

## 論文の内容の要旨

# On the exact WKB analysis of Schrödinger equations

(Schrödinger 方程式の完全 WKB 解析に関して)

神本晋吾

本稿では, 大きなパラメータ  $\eta$  を含み, 解析的なポテンシャル  $Q(x)$  を持つ次の 1 次元定常型 Schrödinger 方程式を考える:

$$(1) \quad \left( \frac{d^2}{dx^2} - \eta^2 Q(x) \right) \psi(x, \eta) = 0.$$

### Part I

On the Borel summability of WKB-theoretic transformation series

(1) の完全 WKB 解析 (Borel 総和法を用いた解析) は A. Voros により始められ, [V] では (1) の単純変わり点から出る Stokes 曲線上での WKB 解  $\psi(x, \eta)$  の接続公式が与えられた. そこでは WKB 解の Borel 和の変わり点での一価性の議論を用いて接続係数が求められた.

この接続公式を別の視点から捉えるべく, 青木-河合-竹井により (1) の次の Airy 方程式への WKB 解析的な変換が導入された ([AKT1]):

$$(2) \quad \left( \frac{d^2}{dx^2} - \eta^2 x \right) \phi(x, \eta) = 0.$$

ここで、WKB 解析的変換とは、 $\eta$  に関する形式冪級数 (WKB 解析的変換級数と呼ぶ) による形式的座標変換、及び gauge 変換のことである。これは、(1) の WKB 解の Borel 変換像  $\psi_B(x, y)$  の持つ「動く特異点」と呼ばれる特異点の解析を目的としたもので、この特異点での  $\psi_B$  の特異性の記述を通して、Voros の接続公式は説明された。(実際、(2) の WKB 解  $\phi(x, \eta)$  の Borel 変換像  $\phi_B(x, y)$  は具体的に超幾何関数を用いて表示することが可能であり、 $\phi_B$  の動く特異点での特異性の記述から、WKB 解析的変換を通して、 $\psi_B$  の動く特異点での特異性の記述が行われていた。) ここで次に注意する: [AKT1] では変わり点の近傍、及び Borel 平面の「基準の特異点」の近傍での、 $\psi_B$  の特異性の局所的な記述は与えられていた。しかし、 $\psi$  の Borel 和、及び Stokes 現象を扱うためには、変わり点から出る Stokes 曲線の近傍、及び基準の特異点から Borel 平面の実軸に平行に延びる直線の近傍での  $\psi_B$  の特異性の記述が必要になる。Part I では、注目している単純変わり点から出る Stokes 曲線が、全て  $Q(x)$  の二位以上の位数の極に流れ込むという Stokes 幾何に関する仮定の下、WKB 解析的変換級数の Borel 総和可能性を示すことにより、上記の場所での  $\psi_B$  の特異性の記述を与えた。この結果から、[AKT1] の立場からの Voros の接続公式の完全な証明が得られたことになる。また、 $Q(x)$  の単純極も変わり点と同様の役割を果たすことが、小池により WKB 解析的変換を用いた方法により明らかにされているが ([Ko1], [Ko2])、そこで用いられた WKB 解析的変換級数に対しても、同様の仮定の下、Borel 総和可能性を示した。

## Part II

### On the WKB theoretic structure of a Schrödinger operator with a merging pair of a simple pole and a simple turning point

Part I では一つの単純変わり点 (あるいは一つの単純極) が考察の対象であった。しかし、このように一つの変わり点に注目しているだけでは捉えることができない、「動かない特異点」と呼ばれる  $\psi_B$  の特異点が知られている。この動かない特異点の解析に関して、青木-河合-竹井による、二つの合流する単純変わり点を持つ Schrödinger 方程式 (MTP 方程式) の完全 WKB 解析が知れている ([AKT2])。簡単のため、まず次の Weber 方程式を考えてみる:

$$(3) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( E - \frac{x^2}{4} \right) \right) \hat{\phi}(x, E, \eta) = 0.$$

ここでは  $E$  が合流パラメータの役割を果たしている。この場合、動かない特異点は基準の特異点から  $2\pi E$  の周期で現れることが知られている。(3)

は  $x = \pm 2\sqrt{E}$  に単純変わり点を持っているが、この動かない特異点は、二つの単純変わり点が Stokes 曲線で結ばれるという、Stokes 幾何の退化と関係しており、動かない特異点での特異性の記述、特に alien derivative の計算は、合流パラメータ  $E$  に関する Stokes 現象を記述する際に重要となる。(3) に関しては、動かない特異点の解析に有効な「Voros 係数」と呼ばれる級数の Bernoulli 数を用いた明示的な表示が得られており ([SS], [T]), これにより動かない特異点の具体的な解析が可能である。[AKT2] では MTP 方程式の変わり点の合流点の近傍での (3) への WKB 解析的変換を通して、合流する二つの単純変わり点に起因する動かない特異点の解析が行われた。

上述したように、単純極も変わり点と同様の振る舞いをする事が知られている。すると、単純変わり点と単純極の組、あるいは、二つの単純極の組に関しても、これらに起因する動かない特異点が現れると期待される。Part II では、単純変わり点と単純極の組に起因する動かない特異点の解析を行うべく、合流する単純変わり点と単純極の組を持つ Schrödinger 方程式 (MPPT 方程式) の完全 WKB 解析を行った。ここでは [AKT2] と同様に、合流点の近傍での Whittaker 方程式への WKB 解析的変換を通して、合流する単純変わり点と単純極の組に起因する動かない特異点の解析を行った。

### Part III

**Exact WKB analysis of a Schrödinger equation with a merging triplet of two simple poles and one simple turning point — its relevance to the Mathieu equation and the Legendre equation**

[AKT2], Part II での結果を踏まえて、次の自然な問いが生じる: 二つの単純極の組に起因する動かない特異点の解析も、同様に合流操作を用いて解析を行えないであろうか? しかし、この場合には次の困難が生じる: 例えば、ポテンシャルとして次のものを考えてみる:

$$(4) \quad Q_1(x, a, \Gamma) = \frac{\Gamma}{x^2 - a^2}.$$

ただし、 $a$  は合流パラメータとする。  $Q_1(x, a, \Gamma)$  は  $x = \pm a$  に  $a = 0$  で合流する単純極を持っているが、これらに起因する動かない特異点の現れる周期は  $a$  に依存せず  $2\pi i\sqrt{\Gamma}$  となる。(実際、この周期は二つの単純極の周りを回る  $Q_1(x, a, \Gamma)$  の周回積分により与えられる。) 一方、MTP 方程式、MPPT 方程式では、解析の対象であった動かない特異点の周期は合流に

伴い 0 となり, この性質は動かない特異点の解析を行う上で重要であった.  
では, 次の形の単純極の合流ではどうなるであろうか:

$$(5) \quad Q_2(x, a) = \frac{p_+}{x-a} + \frac{p_-}{x+a} = \frac{(p_+ + p_-)x + (p_+ - p_-)a}{x^2 - a^2}.$$

すると,  $p_+ - p_- \neq 0$  の場合, 二つの単純極に加えて新たに  $a = 0$  で合流する単純変わり点が現れてしまう. しかし,  $Q_2(x, a)$  の二つの単純極に起因する動かない特異点の現れる周期は  $O(\sqrt{|a|})$  となり,  $Q_1(x, a, \Gamma)$  の動かない特異点よりは扱い易い. これらを踏まえて, Part III では合流する二つの単純極と一つの単純変わり点を持つ Schrödinger 方程式 (M2P1T 方程式) の完全 WKB 解析を行った. ここでは, まず合流点の近傍において, 代数的 Mathieu 方程式

$$(6) \quad \left( \frac{d^2}{dx^2} - \eta^2 \frac{aA + Bx}{x^2 - a^2} \right) \tilde{\phi}(x, a, A, B, \eta) = 0$$

( $A, B \neq 0$  は方程式のパラメータ) への WKB 解析的変換の構成を行った.  
では, (6) の二つの単純極に起因する動かない特異点の解析は可能であろうか? 当然ながら, 単純変わり点と単純極に起因する動かない特異点も混在しているため, これらを取り除く必要が生じる. Part III では, 更に単純変わり点の合流の速度をパラメータ  $B$  により調節し, 単純変わり点の影響を取り除くことにより, 二つの単純極の近傍における  $Q_1(x, a, a\Gamma)$  をポテンシャルに持つ Schrödinger 方程式 (Legendre 方程式) への WKB 解析的変換を通して, 二つの単純極に起因する動かない特異点の解析を行った.

## 謝辞

学部生の頃からご指導下さった片岡清臣先生, 京都での充実した研究環境を提供して下さい下された河合隆裕先生, 竹井義次先生, 研究を行う上での貴重な助言を下された青木貴史先生, 小池達也先生, 共に切磋琢磨した岩木耕平氏, 佐々木真二氏, 廣瀬三平氏に心からの感謝を捧げたい.

## 参考文献

- [AKT1] T. Aoki, T. Kawai and T. Takei: The Bender-Wu analysis and the Voros theory, Special Functions, Springer-Verlag, 1991, pp.1-29.



- [AKT2] ———: The Bender-Wu analysis and the Voros theory. II, *Advanced Studies in Pure Mathematics*, **54**, Math. Soc. Japan, 2009, pp.19–94.
- [Ko1] T. Koike: On a regular singular point in the exact WKB analysis, *Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear*, Kyoto Univ. Press, 2000, pp.39–54.
- [Ko2] ———: On the exact WKB analysis of second order linear ordinary differential equations with simple poles, *Publ. RIMS, Kyoto Univ.*, **36** (2000), 297–319.
- [SS] H. Shen and H. J. Silverstone: Observations on the JWKB treatment of the quadratic barrier, *Algebraic Analysis of Differential Equations*, Springer, 2008, pp.307–319.
- [T] Y. Takei: Sato’s conjecture for the Weber equation and transformation theory for Schrödinger equations with a merging pair of turning points, *RIMS Kôkyûroku Bessatsu*, **B10** (2008), 205–224.
- [V] A. Voros: The return of the quartic oscillator — The complex WKB method. *Ann. Inst. Henri Poincaré*, **39** (1983), 211–338.

# On the exact WKB analysis of Schrödinger equations

(Schrödinger 方程式の完全 WKB 解析に関して)

神本 晋吾

The principal aim of this thesis is to study the analytic structure of Borel transformed WKB solutions of a Schrödinger equation

$$(1) \quad \left( \frac{d^2}{dx^2} - \eta^2 Q(x) \right) \psi = 0 \quad (\eta : \text{a large parameter}).$$

This thesis consists of three parts.

## **Part I**

### **On the Borel summability of WKB-theoretic transformation series**

In Part I, we develop the WKB-theoretic transformation theory, which is first introduced by Aoki-Kawai-Takei in [AKT1] to analyze the singularity structure of Borel transformed WKB solutions of (1) near a simple turning point. In the transformation theory, WKB-theoretic transformation series play a central role. The main purpose of Part I is to show the Borel summability of the series on the Stokes curves that emanate from a simple turning point in question. As an application, we give a complete description of the singularity structure of Borel transformed WKB solutions near a straight line from the reference singular point which is parallel to the real axis of Borel plane. We also show the Borel summability of transformation series that are employed in [Ko1] and [Ko2] to study the WKB-theoretic structure of simple poles of  $Q(x)$ . These results enhance the effectiveness of the transformation theory in the study of global aspects of Borel transformed WKB solutions.

## **Part II**

### **On the WKB theoretic structure of a Schrödinger operator with a merging pair of a simple pole and a simple turning point**

In Part II, we study the WKB-theoretic structure of a Schrödinger equation with a merging pair of a simple pole and a simple turning point (called an MPPT equation for short) in a way parallel to the case of merging-turning-points (MTP) equations studied in [AKT4]. The main purpose of our study is to analyze so-called “fixed singular points” relevant to the pair of a simple pole and a simple turning point. We first construct a WKB-theoretic transformation that brings an MPPT equation to its canonical form, that is, the  $\infty$ -Whittaker

equation in this case. Then, through the transformation, we analyze the singularity structure of Borel transformed WKB solutions of an MPPT equation. In particular, we give the explicit description of alien derivatives at these fixed singular points using the concrete form of the Voros coefficient for the Whittaker equation given in [KoT].

### **Part III**

#### **Exact WKB analysis of a Schrödinger equation with a merging triplet of two simple poles and one simple turning point — its relevance to the Mathieu equation and the Legendre equation**

In Part III, we develop the exact WKB analysis of an M2P1T (merging two simple poles and one simple turning point) Schrödinger equation. Our emphasis is put on the analysis of the singularity structure of its Borel transformed WKB solutions near fixed singular points relevant to the two simple poles contained in the potential of the equation. We first show that the WKB-theoretic canonical form of an M2P1T equation is given by an algebraic Mathieu equation. Then we calculate the alien derivative of its Borel transformed WKB solutions at each fixed singular point relevant to the simple poles through the analysis of Borel transformed WKB solutions of the Legendre equations. In the course of the calculation of the alien derivative, we make full use of microdifferential operators whose symbols are given by the infinite series that appear in the coefficients of the algebraic Mathieu equation and the Legendre equation.

Part I

**On the Borel summability of  
WKB-theoretic transformation  
series**

## 0 Introduction

From the early days of its development, the turning point problem is one of the central issues in WKB theory. Since the approximation by WKB wave functions breaks down near a turning point, Kramers, in his pioneering work [Kr], replaced the potential by a linear variation at a simple zero of the potential, and he connected WKB wave function across a turning point by matching WKB wave function to Airy function. The matching methods of this kind have been widely used in WKB approximation theory (cf. [BW], [F], [W]).

From a viewpoint of exact WKB analysis, i.e., a WKB theory based on the Borel resummation method initiated by Voros ([V]), Aoki, Kawai and Takei interpreted the matching method as a transformation theory near a simple turning point ([AKT1]); they constructed a transformation series

$$(0.1) \quad x(\tilde{x}, \eta) = x_0(\tilde{x}) + \eta^{-1}x_1(\tilde{x}) + \eta^{-2}x_2(\tilde{x}) + \cdots$$

from the stationary Schrödinger equation

$$(0.2) \quad \left( \frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x}) \right) \tilde{\psi} = 0$$

with analytic potential  $Q$  and a complex large parameter  $\eta$  to the Airy equation

$$(0.3) \quad \left( \frac{d^2}{dx^2} - \eta^2 x \right) \psi = 0$$

near a simple turning point of (0.2). By using the transformation series (0.1) WKB solutions of (0.2) can be expressed by those of (0.3) as

$$(0.4) \quad \tilde{\psi}(\tilde{x}, \eta) = \left( \frac{\partial x}{\partial \tilde{x}}(\tilde{x}, \eta) \right)^{-1/2} \psi(x(\tilde{x}, \eta), \eta).$$

Although this relation (0.4) is obtained as a formal relation, it becomes an analytic one after Borel transformation, and they argued that the Voros' connection formula of Borel summed WKB solutions near a simple turning point follows from that of Gauss' hypergeometric functions.

Since Aoki, Kawai and Takei discussed in a general situation, the transformation series  $x(\tilde{x}, \eta)$  was only obtained near a turning point,

and the Borel transform of (0.4) (more precisely an integral representation (1.32) of the Borel transform of  $\psi$ ) holds only near the reference point of the Borel sum. In this article, by assuming that  $Q$  is a rational function and also making some generic assumptions concerning on the Stokes geometry, we will show that the transformation series  $x(\tilde{x}, \eta)$  is Borel summable near a simple turning point and along Stokes curves emanating from it (Theorem 2.1).

From our results it follows that the relation (0.4) itself is now an exact one if we consider  $\psi$ ,  $\tilde{\psi}$  and  $x(\tilde{x}, \eta)$  as their Borel sums in appropriate domain. Our result also completes the proof of Voros' connection formula near a simple turning point in the framework of transformation theory since the Borel transform of (0.4) holds near Stokes curves and near the path of integration to define Borel sum.

Our argument in this article is not specific to a simple turning point as the transformation theory is not. To demonstrate it, we also discussed a connection problem near a simple pole of (0.2) through the transformation ([Ko1, Ko2]). (This is also the case for the studies of so-called "fixed singularities". See [AKT4], [KKKoT1], [KKKoT2], [KKT1] and [KKT2] for details.) In this case (0.2) is transformed to

$$(0.5) \quad \left( \frac{d^2}{dx^2} - \eta^2 \frac{1}{x} \right) \psi = 0,$$

and we will show in Section 3 that the transformation series of this case is also Borel summable.

#### Acknowledgment.

The authors wish to thank Professor T. Aoki, Professor T. Kawai, Professor R. Schäfke and Professor Y. Takei for the valuable discussions with them and their suggestions.

## 1 WKB theoretic transformation — a simple turning point case

The main purpose of this section is to verify the Borel summability of transformation series of (1.1) to the WKB theoretic canonical equation (1.16) near Stokes curves emanating from a simple turning point. To make our discussion simple, we assume that all of Stokes curves emanating from a simple turning point in question run into some irregular

singular points in our discussion. (See Remark 2.5 and Remark 2.6 in the case that Stokes curves run into a double pole of  $Q$ .)

In Section 2.1 we state our main theorem (Theorem 2.1), and review fundamental properties of WKB theoretic transformation to explain the results obtained from Theorem 2.1. In Section 2.2 we show the uniform Borel transformability of transformation series constructed near a simple turning point in question. Finally, in Section 2.3 we prove the Borel summability of the transformation series using its uniform Borel transformability obtained in Section 2.2.

### 1.1 Fundamental properties of WKB theoretic transformation and its application

We consider the following Schrödinger equation

$$(1.1) \quad \left( \frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x}) \right) \tilde{\psi}(\tilde{x}, \eta) = 0$$

with a rational potential  $Q(\tilde{x})$  that has a simple turning point at  $\tilde{x} = 0$ , i.e.,  $Q(\tilde{x})$  is holomorphic at  $\tilde{x} = 0$  and satisfies

$$(1.2) \quad Q(0) = 0, \quad \frac{dQ}{d\tilde{x}}(0) \neq 0.$$

Further we assume the following geometric conditions (1.3) and (1.6): the first assumption is that

$$(1.3) \quad \text{three Stokes curves } \{T_j\}_{j=1}^3 \text{ emanating from } \tilde{x} = 0 \text{ run into} \\ \text{irregular singular points } \{b_j\}_{j=1}^3 \text{ respectively.}$$

(We do not assume that  $b_j$ 's are mutually distinct.) Here Stokes curves are integral curves of  $\text{Im}\sqrt{Q(\tilde{x})}d\tilde{x} = 0$  emanating from  $\tilde{x} = 0$ , that is, a curve defined by

$$(1.4) \quad \text{Im} \int_0^{\tilde{x}} \sqrt{Q(\tilde{x})} d\tilde{x} = 0.$$

To give a second assumption we prepare some notations. Let  $U^{\tilde{\varepsilon}} = \{\tilde{x} \in \mathbb{C} : |\tilde{x}| < \tilde{\varepsilon}\}$ . By taking  $\tilde{\varepsilon} > 0$  sufficiently small, we may assume that  $U^{\tilde{\varepsilon}} \setminus \{T_j\}_{j=1}^3$  is decomposed into three connected components,



which we denote them by  $\{U_j^\varepsilon\}_{j=1}^3$ . Let  $\widehat{U}_{j,\pm}^\varepsilon$  be connected components of

$$(1.5) \quad \bigcup_{\tilde{x}_0 \in U_j^\varepsilon} \left\{ x \in \mathbb{C} : \operatorname{Im} \int_{\tilde{x}_0}^{\tilde{x}} \sqrt{Q(\tilde{x})} d\tilde{x} = 0, \pm \operatorname{Re} \int_{\tilde{x}_0}^{\tilde{x}} \sqrt{Q(\tilde{x})} d\tilde{x} \geq 0 \right\}$$

which contains  $\{U_j^\varepsilon\}_{j=1}^3$ . For  $j_1, j_2 \in \{1, 2, 3\}$ ,  $j_1 \neq j_2$ , we can take a Stokes curve  $T_j$  so that  $\overline{U_{j_1}^\varepsilon} \cap \overline{U_{j_2}^\varepsilon} \subset T_j$ . Then our second assumption is that there exists a sufficiently small positive constant  $\tilde{\varepsilon}$  such that, for any pair of  $j_1, j_2 \in \{1, 2, 3\}$ ,

$$(1.6) \quad \widehat{U}_{j_1,\pm}^{\tilde{\varepsilon}} \text{ and } \widehat{U}_{j_2,\pm}^{\tilde{\varepsilon}} \text{ run into } b_j$$

when

$$(1.7) \quad \pm \operatorname{Re} \int_0^{\tilde{x}} \sqrt{Q(\tilde{x})} d\tilde{x} \geq 0$$

holds for  $\tilde{x} \in T_j$ . Let  $\widehat{U}^{\tilde{\varepsilon}}$  be a union of integral curves that through  $U^\varepsilon$ , i.e.,

$$(1.8) \quad \widehat{U}^{\tilde{\varepsilon}} = \bigcup_{j=1}^3 \left\{ \bigcup_{*= \pm} \widehat{U}_{j,*}^{\tilde{\varepsilon}} \cup T_j \right\}.$$

It follows from the assumptions (1.3) and (1.6) that  $\widehat{U}^{\tilde{\varepsilon}}$  does not contain any poles nor turning points except for a simple turning point at the origin.

Now we state our main theorem.

**Theorem 1.1.** *Let  $Q(\tilde{x})$  be a rational function that satisfies (1.2), (1.3) and (1.6). Then there exists a Borel summable series*

$$(1.9) \quad x(\tilde{x}, \eta) = \sum_{k=0}^{\infty} x_k(\tilde{x}) \eta^{-k}$$

on  $\widehat{U}^{\tilde{\varepsilon}}$  for which the following conditions (1.10)  $\sim$  (1.14) hold:

$$(1.10) \quad \{x_k(\tilde{x})\}_{k=0}^{\infty} \text{ are holomorphic on } \widehat{U}^{\tilde{\varepsilon}},$$

(1.11)  $x_{2k+1}(\tilde{x})$  ( $k = 0, 1, 2, \dots$ ) are identically zero,

$$(1.12) \quad x_0(0) = 0,$$

$$(1.13) \quad \frac{dx_0}{d\tilde{x}} \neq 0 \text{ on } \widehat{U}^\varepsilon,$$

$$(1.14) \quad Q(\tilde{x}) = \left( \frac{dx(\tilde{x}, \eta)}{d\tilde{x}} \right)^2 x(\tilde{x}, \eta) - \frac{1}{2} \eta^{-2} \{x(\tilde{x}, \eta); \tilde{x}\}.$$

Here  $\{x(\tilde{x}, \eta); \tilde{x}\}$  stands for the Schwarzian derivative, i.e.,

$$(1.15) \quad \frac{d^3 x / d\tilde{x}^3}{dx / d\tilde{x}} - \frac{3}{2} \left( \frac{d^2 x / d\tilde{x}^2}{dx / d\tilde{x}} \right)^2.$$

In Section 2.2 and Section 2.3, we will give more detailed properties of  $x(\tilde{x}, \eta)$  in Theorem 1.1 including growth estimates.

