cdots \\
&= (q_0 z_H^{L_0})(w_1^* \Delta_H^{-1}) q_1^* \cdots (w_{l-1}^* \Delta_H^{-1}) q_{l-1}^* (w_l^* \Delta_H^{-1}) q_l^*.
\end{aligned}$$

Now we choose an arbitrary $\mathcal{G}$-positive word expression of each $w_i$, and rewrite them by the generators $\{\mathcal{X}, \mathcal{H}\}$ by using the relation $g_i = x_i \Delta_H$. Then we get a standard factorization $\mathcal{F}(x)$. By construction, all distinguished sub-factorization is derived from $w_i^*$, so all distinguished sub-factorization are not reducible.

□

### 5.1.4 Reducing operation and the proof of Property A

In the proof of Lemma 5.1.6 given in previous section, we mainly use the generating set $\{\mathcal{G}, \mathcal{H}\}$. In this section we give an alternative way to get a reduced standard factorization, which works mainly in the generating set $\{\mathcal{X}, \mathcal{H}\}$.

For a standard factorization $\mathcal{F}(x)$, let $d(\mathcal{F})$ be the number of maximal reducible distinguished subfactorizations. We may regard a pair of non-negative integers $(c(\mathcal{F}), d(\mathcal{F}))$ as a complexity of standard factorization. We say a standard factorization $\mathcal{F}(x) = rp_1q_1 \cdots p_lq_l$ is *pre-reduced* if $1 <_H q_i <_H z_H$ holds for all $i$.

The following Lemma 5.1.7 and Lemma 5.1.8 gives another proof of existence of a reduced standard factorization.

**Lemma 5.1.7** (Existence of pre-reduced standard factorization). *Every element $x \in X$ admits a pre-reduced standard factorization.*

*Proof.* Since $x_i^{-1} = \Delta_H g_i^{-1} = \Delta_H z_H^{-1}(z_G g_i^{-1})$, every element $x$ has a standard factorization

$$\mathcal{F}(x) = rp_1q_1 \cdots p_lq_l.$$

For each $i$, take $M_i \geq 0$ so that $z_H^{M_i} <_H q_i <_H z_H^{M_i+1}$. Let $L_i = \sum_{j\geq i} M_i$ and $q_i^* = z_H^{-L_i} q_i z_H^{L_{i+1}} = z_H^{-L_{i+1}}(z_H^{-M_i} q_i) z_H^{L_{i+1}}$. Since $<_H$ is $z_H$-right invariant, $1 <_H q_i^* <_H z^H$. Therefore, we get a pre-reduced standard factorization

$$\begin{aligned} x &= rp_1q_1\cdots p_lq_l \\ &= rp_1q_1\cdots p_{l-1}q_{l-1}p_l(z_H^{M_l}q_l^*) \\ &= rp_1q_1\cdots p_{l-1}(q_{l-1}z_H^{M_l})p_lq_l^* \\ &= rp_1q_1\cdots p_{l-1}z_H^{-M_l-M_{l-1}}(z_H^{-M_l-M_{l-1}}q_{l-1}z_H^{M_l})p_lq_l^* \\ &= rp_1q_1\cdots p_{l-1}z_H^{-L_{l-1}}q_{l-1}^*p_lq_l^* \\ &= \cdots \\ &= (rz_H^{-L_0})p_1q_1^*\cdots p_lq_l^*. \end{aligned}$$

□

**Lemma 5.1.8** (Reducing operation). *Let $\mathcal{F}(x) = rp_1q_1\cdots p_lq_l$ be a pre-reduced standard factorization of $x \in X$. If $\mathcal{F}(x)$ contains a reducible distinguished subfactorization, then we can find another pre-reduced standard factorization $\mathcal{F}'(x) = r'p_1'q_1'\cdots$ which satisfies either $c(\mathcal{F}') < c(\mathcal{F})$ or $d(\mathcal{F}') < d(\mathcal{F})$. Moreover, if $r >_H 1$ then $r' >_H 1$.*

*Proof.* Let $g$ be a maximal reducible distinguished subfactorization. Then the pre-reduced standard factorization $\mathcal{F}(x)$ is written as

$$\mathcal{F}(x) = rp_1q_1\cdots p_{i-1}q_{i-1}p_i'[g]x_up_s'q_s\cdots p_lq_l$$

Now take $N > 0$ so that $z_G^N <_G gg_u \leq_G z_G^{N+1}$.

$$\begin{aligned} x &= rp_1q_1\cdots p_{i-1}q_{i-1}p_i'[g]x_up_s'q_s\cdots p_lq_l \\ &= rp_1q_1\cdots p_{i-1}q_{i-1}p_i'z_G^N(z_G^{-N}gg_u)\Delta_H^{-1}p_s'q_s\cdots p_lq_l \end{aligned}$$

For $j < i$, let $p_j^* = z_H^{-N}p_jz_H^N$ and let $p_i^* = z_H^{-N}p_i'z_H^{-N}$. Then,

$$x = (rz_H^N)p_1^*q_1\cdots p_{i-1}^*q_{i-1}p_i^*(gg_uz_G^{-N})\Delta_H^{-1}p_s'q_s\cdots p_lq_l$$

First of all, assume that $(z_G^{-N}gg_u) = z_G = z_H$. Then

$$\begin{aligned} x &= (rz_H^N)p_1^*q_1\cdots p_{i-1}^*q_{i-1}(p_i^*z_H\Delta_H^{-1}p_s')q_s\cdots p_lq_l \\ &= (rz_H^N)p_1^*q_1\cdots p_{i-1}^*q_{i-1}(p_i^*h_1p_s')q_s\cdots p_lq_l \end{aligned}$$

In this case, by modifying the last standard factorization into a pre-reduced standard factorization as in the proof of Lemma 5.1.7, we get a new pre-reduced standard factorization $\mathcal{F}'$ of $x$ which satisfies $c(\mathcal{F}') < c(\mathcal{F})$.

