Scattering Theory on Manifolds with Asymptotically Polynomially Growing Ends

多項式増大する無限遠境界を持つ多様体上の散乱理論

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Scattering Theory on Manifolds with Asymptotically Polynomially Growing Ends

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Preface

In this thesis we study scattering theory for Schrödinger equations on non-compact manifolds.

From the beginning of the 20th century, quantum mechanics has been the central research topics in physics. The principles of quantum mechanics are fundamental, and it has branched out into almost every aspect of modern physics such as elementary particles physics and condensed matter physics. In quantum mechanics, a partial differential equation called a Schrödinger equation is used to describe how the quantum state of a physical system changes in time. The time evolution is generated by a partial differential operator with respect to space, which is called a Hamiltonian, or a Schrödinger operator.

Scattering is a physical phenomenon where moving particles and waves are forced to deviate from a straight trajectory by non-uniformities of the medium or forces. Quantum scattering describes how solutions of partial differential equations, propagating freely in the distant past, interact with potentials at the present, and then propagate away in the distant future. Quantum scattering is a basic tool for physicists to study microscopic structure of particles. For example, the Rutherford scattering (also referred as Coulomb scattering) of alpha particles against gold nuclei led to the discovery of the nuclei, or the Rutherford model of atom.

Quantum mechanics and Schrödinger equations have been studied mathematically also. A quantum state is represented by an element of a complex Hilbert space, and a Schrödinger operator is a self-adjoint realization of a partial differential operator. Time evolution is a unitary operator generated by the self-adjoint operator. Mathematical quantum scattering theory is a perturbation theory. If the unperturbed "free" operator H_0 and the perturbed "full" operator H are close in some sense, the information of the free system makes it possible to describe the full system.

We here introduce basic concepts of mathematical quantum scattering theory. Let H be a self-adjoint operator on a Hilbert space $\mathscr{H}$. The Schrödinger equation with initial value f,

$$i\frac{\partial u}{\partial t} = Hu, \quad u(0) = f,$$

has a unique solution $u(t) = \exp(-itH)f$. The solution of the same equation with the operator H_0 is given by $u_0(t) = \exp(-itH_0)f_0$. Scattering theory investigates the asymptotics of $u(t)$ as $t \to \pm\infty$ in terms of $u_0(t)$, and the unitarily equivalence of the absolutely continuous parts $H_0^{(ac)}$ and $H^{(ac)}$. The wave operator $W_\pm(H,H_0)$ for the pair (H,H_0) is defined as follows

$$W_\pm(H,H_0) := \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_0}P_{ac}(H_0)$$

where $P_{ac}(H_0)$ is the orthogonal projection onto the absolutely continuous subspace of H_0. The existence of the limit is equivalent to the condition that for every free trajectory $u_0^\pm(t)$ which starts from the initial data $f_0^\pm \in P_{ac}(H_0)\mathscr{H}$, there exists a trajectory $u^\pm(t)$ which starts from $f^\pm$ such that

$$\|u^\pm(t) - u_0^\pm(t)\| = \|e^{-itH}f^\pm - e^{-itH_0}f_0^\pm\| \to 0 \tag{0.1}$$

as $t \to \pm\infty$. If the range of wave operators $W_\pm$ coincides with the absolute continuous subspace $P_{ac}(H)\mathscr{H}$ of H, then the wave operators $W_\pm$ are said to be complete. If the wave operators

$W_\pm(H,H_0)$ exist, completeness of the wave operators $W_\pm(H,H_0)$ is equivalent to the existence of the inverse wave operators $W_\pm(H,H_0)$. Completeness of the wave operators implies that for every trajectory $u^\pm(t)$ which starts from the initial data $f^\pm \in P_{ac}(H)\mathscr{H}$, there exists a free trajectory $u_0^\pm(t)$ which starts from $f_0^\pm$ such that (0.1) holds. For complete wave operators the operators $H_0^{(ac)}$ and $H^{(ac)}$ are unitarily equivalent each other.

The operator $S = W_+^* W_-$ is called the scattering operator, and is unitary in $P_{ac}(H_0)\mathscr{H}$ when wave operators $W_\pm$ are complete. Since the scattering operator S commutes with H_0, S acts as a multiplication by a operator-valued function $S(\lambda)$, called the scattering matrix, in the diagonal representation for H_0. The scattering operator and scattering matrix are important objects in mathematical physics, because they relate the initial state of a system with the final state, and is "measured" by physical experiments.

We give two typical examples of Schrödinger operator and corresponding scattering theory. Let $H = H_0 + V(x)$ be a Schrödinger operator in the space $L^2(\mathbb{R}^n)$ where $H_0 = -\Delta$ is the Laplacian. We assume that the potential function V is a multiplication operator satisfying the short-range condition $V(x) = O(|x|^{-\mu})$, with $\mu > 1$ as $|x| \to +\infty$. Then it is well-known that the wave operators $W_\pm(H,H_0)$ exist and are complete. The scattering operator and scattering matrix are defined accordingly. If V satisfies the long-range condition $V(x) = O(|x|^{-\mu})$, with $\mu > 0$, the wave operators do not exist in general. But we can show the existence and completeness of wave operators by modifying the free propagation. We note that the Coulomb potential $V(x) = |x|^{-1}$ is a critical case.

We consider Schrödinger operators on manifolds, which are sums of the Laplacian on manifolds and potential functions. Richard Melrose introduced the concept of scattering manifolds and investigated their properties extensively. We also consider manifolds with asymptotically polynomially growing ends. We here explain concepts of these manifolds briefly. First recall the polar coordinates in Euclidean spaces $\mathbb{R}^n$, that is, $(r,\theta) \in \mathbb{R}_+ \times \mathbb{S}^{n-1}$, where r is the radial coordinate and θ is the angular coordinates. Let $\Delta_{\mathbb{S}^{n-1}}$ be the Laplacian on $\mathbb{S}^{n-1}$, then the Laplacian Δ on the Euclidean space can be written as $\Delta = \frac{\partial^2}{\partial r^2} + r^{-2}\Delta_{\mathbb{S}^{n-1}}$. Denote by $d\theta^2$ the Riemannian metric on $\mathbb{S}^{n-1}$, then the Riemannian metric dx^2 on Euclidean space is written as $dx^2 = dr^2 + r^2 d\theta^2$. We can regard the Euclidean space as a cone, the size of which grows proportionally to the radial parameter r. We extend these concepts to general non-compact manifolds. A non-compact Riemannian manifold M with an "end", where the metric g is of the form $g = dr^2 + r^2 d\theta^2$ for large r, is called a manifold with conic ends. Here we also assume that the "end" is decomposed into the product of $\mathbb{R}_+$ and a compact manifold ∂M, called a boundary manifold, and we take the (extended) polar coordinates $(r,\theta) \in \mathbb{R}_+ \times \partial M$, and the complement of the "end" is relatively compact. If we allow metric perturbations of $g = dr^2 + r^2 d\theta^2$ which decays as $r \to +\infty$, then the corresponding manifold is called a manifold with asymptotically conic ends, or, a scattering manifold. If the metric in the end is asymptotically of the form $g = dr^2 + r^{2\alpha} d\theta^2$ with $\alpha > 0$, then the corresponding manifold is called a manifold with asymptotically polynomially growing ends, indeed, the size of this manifold at r grows as r^α.

This paper is constructed by three Parts. In Part I, we show the existence of modified wave operators for Schrödinger equations on scattering manifolds with long-range metric perturbation and long-range potentials. In Part II, we consider Schrödinger equations on manifolds with polynomially growing ends with short-range potentials. We prove the existence and completeness of wave operators. The scattering operator and scattering matrix are defined accordingly. We investigate the properties of the scattering matrix in Part III. We show that the scattering matrix defined on the L^2 space of the boundary manifold does not change the wave front set if $\alpha > 1$. We give a summary and remarks of each of these three parts.

Part I

We consider Schrödinger equations on scattering manifolds with long-range metric perturbation and long-range potentials (see Melrose [25] about scattering manifolds). We employ the formulation of Ito-Nakamura [16], which uses the two-space scattering framework of Kato [21]. Following Hörmander [13] and Dereziński and Gérard [6], we construct exact solutions of the Hamilton-Jacobi equation and show the existence of the modified two-space wave operators $\text{s-lim}_{t\to\pm\infty} e^{itH} J e^{-iS(t,D_r,\theta)}$ using the stationary phase method, where J is an identification operator from the reference system to the original system.

We refer Reed and Simon [31], Dereziński and Gérard [6], and Yafaev [36] for general concepts of wave operators and scattering theory for Schödinger equations. The concept of wave operator was introduced by Møller [27]. The existence of wave operators has long been studied (see Cook [2], and Kuroda [23]) for short range potentials, which decay faster than the Coulomb potential. For the Coulomb potential, it was proved by Dollard [7, 8] that the wave operators do not exist unless the definition is modified. Dollard introduced the concept of the modified wave operators. Hörmander [13] constructed exact solutions of the Hamilton-Jacobi equation (see also [14] vol. IV) for general smooth long-range potentials and showed the existence of modified wave operators.

The spectral properties of Laplace operators on a class of non-compact manifolds were studied by Froese, Hislop and Perry [10, 11], and Donnelly [9] using the Mourre theory (see, the original paper Mourre [28], and Perry, Sigal, and Simon [29]). In early 1990s, Melrose introduced a new framework of scattering theory on a class of Riemannian manifolds with metrics called scattering metrics. Melrose and Zworski [26] showed that the absolute scattering matrix, which is defined through the asymptotic expansion of generalized eigenfunctions, is a Fourier integral operator. Vasy [33] studied Laplace operators on such manifolds with long-range potentials of Coulomb type decay ($|V(r,\theta)| \leq Cr^{-1}$).

Ito and Nakamura [16] studied a time-dependent scattering theory for Schrödinger operators on scattering manifolds. They used the two-space scattering framework of Kato [21] with a simple reference operator $D_r^2/2$ on a space of the form $\mathbb{R}\times\partial M$, where ∂M is the boundary of the scattering manifold M.

We employ the formulation of Ito and Nakamura [16], and consider general long-range metric perturbations and potential perturbations. We assume that the scalar potential decay as $|V(r,\theta)| \leq Cr^{-\varepsilon}, \varepsilon > 0$.

We state some remarks along with the outline of the proof. The time-dependent modifier function $S(t,\rho,\theta)$ is not uniquely determined. Our choice will be a solution of the Hamilton-Jacobi equation on the reference manifold $\mathbb{R}\times\partial M$ with the long-range potential V^L:

$$h(\frac{\partial S}{\partial \rho},\theta,\rho,-\frac{\partial S}{\partial \theta}) = \frac{\partial S}{\partial t},$$
$$h(r,\theta,\rho,\omega) = \frac{1}{2}\rho^2 + \frac{1}{2}a_1\rho^2 + \frac{1}{r}a_2^j\rho\,\omega_j + \frac{1}{r^2}a_3^{jk}\omega_j\omega_k + V^L,$$

for large t and for every ρ in any fixed compact set of $\mathbb{R}\setminus\{0\}$, where h is the corresponding classical Hamiltonian. We choose ρ and θ as variables of S because ρ and θ components of the classical trajectories have limits as t goes to infinity. The time-dependent modifier $e^{-iS(t,D_r,\theta)}$ is a Fourier multiplier in r-variable for each θ and we only need to consider the 1-dimensional Fourier transform with respect to r-variable. We construct solutions of the Hamilton-Jacobi equation mainly following J. Dereziński and C. Gérard [6].

In Section 2.1, we consider the boundary value problem for Newton equation on $\mathbb{R} \times \partial M$ with time-dependent slowly-decaying forces, which decay in time (Definition 2.1). We use an integral equation and Banach's contraction mapping theorem (Proposition 2.4, refer Dereziński [5] and Section 1.5 of [6]). We observe that the classical trajectories will stay in outgoing (incoming) regions as $t \to +\infty(-\infty)$.

In Section 2.2, we consider Newton equations with time-independent long-range forces which decay in space (Definition 2.5) in appropriate outgoing (incoming) regions. By inserting time-dependent cut-off functions, we introduce an effective time-dependent force and reduce the time-independent problem to the time-dependent one (Theorem 2.6). Our model (the Hamiltonian flow induced by the classical Hamiltonian) turns out to fit into this framework (Lemma 2.8). These tricks are also used in [6] for Hamiltonians with long-range potentials on Euclidean spaces.

In Section 2.3, in Theorem 2.10 we construct exact solutions for the Hamilton-Jacobi equation, using the classical trajectories with their dependence on initial data. Here we use the idea by Hörmandor [13] (see also Section 2.7 of [6]). We show that these solutions with their derivatives satisfy "good estimates", which are used to show the existence of the modifiers.

Using the Cook-Kuroda method (see Cook [2], and Kuroda [23]) and 1-dimensional Fourier transform, we deduce the proof of the main theorem to estimates of the integral (Proposition 3.1):

$$\int [h(r,\theta,\rho,-\frac{\partial S}{\partial \theta}(t,\rho,\theta)) - h(\frac{\partial S}{\partial \rho}(t,\rho,\theta),\theta,\rho,-\frac{\partial S}{\partial \theta}(t,\rho,\theta))]$$
$$\cdot e^{ir\rho - iS(t,\rho,\theta)}\hat{u}(\rho,\theta)d\rho.$$

In Section 3, we apply the stationary phase method (Hörmander [14] Section 7.7) to the integral. In the asymptotic expansion of the above integral, the terms in which h is not differentiated vanish since the equation $r = \partial S/\partial \rho$ holds at the stationary points.

Part II

We study Schrödinger operators with long-range potentials on non-compact manifolds with asymptotically polynomially growing ends. We follow the settings in Froese and Hislop [10], and show the spectral properties of Schrödinger operators using Mourre theory (see, the original paper Mourre [28], and Perry, Sigal, and Simon[29]). We show the Kato-smoothness (see Kato [20]) for three types of operators. If the perturbation is "short-range", it admits a factorization into a product of Kato-smooth operators. By virtue of the smooth perturbation theory of Kato, we learn the existence and the asymptotic completeness of wave operators in the one-space scattering framework. By employing the formulation of Ito-Nakamura [16] as described in Part I again, we show the existence and completeness of wave operators in the two-space scattering framework.

Froese and Hislop [10] studied manifolds with exponentially large ends, constant ends (tubes), vanishingly small ends, and polynomially large ends. We follow the formulation of

Froese and Hislop [10], and Theorem 4.1 may be seen as a direct generalization of [10] to long-range pertuabations. We note that the case where the perturbations decay as $\leq Cr^{-2}$ is considerd in [10].

De Bièvre, Hislop, and Sigal [4] studied a time-dependent scattering theory for wave equations, and proved its asymptotic completeness for several classes of manifolds, including manifolds with asymptotically growing ends with short-range perturbations. We consider more general perturbations on angular part of the metric than them. Ito and Nakamura [16] studied a time-dependent scattering theory for Schrödinger operators on scattering manifolds in the two-space scattering framework.

Our proof of the existence and asymptotic completeness of wave operators depends on the smooth perturbation theory of Kato [20] (see Theorem 8.1, we refer Yafaev [34] and [36] also), which shows: "If the perturbation $V = H - H_0$ admits a factorization into a product of Kato-smooth operators, then the wave operators $W_\pm(H, H_0)$ exist and are complete". The definition of Kato-smoothness relies on the unitary evolution generated by the self-adjoint operator H, but this definition is equivalent to estimates of boundary values of resolvents of H (Definition 7.1). The resolvent estimates assured by the Mourre theory (Theorem 4.1) imply the limiting absorption principle via a technique in Section 8 of [29], where the scattering theory for N-body Schrödinger operators with short range scalar potentials on Euclidean spaces is discussed. Then the limiting absorption principle implies the Kato smoothness of $G_0 = \langle r \rangle^{-s}$, $s > \frac{1}{2}$ in Theorem 4.2. The Kato smoothness of $G_1 = \chi_R \langle r \rangle^{-s} D_r$ is obtained in a similar way, but we have to extend the technique in [29] from $\alpha = 1$ to $\alpha = 2$ (Lemma 6.3 (i)). The Kato-smoothness of $G_2 = \chi_R \langle r \rangle^{-\frac{1}{2}} (-r^{-2\alpha} \Delta_{\partial M})^{\frac{1}{2}}$ is called the radiation estimates, where $\Delta_{\partial M}$ is the Laplacian on the boundary manifold ∂M. Our proof in Section 7 is similar to the one in [35], which relies on the commutator method (see Putnam [30] and Kato [22]).

The limiting absorption principle implies the asymptotic completeness in the case of two-particle Hamiltonians with short-range scalar potentials on Euclidean spaces. However, radiation estimates are crucial in scattering for long-range potentials on Euclidean spaces, in order to handle the "angular part" of the perturbations which come from modifiers (see Yafaev [36]). We found that radiation estimates are also useful for handling short-range metric perturbations on non-compact manifolds. We note that in Yafaev [35] the radiation estimates is used to show the asymptotic completeness for N-body Hamiltonians with short-range potentials.

In the two-space scattering, essentially we only need to examine wave operators for the pair $(D_r^2 - k(r)\Delta_{\partial M}, D_r^2)$, where typically $k(r) = r^{-2\alpha}$ if $|r|$ is large, and $r \in \mathbb{R}$. Since $\Delta_{\partial M}$ commutes with both of $D_r^2 - k(r)\Delta_{\partial M}$ and H_0, it reduces to the 1-dimensional scattering through the spectral representation of $-\Delta_{\partial M}$. If $k(r)$ is short-range ($\alpha > 1/2$), we only need a narutal identifier J. If $k(r)$ is long-range, we construct 1-dimensional time-independent modifiers for the corresponding 1-dimensional long-range scattering problem. We employ a class of pseudo-differential operators with oscillating symbols (see Section 10, see Yafaev [37] also) as our 1-dimensional modifiers. We construct approximate solutions to the eikonal equation and employ them as the oscillating factor of the symbol of the pseudo-differential operators. Then we show the existence and asymptotic completeness of the modified wave operators, using the 1-dimensional limiting absorption principle and the composition formula of pseudodifferential operators with oscillating symbols and usual pseudodifferential operators of Hörmander class. We note that both the limiting absorption principle and the radiation estimates are needed for long-range scattering on n-dimensional Euclidean spaces with $n \geq 2$. Here we do not need the radiation estimates since we consider 1-dimensional scattering and there is no “angular part”.

Part III

We consider a manifold M with asymptotically polynomially growing ends of growing rate r^α with a real positive number $\alpha > \frac{1}{2}$, where r is the radial parameter. The case where $\alpha = 1$ corresponds to Euclidean spaces and scattering manifolds. Let P be a Schrödinger operator on M. Then a time-dependent scattering theory for P with a simple reference system is constructed in Part II (see [18]. For the scattering manifolds case $\alpha = 1$ we refer Ito and Nakamura [16] also). We show that if the growing order satisfies $\alpha > 1$, then the scattering matrix do not change the wave front set. We see how the scaling property of the corresponding classical scattering operator determines laws of the propagation of singularities for quantum scattering operators.

Melrose and Zworski [26] showed that, for the scattering manifolds case $\alpha = 1$, the scattering matrices are Fourier integral operators associated to the canonical transform on the boundary manifold generated by the geodesic flow with length π, and hence the scattering matrices propagate the wave front set according to the same canonical map. Their arguments use the asymptotic expansion of generalized eigenfunctions. Ito and Nakamura [17] generalized these results using Egorov-type theorem, which is time-dependent theoretical. We follow the discussions of Ito and Nakamura.

The main idea is to consider the evolution:

$$A(t) = e^{itP_f/h^{1/\alpha}} J^* e^{-itP/h^{1/\alpha}} a(h^{1/\alpha} r, D_r, \theta, hD_\theta) e^{itP/h^{1/\alpha}} J e^{-itP_f/h^{1/\alpha}}$$

with some symbol a. We use a semi-classical Egorov type theorem argument for this time-dependent operator in Section 14 (see the textbook by Martinez [24]). We consider $W(t) = e^{itP/h^{1/\alpha}} J e^{-itP_f/h^{1/\alpha}}$ as a time-evolution, and construct an asymptotic solution of a Heisenberg equation which is very close to $A(t)$. The construction of the asymptotic solution relies on the classical Hamilton flow generated by $p_k = \frac{1}{2}\rho^2 + k(r)q(\theta,\omega)$, where $k(r) = r^{-2\alpha}$. The classical scattering operator has a scaling property, and its semi-classical limit satisfies for $(r_-, \rho_-, \theta_-, \omega_-) \in T^*\mathbb{R}_- \times (T^*\partial M \setminus \{0\})$,

$$\lim_{h\to+0} (\Pi_\theta, h\Pi_\omega) s_k(h^{-1/\alpha} r_-, \rho_-, \theta_-, h^{-1}\omega_-) = (\theta_-, \omega_-),$$

where s_k is the classical scattering operator and Π_* is a projection to $*$-variable (see Section 12). Thus, one may consider our results as a quantization of the classical mechanical scattering on manifolds with polynomially growing ends of growing order $\alpha > 1$. We note that on scattering manifolds ($\alpha = 1$), (θ, ω)-components of classical scattering operator is $\exp(\pi H_{\sqrt{2q}})$, the Hamilton flow generated by $\sqrt{2q}$ where q is the classical hamiltonian on the boundary manifold, and the wave front set is propagated along the same map (see [26] or [17]).

Contents

Part I
Existence of wave operators with time-dependent modifiers for the Schrödinger equations with long-range potentials on scattering manifolds

Abstract

We construct time-dependent wave operators for Schrödinger equations with long-range potentials on a manifold M with asymptotically conic structure. We use the two space scattering theory formalism, and a reference operator on a space of the form $\mathbb{R} \times \partial M$, where ∂M is the boundary of M at infinity. We construct exact solutions to the Hamilton-Jacobi equation on the reference system $\mathbb{R} \times \partial M$, and prove the existence of the modified wave operators.

1 Introduction of Part I

We show the existence of wave operators for the Schödinger equations with long-range potentials on scattering manifolds, which have asymptotically conic structure at infinity (see Melrose [25] about scattering manifolds). We employ the formulation of Ito-Nakamura [16], which uses the two-space scattering framework of Kato [21]. Following Hörmander [13] and Dereziński and Gérard [6], we construct exact solutions to the Hamilton-Jacobi equation and show the existence of the modified two-space wave operators using the stationary phase method.

Let M be an n-dimensional smooth non-compact manifold such that M is decomposed to $M_C \cup M_\infty$, where M_C is relatively compact, and M_∞ is diffeomorphic to $\mathbb{R}_+ \times \partial M$ with a compact manifold ∂M. We fix an identification map:

$$\iota : M_\infty \longrightarrow \mathbb{R}_+ \times \partial M \ni (r, \theta).$$

We suppose $M_c \cap M_\infty \subset (0, 1/2) \times \partial M$ under this identification. We also suppose that ∂M is equipped with a measure $H(\theta)d\theta$ where $H(\theta)$ is a smooth positive density .

Let $\{\phi_\lambda : U_\lambda \to \mathbb{R}^{n-1}\}, U_\lambda \subset \partial M$, be a local coordinate system of ∂M. We set $\{\tilde{\phi}_\lambda : \mathbb{R}_+ \times U_\lambda \to \mathbb{R} \times \mathbb{R}^{n-1}\}$ to be a local coordinate system of $M_\infty \cong \mathbb{R}_+ \times \partial M$, and we denote $(r, \theta) \in \mathbb{R} \times \mathbb{R}^{n-1}$ to represent a point in M_∞. We suppose $G(x)$ is a smooth positive density on M such that

$$G(x)dx = r^{n-1}H(\theta)drd\theta \text{ on } (\frac{1}{2}, \infty) \times \partial M \subset M_\infty,$$

and we set

$$\mathscr{H} = L^2(M, G(x)dx).$$

Let P_0 be a formally self-adjoint second order elliptic operator on $\mathscr{H}$ of the form:

$$P_0 = -\frac{1}{2}G^{-1}(\partial_r, \partial_\theta/r)G \begin{pmatrix} 1+a_1 & a_2 \\ {}^t a_2 & a_3 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta/r \end{pmatrix} \quad \text{on } \tilde{M}_\infty = (1, \infty) \times \partial M$$

where a_1, a_2, and a_3 are real-valued smooth tensors.

Assumption 1.1. *For any $l \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^{n-1}$, there is $C_{l,\alpha}$ such that*

$$|\partial_r^l \partial_\theta^\alpha a_j(r, \theta)| \le C_{l,\alpha} r^{-\mu_j - l}$$

on $\tilde{M}_\infty$, where $\mu_j \ge 0$. Note that we use the coordinate system in M_∞ described above.

We will construct a time-dependent scattering theory for $P_0 + V$ on $\mathscr{H}$ where V is a potential.

Definition 1.2. *Let $\mu_s > 0$. A finite rank differential operator V^S of the form $V^S = \sum_{l,\alpha} V^S_{l,\alpha}(r, \theta)\partial_r^l \partial_\theta^\alpha$ on M_∞ is said to be a short range perturbation of μ_S type if for every l, α the coefficient $V^S_{l,\alpha}$ is a L^2_{loc} tensor and satisfies*

$$\int_{\mathbb{R}_+ \times U_\lambda} |V^S_{l,\alpha}(x)|^2 \langle r \rangle^{-M} G(x)dx < \infty$$

for some M, and almost every $(\rho_0,\theta_0)\in\mathbb{R}\times\partial M$ *has a neighborhood* ω_{ρ_0,θ_0} *such that*

$$\int_1^\infty\Big(\int_{(\rho,\theta)\in\omega_{\rho_0,\theta_0}}|V^S_{l,\alpha}(t\rho,\theta)|^2d\rho H(\theta)d\theta\Big)^{1/2}t^{\mu_S|\alpha|}dt<\infty.$$

Let $\mu_L>0$. V^L *is called a long-range smooth potential if* V^L *is a real-valued* C^∞ *function with support in* $\tilde{M}_\infty$, *and satisfies for any indices* l,α,

$$|D_r^jD_\theta^\alpha V^L(r,\theta)|\le C_{j,\alpha}r^{-\mu_L-j}.$$

A differential operator V on M is called an admissible long-range perturbation of P_0 *if V is of the form* $V=V^S+V^L$ *where* V^S *is a short range perturbation of* μ_S *type and* V^L *is a long-range smooth potential and*

$$\varepsilon=\mu_1=\mu_2=\mu_L>0,\ \ \mu_3=0,\ \ \mu_S=1-\varepsilon.$$

Example 1. If $V^S=V^S(r,\theta)$ is a multiplication operator and $|V^S(r,\theta)|\le Cr^{-1-\eta}$, $\eta>0$, then V^S satisfies the short-range condition above.

If $V^S=\sum_{|\alpha|=1}V^S_\alpha\partial_\theta^\alpha$ and $|V^S_\alpha(r,\theta)|\le Cr^{-1-\mu_S-\eta},\eta>0$, then V^S satisfies the short-range condition above. As the order of the derivative with respect to θ-variable increases, we need more rapid decay conditions on the coefficients.

Remark 2. If V^S is a smooth function, then P_0+V is essentially self-adjoint. More generally, if V^S is at most second-order differential operator with "small" smooth coefficients, then V^S is P_0-bounded with relateve bound less than one, and P_0+V is essentially self-adjoint. We assume that P_0+V is essentially self-adjoint on suitable domains (see Theorem 1.3) and do not investigate the conditions of self-adjointness.

Remark 3. If we assume ∂M is equipped with a positive (2, 0)-tensor $h=(h^{jk}(\theta))$, for some $\varepsilon>0$,

$$|\partial_r^l\partial_\theta^\alpha(a_3(r,\theta)-h(\theta))|\le C_{l,\alpha}r^{-\varepsilon-l},$$

and $V^S=0$, then P_0+V has a self-adjoint extension H and corresponds (via a unitary equivalence) to the Laplacian on Riemannian manifolds with asymptotically conic structure. Since $\varepsilon>0$, our model includes the scattering metric of long-range type described in [15]. Thus our results are generalizations of [16].

We prepare a reference system as follows:

$$M_f=\mathbb{R}\times\partial M,\ \ \mathscr{H}_f=L^2(M_f,H(\theta)drd\theta),\ \ P_f=-\frac{1}{2}\frac{\partial^2}{\partial r^2}\text{ on }M_f$$

Note that P_f is essentially self-adjoint on $C_0^\infty(M_f)$, and we denote the unique self-adjoint extension by the same symbol. Let $j(r)\in C^\infty(\mathbb{R})$ be a real-valued function such that $j(r)=1$ if $r\ge1$ and $j(r)=0$ if $r\le1/2$. We define the identification operator $J:\mathscr{H}_f\to\mathscr{H}$ by

$$(Ju)(r,\theta)=r^{-(n-1)/2}j(r)u(r,\theta)\ \text{ if }\ (r,\theta)\in M_\infty$$

and $Ju(x) = 0$ if $x \notin M_\infty$, where $u \in \mathscr{H}_f$. We denote the Fourier transform with respect to r-variable by $\mathscr{F}$:

$$\mathscr{F}u(\rho,\theta) = \int_{-\infty}^{\infty} e^{-ir\rho}u(r,\theta)dr, \text{ for } u \in C_0^\infty(M_f).$$

We decompose the reference Hilbert space $\mathscr{H}_f$ as $\mathscr{H}_f = \mathscr{H}_f^+ \oplus \mathscr{H}_f^-$, where $\mathscr{H}_f^\pm$ are defined by

$$\mathscr{H}_f^+ = \{u \in \mathscr{H} | \mathrm{supp}(\mathscr{F}u) \subset [0,\infty) \times \partial M\},$$
$$\mathscr{H}_f^- = \{u \in \mathscr{H} | \mathrm{supp}(\mathscr{F}u) \subset (-\infty,0] \times \partial M\}.$$

We use the following notation: For $x \in M$, we write

$$\langle x \rangle = \langle r \rangle = \begin{cases} 1 + rj(r) & \text{for } x \in M_\infty, \\ 1 & \text{for } x \in M_c. \end{cases}$$

We state our main theorem.

Theorem 1.3. *Let $V = V^L + V^S$ be an admissible long-range perturbation of P_0, and V is symmetric on $J\mathscr{F}^{-1}C_0^\infty(M_f)$, and $P_0 + V$ has a self adjoint extension H. Let $S(t,\rho,\theta)$ be a solution to the Hamilton-Jacobi equation which is constructed in Theorem 2.10. Then the modified wave operators*

$$\Omega_\pm = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH} J e^{-iS(t,D_r,\theta)}$$

exist, and are partial isometries from $\mathscr{H}_f^\pm$ into $\mathscr{H}$ intertwining H and P_f:

$$e^{isH}\Omega_\pm = \Omega_\pm e^{isP_f}.$$

We refer Reed and Simon [31], Dereziński and Gérard [6], and Yafaev [36] for general concepts of wave operators and scattering theory for Schödinger equations. We here briefly review the history of wave operators. The concept of wave operator was introduced by Møller [27]. The existence of wave operators has long been studied (see Cook [2] and Kuroda [23]) for short range potentials, which decay faster than the Coulomb potential. For the Coulomb potential, it was proved by Dollard [7, 8] that the wave operators do not exist unless the definition is modified. Dollard introduced the concept of the modified wave operators $\text{s-lim}_{t\to\pm\infty} e^{itH}e^{-iS(t,D_x)}$. Buslaev-Mateev [1] showed the existence of modified wave operators by using stationary phase method and by employing an approximate solution to the Hamilton-Jacobi equation as a modifier function $S(t,\xi)$. Hörmander [13] constructed exact solutions to the Hamilton-Jacobi equation (see also [14] vol. IV).

The spectral properties of Laplace operators on a class of non-compact manifolds were studied by Froese, Hislop and Perry [10, 11], and Donnelly [9] using the Mourre theory (see, the original paper Mourre [28], and Perry, Sigal, and Simon [29]). In early 1990s, Melrose introduced a new framework of scattering theory on a class of Riemannian manifolds with metrics called scattering metrics (see [25] and references therein), and showed that the absolute scattering matrix, which is defined through the asymptotic expansion of generalized eigenfunctions, is a Fourier integral operator. Vasy [33] studied Laplace operators on such manifolds with long-range potentials of Coulomb type decay ($|V(r,\theta)| \leq Cr^{-1}$).

Ito and Nakamura [16] studied a time-dependent scattering theory for Schrödinger operators on scattering manifolds. They used the two-space scattering framework of Kato [21] with a simple reference operator $D_r^2/2$ on a space of the form $\mathbb{R} \times \partial M$, where ∂M is the boundary of the scattering manifold M.

We employ the formulation of Ito and Nakamura [16], and consider general long-range metric perturbations and potential perturbations. We assume that the scalar potential decay as $|V(r,\theta)| \leq Cr^{-\varepsilon}, \varepsilon > 0$.

We make some remarks along with the outline of the proof. The time-dependent modifier function $S(t,\rho,\theta)$ is not uniquely determined. Our choice is a solution to the Hamilton-Jacobi equation on the reference manifold $\mathbb{R} \times \partial M$ with the long-range potential V^L:

$$h(\frac{\partial S}{\partial \rho}, \theta, \rho, -\frac{\partial S}{\partial \theta}) = \frac{\partial S}{\partial t}, \tag{1.1}$$
$$h(r,\theta,\rho,\omega) = \frac{1}{2}\rho^2 + \frac{1}{2}a_1\rho^2 + \frac{1}{r}a_2^j \rho\,\omega_j + \frac{1}{r^2}a_3^{jk}\omega_j\omega_k + V^L,$$

for large t and for every ρ in any fixed compact set of $\mathbb{R} \setminus \{0\}$, where h is the corresponding classical Hamiltonian. We choose ρ and θ as variables of S because ρ and θ components of the classical trajectories have limits as t goes to infinity. The time-dependent modifier $e^{-S(t,D_r,\theta)}$ is a Fourier multiplier in r-variable for each θ and we only need to consider the 1-dimensional Fourier transform with respect to r-variable. We construct solutions to the Hamilton-Jacobi equation mainly following J. Dereziński and C. Gérard [6].

In Section 2.1, we consider the boundary value problem for Newton equation on $\mathbb{R} \times \partial M$ with time-dependent slowly-decaying forces, which decay in time (Definition 2.1). In Theorem 2.2, we construct solutions and show several estimates. We use an integral equation and Banach's contraction mapping theorem (Proposition 2.4, refer Dereziński [5] and Section 1.5 of [6]). In the definition of slowly-decaying forces (Definition 2.1) and the function spaces (Definition 2.3), we assume different decaying rates on different variables r, θ, ρ, and ω. These are efficiently used to show Proposition 2.4. We observe that the classical trajectories will stay in outgoing (incoming) regions as $t \longrightarrow +\infty(-\infty)$.

In Section 2.2, we consider Newton equations with time-independent long-range forces which decay in space (Definition 2.5) in appropriate outgoing (incoming) regions. By inserting time-dependent cut-off functions, we introduce an effective time-dependent force and reduce the time-independent problem to the time-dependent one (Theorem 2.6). Our model (the Hamiltonian flow induced by the classical Hamiltonian) turns out to fit into this framework (Lemma 2.8). These tricks are also used in [6] for Hamiltonians with long-range potentials on Euclidean spaces.

Finally, in Section 2.3, in Theorem 2.10 we construct exact solutions to the Hamilton-Jacobi equation, using the classical trajectories with their dependence on initial data. Here we use the idea by Hörmandor [13], see also Section 2.7 of [6]. We show that these solutions with their derivatives satisfy "good estimates", which are used to show the existence of the modifiers. Once we obtain a suitable modifier $S(t,\rho,\theta)$, we can show the existence of modified wave operators through stationary phase method (Section 3).

Using the Cook-Kuroda method (see Cook [2], and Kuroda [23]) and 1-dimensional Fourier transform, we deduce the proof of the main theorem to estimates of the integral (Proposition 3.1):

$$\int [h(r,\theta,\rho,-\frac{\partial S}{\partial \theta}(t,\rho,\theta))-h(\frac{\partial S}{\partial \rho}(t,\rho,\theta),\theta,\rho,-\frac{\partial S}{\partial \theta}(t,\rho,\theta))]$$
$$\cdot e^{ir\rho-iS(t,\rho,\theta)}\hat{u}(\rho,\theta)d\rho.$$

In Section 3, we apply the stationary phase method (Hörmander [14] Section 7.7). In the asymptotic expansion of the above integral, the terms in which h is not differentiated vanish since the equation $r=\partial S/\partial \rho$ holds at the stationary points. To show the uniformly boundedness of constants which appear in the asymptotic expansions of the integral, we construct diffeomorphisms in small neighborhoods of the stationary points which transform the phase function into quadratic forms there (Lemma 3.2). In the constructions of these diffeomorphisms, we use the estimates on the modifier function S.

Notations We use the following notation. Let $t\in\mathbb{R}$ and s be a parameter. We write $f(t,s)\in g(s)\mathrm{O}(\langle t\rangle^{-m})$ if $f(t,s)\leq Cg(s)\langle t\rangle^{-m}$ uniformly for t and s. We denote $f(t,s)\in g(s)\mathrm{o}(t^0)$ if $\lim_{t\to\infty} f(t,s)/g(s)=0$.

2 Classical mechanics

In this section, we study classical trajectories and solutions to the Hamilton-Jacobi equation.

2.1 Classical trajectories with slowly-decaying time-dependent force

Let $(r,\theta,\rho,\omega)\in T^*(\mathbb{R}\times\mathbb{R}^{n-1})$ and consider the Newton's equation:

$$(\dot{r},\dot{\theta},\dot{\rho},\dot{\omega})(t)=(\rho+F_r,F_\theta,F_\rho,F_\omega)(t,(r,\theta,\rho,\omega)(t)) \tag{2.1}$$

where

$$F=F(t,r,\theta,\rho,\omega)=(F_r,F_\theta,F_\rho,F_\omega)(t,r,\theta,\rho,\omega)$$

is a time-dependent force. Let $\varepsilon>0$ and $\tilde{\varepsilon}=\frac{1}{2}\varepsilon$.

