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Determining nodes for semilinear parabolic evolution equations in Banach spaces

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Determining nodes for semilinear parabolic evolution equations in Banach spaces

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Abstract

We are concerned with the determination of the asymptotic behaviour of strong solutions to the initial-boundary value problem for general semilinear parabolic equations by the asymptotic behaviour of these strong solutions on a finite set. More precisely, if the asymptotic behaviour of the strong solution is known on a suitable finite set which is called determining nodes, then the asymptotic behaviour of the strong solution itself is entirely determined. We prove the above property by the energy method. Moreover, we are concerned with the determination of the asymptotic behaviour of mild solutions to the abstract initial value problem for semilinear parabolic evolution equations in L_p by the asymptotic behaviour of these mild solutions on a finite set. More precisely, if the asymptotic behaviour of the mild solution is known on determining nodes, then the asymptotic behaviour of the mild solution itself is entirely determined. Not only the asymptotic equivalence but also rate of monomial or exponential convergence can be clarified. We prove the above properties by the theory of analytic semigroups on Banach spaces. As an important application of sectorial operators, we give the linearized operator (Stokes operator) associated with the initial-boundary value problem for the Navier-Stokes equations in a multiply-connected bounded domain with the Navier-Dirichlet boundary condition. Furthermore, we study the asymptotic properties of stationary solutions to this problem. As for the existence and uniqueness, this problem has uniquely a stationary solution in $(W_p^2)^n$ satisfying L_p estimates for any $n < p < \infty$. The first result is obtained from resolvent estimates for the Stokes operator in $L_{p,\sigma}$ and the Banach fixed point theorem. On the asymptotic stability, the stationary solutions are asymptotically stable in $L_{p,\sigma}$ if they are small in $(W_p^1)^n$. The second result is proved by the theory of analytic semigroups on Banach spaces.

1 Introduction

Let $n \in \mathbb{Z}$, $n \geq 2$, Ω be a bounded domain in $\mathbb{R}^n$ with its $C^{0,1}$-boundary $\partial\Omega$, H be a closed subspace of $L_2(\Omega)$, $V = H_0^1(\Omega) \cap H$. The first problem of this paper is the following strong formulation of the initial-boundary value problem for the semilinear parabolic equation:

$$\begin{cases} d_t u + Au = F(u) + f & \text{in } L_2((0,\infty); H), \\ u(0) = u_0 & \text{in } V, \end{cases} \tag{1.1}$$

where u is a strong solution to (1.1), A is a densely defined closed linear operator from $D(A)$ to H, $F(u)$ is a nonlinear term, u_0 is an initial data, f is an external force. Moreover, $D(A)$ is a domain of A. As is explained in Section 9, a typical example of the first equation of (1.1) is the following semilinear heat equation:

$$\partial_t u - \kappa\Delta u - |u|^{p-1}u = 0,$$

where u is the absolute temperature, $\kappa > 0$ is the coefficient of heat conductivity, $p > 1$. The existence, uniqueness and regularity of strong solutions to the initial-boundary value problem for the semilinear heat equation has been much studied for fifty years. See, for example, [10] and the references given there on the existence, uniqueness and regularity of strong solutions to the initial-boundary value problem for the semilinear heat equation in $\mathbb{R}^n$ with the Dirichlet boundary condition.

The stationary problem associated with (1.1) is the following boundary value problem for the semilinear elliptic equation:

$$A\bar{u} = F(\bar{u}) + \bar{f} \text{ in } H, \tag{1.2}$$

where $\bar{u}$ is a strong solution to (1.2), $\bar{f}$ is an external force. As is well known in [21], the stationary problem for the semilinear heat equation in $\mathbb{R}^n$ with the Dirichlet boundary condition has a trivial solution and nontrivial solutions for any $1 < p < (n+2)/(n-2)$. It is one of interesting questions whether a strong solution to (1.1) converges to a trivial or nontrivial solution to (1.2). According to the previous result by Foias and Temam [6], the conclusion of the asymptotic properties of strong solutions to (1.1) can be given by the theory of determining nodes. An approach of determining nodes is quite natural from the computational point of view. In general, the asymptotic behaviour of strong solutions to the initial-boundary value problem for semilinear parabolic equations is uniquely determined by determining nodes which can be obtained from finite many measurements. Some problems related to determining nodes for semilinear parabolic equations have been studied in recent years. Foias and Temam [6] first discussed the existence of determining nodes for the Navier-Stokes equations in $\mathbb{R}^2$ and in $\mathbb{R}^3$ with the Dirichlet and periodic boundary conditions. As for partly dissipative reaction diffusion systems in $\mathbb{R}^2$ and in $\mathbb{R}^3$ with the Dirichlet, Neumann and periodic boundary conditions, Lu and Shao [18] obtained the same results as in [6]. Not only the existence of determining nodes but also the number of determining nodes can be deeply studied in the one-dimensional case. See, for example, [5], [15], [19] on the theory of determining nodes for the Kuramoto-Sivashinsky equation, the complex Ginzburg-Landau equation and the semilinear Schrödinger equation respectively in $\mathbb{R}$ with various periodic boundary conditions. As is mentioned above, the semilinear heat equation is a typical example of semilinear parabolic equations, but the theory of determining nodes for it has not been constructed yet. It is necessary to discuss the existence of determining nodes for semilinear parabolic equations such as the first equation of (1.1).

In the first problem of this paper, we are concerned with the determination of the asymptotic behaviour of strong solutions to (1.1) by determining nodes. It is an important consequence of our main results that the theory of determining nodes for the Navier-Stokes equations and the semilinear heat equation can be unified. One of our main results is stated as follows: There exists a finite set E in Ω such that if two strong solutions u and v to (1.1) satisfy $u(x,t) - v(x,t) \to 0$ as $t \to \infty$ for any $x \in E$, then $u(\cdot,t) - v(\cdot,t) \to 0$ in $V \cap C^{0,\gamma}(\overline{\Omega})$ as $t \to \infty$ for any $0 < \gamma < 1/2$. We prove the above property by the argument based on [6], [18]. Note that main results on the L_2-theory of determining nodes were published as a paper by Kakizawa [14].

Let $n \in \mathbb{Z}$, $n \geq 2$, Ω be a bounded domain in $\mathbb{R}^n$ with its $C^{1,1}$-boundary $\partial\Omega$, X_p $(1 < p < \infty)$ be a closed subspace of $L_p(\Omega)$. The second of this paper is concerned with the following abstract initial value problem for the semilinear parabolic evolution equation in X_p:

$$\begin{cases} d_t u + A_p u = F(u) + f & \text{in } (0,\infty), \\ u(0) = u_0, \end{cases} \tag{I}$$

where u is a strong solution to (I), A_p is a densely defined closed linear operator in X_p, $F(u)$ is a nonlinear term, u_0 is an initial data, f is an external force. As is well known, (I) is the abstract initial value problem associated with the semilinear heat equation and the Navier-Stokes equations.

See, for example, [10] and the references given there on the existence, uniqueness and regularity of mild solutions to (I).

Some problems related to determining nodes for semilinear parabolic equations have been studied in recent years. Not only Foias and Temam [6], Lu and Shao [18] but also Kakizawa [14] discussed the existence of determining nodes for the semilinear parabolic equation such as the first equation of (I) in the general closed subspace H of $L_2(\Omega)$ with the Dirichlet boundary condition. The energy method shows their results, but it remains to consider two main difficulties, i.e., variety of boundary conditions and rate of convergence. In fact, the Robin and Navier boundary conditions play an important role in the semilinear heat equation and the Navier-Stokes equations respectively. Even if determining nodes for (I) exist, it is one of serious problems whether the convergence of $u(\cdot,t)-v(\cdot,t)$ to zero as $t\to\infty$ is fast or slow. The new method is required for the theory of determining nodes overcoming the above difficulties.

The second purpose of this paper is to establish the L_p-theory of determining nodes for (I) with the aid of the theory of analytic semigroups on Banach spaces, e.g., [12, Chapter 3], [20, Chapter 6]. One of our main results is stated as follows: there exists a finite set E in Ω such that if $n/2<p<\infty$ and if two mild solutions u and v to (I) satisfy $u(x,t)-v(x,t)\to 0$ as $t\to\infty$ for any $x\in E$, then $\|u(t)-v(t)\|_{X_p^\alpha}=O(t^{-\alpha})$ as $t\to\infty$ for any $0<\alpha<1$. Here A_p^α and X_p^α $(0\le\alpha\le 1)$ are fractional powers of A_p and their domain, i.e., $X_p^\alpha=D(A_p^\alpha)$ respectively. By virtue of our argument, variety of boundary conditions corresponds to the analyticity of the semigroup $\{e^{-tA_p}\}_{t\ge 0}$ generated by $-A_p$. Moreover, X_p^α-estimates for mild solutions to (I), which are established by the similar method to [9, Theorem 2.6], clarify not only the asymptotic equivalence but also rate of monomial or exponential convergence.

As is mentioned above, the semilinear heat equation and the Navier-Stokes equation are typical examples of (I). The linerlized operator of (I), i.e., A_p are the Laplace and Stokes operators with suitable boundary conditions, e.g., the Dirichlet, Neumann [20], no-slip [7] and Navier [24] boundary conditions. From the view point of asymptotic stability of stationary flows, this paper provides an untypical but important application of A_p associated with the following problem: Let $n\in\mathbb{Z}$, $n\ge 2$, Ω be a bounded domain in $\mathbb{R}^n$ with its boundary $\partial\Omega$ which consists of two connected components Γ_0 and Γ_1, i.e., $\partial\Omega=\Gamma_0\cup\Gamma_1$, $0<T\le\infty$. Throughout this paper, we assume that Γ_0 and Γ_1 have the following properties:

- $\Gamma_0=\partial\Omega_0$, where Ω_0 is an exterior domain in $\mathbb{R}^n$ with its $C^{2,1}$-boundary $\partial\Omega_0$.
- $\Gamma_1=\partial\Omega_1$, where Ω_1 is a bounded domain in $\mathbb{R}^n$ with its $C^{1,1}$-boundary $\partial\Omega_1$.
- $\mathbb{R}^n\setminus\overline{\Omega}=\Omega_0\cup\Omega_1$.

Here and hereafter Ω is called a multiply-connected bounded domain with its boundary $\Gamma_0\cup\Gamma_1$ if Ω satisfies the above properties. Motion of incompressible viscous fluids in Ω with the Navier-Dirichlet boundary condition is described by the initial-boundary value problem for the system of $n+1$ equations as follows:

$$\begin{cases}\mathrm{div}u=0 & \text{in } \Omega\times(0,T),\\ \rho\{\partial_t+(u\cdot\nabla)\}u-\mathrm{div}T(u,p)=\rho g & \text{in } \Omega\times(0,T),\\ u|_{t=0}=u_0 & \text{in } \Omega,\\ u_\nu|_{\partial\Omega}=0 & \text{on } \partial\Omega\times(0,T),\\ K(T(u,p)\nu)_\tau+(1-K)u_\tau|_{\Gamma_0}=0 & \text{on } \Gamma_0\times(0,T),\\ u_\tau|_{\Gamma_1}=h & \text{on } \Gamma_1\times(0,T),\end{cases}\tag{1.3}$$

where $u = (u_1, \cdots, u_n)^T$ is the fluid velocity, p is the pressure, ρ is the density, μ is the coefficient of viscosity, $0 \leq K \leq 1$ is a constant, u_0 is the initial fluid velocity, $g = (g_1, \cdots, g_n)^T$ and $h = (h_1, \cdots, h_n)^T$ are external forces, $\nu \in (C^{2,1}(\Gamma_0) \cap C^{1,1}(\Gamma_1))^n$ is the outward unit normal vector on $\partial\Omega$, $u_\nu := \nu \cdot u$, $u_\tau := u - u_\nu \nu$, $T(u,p)$ is the Cauchy stress tensor defined as

$$T(u,p) = -pI_n + 2\mu D(u), \ D(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right),$$

I_n is the n-th identity matrix, $\cdot^T$ is the transposition. Moreover, it is useful to remark that $(T(u,p)\nu)_\tau = T(u,p)\nu - (\nu \cdot T(u,p)\nu)\nu = 2\mu(D(u)\nu)_\tau$. These equations correspond to the laws of conservation of mass and momentum respectively. The fifth equation of (1.3) is called the Navier boundary condition which can be considered as an intermediate between the no-slip ($K = 0$) and slip ($K = 1$) boundary conditions. Throughout this paper, it is required that ρ and μ are positive constants. See, for example, [17, 22] on conservation laws of fluid motion and the derivation of the above equations.

The stability theory of laminar flows between two concentric rotating spheres has been studied from the numerical point of view, which is relevant to the global astrophysical and geophysical processes. As is well known in [4, 23] and the references given there, the spherical Couette flow with the rotating inner sphere is characterized by three parameters, i.e., the Reynolds number, clearance ratio and angular acceleration. Formation of complicated vortices is observed at a high Reynolds number. More precisely, the vortex structure consists of one toroidal vortex near the equator and some pairs of spiral vortices in high-latitude regions whose axes are tilted with respect to the azimuthal direction. The laminar-turbulent transition varies from the range of clearance ratios which are classified by the narrow, medium and wide gap cases. In the case where Ω is axisymmetric defined as in [3, Definition-Lemma 1], g and h can be regarded as gravity in Ω and rotation of Ω_1 respectively. The stability theory of stationary solutions to (1.3) yields that of the generalized spherical Couette flow from the mathematical point of view. Some stationary problems related to (1.3) have been studied in recent years. Solonnikov and Sčadilov [25] first discussed the existence and uniqueness of stationary solutions to (1.3) in the case where $K = 1$. Here L_2 estimates for the stationary solutions were established. Furthermore, Itoh, Tanaka and Tani [13] treated the existence, uniqueness and asymptotic stability of stationary solutions to the initial-boundary value problem similar to (1.3). It was proved that the stationary solutions are asymptotically stable in $L_{2,\sigma}(\Omega)$ if they are small in $(W^1_\infty(\Omega))^n$, where $L_{p,\sigma}(\Omega)$ $(1 < p < \infty)$ is the closed subspace of $(L_p(\Omega))^n$ defined as in Section 6.

As for the analyticity of the semigroup on $L_{p,\sigma}(\Omega)$ generated by the Stokes operator, our argument is essentially based on the following resolvent problem for the system of $n+1$ equations:

$$\begin{cases} \mathrm{div} u = f & \text{in } \Omega, \\ \lambda u - \mathrm{div} T(u,p) = g & \text{in } \Omega, \\ u_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ K(T(u,p)\nu)_\tau + (1-K)u_\tau|_{\Gamma_0} = h^0 & \text{on } \Gamma_0, \\ u_\tau|_{\Gamma_1} = h^1 & \text{on } \Gamma_1, \end{cases} \tag{1.4}$$

where λ is the resolvent parameter, f, $g = (g_1, \cdots, g_n)^T$, $h^0 = (h^0_1, \cdots, h^0_n)^T$ and $h^1 = (h^1_1, \cdots, h^1_n)^T$ are given functions defined in Ω, in Ω, on Γ_0 and on Γ_1 respectively. Note that (1.4) is relevant to not only the analyticity of the semigroup but also the existence and uniqueness of solutions to (1.5) below. Our results concerning (1.4) are the existence and uniqueness of solutions to (1.4) satisfying L_p estimates. The existence and uniqueness of generalized solutions to (1.4) are obtained from the modified method of [25]. (1.4) has a more complicated boundary condition than [7], [24], so the new method is required for L_p estimates for solutions to (1.4). It is well known in [7], [24] that we have L_p estimates for solutions to the Stokes equations in $\mathbb{R}^n$, in $\mathbb{R}^n_+$ and in the bent half-space H^n_ω with the no-slip and Navier boundary conditions. By combining the localization method with the above fact, L_p estimates for solutions to (1.4) are established.

By virtue of our results concerning (1.4), we can study the asymptotic properties of solutions to the following stationary problem for the system of $n+1$ equations associated with (1.3):

$$\begin{cases} \mathrm{div}\bar{u} = 0 & \text{in } \Omega, \\ \rho(\bar{u}\cdot\nabla)\bar{u} - \mathrm{div}T(\bar{u},\bar{p}) = \rho g & \text{in } \Omega, \\ \bar{u}_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ K(T(\bar{u},\bar{p})\nu)_\tau + (1-K)\bar{u}_\tau|_{\Gamma_0} = 0 & \text{on } \Gamma_0, \\ \bar{u}_\tau|_{\Gamma_1} = h & \text{on } \Gamma_1. \end{cases} \tag{1.5}$$

First, we prove the existence and uniqueness of solutions to this problem, i.e., (1.5) has uniquely a solution in $(W_p^2(\Omega))^n$ satisfying L_p estimates for any $n < p < \infty$ if g and h are small in $(L_p(\Omega))^n$ and in $(W_p^{2-1/p}(\Gamma_1))^n$ respectively. The existence and uniqueness of solutions to (1.5) is obtained from L_p estimates for solutions to (1.4) and the Banach fixed point theorem. Second, we proceed to obtain the asymptotic stability of solutions to (1.5) in $L_{p,\sigma}(\Omega)$. Set $v = u - \bar{u}$, $q = p - \bar{p}$. Then we have the following initial-boundary value problem for the system of $n+1$ equations:

$$\begin{cases} \mathrm{div}v = 0 & \text{in } \Omega\times(0,T), \\ \rho\partial_t v - \mathrm{div}T(v,q) = -\rho\{(v\cdot\nabla)v + (v\cdot\nabla)\bar{u} + (\bar{u}\cdot\nabla)v\} & \text{in } \Omega\times(0,T), \\ v|_{t=0} = u_0 - \bar{u} & \text{in } \Omega, \\ v_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega\times(0,T), \\ K(T(v,q)\nu)_\tau + (1-K)v_\tau|_{\Gamma_0} = 0 & \text{on } \Gamma_0\times(0,T), \\ v_\tau|_{\Gamma_1} = 0 & \text{on } \Gamma_1\times(0,T). \end{cases} \tag{1.6}$$

It is proved in this paper that the abstract initial value problem for (1.6) has uniquely a mild solution globally in time if $\bar{u}$ and $u_0 - \bar{u}$ are small in $(W_p^1(\Omega))^n$ and in $L_{p,\sigma}(\Omega)$ respectively. The second result yields that solutions to (1.5) are asymptotically stable in $L_{p,\sigma}(\Omega)$ if they are small in $(W_p^1(\Omega))^n$. The asymptotic stability of solutions to (1.5) is proved by the theory of analytic semigroups on Banach spaces, e.g., [12, Chapter 3], [20, Chapter 6].

This paper is divided into three parts, i.e., Part I (Sections 2 and 3), Part II (Sections 4 and 5) and Part III (Sections 6–8). The existence of determining nodes for (1.1) and (1.2) is stated and proved in Part I. In Part II, we state and prove our main results concerning the existence of determining nodes for (I). Part III is devoted to the analyticity of the semigroup on $L_{p,\sigma}(\Omega)$ generated by the Stokes operator. Finally, in Section 9, we apply the L_p-theory of determining nodes to the semilinear heat equation and the Navier-Stokes equations, and give the proofs of the existence, uniqueness and asymptotic stability of solutions to (1.5) in $L_{p,\sigma}(\Omega)$.

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2 Part I: The L_2-theory of determining nodes

In this part, we are concerned with the L_2-theory of determining nodes for (1.1) and (1.2). This part is organized as follows: In Section 2, we state our main results concerning the existence of determining nodes for (1.1) and (1.2) after setting up notation and terminology used in this part. The proofs of our main results are given in Section 3. Note that main results on the L_2-theory of determining nodes were published as a paper by Kakizawa [14].

2.1 Function spaces

All functions which appear in Sections 2 and 3 are either H or H^n-valued. For the sake of notational simplicity, we will not distinguish them from their values, i.e., H^n will also be simply denoted by H.

Function spaces and basic notation which we use throughout this paper are introduced as follows: The norm in $L_p(\Omega)$ $(1 \leq p \leq \infty)$ and in the Sobolev space $H^k(\Omega)$ $(k \in \mathbb{Z},\ k \geq 0)$ are denoted by $\|\cdot\|_{L_p(\Omega)}$ and $\|\cdot\|_{H^k(\Omega)}$ respectively, $H^0(\Omega) = L_2(\Omega)$. Moreover, the scalar product in $L_2(\Omega)$ and in $H^k(\Omega)$ are denoted by $(\cdot,\cdot)_{L_2(\Omega)}$ and $(\cdot,\cdot)_{H^k(\Omega)}$ respectively. $C_0^\infty(\Omega)$ is the set of all functions which are infinitely differentiable and have compact support in Ω. $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. Note that $H_0^1(\Omega)$ is characterized as $H_0^1(\Omega) = \{u \in H^1(\Omega)\ ;\ u|_{\partial\Omega} = 0\}$. As is well known in the theory of Hilbert spaces, $L_2(\Omega)$ is decomposed into $L_2(\Omega) = H \oplus H^\perp$, where $H^\perp$ is the orthogonal complement of H. Let P be the orthogonal projection of $L_2(\Omega)$ onto H. The norm in $C(\overline{\Omega})$ is denoted by $\|\cdot\|_{C(\overline{\Omega})}$. $C^{0,\gamma}(\overline{\Omega})$ $(0 < \gamma \leq 1)$ is the Banach space of all functions which are uniformly Hölder continuous with the exponent γ on $\overline{\Omega}$. The norm in $C^{0,\gamma}(\overline{\Omega})$ is denoted by $\|\cdot\|_{C^{0,\gamma}(\overline{\Omega})}$, i.e.,

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} := \|u\|_{C(\overline{\Omega})} + [[u]]_{C^{0,\gamma}(\overline{\Omega})},\quad [[u]]_{C^{0,\gamma}(\overline{\Omega})} := \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

Let I be an open interval in $\mathbb{R}$, $(X, \|\cdot\|_X)$ be a Banach space. $L_p(I;X)$ $(1 \leq p < \infty)$ is the Banach space of all X-valued functions u which u is strongly measurable and $\|u\|_X^p$ is integrable in I. $L_\infty(I;X)$ is the Banach space of all X-valued functions u which u is strongly measurable and $\|u\|_X$ is essentially bounded in I. The norm in $L_p(I;X)$ and in $L_\infty(I;X)$ are denoted by $\|\cdot\|_{L_p(I;X)}$ and $\|\cdot\|_{L_\infty(I;X)}$ respectively. In the case where I is a bounded closed interval in $\mathbb{R}$, $C(I;X)$ is the Banach space of all X-valued functions which are continuous on I. If I is not bounded or closed, $C_b(I;X)$ is the Banach space of all X-valued functions which are bounded and continuous in I. The norm in $C(I;X)$ and in $C_b(I;X)$ is denoted by $\|\cdot\|_{C(I;X)}$ and $\|\cdot\|_{C_b(I;X)}$ respectively.

2.2 Strong solutions to (1.1) and (1.2)

In this subsection, we will make the properties of A and $F(u)$ which appeared in (1.1). First, A is the densely defined closed linear operator from $D(A) := H^2(\Omega) \cap V$ to H defined as

$$Au = -P\left\{\sum_{i,j=1}^n \partial_{x_j}(a_{ij}\partial_{x_i}u)\right\}.$$

It is required throughout this paper that A has the following properties (A.1)–(A.4):

(A.1) $a_{ij} \in C^{0,1}(\overline{\Omega})$ for any $i,j = 1,\cdots,n$.

(A.2) $a_{ij} = a_{ji}$ on $\overline{\Omega}$ for any $i,j = 1,\cdots,n$.

(A.3) There exists a positive constant a such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq a|\xi|^2$$

for any $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$.

(A.4) Set $(u,v)_{D(A)} = (Au, Av)_{L_2(\Omega)}$, $\|u\|_{D(A)} = ((u,u)_{D(A)})^{1/2}$. Then $\|\cdot\|_{D(A)}$ is equivalent to the standard norm in $H^2(\Omega)$. Therefore, there exist two positive constants a_1 and a_2 such that

$$a_1\|u\|_{H^2(\Omega)} \leq \|u\|_{D(A)} \leq a_2\|u\|_{H^2(\Omega)}$$

for any $u \in D(A)$.

Note that $A = -\kappa\Delta$ is a typical example of A. The norm $\|-\kappa\Delta\cdot\|_{L_2(\Omega)}$ induced by $-\kappa\Delta$ is equivalent to the standard norm in $H^2(\Omega)$, which follows from [11, Theorem 8.12]. Second, $F(u)$ is the nonlinear term satisfying the following properties (F.1), (F.2):

(F.1) $F(0) = 0$.

(F.2) There exist two constants $C > 0$ and $p > 1$ such that

$$\|F(u) - F(v)\|_{L_2(\Omega)} \leq C(\|u\|_{D(A)}^{p-1} + \|v\|_{D(A)}^{p-1})\|u - v\|_{H^1(\Omega)}$$

for any $u, v \in D(A)$.

It is important for our main results that $F(u) = |u|^{p-1}u$ and $F(u) = -P(u \cdot \nabla)u$ can be considered. By virtue of (A.1)–(A.4), the scalar product and the norm in V can be introduced as follows:

$$(u,v)_a = \sum_{i,j=1}^{n} (a_{ij}\partial_{x_i}u, \partial_{x_j}v)_{L_2(\Omega)}, \quad \|u\|_a = ((u,u)_a)^{1/2}.$$

It follows easily from (A.3) and the Schwarz inequality that $\|\cdot\|_a$ and the standard norm in $H^1(\Omega)$ are equivalent norms in V. Consequently, there exist two positive constants a_3 and a_4 such that

$$a_3\|u\|_{H^1(\Omega)} \leq \|u\|_a \leq a_4\|u\|_{H^1(\Omega)}$$

for any $u \in V$. Finally, strong solutions to (1.1) and (1.2) are defined as follows:

Definition 2.1. Let $u_0 \in V$, $f \in L_2((0,\infty); H)$. Then u is called a strong solution to (1.1) if it satisfies

$$u \in L_2((0,\infty); D(A)) \cap C_b([0,\infty); V), \quad d_t u \in L_2((0,\infty); H)$$

and (1.1). Let $\mathcal{S}(u_0, f)$ be the set of all functions which are strong solutions to (1.1).

Definition 2.2. Let $\bar{f} \in H$. Then $\bar{u}$ is called a strong solution to (1.2) if it satisfies

$$\bar{u} \in D(A)$$

and (1.2). Let $\mathcal{S}(\bar{f})$ be the set of all functions which are strong solutions to (1.2).

2.3 Main results on the L_2-theory of determining nodes

Our main results of Sections 2 and 3 will be stated in this subsection. We begin by formulation of determining nodes. For any $N \in \mathbb{Z}$, $N \geq 1$, $x \in \overline{\Omega}$, $u \in D(A)$, set

$$E_N = \{x_1, \cdots, x_N \; ; \; x_i \in \overline{\Omega}, \; i = 1, \cdots, N\},$$

$$d_N(x) = \min_{i=1,\cdots,N} |x - x_i|,$$

$$d_N = \max_{x\in\overline{\Omega}} d_N(x),$$

$$\eta_N(u) = \max_{i=1,\cdots,N} |u(x_i)|.$$

Note that E_N and d_N can be considered as determining nodes and the density of E_N in Ω respectively. As for strong solutions to (1.1) and (1.2), the following assumptions (H.1)–(H.4) are essentially required for our main results.

(H.1) $\mathcal{S}(\bar{f}) \neq \emptyset$ for any $\bar{f} \in H$.

(H.2) There exists a positive constant $M(\bar{f})$ for any $\bar{f} \in H$ such that

$$\|\bar{u}\|_{D(A)} \leq M(\bar{f})$$

for any $\bar{u} \in \mathcal{S}(\bar{f})$.

(H.3) $\mathcal{S}(u_0, f) \neq \emptyset$ for any $u_0 \in V$, $f \in L_\infty((0,\infty); H)$.

(H.4) There exists a positive constant $M(f, t_0)$ for any $R > 0$, $f \in L_\infty((0,\infty); H)$, $t_0 > 0$ such that

$$\|u\|_{C_b([t_0,\infty); D(A))} \leq M(f, t_0)$$

for any $u \in \mathcal{S}(V(R), f)$, where

$$\mathcal{S}(V(R), f) := \bigcup_{u_0\in V(R)} \mathcal{S}(u_0, f), \quad V(R) := \{u_0 \in V \ ; \ \|u_0\|_a \leq R\}.$$

Compared with Foias and Temam [6], this part is concerned with the L_2-theory of determining nodes for (1.1) and (1.2) which unifies the Navier-Stokes equations and the semilinear heat equation. Our main results are given by the following theorems on the existence of determining nodes for (1.1) and (1.2):

Theorem 2.1. *Let $n = 2, 3$, $\bar{f} \in H$, and assume* (H.1), (H.2). *Then there exists a positive constant δ_1 depending only on Ω, A, F and $M(\bar{f})$ such that if $0 < d_N \leq \delta_1$ and if $\bar{u}, \bar{v} \in \mathcal{S}(\bar{f})$ satisfy*

$$\bar{u}(x_i) = \bar{v}(x_i)$$

for any $i = 1, \cdots, N$, then

$$\bar{u} = \bar{v} \ \textit{in} \ \Omega.$$

Theorem 2.2. *Let $n = 2, 3$, $R > 0$, $f \in L_\infty((0,\infty); H)$, $t_0 > 0$, and assume* (H.2)–(H.4),

$$f(t) \to f_\infty \in H \ \textit{in} \ H \ \textit{as} \ t \to \infty.$$

Then there exists a positive constant δ_2 depending only on Ω, A, F, $M(f, t_0)$ and $M(f_\infty)$ such that if $0 < d_N \leq \delta_2$ and if $u \in \mathcal{S}(V(R), f)$ satisfies

$$u(x_i, t) \to \xi_i \in \mathbb{R} \ \textit{as} \ t \to \infty$$

for any $i = 1, \cdots, N$, then (1.2) *has uniquely a strong solution $u_\infty \in \mathcal{S}(f_\infty)$ satisfying*

$$u(t) \to u_\infty \ \textit{in} \ V \cap C^{0,\gamma}(\overline{\Omega}) \ \textit{as} \ t \to \infty$$

for any $0 < \gamma < 1/2$ and $u_\infty(x_i) = \xi_i$ for any $i = 1, \cdots, N$.

Theorem 2.3. *Let $n = 2, 3$, $R > 0$, $f, g \in L_\infty((0,\infty); H)$, $t_0 > 0$, and assume* (H.3), (H.4),

$$f(t) - g(t) \to 0 \text{ in } H \text{ as } t \to \infty.$$

Then there exists a positive constant δ_3 depending only on Ω, A, F, $M(f, t_0)$ and $M(g, t_0)$ such that if $0 < d_N \leq \delta_3$ and if $u \in \mathcal{S}(V(R), f)$, $v \in \mathcal{S}(V(R), g)$ satisfy

$$u(x_i, t) - v(x_i, t) \to 0 \text{ as } t \to \infty$$

for any $i = 1, \cdots, N$, then

$$u(t) - v(t) \to 0 \text{ in } V \cap C^{0,\gamma}(\overline{\Omega}) \text{ as } t \to \infty$$

for any $0 < \gamma < 1/2$.

2.4 Auxiliary lemmata

In this subsection, we will state three interpolation inequalities concerning the density of determining nodes. The following lemma yields that the standard norms in $C(\overline{\Omega})$, in $L_2(\Omega)$ and in $H^1(\Omega)$ are connected with d_N.

Lemma 2.1. *Let $n = 2, 3$. Then There exists a positive constant C_1 depending only on Ω such that*

$$\|u\|_{C(\overline{\Omega})} \leq \eta_N(u) + C_1 d_N^{1/2} \|u\|_{D(A)} \tag{2.1}$$

for any $u \in D(A)$.

Proof. See [6, Lemma 2.1]. □

Lemma 2.2. *Let $n = 2, 3$. Then There exist two positive constants C_2 and C_3 depending only on Ω such that*

$$\|u\|_{L^2(\Omega)} \leq C_2 \eta_N(u) + C_3 d_N^{1/2} \|u\|_{D(A)} \tag{2.2}$$

for any $u \in D(A)$.