Next assume that $(gg_u z_G^{-N}) \neq z_G$. Then let $g' = (z_G^{-N} gg_u)g_1^{-1}$. Then $1 \leq_G g' < z_G g_1^{-1}$, and we get a new pre-reduced standard factorization

$$\begin{aligned}\mathcal{F}'(x) &= (rz_H^N)p_1^*q_1 \cdots p_{i-1}^* q_{i-1} p_i^* g' g_1 \Delta_H^{-1} p_s' q_s \cdots p_l q_l \\ &= (rz_H^N)p_1^*q_1 \cdots p_{i-1}^* q_{i-1} p_i^* g'(x_1 p_s') q_s \cdots p_l q_l.\end{aligned}$$

$p_i^* = \Delta_H$ if and only if $p_i = \Delta_H$, hence $d(\mathcal{F}') < d(\mathcal{F})$. □

Now we are ready to prove Property A.

**Propotition 5.1.2** (Property A). *If $x$ is expressed as an $\{\mathcal{X},\mathcal{H}\}$-positive word, then $x \neq 1$.*

*Proof.* Assume that $x$ is expressed by an $\{\mathcal{X},\mathcal{H}\}$-positive word. Such a word expression can be modified to a standard factorization which is also an $\{\mathcal{X},\mathcal{H}\}$-positive word. By the proof of Lemma 5.1.7, we can modify such a standard factorization so that it is pre-reduced, preserving the property that it is also an $\{\mathcal{X},\mathcal{H}\}$-positive word. By Lemma 5.1.8, we may modify the $\{\mathcal{X},\mathcal{H}\}$-positive pre-reduced standard expression $\mathcal{F}(x)$ so that it is $\{\mathcal{X},\mathcal{H}\}$-positive reduced standard factorization.

Now let us rewrite $\mathcal{F}(x)$ as a word on $\{\mathcal{G},\mathcal{H}\}$ as follows. First we replace each distinguished subword $[g]$ in $\mathcal{F}(x)$ by the corresponding $\mathcal{G}$-positive word. Then we remove the rest of $x_i$ by using the relation $x_i = g_i\Delta_H^{-1}$. Thus, we write $x = W_0V_1W_1 \cdots V_nW_n$, where $W_i$ is a word on $\mathcal{H}^{\pm 1}$ and $V_i$ is a word on $\mathcal{G}^{\pm 1}$. Since $\mathcal{F}(x)$ is a reduced standard expression, $V_i, W_i \notin \langle z_H \rangle$ for $i > 0$. This implies that $x \neq 1$, since $X = G *_{\langle z_G = z_H \rangle} H$. □

### 5.1.5 Proof of Property C

Next we give a proof of Property C. To being with, we observe a simple, but useful observation.

**Lemma 5.1.9.**
$$h_j^{-1}x_i = N(\mathcal{X},\mathcal{H})\Delta_H^{-1}$$
*where $N(\mathcal{X},\mathcal{H})$ represents a $\{\mathcal{X},\mathcal{H}\}$-negative word.*

*Proof.* Since $z_H = z_G$ and $x_i = g_i\Delta_H^{-1}$, we have

$$z_H = x_i\Delta_H(g_i^{-1}z_G g_1^{-1})x_1\Delta_H.$$

Therefore

$$h_j^{-1}x_i = (h_j^{-1}z_H\Delta_H^{-1})x_1^{-1}(z_G^{-1}g_1g_i)\Delta_H^{-1} = (h_j^{-1}h_1)x_1^{-1}(z_G^{-1}g_1g_i)\Delta_H^{-1}.$$

Since $z_G^{-1}g_i <_G 1$ and $g_1$ is $<_G$-minimal positive, $z_G^{-1}g_i \leq_G g_1^{-1}$, hence $z_G^{-1}g_1g_i \leq_G 1$. Thus, $(h_j^{-1}h_1)x_1^{-1}(z_G^{-1}g_1g_i)$ is written as an $\{\mathcal{X},\mathcal{H}\}$-negative word. □

Now we are ready to prove Property $C$.

**Propotition 5.1.3** (Property C). *Each non-trivial element $x \in X$ is expressed by an $\{\mathcal{X},\mathcal{H}\}$-positive word or an $\{\mathcal{X},\mathcal{H}\}$-negative word.*

*Proof.* Let $x$ be a non-trivial element of $X$ and take a reduced standard factorization of $x$,

$$\mathcal{F}(x) = rp_1q_1\cdots p_lq_l.$$

Recall that each $p_i$ is written as an $\mathcal{X}$-positive word and each $q_i$ is written as an $\mathcal{H}$-positive word. If $r \geq_H 1$, $r$ is also written as an $\mathcal{H}$-positive or an empty word, hence we may express $x$ as an $\{\mathcal{X},\mathcal{H}\}$-positive word.