Definition 2.1. *A time-dependent force F is said to be slow-decaying if F satisfies*

$$\sup_{(r,\rho,\theta,\omega)\in T^*(\mathbb{R}\times\mathbb{R}^{n-1})}|\partial_r^l\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(F_r,F_\theta,F_\rho,F_\omega)(t)|\in\mathrm{O}(\langle t\rangle^{-n_{r,\theta,\rho,\omega}(l,\alpha,k,\beta)}) \tag{2.2}$$

where

$$\begin{aligned}
&n_r(l,\alpha,k,\beta)=m(l,\alpha,k+1,\beta),\ n_\theta(l,\alpha,k,\beta)=m(l,\alpha,k,\beta+e_i),\\
&n_\rho(l,\alpha,k,\beta)=m(l+1,\alpha,k,\beta),\ n_\omega(l,\alpha,k,\beta)=m(l,\alpha+e_i,k,\beta),\\
&m(l,\alpha,0,0)=l+\varepsilon,\ m(l,\alpha,1,0)=l+\varepsilon,\ m(l,\alpha,2,0)=l+\varepsilon,\\
&m(l,\alpha,0,e_i)=l+1+\tilde{\varepsilon},\ m(l,\alpha,1,e_i)=l+1+\varepsilon,\ m(l,\alpha,0,e_i+e_j)=l+2,\\
&m(l,\alpha,k,\beta)=+\infty,\quad \mathit{if}\quad k+|\beta|\geq 3,
\end{aligned} \tag{2.3}$$

$i,j=1,\cdots,n-1$, and $e_i=(0,\cdots,1,0,\cdots,0)\in\mathbb{Z}_+^{n-1}$ is the canonical unit vector, i.e., every component of e_i is 0 except i-th component.

In the next theorem, we show the unique existence of trajectories for the dynamics (2.1) where the boundary conditions are the initial position and the final momentum.

Theorem 2.2. *Assume that F is a time-dependent slowly-decaying force in the sense of Definition 2.1. Then there exists T such that if $T \leq t_1 < t_2 \leq \infty$ and $(r_i, \theta_f, \rho_f, \omega_i) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1})$, there exists a unique trajectory*

$$[t_1, t_2] \ni s \mapsto (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, t_1, t_2, r_i, \theta_f, \rho_f, \omega_i)$$

satisfying

$$\begin{aligned}&\partial_s(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, t_1, t_2, r_i, \theta_f, \rho_f, \omega_i)\\ &= (\rho + F_r, F_\theta, F_\rho, F_\omega)(s, (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, t_1, t_2, r_i, \theta_f, \rho_f, \omega_i)),\\ &(\tilde{r}, \tilde{\omega})(t_1, t_1, t_2, r_i, \theta_f, \rho_f, \omega_i) = (r_i, \omega_i), \quad (\tilde{\theta}, \tilde{\rho})(s, t_1, t_2, r_i, \theta_f, \rho_f, \omega_i) = (\theta_f, \rho_f).\end{aligned}$$

We set $\underline{r}(s), \underline{\theta}(s), \underline{\rho}(s), \underline{\omega}(s)$ by

$$\begin{aligned}&\underline{r}(s) = \tilde{r}(s) - r_i - (s - t_1)\rho_f, \quad \underline{\theta}(s) = \tilde{\theta}(s) - \theta_f,\\ &\underline{\rho}(s) = \tilde{\rho}(s) - \rho_f, \quad \underline{\omega}(s) = \tilde{\omega}(s) - \omega_i.\end{aligned}$$

Moreover the solution satisfies the following estimates uniformly for $T \leq t_1 \leq s \leq t_2 \leq \infty$, $(r_i, \theta_f, \rho_f, \omega_i) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1})$:

$$\begin{aligned}&|\underline{r}(s)| \in \mathrm{o}(s^0)|s - t_1|, \quad |\underline{\theta}(s)| \in \mathrm{O}(s^{-\tilde{\varepsilon}}),\\ &|\underline{\rho}(s)| \in \mathrm{O}(s^{-\tilde{\varepsilon}}), \quad |\underline{\omega}(s)| \in \mathrm{o}(s^0)|s - t_1|^{1-\tilde{\varepsilon}},\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\begin{pmatrix} \partial_{r_i}\underline{r}(s) & \partial_{\theta_f}\underline{r}(s) & \partial_{\rho_f}\underline{r}(s) & \partial_{\omega_i}\underline{r}(s) \\ \partial_{r_i}\underline{\theta}(s) & \partial_{\theta_f}\underline{\theta}(s) & \partial_{\rho_f}\underline{\theta}(s) & \partial_{\omega_i}\underline{\theta}(s) \\ \partial_{r_i}\underline{\rho}(s) & \partial_{\theta_f}\underline{\rho}(s) & \partial_{\rho_f}\underline{\rho}(s) & \partial_{\omega_i}\underline{\rho}(s) \\ \partial_{r_i}\underline{\omega}(s) & \partial_{\theta_f}\underline{\omega}(s) & \partial_{\rho_f}\underline{\omega}(s) & \partial_{\omega_i}\underline{\omega}(s) \end{pmatrix}\\ &\in \begin{pmatrix} \mathrm{o}(s^0)|s - t_1| \\ \mathrm{O}(1) \\ \mathrm{O}(s^{-\tilde{\varepsilon}}) \\ \mathrm{o}(s^0)|s - t_1|^{1-\tilde{\varepsilon}} \end{pmatrix} \otimes (t_1^{-1-\tilde{\varepsilon}}, t_1^{-\tilde{\varepsilon}}, t_1^{-\tilde{\varepsilon}}, t_1^{-1}),\end{aligned} \tag{2.5}$$

$$\partial_{r_i}^l \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta \begin{pmatrix} \underline{r} \\ \underline{\theta} \\ \underline{\rho} \\ \underline{\omega} \end{pmatrix} \in \begin{pmatrix} \mathrm{o}(s^0)|s - t_1| \\ \mathrm{O}(1) \\ \mathrm{O}(s^{-\tilde{\varepsilon}}) \\ \mathrm{o}(s^0)|s - t_1|^{1-\tilde{\varepsilon}} \end{pmatrix} \cdot t_1^{-l-|\beta|}. \tag{2.6}$$

Here $\otimes$ is an outer product and (2.5) *means, for example, $\partial_{r_i}\underline{r}(s) \in \mathrm{o}(s^0)|s - t_1| t_1^{-1-\tilde{\varepsilon}}$, and $\partial_{\theta_f}\underline{\theta}(s) \in \mathrm{O}(1) t_1^{-\tilde{\varepsilon}}$.*

A straightforward computation shows that $(\underline{r}, \underline{\theta}, \underline{\rho}, \underline{\omega})(s)$ satisfies the following integral equation:

$$(\underline{r},\underline{\theta},\underline{\rho},\underline{\omega})(s) = (P_r,P_\theta,P_\rho,P_\omega)(\underline{r},\underline{\theta},\underline{\rho},\underline{\omega})(s) := \tag{2.7}$$
$$\begin{pmatrix} \int_{t_1}^{s}\big(\underline{\rho}(u)+F_r(u,r_i+(u-t_1)\rho_f+\underline{r}(u),\theta_f+\underline{\theta}(u),\rho_f+\underline{\rho}(u),\omega_i+\underline{\omega}(u))\big)du \\ -\int_{s}^{t_2}\big(F_\theta(u,r_i+(u-t_1)\rho_f+\underline{r}(u),\theta_f+\underline{\theta}(u),\rho_f+\underline{\rho}(u),\omega_i+\underline{\omega}(u))\big)du \\ -\int_{s}^{t_2}\big(F_\rho(u,r_i+(u-t_1)\rho_f+\underline{r}(u),\theta_f+\underline{\theta}(u),\rho_f+\underline{\rho}(u),\omega_i+\underline{\omega}(u))\big)du \\ \int_{t_1}^{s}\big(F_\omega(u,r_i+(u-t_1)\rho_f+\underline{r}(u),\theta_f+\underline{\theta}(u),\rho_f+\underline{\rho}(u),\omega_i+\underline{\omega}(u))\big)du \end{pmatrix}$$

where the map $P=(P_r,P_\theta,P_\rho,P_\omega)$ depends on the parameters $t_1,t_2,r_i,\theta_f,\rho_f,\omega_i$. We will apply the fixed point theorem to solve (2.7). We define the Banach space on which the map P is defined as follows:

Definition 2.3. *For $m\geq 0$, we define*

$$Z_T^m := \{z\in C([T,\infty)) : \sup\frac{|z(t)|}{|t-T|^m}<\infty\},\ \ Z_{T,\infty}^m := \{z\in Z_T^m : \lim_{t\to\infty}\frac{|z(t)|}{|t-T|^m}=0\}.$$

For $m<0$, we define

$$Z_T^m := \{z\in C([T,\infty)) : \sup\frac{|z(t)|}{\langle t\rangle^m}<\infty\}.$$

We define

$$Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}} := \{(\underline{r},\underline{\theta},\underline{\rho},\underline{\omega})\in Z_{t_1,\infty}^1\times Z_{t_1}^0\times Z_{t_1}^{-\tilde{\varepsilon}}\times Z_{t_1,\infty}^{1-\tilde{\varepsilon}}\}.$$

Then we have the following Proposition:

Proposition 2.4. *For large enough $T>0$, the map P is a contraction map on $Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}$ for any $T\leq t_1\leq t_2\leq\infty$, $(r_i,\theta_f,\rho_f,\omega_i)\in T^*(\mathbb{R}\times\mathbb{R}^{n-1})$. Indeed, for some constant c which does not depend on $t_1,t_2,(r_i,\theta_f,\rho_f,\omega_i)$ but T, we have*

$$\|\nabla_x P(x)\|_{B\left(Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}\right)}<c<1. \tag{2.8}$$

Proof. We first note that P is well defined as a map of $Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}$ into itself. Indeed, for example, if $x=(\underline{r},\underline{\theta},\underline{\rho},\underline{\omega})\in Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}$,

$$\begin{aligned}&|P_r(x)(s)|\\ &\leq\int_{t_1}^{s}|\big(\underline{\rho}(u)+F_r(u,r_i+(s-t_1)\rho_f+\underline{r}(u),\theta_f+\underline{\theta}(u),\rho_f+\underline{\rho}(u),\omega_i+\underline{\omega}(u))\big)|du\\ &\leq\int_{t_1}^{s}|C\langle u\rangle^{-\tilde{\varepsilon}}+C\langle u\rangle^{-\varepsilon}|du,\end{aligned}$$

which implies $P_r(x)(s)\in Z_{t_1,\infty}^1$. Others are similar to prove.

Now we check that P is a contraction on $Z_{t_1}^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}$. It suffices to show (2.8) for some constant c which does not depend on $t_1,t_2,(r_i,\theta_f,\rho_f,\omega_i)$ but T. Let $v\in Z_{t_1,\infty}^1$. Then

$$\begin{aligned}|s-t_1|^{-1}(\nabla_r P_r(x)v)(s)&\leq|s-t_1|^{-1}\int_{t_1}^{s}\|\nabla_r F_r(u,\cdot)\|_\infty|u-t_1|\|v\|_{Z_{t_1,\infty}^1}du\\ &\leq\|v\|_{Z_{t_1,\infty}^1}\int_{t_1}^{s}|s-t_1|^{-1}|u-t_1|\langle u\rangle^{-1-\varepsilon}du.\end{aligned}$$

If we let $T \to \infty$, the right hand side goes to zero uniformly for $T \leq t_1 \leq t_2 \leq \infty$. Moreover, the right hand side goes to zero as $s \to \infty$. Hence taking T large enough, we may assure that

$$\|\nabla_r P_r\|_{B(Z^1_{t_1,\infty})} < c < 1$$

for some constant c for any $T \leq t_1 \leq t_2 \leq \infty$ and for any $(r_i, \theta_f, \rho_f, \omega_i) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1})$. In a similar way, we can show that for some large enough T, (2.8) holds for any $T \leq t_1 \leq t_2 \leq \infty$ and for any $(r_i, \theta_f, \rho_f, \omega_i) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1})$. □

Proof of Theorem 2.2 . The fixed point theorem together with Proposition 2.4 implies that there exists a unique solution $(\underline{r}, \underline{\theta}, \underline{\rho}, \underline{\omega})(s) \in Z^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}_{t_1}$ for the integral equation (2.7) for each $T \leq t_1 < t_2 \leq \infty$ and $(r_i, \theta_f, \rho_f, \omega_i) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1})$ if T is large enough. $(\underline{r}, \underline{\theta}, \underline{\rho}, \underline{\omega})(s) \in Z^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}_{t_1}$ directly means (2.4).

Let us now prove (2.5). We use the identity

$$(I - \nabla_x P(x))\partial^\gamma(x) = h^\gamma = (h^\gamma_r, h^\gamma_\theta, h^\gamma_\rho, h^\gamma_\omega) \tag{2.9}$$
$$:= \begin{pmatrix} \int_{t_1}^s (\nabla F_r)(u,y)\partial^\gamma(y-x)du \\ -\int_s^{t_1} (\nabla F_\theta)(u,y)\partial^\gamma(y-x)du \\ -\int_s^{t_1} (\nabla F_\rho)(u,y)\partial^\gamma(y-x)du \\ \int_{t_1}^s (\nabla F_\omega)(u,y)\partial^\gamma(y-x)du \end{pmatrix}$$

where $\partial^\gamma = \partial_{r_i}, \partial_{\theta_f}, \partial_{\rho_f}$, or ∂_{ω_i}, $x = (\underline{r}, \underline{\theta}, \underline{\rho}, \underline{\omega})$ is the solution of (2.7), and $y = (r_i + (u - t_1)\rho_f + \underline{r}(u), \theta_f + \underline{\theta}(u), \rho_f + \underline{\rho}(u), \omega_i + \underline{\omega}(u))$. By a straight computation we have

$$(h^{\partial_{r_i}}, h^{\partial_{\theta_i}}, h^{\partial_{\rho_i}}, h^{\partial_{\omega_i}}) \in (\langle t_1\rangle^{-1-\tilde{\varepsilon}}, \langle t_1\rangle^{-\tilde{\varepsilon}}, \langle t_1\rangle^{-\tilde{\varepsilon}}, \langle t_1\rangle^{-1}) Z^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}_{t_1}.$$

(2.8) implies that $I - \nabla_x P(x)$ is invertible on $Z^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}_{t_1}$. Using (2.9), we get

$$(\partial_{r_i}x, \partial_{\theta_f}x, \partial_{\rho_f}x, \partial_{\omega_i}x) \in Z^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}_{t_1},$$

and

$$\|(\partial_{r_i}x, \partial_{\theta_f}x, \partial_{\rho_f}x, \partial_{\omega_i}x)\|_{Z^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}_{t_1}} \in O(\langle t_1\rangle^{-1-\tilde{\varepsilon}}, \langle t_1\rangle^{-\tilde{\varepsilon}}, \langle t_1\rangle^{-\tilde{\varepsilon}}, \langle t_1\rangle^{-1}),$$

which implies (2.5).

Now we prove (2.6) by an induction. Assume that $\partial^\gamma = \partial^l_{r_i}\partial^\alpha_\theta\partial^k_\rho\partial^\beta_\omega$, $l + |\alpha| + k + |\beta| = n \geq 2$, and (2.6) is true for $l + |\alpha| + k + |\beta| \leq n - 1$. We use the identity

$$(I - \nabla_x P(x))\partial^\gamma(x) = h^\gamma = (h^\gamma_r, h^\gamma_\theta, h^\gamma_\rho, h^\gamma_\omega) \tag{2.10}$$
$$:= \begin{pmatrix} \int_{t_1}^s \sum_{q\geq 2}(\nabla^q F_r)(u,y)\partial^{\gamma_1}(y)\partial^{\gamma_2}(y)\cdots\partial^{\gamma_q}(y)du \\ -\int_s^{t_1} \sum_{q\geq 2}(\nabla^q F_\theta)(u,y)\partial^{\gamma_1}(y)\partial^{\gamma_2}(y)\cdots\partial^{\gamma_q}(y)du \\ -\int_s^{t_1} \sum_{q\geq 2}(\nabla^q F_\rho)(u,y)\partial^{\gamma_1}(y)\partial^{\gamma_2}(y)\cdots\partial^{\gamma_q}(y)du \\ \int_{t_1}^s \sum_{q\geq 2}(\nabla^q F_\omega)(u,y)\partial^{\gamma_1}(y)\partial^{\gamma_2}(y)\cdots\partial^{\gamma_q}(y)du \end{pmatrix}$$

where the sum is taken over $\gamma = \sum_{p=1}^q \gamma_p, q \geq 2$. The induction hypothesis with a straight computation shows that

$$(h^\gamma) \in (\langle t_1\rangle^{-l-|\beta|}) Z^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}_{t_1}.$$

Thus we have

$$\|\partial_\gamma x\|_{Z^{1,0,-\tilde{\varepsilon},1-\tilde{\varepsilon}}_{t_1}} \in (\langle t_1\rangle^{-l-|\beta|}),$$

which implies (2.6). □

2.2 Classical trajectories with long-range time-independent force

We denote the outgoing region by $\Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}$:

$$\Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q} := \{(r,\theta,\rho,\omega) \in T^*(\mathbb{R}\times\mathbb{R}^{n-1}) : r > R, \theta \in U, \rho \in J, |\omega| \le Qr^{1-\tilde{\varepsilon}}\}$$

for $R>0, U \subset \mathbb{R}^{n-1}, J \subset \mathbb{R}, Q>0$.

We now consider the dynamics with time-independent long-range forces.

Definition 2.5. *A time-independent force F is said to be a long-range force if it satisfies*

$$\sup_{(r,\theta,\rho,\omega)\in\Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}} |\partial_r^l \partial_\theta^\alpha \partial_\rho^k \partial_\omega^\beta (F_r,F_\theta,F_\rho,F_\omega)(r,\theta,\rho,\omega)| \in \mathrm{O}(\langle R\rangle^{-n_{r,\theta,\rho,\omega}(l,\alpha,k,\beta)}) \tag{2.11}$$

for any $R>0$, $U \Subset \mathbb{R}^{n-1}$, $J \Subset (0,\infty)$, $Q>0$.

As in Theorem 2.2, we show the unique existence of trajectories for the dynamics where the boundary conditions are the initial position and the final momentum.

Theorem 2.6. *Assume that F is a time-independent long-range force in the sense of Definition 2.5. Then for any open $U \Subset \tilde{U} \Subset \mathbb{R}^{n-1}$, open $J \Subset \tilde{J} \Subset (0,\infty)$, and $Q>0$, there exists $R>0$ such that for any $t \ge 0$ and for any $(r_i,\theta_f,\rho_f,\omega_i) \in \Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}$, there exists a unique trajectory*

$$[0,t] \ni s \mapsto (\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(s,t,r_i,\theta_f,\rho_f,\omega_i)$$

satisfying

$$\partial_s(\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(s,t,r_i,\theta_f,\rho_f,\omega_i) = (\rho+F_r,F_\theta,F_\rho,F_\omega)((\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(s,t,r_i,\theta_f,\rho_f,\omega_i)) \tag{2.12}$$

$$(\tilde{r},\tilde{\omega})(0,t,r_i,\theta_f,\rho_f,\omega_i) = (r_i,\omega_i), \quad (\tilde{\theta},\tilde{\rho})(t,t,r_i,\theta_f,\rho_f,\omega_i) = (\theta_f,\rho_f),$$

and the estimates

$$\begin{aligned} &|\underline{r}(s)| \in \mathrm{o}((s+\langle r_i\rangle)^0)|s|, \quad |\underline{\theta}(s)| \in \mathrm{O}((s+\langle r_i\rangle)^{-\tilde{\varepsilon}}), \\ &|\underline{\rho}(s)| \in \mathrm{O}((s+\langle r_i\rangle)^{-\tilde{\varepsilon}}), \quad |\underline{\omega}(s)| \in \mathrm{o}((s+\langle r_i\rangle)^0)|s|^{1-\tilde{\varepsilon}}, \end{aligned} \tag{2.13}$$

and

$$\tilde{\theta}(s,t,r_i,\theta_f,\rho_f,\omega_i) \in \tilde{U}, \quad \tilde{\rho}(s,t,r_i,\theta_f,\rho_f,\omega_i) \in \tilde{J}$$

where

$$\begin{aligned} &\underline{r}(s) = \tilde{r}(s) - r_i - s\rho_f, \quad \underline{\theta}(s) = \tilde{\theta}(s) - \theta_f \\ &\underline{\rho}(s) = \tilde{\rho}(s) - \rho_f, \quad \underline{\omega}(s) = \tilde{\omega}(s) - \omega_i. \end{aligned}$$

Moreover the solution satisfies the following estimates uniformly for $0 \le s \le t \le \infty$, $(r_i, \theta_f, \rho_f, \omega_i) \in \Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}$ *:*

$$\begin{pmatrix} \partial_{r_i}\underline{r}(s) & \partial_{\theta_f}\underline{r}(s) & \partial_{\rho_f}\underline{r}(s) & \partial_{\omega_i}\underline{r}(s) \\ \partial_{r_i}\underline{\theta}(s) & \partial_{\theta_f}\underline{\theta}(s) & \partial_{\rho_f}\underline{\theta}(s) & \partial_{\omega_i}\underline{\theta}(s) \\ \partial_{r_i}\underline{\rho}(s) & \partial_{\theta_f}\underline{\rho}(s) & \partial_{\rho_f}\underline{\rho}(s) & \partial_{\omega_i}\underline{\rho}(s) \\ \partial_{r_i}\underline{\omega}(s) & \partial_{\theta_f}\underline{\omega}(s) & \partial_{\rho_f}\underline{\omega}(s) & \partial_{\omega_i}\underline{\omega}(s) \end{pmatrix}$$
$$\in \begin{pmatrix} \mathrm{o}((s+\langle r_i\rangle)^0)|s| \\ \mathrm{O}(1) \\ \mathrm{O}((s+\langle r_i\rangle)^{-\tilde{\varepsilon}}) \\ \mathrm{o}((s+\langle r_i\rangle)^0)|s|^{1-\tilde{\varepsilon}} \end{pmatrix} \otimes (\langle r_1\rangle^{-1-\tilde{\varepsilon}}, \langle r_1\rangle^{-\tilde{\varepsilon}}, \langle r_1\rangle^{-\tilde{\varepsilon}}, \langle r_1\rangle^{-1}),$$

$$\partial^l_{r_i}\partial^\alpha_\theta\partial^k_\rho\partial^\beta_\omega \begin{pmatrix} \underline{r} \\ \underline{\theta} \\ \underline{\rho} \\ \underline{\omega} \end{pmatrix} \in \begin{pmatrix} \mathrm{o}((s+\langle r_i\rangle)^0)|s-t_1| \\ \mathrm{O}(1) \\ \mathrm{O}((s+\langle r_i\rangle)^{-\tilde{\varepsilon}}) \\ \mathrm{o}((s+\langle r_i\rangle)^0)|s-t_1|^{1-\tilde{\varepsilon}} \end{pmatrix} \cdot \langle r_i\rangle^{-l-|\beta|}.$$

Proof. There exists C_0 such that if $\rho \in J$ and $r > 0$, then

$$|r+(s-t_1)\rho| \ge C_0(|s-t_1|+r). \tag{2.14}$$

We fix constants $\varepsilon_0, \tilde{Q}, \varepsilon_1$ such that

$$0 < \varepsilon_0 < C_0, \ \ \tilde{Q} \ge \frac{2Q}{C_0^{1-\tilde{\varepsilon}}}, \ \ 0 < \varepsilon_1 < \frac{1}{2}\tilde{Q}(C_0-\varepsilon_0)^{1-\tilde{\varepsilon}},$$

and introduce cut-off functions $I_r, I_\theta, I_\rho, I_\omega$ as follows. We take $I_r \in C^\infty(0,\infty)$ such that $I_r = 1$ on a neighborhood of $\{r; r > C_0 - \varepsilon_0\}$, $I_\theta \in C_0^\infty(\mathbb{R}^{n-1})$ such that $I_\theta = 1$ on $\tilde{U}$, $I_\rho \in C_0^\infty(0,\infty)$ such that $I_\rho = 1$ on $\tilde{J}$, and $I_\omega \in C_0^\infty(\mathbb{R}^{n-1})$ such that $I_\omega = 1$ on a neighborhood of $\{\omega : |\omega| < \tilde{Q}\}$. Using these cut-off functions, we define the effective time-dependent force F_{e} by

$$F_{\mathrm{e}}(t,r,\theta,\rho,\omega) = I_r(\frac{r}{t})I_\theta(\theta)I_\rho(\rho)I_\omega(\frac{\omega}{r^{1-\tilde{\varepsilon}}})F(r,\theta,\rho,\omega).$$

It follows from (2.11) that $F_{\mathrm{e}}(t,r,\theta,\rho,\omega)$ is a slowly-decaying force in the sense of Definition 2.1. Therefore, we can find T such that the boundary value problem considered in Theorem 2.2 possesses a unique solution for any $T \le t_1 \le t_2$ and any $r_i, \theta_f, \rho_f, \omega_i$. Let us denote this solution by

$$(\tilde{r}_e, \tilde{\theta}_e, \tilde{\rho}_e, \tilde{\omega}_e)(s, t_1, t_2, r_i, \theta_f, \rho_f, \omega_i).$$

By enlarging T if needed, we can guarantee that

$$|\tilde{r}_e(s,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i) - r_i - (s-t_1)\rho_f| \le \varepsilon_0|s-t_1|, \tag{2.15}$$
$$|\tilde{\theta}_e(s,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i) - \theta_f| \le \mathrm{dist}(U,\tilde{U}^C)$$
$$|\tilde{\rho}_e(s,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i) - \rho_f| \le \mathrm{dist}(J,\tilde{J}^C)$$
$$|\tilde{\omega}_e(s,t_1,t_2,r_i,\theta_f,\rho_f,\omega_i) - \omega_i| \le \varepsilon_1|s-t_1|^{1-\tilde{\varepsilon}}. \tag{2.16}$$

We claim that if $R = T(C_0 - \varepsilon_0)/C_0$ and $(r_i, \theta_f, \rho_f, \omega_i) \in \Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}$, then we can solve our boundary problem by setting

$$(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s,t,r_i,\theta f,\rho_f,\omega_i) := (\tilde{r}_e, \tilde{\theta}_e, \tilde{\rho}_e, \tilde{\omega}_e)(r+s, r, r+t, r_i, \theta_f, \rho_f, \omega_i) \tag{2.17}$$

where $r = |r_i|C_0/(C_0 - \varepsilon)$. Indeed, from (2.14), (2.15), and (2.16) we see that

$$\begin{aligned}
&|\tilde{r}_e(r+s, r, r+t, r_i, \theta_f, \rho_f, \omega_i)| \geq (C_0 - \varepsilon_0)|s+r|,\\
&\tilde{\theta}_e(r+s, r, r+t, r_i, \theta_f, \rho_f, \omega_i) \in \tilde{U},\\
&\tilde{\rho}_e(r+s, r, r+t, r_i, \theta_f, \rho_f, \omega_i) \in \tilde{J},
\end{aligned}$$

and

$$\begin{aligned}
&|\tilde{\omega}_e(r+s, r, r+t, r_i, \theta_f, \rho_f, \omega_i)| \leq \varepsilon_1 |s|^{1-\tilde{\varepsilon}} + \omega_i \leq \varepsilon_1 |s|^{1-\tilde{\varepsilon}} + Q r_i^{1-\tilde{\varepsilon}}\\
&\leq \varepsilon_1 |s|^{1-\tilde{\varepsilon}} + Q\Big(\frac{C_0 - \varepsilon_0}{C_0}\Big)^{1-\tilde{\varepsilon}} r_i^{1-\tilde{\varepsilon}} \leq \tilde{Q}(C_0 - \varepsilon_0)^{1-\tilde{\varepsilon}} |s+r|^{1-\tilde{\varepsilon}}\\
&\leq \tilde{Q}|\tilde{r}_e(r+s, r, r+t, r_i, \theta_f, \rho_f, \omega_i)|^{1-\tilde{\varepsilon}}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&F_e(r+s, (\tilde{r}_e, \tilde{\theta}_e, \tilde{\rho}_e, \tilde{\omega}_e)(r+s, r, r+t, r_i, \theta_f, \rho_f, \omega_i))\\
&= F((\tilde{r}_e, \tilde{\theta}_e, \tilde{\rho}_e, \tilde{\omega}_e)(r+s, r, r+t, r_i, \theta_f, \rho_f, \omega_i)).
\end{aligned}$$

Therefore the function (2.17) solves the boundary problem (2.12) with the initial time-independent force.

The estimates on $(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s,t,r_i,\theta f,\rho_f,\omega_i)$ are obtained directly from those of Theorem 2.2 using the identity (2.17) and replacing s, t_1, t_2 there by $s + \langle r_i \rangle, \langle r_i \rangle, t + \langle r_i \rangle$.

Finally, the uniqueness of the solution comes from the fact that any solution of (2.12) with (2.13) is also a solution of the problem considered in Theorem 2.2 for the force $F_e(t, r, \theta, \rho, \omega)$ if time t is large enough.

□

Now we solve the dynamics with initial conditions.

Theorem 2.7. *Assume F is a time-independent long-range force in the sense of Definition 2.5. Then for any open $U \Subset \tilde{U} \Subset \mathbb{R}^{n-1}$, open $J \Subset \tilde{J} \Subset (0,\infty)$, and $Q > 0$, there exists $R > 0$ such that for any $(r_0, \theta_0, \rho_0, \omega_0) \in \Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}$, there exists a unique trajectory*

$$[0,\infty) \ni s \mapsto (\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, r_0, \theta_0, \rho_0, \omega_0)$$

satisfying

$$\begin{aligned}
&\partial_s(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, r_0, \theta_0, \rho_0, \omega_0) = (\rho + F_r, F_\theta, F_\rho, F_\omega)((\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(s, r_0, \theta_0, \rho_0, \omega_0)),\\
&(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\omega})(0, r_0, \theta_0, \rho_0, \omega_0) = (r_0, \theta_0, \rho_0, \omega_0).
\end{aligned}$$

Set

$$\underline{r}(s) = \tilde{r}(s) - r_0 - s\rho_0, \quad \underline{\theta}(s) = \tilde{\theta}(s) - \theta_0,$$
$$\underline{\rho}(s) = \tilde{\rho}(s) - \rho_0, \quad \underline{\omega}(s) = \tilde{\omega}(s) - \omega_0.$$

Moreover the solution satisfies the following estimates uniformly for $0 \le s \le t \le \infty$, $(r_0, \theta_0, \rho_0, \omega_0) \in \Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}$:

$$\tilde{\theta}(s, r_0, \theta_0, \rho_0, \omega_0) \in \tilde{U}, \quad \tilde{\rho}(s, r_0, \theta_0, \rho_0, \omega_0) \in \tilde{J},$$

$$|\underline{r}(s)| \in \mathrm{o}((s + \langle r_0 \rangle)^0)|s|, \quad |\underline{\theta}(s)| \in \mathrm{O}((\langle r_0 \rangle)^{-\tilde{\varepsilon}}),$$
$$|\underline{\rho}(s)| \in \mathrm{O}((\langle r_0 \rangle)^{-\tilde{\varepsilon}}), \quad |\underline{\omega}(s)| \in \mathrm{o}((s + \langle r_0 \rangle)^0)|s|^{1-\tilde{\varepsilon}},$$
$$\begin{pmatrix} \partial_{r_0}\underline{r}(s) & \partial_{\theta_0}\underline{r}(s) & \partial_{\rho_0}\underline{r}(s) & \partial_{\omega_0}\underline{r}(s) \\ \partial_{r_0}\underline{\theta}(s) & \partial_{\theta_0}\underline{\theta}(s) & \partial_{\rho_0}\underline{\theta}(s) & \partial_{\omega_0}\underline{\theta}(s) \\ \partial_{r_0}\underline{\rho}(s) & \partial_{\theta_0}\underline{\rho}(s) & \partial_{\rho_0}\underline{\rho}(s) & \partial_{\omega_0}\underline{\rho}(s) \\ \partial_{r_0}\underline{\omega}(s) & \partial_{\theta_0}\underline{\omega}(s) & \partial_{\rho_0}\underline{\omega}(s) & \partial_{\omega_0}\underline{\omega}(s) \end{pmatrix}$$
$$\in \begin{pmatrix} \mathrm{o}((s + \langle r_0 \rangle)^0)|s| \\ \mathrm{O}(1) \\ (\langle r_0 \rangle)^{-\tilde{\varepsilon}} \\ \mathrm{o}((s + \langle r_0 \rangle)^0)|s|^{1-\tilde{\varepsilon}} \end{pmatrix} \otimes (\langle r_0 \rangle^{-1-\tilde{\varepsilon}}, \langle r_0 \rangle^{\tilde{\varepsilon}}, \langle r_0 \rangle^{\tilde{\varepsilon}}, \langle r_0 \rangle^{-1}),$$

$$\partial^l_{r_0}\partial^\alpha_{\theta_0}\partial^k_{\rho_0}\partial^\beta_{\omega_0} \begin{pmatrix} \underline{r} \\ \underline{\theta} \\ \underline{\rho} \\ \underline{\omega} \end{pmatrix} \in \begin{pmatrix} \mathrm{o}((s + \langle r_0 \rangle)^0)|s - t_1| \\ \mathrm{O}(1) \\ (\langle r_0 \rangle)^{-\tilde{\varepsilon}} \\ \mathrm{o}((s + \langle r_0 \rangle)^0)|s - t_1|^{1-\tilde{\varepsilon}} \end{pmatrix} \cdot \langle r_0 \rangle^{-l-|\beta|}.$$

Proof. Let $(\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})$ be the solutions in Theorem 2.6 with $t = \infty$:

$$[0, \infty] \ni s \mapsto (\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})(s, \infty, r_i, \theta_f, \rho_f, \omega_i),$$
$$\partial_s(\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})(s, \infty, r_i, \theta_f, \rho_f, \omega_i)$$
$$= (\rho + F_r, F_\theta, F_\rho, F_\omega)((\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})(s, \infty, r_i, \theta_f, \rho_f, \omega_i)),$$
$$(\bar{r}, \bar{\omega})(0, \infty, r_i, \theta_f, \rho_f, \omega_i) = (r_i, \omega_i), \quad (\bar{\theta}, \bar{\rho})(\infty, \infty, r_i, \theta_f, \rho_f, \omega_i) = (\theta_f, \rho_f).$$

Set

$$(r_0, \theta_0, \rho_0, \omega_0)(r_i, \theta_f, \rho_f, \omega_i) := (\bar{r}, \bar{\theta}, \bar{\rho}, \bar{\omega})(0, \infty, r_i, \theta_f, \rho_f, \omega_i).$$

It is clear that

$$(r_0, \omega_0)(r_i, \theta_f, \rho_f, \omega_i) = (r_i, \omega_i).$$

Theorem 2.6 assures the following estimates:

$$|\theta_0(r_i, \theta_f, \rho_f, \omega_i) - \theta_f| \in \mathrm{O}(\langle r_i \rangle^{-\tilde{\varepsilon}}), \quad |\rho_0(r_i, \theta_f, \rho_f, \omega_i) - \rho_f| \in \mathrm{O}(\langle r_i \rangle^{-\tilde{\varepsilon}}),$$

$$\begin{pmatrix} \partial_{r_i}(\theta_0-\theta_f) & \partial_{\theta_f}(\theta_0-\theta_f) & \partial_{\rho_f}(\theta_0-\theta_f) & \partial_{\omega_i}(\theta_0-\theta_f) \\ \partial_{r_i}(\rho_0-\rho_f) & \partial_{\theta_f}(\rho_0-\rho_f) & \partial_{\rho_f}(\rho_0-\rho_f) & \partial_{\omega_i}(\rho_0-\rho_f) \end{pmatrix}$$

$$\in \begin{pmatrix} O(1) \\ (\langle r_i\rangle)^{-\tilde{\varepsilon}} \end{pmatrix} \otimes (\langle r_1\rangle^{-1-\tilde{\varepsilon}}, \langle r_1\rangle^{-\tilde{\varepsilon}}, \langle r_1\rangle^{-\tilde{\varepsilon}}, \langle r_1\rangle^{-1}). \tag{2.18}$$

By taking R large enough, we can assure that the map $(r_0,\theta_0,\rho_0,\omega_0)(r_i,\theta_f,\rho_f,\omega_i)$ is injective. Let $(r_i,\theta_i,\rho_i,\omega_i)(r_0,\theta_0,\rho_0,\omega_0)$ be the inverse function. We will show that

$$(\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(s,r_0,\theta_0,\rho_0,\omega_0) := (\bar{r},\bar{\theta},\bar{\rho},\bar{\omega})(s,\infty,(r_i,\theta_f,\rho_f,\omega_i)(r_0,\theta_0,\rho_0,\omega_0))$$

gives the desired function. (2.18) implies

$$\begin{pmatrix} \partial_{r_0}(\theta_f-\theta_0) & \partial_{\theta_0}(\theta_f-\theta_0) & \partial_{\rho_0}(\theta_f-\theta_0) & \partial_{\omega_0}(\theta_f-\theta_0) \\ \partial_{r_0}(\rho_f-\rho_0) & \partial_{\theta_0}(\rho_f-\rho_0) & \partial_{\rho_0}(\rho_f-\rho_0) & \partial_{\omega_0}(\rho_f-\rho_0) \end{pmatrix}$$

$$\in \begin{pmatrix} O(1) \\ (\langle r_0\rangle)^{-\tilde{\varepsilon}} \end{pmatrix} \otimes (\langle r_1\rangle^{-1-\tilde{\varepsilon}}, \langle r_1\rangle^{-\tilde{\varepsilon}}, \langle r_1\rangle^{-\tilde{\varepsilon}}, \langle r_1\rangle^{-1}). \tag{2.19}$$

Moreover, it is easy to see that

$$\partial_{r_0}^l \partial_{\theta_0}^\alpha \partial_{\rho_0}^k \partial_{\omega_0}^\beta \begin{pmatrix} \theta_f-\theta_0 \\ \rho_f-\rho_0 \end{pmatrix} \in \begin{pmatrix} O(\langle r_0\rangle^{-l-|\beta|}) \\ O(\langle r_0\rangle^{-l-|\beta|-\tilde{\varepsilon}}) \end{pmatrix}. \tag{2.20}$$

(2.19) and (2.20) shows the desired estimates. □

2.3 Solutions to the Hamilton-Jacobi equation

We state a lemma which relates the hamiltonian h with the time-independent force F.