Proof. See [6, Lemma 2.1]. □

Lemma 2.3. *Let $n = 2, 3$. Then There exist two positive constants C_4 and C_5 depending only on Ω such that*

$$\|u\|_{H^1(\Omega)} \leq C_4 d_N^{-1/4} \eta_N(u) + C_5 d_N^{1/4} \|u\|_{D(A)} \tag{2.3}$$

for any $u \in D(A)$.

Proof. See [6, Lemma 2.1]. □

3 Existence of determining nodes for (1.1) and (1.2)

Theorems 2.1–2.3 will be proved in this subsection. The proofs of Theorems 2.1–2.3 are based on the energy method with the aid of Lemma 2.3.

3.1 Proof of Theorem 2.1

Recall that $\bar{v}$ satisfies

$$A\bar{v} = F(\bar{v}) + \bar{f}. \tag{3.1}$$

Then it follows from (1.2), (3.1) that

$$A(\bar{u} - \bar{v}) = F(\bar{u}) - F(\bar{v}). \tag{3.2}$$

By taking the H-norm of (3.2) and (F.2), we obtain

$$\|\bar{u} - \bar{v}\|_{D(A)} \leq 2CM(\bar{f})^{p-1}\|\bar{u} - \bar{v}\|_{H^1(\Omega)}. \tag{3.3}$$

Notice that $\eta(\bar{u} - \bar{v}) = 0$, which follows from $\bar{u}(x_i) = \bar{v}(x_i)$ for any $i = 1, \cdots, N$. Then (2.3) yields

$$\|\bar{u} - \bar{v}\|_{H^1(\Omega)} \leq C_5 d_N^{1/4}\|\bar{u} - \bar{v}\|_{D(A)}. \tag{3.4}$$

Therefore, by (3.3), (3.4), we have

$$\|\bar{u} - \bar{v}\|_{D(A)} \leq 2CC_5 M(\bar{f})^{p-1} d_N^{1/4}\|\bar{u} - \bar{v}\|_{D(A)},$$

$$(1 - 2CC_5 M(\bar{f})^{p-1} d_N^{1/4})\|\bar{u} - \bar{v}\|_{D(A)} \leq 0. \tag{3.5}$$

Assume that

$$1 - 2CC_5 M(\bar{f})^{p-1} d_N^{1/4} > 0,$$

$$0 < d_N < \frac{1}{(2CC_5 M(\bar{f})^{p-1})^4}. \tag{3.6}$$

Then (3.5) implies $\bar{u} = \bar{v}$ in Ω. Consequently, the sufficient condition for (3.6) is

$$0 < \delta_1 < \frac{1}{(2CC_5 M(\bar{f})^{p-1})^4}.$$

This completes the proof of Theorem 2.1.

3.2 Proof of Theorem 2.2

We begin with the energy-type estimate for strong solutions to (1.1). Consider two times t and s satisfying $t < s$, write $s = t + \tau$ for any $\tau > 0$, and set $v(t) = u(t + \tau)$, $g(t) = f(t + \tau)$. Then (1.1) implies that v satisfies

$$d_t v + Av = F(v) + g. \tag{3.7}$$

It is easy to see from (1.1), (3.7) that

$$d_t(u - v) + A(u - v) = F(u) - F(v) + f - g. \tag{3.8}$$

By taking the H-scalar product of (3.8) with $A(u - v)$ and (F.2), we get

$$\frac{1}{2}d_t(\|u-v\|_a^2)+\|u-v\|_{D(A)}^2 \leq 2CM(f, t_0)^{p-1}\|u-v\|_{H^1(\Omega)}\|u-v\|_{D(A)}+\|u-v\|_{D(A)}\|f-g\|_{L^2(\Omega)}. \tag{3.9}$$

Notice that

$$\|u-v\|_{H^1(\Omega)} \le C_4 d_N^{-1/4}\eta_N(u-v) + C_5 d_N^{1/4}\|u-v\|_{D(A)}, \tag{3.10}$$

which follows from (2.3). Then, by (3.9), (3.10) and the Cauchy inequality, we have

$$\begin{aligned} d_t(\|u-v\|_a^2) + (1-4CC_5M(f,t_0)^{p-1}d_N^{1/4})\|u-v\|_{D(A)}^2 \\ \le 8C^2C_4^2M(f,t_0)^{2(p-1)}d_N^{-1/2}\eta_N(u-v)^2 + 2\|f-g\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.11}$$

Assume that

$$\begin{aligned} &1-4CC_5M(f,t_0)^{p-1}d_N^{1/4} > 0, \\ &0 < d_N < \frac{1}{(4CC_5M(f,t_0)^{p-1})^4}, \end{aligned} \tag{3.12}$$

and set

$$\lambda = \frac{a_1^2}{a_4^2}(1-4CC_5M(f,t_0)^{p-1}d_N^{1/4}) > 0,$$

$$h(t) = 8C^2C_4^2M(f,t_0)^{2(p-1)}d_N^{-1/2}\eta_N((u-v)(t))^2 + 2\|(f-g)(t)\|_{L^2(\Omega)}^2.$$

Then (3.11) yields

$$d_t(\|(u-v)(t)\|_a^2) + \lambda\|(u-v)(t)\|_a^2 \le h(t) \tag{3.13}$$

for any $t \ge t_0$. We shall show that $\{u(t)\}_{t\ge t_0}$ is a Cauchy sequence in V with the aid of (3.13). Since $f(t) \to f_\infty$ in H as $t\to\infty$ and $u(x_i,t)\to\xi_i$ as $t\to\infty$ for any $i=1,\cdots,N$, we have $h(t)\to 0$ as $t\to\infty$. Therefore, there exists a positive constant t_ε for any positive constant ε such that $|h(t)|\le\varepsilon$ for any $t\ge t_\varepsilon$. It is easy to see from (3.13) that

$$d_t(\|(u-v)(t)\|_a^2) + \lambda\|(u-v)(t)\|_a^2 \le \varepsilon \tag{3.14}$$

for any $t\ge t_\varepsilon$. The Gronwall lemma and (3.14) imply

$$\|u(t)-u(s)\|_a^2 \le \|(u-v)(t_\varepsilon)\|_a^2 e^{-\lambda(t-t_\varepsilon)} + \frac{\varepsilon}{\lambda}(1-e^{-\lambda(t-t_\varepsilon)}) \tag{3.15}$$

for any $t\ge t_\varepsilon$. By taking t and s to infinity in (3.15), we have

$$\limsup_{t,s\to\infty}\|u(t)-u(s)\|_a^2 \le \frac{\varepsilon}{\lambda}.$$

Since ε is an arbitrary positive constant, we conclude that $u(t)-v(t)\to 0$ in V as $t,s\to\infty$, i.e., $\{u(t)\}_{t\ge t_0}$ is a Cauchy sequence in V. The completeness of V yields that there exists a function $u_\infty\in V$ satisfying

$$u(t)\to u_\infty \text{ in } V \text{ as } t\to\infty. \tag{3.16}$$

As for the function u_∞ which is obtained above, we shall prove that $u_\infty\in\mathcal{S}(f_\infty)$ and $u_\infty(x_i)=\xi_i$ for any $i=1,\cdots,N$. Notice that $\{u(t)\}_{t\ge t_0}$ is bounded in $D(A)$ by virtue of (H.4). Then (3.16) implies $u_\infty\in D(A)$ and

$$u(t)\to u_\infty \text{ in } C^{0,\gamma}(\overline{\Omega}) \text{ as } t\to\infty \tag{3.17}$$

for any $0 < \gamma < 1/2$, which follows from the Rellich-Kondrachov theorem [1, Theorem 6.3]. Since $u(x_i, t) \to \xi_i$ as $t \to \infty$ for any $i = 1, \cdots, N$, it follows from (3.17) that $u_\infty(x_i) = \xi_i$ for any $i = 1, \cdots, N$. By taking t to infinity in the first equation of (1.1), a straightforward argument shows that $u_\infty \in \mathcal{S}(f_\infty)$. Assume that $\delta_2 \leq \delta_1(M(f_\infty))$. Then (1.2) has uniquely a strong solution $u_\infty \in \mathcal{S}(f_\infty)$ satisfying $u_\infty(x_i) = \xi_i$ for any $i = 1, \cdots, N$, which follows from Theorem 2.1. Therefore, the sufficient condition for (3.12) and desired properties of u_∞ is

$$0 < \delta_2 < \min\left\{\delta_1(M(f_\infty)), \frac{1}{(4CC_5 M(f, t_0)^{p-1})^4}\right\},$$

which completes the proof of Theorem 2.2.

3.3 Proof of Theorem 2.3

In the same manner as in Subsection 3.2, we shall establish the energy-type estimate for strong solution to (1.1). Recall that v satisfies

$$d_t v + Av = F(v) + g. \tag{3.18}$$

It follows easily from (1.1), (3.18) that

$$d_t(u - v) + A(u - v) = F(u) - F(v) + f - g. \tag{3.19}$$

By taking the H-scalar product of (3.19) with $A(u - v)$ and (F.2), we have

$$\begin{aligned} &\frac{1}{2} d_t(\|u - v\|_a^2) + \|u - v\|_{D(A)}^2 \\ &\quad \leq C(M(f, t_0)^{p-1} + M(g, t_0)^{p-1})\|u - v\|_{H^1(\Omega)}\|u - v\|_{D(A)} + \|u - v\|_{D(A)}\|f - g\|_{L^2(\Omega)}. \end{aligned} \tag{3.20}$$

Notice that

$$\|u - v\|_{H^1(\Omega)} \leq C_4 d_N^{-1/4}\eta_N(u - v) + C_5 d_N^{1/4}\|u - v\|_{D(A)}, \tag{3.21}$$

which follows from (2.3). Then, by (3.20), (3.21) and the Cauchy inequality, we get

$$\begin{aligned} &d_t(\|u - v\|_a^2) + \{1 - 2CC_5(M(f, t_0)^{p-1} + M(g, t_0)^{p-1})d_N^{1/4}\}\|u - v\|_{D(A)}^2 \\ &\qquad \leq 2C^2 C_4^2(M(f, t_0)^{p-1} + M(g, t_0)^{p-1})^2 d_N^{-1/2}\eta_N(u - v)^2 + 2\|f - g\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.22}$$

Assume that

$$\begin{aligned} &1 - 2CC_5(M(f, t_0)^{p-1} + M(g, t_0)^{p-1})d_N^{1/4} > 0, \\ &0 < d_N < \frac{1}{\{2CC_5(M(f, t_0)^{p-1} + M(g, t_0)^{p-1})\}^4}, \end{aligned} \tag{3.23}$$

and set

$$\lambda = \frac{a_1^2}{a_4^2}\{1 - 2CC_5(M(f, t_0)^{p-1} + M(g, t_0)^{p-1})d_N^{1/4}\} > 0,$$

$$h(t) = 2C^2 C_4^2(M(f, t_0)^{p-1} + M(g, t_0)^{p-1})^2 d_N^{-1/2}\eta_N((u - v)(t))^2 + \|(f - g)(t)\|_{L^2(\Omega)}^2.$$

Then (3.22) gives

$$d_t(\|(u - v)(t)\|_a^2) + \lambda\|(u - v)(t)\|_a^2 \leq h(t) \tag{3.24}$$

for any $t \geq t_0$. By (3.24), we shall prove that $u(t) - v(t) \to 0$ in V as $t \to \infty$. Notice that $f(t) - g(t) \to 0$ in H as $t \to \infty$ and $u(x_j, t) - v(x_j, t) \to 0$ as $t \to \infty$ for any $j = 1, \cdots, N$. Then we have $h(t) \to 0$ as $t \to \infty$. Hence, there exists a positive constant t_ε for any positive constant ε such that $|h(t)| \leq \varepsilon$ for any $t \geq t_\varepsilon$. We can see easily from (3.24) that

$$d_t(\|(u-v)(t)\|_a^2) + \lambda\|(u-v)(t)\|_a^2 \leq \varepsilon \tag{3.25}$$

for any $t \geq t_\varepsilon$. It follows from the Gronwall lemma and (3.25) that

$$\|(u-v)(t)\|_a^2 \leq \|(u-v)(t_\varepsilon)\|_a^2 e^{-\lambda(t-t_\varepsilon)} + \frac{\varepsilon}{\lambda}(1 - e^{-\lambda(t-t_\varepsilon)}) \tag{3.26}$$

for any $t \geq t_\varepsilon$. By taking t to infinity in (3.26), we have

$$\limsup_{t\to\infty} \|(u-v)(t)\|_a^2 \leq \frac{\varepsilon}{\lambda}.$$

Since ε is an arbitrary positive constant, we conclude that

$$u(t) - v(t) \to 0 \text{ in } V \text{ as } t \to \infty. \tag{3.27}$$

It remains to prove that $u(t) - v(t) \to 0$ in $C^{0,\gamma}(\overline{\Omega})$ as $t \to \infty$ for any $0 < \gamma < 1/2$. We can see easily from (H.4) that $\{(u-v)(t)\}_{t \geq t_0}$ is bounded in $D(A)$. By the Rellich-Kondrachov theorem, (3.27) yields

$$u(t) - v(t) \to 0 \text{ in } C^{0,\gamma}(\overline{\Omega}) \text{ as } t \to \infty.$$

Consequently, the sufficient condition for (3.23) is

$$0 < \delta_3 < \frac{1}{\{2CC_5(M(f,t_0)^{p-1} + M(g,t_0)^{p-1})\}^4}.$$

This completes the proof of Theorem 2.3.

4 Part II: The L_p-theory of determining nodes

In this part, we are concerned with the L_p-theory of determining nodes for (I). This part is organized as follows: In Section 4, we state our main results concerning the existence of determining nodes for (I) after setting up notation and terminology used in this part. The proofs of our main results are given in Section 5.

4.1 Function spaces

All functions appearing in Sections 4 and 5 are either X_p or $(X_p)^n$-valued. For the sake of notational simplicity, we will not distinguish them from their values, i.e., $(X_p)^n$ will also be simply denoted by X_p.

Function spaces and basic notation which we use throughout Sections 4 and 5 are introduced as follows: The norm in $L_r(\Omega)$ $(1 \leq r \leq \infty)$ and the norm in the Sobolev space $W_r^k(\Omega)$ $(k \in \mathbb{Z},\ k \geq 0)$ are denoted by $\|\cdot\|_{L_r(\Omega)}$ and $\|\cdot\|_{W_r^k(\Omega)}$ respectively, $W_r^0(\Omega) = L_r(\Omega)$. $C_0^\infty(\Omega)$ is the set of all functions which are infinitely differentiable and have compact support in Ω. $W_{r,0}^k(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $W_r^k(\Omega)$. $C^{k,\gamma}(\overline{\Omega})$ $(0 < \gamma \leq 1)$ is the Hölder space defined as in [1, 1.26–1.29], $C^{k,0}(\overline{\Omega}) = C^k(\overline{\Omega})$, $C^0(\overline{\Omega}) = C(\overline{\Omega})$.

Let I be an interval in $\mathbb{R}$, $(X, \|\cdot\|_X)$ be a Banach space. $C(I; X)$ is the set of all X-valued functions which are continuous in I. $C_b(I; X)$ is the set of all X-valued functions which are bounded continuous in I.

4.2 Sectorial operators in X_p and analytic semigroups on X_p

In this subsection, we will make the properties of A_p and $F(u)$ which appeared in (I). Let $(X, \|\cdot\|_X)$ be a Banach space, A be a densely defined closed linear operator in X. Then the resolvent set and the spectrum of A are denoted by $\rho(A)$ and $\sigma(A)$ respectively, $\mathrm{Re}\sigma(A) := \{\mathrm{Re}\lambda \ ; \ \lambda \in \sigma(A)\}$. First, for any $1 < p < \infty$, A_p is the densely defined closed linear operator in X_p satisfying the following properties (A.5), (A.6):

(A.5) A_p is a sectorial operator in X_p defined as in [12, Definition 1.3.1], $D(A_p) \subset W_p^2(\Omega)$, where $D(A_p)$ is the domain of A_p.

(A.6) $\mathrm{Re}\sigma(A_p) > 0$, where $\mathrm{Re}\sigma(A_p) > 0$ means that $\mathrm{Re}\lambda > 0$ for any $\lambda \in \sigma(A_p)$.

It is well known in [12, Theorem 1.3.4 and Definition 1.4.1], [20, Theorem 2.5.2 and Definition 2.6.7] that $-A_p$ generates a uniformly bounded analytic semigroup $\{e^{-tA_p}\}_{t\geq 0}$ on X_p, fractional powers A_p^α of A_p can be defined for any $\alpha \geq 0$, $A_p^0 = I_p$, where I_p is the identity operator in $L_p(\Omega)$. Let us introduce the Banach space derived from A_p^α. X_p^α is defined as $X_p^\alpha = D(A_p^\alpha)$ with the norm $\|\cdot\|_{X_p^\alpha} = \|A_p^\alpha \cdot\|_{L_p(\Omega)}$, $X_p^0 = X_p$.

We state some lemmata concerning sectorial operators in Banach spaces. See, for example, [12, Chapter 1], [20, Chapter 2] on the theory of analytic semigroups on Banach spaces and fractional powers of sectorial operators.

Lemma 4.1. *Let $1 < p < \infty$, $\alpha \geq 0$, $0 < \lambda_1 < \Lambda_1$, where $\Lambda_1 := \min\{\lambda_1 > 0 \ ; \ \lambda_1 \in \mathrm{Re}\sigma(A_p)\}$. Then there exists a positive constant C_{p,α,λ_1} depending only on n, Ω, p, A_p, α and λ_1 such that*

$$\|A_p^\alpha e^{-tA_p} u\|_{L_p(\Omega)} \leq C_{p,\alpha,\lambda_1} t^{-\alpha} e^{-\lambda_1 t} \|u\|_{L_p(\Omega)} \tag{4.1}$$

for any $u \in X_p$.

Proof. See [12, Theorem 1.4.3]. □

Lemma 4.2. *Let $1 < p < \infty$, $\alpha \geq 0$, $0 \leq \beta \leq \alpha$. Then there exists a positive constant $C_{p,\alpha,\beta}$ depending only on n, Ω, p, A_p, α and β such that*

$$\|u\|_{X_p^\beta} \leq C_{p,\alpha,\beta} \|u\|_{L_p(\Omega)}^{1-\beta/\alpha} \|u\|_{X_p^\alpha}^{\beta/\alpha} \tag{4.2}$$

for any $u \in X_p^\alpha$.

Proof. See [12, Exercise 1.4.5]. □

Lemma 4.3. *Let $1 < p < \infty$, $0 \leq \alpha \leq 1$. Then*

$$X_p^\alpha \hookrightarrow W_r^k(\Omega) \ \text{if} \ k \in \mathbb{Z},\ k \geq 0,\ 1 < r < \infty,\ \frac{1}{p} - \frac{2\alpha - k}{n} \leq \frac{1}{r} \leq \frac{1}{p}, \tag{4.3}$$

$$X_p^\alpha \hookrightarrow C^{k,\gamma}(\overline{\Omega}) \ \text{if} \ k \in \mathbb{Z},\ k \geq 0,\ 0 < \gamma < 1,\ \frac{1}{p} - \frac{2\alpha - (k+\gamma)}{n} \leq 0, \tag{4.4}$$

where $\hookrightarrow$ is the continuous inclusion.

Proof. See [12, Theorem 1.6.1]. □

Second, $F(u)$ is the nonlinear term satisfying the following properties (F.3), (F.4):

(F.3) $F(0) = 0$.

(F.4) There exist three constants $C_p > 0$, $0 < \alpha_1 < 1$ and $q > 1$ such that

$$\|F(u) - F(v)\|_{L_p(\Omega)} \le C_p(\|u\|^{q-1}_{X_p^{\alpha_1}} + \|v\|^{q-1}_{X_p^{\alpha_1}})\|u - v\|_{X_p^{\alpha_1}}$$

for any $u, v \in X_p^{\alpha_1}$.

Finally, we are concerned with mild solutions to (I). As is well known, (I) is reduced to the following abstract integral equation:

$$u(t) = e^{-tA_p}u_0 + \int_0^t e^{-(t-s)A_p}F(u)(s)ds + \int_0^t e^{-(t-s)A_p}f(s)ds \tag{II}$$

for any $t \ge 0$. A mild solution to (I) is defined as follows:

Definition 4.1. Let $1 < p < \infty$, $0 \le \alpha_0 < 1$, $u_0 \in X_p^{\alpha_0}$, $f \in C((0,\infty); X_p)$. Then u is called a mild solution to (I) if it satisfies

$$u \in C_b([0,\infty); X_p^{\alpha_0})$$

and (II). Let $\mathcal{S}(u_0, f)$ be the set of all functions which are mild solutions to (I).

4.3 Main results on the L_p-theory of determining nodes

Our main results of Sections 4 and 5 will be stated in this subsection. We begin with formulation of determining nodes. For any $N \in \mathbb{Z}$, $N \ge 1$, $x \in \overline{\Omega}$, $u \in C(\overline{\Omega})$, set

$$E_N = \{x_1, \cdots, x_N \ ; \ x_i \in \overline{\Omega}, \ i = 1, \cdots, N\},$$

$$d_N(x) = \min_{i=1,\cdots,N} |x - x_i|,$$

$$d_N = \max_{x \in \overline{\Omega}} d_N(x),$$

$$\eta_N(u) = \max_{i=1,\cdots,N} |u(x_i)|.$$

Note that E_N and d_N can be regarded as determining nodes and the density of E_N in Ω respectively. As for mild solutions to (I), the following assumptions (H.5), (H.6) are essentially required for our main results.

(H.5) $\mathcal{S}(u_0, f) \neq \emptyset$ for any $u_0 \in X_p^{\alpha_0}$, $f \in C((0,\infty); X_p)$.

(H.6) There exists a positive constant $M(f, t_0)$ for any $R > 0$, $f \in C((0,\infty); X_p)$, $t_0 > 0$ such that

$$\|u\|_{C_b([t_0,\infty); X_p^{\alpha_1})} \le M(f, t_0)$$

for any $u \in \mathcal{S}(X_p^{\alpha_0}(R), f)$, where

$$\mathcal{S}(X_p^{\alpha_0}(R), f) := \bigcup_{u_0 \in X_p^{\alpha_0}(R)} \mathcal{S}(u_0, f), \ X_p^{\alpha_0}(R) := \{u_0 \in X_p^{\alpha_0} \ ; \ \|u_0\|_{X_p^{\alpha_0}} \le R\}.$$

Compared with Kakizawa [14], this part is devoted to the L_p-theory of determining nodes for (I). Our main results overcome variety of boundary conditions, and clarify not only the asymptotic equivalence but also rate of monomial or exponential convergence. First, the following theorem yields the existence of determining nodes for (I) and rate of monomial convergence.

Theorem 4.1. *Let $n/2 < p < \infty$, $0 \le \alpha_0 < 1$, $R > 0$, $f, g \in C((0,\infty); X_p)$, $t_0 > 0$, and assume* (H.5), (H.6),

$$f(t) - g(t) \to 0 \text{ in } X_p \text{ as } t \to \infty.$$

Then there exists a positive constant δ_1 depending only on n, Ω, p, A_p, F, α_0, $M(f, t_0)$ and $M(g, t_0)$ such that if $0 < d_N \le \delta_1$ and if $u \in \mathcal{S}(X_p^{\alpha_0}(R), f)$, $v \in \mathcal{S}(X_p^{\alpha_0}(R), g)$ satisfy

$$u(x_i, t) - v(x_i, t) \to 0 \text{ as } t \to \infty$$

for any $i = 1, \cdots, N$, then

(i) *For any $\alpha_0 < \alpha < 1$,*

$$\|u(t) - v(t)\|_{X_p^\alpha} = O(t^{\alpha_0 - \alpha}) \text{ as } t \to \infty. \tag{4.5}$$

(ii) *For any $k \in \mathbb{Z}$, $k \ge 0$, $0 < \gamma < 1$, $k + \gamma \le 2\alpha - n/p$,*

$$\|u(t) - v(t)\|_{C^{k,\gamma}(\overline{\Omega})} = O(t^{\alpha_0 - \alpha}) \text{ as } t \to \infty \tag{4.6}$$

provided that $n/(2p) < \alpha < 1$.

Second, we proceed to the existence of determining nodes for (I) and rate of exponential convergence.

Theorem 4.2. *Let $n/2 < p < \infty$, $0 \le \alpha_0 < 1$, $R > 0$, $f, g \in C((0,\infty); X_p)$, $t_0 > 0$, and assume* (H.5), (H.6),

$$\|f(t) - g(t)\|_{L_p(\Omega)} = O(e^{-\lambda_1 t}) \text{ as } t \to \infty$$

for some $0 < \lambda_1 < \Lambda_1$. Then there exists a positive constant δ_2 depending only on n, Ω, p, A_p, F, α_0, $M(f, t_0)$ and $M(g, t_0)$ such that if $0 < d_N \le \delta_2$ and if $u \in \mathcal{S}(X_p^{\alpha_0}(R), f)$, $v \in \mathcal{S}(X_p^{\alpha_0}(R), g)$ satisfy

$$u(x_i, t) - v(x_i, t) = O(e^{-\lambda_1 t}) \text{ as } t \to \infty$$

for any $i = 1, \cdots, N$, then

(i) *For any $\alpha_0 \le \alpha < 1$,*

$$\|u(t) - v(t)\|_{X_p^\alpha} = O(t^{\alpha_0 - \alpha} e^{-\lambda_1 t}) \text{ as } t \to \infty. \tag{4.7}$$

(ii) *For any $k \in \mathbb{Z}$, $k \ge 0$, $0 < \gamma < 1$, $k + \gamma \le 2\alpha - n/p$,*

$$\|u(t) - v(t)\|_{C^{k,\gamma}(\overline{\Omega})} = O(t^{\alpha_0 - \alpha} e^{-\lambda_1 t}) \text{ as } t \to \infty \tag{4.8}$$

provided that $n/(2p) < \alpha < 1$.

4.4 Auxiliary lemmata

In this subsection, we will state three interpolation inequalities concerning the density of determining nodes. The following lemma yields that the standard norms in $C(\overline{\Omega})$, in $L^p(\Omega)$ and in X_p^β are connected with d_N.

Lemma 4.4. *Let $n/2 < p < \infty$, $n/(2p) < \alpha \leq 1$, $0 < \gamma < 1$, $0 < \gamma \leq 2\alpha - n/p$. Then there exists a positive constant C_1 depending only on n, Ω, p, α and γ such that*

$$\|u\|_{C(\overline{\Omega})} \leq \eta_N(u) + C_1 d_N^\gamma \|u\|_{X_p^\alpha} \tag{4.9}$$

for any $u \in X_p^\alpha$.

Proof. It follows from (4.4) that

$$\begin{aligned}
&\|u\|_{C^{0,\gamma}(\overline{\Omega})} \leq C_1 \|u\|_{X_p^\alpha},\\
&|u(x) - u(y)| \leq C_1 |x-y|^\gamma \|u\|_{X_p^\alpha},\\
&|u(x)| \leq |u(y)| + C_1 |x-y|^\gamma \|u\|_{X_p^\alpha}
\end{aligned} \tag{4.10}$$

for any $x, y \in \overline{\Omega}$. By taking the maximum with respect to $y \in E_N$, (4.10) leads to (4.9). □

Lemma 4.5. *Let $n/2 < p < \infty$, $n/(2p) < \alpha \leq 1$, $0 < \gamma < 1$, $0 < \gamma \leq 2\alpha - n/p$. Then there exist two positive constants C_2 and C_3 depending only on n, Ω, p, α and γ such that*

$$\|u\|_{L_p(\Omega)} \leq C_2 \eta_N(u) + C_3 d_N^\gamma \|u\|_{X_p^\alpha} \tag{4.11}$$

for any $u \in X_p^\alpha$.

Proof. It is equivalent to (4.9) that

$$|u(x)| \leq \eta_N(u) + C_1 d_N^\gamma \|u\|_{X_p^\alpha} \tag{4.12}$$

for any $x \in \overline{\Omega}$. By taking the L_p-norm of (4.12), we obtain

$$\|u\|_{L_p(\Omega)} \leq |\Omega|^{1/p} (\eta_N(u) + C_1 d_N^\gamma \|u\|_{X_p^\alpha}). \tag{4.13}$$

Set $C_2 = |\Omega|^{1/p}$, $C_3 = |\Omega|^{1/p} C_1$. Then (4.11) is established by (4.13). □

Lemma 4.6. *Let $n/2 < p < \infty$, $n/(2p) < \alpha \leq 1$, $0 \leq \beta \leq \alpha$, $0 < \gamma < 1$, $0 < \gamma \leq 2\alpha - n/p$. Then there exist two positive constants C_4 and C_5 depending only on n, Ω, p, α, β and γ such that*

$$\|u\|_{X_p^\beta} \leq C_4 d_N^{-\gamma(\beta/\alpha)} \eta_N(u) + C_5 d_N^{\gamma(1-\beta/\alpha)} \|u\|_{X_p^\alpha} \tag{4.14}$$

for any $u \in X_p^\alpha$.

Proof. It follows from (4.2), (4.11) that

$$\begin{aligned}
\|u\|_{X_p^\beta} &\leq C_{p,\alpha,\beta} \|u\|_{L_p(\Omega)}^{1-\beta/\alpha} \|u\|_{X_p^\alpha}^{\beta/\alpha}\\
&\leq C_{p,\alpha,\beta} (C_2 \eta_N(u) + C_3 d_N^\gamma \|u\|_{X_p^\alpha})^{1-\beta/\alpha} \|u\|_{X_p^\alpha}^{\beta/\alpha}\\
&\leq C_{p,\alpha,\beta} (C_2^{1-\beta/\alpha} \eta_N(u)^{1-\beta/\alpha} + C_3^{1-\beta/\alpha} d_N^{\gamma(1-\beta/\alpha)} \|u\|_{X_p^\alpha}^{1-\beta/\alpha}) \|u\|_{X_p^\alpha}^{\beta/\alpha}\\
&\leq C_{p,\alpha,\beta} C_2^{1-\beta/\alpha} \eta_N(u)^{1-\beta/\alpha} \|u\|_{X_p^\alpha}^{\beta/\alpha} + C_{p,\alpha,\beta} C_3^{1-\beta/\alpha} d_N^{\gamma(1-\beta/\alpha)} \|u\|_{X_p^\alpha},
\end{aligned}$$

$$\|u\|_{X_p^\beta} \leq (1-\beta/\alpha) C_{p,\alpha,\beta} C_2 C_3^{-\beta/\alpha} d_N^{-\gamma(\beta/\alpha)} \eta_N(u) + (1+\beta/\alpha) C_{p,\alpha,\beta} C_3^{1-\beta/\alpha} d_N^{\gamma(1-\beta/\alpha)} \|u\|_{X_p^\alpha}. \tag{4.15}$$

Set $C_4 = (1-\beta/\alpha) C_{p,\alpha,\beta} C_2 C_3^{-\beta/\alpha}$, $C_5 = (1+\beta/\alpha) C_{p,\alpha,\beta} C_3^{1-\beta/\alpha}$. Then (4.15) implies (4.14). □

4.5 X_p^α-estimates for mild solutions to (I)

This subsection provides X_p^α-estimates for mild solutions to (I). Note that $u \in \mathcal{S}(X_p^{\alpha_0}(R), f)$ and $v \in \mathcal{S}(X_p^{\alpha_0}(R), g)$ satisfy

$$u(t) = e^{-(t-t_0)A_p}u(t_0) + \int_{t_0}^t e^{-(t-s)A_p}F(u)(s)ds + \int_{t_0}^t e^{-(t-s)A_p}f(s)ds, \tag{4.16}$$

$$v(t) = e^{-(t-t_0)A_p}v(t_0) + \int_{t_0}^t e^{-(t-s)A_p}F(v)(s)ds + \int_{t_0}^t e^{-(t-s)A_p}g(s)ds \tag{4.17}$$

for any $t \geq t_0$. By subtracting (4.17) from (4.16), we obtain

$$\begin{aligned} \|u(t) - v(t)\|_{X_p^\alpha} \leq & C_{p,\alpha-\alpha_0,\lambda_1} t^{\alpha_0-\alpha} e^{-\lambda_1 t} \|u(t_0) - v(t_0)\|_{X_p^{\alpha_0}} \\ & + C_{p,\alpha,\lambda_1} C_p \int_{t_0}^t (t-s)^{-\alpha} e^{-\lambda_1(t-s)} (\|u(s)\|_{X_p^{\alpha_1}}^{q-1} + \|v(s)\|_{X_p^{\alpha_1}}^{q-1}) \|u(s) - v(s)\|_{X_p^{\alpha_1}} ds \\ & + C_{p,\alpha,\lambda_1} \int_{t_0}^t (t-s)^{-\alpha} e^{-\lambda_1(t-s)} \|f(s) - g(s)\|_{L_p(\Omega)} ds \end{aligned} \tag{4.18}$$

for any $\alpha_0 \leq \alpha < 1$, $t \geq t_0$. Here and hereafter we set

$$I^1_{\alpha,\lambda_1}(u-v)(t) = \int_{t_0}^t (t-s)^{-\alpha} e^{-\lambda_1(t-s)} \|u(s) - v(s)\|_{X_p^{\alpha_1}} ds,$$

$$I^2_{\alpha,\lambda_1}(f-g)(t) = \int_{t_0}^t (t-s)^{-\alpha} e^{-\lambda_1(t-s)} \|f(s) - g(s)\|_{L_p(\Omega)} ds$$

for any $\alpha_0 \leq \alpha < 1$, $t \geq t_0$.