By induction on $l = c(\mathcal{F})$, we prove that $x$ is expressed by an $\{\mathcal{X},\mathcal{H}\}$-negative word under the assumption $r <_H 1$.

First assume that $q_1 \neq \Delta_H$. Since $r <_H 1$, we can express $r$ as $r = N'(\mathcal{H})h_1^{-1}$, where $N'(\mathcal{H})$ is an $\mathcal{H}$-negative word or an empty word. Take an $\mathcal{X}$-positive word expression of $p_1 = x_{i_1}x_{i_2}\cdots x_{i_p}$. Then by Lemma 5.1.9,

$$\begin{aligned} rp_1p_2q_2\cdots &= (N'(\mathcal{H})h_1^{-1})(x_{i_1}x_{i_2}\cdots x_{i_p})q_1p_2q_2\cdots \\ &= N(\mathcal{H})(h_1^{-1}x_{i_1})x_{i_2}\cdots x_{i_p}q_1p_2q_2\cdots \\ &= N(\mathcal{X},\mathcal{H})\Delta_H^{-1}x_{i_2}\cdots x_{i_p}q_1p_2q_2\cdots \\ &= N(\mathcal{X},\mathcal{H})(h_1^{-1}x_{i_2})\cdots x_{i_p}q_1p_2q_2\cdots \\ &= \cdots \\ &= N(\mathcal{X},\mathcal{H})\Delta_H^{-1}q_1p_2q_2\cdots. \end{aligned}$$

$N(\mathcal{X},\mathcal{H})$ represents a $\{\mathcal{X},\mathcal{H}\}$-negative word.

Since $S$ is a reduced standard factorization, $q_1 <_H z_H$. By Lemma 5.1.4 $\Delta_H$ is the $<_H$-maximal element of $H$ which is strictly smaller than $z_H$, so $Q_1 \leq_H \Delta_H$. We have assumed that $q_1 \neq \Delta_H$, $(\Delta_H^{-1}q_1) <_H 1$. Therefore, we may write $x$ as

$$x = N(\mathcal{X},\mathcal{H})(\Delta_H^{-1}q_1)p_2q_3\cdots p_{l-1}q_l.$$

Then $(\Delta_H^{-1}q_1)p_2q_3\cdots p_{l-1}q_l$ is a reduced standard factorization with complexity $(l-1)$. By induction, $(\Delta_H^{-1}q_1)p_2q_3\cdots p_{l-1}q_l$ is written as an $\{\mathcal{X},\mathcal{H}\}$-negative word, hence we conclude that $x$ is written as an $\{\mathcal{X},\mathcal{H}\}$-negative word

Next assume that $q_1 = \Delta_H$. let $w = [g]$ be the maximal distinguished subfactorization of $\mathcal{F}(x)$ which contains $q_1$. Thus, the reduced standard factorization $S$ is written as

$$\mathcal{F}(x) = rp'_1[g]x_up'_sq_sp_{s+1}\cdots p_lq_i$$

where $p'_s = x_u^{-1}p_s \in PX \cup \{1\}$.

Then by Lemma 5.1.9,

$$\begin{aligned} x &= rp'_1[g]x_up'_sq_sp_{s+1}\cdots p_lq_i \\ &= N(\mathcal{X},\mathcal{H})h_1^{-1}[g]x_u\Delta_H\Delta_H^{-1}p'_sq_s\cdots p_lq_l \\ &= N(\mathcal{X},\mathcal{H})h_1^{-1}[g]g_u\Delta_H^{-1}p'_sq_s\cdots p_lq_l \\ &= N(\mathcal{X},\mathcal{H})\Delta_H(z_G^{-1}gg_u)\Delta_H^{-1}p'_sq_s\cdots p_lq_l \end{aligned}$$

The distinguished subfactorization $g$ is irreducible, hence $z_G^{-1}gg_u <_G 1$. This implies that $z_G^{-1}gg_u = \Delta_H^{-1}N(\mathcal{X},\mathcal{H})$, hence

$$\begin{aligned} &= N(\mathcal{X},\mathcal{H})\Delta_H\Delta_H^{-1}N(\mathcal{X},\mathcal{H})(\Delta_H^{-1}p_sq_s\cdots p_lq_l) \\ &= N(\mathcal{X},\mathcal{H})(\Delta_H^{-1}p'_sq_s\cdots p_lq_l) \end{aligned}$$

If $p'_s \neq 1$, then $(\Delta_H^{-1}p'_sq_s\cdots p_lq_l)$ is a reduced standard factorization. Hence by induction, $(\Delta_H^{-1}p'_sq_s\cdots p_lq_l)$ is expressed by an $\{\mathcal{X},\mathcal{H}\}$-negative word.

Since we have chosen the maximal distinguished sub-factorization $g$, if $p'_s = 1$ then $q_s \neq \Delta_H$. Thus $q_s <_H \Delta_H$, and $(\Delta_H^{-1}q_s)p_{q+1}\cdots p_lq_l$ is a reduced standard factorization. By induction, $(\Delta_H^{-1}q_s)p_{q+1}\cdots p_lq_l$ is expressed by an $\{\mathcal{X},\mathcal{H}\}$-negative word.