Lemma 2.8. *Let*

$$h(r,\theta,\rho,\omega) = \frac{1}{2}\rho^2 + \tilde{h}(r,\theta,\rho,\omega)$$
$$\tilde{h}(r,\theta,\rho,\omega) = \frac{1}{2}a_1(r,\theta)\rho^2 + \frac{1}{r}a_2^j\rho\,\omega_j + \frac{1}{2r^2}a_3^{jk}\omega_j\omega_k + V^L(r,\theta).$$

Assume

$$|\partial_r^l\partial_\theta^\alpha a_j(r,\theta)| \le C_{l,\alpha}r^{-\mu_j-l}, \ \ |D_r^j D_\theta^\alpha V^L(r,\theta)| \le C_j r^{-\mu_L-j},$$

with

$$\mu_1 = \mu_2 = \mu_L = \varepsilon > 0, \ \ \mu_3 = 0.$$

Then for any $U \Subset \mathbb{R}^{n-1}, J \Subset \mathbb{R}$, and $Q > 0$,

$$\sup_{(r,\theta,\rho,\omega)\in\Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}} |\partial_r^l\partial_\theta^\alpha\partial_\rho^k\partial_\omega^\beta(\tilde{h})(r,\theta,\rho,\omega)| \in O(\langle R\rangle^{-m(l,\alpha,k,\beta)}). \tag{2.21}$$

This immediately implies that setting

$$(F_r,F_\theta,F_\rho,F_\omega) = (\partial_\rho\tilde{h}, \partial_\omega\tilde{h}, -\partial_r\tilde{h}, -\partial_\theta\tilde{h}), \tag{2.22}$$

we have (2.11), i.e., h defines a long-range time-independent force via (2.22).

Combining Theorem 2.7 and Lemma 2.8, we obtain solutions to the Hamilton-Jacobi equation:

Theorem 2.9. *Let $h, \tilde{h}$ be as in Lemma 2.8. For any $\tilde{U} \Subset U \Subset \mathbb{R}^{n-1}, \tilde{J} \Subset J \Subset (0,\infty)$, $C_{j,\alpha} > 0$, there exists $T > 0$ such that if a smooth function $\psi(\rho,\theta)$ defined on $J \times U$ satisfies*

$$|\partial_\rho^j \partial_\theta^\alpha (\psi(\rho,\theta) - \frac{1}{2}s\rho^2)| \leq C_{j,\alpha}\langle s\rangle^{1-\tilde{\varepsilon}} \tag{2.23}$$

for some $s > T$, then there exists a unique function $S(t,\rho,\theta)$ defined on a region $\Theta \subset (0,\infty) \times (0,\infty) \times \mathbb{R}^{n-1}$ (which will be defined in the proof), with $\Theta \supset (0,\infty) \times \tilde{J} \times \tilde{U}$, satifying the Hamilton-Jacobi equation:

$$(\partial_t S)(t,\rho,\theta) = h((\partial_\rho S)(\rho,\theta),\theta,\rho,-(\partial_\theta S)(\rho,\theta))$$

with the initial value

$$S(0,\rho,\theta) = \psi(\rho,\theta).$$

Moreover the function S satisfies the following estimates:

$$|\partial_\rho^j \partial_\theta^\alpha (S(t,\rho,\theta) - \frac{1}{2}t\rho^2)| \leq \tilde{C}_{j,\alpha}\langle t\rangle^{1-\varepsilon}. \tag{2.24}$$

Proof. Let

$$[0,\infty) \ni t \mapsto (\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(t,r_0,\theta_0,\rho_0,\omega_0)$$

be the unique trajectory of the Hamilton equations with initial value problem as in the Theorem 2.7:

$$\partial_t(\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(t,r_0,\theta_0,\rho_0,\omega_0) = (\rho + F_r, F_\theta, F_\rho, F_\omega)((\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(t,r_0,\theta_0,\rho_0,\omega_0)),$$
$$(\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(0,r_0,\theta_0,\rho_0,\omega_0) = (r_0,\theta_0,\rho_0,\omega_0),$$

for $(r_0,\theta_0,\rho_0,\omega_0) \in \Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}$, where we took $R > 0$ such that

$$\{((\partial_\rho\psi)(\rho_0,\theta_0),\theta_0,\rho_0,-(\partial_\theta\psi)(\rho_0,\theta_0)) : (\theta_0,\rho_0) \in U \times J\} \subset \Gamma^{+,\tilde{\varepsilon}}_{R,U,J,Q}.$$

Set

$$(r,\theta,\rho,\omega)(t;\rho_0,\theta_0) := (\tilde{r},\tilde{\theta},\tilde{\rho},\tilde{\omega})(t,(\partial_\rho\psi)(\rho_0,\theta_0),\theta_0,\rho_0,-(\partial_\theta\psi)(\rho_0,\theta_0)).$$

and consider the map

$$(\rho_0,\theta_0) \mapsto (\rho,\theta)(t;\rho_0,\theta_0) \tag{2.25}$$

and its first derivatives. We set $\Theta := \{(t,(\rho,\theta)(t;\rho_0,\theta_0))|(\rho_0,\theta_0) \in J \times U\}$. By a straight computation, we obtain

$$\left|\frac{\partial(\rho,\theta)(t,\rho_0,\theta_0)}{\partial(\rho_0,\theta_0)} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = C\langle s\rangle^{-\tilde{\varepsilon}}.$$

where C depends on $C_{j,\alpha}$ and not on the choice of ψ as long as ψ satisfies (2.23) for some s. We fix a large enough $T > 0$ so that for any $s > T$ we have

$$\left| \frac{\partial(\rho,\theta)(t,\rho_0,\theta_0)}{\partial(\rho_0,\theta_0)} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| \ll 1.$$

Now (2.25) becomes an injective map for every $t > 0$. We denote its inverse by

$$(\rho,\theta) \mapsto (\rho_0,\theta_0)(t;\rho,\theta).$$

Let

$$\begin{aligned} &Q(t;\rho_0,\theta_0) \\ &= \psi(\rho_0,\theta_0) + \int_0^t [h((r,\theta,\rho,\omega)(u;\rho_0\omega_0) \\ &+ \langle r(t;\rho_0,\theta_0), (\partial_u\rho)(u;\rho,\theta)\rangle - \langle \omega(t;\rho_0,\theta_0), (\partial_u\theta)(u;\rho,\theta)\rangle] du. \end{aligned}$$

Then the function

$$S(t,\rho,\theta) = Q(t;(\rho_0,\theta_0)(t;\rho,\theta))$$

defined on Θ is the desired solution to the Hamilton-Jacobi equation (see, for example, [6] Appendix A.3). Moreover

$$\begin{aligned} &(\partial_\rho S)(t,\rho,\theta) = r(t;\rho_0(t,\rho,\theta),\theta_0(t,\rho,\theta)), \\ &-(\partial_\theta S)(t,\rho,\theta) = \omega(t;\rho_0(t,\rho,\theta),\theta_0(t,\rho,\theta)). \end{aligned}$$

The derivatives of $S(t,\rho,\theta)$

$$\partial_\rho^j \partial_\theta^\alpha \partial_t (S(t,\rho,\theta) - \frac{1}{2}t\rho^2) = \partial_\rho^j \partial_\theta^\alpha \tilde{h}(\partial_\rho S(t,\rho,\theta),\theta,\rho,-\partial_\theta S(t,\rho,\theta))$$

is a summation of the terms of the type

$$\begin{aligned} &(\partial_r^l \partial_\theta^\beta \partial_\rho^k \partial_\omega^\gamma \tilde{h})(\partial_\rho S(t,\rho,\theta),\theta,\rho,-\partial_\theta(t,\rho,\theta)) \times \\ &\prod_{i=1}^{l} \partial_\rho^{k_i} \partial_\theta^{\beta_i} (\partial_\rho S)(t,\rho,\theta) \times \prod_{d=1}^{n-1} \prod_{j=1}^{\gamma_d} \partial_\rho^{k_{d,j}} \partial_\theta^{\beta_{d,j}} (-\partial_{\theta_d} S)(t,\rho,\theta)), \end{aligned}$$

which belongs to $\mathrm{O}(\langle t\rangle^{-m(l,\beta,k,\gamma)+l+(1-\tilde{\varepsilon})|\gamma|}) \subset \mathrm{O}(\langle t\rangle^{-\varepsilon})$. This shows (2.24).

□

Finally we extend $S(t,\rho,\theta)$ to a globally defined function on $\mathbb{R} \times (0,\infty) \times \partial M$ which satisfies the same kind of estimates locally.

Theorem 2.10. *Let $h, \tilde{h}$ be as in Lemma 2.8 defined on $T^*\mathbb{R} \times T^*\partial M$. Then there exists a function $S(t,\rho,\theta)$ defined on $T^*\mathbb{R} \times T^*\partial M$ such that for every $J \Subset \mathbb{R} \setminus \{0\}$, there exists $T > 0$ such that the Hamilton-Jacobi equation:*

$$(\partial_t S)(t,\rho,\theta) = h((\partial_\rho S)(\rho,\theta),\theta,\rho,-(\partial_\theta S)(\rho,\theta)) \tag{2.26}$$

is satisfied for $t > |T|$, $\rho \in J$, and $\theta \in \partial M$. Moreover the function S satisfies the following estimates:

$$|\partial_\rho^j \partial_\theta^\alpha (S(t,\rho,\theta) - \frac{1}{2}t\rho^2)| \le \tilde{C}_{j,\alpha} \langle t\rangle^{1-\varepsilon}. \tag{2.27}$$

Proof. First note that since ∂M is compact and the Hamilton-Jacobi equation is defined in a coordinate invariant manner, we can extend U in the Théorem 2.9 to ∂M. It is sufficient to consider the case $J \Subset (0,\infty)$ and $t > T$, since we can extend the function S in a C^∞-fashion.

Take a sequence of open sets in $(0,\infty)$ such that

$$J_0 \Subset J_1 \Subset J_2 \Subset J_3 \Subset \dots, \bigcup_{n=0}^{\infty} J_n = (0,\infty).$$

First we solve the Cauchy problem for the Hamilton-equation with initial data

$$S(t,\rho,\theta) = \frac{1}{2}t\rho^2 \quad \text{when} \quad \rho \in J_1, t = T_1 > 0$$

for a big enough T_1 by Theorem 2.9 with U replaced by ∂M. We denote the solution by S_1. We can assume that S_1 is defined on $(T_1,\infty) \times J_0 \times \partial M$. S_1 also satisfies (2.24) for $\rho \in J_1$ and $t \geq T_1$.

Next we take $\chi_1 \in C_0^\infty(J_1)$ equal to 1 in a neighborhood of $\overline{J_0}$ (the closure of J_0). We solve the Cauchy Problem with initial data

$$S(t,\rho,\theta) = \chi_1 S_1 + (1-\chi_1)\frac{t}{2}\rho^2 \quad \text{when} \quad \rho \in J_2, t = T_2.$$

By taking $T_2 > T_1$ large enough, the right hand side satisfies the conditions for T_2 in Theorem 2.9. So we can solve the Cauchy Problem for such T_2. We denote the solution by S_2.

Repeating this procedure, we obtain a sequence S_n of functions and a sequence $T_1 < T_2 < \dots$ such that S_n is defined on $J_n \times [T_n,\infty) \times \partial M$,

$$S_{n+1} = S_m \text{ for } m \geq n+1 \text{ on } J_n \times [T_m,\infty) \times \partial M,$$

and satisfies (2.24). Thus by extending in a C^∞ fashion, we can construct a C^∞ function S which satisfies (2.26), and (2.24) for large enough t and ρ in any fixed compact subset of $(0,\infty)$. □

3 Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. First we give the outline of the proof.

Outline of the Proof. We consider the $t \to +\infty$ case. By the density argument, it is sufficient to show the existence of the norm limit

$$\lim_{t\to\infty} e^{itH} J e^{-iS(t,D_r,\theta)} u$$

for all $\hat{u} \in C_0^\infty((\mathbb{R}\setminus\{0\}) \times U_\lambda)$ for all λ. For such u, we have

$$\frac{1}{i} e^{-itH} \frac{\partial}{\partial t}[e^{itH} J e^{-iS(t,D_r,\theta)} u] = [HJ - J\frac{\partial S}{\partial t}(t,D_r,\theta)] e^{-iS(t,D_r,\theta)} u.$$

By the Cook-Kuroda method we only need to show that

$$\|[HJ - J\frac{\partial S}{\partial t}(t,D_r,\theta)] e^{-iS(t,D_r,\theta)} u\|_{\mathscr{H}} \in L_t^1(1,\infty).$$

We decompose

$$[HJ - J\frac{\partial S}{\partial t}(t,D_r,\theta)]$$
$$= [P_0J - JP_0] + V_SJ + [V^LJ - JV^L] + J[P_0 + V^L(r) - \frac{\partial S}{\partial t}(t,D_r,\theta)].$$

The fisrt three terms are essentially short range terms. It is easy to check

$$\|[P_0J - JP_0] + V_SJ + [V^LJ - JV^L]e^{-iS(t,D_r,\theta)}u\|_{\mathscr{H}} \in L^1_t(1,\infty). \tag{3.1}$$

We examine the last term:

$$[P_0 + V^L(r) - (\partial_t S)(t,D_r,\theta)]e^{-iS(t,D_r,\theta)}u$$
$$= h^r(r,\theta,D_r, -\frac{\partial S}{\partial \theta}(t,D_r,\theta))e^{-iS(t,D_r,\theta)}u$$
$$-h((\partial_\rho S)(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial \theta}(t,D_r,\theta))e^{-iS(t,D_r,\theta)}u$$
$$+(\frac{1}{r}a_2^jD_r + \frac{1}{2r^2}a_3^{jk}\frac{\partial S}{\partial \theta^k}(t,D_r,\theta))e^{-iS(t,\rho,\theta))}(\partial_{\theta^j}u)$$
$$-\frac{1}{2r^2}a_3^{jk}e^{-iS(t,\rho,\theta))}(\partial_{\theta^j}\partial_{\theta^k}u)$$
$$+ [\text{ short range terms }].$$

We apply the stationary phase method to the first two terms. Then the first terms which appear in the asymptotic expansion will vanish since the relation

$$(\partial_\rho S)(t,\rho,\theta) = r$$

gives the stationary point with respect to the ρ-variable. Therefore we obtain

$$\|[P_0 + V^L(r) - (\partial_t S)(t,D_r,\theta)]e^{-iS(t,D_r,\theta)}u\|_{\mathscr{H}} \in L^1_t(1,\infty). \tag{3.2}$$

We give a detailed proof of (3.1) and (3.2) in the remaining of this section. □

First we consider the long-range term (3.2). The next proposition is our key estimate.

Proposition 3.1. *Assume the assumptions of Theorem 1.3. Suppose u satisfies* $\hat{u} \in C_0^\infty((\mathbb{R}\setminus\{0\}) \times U_\lambda)$ *and* $J \times U$ *is a neighborhood of supp* $\hat{u}$. *Then we have*

$$|[\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial \theta}(t,D_r,\theta)) - \tilde{h}(\frac{\partial S}{\partial \rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial \theta}(t,D_r,\theta))]$$
$$\cdot e^{-iS(t,D_r,\theta)}u(r,\theta)| \le Ct^{-\frac{1}{2}-1-\varepsilon} \tag{3.3}$$

for $(\frac{r}{t},\theta) \in J \times U \Subset (0,\infty) \times \partial M$, *and*

$$|[\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial \theta}(t,D_r,\theta)) - \tilde{h}(\frac{\partial S}{\partial \rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial \theta}(t,D_r,\theta))]$$
$$\cdot e^{-iS(t,D_r,\theta)}u(r,\theta)| \le C_N(1+|r|+|t|)^{-N} \tag{3.4}$$

for any N *and for* $(\frac{r}{t},\theta) \notin J \times U$.

Proof of (3.2) **.** We fix a neighborhood $J\times U$ of supp$\hat{u}$ which appears in Proposition 3.1. Then

$$\int_1^\infty \|J|[\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))-\tilde{h}(\frac{\partial S}{\partial\rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))]$$
$$e^{-iS(t,D_r,\theta)}u\|_{\mathscr{H}}dt$$
$$=\int_1^\infty (\int_{\mathbb{R}_+\times\partial M}|j(r)[\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))$$
$$-\tilde{h}(\frac{\partial S}{\partial\rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))]e^{-iS(t,D_r,\theta)}u(r,\theta)|^2drH(\theta)d\theta)^{\frac{1}{2}}dt$$
$$\le\int_1^\infty (\int_{\frac{r}{t}\in J}|j(r)[\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))$$
$$-\tilde{h}(\frac{\partial S}{\partial\rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))]e^{-iS(t,D_r,\theta)}u(r,\theta)|^2drH(\theta)d\theta)^{\frac{1}{2}}$$
$$+(\int_{\frac{r}{t}\notin J}|j(r)[\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))$$
$$-\tilde{h}(\frac{\partial S}{\partial\rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))]e^{-iS(t,D_r,\theta)}u(r,\theta)|^2drH(\theta)d\theta)^{\frac{1}{2}}dt.$$

By (3.3), the first term is finite:

$$\int_1^\infty (\int_{\frac{r}{t}\in J}|j(r)[[\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))-\tilde{h}(\frac{\partial S}{\partial\rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))]$$
$$e^{-iS(t,D_r,\theta)}u(r,\theta)|^2drH(\theta)d\theta)^{\frac{1}{2}}dt$$
$$\le\int_1^\infty (\int_{R\in J}|Ct^{-\frac{1}{2}-1-\varepsilon}|^2tdR)^{\frac{1}{2}}dt \quad \le C\int_1^\infty t^{-1-\varepsilon}dt \quad <\infty.$$

By (3.4), the second term is also finite:

$$\int_1^\infty (\int_{\frac{r}{t}\notin J}|j(r)[[\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))-\tilde{h}(\frac{\partial S}{\partial\rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))]$$
$$e^{-iS(t,D_r,\theta)}u(r,\theta)|^2drH(\theta)d\theta)^{\frac{1}{2}}dt$$
$$\le\int_1^\infty (\int_{\frac{r}{t}\notin J}C(1+|r|+|t|)^{-N}dr)^{\frac{1}{2}}dt \quad <\infty$$

Therefore

$$\|J[V^L(r,D_r,\theta)-V^L(\frac{\partial W}{\partial\rho}(D_r,\theta,t),D_r,\theta)]e^{-iW(D_r,\theta,t)}u\|_{\mathscr{H}}\in L^1_t(1,\infty).$$

□

In order to prove Proposition 3.1, we prepare a lemma.

Lemma 3.2. *Let $S(t,\rho,\theta)$ satisfy the properties listed in Theorem 2.10. Set*

$$f_{r,\theta,t}(\rho):=\frac{1}{t}(r\rho-S(t,\rho,\theta)).$$

For ρ in any fixed compact subset of $\mathbb{R}\setminus\{0\}$ and for large enough $|t|$, there exists a function $\Xi_{\theta,t}(r)$ which gives the critical point of $f_{r,\theta,t}(\rho)$:

$$\partial_\rho f_{r,\theta,t}(\rho)=0 \iff \rho=\Xi_{\theta,t}(r).$$

Set $\Omega_d:=(-d,d)$. Then there exist $0<\tilde{d}<d$ and a function $\phi_{r,\theta,t}\in C^\infty(\Omega_d;\mathbb{R})$ such that $\Omega_{2\tilde{d}}\Subset\phi_{r,\theta,t}(\Omega_d)$. Setting

$$\psi_{r,\theta,t}(y):=\Xi_{\theta,t}(r)+\phi_{r,\theta,t}(y),$$
$$(f_{r,\theta,t}\circ\psi_{r,\theta,t})(y)=f_{r,\theta,t}(\Xi_{\theta,t}(r))+\langle A_{r,\theta,t}y,y\rangle/2,$$

where

$$A_{r,\theta,t}=(\partial_\rho^2 f_{r,\theta,t})(\Xi_{\theta,t}(r)),$$

we have

$$|\partial_y^k\psi_{r,\theta,t}(0)|\le Ct^{-\varepsilon}\quad(k\ge 2),\ \ \partial_y\psi_{r,\theta,t}(0)=1. \tag{3.5}$$

Proof. We only consider the $t>0$ and $\rho>0$ case. First we prove that $\Xi_{\theta,t}(r)$ is well-defined. Compute

$$0=\partial_\rho f_{r,\theta,t}(\rho)\ =\frac{1}{t}\big[r-\frac{\partial S}{\partial\rho}(t,\rho,\theta)\big]$$

We note that by (2.27),

$$\Big|\frac{1}{t}\frac{\partial^2 S}{\partial\rho^2}(t,\rho,\theta)-1\Big|\le Ct^{-\varepsilon}.$$

This implies that $\frac{1}{t}\frac{\partial S}{\partial\rho}$ is monotonously increasing with respect to ρ for large enough t. Thus there is a unique inverse function $\Xi_{\theta,t}(r)$ such that

$$(\partial_\rho f_{r,\theta,t})(\Xi_{\theta,t}(r))=0$$

for large enough t and $\frac{r}{t}\in J$, a fixed compact subset of $(0,\infty)$.

Now we construct $\phi_{r,\theta,t}$ and $\psi_{r,\theta,t}$. We set

$$A_{r,\theta,t}:=f''_{r,\theta,t}(\Xi_{\theta,t}(r))\ =-\frac{1}{t}\frac{\partial^2 S}{\partial\rho^2}(t,\Xi_{\theta,t}(r),\theta).$$

Then (2.27) implies that

$$|A_{r,\theta,t}+1|\le Ct^{-\varepsilon}.$$

Hence we have $A_{r,\theta,t}\to -1$ uniformly for $r/t\in J$. If we set

$$g_{r,\theta,t}(\rho):=f_{r\theta,t}(\Xi_{\theta,t}(r)+\rho),$$

then

$$g'_{r,\theta,t}(0) := f'_{r\theta,t}(\Xi_{\theta,t}(r)) = 0,\ g''_{r,\theta,t}(0) := f''_{r\theta,t}(\Xi_{\theta,t}(r)) = A_{r,\theta,t},$$
$$g_{r,\theta,t}(\rho) - g_{r,\theta,t}(0) = \langle B_{r,\theta,t}(\rho)\rho,\rho\rangle/2,$$

where

$$B_{r,\theta,t}(\rho) := 2\int_0^1 g_{r,\theta,t}(s\rho)(1-s)ds,\ B_{r,\theta,t}(0) = A_{r,\theta,t}$$

by Taylor's formula. Now we compute

$$\begin{aligned}|B_{r,\theta,t}(\rho) - A_{r,\theta,t}| &= |B_{r,\theta,t}(\rho) - B_{r,\theta,t}(0)| = 2|\int_0^1 (g''_{r,\theta,t}(s\rho) - g''_{r,\theta,t}(0))(1-s)ds| \\ &\le 2\sup_{0\le s\le 1} |g''_{r,\theta,t}(s\rho) - g''_{r,\theta,t}(0)| \\ &\le 2\sup_{0\le s\le 1} |\frac{1}{t}\frac{\partial^2 S}{\partial\rho^2}(t,\Xi_{\theta,t}(r)+s\rho,\theta) - \frac{1}{t}\frac{\partial^2 S}{\partial\rho^2}(t,\Xi_{\theta,t}(r),\theta)| \\ &\le Ct^{-\varepsilon} \to 0\end{aligned}$$

as $t\to\infty$, uniformly for $\frac{r}{t},\rho\in J$ by (2.27). Hence by taking t sufficiently large, we may assume $|\frac{B_{r,\theta,t}(\rho)}{A_{r,\theta,t}(\rho)} - 1| < 1/2$ is uniformly very small. For such t, $\frac{r}{t}$,and ρ, we set

$$X_{r,\theta,t}(\rho) := \sqrt{\frac{B_{r,\theta,t}(\rho)}{A_{r,\theta,t}}}\cdot\rho.$$

Then we have

$$g_{r,\theta,t}(\rho) - g_{r,\theta,t}(0) = \langle A_{r,\theta,t}X_{r,\theta,t}(\rho), X_{r,\theta,t}(\rho)\rangle/2.$$

Now we compute

$$\begin{aligned}(\partial_\rho X_{t,\theta,t})(\rho) &= (\sqrt{\frac{B_{r,\theta,t}(\rho)}{A_{r,\theta,t}}})'\cdot\rho + \sqrt{\frac{B_{r,\theta,t}(\rho)}{A_{r,\theta,t}}}\cdot 1 \\ &= \frac{1}{\sqrt{A_{r,\theta,t}}} 2\sqrt{B_{r,\theta,t}(\rho)}\cdot B'_{r,\theta,t}(\rho)\cdot\rho + \sqrt{\frac{B_{r,\theta,t}(\rho)}{A_{r,\theta,t}}}\cdot 1,\\ (\partial_\rho B_{r,\theta,t})(\rho) &= 2\int_0^1 g'''_{r,\theta,t}(s\rho)s(1-s)ds,\\ |g'''_{r,\theta,t}(s\rho)| &= |-\frac{1}{t}(\partial_\rho^3 S)(t,\Xi_{\theta,t}(r)+s\rho,\theta)| \le Ct^{-\varepsilon},\\ |\partial_\rho X_{r,\theta,t}(\rho) - 1| &\le Ct^{-\varepsilon}.\end{aligned}$$

Hence for small enough $d_0 > 0$ and for $|\rho| \le d_0$, we have $|\partial_\rho X_{r,\theta,t}(\rho) - 1|$ arbitrary small for all large enough t, and $X_{r,\theta,t} : \Omega_{d_0} \to X_{r,\theta,t}(\Omega_{d_0})$ is a C^∞-diffeomorphism. We can pick $d > 0$ such that, $\Omega_d \subset X_{r,\theta,t}(\Omega_{d_0})$ for all r,θ, large enough t, $\frac{r}{t}\in J$. Let $\phi_{r,\theta,t}$ be the inverse map of

$X_{r,\theta,t}$ with domain Ω_d. Then we can also pick $\tilde{d} > 0$ such that $\Omega_{\tilde{d}} \subset \phi_{r,\theta,t}(\Omega_d)$ for all r, θ, large enough $t, \frac{r}{t} \in J$. We note that

$$g_{r,\theta,t} \circ \phi_{r,\theta,t}(y) - g_{r,\theta,t}(0) = \langle A_{r,\theta,t} X_{r,\theta,t} \circ \phi_{r,\theta,t}(y), X_{r,\theta,t} \circ \phi_{r,\theta,t}(y) \rangle / 2. \\ = \langle A_{r,\theta,t} y, y \rangle / 2.$$

We set

$$\psi_{r,\theta,t}(y) = \phi_{r,\theta,t}(y) + \Xi_{\theta,t}(r).$$

Then we have

$$f_{r,\theta,t} \circ \psi_{r,\theta,t}(y) = f_{r,\theta,t}(\Xi_{\theta,t}(r)) + \langle A_{r,\theta,t} y, y \rangle / 2.$$

Lastly, we prove the estimates (3.5). For $k \geq 1$,

$$|\partial_\rho^k B_{r,\theta,t}(\rho)| = 2 \left| \int_0^1 g_{r,\theta,t}^{2+k}(s\rho) s^k (1-s) ds \right| \leq 2 \sup |g_{r,\theta,t}^{2+k}(s\rho)| \\ \leq 2 \sup \left| \frac{1}{t} \partial_\rho^{2+k} S(t, \Xi_{\theta,t}(r) + s\rho, \theta) \right| \leq C t^{-\varepsilon}$$

by (2.27). We also have

$$|\partial_\rho^k \sqrt{B_{r,\theta,t}}(\rho)| \leq C t^{-\varepsilon}.$$

Therefore

$$|\partial_\rho^k X_{r,\theta,t}(\rho)| \leq C t^{-\varepsilon} \quad (k \geq 2), \; |\partial_\rho X_{r,\theta,t}(0) - 1| = C t^{-\varepsilon},$$

and we have

$$|\partial_y^k \psi_{r,\theta,t}(y)| = |\partial_y^k \phi_{r,\theta,t}(y)| \leq C t^{-\varepsilon} \quad (k \geq 2), \\ |\partial_y \psi_{r,\theta,t}(0) - 1| = |\partial_y \phi_{r,\theta,t}(0) - 1| \leq C t^{-\varepsilon}.$$

Then we complete the proof of Lemma 3.2.

□

Proof of Proposition 3.1. First we prove (3.3) for $\frac{r}{t} \in J$. We fix $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(x) = 1$ if $|x| \leq \frac{1}{2}$, and $\chi(x) = 0$ if $|x| \geq 1$. We split u into two terms depending on r, θ, and t:

$$\hat{u}_{r,\theta,t}^c(\rho, \theta) = \hat{u}(\rho, \theta) \chi\left(\frac{\rho - \Xi_{\theta,t}(r)}{\tilde{d}}\right), \\ \hat{u}_{r,\theta,t}^d(\rho, \theta) = \hat{u}(\rho, \theta) \left[1 - \chi\left(\frac{\rho - \Xi_{\theta,t}(r)}{\tilde{d}}\right)\right]$$

where we use notations defined in Lemma 3.2. The support of $\hat{u}_{r,\theta,t}^c$ is close to the critical point of $r\rho - S(t, \rho, \theta)$, while that of $\hat{u}_{r,\theta,t}^d$ is apart from it. Note that

$$\mathrm{supp} \hat{u}_{r,\theta,t}^c \subset \Xi_{\theta,t}(r) + \Omega_{\tilde{d}} \subset \mathrm{Ran}(\psi_{r,\theta,t}).$$

By a change of variables we have

$$\begin{aligned}
&\tilde{h}(\frac{\partial S}{\partial \rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial \theta}(t,D_r,\theta))e^{-iS(t,D_r,\theta)}u^c_{r,\theta,t}(r,\theta)\\
=&\frac{1}{2\pi}\int_{\mathbb{R}}\tilde{h}(\frac{\partial S}{\partial \rho}(t,\rho,\theta),\theta,\rho,-\frac{\partial S}{\partial \theta}(t,\rho,\theta))e^{ir\rho-iS(t,\rho,\theta)}\hat{u}^c_{r,\theta,t}(\rho,\theta)d\rho\\
=&\frac{1}{2\pi}\int_{\Omega_d}\tilde{h}(\frac{\partial S}{\partial \rho}(t,\psi_{r,\theta,t}(y),\theta),\theta,\psi_{r,\theta,t}(y),-\frac{\partial S}{\partial \theta}(t,\psi_{r,\theta,t}(y),\theta))\hat{u}^c_{r,\theta,t}(\psi_{r,\theta,t}(y),\theta)\\
&\cdot J_{r,\theta,t}(y)e^{itf_{r,\theta,t}(\Xi_{\theta,t}(r))}e^{it\langle A_{r,\theta,t}y,y\rangle/2}dy
\end{aligned}$$

where $J_{r,\theta,t}(y)=|\psi'_{r,\theta,t}(y)|$ is the Jacobian. Since

$$|D^j_\rho\tilde{h}(\frac{\partial S}{\partial \rho}(t,\rho,\theta),\theta,\rho,-\frac{\partial S}{\partial \theta}(t,\rho,\theta))|\le Ct^{-\varepsilon},$$

we have

$$\begin{aligned}
&|D^j_y\tilde{h}(\frac{\partial S}{\partial \rho}(t,\psi_{r,\theta,t}(y),\theta),\theta,\psi_{r,\theta,t}(y),-\frac{\partial S}{\partial \theta}(t,\psi_{r,\theta,t}(y),\theta))|\le Ct^{-\varepsilon},\\
&|\tilde{h}(r,\theta,\psi_{r,\theta,t}(y),-\frac{\partial S}{\partial \theta}(t,\psi_{r,\theta,t}(y),\theta))|\le Ct^{-\varepsilon},\\
&|D^j_y\hat{u}^c_{r,\theta,t}(\psi_{r,\theta,t}(y),\theta)|\le C,\\
&|D^j_yJ_{r,\theta,t}(y)|\le C,
\end{aligned}$$

for $y\in\Omega_d$, $\frac{r}{t}\in J$. Now we apply the stationary phase method (see Hörmander [14] Section 7.7) to the integral. In the asymptotic expansion of

$$\begin{aligned}
&[\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial \theta}(t,D_r,\theta))-\tilde{h}(\frac{\partial S}{\partial \rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial \theta}(t,D_r,\theta))]\\
&\cdot e^{-iS(t,D_r,\theta)}u^c_{r,\theta,t}(r,\theta)\\
=&\frac{1}{2\pi}\int_{\Omega_d}[\tilde{h}(r,\theta,\psi_{r,\theta,t}(y),-\frac{\partial S}{\partial \theta}(t,\psi_{r,\theta,t}(y),\theta))\\
&-\tilde{h}(\frac{\partial S}{\partial \rho}(t,\psi_{r,\theta,t}(y),\theta),\theta,\psi_{r,\theta,t}(y),-\frac{\partial S}{\partial \theta}(t,\psi_{r,\theta,t}(y),\theta))]\cdot\\
&\hat{u}^c_{r,\theta,t}(\psi_{r,\theta,t}(y),\theta)\cdot J_{r,\theta,t}(y)e^{itf_{r,\theta,t}(\Xi_{\theta,t}(r))}e^{it\langle A_{r,\theta,t}y,y\rangle/2}dy,
\end{aligned}$$

the terms in which $\tilde{h}$ is not differentiated will vanish since

$$\frac{\partial S}{\partial \rho}(t,\psi_{r,\theta,t}(0),\theta)=r.$$

Especially, in the first step of the asymptotic expansion, we need to estimate only the remainder terms. Therefore we have

$$([\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))-\tilde{h}(\frac{\partial S}{\partial\rho}(t,D_r,\theta),\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))]$$
$$\cdot e^{-iS(t,D_r,\theta)}u^c_{r,\theta,t}(r,\theta)$$
$$\leq Ct^{-\frac{1}{2}-1}\sum_{|k|\leq 3}\sup||D^k_y[\tilde{h}(r,\theta,\psi_{r,\theta,t}(y),-\frac{\partial S}{\partial\theta}(t,\psi_{r,\theta,t}(y),\theta))$$
$$-\tilde{h}(\frac{\partial S}{\partial\rho}(t,\psi_{r,\theta,t}(y),\theta),\theta,\psi_{r,\theta,t}(y),-\frac{\partial S}{\partial\theta}(t,\psi_{r,\theta,t}(y),\theta))]\cdot$$
$$\hat{u}^c_{r,\theta,t}(\psi_{r,\theta,t}(y),\theta)\cdot J_{r,\theta,t}(y)||_{L^2}$$
$$\leq Ct^{-\frac{1}{2}-1-\varepsilon}.$$

We now consider $u^d_{r,\theta,t}$ term.

$$\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))e^{-iS(t,D_r,\theta)}u^d_{r,\theta,t}(r,\theta)$$
$$=\frac{1}{2\pi}(\int_{-\infty}^{\Xi_{\theta,t}(r)-\frac{1}{2}\tilde{d}}+\int_{\Xi_{\theta,t}(r)+\frac{1}{2}\tilde{d}}^{\infty})$$
$$\tilde{h}(\frac{\partial S}{\partial\rho}(t,\rho,\theta),\theta,\rho,-\frac{\partial S}{\partial\theta}(t,\rho,\theta))e^{ir\rho-iS(t,\rho,\theta)}\hat{u}^d_{r,\theta,t}(\rho,\theta)d\rho$$

We consider integration over $\geq\Xi_{\theta,t}(r)+\frac{1}{2}\tilde{d}$ only (the other part is similar to prove). (2.27) implies

$$\partial_\rho f_{r,\theta,t}(\rho)\leq -C<0,$$
$$|\partial^j_\rho f_{r,\theta,t}(\rho)|\leq Ct^{-\varepsilon},\ j\geq 2$$

in this region. Let $y\mapsto h_{r,\theta,t}(y)$ be the inverse of $\rho\mapsto f_{r,\theta,t}(\rho)$. Then

$$|\partial_y h_{r,\theta,t}(y)|\leq C,$$
$$|\partial^j_y h_{r,\theta,t}(y)|\leq Ct^{-\varepsilon},\ j\geq 2.$$

By a change of the variables we obtain

$$|\int_{\Xi_{\theta,t}(r)+\frac{1}{2}\tilde{d}}^{\infty}\tilde{h}(\frac{\partial S}{\partial\rho}(t,\rho,\theta),\theta,\rho,-\frac{\partial S}{\partial\theta}(t,\rho,\theta))e^{itf_{r,\theta,t}(\rho)}\hat{u}^d_{r,\theta,t}(\rho,\theta)d\rho|$$
$$=|\int e^{ity}\tilde{h}(\frac{\partial S}{\partial\rho}(t,h_{r,\theta,t}(y),\theta),\theta,h_{r,\theta,t}(y),-\frac{\partial S}{\partial\theta}(t,h_{r,\theta,t}(y),\theta))\hat{u}^d_{r,\theta,t}(h_{r,\theta,t}(y),\theta)\cdot$$
$$|h'_{r,\theta,t}(y)|dy|$$
$$\leq Ct^{-N}.$$

We can show the same kind of estimations for $\tilde{h}(r,\theta,D_r,-\frac{\partial S}{\partial\theta}(t,D_r,\theta))e^{-iS(t,D_r,\theta)}u^d_{r,\theta,t}(r,\theta)$. We have ended the proof of (3.3).