The series  $x(\tilde{x}, \eta)$  in Theorem 1.1 is the same transformation series as that in [AKT1], which transforms (1.1) to

$$(1.16) \quad \left( \frac{d^2}{dx^2} - \eta^2 x \right) \psi = 0.$$

A new result in Theorem 1.1 is the Borel summability of the transformation series. In the remaining of this section, we explain the consequences of Theorem 1.1 following [KT2]. (See [AKT1] for details.) One of basic properties of  $x(\tilde{x}, \eta)$  is that it relates solutions of Riccati equations associated with (1.1) and (1.16):

**Theorem 1.2.** ([KT2, Theorem 2.16]) *The transformation series  $x(\tilde{x}, \eta)$  in Theorem 1.1 satisfies*

$$(1.17) \quad \tilde{S}(\tilde{x}, \eta) = \left( \frac{dx}{d\tilde{x}} \right) S(x(\tilde{x}, \eta), \eta) - \frac{1}{2} \left( \frac{d^2 x}{d\tilde{x}^2} \right) / \left( \frac{dx}{d\tilde{x}} \right).$$

Here formal power series

$$(1.18) \quad \tilde{S}(\tilde{x}, \eta) = \sum_{k=-1}^{\infty} \tilde{S}_k(\tilde{x}) \eta^{-k} \quad \text{and} \quad S(x, \eta) = \sum_{k=-1}^{\infty} S_k(x) \eta^{-k}$$

are respectively solutions of Riccati equations

$$(1.19) \quad \tilde{S}^2 + \frac{d\tilde{S}}{d\tilde{x}} = \eta^2 Q(\tilde{x})$$

and

$$(1.20) \quad S^2 + \frac{dS}{dx} = \eta^2 x$$

such that  $\tilde{S}_{-1}(\tilde{x})$  and  $S_{-1}(x)$  satisfy

$$(1.21) \quad \tilde{S}_{-1}(\tilde{x}) = \left( \frac{dx_0}{d\tilde{x}} \right) S_{-1}(x_0(\tilde{x})).$$

Let  $\tilde{S}^{(\pm)}$  respectively denote the solutions of (1.19) that are determined so that they satisfy  $\tilde{S}_{-1}^{(\pm)}(\tilde{x}) = \pm \sqrt{Q(\tilde{x})}$ . Then the odd part  $\tilde{S}_{\text{odd}}$  of  $\tilde{S}$  is defined by

$$(1.22) \quad \tilde{S}_{\text{odd}} = \frac{1}{2} \left( \tilde{S}^{(+)} - \tilde{S}^{(-)} \right).$$

In the same manner, we also define the odd part  $S_{\text{odd}}$  of  $S$ . From Theorem 1.2, we immediately obtain

**Corollary 1.3.** ([KT2, Corollary 2.17]) *If the branches of  $\tilde{S}_{-1}$  and  $S_{-1}$  are taken so that they satisfy (1.21), then we have*

$$(1.23) \quad \tilde{S}_{\text{odd}}(\tilde{x}, \eta) = \left( \frac{dx(\tilde{x}, \eta)}{d\tilde{x}} \right) S_{\text{odd}}(x(\tilde{x}, \eta), \eta).$$

Let  $\tilde{\psi}_{\pm}(\tilde{x}, \eta)$  denote WKB solutions of (1.1) normalized at a simple turning point  $\tilde{x} = 0$ , i.e.,

$$(1.24) \quad \tilde{\psi}_{\pm}(\tilde{x}, \eta) = \frac{1}{\sqrt{\tilde{S}_{\text{odd}}}} \exp \left( \pm \int_0^{\tilde{x}} \tilde{S}_{\text{odd}}(\tilde{x}, \eta) d\tilde{x} \right).$$

By the same way, we define WKB solutions  $\psi_{\pm}(x, \eta)$  of (1.16) normalized at a simple turning point  $x = 0$ . The relation for  $\tilde{S}_{\text{odd}}$  and  $S_{\text{odd}}$  in Corollary 1.3 gives

**Theorem 1.4.** ([KT2, Corollary 2.18]) *Let  $\tilde{\psi}_{\pm}(\tilde{x}, \eta)$  and  $\psi_{\pm}(x, \eta)$  respectively be WKB solutions of (1.1) and (1.16) normalized at their simple turning points  $\tilde{x} = 0$  and  $x = 0$ . Then they satisfy the following relation:*

$$(1.25) \quad \tilde{\psi}_{\pm}(\tilde{x}, \eta) = \left( \frac{dx(\tilde{x}, \eta)}{d\tilde{x}} \right)^{-1/2} \psi_{\pm}(x(\tilde{x}, \eta), \eta).$$

For simplicity, we take  $x_0 = x_0(\tilde{x})$  as a new coordinate variable (cf (1.13)). We also let  $\tilde{x}(x_0)$  be the inverse function of  $x_0(\tilde{x})$ . Then the precise meaning of the right hand side of (1.25) is

$$(1.26) \quad \left( \frac{d\tilde{x}}{dx_0} \right)^{1/2} \left( 1 + \frac{dX(x_0, \eta)}{dx_0} \right)^{-1/2} \sum_{n=0}^{\infty} \frac{(X(x_0, \eta))^n}{n!} \frac{d\psi_{\pm}}{dx_0}(x_0, \eta),$$

where  $X(x_0, \eta)$  is

$$(1.27) \quad X(x_0, \eta) = x(\tilde{x}(x_0), \eta) - x_0.$$

Let  $\tilde{\psi}_{\pm, B}$  and  $\psi_{\pm, B}$  respectively denote the Borel transforms of  $\tilde{\psi}_{\pm}$  and  $\psi_{\pm}$ . Through the Borel transformation, (1.26) can be rewritten as  $\mathcal{X}\psi_{\pm, B}$ , where  $\mathcal{X}$  is a microdifferential operator defined by

$$(1.28) \quad \mathcal{X} =: \left( \frac{\partial \tilde{x}}{\partial x_0} \right)^{1/2} \left( 1 + \frac{\partial X}{\partial x_0} \right)^{-1/2} \exp[X(x_0, \eta)\xi] :.$$

Here  $\xi$  stands for the symbol of  $\partial_{x_0}$  and  $: \cdot :$  designates the normal ordered product. (See [A1] and [AY] for details.) Since  $\tilde{\psi}_{\pm}$  and  $\psi_{\pm}$  satisfy (1.25), we can represent  $\tilde{\psi}_{\pm, B}$  by  $\psi_{\pm, B}$  through the action of  $\mathcal{X}$ . As we will see in Appendix B, the action of  $\mathcal{X}$  can be written as an action of an integro-differential operator and the Borel summability of  $X(x_0, \eta)$ , more precisely Theorem 1.9, guarantees that this representation of  $\tilde{\psi}_{\pm, B}$  holds on  $\hat{V}^{\varepsilon} \times E_{\pm y_0}^{\delta}$  for some  $\varepsilon, \delta > 0$ , where

$$(1.29) \quad y_0(x_0) = \int_0^{x_0} \sqrt{x_0} dx_0,$$

$$(1.30) \quad \hat{V}^{\varepsilon} = \{x_0 \in \mathbb{C} : |\operatorname{Im} y_0(x_0)| < \varepsilon\},$$

$$(1.31) \quad E_{\pm y_0}^{\delta} = \bigcup_{s \in \mathbb{R}} \{y \in \mathbb{C} : |y - s \pm y_0(x_0)| < \delta\}.$$

*Remark 1.1.* Since  $x_0(\tilde{x})$  maps an integral curve of  $\text{Im}\sqrt{Q(\tilde{x})}d\tilde{x} = 0$  that passes through  $\tilde{x}$  to that of  $\text{Im}\sqrt{x}dx = 0$  that passes through  $x_0(\tilde{x})$  bijectively, by taking  $\varepsilon > 0$  sufficiently small, we can assume that  $\widehat{V}^\varepsilon$  is contained in  $x_0(\widehat{U}^\varepsilon)$ .

Concretely we have

**Theorem 1.5.**  $\tilde{\psi}_{\pm,B}$  and  $\psi_{\pm,B}$  satisfy the following relation on  $\widehat{V}^\varepsilon \times E_{\pm y_0}^\delta$  for sufficiently small  $\varepsilon, \delta > 0$ :

$$(1.32) \quad \tilde{\psi}_{\pm,B}(\tilde{x}(x_0), y) = \left( \frac{\partial \tilde{x}}{\partial x_0} \right)^{1/2} \psi_{\pm,B}(x_0, y) + \int_{\mp y_0(x_0)}^y K(x_0, y - y', \partial_{x_0}) \psi_{\pm,B}(x_0, y') dy',$$

where  $K(x, y, \partial_x)$  is a differential operator of infinite order on  $\widehat{V}^\varepsilon \times E_{\pm y_0}^\delta$ .

See [SKK] for the notion of a differential operator of infinite order.

*Remark 1.2.* It follows from the construction of  $x(\tilde{x}, \eta)$  given in the proof of Proposition 1.6 (cf. (1.37)) that  $x_0(\tilde{x})$  satisfies the following relation:

$$(1.33) \quad y_0(x_0(\tilde{x})) = \int_0^{\tilde{x}} \sqrt{Q(\tilde{x})} d\tilde{x}.$$

Now we note that the Borel transforms of WKB solutions of the canonical equation (1.16) are explicitly given by

$$(1.34) \quad \begin{cases} \psi_{+,B}(x, y) = \frac{\sqrt{3}}{2\sqrt{\pi x}} s^{-1/2} F\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{2}; s\right), \\ \psi_{-,B}(x, y) = \frac{\sqrt{3}}{2\sqrt{\pi x}} (s-1)^{-1/2} F\left(\frac{5}{6}, \frac{1}{6}, \frac{1}{2}; 1-s\right), \end{cases}$$

where  $F(\alpha, \beta, \gamma; z)$  is Gauss' hypergeometric function and

$$(1.35) \quad s = \frac{3y}{4x^{3/2}} + \frac{1}{2}.$$

(See [KT2, Section 2.2].) Therefore we conclude from Theorem 1.5 that  $\tilde{\psi}_{+,B}(\tilde{x}(x_0), y)$  (resp.,  $\tilde{\psi}_{+,B}(\tilde{x}(x_0), y)$ ) can be analytically continued into the universal covering of  $(\widehat{V}^\varepsilon \setminus \{0\}) \times (E_{-y_0}^\delta \setminus \{\pm y_0(x_0)\})$  (resp.,  $(\widehat{V}^\varepsilon \setminus \{0\}) \times (E_{+y_0}^\delta \setminus \{\pm y_0(x_0)\})$ ). We also conclude that the computation of the alien derivative of  $\tilde{\psi}_+$  (resp.,  $\tilde{\psi}_-$ ) at  $y_0$  (resp.,  $-y_0$ ) by Aoki-Kawai-Takei ([KT2, Theorem 2.21]) works for all  $x_0$  in  $\widehat{V}^\varepsilon \setminus \{0\}$  (not  $V^\varepsilon \setminus \{0\}$ , in which Aoki-Kawai-Takei did). Thus the proof of the connection formula by Aoki-Kawai-Takei from a viewpoint of transformation theory completes.

## 1.2 Uniform Borel transformability of transformation series

As a first step to proving the Borel summability of transformation series  $x(\tilde{x}, \eta)$  introduced in Theorem 1.1, we show the uniform Borel transformability of  $x(\tilde{x}, \eta)$  on  $\widehat{U}^\varepsilon$  in this subsection. Concretely we prove

**Proposition 1.6.** *Let  $Q(\tilde{x})$  be a rational function that satisfies (1.2), (1.3) and (1.6). Then there exist  $\tilde{\varepsilon} > 0$  and formal series  $x(\tilde{x}, \eta) = x_0(\tilde{x}) + \eta^{-1}x_1(\tilde{x}) + \cdots$  that satisfies (1.10)  $\sim$  (1.14) and the following estimates: there exist positive constants  $C_0$  and  $A$  such that for all  $n \geq 1$  and  $\tilde{x} \in \widehat{U}^{\tilde{\varepsilon}}$ ,  $x_n(\tilde{x})$  satisfies*

$$(1.36) \quad |x_n(\tilde{x})| \leq (|x_0(\tilde{x})| + 1)C_0 n! A^n.$$

*Proof.* We first remind us the construction of  $x(\tilde{x}, \eta)$ . We determine  $x_k(\tilde{x})$  ( $k = 0, 1, 2, \dots$ ) inductively by comparing the coefficients of  $\eta^{-k}$  of (1.14). First, by comparing the coefficients of  $\eta^0$  of (1.14), we find that  $x_0(\tilde{x})$  should satisfy

$$(1.37) \quad Q(\tilde{x}) = \left( \frac{dx_0(\tilde{x})}{d\tilde{x}} \right)^2 x_0(\tilde{x}).$$

Therefore we determine  $x_0(\tilde{x})$  by

$$(1.38) \quad x_0(\tilde{x}) = \left( \frac{3}{2} \int_0^{\tilde{x}} \sqrt{Q(\tilde{x})} d\tilde{x} \right)^{2/3}.$$

Since  $Q(\tilde{x})$  satisfies (1.2), (1.3) and (1.6), we immediately find that  $x_0(\tilde{x})$  is holomorphic on  $\widehat{U}^{\tilde{\varepsilon}}$  for some  $\tilde{\varepsilon} > 0$  and satisfies (1.12). Further,

from (1.37), we find the following relation holds:

$$(1.39) \quad \sqrt{Q(\tilde{x})}d\tilde{x} = \sqrt{x}dx|_{x=x_0(\tilde{x})}.$$

Therefore  $x_0$  maps the integral curves of  $\text{Im}\sqrt{Q(\tilde{x})}d\tilde{x}$  that start from  $\overset{\circ}{x} \in \widehat{U}^{\varepsilon}$  to those of  $\text{Im}\sqrt{x}dx$  that start from  $x_0(\overset{\circ}{x})$  in a bijective manner. Now it is clear that  $x_0$  maps  $\widehat{U}^{\varepsilon}$  to  $x_0(\widehat{U}^{\varepsilon})$  bijectively and  $x_0$  satisfies (1.13). We take  $z = x_0(\tilde{x})$  as a new coordinate variable on  $x_0(\widehat{U}^{\varepsilon})$ .

Next we determine  $x_k$  ( $k \geq 1$ ). For simplicity, we use the following notation:

$$(1.40) \quad \sum_{\substack{\mu_1+\dots+\mu_l=k, \\ \mu_1, \dots, \mu_l \geq 1}}^* \frac{dx_{\mu_1}}{dz} \dots \frac{dx_{\mu_l}}{dz} := \begin{cases} 1 & (l=0), \\ \sum_{\substack{\mu_1+\dots+\mu_l=k, \\ \mu_1, \dots, \mu_l \geq 1}} \frac{dx_{\mu_1}}{dz} \dots \frac{dx_{\mu_l}}{dz} & (l \geq 1). \end{cases}$$

By comparing the coefficients of  $\eta^{-k}$  of (1.14), we find that  $x_k$  should satisfy the following relations:

$$(1.41) \quad 2z \frac{dx_k}{dz} + x_k = \Phi_k(z),$$

where  $\Phi_k(z)$  is

$$(1.42) \quad \begin{aligned} \Phi_k(z) = & - \sum_{\substack{k_1+k_2+k_3=k, \\ k_1, k_2, k_3 \leq k-1}} \frac{dx_{k_1}}{dz} \frac{dx_{k_2}}{dz} x_{k_3} \\ & + \frac{1}{2} \sum_{k_1+k_2=k-2} \left( \frac{d\tilde{x}}{dz} \right)^3 \frac{d^3 x_{k_1}}{d\tilde{x}^3} \\ & \times \sum_{l=\min\{1, k_2\}}^{k_2} (-1)^l \sum_{\substack{\mu_1+\dots+\mu_l=k_2, \\ \mu_1, \dots, \mu_l \geq 1}}^* \frac{dx_{\mu_1}}{dz} \dots \frac{dx_{\mu_l}}{dz} \\ & - \frac{3}{4} \sum_{k_1+k_2+k_3=k-2} \left( \frac{d\tilde{x}}{dz} \right)^4 \frac{d^2 x_{k_1}}{d\tilde{x}^2} \frac{d^2 x_{k_2}}{d\tilde{x}^2} \\ & \times \sum_{l=\min\{1, k_3\}}^{k_3} (-1)^l (l+1) \sum_{\substack{\mu_1+\dots+\mu_l=k_3, \\ \mu_1, \dots, \mu_l \geq 1}}^* \frac{dx_{\mu_1}}{dz} \dots \frac{dx_{\mu_l}}{dz}. \end{aligned}$$

Since  $\Phi_k$  does not contain  $x_n$  ( $n \geq k$ ), we can inductively determine  $x_k$  by (1.41). Concretely we take  $x_k$  as

$$(1.43) \quad x_k(z) = \frac{z^{-1/2}}{2} \int_0^z z^{-1/2} \Phi_k(z) dz$$

so that  $x_k$  is holomorphic at  $z = 0$  and satisfies (1.41). We can easily find (1.11) since we can inductively check that  $\Phi_{2k+1}$  ( $k \geq 0$ ) are identically zero.

Now we confirm the estimation of  $x_k$ . First, for sufficiently small  $r > 0$ , we define  $D_r^1$  and  $D_r^2$  by

$$(1.44) \quad D_r^1 = \bigcup_{0 \leq s \leq 1} \{z \in \mathbb{C} : |z - s| \leq r\}$$

$$(1.45) \quad D_r^2 = \bigcup_{s \geq 1} \left\{ z \in \mathbb{C} : |z - s| \leq \frac{r}{\sqrt{s}} \right\}.$$

Since  $\text{Im}((2/3)z^{3/2})$  is expanded to

$$(1.46) \quad \sqrt{\text{Re}z} \cdot \text{Im}z + \frac{1}{24}(\text{Re}z)^{-3/2}(\text{Im}z)^3 + \dots$$

in  $D_r^2$ , we find that  $\text{Im} \int_0^z \sqrt{z} dz$  behaves like  $\sqrt{\text{Re}z} \cdot \text{Im}z$  in  $D_r^2$  for sufficiently large  $\text{Re}z$ . Therefore we can take  $r > 0$  so that  $D_r^1 \cup D_r^2 \subset x_0(\widehat{U}^\varepsilon)$ .

Then we show that  $x_k(z)$  ( $k \geq 1$ ) satisfy the following estimates: there exist positive constants  $C_0 < 1$  and  $A > 1$  such that for all  $\delta$  with  $0 < \delta < r/3$  and

1)  $z \in D_{r-\delta}^1$

$$(1.47) \quad |x_k(z)| \leq C_0 k! \delta^{-k} A^k$$

$$(1.48) \quad \left| \frac{dx_k}{dz}(z) \right| \leq C_0 k! \delta^{-k} A^k,$$

2)  $z \in D_{r-\delta}^2$

$$(1.49) \quad |x_k(z)| \leq |z| C_0 k! \delta^{-k} A^k$$



$$(1.50) \quad \left| \frac{dx_k}{dz}(z) \right| \leq C_0 k! \delta^{-k} A^k$$

hold. Since (1.47) and (1.48) can be verified by the same discussion as in [AKT1], we only confirm (1.49) and (1.50) here.

*Remark 1.3.* The condition  $0 < \delta < r/3$  is used in the proof of (1.47) and (1.48).

We inductively show that  $x_k$  ( $k = 1, 2, \dots$ ) satisfy (1.49) and (1.50). First we immediately find that  $x_1$  satisfies (1.49) and (1.50) since  $x_{2k+1}$  ( $k \geq 0$ ) are identically zero. Just to be sure, we check that  $x_2$  satisfies (1.49) and (1.50). From the assumption (1.3), the inverse image  $x_0^{-1}(z)$  of  $z$  tends to a irregular singular point  $b_j$  of (1.1) when  $z$  tends to  $+\infty$  along positive real axis. Let  $Q(\tilde{x})$  have a pole of order  $p(\geq 3)$  at  $\tilde{x} = b_j$ . Then, from (1.37) and (1.38), we find that  $x_0(\tilde{x})$  and  $dx_0/d\tilde{x}$  behave as

$$(1.51) \quad x_0(\tilde{x}) = O((\tilde{x} - b_j)^{(-p+2)/3}),$$

$$(1.52) \quad \frac{dx_0}{d\tilde{x}}(\tilde{x}) = O((\tilde{x} - b_j)^{-(p+1)/3})$$

when  $\tilde{x}$  tends to  $b_j$ . Therefore we can take positive constants  $M_1$  and  $M_2$  so that the following holds on  $D_r^2$ :

$$(1.53) \quad M_1 |x_0|^{(p+1)/(p-2)} \leq \left| \frac{dx_0}{d\tilde{x}} \right| \leq M_2 |x_0|^{(p+1)/(p-2)}.$$

Now we derive the estimation of  $x_2$  using the representation (1.43). From (1.42), we immediately find that  $\Phi_2(z)$  is given by

$$(1.54) \quad \Phi_2(z) = \frac{1}{2} \left( \frac{d\tilde{x}}{dz} \right)^3 \frac{d^3 x_0}{d\tilde{x}^3} - \frac{3}{4} \left( \frac{d\tilde{x}}{dz} \right)^4 \left( \frac{d^2 x_0}{d\tilde{x}^2} \right)^2.$$

In order to rewrite  $\Phi_2(z)$  by  $dx_0/d\tilde{x}$  and its derivative in  $z$  variable, we use the following relation for a function  $f(\tilde{x})$  of  $\tilde{x}$ :

$$(1.55) \quad \frac{d^2 f}{d\tilde{x}^2}(\tilde{x}(z)) = \left( \frac{d\tilde{x}}{dz}(z) \right)^{-2} \frac{d^2}{dz^2} f(\tilde{x}(z)) + \frac{1}{2} \frac{d}{dz} \left( \frac{d\tilde{x}}{dz}(z) \right)^{-2} \frac{d}{dz} f(\tilde{x}(z)),$$

(1.56)

$$\begin{aligned} \frac{d^3 f}{d\tilde{x}^3}(\tilde{x}(z)) &= \left(\frac{d\tilde{x}}{dz}(z)\right)^{-3} \frac{d^3}{dz^3} f(\tilde{x}(z)) + \frac{d}{dz} \left(\frac{d\tilde{x}}{dz}(z)\right)^{-3} \frac{d^2}{dz^2} f(\tilde{x}(z)) \\ &\quad + \frac{1}{2} \left(\frac{d\tilde{x}}{dz}(z)\right)^{-1} \frac{d^2}{dz^2} \left(\frac{d\tilde{x}}{dz}(z)\right)^{-2} \frac{d}{dz} f(\tilde{x}(z)). \end{aligned}$$

Therefore  $\Phi_2(z)$  can be rewritten to

(1.57)

$$\Phi_2(z) = \frac{1}{4} \left(\frac{d\tilde{x}}{dz}(z)\right)^2 \frac{d^2}{dz^2} \left(\frac{d\tilde{x}}{dz}(z)\right)^{-2} - \frac{3}{16} \left[ \left(\frac{d\tilde{x}}{dz}\right)^2 \frac{d}{dz} \left(\frac{d\tilde{x}}{dz}(z)\right)^{-2} \right]^2.$$

In order to derive the estimation of  $\Phi_2$  from that of  $dx_0/d\tilde{x}$ , we use Cauchy's formula as follows: for a holomorphic function  $g(z)$  on  $D_r^2$ , we have the following representation for  $z \in D_{r-\delta}^2$ :

(1.58)

$$\frac{d^j}{dz^j} g(z) = \frac{j!}{2\pi i} \int_{|\tilde{z}-z|=d} \frac{g(\tilde{z})}{(\tilde{z}-z)^{j+1}} d\tilde{z},$$

where we take  $d > 0$  as

(1.59)

$$d = \delta(2|z|)^{-1/2}.$$

Then we immediately find that the integral path of (1.58) is contained in  $D_r^2$ . Applying (1.58) to  $(d\tilde{x}/dz)^{-2}$ , we obtain the following estimates of  $\Phi_2$ : there exists a positive constant  $M$  such that, for  $z \in D_{r-\delta}^2$ ,  $\Phi_2(z)$  satisfies

(1.60)

$$|\Phi_2(z)| \leq M|z|\delta^{-2}.$$

Actually, for example, the estimation of the first term of (1.57) is given as follows: first, from (1.53), we find that, for  $\tilde{z} \in \{\tilde{z} : |\tilde{z} - z| = \delta(2|z|)^{-1/2}\}$ ,  $(d\tilde{x}/dz(\tilde{z}))^{-2}$  is dominated as follows:

(1.61)

$$\begin{aligned} \left| \left(\frac{d\tilde{x}}{dz}(\tilde{z})\right)^{-2} \right| &\leq M_2^2 |\tilde{z}|^{2(p+1)/(p-2)} \\ &\leq M_2^2 (|z| + \delta(2|z|)^{-1/2})^{2(p+1)/(p-2)} \\ &\leq M_2^2 (2|z|)^{2(p+1)/(p-2)}. \end{aligned}$$

Using the representation (1.58) for  $j = 2$ , we obtain the following estimates from (1.61):

$$(1.62) \quad \left| \frac{d^2}{dz^2} \left( \frac{d\tilde{x}}{dz}(z) \right)^{-2} \right| \leq \frac{1}{\pi} 2|z|\delta^{-2} M_2^2 (2|z|)^{2(p+1)/(p-2)}.$$

Therefore, using (1.53) again, we immediately find

$$(1.63) \quad \left| \frac{1}{4} \left( \frac{d\tilde{x}}{dz}(z) \right)^2 \frac{d^2}{dz^2} \left( \frac{d\tilde{x}}{dz}(z) \right)^{-2} \right| \leq \frac{1}{\pi} 2^{-1+2(p+1)/(p-2)} M_1^{-2} M_2^2 |z|\delta^{-2}.$$

In the same way, we have

$$(1.64) \quad \left| \frac{3}{16} \left[ \left( \frac{d\tilde{x}}{dz} \right)^2 \frac{d}{dz} \left( \frac{d\tilde{x}}{dz}(z) \right)^{-2} \right]^2 \right| \leq \frac{3}{\pi^2} 2^{-5+4(p+1)/(p-2)} M_1^{-4} M_2^4 |z|\delta^{-2}.$$

Combining (1.63) and (1.64), we arrive at (1.60).