Thus we conclude $x$ is expressed by an $\{\mathcal{X},\mathcal{H}\}$-negative word. □

### 5.1.6 Proof of Theorem 5.0.6

Now we are ready to prove Theorem 5.0.6.

*Proof of Theorem 5.0.6.* In Proposition 5.1.2 and Proposition 5.1.3, we have already confirmed the Property $A$ and $C$ for the generating set $\{\mathcal{X},\mathcal{H}\}$. Hence by Corollary 5.1.1 the generating set $\{\mathcal{X},\mathcal{H}\}$ indeed defines an isolated left ordering $<_X$ of $X$.

Now we show the ordering $<_X$ is independent of the choice of generating sets $\mathcal{G}$ and $\mathcal{H}$. Let $\mathcal{G}' = \{g'_1,\ldots\}$ and $\mathcal{H}' = \{h'_1,\ldots\}$ be other generating sets of $G$ and $H$ satisfying **[CF(G)]** and **[CF(H)]**. Recall that $\Delta_H = z_Hh_1^{-1}$ does not depend on a choice of a generating set $\mathcal{H}$. Let $x_i = g_i\Delta_H^{-1}$, $x'_i = g'_i\Delta_H^{-1}$, $\mathcal{X} = \{x_1,\ldots,\}$, and $\mathcal{X}' = \{x'_1,\ldots,\}$.

Since $\mathcal{H}$ and $\mathcal{H}'$ are generator of the same semigroup, we may write $h_i$ as an $\mathcal{H}'$-positive word. Similarly, since $\mathcal{G}$ and $\mathcal{G}'$ are generator of the same semigroup, we may write $g_i$ as a $\mathcal{G}'$-positive word $g_i = g'_{i_1}g'_{i_2}\cdots g'_{i_l}$. Thus,

$$x_i = g_i\Delta_H^{-1} = g'_{i_1}g'_{i_2}\cdots g'_{i_l}\Delta_H^{-1} = x'_{i_1}\Delta_Hx'_{i_2}\Delta_H\cdots x'_{i_{l-1}}\Delta_Hx'_{i_l}$$

so $x_i$ is written as an $\{\mathcal{X}',\mathcal{H}'\}$-positive word. Thus, if $x \in X$ is expressed by an $\{\mathcal{X},\mathcal{H}\}$-positive word, then $x$ is also represented by an $\{\mathcal{X}',\mathcal{H}'\}$-positive word. By interchanging the role of $(\mathcal{G},\mathcal{H})$ and $(\mathcal{G}',\mathcal{H}')$, we conclude that $\{\mathcal{X},\mathcal{H}\}$ and $\{\mathcal{X}',\mathcal{H}'\}$ generates the same sub-semigroup of $X$. Hence they define the same isolated ordering of $X$.

(3) is obvious from the definition of $<_X$.

The inequality $h_1 <_X h_2 <_X \cdots <_X h_p$ is obvious from the definition. By Lemma 5.1.9, $x_i <_X h_1$ for all $i$. Now we show $x_i <_X x_j$ if $i < j$. Since $g_i <_G g_j$ if $i < j$, $g_i^{-1}g_j$ is written as a $\mathcal{G}$-positive word. Now by definition $g_i = x_i\Delta_H$, so we may express $\mathcal{G}$-positive word expression of $g_i^{-1}g_j$ as an $\{\mathcal{X},\mathcal{H}\}$-positive word expression of the form $g_i^{-1}g_j = P(\mathcal{X},\mathcal{H})\Delta_H$. Therefore $x_i^{-1}x_j = \Delta_Hg_i^{-1}g_j\Delta_H^{-1} = P(\mathcal{X},\mathcal{H})$, so $x_i < x_j$. The assertion that $z = z_G = z_H$ is

$<_X$-positive cofinal is obvious. To see $<_X$ is $z$-right invariant, we observe that $z^{-1}x_i z = x_i >_X 1$ and $z^{-1}h_j z >_X 1$. Now for $x, x' \in X$, assume $x <_X x'$, so $x^{-1}x'$ is written as $\{\mathcal{X}, \mathcal{H}\}$-positive word $w = s_1 \cdots s_m$, where $s_i$ denotes $x_j$ or $h_j$. Then $z^{-1}(x^{-1}x)z = (z^{-1}s_1 z) \cdots (z^{-1}s_m z) >_X 1$, hence $xz <_X x'z$. This completes the proof of (4).

To show (5), recall that by Lemma 5.1.3, we may choose the generating sets $\mathcal{G}$ and $\mathcal{H}$ so that the cardinal of $\mathcal{G}$, $\mathcal{H}$ are equal to $r(<_G)$, $r(<_H)$ respectively. Thus, $r(<_X) \leq r(<_G) + r(<_H)$.

Finally, we prove that $\langle x_1 \rangle$ is the unique $<_X$-convex non-trivial proper subgroup of $X$. Recall by (2), (4) and Lemma 5.1.1, $x_1$ is the minimal $<_X$-positive element of $X$, hence $x_1$ does not depend on a choice of $\mathcal{G}$ and $\mathcal{H}$. In particular, $\langle x_1 \rangle$ is a non-trivial $<_X$-convex subgroup.