Now we show (3.4). (2.27) implies that there exists $\tilde{J}$ such that $\frac{1}{t}\frac{\partial S}{\partial \rho} \in \tilde{J} \Subset J$ for large enough t. Thus the absolute value of the derivative of
$\rho \mapsto (r\rho - S(t,\rho,\theta))/(|r|+|t|)$ is bounded below for $\frac{r}{t} \notin J$, large enough t, and $\rho \in \operatorname{supp}\hat{u}$. Thus we can apply the non-stationary phase method and obtain (3.4). □

Proof of (3.1)**, partial isometry, and intertwining property.** First we consider the short-range terms. On $\tilde{M}_0$,

$$P_0J - JP_0 + V^LJ - JV^L = O(r^{-\frac{n-1}{2}}r^{-1-\varepsilon})\partial_r\partial_\theta + O(r^{-\frac{n-1}{2}}r^{-2})\partial_\theta^2 + \sum_j O(r^{-\frac{n-1}{2}}r^{-1-\varepsilon})\partial_r^j.$$

These terms can be treated as a short range perturbation of $(1-\varepsilon) = \mu_S$ type. Hence on $\tilde{M}_0$, $P_0J - JP_0 + V^LJ - JV^L + V^SJ$ is a finite sum of terms of the form $v_{j,\alpha}(r,\theta)r^{-\frac{n-1}{2}}D_r^j\partial_\theta^\alpha$ where $v_{j,\alpha}$ satisfy

$$\int_{\mathbb{R}_+\times U_\lambda} |v_{j,\alpha}^S(x)|^2\langle x\rangle^{-M}G(x)dx < \infty,$$

$$\int_1^\infty \Big(\int_{(\rho,\theta)\in J\times U} |v_{j,\alpha}^S(t\rho,\theta)|^2 d\rho d\theta\Big)^{1/2} t^{(1-\varepsilon)|\alpha|}dt < \infty,$$

for some neighborhood $J\times U$ of almost every $(\rho_0,\theta_0)\in\mathbb{R}\times\partial M$. We assume supp $\hat{u}\subset J\times U$.

We consider the differential operators with respect to θ-variable. $\partial_\theta e^{-iS(t,D_r,\theta)}$ yields $(\partial_\theta S)(t,D_r,\theta)$ terms which increase as $t^{1-\varepsilon}$. Hence, as in the long-range case, the inequalities (2.27) implies

$$|D_r^j\partial_\theta^\alpha e^{-iS(t,D_r,\theta)}u(r,\theta)| = \frac{1}{2\pi}\Big|\int_{\mathbb{R}}\partial_\theta^\alpha[e^{i(r\rho-S(t,\rho,\theta))}\rho^j\hat{u}(\rho,\theta)]d\rho\Big| \le Ct^{-\frac{1}{2}+|\alpha|(1-\varepsilon)} \tag{3.6}$$

for $(\frac{r}{t},\theta)\in J\times U$, and

$$|\partial_r^j\partial_\theta^\alpha e^{-iS(t,D_r,\theta)}u(r,\theta)| \le C_N(1+|r|+|t|)^{-N} \tag{3.7}$$

for any N for $(\frac{r}{t},\theta)\notin J\times U$. Thus we obtain for such $v_{j,\alpha}$

$$\|v_{j,\alpha}(r,\theta)r^{-\frac{n-1}{2}}D_r^j\partial_\theta^\alpha e^{-iS(t,D_r,\theta)}u\|_{\mathscr{H}} \in L_t^1(1,\infty),$$

which proves

$$\|([P_0J - JP_f] + V^SJ + [V^LJ - JV^L])e^{-iS(t,D_r,\theta)}u\|_{\mathscr{H}} \in L_t^1(1,\infty).$$

We have proved the existence of the modified wave operators.

(3.3), (3.4), (3.6), and (3.7) also show that $W_\pm$ are partial isometries from $\mathscr{H}_f$ into $\mathscr{H}$.

The intertwining property follows from

$$\operatorname*{s-lim}_{t\to\infty}(e^{-iS(t+s,D_r,\theta)} - e^{-is(D_r,\theta,t)}e^{-isP_f}) = 0 \tag{3.8}$$

which can be proved using (2.27) and the dominated convergence theorem. The proof of the theorem is complete. □

Part II
Scattering theory for Schrödinger equations on manifolds with asymptotically polynomially growing ends

Abstract

We study a time-dependent scattering theory for Schrödinger operators on a manifold with asymptotically polynomially growing ends. We use the Mourre theory to show the spectral properties of self-adjoint second-order elliptic operators. We prove the existence and the asymptotic completeness of wave operators using the smooth perturbation theory of Kato. We also consider a two-space scattering with a simple reference system.

4 Introduction of Part II

We study a class of self-adjoint second-order elliptic operators, which includes Laplacians with long-range potentials on non-compact manifolds which are asymptotically polynomially growing at infinity. We prove Mourre estimate and apply the Mourre theory to these operators. Then we show that there are no accumulation points of embedded eigenvalues except for the zero energy. We obtain resolvent estimates which imply the absence of singular spectrum. We also show the Kato-smoothness for three types of operators. We construct a time-dependent scattering theory for two operators in our class. If the perturbation is "short-range", it admits a factorization into a product of Kato-smooth operators. By virtue of the smooth perturbation theory of Kato, we learn the existence and the asymptotic completeness of wave operators. Lastly, we consider a two-space scattering with a simple reference system. We follow the settings by Ito and Nakamura.

We now describe our model. Let M be an n-dimensional smooth non-compact manifold such that $M = M_C \cup M_\infty$, where M_C is pre-compact and M_∞ is the non-compact end as follows: We assume that M_∞ has the form $\mathbb{R}_+ \times N$ where N is a $n-1$-dimensional compact manifold, and $\mathbb{R}_+ = (0,\infty)$ is the real half line. Let ω be a positive C^∞ density ω on M such that on M_∞,

$$\omega = dr \cdot \mu$$

where r is a coordinate in $\mathbb{R}_+$ and μ is a smooth positive density on N. We set $\mathscr{H} = L^2(M,\omega)$ be our function space. We set our "free operator" a self-adjoint second-order elliptic operator L_0 which has the form:

$$L_0 = D_r^2 + k(r)P \qquad \text{on } (1,\infty)\times N.$$

Here $D_r = i^{-1}\partial_r$, P is a positive self-adjoint second-order elliptic operator acting on $L^2(N,\mu)$, and k is a positive smooth function of r such that the derivatives of k satisfy the following estimates for some $c_0, C > 0$,

$$\begin{aligned} &c_0 r^{-1}k \le -k' \le Cr^{-1}k, \\ &|k''| \le Cr^{-2}k. \end{aligned} \tag{4.1}$$

For example $k(r) = r^{-\alpha}$, with $\alpha > 0$ satisfies the above conditions.

We assume that L is a second-order elliptic operator on M, essentially self-adjoint on $C_0^\infty(M)$, such that

$$L = L_0 + E,$$

with E having the following properties: There are finitely many coordinate charts $(r,\theta_1,\cdots,\theta_{n-1})$ on M_∞ such that in each chart E has the form

$$E = (1, D_r, \sqrt{k}\tilde{D}_\theta)\begin{pmatrix} V & b_1 & b_2 \\ b_1 & a_1 & a_2 \\ {}^t b_2 & {}^t a_2 & a_3 \end{pmatrix}\begin{pmatrix} 1 \\ D_r \\ \sqrt{k}\tilde{D}_\theta \end{pmatrix} \tag{4.2}$$

where $\mu(\theta)$ is defined by $\mu = \mu(\theta)d_{\theta_1}\cdots d_{\theta_{n-1}}$ and $\tilde{D}_\theta = \mu(\theta)^{-\frac{1}{2}}D_\theta\mu(\theta)^{\frac{1}{2}}$ is self-adjoint on $L^2(N,\mu)$. The coefficients a_1,a_2,b_1,b_2, and V have support in M_∞ and are smooth real-valued functions of $(r,\theta_1,\cdots,\theta_{n-1})$ such that

$$
\begin{aligned}
&|\partial_r^l\partial_\theta^\alpha a_j(r,\theta)| \le C_{l,\alpha}r^{-\nu_{a_j}-l},\\
&|\partial_r^l\partial_\theta^\alpha b_j(r,\theta)| \le C_{l,\alpha}r^{-\nu_{b_j}-l},\\
&|\partial_r^l\partial_\theta^\alpha V(r,\theta)| \le C_{l,\alpha}r^{-\nu_V-l}.
\end{aligned}
\tag{4.3}
$$

Let $\chi(r) \in C^\infty(\mathbb{R})$ be a $\mathbb{R}_+$-valued function such that $\chi(r) = 1$ if $r \ge 1$ and $\chi(r) = 0$ if $r \le \frac{1}{2}$, and set $\chi_R(r) = \chi(\frac{r}{R})$ with $R > 0$. We set our dilation generator by:

$$
A = \frac{1}{2}(\chi_R^2 rD_r + D_r r\chi_R^2). \tag{4.4}
$$

Now we state the main results.

Theorem 4.1. *Suppose $L = L_0 + E$, where k satisfies* (4.1) *and the coefficients in E obey the bounds* (4.3) *with $\nu = \min\{\nu_{a_i},\nu_{b_j},\nu_V\} > 0$. Then $\sigma_{ess}(L) = \mathbb{R}_+\cup\{0\}$ and L satisfies a Mourre estimate at each point in $\mathbb{R}_+$ with conjugate operator A in the sense of Definition 5.3. In particular, eigenvalues of L do not accumulate in $\mathbb{R}_+$, and $\sigma_{sc}(L) = \emptyset$. We also obtain the resolvent estimates:*

$$
\sup_{z\in\Lambda_\pm=\Lambda\pm i\mathbb{R}_+} \|(|A|+1)^{-s}(L-z)^{-1}(|A|+1)^{-s}\| < \infty
$$

if $\Lambda \Subset \mathbb{R}\setminus\sigma_{pp}(L)$ and $s > \frac{1}{2}$.

We prove Theorem 4.1 in Section 5.

Theorem 4.2. *Under the hypotheses of Theorem 4.1, the operators*

$$
\begin{aligned}
&G_0 = \langle r\rangle^{-s},\\
&G_1 = \chi_R\langle r\rangle^{-s}D_r,\\
&G_2 = \chi_R\langle r\rangle^{-\frac{1}{2}}(kP)^{\frac{1}{2}}
\end{aligned}
$$

are L-smooth on Λ if $\Lambda \Subset \mathbb{R}\setminus\sigma_{pp}(L)$ and $s > \frac{1}{2}$.

We prove Theorem 4.2 in Section 6 and Section 7.

Theorem 4.3. *Suppose the short-range condition for E, that is, $\nu_{a_1} = \nu_{a_2} = \nu_{b_1} = \nu_{b_1} = \nu_V > 1$, and $\nu_{a_3} = 1$. Then the wave operators*

$$
W^\pm(L,L_0) := s\text{-}\lim_{t\to\pm\infty} e^{itL}e^{-itL_0}P_{ac}(L_0)
$$

and $W^\pm(L_0,L)$ exist and are adjoint each other. They are complete and give the unitarily equivalence between $L_0^{(ac)}$ and $L^{(ac)}$.

We prove Theorem 4.3 in Section 8. We note that the wave operators $W^{\pm}(L_2, L_1)$ exist and are asymptotically complete if both L_1 and L_2 satisfy the hypotheses of Theorem 4.1 (long-range) but the difference $L_2 - L_1$ is short-range in the sense of Theorem 4.3.

Next we consider a two-space scattering. We prepare a reference system as follows:

$$M_f = \mathbb{R} \times N, \quad \mathscr{H}_f = L^2(M_f, H(\theta)drd\theta),$$
$$H_0 = D_r^2 \text{ on } M_f,$$
$$H_k = D_r^2 + k(r)P \text{ on } M_f.$$

Note that H_0 and H_k are essentially self-adjoint on $C_0^\infty(M_f)$, and we denote the unique self-adjoint extensions by the same symbols. We define the identification operator $J : \mathscr{H}_f \to \mathscr{H}$ by

$$(Ju)(r,\theta) = \chi(r)u(r,\theta) \text{ if } (r,\theta) \in M_\infty$$

and $Ju(x) = 0$ if $x \notin M_\infty$, where $u \in \mathscr{H}_f$. We denote the Fourier transform with respect to r variable by $\mathscr{F}$:

$$(\mathscr{F}u)(\rho,\theta) = \frac{1}{\sqrt{2\pi}} \int e^{ir\rho} u(r,\theta)dr.$$

We set

$$\mathscr{H}_f^{\pm} := \mathscr{F}^{-1}[1_{\mathbb{R}_\pm}(\rho)L^2(\mathbb{R} \times N : d\rho \cdot \mu)].$$

Then $\mathscr{H}_f = \mathscr{H}_f^+ \oplus \mathscr{H}_f^-$.

In the two-space scattering, we need additional conditions on k:

Definition 4.4. *Suppose that k is a positive smooth function of r satisfying* (4.1). *k is said to be short-range if*

$$|k(r)| \leq C\langle r\rangle^{-\nu_k} \tag{4.5}$$

with $\nu_k > 1$. k is said to be smooth long-range if

$$|\partial_r^l k(r)| \leq C\langle r\rangle^{-\nu_k - l} \tag{4.6}$$

with $l \in \mathbb{N}$, and $\nu_k > 0$.

For short-range k, we have the following.

Theorem 4.5. *Suppose the hypotheses of Theorem 4.3 and that k is short-range. Then the wave operators $W^{\pm}(L, H_0; J)$ and $W^{\pm}(H_0, L; J^*)$ exist and are adjoint each other. The asymptotic completeness*

$$W^{\pm}(L, H_0; J)\mathscr{H}_f^{\pm} = P_{ac}(L)\mathscr{H}$$

holds.

For long-range k, we need to modify the identifier.

Theorem 4.6. *Suppose k is smooth long-range in the sense of Definition 4.4. Fix $\Lambda \Subset \mathbb{R}$. Then there exists suitable operators $J^{\pm} \in B(\mathscr{H}_f)$ such that the wave operators $W^{\pm}(L, H_0; JJ^{\pm})$ and $W^{\pm}(H_0, L; (JJ^{\pm})^*)$ exist and are isometric on $E_\Lambda(H_0)\mathscr{H}_f^{\pm}$ and $E_\Lambda(L)P_{ac}(L)\mathscr{H}$, respectively, $W^{\pm}(L, H_0; JJ^{\pm})\mathscr{H}_f^{\mp} = 0$, and the asymptotic completeness*

$$W^{\pm}(L, H_0; JJ^{\pm})E_\Lambda(H_0)\mathscr{H}_f^{\pm} = E_\Lambda(L)P_{ac}(L)\mathscr{H}$$

holds.

The construction of modifiers $J^{\pm}$ will be given in Section 9. We can also admit a_1 to have a long-range part which depends only on r. For details, see Section 9.

There is a long history on spectral and scattering theory for Schrödinger operators (see, for example, [31], [36] and references therein). Much of works are connected to differential operators on a Euclidean space. The spectral properties of Laplace operators on a class of non-compact manifolds were studied by Froese, Hislop and Perry [10, 11], and Donnelly [9] using the Mourre theory (see, the original paper Mourre [28], and Perry, Sigal, and Simon[29]). We follow the settings in Froese and Hislop [10], and Theorem 4.1 may be seen as a direct generalization of [10]. We note that only the case with $\nu = 1$ is treated in [10].

In early 1990s, Melrose introduced a new framework of scattering theory on a class of Riemannian manifolds with metrics called scattering metrics (see [25] and references therein). He and the other authors have studied Laplace operators on such manifolds. They also studied the absolute scattering matrix, which is defined through the asymptotic expansion of generalized eigenfunctions.

De Bièvre, Hislop, and Sigal [4] studied a time-dependent scattering theory and proved its asymptotic completeness for some classes of manifolds, including manifolds with asymptotically growing ends with $\nu > 1$.

Ito and Nakamura [16] studied a time-dependent scattering theory for Schrödinger operators on scattering manifolds. They used the two-space scattering framework of Kato [21] with a simple reference operator D_r^2 on a space of the form $\mathbb{R} \times N$, where N is the boundary of the scattering manifold M.

The case where $M = M_C \cup M_\infty$ is a Riemannian manifold, the metric on M_∞ is "close" to a warped product of $\mathbb{R}_+$ and a compact manifold N, and L is the Laplace operator, fits into our framework. The function $\sqrt{k(r)}$, varies inversely with the size of $M_\infty = \mathbb{R}_+ \times N$. A typical exapmle of k which satisfies (4.1) is given by $k(r) = r^{-\alpha}, \alpha > 0$. The case $\alpha = 2$ corresponds to scattering manifolds including asymptotically Euclidean spaces. By using results of Ito and Nakamura [16] twice, and by applying the chain rule for wave operators, we can show the existence and the asymptotic completeness of wave operators on scattering manifolds in the one-space scattering framework. Therefore our results can be considered as a generalization of [16] for all $\alpha > 0$. In [16], assumptions on a_2 and a_3 are weakend to long-range perturbations.

Our proof of the existence and the asymptotic completeness of wave operators depends on the smooth perturbation theory of Kato [20] (see also Yafaev [34] and [36]). The Kato smoothness of $G_0 = \langle r \rangle^{-s}, s > \frac{1}{2}$ in Theorem 4.2 is closely related to the limiting absorption principle. The resolvent estimates in the Mourre theory (Theorem 4.1) imply the limiting absorption principle via a technique in Section 8 of [29]. The Kato smoothness of $G_1 = \chi_R \langle r \rangle^{-s} D_r$ will be obtained in a similar way, but we have to extend the technique in [29] from $\alpha = 1$ to $\alpha = 2$ (Lemma 6.3 (i)). The Kato-smoothness of $G_2 = \chi_R \langle r \rangle^{-\frac{1}{2}} (kP)^{\frac{1}{2}}$ is called the radiation estimates.

Our proof is quite similar to the one [35], which relies on the commutator method (see Putnam [30] and Kato [22]).

The limiting absorption principle suffices to show the asymptotic completeness in the case of two-particle Hamiltonians with short-range scalar potentials. However, radiation estimates are crucial in scattering for long-range potentials on a Euclidean space (see Yafaev [36]). We found that radiation estimates are also useful for handling short-range metric perturbations. We hope that we can also construct appropriate modifiers so that the technique of Yafaev can be applied to show the existence and the asymptotic completeness of modified wave operators with long range perturbations in our settings.

In the two-space scattering, essentially we only need to examine wave operators for the pair (H_k, H_0). However, since P commutes with both of H_k and H_0, it reduces to the 1-dimensional scattering. When k is short-range, we only need a narutal identifier. When k is long-range, we will construct 1-dimensional modifiers for the corresponding 1-dimensional long-range scattering.

5 Application of Mourre Theory

In this section, we prove Theorem 4.1. For the sake of completeness, we give a detailed proof. But methods and proofs used here are almost the same as those in [10] and [4], where $\nu = 2$ and $\nu = 1$, respectively, are assumed. We will prove the Mourre estimate under the condition $\nu > 0$. The index ν will explicitly appear, for example, in Lemma 5.12 and Lemma 5.7.

We first quote the Mourre theory. We define a scale of spaces associated to a self-adjoint operator L.

Definition 5.1 (Scale of spaces). *Let L be a self-adjoint operator on a Hilbert space $\mathscr{H}$. For $s \geq 0$ define $\mathscr{H}_s = D((1+|L|)^{\frac{s}{2}}))$ with the graph norm*

$$\|\psi\|_s := \|(1+|L|)^{\frac{s}{2}})\psi\|.$$

Define $\mathscr{H}_{-s}$ to be the dual spaces of $\mathscr{H}_s$ thought of as the closure of $\mathscr{H}$ in the norm

$$\|\psi\|_{-s} := \|(1+|L|)^{-\frac{s}{2}})\psi\|.$$

Definition 5.2 (Conjugate Operators). *Let L be a self-adjoint operator on a Hilbert space $\mathscr{H}$ and $\mathscr{H}_s$ be the scale of spaces associated to L. A self adjoint operator A is called a conjugate operator of L if*

(i). $D(A) \cap \mathscr{H}_2$ is dense in $\mathscr{H}_2$,

(ii). the form $[L, iA]$ defined on $D(A) \cap \mathscr{H}_2$ is bounded below and extends to a bounded operator from $\mathscr{H}_2$ to $\mathscr{H}_{-1}$,

(iii). there is a self-adjoint operator L_0 with $D(L_0) = D(L)$ such that $[L_0, iA]$ extends to a bounded map from $\mathscr{H}_2$ to $\mathscr{H}$, and $D(A) \cap D(L_0 A)$ is a core for L_0,

(iv). the form $[[L, iA], iA]$ extends from $\mathscr{H}_2 \cap D(LA)$ to a bounded operator from $\mathscr{H}_2$ to $\mathscr{H}_{-2}$.

(v). e^{itA} leaves $\mathscr{H}_2$ invariant and for each $\psi \in \mathscr{H}_2$, $\sup_{|t| \leq 1} \|e^{itA}\psi\|_2 < \infty$.

Definition 5.3 (Mourre Estimate). *A self-adjoint operator L satisfies a Mourre estimate on an interval $\Lambda \subset \mathbb{R}$ with conjugate operator A if A is a conjugate operator of L such that there exist a positive constant α and a compact operator K such that*

$$E_\Lambda [L, iA] E_\Lambda \geq \alpha E_\Lambda + K.$$

Here $E_\Lambda = E_\Lambda(L)$ is the spectral projection for L. We say that L satisfies a Mourre estimate at a point $\lambda \in \mathbb{R}$ if there exists an interval Λ containing λ such that L satisfies a Mourre estimate on Λ.

Now we state the Mourre theory.

Theorem 5.4 (Mourre). *Suppose that a self-adjoint operator L satisfies a Mourre estimate at $\lambda \in \mathbb{R}$ with a conjugate operator A. Then there exists an open interval Λ containing λ such that L has finitely many eigenvalues in Λ and each eigenvalue has finite multiplicity. If $\lambda \notin \sigma_{pp}(L)$, then there exists an open interval Λ containing λ such that L has no singular continuous spectrum in Λ and for $s > \frac{1}{2}$,*

$$\sup_{z \in \Lambda_\pm = \Lambda \pm i\mathbb{R}_+} \|(|A|+1)^{-s}(L-z)^{-1}(|A|+1)^{-s}\| < \infty.$$

We refer to [28] and [29] for the proof of this theorem.

In the following of this section we will show that the hypotheses of the Theorem 5.4 will be satisfied for our case.

Lemma 5.5. *Suppose $f \in C_0^\infty(\mathbb{R})$, D is a differential operator with smooth coefficients, and χ is a smooth cut-off function with compact support. Then $\chi D f(L)$ and $\chi D f(L_0)$ are compact operators from $L^2(M)$ to $L^2(M)$.*

Proof. Let Ω be a bounded domain with smooth boundary which contains supp χ. Then $\chi D f(L_0)$ and $\chi D f(L)$ map $L^2(M)$ to a Sobolev space $H^s(\Omega)$ for any $s > 0$. But $H^s \hookrightarrow L^2(\Omega) \hookrightarrow L^2(M)$, and the first embedding is compact by Rellich's theorem. □

Lemma 5.6. *Let $\mathscr{H}_s$ be the scale of spaces associated with L_0. Then*

(i). $\chi_R D_r : \mathscr{H}_s \to \mathscr{H}_{s-1}$ is bounded for $s \in [-1, 2]$,

(ii). $\chi_R D_r^2 : \mathscr{H}_s \to \mathscr{H}_{s-2}$ is bounded for $s \in [0, 2]$,

(iii). $\chi_R (kP+1)^{\frac{1}{2}} : \mathscr{H}_s \to \mathscr{H}_{s-1}$ is bounded for $s \in [-1, 2]$,

(iv). $\chi_R (kP+1) : \mathscr{H}_s \to \mathscr{H}_{s-2}$ is bounded for $s \in [0, 2]$.

Proof of Lemma 5.6. We begin by proving that

$$\|\chi_R D_r (L_0 + C)^{-\frac{1}{2}}\| \leq 1 \tag{5.1}$$

for some constant C. Choose a positive constant C_1 so that $L_0 + C_1$ is a positive operator. Let $\tilde{\chi}_R = (1-\chi_R)^{-\frac{1}{2}}$. The IMS localozation formula gives

$$L_0 + C_1 = \chi_R (L_0 + C_1) \chi_R + \tilde{\chi}_R (L_0 + C_1) \tilde{\chi}_R - (\chi_R')^2 - (\tilde{\chi}_R')^2.$$

This implies

$$\begin{aligned} L_0 + C_1 &\geq \chi_R D_r^2 \chi_R - (\chi_R')^2 - (\tilde{\chi}_R')^2 \\ &\geq D_r \chi_R^2 D_r - \chi_R'' \chi_R - (\chi_R')^2 - (\tilde{\chi}_R')^2 \end{aligned}$$

as form inequalities on C_0^∞. Since

$$|\chi_R'' \chi_R - (\chi_R')^2 - (\tilde{\chi}_R')^2| \leq \frac{C}{R^2},$$

this implies

$$D_r \chi_R^2 D_r \leq L_0 + C.$$

This shows for $\phi \in C_0^\infty$,

$$\|\chi_R D_r \phi\| \leq \|(L_0 + C)^{\frac{1}{2}} \phi\|. \tag{5.2}$$

Since C_0^∞ is a core for $(L_0+C)^{\frac{1}{2}}$, we can find for every $\phi \in D((L_0+C)^{\frac{1}{2}})$, a sequence $\phi_n \in C_0^\infty$ such that $\phi_n \to \phi$ and $(L_0+C)^{\frac{1}{2}}\phi_n \to (L_0+C)^{\frac{1}{2}}\phi$. Then

$$\begin{aligned} |(D_r \chi_R \psi, \phi)| &= \lim_{n\to\infty} |(D_r \chi_R \psi, \phi)| \\ &= \lim_{n\to\infty} |(\psi, \chi_R D_r \phi)| \\ &= \limsup_{n\to\infty} \|\psi\| \|(L_0 + C)^{\frac{1}{2}} \phi_n\| \\ &\leq \|\psi\| \|(L_0 + C)^{\frac{1}{2}} \phi\| \end{aligned}$$

which shows that $\phi \in D(\chi_R D_r)$ and that (5.2) holds for any $\phi \in D((L_0+C)^{\frac{1}{2}})$. Writing $\phi = (L_0+C)^{-\frac{1}{2}}\psi$ for $\psi \in L^2$, we see that this implies (5.1).

Next we will prove that

$$\|\chi_R D_r^2 (L_0 + C)^{-1}\| \leq C. \tag{5.3}$$

With C_1 as above,

$$\begin{aligned} (L_0 + C_1)^2 &\geq (L_0 + C_1)\chi_R^2 (L_0 + C_1) \\ &= D_r^2 \chi_R^2 D_r^2 + D_r^2 \chi_R^2 (kP + C_1) + (kP + C_1)\chi_R^2 D_r^2 + (kP + C_1)^2 \chi_R^2 \\ &= D_r^2 \chi_R^2 D_r^2 + 2 D_r \chi_R^2 (kP + C_1) D_r - (\chi_R^2 (kP + C_1))'' + (kP + C_1)^2 \chi_R^2. \end{aligned}$$

Using the fact that $|k'|$ and $|k''|$ are bounded by a constant times k, we see that

$$(\chi_R^2 (kP + C_1))'' \leq C \chi (kP + C_1) \chi$$

for some cut-off function χ. Using the IMS formula again, this implies

$$(\chi_R^2 (kP + C_1))'' \leq C (L_0 + C)$$

for some C. Since $2D_r\chi_R^2(kP+C_1)D_r+(kP+C_1)^2\chi_R^2\geq 0$, we obtain

$$(L_0+C_1)^2\geq D_r^2\chi_R^2D_r^2-C(L_0+C),$$

which implies for some C,

$$D_r^2\chi_R^2D_r^2\leq C(L_0+C)^2,$$

which leads to (5.3) as in the proof of (5.1).

A similar argument shows that

$$\|D_r^2\chi_R(L_0+C)^{-1}\|\leq C. \tag{5.4}$$

Now by complex interpolation, (5.3) and (5.4) implies

$$\|(L_0+C)^{-1+z}D_r^2\chi_R(L_0+C)^{-z}\|\leq C.$$

for Rez in $[0,1]$, which implies (ii) of Lemma.

To prove (i) using complex interpolation, one need to prove

$$\|(L_0+C)^{-1}\chi_RD_r(L_0+C)^{\frac{1}{2}}\|\leq C, \tag{5.5}$$

$$\|(L_0+C)^{\frac{1}{2}}\chi_RD_r(L_0+C)^{-1}\|\leq C. \tag{5.6}$$

Examining (5.5), we see that

$$\begin{aligned}&\|(L_0+C)^{-1}\chi_RD_r(L_0+C)^{\frac{1}{2}}\|\\ &\leq\|\chi_RD_r(L_0+C)^{\frac{1}{2}}\|+\|(L_0+C)^{-1}[L_0,\chi_RD_r](L_0+C)^{-\frac{1}{2}}\|.\end{aligned}$$

The first term on the right hand side is bounded by (5.1). The second term can be decomposed into two terms according to the following equation:

$$[L_0,\chi_RD_r]=[D_r^2,\chi_RD_r]+[kP,\chi_RD_r].$$

The first one is bounded using (ii) of Lemma; the second is bounded by

$$[\chi_R,i\chi_RD_r]=\chi_RkP\leq C\chi_RkP\leq C(L_0+C)$$

and an argument similar to the one above. This gives (5.5). (5.6) follows similarly.

(iii) and (iv) follow in a similar way. □

Lemma 5.7. *Let L,L_0 and E be as in Theorem 4.1 and let $\mathscr{H}_s$ be the scale of spaces associated with L_0. Then*

(i). $\langle r\rangle^\nu E:\mathscr{H}_s\to\mathscr{H}_{s-2}$ is bounded for $s\in[0,2]$,

(ii). by taking R large enough in the definition of E, we may assume that the relative L_0-bound of E is less than 1,

(iii). $(L_0-z)^{-1}-(L-z)^{-1}$ is compact for $Im(z)\neq 0$.

Proof of Lemma5.7. (i) will follow by complex interpolation if we can show

$$\|\langle r\rangle^{\nu} E(L_0+i)^{-1}\| \leq C,$$
$$\|(L_0+i)^{-1}\langle r\rangle^{\nu} E\| \leq C.$$

As a typical exapmle, we choose $\sqrt{k}\tilde{D}_\theta a_2 D_r$. Then

$$\begin{aligned}
&\|\langle r\rangle^{\nu}\chi_R\sqrt{k}\tilde{D}_\theta a_2 D_r \chi_R(L_0+i)^{-1}\| \\
&\leq \|\langle r\rangle^{\nu}\sqrt{k}a_2\tilde{D}_\theta\chi_R D_r(L_0+i)^{-1}\| + \|\langle r\rangle^{\nu}\sqrt{k}(\tilde{D}_\theta a_2)\chi_R D_r(L_0+i)^{-1}\| \\
&\leq \sup_r\{\langle r\rangle^{\nu}|a_2|\}\cdot\|\sqrt{k}\tilde{D}_\theta(kP+1)^{-\frac{1}{2}}\|\cdot\|\chi_R(kP+1)^{\frac{1}{2}}D_r(L_0+i)^{-1}\| \\
&+\sup_r\{\langle r\rangle^{\nu}|\tilde{D}_\theta a_2|\}\cdot\|\sqrt{k}\chi_R D_r(L_0+i)^{-1}\| \\
&\leq C,
\end{aligned}$$

by assumptions on a_2 and Lemma 5.6.

To prove (iii), we use the resolvent formula

$$\begin{aligned}
&(L_0-z)^{-1}-(L-z)^{-1} \\
&=(L-z)^{-1}E(L_0-z)^{-1} \\
&=(L-z)^{-1}\langle r\rangle^{-\nu}\chi_R\langle r\rangle^{\nu}E(L_0-z)^{-1}.
\end{aligned}$$

The operator $(L-z)^{-1}\langle r\rangle^{-\nu}\chi_R$ can be approximated in norm by operators $f(L)\chi$ considerd in Lemma 5.5, and thus is compact while $\langle r\rangle^{\nu}E(L_0-z)^{-1}$ is bounded by (i). This proves (iii). □

Lemma 5.8. *$D(L)=D(L_0)$ and the scale of spaces associated to L and L_0 are the same. If $f\in C_0^\infty$, then $f(L)-f(L_0)$ is compact.*

Proof of Lemma 5.8. The first statement is obtained by the relative boundedness, and the second follows from (iii) of Lemma 5.7 and a Stone-Weierstrass argument. □

Lemma 5.9. $\sigma_{ess}(L)=[0,\infty)$.

Proof. The Persson's formula (see, for exapmle, [3])

$$\inf\sigma_{\mathrm{ess}}(L)=\sup_{K\Subset M}\inf_{\phi\in C_0^\infty(M\setminus K),\|\phi\|=1}\langle\phi,L\phi\rangle$$

and a Weyl sequene argument give the desired result. □

Lemma 5.10. *Let L_0 be as in Theorem 4.1. Then for large enough R,*

(i). $[L_0,iA]$ *extends from* C_0^∞ *to a bounded operator* $\mathscr{H}_{+2}\to\mathscr{H}$,

(ii). $[[L_0,iA],iA]$ *extends from* C_0^∞ *to a bounded operator* $\mathscr{H}_{+2}\to\mathscr{H}_{-2}$.

Proof. To begin, we show that $[D_r^2, iA]$ is bounded from $\mathscr{H}_{+2}$ to $\mathscr{H}$. A brief calculation shows

$$[D_r^2, iA] = 2(\chi_R^2 r)' D_r^2 + \frac{2}{i}(\chi_R^2 r)'' D_r - \frac{1}{2}(\chi_R^2 r)'''.$$

The coefficients $(\chi_R^2 r)', (\chi_R^2 r)''$, and $(\chi_R^2 r)'''$ are bounded. By taking R large enough, the boundedness of $[D_r^2, iA]$ from $\mathscr{H}_{+2}$ to $\mathscr{H}$ follows from that of $\chi_R D_r^2$ and $\chi_R D_r$, which is ensured by Lemma 5.6.

Next we consider the term

$$[kP, iA] = -\chi_R^2 r k' P.$$

By (4.1), $|rk'| \leq Ck$. Using Lemma 5.6, it follows that $[kP, iA]$ is bounded from $\mathscr{H}_{+2}$ to $\mathscr{H}$. This completes the proof of (i).

The boundedness of the double commutator in (ii) is proven using similar arguments. We can use Lemma 5.6 to prove the boundedness of $[[D_r^2, iA], iA]$ from $\mathscr{H}_{+2}$ to $\mathscr{H}_{-2}$. Since

$$[[kP, iA], iA] = \chi_R^2 r (\chi_R^2 r k')' P,$$

we need the estimates (4.1) on the second derivative of k for r large

$$|r^2 k''| \leq Ck$$

to prove the boundedness of $[[kP, iA], iA]$. □

Now we prove the Mourre estimate for unperturbated system L_0.

Lemma 5.11. *Let L_0 be as in Theorem 4.1 and A given by* (4.4). *Suppose $\lambda_0 > 0$. Then for every $\varepsilon > 0$ there exist an interval Λ about λ_0 and a compact operator K such that for R large,*

$$E_\Lambda(L_0)[L_0, iA]E_\Lambda(L_0) \geq \min(2, c_0)(\lambda_0 - \varepsilon)E_\Lambda(L_0) + K.$$

Here $E_\Lambda(L_0)$ is the spectral projection for L_0 corresponding to Λ, and c_0 is the constant which appears in (4.1).

Proof. Choosing R large, we have

$$\begin{aligned}[D_r^2, iA] &= 2D_r(\chi_R^2 r)' D_r - \frac{1}{2}(\chi_R^2 r)''' \\ &\geq 2D_r \chi_R^2 D_r - \frac{\varepsilon}{4}\min\{2, c_0\} \\ &\geq 2\chi_R D_r^2 \chi_R - \frac{\varepsilon}{2}\min\{2, c_0\}.\end{aligned}$$

Also,

$$[kP, iA] = -\chi_R^2 r k' P \geq c_0 \chi_R^2 k P.$$

Combining these two inequalities, we obtain

$$\begin{aligned}[L_0, iA] &\geq \chi_R(2D_r^2 + c_0 kP)\chi_R - \frac{\varepsilon}{2}\min\{2, c_0\} \\ &\geq \min\{2, c_0\}(\chi_R L_0 \chi_R - \frac{\varepsilon}{2}).\end{aligned}$$

We now multiply this estimate on both sides with $f(L_0)$ where f is a smooth compactly supported characteristic function of an interval about λ_0. This gives

$$f(L_0)[L_0,iA]f(L_0) \geq \min\{2,c_0\}(f(L_0)\chi_R L_0 \chi_R f(L_0) - \frac{\varepsilon}{2} f^2(L_0)).$$

Now

$$\begin{aligned} & f(L_0)\chi_R L_0 \chi_R f(L_0) \\ =& f(L_0)L_0(\chi_R - 1)f(L_0) + f(L_0)(\chi_R - 1)L_0\chi_R f(L_0) + f(L_0)L_0 f(L_0). \end{aligned}$$

$f \in C_0^\infty$ implies $f(L_0)L_0$ is bounded. It is not difficult to see that $L_0\chi_R f(L_0)$ is bounded using Lemma 5.6. Since $\chi_R - 1$ has compact support, $(\chi_R - 1)f(L_0)$ is compact by Lemma 5.5. Thus, if the support of f is within $\frac{\varepsilon}{2}$ of λ_0, we have

$$f(L_0)\chi_R L_0 \chi_R f(L_0) \geq (\lambda_0 - \frac{\varepsilon}{2})f^2(L_0) + K.$$

where K is a compact operator. Therefore we have

$$f(L_0)[L_0,iA]f(L_0) \geq \min\{2,c_0\}(\lambda_0 - \varepsilon)f^2(L_0) + K. \tag{5.7}$$

Taking $f = 1$ in a neighbourhood of λ_0 and multiplying from both sides with $E_\Lambda(L_0)$, with Λ small enough to ensure $E_\Lambda(L_0)f(L_0) = E_\Lambda(L_0)$, this inequality gives the desired Mourre estimate.