Then, from (1.60), we find that, for arbitrarily small  $C_0 > 0$ , we can take  $A > 0$  so that  $\Phi_2(z)$  satisfies

$$(1.65) \quad |\Phi_2(z)| \leq C_0^2 |z|\delta^{-2} A^2$$

on  $D_{r-\delta}^2$ . Actually it suffices to set  $A = \sqrt{M} C_0^{-1}$ . Similarly we can show that

$$(1.66) \quad |\Phi_2(z)| \leq C_0^2 \delta^{-2} A^2$$

holds on  $D_{r-\delta}^1$ .

Now we divide the integral path of (1.43) as follows:

$$(1.67) \quad x_2(z) = \frac{z^{-1/2}}{2} \int_0^1 z^{-1/2} \Phi_2(z) dz + \frac{z^{-1/2}}{2} \int_1^z z^{-1/2} \Phi_2(z) dz.$$

Here we take the integral path of the first term of (1.67) as a straight line joining 0 and 1 so that the path is contained in  $D_{r-\delta}^1$ . And we take the path of the second term as a straight line joining 1 and  $z$  so that

the path is contained in  $D_{r-\delta}^2$ . Then we obtain the following estimates of the first term of (1.67) for  $z \in D_{r-\delta}^2$  from (1.66):

$$(1.68) \quad \left| \frac{z^{-1/2}}{2} \int_0^1 z^{-1/2} \Phi_2(z) dz \right| \leq \frac{|z|^{-1/2}}{2} \int_0^1 |z|^{-1/2} C_0^2 \delta^{-2} A^2 |dz| \\ \leq |z|^{-1/2} C_0^2 \delta^{-2} A^2.$$

Similarly we find the following estimates of the second term of (1.67) for  $z \in D_{r-\delta}^2$  from (1.65):

$$(1.69) \quad \left| \frac{z^{-1/2}}{2} \int_1^z z^{-1/2} \Phi_2(z) dz \right| \leq \frac{|z|^{-1/2}}{2} \int_1^z |z|^{1/2} C_0^2 \delta^{-2} A^2 |dz| \\ \leq \frac{|z|^{-1/2}}{3} (|z|^{3/2} + 1) C_0^2 \delta^{-2} A^2.$$

Since  $z \in D_{r-\delta}^2$ , by taking  $r < 1/2$ , we find  $|z|^{-1/2} \leq 2\sqrt{2}|z|$ . Therefore, combining (1.68) and (1.69), we obtain

$$(1.70) \quad |x_2(z)| \leq \frac{1+8\sqrt{2}}{3} |z| C_0^2 \delta^{-2} A^2$$

for  $z \in D_{r-\delta}^2$ . Further, from (1.41), we immediately find the following estimates for  $z \in D_{r-\delta}^2$ :

$$(1.71) \quad \left| \frac{dx_2}{dz} \right| \leq \frac{|x_2| + |\Phi_2(z)|}{2|z|} \\ \leq \frac{2+4\sqrt{2}}{3} C_0^2 \delta^{-2} A^2.$$

Finally, by taking  $C_0$  so that

$$(1.72) \quad \frac{1+8\sqrt{2}}{3} C_0 < 1,$$

we are convinced that  $x_2$  satisfies (1.49) and (1.50).

Next we show that  $x_k$  ( $k \geq 2$ ) satisfies (1.49) and (1.50) under the assumption that  $x_n$  ( $1 \leq n \leq k-1$ ) satisfy them. As in the case of the

estimation of  $x_2$ , we first examine that of  $\Phi_k$  on  $D_{r-\delta}^2$ . The first term of (1.42) is directly estimated from the induction hypothesis as follows:

$$(1.73) \quad \sum_{\substack{k_1+k_2+k_3=k, \\ k_1, k_2, k_3 \leq k-1}} \left| \frac{dx_{k_1}}{dz} \right| \left| \frac{dx_{k_2}}{dz} \right| |x_{k_3}| \leq |z| C_0^2 \delta^{-k} A^k \sum_{\substack{k_1+k_2+k_3=k, \\ k_1, k_2, k_3 \leq k-1}} k_1! k_2! k_3! \\ \leq |z| C_0^2 \left( \frac{4^2}{k-1} + 12 \right) \delta^{-k} A^k (k-1)!.$$

Here we use the following

**Lemma 1.7** ([AKT4]). *For  $k, l \in \mathbb{N} = \{1, 2, 3, \dots\}$  with  $l \leq k$ , the following inequality holds:*

$$(1.74) \quad \sum_{\substack{\mu_1 + \dots + \mu_l = k, \\ \mu_1, \dots, \mu_l \geq 1}} \mu_1! \dots \mu_l! \leq 4^{l-1} (k-l+1)!.$$

In fact, we apply Lemma 1.7 as follows:

$$(1.75) \quad \sum_{\substack{k_1+k_2+k_3=k, \\ k_1, k_2, k_3 \leq k-1}} k_1! k_2! k_3! = \sum_{\substack{k_1+k_2+k_3=k, \\ 1 \leq k_1, k_2, k_3 \leq k-1}} k_1! k_2! k_3! + 3 \sum_{\substack{k'_1+k'_2=k, \\ 1 \leq k'_1, k'_2 \leq k-1}} k'_1! k'_2! \\ \leq 4^2 (k-2)! + 12 (k-1)!.$$

*Remark 1.4.* We have to care that (1.47)  $\sim$  (1.50) hold for  $k \geq 1$ , on the other hand,  $x_0$  satisfies  $|x_0| = |z|$  and  $|dx_0/dz| = 1$ . Therefore the estimates  $|x_0| \leq C_0|z|$  and  $|dx_0/dz| \leq C_0$  that is obtained from (1.49) and (1.50) by letting  $k = 0$  does not hold for sufficiently small  $C_0$ . Hence, to simplify the discussion, when  $x_0$  and  $x_k$  ( $k \geq 1$ ) appear at the same time and the extra factor  $C_0$  is not important in the estimation, we neglect the factor  $C_0$  that appears in (1.47)  $\sim$  (1.50).

Then we consider the second term of (1.42), which is the most important term in (1.42) in the sense that  $k!$  in the estimation of  $x_k$  originates from this term. First we rewrite the third derivative of  $x_{k_1}$  in  $\tilde{x}$  variable to that of  $x_{k_1}$  in  $z$  variable using the relation (1.56). And, multiplying  $(d\tilde{x}/dz)^3$ , we obtain the following relation:

$$(1.76) \quad \left( \frac{d\tilde{x}}{dz} \right)^3 \frac{d^3 x_{k_1}}{d\tilde{x}^3} = \frac{d^3 x_{k_1}}{dz^3} + \left( \frac{d\tilde{x}}{dz} \right)^3 \frac{d}{dz} \left( \frac{d\tilde{x}}{dz} \right)^{-3} \frac{d^2 x_{k_1}}{dz^2}$$

$$+ \frac{1}{2} \left( \frac{d\tilde{x}}{dz} \right)^2 \frac{d^2}{dz^2} \left( \frac{d\tilde{x}}{dz} \right)^{-2} \frac{dx_{k_1}}{dz}.$$

Since  $k_1 \leq k - 2$ ,  $dx_{k_1}/dz$  satisfies (1.50) for all  $\delta$  with  $0 < \delta < r/3$  from the induction hypothesis. Now we derive the estimates of the second and the third derivative of  $x_{k_1}$  from that of  $dx_{k_1}/dz$  through the representation (1.58). In this case, we take

$$(1.77) \quad d = \frac{\delta}{(k_1 + 1)\sqrt{2|z|}}.$$

Then, for  $z \in D_{r-\delta}^2$ , if  $\tilde{z}$  satisfies  $|\tilde{z} - z| \leq \delta/(k_1 + 1)\sqrt{2|z|}$ , we find that  $\tilde{z} \in D_{r-k_1\delta/(k_1+1)}^2$ . Indeed, since  $z \in D_{r-\delta}^2$ , we can take  $s \geq 1$  so that  $|z - s| \leq (r - \delta)/\sqrt{s}$ . Therefore  $|z| \geq s/2$  holds and  $\tilde{z}$  satisfies

$$(1.78) \quad \begin{aligned} |\tilde{z} - s| &\leq \frac{r - \delta}{\sqrt{s}} + \frac{\delta}{(k_1 + 1)\sqrt{2|z|}} \\ &\leq \frac{r - \delta}{\sqrt{s}} + \frac{\delta}{(k_1 + 1)\sqrt{s}} \\ &= \frac{1}{\sqrt{s}} \left( r - \frac{k_1}{k_1 + 1} \delta \right). \end{aligned}$$

Substituting  $\delta$  in (1.50) for  $k = k_1$  to  $k_1\delta/(k_1 + 1)$ , we find that  $dx_{k_1}/dz$  satisfies the following estimates for  $\tilde{z} \in D_{r-k_1\delta/(k_1+1)}^2$ :

$$(1.79) \quad \begin{aligned} \left| \frac{dx_{k_1}}{dz}(\tilde{z}) \right| &\leq k_1! \left( 1 + \frac{1}{k_1} \right)^{k_1} \delta^{-k_1} A^{k_1} \\ &\leq k_1! e \delta^{-k_1} A^{k_1} \end{aligned}$$

Hence, using the representation (1.58), we obtain

$$(1.80) \quad \left| \frac{d^j}{dz^j} \frac{dx_{k_1}}{dz}(z) \right| \leq \frac{j!}{2\pi} \left( \frac{\delta}{(k_1 + 1)\sqrt{2|z|}} \right)^{-j} e k_1! \delta^{-k_1} A^{k_1}.$$

The estimation of the coefficient of  $dx_{k_1}/dz$  is given from (1.63). By the same reasoning, the coefficient of  $d^2x_{k_1}/dz^2$  satisfies

$$(1.81) \quad \left| \left( \frac{d\tilde{x}}{dz}(z) \right)^3 \frac{d}{dz} \left( \frac{d\tilde{x}}{dz}(z) \right)^{-3} \right| \leq \frac{1}{2\pi} 2^{3(p+1)/(p-2)} M_1^{-3} M_2^3 \sqrt{2|z|} \delta^{-1}.$$

In conclusion, we gain the following estimation: we can take some positive constant  $M$  that is independent of  $z, k_1, C_0, \delta$  and  $A$  so that

$$(1.82) \quad \left| \left( \frac{d\tilde{x}}{dz} \right)^3 \frac{d^3 x_{k_1}}{d\tilde{x}^3} \right| \leq |z| M (k_1 + 2)! \delta^{-k_1-2} A^{k_1}$$

holds on  $D_{r-\delta}^2$ . Actually the estimates of the first term of (1.76) immediately follows from (1.80):

$$(1.83) \quad \left| \frac{d^3 x_{k_1}}{dz^3} \right| \leq \frac{2!}{2\pi} \left( \frac{\delta}{(k_1 + 1)\sqrt{2|z|}} \right)^{-2} e k_1! \delta^{-k_1} A^{k_1} \\ \leq |z| 2e\pi^{-1} (k_1 + 2)! \delta^{-k_1-2} A^{k_1}.$$

Similarly the estimates of the second and the third term of (1.76) is obtained from (1.63) and (1.81) as follows:

$$(1.84) \quad \left| \left( \frac{d\tilde{x}}{dz} \right)^3 \frac{d}{dz} \left( \frac{d\tilde{x}}{dz} \right)^{-3} \frac{d^2 x_{k_1}}{dz^2} \right| \leq \frac{|z| M_2^3}{2\pi^2 M_1^3} 2^{3(p+1)/(p-2)} (k_1 + 1)! \delta^{-k_1-2} A^{k_1},$$

$$(1.85) \quad \left| \frac{1}{2} \left( \frac{d\tilde{x}}{dz} \right)^2 \frac{d^2}{dz^2} \left( \frac{d\tilde{x}}{dz} \right)^{-2} \frac{dx_{k_1}}{dz} \right| \leq \frac{|z| M_2^2}{\pi M_1^2} 2^{2(p+1)/(p-2)} k_1! \delta^{-k_1-2} A^{k_1}.$$

Combining (1.83), (1.84) and (1.85), we find that (1.82) holds.

Now (1.50) and (1.82) enable us to estimate the second term of (1.74) as follows:

$$(1.86) \quad \left| \frac{1}{2} \sum_{k_1+k_2=k-2} \left( \frac{d\tilde{x}}{dz} \right)^3 \frac{d^3 x_{k_1}}{d\tilde{x}^3} \sum_{l=\min\{1, k_2\}}^{k_2} \sum_{\substack{\mu_1+\dots+\mu_l=k_2, \\ \mu_1, \dots, \mu_l \geq 1}}^* (-1)^l \frac{dx_{\mu_1}}{dz} \dots \frac{dx_{\mu_l}}{dz} \right| \\ \leq |z| \frac{M}{2} \delta^{-k} A^{k-2} \sum_{k_1+k_2=k-2} (k_1 + 2)! \sum_{l=\min\{1, k_2\}}^{k_2} C_0^l \sum_{\substack{\mu_1+\dots+\mu_l=k_2, \\ \mu_1, \dots, \mu_l \geq 1}}^* \mu_1! \dots \mu_l! \\ \leq |z| \frac{M}{2} \delta^{-k} A^{k-2} \sum_{k_1+k_2=k-2} (k_1 + 2)! \left( 1 + \sum_{l=1}^{k_2} C_0^l 4^{l-1} (k_2 - l + 1)! \right)$$

$$\begin{aligned}
&\leq |z| \frac{M}{2} \delta^{-k} A^{k-2} \sum_{k_1+k_2=k-2} (k_1+2)! k_2! \left( 1 + \sum_{l=1}^{\infty} \frac{C_0^l 4^{l-1}}{(l-1)!} \right) \\
&\leq |z| \frac{M(1+C_0 e^{4C_0})}{2} \left( 1 + \frac{4}{k} \right) k! \delta^{-k} A^{k-2}.
\end{aligned}$$

Here we applied (1.74) to the second line of (1.86). Then, since we can take some positive constant  $M$  as (1.82) so that

$$(1.87) \quad \left| \left( \frac{d\tilde{x}}{dz} \right)^2 \frac{d^2 x_{k_1}}{d\tilde{x}^2} \right| \leq \sqrt{|z|} M (k_1+1)! \delta^{-k_1-1} A^{k_1}$$

holds for  $k_1 \leq k-1$ , by the similar discussion, we find the following estimates for the third term of (1.42):

$$\begin{aligned}
(1.88) \quad &\left| \frac{3}{4} \sum_{k_1+k_2+k_3=k-2} \left( \frac{d\tilde{x}}{dz} \right)^4 \frac{d^2 x_{k_1}}{d\tilde{x}^2} \frac{d^2 x_{k_2}}{d\tilde{x}^2} \right. \\
&\quad \times \sum_{k_3}^{k_3} \sum_{\substack{l=\min\{1,k_3\} \\ \mu_1+\dots+\mu_l=k_3, \\ \mu_1,\dots,\mu_l \geq 1}}^* (-1)^l (l+1) \frac{dx_{\mu_1}}{dz} \dots \frac{dx_{\mu_l}}{dz} \Big| \\
&\leq 9|z| M^2 (1+4C_0^2 e^{4C_0}) \left( 1 + \frac{4}{k-1} \right) (k-1)! \delta^{-k} A^{k-2}.
\end{aligned}$$

In conclusion, we obtain the following estimates for  $\Phi_k(z)$ : there exists some positive constant  $M$  that is independent of  $C_0 (< 1)$  and  $A$  such that, for  $k \geq 2$ ,  $0 < \delta < r/3$  and  $z \in D_{r-\delta}^2$ ,

$$(1.89) \quad |\Phi_k(z)| \leq |z| M (C_0^2 + A^{-2}) k! \delta^{-k} A^k$$

holds under the assumption that  $x_n$  ( $1 \leq n \leq k-1$ ) satisfy (1.49) and (1.50). Similarly we can show the following: there exists some positive constant  $M$  that is independent of  $C_0 (< 1)$  and  $A$  such that, for  $k \geq 2$ ,  $0 < \delta < r/3$  and  $z \in D_{r-\delta}^1$ ,

$$(1.90) \quad |\Phi_k(z)| \leq M (C_0^2 + A^{-2}) k! \delta^{-k} A^k$$

holds under the assumption that  $x_n$  ( $1 \leq n \leq k-1$ ) satisfy (1.47) and (1.48). Then, by the same discussion with the case of  $k=2$ , we find



that  $x_k(z)$  satisfies

$$(1.91) \quad |x_k(z)| \leq \frac{1+8\sqrt{2}}{3} |z| M(C_0^2 + A^{-2}) k! \delta^{-k} A^k,$$

$$(1.92) \quad \left| \frac{dx_k}{dz}(z) \right| \leq \frac{2+4\sqrt{2}}{3} M(C_0^2 + A^{-2}) k! \delta^{-k} A^k$$

for  $z \in D_{r-\delta}^2$ . Therefore, by taking  $C_0$  sufficiently small so that

$$(1.93) \quad \frac{1+8\sqrt{2}}{3} M C_0 < \frac{1}{2}$$

and then  $A$  sufficiently large so that

$$(1.94) \quad \frac{1+8\sqrt{2}}{3} M A^{-2} < \frac{1}{2} C_0,$$

we find that  $x_k$  satisfies (1.49) and (1.50). Thus the induction proceeds and (1.49) and (1.50) holds for all  $k \geq 1$ . By fixing  $\delta = r/6$  and combining (1.47) and (1.49), we obtain the following estimates: there exist positive constants  $r$  and  $A$  such that

$$(1.95) \quad |x_k(z)| \leq (|z| + 1) k! A^k$$

for  $k \geq 1$  and  $z \in D_r^0 = D_r^1 \cup D_r^2$ .

By the same discussion, we can show that (1.95) holds on

$$(1.96) \quad D_r^\pm = \{z \in \mathbb{C} : e^{\mp 2\pi i/3} z \in D_r^0\}$$

for some  $r$  and  $A$ . Bearing in mind that we can take  $\varepsilon > 0$  so that

$$(1.97) \quad \widehat{V}^\varepsilon = \left\{ z \in \mathbb{C} : \left| \operatorname{Im} \int_0^z \sqrt{z} dz \right| < \varepsilon \right\} \subset \bigcup_{* \in \{0, \pm\}} D_r^*,$$

we immediately see that we can take a neighborhood  $U^\varepsilon$  of  $\tilde{x} = 0$  such that (1.36) holds on  $\widehat{U}^\varepsilon$ .  $\square$

*Remark 1.5.* Since we can take the constant  $A$  in (1.36) independent of  $\tilde{x}$ , we use the phrase “uniform Borel transformable”. This uniform Borel transformability guarantees that the Borel transform of  $x - x_0$  is holomorphic on  $\widehat{U}^\varepsilon \times \{y \in \mathbb{C} : |y| < A^{-1}\}$ .

*Remark 1.6.* By the same discussion with the proof of Proposition 1.6, we can show that the transformation series  $x(\tilde{x}, \eta)$  satisfies (1.95) when  $x_0^{-1}(D_r^0)$  runs into some double pole  $b_1$  of  $Q(\tilde{x})$ , i.e.,  $Q(\tilde{x})$  has the following expansion at  $\tilde{x} = b_1$ :

$$(1.98) \quad Q(\tilde{x}) = \frac{\alpha}{(\tilde{x} - b_1)^2} + \frac{\beta}{\tilde{x} - b_1} + f(\tilde{x}),$$

where  $\alpha, \beta \in \mathbb{C}$ . In fact, under the assumption, we find that

$$(1.99) \quad \int_0^{\tilde{x}} \sqrt{Q(\tilde{x})} d\tilde{x} = \sqrt{\alpha} \log(\tilde{x} - b_1) + f(\tilde{x})$$

holds around  $\tilde{x} = b_1$ , where the integral path is taken along the Stokes curve emanating from the simple turning point  $\tilde{x} = 0$  and  $f(\tilde{x})$  is multi-valued analytic function that is bounded at  $\tilde{x} = b_1$ . Therefore, from (1.38), we obtain the following estimates of  $\sqrt{Q(\tilde{x}(z))}$ : there exists positive constants  $M_1$  and  $M_2$  such that

$$(1.100) \quad M_1 \left| \exp \left[ \frac{-2}{3\sqrt{\alpha}} z^{3/2} \right] \right| \leq \left| \sqrt{Q(\tilde{x}(z))} \right| \leq M_2 \left| \exp \left[ \frac{-2}{3\sqrt{\alpha}} z^{3/2} \right] \right|$$

holds on  $D_r^2$ . Then, from (1.37), we find that  $dx_0/d\tilde{x}$  satisfies

$$(1.101) \quad \frac{M_1}{\sqrt{|z|}} \left| \exp \left[ \frac{-2}{3\sqrt{\alpha}} z^{3/2} \right] \right| \leq \left| \frac{dx_0}{d\tilde{x}} \right| \leq \frac{M_2}{\sqrt{|z|}} \left| \exp \left[ \frac{-2}{3\sqrt{\alpha}} z^{3/2} \right] \right|$$

on  $D_r^2$ . Since we can take some positive constant  $M$  so that

$$(1.102) \quad \left| \left( z + \frac{e^{i\theta}\delta}{\sqrt{2|z|}} \right)^{3/2} - z^{3/2} \right| \leq M$$

holds for  $\theta \in \mathbb{R}$ , sufficiently small  $\delta > 0$  and  $z \in D_r^2$ , by the same way with the derivation of (1.63) and (1.64), we obtain

$$(1.103) \quad \left| \left( \frac{d\tilde{x}}{dz}(z) \right)^2 \frac{d^j}{dz^j} \left( \frac{d\tilde{x}}{dz}(z) \right)^{-2} \right| \leq \frac{j!}{2\pi} 2^{-1/2} e^M M_1^{-2} M_2^2 |z|^{j/2} \delta^{-j}$$

for  $z \in D_{r-\delta}^2$ . Therefore the estimates (1.95) follows from exactly the same discussion with the proof of Proposition 1.6.