Let $C$ be a $<_X$-convex subgroup of $X$. Assume that $C \supset \langle x_1 \rangle$. Let $y \in C - \langle x_1 \rangle$ be an $<_X$-positive element. Then $y$ is written as $y = x_1^m x_j P(\mathcal{X}, \mathcal{H})$ or $y = x_1^m h_l P(\mathcal{X}, \mathcal{H})$ where $m \geq 0$, $l > 0$, $j > 1$ and $P(\mathcal{X}, \mathcal{H})$ is an $\{\mathcal{X}, \mathcal{H}\}$-positive word. Since $x_1 \in C$, we may choose $y$ so that $m = 0$.

First we consider the case $\mathcal{X} \not\subset \langle x_1 \rangle$. Then we may choose $y$ so that $1 < x_2 \leq_X y$ holds, so the convexity assumption implies $x_2 \in C$. Now observe that $x_1^{-1}x_2 = \Delta_H g_1^{-1} g_2 \Delta_H^{-1} = \Delta_H P(\mathcal{X}, \mathcal{H})$, hence

$$1 <_X h_p \leq z_H h_1^{-1} = \Delta_H <_X \Delta_H P(\mathcal{X}, \mathcal{H}) = x_1^{-1}x_2$$

Since $x_1^{-1}x_2 \in C$, this implies $\mathcal{X} \cup \mathcal{H} = \{x_1, \ldots, x_k, h_1, \ldots, h_p\} \subset C$. Therefore we conclude $C = X$.

Next we consider the case $\mathcal{X} \subset \langle x_1 \rangle$. This happens only when $G = \mathbb{Z} = \langle g_1 \rangle$ and $z_G = g_1^N$. Then we may choose $y$ so that $1 < h_1 \leq_X y$ holds, so $h_1 \in C$. Then $x_1^{-1}h_1 = \Delta_H g_1^{-1} h_1 = h_1^{-1} z_G g_1^{-1} h_1$ so $z_G g_1^{-1} = g_1^{N-1} \in C$. This implies $z_G = z_H \in C$, so $C = X$.

□

### 5.1.7 Computational issues

In this section we briefly mention the computational issue concerning the isolated ordering $<_X$. Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a group presentation and $<_G$ be a left ordering of $G$. The *order-decision problem* for $<_G$ is the algorithmic problem of deciding for an element $g \in G$ given as a word on $\mathcal{S} \cup \mathcal{S}^{-1}$ whether $1 <_G g$ holds or not. Clearly, the order-decision problem is harder than the word problem, since $1 <_G g$ implies $1 \neq g$. It is interesting to find an example of a left ordering $<_G$ of a group $G$, such that the order-decision problem for $<_G$ is unsolvable but the word problem for $G$ is solvable.

There is another algorithmic problem which is related to the order-decision problem of isolated orderings. We say a word on $\mathcal{G} \cup \mathcal{G}^{-1}$ is $\mathcal{G}$*-definite* if $w$ is $\mathcal{G}$-positive or $\mathcal{G}$-negative, or empty. If $\mathcal{G}$ defines an isolated ordering of $G$, then every $g \in G$ admits a $\mathcal{G}$-definite word expression. The $\mathcal{G}$*-definite search problem* is a problem to find a $\mathcal{G}$-definite word expression of a given element of $G$.

**Theorem 5.1.1.** *Let us take $G, H, X, <_G, <_H, z_G, z_H, \mathcal{G}, \mathcal{H}, \mathcal{X}$ as in Theorem 5.0.6.*

1. *The order-decision problem for $<_X$ is solvable if and only if the order-decision problem for $<_G$ and $<_H$ are solvable.*
2. *The $\{\mathcal{X}, \mathcal{H}\}$-definite search problem is solvable if and only if the $\mathcal{G}$-definite search problem and the $\mathcal{H}$-search problem are solvable.*

*Proof.* Observe that since the restriction of $<_X$ to $G$ and $H$ yields the ordering $<_G$ and $<_H$ respectively, if the order-decision problem for $<_X$ is solvable, then so is for $<_G$ and $<_H$. Assume that $\{\mathcal{X}, \mathcal{H}\}$-definite search problem is solvable. It is easy to see that this implies $\mathcal{H}$-definite search problem is solvable. Since if $x \in G \subset X$, then $\{\mathcal{X}, \mathcal{H}\}$-definite word expression of $x$ is naturally transformed into $\mathcal{G}$-positive word by using $g_i = x_i \Delta_H$ and $z_H = z_G$, hence $\mathcal{G}$-definite search problem is also solvable.

The proof of converse is implicit in the proof of Theorem 5.0.6 (1). Recall that in the proof of Property C (Proposition 5.1.3), we have shown that for a reduced standard factorization $\mathcal{F}(x) = rp_1q_1 \cdots p_lq_l$, $x >_X 1$ if $r >_H 1$ and $x <_X 1$ if $r_H < 1$. Moreover, the proof of Property C (Proposition 5.1.3) is constructive, hence we can algorithmically compute an $\{\mathcal{X}, \mathcal{H}\}$-negative word expression of $x$ if $r <_H 1$ if the $\mathcal{G}$-definite search problem and the $\mathcal{H}$-search problem is solvable.