□

Lemma 5.12. *Under the hypotheses of Theorem 4.1,*

(i). $[E,iA] : \mathscr{H}_{+2} \longrightarrow \mathscr{H}_0$ *is bounded,*

(ii). $f(L)[E,iA]f(L)$*: is compact for* $f \in C_0^\infty$*,*

(iii). $[[E,iA],iA] : \mathscr{H}_{+2} \longrightarrow \mathscr{H}_{-2}$ *is bounded.*

Proof. It is easy to see that $[E,iA]$ has the following form:

$$[E,iA] = (1, D_r, \sqrt{k}\tilde{D}_\theta)\chi_R \begin{pmatrix} \tilde{V} & \tilde{b}_1 & \tilde{b}_2 \\ \tilde{b}_1 & \tilde{a}_1 & \tilde{a}_2 \\ {}^t\tilde{b}_2 & {}^t\tilde{a}_2 & \tilde{a}_3 \end{pmatrix} \chi_R \begin{pmatrix} 1 \\ D_r \\ \sqrt{k}\tilde{D}_\theta \end{pmatrix} \tag{5.8}$$

where $\tilde{e} = \tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_1$ and $\tilde{V}$ satisfy

$$|\tilde{e}(r,\theta)| \leq Cr^{-\nu}, \quad \nu > 0. \tag{5.9}$$

Here we used the estimates (4.3) on the first derivatives with respect to r of coefficients in E and the estimates (4.1) on the first derivative of k. By Lemma 5.6, (5.8) and (5.9) imply $[E,iA]$ is bounded from $\mathscr{H}_{+2} \longrightarrow \mathscr{H}_0$, which is (i).

The boundedness of the double commutator in (iii) is proven using similar arguments. We need the estimates on second derivatives with respect to r of coefficients in E and k.

To prove (ii), note that $\chi_R(1, D_r, \sqrt{k}\tilde{D}_\theta)f(L)$ is bounded and $\chi_R r^{-\nu}(1, D_r, \sqrt{k}\tilde{D}_\theta)f(L)$ is compact by Lemma 5.5. Hence (5.8) implies (ii). □

Proof of Theorem 4.1. We first show that L and A satisfy the conditions in Definition 5.2, that is, A is a conjugate operator of L. Since $C_0^\infty \subset D(A) \cap \mathscr{H}_2$ is a core for L by hypothesis, condition (i) in Definition 5.2 is satisfied. Condition (ii) follows from (i) of Lemma 5.10 and (i) of Lemma 5.12. The first statement of (iii) follows from Lemma 5.8 and (i) of Lemma 5.10. The second statement follows from the inclusion $C_0^\infty \subset D(A) \cap D(L_0 A)$. Condition (iv) follows from (ii) of Lemma 5.10 and (iii) of Lemma 5.12. Let X be a vector field on M such that

$$X = r\chi_R^2(r)\frac{\partial}{\partial r} \text{ on} M_\infty,$$

and $X = 0$ on M_C. Let $\{\exp[tX] | t \in \mathbb{R}\}$ be the flow generated by X. The flow induces a one-parameter unitary group defined by

$$U(t)\phi(x) = \Phi(t,x)\phi(\exp[-tX]x)$$

for $\phi \in \mathscr{H}$, where $\Phi(t,x)$ is a weight function to make the dilation operator $U(t)$ unitary. By simple calculation, we find that

$$A = \frac{1}{2}(\chi_R^2 r D_r + D_r r \chi_R^2)$$

is the generator of the dilation operator $U(t)$, that is, $U(t) = e^{-itA}$. Now it is easy to see e^{-itA} leaves $D(L) = \mathscr{H}_2$ invariant and to show (v) as in the Euclidean case.

Now we show the Mourre estimate. We replce L_0 with L in (5.7). By Lemma 5.8, $f^2(L) - f^2(L_0)$ and $f(L) - f(L_0)$ are compact. By Lemma 5.10, we can see that $[L_0, iA]f(L)$ and $f(L_0)[L_0, iA]$ are bounded. $f(L)[E, iA]f(L)$ is compact by (ii) of Lemma 5.12. Using these facts, it is easily seen that replacing L_0 with L in (5.7) introduces a compact error, which can be handled in K. Making this replacement and multiplying the resulting equation from both sides with $E_\Lambda(L_0)$, with Λ small enough to ensure $E_\Lambda(L_0)f(L_0) = E_\Lambda(L_0)$, give the Mourre estimate

$$E_\Lambda(L)[L, iA]E_\Lambda(L) \geq \min(2, c_0)(\lambda_0 - \varepsilon)E_\Lambda(L) + K.$$

We have showed that L satisfies a Mourre estimate at any point $\lambda_0 > 0$ with conjugate operator A, which completes the proof of Theorem 4.1. □

6 Limiting Absorption Principle

In this section we show the limiting absorption principle, which leads to the Kato-smoothness of G_0 and G_1 in Theorem 4.2. We extend the discussion in [29].

We will prove

Theorem 6.1. *Let L be as in Theorem 4.1, $s > \frac{1}{2}$, and $\Lambda \Subset \mathbb{R}_+ \setminus \sigma_{pp}(L)$. Then*

$$\sup_{z \in \Lambda_\pm = \Lambda \pm i\mathbb{R}_+} \|(|r|+1)^{-s}(L-z)^{-1}(|r|+1)^{-s}\| < \infty$$

$$\sup_{z \in \Lambda_\pm = \Lambda \pm i\mathbb{R}_+} \|(|r|+1)^{-1-s}A(L-z)^{-1}A(|r|+1)^{-1-s}\| < \infty.$$

The first estimate is called the limiting absorption principle.
As a preliminary we prove

Lemma 6.2. *Let L be as in Theorem 4.1. Then*

(i). $[E, ir\chi_R](L+i)^{-1}$ *is bounded,*

(ii). $[[E, ir\chi_R], ir\chi_R](L+i)^{-1}$ *is bounded,*

(iii). $\chi_R D_r r \chi_R (L+i)^{-1}\langle r\rangle^{-1}$ *is bounded,*

(iv). $\chi_R D_r^2 r^2 \chi_R (L+i)^{-1}\langle r\rangle^{-2}$ *is bounded.*

Proof. By (4.1), (4.3) and Lemma 5.6, we can show (i) and (ii).
Now we compute (iii).

$$\begin{aligned}&\chi_R D_r r\chi_R(L+i)^{-1}\langle r\rangle^{-1}\\ =&\chi_R D_r(L+i)^{-1} r\chi_R\langle r\rangle^{-1}+\chi_R D_r(L+i)^{-1}[L, r\chi_R](L+i)^{-1}\langle r\rangle^{-1}\end{aligned}$$

The first term in the right hand side is bounded by (i) of Lemma 5.6. We have

$$\begin{aligned}[L, r\chi_R] &= [D_r^2, r\chi_R] + [E, r\chi_R]\\ &= 2i^{-1}(r\chi_R)' D_r - (r\chi_R)'' + [E, r\chi_R].\end{aligned}$$

Using (i) of Lemma 5.6 and (i) of Lemma 6.2, we obtain the boundedness of $[L, r\chi_R](L+i)^{-1}$, which implies the boundedness of the second term.

Next we will show (iv). we begin with the equality

$$\begin{aligned}&\chi_R D_r^2 r^2\chi_R(L+i)^{-1}\langle r\rangle^{-2}\\ =&\ \chi_R D_r^2(L+i)^{-1} r^2\chi_R\langle r\rangle^{-2}+\chi_R D_r^2(L+i)^{-1}[L, r^2\chi_R](L+i)^{-1}\langle r\rangle^{-2}.\end{aligned}$$

The first term in the right hand side is bounded by (ii) of Lemma 5.6. The second term can be decomposed into two terms according to the following:

$$[L, r^2\chi_R] = [D_r^2, r^2\chi_R] + [E, r^2\chi_R].$$

It is easy to see that the first one is bounded using (iii). Replacing χ_R by χ_R^2, the second can be decomposed in the following way:

$$\begin{aligned}&[E, r^2\chi_R^2](L+i)^{-1}\langle r\rangle^{-1}\\ =&2[E, r\chi_R]r\chi_R(L+i)^{-1}\langle r\rangle^{-1}+[r\chi_R,[E, r\chi_R]](L+i)^{-1}\langle r\rangle^{-1}\\ =&2[E, r\chi_R](L+i)^{-1}r\chi_R\langle r\rangle^{-1}+2[E, r\chi_R](L+i)^{-1}[L, r\chi_R](L+i)^{-1}\langle r\rangle^{-1}\\ &+[r\chi_R,[E, r\chi_R]](L+i)^{-1}\langle r\rangle^{-1},\end{aligned}$$

which is bounded using (i), (ii) and the boundedness of $[D_r^2, r\chi_R](L+i)^{-1}\langle r\rangle^{-1}$, which can be shown by the argument in (iii). This proves (iv). □

Lemma 6.3. *Let L be as in Theorem 4.1. Then* *(i).* $\langle |A|\rangle^{\alpha}(L+i)^{-1}\langle r\rangle^{-\alpha}$ *is bounded for* $0 \leq \alpha \leq 2$

(ii). $\langle |A|\rangle^{s}[A,(L+i)^{-1}]\langle r\rangle^{-1-s}$ *is bounded for* $0 \leq s \leq 1$.

Proof of Lemma 6.3. By interpolation, it is enough to prove for $\alpha = 0, 2$ and $s = 0, 1$. The case $\alpha = 0$ is obvious. The case $\alpha = 2$ and $s = 1$ follows from (iii) and (iv) of Lemma 6.2.
The case $s = 0$ follows from Lemma 5.10 and Lemma 5.12. □

Proof of Theorem 6.1. Writing

$$\begin{aligned}&\langle r\rangle^{-s}(L+i)^{-1}(L-z)^{-1}(L+i)^{-1}\langle r\rangle^{-s}\\ &= \langle r\rangle^{-s}(L+i)^{-1}\langle |A|\rangle^{-s}\cdot\langle |A|\rangle^{s}(L-z)^{-1}\langle |A|\rangle^{s}\cdot\langle |A|\rangle^{-s}(L+i)^{-1}\langle r\rangle^{-s}\end{aligned}$$

and using Theorem 4.1 and Lemma 6.3, we see that

$$\langle r\rangle^{-s}(L+i)^{-1}(L-z)^{-1}(L+i)^{-1}\langle r\rangle^{-s}$$

is bounded.
Also, writing

$$\begin{aligned}&\langle r\rangle^{-1-s}A(L+i)^{-1}(L-z)^{-1}(L+i)^{-1}A\langle r\rangle^{-1-s}\\ &= \langle r\rangle^{-1-s}A(L+i)^{-1}\langle |A|\rangle^{-s}\cdot\langle |A|\rangle^{s}(L-z)^{-1}\langle |A|\rangle^{s}\cdot\langle |A|\rangle^{-s}(L+i)^{-1}A\langle r\rangle^{-1-s}\end{aligned}$$

and using Theorem 4.1 and Lemma 6.3, we see that

$$\langle r\rangle^{-1-s}A(L+i)^{-1}(L-z)^{-1}(L+i)^{-1}A\langle r\rangle^{-1-s}$$

is bounded.
Since

$$(L-z)^{-1} = (L+i)^{-1} + (z+i)(L+i)^{-2} + (z+i)^{2}(L+i)^{-1}(L-z)^{-1}(L+i)^{-1},$$

we obtain the desired result. □

7 Radiation Estimates

In this section, we prove the radiation estimates. We want first to recall general definitions of Kato-smoothness and the commutator method which allow us to find new Kato-smooth operators K given Kato-smooth operators G. For details, we refer the textbooks Yafaev [34] and [36].

Definition 7.1. *An H-bounded operator G is called H-smooth in the sense of Kato if*

$$\begin{aligned}&\sup_{f\in D(H),\|f\|=1}\int_{-\infty}^{\infty}\|Ge^{-iHt}f\|^{2}dt\\ &= \sup_{z\in\mathbb{R}+i\mathbb{R}}\|G((H-z)^{-1}-(H-\bar{z})^{-1})G^{*}\|\\ &<\infty.\end{aligned}$$

An operator G is called H-smooth on a Borel set Λ if $GE(\Lambda)$ is H-smooth, which is equivalent to the condition

$$\sup_{z\in\Lambda+i\mathbb{R}} \|G((H-z)^{-1}-(H-\bar{z})^{-1})G^*\| < \infty,$$

where $E(\Lambda)$ is the spectral projection of H on Λ.

Proposition 7.2. *Suppose that*

$$G^*G \le i[H,M]+K^*K,$$

where M is a H-bounded operator and K is H-smooth on a Borel set Λ. Then G is also H-smooth on Λ.

For the proof of Proposition 7.2, see Proposition 1.19 in [36].
Now we return to our problem.

Theorem 7.3. *Let L be as in Theorem 4.1. Then for large enough R,*

$$\chi_R r^{-\frac{1}{2}}(kP)^{\frac{1}{2}}$$

is L-smooth on Λ if $\Lambda \Subset \mathbb{R}\setminus\sigma_{pp}(L)$.

We prepare the following lemma.

Lemma 7.4. *For every $\varepsilon>0$, there exist a constant $C>0$ such that*

$$(c_0-\varepsilon)G_2^*G_2 \le [L,iM]+C\sum_{j,k=0,1} G_j^*G_j \tag{7.1}$$

where

$$\begin{aligned} M &= \frac{1}{2}(\chi_R D_r + D_r\chi_R) \\ G_0 &= \langle r\rangle^{-s}, \\ G_1 &= \chi_R\langle r\rangle^{-s}D_r, \\ G_2 &= \chi_R\langle r\rangle^{-\frac{1}{2}}(kP)^{\frac{1}{2}} \\ s &= \frac{1}{2}(1+\nu) > \frac{1}{2} \end{aligned}$$

and c_0 is the constant which appears in (4.1).

Proof of Lemma 7.4. To calculate the commutator $[L,iM]$, we first remark that

$$[D_r^2, iM] = 2D_r\chi_R' D_r \tag{7.2}$$
$$[k(r)P, iM] = -\chi_R k' P \ge c_0\chi_R r^{-1}kP. \tag{7.3}$$

Here we used the inequality (4.1).

For the perturbation term $[E, iM]$. we can prove

$$|([E, iM]u, u)| \leq C\|G_0 u\|^2 + \|G_1 u\| + C\|\chi_R r^{-\nu}\|\|G_2 u\|^2. \tag{7.4}$$

It suffices to prove this estimate for each term of E in the sum (4.2). First we consider the terms involving V.

$$[V, iM] = -\chi_R V',$$
$$|([V, iM]u, u)| \leq C\|\chi_R r^{-\frac{1+\nu}{2}} u\|^2 \leq C\|G_0 u\|^2..$$

For the a_1 part, we have that

$$[D_r a_1 D_r, iM] = -D_r(\chi_R a_1)' D_r,$$
$$|([D_r a_1 D_r, iM]u, u)| \leq C\|G_1 u\|^2.$$

For the a_3 part, we have that

$$[\tilde{D}_\theta k a_3 \tilde{D}_\theta, iM] = -\tilde{D}_\theta \chi_R (k a_3)' \tilde{D}_\theta$$
$$|([\tilde{D}_\theta k a_3 \tilde{D}_\theta, iM]u, u)| \leq C\|\chi_R r^{-\nu}\| \cdot \|\chi_R r^{-\frac{1}{2}} (kP)^{\frac{1}{2}} u\|^2 = C\|\chi_R r^{-\nu}\|\|G_2 u\|^2.$$

Other terms can be handled in a similar way.

Combinig the inequalities (7.2), (7.3) and (7.4), we arrive at the estimate

$$([L, iM]u, u) \geq c_0\|G_2 u\|^2 - C\|G_0 u\|^2 - C\|G_1 u\|^2 - \|\chi_R r^{-\nu}\|\|G_2 u\|^2$$
$$\geq (c_0 - \varepsilon)\|G_2 u\|^2 - C\|G_0 u\|^2 - C\|G_1 u\|^2$$

for an arbitrary $\varepsilon > 0$ by taking $R > 0$ large enough. This gives the desired estimate (7.1). □

Proof of Theorem 7.3. Fix $\Lambda \Subset \mathbb{R} \setminus \sigma_{\mathrm{pp}}(L)$ and consider (7.1). The operators G_0 and G_1 are L-smooth on Λ by Theorem 6.1 and G_2 is L-bounded. The commutator method Proposition 7.2 implies that the operator G_2 is also L-smooth on Λ. □

Theorem 6.1 and Theorem 7.3 directly mean Theorem4.2.

8 One-space scattering

We recall the smooth method of Kato which assures the existence of wave operators for perturbations that are smooth locally. For more details, see Corollary 4.5.7. in [34].

Theorem 8.1. *Suppose that H and H_0 are self-adjoint operators on Hilbert spaces $\mathscr{H}$ and $\mathscr{H}_0$ respectively, $J \in B(\mathscr{H}_0, \mathscr{H})$ is the identifier, and the pertuabation $HJ - JH_0$ admits a factorization*

$$HJ - JH_0 = G^* G_0,$$

where G_0 is H_0-bounded and G is H-bounded. Suppose $\{\Lambda_n\}$ is a set of intervals which exhausts the core of the spectra of the operators H_0 and H up to a set of Lebesgue measure zero. If on each of the intervals Λ_n the operator G_0 is H_0-smooth and G is H-smooth, then the wave operators $W^\pm(H, H_0; J)$ and $W^\pm(H_0, H; J^*)$ exist.

Now we apply Theorem 8.1 to our model.

Proof of Theorem4.3. First we note that any first order differential operator with compactly supported smooth coefficient function is L_0- and $L-$ locally smooth. This fact can be easily proved as in Section6.

The perturbation term E admits a factorization of the following form

$$E = \sum_{l,m=0,1,2} G_l^* B_{l,m} G_m + E_C$$

where G_l are L_0-smooth on any $\Lambda \Subset \mathbb{R} \setminus \sigma_{pp}(L_0)$ and L-smooth on any $\Lambda \Subset \mathbb{R} \setminus \sigma_{pp}(L)$ and E_C is a second-order differential operator with compactly supported coefficient function. Then the smooth perturbation theory of Kato shows the existence of the wave operators $W^{\pm}(L, L_0)$ and $W^{\pm}(L_0, L)$, which proves the Theorem. □

9 Two-space scattering

In this section, we consider a two-space scattering.

First we treat the short-range case.

Proposition 9.1. *Suppose that k is short-range. Then the wave operators $W^{\pm}(H_k, H_0)$ and $W^{\pm}(H_0, H_k)$ exist and are adjoint each other. They are asymptotically complete:*

$$W^{\pm}(H_k, H_0)\mathscr{H}_f = P_{ac}(H_k)\mathscr{H}.$$

Proof. Let $E_P(\Lambda)$ be the spectral projections of P on Λ with $\Lambda \Subset \mathbb{R}$. We decompose the perturbation term with identifier $E_P(\Lambda)$ as follows:

$$H_k E_P(\Lambda) - E_P(\Lambda) H_0 = \sqrt{k} P E_P(\Lambda) \sqrt{k}.$$

The limiting absorption principle implies that $\sqrt{k}$ is locally H_0- and H_k- smooth. $PE_P(\Lambda)$ is bounded. The smooth perturbation theory of Kato implies that the wave operators $W^{\pm}(H_k, H_0; E_P(\Lambda))$ and $W^{\pm}(H_0, H_k; E_P(\Lambda))$ exist and are adjoint each other.

Since P commutes with H_0 and H_k,

$$W^{\pm}(H_k, H_0; E_P(\Lambda)) = W^{\pm}(H_k, H_0) E_P(\Lambda),$$
$$W^{\pm}(H_0, H_k; E_P(\Lambda)) = W^{\pm}(H_0, H_k) E_P(\Lambda).$$

Hence $W^{\pm}(H_k, H_0)$ and $W^{\pm}(H_0, H_k)$ exist and are adoint each other. □

Proposition 9.2. *Suppose that k is short-range or long-range. Then the wave operators $W^{\pm}(L_0, H_k; J)$ and $W^{\pm}(H_k, L_0; J^*)$ exist and are adjoint each other.*

Proof. The perturbation $L_0 J - J(D_r^2 + k(r)P)$ can be decomposed into a sum of products of first-order differential operator with smooth compactly supported coefficients. Hence we can apply the smooth method of Kato. □

Now we obtain the following:

Theorem 9.3. *Suppose that k is short-range. Then the wave operators* $W^{\pm}(L_0,H_0;J)$ *and* $W^{\pm}(H_0,L_0;J^*)$ *exist and are adjoint each other.* $W^{\pm}(L_0,H_0;J)\mathscr{H}_f^{\mp}=0$. $W^{\pm}(L_0,H_0;J)$ *and* $W^{\pm}(H_0,L_0;J^*)$ *are isometric on* $\mathscr{H}_f^{\pm}$ *and* $P_{ac}(L_0)\mathscr{H}$*, respectively, and the asymptotic completeness*

$$W^{\pm}(L_0,H_0;J)\mathscr{H}_f^{\pm}=P_{ac}(L_0)\mathscr{H}$$

holds.

Proof. It follows from Proposition 9.1 and Proposition 9.2 that the wave operators $W^{\pm}(L_0,H_0;J)$ and $W^{\pm}(H_0,L_0;J^*)$ exist and are adjoint each other.

For $u\in\mathscr{H}_f^{\pm}$,

$$\lim_{t\to\pm\infty}\|Je^{-itH_0}u\|=\|u\|,$$
$$\lim_{t\to\mp\infty}\|Je^{-itH_0}u\|=0.$$

Hence $W^{\pm}(L_0,H_0;J)\mathscr{H}_f^{\mp}=0$, and $W^{\pm}(L_0,H_0;J)$ is isometric on $\mathscr{H}_f^{\pm}$.

To show the isometricity of $W^{\pm}(H_0,L_0;J^*)$, it is enough to check that

$$\lim_{t\to\pm\infty}\|(1-\chi)e^{-itL_0}u\|=0$$

for $u\in P_{ac}(L_0)\mathscr{H}$. This follows from the local L_0-smoothness of $1-\chi$. □

Combining Theorem 4.3 and Theorem 9.3, we obtain Theorem 4.5 by virtue of the chain rule of wave operators. Conversely, Theorem 9.3 and Theorem 4.5 imply Theorem 4.3. Theorem 4.5 is essentially solved in [16]. Hence Theorem 4.3 with $k(r)=r^{-2}$ is essentially solved in [16]. Our result may be considered as an extention of [16].

In the following of this section, we consider smooth long-range k. We also suppose that the coefficient a_1 in E is separated into two parts, long-range θ- independent term and short-range term:

$$a_1=a_1^L(r)+a_1^S(r,\theta) \tag{9.1}$$
$$|\partial_r^l a_1^L(r)|\le C_l\langle r\rangle^{-\nu_{a_1^L}-l},\nu_{a_1^L}>0 \tag{9.2}$$
$$|\partial_r^l\partial_\theta^\alpha a_1^S(r,\theta)|\le C_{l,\alpha}\langle r\rangle^{-\nu_{a_1^S}-l},\nu_{a_1^S}>1. \tag{9.3}$$

Set

$$H_L=D_r(1+a_1^L(r))D_r+k(r)P.$$

We formulate a long-range scattering theory for the triplet $(H_L,H_0;J^{\pm})$ with modified identifiers $J^{\pm}\in B(\mathscr{H}_f,\mathscr{H}_f)$. Since P commutes with H_0 and H_L, it is natural to choose $J^{\pm}$ as

$$J^{\pm}=\int J_\lambda^{\pm}dE_P(\lambda) \tag{9.4}$$

where

$$P = \int \lambda dE_P(\lambda)$$

is the spectral decomposition of P, and $J_\lambda^\pm$ are bounded operators $L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Through this decomposition, the problem reduces to the long-range scattering for the triplet $(H_{L,\lambda}, H_{0,\lambda}; J_\lambda^\pm)$ on the real line, where $H_{L,\lambda} = D_r(1+a_1^L)D_r + \lambda k(r)$ and $H_{0,\lambda} = D_r^2$ are self-adjoint operators on $L^2(\mathbb{R})$. We choose $J_\lambda^\pm$ as a pseudo-differential operator with oscillating symbols

$$J_\lambda^\pm = \chi_\lambda^\pm(D_r)J(\Phi_\lambda^\pm, a^\pm) \tag{9.5}$$
$$J(\Phi_\lambda^\pm, a^\pm)u(r) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{ir\rho + i\Phi_\lambda^\pm(r,\rho)} a^\pm(r,\rho)\hat{u}(\rho)d\rho$$
$$a^\pm(r,\rho) = \eta(r)\psi(\rho^2)\sigma^\pm(r,\rho). \tag{9.6}$$

Here $\eta \in C^\infty(\mathbb{R})$ such that $\eta(r) = 0$ near $r = 0$ and $\eta(r) = 1$ for large $|r|$, $\psi \in C_0^\infty(\mathbb{R}_+)$, $\chi_\lambda^\pm \in C_0^\infty(\mathbb{R})$ and $\sigma^\pm = 1$ if $\pm r\rho > 0$ and $\sigma^\pm = 0$ if $\pm r\rho \le 0$. We seach for a PDO $J_\lambda^\pm$ such that the perturbation

$$T_\lambda^\pm = H_{L,\lambda}J_\lambda^\pm - J_\lambda^\pm H_{0,\lambda}$$

admits a factorization into a product of $H_{L,\lambda}$- and $H_{0,\lambda}$- smooth operators.

Roughly speaking, up to compact terms, $T_\lambda^\pm$ is also a PDO with symbol

$$t_\lambda^\pm(r,\rho) = ((1+a_1^L(r))(D_r+\rho)^2 - \rho^2 + \lambda k(r))e^{i\Phi_\lambda^\pm(r,\rho)}a^\pm(r,\rho).$$

Let us compute

$$\begin{aligned} &e^{-i\Phi_\lambda^\pm}(r,\rho)((1+a_1^L(r))(D_r+\rho)^2 - \rho^2 + \lambda k(r))e^{i\Phi_\lambda^\pm}(r,\rho) \\ =&(1+a_1^L(r))(\nabla\Phi_\lambda^\pm + \rho)^2 + \lambda k(r) - \rho^2 - i(1+a_1^L(r))\Delta\Phi_\lambda^\pm. \end{aligned}$$

We want to find $\Phi_\lambda^\pm$ such that

$$(1+a_1^L(r))(\nabla\Phi_\lambda^\pm + \rho)^2 + \lambda k(r) - \rho^2$$

is "small". In the case $a_1^L = 0$, and $\nu_k > \frac{1}{2}$, it is enough to set

$$\Phi_\lambda^\pm(r,\rho) = -\frac{1}{2\rho}\int_0^r \lambda k(s)ds.$$

For general a_1^L and $\nu_k > 0$, we need to apply the method of succesive approximations and to keep $[\nu_k^{-1}]$ (the largest integer which does not exceed ν_k^{-1}) iterations:

Lemma 9.4. *Let $a_1^L(r), k(r) \in C^\infty(\mathbb{R})$ satisfy the smooth long-range condition:*

$$|\partial_r^l a_1^L(r)| \le C\langle r\rangle^{-\nu_{a_1^L}-l}$$
$$|\partial_r^l k(r)| \le C\langle r\rangle^{-\nu_k-l} \tag{9.7}$$

with $l \in \mathbb{N}$, and $\nu = \max\{\nu_{a_1^L}, \nu_k\} > 0$. We assume that ν^{-1} is not an integer. Let $\Lambda \Subset \mathbb{R}\setminus\{0\}$. Then for large enough R, there exists a C^∞-function $\Phi^\pm(r,\rho)$ defined on $(r,\rho) \in \Gamma^\pm(R,\Lambda) = \{(r,\rho) | |r| > R, \rho \in \Lambda, \pm r\rho > 0\}$ such that

$$|\partial_r^l \partial_\rho^k \Phi^\pm(r,\rho)| \le C(1+|r|)^{1-\nu-l} \tag{9.8}$$
$$R[\Phi^\pm] := (1+a_1^L)|\nabla\Phi^\pm + \rho|^2 + k(r) - \rho^2$$
$$|\partial_r^l \partial_\rho^k R[\Phi](r,\rho)| \le C(1+|r|)^{-1-\varepsilon-l}$$

where $\nabla = \partial_r$ and $\varepsilon = \nu([\nu^{-1}]+1) - 1 > 0$.

Proof. We only consider the case Φ^+ with $\Lambda \subset \mathbb{R}_+$, and abbreviate "+" . Other cases are similar to prove.

We fix $R > 0$ large enough such that $|a_1^L(r)| < \frac{1}{2}$ for $|r| > R$. Set

$$\Phi^{(0)}(r,\rho) := 0,$$
$$\Phi^{(1)}(r,\rho) := -\int_R^r \frac{k(s) + a_1^L(s)\rho^2}{2(1+a_1^L(s)\rho)} ds,$$
$$\Phi^{(N+1)} := \Phi^{(N)} + \phi^{(N+1)}$$
$$\phi^{(N+1)}(r,\rho) := -\frac{1}{2\rho}\int_R^r \left(|\nabla\Phi^{(N)})(s,\rho)|^2 - |\nabla\Phi^{(N-1)}(s,\rho)|^2\right) ds.$$

with $N \ge 1$.

A simple computation gives

$$R[\Phi^{(2)}] = (1+a_1^L)(|\nabla\Phi^{(2)}|^2 - |\nabla\Phi^{(1)}|^2),$$
$$R[\Phi^{(N+1)}] = (1+a_1^L)(|\nabla\Phi^{(N+1)}|^2 - |\nabla\Phi^{(N)}|^2) + R[\Phi^{(N)}] + 2(1+a_1)\langle\nabla\phi^{(N+1)}, \rho\rangle.$$

Hence by inducetion we have

$$R[\Phi^{(N+1)}] = (1+a_1^L)(|\nabla\Phi^{(N+1)}|^2 - |\nabla\Phi^{(N)}|^2).$$

We have uniformly for $\rho \in \Lambda$,

$$|\partial_r^l \partial_\rho^k \Phi^{(N)}| \le C(1+|r|)^{1-\nu-l},$$
$$|\partial_r^l \partial_\rho^k \phi^{(N)}| \le C(1+|r|)^{1-N\nu-l},$$
$$|\partial_r^l \partial_\rho^k R[\Phi^{(N)}]| \le C(1+|r|)^{-(1+N)\nu-l}.$$

It is now sufficient to set $\Phi = \Phi^{([\nu^{-1}])}$. □

From now on, we assume that $\Phi_\lambda^\pm$ satisfy the conclusions of Lemma 9.4 with k replaced by λk. We also assume that $\eta(r) = 0$ if $|r| < R$ and $\chi_\lambda^\pm(\rho) = 1$ near $\{\rho + \nabla_r\Phi_\lambda^\pm(r,\rho) : \rho^2 \in \operatorname{supp}\psi, |r| > R\}$. Now we state the existence of modified wave operators:

Lemma 9.5. *The wave operators*

$$W^{\pm}(H_L, H_0; J^{\pm}), W^{\pm}(H_0, H_L; (J^{\pm})^*) \tag{9.9}$$

and

$$W^{\pm}(H_L, H_0; J^{\mp}), W^{\pm}(H_0, H_L; (J^{\mp})^*) \tag{9.10}$$

exist. Operators (9.9) *as well as* (9.10) *are adjoint each other.*

Proof. It is enough to consider the scattering theory for the triplets $(H_{L,\lambda}, H_{0,\lambda}, J_{\lambda}^{\pm})$.

Set $b = (i^{-1}(\partial_r a_1^L)\rho + (1+a_1^L)\rho^2 + \lambda k(r))\chi_{\lambda}^{\pm}(\rho)$. a and b are in $\mathscr{S}^0$. By Theorem 10.3, there exists $d \in \mathscr{S}^{m_d}$ with $m_d = 0$ such that

$$\begin{aligned} H_{L,\lambda} J_{\lambda}^{\pm} &= (D_r(1+a_1^L)D_r + \lambda k(r))\chi_{\lambda}^{\pm}(D_r) J(\Phi_{\lambda}^{\pm}, a^{\pm}) = b(x, D_r) J(\Phi_{\lambda}^{\pm}, a^{\pm}) \\ &= J(\Phi_{\lambda}^{\pm}, d) \end{aligned}$$

and admits the asymptotic expansion

$$\begin{aligned} d &= \sum_{l \geq 0} \frac{1}{l!} d_l, \\ d_l(r,\rho) &= (\partial_{\tau}^l D_s^l p)(0,0,;r,\rho) \end{aligned}$$

where

$$p(s,\tau;r,\rho) = b(r, \rho + \tau + \delta(r, r+s, \rho)) a(r+s, \rho)$$

and

$$\delta(r,q,\rho) = \int_0^1 (\nabla_r \Phi_{\lambda}^{\pm})((1-t)r + tq), \rho) dt.$$

In particular, $d_l \in \mathscr{S}^{m_d - l} = \mathscr{S}^{-l}$ and

$$\begin{aligned} d_0(r,\rho) &= b(r, \rho + (\nabla_r \Phi_{\lambda}^{\pm})(r,\rho)) a(r,\rho), \\ d_1(r,\rho) &= (\partial_{\rho} b)(r, \rho + (\nabla_r \Phi_{\lambda}^{\pm})(r,\rho))(D_r a)(r,\rho) \\ &+ \langle \partial_{\rho}^2 b(r, \rho + (\nabla_r \Phi_{\lambda}^{\pm})(r,\rho)), \frac{1}{2}(\partial_r D_r \Phi_{\lambda}^{\pm})(r,\rho) \rangle \, a(r,\rho). \end{aligned}$$

(9.8) implies that $d_1 \in \mathscr{S}^{-1-\nu}$ where $\nu = \max\{\nu_k, \nu_{a_1}^L\}$. Hence $d - d_0 \in \mathscr{S}^{-1-\nu}$.

Set $c(r,\rho) = \rho^2$. Then by Theorem 10.3 and Theorem 10.6, there exists $e \in \mathscr{S}^{m_e}$ with $m_e = 0$ such that

$$\begin{aligned} J_{\lambda}^{\pm} H_{0,\lambda} &= \chi_{\lambda}^{\pm}(D_r) J(\Phi_{\lambda}^{\pm}, a^{\pm})(D_r^2) \\ &= J(\Phi_{\lambda}^{\pm}, e) \end{aligned}$$

and admits the asymptotic expansion

$$\begin{aligned} e &= \sum_{l \geq 0} \frac{1}{l!} e_l, \\ e_l(r,\rho) &= (\partial_{\tau}^l D_s^l q)(0,0,;r,\rho) \end{aligned}$$

where

$$q(s,\tau;r,\rho) = a(r,\rho+\tau)\bar{c}(r+s+\gamma(r,\rho+\tau,\rho),\rho+\tau)$$

and

$$\gamma(r,\rho,\sigma) = \int_0^1 (\nabla_\rho \Phi_\lambda^\pm)(r,(1-t)\rho+t\sigma)dt.$$

In particular, $e_l \in \mathscr{S}^{m_e-l} = \mathscr{S}^{-l}$ and

$$\begin{aligned} e_0(r,\rho) &= a(r,\rho)\bar{c}(r+(\nabla_\rho\Phi_\lambda^\pm)(r,\rho),\rho),\\ e_1(r,\rho) &= (\partial_\rho a)(r,\rho)(D_r\bar{c})(r+(\nabla_\rho\Phi_\lambda^\pm)(r,\rho),\rho)\\ &\quad + a(r,\rho)\big((D_r\partial_\rho\bar{c})(r+(\nabla_\rho\Phi_\lambda^\pm)(r,\rho),\rho)\\ &\quad + \langle(\nabla_r D_r\bar{c})(r+(\nabla_\rho\Phi_\lambda^\pm)(r,\rho),\rho),\frac{1}{2}\nabla_\rho\partial_\rho\Phi_\lambda^\pm(r,\rho)\rangle\big). \end{aligned}$$

Since $c(r,\rho)=\rho^2$, $e_1=0$. Hence $e-e_1\in\mathscr{S}^{-2}$.