### 1.3 Borel summability of transformation series

Now we show the Borel summability of transformation series. For simplicity, we discuss in  $z$  variable as in the proof of Proposition 1.6 and we assume that

$$(1.104) \quad x_0^{-1}(e^{2(j-1)\pi i/3} \mathbb{R}_{\geq 0}) = T_j \quad (j = 1, 2, 3).$$

We take  $\varepsilon > 0$  so that  $\widehat{V}^\varepsilon$  is contained in  $x_0(\widehat{U}^\varepsilon)$  and (1.95) holds there. Let  $\widehat{V}_j^\varepsilon$  ( $j = 1, 2, 3$ ) denote

$$(1.105) \quad \widehat{V}_j^\varepsilon = \{z \in \mathbb{C} : \operatorname{Re}(e^{-2(j-1)\pi i/3} z) > 0\} \cap \widehat{V}^\varepsilon$$

and  $p_j (\geq 3)$  ( $j = 1, 2, 3$ ) be the orders of poles of  $Q(\tilde{x})$  at  $b_j$ . Then, from [KoS1] (and also [DLS]), we immediately find the following

**Theorem 1.8** ([KoS1]). *There exist some positive constants  $C_1, C_2$  and  $\delta$  such that*

$$(1.106) \quad |\tilde{R}_B(z, y)| \leq C_1 |z|^{-3(p_j-4)/2(p_j-2)} \exp[C_2 |y|],$$

$$(1.107) \quad |R_B(z, y)| \leq C_1 |z|^{-5/2} \exp[C_2 |y|]$$

hold on  $(\widehat{V}_j^{2\varepsilon/3} \setminus \widehat{V}_j^{\varepsilon/3}) \times E_\delta^+$  ( $j = 1, 2, 3$ ), where  $\tilde{R}_B$  and  $R_B$  are the Borel transform of  $\tilde{R} = \eta^{-1} \tilde{S}_{\text{odd}}(\tilde{x}(z), \eta) - S_{-1}(\tilde{x}(z))$  and  $R = \eta^{-1} S_{\text{odd}}(z, \eta) - S_{-1}(z)$  respectively and

$$(1.108) \quad E_\delta^+ = \bigcup_{s \geq 0} \{y \in \mathbb{C} : |y - s| \leq \delta\}.$$

Now we apply Theorem A.1 to

$$(1.109) \quad F(z, X, \eta) = \int_0^{z+X} \eta^{-1} S_{\text{odd}}(z, \eta) dz - \int_0^{\tilde{x}(z)} \eta^{-1} \tilde{S}_{\text{odd}}(\tilde{x}, \eta) d\tilde{x}$$

in  $X$  variable. Indeed, (1.23) guarantees that the transformation series  $x(\tilde{x}, \eta)$  satisfies

$$(1.110) \quad \int_0^x \eta^{-1} S_{\text{odd}}(x, \eta) dx \Big|_{x=x(z, \eta)} - \int_0^{\tilde{x}(z)} \eta^{-1} \tilde{S}_{\text{odd}}(\tilde{x}, \eta) d\tilde{x} = 0$$

and the Borel summability of  $X(z, \eta) = x(\tilde{x}(z), \eta) - z$  on a neighborhood  $W^\varepsilon$  of  $\partial\widehat{V}^{\varepsilon/2}$  follows from that of  $F(z, X, \eta)$ . Hence our task is to confirm that  $F(z, X, \eta)$  satisfies the conditions corresponding to (A.1), (A.2) and (A.3). First, bearing the shape of  $\widehat{V}^\varepsilon$  in mind, we easily see that we can take some positive constant  $r$  so that  $z + X \in \widehat{V}^{2\varepsilon/3} \setminus \widehat{V}^{\varepsilon/3}$  for  $(z, X)$  in

$$(1.111) \quad D_r^\varepsilon = \left\{ (z, X) \in W^\varepsilon \times \mathbb{C} : |X| \leq \frac{r}{\sqrt{|z|}} \right\}$$

and the coefficients of  $F(z, X, \eta)$  are holomorphic there. Since  $\tilde{S}_{-1}(\tilde{x})$  and  $S_{-1}(z) = \sqrt{z}$  satisfy (1.21), we find that

$$(1.112) \quad F_0(z, X) = \frac{2}{3} ((z + X)^{3/2} - z^{3/2}).$$

Therefore we immediately find that  $F_0(z, 0) = 0$  and, by taking  $r$  sufficiently small, we can take some positive constant  $M$  so that  $F_0$  satisfies

$$(1.113) \quad \left| \frac{F_0(z, X)}{X} \right| \geq M\sqrt{|z|}$$

for  $(z, X) \in D_r^\varepsilon$ . Finally, the Borel summability of  $\tilde{F} = F - F_0$  on  $D_r^\varepsilon$  is derived from (1.106) and (1.107) as follows: we first remind us that the integration in (1.109) is defined by a contour integral around  $z = 0$ . Let  $z + X$  be in  $\partial\widehat{V}^{\varepsilon_0}$  with  $\varepsilon/3 < \varepsilon_0 < 2\varepsilon/3$ . Since (1.107) guarantees the integrability of  $R_B$  at infinity, by deforming the contour along  $\partial\widehat{V}^{\varepsilon_0}$ , we find that the following estimates holds on  $D_r^\varepsilon \times E_\delta^+$ : there exists some positive constants  $C_1$  and  $C_2$  such that

$$(1.114) \quad \left| \int_0^{z+X} R_B(z, y) dz \right| \leq C_1 \exp[C_2|y|].$$

Further, since  $dx_0/d\tilde{x}$  satisfies (1.53), we find that, for some positive constants  $C_1$  and  $C_2$ ,

$$(1.115) \quad \left| \tilde{R}_B(z, y) \frac{d\tilde{x}}{dz} \right| \leq C_1 |z|^{-5/2} \exp[C_2|y|]$$

holds on  $W^\varepsilon \times E_\delta^+$  and, by the same discussion as in the derivation of (1.114), we obtain

$$(1.116) \quad \left| \int_0^{\tilde{x}(z)} \tilde{R}_B(\tilde{x}, y) d\tilde{x} \right| \leq C_1 \exp[C_2|y|]$$

for  $(z, y) \in W^\varepsilon \times E_\delta$ .

In conclusion, applying Theorem A.1 to  $F(z, X, \eta)$ , we find that  $X(z, \eta)$  is Borel summable on  $W^\varepsilon$ . More precisely, from (1.113), (1.114) and (1.116), we obtain the following estimates, which is corresponding to (A.11), of the Borel transform  $X_B$  of  $X$  on  $W^\varepsilon \times E_\delta^+$ :

$$(1.117) \quad |X_B(z, y)| \leq \frac{C_1}{M\sqrt{|z|}} \exp \left[ \left( \frac{4C_1}{Mr} + C_2 \right) |y| \right].$$

Finally, combining (1.95) and (1.117), we validate the Borel summability of  $X(z, \eta)$  on  $\widehat{V}^{\varepsilon/2}$ . First, from Cauchy's formula, we obtain the following integral representation of  $X_B(z, y)$  in a neighborhood of  $(z, y) = (0, 0)$ :

$$(1.118) \quad X_B(z, y) = \frac{(z+1)^2}{2\pi i} \oint_{|\tilde{z}-z|=\varepsilon_0} \frac{1}{(\tilde{z}+1)^2} \frac{X_B(\tilde{z}, y)}{\tilde{z}-z} d\tilde{z}.$$

Here  $\varepsilon_0 > 0$  is taken so that the integral path is contained in  $\widehat{V}^{\varepsilon/2}$  and we assume that, by taking  $\varepsilon$  sufficiently small,  $z = -1$  is not contained in  $\widehat{V}^\varepsilon$ . Then we deform the integral path along  $\partial\widehat{V}^{\varepsilon/2}$  and find that  $X_B(z, y)$  is holomorphic on  $\widehat{V}^{\varepsilon/2} \times \{y \in \mathbb{C} : |y| < 1/4\}$ . In fact, (1.95) guarantees that the integrand of (1.118) is integrable along  $\partial\widehat{V}^{\varepsilon/2}$  and holomorphic in a fixed neighborhood of the origin under the deformation of the integral path, i.e., this integral representation gives the analytic continuation of  $X_B(z, y)$ . Then this integral representation tells us that  $X(z, \eta)$  is Borel summable on  $\widehat{V}^{\varepsilon/2}$ . Further, since  $x(\tilde{x}, \eta)$  satisfies (1.11), we find that  $X_B(z, y)$  satisfies

$$(1.119) \quad X_B(z, -y) = -X_B(z, y).$$

In conclusion, we obtain the following

**Theorem 1.9.** *There exist positive constants  $C_1, C_2, \delta$  and  $\varepsilon$  such that  $X_B(z, y)$  is holomorphic on  $\widehat{V}^\varepsilon \times E_\delta$  and satisfies the following estimates there:*

$$(1.120) \quad |X_B(z, y)| \leq C_1(|z| + 1)^2 \exp[C_2|y|].$$

Here

$$(1.121) \quad E_\delta = \bigcup_{s \in \mathbb{R}} \{y \in \mathbb{C} : |y - s| \leq \delta\}.$$

*Remark 1.7.* We can also show the Borel summability of the transformation series with a minor change of discussion when the Stokes curves run into some double poles of  $Q$ . For simplicity, we consider the case that  $x_0^{-1}(\widehat{V}_1^\varepsilon)$  runs into a double pole  $b_1$  and  $\widehat{V}_j^\varepsilon$  ( $j = 2, 3$ ) run into irregular singular points  $b_j$  ( $j = 2, 3$ ) respectively. Then, instead of (1.116), we find from [KoSl] that

$$(1.122) \quad \left| \int_0^{\tilde{x}(z)} \tilde{R}_B(\tilde{x}, y) d\tilde{x} \right| \leq C_1|z|^{3/2} \exp[C_2|y|]$$

holds for  $(z, y) \in (W^\varepsilon \cap \widehat{V}_1^\varepsilon) \times E_\delta^+$ . Therefore, applying Theorem A.1 to  $F(z, X, \eta)$ , we obtain the following estimates on  $(W^\varepsilon \cap \widehat{V}_1^\varepsilon) \times E_\delta^+$ :

$$(1.123) \quad |X_B(z, y)| \leq \frac{C_1|z|}{M} \exp \left[ \left( \frac{4C_1|z|^{3/2}}{Mr} + C_2 \right) |y| \right].$$

Similarly, we find that (1.117) holds on  $(W^\varepsilon \cap \widehat{V}_j^\varepsilon) \times E_\delta^+$  ( $j = 2, 3$ ). In conclusion, applying the same technique as in the proof of Theorem 1.9 to the integral representation

$$(1.124) \quad X_B(z, y) = \frac{\exp[C_3(z+1)^{3/2}y]}{2\pi i} \times \oint_{|\tilde{z}-z|=\varepsilon_0} \frac{\exp[-C_3(\tilde{z}+1)^{3/2}y] X_B(\tilde{z}, y)}{\tilde{z} - z} d\tilde{z},$$

where a positive constant  $C_3$  is taken as

$$(1.125) \quad C_3 > \frac{4C_1}{Mr},$$

we find the following

**Theorem 1.10.** *There exist positive constants  $C_1, C_2, C_3, \delta$  and  $\varepsilon$  such that  $X_B(z, y)$  is holomorphic on  $\widehat{V}^\varepsilon \times E_\delta$  and satisfies the following estimates there:*

$$(1.126) \quad |X_B(z, y)| \leq C_1 \exp \left[ (C_2 + C_3 |z|^{3/2}) |y| \right].$$

*Remark 1.8.* It is difficult to derive the Borel summability of  $X(z, \eta)$  from (1.14) directly. Therefore we appealed to the implicit function theorem for Borel summable series. However, since  $S_{\text{odd}}$  and  $\tilde{S}_{\text{odd}}$  are not Borel summable on the Stokes curves, we can not show the Borel summability of  $X(z, \eta)$  there only by the implicit function theorem. Hence we used a kind of Hartogs' phenomenon to extend the region where  $X(z, \eta)$  is Borel summable. We can find a similar discussion in [D1]. There, the convergence of inverse factorial series solution on a neighborhood of a simple turning point was examined. And he used the maximum modulus theorem on the set like  $W^\varepsilon$  to avoid a direct discussion at the simple turning point. This similarity of the discussion was suggested by Professor R. Schäfke. In [D2], we can also find a similar discussion used in Section 3.3.

## 2 WKB theoretic transformation — a simple pole case

The main purpose of this section is to show the Borel summability of transformation series, which is given in [Ko1], of (2.2) to the WKB theoretic canonical equation (2.7) near a Stokes curve emanating from a simple pole of  $Q(\tilde{x})$  when it runs into some irregular singular points. (See Remark 3.1 and Remark 3.2 in the case that it runs into a double pole of  $Q(\tilde{x})$ .) Discussions in this section proceed in the same way as in Section 2.

### 2.1 Fundamental properties of WKB theoretic transformation and its application

Let  $Q(\tilde{x})$  be a rational function that has a simple pole at the origin, i.e.,  $\tilde{x}Q(\tilde{x})$  is holomorphic at  $\tilde{x} = 0$  and satisfies

$$(2.1) \quad \tilde{x}Q(\tilde{x})|_{\tilde{x}=0} \neq 0.$$

Then we consider the Schrödinger equation

$$(2.2) \quad \left( \frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x}) \right) \tilde{\psi}(\tilde{x}, \eta) = 0.$$

with the following geometric assumptions (2.3) and (2.4): at first, we assume that

(2.3) a Stokes curve  $T$  emanating from  $\tilde{x} = 0$  runs into an irregular singular point  $b$ .

Let  $\widehat{U}_\pm^{\tilde{\varepsilon}}$  be unions of integral curves of  $\text{Im}\sqrt{Q(\tilde{x})}d\tilde{x} = 0$  that pass through some  $\tilde{x}_0 \in U^{\tilde{\varepsilon}} \setminus T$  and  $\pm \text{Re} \int_{\tilde{x}_0}^{\tilde{x}} \sqrt{Q(\tilde{x})}d\tilde{x} \geq 0$  there. Here  $U^{\tilde{\varepsilon}}$  denotes a disk  $U^{\tilde{\varepsilon}} = \{\tilde{x} \in \mathbb{C} : |\tilde{x}| < \tilde{\varepsilon}\}$  and  $\tilde{\varepsilon}$  is a sufficiently small positive constant. Then we assume that we can take  $\tilde{\varepsilon} > 0$  so that

(2.4)  $\widehat{U}_+^{\tilde{\varepsilon}}$  and  $\widehat{U}_-^{\tilde{\varepsilon}}$  run into  $b$ .

We remark here that (2.3) and (2.4) guarantee that

$$(2.5) \quad \widehat{U}^{\tilde{\varepsilon}} = \widehat{U}_+^{\tilde{\varepsilon}} \cup \widehat{U}_-^{\tilde{\varepsilon}} \cup T$$

does not contain any poles nor turning points except for a simple pole at the origin.

Now, we consider a transformation series

$$(2.6) \quad x(\tilde{x}, \eta) = \sum_{k=0}^{\infty} x_k(\tilde{x}) \eta^{-k}$$

of (2.1) to the following canonical equation on  $\widehat{U}^{\tilde{\varepsilon}}$ :

$$(2.7) \quad \left( \frac{d^2}{dx^2} - \eta^2 \frac{1}{x} \right) \psi = 0.$$

In parallel with Theorem 1.1, we have the following

**Theorem 2.1.** *Let  $Q(\tilde{x})$  be a rational function that satisfies (2.1), (2.3) and (2.4). Then there exists a Borel summable series  $x(\tilde{x}, \eta)$  on  $\widehat{U}^{\tilde{\varepsilon}}$  such that*

$$(2.8) \quad \{x_k(\tilde{x})\}_{k=0}^{\infty} \text{ are holomorphic on } \widehat{U}^{\tilde{\varepsilon}},$$



(2.9)  $x_{2k+1}(\tilde{x})$  ( $k = 0, 1, 2, \dots$ ) are identically zero,

$$(2.10) \quad x_0(0) = 0,$$

$$(2.11) \quad \frac{dx_0}{d\tilde{x}} \neq 0 \text{ on } \widehat{U}^\varepsilon$$

and satisfies the following relation:

$$(2.12) \quad Q(\tilde{x}) = \left( \frac{dx(\tilde{x}, \eta)}{d\tilde{x}} \right)^2 \frac{1}{x(\tilde{x}, \eta)} - \frac{1}{2} \eta^{-2} \{x(\tilde{x}, \eta); \tilde{x}\}.$$

The proof of Theorem 2.1 is given in Section 3.2 and Section 3.3.

Then  $x(\tilde{x}, \eta)$  gives the following relations (See [Kol]):

**Theorem 2.2.** *Let  $\tilde{S}(\tilde{x}, \eta)$  and  $S(x, \eta)$  respectively be solutions of Riccati equations*

$$(2.13) \quad \tilde{S}^2 + \frac{d\tilde{S}}{d\tilde{x}} = \eta^2 Q(\tilde{x})$$

and

$$(2.14) \quad S^2 + \frac{dS}{dx} = \eta^2 \frac{1}{x},$$

where  $\tilde{S}_{-1}(\tilde{x})$  and  $S_{-1}(x)$  are taken so that they satisfy

$$(2.15) \quad \tilde{S}_{-1}(\tilde{x}) = \left( \frac{dx_0}{d\tilde{x}} \right) S_{-1}(x_0(\tilde{x})).$$

Then  $x(\tilde{x}, \eta)$  in Theorem 2.1 satisfies the following relation:

$$(2.16) \quad \tilde{S}(\tilde{x}, \eta) = \left( \frac{dx}{d\tilde{x}} \right) S(x(\tilde{x}, \eta), \eta) - \frac{1}{2} \left( \frac{d^2x}{d\tilde{x}^2} \right) / \left( \frac{dx}{d\tilde{x}} \right).$$

**Corollary 2.3.** *Let  $\tilde{S}_{\text{odd}}$  and  $S_{\text{odd}}$  respectively be the odd part of  $\tilde{S}$  and  $S$ . And assume that  $\tilde{S}_{-1}$  and  $S_{-1}$  are taken so that they satisfy (2.15). Then the following relation holds:*

$$(2.17) \quad \tilde{S}_{\text{odd}}(\tilde{x}, \eta) = \left( \frac{dx(\tilde{x}, \eta)}{d\tilde{x}} \right) S_{\text{odd}}(x(\tilde{x}, \eta), \eta).$$

Now we consider WKB solutions  $\tilde{\psi}_{\pm}(\tilde{x}, \eta)$  of (2.2) normalized at a simple pole at the origin, i.e.,

$$(2.18) \quad \tilde{\psi}_{\pm}(\tilde{x}, \eta) = \frac{1}{\sqrt{\tilde{S}_{\text{odd}}}} \exp\left(\pm \int_0^{\tilde{x}} \tilde{S}_{\text{odd}}(\tilde{x}, \eta) d\tilde{x}\right)$$

and WKB solutions  $\psi_{\pm}(x, \eta)$  of (2.7) normalized at a simple pole in the same manner. Then they satisfy the following

**Theorem 2.4.** *Let  $\tilde{\psi}_{\pm}(\tilde{x}, \eta)$  and  $\psi_{\pm}(x, \eta)$  respectively be WKB solutions of (2.2) and (2.7) normalized at their simple pole at the origin. Then the following relation holds:*

$$(2.19) \quad \tilde{\psi}_{\pm}(\tilde{x}, \eta) = \left(\frac{dx(\tilde{x}, \eta)}{d\tilde{x}}\right)^{-1/2} \psi_{\pm}(x(\tilde{x}, \eta), \eta).$$

In order to simplify the discussion, we employ  $x_0 = x_0(\tilde{x})$  as a new coordinate variable. Then, applying the Borel transform to the relation (2.19), we find that the Borel transform  $\tilde{\psi}_{\pm,B}(\tilde{x}, y)$  of  $\tilde{\psi}_{\pm}(\tilde{x}, \eta)$  can be described by  $\psi_{\pm,B}(x, y)$  through the action of the following microdifferential operator  $\mathcal{X}$ :

$$(2.20) \quad \mathcal{X} =: \left(\frac{\partial \tilde{x}}{\partial x_0}\right)^{1/2} \left(1 + \frac{\partial X}{\partial x_0}\right)^{-1/2} \exp[X(x_0, \eta)\xi] :,$$

where  $X(x_0, \eta)$  is

$$(2.21) \quad X(x_0, \eta) = x(\tilde{x}(x_0), \eta) - x_0$$

and  $\xi$  designates the symbol of  $\partial_{x_0}$ . By the same reasoning as in Section 2.1, we obtain the following

**Theorem 2.5.**  *$\tilde{\psi}_{\pm,B}$  and  $\psi_{\pm,B}$  satisfy the following relation on  $\widehat{V}^{\varepsilon} \times E_{\pm y_0}^{\delta}$  for sufficiently small  $\varepsilon, \delta > 0$ :*

$$(2.22) \quad \begin{aligned} \tilde{\psi}_{\pm,B}(\tilde{x}(x_0), y) &= \left(\frac{\partial \tilde{x}}{\partial x_0}\right)^{1/2} \psi_{\pm,B}(x_0, y) \\ &\quad + \int_{\mp y_0}^y K(x_0, y - y', \partial_{x_0}) \psi_{\pm,B}(x_0, y') dy', \end{aligned}$$

where

$$(2.23) \quad y_0(x_0) = \int_0^{x_0} \frac{1}{\sqrt{x_0}} dx_0,$$

$$(2.24) \quad \widehat{V}^\varepsilon = \{x_0 \in \mathbb{C} : |\operatorname{Im} y_0(x_0)| < \varepsilon\},$$

$$(2.25) \quad E_{\pm y_0}^\delta = \bigcup_{s \in \mathbb{R}} \{y \in \mathbb{C} : |y - s \pm y_0(x_0)| < \delta\}.$$

and  $K(x, y, \partial_x)$  is a differential operator of infinite order on  $\widehat{V}^\varepsilon \times E_{\pm y_0}^\delta$ .