5 Existence of determining nodes for (I)

Theorems 4.1 and 4.2 will be proved in this subsection. The proofs of Theorems 4.1 and 4.2 are based on the similar method to [9, Theorem 2.6] with the aid of Lemma 4.6.

5.1 Proof of Theorem 4.1

As is mentioned above, we shall establish X_p^α-estimates for $u - v$. Set

$$E_\alpha(u-v)(t) = \max_{t_0 \leq s \leq t} t^{\alpha-\alpha_0} \|u(s) - v(s)\|_{X_p^\alpha},$$

$$E(f-g)(t) = \max_{t_0 \leq s \leq t} \|f(s) - g(s)\|_{L_p(\Omega)},$$

$$H_N(u-v)(t) = \max_{t_0 \leq s \leq t} \eta_N(u(s) - v(s))$$

for any $\alpha_0 \leq \alpha < 1$, $t \geq t_0$. Then (4.18) yields

$$\begin{aligned} E_\alpha(u-v)(t) \leq & C_{p,\alpha-\alpha_0,\lambda_1} \|u(t_0) - v(t_0)\|_{X_p^{\alpha_0}} \\ & + C_{p,\alpha,\lambda_1} C_p (M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) t^{\alpha-\alpha_0} I^1_{\alpha,\lambda_1}(u-v)(t) \\ & + C_{p,\alpha,\lambda_1} t^{\alpha-\alpha_0} I^2_{\alpha,\lambda_1}(f-g)(t) \end{aligned} \tag{5.1}$$

for any $\alpha_0 \le \alpha < 1$, $t \ge t_0$. By choosing $\max\{\alpha_0, \alpha_1, n/(2p)\} < \alpha_2 < 1$ and applying (4.14) with $\alpha = \alpha_2$ and $\beta = \alpha_1$ to I^1_{α,λ_1}, we have

$$\begin{aligned} t^{\alpha-\alpha_0} I^1_{\alpha,\lambda_1}(u-v)(t) \le & C_4 d^{-\gamma(\alpha_1/\alpha_2)} t^{\alpha-\alpha_0} \int_{t_0}^t (t-s)^{-\alpha} e^{-\lambda_1(t-s)} \eta_N(u(s)-v(s))ds \\ & + C_5 d_N^{\gamma(1-\alpha_1/\alpha_2)} t^{\alpha-\alpha_0} \int_{t_0}^t (t-s)^{-\alpha} e^{-\lambda_1(t-s)} \|u(s)-v(s)\|_{X_p^{\alpha_2}} ds \end{aligned}$$

for any $\alpha_0 \le \alpha < 1$, $t \ge t_0$, which implies

$$\begin{aligned} t^{\alpha-\alpha_0} I^1_{\alpha,\lambda_1}(u-v)(t) \le & C_4 d^{-\gamma(\alpha_1/\alpha_2)} t^{\alpha-\alpha_0} H_N(u-v)(t) \int_{t_0}^t (t-s)^{-\alpha} e^{-\lambda_1(t-s)} ds \\ & + C_5 d_N^{\gamma(1-\alpha_1/\alpha_2)} t^{\alpha-\alpha_0} E_{\alpha_2}(u-v)(t) \int_{t_0}^t (t-s)^{-\alpha} s^{\alpha_0-\alpha_2} e^{-\lambda_1(t-s)} ds \end{aligned}$$

for any $\alpha_0 \le \alpha < 1$, $t \ge t_0$. Let $0 < \delta < 1 - \alpha_0$ be taken arbitrarily, and recall that there exists a positive constant C_{δ,λ_1} depending only on F, α_0, δ and λ_1 such that

$$\begin{aligned} & t^{1-\alpha_0}(1-\tau)^\delta e^{-\lambda_1 t(1-\tau)} \le C_{\delta,\lambda_1}, \\ & t^{1-\alpha_2}(1-\tau)^\delta e^{-\lambda_1 t(1-\tau)} \le C_{\delta,\lambda_1} \end{aligned}$$

for any $t \ge 0$, $0 \le \tau \le 1$. Then it follows that

$$\begin{aligned} t^{\alpha-\alpha_0} I^1_{\alpha,\lambda_1}(u-v)(t) \le & C_4 C_{\delta,\lambda_1} d^{-\gamma(\alpha_1/\alpha_2)} H_N(u-v)(t) \int_0^1 (1-\tau)^{-(\alpha+\delta)} d\tau \\ & + C_5 C_{\delta,\lambda_1} d_N^{\gamma(1-\alpha_1/\alpha_2)} E_{\alpha_2}(u-v)(t) \int_0^1 (1-\tau)^{-(\alpha+\delta)} \tau^{\alpha_0-\alpha_2} d\tau, \end{aligned}$$

$$\begin{aligned} t^{\alpha-\alpha_0} I^1_{\alpha,\lambda_1}(u-v)(t) \le & C_4 C_{\delta,\lambda_1} (1-\alpha-\delta)^{-1} d^{-\gamma(\alpha_1/\alpha_2)} H_N(u-v)(t) \\ & + C_5 C_{\delta,\lambda_1} B(1-\alpha-\delta, 1+\alpha_0-\alpha_2) d_N^{\gamma(1-\alpha_1/\alpha_2)} E_{\alpha_2}(u-v)(t) \end{aligned} \tag{5.2}$$

for any $\alpha_0 \le \alpha < 1 - \delta$, $t \ge t_0$, where $B(x,y)$ $(x > 0,\ y > 0)$ is the beta function. As for I^2_{α,λ_1}, we have

$$t^{\alpha-\alpha_0} I^2_{\alpha,\lambda_1}(f-g)(t) \le t^{\alpha-\alpha_0} E(f-g)(t) \int_{t_0}^t (t-s)^{-\alpha} e^{-\lambda_1(t-s)} ds$$

for any $\alpha_0 \le \alpha < 1$, $t \ge t_0$. By the same argument as in (5.2), we obtain

$$t^{\alpha-\alpha_0} I^2_{\alpha,\lambda_1}(f-g)(t) \le C_{\delta,\lambda_1} E(f-g) \int_0^1 (1-\tau)^{-(\alpha+\delta)} d\tau,$$

$$t^{\alpha-\alpha_0} I^2_{\alpha,\lambda_1}(f-g)(t) \le C_{\delta,\lambda_1} (1-\alpha-\delta)^{-1} E(f-g)(t) \tag{5.3}$$

for any $\alpha_0 \le \alpha < 1 - \delta$, $t \ge t_0$. Consequently, it follows from (5.1)–(5.3) that

$$\begin{aligned} E_\alpha(u-v)(t) \le & C_{p,\alpha-\alpha_0,\lambda_1} \|u(t_0) - v(t_0)\|_{X_p^{\alpha_0}} \\ & + C_{p,\alpha,\lambda_1} C_p C_4 C_{\delta,\lambda_1} (1-\alpha-\delta)^{-1} (M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) d^{-\gamma(\alpha_1/\alpha_2)} H_N(u-v)(t) \\ & + C_{p,\alpha,\lambda_1} C_p C_5 C_{\delta,\lambda_1} B(1-\alpha-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)} E_{\alpha_2}(u-v)(t) \\ & + C_{p,\alpha,\lambda_1} C_{\delta,\lambda_1} (1-\alpha-\delta)^{-1} E(f-g)(t) \end{aligned}$$

$$(5.4)$$

for any $\alpha_0 \leq \alpha < 1-\delta$, $t \geq t_0$. Assume that

$$0 < d_N < \frac{1}{\{C_{p,\alpha_2,\lambda_1} C_p C_5 C_{\delta,\lambda_1} B(1-\alpha_2-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1}+M(g,t_0)^{q-1})\}^{1/\{\gamma(1-\alpha_1/\alpha_2)\}}}, \tag{5.5}$$

and set

$$K^1_{\lambda_1} = \frac{C_{p,\alpha_2-\alpha_0,\lambda_1}}{1-C_{p,\alpha_2,\lambda_1} C_p C_5 C_{\delta,\lambda_1} B(1-\alpha_2-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1}+M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)}},$$

$$K^2_{\lambda_1} = \frac{C_{p,\alpha_2,\lambda_1} C_p C_4 C_{\delta,\lambda_1} (1-\alpha_2-\delta)^{-1}(M(f,t_0)^{q-1}+M(g,t_0)^{q-1}) d^{-\gamma(\alpha_1/\alpha_2)}}{1-C_{p,\alpha_2,\lambda_1} C_p C_5 C_{\delta,\lambda_1} B(1-\alpha_2-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1}+M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)}},$$

$$K^3_{\lambda_1} = \frac{C_{p,\alpha_2,\lambda_1} C_{\delta,\lambda_1} (1-\alpha_2-\delta)^{-1}}{1-C_{p,\alpha_2,\lambda_1} C_p C_5 C_{\delta,\lambda_1} B(1-\alpha_2-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1}+M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)}}.$$

Then (5.4) yields

$$E_{\alpha_2}(u-v)(t) \leq K^1_{\lambda_1} \|u(t_0)-v(t_0)\|_{X_p^{\alpha_0}} + K^2_{\lambda_1} H_N(u-v)(t) + K^3_{\lambda_1} E(f-g)(t) \tag{5.6}$$

for any $t \geq t_0$. Set

$$\begin{aligned}
K^1_{\alpha,\lambda_1} =& C_{p,\alpha-\alpha_0,\lambda_1} \\
&+ C_{p,\alpha,\lambda_1} C_p C_5 C_{\delta,\lambda_1} B(1-\alpha-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1}+M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)} K^1_{\lambda_1}, \\
K^2_{\alpha,\lambda_1} =& C_{p,\alpha,\lambda_1} C_p C_4 C_{\delta,\lambda_1} (1-\alpha-\delta)^{-1}(M(f,t_0)^{q-1}+M(g,t_0)^{q-1}) d^{-\gamma(\alpha_1/\alpha_2)} \\
&+ C_{p,\alpha,\lambda_1} C_p C_5 C_{\delta,\lambda_1} B(1-\alpha-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1}+M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)} K^2_{\lambda_1}, \\
K^3_{\alpha,\lambda_1} =& C_{p,\alpha,\lambda_1} C_{\delta,\lambda_1} (1-\alpha-\delta)^{-1} \\
&+ C_{p,\alpha,\lambda_1} C_p C_5 C_{\delta,\lambda_1} B(1-\alpha-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1}+M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)} K^3_{\lambda_1}.
\end{aligned}$$

for any $\alpha_0 \leq \alpha < 1-\delta$. Then it follows from (5.4), (5.6) that

$$E_\alpha(u-v)(t) \leq K^1_{\alpha,\lambda_1} \|u(t_0)-v(t_0)\|_{X_p^{\alpha_0}} + K^2_{\alpha,\lambda_1} H_N(u-v)(t) + K^3_{\alpha,\lambda_1} E(f-g)(t) \tag{5.7}$$

for any $\alpha_0 \leq \alpha < 1-\delta$, $t \geq t_0$, which leads to (4.5). Furthermore, the arbitrariness of the choice of δ allows us to assume that $\alpha_0 \leq \alpha < 1$. In the case where $n/(2p) < \alpha < 1$, we can see easily from (4.4), (5.7) that (4.6) holds for any $k \in \mathbb{Z}$, $k \geq 0$, $0 < \gamma < 1$, $k+\gamma \leq 2\alpha - n/p$. Therefore, the sufficient condition for (4.5), (4.6) is

$$0 < \delta_1 < \frac{1}{\{C_{p,\alpha_2,\lambda_1} C_p C_5 C_{\delta,\lambda_1} B(1-\alpha_2-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1}+M(g,t_0)^{q-1})\}^{1/\{\gamma(1-\alpha_1/\alpha_2)\}}}.$$

5.2 Proof of Theorem 4.2

In the same manner as in Subsection 5.1, we shall obtain exponentially weighted X_p^α-estimates for $u-v$. Let $\lambda_1 < \lambda_2 < \Lambda_1$, and set

$$E_{\alpha,\lambda_1}(u-v)(t) = \max_{t_0\le s\le t} t^{\alpha-\alpha_0}e^{\lambda_1 s}\|u(s)-v(s)\|_{X_p^\alpha},$$

$$E_{\lambda_1}(f-g)(t) = \max_{t_0\le s\le t} e^{\lambda_1 s}\|f(s)-g(s)\|_{L_p(\Omega)},$$

$$H_{N,\lambda_1}(u-v)(t) = \max_{t_0\le s\le t} e^{\lambda_1 s}\eta_N(u(s)-v(s))$$

for any $\alpha_0 \le \alpha < 1$, $t \ge t_0$. Then (4.18) implies

$$\begin{aligned} E_{\alpha,\lambda_1}(u-v)(t) \le & C_{p,\alpha-\alpha_0,\lambda_1}\|u(t_0)-v(t_0)\|_{X_p^{\alpha_0}} \\ & + C_{p,\alpha,\lambda_2}C_p(M(f,t_0)^{q-1}+M(g,t_0)^{q-1})t^{\alpha-\alpha_0}e^{\lambda_1 t}I^1_{\alpha,\lambda_2}(u-v)(t) \\ & + C_{p,\alpha,\lambda_2}t^{\alpha-\alpha_0}e^{\lambda_1 t}I^2_{\alpha,\lambda_2}(f-g)(t) \end{aligned} \tag{5.8}$$

for any $\alpha_0 \le \alpha < 1$, $t \ge t_0$. By choosing $\max\{\alpha_0,\alpha_1,n/(2p)\} < \alpha_2 < 1$ and applying (4.14) with $\alpha=\alpha_2$ and $\beta=\alpha_1$ to I^1_{α,λ_2}, we obtain

$$\begin{aligned} t^{\alpha-\alpha_0}e^{\lambda_1 t}I^1_{\alpha,\lambda_2}(u-v)(t) \le & C_4 d^{-\gamma(\alpha_1/\alpha_2)}t^{\alpha-\alpha_0}\int_{t_0}^t (t-s)^{-\alpha}e^{-(\lambda_2-\lambda_1)(t-s)}e^{\lambda_1 s}\eta_N(u(s)-v(s))ds \\ & + C_5 d_N^{\gamma(1-\alpha_1/\alpha_2)}t^{\alpha-\alpha_0}\int_{t_0}^t (t-s)^{-\alpha}e^{-(\lambda_2-\lambda_1)(t-s)}e^{\lambda_1 s}\|u(s)-v(s)\|_{X_p^{\alpha_2}}ds \end{aligned}$$

for any $\alpha_0 \le \alpha < 1$, $t \ge t_0$, which gives

$$\begin{aligned} t^{\alpha-\alpha_0}e^{\lambda_1 t}I^1_{\alpha,\lambda_2}(u-v)(t) \le & C_4 d^{-\gamma(\alpha_1/\alpha_2)}t^{\alpha-\alpha_0}H_{N,\lambda_1}(u-v)(t)\int_{t_0}^t (t-s)^{-\alpha}e^{-(\lambda_2-\lambda_1)(t-s)}ds \\ & + C_5 d_N^{\gamma(1-\alpha_1/\alpha_2)}t^{\alpha-\alpha_0}E_{\alpha_2,\lambda_1}(u-v)(t)\int_{t_0}^t (t-s)^{-\alpha}s^{\alpha_0-\alpha_2}e^{-(\lambda_2-\lambda_1)(t-s)}ds \end{aligned}$$

for any $\alpha_0 \le \alpha < 1$, $t \ge t_0$. Let $0 < \delta < 1-\alpha_0$ be taken arbitrarily, and recall that

$$t^{1-\alpha_0}(1-\tau)^\delta e^{-(\lambda_2-\lambda_1)t(1-\tau)} \le C_{\delta,\lambda_2-\lambda_1},$$

$$t^{1-\alpha_2}(1-\tau)^\delta e^{-(\lambda_2-\lambda_1)t(1-\tau)} \le C_{\delta,\lambda_2-\lambda_1}$$

for any $t \ge 0$, $0 \le \tau \le 1$. Then it follows that

$$\begin{aligned} t^{\alpha-\alpha_0}e^{\lambda_1 t}I^1_{\alpha,\lambda_2}(u-v)(t) \le & C_4 C_{\delta,\lambda_2-\lambda_1}d^{-\gamma(\alpha_1/\alpha_2)}H_{N,\lambda_1}(u-v)(t)\int_0^1 (1-\tau)^{-(\alpha+\delta)}d\tau \\ & + C_5 C_{\delta,\lambda_2-\lambda_1}d_N^{\gamma(1-\alpha_1/\alpha_2)}E_{\alpha_2,\lambda_1}(u-v)(t)\int_0^1 (1-\tau)^{-(\alpha+\delta)}\tau^{\alpha_0-\alpha_2}d\tau, \end{aligned}$$

$$\begin{aligned} t^{\alpha-\alpha_0}e^{\lambda_1 t}I^1_{\alpha,\lambda_2}(u-v)(t) \le & C_4 C_{\delta,\lambda_2-\lambda_1}(1-\alpha-\delta)^{-1}d^{-\gamma(\alpha_1/\alpha_2)}H_{N,\lambda_1}(u-v)(t) \\ & + C_5 C_{\delta,\lambda_2-\lambda_1}B(1-\alpha-\delta,1+\alpha_0-\alpha_2)d_N^{\gamma(1-\alpha_1/\alpha_2)}E_{\alpha_2,\lambda_1}(u-v)(t) \end{aligned} \tag{5.9}$$

for any $\alpha_0 \le \alpha < 1-\delta$, $t \ge t_0$. Concerning I^2_{α,λ_2}, we have

$$t^{\alpha-\alpha_0}e^{\lambda_1 t}I^2_{\alpha,\lambda_2}(f-g)(t) \le t^{\alpha-\alpha_0}E_{\lambda_1}(f-g)(t)\int_{t_0}^t (t-s)^{-\alpha}e^{-(\lambda_2-\lambda_1)(t-s)}ds$$

for any $\alpha_0 \le \alpha < 1$, $t \ge t_0$. The same argument as in (5.9) shows that

$$t^{\alpha-\alpha_0} e^{\lambda_1 t} I^2_{\alpha,\lambda_2}(f-g)(t) \le C_{\delta,\lambda_2-\lambda_1} E_{\lambda_1}(f-g) \int_0^1 (1-\tau)^{-(\alpha+\delta)} d\tau,$$

$$t^{\alpha-\alpha_0} e^{\lambda_1 t} I^2_{\alpha,\lambda_2}(f-g)(t) \le C_{\delta,\lambda_2-\lambda_1} (1-\alpha-\delta)^{-1} E_{\lambda_1}(f-g)(t) \tag{5.10}$$

for any $\alpha_0 \le \alpha < 1-\delta$, $t \ge t_0$. Therefore, we conclude from (5.8)–(5.10) that

$$\begin{aligned}
&E_{\alpha,\lambda_1}(u-v)(t) \le C_{p,\alpha-\alpha_0,\lambda_1} \|u(t_0)-v(t_0)\|_{X_p^{\alpha_0}} \\
&+ C_{p,\alpha,\lambda_2} C_p C_4 C_{\delta,\lambda_2-\lambda_1} (1-\alpha-\delta)^{-1} (M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) d^{-\gamma(\alpha_1/\alpha_2)} H_{N,\lambda_1}(u-v)(t) \\
&+ C_{p,\alpha,\lambda_2} C_p C_5 C_{\delta,\lambda_2-\lambda_1} B(1-\alpha-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)} E_{\alpha_2,\lambda_1}(u-v)(t) \\
&+ C_{p,\alpha,\lambda_2} C_{\delta,\lambda_2-\lambda_1} (1-\alpha-\delta)^{-1} E_{\lambda_1}(f-g)(t)
\end{aligned} \tag{5.11}$$

for any $\alpha_0 \le \alpha < 1-\delta$, $t \ge t_0$. Assume that

$$0 < d_N < \frac{1}{\{C_{p,\alpha_2,\lambda_2} C_p C_5 C_{\delta,\lambda_2-\lambda_1} B(1-\alpha_2-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1} + M(g,t_0)^{q-1})\}^{1/\{\gamma(1-\alpha_1/\alpha_2)\}}}, \tag{5.12}$$

and set

$$\begin{aligned}
L^1_{\lambda_1} &= \frac{C_{p,\alpha_2-\alpha_0,\lambda_1}}{1 - C_{p,\alpha_2,\lambda_2} C_p C_5 C_{\delta,\lambda_2-\lambda_1} B(1-\alpha_2-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)}}, \\
L^2_{\lambda_1} &= \frac{C_{p,\alpha_2,\lambda_2} C_p C_4 C_{\delta,\lambda_2-\lambda_1} (1-\alpha_2-\delta)^{-1} (M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) d^{-\gamma(\alpha_1/\alpha_2)}}{1 - C_{p,\alpha_2,\lambda_2} C_p C_5 C_{\delta,\lambda_2-\lambda_1} B(1-\alpha_2-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)}}, \\
L^3_{\lambda_1} &= \frac{C_{p,\alpha_2,\lambda_2} C_{\delta,\lambda_2-\lambda_1} (1-\alpha_2-\delta)^{-1}}{1 - C_{p,\alpha_2,\lambda_2} C_p C_5 C_{\delta,\lambda_2-\lambda_1} B(1-\alpha_2-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)}}.
\end{aligned}$$

Then (5.11) implies

$$E_{\alpha_2,\lambda_1}(u-v)(t) \le L^1_{\lambda_1} \|u(t_0)-v(t_0)\|_{X_p^{\alpha_0}} + L^2_{\lambda_1} H_{N,\lambda_1}(u-v)(t) + L^3_{\lambda_1} E_{\lambda_1}(f-g)(t) \tag{5.13}$$

for any $t \ge t_0$. Set

$$\begin{aligned}
L^1_{\alpha,\lambda_1} =& C_{p,\alpha-\alpha_0,\lambda_1} \\
&+ C_{p,\alpha,\lambda_2} C_p C_5 C_{\delta,\lambda_2-\lambda_1} B(1-\alpha-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)} L^1_{\lambda_1}, \\
L^2_{\alpha,\lambda_1} =& C_{p,\alpha,\lambda_2} C_p C_4 C_{\delta,\lambda_2-\lambda_1} (1-\alpha-\delta)^{-1} (M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) d^{-\gamma(\alpha_1/\alpha_2)} \\
&+ C_{p,\alpha,\lambda_2} C_p C_5 C_{\delta,\lambda_2-\lambda_1} B(1-\alpha-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)} L^2_{\lambda_1}, \\
L^3_{\alpha,\lambda_1} =& C_{p,\alpha,\lambda_2} C_{\delta,\lambda_2-\lambda_1} (1-\alpha-\delta)^{-1} \\
&+ C_{p,\alpha,\lambda_2} C_p C_5 C_{\delta,\lambda_2-\lambda_1} B(1-\alpha-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1} + M(g,t_0)^{q-1}) d_N^{\gamma(1-\alpha_1/\alpha_2)} L^3_{\lambda_1}.
\end{aligned}$$

Then it follows from (5.11), (5.13) that

$$E_{\alpha,\lambda_1}(u-v)(t) \le L^1_{\alpha,\lambda_1} \|u(t_0)-v(t_0)\|_{X_p^{\alpha_0}} + L^2_{\alpha,\lambda_1} H_{N,\lambda_1}(u-v)(t) + L^3_{\alpha,\lambda_1} E_{\lambda_1}(f-g)(t) \tag{5.14}$$

for any $\alpha_0 \leq \alpha < 1-\delta$, $t \geq t_0$, which yields (4.7). Moreover, the arbitrariness of the choice of δ allows us to assume that $\alpha_0 \leq \alpha < 1$. It is easy to see from (4.4), (5.14) in the case where $n/(2p) < \alpha < 1$ that (4.8) holds for any $k \in \mathbb{Z}$, $k \geq 0$, $0 < \gamma < 1$, $k+\gamma \leq 2\alpha - n/p$. Hence, the sufficient condition for (4.7), (4.8) is

$$0 < \delta_2 < \frac{1}{\{C_{p,\alpha_2,\lambda_2} C_p C_5 C_{\delta,\lambda_2-\lambda_1} B(1-\alpha_2-\delta, 1+\alpha_0-\alpha_2)(M(f,t_0)^{q-1} + M(g,t_0)^{q-1})\}^{1/\{\gamma(1-\alpha_1/\alpha_2)\}}}.$$

6 Part III: The resolvent problem for the Stokes equations

In this part, we are concerned with the existence and uniqueness of solutions to (1.4) satisfying L_p estimates. This part is organized as follows: In Section 6, we define basic notation in this part, and state our results concerning the analyticity of the semigroup on $L_{p,\sigma}(\Omega)$ generated by the Stokes operator. It is stated in Section 7 that we have L_p estimates for solutions to the Stokes equations in $\mathbb{R}^n$, in $\mathbb{R}^n_+$ and in the bent-half space H^n_ω. Section 8 is devoted to the study of (1.4), i.e., the existence and uniqueness of solutions to (1.4) and resolvent estimates for the Stokes operator in $L_{p,\sigma}(\Omega)$.

6.1 Function spaces

Function spaces and basic notation which we use throughout Sections 6–8 are introduced as follows: Let G be an open set in $\mathbb{R}^n$ with its boundary ∂G, ν be the outward unit normal vector on ∂G, Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$. $(L_p(G), \|\cdot\|_{L_p(G)})$ and $(W_p^k(G), \|\cdot\|_{W_p^k(G)})$ $(1 \leq p \leq \infty,\ k \in \mathbb{Z},\ k \geq 0)$ are the Lebesgue and Sobolev spaces respectively, $W_p^0(G) = L_p(G)$. $W^k_{p,\nu}(G) := \{u \in (W_p^k(G))^n\ ;\ u_\nu|_{\partial G} = 0\}$. $(\dot{W}_p^1(G), \|\cdot\|_{\dot{W}_p^1(G)})$ is the homogeneous Sobolev space defined as

$$\dot{W}_p^1(G) = \{p \in L_{p,loc}(\overline{G})\ ;\ \nabla p \in (L_p(G))^n\},\ \|p\|_{\dot{W}_p^1(G)} = \|\nabla p\|_{(L_p(G))^n}.$$

In the case where $G = \Omega$, $\dot{W}_p^1(\Omega)$ is characterized as

$$\dot{W}_p^1(\Omega) = \left\{p \in W_p^1(\Omega)\ ;\ \int_\Omega p(x)dx = 0\right\}.$$

$(W_p^s(G), \|\cdot\|_{W_p^s(G)})$ $(s > 0,\ s \notin \mathbb{Z})$ is the Sobolev-Slobodetskiĭ space defined as $W_p^s(G) = (L_p(G), W_p^{\langle s\rangle}(G))_{s/\langle s\rangle,p}$, where $(X_0, X_1)_{\theta,p}$ $(0 < \theta < 1)$ is an interpolation space between two Banach spaces X_0 and X_1 by the K or J-method, $\langle s\rangle := \min\{k \in \mathbb{Z}\ ;\ k \geq s\}$. Let us introduce solenoidal function spaces. $C^\infty_{0,\sigma}(\Omega) := \{u \in (C_0^\infty(\Omega))^n\ ;\ \mathrm{div} u = 0\}$. $L_{p,\sigma}(\Omega)$ $(1 < p < \infty)$ is the completion of $C^\infty_{0,\sigma}(\Omega)$ in $(L_p(\Omega))^n$. It follows from [7, Lemma 5.3] that $(L_p(\Omega))^n$ is decomposed into $(L_p(\Omega))^n = L_{p,\sigma}(\Omega) \oplus L_{p,\pi}(\Omega)$, where $L_{p,\pi}(\Omega) := \{\nabla p\ ;\ p \in \dot{W}_p^1(\Omega)\}$. Let P_p be the projection of $(L_p(\Omega))^n$ onto $L_{p,\sigma}(\Omega)$. As for generalized solutions to (1.2), we define the following function spaces:

$$H_p^1(\Omega) = \{u \in (W_p^1(\Omega))^n\ ;\ u_\nu|_{\partial\Omega} = 0,\ u_\tau|_{\Gamma_1} = 0\},\ J_p^1(\Omega) = H_p^1(\Omega) \cap L_{p,\sigma}(\Omega),$$

$$\dot{L}_p(\Omega) = \left\{p \in L_p(\Omega)\ ;\ \int_\Omega p(x)dx = 0\right\}.$$

It is useful to remark that $u \in W^{k+1}_{p,\nu}(\Omega)$ yields $\mathrm{div} u \in \dot{W}_p^k(\Omega)$ for $k = 0, 1$.

Let I be an interval in $\mathbb{R}$, $(X, \|\cdot\|_X)$ be a Banach space. $C(I; X)$ is the set of all X-valued functions which are continuous in I. $C_b(I; X)$ is the set of all X-valued functions which are bounded continuous in I.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. $(\mathcal{B}(X;Y), \|\cdot\|_{\mathcal{B}(X;Y)})$ is the Banach space of all bounded linear operators from X to Y, $\mathcal{B}(X) := \mathcal{B}(X;X)$,

$$\|A\|_{\mathcal{B}(X;Y)} = \sup_{x \in X \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X}.$$

$(\dot{W}_p^{-1}(G), \|\cdot\|_{\dot{W}_p^{-1}(G)})$ $(1 < p < \infty)$ is the dual space of $\dot{W}_{p^*}^1(G)$ with respect to the pivot space $L_{p^*}(G)$,

$$\|f\|_{\dot{W}_p^{-1}(G)} = \sup_{v \in \dot{W}_{p^*}^1(G) \setminus \{0\}} \frac{|\langle f, v \rangle_{L_{p^*}(G)}|}{\|v\|_{\dot{W}_{p^*}^1(G)}},$$

where p^* is the dual exponent to p defined as $1/p + 1/p^* = 1$. Note that $(H_p^{-1}(\Omega), \|\cdot\|_{H_p^{-1}(\Omega)})$ is similarly defined.