Now we have $T_\lambda^\pm = J(\Phi_\lambda^\pm, d-e)$ where $(d-e)-(d_0-e_0)\in\mathscr{S}^{-1-\nu}$ and

$$(d_0-e_0)(r,\rho) = \big(i^{-1}(\partial_r a_1^L)(r)(\rho+\nabla_r\Phi_\lambda^\pm(r,\rho)) + R[\Phi_\lambda^\pm](r,\rho)\big)a(r,\rho),$$

where

$$R[\Phi_\lambda^\pm](r,\rho) = (1+a_1^L(r))|\rho+|\Phi_\lambda^\pm(r,\rho)|^2+\lambda k(r)-\rho^2.$$

As in Lemma 9.4, we chose $\Phi_\lambda^\pm$ so that $R[\Phi_\lambda^\pm](r,\rho)a(r,\rho)\in\mathscr{S}^{-1-\varepsilon}$ with some $\varepsilon>0$. Therefore $T_\lambda^\pm = J(\Phi_\lambda^\pm, d-e)$ with $d-e\in\mathscr{S}^{-1-\varepsilon}$ and hence $\langle r\rangle^{\frac{1+\varepsilon}{2}}T_\lambda^\pm\langle r\rangle^{\frac{1+\varepsilon}{2}}$ is bounded. The operator $\langle r\rangle^{-\frac{1+\varepsilon}{2}}$ is $H_{0,\lambda}$- and $H_{L,\lambda}$-smooth on any positive bounded interval disjoint from eigenvalues of $H_{L,\lambda}$. So the smooth perturbation theory of Kato yields the Lemma. □

Now we show that these wave operators are isometric on suitable subspaces.

Lemma 9.6.

$$\operatorname*{s-lim}_{t\to\pm\infty}((J^\pm)^*J^\pm - \psi(H_0))e^{-iH_0t} = 0 \tag{9.11}$$

$$\operatorname*{s-lim}_{t\to\mp\infty}(J^\pm)^*J^\pm e^{-iH_0t} = 0. \tag{9.12}$$

In particular, if $\Lambda\Subset\mathbb{R}_+$ and $\psi\in C_0^\infty(\mathbb{R})$ such that $\psi=1$ on Λ, then the wave operators $W^\pm(H_L,H_0;J^\pm)$ are isometric on the subspace $E_{H_0}(\Lambda)\mathscr{H}_f$ and $W^\pm(H_L,H_0;J^\mp)=0$.

Proof. Up to a compact term, $(J_\lambda^\pm)^*J_\lambda^\pm$ is a PDO $Q_\lambda^\pm$ with symbol

$$\eta^2(r)\psi^2(\rho^2)(\sigma^\pm)^2(r,\rho).$$

If $t\to\mp\infty$, then the stationary point $\rho=\frac{r}{2t}$ of the integral

$$(Q^\pm e^{-iH_{0,\lambda}t}u)(r) = \frac{1}{(2\pi)^{\frac{1}{2}}}\eta^2(r)^2\int_{\mathbb{R}} e^{ir\rho-i\rho^2t}\psi^2(\rho^2)(\sigma^\pm)^2(r,\rho)\hat{u}(\rho)d\rho.$$

does not belong to the support of the function $\sigma^\pm$. Therefore supposing $\hat{u}\in C_0^\infty(\mathbb{R})$ and integrating by parts, we estimate this integral by $C_N(1+|r|+|t|)^{-N}$ for an arbitrary N. This proves (9.12). We apply the same argument to the PDO with sumbol $\eta^2(r)\psi^2(\rho^2)(\sigma^\pm)^2(r,\rho)-\psi^2(\rho^2)$ to prove (9.11). □

From now on, fix Λ and ψ as in Lemma 9.6.

Lemma 9.7. *The wave operators $W^{\pm}(H_0,H_L;(J^{\pm})^*)$ are isometric on $E_{H_L}(\Lambda)\mathscr{H}_f$.*

Proof. By Lemma 9.6, $W^{\pm}(H_0,H_L;(J^{\mp})^*) = W^{\pm}(H_L,H_0;J^{\mp})^* = 0$. This implies

$$\lim_{t\to\pm\infty} \|J^{\mp *}e^{-iH_Lt}u\| = 0, \; u \in E_{H_L}(\Lambda)\mathscr{H}_f. \tag{9.13}$$

Moreover, $J_\lambda^+(J_\lambda^+)^* + J_\lambda^-(J_\lambda^-)^* - \psi^2(H_{0,\lambda})$ and $\psi^2(H_{0,\lambda}) - \psi^2(H_{L,\lambda})$ are compact, and (9.13) implies that

$$\lim_{t\to\pm\infty} \|(J^{\pm})^*e^{-iH_Lt}u\| = \|u\|, \; u \in E_{H_L}(\Lambda)\mathscr{H}_f.$$

This implies the Lemma. □

Lemma 9.8. *The wave operators $W^{\pm}(H_L,L_0+D_ra_1^LD_r;J^*)$ are isometric on $P_{ac}(L_0)\mathscr{H}$.*

Proof. Use the local $L_0+D_ra_1^LD_r$-smoothness of $1-\chi$. □

Lemma 9.9. *The wave operators $W^{\pm}(L_0+D_ra_1^LD_r,H_0;JJ^{\pm})$ are isometric on $E_\Lambda(H_0)\mathscr{H}_f^{\pm}$ and $W^{\pm}(L_0+D_ra_1^LD_r,H_0;JJ^{\pm})\mathscr{H}_f^{\mp} = 0$*

Proof. It is enough to show that

$$\operatorname*{s-lim}_{t\to\pm\infty}[(JJ^{\pm})^*JJ^{\pm} - \psi(H_0)]e^{-iH_0t}P_{\pm} = 0 \tag{9.14}$$

$$\operatorname*{s-lim}_{t\to\pm\infty}(JJ^{\pm})^*JJ^{\pm}e^{-iH_0t}P_{\mp} = 0 \tag{9.15}$$

where $P_{\pm} = 0$ are projections onto the subspaces $\mathscr{H}_f^{\pm}$. Again up to a compact term, $(JJ^{\pm})^*JJ^{\pm}P_{\mp}$ is a PDO with symbol

$$\chi^2(r)\eta^2(r)\psi^2(\rho^2)(\sigma^{\pm})^2(r,\rho)1_{\mathbb{R}_{\mp}}(\rho) = 0.$$

This implies (9.15). Similarly, up to a compact term, $(JJ^{\pm})^*JJ^{\pm}P_{\pm}$ is a PDO with symbol

$$[(\chi^2(r)\eta^2(r)(\sigma^{\pm})^2(r,\rho)-1]\psi^2(\rho^2)1_{\mathbb{R}_{\pm}}(\rho).$$

We apply the same argument as in Lemma 9.6 to prove (9.14). □

Combinig these results, we obtain the following theorem.

Theorem 9.10. *Suppose $v_{a_2} = v_{b_1} = v_{b_2} = v_V > 1$, $v_{a_3} = 1$ and a_1 can be separated into two parts as in* (9.1) - (9.3). *Suppose k is smooth long-range in the sense of Definition 4.4 and let the operators $J^{\pm}$ be defined by* (9.4), (9.5), *and* (9.6) *with $\Phi_\lambda^{\pm}$ satisfying the properties listed in Lemma 9.4 with k replaced by λk. We also assume that $\psi(\lambda) = 1$ on $\Lambda \Subset \mathbb{R}_+$, $\eta(r) = 0$ if $|r| < R$ for large enough R as is taken in Lemma 9.4. Then the wave operators $W^{\pm}(L,H_0;JJ^{\pm})$ and $W^{\pm}(H_0,L;(JJ^{\pm})^*)$ exist, are adjoint each other, are isometric on $E_\Lambda(H_0)\mathscr{H}_f^{\pm}$ and $E_\Lambda(L)P_{ac}(L)\mathscr{H}$, respectively, $W^{\pm}(L,H_0;JJ^{\pm})\mathscr{H}_f^{\mp} = 0$, and the asymptotic completeness*

$$W^{\pm}(L,H_0;JJ^{\pm})E_\Lambda(H_0)\mathscr{H}_f^{\pm} = E_\Lambda(L)P_{ac}(L)\mathscr{H}$$

holds.

10 Appendix: PDOs with oscillating symbols

In this appendix, we describe a class of pseudo-differential operators with oscillating symbols.

We recall the Hörmander classes $\mathscr{S}^m_{\rho,\delta}$ for $m \in \mathbb{R}, \rho > 0, \delta < 1$. We set $\mathscr{S}^m_{\rho,\delta} = \mathscr{S}^m_{\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ consists of functions $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that, for all multi-indices α, β, there exist $C_{\alpha,\beta}$ such that

$$|(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi)| \leq C_{\alpha,\beta}(1+|x|)^{m-|\alpha|\rho+|\beta|\delta}$$

for all $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$. The best $C_{\alpha,\beta}$ are the semi-norms of the symbol a. We denote $\mathscr{S}^m = \mathscr{S}^m_{1,0}$. We say a symbol $a(x,\xi)$ is compactly supported in the variable ξ if there is a compact set $K \Subset \mathbb{R}^d$ such that

$$a(x,\xi) = 0$$

for all $x \in \mathbb{R}^d$ if $\xi \notin K$. We denote the pseudo-differential operator (PDO) with symbol $a(x,\xi)$ by $a(x,D)$

$$(a(x,D)u)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} a(x,\xi)\hat{u}(\xi)d\xi$$

where $\hat{u}$ is the Fourier transform of u

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} u(x)dx.$$

The following is elementary.

Lemma 10.1. *Suppose that $a \in \mathscr{S}^m$ and a is compactly supported in the variable ξ. Then $a(x,D)\langle x\rangle^{-m}$ is bounded in the space $L^2(\mathbb{R}^d)$ and $a(x,D)\langle x\rangle^{-m'}$ is compact if $m' > m$.*

Now we define a class of symbols with oscillating factor. Let $\varepsilon > 0, m \in \mathbb{R}$, $\Phi \in \mathscr{S}^{1-\varepsilon}$, and $a \in \mathscr{S}^m$. We denote classes of symbols of the form

$$e^{i\Phi(x,\xi)}a(x,\xi)$$

by $C^m(\Phi)$. We denote the PDO with symbol $e^{i\Phi}a$ by $J(\Phi,a)$

$$J(\Phi,a) = (e^{i\Phi}a)(x,D).$$

Clearly $C^m(\Phi) \subset \mathscr{S}^m_{\varepsilon,1-\varepsilon}$ so that $C^m(\Phi)$ are good classes if $\varepsilon > \frac{1}{2}$. On the other hand, the standard calculus fails for operators from these classes if $\varepsilon \leq \frac{1}{2}$. However as is shown in [37], $J(\Phi,a_1)J(\Phi,a_2)^*$ and $J(\Phi,a_1)^*J(\Phi,a_2)$ become usual PDO and admit asymptotic expansions.

Theorem 10.2. *Suppose that $\Phi \in \mathscr{S}^{1-\varepsilon}$ with $\varepsilon > 0$, and $a_j \in \mathscr{S}^{m_j}$ for $j = 1,2$ and some numbers m_j. Suppose that a_j are compactly supported in the variable ξ. Then the following holds.*

(i). $G = J(\Phi, a_1)J(\Phi, a_2)^*$ *is a PDO with symbol* $g \in \mathscr{S}^m$ *for* $m = m_1 + m_2$ *and* $g(x, \xi)$ *admits the asymptotic expansion*

$$g = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} g_\alpha,$$
$$g_\alpha(x, \xi) = \partial_\xi^\alpha (e^{i\Phi(x,\xi)} a_1(x, \xi) D_x^\alpha (e^{-i\Phi(x,\xi)} \bar{a}_2(x, \xi)));$$

in particular, $g_\alpha \in \mathscr{S}^{m-|\alpha|\varepsilon}$.

(ii). $H = J(\Phi, a_2)^* J(\Phi, a_1)$ *is a PDO with symbol* $h \in \mathscr{S}^m$ *for* $m = m_1 + m_2$ *and* $h(x, \xi)$ *admits the asymptotic expansion*

$$h = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} h_\alpha,$$
$$h_\alpha(x, \xi) = D_x^\alpha (e^{i\Phi(x,\xi)} a_1(x, \xi) \partial_\xi^\alpha (e^{-i\Phi(x,\xi)} \bar{a}_2(x, \xi)));$$

in particular, $h_\alpha \in \mathscr{S}^{m-|\alpha|\varepsilon}$.

(iii). $J(\Phi, a_1)$ *is bounded in the space* $L^2(\mathbb{R}^d)$ *if* $m_1 = 0$ *and it is compact if* $m_1 < 0$.

(iv). *Suppose* $m_1 = m_2 = 0$. *Denote by A the PDO with symbol*

$$a(x, \xi) = a_1(x, \xi) \bar{a}_2(x, \xi) \in \mathscr{S}^0.$$

Then $J(\Phi, a_1)J(\Phi, a_2)^* - A$ *and* $J(\Phi, a_2)^* J(\Phi, a_1) - A$ *are compact in* $L^2(\mathbb{R}^d)$.

For the proof of Theorem 10.2, we refer Yafaev [37].

Next we consider the product of a *PDO* with oscillating symbol and a usual pseudo-differential operator. The situation is different whether the pseudo-differential operator is on the left and on the right.

Theorem 10.3. *Suppose that* $\Phi \in \mathscr{S}^{1-\varepsilon}$, $a \in \mathscr{S}^{m_a}$, *and* $b \in \mathscr{S}^{m_b}$ *for* $\varepsilon > 0$ *and some* $m_a, m_b \in \mathbb{R}$. *Suppose a and b are compactly supported in the variable* ξ. *Then there exists a symbol* $d \in \mathscr{S}^{m_d}$ *for* $m_d = m_a + m_b$ *such that d is compactly supported in the variable* ξ,

$$b(x, D)J(\Phi, a) = J(\Phi, d),$$

and admits the asymptotic expansion

$$d = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} d_\alpha,$$
$$d_\alpha(x, \eta) = (\partial_\zeta^\alpha D_z^\alpha p)(0, 0, ; x, \eta)$$

where

$$p(z, \zeta; x, \eta) = b(x, \eta + \zeta + r(x, x + z, \eta)) a(x + z, \eta)$$

and

$$r(x,y,\eta)=\int_0^1(\nabla_x\Phi)((1-\tau)x+\tau y),\eta)d\tau.$$

In particular, $d_\alpha\in\mathscr{S}^{m_d-|\alpha|}$ *and*

$$\begin{aligned}&d_0(x,\eta)=b(x,\eta+(\nabla_x\Phi)(x,\eta))a(x,\eta),\\&d_\alpha(x,\eta)=(\partial_\eta^\alpha b)(x,\eta+(\nabla_x\Phi)(x,\eta))(D_x^\alpha a)(x,\eta)\\&+\langle\nabla_\eta\partial_\eta^\alpha b(x,\eta+(\nabla_x\Phi)(x,\eta)),\frac{1}{2}(\nabla_x D_x^\alpha\Phi)(x,\eta)\rangle a(x,\eta)\end{aligned}$$

if $|\alpha|=1$.

Proof. We compute

$$\begin{aligned}&(b(x,D)J(\Phi,a)u)(x)\\=&(2\pi)^{-\frac{3n}{2}}\int e^{i\langle x,\xi\rangle-i\langle y,\xi\rangle-+i\langle y,\eta\rangle+i\Phi(y,\eta)}b(x,\xi)a(y,\eta)\hat{u}(\eta)d\eta dyd\xi\\=&(2\pi)^{-\frac{3n}{2}}\int e^{i\langle x,\eta\rangle+i\Phi(x,\eta)}\hat{u}(\eta)\Big(\int e^{i\langle x-y,\xi-\eta\rangle+i(\Phi(y,\eta)-\Phi(x,\eta))}b(x,\xi)a(y,\eta)dyd\xi\Big)d\eta\\=&(2\pi)^{-\frac{n}{2}}\int e^{i\langle x,\eta\rangle+i\Phi(x,\eta)}\hat{u}(\eta)d(x,\eta)d\eta\end{aligned}$$

where

$$d(x,\eta)=(2\pi)^{-n}\int e^{i\langle x-y,\xi-\eta\rangle+i(\Phi(y,\eta)-\Phi(x,\eta))}b(x,\xi)a(y,\eta)dyd\xi.$$

We set

$$r(x,y,\eta)=\int_0^1(\nabla_x\Phi)((1-\tau)x+\tau y,\eta)d\tau.$$

Then

$$\Phi(y,\eta)-\Phi(x,\eta)=\langle y-x,r(x,y,\eta)\rangle.$$

By changing variables, we compute

$$\begin{aligned}&d(x,\eta)\\=&(2\pi)^{-n}\int e^{i\langle x-y,\xi-\eta-r(x,y,\eta)\rangle}b(x,\xi)a(y,\eta)dyd\xi\\=&(2\pi)^{-n}\int e^{i\langle x-y,\tilde{\xi}-\eta\rangle}b(x,\tilde{\xi}+r(x,y,\eta))a(y,\eta)dyd\xi\\=&(2\pi)^{-n}\int e^{-i\langle z,\zeta\rangle}b(x,\eta+\zeta+r(x,x+z,\eta))a(x+z,\eta)dyd\xi.\end{aligned}$$

Set

$$p(z,\zeta;x,\eta)=b(x,\eta+\zeta+r(x,x+z,\eta))a(x+z,\eta).$$

Then by Taylor's expansion formula, we obtain the folowing:

$$d(x,\eta) = \sum_{0\le|\alpha|\le N-1} \frac{1}{\alpha!}(\partial_\zeta^\alpha D_\zeta^\alpha p)(0,0;x,\eta) + p^{(N)}(x,\eta)$$

where

$$p^{(N)}(x,\eta) = (2\pi)^{-n} N \sum_{|\alpha|=N} \frac{1}{\alpha!}\int_0^1 (1-t)^{N-1} \int\int (\partial_z^\alpha p)(tz,\zeta;x.\eta) z^\alpha e^{-i\langle z,\zeta\rangle} dz d\zeta dt.$$

Set

$$R^{(\alpha)}(x,\eta;t) = \int\int (\partial_z^\alpha D_\zeta^\alpha p)(tz,\zeta;x.\eta) e^{-i\langle z,\zeta\rangle} dz d\zeta.$$

Now it is enough to show that $R^{(\alpha)} \in S^{m_d-|\alpha|}$ and the seminorms are bounded uniformly with respect to the variable t. This obeys from the following two elementary lemmas.

Lemma 10.4. *Fix $C>0$. If $|z| \ge C|x|$, then for any n,*

$$|\int (\partial_z^\alpha D_\zeta^\alpha p)(tz,\zeta;x.\eta) e^{-i\langle z,\zeta\rangle} d\zeta| \le C\langle z\rangle^{-n}.$$

Lemma 10.5. *There exists $C>0$ such that*

$$|\int\int_{|z|\le C|x|} (\partial_z^\alpha D_\zeta^\alpha p)(tz,\zeta;x.\eta) e^{-i\langle z,\zeta\rangle} dz d\zeta| \le C\langle x\rangle^{m_d-|\alpha|}.$$

By integrating by parts, we can show these lemmas. □

Theorem 10.6. *Suppose that $\Phi \in \mathcal{S}^{1-\varepsilon}, a \in \mathcal{S}^{m_a}$, and $c \in \mathcal{S}^{m_c}$ for $\varepsilon > 0$ and some $m_a, m_c \in \mathbb{R}$. Suppose a is compactly supported in the variable ξ. Then there exists a symbol $e \in \mathcal{S}^{m_e}$ for $m_e = m_a + m_c$ such that*

$$J(\Phi,a)c(x,D)^* = J(\Phi,e),$$

and admits the asymptotic expansion

$$e = \sum_{|\alpha|\ge 0} \frac{1}{\alpha!} e_\alpha,$$
$$e_\alpha(x,\eta) = (\partial_\zeta^\alpha D_z^\alpha q)(0,0,;x,\eta)$$

where

$$q(z,\zeta;x,\eta) = a(x,\eta+\zeta)\bar{c}(x+z+s(x,\eta+\zeta,\eta),\eta+\zeta)$$

and

$$s(x,\zeta,\eta) = \int_0^1 (\nabla_\xi \Phi)(x,(1-\tau)\eta + \tau\xi) d\tau.$$

In particular, $e_\alpha \in \mathscr{S}^{m_e-|\alpha|}$ *and*

$$\begin{aligned} e_0(x,\eta) &= a(x,\eta)\bar{c}(x+(\nabla_\eta\Phi)(x,\eta),\eta), \\ e_\alpha(x,\eta) &= (\partial_\eta^\alpha a)(x,\eta)(D_x^\alpha\bar{c})(x+(\nabla_\eta\Phi)(x,\eta),\eta) \\ &\quad + a(x,\eta)\big((D_x^\alpha\partial_\eta\bar{c}(x+(\nabla_\eta\Phi)(x,\eta),\eta) \\ &\quad + \langle(\nabla_x D_x^\alpha\bar{c})(x+(\nabla_\eta\Phi)(x,\eta),\eta),\frac{1}{2}\nabla_\xi\partial_\xi^\alpha\Phi(x,\eta)\rangle\big) \end{aligned}$$

if $|\alpha| = 1$.

Proof is similar to Theorem 10.3.

Part III
Non-propagation of singularities of scattering matrices for Schrödinger equations on manifolds with polynomially growing ends

Abstract

Let M be a manifold with asymptotically polynomially growing ends of growing rate r^{α} with a real positive number $\alpha > 1$, where r is the radial coordinate. Let P be a Schrödinger operator on M. A time-dependent scattering theory for P with a simple reference system is constructed in [18] (Part II of this paper), and the scattering matrix is defined. We here show that the scattering matrices do not change the wave front set. In particular, the scattering matrix has a smooth kernel away from the diagonal set. Physically this corresponds to the fact that incoming waves are almost vertically reflected.

11 Introduction of Part III

Consider a manifold M with asymptotically polynomially growing ends of growing rate r^{α} with a real positive number $\alpha > \frac{1}{2}$, where r is the radial coordinate. The case where $\alpha = 1$ corresponds to Euclidean spaces and scattering manifolds, i.e., Riemannian manifolds with asymptotically conic structure (see Melrose [25]). Let P be a Schrödinger operator on M. We can construct a time-dependent scattering theory for P with a simple reference system (see [18], for the scattering manifolds case we refer Ito and Nakamura [16] also). We consider the scattering operator and scattering matrix. Melrose and Zworski [26] showed that, for the scattering manifolds case, the scattering matrices are Fourier integral operators associated to the canonical transform on the boundary manifold generated by the geodesic flow with length π. The scattering matrices propagate the wave front set according to the same canonical map. They use the asymptotic expansion of generalized eigenfunctions. Ito and Nakamura [17] generalized these results using Egorov-type theorem, which is time-dependent theoretical. We here show that if the growing order satisfies $\alpha > 1$, then the scattering matrices no longer change the wave front set. We see how the scaling property of the corresponding classical scattering operator determines laws of the propagation of singularities for quantum scattering operators.

We describe our model. Let M be an n-dimensional smooth non-compact manifold such that $M = M_C \cup M_\infty$, where M_C is pre-compact and M_∞ is the non-compact end as follows: We assume that M_∞ has the form $\mathbb{R}_+ \times \partial M$ where ∂M is a $n-1$-dimensional compact manifold, and $\mathbb{R}_+ = (0,\infty)$ is the real half line. We identify M_∞ with $\mathbb{R}_+ \times \partial M$ and suppose $M_C \cap M_\infty \subset (0, \frac{1}{2}) \times \partial M$. Let $\{\psi_\lambda : U_\lambda \to \mathbb{R}^{n-1}\}, U_\lambda \subset \partial M$ be a local coordinate system of ∂M. We set $\{I \otimes \psi_\lambda : \mathbb{R}_+ \times U_\lambda \to \mathbb{R} \times \mathbb{R}^{n-1}\}$ be a local coordinate system of M_∞, and we denote $(r,\theta) \in \mathbb{R} \times \mathbb{R}^{n-1}$ to represent a point in ∂M.

We suppose that ∂M is equipped with a smooth strictly positive density $H = H(\theta)$ and a positive $(2,0)$-tensor $h = (h^{jk}(\theta))$ on ∂M. We set

$$Q = -\frac{1}{2}\sum_{j,k} H(\theta)^{-1}\frac{\partial}{\partial\theta_j}H(\theta)h^{jk}(\theta)\frac{\partial}{\partial\theta_k} \quad \text{on } \mathscr{H}_b = L^2(\partial M, H(\theta)d\theta).$$

Q is essentially self-adjoint on $\mathscr{H}_b$, and we denote its unique extension by Q.

We suppose that M is equipped with a smooth strictly positive density $G = G(x)$ such that

$$G(x)dx = H(\theta)drd\theta \quad \text{on } M_\infty$$

and set our Hilbert space $\mathscr{H} = L^2(M, G(x)dx)$. We set k as

$$k(r) = r^{-2\alpha}$$

with growing order $\alpha > 1/2$. Let P be a formally self-adjoint second-order elliptic operator on M such that

$$P = -\frac{1}{2}G^{-1}(\partial_r, \sqrt{k}\partial_\theta)\begin{pmatrix} 1+a_1 & a_2 \\ {}^t a_2 & h+a_3 \end{pmatrix}\begin{pmatrix} \partial_r \\ \sqrt{k}\partial_\theta \end{pmatrix} + V \quad \text{on } (1,\infty)\times\partial M.$$

where $\begin{pmatrix} 1+a_1 & a_2 \\ {}^t a_2 & h+a_3 \end{pmatrix}$ defines a real-valued smooth tensor, and V is a real-valued smooth function. We suppose

Assumption 11.1. *There is* $\mu_1 > 1, \mu_2 > 1, \mu_3 \geq 1$ *and* $\mu_V > 1$ *such that for any* $l \in \mathbb{Z}_+$, $\gamma \in \mathbb{Z}_+^{n-1}$, *there is* $C_{l,\alpha}$ *such that*

$$|\partial_r^l \partial_\theta^\gamma a_j(r,\theta)| \leq C_{l,\gamma} r^{-\mu_j - l} \text{ for } j = 1,2,3,$$
$$|\partial_r^l \partial_\theta^\gamma V(r,\theta)| \leq C_{l,\gamma} r^{-\mu_V - l}.$$

P is essentially self-adjoint, $\sigma_{\text{ess}}(P) = [0,\infty)$ and P is absolutely continuous except for a countable discrete spectrum with the only accumulation point 0 (see [18] and references therein). In a sense, P is a "short-range" perturbation of $-\frac{1}{2}\partial_r^2 + k(r)Q$ and we can construct a one-space scattering theory for them ([18]). We here consider a two-space time-dependent scattering theory for P with the following reference system: We set

$$M_f = \mathbb{R} \times \partial M,$$
$$\mathscr{H}_f = L^2(M_f, H(\theta)drd\theta),$$
$$P_f = -\frac{1}{2}\frac{\partial^2}{\partial r^2} \quad \text{on } M_f.$$

P_f is the one-dimensional free Schrödinger operator, and it is self-adjoint on $H^2(\mathbb{R}) \otimes \mathscr{H}_b$, where, H^2 is the Sobolev space of order 2. Let $\mathscr{F}$ be the Fourier transform in r-variable, i.e.,

$$(\mathscr{F}u)(\rho,\theta) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ir\rho} u(r,\theta)dr$$

$\mathscr{F}$ is defined for $u \in C_0^\infty(M_f)$, and extends to a unitary operator on $L^2(M_f)$. We decompose

$$\mathscr{H}_{f,\pm} := \{u \in \mathscr{H}_f | \operatorname{supp}(\mathscr{F}u) \subset \bar{\mathbb{R}}_\pm \times \partial M\}$$

then $\mathscr{H}_f = \mathscr{H}_{f,+} \oplus \mathscr{H}_{f,-}$. Let $j(r) \in C^\infty(\mathbb{R})$ such that $j(r) = 0$ if $r \leq \frac{1}{2}$ and $j(r) = 1$ if $r \geq 1$. We set an identification operator $J : \mathscr{H}_f \to \mathscr{H}$ by

$$(Ju)(r,\theta) = j(r)u(r,\theta) \quad \text{if } (r,\theta) \in M_\infty$$

and $(Ju)(x) = 0$ if $x \notin M_\infty$. We define the two-space wave operators by

$$W_\pm := W_\pm(P, P_f; J) := \operatorname*{s-lim}_{t \to \pm\infty} e^{itP} J e^{-itP_f}.$$

It is shown in Theorem 5 in [18] that under Assumption 11.1 the wave operators $W_\pm$ exist, are isometric on $\mathscr{H}_{f,\pm}$, and the asymptotic completeness holds:

$$W_\pm \mathscr{H}_{f,\pm} = \mathscr{H}_{ac}(P),$$

where $\mathscr{H}_{ac}(P)$ is the absolutely continuous subspace of P. Then the scattering operator is defined by

$$S = W_+^* W_- \ : \ \mathscr{H}_{f,-} \to \mathscr{H}_{f,+}$$

and is a unitary operator there. By the intertwining property: $P_f S = S P_f$, there is $S(\lambda) \in B(\mathscr{H}_b)$ for $\lambda > 0$ such that

$$(\mathscr{F} S \mathscr{F}^{-1} u)(\rho,\cdot) = S(\rho^2/2) u(-\rho,\cdot) \text{ for } \rho > 0, u \in \mathscr{F}\mathscr{H}_{f,-}.$$

$S(\lambda)$ is the scattering matrix and we study its properties.

We state our main Theorem.

Theorem 11.2. *Suppose Assumption 11.1 and the growing order α satisfies $\alpha > 1$. Then for $u \in \mathscr{H}_b$ and $\lambda \notin \sigma_{pp}(P)$ (the pure point spectrum of P),*

$$\mathrm{WF}(S(\lambda)u) = \mathrm{WF}(u),$$

where $\mathrm{WF}(u)$ *denotes the wave front set of u.*

Remark 4. The case where $M = M_C \cup M_\infty$ is a Riemannian manifold, the metric on M_∞ is "close" to a warped product of $\mathbb{R}_+$ and a compact manifold ∂M, and P is the Laplace operator, fits into our framework. The corresponding metric near infinity is $dr^2 + r^{2\alpha} d\theta^2$, and the function r^α varies proportionate to the size of $M_\infty = \mathbb{R}_+ \times \partial M$ at r.

Remark 5. The case where $\alpha = 1$ corresponds to scattering manifolds including asymptotically Euclidean spaces. Melrose-Zworski [26] and Ito-Nakamura [17] showed the propagation of wave front set by scattering matrices for Schrödinger equations on scattering manifolds: Let

$$q(\theta, \omega) = \frac{1}{2} \sum_{j,k} h^{jk}(\theta) \omega_j \omega_k$$

be the classical Hamiltonian associated to Q on $T^*(\partial M)$. We denote the Hamilton flow generated by b by $\exp(tH_b)$ for $t \in \mathbb{R}$. Then

$$\mathrm{WF}(S(\lambda)u) = \exp(\pi H_{\sqrt{2q}}) \mathrm{WF}(u),$$

when $\alpha = 1$. If $M = \mathbb{R}^n$ and the Hamiltonian P is a short-range perturbation of the Laplacian $-\frac{1}{2}\Delta$, then the canonical map $\exp(\pi H_{\sqrt{2q}})$ is the antipodal map on $T^*(S^{n-1})$.

The main idea to prove Theorem 11.2 is to consider the evolution:

$$A(t) = e^{itP_f/h^{1/\alpha}} J^* e^{-itP/h^{1/\alpha}} a(h^{1/\alpha} r, D_r, \theta, hD_\theta) e^{itP/h^{1/\alpha}} J e^{-itP_f/h^{1/\alpha}}$$

with some symbol a. We use an semi-classical Egorov type theorem argument for this time-dependent operator (see [17], also see the textbook by Martinez [24]). We consider $W(t) = e^{itP/h^{1/\alpha}} J e^{-itP_f/h^{1/\alpha}}$ as a time-evolution, and construct an asymptotic solution of a Heisenberg equation which is very close to $A(t)$. The construction of the asymptotic solution relies on the classical Hamilton flow generated by $p_k = \frac{1}{2}\rho^2 + k(r)q(\theta, \omega)$. The classical scattering operator has a scaling property, and its semi-classical limit satisfies for $(r_-, \rho_-, \theta_-, \omega_-) \in T^*\mathbb{R}_- \times (T^*\partial M \setminus \{0\})$,

$$\lim_{h \to +0} (\Pi_\theta, h\Pi_\omega) s_k(h^{-1/\alpha} r_-, \rho_-, \theta_-, h^{-1}\omega_-) = (\theta_-, \omega_-),$$

where s_k is the classical scattering operator and Π_* is a projection to $*$-variable. Thus, one may consider our results as a quantization of the classical mechanical scattering on manifolds with polynomially growing ends of growing order $\alpha > 1$. We note that on scattering manifolds, (θ, ω)-components of classical scattering operator is $\exp(\pi H_{\sqrt{2q}})$ and the wave front set is propagated along this map.

The paper is constructed as follows. In Section 12, we discuss Hamilton flows generated by $p_k = \frac{1}{2}\rho^2 + k(r)q(\theta, \omega)$ and classical scattering theory. We show scaling properties for solutions of classical trajectories and for classical scattering operators. In section 13, we prepare symbol calculus and Weyl quantization on manifolds. In Section 14, we discuss an Egorov type theorem and the construction of asymptotic solutions. We prove Theorem 11.2 in Section 15.

12 Classical scattering and scaling properties

We consider the classical mechanics for the Hamiltonian with "polynomially growing ends" structure on $T^*(M_\infty)$ where $M_\infty = \mathbb{R} \times \partial M$. We assume $\alpha > \frac{1}{2}$ in this section.

12.1 Classical trajectories

We set

$$p_k(r,\rho,\theta,\omega) = \frac{1}{2}\rho^2 + k(r)q(\theta,\omega),$$
$$q(\theta,\omega) = \frac{1}{2}\sum_{j,k} h^{jk}(\theta)\omega_j\omega_k,$$
$$k(r) = r^{-2\alpha},$$

on $T^*M_\infty \cong T^*\mathbb{R}_+ \times T^*\partial M$. We consider the Hamilton flow

$$(r(t),\rho(t),\theta(t),\omega(t)) = \exp(tH_{p_k})(r_0,\rho_0,\theta_0,\omega_0)$$

which starts from $(r_0,\rho_0,\theta_0,\omega_0) \in T^*\mathbb{R}_+ \times (T^*\partial M \setminus 0)$. We assume $\omega_0 \neq 0$. It satisfies the Hamilton equation:

$$r'(t) = \frac{\partial p_k}{\partial \rho} = \rho(t),$$
$$\rho'(t) = -\frac{\partial p_k}{\partial r} = -k'(r(t))q(\theta(t),\omega(t)) = 2\alpha r(t)^{-1-2\alpha}q(\theta(t),\omega(t)),$$
$$\theta'(t) = \frac{\partial p_k}{\partial \omega} = k(r(t))\frac{\partial q}{\partial \omega}(\theta(t),\omega(t)) = r(t)^{-2\alpha}\frac{\partial q}{\partial \omega}(\theta(t),\omega(t)),$$
$$\omega'(t) = -\frac{\partial p_k}{\partial \theta} = -k(r(t))\frac{\partial q}{\partial \theta}(\theta(t),\omega(t)) = r(t)^{-2\alpha}\frac{\partial q}{\partial \theta}(\theta(t),\omega(t)).$$

The solution has two invariants: the total energy $E_0 = p_k(r_0,\rho_0,\theta_0,\omega_0)$ and the angular momentum $q_0 = q(\theta_0,\omega_0)$. Then $(r(t),\rho(t))$ satisfies

$$r'(t) = \rho(t),$$
$$\rho'(t) - k'(r(t))q(\theta(t),\omega(t)) = 2\alpha r(t)^{-1-2\alpha}q_0,$$

which depends on $(\theta(t),\omega(t))$ only through q_0. Or we can say that $(r(t),\rho(t))$ obeys the reduced 1-dimensional classical Hamiltonian $\frac{1}{2}\rho^2 + k(r)q_0$. We set

$$\tau(t) := \tau(t,r_0,\rho_0,\theta_0,\omega_0) := \int_0^t k(\Pi_r \exp(sH_{p_k})(r_0,\rho_0,\theta_0,\omega_0))ds.$$

Then $(\theta(t),\omega(t))$ satisfies

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial q}{\partial\omega}(\theta,\omega), \quad \frac{\partial\omega}{\partial\tau} = -\frac{\partial q}{\partial\theta}(\theta,\omega).$$

and $(\theta(t),\omega(t)) = \exp(\tau(t)H_q)(\theta_0,\omega_0)$. We set

$$\sigma(t) := \sigma(t;r_0,\rho_0,\theta_0,\omega_0) = \sqrt{2q(\theta_0,\omega_0)}\,\tau(t,r_0,\rho_0,\theta_0,\omega_0).$$

Then

$$\frac{\partial\theta}{\partial\sigma} = \frac{\partial\sqrt{2q}}{\partial\omega}(\theta,\omega),\quad \frac{\partial\omega}{\partial\sigma} = -\frac{\partial\sqrt{2q}}{\partial\theta}(\theta,\omega).$$

and $(\theta(t),\omega(t)) = \exp(\sigma(t)H_{\sqrt{2q}})(\theta_0,\omega_0)$. Note that $\exp(\sigma(t)H_{\sqrt{2q}})$ is the geodesic flow on ∂M with respect to the (co)metric $(h^{jk}(\theta))$ on $T^*(\partial M)$.