*Remark 2.1.* It follows from the construction of  $x(\tilde{x}, \eta)$  given in the proof of Proposition 2.6 (cf. (2.33)) that  $x_0(\tilde{x})$  satisfies the following relation:

$$(2.26) \quad y_0(x_0(\tilde{x})) = \int_0^{\tilde{x}} \sqrt{Q(\tilde{x})} d\tilde{x}.$$

As is given in [Ko1], the Borel transforms of WKB solutions of the canonical equation (2.7) are explicitly written as follows:

$$(2.27) \quad \begin{cases} \psi_{+,B}(x, y) = \frac{1}{\sqrt{4\pi}} s^{-1/2} F\left(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}; s\right), \\ \psi_{-,B}(x, y) = \frac{1}{\sqrt{-4\pi}} (1-s)^{-1/2} F\left(-\frac{1}{2}, \frac{3}{2}, \frac{1}{2}; 1-s\right), \end{cases}$$

where

$$(2.28) \quad s = \frac{y}{4x^{1/2}} + \frac{1}{2}.$$

Therefore we conclude from Theorem 2.5 that  $\tilde{\psi}_{+,B}(\tilde{x}(x_0), y)$  (resp.,  $\tilde{\psi}_{-,B}(\tilde{x}(x_0), y)$ ) can be analytically continued into the universal covering of  $(\widehat{V}^\varepsilon \setminus \{0\}) \times (E_{-y_0}^\delta \setminus \{\pm y_0(x_0)\})$  (resp.,  $(\widehat{V}^\varepsilon \setminus \{0\}) \times (E_{+y_0}^\delta \setminus \{\pm y_0(x_0)\})$ ). Therefore, by the same reasoning with Section 2.2, the proof of the connection formula in [Ko1] from a viewpoint of transformation theory completes.

## 2.2 Uniform Borel transformability of transformation series

The purpose of Section 3.2 is to verify the following

**Proposition 2.6.** *Let  $Q(\tilde{x})$  be a rational function that satisfies (2.1), (2.3) and (2.4). Then there exist  $\tilde{\varepsilon} > 0$  and formal series  $x(\tilde{x}, \eta) = x_0(\tilde{x}) + \eta^{-1}x_1(\tilde{x}) + \cdots$  that satisfies (2.8)  $\sim$  (2.12) and the following estimates: there exist positive constants  $C_0$  and  $A$  such that for all  $n \geq 1$  and  $\tilde{x} \in \widehat{U}^{\tilde{\varepsilon}}$ ,  $x_n(\tilde{x})$  satisfies*

$$(2.29) \quad |x_n(\tilde{x})| \leq (|x_0(\tilde{x})| + 1)C_0 n! A^n.$$

*Proof.* Proof of Proposition 2.6 proceeds in the same process as in the proof of Proposition 1.9.

We first review the construction of  $x(\tilde{x}, \eta)$ . We inductively determine  $x_k(\tilde{x})$  ( $k = 0, 1, 2, \dots$ ) by comparing the coefficients of  $\eta^{-k}$  of (2.12). First, by comparing the coefficients of  $\eta^0$  of (2.12), we find that  $x_0(\tilde{x})$  should satisfy the following relation:

$$(2.30) \quad Q(\tilde{x}) = \left( \frac{dx_0(\tilde{x})}{d\tilde{x}} \right)^2 \frac{1}{x_0(\tilde{x})}.$$

Then we determine  $x_0(\tilde{x})$  so that it becomes holomorphic at the origin as follows:

$$(2.31) \quad x_0(\tilde{x}) = \left( \frac{1}{2} \int_0^{\tilde{x}} \sqrt{Q(\tilde{x})} d\tilde{x} \right)^2.$$

From the assumptions (2.1), (2.3) and (2.4), we immediately find that  $x_0(\tilde{x})$  satisfies (2.8) and (2.10). Further, by the same reasoning as in the proof of Proposition 1.9, we see that  $x_0$  maps  $\widehat{U}^{\tilde{\varepsilon}}$  to  $x_0(\widehat{U}^{\tilde{\varepsilon}})$  bijectively and satisfies (2.11). From now, we employ  $z = x_0(\tilde{x})$  as a new coordinate variable on  $x_0(\widehat{U}^{\tilde{\varepsilon}})$ .

Next we determine  $x_k$  ( $k \geq 1$ ). By comparing the coefficients of  $\eta^{-k}$  of (2.12), we find that  $x_k$  should satisfy the following recurrence relation:

$$(2.32) \quad 2z \frac{dx_k}{dz} - x_k = z\Phi_k(z),$$

where  $\Phi_k(z)$  is

(2.33)

$$\begin{aligned}
\Phi_k(z) = & - \sum_{l=2}^k \frac{(-1)^l}{z^l} \sum_{\substack{\mu_1+\dots+\mu_l=k, \\ \mu_1, \dots, \mu_l \geq 1}}^* x_{\mu_1} \cdots x_{\mu_l} \\
& - \sum_{\substack{k_1+k_2+k_3=k, \\ k_1, k_2, k_3 \leq k-1}} \frac{dx_{k_1}}{dz} \frac{dx_{k_2}}{dz} \sum_{l=\min\{1, k_3\}}^{k_3} \frac{(-1)^l}{z^l} \sum_{\substack{\mu_1+\dots+\mu_l=k_3, \\ \mu_1, \dots, \mu_l \geq 1}}^* x_{\mu_1} \cdots x_{\mu_l} \\
& + \frac{z}{2} \sum_{k_1+k_2=k-2} \left( \frac{d\tilde{x}}{dz} \right)^3 \frac{d^3 x_{k_1}}{d\tilde{x}^3} \\
& \quad \times \sum_{l=\min\{1, k_2\}}^{k_2} (-1)^l \sum_{\substack{\mu_1+\dots+\mu_l=k_2, \\ \mu_1, \dots, \mu_l \geq 1}}^* \frac{dx_{\mu_1}}{dz} \cdots \frac{dx_{\mu_l}}{dz} \\
& - \frac{3z}{4} \sum_{k_1+k_2+k_3=k-2} \left( \frac{d\tilde{x}}{dz} \right)^4 \frac{d^2 x_{k_1}}{d\tilde{x}^2} \frac{d^2 x_{k_2}}{d\tilde{x}^2} \\
& \quad \times \sum_{l=\min\{1, k_3\}}^{k_3} (-1)^l (l+1) \sum_{\substack{\mu_1+\dots+\mu_l=k_3, \\ \mu_1, \dots, \mu_l \geq 1}}^* \frac{dx_{\mu_1}}{dz} \cdots \frac{dx_{\mu_l}}{dz}.
\end{aligned}$$

Since  $\Phi_k$  does not contain  $x_n$  ( $n \geq k$ ), we inductively determine  $x_k$  by

$$(2.34) \quad x_k(z) = \frac{z^{1/2}}{2} \int_0^z z^{-1/2} \Phi_k(z) dz$$

so that  $x_k$  is holomorphic at  $z = 0$  and satisfies (2.33). We easily see that we can inductively confirm that  $\Phi_{2k+1}$  ( $k \geq 0$ ) identically vanish and hence (2.9) holds.

Next we validate the estimates that  $x_k$  ( $k = 1, 2, \dots$ ) satisfy. We first set

$$(2.35) \quad D_r^1 = \bigcup_{0 \leq s \leq 1} \{z \in \mathbb{C} : |z - s| \leq r\}$$

$$(2.36) \quad D_r^2 = \bigcup_{s \geq 1} \{z \in \mathbb{C} : |z - s| \leq r\sqrt{s}\}$$

for sufficiently small  $r > 0$ . Then, taking account of the fact that  $2\text{Im}z^{1/2}$  is expanded to

$$(2.37) \quad \frac{\text{Im}z}{(\text{Re}z)^{1/2}} - \frac{1}{8} \frac{(\text{Im}z)^3}{(\text{Re}z)^{5/2}} + \dots$$

on  $D_r^2$ , we find that  $\text{Im} \int_0^z z^{-1/2} dz$  behaves as  $\text{Im}z \cdot (\text{Re}z)^{-1/2}$  for sufficiently large  $\text{Re}z$ . Therefore, by taking  $r > 0$  sufficiently small, we can assume that  $D_r^1 \cup D_r^2$  is included in  $x_0(\widehat{U}^\varepsilon)$ .

Now we inductively confirm that  $x_k(z)$  ( $k \geq 1$ ) satisfy the following estimates: there exist positive constants  $C_0 < 1$  and  $A > 1$  such that, for all  $\delta$  with  $0 < \delta < r/3$  and  $z \in D_{r-\delta}^1 \cup D_{r-\delta}^2$ ,

$$(2.38) \quad |x_k(z)| \leq |z| C_0 k! \delta^{-k} A^k$$

$$(2.39) \quad \left| \frac{dx_k}{dz}(z) \right| \leq C_0 k! \delta^{-k} A^k$$

hold. Since  $x_1$  is identically zero, it is obvious that (2.38) and (2.39) hold when  $k = 1$ . Therefore our task is to show that  $x_k$  satisfies (2.38) and (2.39) under the assumption that  $x_n$  ( $1 \leq n \leq k-1$ ) satisfy them.

We first note the behavior of  $x_0(\tilde{x})$  when  $\tilde{x}$  tends to the irregular singular point  $b$  along the Stokes curve  $T$ . Let  $Q(\tilde{x})$  have a pole of order  $p(\geq 3)$  at  $\tilde{x} = b$ . Then (2.30) and (2.31) tells us that  $x_0(\tilde{x})$  and  $dx_0/d\tilde{x}$  behave as

$$(2.40) \quad x_0(\tilde{x}) = O((\tilde{x} - b)^{-p+2}),$$

$$(2.41) \quad \frac{dx_0}{d\tilde{x}}(\tilde{x}) = O((\tilde{x} - b)^{-p+1})$$

when  $\tilde{x}$  tends to  $b$ . Therefore  $dx_0/d\tilde{x}$  can be estimated by  $x_0$  as follows: there exist positive constants  $M_1$  and  $M_2$  such that

$$(2.42) \quad M_1 |x_0|^{(p-1)/(p-2)} \leq \left| \frac{dx_0}{d\tilde{x}} \right| \leq M_2 |x_0|^{(p-1)/(p-2)}$$

holds on  $D_r^2$ .

Next we derive the estimates that  $\Phi_k$  satisfies. Precisely we show the following estimates: there exists some positive constant  $M$  that is

independent of  $C_0(< 1)$  and  $A$  such that, for  $k \geq 2$ ,  $0 < \delta < r/3$  and  $z \in D_{r-\delta}^1 \cup D_{r-\delta}^2$ ,

$$(2.43) \quad |\Phi_k(z)| \leq M(C_0^2 + A^{-2})k!\delta^{-k}A^k$$

holds. Since (2.43) is obtained for  $z \in D_{r-\delta}^1$  by the same discussion given in [Ko1], we prove (2.43) only for  $z \in D_{r-\delta}^2$  here. From Lemma 1.7 and (2.38), we find that the first term of (2.33) is estimated as follows:

$$(2.44) \quad \begin{aligned} & \left| \sum_{l=2}^k \frac{(-1)^l}{z^l} \sum_{\substack{\mu_1+\dots+\mu_l=k, \\ \mu_1, \dots, \mu_l \geq 1}}^* x_{\mu_1} \cdots x_{\mu_l} \right| \\ & \leq \sum_{l=2}^k \frac{1}{|z|^l} \sum_{\substack{\mu_1+\dots+\mu_l=k, \\ \mu_1, \dots, \mu_l \geq 1}}^* |z|^l C_0^l \mu_1! \cdots \mu_l! \delta^{-k} A^k \\ & \leq 4C_0^2 k! \delta^{-k} A^k \sum_{l=2}^k \frac{C_0^{l-2} 4^{l-2}}{(l-1)!} \\ & \leq 4e^{4C_0} C_0^2 k! \delta^{-k} A^k. \end{aligned}$$

Similarly we obtain the following estimates of the second term of (2.33):

$$(2.45) \quad \begin{aligned} & \left| \sum_{\substack{k_1+k_2+k_3=k, \\ k_1, k_2, k_3 \leq k-1}} \frac{dx_{k_1}}{dz} \frac{dx_{k_2}}{dz} \sum_{l=\min\{1, k_3\}}^{k_3} \frac{(-1)^l}{z^l} \sum_{\substack{\mu_1+\dots+\mu_l=k_3, \\ \mu_1, \dots, \mu_l \geq 1}}^* x_{\mu_1} \cdots x_{\mu_l} \right| \\ & \leq C_0^2 (1 + e^{4C_0}) \left( \frac{4^2}{k-1} + 12 \right) (k-1)! \delta^{-k} A^k. \end{aligned}$$

Now we examine the third term of (2.33). We first estimate the most important factor

$$(2.46) \quad \begin{aligned} \left( \frac{d\tilde{x}}{dz} \right)^3 \frac{d^3 x_{k_1}}{d\tilde{x}^3} &= \frac{d^3 x_{k_1}}{dz^3} + \left( \frac{d\tilde{x}}{dz} \right)^3 \frac{d}{dz} \left( \frac{d\tilde{x}}{dz} \right)^{-3} \frac{d^2 x_{k_1}}{dz^2} \\ &\quad + \frac{1}{2} \left( \frac{d\tilde{x}}{dz} \right)^2 \frac{d^2}{dz^2} \left( \frac{d\tilde{x}}{dz} \right)^{-2} \frac{dx_{k_1}}{dz}. \end{aligned}$$

We consider the first term of (2.46). We use the following representation of the derivative of  $dx_{k_3}/dz$  on  $D_{r-\delta}^2$ :

$$(2.47) \quad \frac{d^j}{dz^j} \frac{dx_{k_3}}{dz}(z) = \frac{j!}{2\pi i} \int_{|\tilde{z}-z|=d} \frac{1}{(\tilde{z}-z)^{j+1}} \frac{dx_{k_3}}{dz}(\tilde{z}) d\tilde{z},$$

where  $d > 0$  is taken as

$$(2.48) \quad d = \frac{\delta}{\sqrt{2}(k_1+1)} \sqrt{|z|}.$$

Then we find that the integral path of (2.47) is contained in  $D_{r-k_1\delta/(k_1+1)}^2$ . Actually, if we take  $s \geq 1$  so that  $|z-s| \leq (r-\delta)\sqrt{s}$ , we find that  $|z| \leq 2s$  holds and that  $\tilde{z}$  on the integral path satisfies

$$(2.49) \quad \begin{aligned} |\tilde{z}-s| &\leq (r-\delta)\sqrt{s} + \frac{\delta}{\sqrt{2}(k_1+1)} \sqrt{|z|} \\ &\leq (r-\delta)\sqrt{s} + \frac{\delta}{\sqrt{2}(k_1+1)} \sqrt{2s} \\ &= \sqrt{s} \left( r - \frac{k_1}{k_1+1} \delta \right). \end{aligned}$$

Therefore, substituting  $\delta$  in (2.39) for  $k = k_1$  to  $k_1\delta/(k_1+1)$ , we find that  $dx_{k_1}/dz$  satisfies the following estimates for  $\tilde{z} \in D_{r-k_1\delta/(k_1+1)}^2$ :

$$(2.50) \quad \left| \frac{dx_{k_1}}{dz}(\tilde{z}) \right| \leq k_1! e \delta^{-k_1} A^{k_1}.$$

Then, from (2.47), we obtain

$$(2.51) \quad \left| \frac{d^j}{dz^j} \frac{dx_{k_1}}{dz}(z) \right| \leq \frac{j!}{2\pi} \left( \frac{\delta \sqrt{|z|}}{\sqrt{2}(k_1+1)} \right)^{-j} e k_1! \delta^{-k_1} A^{k_1}.$$

In the same way, we find from (2.42) that the following estimates hold for  $j = 1, 2$ :

$$(2.52) \quad \left| \left( \frac{d\tilde{x}}{dz}(z) \right)^{4-j} \frac{d^j}{dz^j} \left( \frac{d\tilde{x}}{dz}(z) \right)^{-4+j} \right|$$



$$\leq \frac{2^{(4-j)(p-1)/(p-2)}}{2\pi} \left( \frac{M_2}{M_1} \right)^{4-j} \left( \frac{\sqrt{2}}{\sqrt{|z|}} \right)^j \delta^{-j}.$$

In conclusion, we obtain the following estimates: there exists some positive constant  $M$  that is independent of  $z, k_1, C_0, \delta$  and  $A$  such that

$$(2.53) \quad \left| \left( \frac{d\tilde{x}}{dz} \right)^3 \frac{d^3 x_{k_1}}{d\tilde{x}^3} \right| \leq \frac{M}{|z|} (k_1 + 2)! \delta^{-k_1-2} A^{k_1}$$

holds on  $D_{r-\delta}^2$ . Therefore, by the same calculation with (1.86), we gain the following estimates

$$(2.54) \quad \left| \frac{z}{2} \sum_{k_1+k_2=k-2} \left( \frac{d\tilde{x}}{dz} \right)^3 \frac{d^3 x_{k_1}}{d\tilde{x}^3} \sum_{l=\min\{1,k_2\}}^{k_2} (-1)^l \sum_{\substack{\mu_1+\dots+\mu_l=k_2, \\ \mu_1, \dots, \mu_l \geq 1}}^* \frac{dx_{\mu_1}}{dz} \dots \frac{dx_{\mu_l}}{dz} \right| \\ \leq \frac{M(1+C_0 e^{4C_0})}{2} \left( 1 + \frac{4}{k} \right) k! \delta^{-k} A^{k-2}.$$

Similarly, we find that the fourth term of (2.33) is dominated as follows:

$$(2.55) \quad \left| \frac{3z}{4} \sum_{k_1+k_2+k_3=k-2} \left( \frac{d\tilde{x}}{dz} \right)^4 \frac{d^2 x_{k_1}}{d\tilde{x}^2} \frac{d^2 x_{k_2}}{d\tilde{x}^2} \right. \\ \left. \times \sum_{l=\min\{1,k_3\}}^{k_3} \sum_{\substack{\mu_1+\dots+\mu_l=k_3, \\ \mu_1, \dots, \mu_l \geq 1}}^* (-1)^l (l+1) \frac{dx_{\mu_1}}{dz} \dots \frac{dx_{\mu_l}}{dz} \right| \\ \leq 9M^2(1+4C_0^2 e^{4C_0}) \left( 1 + \frac{4}{k-1} \right) (k-1)! \delta^{-k} A^{k-2}.$$

Summing up (2.44), (2.45), (2.54) and (2.55), we obtain (2.43).

It is now clear from (2.34) and (2.43) that  $x_k(z)$  satisfies

$$(2.56) \quad |x_k(z)| \leq |z| M(C_0^2 + A^{-2}) k! \delta^{-k} A^k$$

on  $D_{r-\delta}^1 \cup D_{r-\delta}^2$ . Further, combining (2.43) and (2.56), we find from (2.32) that

$$(2.57) \quad \left| \frac{dx_k}{dz}(z) \right| \leq \frac{3M}{2} (C_0^2 + A^{-2}) k! \delta^{-k} A^k$$

also holds there. Therefore, by taking  $C_0$  sufficiently small so that

$$(2.58) \quad \frac{3M}{2}C_0 < \frac{1}{2}$$

and then  $A$  sufficiently large so that

$$(2.59) \quad \frac{3M}{2}A^{-2} < \frac{1}{2}C_0,$$

we find that  $x_k$  satisfies (2.38) and (2.39), and hence the induction proceeds. Now, fixing  $\delta = r/6$ , we obtain the following estimates from (2.38): there exist positive constants  $r$  and  $A$  such that

$$(2.60) \quad |x_k(z)| \leq |z|k!A^k$$

for  $k \geq 1$  and  $z \in D_r^1 \cup D_r^2$ .

In conclusion, by taking  $\varepsilon, \tilde{\varepsilon} > 0$  so that

$$(2.61) \quad \widehat{V}^\varepsilon = \left\{ z \in \mathbb{C} : \left| \operatorname{Im} \int_0^z \frac{1}{\sqrt{z}} dz \right| < \varepsilon \right\} \subset D_r^1 \cup D_r^2$$

and  $x_0(U^\varepsilon) \subset \widehat{V}^\varepsilon$  are satisfied, we obtain (2.29).  $\square$

*Remark 2.2.* As in Section 2.2, we can show that the transformation series  $x(\tilde{x}, \eta)$  satisfies (2.60) when  $x_0^{-1}(D_r^2)$  runs into some double pole  $b_1$  of  $Q(\tilde{x})$ . We assume that  $Q(\tilde{x})$  is expanded as (1.98). Then, from (1.99), (2.30) and (2.31), we find the following estimates holds on  $D_r^2$ : there exists positive constants  $M_1$  and  $M_2$  such that

$$(2.62) \quad M_1 \sqrt{|z|} \left| \exp \left[ \frac{-2}{\sqrt{\alpha}} z^{1/2} \right] \right| \leq \left| \frac{dx_0}{d\tilde{x}} \right| \leq M_2 \sqrt{|z|} \left| \exp \left[ \frac{-2}{\sqrt{\alpha}} z^{1/2} \right] \right|.$$

Bearing in mind that we can take some positive constant  $M$  so that

$$(2.63) \quad \left| \left( z + e^{i\theta} \delta \sqrt{|z|} \right)^{1/2} - z^{1/2} \right| \leq M$$

holds for  $\theta \in \mathbb{R}$ , sufficiently small  $\delta > 0$  and  $z \in D_r^2$ , we find from the same discussion with the proof of Proposition 2.6 that  $x(\tilde{x}, \eta)$  satisfies (2.60).