It is useful to remark that $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ and $\nabla_x = (\partial_{x_1}, \cdots, \partial_{x_n})^T$ are simply denoted by $x = (x', x_n)^T$, $x' = (x_1, \cdots, x_{n-1})^T \in \mathbb{R}^{n-1}$ and $\nabla_x = (\nabla'_{x'}, \partial_{x_n})^T$, $\nabla'_{x'} = (\partial_{x_1}, \cdots, \partial_{x_{n-1}})^T$ respectively. Note that y, z, ξ and so on are similarly defined. Moreover, simplified notation is given as follows: We simply denote a generic positive constant depending only on n, Ω, p, ε, ρ, μ and K by C. As for boundary conditions, $h^0 \in (W_p^{1-1/p}(\Gamma_0))^n$ and $h^1 \in (W_p^{2-1/p}(\Gamma_1))^n$ are suitably extended to $\tilde{h}^0 \in (W_p^{1-1/p}(\partial\Omega))^n$ and $\tilde{h}^1 \in (W_p^{2-1/p}(\partial\Omega))^n$ respectively, which yields $\tilde{h}^0 \in (W_p^1(\Omega))^n$ and $\tilde{h}^1 \in (W_p^2(\Omega))^n$. It is sufficient to be assumed that $\tilde{h}^0$ and $\tilde{h}^1$ are extensions of h^0 and h^1 to $\partial\Omega$ defined as

$$\tilde{h}^0(x) = \begin{cases} h^0(x) & \text{if } x \in \Gamma_0, \\ 0 & \text{if } x \in \Gamma_1 \end{cases}$$

and

$$\tilde{h}^1(x) = \begin{cases} 0 & \text{if } x \in \Gamma_0, \\ h^1(x) & \text{if } x \in \Gamma_1 \end{cases}$$

respectively. For the sake of notational simplicity, we will not distinguish (h^0, h^1) and $(\tilde{h}^0, \tilde{h}^1)$, i.e., $(\tilde{h}^0, \tilde{h}^1)$ will also be simply denoted by (h^0, h^1).

6.2 Results on the resolvent problem for the Stokes equations

This subsection provides our results of Sections 6–8 after some preliminaries. We begin with resolvent estimates for the Stokes operator in $L_{p,\sigma}(\Omega)$. Set

$$S_\varepsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \ ; \ |\arg\lambda| \leq \pi - \varepsilon\}$$

for any $0 < \varepsilon < \pi/2$. For the sake of notational simplicity, we introduce the following norms in $W_p^1(G) \cap \dot{W}_p^{-1}(G)$ and in $W_p^k(G)$ respectively:

$$I_{p,\lambda,G}^{1,-1}(f) = |\lambda| \|f\|_{\dot{W}_p^{-1}(G)} + |\lambda|^{1/2} \|f\|_{L_p(G)} + \sum_{\alpha \in \mathbb{Z}^n, \alpha \geq 0, |\alpha| = 1} \|\partial^\alpha f\|_{L_p(G)},$$

$$I_{p,\lambda,G}^k(g) = \sum_{i=0}^{k} |\lambda|^{(k-i)/2} \sum_{\alpha \in \mathbb{Z}^n, \alpha \geq 0, |\alpha| = i} \|\partial^\alpha g\|_{L_p(G)}$$

for any open set G in $\mathbb{R}^n$.

Compared with Farwig and Sohr [7] and Shibata and Shimada [24], this part is concerned with (1.4) which has more complicated boundary condition than [7], [24]. The following theorem yields the existence and uniqueness of solutions to (1.4) in $(W_p^2(\Omega))^n$.

Theorem 6.1. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $0 < K \leq 1$, $1 < p < \infty$, $0 < \varepsilon < \pi/2$, $f \in \dot{W}_p^1(\Omega)$, $g \in (L_p(\Omega))^n$, $h^0 \in (W_p^{1-1/p}(\Gamma_0))^n$, $h_\nu^0|_{\Gamma_0} = 0$, $h^1 \in (W_p^{2-1/p}(\Gamma_1))^n$, $h_\nu^1|_{\Gamma_1} = 0$. Then (1.4) has uniquely a solution $(u, p) \in (W_p^2(\Omega))^n \times \dot{W}_p^1(\Omega)$ satisfying*

$$I_{p,\lambda,\Omega}^2(u) + \|p\|_{W_p^1(\Omega)} \leq C_{p,\varepsilon}(I_{p,\lambda,\Omega}^{1,-1}(f) + \|g\|_{(L_p(\Omega))^n} + I_{p,\lambda,\Omega}^1(h^0) + I_{p,\lambda,\Omega}^2(h^1)) \tag{6.1}$$

for any $\lambda \in S_\varepsilon \cup \{0\}$, where $C_{p,\varepsilon}$ is a positive constant depending only on n, Ω, p, ε, μ and K.

Let $(X, \|\cdot\|_X)$ be a Banach space, A be a densely defined closed linear operator in X. Then the resolvent set and the spectrum of A are denoted by $\rho(A)$ and $\sigma(A)$ respectively, $\mathrm{Re}\sigma(A) := \{\mathrm{Re}\lambda \,;\, \lambda \in \sigma(A)\}$. The Stokes operator A_p and its domain $D(A_p)$ are introduced as follows:

$$A_p u = -P_p \mathrm{div} T(u, p),$$

$$D(A_p) = \{u \in (W_p^2(\Omega))^n \cap L_{p,\sigma}(\Omega) \,;\, K(T(u,p)\nu)_\tau + (1-K)u_\tau|_{\Gamma_0} = 0,\ u_\tau|_{\Gamma_1} = 0\}$$

for any $1 < p < \infty$. It is useful to remark that $A_p u = -P_p(-\nabla p + \mu \Delta u) = -\mu P_p \Delta u$. Theorem 6.1 allows us to establish resolvent estimates for $-A_p$, consequently, $-A_p$ is a sectorial operator in $L_{p,\sigma}(\Omega)$ defined as in [12, Definition 1.3.1].

Theorem 6.2. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $0 < K \leq 1$, $1 < p < \infty$, $0 < \varepsilon < \pi/2$. Then $\rho(-A_p) \supset S_\varepsilon \cup \{0\}$, and*

$$\|(\lambda I_p + A_p)^{-1}\|_{\mathcal{B}(L_{p,\sigma}(\Omega))} \leq \frac{C_{p,\varepsilon}}{|\lambda| + 1} \tag{6.2}$$

for any $\lambda \in S_\varepsilon \cup \{0\}$, where $C_{p,\varepsilon}$ is a positive constant depending only on n, Ω, p, ε, μ and K. Therefore, $-A_p$ is the infinitesimal generator of a uniformly bounded analytic semigroup $\{e^{-tA_p}\}_{t\geq 0}$ on $L_{p,\sigma}(\Omega)$, and there exists a positive constant $C_{p,\varepsilon,\lambda_1}$ depending only on n, Ω, p, ε, μ, K and λ_1 for any $0 < \lambda_1 < \Lambda_1$ such that

$$\|e^{-tA_p}\|_{\mathcal{B}(L_{p,\sigma}(\Omega))} \leq C_{p,\varepsilon,\lambda_1} e^{-\lambda_1 t}$$

for any $t \geq 0$, where $\Lambda_1 := \min\{\lambda_1 > 0 \,;\, \lambda_1 \in \mathrm{Re}\sigma(A_p)\}$.

Fractional powers A_p^α can be defined for any $\alpha \geq 0$, $A_p^0 = I_p$, which follows from Theorem 6.2. Let us introduce the Banach space derived from A_p^α. X_p^α is defined as $X_p^\alpha = D(A_p^\alpha)$ with the norm $\|\cdot\|_{X_p^\alpha} = \|A_p^\alpha \cdot\|_{(L_p(\Omega))^n}$, $X_p^0 = L_{p,\sigma}(\Omega)$.

6.3 Auxiliary lemmata

In this subsection, we will state and prove some lemmata which play an important role throughout Sections 6–8. The following lemma yields the existence of solutions to the auxiliary boundary value problem.

Lemma 6.1. *Let Ω be a bounded domain with its $C^{0,1}$-boundary $\partial\Omega$, $1 < p < \infty$, $f \in \dot{L}_p(\Omega)$, $h \in (W_p^{1-1/p}(\Gamma_1))^n$, $h_\nu|_{\Gamma_1} = 0$, and consider the following boundary value problem:*

$$\begin{cases} \mathrm{div} u = f & \text{in } \Omega, \\ u_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ u_\tau|_{\Gamma_1} = h & \text{on } \Gamma_1. \end{cases} \tag{6.3}$$

Then (6.3) has a solution $u \in (W_p^1(\Omega))^n$ satisfying

$$\|u\|_{(W_p^1(\Omega))^n} \le C_p(\|f\|_{L_p(\Omega)} + \|h\|_{(W_p^{1-1/p}(\Gamma_1))^n}), \tag{6.4}$$

where C_p is a positive constant depending only on n, Ω and p.

Proof. Let $\tilde{h} \in (W_p^{1-1/p}(\partial\Omega))^n$ be the extension of h to $\partial\Omega$ defined as

$$\tilde{h}(x) = \begin{cases} 0 & \text{if } x \in \Gamma_0, \\ h(x) & \text{if } x \in \Gamma_1, \end{cases}$$

and consider the following boundary value problem:

$$\begin{cases} \mathrm{div} u = f & \text{in } \Omega, \\ u_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ u_\tau|_{\partial\Omega} = \tilde{h} & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Then [2, Theorem 1] yields that (6.5) has uniquely a solution $u \in (W_p^1(\Omega))^n$ satisfying (6.4). This completes the proof of Lemma 6.1. $\square$

As for the existence and uniqueness of solutions to the auxiliary problem, Lemma 6.1 admits that we have the following lemma:

Lemma 6.2. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $u \in H_2^1(\Omega)$, and consider the following problem:*

$$(p, \mathrm{div} v)_{L_2(\Omega)} = (u, v)_{(W_2^1(\Omega))^n} \tag{6.6}$$

for any $v \in H_2^1(\Omega)$. Then (6.6) has uniquely a solution $p \in \dot{L}_2(\Omega)$ satisfying

$$\|p\|_{L_2(\Omega)} \le C\|u\|_{(W_2^1(\Omega))^n},$$

where C is a positive constant depending only on n and Ω.

Proof. There exists uniquely a function $u = Ap \in H_2^1(\Omega)$ for any $p \in \dot{L}_2(\Omega)$ such that we have (6.6) for any $v \in H_2^1(\Omega)$, which follows from the Riesz representation theorem. By substituting $v = Ap$ into (6.6), we have

$$\|Ap\|^2_{(W_2^1(\Omega))^n} \le |(p, \mathrm{div} Ap)_{L_2(\Omega)}| \le C\|p\|_{L_2(\Omega)}\|Ap\|_{(W_2^1(\Omega))^n}.$$

This inequality yields $\|Ap\|_{(W_2^1(\Omega))^n} \le C\|p\|_{L_2(\Omega)}$, i.e., $A \in \mathcal{B}(\dot{L}_2(\Omega); H_2^1(\Omega))$. Furthermore, the inverse operator of A is also bounded. Indeed, Lemma 6.1 admits that (6.3) with $f = p$ and $h = 0$ has a solution $v \in H_2^1(\Omega)$ satisfying (6.4) for any $p \in \dot{L}_2(\Omega)$. By substituting the solution v into (6.6), we obtain

$$\|p\|^2_{L_2(\Omega)} \le |(Ap, v)_{(W_2^1(\Omega))^n}| \le \|Ap\|_{(W_2^1(\Omega))^n}\|v\|_{(W_2^1(\Omega))^n} \le C\|Ap\|_{(W_2^1(\Omega))^n}\|p\|_{L_2(\Omega)},$$

which gives $\|p\|_{L_2(\Omega)} \le C\|Ap\|_{(W_2^1(\Omega))^n}$. It follows from the inequality that A^{-1} is defined as a bounded linear operator $A^{-1} \in \mathcal{B}(H_2^1(\Omega); \dot{L}_2(\Omega))$. This completes the proof of Lemma 6.2. $\square$

We are devoted to the basic property of $H_2^1(\Omega)$ with the aid of Lemma 6.2. The orthogonal decomposition of $H_2^1(\Omega)$ is given by the following lemma:

Lemma 6.3. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $A \in \mathcal{B}(\dot{L}_2(\Omega); H_2^1(\Omega))$ be a bounded linear operator defined as in Lemma 6.2. Then $H_2^1(\Omega)$ is decomposed into $H_2^1(\Omega) = J_2^1(\Omega) \oplus R(A)$.*

Proof. Let $v \in H_2^1(\Omega) \setminus R(A)$. Then it follows that

$$(p, \mathrm{div} v)_{L_2(\Omega)} = (Ap, v)_{(W_2^1(\Omega))^n} = 0$$

for any $p \in \dot{L}_2(\Omega)$, which implies $\mathrm{div} v = 0$. Consequently, we have $v \in J_2^1(\Omega)$. □

We proceed to the Korn inequality for functions in $H_2^1(\Omega)$. In contrast to the case where $\Gamma_0 = \partial\Omega$ and $\Gamma_1 = \emptyset$, Theorems 6.1 and 6.2 are valid for any $0 < K \leq 1$, which is essentially depending on the following lemma:

Lemma 6.4. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$. Then*

$$\|u\|^2_{(W_2^1(\Omega))^n} \leq C\|D(u)\|^2_{(L_2(\Omega))^{n^2}} \tag{6.7}$$

for any $u \in H_2^1(\Omega)$, where C is a positive constant depending only on n and Ω.

Proof. See [25, Lemma 4]. □

The following lemma yields the generalized Poincaré inequality which is required for $\dot{W}_p^{-1}$ estimates for the external force.

Lemma 6.5. *Let Ω be a bounded domain with its boundary $\partial\Omega$, $1 < p < \infty$, $f \in W_p^1(\Omega)$, $\varphi \in L_{p^*}(\Omega) \setminus \{0\}$, and set*

$$c_f = \int_\Omega f(x)\varphi(x)dx \Big/ \int_\Omega \varphi(x)dx.$$

Then

$$\|f - c_f\|_{L_p(\Omega)} \leq C_p\|f\|_{\dot{W}_p^1(\Omega)}, \tag{6.8}$$

where C_p is a positive constant depending only on n, Ω and p.

Proof. Assume that (6.8) does not hold for any $f \in W_p^1(\Omega)$. Then there exists a function $f_m \in W_p^1(\Omega)$ for any $m \in \mathbb{N}$ such that

$$\|f_m - c_{f_m}\|_{L_p(\Omega)} > m\|f\|_{\dot{W}_p^1(\Omega)}. \tag{6.9}$$

Set

$$g_m = \frac{f_m - c_{f_m}}{\|f_m - c_{f_m}\|_{L_p(\Omega)}}.$$

Then it is easy to see from (6.9) that $\|g_m\|_{L_p(\Omega)} = 1$ and $\|g_m\|_{\dot{W}_p^1(\Omega)} < 1/m$. Since $\|g_m\|_{W_p^1(\Omega)} < 2$ holds for any $m \in \mathbb{N}$, i.e., $\{g_m\}_m$ is bounded in $W_p^1(\Omega)$, we can choose a subsequence of $\{g_m\}_m$ which converges weakly to a function $g \in W_p^1(\Omega)$ in $W_p^1(\Omega)$. Note that $\|g\|_{L_p(\Omega)} = 1$ and $g = 1/|\Omega|^{1/p}$. On the other hand, we have

$$\int_\Omega g_m(x)\varphi(x)dx = \frac{1}{\|f_m - c_{f_m}\|_{L_p(\Omega)}}\left(\int_\Omega f_m(x)\varphi(x)dx - c_{f_m}\int_\Omega \varphi(x)dx\right) = 0.$$

The strong convergence in $L_p(\Omega)$ implies that

$$\int_\Omega g\varphi(x)dx = g\int_\Omega \varphi(x)dx = 0.$$

This contradicts $g \neq 0$ and $\varphi \neq 0$, which completes the proof of Lemma 6.5. □

In order to obtain L_p estimates for the pressure, we utilize the following lemma on the existence and uniqueness of solutions to the auxiliary boundary value problem.

Lemma 6.6. *Let Ω be a bounded domain with its $C^{1,1}$-boundary $\partial\Omega$, $1 < p < \infty$, $f \in \dot{L}_p(\Omega)$, and consider the following boundary value problem:*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.10}$$

Then (6.10) *has uniquely a solution $u \in W_p^2(\Omega) \cap \dot{L}_p(\Omega)$ satisfying*

$$\|u\|_{W_p^2(\Omega)} \leq C_p\|f\|_{L_p(\Omega)}, \tag{6.11}$$

where C_p is a positive constant depending only on n, Ω and p.

Proof. See [24, Proposition 5.5]. □

7 L_p estimates for solutions to (1.4) in unbounded domains

We will state and prove some theorems which are essential for Theorems 6.1 and 6.2. In proving our results, simplified notation is given as follows: We simply denote a generic positive constant depending only on n, Ω, p, ε, ρ, μ and K by C. It is useful to remark that a generic positive constant depending only on the above elements and additive elements (e.g., λ_0, α and so on) is simply denoted by C_{λ_0}, C_α and so on respectively. As for boundary conditions, $h^0 \in (W_p^{1-1/p}(\Gamma_0))^n$ and $h^1 \in (W_p^{2-1/p}(\Gamma_1))^n$ are suitably extended to $h^0 \in (W_p^{1-1/p}(\partial\Omega))^n$ and $h^1 \in (W_p^{2-1/p}(\partial\Omega))^n$ respectively, which yields $h^0 \in (W_p^1(\Omega))^n$ and $h^1 \in (W_p^2(\Omega))^n$.

In this section, we will state some theorems concerning L_p estimates for solutions to (1.4) in unbounded domains, e.g., $\mathbb{R}^n$, $\mathbb{R}^n_+$ and H^n_ω.

7.1 L_p estimates for solutions to (1.4) in $\mathbb{R}^n$

In this subsection, we consider the following resolvent problem in $\mathbb{R}^n$:

$$\begin{cases} \mathrm{div} u = f & \text{in } \mathbb{R}^n, \\ \lambda u - \mathrm{div} T(u,p) = g & \text{in } \mathbb{R}^n. \end{cases} \tag{7.1}$$

By applying the Fourier multiplier theorem to the solution formula of (7.1), we have the following theorem on L_p estimates for solutions to (7.1):

Theorem 7.1. *Let $1 < p < \infty$, $0 < \varepsilon < \pi/2$, $f \in W_p^1(\mathbb{R}^n) \cap \dot{W}_p^{-1}(\mathbb{R}^n)$, $g \in (L_p(\mathbb{R}^n))^n$. Then (7.1) has uniquely a solution $(u,p) \in (W_p^2(\mathbb{R}^n))^n \times \dot{W}_p^1(\mathbb{R}^n)$ satisfying*

$$I^2_{p,\lambda,\mathbb{R}^n}(u) + \|p\|_{\dot{W}_p^1(\mathbb{R}^n)} \le C_{p,\varepsilon}(I^{1,-1}_{p,\lambda,\mathbb{R}^n}(f) + \|g\|_{(L_p(\mathbb{R}^n))^n}) \tag{7.2}$$

for any $\lambda \in S_\varepsilon$, where $C_{p,\varepsilon}$ is a positive constant depending only on n, p, ε and μ.

Proof. See [7, Theorem 1.3], [24, Theorem 2.1]. □

7.2 L_p estimates for solutions to (1.4) in $\mathbb{R}^n_+$

This subsection is devote to the study of (1.4) in the case of the half-space. First, we discuss the following resolvent problem in $\mathbb{R}^n_+$:

$$\begin{cases} \mathrm{div} u = f & \text{in } \mathbb{R}^n_+, \\ \lambda u - \mathrm{div} T(u,p) = g & \text{in } \mathbb{R}^n_+, \\ u|_{\partial\mathbb{R}^n_+} = h & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{7.3}$$

The Fourier multiplier theorem admits that we have the following theorem on L_p estimates for solutions to (7.3):

Theorem 7.2. *Let $1 < p < \infty$, $0 < \varepsilon < \pi/2$, $f \in W_p^1(\mathbb{R}^n_+) \cap \dot{W}_p^{-1}(\mathbb{R}^n_+)$, $g \in (L_p(\mathbb{R}^n_+))^n$, $h \in (W_p^2(\mathbb{R}^n_+))^n$. Then (7.3) has uniquely a solution $(u,p) \in (W_p^2(\mathbb{R}^n_+))^n \times \dot{W}_p^1(\mathbb{R}^n_+)$ for any $\lambda \in S_\varepsilon$, and there exists a positive constant $C_{p,\varepsilon,\lambda_0}$ depending only on n, p, ε, μ and λ_0 for any $\lambda_0 > 0$ such that*

$$I^2_{p,\lambda,\mathbb{R}^n_+}(u) + \|p\|_{\dot{W}_p^1(\mathbb{R}^n_+)} \le C_{p,\varepsilon,\lambda_0}(I^{1,-1}_{p,\lambda,\mathbb{R}^n_+}(f) + \|g\|_{(L_p(\mathbb{R}^n_+))^n} + I^2_{p,\lambda,\mathbb{R}^n_+}(h)) \tag{7.4}$$

for any $\lambda \in S_\varepsilon$, $|\lambda| \ge \lambda_0$.

Proof. See [7, Theorem 1.3]. □

Second, we proceed to establish L_p estimates for solutions to the following resolvent problem in $\mathbb{R}^n_+$:

$$\begin{cases} \mathrm{div} u = f & \text{in } \mathbb{R}^n_+, \\ \lambda u - \mathrm{div} T(u,p) = g & \text{in } \mathbb{R}^n_+, \\ u_n|_{\partial\mathbb{R}^n_+} = 0 & \text{on } \partial\mathbb{R}^n_+, \\ -K(\mu\partial_{x_n} u') + (1-K)u'|_{\partial\mathbb{R}^n_+} = h' & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{7.5}$$

Similarly to Theorem 7.2, we obtain the following theorem:

Theorem 7.3. *Let $0 < K \le 1$, $1 < p < \infty$, $0 < \varepsilon < \pi/2$, $f \in W_p^1(\mathbb{R}^n_+) \cap \dot{W}_p^{-1}(\mathbb{R}^n_+)$, $g \in (L_p(\mathbb{R}^n_+))^n$, $h \in (W_p^1(\mathbb{R}^n_+))^n$, $h_n = 0$. Then (7.5) has uniquely a solution $(u,p) \in (W_p^2(\mathbb{R}^n_+))^n \times \dot{W}_p^1(\mathbb{R}^n_+)$ for any $\lambda \in S_\varepsilon$, and there exists a positive constant $C_{p,\varepsilon,\lambda_0}$ depending only on n, p, ε, μ and λ_0 for any $\lambda_0 > 0$ such that*

$$I^2_{p,\lambda,\mathbb{R}^n_+}(u) + \|p\|_{\dot{W}_p^1(\mathbb{R}^n_+)} \le C_{p,\varepsilon,\lambda_0}(I^{1,-1}_{p,\lambda,\mathbb{R}^n_+}(f) + \|g\|_{(L_p(\mathbb{R}^n_+))^n} + I^1_{p,\lambda,\mathbb{R}^n_+}(h)) \tag{7.6}$$

for any $\lambda \in S_\varepsilon$, $|\lambda| \ge \lambda_0$.

Proof. See [7, Corollary 1.5], [24, Theorem 3.1]. □

7.3 L_p estimates for solutions to (1.4) in H^n_ω

Let $\omega \in C^{k,1}(\mathbb{R}^{n-1})$ satisfy

$$\|\nabla'\omega\|_{(W^k_\infty(\mathbb{R}^{n-1}))^{n-1}} < \infty, \tag{7.7}$$

and set

$$H^n_\omega = \{x = (x', x_n)^T \in \mathbb{R}^n \ ; \ x_n > \omega(x'),\ x' \in \mathbb{R}^{n-1}\},$$
$$\partial H^n_\omega = \{x = (x', x_n)^T \in \mathbb{R}^n \ ; \ x_n = \omega(x'),\ x' \in \mathbb{R}^{n-1}\},$$
$$M_j = \|\nabla'\omega\|_{(W^j_\infty(\mathbb{R}^{n-1}))^{n-1}}$$

for any $j = 0, \cdots, k$. Then the outward unit normal vector $\nu_\omega \in (C^{k,1}(\partial H^n_\omega))^n$ on ∂H^n_ω is characterized as

$$\nu_\omega(x') = \frac{(\nabla'\omega(x'), -1)^T}{(1 + |\nabla'\omega(x')|^2)^{1/2}}.$$

In this subsection, we are concerned with (1.4) in the case of the bent half-space. The first result is L_p estimates for solutions to the following resolvent problem in H^n_ω:

$$\begin{cases} \mathrm{div} u = f & \text{in } H^n_\omega, \\ \lambda u - \mathrm{div} T(u,p) = g & \text{in } H^n_\omega, \\ u|_{\partial H^n_\omega} = h & \text{on } \partial H^n_\omega. \end{cases} \tag{7.8}$$

It is easy to see from Theorem 7.2 and the Banach fixed point theorem that we have the following theorem:

Theorem 7.4. *Let $\omega \in C^{1,1}(\mathbb{R}^{n-1})$ satisfy (7.7), $1 < p < \infty$, $0 < \varepsilon < \pi/2$, $f \in W^1_p(H^n_\omega) \cap \dot{W}^{-1}_p(H^n_\omega)$, $g \in (L_p(H^n_\omega))^n$, $h \in (W^2_p(H^n_\omega))^n$. Then there exist two positive constants M depending only on n, p, ε and μ and $\lambda_1 \geq 1$ depending only on n, p, ε, μ and M_1 such that if $M_0 \leq M$, then (7.8) has uniquely a solution $(u,p) \in (W^2_p(H^n_\omega))^n \times \dot{W}^1_p(H^n_\omega)$ satisfying*

$$I^2_{p,\lambda,H^n_\omega}(u) + \|p\|_{\dot{W}^1_p(H^n_\omega)} \leq C_{p,\varepsilon}(I^{1,-1}_{p,\lambda,H^n_\omega}(f) + \|g\|_{(L_p(H^n_\omega))^n} + I^2_{p,\lambda,H^n_\omega}(h)) \tag{7.9}$$

for any $\lambda \in S_\varepsilon$, $|\lambda| \geq \lambda_1$, where $C_{p,\varepsilon}$ is a positive constant depending only on n, p, ε and μ.

Proof. See [7, Theorem 3.1]. □

The second result is similar to the first, i.e., L_p estimates for solutions to the following resolvent problem in H^n_ω:

$$\begin{cases} \mathrm{div} u = f & \text{in } H^n_\omega, \\ \lambda u - \mathrm{div} T(u,p) = g & \text{in } H^n_\omega, \\ u_{\nu_\omega}|_{\partial H^n_\omega} = 0 & \text{on } \partial H^n_\omega, \\ K(T(u,p)\nu_\omega)_{\tau_\omega} + (1-K)u_{\tau_\omega}|_{\partial H^n_\omega} = h & \text{on } \partial H^n_\omega, \end{cases} \tag{7.10}$$

where $u_{\nu_\omega} := \nu_\omega \cdot u$, $u_{\tau_\omega} := u - u_{\nu_\omega}\nu_\omega$. In the same manner as in Theorem 5.4, we obtain the following theorem:

Theorem 7.5. *Let $\omega \in C^{2,1}(\mathbb{R}^{n-1})$ satisfy (7.7), $0 < K \leq 1$, $1 < p < \infty$, $0 < \varepsilon < \pi/2$, $f \in W_p^1(H_\omega^n) \cap \dot{W}_p^{-1}(H_\omega^n)$, $g \in (L_p(H_\omega^n))^n$, $h \in W_{p,\nu_\omega}^1(H_\omega^n)$. Then there exist two positive constants M depending only on n, p, ε, μ and K and $\lambda_1 \geq 1$ depending only on n, p, ε, μ, K and M_2 such that if $M_0 \leq M$, then (7.10) has uniquely a solution $(u,p) \in (W_p^2(H_\omega^n))^n \times \dot{W}_p^1(H_\omega^n)$ satisfying*

$$I_{p,\lambda,H_\omega^n}^2(u) + \|p\|_{\dot{W}_p^1(H_\omega^n)} \leq C_{p,\varepsilon}(I_{p,\lambda,H_\omega^n}^{1,-1}(f) + \|g\|_{(L_p(H_\omega^n))^n} + I_{p,\lambda,H_\omega^n}^1(h)) \tag{7.11}$$

for any $\lambda \in S_\varepsilon$, $|\lambda| \geq \lambda_1$, where $C_{p,\varepsilon}$ is a positive constant depending only on n, p, ε, μ and K.

Proof. See [24, Theorem 4.1]. □

8 L_p estimates for solutions to (1.4) in a bounded domain

8.1 Existence and uniqueness of solutions to (1.4)

This subsection deals with the existence and uniqueness of solutions to (1.4). Let G be an open set in $\mathbb{R}^n$ with its boundary ∂G, and set $(u,w)_G = (u,w)_{L_2(G)}$, $(u,w)_{\partial G} = (u,w)_{L_2(\partial G)}$. Then we are devote to the following variational formula in Ω:

$$\lambda(u,w)_\Omega - (p, \mathrm{div} w)_\Omega + 2\mu(D(u), D(w))_\Omega + (K^{-1}-1)(u,w)_{\Gamma_0} = (g,w)_\Omega + K^{-1}(h^0, w)_{\Gamma_0} \tag{8.1}$$

for any $w \in H_2^1(\Omega)$. It is important to remark that (8.1) is the variational problem associated with (1.4). Let us introduce the sesquilinear form $B_{\lambda,\Omega}$ in $H_2^1(\Omega)$ as follows:

$$B_{\lambda,\Omega}(u,w) = \lambda(u,w)_\Omega + 2\mu(D(u), D(w))_\Omega + (K^{-1}-1)(u,w)_{\Gamma_0}.$$

Note that $B_{\lambda,\mathbb{R}^n}$ and B_{λ,H_ω^n} are similarly defined. Then (8.1) yields

$$B_{\lambda,\Omega}(u,w) - (p, \mathrm{div} w)_\Omega = (g,w)_\Omega + K^{-1}(h^0, w)_{\Gamma_0} \tag{8.2}$$

for any $w \in H_2^1(\Omega)$. It follows easily that $B_{\lambda,\Omega}$ is bounded in $H_2^1(\Omega)$. As for the existence and uniqueness of generalized solutions to (8.1) in $J_2^1(\Omega)$, we have only to prove that $B_{\lambda,\Omega}$ is coercive in $J_2^1(\Omega)$.

Lemma 8.1. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $0 < K \leq 1$, $0 < \varepsilon < \pi/2$. Then*

$$|B_{\lambda,\Omega}(u,u)| \geq \sin(\varepsilon/2)\{|\lambda|\|u\|_{(L_2(\Omega))^n}^2 + 2\mu\|D(u)\|_{(L_2(\Omega))^{n^2}}^2 + (K^{-1}-1)\|u\|_{(L_2(\Gamma_0))^n}^2\} \tag{8.3}$$

for any $u \in H_2^1(\Omega)$, $\lambda \in S_\varepsilon \cup \{0\}$.