12.2 Classical wave operators and scattering operators

We set

$$w_k^{-1}(t) = \exp(-tH_{p_f})\circ\exp(tH_{p_k})$$

where $p_f(r,\rho,\theta,\omega) = \frac{1}{2}\rho^2$ is the classical free Hamiltonian on $T^*M_f = T^*\mathbb{R}\times T^*\partial M$ and we consider the limit $t\longrightarrow\pm\infty$. Here we naturally identify $\mathbb{R}_+\times\mathbb{R}\times(T^*\partial M\setminus 0)$ as a subset of T^*M_f. Since $(r(t),\rho(t))$ obeys the reduced 1-dimensional classical Hamiltonian with short-range potential $\frac{1}{2}\rho^2+k(r)q_0$, it is easy to see the inverse wave operators $\lim_{t\to\pm\infty}\Pi_{r,\rho}w_k^{-1}(t)$ exist, where, Π_* denotes a projection to $*$-variable. $r(t)$ satisfies $r(t)\geq c\langle t\rangle$ for some $c>0$, which implies $|k(r(t))|\leq C\langle t\rangle^{-2\alpha}$. Hence $k(r(t))$ is absolutely integrable over $\mathbb{R}$ and the limit

$$\tau_\pm(r_0,\rho_0,\theta_0,\omega_0) = \lim_{t\to\pm\infty}\tau(t,r_0,\rho_0,\theta_0,\omega_0) = \int_0^{\pm\infty}k(\Pi_r\exp(sH_{p_k})(r_0,\rho_0,\theta_0,\omega_0))ds$$
$$\sigma_\pm(r_0,\rho_0,\theta_0,\omega_0) = \lim_{t\to\pm\infty}\sigma(t,r_0,\rho_0,\theta_0,\omega_0) = \sqrt{2q(\theta_0,\omega_0)}\,\tau_\pm(r_0,\rho_0,\theta_0,\omega_0)$$

exist. Hence the classical inverse wave operator

$$w_{k,\pm}^{-1} = \lim_{t\to\infty}w_k^{-1}(t)$$

exist and are diffeomorphic from $\mathbb{R}_+\times\mathbb{R}\times(T^*\partial M\setminus 0)$ to $\mathbb{R}\times\mathbb{R}_\pm\times(T^*\partial M\setminus 0)$, and the following formula holds:

$$\Pi_{\theta,\omega}w_{k,\pm}^{-1}(r_0,\rho_0,\theta_0,\omega_0) = \exp(\sigma_\pm(r_0,\rho_0,\theta_0,\omega_0)H_{\sqrt{2q}})(\theta_0,\omega_0).$$

Let $U\subset\mathbb{R}_+\times\mathbb{R}\times(T^*\partial M\setminus 0)$ be a compact domain. Then the convergence of $w_k^{-1}(t)$ to $w_\pm^{-1}$ is uniform on U with all the derivatives. Since the limit is diffeomorphic, its inverse $w_k(t)$ has the same property on $w_k^{-1}(t)U$. In particular, $w_k^{-1}(t)$ on U, and $w_k(t)$ on $w_k^{-1}(t)U$ are uniformly bounded in t with all the derivatives.

Note that we here consider a two-space classical scattering with a natural inclusion $\mathbb{R}_+\times\mathbb{R}\times(T^*\partial M\setminus 0)\hookrightarrow T^*M_f$. $w_k(t)$ is well-defined only for t near $+\infty$ if $\rho>0$, and for t near $-\infty$ if $\rho<0$. This is the reason why above discussions about domains are a little bit delicate.

We set the classical scattering operator

$$s_k = w_{k,+}^{-1}\circ w_{k,-}.$$

We set

$$r_\infty(r_-,\rho_-,\theta_-,\omega_-) = r_- + \Pi_r s_k(r_-,\rho_-,\theta_-,\omega_-),$$

and

$$\sigma_\infty(r_-,\rho_-,\theta_-,\omega_-) = \sigma_+(w_{k,-}(r_-,\rho_-,\theta_-,\omega_-)) - \sigma_-(w_{k,-}(r_-,\rho_-,\theta_-,\omega_-)),$$

for $(r_-,\rho_-,\theta_-,\omega_-) \in \mathbb{R}\times\mathbb{R}_- \times (T^*\partial M\setminus 0)$. Then we have

$$\Pi_{\theta,\omega}s_k(r_-,\rho_-,\theta_-,\omega_-) = \exp(\sigma_\infty(r_-,\rho_-,\theta_-,\omega_-)H_{\sqrt{2q}})(\theta_-,\omega_-).$$

Proposition 12.1. *$r_\infty(r_-,\rho_-,\theta_-,\omega_-)$ and $\sigma_\infty(r_-,\rho_-,\theta_-,\omega_-)$ take the same values as long as the ratio of the angular momentum to the total energy $q(\theta_-,\omega_-)/\frac{1}{2}\rho_-^2$ is a constant.*

Proof. Let $(r_-,\rho_-,\theta_-,\omega_-) \in \mathbb{R}\times\mathbb{R}_- \times (T^*\partial M\setminus 0)$ and

$$(r_0,\rho_0,\theta_0,\omega_0) = w_{k,-}(r_-,\rho_-,\theta_-,\omega_-),$$
$$(r(t),\rho(t),\theta(t),\omega(t)) = \exp(tH_{p_k})(r_0,\rho_0,\theta_0,\omega_0).$$

Then we have the formula

$$\sigma_\infty(r_-,\rho_-,\theta_-,\omega_-) = \int_{\mathbb{R}} k(r(t))dt.$$

Consider another $(\tilde{r}_-,\tilde{\rho}_-,\tilde{\theta}_-,\tilde{\omega}_-) \in \mathbb{R}\times\mathbb{R}_- \times (T^*\partial M\setminus 0)$ but with the same angular momentum $q(\theta_0,\omega_0) = q(\theta_-,\omega_-) = q(\tilde{\theta}_-,\tilde{\omega}_-)$ and the same total energy $p_k(r_0,\rho_0,\theta_0,\omega_0) = \rho_-^2/2 = \tilde{\rho}_-^2/2$. The corresponding classical trajectory $\tilde{r}(t)$ can be written as a time translation of $r(t)$: $\tilde{r}(t) = r(t+c)$ for some c. By changing the variables in the above formula, we learn that r_∞ and σ_∞ depend only on the angular momentum $q(\theta_0,\omega_0) = q(\theta_-,\omega_-)$ and the total energy $p_k(r_0,\rho_0,\theta_0,\omega_0) = \rho_-^2/2$.

Every trajectory has a unique time t_* when the radial momentum $\rho(t_*)$ becomes zero. We call $r_* = r(t_*)$ the turning point of the trajectory. We see that

$$r_\infty(r_-,\rho_-,\theta_-,\omega_-) = 2\Pi_r w_{k,+}^{-1}(r_*,0,\theta_0,\omega_0) =: 2r_{+,\infty}(r_*,\theta_0,\omega_0),$$
$$\sigma_+(r_*,0,\theta_0,\omega_0) = -\sigma_-(r_*,0,\theta_0,\omega_0),\ \sigma_\infty(r_-,\rho_-,\theta_-,\omega_-) = 2\sigma_+(r_*,0,\theta_0,\omega_0).$$

There is a one to one correspondence between the turning point r_* and the ratio of the angular momentum to the total energy because the relationship $k(r_*)q = p_k$ holds. It is now enough to show that $r_{+,\infty}(r_*,\theta_0,\omega_0)$ and $\sigma_\pm(r_*,0,\theta_0,\omega_0)$ are constants for every $(\theta,\omega) \in T^*\partial M\setminus 0$ if the turning point $r_* > 0$ is fixed. So the Proposition follows from the following lemma. □

Lemma 12.2. *Fix the turning point $r_* > 0$. Then*

$$r_{+,\infty}(r_*,\theta_0,\omega_0) = r_{+,\infty}(r_*,\tilde{\theta}_0,\tilde{\omega}_0),\ \sigma_\pm(r_*,0,\theta_0,\omega_0) = \sigma_\pm(r_*,0,\tilde{\theta}_0,\tilde{\omega}_0)$$

for any $(\theta_0,\omega_0),(\tilde{\theta}_0,\tilde{\omega}_0) \in T^*\partial M\setminus 0$.

Proof. Set $q_0 = q(\theta_0, \omega_0)$ and $\tilde{q}_0 = q(\tilde{\theta}_0, \tilde{\omega}_0)$. Take $\lambda > 0$ such that $\tilde{q}_0 = \lambda^2 q_0$. Denote $r(t)$ be the solution starting at $(r_*, 0, \theta_0, \omega_0)$ and set $\tilde{r}(t) = r(\lambda t)$. Then

$$\tilde{r}''(t) = \lambda^2 r''(\lambda t) = \lambda^2(-\frac{\partial k}{\partial r}(r(\lambda t)))q_0 = -\frac{\partial k}{\partial r}(\tilde{r}(\lambda t))\tilde{q}_0.$$

Hence $\tilde{r}(t)$ is the solution starting at $(r_*, 0, \tilde{\theta}_0, \tilde{\omega}_0)$. We note that $\tilde{\rho}_+ = \lambda \rho_+$, where, ρ_+ and $\tilde{\rho}_+$ are the corresponding radial momentum of the scattering data. Recall that $\Pi_r w_{k,+}^{-1}(r_*, 0, \theta_0, \omega_0) = r_{+,\infty}(r_*, \theta_0, \omega_0)$ implies

$$r(t) - (r_{+,\infty}(r_*, \theta_0, \omega_0) + \rho_+ t) \to 0 \quad \text{as } t \to \infty.$$

Since

$$\tilde{r}(t) - (r_{+,\infty}(r_*, \theta_0, \omega_0) + \tilde{\rho}_+ t) = r(\lambda t) - (r_{+,\infty}(r_*, \theta_0, \omega_0) + \lambda \rho_+ t) \to 0 \quad \text{as } t \to \infty,$$

we conclude $r_{+,\infty}(r_*, \theta_0, \omega_0) = r_{+,\infty}(r_*, \tilde{\theta}_0, \tilde{\omega}_0)$. Compute

$$\begin{aligned} &\sigma_\pm(r_*, 0, \tilde{\theta}_0, \tilde{\omega}_0) = \sqrt{2\tilde{q}_0}\int_0^\infty k(\tilde{r}(s))ds = \lambda\sqrt{2q_0}\int_0^\infty k(r(\lambda s))ds \\ &= \sqrt{2q_0}\int_0^\infty k(r(u))du = \sigma_\pm(r_*, 0, \theta_0, \omega_0). \end{aligned}$$

Here we used the change of variables $\lambda s = u$. □

12.3 Scaling property

We here show that classical trajectories $\exp(tH_{p_k})$, classical wave operators $w_{k,\pm}$, and classical scattering operators s_k have a scaling property with the (semi-classical) scaling parameter $h > 0$. We will consider the scaling

$$(r, \omega) \mapsto (h^{-1/\alpha} r, h^{-1}\omega).$$

We note that this scaling preserves the total energy p_k. We choose this scaling so that ω is scaled in a standard way, because we investigate the asymptotic behavior of quantum scattering matrix as $|\omega| \to \infty$.

Proposition 12.3. *Let $h > 0$ and $(r_0, \rho_0, \theta_0, \omega_0) \in T^*\mathbb{R}_+ \times (T^*\partial M \setminus 0)$. Then the classical trajectories satisfy the following scaling properties:*

$$\begin{aligned} &(\Pi_r, \Pi_\rho)(\exp(h^{-1/\alpha} t H_{p_k})(h^{-1/\alpha} r_0, \rho_0, \theta_0, h^{-1}\omega_0)) \\ &= (h^{-1/\alpha}\Pi_r, \Pi_\rho)(\exp(tH_{p_k})(r_0, \rho_0, \theta_0, \omega_0)), \\ &\tau(h^{-1/\alpha} t, h^{-1/\alpha} r_0, \rho_0, \theta_0, h^{-1}\omega_0) = h^{2-1/\alpha}\tau(t, r_0, \rho_0, \theta_0, \omega_0), \\ &\sigma(h^{-1/\alpha} t, h^{-1/\alpha} r_0, \rho_0, \theta_0, h^{-1}\omega_0) = h^{1-1/\alpha}\sigma(t, r_0, \rho_0, \theta_0, \omega_0), \\ &(\Pi_\theta, \Pi_\omega)(\exp(h^{-1/\alpha} t H_{p_k})(h^{-1/\alpha} r_0, \rho_0, \theta_0, h^{-1}\omega_0)) \\ &= (\Pi_\theta, h^{-1}\Pi_\omega)(h^{1-1/\alpha}\sigma(t, r_0, \rho_0, \theta_0, \omega_0)\exp(tH_{\sqrt{2q}})(\theta_0, \omega_0)). \end{aligned}$$

Proof. Set $q_0 = q(\theta_0, \omega_0) > 0$. Since q is homogeneous of order 2 with respect to ω variable, we have

$$q(\theta_0, h^{-1}\omega) = h^{-2}q_0.$$

Compute

$$\partial_t h^{-1/\alpha}\Pi_r \exp(tH_{p_k})(r_0,\rho_0,\theta_0,\omega_0) = h^{-1/\alpha}\Pi_\rho \exp(tH_{p_k})(r_0,\rho_0,\theta_0,\omega_0),$$

and

$$\begin{aligned}
&\partial_t\Pi_\rho \exp(tH_{p_k})(r_0,\rho_0,\theta_0,\omega_0) = (-k')(\Pi_r \exp(tH_{p_k})(r_0,\rho_0,\theta_0,\omega_0))q_0\\
&= 2\alpha(\Pi_r \exp(tH_{p_k})(r_0,\rho_0,\theta_0,\omega_0))^{-1-2\alpha}q_0\\
&= h^{-1/\alpha}2\alpha(h^{-1/\alpha}\Pi_r \exp(tH_{p_k})(r_0,\rho_0,\theta_0,\omega_0))^{-1-2\alpha}h^{-2}q_0\\
&= h^{-1/\alpha}(-k')(h^{-1/\alpha}\Pi_r \exp(tH_{p_k})(r_0,\rho_0,\theta_0,\omega_0))h^{-2}q_0.
\end{aligned}$$

This implies the first equation.

We compute τ.

$$\begin{aligned}
&\tau(h^{-1/\alpha}t, h^{-1/\alpha}r_0,\rho_0,\theta_0,h^{-1}\omega_0)\\
&= \int_0^{h^{-1/\alpha}t} k(\Pi_r \exp(sH_{p_k})(h^{-1/\alpha}r_0,\rho_0,\theta_0,h^{-1}\omega_0)ds\\
&= \int_0^{h^{-1/\alpha}t} (h^{-1/\alpha}\Pi_r \exp(h^{1/\alpha}sH_{p_k})(r_0,\rho_0,\theta_0,\omega_0))^{-2\alpha}ds\\
&= \int_0^{t} h^2(\Pi_r \exp(uH_{p_k})(r_0,\rho_0,\theta_0,\omega_0))^{-2\alpha}h^{-1/\alpha}du\\
&= \int_0^{t} h^{2-1/\alpha}k(\Pi_r \exp(uH_{p_k})(r_0,\rho_0,\theta_0,\omega_0))du\\
&= h^{2-1/\alpha}\tau(t,r_0,\rho_0,\theta_0,\omega_0).
\end{aligned}$$

We compute σ.

$$\begin{aligned}
&\sigma(h^{-1/\alpha}t, h^{-1/\alpha}r_0,\rho_0,\theta_0,h^{-1}\omega_0) = \sqrt{q(\theta_0,h^{-1}\omega_0)}\,\tau(h^{-1/\alpha}t,h^{-1/\alpha}r_0,\rho_0,\theta_0,h^{-1}\omega_0)\\
&= h^{-1}h^{2-1/\alpha}q_0\tau(t,r_0,\rho_0,\theta_0,\omega_0) = h^{1-1/\alpha}\sigma(t,r_0,\rho_0,\theta_0,\omega_0).
\end{aligned}$$

□

Proposition 12.4. *Let $h > 0$, $(r_0,\rho_0,\theta_0,\omega_0) \in T^*\mathbb{R}_+ \times (T^*\partial M \setminus 0)$, and $(r_-,\rho_-,\theta_-,\omega_-) \in \mathbb{R} \times \mathbb{R}_- \times (T^*\partial M \setminus 0)$. Then the classical wave operators and scattering operators satisfy the following scaling properties:*

$$\begin{aligned}
&w_k^{-1}(h^{-1/\alpha}t)(h^{-1/\alpha}r_0,\rho_0,\theta_0,h^{-1}\omega_0)\\
&= ((h^{-1/\alpha}\Pi_r,\Pi_\rho)w_k^{-1}(t)(r_0,\rho_0,\theta_0,\omega_0),\\
&(\Pi_\theta,h^{-1}\Pi_\omega)\exp(h^{1-1/\alpha}\sigma(t,r_0,\rho_0,\theta_0,\omega_0)H_{\sqrt{2q}})(\theta_0,\omega_0)).
\end{aligned}$$

$$\begin{aligned}
& w_{k,\pm}^{-1}(h^{-1/\alpha}r_0,\rho_0,\theta_0,h^{-1}\omega_0) \\
&= ((h^{-1/\alpha}\Pi_r,\Pi_\rho)w_{k,\pm}^{-1}(r_0,\rho_0,\theta_0,\omega_0), \\
&(\Pi_\theta,h^{-1}\Pi_\omega)\exp(h^{1-1/\alpha}\sigma_\pm(r_0,\rho_0,\theta_0,\omega_0)H_{\sqrt{2q}})(\theta_0,\omega_0)).
\end{aligned}$$

$$\begin{aligned}
& s_k(h^{-1/\alpha}r_-,\rho_-,\theta_-,h^{-1}\omega_-) \\
&= (h^{-1/\alpha}(r_\infty(r_-,\rho_-,\theta_-,\omega_-)-r_-),-\rho_-, \\
&(\Pi_\theta,h^{-1}\Pi_\omega)\exp(h^{1-1/\alpha}\sigma_\infty(r_-,\rho_-,\theta_-,\omega_-)H_{\sqrt{2q}})(\theta_-,\omega_-)).
\end{aligned}$$

Proof. The proof is straightforward. □

13 Symbol classes and their quantization

We prepare pseudodifferential operator calculus. Most of the discussions and notations here are the same as in [18]. But since we assume a different relationships between G and H, we have different formulas for Weyl quantization. We also refer textbooks by Höremander [14] for the standard theory of microlocal analysis.

We employ symbol calculus on T^*M, but we always suppose that the symbol is supported in T^*M_∞, and choose the local coordinate system as in Section 11. We also consider symbols on T^*M_f.

13.1 Symbol classes

We set a metric on T^*M_∞(or T^*M_f)defined by

$$g_1 = \frac{dr^2}{\langle r\rangle^2} + d\rho^2 + d\theta^2 + \frac{d\omega^2}{\langle\omega\rangle^2},$$

and consider symbols in $S(m,g_1)$ with a weight function m, i.e., $a \in S(m,g_1)$ if $a \in C^\infty$ and for any indices k,l,γ,δ, there is $C_{k,l,\gamma,\delta}$ such that

$$|\partial_r^k\partial_\rho^l\partial_\theta^\gamma\partial_\omega^\delta a(r,\rho,\theta,\omega)| \le C_{k,l,\gamma,\delta}m(r,\rho,\theta,\omega)\langle r\rangle^{-k}\langle\omega\rangle^{-\delta}.$$

We will consider symbol calculus for symbols supported on sets of the form:

$$\Omega^{h,\alpha} = \{(r,\rho,\theta,\omega)|(h^{1/\alpha}r,\rho,\theta,h\omega) \in \Omega\}$$

with a fixed compact set $\Omega \subset T^*\mathbb{R}_+ \times (T^*\partial M \setminus 0)$ and a semiclassical small parameter $h > 0$. In such cases, the metric is

$$g_1^{h,\alpha} = h^{2/\alpha}dr^2 + d\rho^2 + d\theta^2 + h^2 d\omega^2.$$

The h-dependent symbol $a^h \in S(m, g_1^{h,\alpha})$ satisfies

$$|\partial_r^k\partial_\rho^l\partial_\theta^\gamma\partial_\omega^\delta a^h(r,\rho,\theta,\omega)| \le C_{k,l,\gamma,\delta}m(h)h^{k/\alpha+|\delta|}.$$

13.2 Weyl quantization

Let $\{\chi_\lambda\}$ be a partition of unity on ∂M compatible with our coordinate system $\{\psi_\lambda : U_\lambda\}$, i.e., $\chi_\lambda \in C_0^\infty(U_\lambda)$ and $\sum_\lambda \chi_\lambda(\theta)^2 = 1$ on ∂M. Let $a \in S(m, g_1)$ be a symbol on $T^*(M_f)$, and let $u \in C_0^\infty(M_f)$. We denote by a_α and H_α the representation of u and H in the local coordinate $(1 \otimes \psi_\lambda, \mathbb{R} \times U_\lambda)$, respectively. We quantize it by

$$\mathrm{Op}_{M_f}^W(a)u = \sum_\lambda \chi_\lambda H_\lambda^{-1/2} a_\lambda^W(r, D_r, \theta, D_\theta) H_\lambda^{1/2} \chi_\lambda u.$$

Here $a_\lambda^W(r, D_r, \theta, D_\theta)$ denotes the usual Weyl quantization on the Euclidean space $\mathbb{R}^n$, and we identify $\mathbb{R} \times U_\lambda$ with $\mathbb{R} \times \psi_\lambda(U_\lambda)$ and denote pull-backs and push-forwards of symbols and functions by the same notations.

Similarly, for a symbol a on T^*M_∞, we quantize it by

$$\mathrm{Op}_M^W(a)u = \sum_\lambda j\chi_\lambda G_\lambda^{-1/2} a_\lambda^W(r, D_r, \theta, D_\theta) G_\lambda^{1/2} j\chi_\lambda u.$$

Here $u \in C_0^\infty(M)$, G_λ is the representation of G in the local coordinate $\{\psi_\lambda : U_\lambda\}$, and we identify the symbol a with a symbol on T^*M_f by the obvious way and denote it by the same symbol. Since $G(x)dx = H(\theta)drd\theta$ on M_∞, we have

$$\begin{aligned}\mathrm{Op}_M^W(a)u &= \sum_\lambda j\chi_\lambda H_\lambda^{-1/2} a_\lambda^W(r, D_r, \theta, D_\theta) H_\lambda^{1/2} j\chi_\lambda u \\ &= J\,\mathrm{Op}_{M_f}^W(a)J^*u.\end{aligned}$$

So we may identify these quantizations by using J. We omit the subscript M or M_f in these Weyl quantizations for such symbols a supported in $\{r > 0\}$.

We may consider $\mathrm{Op}^W(a)$ as an operator from $\mathscr{H}$ to $\mathscr{H}_f$ by multiplying J^* from the right: $\mathrm{Op}_{M_f}^W(a)J^* : C_0^\infty(M) \to \mathscr{H}_f$. We may also consider $\mathrm{Op}^W(a)$ as an operator from $\mathscr{H}_f$ to $\mathscr{H}$ by multiplying J^* from the left: $J\,\mathrm{Op}_{M_f}^W(a) : C_0^\infty(M_f) \to \mathscr{H}$.

By virtue of the weights put around the locally defined pseudo differential operators, $\mathrm{Op}^W(a)$ is symmetric if a is real-valued.

If $A = \mathrm{Op}^W(a)$, then we denote the Weyl symbol of A by $a = \Sigma(A)$.

We denote the class of pseudodifferential operators with Weyl symbols in $S(m, g)$ by $\Psi(m, g)$. The class $\Psi(h^m, g_1^{h,\alpha})$ has the following properties:

Proposition 13.1. *Suppose* $A_i = \mathrm{Op}^W(a_i^h) \in \Psi(h^{m_i}, g_1^{h,\alpha})$, $i = 1, 2$. *Then*

$$\begin{aligned}&A_1A_2 \in \Psi(h^{m_1+h_2}, g_1^{h,\alpha}),\\ &[A_1, A_2] \in \Psi(h^{m_1+h_2-1/\alpha}, g_1^{h,\alpha}),\\ &[A_1, A_2] - i^{-1}\mathrm{Op}^W(\{a_1^h, a_2^h\}) \in \Psi(h^{m_1+m_2-2/\alpha}, g_1^{h,\alpha}),\end{aligned}$$

where $\{\cdot,\cdot\}$ *is the Poisson bracket:*

$$\{a_1, a_2\} = \frac{\partial a_1}{\partial \rho}\frac{\partial a_2}{\partial r} - \frac{\partial a_1}{\partial r}\frac{\partial a_2}{\partial \rho} + \frac{\partial a_1}{\partial \omega}\frac{\partial a_2}{\partial \theta} - \frac{\partial a_1}{\partial \theta}\frac{\partial a_2}{\partial \omega}. \tag{13.1}$$

13.3 Hamiltonians

Now we consider the Weyl quantization of classical hamiltonians and related Schrödinger operators.

We first note that if $a(x,\xi) = \sum_{j,k} a_{jk}(x)\xi_j\xi_k$, then

$$a^W(x,D_x) = \sum_{j,k} D_j a_{jk}(x) D_k - \frac{1}{4}\sum_{j,k}(\partial_j\partial_k a_{jk})(x).$$

We set the symbol of the full Hamiltonian

$$p(r,\rho,\theta,\omega) = \frac{1}{2}(\partial_r, \sqrt{k}\omega)\begin{pmatrix} 1+a_1 & a_2 \\ {}^t a_2 & h+a_3 \end{pmatrix}\begin{pmatrix} \partial_r \\ \sqrt{k}\omega \end{pmatrix} + V$$

on T^*M_∞.

Then we have

$$\mathrm{Op}^W(p) = P + f,$$

with $f \in C^\infty$ such that

$$|\partial_r^k \partial_\theta^\gamma f(r,\theta)| \le C_{k,\gamma}\langle r\rangle^{-\min\{2+\mu_1, 1+\mu_2+\alpha, \mu_3+2\alpha\}-k}.$$

So we can include this error term f into the potential perturbation term V, or even we may assume that $P = \mathrm{Op}^W(a)$.

14 Egorov theorem

In this section, we prove Egorov-type theorem. Our discussion is almost the same as [17] but the choice of the scaling for r, ω, and t is different. we employ the same scaling as in Section 12.

Suppose $a(h;r,\rho,\theta,\omega)$ is supported in a common compact subset $\Omega \subset T^*\mathbb{R}_+ \times (T^*\partial M \setminus 0)$ and is uniformly bounded in C_0^∞, namely, there is $C_{k,l,\gamma,\delta}$ such that for every $h > 0$,

$$|\partial_r^k \partial_\rho^l \partial_\theta^\gamma \partial_\omega^\delta a(h;r,\rho,\theta,\omega)| \le C_{k,l,\gamma,\delta}.$$

We set

$$a^h(r,\rho,\theta,\omega) = a(h; h^{1/\alpha}r, \rho, \theta, h\omega)$$

Then a^h is supported in $\Omega^{h,\alpha}$ and $a^h \in S(1, g_1^{h,\alpha})$. We set

$$A_0 = \mathrm{Op}^W(a^h).$$

We take $\varepsilon > 0$ such that

$$\exp(tH_{p_k})\Omega \cap \{r : r \le \varepsilon\langle t\rangle\} = \emptyset.$$

for all $t \in \mathbb{R}$. Then by the scaling property,

$$\exp(h^{-1/\alpha} t H_{p_k})\Omega^{h,\alpha} \cap \{r : r \leq \varepsilon\langle h^{-1/\alpha} t\rangle\} = \emptyset.$$

for $0 < h < 1$. We choose $\eta \in C^\infty(\mathbb{R})$ such that $\eta = 1$ for $r \geq 1$ and $\eta = 0$ for $r \leq 1/2$, and we set

$$Y = \eta\left(\frac{r}{\varepsilon\langle h^{-1/\alpha} t\rangle}\right).$$

We define

$$A(t) = e^{itP_f/h^{1/\alpha}} J^* Y e^{-itP/h^{1/\alpha}} A_0 e^{itP/h^{1/\alpha}} Y J e^{-itP_f/h^{1/\alpha}}$$

for $t \in \mathbb{R}$. The purpose of this section is to obtain the symbols of $A(t)$ as a pseudo differential operator, and to study its behavior as $t \to \pm\infty$.

We compute

$$\frac{d}{dt}\left(e^{itP/h^{1/\alpha}} Y J e^{-itP_f/h^{1/\alpha}}\right) = \frac{i}{h^{1/\alpha}}\left(e^{itP/h^{1/\alpha}} Y J e^{-itP_f/h^{1/\alpha}}\right) L(t) + R_1(t)$$

where

$$\begin{aligned} L(t) &= e^{itP_f/h^{1/\alpha}} J^* T(t) e^{-itP_f/h^{1/\alpha}}, \\ T(t) &= PYJ - JYP_f + i^{-1} h^{1/\alpha}\left(\frac{d}{dt}Y\right)J, \\ R_1(t) &= \frac{i}{h^{1/\alpha}} e^{itP/h^{1/\alpha}}(1 - YJJ^*)T(t)e^{-itP_f/h^{1/\alpha}}. \end{aligned}$$

We compute bounds for symbols of $T(t)$ and $L(t)$. We have

$$\begin{aligned} &|\partial_r^k \partial_\rho^l \partial_\theta^\gamma \partial_\omega^\delta \Sigma(T(t))(r,\rho,\theta,\omega)| \\ &\leq C\left(\langle r\rangle^{-\mu_V} + \langle r\rangle^{-\mu_1}\langle\rho\rangle^2 + \langle r\rangle^{-\mu_2-\alpha}\langle\rho\rangle\langle\omega\rangle + \langle r\rangle^{-2\alpha}\langle\omega\rangle^2\right)\langle r\rangle^{-k}\langle\rho\rangle^{-l}\langle\omega\rangle^{-|\delta|}. \end{aligned}$$

We also have

$$\begin{aligned} &|\partial_r^k \partial_\rho^l \partial_\theta^\gamma \partial_\omega^\delta \Sigma(T(t))(r,\rho,\theta,\omega) - Yk(r)q(\theta,\omega)| \\ &\leq C\left(\langle r\rangle^{-\mu_V} + \langle r\rangle^{-\mu_1}\langle\rho\rangle^2 + \langle r\rangle^{-\mu_2-\alpha}\langle\rho\rangle\langle\omega\rangle + \langle r\rangle^{-\mu_3-2\alpha}\langle\omega\rangle^2\right)\langle r\rangle^{-k}\langle\rho\rangle^{-l}\langle\omega\rangle^{-|\delta|}. \end{aligned}$$

In particular,

$$\begin{aligned} &|\partial_r^k \partial_\rho^l \partial_\theta^\gamma \partial_\omega^\delta \Sigma(T(t))(r,\rho,\theta,\omega) - Yk(r)q(\theta,\omega)| \\ &\leq C\langle t\rangle^{-\min\{\mu_V,\mu_1,\mu_2+\alpha,\mu_3+2\alpha\}-k} h^{\min\{\mu_V,\mu_1,\mu_2,\mu_3\}/\alpha + k/\alpha + |\delta|} \end{aligned}$$

on $\exp((t/h^{1/\alpha})H_{p_k})\Omega^{h,\alpha}\left(\supset \exp((t/h^{1/\alpha})H_{p_k})\operatorname{supp} a^h\right)$, uniformly for t and h, since $\langle r\rangle \sim h^{-1/\alpha}\langle t\rangle$ and $\langle\omega\rangle \sim h^{-1}$ there.

By the Weyl calculus, we have

$$\Sigma(L(t)) = \Sigma(J^*T(t))(r + (t/h^{1/\alpha})\rho, \rho, \theta, \omega).$$

Hence we have

$$\begin{aligned}&|\partial_r^k \partial_\rho^l \partial_\theta^\gamma \partial_\omega^\delta \Sigma(L(t))(r,\rho,\theta,\omega)| \\ &\leq C\left(\langle\tilde{r}\rangle^{-\mu_V} + \langle\tilde{r}\rangle^{-\mu_1}\langle\rho\rangle^2 + \langle\tilde{r}\rangle^{-\mu_2-\alpha}\langle\rho\rangle\langle\omega\rangle + \langle\tilde{r}\rangle^{-2\alpha}\langle\omega\rangle^2\right)\langle\tilde{r}\rangle^{-k}\langle\rho\rangle^{-l}\langle\omega\rangle^{-|\delta|}.\end{aligned}$$

with $\tilde{r} = r + (t/h^{1/\alpha})\rho$. We also have

$$\begin{aligned}&|\partial_r^k \partial_\rho^l \partial_\theta^\gamma \partial_\omega^\delta \Sigma(L(t))(r,\rho,\theta,\omega) - Yk(r + (t/h^{1/\alpha})\rho)q(\theta,\omega)| \\ &\leq C\langle t\rangle^{-\min\{\mu_V,\mu_1,\mu_2+\alpha,\mu_3+2\alpha\}-k} h^{\min\{\mu_V,\mu_1,\mu_2,\mu_3\}/\alpha+k/\alpha+|\delta|}\end{aligned} \tag{14.1}$$

on $\exp(-(t/h^{1/\alpha})H_f)\circ\exp(-(t/h^{1/\alpha})H_k)\Omega^{h,\alpha} = w_k^{-1}((t/h^{1/\alpha}))\Omega^{h,\alpha} \supset \mathrm{supp}(a^h \circ w_k(t/h^{1/\alpha}))$.

We set

$$l_0(t;r,\rho,\theta,\omega) = \eta\left(\frac{r+t\rho}{\varepsilon\langle t\rangle}\right)k(r+t\rho)q(\theta,\omega)$$

be the principal symbol of $L(t)$ with $h = 1$. We also set

$$b_0^h(t) = a^h \circ w_k\left(\frac{t}{h^{1/\alpha}}\right).$$

Then b_0^h have the following properties:

Lemma 14.1. *b_0^h satisfies the equation*

$$\frac{d}{dt}b_0^h + h^{-1/\alpha}\{l_0(h^{-1/\alpha}t), b_0^h(t)\} = 0$$

and $b_0^h \in S(1, g_1^{h,\alpha})$, i.e.,

$$|\partial_r^k \partial_\rho^l \partial_\theta^\gamma \partial_\omega^\delta b_0^h(r,\rho,\theta,\omega)| \leq C_{k,l,\gamma,\delta} h^{k/\alpha+|\delta|}.$$

Proof. By a standard calculation, we see that $l_0(t)$ is the generator of $w_k^{-1}(t)$ on Ω, i.e.,

$$\frac{d}{dt}w_k^{-1}(t)(r,\rho,\theta,\omega) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix}\partial_r \\ \partial_\rho \\ \partial_\theta \\ \partial_\omega\end{pmatrix} l_0(t)(w_k^{-1}(t)(r,\rho,\theta,\omega)). \tag{14.2}$$

for $(r,\rho,\theta,\omega) \in \Omega$. We will prove r- component of this equation. Other components are similar to prove. We compute

$$\begin{aligned}&\frac{d}{dt}\Pi_r w_k^{-1}(t)(\cdot) = \frac{d}{dt}\left[\Pi_r \exp(tH_{p_k})(\cdot) - t\Pi_\rho \exp(tH_{p_k})(\cdot)\right] \\ &= \frac{\partial p_k}{\partial \rho}(\cdot) - \Pi_\rho \exp(tH_{p_k})(\cdot) + t\frac{\partial p_k}{\partial r}(\exp(tH_{p_k}(\cdot)) = t\frac{\partial p_k}{\partial r}(\exp(tH_{p_k}(\cdot)).\end{aligned}$$

Since

$$\frac{\partial l_0}{\partial \rho}(t;r,\rho,\theta,\omega) = \frac{\partial p_k}{\partial r}(r+t\rho,\rho,\theta,\omega)\cdot t,$$

we have

$$\frac{\partial l_0}{\partial \rho}(t;w_k^{-1}(t)(\cdot)) = t\frac{\partial p_k}{\partial r}(\exp(tH_{p_k})(\cdot)).$$

And the cut-off $\eta = 1$ on $w_k^{-1}(t)\Omega$. This ends the proof of (14.2).

Now we differentiate the equation

$$a^h = b_0^h(t)\circ w_k^{-1}(h^{-1/\alpha}t)$$

with respect to t:

$$0 = \frac{d}{dt}a^h(\cdot) = \frac{d}{dt}b_0^h(t;w_k^{-1}(h^{-1/\alpha}t)(\cdot)) + h^{-1/\alpha}\left(\partial_r\partial_\rho\partial_\theta\partial_\omega\right)b_0^h(t,w_k^{-1}(h^{-1/\alpha}t)(\cdot))$$
$$\cdot\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}\begin{pmatrix}\partial_r\\ \partial_\rho\\ \partial_\theta\\ \partial_\omega\end{pmatrix} l_0(h^{-1/\alpha}t)(w_k^{-1}(h^{-1/\alpha}t)(\cdot)).$$

Hence we have

$$\frac{d}{dt}b_0^h + h^{-1/\alpha}\{l_0(h^{-1/\alpha}t), b_0^h(t)\} = 0.$$

Now let $b_0(t;h;h^{1/\alpha}r,\rho,\theta,h\omega) = b_0^h(t;r,\rho,\theta,\omega)$. Then we have

$$b_0(t;h;r,\rho,\theta,\omega) = a(\Pi_{r,\rho}w_k(t)(r,\rho,\theta,\omega),\exp(-h^{1-1/\alpha}\sigma(t,r+t\rho,\rho,\theta,\omega)H_{\sqrt{2q}}(\theta,\omega)).$$

Hence we have

$$|\partial_r^k\partial_\rho^l\partial_\theta^\gamma\partial_\omega^\delta b_0(t;h;r,\rho,\theta,\omega)| \leq C_{k,l,\gamma,\delta}$$

uniformly in t and h, which shows the desired estimates. □

Using these estimates and properties of classical trajectories, we will construct an asymptotic solution to the Heisenberg equation:

$$\frac{d}{dt}B(t) = -\frac{1}{h^{1/\alpha}}[L(t),B(t)], B(0) = J^*A_0J.$$

Lemma 14.2. *Let* $\mu_h = \min\{\mu_V,\mu_1,\mu_2,\mu_3\}/\alpha > 0$ *and* $\mu_t = \min\{\mu_V,\mu_1,\mu_2+\alpha,\mu_3+2\alpha\} - 1 > 0$. *There exists* $b^h(t;r,\rho,\theta,\omega) \in C_0^\infty(T^*M_f)$ *such that*

(i). $b^h(0) = a^h$.