### 2.3 Borel summability of transformation series

As in the proof of Theorem 1.9, we first show the Borel summability of  $X(z, \eta) = x(z, \eta) - z$  on a neighborhood  $W^\varepsilon$  of  $\partial\widehat{V}^{\varepsilon/2}$  by applying Theorem A.1 to

$$(2.64) \quad F(z, X, \eta) = \int_0^{z+X} \eta^{-1} S_{\text{odd}}(z, \eta) dz - \int_0^{\tilde{x}(z)} \eta^{-1} \tilde{S}_{\text{odd}}(\tilde{x}, \eta) d\tilde{x}$$

in  $X$  variable. We note the following

**Theorem 2.7** ([KoS2]). *There exist some positive constants  $C_1, C_2$  and  $\delta$  such that*

$$(2.65) \quad \left| \tilde{R}_B(z, y) \right| \leq C_1 |z|^{-(p-4)/2(p-2)} \exp[C_2 |y|],$$

$$(2.66) \quad |R_B(z, y)| \leq C_1 |z|^{-3/2} \exp[C_2 |y|]$$

hold on  $(\widehat{V}^{2\varepsilon/3} \setminus \widehat{V}^{\varepsilon/3}) \times E_\delta^+$ , where  $\tilde{R}_B$  and  $R_B$  are the Borel transform of  $\tilde{R} = \eta^{-1} \tilde{S}_{\text{odd}}(\tilde{x}(z), \eta) - S_{-1}(\tilde{x}(z))$  and  $R = \eta^{-1} S_{\text{odd}}(z, \eta) - S_{-1}(z)$  respectively and

$$(2.67) \quad E_\delta^+ = \bigcup_{s \geq 0} \{y \in \mathbb{C} : |y - s| \leq \delta\}.$$

We take  $r > 0$  so that  $z + X \in \widehat{V}^{2\varepsilon/3} \setminus \widehat{V}^{\varepsilon/3}$  for  $(z, X)$  in

$$(2.68) \quad D_r^\varepsilon = \left\{ (z, X) \in W^\varepsilon \times \mathbb{C} : |X| \leq r \sqrt{|z|} \right\}.$$

Since  $\tilde{S}_{-1}(\tilde{x})$  and  $S_{-1}(z) = z^{-1/2}$  satisfy (2.15), we find that

$$(2.69) \quad F_0(z, X) = 2 \left( (z + X)^{1/2} - z^{1/2} \right).$$

Therefore we can take some positive constant  $M$  so that

$$(2.70) \quad \left| \frac{F_0(z, X)}{X} \right| \geq \frac{M}{\sqrt{|z|}}$$

holds on  $D_r^\varepsilon$ . Then, by the same discussion with Section 2.3, we find that the Borel transform  $X_B$  of  $X$  is holomorphic on  $W^\varepsilon \times E_\delta^+$  and

satisfies the following estimates: there exist some positive constants  $C_1$  and  $C_2$  such that

$$(2.71) \quad |X_B(z, y)| \leq C_1 \sqrt{|z|} \exp[C_2|y|]$$

holds on  $W^\varepsilon \times E_\delta^+$ . Hence, applying the same technique used in Section 2.3 to the integral representation

$$(2.72) \quad X_B(z, y) = \frac{(z+1)^2}{2\pi i} \oint_{|\tilde{z}-z|=\varepsilon_0} \frac{1}{(\tilde{z}+1)^2} \frac{X_B(\tilde{z}, y)}{\tilde{z}-z} d\tilde{z},$$

we finally obtain the following

**Theorem 2.8.** *There exist positive constants  $C_1, C_2, \delta$  and  $\varepsilon$  such that  $X_B(z, y)$  is holomorphic on  $\hat{V}^\varepsilon \times E_\delta$  and satisfies the following estimates there:*

$$(2.73) \quad |X_B(z, y)| \leq C_1(|z|+1)^2 \exp[C_2|y|].$$

Here

$$(2.74) \quad E_\delta = \bigcup_{s \in \mathbb{R}} \{y \in \mathbb{C} : |y-s| \leq \delta\}.$$

*Remark 2.3.* As in Section 2.3, we can also show the Borel summability of the transformation series when  $x_0^{-1}(\hat{V}^\varepsilon)$  runs into a double pole  $b$  of  $Q$ . In this case, we find that

$$(2.75) \quad \left| \int_0^{\tilde{x}(z)} \tilde{R}_B(\tilde{x}, y) d\tilde{x} \right| \leq C_1 |z|^{1/2} \exp[C_2|y|]$$

holds for  $(z, y) \in W^\varepsilon \times E_\delta^+$ . (See [KoS2].) Then, applying Theorem A.1 to  $F(z, X, \eta)$ , we obtain the following estimates on  $(z, y) \in W^\varepsilon \times E_\delta^+$ :

$$(2.76) \quad |X_B(z, y)| \leq \frac{C_1|z|}{M} \exp \left[ \left( \frac{4C_1|z|^{1/2}}{Mr} + C_2 \right) |y| \right].$$

Therefore, by the same discussion as in the proof of Theorem 1.9 to the integral representation

$$(2.77) \quad X_B(z, y) = \frac{\exp[C_3(z+1)^{1/2}y]}{2\pi i} \times \oint_{|\tilde{z}-z|=\varepsilon_0} \frac{\exp[-C_3(\tilde{z}+1)^{1/2}y] X_B(\tilde{z}, y)}{\tilde{z}-z} d\tilde{z},$$

where a positive constant  $C_3$  is taken as (1.125), we find the following

**Theorem 2.9.** *There exist positive constants  $C_1, C_2, C_3, \delta$  and  $\varepsilon$  such that  $X_B(z, y)$  is holomorphic on  $\widehat{V}^\varepsilon \times E_\delta$  and satisfies the following estimates there:*

$$(2.78) \quad |X_B(z, y)| \leq C_1 \exp [(C_2 + C_3|z|^{1/2}) |y|].$$

## A Implicit function theorem for Borel summable series

The implicit resurgent function theorem was given by Pham ([P1]). For the readers' convenience we show the implicit function theorem for Borel summable series in this appendix. Concretely we prove the following

**Theorem A.1.** *Let  $F(\alpha, \eta) = \sum_{n=0}^{\infty} \eta^{-n} F_n(\alpha)$  be formal series in  $\eta$  that satisfies*

$$(A.1)$$

*$F_k(\alpha)$  ( $k = 0, 1, 2, \dots$ ) are holomorphic on  $D_r$ ,*

$$(A.2)$$

$$F_0(\alpha_0) = 0, \quad \frac{\partial F_0}{\partial \alpha}(\alpha_0) \neq 0,$$

$$(A.3)$$

*$\tilde{F}(\alpha, \eta) := F(\alpha, \eta) - F_0(\alpha)$  is uniformly Borel summable on  $D_r$*

*where  $\alpha_0 \in \mathbb{C}$  and  $D_r$  is*

$$(A.4) \quad D_r = \{\alpha : |\alpha - \alpha_0| \leq r\}.$$

*Then the formal solution  $\alpha(\eta) = \alpha_0 + \eta^{-1}\alpha_1 + \eta^{-2}\alpha_2 + \dots$  of*

$$(A.5) \quad F(\alpha(\eta), \eta) = 0$$

*that starts from  $\alpha_0$  uniquely exists and  $\tilde{\alpha}(\eta) := \alpha(\eta) - \alpha_0$  is Borel summable.*

*Proof.* First, by expanding  $F(\alpha(\eta), \eta)$  at  $\alpha = \alpha_0$ , we can rewrite (A.5) to the following equality:

$$(A.6) \quad \sum_{n=0}^{\infty} \eta^{-n} \left[ F_n(\alpha_0) + \sum_{\substack{\mu+k+l=n, \\ k \geq 0, \mu \geq l \geq 1}} \sum_{\substack{\mu_1+\dots+\mu_l=\mu, \\ \mu_1, \dots, \mu_l \geq 1}} \frac{\alpha_{\mu_1} \cdots \alpha_{\mu_l}}{l!} \frac{\partial^l F_k}{\partial \alpha^l}(\alpha_0) \right] = 0.$$

Therefore, by comparing the coefficients of  $\eta^{-n}$  of (A.6), we find the following equalities hold:

$$(A.7) \quad \alpha_n \frac{\partial F_0}{\partial \alpha}(\alpha_0) = R_n(\alpha_0, \dots, \alpha_{n-1}) \quad (n \geq 1).$$

Here  $R_n$  are the remainder terms of the coefficients of  $\eta^{-n}$  of (A.6) that are determined only by  $\alpha_0, \dots, \alpha_{n-1}$  and  $F$ . Since  $\partial F_0 / \partial \alpha$  does not vanish at  $\alpha = \alpha_0$ , we can inductively determine  $\alpha_n$  and hence the uniqueness of  $\alpha(\eta)$  immediately follows.

Now we show the Borel summability of the solution  $\alpha(\eta)$  of (A.5). At first, we rewrite the assumptions on  $F$  more concrete form as follows: first, from (A.2), we can take  $M > 0$  such that

$$(A.8) \quad \inf_{\alpha \in D_r} \left| \frac{F_0(\alpha)}{\alpha - \alpha_0} \right| \geq M.$$

Next the Borel summability of  $\tilde{F}$  guarantees that the Borel transform  $\tilde{F}_B$  of  $\tilde{F}$  satisfies

$$(A.9) \quad \left| \tilde{F}_B(\alpha, y) \right| \leq C_1 \exp [C_2 |y|]$$

for  $(\alpha, y) \in D_r \times E_\delta$  where  $C_1$  and  $C_2$  are positive constants and

$$(A.10) \quad E_\delta = \bigcup_{s \geq 0} \{y \in \mathbb{C} : |y - s| \leq \delta\}.$$

Now our task is to prove that  $\tilde{\alpha}_B(y)$  is holomorphic on  $E_\delta$  and satisfy the following estimates there:

$$(A.11) \quad |\tilde{\alpha}_B(y)| \leq \frac{C_1}{M} \exp \left[ \left( \frac{4C_1}{Mr} + C_2 \right) |y| \right].$$

Since  $F_0(\alpha_0) = 0$ ,  $F_0$  can be written as  $F_0(\alpha_0 + \tilde{\alpha}) = \tilde{\alpha}\tilde{F}_0(\tilde{\alpha})$  where  $\tilde{F}_0(\tilde{\alpha})$  is a holomorphic function on  $\tilde{D}_r = \{\tilde{\alpha} \in \mathbb{C} : |\tilde{\alpha}| \leq r\}$  and satisfies

$$(A.12) \quad \inf_{\alpha \in \tilde{D}_r} |\tilde{F}_0(\tilde{\alpha})| \geq M.$$

Therefore (A.5) can be rewritten as follows:

$$(A.13) \quad \tilde{\alpha}(\eta) = -\frac{1}{\tilde{F}_0(\tilde{\alpha}(\eta))} \tilde{F}(\alpha_0 + \tilde{\alpha}(\eta), \eta).$$

Let  $G(\tilde{\alpha}, \eta)$  denote the right hand side of (A.13). We expand  $G(\tilde{\alpha}, \eta)$  in  $\tilde{\alpha}$ :

$$(A.14) \quad G(\tilde{\alpha}, \eta) = \sum_{l=0}^{\infty} G^{(l)}(\eta) \tilde{\alpha}^l.$$

In order to obtain the estimation of  $\tilde{\alpha}_B(y)$ , we rewrite  $\tilde{\alpha}(\eta)$  as follows:

$$(A.15) \quad \tilde{\alpha}(\eta) = \sum_{l=0}^{\infty} \tilde{\alpha}^{(l)}(\eta),$$

where  $\tilde{\alpha}^{(l)}(\eta)$  ( $l = 0, 1, 2, \dots$ ) are formal series that are inductively determined by

$$(A.16.0) \quad \tilde{\alpha}^{(0)}(\eta) = G^{(0)}(\eta),$$

$$(A.16.l) \quad \tilde{\alpha}^{(l)}(\eta) = \sum_{\substack{\mu_1 + \dots + \mu_j + j = l, \\ \mu_1, \dots, \mu_j \geq 0, j \geq 1}} G^{(j)}(\eta) \cdot \tilde{\alpha}^{(\mu_1)}(\eta) \dots \tilde{\alpha}^{(\mu_j)}(\eta) \quad (l \geq 1).$$

We immediately find that  $\tilde{\alpha}^{(l)}(\eta)$  has the following shape:

$$(A.17) \quad \tilde{\alpha}^{(l)}(\eta) = \tilde{\alpha}_{l+1}^{(l)} \eta^{-l-1} + \tilde{\alpha}_{l+2}^{(l)} \eta^{-l-2} + \dots$$

Therefore  $\tilde{\alpha}^{(0)}(\eta) + \tilde{\alpha}^{(1)}(\eta) + \dots$  actually defines a formal series that satisfies (A.12) and this gives another representation of  $\tilde{\alpha}(\eta)$ . Applying the Borel transformation to (A.16), we find that the Borel transform  $\tilde{\alpha}_B^{(l)}(y)$  of  $\tilde{\alpha}^{(l)}(\eta)$  satisfies

$$(A.18.0) \quad \tilde{\alpha}_B^{(0)}(y) = G_B^{(0)}(y),$$

$$(A.18.l) \quad \tilde{\alpha}_B^{(l)}(y) = \sum_{\substack{\mu_1 + \dots + \mu_j + j = l, \\ \mu_1, \dots, \mu_j \geq 0, j \geq 1}} G_B^{(j)} * \tilde{\alpha}_B^{(\mu_1)} * \dots * \tilde{\alpha}_B^{(\mu_j)}(y) \quad (l \geq 1),$$

where  $f * g(y)$  denotes the convolution of  $f(y)$  and  $g(y)$ , i.e.,

$$(A.19) \quad f * g(y) = \int_0^y f(y - y')g(y')dy'.$$

Now we confirm that  $\tilde{\alpha}_B^{(l)}(y)$  ( $l = 0, 1, 2, \dots$ ) are holomorphic on  $E_\delta$  and derive the estimation that they satisfy by using (A.18). First, from Cauchy's integral formula, we find that  $G_B^{(l)}(y)$  has the following representation:

$$(A.20) \quad G_B^{(l)}(y) = \frac{-1}{2\pi i} \oint_{|\tilde{\alpha}|=r} \frac{\tilde{F}_B(\alpha_0 + \tilde{\alpha}, y)}{\tilde{F}_0(\tilde{\alpha})} \frac{d\tilde{\alpha}}{\tilde{\alpha}^{l+1}}.$$

Since  $\tilde{F}_0(\tilde{\alpha})$  satisfies (A.12), we immediately find from (A.9) and (A.20) that  $G_B^{(l)}(y)$  is holomorphic on  $E_\delta$  and satisfies the following estimates there:

$$(A.21) \quad |G_B^{(l)}(y)| \leq \frac{C_1}{Mr^l} \exp[C_2|y|].$$

Then it is clear from (A.18.0) that  $\tilde{\alpha}_B^{(0)}(y)$  is holomorphic on  $E_\delta$  and we can inductively confirm from the recurrence relation (A.18.l) that  $\tilde{\alpha}_B^{(l)}(y)$  ( $l = 1, 2, \dots$ ) are also holomorphic on  $E_\delta$ .

Next, we determine positive constants  $B_l$  ( $l = 0, 1, 2, \dots$ ) so that they satisfy

$$(A.22.l) \quad |\tilde{\alpha}_B^{(l)}(y)| \leq B_l \frac{|y|^l}{l!} \exp[C_2|y|].$$

on  $E_\delta$ . Actually, since  $\tilde{\alpha}_B^{(0)}(y) = G_B^{(0)}(y)$  satisfies

$$(A.23) \quad |\tilde{\alpha}_B^{(0)}(y)| \leq \frac{C_1}{M} \exp[C_2|y|],$$

we can take  $B_0$  as

$$(A.24) \quad B_0 = \frac{C_1}{M}.$$



Further, when (A.22.m) holds for  $0 \leq m \leq l-1$ , applying these estimates to (A.18.l), we obtain the following estimates for  $\tilde{\alpha}_B^{(l)}(y)$ :

$$\begin{aligned}
(A.25) \quad \left| \tilde{\alpha}_B^{(l)}(y) \right| &\leq \sum_{\substack{\mu_1 + \dots + \mu_j + j = l, \\ \mu_1, \dots, \mu_j \geq 0, j \geq 1}} \frac{C_1}{M r^j} B_{\mu_1} \dots B_{\mu_j} \frac{|y|^{\mu_1 + \dots + \mu_j + j}}{(\mu_1 + \dots + \mu_j + j)!} \exp[C_2|y|] \\
&= \sum_{\substack{\mu_1 + \dots + \mu_j + j = l, \\ \mu_1, \dots, \mu_j \geq 0, j \geq 1}} \frac{C_1}{M r^j} B_{\mu_1} \dots B_{\mu_j} \frac{|y|^l}{l!} \exp[C_2|y|].
\end{aligned}$$

Here we repeatedly used the following estimation:

$$\begin{aligned}
(A.26) \quad &\int_0^{|y|} \frac{|y - y'|^{\mu_1} |y'|^{\mu_2}}{\mu_1! \mu_2!} \exp[C_2(|y - y'| + |y'|)] |dy'| \\
&\leq \frac{|y|^{\mu_1 + \mu_2 + 1}}{(\mu_1 + \mu_2 + 1)!} \exp[C_2|y|].
\end{aligned}$$

Hence we recursively determine  $B_l$  ( $l = 1, 2, \dots$ ) by

$$(A.27.l) \quad B_l = \sum_{\substack{\mu_1 + \dots + \mu_j + j = l, \\ \mu_1, \dots, \mu_j \geq 0, j \geq 1}} \frac{C_1}{M r^j} B_{\mu_1} \dots B_{\mu_j}.$$

Then we find that  $B_l$  satisfies (A.22.l).

Now we derive explicit form of  $B_l$  from (A.27.l). Let  $b_l$  ( $l = 0, 1, 2, \dots$ ) be taken so that they satisfy

$$(A.28) \quad B_l = \left( \frac{C_1}{M} \right)^{l+1} \frac{1}{r^l} b_l.$$

Then the recurrence relation (A.27.l) can be rewritten to that for  $b_l$  ( $l = 1, 2, \dots$ ) as follows:

$$(A.29.l) \quad b_l = \sum_{\substack{\mu_1 + \dots + \mu_j + j = l, \\ \mu_1, \dots, \mu_j \geq 0, j \geq 1}} b_{\mu_1} \dots b_{\mu_j}.$$

We define  $b(t)$  by

$$(A.30) \quad b(t) = \sum_{l=0}^{\infty} b_l t^l.$$

Multiplying both hand side of (A.29.l) by  $t^l$  for  $l \geq 1$  and summing up all of them, we obtain

$$(A.31) \quad \begin{aligned} \sum_{l=1}^{\infty} b_l t^l &= \sum_{l=1}^{\infty} \sum_{\substack{\mu_1 + \dots + \mu_j + j = l, \\ \mu_1, \dots, \mu_j \geq 0, j \geq 1}} (b_{\mu_1} t^{\mu_1+1}) \dots (b_{\mu_j} t^{\mu_j+1}) \\ &= \sum_{j=1}^{\infty} \sum_{\mu=0}^{\infty} \sum_{\substack{\mu_1 + \dots + \mu_j = \mu, \\ \mu_1, \dots, \mu_j \geq 0}} (b_{\mu_1} t^{\mu_1+1}) \dots (b_{\mu_j} t^{\mu_j+1}) \\ &= \sum_{j=1}^{\infty} (tb(t))^j. \end{aligned}$$

Since  $b_0 = 1$ , we find from (A.31) that  $b(t)$  satisfies

$$(A.32) \quad b(t) - 1 = \frac{tb(t)}{1 - tb(t)}.$$

Therefore  $b(t)$  is explicitly given by

$$(A.33) \quad b(t) = \frac{1 - \sqrt{1 - 4t}}{2t} = \frac{1}{\Gamma(1/2)} \sum_{l=0}^{\infty} \frac{\Gamma(l + 1/2)}{(n+1)!} 4^l t^l.$$

And, from the definition of  $b_l$ , we find that  $b_l$  ( $l = 0, 1, 2, \dots$ ) are given by

$$(A.34) \quad b_l = \frac{\Gamma(l + 1/2)}{\Gamma(1/2)(n+1)!} 4^l.$$

Obviously  $b_l$  satisfies  $b_l \leq 4^l$ . Hence, in conclusion, we obtain the following estimates for  $\tilde{\alpha}_B^{(l)}(y)$  on  $E_\delta$ :

$$(A.35) \quad \left| \tilde{\alpha}_B^{(l)}(y) \right| \leq \frac{C_1}{M} \frac{1}{l!} \left( \frac{4C_1|y|}{rM} \right)^l \exp [C_2|y|].$$

Then (A.11) immediately follows from (A.35). Actually

$$(A.36) \quad |\tilde{\alpha}_B(y)| \leq \sum_{l=0}^{\infty} |\tilde{\alpha}_B^{(l)}(y)| \leq \frac{C_1}{M} \exp \left[ \left( \frac{4C_1|y|}{rM} + C_2 \right) |y| \right].$$

This is the end of the proof.  $\square$

*Remark A.1.* In Section 2.3 and Section 3.3, we use the concrete form of estimation (A.11) of  $\tilde{\alpha}_B$  to prove the Borel summability of transformation series.

## B Representation of the action of transformation as an integro-differential operator

The main purpose of this appendix is to derive the properties of microdifferential operators  $\mathcal{X}$  suggested in Theorem 1.5 and Theorem 2.5 from the Borel summability of the transformation series  $x(\tilde{x}, \eta)$ . Although the situations considered in Section 2 and Section 3 are different, in the both cases, the microdifferential operators  $\mathcal{X}$  have the following form

$$(B.1) \quad \mathcal{X} =: \left( \frac{\partial \tilde{x}}{\partial x_0} \right)^{1/2} \left( 1 + \frac{\partial X}{\partial x_0} \right)^{-1/2} \exp(X\xi) :.$$

Here  $x_0$  is the top order term of  $x$  that is taken as a new coordinate variable,  $X$  denotes  $x - x_0$  and  $\xi$  stands for the symbol of  $\partial_{x_0}$ . Therefore it suffices to show the following proposition in order to attain our aim:

**Proposition B.1.** *Let  $f(x, \eta)$  and  $g(x, \eta)$  be formal series in  $\eta$  that have the following shape*

$$(B.2) \quad f(x, \eta) = \sum_{n=1}^{\infty} f_n(x) \eta^{-n},$$

$$(B.3) \quad g(x, \eta) = \sum_{n=0}^{\infty} g_n(x) \eta^{-n},$$

where  $f_n(x)$  ( $n = 1, 2, \dots$ ) and  $g_n(x)$  ( $n = 0, 1, \dots$ ) are holomorphic on a domain  $U$  of  $\mathbb{C}_x$ . Further we assume that  $f$  and  $\tilde{g} = g - g_0$  are uniformly Borel summable on  $U$ , i.e., the Borel transform of them

$f_B(x, y)$  and  $\tilde{g}_B(x, y)$  are holomorphic on  $U \times E_\varepsilon$  and satisfy the following estimates there:

$$(B.4) \quad \max \{|f_B(x, y)|, |\tilde{g}_B(x, y)|\} \leq C_1 \exp[C_2|y|],$$

where  $E_\varepsilon$  is

$$(B.5) \quad E_\varepsilon = \bigcup_{s \geq 0} \{y \in \mathbb{C} : |y - s| \leq \varepsilon\}$$

and  $C_1, C_2$  and  $\varepsilon$  are positive constants. We consider a microdifferential operator  $\mathcal{P} = \mathcal{P}(x, \partial_x, \partial_y)$  on

$$(B.6) \quad \Omega = \{(x, y; \xi, \eta) \in T^*(U \times \mathbb{C}_y) : \eta \neq 0\}$$

defined by

$$(B.7) \quad \mathcal{P} =: g \exp[\xi f] :.$$

Then the action of  $\mathcal{P}$  upon a (multi-valued) analytic function  $\phi(x, y)$  has the following representation:

$$(B.8) \quad (\mathcal{P}\phi)(x, y) = g_0(x)\phi(x, y) + \int_{y_0}^y K(x, y - y', \partial_x)\phi(x, y')dy',$$

where  $K(x, y, \partial_x)$  is a differential operator of infinite order on  $U \times E_\varepsilon$ .