Proof. An elementary calculation shows that

$$|\lambda a + b| \geq \sin(\varepsilon/2)(|\lambda| a + b) \tag{8.4}$$

for any $a \geq 0$, $b \geq 0$, $\lambda \in S_\varepsilon \cup \{0\}$. By substituting $a = \|u\|_{(L_2(\Omega))^n}^2$, $b = 2\mu\|D(u)\|_{(L_2(\Omega))^{n^2}}^2 + (K^{-1}-1)\|u\|_{(L_2(\Gamma_0))^n}^2$ into (8.4), we have (8.3). □

It is easy to see from Lemma 8.1 and the Lax-Milgram theorem that we have the following lemma:

Lemma 8.2. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $0 < K \leq 1$, $0 < \varepsilon < \pi/2$, $g \in H_2^{-1}(\Omega)$, $h^0 \in (L_2(\Gamma_0))^n$. Then (8.1) has uniquely a generalized solution $u \in J_2^1(\Omega)$ for any $\lambda \in S_\varepsilon \cup \{0\}$, i.e., u satisfies (8.1) for any $w \in J_2^1(\Omega)$.*

Proof. It follows from Lemmata 6.4 and 8.1 that $B_{\lambda,\Omega}$ is coercive in $J_2^1(\Omega)$. Therefore, the Lax-Milgram theorem admits that (8.1) has uniquely a generalized solution $u \in J_2^1(\Omega)$ for any $\lambda \in S_\varepsilon \cup \{0\}$. □

Lemmata 6.2 and 8.2 imply that we obtain the following lemma on the existence and uniqueness of generalized solutions to (8.1) in $J_2^1(\Omega) \times \dot{L}_2(\Omega)$:

Lemma 8.3. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $0 < K \leq 1$, $0 < \varepsilon < \pi/2$, $g \in H_2^{-1}(\Omega)$, $h^0 \in (L_2(\Gamma_0))^n$. Then* (8.1) *has uniquely a generalized solution $(u,p) \in J_2^1(\Omega) \times \dot{L}_2(\Omega)$ for any $\lambda \in S_\varepsilon \cup \{0\}$, i.e., (u,p) satisfies* (8.1) *for any $w \in H_2^1(\Omega)$.*

Proof. Set

$$D_{\lambda,\Omega}(u,w) = \lambda(u,w)_\Omega + 2\mu(D(u),D(w))_\Omega + (K^{-1}-1)(u,w)_{\Gamma_0} - (g,w)_\Omega - K^{-1}(h^0,w)_{\Gamma_0},$$

$$D_\Omega(u,w) = 2\mu(D(u),D(w))_\Omega + (K^{-1}-1)(u,w)_{\Gamma_0} - (g,w)_\Omega - K^{-1}(h^0,w)_{\Gamma_0},$$

and consider the following variational problem:

$$D_{\lambda,\Omega}(u,w) = (p,\mathrm{div}w)_\Omega \tag{8.5}$$

for any $w \in H_2^1(\Omega)$. Then Lemma 8.2 implies that we have only to determine uniquely a generalized solution $p \in \dot{L}_2(\Omega)$ corresponding to a generalized solution $u \in J_2^1(\Omega)$ obtained in Lemma 8.2. There exist two functions $w^1 \in J_2^1(\Omega)$ and $w^2 \in R(A)$ for any $w \in H_2^1(\Omega)$ such that $w = w^1 + w^2$, which follows from Lemma 6.3. By substituting $w = w^1 + w^2$ into (8.5), we have $D_\Omega(u,w^2) = (p,\mathrm{div}w^2)_\Omega$. Since D_Ω is bounded in $H_2^1(\Omega)$, the Riesz representation theorem yields that there exists uniquely a function $v \in R(A)$ such that $D_\Omega(u,w^2) = (v,w^2)_{(W_2^1(\Omega))^n}$. Therefore, (8.5) is reduced to the following variational problem:

$$(v,w)_{(W_2^1(\Omega))^n} = (p,\mathrm{div}w)_{L_2(\Omega)} \tag{8.6}$$

for any $w \in H_2^1(\Omega)$. It follows from Lemma 6.2 that (8.6) has uniquely a generalized solution $p \in \dot{L}_2(\Omega)$, which completes the proof of Lemma 8.3. □

We proceed to discuss the existence and uniqueness of generalized solutions to the following variational problem:

$$\begin{cases} \mathrm{div}u = f \text{ in } \Omega, \\ B_{\lambda,\Omega}(u,w) - (p,\mathrm{div}w)_\Omega = (g,w)_\Omega + K^{-1}(h^0,w)_{\Gamma_0}, \\ u_\tau|_{\Gamma_1} = h^1 \text{ on } \Gamma_1 \end{cases} \tag{8.7}$$

for any $w \in H_2^1(\Omega)$. It follows easily from Lemmata 6.1 and 8.3 that we have the following lemma:

Lemma 8.4. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $0 < K \leq 1$, $0 < \varepsilon < \pi/2$, $f \in \dot{L}_2(\Omega)$, $g \in H_2^{-1}(\Omega)$, $h^0 \in (L_2(\Gamma_0))^n$, $h^1 \in (W_2^{1/2}(\Gamma_1))^n$, $h_\nu^1|_{\Gamma_1} = 0$. Then* (8.7) *has uniquely a generalized solution $(u,p) \in H_2^1(\Omega) \times \dot{L}_2(\Omega)$ for any $\lambda \in S_\varepsilon \cup \{0\}$, i.e., (u,p) satisfies* (8.7) *for any $w \in H_2^1(\Omega)$.*

Proof. Consider the following boundary value problem:

$$\begin{cases} \mathrm{div}u^1 = f & \text{in } \Omega, \\ u_\nu^1|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ u_\tau^1|_{\Gamma_1} = h^1 & \text{on } \Gamma_1. \end{cases} \tag{8.8}$$

Then (8.8) has a solution $u^1 \in (W_2^1(\Omega))^n$, which follows from Lemma 6.1. Let $u = u^1 + u^2$, where u^2 is a generalized solution to the following variational problem:

$$B_{\lambda,\Omega}(u^2, w) - (p, \mathrm{div}w)_\Omega = (g, w)_\Omega + K^{-1}(h^0, w)_{\Gamma_0} - B_{\lambda,\Omega}(u^1, w) \tag{8.9}$$

for any $w \in H_2^1(\Omega)$. We can see easily that $u^1 \in (W_2^1(\Omega))^n$ yields $B_{\lambda,\Omega}(u^1, \cdot) \in H_2^{-1}(\Omega)$. Lemma 8.3 admits that (8.9) has uniquely a generalized solution $(u^2, p) \in J_2^1(\Omega) \times \dot{L}_2(\Omega)$ for any $\lambda \in S_\varepsilon \cup \{0\}$. Therefore, $(u^1 + u^2, p)$ is a generalized solution to (8.7). The uniqueness of generalized solutions to (8.7) follows easily from Lemmata 6.2, 6.4 and 8.1, which completes the proof of Lemma 8.4. □

We shall obtain the existence and uniqueness of solutions to (1.4) with the aid of Theorems 7.1, 7.4, 7.5 and Lemma 8.4. In order to reduce (1.4) to the resolvent problem in $\mathbb{R}^n$ and in H_ω^n, the localization method is carried out. First, we derive the resolvent problem in $\mathbb{R}^n$ from (1.4) in the interior of Ω. Set $\Omega_\delta = \{x \in \Omega \ ; \ \mathrm{dist}(x, \partial\Omega) > \delta\}$ for any $\delta > 0$. Then $\varphi \in C_0^\infty(\mathbb{R}^n)$ can be defined as

$$\begin{cases} \varphi(x) = 1 & \text{if } x \in \Omega_\delta, \\ 0 < \varphi(x) < 1 & \text{if } x \in \Omega_{\delta/2} \setminus \overline{\Omega_\delta}, \\ \varphi(x) = 0 & \text{if } x \in \mathbb{R}^n \setminus \overline{\Omega_{\delta/2}}. \end{cases}$$

We can see easily from (1.4) that $(\varphi u, \varphi p)$ satisfies the following resolvent problem in $\mathbb{R}^n$:

$$\begin{cases} \mathrm{div}(\varphi u) = f_\delta & \text{in } \mathbb{R}^n, \\ \lambda(\varphi u) - \mathrm{div}T(\varphi u, \varphi p) = g_\delta & \text{in } \mathbb{R}^n, \end{cases} \tag{8.10}$$

where

$$f_\delta := (\varphi f) + \nabla\varphi \cdot u,$$

$$g_\delta := (\varphi g) + (\nabla\varphi)p - \mathrm{div}(2\mu D(u, \varphi)),$$

$$D(u, \varphi) := \frac{1}{2}(u(\nabla\varphi)^T + \nabla\varphi u^T).$$

Set $(v, q) = (\varphi u, \varphi p)$. Then it follows from (8.10) that (v, q) satisfies the following resolvent problem in $\mathbb{R}^n$:

$$\begin{cases} \mathrm{div}v = f_\delta & \text{in } \mathbb{R}^n, \\ \lambda v - \mathrm{div}T(v, q) = g_\delta & \text{in } \mathbb{R}^n. \end{cases} \tag{8.11}$$

Second, we proceed to derive the resolvent problem in H_ω^n from (1.4) near Γ_0. Set $B_\delta(x_0) = \{x \in \mathbb{R}^n \ ; \ |x - x_0| < \delta\}$ for any $x_0 \in \Gamma_0$, $\delta > 0$. Then $\varphi \in C_0^\infty(\mathbb{R}^n)$ can be defined as

$$\begin{cases} \varphi(x) = 1 & \text{if } x \in B_\delta(x_0), \\ 0 < \varphi(x) < 1 & \text{if } x \in B_{2\delta}(x_0) \setminus \overline{B_\delta(x_0)}, \\ \varphi(x) = 0 & \text{if } x \in \mathbb{R}^n \setminus \overline{B_{2\delta}(x_0)}. \end{cases}$$

It is easy to see from (1.4) that $(\varphi u, \varphi p)$ satisfies the following resolvent problem in Ω:

$$\begin{cases} \mathrm{div}(\varphi u) = f_{x_0,\delta} & \text{in } \Omega, \\ \lambda(\varphi u) - \mathrm{div}T(\varphi u, \varphi p) = g_{x_0,\delta} & \text{in } \Omega, \\ (\varphi u)_\nu|_{\Gamma_0} = 0 & \text{on } \Gamma_0, \\ K(2\mu D(\varphi u)\nu)_\tau + (1 - K)(\varphi u)_\tau|_{\Gamma_0} = h^0_{x_0,\delta} & \text{on } \Gamma_0, \end{cases} \tag{8.12}$$

where

$$f_{x_0,\delta} := (\varphi f) + \nabla\varphi \cdot u,$$

$$g_{x_0,\delta} := (\varphi g) + (\nabla\varphi)p - \mathrm{div}(2\mu D(u,\varphi)),$$
$$h^0_{x_0,\delta} := (\varphi h^0) + K(2\mu D(u,\varphi)\nu)_\tau,$$
$$D(u,\varphi) := \frac{1}{2}(u(\nabla\varphi)^T + \nabla\varphi u^T).$$

Let O be the orthogonal matrix such that $O^T\nu(x_0) = (0,\cdots,0,-1)^T$, and set $x = x_0 + Oy$, $\widetilde{\Omega} = \{O^T(x - x_0)\ ;\ x \in \Omega\}$, $\widetilde{\Gamma}_0 = \{O^T(x - x_0)\ ;\ x \in \Gamma_0\}$, $v(y) = O^T(\varphi u)(x)$, $q(y) = (\varphi p)(x)$, $\tilde{f}_{x_0,\delta}(y) = f_{x_0,\delta}(x)$, $\tilde{g}_{x_0,\delta}(y) = O^T g_{x_0,\delta}(x)$, $\tilde{h}^0_{x_0,\delta}(y) = O^T h^0_{x_0,\delta}(x)$, $\tilde{\nu}(y) = O^T\nu(x)$. Then it follows from (8.12) that (v,q) satisfies the following resolvent problem in $\widetilde{\Omega}$:

$$\begin{cases} \mathrm{div} v = \tilde{f}_{x_0,\delta} & \text{in } \widetilde{\Omega}, \\ \lambda v - \mathrm{div} T(v,q) = \tilde{g}_{x_0,\delta} & \text{in } \widetilde{\Omega}, \\ v_{\tilde{\nu}}|_{\widetilde{\Gamma}_0} = 0 & \text{on } \widetilde{\Gamma}_0, \\ K(2\mu D(v)\tilde{\nu})_{\tilde{\tau}} + (1-K)v_{\tilde{\tau}}|_{\widetilde{\Gamma}_0} = \tilde{h}^0_{x_0,\delta} & \text{on } \widetilde{\Gamma}_0, \end{cases} \tag{8.13}$$

where $v_{\tilde{\nu}} := \tilde{\nu} \cdot v$, $v_{\tilde{\tau}} := v - v_{\tilde{\nu}}\tilde{\nu}$. Take a small positive constant δ for any small positive constant ε_0 such that $\mathrm{supp} v \subset \widetilde{B}_{\varepsilon_0}(0)$, where $\widetilde{B}_{\varepsilon_0}(0) := \{y \in \mathbb{R}^n\ ;\ |y| < \varepsilon_0\}$. Then there exist a positive constant $\varepsilon_1 > \varepsilon_0$ and $\chi \in C^{2,1}(\widetilde{B}'_{\varepsilon_1}(0))$ such that

$$\widetilde{B}_{\varepsilon_0}(0) \cap \widetilde{\Omega} \subset \{y = (y', y_n)^T \in \mathbb{R}^n\ ;\ y_n > \chi(y'),\ |y'| < \varepsilon_1\},$$

$$\widetilde{B}_{\varepsilon_0}(0) \cap \partial\widetilde{\Omega} \subset \{y = (y', y_n)^T \in \mathbb{R}^n\ ;\ y_n = \chi(y'),\ |y'| < \varepsilon_1\},$$

where $\widetilde{B}'_{\varepsilon_1}(0) := \{y' \in \mathbb{R}^{n-1}\ ;\ |y'| < \varepsilon_1\}$. Since $\tilde{\nu}(0) = (0,\cdots,0,-1)^T$, we can see easily that $\chi(0) = 0$, $\nabla'\chi(0) = 0$ and $\tilde{\nu}$ is characterized as

$$\tilde{\nu}(y') = \frac{(\nabla'\chi(y'), -1)^T}{(1 + |\nabla'\chi(y')|^2)^{1/2}}.$$

Take a small positive constant $\delta_0 \geq \delta$ satisfying $0 < \varepsilon_0 < \varepsilon_1/2$, let $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ be defined as

$$\begin{cases} \psi(y') = 1 & \text{if } |y'| \leq 1, \\ 0 < \psi(y') < 1 & \text{if } 1 < |y'| < 2, \\ \psi(y') = 0 & \text{if } |y'| \geq 2, \end{cases}$$

$\omega(y') = \psi(y'/\varepsilon_0)\chi(y')$, and set

$$H^n_\omega = \{y = (y', y_n)^T \in \mathbb{R}^n\ ;\ y_n > \omega(y'),\ y' \in \mathbb{R}^{n-1}\},$$

$$\partial H^n_\omega = \{y = (y', y_n)^T \in \mathbb{R}^n\ ;\ y_n = \omega(y'),\ y' \in \mathbb{R}^{n-1}\}.$$

Then it follows from (8.13) that (v,q) satisfies the following resolvent problem in H^n_ω:

$$\begin{cases} \mathrm{div} v = \tilde{f}_{x_0,\delta} & \text{in } H^n_\omega, \\ \lambda v - \mathrm{div} T(v,q) = \tilde{g}_{x_0,\delta} & \text{in } H^n_\omega, \\ v_{\nu_\omega}|_{\partial H^n_\omega} = 0 & \text{on } \partial H^n_\omega, \\ K(2\mu D(v)\nu_\omega)_{\tau_\omega} + (1-K)u_{\tau_\omega}|_{\partial H^n_\omega} = \tilde{h}^0_{x_0,\delta} & \text{on } \partial H^n_\omega, \end{cases} \tag{8.14}$$

where $v_{\nu_\omega} := \nu_\omega \cdot v$, $v_{\tau_\omega} := v - v_{\nu_\omega}\nu_\omega$. Since $\chi(0) = 0$ and $\nabla'\chi(0) = 0$, the Taylor formula yields

$$\|\nabla'\omega\|_{(L_\infty(\mathbb{R}^{n-1}))^{n-1}} \le C \max_{|y'|\le\varepsilon_1} |\nabla'^2\chi(y')|\varepsilon_0. \tag{8.15}$$

Set

$$M_j = \|\nabla'\omega\|_{(W^j_\infty(\mathbb{R}^{n-1}))^{n-1}}$$

for any $j = 0, 1, 2$. Then (8.15) implies

$$M_0 \le C \max_{|y'|\le\varepsilon_1} |\nabla'^2\chi(y')|\varepsilon_0, \tag{8.16}$$

consequently, there exists a small positive constant ε_0 such that $M_0 \le M$, where M is a positive constant which appeared in Theorem 7.5. In the same manner as above, we can derive the resolvent problem in H^n_ω from (1.4) near Γ_1.

By virtue of the localization method, Theorems 7.1, 7.4, 7.5 and Lemma 8.4, we prove the following theorem on the existence and uniqueness of solutions to (1.4):

Theorem 8.1. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $0 < K \le 1$, $1 < p < \infty$, $0 < \varepsilon < \pi/2$, $f \in C^1(\overline{\Omega}) \cap \dot{L}_p(\Omega)$, $g \in (C(\overline{\Omega}))^n$, $h^0 \in (C^1(\Gamma_0))^n$, $h^0_\nu|_{\Gamma_0} = 0$, $h^1 \in (C^2(\Gamma_1))^n$, $h^1_\nu|_{\Gamma_1} = 0$. Then (1.4) has uniquely a solution $(u,p) \in (W^2_p(\Omega))^n \times \dot{W}^1_p(\Omega)$ for any $\lambda \in S_\varepsilon \cup \{0\}$.*

Proof. Lemma 8.4 admits that (8.7) has uniquely a generalized solution $(u,p) \in H^1_2(\Omega) \times \dot{L}_2(\Omega)$. It is sufficient to obtain the regularity of generalized solutions to (8.7). In the case where $1 < p \le 2$, (8.7) and the localization method in Ω_δ yield

$$\begin{cases} \mathrm{div} v = f_\delta \text{ in } \mathbb{R}^n, \\ B_{\lambda_0,\mathbb{R}^n}(v,w) - (q, \mathrm{div} w)_{\mathbb{R}^n} = (\lambda_0 - \lambda)(v,w)_{\mathbb{R}^n} + (g_\delta, w)_{\mathbb{R}^n} \end{cases} \tag{8.17}$$

for any $w \in (W^1_2(\mathbb{R}^n))^n$, $\lambda_0 > 0$. Since $(u,p) \in H^1_2(\Omega) \times \dot{L}_2(\Omega)$, we have $f_\delta \in W^1_2(\mathbb{R}^n) \cap \dot{W}^{-1}_2(\mathbb{R}^n)$, $(\lambda_0 - \lambda)v \in (W^1_2(\mathbb{R}^n))^n$, $g_\delta \in (L_2(\mathbb{R}^n))^n$. Theorem 7.1 implies that (8.17) has uniquely a solution $(v^1, q^1) \in (W^2_2(\mathbb{R}^n))^n \times \dot{W}^1_2(\mathbb{R}^n)$ satisfying

$$\begin{cases} \mathrm{div} v^1 = f_\delta \text{ in } \mathbb{R}^n, \\ B_{\lambda_0,\mathbb{R}^n}(v^1,w) - (q^1, \mathrm{div} w)_{\mathbb{R}^n} = (\lambda_0 - \lambda)(v,w)_{\mathbb{R}^n} + (g_\delta, w)_{\mathbb{R}^n} \end{cases} \tag{8.18}$$

for any $w \in (W^1_2(\mathbb{R}^n))^n$, $\lambda_0 > 0$. By subtracting (8.18) from (8.17) and setting $(v^2, q^2) = (v - v^1, q - q^1)$, it follows that $(v^2, q^2) \in (W^1_2(\mathbb{R}^n))^n \times \dot{W}^1_2(\mathbb{R}^n)$ and

$$\begin{cases} \mathrm{div} v^2 = 0 \text{ in } \mathbb{R}^n, \\ B_{\lambda_0,\mathbb{R}^n}(v^2,w) - (q^2, \mathrm{div} w)_{\mathbb{R}^n} = 0 \end{cases} \tag{8.19}$$

for any $w \in (W^1_2(\mathbb{R}^n))^n$, $\lambda_0 > 0$. Let $w = v^2$ in (8.19). Then $B_{\lambda_0,\mathbb{R}^n}(v^2, v^2) = 0$ for any $\lambda_0 > 0$. We can see easily from Lemma 8.1 that $v^2 = 0$, i.e., $v = v^1 \in (W^2_2(\mathbb{R}^n))^n$. It follows from (8.19) that $-(q^2, \mathrm{div} w)_{\mathbb{R}^n} = (\nabla q^2, w)_{\mathbb{R}^n} = 0$ for any $w \in (W^1_2(\mathbb{R}^n))^n$, which yields that $\nabla q^2 = 0$, i.e., $\nabla q = \nabla q^1$. This means that $(u,p) \in (W^2_2(\Omega_\delta))^n \times W^1_2(\Omega_\delta)$ for any $\delta > 0$. In the same manner as above, (8.7) and the localization method near Γ_0 imply

$$\begin{cases} \mathrm{div} v = \tilde{f}_{x_0,\delta} \text{ in } H^n_\omega, \\ B_{\lambda_1,H^n_\omega}(v,w) - (q, \mathrm{div} w)_{H^n_\omega} = (\lambda_1 - \lambda)(v,w)_{H^n_\omega} + (\tilde{g}_{x_0,\delta}, w)_{H^n_\omega} + K^{-1}(\tilde{h}^0_{x_0,\delta}, w)_{\partial H^n_\omega} \end{cases} \tag{8.20}$$

for any $w \in W^1_{2,\nu_\omega}(H^n_\omega)$, where λ_1 is a positive constant which appeared in Theorem 7.5. Since $(u,p) \in H^1_2(\Omega) \times \dot{L}_2(\Omega)$, we have $\tilde{f}_{x_0,\delta} \in W^1_2(H^n_\omega) \cap \dot{W}^{-1}_2(H^n_\omega)$, $(\lambda_1 - \lambda)v \in W^1_{2,\nu_\omega}(H^n_\omega)$, $\tilde{g}_{x_0,\delta} \in L_2(H^n_\omega)$, $\tilde{h}^0_{x_0,\delta} \in W^1_{2,\nu_\omega}(H^n_\omega)$. Theorem 7.5 yields that (8.20) has uniquely a solution $(v^1, q^1) \in (W^2_2(H^n_\omega))^n \times \dot{W}^1_2(H^n_\omega)$ satisfying

$$\begin{cases} \mathrm{div} v^1 = \tilde{f}_{x_0,\delta} \text{ in } H^n_\omega, \\ B_{\lambda_1, H^n_\omega}(v^1, w) - (q^1, \mathrm{div} w)_{H^n_\omega} = (\lambda_1 - \lambda)(v, w)_{H^n_\omega} + (\tilde{g}_{x_0,\delta}, w)_{H^n_\omega} + K^{-1}(\tilde{h}^0_{x_0,\delta}, w)_{\partial H^n_\omega} \end{cases} \tag{8.21}$$

for any $w \in W^1_{2,\nu_\omega}(H^n_\omega)$. By subtracting (8.21) from (8.20) and setting $(v^2, q^2) = (v - v^1, q - q^1)$, we obtain $(v^2, q^2) \in (W^2_2(H^n_\omega))^n \times \dot{W}^1_2(H^n_\omega)$ and

$$\begin{cases} \mathrm{div} v^2 = 0 \text{ in } H^n_\omega, \\ B_{\lambda_1, H^n_\omega}(v^2, w) - (q^2, \mathrm{div} w)_{H^n_\omega} = 0 \end{cases} \tag{8.22}$$

for any $w \in W^1_{2,\nu_\omega}(H^n_\omega)$. Let $w = v^2$ in (8.22). Then $B_{\lambda_1, H^n_\omega}(v^2, v^2) = 0$. It is easy to see from Lemma 8.1 that $v^2 = 0$, i.e., $v = v^1 \in W^1_{2,\nu_\omega}(H^n_\omega)$. It follows from (8.22) that $-(q^2, \mathrm{div} w)_{H^n_\omega} = (\nabla q^2, w)_{H^n_\omega} = 0$ for any $w \in W^1_{2,\nu_\omega}(H^n_\omega)$, which implies that $\nabla q^2 = 0$, i.e., $\nabla q = \nabla q^1$. Hence, there exists a positive constant δ_0 for any $x_0 \in \Gamma_0$ such that $(u,p) \in (W^2_2(B_\delta(x_0) \cap \Omega))^n \times W^1_2(B_\delta(x_0) \cap \Omega)$ for any $0 < \delta \le \delta_0$. Set $\Gamma^\delta_0 = \{x \in \Omega \ ; \ \mathrm{dist}(x, \Gamma_0) < \delta\}$ for any $\delta > 0$. Then, since Γ_0 is compact in $\mathbb{R}^n$, there exists a positive constant δ_1 such that $(u,p) \in (W^2_2(\Gamma^\delta_0))^n \times W^1_2(\Gamma^\delta_0)$, $u_\nu|_{\Gamma_0} = 0$ for any $0 < \delta \le \delta_1$. By the same argument as above, which follows from the localization method near Γ_1 and Theorem 7.4, we have $(u,p) \in W^2_{2,\nu}(\Omega) \times \dot{W}^1_2(\Omega)$. In the case where $2 < p < \infty$, let us introduce a sequence $\{p_i\}_{i \in \mathbb{Z}, i \ge 0}$ of exponents as follows:

$$p_0 = 2, \ 0 < \frac{1}{p_i} - \frac{1}{p_{i+1}} \le \frac{1}{n}$$

for any $i \in \mathbb{Z}$, $i \ge 0$. Then it is easy to see that $p_i < p_{i+1}$ for any $i \in \mathbb{Z}$, $i \ge 0$. Since $(u,p) \in (W^2_2(\Omega))^n \times \dot{W}^1_2(\Omega)$, the Sobolev embedding theorem implies $(u,p) \in (W^1_{p_1}(\Omega))^n \times \dot{L}_{p_1}(\Omega)$. Hence, we have $f_\delta \in W^1_{p_1}(\mathbb{R}^n) \cap \dot{W}^{-1}_{p_1}(\mathbb{R}^n)$, $(\lambda_0 - \lambda)v \in (W^1_{p_1}(\mathbb{R}^n))^n$, $g_\delta \in (L_{p_1}(\mathbb{R}^n))^n$ in (8.17). Concerning (8.20), we obtain $\tilde{f}_{x_0,\delta} \in W^1_{p_1}(H^n_\omega) \cap \dot{W}^{-1}_{p_1}(H^n_\omega)$, $(\lambda_1 - \lambda)v \in W^1_{p_1,\nu_\omega}(H^n_\omega)$, $\tilde{g}_{x_0,\delta} \in L_{p_1}(H^n_\omega)$, $\tilde{h}^0_{x_0,\delta} \in W^1_{p_1,\nu_\omega}(H^n_\omega)$. The same argument as above shows that $(u,p) \in (W^2_{p_1}(\Omega))^n \times \dot{W}^1_{p_1}(\Omega)$. Moreover, it follows from the induction with respect to i that $(u,p) \in (W^2_{p_i}(\Omega))^n \times \dot{W}^1_{p_i}(\Omega)$ implies $(u,p) \in (W^2_{p_{i+1}}(\Omega))^n \times \dot{W}^1_{p_{i+1}}(\Omega)$ for any $i \in \mathbb{Z}$, $i \ge 0$. Since there exists an exponent p_i such that $p_i \ge p$, we have $(u,p) \in (W^2_p(\Omega))^n \times \dot{W}^1_p(\Omega)$. As for the uniqueness of solutions to (1.4), we have only to prove that $(u,p) \in (W^2_p(\Omega))^n \times \dot{W}^1_p(\Omega)$ satisfying

$$\begin{cases} \mathrm{div} u = 0 & \text{in } \Omega, \\ \lambda u - \mathrm{div} T(u,p) = 0 & \text{in } \Omega, \\ u_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ K(T(u,p)\nu)_\tau + (1-K)u_\tau|_{\Gamma_0} = 0 & \text{on } \Gamma_0, \\ u_\tau|_{\Gamma_1} = 0 & \text{on } \Gamma_1 \end{cases} \tag{8.23}$$

yields $(u,p) = (0,0)$. It is easy to see that $\lambda \in S_\varepsilon \cup \{0\}$ implies $\bar{\lambda} \in S_\varepsilon \cup \{0\}$. Let p^* be the dual exponent of p defined as $1/p + 1/p^* = 1$, $w \in (C(\overline{\Omega}))^n$. Then the above result admits that

$$\begin{cases} \mathrm{div} v = 0 & \text{in } \Omega, \\ \bar{\lambda} v - \mathrm{div} T(v, q) = w & \text{in } \Omega, \\ v_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ K(T(v, q)\nu)_\tau + (1-K) v_\tau|_{\Gamma_0} = 0 & \text{on } \Gamma_0, \\ v_\tau|_{\Gamma_1} = 0 & \text{on } \Gamma_1 \end{cases} \tag{8.24}$$

has uniquely a solution $(v, q) \in (W^2_{p^*}(\Omega))^n \times \dot{W}^1_{p^*}(\Omega)$. It follows from (8.23), (8.24) that

$$(u, w)_\Omega = (u, \bar{\lambda} v - \mathrm{div} T(v, q))_\Omega = B_{\lambda,\Omega}(u, v) = 0 \tag{8.25}$$

for any $\lambda \in S_\varepsilon \cup \{0\}$. Since $C(\overline{\Omega})$ is dense in $L_p(\Omega)$, (8.25) and the arbitrariness of the choice of w yield $u = 0$. Furthermore, we can see easily from (8.23) that $\nabla p = 0$. Recall that $p \in \dot{W}^1_{p^*}(\Omega)$. Then we have $p = 0$. This completes the proof of Theorem 8.1. $\square$

8.2 L_p estimates for solutions to (1.4) in Ω

The main purpose of this subsection is to prove Theorem 6.1. The proof of L_p estimates for solutions to (1.4) is essentially based on the following lemma:

Lemma 8.5. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $0 < K \leq 1$, $1 < p < \infty$, $0 < \varepsilon < \pi/2$, $\lambda_0 > 0$, $f \in \dot{W}^1_p(\Omega)$, $g \in (L_p(\Omega))^n$, $h^0 \in (W^{1-1/p}_p(\Gamma_0))^n$, $h^0_\nu|_{\Gamma_0} = 0$, $h^1 \in (W^{2-1/p}_p(\Gamma_1))^n$, $h^1_\nu|_{\Gamma_1} = 0$. Then a solution $(u, p) \in (W^2_p(\Omega))^n \times \dot{W}^1_p(\Omega)$ to (1.4) satisfies*

$$I^2_{p,\lambda,\Omega}(u) + \|p\|_{W^1_p(\Omega)} \leq C_{p,\varepsilon,\lambda_0}(I^{1,-1}_{p,\lambda,\Omega}(f) + \|g\|_{(L_p(\Omega))^n} + I^1_{p,\lambda,\Omega}(h^0) + I^2_{p,\lambda,\Omega}(h^1) + I^{1,-1}_{p,\lambda,\Omega}(u) + \|p\|_{L_p(\Omega)}) \tag{8.26}$$

for any $\lambda \in S_\varepsilon$, $|\lambda| \geq \lambda_0$, where $C_{p,\varepsilon,\lambda_0}$ is a positive constant depending only on n, Ω, p, ε, μ, K and λ_0.