(ii). $b^h(t)$ *is supported in* $w_k(t/h^{1/\alpha})^{-1}\operatorname{supp} a^h$.

(iii). $b^h(t) \in S(1, g_1^{h,\alpha})$, *and it is bounded uniformly in* $t \in \mathbb{R}$.

(iv). $b^h(t) - a^h \circ w_k(t/h^{1/\alpha}) \in S(h^{\mu_h}, g_1^{\alpha})$, *i.e., the princpal symbol of* $b^h(t)$ *is given by* $a^h \circ w_k(t/h^{1/\alpha})$, *and the remainder is bounded uniformly in* t.

(v). If we set $B(t) = \mathrm{Op}^W(b^h(t))$, *then for any* N, *there exists* $C_N > 0$ *such that*

$$\|\frac{d}{dt}B(t) + \frac{i}{h^{1/\alpha}}[L(t), B(t)]\| \le C_N \langle t\rangle^{-1-\mu_t} h^N, \ h > 0.$$

(vi). $B(t)$ *converges to* $B_\pm$ *as* $t \to \pm\infty$ *in* $B(\mathscr{H}_f)$, *and the symbols* $b_\pm^h := \Sigma(B_\pm)$ *satisfy*

$$b_\pm^h - a^h \circ w_{k,\pm} \in S(h^{\mu_h}, g_1^{h,\alpha}).$$

Proof. We set $B_0(t) = \mathrm{Op}^W(b_0^h(t))$ and write

$$R_0(t) = \frac{d}{dt}B_0 + \frac{i}{h^{1/\alpha}}[L(t), B_0(t)], \ r_0^h(t) = \Sigma(R_0(t)).$$

Then by (14.1), Lemma 14.1, and the symbol calculus (Proposition 13.1), $r_0^h(t)$ is supported in $w_k^{-1}(t/h^{1/\alpha})[\mathrm{supp}\, a^h]$ modulo $O(h^\infty)$-terms, and

$$r_0^h(t) \in S(\langle t\rangle^{-1-\mu_t} h^{\mu_h}, g_1^{h,\alpha}). \tag{14.3}$$

We set

$$b_1^h(t;\cdot) = -\int_0^t r_0^h(w_k^{-1}(s/h^{1/\alpha}) \circ w_k(t/h^{1/\alpha})(\cdot), s)ds. \tag{14.4}$$

Then b_1^h is the solution of the transport equation:

$$\frac{d}{dt}b_1^h(t) + h^{1/\alpha}\{l_0(t)(t/h^{1/\alpha}), b_1^h(t)\} = -r_0^h(t), \ b_1^h(0) = 0.$$

By (14.3) and (14.4), we observe

$$b_1^h(t) \in S(h^{\mu_h}, g_1^{h,\alpha})$$

uniformly in t, and b_1^h is supported in $w_k^{-1}(t/h^{1/\alpha})[\mathrm{supp}\, a^h]$ modulo $O(h^\infty)$-terms. Moreover, b_1^h converges to a symbol supported in $w_{k,\pm}^{-1}[\mathrm{supp}\, a^h]$ in C_0^∞- topology as $t \to \pm\infty$.

Repeating this procedure, we can inductively construct symbols $b_j^h \in S(h^{j\mu_h}, g_1^{h,\alpha})$ and $r_j^h \in S(\langle t\rangle^{-1-\mu_t} h^{(j+1)\mu_h}, g_1^{h,\alpha})$. We set b^h as an asymptotic sum:

$$b^h(t) \sim \sum_{j=0}^\infty b_j^h(t), \ B(t) = \mathrm{Op}^W(b^h(t)).$$

By the construction, $b^h(t)$ and $\mathrm{Op}^W(b^h(t))$ satisfy the assertion. □

We then observe that $A(t)$ is very close to $B(t)$ constructed above.

Lemma 14.3. *For any N, there is C_N such that*

$$\|A(t) = B(t)\| \le C_N h^N, \quad t \in \mathbb{R}.$$

In particular,

$$A_\pm := \operatorname*{w-lim}_{t \to \pm\infty} A(t)$$

have the symbols $b^h_\pm$ as pseudodifferential operators.

Proof. We first observe

$$\begin{aligned}
&\|A(t) - B(t)\| = \|e^{itP_f/h^{1/\alpha}} J^* Y e^{-itP/h^{1/\alpha}} A_0 e^{itP/h^{1/\alpha}} Y J e^{-itP_f/h^{1/\alpha}} - B(t)\| \\
&= \|J^* Y e^{-itP/h^{1/\alpha}} A_0 e^{itP/h^{1/\alpha}} Y J - e^{-itP_f/h^{1/\alpha}} B(t) e^{itP_f/h^{1/\alpha}}\| \\
&\le \|Y J J^* Y e^{-itP/h^{1/\alpha}} A_0 e^{itP/h^{1/\alpha}} Y J J^* Y - Y J e^{-itP_f/h^{1/\alpha}} B(t) e^{itP_f/h^{1/\alpha}} J^* Y\| \\
&\le \|e^{-itP/h^{1/\alpha}} A_0 e^{itP/h^{1/\alpha}} - Y J e^{-itP_f/h^{1/\alpha}} B(t) e^{itP_f/h^{1/\alpha}} J^* Y\| + R_2 \\
&= \|A_0 - \tilde{B}(t)\| + R_2
\end{aligned}$$

where

$$R_2 = 2\|(1 - Y J J^* Y) e^{-itP/h^{1/\alpha}} A_0\|$$

and

$$\tilde{B}(t) = e^{itP/h^{1/\alpha}} Y J e^{-itP_f/h^{1/\alpha}} B(t) e^{itP_f/h^{1/\alpha}} J^* Y e^{-itP/h^{1/\alpha}}.$$

By local decay estimates for P, we learn $R_2 = O(\langle t \rangle^{-N} h^N)$ for any N (see [17]). We then show $\tilde{B}(t)$ is very close to A_0 uniformly in t. We compute

$$\begin{aligned}
\frac{d}{dt}\tilde{B}_0(t) &= \left(e^{itP/h^{1/\alpha}} Y J e^{-itP_f/h^{1/\alpha}}\right) \frac{d}{dt} B(t) \left(e^{itP_f/h^{1/\alpha}} J^* Y e^{-itP/h^{1/\alpha}}\right) \\
&+ \frac{i}{h^{1/\alpha}} \left(e^{itP/h^{1/\alpha}} Y J e^{-itP_f/h^{1/\alpha}}\right) L(t) B(t) \left(e^{itP_f/h^{1/\alpha}} J^* Y e^{-itP/h^{1/\alpha}}\right) \\
&- \frac{i}{h^{1/\alpha}} \left(e^{itP/h^{1/\alpha}} Y J e^{-itP_f/h^{1/\alpha}}\right) B(t) L(t)^* \left(e^{itP_f/h^{1/\alpha}} J^* Y e^{-itP/h^{1/\alpha}}\right) \\
&+ R_1 B(t) \left(e^{itP_f/h^{1/\alpha}} J^* Y e^{-itP/h^{1/\alpha}}\right) - \frac{i}{h^{1/\alpha}} \left(e^{itP/h^{1/\alpha}} Y J e^{-itP_f/h^{1/\alpha}}\right) B(t) R_1^* \\
&= \left(e^{itP/h^{1/\alpha}} Y J e^{-itP_f/h^{1/\alpha}}\right) \left(\frac{d}{dt} B(t) + \frac{i}{h^{1/\alpha}} [L(t), B(t)]\right) \\
&\quad \left(e^{itP_f/h^{1/\alpha}} J^* Y e^{-itP/h^{1/\alpha}}\right) + R_3(t)
\end{aligned}$$

where

$$\begin{aligned}
R_3(t) &= R_1 B(t) \left(e^{itP_f/h^{1/\alpha}} J^* Y e^{-itP/h^{1/\alpha}}\right) - \frac{i}{h^{1/\alpha}} \left(e^{itP/h^{1/\alpha}} Y J e^{-itP_f/h^{1/\alpha}}\right) B(t) R_1^* \\
&+ \frac{i}{h^{1/\alpha}} \left(e^{itP/h^{1/\alpha}} Y J e^{-itP_f/h^{1/\alpha}}\right) B(t) (L(t) - L(t)^*) \left(e^{itP_f/h^{1/\alpha}} J^* Y e^{-itP/h^{1/\alpha}}\right)
\end{aligned}$$

We can show $\|R_3(t)\| = \mathcal{O}(\langle t\rangle^{-N}h^N)$. Combining this with Lemma 14.2 (iv), we learn that

$$\|\frac{d}{dt}\tilde{B}(t)\| \leq C_N\langle t\rangle^{-1-\mu_t}h^N$$

with any N, and hence $\|\tilde{B}(t)-\tilde{B}(0)\| \leq C_N h^N$. We note

$$\tilde{B}(0) = \eta\left(\frac{r}{t}\right)JJ^*A_0JJ^*\eta\left(\frac{r}{t}\right) = A_0 + \mathrm{O}(h^N)$$

by the choice of $\varepsilon > 0$. Combining these, we conclude the assertion. □

15 Proof of Theorem 11.2

Suppose $a(h;r,\rho,\theta,\omega)$ is supported in a common compact subset $\Omega \subset T^*\mathbb{R}_+ \times (T^*\partial M\setminus 0)$ and is uniformly bounded in C_0^∞. We set

$$A_0 = \mathrm{Op}^W(a^h),\ \ a^h(r,\rho,\theta,\omega) = a(h;h^{1/\alpha}r,\rho,\theta,h\omega).$$

Let ε also as in the last section.

Lemma 15.1. *If $\delta > 2\varepsilon^2$, then*

$$\operatorname*{w-lim}_{t\to\pm\infty}\eta(P_f/\delta)A(t)\eta(P_f/\delta) = \eta(P_f/\delta)W_\pm^*A_0W_\pm\eta(P_f/\delta).$$

Proof. It is easy to show by the non-stationary phase method that

$$\operatorname*{s-lim}_{t\to\pm\infty}\left(1-\eta\left(\frac{r}{\varepsilon\langle h^{-1/\alpha}t\rangle}\right)\right)Je^{-itP_f/h^{1/\alpha}}\eta(P_f/\delta),$$

and the claim follows easily from this. □

This implies, combined with Lemmas 14.2 and 14.3:

Lemma 15.2. *Let A_0 be as above. Then $W_\pm^*A_0W_\pm$ are pseudodifferential operators with the symbols $b_\pm^h$ given in Lemma 14.2. In particular, $\Sigma(W_\pm^*A_0W_\pm)$ are supported in $w_{k,\pm}^{-1}[\operatorname{supp} a^h]$ modulo $\mathrm{O}(h^\infty)$-terms, and the principal symbol modulo $S(h^{\mu_h}, g_1^{h,\alpha})$ are given by $a^h\circ w_{k,\pm}$.*

The converse of Lemma 15.2 is given as follows:

Lemma 15.3. *Suppose $(r_-,\rho_-,\theta_-,\omega_-) \in \mathbb{R}\times\mathbb{R}_-\times(T^*\partial M\setminus 0)$ with $\frac{1}{2}\rho_-^2 \in \sigma_{pp}(P)$ and $\tilde{a} \in C_0^\infty(\mathbb{R}\times\mathbb{R}_-\times(T^*\partial M\setminus 0))$ is supported in a small neighborhood of $(r_-,\rho_-,\theta_-,\omega_-)$. We set*

$$\tilde{A} = \mathrm{Op}^W(\tilde{a}^h),\ \ \tilde{a}^h(r,\rho,\theta,\omega) = \tilde{a}(h^{1/\alpha}r,\rho,\theta,h\omega),$$

then $W_-\tilde{A}W_-^*$ is a pseudodifferential operator with a symbol supported in $w_{k,-}[\operatorname{supp}\tilde{a}^h]$, and has the principal symbol modulo $S(h^{\mu_h}, g_1^{h,\alpha})$ given by $\tilde{a}^h\circ w_{k,-}^{-1}$.

Proof. We set $a^h_{0,0} = \tilde{a}^h \circ w^{-1}_{k,-}$. By Proposition 12.4, we have

$$\begin{aligned}\operatorname{supp} a^h_{0,0} &= \operatorname{supp} \tilde{a}^h \circ w^{-1}_{k,-} \\ &= \{(h^{-1/\alpha}\Pi_r, \Pi_\rho) w_{k,-}(r,\rho,\theta,\omega), (\Pi_\theta, h^{-1}\Pi_\omega)\exp(-h^{1-1/\alpha}\sigma_-(w^{-1}_{k,-}(r,\rho,\theta,\omega))H_{\sqrt{2q}}(\theta,\omega)) | \\ &\quad (r,\rho,\theta,\omega) \in \operatorname{supp} a\}.\end{aligned}$$

We set $a_{0,0}(h; h^{1/\alpha}r, \rho, \theta, h\omega) = a^h_{0,0}(r,\rho,\theta,\omega)$ and

$$\begin{aligned}\Omega = \{&(\Pi_{r,\rho}) w_{k,-}(r,\rho,\theta,\omega), \exp(-h^{1-1/\alpha}\sigma_-(w^{-1}_{k,-}(r,\rho,\theta,\omega))H_{\sqrt{2q}}(\theta,\omega)) | 0 < h \ll 1, \\ &(r,\rho,\theta,\omega) \in \operatorname{supp} a\}.\end{aligned}$$

Then $\operatorname{supp} a_{0,0}(h;\cdot)$ is supported in a compact set $\Omega \Subset T^*\mathbb{R}_+ \times (T^*\partial M \setminus 0)$ and $a_{0,0}(h;\cdot)$ is bounded in C_0^∞ uniformly for h. Hence $a_{0,0}(h;\cdot)$ satisfies the assumptions of Lemma 15.2. We have

$$a^h_{-,1} := \Sigma(\tilde{A} - W_-^* \operatorname{Op}^W(a^h_{0,0}) W_-) \in S(h^{\mu_h}, g_1^{h,\alpha}).$$

and it is supported in $\operatorname{supp}[\tilde{a}^h]$ modulo $O(h^\infty)$-terms. Then we set $a^h_{0,1} = a^h_{-,1} \circ w^{-1}_{k,1}$ and set

$$a_{-,2} = \Sigma(\tilde{A} - W_-^* \operatorname{Op}^W(a^h_{0,0} + a^h_{0,1}) W_-) \in S(h^{2\mu_h}, g_1^{h,\alpha}).$$

We construct $a^h_{-,j}$, $j = 2, 3, \cdots$, inductively by

$$a_{-,j} = \Sigma(\tilde{A} - W_-^* \operatorname{Op}^W(a^h_{0,0} + \cdots + a^h_{0,j-1}) W_-) \in S(h^{2\mu_h}, g_1^{h,\alpha}).$$

$a^h_{0,j} = a^h_{-,j} \circ w^{-1}_{k,-}$, and we set a^h_0 as an asymptotic sum:

$$a^h_0 \sim \sum_{j=0}^{\infty} a^h_{0,j}.$$

Then by the construction, we have

$$\Sigma(\tilde{A} - W_-^* \operatorname{Op}^W(a^h_0) W_-) \in S(h^\infty, g_1^{h,\alpha}).$$

By multiplying W_- and W_-^*, we have

$$\Sigma(W_- \tilde{A} W_-^* - W_- W_-^* \operatorname{Op}^W(a^h_0) W_- W_-^*) \in S(h^\infty, g_1^{h,\alpha}).$$

Take $f \in C_0^\infty(\mathbb{R}_+ \setminus \sigma_{pp}(P))$ such that $f(\lambda) = 1$ in a neighborhood of $\lambda_- = \frac{1}{2}\rho_-^2$. Denote by P_{ac} the orthogonal projection on the absolutely continuous subspace of P. Since $P_{ac} = W_- W_-^*$, we have

$$W_- \tilde{A} W_-^* - W_- W_-^* \operatorname{Op}^W(a^h_0) W_- W_-^* = W_- \tilde{A} W_-^* - P_{ac} \operatorname{Op}^W(a^h_0) P_{ac} \in S(h^\infty, g_1^{h,\alpha}).$$

We compute

$$\operatorname{Op}^W(a^h_0) - P_{ac} \operatorname{Op}^W(a^h_0) P_{ac} = (1 - P_{ac}) \operatorname{Op}^W(a^h_0) + P_{ac} \operatorname{Op}^W(a^h_0)(1 - P_{ac}),$$

and

$$(1-P_{ac})\operatorname{Op}^W(a_0^h) = (1-P_{ac})f(P)\operatorname{Op}^W(a_0^h) + (1-P_{ac})(1-f(P))\operatorname{Op}^W(a_0^h)$$
$$= (1-P_{ac})(1-f(P))\operatorname{Op}^W(a_0^h).$$

We note that $\Sigma((1-f(P))\operatorname{Op}^W(a_0^h)) \in S(h^\infty, g_1^{h,\alpha})$ if $f(\lambda)=1$ in a neighborhood of $\lambda_- = \frac{1}{2}\rho_-^2$ and a is supported in a sufficiently small neighborhood of $(r_-,\rho_-,\theta_-,\omega_-)$. Combining these we have

$$\Sigma(W_-\tilde{A}W_-^* - \operatorname{Op}^W(a_0^h)) \in S(h^\infty, g_1^{h,\alpha}).$$

□

Combining Lemmas 15.2 and 15.3, we learn the following Lemma.

Lemma 15.4. *Let* $(r_-,\rho_-,\theta_-,\omega_-) \in \mathbb{R}\times\mathbb{R}_-\times(T^*\partial M\setminus 0)$, $\tilde{a}\in C_0^\infty(\mathbb{R}\times\mathbb{R}_-\times(T^*\partial M\setminus 0))$ *be supported in a small neighborhood of* $(r_-,\rho_-,\theta_-,\omega_-)$ *with* $\rho_-^2/2 \notin \sigma_{pp}(P)$, *and let*

$$\tilde{A} = \operatorname{Op}^W(\tilde{a}^h), \quad \tilde{a}^h(r,\rho,\theta,\omega) = \tilde{a}(h^{1/\alpha}r,\rho,\theta,h\omega).$$

Then $S\tilde{A}S^*$ *is a pseudodifferential operator with a symbol supported in* $s_k[\operatorname{supp}\tilde{a}^h]$, *and the principal symbol modulo* $S(h^{\mu_h}, g_1^{h,\alpha})$ *is given by* $\tilde{a}^h \circ s_k^{-1}$.

We set $\hat{\mathscr{H}}_{f,\pm} = \mathscr{F}\mathscr{H}_{f,\pm}$. Then $\mathscr{F}S\mathscr{F}^{-1}$ is a unitary map from $\hat{H}_{f,-}$ to $\hat{H}_{f,+}$. We set

$$\Pi u(r,\theta) = u(-r,\theta) \quad \text{for } u\in\mathscr{H}_{f,\pm},$$

so that $\mathscr{F}(S\Pi)\mathscr{F}^{-1}$ is a unitary map on $\hat{\mathscr{H}}_{f,+}$. By the intertwining property of the scattering operator, $\mathscr{F}(S\Pi)\mathscr{F}^{-1}$commutes with ρ and hence is decomposed to

$$\mathscr{F}(S\Pi)\mathscr{F}^{-1} = \int^{\oplus} S(\rho^2/2)d\rho \quad \text{on } \hat{\mathscr{H}}_{f,\pm} \cong L^2(\mathbb{R};L^2(\partial M)),$$

where $S(\lambda)\in B(L^2(\partial M))$ is the scattering matrix.

We here introduce the α-wave front set as follows:

Definition 15.5. *Let* $\alpha > 1$ *and* $g(\rho,\theta)\in\mathscr{D}'(\mathbb{R}_+\times\partial M)$. *We say* $(\rho_0,\theta_0,r_0,\omega_0)$ *is not in the* α*-wave front set of* g *if there exists* $a\in C_0^\infty(T^*(\mathbb{R}_+\times\partial M))$ *such that* $a(\rho_0,\theta_0,r_0,\omega_0)\neq 0$ *and*

$$\|a(\rho,\theta,h^{1/\alpha}D_\rho,hD_\theta)g\| = \mathscr{O}(h^\infty) \quad \text{as } h\to +0,$$

where $\|\cdot\|$ *denotes the* $L^2(\mathbb{R}_+\times\partial M)$*-norm. We denote* $(\rho_0,\theta_0,r_0,\omega_0)\notin \mathrm{WF}^\alpha(g)$ *if this condition is satisfied, and denote the complement by* $\mathrm{WF}^\alpha(g)$.

Note that if $(\rho,\theta,r,\omega)\in\mathrm{WF}^\alpha(g)$ then $(\rho,\theta,\lambda^{1/\alpha}r,\lambda\omega)\in\mathrm{WF}^\alpha(g)$ for all $\lambda>0$. Also note that we may replace a by h-dependent symbol with a principal symbol which does not vanish at $(\rho_0,\theta_0,r_0,\omega_0)$.

We recall the semiclassical type characterization of the usual wave front set. $(\rho_0,\theta_0,r_0,\omega_0)\notin \mathrm{WF}(g)$ if and only if there is $a\in C_0^\infty(T^*(\mathbb{R}_+\times\partial M))$ such that $a(\rho_0,\theta_0,r_0,\omega_0)\neq 0$ and

$$\|a(\rho,\theta,hD_\rho,hD_\theta)g\| = \mathscr{O}(h^\infty) \quad \text{as } h\to +0.$$

We present basic relationships between the WF set and the WF^α set, which follow from the definition.

Proposition 15.6. *Let $(\rho_0,\theta_0)\in\mathbb{R}_+\times\partial M$. Then the following holds:*

(i). Let $r_0\neq 0$. $(\rho_0,\theta_0,r_0,0)\notin \mathrm{WF}^\alpha(g)$ if and only if $(\rho_0,\theta_0,r_0,\omega_0)\notin \mathrm{WF}(g)$ for every $\omega_0\in T^*_{\theta_0}(\partial M)$.

(ii). Let $\omega_0\neq 0$. $(\rho_0,\theta_0,0,\omega_0)\notin \mathrm{WF}(g)$ if and only if $(\rho_0,\theta_0,r_0,\omega_0)\notin \mathrm{WF}^\alpha(g)$ for every $r_0\in T^*_{\rho_0}(\mathbb{R}_+)$.

We give the proof of the main Theorem. We first show the propagation of the WF^α set. Using Proposition above, we show the propagation of the usual WF set.

Proof of Theorem 11.2. We fix $\lambda_0=\rho_0^2/2\notin\sigma_{pp}(P)$ with $\rho_0>0$ and consider $S(\lambda)$ where λ is in a small neighborhood of λ_0. Let $u\in L^2(\partial M)$ and let $v\in C_0^\infty(\mathbb{R}_+)$ supported in a small neighborhood of λ_0. Then

$$\mathrm{WF}(v(\rho)u(\theta)) = \mathrm{WF}^\alpha(v(\rho)u(\theta)) = \{(\rho,\theta,0,\omega)|\rho\in\operatorname{supp} v,(\theta,\omega)\in \mathrm{WF}(u)\}.$$

Now let $(\bar\rho_-,\bar\theta_-,\bar r_-,\bar\omega_-)\notin \mathrm{WF}^\alpha(v(\rho)u(\theta))$ and take $a\in C_0^\infty(T^*(\mathbb{R}_+\times\partial M))$ such that $a(\bar\rho_-,\bar\theta_-,\bar r_-,\bar\omega_-)\neq 0$ and

$$\|a(\rho,\theta,h^{1/\alpha}D_\rho,hD_\theta)v(\rho)u(\theta)\| = \mathscr{O}(h^\infty) \quad \text{as } h\to +0.$$

Without loss of generality, we may assume that $a(\cdot)\geq\delta>0$ in a neighborhood of $(\bar\rho_-,\bar\theta_-,\bar r_-,\bar\omega_-)$. Let

$$\tilde A = \mathrm{Op}^W(\tilde a^h),\quad \tilde a^h(\rho,\theta,r,\omega)=\tilde a(\rho,\theta,h^{1/\alpha}r,h\omega).$$

Then by Lemma 15.4, there exists $b^h\in S(1,g_1^{h,\alpha})$ such that

$$\|\mathscr{F}(S\Pi)\mathscr{F}^{-1}\tilde A\mathscr{F}(\Pi S^*)\mathscr{F}^{-1}-\mathrm{Op}^W(b^h)\| = \mathscr{O}(h^\infty)$$

and the principal symbol of b^h modulo $S(h^{\mu_h},g_1^{h,\alpha})$ is given by $a^h\circ\pi\circ s_k^{-1}$, where $\pi(r,\rho)=(-r,-\rho)$. Combinig these, we have

$$\|\mathrm{Op}^W(b^h)\mathscr{F}(\Pi S)\mathscr{F}^{-1}v(\rho)u(\theta)\| = \mathscr{O}(h^\infty).$$

Let $b(h;\rho,\theta,h^{1/\alpha}r,h\omega)=b^h(\rho,\theta,r,\omega)$, and $b_0(h;\rho,\theta,h^{1/\alpha}r,h\omega)=a^h\circ\pi\circ s_k^{-1}(\rho,\theta,r,\omega)$. Then $b_0(h;\cdot)$ is the principal symbol of $b(h;\cdot)$ modulo $S(h^{\mu_h},g_1)$. Recall that by the scaling property (Proposition 12.4), we have

$$\begin{aligned}
&s_k\circ\pi(h^{-1/\alpha}\bar r_-,\bar\rho_-,\bar\theta_-,h^{-1}\bar\omega_-)\\
&=(h^{-1/\alpha}(r_\infty\circ\pi(\bar r_-,\bar\rho_-,\bar\theta_-,\bar\omega_-)+\bar r),\bar\rho_-,\\
&(\Pi_\theta,h^{-1}\Pi_\omega)\exp(h^{1-1/\alpha}\sigma_\infty(h^{-1/\alpha}\bar r_-,\bar\rho_-,\bar\theta_-,h^{-1}\bar\omega_-)H_{\sqrt{2q}})(\bar\theta_-,\bar\omega_-)),
\end{aligned}$$

and

$$(h^{1/\alpha}\Pi_r, \Pi_\rho, \Pi_\theta, h\Pi_\omega)s_k \circ \pi(h^{-1/\alpha}\bar{r}_-, \bar{\rho}_-, \bar{\theta}_-, h^{-1}\bar{\omega}_-) \to (r_\infty + \bar{r}_-, \bar{\rho}_-, \bar{\theta}_-, \bar{\omega}_-)$$

as $h \to 0$, where $r_\infty = r_\infty \circ \pi(\bar{r}_-, \bar{\rho}_-, \bar{\theta}_-, \bar{\omega}_-)$. Hence $b_0(h;\cdot) \geq \delta > 0$ in a neighborhood of $(r_\infty + \bar{r}_-, \bar{\rho}_-, \bar{\theta}_-, \bar{\omega}_-)$ for small enough $h > 0$. By the semiclassical characterization of the α-wave front set, we learn

$$(\bar{\rho}_-, \bar{\theta}_-, r_\infty + \bar{r}_-, \bar{\omega}_-) \notin WF^\alpha(\mathscr{F}(\Pi S)\mathscr{F}^{-1}v(\rho)u(\theta)).$$

Since we can take any $\bar{r}_- \in \mathbb{R} \setminus 0$, we have

$$\mathrm{WF}^\alpha(\mathscr{F}(\Pi S)\mathscr{F}^{-1}v(\rho)u(\theta)) \subset \{(\rho, \theta, r_\infty(0, -\rho, \theta, \omega), \omega); \rho \in \operatorname{supp} v, (\theta, \omega) \in WF(u)\}.$$

By Proposition 15.6, we learn that

$$\mathrm{WF}(\mathscr{F}(\Pi S)\mathscr{F}^{-1}v(\rho)u(\theta)) \subset \{(\rho, \theta, 0, \omega); \rho \in \operatorname{supp} v, (\theta, \omega) \in \mathrm{WF}(u)\}.$$

By the definition of the scattering matrix, this implies

$$WF(S(\lambda)u) \subset WF(u)$$

for $u \in \operatorname{supp} v$. Since this argument works for S^{-1} also, the above inclusion is actually an equality, and we complete the proof of Theorem 11.2. □

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論文の内容の要旨

論文題目：

Scattering Theory on Manifolds with Asymptotically Polynomially Growing Ends

多項式増大する無限遠境界を持つ多様体上の散乱理論

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本論文では非コンパクト多様体でのシュレディンガー方程式の散乱理論について考察する．

20世紀初頭より量子力学は物理学において中心的な研究対象であった．量子力学の理論は基礎的かつ一般的なもので，量子力学をもとに多くの現代物理学の分野が発展してきた．量子力学ではシュレディンガー方程式と呼ばれる偏微分方程式が量子力学的な物理状態の時間発展を支配している．

散乱とは直進している波や粒子が媒質の非一様性や力によって曲げられることをいう．量子力学的散乱とは上述の偏微分方程式の解が，遠い過去には自由に進んでいたものが，ポテンシャルと相互作用し，遠い未来へ進んでいく様を記述するものである．量子散乱は物理において微視的な粒子の性質を調べる上で基本的な観測対象である．

量子力学やシュレディンガー方程式に関しては数学的な構造についての多くの研究がなされてきた．量子的物理状態は複素ヒルベルト空間の点と対応し，シュレディンガー作用素は偏微分作用素の自己共役な実現として表現される．時間発展は自己共役作用素によって生成されるユニタリ作用素である．量子力学的散乱理論は摂動理論としての一面も持っている．非摂動系である自由なハミルトニアンと摂動を加えた全ハミルトニアンとが近ければ自由な系の情報を用いて全体の系を記述できる．

ここで数学的な量子散乱理論の基本的な概念を導入する．自由な運動に対して，時間無限大で自由な運動に漸近する摂動された系の運動があるとき波動作用素が存在するとい

ここで数学的な量子散乱理論の基本的な概念を導入する．自由な運動に対して，時間無限大で自由な運動に漸近する摂動された系の運動があるとき波動作用素が存在するといい，自由な運動の始状態を摂動された系の運動の始状態に対応させる作用素として波動作用素は定義される．時間の方向に応じて二種類の波動作用素がある．波動作用素の値域と摂動されたハミルトニアンの絶対連続部分空間とが一致するとき波動作用素は完全であるという．二つの方向の波動作用素を組み合わせて散乱作用素が定義され，波動作用素が存在して完全なときユニタリとなる．遠い過去に自由に運動していた粒子が相互作用をしてまた遠い未来に自由に飛んでいくとき，この過去の状態から未来の状態への対応を与えるのが散乱作用素である．散乱作用素は自由なハミルトニアンと交換するため，自由なハミルトニアンと同時対角化でき，自由なハミルトニアンのスペクトル分解に従って作用素値の関数として表される．この作用素を散乱行列と呼ぶ．これは定エネルギーでの散乱の様子を記述するものである．

典型的なシュレディンガー方程式はユークリッド空間上のラプラシアンとポテンシャル関数の和で書けるものである．遠方での減少がクーロンポテンシャルよりも早いポテンシャルは短距離型であるといい，波動作用素が存在して完全であることが知られている．遠方での減少がクーロンポテンシャルよりもゆるやかなポテンシャルは長距離型であるという．このとき一般には波動作用素は存在しないが，波動作用素の定義を修正することで同様の理論が構築できることが知られている．

多様体上でラプラシアンとポテンシャル関数の和として書けるシュレディンガー作用素を考える．Melrose は散乱多様体という散乱理論を展開するのに適した多様体の族を定義した．我々は漸近的に多項式増大する無限遠境界をもつ多様体を考察する．ここではその性質を簡単に述べる．ユークリッド空間の極座標表示を思い出すと，ユークリッド空間は球面と半直線との直積として表すことができ，球面の大きさは動径座標と比例して，円錐のように広がっていくと考えることができる．これを拡張して，一般に非コンパクトな多様体であって漸近的に錐型な無限遠境界を持つ多様体を散乱多様体と呼ぶ．ここでは球面の代わりに任意のコンパクト多様体を考えることができ，そのコンパクト多様体は境界多様体と呼ばれる．さらに，動径座標の正の実数乗に比例して増大する多様体のことを漸近的に多項式増大する無限遠境界を持つ多様体と呼ぶことにし，この正実数を増大度と呼ぶことにする．散乱多様体は増大度１の漸近的に多項式増大する無限遠境界を持つ多様体である．

本論文は三部からなっている．第一部では，散乱多様体上で長距離摂動をもつシュレディンガー方程式に対して修正型波動作用素の存在を示す．第二部では，漸近的に多項式増大する無限遠境界を持つ多様体上で短距離摂動を持つシュレディンガー方程式に対して波動作用素の存在と完全性を示す．第三部において，増大度が１より大きいとき散乱行列は波面集合を変えないことを示す．波面集合とは関数の特異性を特異性の方向も込めて考えた集合である．

以下，各部について簡単な要旨と注意を与える.

・第一部

散乱多様体上の長距離摂動をもつシュレディンガー方程式を考える．我々は加藤(1967)の*2*空間散乱理論を用いる伊藤・中村 (2010) の定式化を採用する．これは自由なハミルトニアンとして１次元の自由ラプラシアンを境界多様体と実軸の直積上で考えるものである．Hörmander (1976)およびDereziński-Gerard (1997)の手法を用いて，対応する古典力学的軌道を計算しハミルトン・ヤコビ方程式の厳密解を構成する．この解を修正関数とする修正波動作用素の存在を，Cook (1957)・黒田 (1959) の方法と停留位相法を用いて示す.

Melrose-Zworski (1996) は，散乱多様体上の散乱行列の性質を調べている．長距離型の先行結果としては，Vasy (1998) がクーロン型の減衰をする長距離摂動に対して，散乱行列の性質を調べている．我々は一般の滑らかな長距離摂動を扱っている.

証明の鍵はハミルトン・ヤコビ方程式の解の構成である．我々は Dereziński-Gerard (1997) あるいは Dereziński(1991)の手法を用いる．まず時間に依存する時間について緩減少する力のもとでニュートン方程式の境界値問題を解き，この解が運動量と空間座標が平行（または半平行）となる相空間上の領域に入ることを示す．次に時間に依存しない空間について長距離減少する力のもとでのニュートン方程式に対し，上述の領域へのカットオフ関数を長距離力に掛けることで問題を時間に依存する場合に帰着させる．得られた古典力学的軌道を用いればハミルトン・ヤコビ方程式の解が得られる.

・第二部

漸近的に多項式増大する無限遠境界を持つ多様体上のシュレディンガー作用素を考える．Froese-Hislop(1989) は，Mourre 理論 (Mourre (1981), Perry-Sigal-Simon(1981)も参照) を用いてシュレディンガー作用素のスペクトルの性質を短距離摂動の時に示している．これを拡張して一般の長距離摂動に対しシュレディンガー作用素のスペクトルの性質を示す．さらに，三種類の作用素が加藤のなめらかな摂動作用素になっていることを示す．摂動が短距離型のとき摂動は加藤のなめらかな摂動作用素の積の形に分解することができ，加藤のなめらかな摂動理論 (1966) を適用することで波動作用素の存在と完全性が示される．我々は第一部と同様に伊藤・中村の定式化を用いることで，2空間散乱理論の枠組みでも波動作用素の存在と完全性を示す.

De Bièvre-Hislop-Sigalは波動方程式の時間に依存する散乱理論をより一般的なクラスの無限遠境界を持つ多様体に対して示していて，漸近的に多項式増大する多様体もその一種である．我々は角度方向の摂動に関してより一般的な場合を扱っている．伊藤・中村 (2010) は，散乱多様体に対して時間に依存する散乱理論を構築している.

ある作用素が加藤のなめらかな摂動作用素であることはハミルトニアンのレゾルベントの境界値を用いて特徴付けられる．Mourre 理論で得られたレゾルベントの評価式に Perry-Sigal-Simon (1981) の手法を適用して極限吸収原理が示される．Yafaev (1993)の手法による放射条件評価も用いる．

2空間散乱の理論の構成は1 次元の散乱理論に帰着されることがわかる．多様体の無限遠境界の増大度が２分の１より大きいときは短距離型，２分の１以下の時は長距離型の理論が用いられる．

・第三部

漸近的に多項式増大する無限遠境界を持つ多様体上のシュレディンガー方程式の散乱問題を考える．第二部の結果より散乱作用素および散乱行列が定義される．散乱行列は境界多様体上の作用素となる．我々は多様体の無限遠境界での増大度が１より大きいとき，散乱行列は波面集合を変えないことを示す．これは散乱行列の核が対角集合から離れたところでは滑らかなことを示し，物理的には入射した波がほとんど全反射されることに対応する．

散乱多様体の場合，すなわち増大度が１の場合に先行結果があり，Melrose-Zworski (1996)は，散乱行列はフーリエ積分作用素であり，付随する正準変換は境界多様体上の距離 π の測地流であることを示した．この系として波面集合は同じ正準変換に従って伝播することが示される．彼らの議論は一般化固有関数の漸近展開を用いて散乱行列を定義するものである．伊藤・中村 (2011) は，Egorov 型の定理と Beals 型のフーリエ積分作用素の特徴付けを用いることで時間に依存する散乱理論を用いた別証明と一般化を与えている．

多項式増大する多様体上の古典力学を考える．増大度が１のとき，古典力学的な散乱行列，つまり散乱作用素の境界多様体上の成分は 距離 π の測地流となる．増大度が１より大きいときは，インパクトパラメーターが大きいとき，散乱行列は漸近的に恒等写像に近づく．我々の結果および Melrose-Zworski, 伊藤・中村の結果は，古典力学的散乱を量子化したものととらえることができる．