*Proof.* Let  $P_n(x, \eta)$  be the coefficients of  $\xi^n$  of  $g \exp[\xi f]$ , i.e.,

$$(B.9) \quad P_n(x, \eta) = g(x, \eta) \frac{(f(x, \eta))^n}{n!}.$$

Then the action of  $\mathcal{P}$  can be written as follows:

$$(B.10) \quad (\mathcal{P}\phi)(x, y) = \int_{y_0}^y \sum_{n=0}^{\infty} P_{n,B}(x, y - y') \frac{\partial^n \phi}{\partial x^n}(x, y') dy'.$$

From (B.9), we immediately find that  $P_{n,B}$  ( $n = 0, 1, 2, \dots$ ) are given by

$$(B.11) \quad P_{0,B}(x, y) = g_0(x)\delta(y) + \tilde{g}_B(x, y),$$

$$(B.12) \quad P_{n,B}(x, y) = \frac{1}{n!} (g_0(x)f_B^{*n}(x, y) + \tilde{g}_B * f_B^{*n}(x, y)) \quad (n \geq 1)$$

where  $f_B^{*n}$  is

$$(B.13) \quad f_B^{*n} = \overbrace{f_B * \cdots * f_B}^n.$$

Therefore, when we write the action of  $\mathcal{P}$  in the form (B.8),  $K(x, y, \partial_x)$  is given by

$$(B.14) \quad K(x, y, \partial_x) = \tilde{g}_B(x, y) + \sum_{n=1}^{\infty} P_{n,B}(x, y) \partial_x^n.$$

Now we confirm that  $K(x, y, \partial_x)$  given by (B.14) defines a differential operator of infinite order on  $U \times E_\epsilon$ . First, by repeated use of (A.26), we obtain the following estimates from (B.4) on  $U \times E_\epsilon$  for  $n \geq 1$ :

$$(B.15) \quad |P_{n,B}(x, y)| \leq \frac{1}{n!} \left( |g_0(x)| C_1^n \frac{|y|^{n-1}}{(n-1)!} + C_1^{n+1} \frac{|y|^n}{n!} \right) \exp[C_2|y|].$$

Hence the symbol  $\sigma(K)(x, y, \xi)$  of  $K$  satisfies the following estimates for  $(x, y, \xi) \in U \times E_\epsilon \times \mathbb{C}_\xi$ :

$$(B.16) \quad \begin{aligned} & |\sigma(K)(x, y, \xi)| \\ & \leq |\tilde{g}_B(x, y)| + \sum_{n=1}^{\infty} |\xi|^n |P_{n,B}(x, y)| \\ & \leq \sum_{n=0}^{\infty} \left( |g_0(x)| C_1^{n+1} \frac{|\xi|^{n+1} |y|^n}{(n+1)! n!} + C_1^{n+1} \frac{|\xi|^n |y|^n}{(n!)^2} \right) \exp[C_2|y|] \\ & \leq C_1 (|g_0(x)| |\xi| + 1) \exp \left[ 2\sqrt{C_1|y||\xi|} + C_2|y| \right]. \end{aligned}$$

This estimates (B.16) guarantees that  $K$  defines a differential operator of infinite order on  $U \times E_\epsilon$ .  $\square$

In conclusion, if

$$(B.17) \quad \partial \tilde{x} / \partial x_0 \neq 0 \text{ and } X \text{ is Borel summable}$$

in a neighborhood  $\overset{\circ}{x}_0 \in \mathbb{C}$ , then we find that  $(\partial \tilde{x} / \partial x_0)^{1/2} (1 + \partial X / \partial x_0)^{-1/2}$  and  $X$  respectively satisfy the assumption on  $f$  and  $g$  in Theorem B.1. Therefore we can apply Theorem B.1 to a microdifferential operator  $\mathcal{X}$  defined by (B.1) and we attain our purpose.

Part II

**On the WKB theoretic structure  
of a Schrödinger operator with a  
merging pair of a simple pole and  
a simple turning point**

## 0 Introduction

The principal aim of this paper is to form a basis for the exact WKB analysis of a Schrödinger equation

$$(0.1) \quad \left( \frac{d^2}{dx^2} - \eta^2 Q(x, \eta) \right) \psi = 0 \quad (\eta : \text{a large parameter})$$

with one simple turning point and with one simple pole in the potential  $Q$ . As [Ko1] and [Ko2] emphasize, the Borel transform of a WKB solution of (0.1) displays, near the simple pole singularity, behavior similar to that near a simple turning point. Hence it is natural to expect that such an equation plays an important role in the exact WKB analysis in the large. Such an expectation has recently been enhanced by the discovery ([KoT]) that the Voros coefficient of a WKB solution of (0.1) with

$$(0.2) \quad Q = \frac{1}{4} + \frac{\alpha}{x} + \eta^{-2} \frac{\gamma}{x^2} \quad (\alpha, \gamma : \text{fixed complex numbers})$$

can be explicitly written down with the help of the Bernoulli numbers. The potential  $Q$  given by (0.2) will play an important role in Section 2; the Schrödinger equation with the potential  $Q$  of the form (0.2), that is, the Whittaker equation with a large parameter  $\eta$ , gives us a WKB theoretic canonical form of a Schrödinger equation with one simple turning point and with one simple pole in its potential. We note that the parameter  $\alpha$  contained in the Whittaker equation in Section 2 is an infinite series  $\alpha(\eta) = \sum_{k \geq 0} \alpha_k \eta^{-k}$  ( $\alpha_k$ : a constant), and we call such an

equation the  $\infty$ -Whittaker equation when we want to emphasize that  $\alpha$  is not a genuine constant but an infinite series as above.

In order to make a semi-global study of a Schrödinger equation with one simple turning point and with a simple pole in its potential, we let the simple pole singular point merge with the turning point and observe what kind of equation appears. For example, what if we let  $\alpha$  tend to 0 in (0.2) with  $\gamma$  being kept intact? Interestingly enough, the resulting equation is what we call a ghost equation ([Ko3]); we have been worrying where we should place the class of ghost equations in regard to the whole WKB analysis. A ghost equation has no turning point by its definition (cf. Remark 1.1 in Section 1); still a WKB solution

of a ghost equation displays singularity similar to that which a WKB solution normally has near a turning point. The singularity is due to the singularities contained in the coefficients of  $\eta^{-k}$  ( $k \geq 1$ ) in the potential  $Q$ . (See [Ko3] for details; there a ghost (point) is tentatively called a "new" turning point.) In view of the above observation, we regard a Schrödinger equation with one simple turning point and with one simple pole in its potential as an equation obtained through perturbation of a ghost equation by a simple pole term  $aq(x, a)/x$ , where  $a$  is a complex parameter and  $q(x, a)$  is a holomorphic function defined on a neighborhood of  $(x, a) = (0, 0)$ . An equation obtained by such a procedure is called an equation with a merging pair of a simple pole and a simple turning point, or, for short, an MPPT equation. Precisely speaking, we call a Schrödinger equation (0.1) an MPPT equation if its potential  $Q$  depends also on an auxiliary parameter  $a$  and has the following form

$$(0.3) \quad Q = \frac{Q_0(x, a)}{x} + \eta^{-1} \frac{Q_1(x, a)}{x} + \eta^{-2} \frac{Q_2(x, a)}{x^2},$$

where  $Q_j(x, a)$  ( $j = 0, 1, 2$ ) are holomorphic near  $(x, a) = (0, 0)$  and  $Q_0(x, a)$  satisfies the following conditions (0.4) and (0.5):

$$(0.4) \quad Q_0(0, a) \neq 0 \text{ if } a \neq 0,$$

$$(0.5) \quad Q_0(x, 0) = c_0^{(0)}x + O(x^2) \text{ holds with } c_0^{(0)} \text{ being a constant different from 0.}$$

Clearly we find a ghost equation at  $a = 0$ ; furthermore the implicit function theorem together with the assumption (0.5) guarantees the existence of a unique holomorphic function  $x(a)$  that satisfies

$$(0.6) \quad Q_0(x(a), a) = 0.$$

The assumption (0.4) entails

$$(0.7) \quad x(a) \neq 0 \text{ if } a \neq 0,$$

and the assumption (0.5) guarantees that, for a sufficiently small  $a (\neq 0)$ ,  $x = x(a)$  is a simple turning point of the operator in question.

As the above naming "an MPPT equation" indicates, it is a counterpart of an MTP equation in our context. An MTP equation, i.e.,



a merging-turning-points equation introduced in [AKT4] contains, by definition, two simple turning points that merge into one double turning point as the parameter  $t$  tends to 0, whereas, in an MPPT equation, a simple pole and a simple turning point merge into a ghost point where neither zero nor singularity is observed in the highest degree (i.e., degree 0) in  $\eta$  part of the potential. The parallelism of these two notions is not a superficial one. The reduction of an MPPT equation to a canonical one is achieved in Sections 1 and 2 below in a way parallel to that used in the reduction of MTP equation to a canonical one; first, in Section 1 we construct a WKB theoretic transformation that brings an MPPT equation with the parameter  $a$  being 0 to a particular  $\infty$ -Whittaker equation, that is, the  $\infty$ -Whittaker equation with the top degree part of the parameter  $\alpha(\eta)$  being 0 (i.e.,  $\alpha(\eta) = \sum_{k \geq 1} \alpha_k \eta^{-k}$ ), and

then in Section 2 we construct the transformation of a generic (i.e.,  $a \neq 0$ ) MPPT equation to the  $\infty$ -Whittaker equation in the form of a perturbation series in  $a$ , starting with the transformation constructed in Section 1. In Sections 1 and 2 we focus our attention on the formal aspect of the problem, and the estimation of the growth order of the coefficients that appear in several formal series is given separately in Appendices A and B. One important implication of the estimates given in Appendix B is that they endow the formal transformation with an analytic meaning as a microdifferential operator through the Borel transformation. Furthermore, as is shown in Theorem 1.7 and Theorem 2.7, the action of the resulting microdifferential operator upon multi-valued analytic functions such as Borel transformed WKB solutions, is described in terms of an integro-differential operator of particular type; its kernel function contains a differential operator of infinite order in  $x$ -variable. Thus it is of local character in  $x$ -variable, whereas it is suited for the global study related to the resurgence phenomena in  $y$ -variable. (See e.g. [SKK] and [K] for the notion of a differential operator of infinite order. See also [AKT4] that has first used a differential operator of infinite order in exact WKB analysis.) As the domain of definition of the integro-differential operator may be chosen to be uniform with respect to the parameter  $a$  (Remark 2.3), our results in Section 2 are of semi-global character, as is noted in Remark 4.1. This uniformity is one of the most important advantages in introducing the notion of an MPPT operator. It is worth emphasizing

that the uniformity becomes clearly visible through the Borel transformation. In order to use the results obtained in Section 2 for the detailed study of the structure of Borel transformed WKB solutions of an MPPT equation, we first study in Section 3 analytic properties of Borel transformed WKB solutions of the Whittaker equation, and then in Section 4 we analyze Borel transformed WKB solutions of the  $\infty$ -Whittaker equation using the results obtained in Section 3. The basis of the study in Section 3 is a recent result of Koike ([KoT]), and the analysis in Section 4 makes essential use of the estimate (B.3) of the coefficients  $\{\alpha_k(a)\}_{k \geq 0}$  of the parameter  $\alpha(a, \eta) = \sum_{k \geq 0} \alpha_k(a) \eta^{-k}$ ; the

effect of this infinite series that appears in the  $\infty$ -Whittaker equation is grasped as a microdifferential operator acting on Borel transformed WKB solutions of the Whittaker equation. Combining all the results obtained in Sections 2 and 4 we summarize in Section 5 basic properties of Borel transformed WKB solutions of an MPPT equation with  $a \neq 0$ .

#### Acknowledgment.

We sincerely thank Professor T. Aoki for the stimulating discussions with him on the subject discussed in this paper.

## 1 Construction of the transformation to the canonical form, I. — the case where $a = 0$

The purpose of this section is to show how to construct the Borel transformable series

$$(1.1) \quad x^{(0)}(\tilde{x}, \eta) = \sum_{k \geq 0} x_k^{(0)}(\tilde{x}) \eta^{-k}$$

and

$$(1.2) \quad \alpha^{(0)}(\eta) = \sum_{k \geq 0} \alpha_k^{(0)} \eta^{-k}$$

with  $\alpha_0^{(0)}$  being 0, i.e.,

$$(1.2') \quad \alpha^{(0)}(\eta) = \sum_{k \geq 1} \alpha_k^{(0)} \eta^{-k}$$

so that the Schrödinger equation

$$(1.3) \quad \left( \frac{d^2}{d\tilde{x}^2} - \eta^2 \left( \frac{\tilde{Q}_0(\tilde{x}, 0)}{\tilde{x}} + \eta^{-1} \frac{\tilde{Q}_1(\tilde{x}, 0)}{\tilde{x}} + \eta^{-2} \frac{\tilde{Q}_2(\tilde{x}, 0)}{\tilde{x}^2} \right) \right) \tilde{\psi}(\tilde{x}, \eta) = 0$$

with  $\tilde{Q}_j(\tilde{x}, 0)$  ( $j = 0, 1, 2$ ) being holomorphic functions near the origin that satisfy (1.5) below may be brought to a particular  $\infty$ -Whittaker equation

$$(1.4) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{1}{4} + \frac{\alpha^{(0)}(\eta)}{x} + \eta^{-2} \frac{\tilde{Q}_2(0, 0)}{x^2} \right) \right) \psi(x, \eta) = 0.$$

Here the adjective “particular” refers to the vanishing of  $\alpha_0^{(0)}$ . The Borel transformability of  $x^{(0)}$  and  $\alpha^{(0)}$ , i.e., the growth order conditions on their coefficients will be separately discussed in Appendix B. Thus the first task is to establish Theorem 1.1 below, which relates the potentials in (1.3) and (1.4); the relation (1.6) enables us to relate (1.3) and (1.4) in an appropriate way, as we will expound after proving Theorem 1.1.

**Theorem 1.1.** *Let  $\tilde{Q}_j(\tilde{x}, a)$  ( $j = 0, 1, 2$ ) be holomorphic functions defined on a neighborhood of  $(\tilde{x}, a) = (0, 0)$ , and suppose that the following condition is satisfied:*

$$(1.5) \quad \tilde{Q}_0(\tilde{x}, 0) = c_0^{(0)} \tilde{x} + O(\tilde{x}^2) \text{ with } c_0^{(0)} \text{ being a constant different from 0.}$$

*Then there exist Borel transformable series  $x^{(0)}(\tilde{x}, \eta)$  and  $\alpha^{(0)}(\eta)$  respectively given in (1.1) and (1.2') so that the following relations (1.6)  $\sim$  (1.9) hold on an open neighborhood  $U$  of the origin  $\tilde{x} = 0$ :*

$$(1.6) \quad \begin{aligned} & \tilde{x}^{-1} \tilde{Q}_0(\tilde{x}, 0) + \eta^{-1} \tilde{x}^{-1} \tilde{Q}_1(\tilde{x}, 0) + \eta^{-2} \tilde{x}^{-2} \tilde{Q}_2(\tilde{x}, 0) \\ &= \left( \frac{dx^{(0)}(\tilde{x}, \eta)}{d\tilde{x}} \right)^2 \left( \frac{1}{4} + \frac{\alpha^{(0)}(\eta)}{x^{(0)}(\tilde{x}, \eta)} + \eta^{-2} \frac{\tilde{Q}_2(0, 0)}{x^{(0)}(\tilde{x}, \eta)^2} \right) \\ & \quad - \frac{1}{2} \eta^{-2} \{x^{(0)}(\tilde{x}, \eta); \tilde{x}\}, \end{aligned}$$

(1.7)  $x_k^{(0)}(\tilde{x})$  ( $k = 0, 1, 2, \dots$ ) is holomorphic on  $U$ ,

(1.8)  $x_k^{(0)}(0) = 0$  ( $k = 0, 1, 2, \dots$ ),

(1.9)  $(dx_0^{(0)}/d\tilde{x})(0) \neq 0$ .

Here  $\{x^{(0)}(\tilde{x}, \eta); \tilde{x}\}$  stands for the Schwarzian derivative, i.e.,

$$(1.10) \quad \frac{d^3 x^{(0)}/d\tilde{x}^3}{dx^{(0)}/d\tilde{x}} - \frac{3}{2} \left( \frac{d^2 x^{(0)}/d\tilde{x}^2}{dx^{(0)}/d\tilde{x}} \right)^2.$$

*Remark 1.1.* The assumption (1.5) entails that  $\tilde{x}^{-1}\tilde{Q}_0(\tilde{x}, 0)$  is holomorphic near  $\tilde{x} = 0$  and that it does not vanish there. Thus MPPT operator restricted to  $\{a = 0\}$  is exactly of the form of a ghost operator ([Ko3]). Hence the content of Theorem 1.1 is essentially the same as [Ko3, Proposition 2.1].

*Proof.* We construct  $x_k^{(0)}$  inductively, and to facilitate the required computation we introduce a series  $z^{(0)}(\tilde{x}, \eta)$  given by

$$(1.11) \quad \tilde{x}^{-1}x^{(0)}(\tilde{x}, \eta).$$

By setting

$$(1.12) \quad \gamma = \tilde{Q}_2(0, 0),$$

we define  $\tilde{R}_2 = \tilde{R}_2(\tilde{x})$  by

$$(1.13) \quad \tilde{x}^{-1}(\tilde{Q}_2(\tilde{x}, 0) - \gamma).$$

Then we find

$$(1.14) \quad \begin{aligned} & \tilde{x}^{-2}\tilde{Q}_2(\tilde{x}, 0) - \gamma(dx^{(0)}/d\tilde{x})^2(x^{(0)})^{-2} \\ &= \tilde{x}^{-1} \left[ \tilde{R}_2 - 2\gamma(dz^{(0)}/d\tilde{x})(z^{(0)})^{-1} - \gamma\tilde{x}(dz^{(0)}/d\tilde{x})^2(z^{(0)})^{-2} \right]. \end{aligned}$$

Hence our task is to construct series  $x^{(0)}(\tilde{x}, \eta)$  and  $\alpha^{(0)}(\eta)$  so that they satisfy

$$(1.15) \quad \tilde{Q}_0(\tilde{x}, 0) + \eta^{-1}\tilde{Q}_1(\tilde{x}, 0)$$

$$\begin{aligned}
&= \left( \frac{dx^{(0)}}{d\tilde{x}} \right)^2 \left( \frac{\tilde{x}}{4} + \frac{\alpha^{(0)}}{z^{(0)}} \right) + \eta^{-2} \left[ -\tilde{R}_2(\tilde{x}) + 2\gamma(dz^{(0)}/d\tilde{x})(z^{(0)})^{-1} \right. \\
&\quad \left. + \gamma\tilde{x}(dz^{(0)}/d\tilde{x})^2(z^{(0)})^{-2} - \frac{1}{2}\tilde{x}\{x^{(0)}; \tilde{x}\} \right].
\end{aligned}$$

Since we will choose  $z_0^{(0)}(\tilde{x})$  so that it does not vanish at the origin the following relations (1.16) and (1.17) guarantee that the right-hand side of (1.15) is well-defined on a sufficiently small neighborhood  $U$  of the origin:

(1.16)

$$\begin{aligned}
&(z^{(0)})^{-1} \\
&= \frac{1}{z_0^{(0)}(\tilde{x})} \left( 1 - \frac{z_1^{(0)}(\tilde{x})}{z_0^{(0)}(\tilde{x})} \eta^{-1} + \frac{z_1^{(0)}(\tilde{x})^2 - z_0^{(0)}(\tilde{x})z_2^{(0)}(\tilde{x})}{z_0^{(0)}(\tilde{x})^2} \eta^{-2} + \dots \right), \\
(1.17) \quad &\left( \frac{dx^{(0)}}{d\tilde{x}} \right)^{-1}
\end{aligned}$$

$$= \frac{1}{z_0^{(0)}(\tilde{x}) + \tilde{x}dz_0^{(0)}/d\tilde{x}} \left( 1 - \frac{z_1^{(0)}(\tilde{x}) + \tilde{x}dz_1^{(0)}/d\tilde{x}}{z_0^{(0)}(\tilde{x}) + \tilde{x}dz_0^{(0)}/d\tilde{x}} \eta^{-1} + \dots \right).$$

Let us now compare the coefficients of  $\eta^0$  in (1.15). Then we find

$$(1.18) \quad \tilde{Q}_0(\tilde{x}, 0) = \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \left( \frac{\tilde{x}}{4} + \frac{\alpha_0^{(0)}}{z_0^{(0)}} \right),$$

and hence we choose

$$(1.19) \quad \alpha_0^{(0)} = 0$$

and

$$(1.20) \quad x_0^{(0)}(\tilde{x}) = 2 \int_0^{\tilde{x}} \sqrt{\tilde{x}^{-1} \tilde{Q}_0(\tilde{x}, 0)} d\tilde{x}.$$

It then follows from (1.5) that

$$(1.21) \quad z_0^{(0)}(0) = 2\sqrt{c_0^{(0)}} \neq 0.$$

Next, using (1.19) we obtain the following relation (1.22) by comparing the coefficients of  $\eta^{-1}$  in (1.15):

$$(1.22) \quad \tilde{Q}_1(\tilde{x}, 0) = 2 \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_1^{(0)}}{d\tilde{x}} \frac{\tilde{x}}{4} + \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \left( \frac{\alpha_1^{(0)}}{z_0^{(0)}} \right).$$

Setting  $\tilde{x} = 0$  in (1.22) we find that  $\alpha_1^{(0)}$  should satisfy

$$(1.23) \quad \alpha_1^{(0)} = \tilde{Q}_1(0, 0)/z_0^{(0)}(0).$$

Then we can find a holomorphic function  $f_1(\tilde{x})$  which satisfies

$$(1.24) \quad \tilde{Q}_1(\tilde{x}, 0) - \left( \frac{dx_0^{(0)}(\tilde{x})}{d\tilde{x}} \right)^2 \frac{\alpha_1^{(0)}}{z_0^{(0)}(\tilde{x})} = \tilde{x} f_1(\tilde{x}).$$

Thus it suffices to solve

$$(1.25) \quad \frac{dx_1^{(0)}}{d\tilde{x}} = 2 \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^{-1} f_1(\tilde{x})$$

to find  $x_1^{(0)}$  that satisfies (1.22). If we solve (1.25) with the initial condition at  $\tilde{x} = 0$  being 0 on a sufficiently small disc  $U$  centered at the origin, we obtain  $x_1^{(0)}(\tilde{x})$  that also satisfies the condition (1.8). The construction of  $x_k^{(0)}$  and  $\alpha_k^{(0)}$  ( $k \geq 2$ ) can be inductively done on the same disc  $U$  in a similar manner. For example, the comparison of the coefficients of  $\eta^{-2}$  in (1.15) results in the following:

$$(1.26) \quad 0 = \left( 2 \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_2^{(0)}}{d\tilde{x}} + \left( \frac{dx_1^{(0)}}{d\tilde{x}} \right)^2 \right) \frac{\tilde{x}}{4} + 2 \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_1^{(0)}}{d\tilde{x}} \frac{\alpha_1^{(0)}}{z_0^{(0)}} \\ + \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \left( \frac{\alpha_2^{(0)}}{z_0^{(0)}} - \frac{\alpha_1^{(0)} z_1^{(0)}}{z_0^{(0)2}} \right) - \tilde{R}_2(\tilde{x}) + \frac{2\gamma \frac{dz_0^{(0)}}{d\tilde{x}}}{z_0^{(0)}} \\ + \gamma \tilde{x} \left( \frac{\frac{dz_0^{(0)}}{d\tilde{x}}}{z_0^{(0)}} \right)^2 - \frac{1}{2} \tilde{x} \{x_0^{(0)}; \tilde{x}\}.$$

Then we set  $\tilde{x} = 0$  in (1.26) to find  
(1.27)

$$\alpha_2^{(0)} = (z_0^{(0)}(0))^{-1} \left[ \alpha_1^{(0)}(z_1^{(0)}(0) - 2z_1^{(0)}(0)) + \tilde{R}_2(0) - \frac{2\gamma \frac{dz_0^{(0)}}{d\tilde{x}}(0)}{z_0^{(0)}(0)} \right].$$

After choosing  $\alpha_2^{(0)}$  as in (1.27) we can divide (1.26) by  $\tilde{x}$  to find a differential equation of the form

$$(1.28) \quad \frac{dx_2^{(0)}}{d\tilde{x}} = f_2(\tilde{x}),$$

where  $f_2(\tilde{x})$  is holomorphic on  $U$ . Thus we can find the required  $x_2^{(0)}(\tilde{x})$  by solving (1.28) with the initial condition  $x_2^{(0)}(0) = 0$ . The construction of  $\alpha_k^{(0)}$  and  $x_k^{(0)}(\tilde{x})$  can be performed in exactly the same manner: first compute the coefficients of  $\eta^{-k}$  in (1.15), set  $\tilde{x}$  to be 0 to find  $\alpha_k^{(0)}$  so that we may divide the sum of the coefficients by  $\tilde{x}$  to find a first order equation of normal form for  $x_k^{(0)}(\tilde{x})$  with holomorphic coefficients on  $U$ , and finally solve the differential equation with the initial condition  $x_k^{(0)}(0) = 0$ .