Proof. It follows from the localization method in Ω_δ and Theorem 7.1 that

$$I^2_{p,\lambda,\mathbb{R}^n}(v) + \|q\|_{\dot{W}^1_p(\mathbb{R}^n)} \leq C(I^{1,-1}_{p,\lambda,\mathbb{R}^n}(f_\delta) + \|g_\delta\|_{(L_p(\mathbb{R}^n))^n}) \tag{8.27}$$

for any $\lambda \in S_\varepsilon$, $\delta > 0$. We can see easily that

$$I^1_{p,\lambda,\mathbb{R}^n}(f_\delta) \leq C_\delta(I^1_{p,\lambda,\Omega}(f) + I^1_{p,\lambda,\Omega}(u)), \tag{8.28}$$

$$\|g_\delta\|_{(L_p(\mathbb{R}^n))^n} \leq C_\delta(\|g\|_{(L_p(\Omega))^n} + I^1_{p,\lambda,\Omega}(u) + \|p\|_{L_p(\Omega)}) \tag{8.29}$$

for any $\lambda \in S_\varepsilon$, $\delta > 0$. It remains to establish the estimate for $\|f_\delta\|_{\dot{W}^{-1}_p(\Omega)}$. For any $w \in C^\infty_0(\mathbb{R}^n)$, set

$$c_w = \int_\Omega w(x)\varphi(x)dx \Big/ \int_\Omega \varphi(x)dx.$$

Then $(f_\delta, w)_{\mathbb{R}^n} = (\mathrm{div}(\varphi u), w)_{\mathbb{R}^n} = -(\varphi u, \nabla(w - c_w))_{\mathbb{R}^n} = (\mathrm{div}(\varphi u), w - c_w)_{\mathbb{R}^n} = (f_\delta, w - c_w)_{\mathbb{R}^n}$. By applying Lemma 6.5 to the identity, we obtain

$$\begin{aligned} |(f_\delta, w)_{\mathbb{R}^n}| &\leq |(f, \varphi(w - c_w))_\Omega| + |(u, \nabla\varphi(w - c_w))_\Omega| \\ &\leq \|f\|_{\dot{W}^{-1}_p(\Omega)} \|\varphi(w - c_w)\|_{\dot{W}^1_{p^*}(\Omega)} + \|u\|_{(\dot{W}^{-1}_p(\Omega))^n} \|\nabla\varphi(w - c_w)\|_{(\dot{W}^1_{p^*}(\Omega))^n}, \end{aligned}$$

$$|(f_\delta, w)_{\mathbb{R}^n}| \leq C_\delta(\|f\|_{\dot{W}_p^{-1}(\Omega)} + \|u\|_{(\dot{W}_p^{-1}(\Omega))^n})\|w\|_{\dot{W}^1_{p^*}(\mathbb{R}^n)} \tag{8.30}$$

for any $\delta > 0$. (8.30) gives

$$\|f_\delta\|_{\dot{W}_p^{-1}(\mathbb{R}^n)} \leq C_\delta(\|f\|_{\dot{W}_p^{-1}(\Omega)} + \|u\|_{(\dot{W}_p^{-1}(\Omega))^n}) \tag{8.31}$$

for any $\delta > 0$. It follows from (8.27)–(8.29), (8.31) that

$$I^2_{p,\lambda,\Omega_\delta}(u) + \|p\|_{W^1_p(\Omega_\delta)} \leq C_\delta(I^{1,-1}_{p,\lambda,\Omega}(f) + \|g\|_{(L_p(\Omega))^n} + I^{1,-1}_{p,\lambda,\Omega}(u) + \|p\|_{L_p(\Omega)}) \tag{8.32}$$

for any $\lambda \in S_\varepsilon$, $\delta > 0$. We proceed to obtain (8.26) near $\partial\Omega$. By the localization method near Γ_0 and Theorem 7.5, there exist three positive constants δ_{x_0}, C_{x_0} and λ_{1,x_0} for any $x_0 \in \Gamma_0$ such that

$$I^2_{p,\lambda,H^n_\omega}(v) + \|q\|_{W^1_p(H^n_\omega)} \leq C_{x_0}(I^{1,-1}_{p,\lambda,H^n_\omega}(\tilde{f}_{x_0,\delta_{x_0}}) + \|\tilde{g}_{x_0,\delta_{x_0}}\|_{(L_p(H^n_\omega))^n} + I^1_{p,\lambda,H^n_\omega}(\tilde{h}^0_{x_0,\delta_{x_0}})) \tag{8.33}$$

for any $\lambda \in S_\varepsilon$, $|\lambda| \geq \lambda_{1,x_0}$. It is easy to see that

$$I^1_{p,\lambda,H^n_\omega}(\tilde{f}_{x_0,\delta_{x_0}}) \leq C_{x_0}(I^1_{p,\lambda,\Omega}(f) + I^1_{p,\lambda,\Omega}(u)), \tag{8.34}$$

$$\|\tilde{g}_{x_0,\delta_{x_0}}\|_{(L_p(H^n_\omega))^n} \leq C_{x_0}(\|g\|_{(L_p(\Omega))^n} + I^1_{p,\lambda,\Omega}(u) + \|p\|_{L_p(\Omega)}), \tag{8.35}$$

$$I^1_{p,\lambda,H^n_\omega}(\tilde{h}^0_{x_0,\delta_{x_0}}) \leq C_{x_0}(I^1_{p,\lambda,\Omega}(h^0) + I^1_{p,\lambda,\Omega}(u)) \tag{8.36}$$

for any $\lambda \in S_\varepsilon$, $|\lambda| \geq \lambda_{1,x_0}$. The remainder is the estimate for $\|\tilde{f}_{x_0,\delta_{x_0}}\|_{\dot{W}_p^{-1}(H^n_\omega)}$. For any $\tilde{w} = \tilde{w}(y) \in \dot{W}^1_{p^*}(H^n_\omega)$, set $x = x_0 + Oy$, $w(x) = \tilde{w}(y)$. Then $(\tilde{f}_{x_0,\delta_{x_0}}, \tilde{w})_{H^n_\omega} = (f_{x_0,\delta_{x_0}}, w)_\Omega = (f_{x_0,\delta_{x_0}}, w - c_w)_\Omega$. Consequently, we have

$$\begin{aligned}|(\tilde{f}_{x_0,\delta_{x_0}}, \tilde{w})_{H^n_\omega}| &\leq |(f, \varphi(w - c_w))_\Omega| + |(u, \nabla\varphi(w - c_w))_\Omega| \\ &\leq \|f\|_{\dot{W}_p^{-1}(\Omega)}\|\varphi(w - c_w)\|_{\dot{W}^1_{p^*}(\Omega)} + \|u\|_{(\dot{W}_p^{-1}(\Omega))^n}\|\nabla\varphi(w - c_w)\|_{(\dot{W}^1_{p^*}(\Omega))^n},\end{aligned}$$

$$|(\tilde{f}_{x_0,\delta_{x_0}}, \tilde{w})_{H^n_\omega}| \leq C_{x_0}(\|f\|_{\dot{W}_p^{-1}(\Omega)} + \|u\|_{(\dot{W}_p^{-1}(\Omega))^n})\|\tilde{w}\|_{\dot{W}^1_{p^*}(H^n_\omega)}, \tag{8.37}$$

which yields

$$\|\tilde{f}_{x_0,\delta_{x_0}}\|_{\dot{W}_p^{-1}(H^n_\omega)} \leq C_{x_0}(\|f\|_{\dot{W}_p^{-1}(\Omega)} + \|u\|_{(\dot{W}_p^{-1}(\Omega))^n}). \tag{8.38}$$

It follows from (8.33)–(8.36), (8.38) that

$$I^2_{p,\lambda,B_{\delta_{x_0}}(x_0)\cap\Omega}(u) + \|p\|_{W^1_p(B_{\delta_{x_0}}(x_0)\cap\Omega)} \leq C_{x_0}(I^{1,-1}_{p,\lambda,\Omega}(f) + \|g\|_{(L_p(\Omega))^n} + I^1_{p,\lambda,\Omega}(h^0) + I^{1,-1}_{p,\lambda,\Omega}(u) + \|p\|_{L_p(\Omega)}) \tag{8.39}$$

for any $\lambda \in S_\varepsilon$, $|\lambda| \geq \lambda_{1,x_0}$. Since Γ_0 is compact in $\mathbb{R}^n$, we can choose a finite number of points $\{x_i\}_{i=1,\cdots,N}$ on Γ_0 such that

$$\Gamma_0 \subset \bigcup_{i=1}^N B_{\delta_{x_i}}(x_i).$$

(8.39) implies that there exist three positive constants δ_0, C_0 and $\lambda_{1,0}$ such that

$$I^2_{p,\lambda,\Gamma^\delta_0}(u) + \|p\|_{W^1_p(\Gamma^\delta_0)} \leq C_0(I^{1,-1}_{p,\lambda,\Omega}(f) + \|g\|_{(L_p(\Omega))^n} + I^1_{p,\lambda,\Omega}(h^0) + I^{1,-1}_{p,\lambda,\Omega}(u) + \|p\|_{L_p(\Omega)}) \tag{8.40}$$

for any $\lambda \in S_\varepsilon$, $|\lambda| \geq \lambda_{1,0}$, $0 < \delta \leq \delta_0$. In the same manner as in (8.40), there exist three positive constants δ_1, C_1 and $\lambda_{1,1}$ such that

$$I^2_{p,\lambda,\Gamma_1^\delta}(u) + \|p\|_{W^1_p(\Gamma_1^\delta)} \leq C_1(I^{1,-1}_{p,\lambda,\Omega}(f) + \|g\|_{(L_p(\Omega))^n} + I^2_{p,\lambda,\Omega}(h^1) + I^{1,-1}_{p,\lambda,\Omega}(u) + \|p\|_{L_p(\Omega)}) \tag{8.41}$$

for any $\lambda \in S_\varepsilon$, $|\lambda| \geq \lambda_{1,1}$, $0 < \delta \leq \delta_1$, which follows from the localization method near Γ_1 and Theorem 7.4. Therefore, it follows from (8.32), (8.40), (8.41) that we have (8.26) for any $\lambda \in S_\varepsilon$, $|\lambda| \geq \lambda_1 := \max\{\lambda_{1,0}, \lambda_{1,1}\}$. In the case where $\lambda \in S_\varepsilon$, $\lambda_0 \leq |\lambda| \leq \lambda_1$, we consider the following resolvent problem in Ω:

$$\begin{cases} \mathrm{div} u = f & \text{in } \Omega, \\ \lambda_1 u - \mathrm{div} T(u,p) = (\lambda_1 - \lambda)u + g & \text{in } \Omega, \\ u_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ K(T(u,p)\nu)_\tau + (1-K)u_\tau|_{\Gamma_0} = h^0 & \text{on } \Gamma_0, \\ u_\tau|_{\Gamma_1} = h^1 & \text{on } \Gamma_1. \end{cases} \tag{8.42}$$

By applying the above result to (8.42), we obtain (8.26) for any $\lambda \in S_\varepsilon$, $\lambda_0 \leq |\lambda| \leq \lambda_1$. □

The following theorem yields L_p estimates for solutions to (1.4).

Theorem 8.2. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $0 < K \leq 1$, $1 < p < \infty$, $0 < \varepsilon < \pi/2$, $f \in \dot{W}^1_p(\Omega)$, $g \in (L_p(\Omega))^n$, $h^0 \in (W^{1-1/p}_p(\Gamma_0))^n$, $h^0_\nu|_{\Gamma_0} = 0$, $h^1 \in (W^{2-1/p}_p(\Gamma_1))^n$, $h^1_\nu|_{\Gamma_1} = 0$. Then a unique solution $(u,p) \in (W^2_p(\Omega))^n \times \dot{W}^1_p(\Omega)$ to (1.4) satisfies*

$$I^2_{p,\lambda,\Omega}(u) + \|p\|_{W^1_p(\Omega)} \leq C_{p,\varepsilon}(I^{1,-1}_{p,\lambda,\Omega}(f) + \|g\|_{(L_p(\Omega))^n} + I^1_{p,\lambda,\Omega}(h^0) + I^2_{p,\lambda,\Omega}(h^1)) \tag{8.43}$$

for any $\lambda \in S_\varepsilon \cup \{0\}$, where $C_{p,\varepsilon}$ is a positive constant depending only on n, Ω, p, ε, μ and K.

Proof. For any $v \in C^\infty_0(\Omega)$, set

$$\tilde{v} = v - \frac{1}{|\Omega|}\int_\Omega v(x)dx.$$

Then Lemma 6.6 implies that

$$\begin{cases} \Delta w = \tilde{v} & \text{in } \Omega, \\ \partial_\nu w = 0 & \text{on } \partial\Omega \end{cases} \tag{8.44}$$

has uniquely a solution $w \in W^2_{p^*}(\Omega) \cap \dot{L}_{p^*}(\Omega)$ satisfying

$$\|w\|_{W^2_{p^*}(\Omega)} \leq C\|\tilde{v}\|_{L_{p^*}(\Omega)}. \tag{8.45}$$

It is easy to see from $p \in \dot{W}^1_p(\Omega)$ and (8.44) that $(p,v)_\Omega = (p,\tilde{v})_\Omega = -(\nabla p, \nabla w)_\Omega$. In order to establish L_p estimates for the pressure, we obtain

$$(p,v)_\Omega = -\lambda(f,w)_\Omega + 2\mu(D(u), \nabla^2 w)_\Omega - 2\mu(D(u)\nu, \nabla w)_{\partial\Omega} - (g, \nabla w)_\Omega, \tag{8.46}$$

which follows from the identity and (1.4). We can see easily from the Hölder inequality that

$$|(f,w)_\Omega| \leq \|f\|_{\dot{W}^{-1}_p(\Omega)}\|w\|_{W^1_{p^*}(\Omega)},$$

$$|(D(u), \nabla^2 w)_\Omega| \leq C\|u\|_{(W^1_p(\Omega))^n}\|w\|_{W^2_{p^*}(\Omega)},$$

$$|(D(u)\nu, \nabla w)_{\partial\Omega}| \leq C\|u\|_{(W^1_p(\partial\Omega))^n}\|w\|_{W^1_{p^*}(\partial\Omega)},$$

$$|(g, \nabla w)_\Omega| \leq \|g\|_{(L_p(\Omega))^n}\|w\|_{W^1_{p^*}(\Omega)}.$$

It follows from (8.45), (8.46) that

$$|(p, v)_\Omega| \leq C(|\lambda|\|f\|_{\dot{W}^{-1}_p(\Omega)} + \|g\|_{(L_p(\Omega))^n} + \|u\|_{(W^1_p(\Omega))^n} + \|u\|_{(W^1_p(\partial\Omega))^n})\|v\|_{L_{p^*}(\Omega)}. \tag{8.47}$$

(8.47) yields

$$\|p\|_{L_p(\Omega)} \leq C(|\lambda|\|f\|_{\dot{W}^{-1}_p(\Omega)} + \|g\|_{(L_p(\Omega))^n} + \|u\|_{(W^1_p(\Omega))^n} + \|u\|_{(W^1_p(\partial\Omega))^n}). \tag{8.48}$$

We proceed to obtain $\dot{W}^{-1}_p$ estimates for the fluid velocity. Let $v \in (W^1_{p^*}(\Omega))^n$. Then (1.4) is rewritten as follows:

$$\lambda(u, v)_\Omega = (p, \mathrm{div} v)_\Omega - 2\mu(D(u), D(v))_\Omega + (T(u,p)\nu, v)_{\partial\Omega} + (g, v)_\Omega. \tag{8.49}$$

In the same manner as above, we have

$$|(p, \mathrm{div} v)_\Omega| \leq C\|p\|_{L_p(\Omega)}\|v\|_{(W^1_{p^*}(\Omega))^n},$$

$$|(D(u), D(v))_\Omega| \leq C\|u\|_{(W^1_p(\Omega))^n}\|v\|_{(W^1_{p^*}(\Omega))^n},$$

$$|(T(u,p)\nu, v)_{\partial\Omega}| \leq C(\|u\|_{(W^1_p(\partial\Omega))^n} + \|p\|_{L_p(\partial\Omega)})\|v\|_{(L_{p^*}(\partial\Omega))^n},$$

$$|(g, v)_\Omega| \leq \|g\|_{(L_p(\Omega))^n}\|v\|_{(L_{p^*}(\Omega))^n}.$$

Consequently, (8.48), (8.49) imply

$$|\lambda||(u, v)_\Omega| \leq C(|\lambda|\|f\|_{\dot{W}^{-1}_p(\Omega)} + \|g\|_{(L_p(\Omega))^n} + \|u\|_{(W^1_p(\Omega))^n} + \|u\|_{(W^1_p(\partial\Omega))^n} + \|p\|_{L_p(\partial\Omega)})\|v\|_{(W^1_{p^*}(\Omega))^n}, \tag{8.50}$$

which is equivalent to

$$|\lambda|\|u\|_{(\dot{W}^{-1}_p(\Omega))^n} \leq C(|\lambda|\|f\|_{\dot{W}^{-1}_p(\Omega)} + \|g\|_{(L_p(\Omega))^n} + \|u\|_{(W^1_p(\Omega))^n} + \|u\|_{(W^1_p(\partial\Omega))^n} + \|p\|_{L_p(\partial\Omega)}). \tag{8.51}$$

Recall that

$$\|u\|_{(W^1_p(\partial\Omega))^n} \leq a\|u\|_{(W^2_p(\Omega))^n} + C_a\|u\|_{(W^1_p(\Omega))^n},$$

$$\|p\|_{L_p(\partial\Omega)} \leq b\|p\|_{W^1_p(\Omega)} + C_b\|p\|_{L_p(\Omega)}$$

for any $a > 0$, $b > 0$. Then it follows from (8.48), (8.51) that

$$\begin{aligned} |\lambda|\|u\|_{(\dot{W}^{-1}_p(\Omega))^n} + \|p\|_{L_p(\Omega)} \leq & C_{a,b}(|\lambda|\|f\|_{\dot{W}^{-1}_p(\Omega)} + \|g\|_{(L_p(\Omega))^n} + \|u\|_{(W^1_p(\Omega))^n}) \\ & + Ca\|u\|_{(W^2_p(\Omega))^n} + C_a b\|p\|_{W^1_p(\Omega)} \end{aligned} \tag{8.52}$$

for any $a > 0$, $b > 0$. By combining Lemma 8.5 with (8.52), we obtain

$$\begin{aligned} I^2_{p,\lambda,\Omega}(u) + \|p\|_{W^1_p(\Omega)} \leq & C_{a,b}(I^{1,-1}_{p,\lambda,\Omega}(f) + \|g\|_{(L_p(\Omega))^n} + I^1_{p,\lambda,\Omega}(h^0) + I^2_{p,\lambda,\Omega}(h^1) + I^1_{p,\lambda,\Omega}(u)) \\ & + Ca\|u\|_{(W^2_p(\Omega))^n} + C_a b\|p\|_{W^1_p(\Omega)} \end{aligned} \tag{8.53}$$

for any $\lambda \in S_\varepsilon$, $|\lambda| \geq 1$, $a > 0$, $b > 0$. Assume that $Ca < 1$ and $C_a b < 1$. Then it follows from (8.53) that

$$I^2_{p,\lambda,\Omega}(u) + \|p\|_{\dot{W}^1_p(\Omega)} \leq C(I^{1,-1}_{p,\lambda,\Omega}(f) + \|g\|_{(L_p(\Omega))^n} + I^1_{p,\lambda,\Omega}(h^0) + I^2_{p,\lambda,\Omega}(h^1) + |\lambda|^{-1/2} I^2_{p,\lambda,\Omega}(u)) \quad (8.54)$$

for any $\lambda \in S_\varepsilon$, $|\lambda| \geq 1$. Let $\lambda_1 \geq 1$ be a positive constant satisfying $C\lambda_1^{-1/2} < 1$. Then we can see easily from (8.44) that (8.43) holds for any $\lambda \in S_\varepsilon$, $|\lambda| \geq \lambda_1$. In the case where $\lambda \in S_\varepsilon \cup \{0\}$, $|\lambda| \leq \lambda_1$, it is sufficient to establish

$$\|u\|_{(W^2_p(\Omega))^n} + \|p\|_{\dot{W}^1_p(\Omega)} \leq C(\|f\|_{W^1_p(\Omega)} + \|g\|_{(L_p(\Omega))^n} + \|h^0\|_{(W^{1-1/p}_p(\Gamma_0))^n} + \|h^1\|_{(W^{2-1/p}_p(\Gamma_1))^n}) \quad (8.55)$$

for any $\lambda \in S_\varepsilon \cup \{0\}$, $|\lambda| \leq \lambda_1$. Recall that (1.4) is written by the following resolvent problem in Ω:

$$\begin{cases} \mathrm{div} u = f & \text{in } \Omega, \\ u - \mathrm{div} T(u,p) = (1-\lambda)u + g & \text{in } \Omega, \\ u_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ K(T(u,p)\nu)_\tau + (1-K)u_\tau|_{\Gamma_0} = h^0 & \text{on } \Gamma_0, \\ u_\tau|_{\Gamma_1} = h^1 & \text{on } \Gamma_1. \end{cases} \quad (8.56)$$

Then it follows from Lemma 8.5 that

$$\begin{aligned} \|u\|_{(W^2_p(\Omega))^n} + \|p\|_{\dot{W}^1_p(\Omega)} \leq & C(\|f\|_{W^1_p(\Omega)} + \|g\|_{(L_p(\Omega))^n} + \|h^0\|_{(W^{1-1/p}_p(\Gamma_0))^n} + \|h^1\|_{(W^{2-1/p}_p(\Gamma_1))^n} \\ & + \|u\|_{(W^1_p(\Omega))^n} + \|p\|_{L_p(\Omega)}) \end{aligned} \quad (8.57)$$

for any $\lambda \in S_\varepsilon \cup \{0\}$, $|\lambda| \leq \lambda_1$. By virtue of (8.57), the compactness-uniqueness argument shows that we have (8.55) for any $\lambda \in S_\varepsilon \cup \{0\}$, $|\lambda| \leq \lambda_1$. □

Set $W^{1-1/p}_{p,\nu}(\Gamma_0) = \{u \in (W^{1-1/p}_p(\Gamma_0))^n \; ; \; u_\nu|_{\Gamma_0} = 0\}$, $W^{2-1/p}_{p,\nu}(\Gamma_1) = \{u \in (W^{2-1/p}_p(\Gamma_1))^n \; ; \; u_\nu|_{\Gamma_1} = 0\}$, and let us introduce the linear mapping $A_{p,\lambda}$ from $W^2_{p,\nu}(\Omega) \times \dot{W}^1_p(\Omega)$ to $\dot{W}^1_p(\Omega) \times (L_p(\Omega))^n \times (W^{1-1/p}_{p,\nu}(\Gamma_0) \cap W^{2-1/p}_{p,\nu}(\Gamma_1))$, its domain $D(A_{p,\lambda})$ and range $R(A_{p,\lambda})$ as follows:

$$A_{p,\lambda}(u,p) = \begin{pmatrix} \mathrm{div} u \\ \lambda u - \mathrm{div} T(u,p) \\ K(T(u,p)\nu)_\tau + (1-K)u_\tau|_{\Gamma_0}, \; u_\tau|_{\Gamma_1} \end{pmatrix},$$

$$D(A_{p,\lambda}) = W^2_{p,\nu}(\Omega) \times \dot{W}^1_p(\Omega),$$

$$R(A_{p,\lambda}) = \{A_{p,\lambda}(u,p) \; ; \; (u,p) \in D(A_{p,\lambda})\}.$$

Then it follows from Theorem 8.2 that $R(A_{p,\lambda})$ is closed in $\dot{W}^1_p(\Omega) \times (L_p(\Omega))^n \times (W^{1-1/p}_{p,\nu}(\Gamma_0) \cap W^{2-1/p}_{p,\nu}(\Gamma_1))$. Since $C^k(\overline{\Omega})$ and $C^1(\overline{\Omega}) \cap \dot{L}_p(\Omega)$ are dense in $W^k_p(\Omega)$ and $\dot{W}^1_p(\Omega)$ respectively, it is easy to see from Theorem 8.1 (Existence) that $R(A_{p,\lambda})$ is dense in $\dot{W}^1_p(\Omega) \times (L_p(\Omega))^n \times (W^{1-1/p}_{p,\nu}(\Gamma_0) \cap W^{2-1/p}_{p,\nu}(\Gamma_1))$. Therefore, $R(A_{p,\lambda}) = \dot{W}^1_p(\Omega) \times (L_p(\Omega))^n \times (W^{1-1/p}_{p,\nu}(\Gamma_0) \cap W^{2-1/p}_{p,\nu}(\Gamma_1))$. Moreover, it follows from Theorem 8.1 (Uniqueness) that $A_{p,\lambda}$ is bijective for any $\lambda \in S_\varepsilon \cup \{0\}$. As is well known in Theorem 8.2, we have (6.1). This completes the proof of Theorem 6.1.

8.3 Resolvent estimates for $-A_p$

The proof of Theorem 6.2 is given as follows: Let $g \in L_{p,\sigma}(\Omega)$, and consider the following resolvent problem:

$$\lambda u + A_p u = g. \tag{8.58}$$

Then (8.58) is reduced to the following resolvent problem in Ω:

$$\begin{cases} \mathrm{div} u = 0 & \text{in } \Omega, \\ \lambda u - \mathrm{div} T(u,p) = g & \text{in } \Omega, \\ u_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ K(T(u,p)\nu)_\tau + (1-K)u_\tau|_{\Gamma_0} = 0 & \text{on } \Gamma_0, \\ u_\tau|_{\Gamma_1} = 0 & \text{on } \Gamma_1. \end{cases} \tag{8.59}$$

Theorem 6.1 admits that (8.59) has uniquely a solution $(u,p) \in (W_p^2(\Omega))^n \times \dot{W}_p^1(\Omega)$ satisfying

$$I^2_{p,\lambda,\Omega}(u) \le C\|g\|_{(L_p(\Omega))^n} \tag{8.60}$$

for any $\lambda \in S_\varepsilon \cup \{0\}$. (8.60) implies

$$\|u\|_{(L_p(\Omega))^n} \le \frac{C}{|\lambda|+1}\|g\|_{(L_p(\Omega))^n} \tag{8.61}$$

for any $\lambda \in S_\varepsilon \cup \{0\}$, which is equivalent to (6.2). This completes the proof of Theorem 6.2.

9 Applications

In this section, we will study various semilinear parabolic evolution equations in Banach spaces and the Navier-Stokes equations in a multiply-connected bounded domain with the aid of our main results. First, the L_p-theory of determining nodes is applied to the semilinear heat equation and the Navier-Stokes equations in Subsections 9.1, 9.2 and 9.3, 9.4 respectively. Second, we proceed to be concerned with the asymptotic properties of solutions to (1.5) in Subsections 9.5–9.7.

9.1 Determining nodes for the semilinear heat equation I

The initial-boundary value problem for the semilinear heat equation is described as follows:

$$\begin{cases} \partial_t u - \kappa\Delta u - |u|^{p-1}u = f & \text{in } \Omega \times (0,\infty), \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{9.1}$$

where u is the absolute temperature, $\kappa > 0$ is the coefficient of heat conductivity, $p > 1$, u_0 is the initial temperature, f is the external force.

Set $H = L_2(\Omega)$, $V = H_0^1(\Omega)$, $P = I_2$. Then we have the following strong formulation of (9.1):

$$\begin{cases} d_t u + Au = f(u) + f & \text{in } L_2((0,\infty); L_2(\Omega)), \\ u(0) = u_0 & \text{in } H_0^1(\Omega), \end{cases} \tag{9.2}$$

where $Au = -\kappa\Delta u$, $f(u) = |u|^{p-1}u$. It is well known in [11, Theorem 8.12] that A satisfies (A.1)–(A.4). Moreover, an elementary calculation shows that $f(u)$ has the following properties (f.1), (f.2):

(f.1) $f(0) = 0$.

(f.2) There exists a positive constant C_1 depending only on p such that

$$|f(u) - f(v)| \leq C_1(|u|^{p-1} + |v|^{p-1})|u - v|$$

for any $u, v \in \mathbb{R}$.

It is assured by the following lemma that $F(u) = f(u)$ satisfies (F.1), (F.2).

Lemma 9.1. *Let $n = 2, 3$, $1 < p \leq n/(n-2)$. Then there exists a positive constant C depending only on Ω and p such that*

$$\|F(u) - F(v)\|_{L_2(\Omega)} \leq C(\|u\|_{H^1(\Omega)}^{p-1} + \|v\|_{H^1(\Omega)}^{p-1})\|u - v\|_{H^1(\Omega)} \tag{9.3}$$

for any $u, v \in H^1(\Omega)$.

Proof. After taking the L_2-norm of (f.2), the Hölder and Minkowski inequalities imply

$$\|F(u) - F(v)\|_{L_2(\Omega)} \leq C_1(\|u\|_{L_{2p}(\Omega)}^{p-1} + \|v\|_{L_{2p}(\Omega)}^{p-1})\|u - v\|_{L_{2p}(\Omega)} \tag{9.4}$$

for any $u, v \in H^1(\Omega)$. It is easy to see from (9.4) and the Sobolev embedding theorem that we have (9.3). $\square$

The following theorems yield that (H.3), (H.4) hold for (9.2) under suitable assumptions for p, u_0 and f.

Theorem 9.1. *Let $n = 2, 3$, $1 < p \leq n/(n-2)$, $u_0 \in H_0^1(\Omega)$, $f \in L_2((0, \infty); L_2(\Omega))$. Then there exist two positive constants ε_1 and ε_2 depending only on Ω, κ and p such that* (9.2) *has uniquely a strong solution satisfying*

$$\|u\|_{C_b([0,\infty); H_0^1(\Omega))} \leq \varepsilon_1$$

provided that

$$\|u_0\|_a \leq \varepsilon_1, \ \|f\|_{L_\infty((0,\infty); L_2(\Omega))} \leq \varepsilon_2.$$

Proof. Let $\tilde{u}_0 \in H_0^1(\Omega)$, $\tilde{f} \in L_2((0,T); L_2(\Omega))$, $T > 0$. Then

$$\begin{cases} d_t u + Au = \tilde{f} & \text{in } L_2((0,T); L_2(\Omega)), \\ u(0) = \tilde{u}_0 & \text{in } H_0^1(\Omega) \end{cases} \tag{9.5}$$

has uniquely a strong solution u satisfying

$$u \in L_2((0,T); D(A)) \cap C([0,T]; H_0^1(\Omega)), \ d_t u \in L_2((0,T); L_2(\Omega)),$$

$$\kappa\|\nabla u\|_{C([0,T]; L_2(\Omega))}^2 + \|d_t u\|_{L_2((0,T); L_2(\Omega))}^2 + \|u\|_{L_2((0,T); D(A))}^2 \leq \kappa\|\nabla \tilde{u}_0\|_{L_2(\Omega)}^2 + \|\tilde{f}\|_{L_2((0,T); L_2(\Omega))}^2, \tag{9.6}$$

which is well known in [16, Theorem 3.2.1]. A fixed point argument with the aid of (9.5), (9.6) and the Banach fixed point theorem shows that there exists a positive constant $T_* \leq T$ depending only on Ω, κ, p, u_0 and f such that

$$\begin{cases} d_t u + Au = F(u) + f & \text{in } L^2((0,T); L^2(\Omega)), \\ u(0) = u_0 & \text{in } H_0^1(\Omega) \end{cases} \tag{9.7}$$

has uniquely a strong solution u satisfying

$$u \in L_2((0,T_*); D(A)) \cap C([0,T_*]; H_0^1(\Omega)), \ d_t u \in L_2((0,T_*); L_2(\Omega)).$$

By taking the L_2-scalar product of (9.7) with Au and the Poincaré inequality, a priori estimate for strong solutions to (9.2) is established as follows:

$$d_t(\|\nabla u(t)\|^2_{L_2(\Omega)}) \leq -\kappa\lambda_1 \|\nabla u(t)\|^2_{L_2(\Omega)} + 2\kappa^{-1} C_2^{2p} \|\nabla u(t)\|^{2p}_{L_2(\Omega)} + 2\kappa^{-1}\|f(t)\|^2_{L_2(\Omega)} \tag{9.8}$$

for any $t > 0$, where λ_1 is the first eigenvalue of $-\Delta$ with the zero Dirichlet boundary condition, C_2 is a positive constant depending only on Ω. Assume that

$$\|\nabla u_0\|^2_{L_2(\Omega)} \leq \left(\frac{\kappa^2\lambda_1}{4C_2^{2p}}\right)^{1/(p-1)}, \ \|f\|^2_{L_\infty((0,\infty);L_2(\Omega))} \leq \left(\frac{\kappa^2\lambda_1}{4C_2^{2p}}\right)^{p/(p-1)}. \tag{9.9}$$

Then (9.8), (9.9) give $d_t(\|\nabla u(t)\|^2_{L_2(\Omega)}) \leq 0$ for any $t > 0$, consequently,

$$\|\nabla u\|^2_{C_b([0,\infty);L_2(\Omega))} \leq \left(\frac{\kappa^2\lambda_1}{4C_2^{2p}}\right)^{1/(p-1)}. \tag{9.10}$$

By applying (9.10) to the existence and uniqueness of solutions to (9.7), therefore, (9.2) has uniquely a strong solution satisfying (9.10) provided that u_0 and f satisfy (9.9). □

Theorem 9.2. *Let $n = 2,3$, $1 < p \leq n/(n-2)$, $0 < \alpha \leq 1$, $R > 0$, $f \in L^\infty((0,\infty); D(A^\alpha))$, $t_0 > 0$. Then there exists a positive constant $M_\alpha(f,t_0)$ depending only on Ω, κ, p, R, f, t_0 and α such that*

$$\|u\|_{C_b([t_0,\infty);D(A))} \leq M_\alpha(f,t_0)$$

for any $u \in \mathcal{S}(H_0^1(\Omega)(R), f)$ satisfying $\|u\|_{C_b([0,\infty);H_0^1(\Omega))} \leq R$.