Thus, to solve the order-decision problem or $\{\mathcal{X}, \mathcal{H}\}$-definite search problem, it is sufficient to compute a reduced standard factorization. We have established two different method to obtain a reduced standard factorization, in the proof of Lemma 5.1.6 and Lemma 5.1.8. Both proofs are constructive, hence we can algorithmically compute a reduced standard expression. □

It is not difficult to analyze a computational complexity of order-decision problem or the $\{\mathcal{X}, \mathcal{H}\}$-definite search problems based on the algorithm obtained from the proof of Proposition 5.1.3, Lemma 5.1.6 and Lemma 5.1.8. In particular, we get tthe following result on the computational complexities.

**Propotition 5.1.4.** *Let us take $G, H, X, <_G, <_H, z_G, z_H, \mathcal{G}, \mathcal{H}, \mathcal{X}$ as in Theorem 5.0.6.*

1. *If the order-decision problem for $<_G$ and $<_H$ are solvable in polynomial time with respect to the input of the word length, then the order-decision problem for $<_X$ is also solvable in polynomial time.*
2. *If the $\mathcal{G}$-definite search problem and the $\mathcal{H}$-definite search problem is solvable in polynomial time, then $\{\mathcal{X}, \mathcal{H}\}$-definite search problem is also solvable in polynomial time.*
3. *Moreover, if one can always find a $\mathcal{G}$-definite and a $\mathcal{H}$-definite word expression whose length are polynomial with respect to the length of the input word, then one can always find $\{\mathcal{X}, \mathcal{H}\}$-definite word expression whose length is polynomial with respect to the length of the input word.*

## 5.2 Examples

In this section we give examples of isolated left orderings produced by Theorem 5.0.6.

### 5.2.1 Examples and new phenomenons

First we give examples of isolated left orderings produced by Theorem 5.0.6. All examples in this section are new, and have various properties which previously known isolated orderings do not have. For the sake of simplicity, in the following examples we only use the infinite cyclic group $\mathbb{Z}$, the most fundamental example of group having isolated orderings, as the basic building blocks. Other groups and isolated orderings, such as groups having only finitely many left-orderings, or the braid group $B_n$ with the Dubrovina-Dubrovin ordering $<_{DD}$, also can be used to construct new examples of isolated orderings.

**Example 5.2.1.** Let $a_1, \ldots, a_m$ $(m > 1)$ be positive integers bigger than one and consider the group obtained as a central cyclic amalgamated free product of $m$ infinite cyclic groups $\mathbb{Z}^{(i)}$ $(i = 1, \ldots, n)$,

$$\begin{aligned} G = G_{a_1,\ldots,a_m} &= *_{\mathbb{Z}}\mathbb{Z}^{(i)} = \mathbb{Z}^{(1)} *_{\mathbb{Z}} (\mathbb{Z}^{(2)} *_{\mathbb{Z}} (\cdots (\mathbb{Z}^{(m-1)} *_{\mathbb{Z}} \mathbb{Z}^{(m)}) \cdots) \\ &= \langle x_1, \ldots, x_m \mid x_1^{a_1} = x_2^{a_2} = \cdots = x_m^{a_m} \rangle \end{aligned}$$

By Theorem 5.0.6, the group $G$ has an isolated left ordering $<_G$.

The group $G$ is the simplest example of groups and isolated orderings constructed by Theorem 5.0.6, but nevertheless has various interesting properties which have not appeared in the previous examples:

*(1):* $r(<_G) = m$.

Since $G$ is an amalgamated free product of $m$ infinite cyclic groups, the rank of $G$ is $m$. On the other hand, Theorem 5.0.6 (5) says $c(<_G) \leq m$. Hence the rank of isolated ordering $<_G$ is $m$.

*(2): The isolated orderings $<_G$ of $G$ is not derived from Dehornoy-like orderings if $m > 2$.*

As we have seen in Theorem 4.1.1, a Dehornoy-like ordering having Property F produces isolated orderings and vise versa. Now we observe that $<_G$ cannot be obtained from Dehornoy-like ordering if $m > 2$. Assume that $<_G$ is obtained from the Dehornoy-like ordering $<_D$. Then by Proposition 4.1.3 there exists at least $r(<_H) - 1$ proper, $<_H$-convex nontrivial subgroups. However, we have seen that in Theorem 5.0.6 (6) the isolated orderings $<_G$ has only one proper, $<_G$-convex nontrivial subgroups.

*(3): The natural right $G$-action on $LO(G)$ has at least $2^m$ distinct orbits derived from isolated orderings.*

Recall that there exists a natural, continuous right $G$-action on LO($G$), Observe that we have two choices of isolated orderings for each infinite cyclic group factor $\mathbb{Z}^{(i)}$. Thus by Theorem 5.0.6 we can construct $2^m$ distinct isolated left orderings of $G$. It is easy to see all of them belong to distinct $G$-orbits. Hence, we have at least $2^m$ different $G$-orbits derived from isolated orderings. Recall that $m = r(<_G)$ is equal to $r(G)$, the rank of $G$.

*(4): The natural right Aut(G)-action on LO(G) has at least $2^m$ distinct orbits derived from isolated orderings if all $a_1, \dots, a_m$ are distinct.*

As in the group $G$ itself, there is a natural right Aut($G$)-action on LO($G$). As in (3), if all $a_1, \dots, a_m$ are distinct, then we have $2^m$ distinct Aut($G$)-orbit derived from isolated orderings.