Proof. By virtue of [20, Theorems 2.5.2 and 7.3.6], A is a sectorial operator in $L_2(\Omega)$ satisfying $\mathrm{Re}\sigma(A) > 0$. Since $u \in C_b([0,\infty); H_0^1(\Omega))$, it follows from [12, Lemma 3.3.2] that

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}F(u)(s)ds + \int_0^t e^{-(t-s)A}f(s)ds \tag{9.11}$$

for any $t \geq 0$,

$$u(t) = e^{-tA}u(t_0) + \int_{t_0}^t e^{-(t-s)A}F(u)(s)ds + \int_{t_0}^t e^{-(t-s)A}f(s)ds \tag{9.12}$$

for any $t \geq t_0$. In the case where $1/2 < \beta < 1$, we can see easily from (4.1), (9.11) that

$$\begin{aligned} \|u(t)\|_{D(A^\beta)} \leq & C_{2,\beta-1/2,\lambda_1} t^{-\beta+1/2} e^{-\lambda_1 t}\|u_0\|_{D(A^{1/2})} \\ & + C_{2,\beta,\lambda_1}\int_0^t (t-s)^{-\beta} e^{-\lambda_1(t-s)}\|F(u)(s)\|_{L^2(\Omega)}ds \\ & + C_{2,\beta,\lambda_1}\int_0^t (t-s)^{-\beta} e^{-\lambda_1(t-s)}\|f(s)\|_{L^2(\Omega)}ds \end{aligned} \tag{9.13}$$

for any $t > 0$. Notice that $D(A^{1/2}) = H_0^1(\Omega)$ and $(u,v)_{D(A^{1/2})} = (u,v)_a$. Then, by (9.3), (9.13), we obtain

$$\|u\|_{C_b([t_0,\infty);D(A^\beta))} \leq C_{2,\beta-1/2,\lambda_1} t_0^{-\beta+1/2} R + C_{2,\beta,\lambda_1}\lambda_1^{1+\beta}\Gamma(1-\beta)(CR^p + \|f\|_{L_\infty((0,\infty);L_2(\Omega))}), \tag{9.14}$$

where $\Gamma(x)$ $(x > 0)$ is the gamma function. Let $n/4 < \beta < 1$, and set

$$M(f, t_0) = C_{2,\beta-1/2,\lambda_1} t_0^{-\beta+1/2} R + C_{2,\beta,\lambda_1} \lambda_1^{1+\beta} \Gamma(1-\beta)(CR^p + \|f\|_{L_\infty((0,\infty);L_2(\Omega))}).$$

Then it follows from (4.4), (9.14) that

$$\|F(u)(t)\|_{D(A^{1/2})} = \|F(u)(t)\|_a \le \kappa^{1/2} p C_3^p M(f, t_0)^p \tag{9.15}$$

for any $t \ge t_0$, where C_3 is a positive constant depending only on Ω. In the case where $\beta = 1$, it is easy to see from (4.1), (9.12) that

$$\begin{aligned}\|u(t)\|_{D(A)} \le & C_{2,1/2,\lambda_1}(t-t_0)^{-1/2} e^{-\lambda_1(t-t_0)} \|u(t_0)\|_{D(A^{1/2})} \\ & + C_{2,1/2,\lambda_1} \int_{t_0}^t (t-s)^{-1/2} e^{-\lambda_1(t-s)} \|F(u)(s)\|_{D(A^{1/2})} ds \\ & + C_{2,1-\alpha,\lambda_1} \int_{t_0}^t (t-s)^{-1+\alpha} e^{-\lambda_1(t-s)} \|f(s)\|_{D(A^\alpha)} ds\end{aligned} \tag{9.16}$$

for any $t > t_0$. By (9.15), (9.16) and the same argument as in (9.14), we have

$$\begin{aligned}\|u\|_{C_b([2t_0,\infty);D(A))} \le & C_{2,1/2,\lambda_1} t_0^{-1/2} R + C_{2,1/2,\lambda_1} \lambda_1^{3/2} \pi^{1/2} \kappa^{1/2} p C_3^p M(f, t_0)^p \\ & + C_{2,1-\alpha,\lambda_1} \lambda_1^{2-\alpha} \Gamma(\alpha) \|f\|_{L_\infty((0,\infty);D(A^\alpha))}.\end{aligned} \tag{9.17}$$

Set

$$\begin{aligned}M_\alpha(f, 2t_0) = & C_{2,1/2,\lambda_1} t_0^{-1/2} R + C_{2,1/2,\lambda_1} \lambda_1^{3/2} \pi^{1/2} \kappa^{1/2} p C_3^p M(f, t_0)^p \\ & + C_{2,1-\alpha,\lambda_1} \lambda_1^{2-\alpha} \Gamma(\alpha) \|f\|_{L_\infty((0,\infty);D(A^\alpha))}.\end{aligned}$$

Then the conclusion follows immediately from (9.17). □

9.2 Determining nodes for the semilinear heat equation II

The initial-boundary value problem for the semilinear heat equation is described as follows:

$$\begin{cases} \partial_t u - \kappa\Delta u - |u|^{q-1}u = f & \text{in } \Omega \times (0,\infty), \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{9.18}$$

where u is the absolute temperature, $\kappa > 0$ is the coefficient of heat conductivity, $q > 1$, u_0 is the initial temperature, f is the external force.

Let $1 < p < \infty$, $X_p = L_p(\Omega)$, $A_p = -\kappa\Delta$, $D(A_p) = W_p^2(\Omega) \cap W_{p,0}^1(\Omega)$, $F(u) = |u|^{q-1}u$. Then it follows from [20, Theorems 2.5.2 and 7.3.6] that A_p is a sectorial operator in $L_p(\Omega)$ satisfying $\mathrm{Re}\sigma(A_p) > 0$. Moreover, F satisfies (F.1), (F.2) by virtue of the following lemma:

Lemma 9.2. *Let* $1 < p < \infty$,

$$0 \le \delta < \frac{n}{2p}, \quad q\alpha + (q-1)\delta \ge \frac{n}{2p}(q-1).$$

Then there exists a positive constant $C_{p,\delta}$ *depending only on* n, Ω, p, q, κ, α *and* δ *such that*

$$\|A_p^{-\delta}(F(u) - F(v))\|_{L_p(\Omega)} \le C_{p,\delta}(\|u\|_{X_p^\alpha}^{q-1} + \|v\|_{X_p^\alpha}^{q-1})\|u - v\|_{X_p^\alpha} \tag{9.19}$$

for any $u, v \in X_p^\alpha$.

Proof. Set $1/p' = 1/p - 2\delta/n$. Then it follows from (4.3) that $A_p^{-\delta} : L_p(\Omega) \to X_p^\delta \to L_{p'}(\Omega)$ is a bounded linear operator. Hence, (9.2) is established by the Hölder inequality and $X_p^\alpha \hookrightarrow L_{p'q}(\Omega)$. □

By the successive approximation method based on [9], we obtain the following theorem which gives (H.5), (H.6) to (9.18) under suitable assumptions for u_0 and f.

Theorem 9.3. *Let $1 < p < \infty$, α_0 and δ be chosen as follows:*

$$\max\left\{0, \frac{n}{2p} - \frac{1}{q-1} - \frac{q-2}{q-1}\delta\right\} \le \alpha_0 < 1, \ \delta \ge 0, \ -\alpha_0 < \delta < \min\left\{\frac{n}{2p}, 1-\alpha_0\right\}.$$

Assume that $u_0 \in X_p^{\alpha_0}$ and $\min\{t,1\}^{1-\alpha_0-\delta} A_p^{-\delta} f \in C_b([0,\infty); L_p(\Omega))$ satisfies

$$\|A_p^{-\delta} f(t)\|_{L_p(\Omega)} = o(t^{\alpha_0+\delta-1}) \text{ as } t \to +0.$$

Then there exists a positive constant ε depending only on n, Ω, p, q, κ, α_0 and δ such that (9.18) *has uniquely a mild solution u satisfying the following continuity properties and estimates:*

(i) *For any $\alpha_0 \le \alpha < 1-\delta$, $t > 0$,*

$$\min\{t,1\}^{\alpha-\alpha_0} u \in C_b([0,\infty); X_p^\alpha),$$

$$\|u(t)\|_{X_p^\alpha} \le C \min\{t,1\}^{\alpha_0-\alpha}\left(\|u_0\|_{X_p^{\alpha_0}} + \sup_{s>0} \min\{s,1\}^{1-\alpha_0-\delta} \|A_p^{-\delta} f(s)\|_{L_p(\Omega)}\right), \tag{9.20}$$

where C is a positive constant depending only on n, Ω, p, q, κ, α_0 and δ provided that u_0 and f satisfy

$$\|u_0\|_{X_p^{\alpha_0}} + \sup_{t>0} \min\{t,1\}^{1-\alpha_0-\delta} \|A_p^{-\delta} f(t)\|_{L_p(\Omega)} \le \varepsilon.$$

(ii) *For any $\alpha_0 < \alpha < 1-\delta$,*

$$\|u(t)\|_{X_p^\alpha} = o(t^{\alpha_0-\alpha}) \text{ as } t \to +0.$$

Proof. The proof is similar to that of [9, Theorem 2.6]. □

Remark 9.1. It is easy to see from [20, Theorems 2.5.2 and 7.3.6] that Theorem 9.3 is still valid, instead of (9.18), for the following initial-boundary value problem:

$$\begin{cases} \partial_t u - \kappa\Delta u - |u|^{q-1}u = f & \text{in } \Omega \times (0,\infty), \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ \kappa\partial_\nu u + \kappa_s u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{9.21}$$

where $\kappa_s > 0$.

9.3 Determining nodes for the Navier-Stokes equations I

The initial-boundary value problem for the Navier-Stokes equations is described as follows:

$$\begin{cases} \mathrm{div} u = 0 & \text{in } \Omega \times (0, \infty), \\ \partial_t u + (u \cdot \nabla)u + \nabla p - \mu \Delta u = f & \text{in } \Omega \times (0, \infty), \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{9.22}$$

where $u = (u_1, \cdots, u_n)^T$ is the fluid velocity, p is the pressure, $\mu > 0$ is the coefficient of viscosity, u_0 is the initial fluid velocity, $f = (f_1, \cdots, f_n)^T$ is the external force field, $\cdot^T$ is the transposition.

Set $H^n = L_{2,\sigma}(\Omega)$, $V^n = H^1_{0,\sigma}(\Omega)$, $P = P_2$, where $H^1_{0,\sigma}(\Omega) := (H^1_0(\Omega))^n \cap L_{2,\sigma}(\Omega)$. Then the strong formulation of (9.22) is given by

$$\begin{cases} d_t u + Au = f(u) + f & \text{in } L^2((0, \infty); L^2_\sigma(\Omega)), \\ u(0) = u_0 & \text{in } H^1_{0,\sigma}(\Omega), \end{cases} \tag{9.23}$$

where $Au = -P_2(\mu \Delta u)$, $f(u) = -P_2(u \cdot \nabla)u$. It follows from [26, Lemma 3.3.7] that A satisfies (A.1)–(A.4). The following lemma admits that $F(u) = f(u)$ satisfies (F.1), (F.2).

Lemma 9.3. *Let $n = 2, 3$. Then there exists a positive constant C depending only on Ω such that*

$$\|F(u) - F(v)\|_{(L_2(\Omega))^n} \leq C(\|u\|_{(H^2(\Omega))^n} + \|v\|_{(H^2(\Omega))^n})\|u - v\|_{(H^1(\Omega))^n} \tag{9.24}$$

for any $u, v \in (H^2(\Omega))^n$.

Proof. It is easy to see that

$$F(u) - F(v) = P_2(u \cdot \nabla)(u - v) + P_2((u - v) \cdot \nabla)v$$

for any $u, v \in (H^2(\Omega))^n$. Notice that $H^2(\Omega) \hookrightarrow L_\infty(\Omega)$, which follows from the Sobolev embedding theorem. Then we have

$$\|P_2(u \cdot \nabla)(u - v)\|_{(L_2(\Omega))^n} \leq C_1\|u\|_{(H^2(\Omega))^n}\|u - v\|_{(H^1(\Omega))^n} \tag{9.25}$$

for any $u, v \in (H^2(\Omega))^n$, where C_1 is a positive constant depending only on Ω. Since $H^1(\Omega) \hookrightarrow L_6(\Omega) \hookrightarrow L_3(\Omega)$, which follows from the Sobolev embedding theorem, we can see easily from the Hölder inequality that

$$\|P_2((u - v) \cdot \nabla)v\|_{(L_2(\Omega))^n} \leq C_2\|v\|_{(H^2(\Omega))^n}\|u - v\|_{(H^1(\Omega))^n} \tag{9.26}$$

for any $u, v \in (H^2(\Omega))^n$, where C_2 is a positive constant depending only on Ω. Consequently, (9.25), (9.26) yield clearly that we have (9.24). □

9.4 Determining nodes for the Navier-Stokes equations II

The initial-boundary value problem for the Navier-Stokes equations is described as follows:

$$\begin{cases} \mathrm{div} u = 0 & \text{in } \Omega \times (0, \infty), \\ \partial_t u + (u \cdot \nabla)u + \nabla p - \mu \Delta u = f & \text{in } \Omega \times (0, \infty), \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{9.27}$$

where $u = (u_1, \cdots, u_n)^T$ is the fluid velocity, p is the pressure, $\mu > 0$ is the coefficient of viscosity, u_0 is the initial fluid velocity, $f = (f_1, \cdots, f_n)^T$ is the external force field, $\cdot^T$ is the transposition.

Set $(X_p)^n = L_{p,\sigma}(\Omega)$, $B_p = -\mu\Delta$, $D(B_p) = W_p^2(\Omega) \cap W_{p,0}^1(\Omega)$, $F(u) = -P_p(u \cdot \nabla)u$. Then A_p is the Stokes operator in $L_{p,\sigma}(\Omega)$ defined as $A_p = -\mu P_p \Delta$, $D(A_p) = (D(B_p))^n \cap L_{p,\sigma}(\Omega)$. It follows from [7, Corollary 1.6] that A_p is a sectorial operator in $L_{p,\sigma}(\Omega)$ satisfying $\mathrm{Re}\sigma(A_p) > 0$. Moreover, [8, Theorem 3] implies that $D(A_p^\alpha)$ is characterized as $D(A_p^\alpha) = (D(B_p^\alpha))^n \cap L_{p,\sigma}(\Omega)$ for any $0 \le \alpha \le 1$. The following lemma admits that F satisfies (F.3), (F.4).

Lemma 9.4. *Let* $1 < p < \infty$,

$$\alpha > 0,\ 0 \le \delta < \frac{1}{2} + \frac{n}{2}\left(1 - \frac{1}{p}\right),\ \alpha + \delta > \frac{1}{2},\ 2\alpha + \delta \ge \frac{n}{2p} + \frac{1}{2}.$$

Then there exists a positive constant $C_{p,\delta}$ *depending only on* n, Ω, p, μ, α *and* δ *such that*

$$\|A_p^{-\delta}(F(u) - F(v))\|_{(L_p(\Omega))^n} \le C_{p,\delta}(\|u\|_{X_p^\alpha} + \|v\|_{X_p^\alpha})\|u - v\|_{X_p^\alpha} \tag{9.28}$$

for any $u, v \in X_p^\alpha$.

Proof. See [9, Lemma 2.2]. □

As is well known in [9], the following theorem yields that (H.5), (H.6) hold for (9.27) under suitable assumptions for u_0 and f.

Theorem 9.4. *Let* $1 < p < \infty$, α_0 *and* δ *be chosen as follows:*

$$\max\left\{0, \frac{n}{2p} - \frac{1}{2}\right\} \le \alpha_0 < 1,\ \delta \ge 0,\ -\alpha_0 < \delta < 1 - \alpha_0.$$

Assume that $u_0 \in X_p^{\alpha_0}$ *and* $\min\{t,1\}^{1-\alpha_0-\delta} A_p^{-\delta} P_p f \in C_b([0,\infty); L_{p,\sigma}(\Omega))$ *satisfies*

$$\|A_p^{-\delta} P_p f(t)\|_{(L_p(\Omega))^n} = o(t^{\alpha_0+\delta-1}) \text{ as } t \to +0.$$

Then there exists a positive constant ε *depending only on* n, Ω, p, μ, α_0 *and* δ *such that* (9.27) *has uniquely a mild solution* u *satisfying the following continuity properties and estimates:*

(i) *For any* $\alpha_0 \le \alpha < 1 - \delta$, $t > 0$,

$$\min\{t,1\}^{\alpha-\alpha_0} u \in C_b([0,\infty); X_p^\alpha),$$

$$\|u(t)\|_{X_p^\alpha} \le C \min\{t,1\}^{\alpha_0-\alpha}\left(\|u_0\|_{X_p^{\alpha_0}} + \sup_{s>0} \min\{s,1\}^{1-\alpha_0-\delta}\|A_p^{-\delta} P_p f(s)\|_{(L_p(\Omega))^n}\right), \tag{9.29}$$

where C *is a positive constant depending only on* n, Ω, p, μ, α_0 *and* δ *provided that* u_0 *and* f *satisfy*

$$\|u_0\|_{X_p^{\alpha_0}} + \sup_{t>0} \min\{t,1\}^{1-\alpha_0-\delta}\|A_p^{-\delta} P_p f(t)\|_{(L_p(\Omega))^n} \le \varepsilon.$$

(ii) *For any* $\alpha_0 < \alpha < 1 - \delta$,

$$\|u(t)\|_{X_p^\alpha} = o(t^{\alpha_0-\alpha}) \text{ as } t \to +0.$$

Proof. The proof is similar to that of [9, Theorem 2.6]. □

Remark 9.2. We can see easily from [24, Theorem 1.3] and Theorem 6.2 that Theorem 9.4 is still valid, instead of (9.27), for the following initial-boundary value problem:

$$\begin{cases} \mathrm{div} u = 0 & \text{in } \Omega \times (0,\infty), \\ \partial_t u + (u \cdot \nabla)u + \nabla p - \mu \Delta u = f & \text{in } \Omega \times (0,\infty), \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ u_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ B(u)_\tau|_{\partial\Omega} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{9.30}$$

where $B(u)$ is one of the following boundary conditions:

- $B(u) = K(T(u,p)\nu) + (1-K)u$, $0 < K < 1$.
- $B(u) = \begin{cases} K(T(u,p)\nu) + (1-K)u & \text{on } \Gamma_0, \\ u & \text{on } \Gamma_1. \end{cases}$

9.5 Existence and uniqueness of solutions to (1.5)

As for the asymptotic properties of solutions to (1.5), the first result is the existence and uniqueness of solutions to (1.5) in $(W_p^2(\Omega))^n$. Theorem 6.1 admits that a fixed point argument is applied to (1.4). We shall prove the following theorem:

Theorem 9.5. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $0 < K \le 1$, $n < p < \infty$, $g \in (L_p(\Omega))^n$, $h \in (W_p^{2-1/p}(\Gamma_1))^n$, $h_\nu|_{\Gamma_1} = 0$. Then there exists a positive constant ε_p depending only on n, Ω, p, ρ, μ and K such that (1.5) has uniquely a solution $(\bar{u}, \bar{p}) \in (W_p^2(\Omega))^n \times \dot{W}_p^1(\Omega)$ satisfying*

$$\|\bar{u}\|_{(W_p^2(\Omega))^n} + \|\bar{p}\|_{W_p^1(\Omega)} \le C_p(\|g\|_{(L_p(\Omega))^n} + \|h\|_{(W_p^{2-1/p}(\Gamma_1))^n}), \tag{9.31}$$

where C_p is a positive constant depending only on n, Ω, p, ρ, μ and K provided that

$$\|g\|_{(L_p(\Omega))^n} + \|h\|_{(W_p^{2-1/p}(\Gamma_1))^n} \le \varepsilon_p.$$

Proof. Recall that (1.5) is written by the following boundary value problem in Ω:

$$\begin{cases} \mathrm{div} \bar{u} = 0 & \text{in } \Omega, \\ -\mathrm{div} T(\bar{u}, \bar{p}) = g + G(\bar{u}) & \text{in } \Omega, \\ \bar{u}_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ K(T(\bar{u},\bar{p})\nu)_\tau + (1-K)\bar{u}_\tau|_{\Gamma_0} = 0 & \text{on } \Gamma_0, \\ \bar{u}_\tau|_{\Gamma_1} = h & \text{on } \Gamma_1, \end{cases} \tag{9.32}$$

where $G(\bar{u}) := -(\bar{u} \cdot \nabla)\bar{u}$. Since $n < p < \infty$, the Sobolev embedding theorem implies that there exists a positive constant C_1 depending only on n, Ω and p such that

$$\|G(\bar{u})\|_{(L_p(\Omega))^n} \le C_1 \|\bar{u}\|^2_{(W_p^1(\Omega))^n} \tag{9.33}$$

for any $\bar{u} \in (W_p^1(\Omega))^n$. We shall apply the Banach fixed point theorem to (7.10) in the following Banach space:

$$X_p(\Omega) = \{(\bar{v}, \bar{q}) \in (W_p^2(\Omega))^n \times \dot{W}_p^1(\Omega) \ ; \ \|(\bar{v},\bar{q})\|_{X_p(\Omega)} \le 2C_2(\|g\|_{(L_p(\Omega))^n} + \|h\|_{(W_p^{2-1/p}(\Gamma_1))^n})\},$$

$$\|(\bar{v},\bar{q})\|_{X_p(\Omega)} = \|\bar{v}\|_{(W_p^2(\Omega))^n} + \|\bar{q}\|_{W_p^1(\Omega)},$$

where C_2 is a positive constant which appeared in Theorem 6.1 with $\lambda = 0$. Let S be the solution mapping defined as $S(\bar{v},\bar{q}) = (\bar{u},\bar{p})$, where $(\bar{u},\bar{p})$ is a solution to (9.32) with $G = G(\bar{v})$. Then it follows from Theorem 6.1 and (9.33) that

$$\begin{aligned}\|(\bar{u},\bar{p})\|_{X_p(\Omega)} &\le C_2(\|g\|_{(L_p(\Omega))^n} + \|G(\bar{v})\|_{(L_p(\Omega))^n} + \|h\|_{(W_p^{2-1/p}(\Gamma_1))^n})\\ &\le C_2(\|g\|_{(L_p(\Omega))^n} + \|h\|_{(W_p^{2-1/p}(\Gamma_1))^n} + C_1\|\bar{v}\|^2_{(W_p^1(\Omega))^n}),\end{aligned}$$

$$\|(\bar{u},\bar{p})\|_{X_p(\Omega)} \le C_2\{1+4C_1C_2^2(\|g\|_{(L_p(\Omega))^n}+\|h\|_{(W_p^{2-1/p}(\Gamma_1))^n})\}(\|g\|_{(L_p(\Omega))^n}+\|h\|_{(W_p^{2-1/p}(\Gamma_1))^n}). \tag{9.34}$$

Therefore, S is a mapping in $X_p(\Omega)$ provided that

$$\|g\|_{(L_p(\Omega))^n} + \|h\|_{(W_p^{2-1/p}(\Gamma_1))^n} \le \frac{1}{4C_1C_2^2}. \tag{9.35}$$

Set $S(\bar{v}^i,\bar{q}^i) = (\bar{u}^i,\bar{p}^i)$ for any $(\bar{v}^i,\bar{q}^i) \in X_p(\Omega)$, $i = 1, 2$. Then (9.32) implies the following boundary value problem in Ω:

$$\begin{cases}\mathrm{div}(\bar{u}^2-\bar{u}^1) = 0 & \text{in } \Omega,\\ -\mathrm{div}T(\bar{u}^2-\bar{u}^1,\bar{p}^2-\bar{p}^1) = G(\bar{v}^2) - G(\bar{v}^1) & \text{in } \Omega,\\ (\bar{u}^2-\bar{u}^1)_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega,\\ K(T(\bar{u}^2-\bar{u}^1,\bar{p}^2-\bar{p}^1)\nu)_\tau + (1-K)(\bar{u}^2-\bar{u}^1)_\tau|_{\Gamma_0} = 0 & \text{on } \Gamma_0,\\ (\bar{u}^2-\bar{u}^1)_\tau|_{\Gamma_1} = 0 & \text{on } \Gamma_1.\end{cases} \tag{9.36}$$

By Theorem 6.1 and (9.33), we have

$$\begin{aligned}\|(\bar{u}^2-\bar{u}^1,\bar{p}^2-\bar{p}^1)\|_{X_p(\Omega)} &\le C_2\|G(\bar{v}^2)-G(\bar{v}^1)\|_{(L_p(\Omega))^n}\\ &\le C_1C_2(\|\bar{v}^1\|_{(W_p^1(\Omega))^n} + \|\bar{v}^2\|_{(W_p^1(\Omega))^n})\|\bar{v}^2-\bar{v}^1\|_{(W_p^1(\Omega))^n},\end{aligned}$$

$$\|(\bar{u}^2-\bar{u}^1,\bar{p}^2-\bar{p}^1)\|_{X_p(\Omega)} \le 4C_1C_2^2(\|g\|_{(L_p(\Omega))^n} + \|h\|_{(W_p^{2-1/p}(\Gamma_1))^n})\|\bar{v}^2-\bar{v}^1\|_{(W_p^1(\Omega))^n}. \tag{9.37}$$

Assume that

$$\|g\|_{(L_p(\Omega))^n} + \|h\|_{(W_p^{2-1/p}(\Gamma_1))^n} < \frac{1}{4C_1C_2^2}. \tag{9.38}$$

Then S is a contraction mapping in $X_p(\Omega)$. Consequently, the Banach fixed point theorem admits that (9.32) has uniquely a solution $(\bar{u},\bar{p}) \in X_p(\Omega)$. The uniqueness of solutions to (9.32) can be proved by L_p estimates for the solutions similar to (9.37), which completes the proof of Theorem 9.5. □

9.6 Abstract initial value problem for (1.6)

The second result is the asymptotic stability of solutions to (1.5) in $L_{p,\sigma}(\Omega)$. As is well known in [12, Chapter 3], [20, Chapter 6], not only strong solutions but also mild solutions to (1.6) can be considered with the aid of Theorem 6.2. We shall prove the following theorem:

Theorem 9.6. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $0 < K \le 1$, $n < p < \infty$, $u_0 \in L_{p,\sigma}(\Omega)$, $(\bar{u},\bar{p}) \in (W_p^2(\Omega))^n \times \dot{W}_p^1(\Omega)$ be a solution to (1.5). Then there exist two positive constants δ_{p,λ_1} and $\varepsilon_{p,\lambda_1}$ depending only on n, Ω, p, ρ, μ, K and λ_1 for any $0 < \lambda_1 < \Lambda_1$ such that (1.6) has uniquely a mild solution $v \in C_b([0,\infty); L_{p,\sigma}(\Omega))$ satisfying the following continuity property and estimate:*

(i) *For any* $0 \le \alpha < 1$, $t > 0$,

$$t^{\alpha}e^{\lambda_1 t}v \in C_b([0,\infty); X_p^{\alpha}),$$

$$\|v(t)\|_{X_p^{\alpha}} \le C_{p,\alpha,\lambda_1} t^{-\alpha} e^{-\lambda_1 t}\|u_0 - \bar{u}\|_{(L_p(\Omega))^n}, \tag{9.39}$$

where C_{p,α,λ_1} *is a positive constant depending only on* n, Ω, p, ρ, μ, K, α *and* λ_1 *provided that*

$$\|u_0 - \bar{u}\|_{(L_p(\Omega))^n} \le \delta_{p,\lambda_1}, \ \|\bar{u}\|_{(W_p^1(\Omega))^n} \le \varepsilon_{p,\lambda_1}.$$

(ii) *For any* $0 < \alpha < 1$,

$$\|u(t)\|_{X_p^{\alpha}} = o(t^{-\alpha}) \text{ as } t \to +0.$$

This subsection provides some preliminaries concerning the theory of analytic semigroups on Banach spaces. First, we state the basic properties of the Stokes operator A_p in $L_{p,\sigma}(\Omega)$.

Lemma 9.5. *Let* $1 < p < \infty$, $\alpha \ge 0$, $0 < \lambda_1 < \Lambda_1$. *Then*

$$\|A_p^{\alpha} e^{-tA_p}\|_{\mathcal{B}(L_{p,\sigma}(\Omega))} \le C_{p,\alpha,\lambda_1} t^{-\alpha} e^{-\lambda_1 t} \tag{9.40}$$

for any $t \ge 0$, *where* C_{p,α,λ_1} *is a positive constant depending only on* n, Ω, p, μ, K, α *and* λ_1.

Proof. See [12, Theorem 1.4.3]. □

Lemma 9.6. *Let* $1 < p < \infty$, $0 \le \alpha \le 1$. *Then* $X_p^{\alpha} = [L_{p,\sigma}(\Omega), D(A_p)]_{\alpha}$, *where* $[X_0, X_1]_{\theta}$ $(0 \le \theta \le 1)$ *is an interpolation space between two Banach spaces* X_0 *and* X_1 *by the complex method,* $[X_0, X_1]_0 := X_0$, $[X_0, X_1]_1 := X_1$.

Proof. The proof is similar to that of [8, Theorem 2]. □

Lemma 9.7. *Let* $1 < p < \infty$, $0 \le \alpha \le 1$. *Then* X_p^{α} *is continuously embedded in* $(H_p^{2\alpha}(\Omega))^n \cap L_{p,\sigma}(\Omega)$, *where* $H_p^s(\Omega)$ $(s \ge 0)$ *is the Bessel-potential space defined as* $H_p^s(\Omega) = [L_p(\Omega), W_p^{\langle s\rangle}(\Omega)]_{s/\langle s\rangle}$, $\langle s\rangle := \min\{k \in \mathbb{Z}\ ;\ k \ge s\}$.

Proof. The lemma follows immediately from Lemma 9.6 and the basic property of interpolation spaces by the complex method, e.g., [27, Theorems 1.9.3]. □

Second, we proceed to the abstract initial value problem for (1.6). Let $0 < T < \infty$, $1 < p < \infty$, $u_0 \in L_{p,\sigma}(\Omega)$. Then, by applying P_p to the second equation of (1.6), we get the following abstract initial value problem for the evolution equation:

$$\begin{cases} d_t v + A_p v = G_1(v) + G_2(v) + G_3(v) \text{ in } (0,T], \\ v(0) = u_0 - \bar{u}, \end{cases} \tag{III}$$

where

$$G_1(v) := -P_p(v \cdot \nabla)v,$$
$$G_2(v) := -P_p(v \cdot \nabla)\bar{u},$$
$$G_3(v) := -P_p(\bar{u} \cdot \nabla)v.$$

In order to deal with (III), we shall find a mild solution $v \in C([0,T]; L_{p,\sigma}(\Omega))$ which satisfies the following abstract integral equation related to (III):

$$v(t) = e^{-tA_p}(u_0 - \bar{u}) + \int_0^t e^{-(t-s)A_p} G_1(v)(s)ds + \int_0^t e^{-(t-s)A_p} G_2(v)(s)ds + \int_0^t e^{-(t-s)A_p} G_3(v)(s)ds \tag{IV}$$

for any $0 \le t \le T$. Concerning X_p^{α}-estimates for nonlinear terms which appeared in (IV), the following lemmata on L_p-estimates for $G_1(v)$, $G_2(v)$ and $G_3(v)$ are required.

Lemma 9.8. *Let $1 < p < \infty$,*

$$\alpha_1 > 0,\ \alpha_2 > 0,\ 0 \le \delta < \frac{1}{2} + \frac{n}{2}\left(1 - \frac{1}{p}\right),\ \alpha_2 + \delta > \frac{1}{2},\ \alpha_1 + \alpha_2 + \delta \ge \frac{n}{2p} + \frac{1}{2}.$$

Then

$$\|A_p^{-\delta} G_1(v)\|_{(L_p(\Omega))^n} \le C_1 \|v\|_{X_p^{\alpha_1}} \|v\|_{X_p^{\alpha_2}} \tag{9.41}$$

for any $v \in X_p^{\max\{\alpha_1,\alpha_2\}}$, where C_1 is a positive constant depending only on n, Ω, p, μ, K, α_1, α_2 and δ.

Proof. See [9, Lemma 2.2]. □

Lemma 9.9. *Let $1 < p < \infty$,*

$$\alpha_1 > 0,\ 0 < \delta < \frac{1}{2} + \frac{n}{2}\left(1 - \frac{1}{p}\right),\ \alpha_1 + \delta \ge \frac{n}{2p}.$$

Then

$$\|A_p^{-\delta} G_2(v)\|_{(L_p(\Omega))^n} \le C_2 \|v\|_{X_p^{\alpha_1}} \|\bar{u}\|_{(W_p^1(\Omega))^n} \tag{9.42}$$

for any $v \in X_p^{\alpha_1}$, where C_2 is a positive constant depending only on n, Ω, p, μ, K, α_1 and δ.

Proof. The proof is similar to that of [9, Lemma 2.2]. Here the embedding theorem for Bessel-potential spaces, e.g., [27, Theorem 4.6.1] is required. □

Lemma 9.10. *Let $1 < p < \infty$,*

$$\alpha_2 > 0,\ 0 \le \delta < \frac{1}{2} + \frac{n}{2}\left(1 - \frac{1}{p}\right),\ \alpha_2 + \delta > \frac{1}{2}.$$

Then

$$\|A_p^{-\delta} G_3(v)\|_{(L_p(\Omega))^n} \le C_3 \|\bar{u}\|_{(W_p^1(\Omega))^n} \|v\|_{X_p^{\alpha_2}} \tag{9.43}$$

for any $v \in X_p^{\alpha_2}$, where C_3 is a positive constant depending only on n, Ω, p, μ, K, α_2 and δ.

Proof. The proof is similar to that of [9, Lemma 2.2]. Here the embedding theorem for Bessel-potential spaces, e.g., [27, Theorem 4.6.1] is required. □

The proof of Theorem 9.6 is essentially depending on the following choice of α_1 in Lemmata 9.8 and 9.9 and α_2 in Lemmata 9.8 and 9.10:

$$\frac{n}{2p} \le \alpha_1 + \delta < 1,\ \frac{1}{2} \le \alpha_2 < 1 - \delta,\ \frac{n}{2p} + \frac{1}{2} \le \alpha_1 + \alpha_2 + \delta \le 1 \tag{9.44}$$

for any $\delta > 0$. It is easy to see that Lemmata 9.8–9.10 are valid for α_1, α_2 and δ chosen as in (9.44). Let $0 \le \alpha < 1 - \delta$, $0 < \lambda_1 < \Lambda_1$, and set

$$\mathcal{G}_i(v)(t) = \int_0^t e^{-(t-s)A_p} G_i(v)(s)ds$$

for $i = 1, 2, 3$. Then it follows from Lemmata 9.5, 9.8–9.10 that

$$\|\mathcal{G}_1(v)(t)\|_{X_p^\alpha} \le C_{p,\alpha+\delta,\lambda_1} C_1 \int_0^t (t-s)^{-(\alpha+\delta)} e^{-\lambda_1(t-s)} \|v(s)\|_{X_p^{\alpha_1}(\Omega)} \|v(s)\|_{X_p^{\alpha_2}} ds, \tag{9.45}$$

$$\|\mathcal{G}_2(v)(t)\|_{X_p^\alpha} \le C_{p,\alpha+\delta,\lambda_1} C_2 \|\bar{u}\|_{(W_p^1(\Omega))^n} \int_0^t (t-s)^{-(\alpha+\delta)} e^{-\lambda_1(t-s)} \|v(s)\|_{X_p^{\alpha_1}} ds, \tag{9.46}$$

$$\|\mathcal{G}_3(v)(t)\|_{X_p^\alpha} \le C_{p,\alpha+\delta,\lambda_1} C_3 \|\bar{u}\|_{(W_p^1(\Omega))^n} \int_0^t (t-s)^{-(\alpha+\delta)} e^{-\lambda_1(t-s)} \|v(s)\|_{X_p^{\alpha_2}} ds \tag{9.47}$$

for any $0 \le t \le T$.

9.7 Asymptotic stability of solutions to (1.5)

This subsection is concerned with the asymptotic stability of solutions to (1.5). Since the proof of Theorem 9.6 is similar to that of [9, Theorem 2.6], we have only to obtain X_p^α-estimates for solutions to (1.6) globally in time. For any $0 < \lambda_1 < \Lambda_1$, $\lambda_1 < \lambda_2 < \Lambda_1$, set

$$E_\alpha(t) = \sup_{0<s\le t} s^\alpha e^{\lambda_1 s} \|v(s)\|_{X_p^\alpha}.$$

Then (9.39) is established by the following lemma:

Lemma 9.11. *Let Ω be a multiply-connected bounded domain with its boundary $\Gamma_0 \cup \Gamma_1$, $0 < K \le 1$, $n < p < \infty$, $u_0 \in L_{p,\sigma}(\Omega)$, $(\bar{u}, \bar{p}) \in (W_p^2(\Omega))^n \times \dot{W}_p^1(\Omega)$ be a solution to (1.5). Then there exist two positive constants δ_{p,λ_1} and $\varepsilon_{p,\lambda_1}$ depending only on n, Ω, p, μ, K and λ_1 for any $0 < \lambda_1 < \Lambda_1$ such that*

$$E_\alpha(t) \le C_{p,\alpha,\lambda_1} \|u_0 - \bar{u}\|_{(L_p(\Omega))^n} \tag{9.48}$$

for any $0 \le \alpha < 1$, $t > 0$, where C_{p,α,λ_1} is a positive constant depending only on n, Ω, p, μ, K, α and λ_1 provided that

$$\|u_0 - \bar{u}\|_{(L_p(\Omega))^n} \le \delta_{p,\lambda_1}, \quad \|\bar{u}\|_{(W_p^1(\Omega))^n} \le \varepsilon_{p,\lambda_1}.$$

Proof. It follows from (IV), (9.45)–(9.47) that

$$\begin{aligned} t^\alpha e^{\lambda_1 t} \|v(s)\|_{X_p^\alpha} \le & C_{p,\alpha,\lambda_1} \|u_0 - \bar{u}\|_{(L_p(\Omega))^n} \\ & + C_{p,\alpha+\delta,\lambda_2} C_1 t^\alpha E_{\alpha_1}(t) E_{\alpha_2}(t) \int_0^t (t-s)^{-(\alpha+\delta)} s^{-(\alpha_1+\alpha_2)} e^{-(\lambda_2-\lambda_1)(t-s)} e^{-\lambda_1 s} ds \\ & + C_{p,\alpha+\delta,\lambda_2} C_2 \|\bar{u}\|_{(W_p^1(\Omega))^n} t^\alpha E_{\alpha_1}(t) \int_0^t (t-s)^{-(\alpha+\delta)} s^{-\alpha_1} e^{-(\lambda_2-\lambda_1)(t-s)} ds \\ & + C_{p,\alpha+\delta,\lambda_2} C_3 \|\bar{u}\|_{(W_p^1(\Omega))^n} t^\alpha E_{\alpha_2}(t) \int_0^t (t-s)^{-(\alpha+\delta)} s^{-\alpha_2} e^{-(\lambda_2-\lambda_1)(t-s)} ds \end{aligned}$$

for any $0 \le \alpha < 1 - \delta$, $t > 0$. Recall that $t^{1-(\alpha_1+\alpha_2+\delta)}(1-\tau)^\delta e^{-(\lambda_2-\lambda_1)t(1-\tau)}$, $t^{1-(\alpha_1+\delta)}(1-\tau)^\delta e^{-(\lambda_2-\lambda_1)t(1-\tau)}$ and $t^{1-(\alpha_2+\delta)}(1-\tau)^\delta e^{-(\lambda_2-\lambda_1)t(1-\tau)}$ are bounded with respect to $t > 0$ and $0 \le \tau \le 1$. Then there exists a positive constant C_{δ,λ_1} such that

$$\begin{aligned} t^\alpha e^{\lambda_1 t} \|v(s)\|_{X_p^\alpha} \le & C_{p,\alpha,\lambda_1} \|u_0 - \bar{u}\|_{(L_p(\Omega))^n} \\ & + C_{p,\alpha+\delta,\lambda_2} C_1 C_{\delta,\lambda_1} E_{\alpha_1}(t) E_{\alpha_2}(t) \int_0^1 (1-\tau)^{-(\alpha+2\delta)} \tau^{-(\alpha_1+\alpha_2)} d\tau \\ & + C_{p,\alpha+\delta,\lambda_2} C_2 C_{\delta,\lambda_1} \|\bar{u}\|_{(W_p^1(\Omega))^n} E_{\alpha_1}(t) \int_0^1 (1-\tau)^{-(\alpha+2\delta)} \tau^{-\alpha_1} d\tau \\ & + C_{p,\alpha+\delta,\lambda_2} C_3 C_{\delta,\lambda_1} \|\bar{u}\|_{(W_p^1(\Omega))^n} E_{\alpha_2}(t) \int_0^1 (1-\tau)^{-(\alpha+2\delta)} \tau^{-\alpha_2} d\tau, \end{aligned}$$

$$E_\alpha(t) \le C_{\alpha,\delta,\lambda_1}\{\|u_0-\bar{u}\|_{(L_p(\Omega))^n} + E_{\alpha_1}(t)E_{\alpha_2}(t) + \|\bar{u}\|_{(W_p^1(\Omega))^n}(E_{\alpha_1}(t)+E_{\alpha_2}(t))\} \tag{9.49}$$

for any $0 \le \alpha < 1-2\delta$, $t>0$. By setting $E(t)=\max\{E_{\alpha_1}(t), E_{\alpha_2}(t)\}$, we have

$$E(t) \le C_{\lambda_1}(\|u_0-\bar{u}\|_{(L_p(\Omega))^n} + E^2(t) + \|\bar{u}\|_{(W_p^1(\Omega))^n}E(t)) \tag{9.50}$$

for any $t>0$. Assume that

$$\|\bar{u}\|_{(W_p^1(\Omega))^n} \le \frac{1}{2C_{\lambda_1}}. \tag{9.51}$$

Then (9.50) yields

$$E(t) \le 2C_{\lambda_1}(\|u_0-\bar{u}\|_{(L_p(\Omega))^n} + E^2(t)) \tag{9.52}$$

for any $t>0$. An elementary calculation shows that

$$E(t) \le \frac{8C_{\lambda_1}^2}{4C_{\lambda_1}-1}\|u_0-\bar{u}\|_{(L_p(\Omega))^n} \tag{9.53}$$

for any $t>0$ provided that

$$\|u_0-\bar{u}\|_{(L_p(\Omega))^n} \le \frac{1}{16C_{\lambda_1}^2}. \tag{9.54}$$

It follows from (9.49), (9.51), (9.53), (9.54) that

$$E_\alpha(t) \le C_{\alpha,\delta,\lambda_1}\|u_0-\bar{u}\|_{(L_p(\Omega))^n} \tag{9.55}$$

for any $0 \le \alpha < 1-2\delta$, $t>0$. Moreover, the arbitrariness of the choice of δ allows us to assume that $0 \le \alpha < 1$. This completes the proof of Lemma 9.11. □

Remark 9.3. In the case where Ω is a bounded domain in $\mathbb{R}^n$ with its $C^{1,1}$-boundary $\partial\Omega$ and $K=0$, the same results as in Theorems 9.5 and 9.6 can be proved by our argument with the aid of [7, Theorem 1.2 and Corollary 1.6].

Remark 9.4. According to the same argument as in [9, Section 3], a mild solution to (1.6) can be a strong solution under assumptions for Theorem 9.6.

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論文の内容の要旨

博士論文題目

Determining nodes for semilinear parabolic evolution equations in Banach spaces

(バナッハ空間上の半線型放物型発展方程式に対する確定節点)

氏名　柿澤 亮平

$\mathbb{R}^n$ $(n \in \mathbb{Z},\ n \geq 2)$ の有界領域 Ω 上の半線型熱伝導方程式, Navier-Stokes 方程式などに対する初期境界値問題について, Determining nodes と呼ばれる有限個の点からなる集合の存在を考察した. Determining nodes は時間大域的な解の漸近挙動の観測点からなる集合のことであり, もし存在すれば, Determining nodes E_N での解の漸近挙動というデータから領域 Ω での解の漸近挙動を一意に決定することができる. ただし, $E_N := \{x_1, \cdots, x_N\ ;\ x_i \in \overline{\Omega},\ i = 1, \cdots, N\}$. 本論文では, 導入部 (第 1 節) を経て次の三つのテーマ

第 I 部: Determining nodes の L_2 理論 (第 2 節, 第 3 節)

第 II 部: Determining nodes の L_p 理論 (第 4 節, 第 5 節)

第 III 部: 多重連結有界領域上の Stokes 方程式に対するレゾルベント問題 (第 6 節, 第 7 節, 第 8 節)

について論じ, それらの結果を半線型熱伝導方程式と Navier-Stokes 方程式に応用 (第 9 節) した.

第 I 部では, $n = 2$ または 3 とし, 次の Ω 上の半線型放物型方程式に対する初期境界値問題 (1.1) とその定常問題 (1.2) について, エネルギー法を用いて Determining nodes の存在を考察した. ただし, H は $L_2(\Omega)$ の閉部分空間, $V = H_0^1(\Omega) \cap H$ である.

$$\begin{cases} \dfrac{du}{dt} + Au = F(u) + f & \text{in } L_2((0,\infty); H), \\ u(0) = u_0 & \text{in } V, \end{cases} \tag{1.1}$$

$$A\bar{u} = F(\bar{u}) + \bar{f} \text{ in } H. \tag{1.2}$$

第 I 部の研究内容は, 半線型熱伝導方程式と Navier-Stokes 方程式を含む (1.1) の強解 u, v について, 次の定理 2.2, 定理 2.3 を証明することによって Hilbert 空間上の半線型放物型方程式に対する Determining nodes の L_2 理論を構築したことである.

定理 2.2. $n=2$ または 3, $R>0$, $f \in L_\infty((0,\infty);H)$, $t_0>0$ とし, (H.2)–(H.4) が成り立つことと, $f_\infty \in H$ が存在し,

$$f(t) \to f_\infty \ in\ H\ as\ t \to \infty$$

となることを仮定する．このとき, Ω, A, F, $M(f,t_0)$ と $M(f_\infty)$ のみに依存する正定数 δ_2 が存在し, $0<d_N \le \delta_2$ かつ, $u \in \mathcal{S}(V(R),f)$ とするとき, 任意の $i=1,\cdots,N$ に対して $\xi_i \in \mathbb{R}$ が存在し, u が

$$u(x_i,t) \to \xi_i\ as\ t \to \infty$$

となれば, (1.2) は任意の $0<\gamma<1/2$ に対して

$$u(t) \to u_\infty\ in\ V \cap C^{0,\gamma}(\overline{\Omega})\ as\ t \to \infty$$

かつ, 任意の $i=1,\cdots,N$ に対して $u_\infty(x_i)=\xi_i$ が成り立つような強解 $u_\infty \in \mathcal{S}(f_\infty)$ を一意に持つ．

定理 2.3. $n=2$ または 3, $R>0$, $f,g \in L_\infty((0,\infty);H)$, $t_0>0$ とし, (H.3), (H.4) が成り立つことと,

$$f(t) - g(t) \to 0\ in\ H\ as\ t \to \infty$$

となることを仮定する．このとき, Ω, A, F, $M(f,t_0)$ と $M(g,t_0)$ のみに依存する正定数 δ_3 が存在し, $0<d_N \le \delta_3$ かつ, $u \in \mathcal{S}(V(R),f)$, $v \in \mathcal{S}(V(R),g)$ とするとき, 任意の $i=1,\cdots,N$ に対して u,v が

$$u(x_i,t)-v(x_i,t) \to 0\ as\ t \to \infty$$

となれば, 任意の $0<\gamma<1/2$ に対して

$$u(t)-v(t) \to 0\ in\ V \cap C^{0,\gamma}(\overline{\Omega})\ as\ t \to \infty$$

となる．

初めて Determining nodes の存在を考察したのは Foias-Temam [6] である．彼らは Navier-Stokes 方程式に対する初期境界値問題の強解 u,v について, L_2 の node 補間不等式とエネルギー法を用いて定理 2.2, 定理 2.3 と同様の定理を証明した．反応拡散方程式系に対する初期境界値問題については, Lu-Shao [18] によって Foias-Temam と同様の結果が得られている．ところが, Foias-Temam と Lu-Shao だけでなく, $n=1$ における Foias-Kukavica [5], Kukavica [15], Oliver-Titi [19] の結果を見ても, 個々の方程式に対する Determining nodes の存在を扱っており, 理論の統一性に欠けることがこれまでの問題点であった．定理 2.2, 定理 2.3 を証明するために, まず, Hilbert 空間の直交分解 $L_2(\Omega)=H \oplus H^\perp$ と $L_2(\Omega)$ から H への直交射影 P を用いて (1.1) を定式化する．そして, L_2 の node 補間不等式とエネルギー法を用いて上の性質を持つ Determining nodes の存在を証明することが第 I 部の研究方法である．

第 II 部では, 次の X_p 上の半線型放物型発展方程式に対する初期値問題 (I) について, 解析半群の L_p 理論を用いて Determining nodes の存在を考察した．ただし, X_p は $L_p(\Omega)$ $(1<p<\infty)$ の閉部分空間である．

$$\begin{cases} \dfrac{du}{dt} + A_p u = F(u) + f \quad \text{in } (0,\infty), \\ u(0)=u_0. \end{cases} \tag{I}$$

第 II 部の研究内容は, 半線型熱伝導方程式と Navier-Stokes 方程式を含む (I) のマイルド解 u,v について, 次の定理 4.1, 定理 4.2 を証明することによって Banach 空間上の半線型放物型発展方程式に対する Determining nodes の L_p 理論を構築したことである．ただし, $X_p^\alpha = D(A_p^\alpha)$ $(\alpha \ge 0)$ である．

定理 4.1. $n/2 < p < \infty$, $0 \leq \alpha_0 < 1$, $R > 0$, $f, g \in C((0,\infty); X_p)$, $t_0 > 0$ とし, (H.5), (H.6) が成り立つことと,

$$f(t) - g(t) \to 0 \ in \ X_p \ as \ t \to \infty$$

となることを仮定する. このとき, n, Ω, p, A_p, F, α_0, $M(f, t_0)$ と $M(g, t_0)$ のみに依存する正定数 δ_1 が存在し, $0 < d_N \leq \delta_1$ かつ, $u \in \mathcal{S}(X_p^{\alpha_0}(R), f)$, $v \in \mathcal{S}(X_p^{\alpha_0}(R), g)$ とするとき, 任意の $i = 1, \cdots, N$ に対して u, v が

$$u(x_i, t) - v(x_i, t) \to 0 \ as \ t \to \infty$$

となれば,

(i) 任意の $\alpha_0 < \alpha < 1$ に対して

$$\|u(t) - v(t)\|_{X_p^\alpha} = O(t^{\alpha_0 - \alpha}) \ as \ t \to \infty$$

である.

(ii) $n/(2p) < \alpha < 1$ ならば, 任意の $k \in \mathbb{Z}$, $k \geq 0$, $0 < \gamma < 1$, $k + \gamma \leq 2\alpha - n/p$ に対して

$$\|u(t) - v(t)\|_{C^{k,\gamma}(\overline{\Omega})} = O(t^{\alpha_0 - \alpha}) \ as \ t \to \infty$$

である.

定理 4.2. $n/2 < p < \infty$, $0 \leq \alpha_0 < 1$, $R > 0$, $f, g \in C((0,\infty); X_p)$, $t_0 > 0$ とし, (H.5), (H.6) が成り立つことと, ある $0 < \lambda_1 < \Lambda_1$ に対して

$$\|f(t) - g(t)\|_{L_p(\Omega)} = O(e^{-\lambda_1 t}) \ as \ t \to \infty$$

であることを仮定する. このとき, n, Ω, p, A_p, F, α_0, $M(f, t_0)$ と $M(g, t_0)$ のみに依存する正定数 δ_2 が存在し, $0 < d_N \leq \delta_2$ かつ, $u \in \mathcal{S}(X_p^{\alpha_0}(R), f)$, $v \in \mathcal{S}(X_p^{\alpha_0}(R), g)$ とするとき, 任意の $i = 1, \cdots, N$ に対して u, v が

$$u(x_i, t) - v(x_i, t) = O(e^{-\lambda_1 t}) \ as \ t \to \infty$$

ならば,

(i) 任意の $\alpha_0 \leq \alpha < 1$ に対して

$$\|u(t) - v(t)\|_{X_p^\alpha} = O(t^{\alpha_0 - \alpha} e^{-\lambda_1 t}) \ as \ t \to \infty$$

である.

(ii) $n/(2p) < \alpha < 1$ ならば, 任意の $k \in \mathbb{Z}$, $k \geq 0$, $0 < \gamma < 1$, $k + \gamma \leq 2\alpha - n/p$ に対して

$$\|u(t) - v(t)\|_{C^{k,\gamma}(\overline{\Omega})} = O(t^{\alpha_0 - \alpha} e^{-\lambda_1 t}) \ as \ t \to \infty$$

である.

これまで Kakizawa [14] の結果より, 基礎的な Determining nodes の L_2 理論は完成したように思われたが, 数値解析や工学・産業への応用の観点から, いくつかの問題点が浮き彫りになった.

- (境界条件の多様性) Dirichlet 境界条件や周期境界条件を除き, 例えば, Navier-Stokes 方程式に重要な役割を果たす Navier 境界条件などの物理学的に重要な境界条件が考察されていない.

- (漸近挙動の収束率) たとえ Determining nodes が存在したとしても, 時間大域的な解同士の漸近挙動がどのような収束率で一致するかどうかは不明である.

定理 4.1, 定理 4.2 を証明するために, まず, X_p^α の補間不等式を用いて L_p の node 補間不等式 (補題 4.6) を証明する. これより, $n/2 < p < \infty$ とし, 補題 4.6 と Giga-Miyakawa [9] と類似の方法を用いて $t^{\alpha-\alpha_0}\|u(t)-v(t)\|_{X_p^\alpha}$ が t に関して一様有界となるような Determining nodes の存在を証明することが第 II 部の研究方法である. 境界条件の多様性については, 適当な境界条件を伴った線型化作用素が X_p でセクトリアルかどうかを確認すれば十分となった. 漸近挙動の収束率についても, $t^{\alpha-\alpha_0}\|u(t)-v(t)\|_{X_p^\alpha}$ の t に関する一様有界性を用いて明らかにできた. このように, 本研究は解析半群の L_p 理論を用いて初めて Determining nodes の L_p 理論を構築したものであり, 本論文の中で最も重要かつ主要な結果である.

第 III 部では, Ω を外側の境界 Γ_0 と内側の境界 Γ_1 で囲まれた $\mathbb{R}^n$ ($n \in \mathbb{Z},\ n \geq 2$) の有界領域とし, 次の Ω 上の Stokes 方程式に対するレゾルベント問題 (1.4) について, L_p 評価を満たす解の存在と一意性を考察した.

$$\begin{cases} \mathrm{div} u = f & \text{in } \Omega, \\ \lambda u - T(u,p) = g & \text{in } \Omega, \\ u_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\ K(T(u,p)\nu)_\tau + (1-K)u_\tau|_{\Gamma_0} = h^0 & \text{on } \Gamma_0, \\ u_\tau|_{\Gamma_1} = h^1 & \text{on } \Gamma_1. \end{cases} \tag{1.4}$$

第 III 部の研究内容は, 次の定理 6.1, 定理 6.2 を証明することによって,

$$A_p u = -P_p \mathrm{div} T(u,p),$$

$$D(A_p) = \{u \in (W_p^2(\Omega))^n \cap L_{p,\sigma}(\Omega)\ ;\ K(T(u,p)\nu)_\tau + (1-K)u_\tau|_{\Gamma_0} = 0,\ u_\tau|_{\Gamma_1} = 0\}$$

によって定義される Stokes 作用素 A_p を任意の次元に対して Determining nodes の L_p 理論に応用したことである.

定理 6.1. $0 < K \leq 1,\ 1 < p < \infty,\ 0 < \varepsilon < \pi/2,\ f \in \dot{W}_p^1(\Omega),\ g \in (L_p(\Omega))^n,\ h^0 \in (W_p^{1-1/p}(\Gamma_0))^n,\ h_\nu^0|_{\Gamma_0} = 0,\ h^1 \in (W_p^{2-1/p}(\Gamma_1))^n,\ h_\nu^1|_{\Gamma_1} = 0$ とする. このとき, (1.4) は任意の $\lambda \in S_\varepsilon \cup \{0\}$ に対して

$$I_{p,\lambda,\Omega}^2(u) + \|p\|_{W_p^1(\Omega)} \leq C_{p,\varepsilon}(I_{p,\lambda,\Omega}^{1,-1}(f) + \|g\|_{(L_p(\Omega))^n} + I_{p,\lambda,\Omega}^1(h^0) + I_{p,\lambda,\Omega}^2(h^1))$$

が成り立つような解 $(u,p) \in (W_p^2(\Omega))^n \times \dot{W}_p^1(\Omega)$ を一意に持つ. ただし, $C_{p,\varepsilon}$ は $n,\ \Omega,\ p,\ \varepsilon,\ \mu$ と K のみに依存する正定数である.

定理 6.2. $0 < K \leq 1,\ 1 < p < \infty,\ 0 < \varepsilon < \pi/2$ とする. このとき, $\rho(-A_p) \supset S_\varepsilon \cup \{0\}$ であり, 任意の $\lambda \in S_\varepsilon \cup \{0\}$ に対して

$$\|(\lambda I_p + A_p)^{-1}\|_{\mathcal{B}(L_{p,\sigma}(\Omega))} \leq \frac{C_{p,\varepsilon}}{|\lambda|+1} \tag{0.1}$$

が成り立つ. ただし, $C_{p,\varepsilon}$ は $n,\ \Omega,\ p,\ \varepsilon,\ \mu$ と K のみに依存する正定数である. したがって, A_p は $L_{p,\sigma}(\Omega)$ でセクトリアルであり, 任意の $0 < \lambda_1 < \Lambda_1$ に対して $n,\ \Omega,\ p,\ \varepsilon,\ \mu,\ K$ と λ_1 のみに依存する正定数 $C_{p,\varepsilon,\lambda_1}$ が存在し, 任意の $t \geq 0$ に対して

$$\|e^{-tA_p}\|_{\mathcal{B}(L_{p,\sigma}(\Omega))} \leq C_{p,\varepsilon,\lambda_1} e^{-\lambda_1 t}$$

が成り立つ. ただし, $\Lambda_1 := \min\{\lambda_1 > 0\ ;\ \lambda_1 \in \mathrm{Re}\sigma(A_p)\}$.

非有界領域 (全空間 $\mathbb{R}^n$, 半空間 $\mathbb{R}^n_+$, 曲がった半空間 H^n_ω) 上の Stokes 方程式に対するレゾルベント問題について, Dirichlet 境界条件においては Farwig-Sohr [7] によって, Navier 境界条件においては Shibata-Shimada [24] によって L_p 評価を満たす解の存在と一意性が得られている. これより, (1.4) については, Solonnikov-Ščadilov [25] と類似の方法を用いて (1.4) の広義解の存在と一意性を証明すれば十分と思われた. ところが, 次の発散問題 (6.3) について, 彼らは $p=2$ とし, $n=3$ における Helmholtz の定理を用いて次の補題 6.1 を証明しており, 次元の任意性を失うことがこれまでの問題点であった.

$$
\begin{cases}
\mathrm{div} u = f & \text{in } \Omega, \\
u_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega, \\
u_\tau|_{\Gamma_1} = h & \text{on } \Gamma_1.
\end{cases}
\tag{6.3}
$$

補題 6.1. $1<p<\infty$, $f \in \dot{L}_p(\Omega)$, $h \in (W_p^{1-1/p}(\Gamma_1))^n$, $h_\nu|_{\Gamma_1}=0$ とする. このとき, (6.3) は

$$
\|u\|_{(W_p^1(\Omega))^n} \le C_p(\|f\|_{L_p(\Omega)} + \|h\|_{(W_p^{1-1/p}(\Gamma_1))^n}),
$$

を満たす解 $u \in (W_p^1(\Omega))^n$ を持つ. ただし, C_p は n, Ω と p のみに依存する正定数である.

定理 6.1, 定理 6.2 を証明するために, まず, Bogovskiĭ [2] の結果を用いて Solonnikov-Ščadilov の方法を改良し, 任意の次元に対して (1.4) の広義解の存在と一意性を証明する. そして, 局所化法を用いて (1.4) を $\mathbb{R}^n$ と H^n_ω 上の Stokes 方程式に対するレゾルベント問題に帰着させ, Farwig-Sohr と Shibata-Shimada の結果を用いて L_p 評価を満たす (1.4) の解の存在と一意性を得ることが第 III 部の研究方法である. Determining nodes の L_p 理論への応用に関しては, 定理 6.2 より第 II 部の研究結果が境界条件の多様性に耐えうる理論であると裏づけることができた.

第 9 節では, Determining nodes の理論を半線型熱伝導方程式と Navier-Stokes 方程式に応用した. 他にも Determining nodes の理論と関連した研究課題として, 次の Ω 上の Navier-Stokes 方程式に対する初期境界値問題 (1.3) について, 定常解の漸近的性質を考察した.

$$
\begin{cases}
\mathrm{div} u = 0 & \text{in } \Omega \times (0,T), \\
\rho\{\partial_t + (u\cdot\nabla)\}u - \mathrm{div}T(u,p) = \rho g & \text{in } \Omega \times (0,T), \\
u|_{t=0} = u_0 & \text{in } \Omega, \\
u_\nu|_{\partial\Omega} = 0 & \text{on } \partial\Omega \times (0,T), \\
K(T(u,p)\nu)_\tau + (1-K)u_\tau|_{\Gamma_0} = 0 & \text{on } \Gamma_0 \times (0,T), \\
u_\tau|_{\Gamma_1} = h & \text{on } \Gamma_1 \times (0,T).
\end{cases}
\tag{1.3}
$$

第 9 節の研究内容は, 任意の $n<p<\infty$ に対して (1.3) の小さな定常解 $(\bar{u},\bar{p}) \in (W_p^2(\Omega))^n \times \dot{W}_p^1(\Omega)$ が一意に存在すること (定理 9.5) と, $(\bar{u},\bar{p})$ が $L_{p,\sigma}(\Omega)$ で漸近安定, つまり $u_0-\bar{u}$ を $L_{p,\sigma}(\Omega)$ で十分小さくするとき, 任意の $0 \le \alpha < 1$ に対して

$$
\|u(t)-\bar{u}\|_{X_p^\alpha} \le C_{p,\alpha,\lambda_1} t^{-\alpha} e^{-\lambda_1 t} \|u_0-\bar{u}\|_{(L_p(\Omega))^n}
$$

が成り立つこと (定理 9.6) を証明したことである. (1.3) と類似の問題の定常解の存在, 一意性と漸近安定性については, Itoh-Tanaka-Tani [13] によって L_2-Sobolev-Slobodetskiĭ空間という複雑な関数空間を用いた結果が得られている. 定理 9.5, 定理 9.6 を証明するために, まず, 定理 6.1 と Banach の不動点定理を用いて定理 9.5 を証明する. そして, 定理 6.2 と Giga-Miyakawa と類似の方法を用いて定理 9.6 を証明することが第 9 節の研究方法である. これより, L_p-Sobolev 空間という簡単な関数空間を用いて (1.3) の定常解の存在, 一意性と漸近安定性を得ることができた.