**Example 5.2.2.** Next we consider the construction of the case $z_H$ is non-central. First of all, let $G_{m,n} = \langle b, c \mid b^m = c^n \rangle$. By Example 5.2.1, $G_{m,n}$ has an isolated left ordering $<_{m,n}$ which is defined by $\{bc^{1-n}, c\}$.

Then $bc^{1-n} \cdot b^m = bc$ is non-central element, but is $<_{m,n}$-positive cofinal. $<_{m,n}$ is $(bc^{1-n})$-right invariant by Lemma 5.1.1, and $<_{m,n}$ is also $b^m$-right invariant since $b^m$ is central. Thus, $<_{m,n}$ is $(bc)$-right invariant.

Thus, we can take $bc$ as an element $z_H$ in Theorem 5.0.6. Now we consider the group $G'_{p,q,m,n} = \mathbb{Z} *_{\mathbb{Z}} G_{m,n} = \mathbb{Z} *_{\mathbb{Z}} (\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z})$ defined by

$$\langle a, b, c \mid b^m = c^n, a^p = (bc)^q \rangle$$

This group has an isolated left ordering $<_G$, defined by $\{a(bc)^{1-q}, bc^{1-n}, c\}$. Let us put $x = a(bc)^{1-q}$, $y = (bc)^{1-n}$, and $z = c$. Then the group $G_{p,q,m,n}$ is presented as

$$G'_{p,q,m,n} = \langle x, y, z \mid (yz^{n-1})^m = z^n, (x(yz^n)^{q-1})^p = (yz^n)^q \rangle$$

Now we observe a remarkable feature of the group $G'_{p,q,m,n}$.

*(5): The center of $G' = G'_{p,q,m,n}$ is trivial.*

Since $G'$ is an amalgamated free product of $G_{m,n}$ and $\mathbb{Z}$, central element in $G'$ is written as $a^{pN} = (bc)^{qN}$ for some $N$. However, if $N \neq 0$, $(bc)^{qN}$ do not commute with $b$, hence it is not central.

This is the first example of group having isolated left ordering with trivial center. In a similar manner, we can construct many new examples of groups having isolated ordering with trivial center.

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# 論文の内容の要旨

## 論文題目：Construction of invariant group orderings from topological point of view

## （位相幾何の視点からの群の不変順序の構成）

## 氏名：伊藤哲也

群 $G$ 上の、$G$ 自身の左（右）作用により保たれる全順序 $<_G$ を $G$ の**左（右）不変順序**と呼ぶ。全順序 $<_G$ が左不変かつ右不変であるとき、$<_G$ を両側不変順序と呼び、これらをまとめて以下**不変順序**と呼ぶ。群 $G$ の不変順序 $<_G$ に対し、単位元よりも真に大きい $G$ の元のなす $G$ の部分半群 $P(<_G)$ を $<_G$ の *Positive cone* と呼ぶ。群上の不変順序は、一次元力学系と密接に関連し、近年トポロジー・幾何学などの観点から着目されるようになった対象である。

本論文では、位相幾何の手法および観点から、群 $G$ 上の非自明な不変順序の構成について考察した。論文の前半では、代数的位相幾何学の手法を用いて両側不変順序について考察し、反復積分を用いた幾何的な両側不変順序の構成および三次元多様体の基本群上の両側不変順序の存在とねじれアレキサンダー不変量との関連について調べた。主となる論文の後半では、孤立順序と呼ばれる位相的に特別な左不変順序の構成について研究し孤立順序の新しい例を多数構成することに成功した。

$G$ 上の左不変順序全体のなす集合 $\mathrm{LO}(G)$ 上には自然な位相が定まり、位相空間となることが知られている。Sikora[4] により、位相空間 $\mathrm{LO}(G)$ は完全不連結、コンパクト、距離付け可能であり、Cantor 集合と近い性質を持つことが知られている。Cantor 集合との違いは $\mathrm{LO}(G)$ は孤立点を持ちうることである。

**定義 1.** 群 $G$ の左不変順序 $<$ が左不変順序全体のなす位相空間 $\mathrm{LO}(G)$ の孤立点であるとき、$<$ を $G$ の孤立順序であると呼ぶ。

孤立順序は群 $G$ の性質を強く反映していることが期待される。孤立順序の性質についてはこれまでに様々な研究がなされている。例えば、Navas[3] は Positive cone が有限生成半群であるような不変順序 $<_G$ は孤立順序であることを示した。しかし、現在のところ具体的に知られている孤立順序の例は非常に少なく、孤立順序の具体例の発見および一般的な構成についてが大きな問題となっている。

本論文では、二つの異なる手法での孤立順序の一般的構成法を与えるともに孤立順序の新しい具体例を多数構成した。

群の左不変順序 $<$ の重要な例として、Dehornoy により発見された Braid 群 $B_n$ 上の Dehornoy 順序が知られている。この順序は非常に豊富な組み合わせ・代数的な構造を持ち、多くの異なる定義が知られている [1]。Dehornoy 順序は孤立順序ではないが、Dehornoy 順序を代数的に変形することにより、Dubrovina-Dubrovin 順序と呼ばれる Braid 群の孤立順序が得られる [2] など、Dehornoy 順序は多くの興味深い性質を持つことが知られている。

本論文では Dehornoy 順序の定義を一般化した Dehornoy-like 順序と呼ばれる順序を導入し、その一般的な性質を調べた。

**定義 2** (Dehornoy-like 順序). $<_G$ を有限生成群 $G$ 上の左順序とする。ある $G$ の順序のついた有限生成系 $\mathcal{S} = \{s_1, \dots, s_n\}$ が存在し、$g$ が下の条件 $\sigma(\mathcal{S})$-positive を満たす事と $1 <_G g$ が同値であるとき、$<_G$ を $\mathcal{S}$ の定める Dehornoy-like 順序であると呼ぶ。

**$\sigma(\mathcal{S})$-positive:** ある $i$ が存在し、$g$ は正の生成元 $s_i$ を少なくても一つ含むような $s_i, s_{i+1}^{\pm 1}, \dots, s_n^{\pm 1}$ のみを用いた word で表される。

群 $G$ を Braid 群 $B_n$、生成系 $\mathcal{S}$ を標準生成系 $\mathcal{S} = \{\sigma_1, \dots, \sigma_{n-1}\}$ としたとき、$\mathcal{S}$ の定める Dehornoy-like 順序は Dehornoy 順序に他ならない。

生成系 $\mathcal{S}$ が Property $F$ というある性質を満たすとき、Dehornoy-like 順序は多くの点で Dehornoy 順序と同様の興味深い性質を持つこと (Dehornoy 順序で成り立つ多くの結果が拡張されること) を示し、特に、Dehornoy-like 順序を用いて孤立順序の一般的な構成法を与えた。

**定理 1.** $<_G$ を $(\mathcal{S})$ の定める群 $G$ の *Dehornoy-like* 順序とする。生成系 $\mathcal{S}$ が *Property* $F$ を持つとき、*Dehornoy-like* 順序を変形することで $G$ の孤立順序が得られる。

また、Dehornoy-like 順序の非自明な例を構成した。

**定理 2.** $m, n > 0$ について $G_{m,n}$ を二つの無限巡回群 $\mathbb{Z}$ の融合積 $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} = \langle x, y \mid x^m = y^n \rangle$ とし、生成元 $s_1 = xyx^{-m+1}, s_2 = x^{m-1}y^{-1}$ とする。このとき、生成系 $\mathcal{S} = \{s_1, s_2\}$ は $G_{m,n}$ 上の *Dehornoy-like* 順序を定める。

系として、これらの群が孤立順序を持つことが示される。

Dehornoy-like 順序の定義は組み合わせ・代数的なものであるが、定理 2 は、群 $G_{m,n}$ が融合積であることを利用し、$G_{m,n}$ の Bass-Serre Tree への作用を用いた幾何的な議論により証明される。

また、孤立順序の別の一般的な構成法として、融合積上の孤立順序について考察し次の結果を得た。

**定理 3.** $<_G$、$<_H$ をそれぞれ群 $G$、$H$ 上の *positive cone* が有限生成となるような孤立順序とする。$z_G$ を $G$ の非自明な中心の元、$z_H$ を $H$ の非自明な元とする。$<_G, <_H, z_G, z_H$ が次の 3 つの条件を満たすと仮定する。

**Inv(H)** 順序 $<_H$ は $z_H$ の右からの作用で不変である。つまり、$a <_H b$ であれば $az_H <_H bz_H$ が常に成り立つ。

**Cof(H)** 任意の $h \in H$ に対しある自然数 $N$ が存在し $z_H^{-N} <_H h <_H z_H^N$ が成り立つ。

**Cof(G)** 任意の $g \in G$ に対しある自然数 $N$ が存在し $z_G^{-N} <_G h <_G z_G^N$ が成り立つ。

この時融合積 $X = G *_{\langle z_G = z_H \rangle} H$ 上に孤立順序 $<_X$ が存在し、次が成り立つ。

*1.* $<_X$ の *Positive cone* は有限生成である。

*2.* $h, h' \in H \subset X$ について $h <_H h'$ であれば、$h <_X h'$ が成り立つ。

*3.* $g, g' \in G \subset X$ について $g <_G g'$ であれば、$g <_X g'$ が成り立つ。

この定理は既存の孤立順序二つを混合して新しい孤立順序を構成するものであり、定理を用いる事で孤立順序を持つ群を多数構成できる。

**例 1.** 1. 自然数 $n_1, \ldots, n_m$ に対し、群 $G = \langle a_1, \ldots a_m \mid a_1^{n_1} = a_2^{n_2} = \cdots = a_m^{n_m} \rangle$ は孤立順序を持つ。

2. 自然数 $m, n, p, q$ に対して、群 $G = \langle x, y, z \mid (yz^{n-1})^m = z_n, x(yz^n)^{q-1})^p = (yz^n)^q \rangle$ は $x, y, z$ で生成される部分半群を positive cone とするような孤立順序を持つ。

これまでに知られている孤立順序を持つ群はすべて非自明な中心を持っていた。上の例 (2) で現れる群は、中心が自明となるような孤立順序を持つ群の最初の例を与えている。また、定理 3 で得られる孤立順序の多くが Dehornoy-like 順序の変形として得られないことを示し、孤立順序と Dehornoy-like 順序が完全に対応するわけではない事を示した。

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