□

As is well known in the exact WKB analysis (e.g. [KT2, Theorem 2.16 and Corollary 2.18]), the relation (1.6) between potentials enables us to clarify the structure of WKB solutions of a general MPPT equation restricted to  $\{a = 0\}$  in terms of WKB solutions of a particular (i.e.,  $\alpha_0^{(0)} = 0$ )  $\infty$ -Whittaker equation; the concrete statements are as follows:

**Theorem 1.2.** *In the situation considered in Theorem 1.1, the infinite series  $x^{(0)}(\tilde{x}, \eta)$  and  $\alpha^{(0)}(\eta)$  satisfy*

$$(1.29) \quad \tilde{S}(\tilde{x}, \eta) = \left( \frac{dx^{(0)}}{d\tilde{x}} \right) S(x^{(0)}(\tilde{x}, \eta), \alpha^{(0)}(\eta), \eta) \\ - \frac{1}{2} \left( \frac{d^2 x^{(0)}(\tilde{x}, \eta)}{d\tilde{x}^2} \right) / \left( \frac{dx^{(0)}(\tilde{x}, \eta)}{d\tilde{x}} \right),$$

where  $\tilde{S}$  and  $S$  are formal series in  $\eta^{-1}$  respectively beginning with  $\tilde{S}_{-1}(x)\eta$  and  $S_{-1}(x)\eta$  which solve the Riccati equations

$$(1.30) \quad \tilde{S}^2 + \frac{d\tilde{S}}{dx} = \eta^2 \left( \frac{\tilde{Q}_0(\tilde{x}, 0)}{\tilde{x}} + \eta^{-1} \frac{\tilde{Q}_1(\tilde{x}, 0)}{\tilde{x}} + \eta^{-2} \frac{\tilde{Q}_2(\tilde{x}, 0)}{\tilde{x}^2} \right)$$

and

$$(1.31) \quad S^2 + \frac{dS}{dx} = \eta^2 \left( \frac{1}{4} + \frac{\alpha^{(0)}(\eta)}{x} + \eta^{-2} \frac{\tilde{Q}_2(0, 0)}{x^2} \right),$$

and for which

$$(1.32) \quad \arg \tilde{S}_{-1}(\tilde{x}) = \arg \left( \frac{dx_0^{(0)}}{d\tilde{x}} S_{-1}(x_0^{(0)}(\tilde{x})) \right)$$

holds (and hence  $\tilde{S}_{-1}(\tilde{x})$  and  $(dx_0^{(0)}/d\tilde{x}) S_{-1}(x_0^{(0)}(\tilde{x}))$  themselves coincide.)

**Theorem 1.3.** *Let us consider the situation assumed in Theorem 1.1, and let  $\psi$  be a WKB solution of the  $\infty$ -Whittaker equation*

$$(1.33) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{1}{4} + \frac{\alpha^{(0)}(\eta)}{x} + \eta^{-2} \frac{\tilde{Q}_2(0, 0)}{x^2} \right) \right) \psi = 0,$$

where  $\alpha^{(0)}(\eta)$  is the infinite series constructed there; in particular

$$(1.34) \quad \alpha_0^{(0)} = 0.$$

Then for the infinite series  $x^{(0)}(\tilde{x}, \eta)$  constructed there we find

$$(1.35) \quad \tilde{\psi}(\tilde{x}, \eta) = \left( \frac{dx^{(0)}(\tilde{x}, \eta)}{d\tilde{x}} \right)^{-1/2} \psi(x^{(0)}(\tilde{x}, \eta), \eta)$$

satisfies the following MPPT equation restricted to  $\{a = 0\}$ :

$$(1.36) \quad \left( \frac{d^2}{d\tilde{x}^2} - \eta^2 \left( \frac{\tilde{Q}_0(\tilde{x}, 0)}{\tilde{x}} + \eta^{-1} \frac{\tilde{Q}_1(\tilde{x}, 0)}{\tilde{x}} + \eta^{-2} \frac{\tilde{Q}_2(\tilde{x}, 0)}{\tilde{x}^2} \right) \right) \tilde{\psi}(\tilde{x}, \eta) = 0.$$

See [KT2, Section 2] for the derivation of Theorems 1.2 and 1.3 from Theorem 1.1; although the situation considered in [KT2] is a



much simpler one (the situation where only one simple turning point is relevant) the logical structure of the derivation is exactly the same.

The analytic meaning of Theorem 1.3 becomes much more transparent if we apply the Borel transformation to all the relevant functions and equations; for example, the Borel transformed  $\infty$ -Whittaker equation turns out to be a microdifferential equation

$$(1.37) \quad \left( \frac{\partial^2}{\partial x^2} - \left( \frac{1}{4} + \frac{1}{x} \alpha^{(0)} \left( \frac{\partial}{\partial y} \right) \right) \frac{\partial^2}{\partial y^2} - \frac{\tilde{Q}_2(0,0)}{x^2} \right) \psi_B(x, y) = 0,$$

thanks to the estimate (B.3) in Appendix B of the growth order of  $\alpha_k^{(0)}$  ( $k \geq 1$ ). Before embarking on the analytic study of the Borel transformed relations, we present an important relation between the infinite series  $\alpha^{(0)}(\eta)$  and  $\tilde{S}(\tilde{x}, \eta)$  in Theorem 1.2. For that purpose we recall the definition of the odd part  $S_{\text{odd}}$  of a solution  $S$  of the Riccati equation with  $\eta$ -dependent potential.

**Definition 1.1.** ([AKT3, Definition 2.1]) Consider the following Riccati equation with  $\eta$ -dependent potential:

$$(1.38) \quad S(x, \eta) + \frac{dS}{dx}(x, \eta) = \eta^2 \left( \sum_{k \geq 0} Q_k(x) \eta^{-k} \right).$$

Let  $S^{(\pm)}$  respectively denote the solution of (1.38) that begins with  $\pm \eta \sqrt{Q_0(x)}$ . Then the odd part  $S_{\text{odd}}$  of  $S$  is, by definition, given by

$$(1.39) \quad S_{\text{odd}} = \frac{1}{2}(S^{(+)} - S^{(-)}).$$

With the help of Definition 1.1, Theorem 1.2 immediately entails the following

**Corollary 1.4.** For  $S$  and  $\tilde{S}$  in Theorem 1.2 their odd parts satisfy the following relation

$$(1.40) \quad \tilde{S}_{\text{odd}}(\tilde{x}, \eta) = \left( \frac{dx^{(0)}}{d\tilde{x}} \right) S_{\text{odd}}(x^{(0)}(\tilde{x}, \eta), \alpha^{(0)}(\eta), \eta),$$

if the branches of  $\tilde{S}_{-1}$  and  $S_{-1}$  are chosen so that (1.32) is satisfied.

Using this result we find the following

**Proposition 1.5.** ([Ko3, Proposition 2.1]) *Let  $\tilde{S}_{\text{odd}}$  denote the odd part of  $\tilde{S}$  in Theorem 1.2. Then we find*

$$(1.41) \quad \text{Res}_{\tilde{x}=0} \tilde{S}_{\text{odd}} = \eta \alpha^{(0)}.$$

*Proof.* In view of the relation (1.40) it suffices to prove (1.41) for  $S$  in Theorem 1.2. To verify (1.41) for  $S_{\text{odd}}$ , we study the concrete form of solutions  $S^{(+)}$  and  $S^{(-)}$  of (1.31) whose top degree (i.e., degree 1 in  $\eta$ ) parts are respectively given by  $+\eta/2$  and  $-\eta/2$ . One can then immediately see

$$(1.42) \quad S_0^{(\pm)} = \pm \frac{\alpha_1^{(0)}}{x}.$$

Here, and in what follows, the sign  $\pm$  is chosen correspondingly in each formula. Next

$$(1.43) \quad 2S_{-1}^{(\pm)} S_1^{(\pm)} + \left(S_0^{(\pm)}\right)^2 + \frac{d}{dx} S_0^{(\pm)} = \frac{\alpha_2^{(0)}}{x} + \frac{\tilde{Q}_2(0, 0)}{x^2}$$

entails

$$(1.44) \quad \pm S_1^{(\pm)} = \frac{\alpha_2^{(0)}}{x} + \frac{\beta_1^{(\pm)}}{x^2}$$

with constants  $\beta_1^{(\pm)}$ . Similarly the computation of the coefficients of  $\eta^{-l}$  ( $l \geq 1$ ) in (1.31) entails

$$(1.45) \quad \pm S_{l+1}^{(\pm)} + \sum_{\substack{j+k=l \\ j, k \geq 0}} S_j^{(\pm)} S_k^{(\pm)} + \frac{d}{dx} S_l^{(\pm)} = \frac{\alpha_{l+2}^{(0)}}{x}.$$

Since each  $S_j^{(\pm)}$  ( $j \geq 0$ ) is a sum of pole terms, (1.45) implies

$$(1.46) \quad \pm S_{l+1}^{(\pm)} = \frac{\alpha_{l+2}^{(0)}}{x} + (\text{multiple pole terms}).$$

Thus the residue of  $S_{\text{odd}} = \frac{1}{2}(S^{(+)} - S^{(-)})$  at the origin is  $\alpha^{(0)}$ , as is expected. This completes the proof of the proposition. □

We have so far studied the formal aspect of the problem; the growth order conditions (B.3) and (B.4) (with  $a = 0$ ) that  $\{x_k^{(0)}(\tilde{x})\}_{k \geq 0}$  and  $\{\alpha_k^{(0)}\}_{k \geq 0}$  respectively satisfy enable us to obtain much deeper analytic results. Applying the Borel transformation ([KT2]) to (1.35), we find that  $\tilde{\psi}_B(\tilde{x}, y)$ , the Borel transform of  $\tilde{\psi}(\tilde{x}, \eta)$ , and  $\psi_B(x_0^{(0)}(\tilde{x}), y)$ , the Borel transform of  $\psi(x_0^{(0)}(\tilde{x}), \eta)$ , are related by a microdifferential operator. This is one of the most important observations made in [AKT1, Section 2], where a simple turning point problem was studied. Following the presentation of [AY] and [AKT4], we formulate this fact in Theorem 1.6 below as the existence of intertwining operators of a Borel transformed MPPT operator with  $a = 0$  and the Borel transformed particular (i.e.,  $\alpha_0^{(0)} = 0$ )  $\infty$ -Whittaker operator; furthermore the intertwining operators enjoy beautiful expressions which are most amenable to the study of the exact WKB analysis. (Theorem 1.7.)

To state Theorem 1.6 and Theorem 1.7 we make some notational preparations. First we let  $g(x)$  denote the inverse function of

$$(1.47) \quad x = x_0^{(0)}(\tilde{x}),$$

where  $x_0^{(0)}(\tilde{x})$  is the function given by (1.20), that is,

$$(1.48) \quad x = x_0^{(0)}(g(x)), \quad \tilde{x} = g(x_0^{(0)}(\tilde{x})).$$

The existence of  $g(x)$  is guaranteed by the condition (1.9). Then, by rewriting the Borel transform  $\tilde{A}$  of an MPPT operator restricted to  $\{a = 0\}$ , i.e.,

$$(1.49) \quad \tilde{A} = \frac{\partial^2}{\partial \tilde{x}^2} - \frac{\tilde{Q}_0(\tilde{x}, 0)}{\tilde{x}} \frac{\partial^2}{\partial y^2} - \frac{\tilde{Q}_1(\tilde{x}, 0)}{\tilde{x}} \frac{\partial}{\partial y} - \frac{\tilde{Q}_2(\tilde{x}, 0)}{\tilde{x}^2},$$

in  $(x, y)$ -coordinate, we find by (1.18) and (1.19)

$$(1.50) \quad \begin{aligned} \tilde{A}|_{\tilde{x}=g(x)} &= \left(\frac{dg}{dx}\right)^{-2} \left[ \frac{\partial^2}{\partial x^2} - \left(\frac{d^2g/dx^2}{dg/dx}\right) \frac{\partial}{\partial x} \right] \\ &\quad - \frac{\tilde{Q}_0(g(x), 0)}{g(x)} \frac{\partial^2}{\partial y^2} - \frac{\tilde{Q}_1(g(x), 0)}{g(x)} \frac{\partial}{\partial y} - \frac{\tilde{Q}_2(g(x), 0)}{g(x)^2} \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{dg}{dx}\right)^{-2} \left[ \frac{\partial^2}{\partial x^2} - \left(\frac{d^2g/dx^2}{dg/dx}\right) \frac{\partial}{\partial x} - \frac{1}{4} \frac{\partial^2}{\partial y^2} \right. \\
&\quad \left. - \frac{(dg/dx)^2}{g(x)} \tilde{Q}_1(g(x), 0) \frac{\partial}{\partial y} - \frac{(dg/dx)^2}{g(x)^2} \tilde{Q}_2(g(x), 0) \right].
\end{aligned}$$

We now define microdifferential operators  $L$  and  $M$  respectively by

$$\begin{aligned}
(1.51) \quad L &= \frac{\partial^2}{\partial x^2} - \left(\frac{d^2g/dx^2}{dg/dx}\right) \frac{\partial}{\partial x} \\
&\quad - \frac{1}{4} \frac{\partial^2}{\partial y^2} - \frac{(dg/dx)^2}{g(x)} \tilde{Q}_1(g(x), 0) \frac{\partial}{\partial y} - \frac{(dg/dx)^2}{g(x)^2} \tilde{Q}_2(g(x), 0)
\end{aligned}$$

and

$$(1.52) \quad M = \frac{\partial^2}{\partial x^2} - \left(\frac{1}{4} + \frac{\alpha^{(0)}(\partial/\partial y)}{x}\right) \frac{\partial^2}{\partial y^2} - \frac{\tilde{Q}_2(0, 0)}{x^2}.$$

Then we have the following

**Theorem 1.6.** *Let  $\omega_0$  be an open neighborhood of  $x = 0$ , and set*

$$(1.53) \quad \Omega_0 = \{(x, y; \xi, \eta) \in T^*\mathbb{C}_{(x,y)}^2; x \in \omega_0, \eta \neq 0\}$$

and

$$(1.54) \quad \Omega_0^* = \{(x, y; \xi, \eta) \in \Omega_0; x \neq 0\}.$$

*Then there exist microdifferential operators  $\mathcal{X}$  and  $\mathcal{Y}$  defined on  $\Omega_0$  that satisfy*

$$(1.55) \quad L\mathcal{X} = \mathcal{Y}M$$

*on  $\Omega_0^*$  and that are invertible on  $\Omega_0$ .*

*Proof.* In this proof, and in what follows, we follow [A1] in the usage of terminologies and ideograms in symbol calculus; for example, for a microdifferential operator  $\mathcal{X}$ ,  $\sigma(\mathcal{X})$  stands for its symbol, and for a

symbol  $s(x, y, \xi, \eta)$ ,  $: s(x, y, \xi, \eta) :$  designates the corresponding normal ordered product operator, and so on. As was first emphasized by [AKT1],

$$(1.56) \quad \psi(x^{(0)}(\tilde{x}, \eta), \eta) = \psi(x_0^{(0)}(\tilde{x}) + x_1^{(0)}(\tilde{x})\eta^{-1} + x_2^{(0)}(\tilde{x})\eta^{-2} + \cdots, \eta)$$

that appears in the right-hand side of (1.35) can be formally rewritten as

$$(1.57) \quad \sum_{n \geq 0} \frac{1}{n!} \left( \sum_{k \geq 1} x_k^{(0)}(\tilde{x}) \eta^{-k} \right)^n \left( \frac{\partial^n}{\partial x^n} \psi(x, \eta) \right) \Big|_{x=x_0^{(0)}(\tilde{x})},$$

and hence its Borel transform is expressed in  $(x, y)$ -coordinate as

$$(1.58) \quad \left( \sum_{n \geq 0} \frac{1}{n!} \left( \sum_{k \geq 1} x_k^{(0)}(g(x)) \left( \frac{\partial}{\partial y} \right)^{-k} \right)^n \frac{\partial^n}{\partial x^n} \right) \psi_B(x, y) \\ =: \exp \left( \left( \sum_{k \geq 1} x_k^{(0)}(g(x)) \eta^{-k} \right) \xi \right) : \psi_B(x, y).$$

Having this expression in mind, we try to find operators  $\mathcal{X}$  and  $\mathcal{Y}$  in the following form:

$$(1.59) \quad \mathcal{X} =: C(x, \eta) \exp(r(x, \eta)\xi) :,$$

$$(1.60) \quad \mathcal{Y} =: C^*(x, \eta) \exp(r(x, \eta)\xi) :,$$

where  $C(x, \eta)$ ,  $C^*(x, \eta)$  and  $r(x, \eta)$  are symbols of microdifferential operators respectively of order 0, 0 and  $-1$ . As the notation indicates we suppose they are free from  $(y, \xi)$ . Let  $r_k(x)$  denote the coefficient of  $\eta^{-k}$  in  $r$ ; that is,

$$(1.61) \quad r(x, \eta) = \sum_{k \geq 1} r_k(x) \eta^{-k}.$$

Then, by the symbol calculus of the composition of operators, we find

$$(1.62) \quad \sigma(L\mathcal{X}) = \sigma(L)\sigma(\mathcal{X}) + \sigma_\xi(L)\sigma_x(\mathcal{X}) + \frac{1}{2!} \sigma_{\xi\xi}(L)\sigma_{xx}(\mathcal{X}).$$

Note that  $\mathcal{X}$  is free from  $y$  and that

$$(1.63) \quad \frac{\partial^p}{\partial \xi^p} \sigma(L) = 0 \quad \text{if } p \geq 3.$$

Here and in what follows we use the subscripts  $x$  (resp.,  $\xi$ ) to designate the differentiation by  $x$  (resp.,  $\xi$ ):  $r_x = dr/dx$ ,  $r_{xx} = d^2r/dx^2$ , etc. We also use the letter  $E$  as an abbreviation of  $\exp(r(x, \eta)\xi)$ . Under these conventions we find

$$(1.64) \quad \begin{aligned} & \sigma(L\mathcal{X}) \\ &= \left[ \xi^2 - \frac{1}{4}\eta^2 - \frac{g_{xx}}{g_x}\xi - \frac{(g_x)^2}{g}\tilde{Q}_1(g(x), 0)\eta - \frac{(g_x)^2}{g^2}\tilde{Q}_2(g(x), 0) \right] CE \\ & \quad + \left( 2\xi - \frac{g_{xx}}{g_x} \right) (C_x E + r_x \xi C E) \\ & \quad + \frac{1}{2!} (2) (C_{xx} E + 2C_x r_x \xi E + C r_{xx} \xi E + C(r_x \xi)^2 E) \\ &= (1 + r_x)^2 C \xi^2 E + \left[ 2(1 + r_x) C_x - \frac{g_{xx}}{g_x} (1 + r_x) C + r_{xx} C \right] \xi E \\ & \quad + \left[ \left( -\frac{1}{4}\eta^2 - \frac{(g_x)^2}{g}\tilde{Q}_1(g(x), 0)\eta - \frac{(g_x)^2}{g^2}\tilde{Q}_2(g(x), 0) \right) C \right. \\ & \quad \left. - \frac{g_{xx}}{g_x} C_x + C_{xx} \right] E. \end{aligned}$$

In parallel with (1.64), by setting

$$(1.65) \quad \beta(\eta) = \eta \alpha^{(0)}(\eta) = \sum_{k \geq 1} \alpha_k^{(0)} \eta^{-k+1}$$

and

$$(1.66) \quad \gamma = \tilde{Q}_2(0, 0),$$

we find

$$(1.67) \quad \sigma(\mathcal{Y}M)$$

$$\begin{aligned}
&= \sum_{n \geq 0} \frac{1}{n!} \left( \frac{\partial^n}{\partial \xi^n} \sigma(\mathcal{Y}) \right) \left( \frac{\partial^n}{\partial x^n} \sigma(M) \right) \\
&= (C^* E) \left( \xi^2 - \frac{1}{4} \eta^2 - \frac{\beta(\eta)\eta}{x} - \frac{\gamma}{x^2} \right) \\
&\quad + \sum_{n \geq 1} \frac{1}{n!} (r^n C^* E) \left( \frac{(-1)^{n+1} n! \beta(\eta)\eta}{x^{n+1}} + \frac{(-1)^{n+1} (n+1)! \gamma}{x^{n+2}} \right) \\
&= (C^* E) \left( \xi^2 - \frac{1}{4} \eta^2 \right) \\
&\quad - (C^* E) \left[ \sum_{n \geq 0} \frac{\beta(\eta)\eta}{x} \left( \frac{-r}{x} \right)^n + \sum_{n \geq 0} \frac{(n+1)\gamma}{x^2} \left( \frac{-r}{x} \right)^n \right] \\
&= (C^* E) \left( \xi^2 - \frac{1}{4} \eta^2 \right) - (C^* E) \left[ \frac{\beta(\eta)\eta}{x} \left( 1 + \frac{r}{x} \right)^{-1} + \frac{\gamma}{x^2} \left( 1 + \frac{r}{x} \right)^{-2} \right] \\
&= (C^* E) \left( \xi^2 - \frac{1}{4} \eta^2 - \frac{\beta(\eta)\eta}{x+r} - \frac{\gamma}{(x+r)^2} \right).
\end{aligned}$$

Hence we obtain the following relations by comparing the coefficients of  $\xi^l E$  ( $l = 2, 1, 0$ ) in (1.64) and (1.67):

$$(1.68) \quad (1 + r_x)^2 C = C^*$$

$$(1.69) \quad (1 + r_x) \left( 2C_x - \frac{g_{xx}}{g_x} C \right) + r_{xx} C = 0$$

$$\begin{aligned}
(1.70) \quad &\left[ -\frac{1}{4} \eta^2 - \frac{(g_x)^2}{g} \tilde{Q}_1(g(x), 0) \eta - \frac{(g_x)^2}{g^2} \tilde{Q}_2(g(x), 0) \right] C - \frac{g_{xx}}{g_x} C_x + C_{xx} \\
&= C^* \left( -\frac{1}{4} \eta^2 - \frac{\beta(\eta)\eta}{x+r} - \frac{\gamma}{(x+r)^2} \right).
\end{aligned}$$

If we set

$$(1.71) \quad s(x, \eta) = x + r(x, \eta),$$

(1.69) is rewritten as follows:

$$(1.72) \quad \frac{C_x}{C} = \frac{1}{2} \left( \frac{g_{xx}}{g_x} - \frac{s_{xx}}{s_x} \right).$$

Hence  $C$  is fixed by  $g$  and  $s$  aside from a constant multiple  $\Gamma$ :

$$(1.73) \quad C = \Gamma(g_x)^{1/2}(s_x)^{-1/2}.$$

As the arbitrariness of  $\Gamma$  is absorbed by the freedom in choosing the constant multiple of  $C^*$  if we define it by (1.68), i.e.,

$$(1.74) \quad C^* = s_x^2 C.$$

Thus we may choose  $\Gamma = 1$  in (1.73) without loss of generality. Substituting (1.74) into (1.70), we obtain

$$(1.75) \quad \begin{aligned} & \frac{1}{4}\eta^2 + \frac{(g_x)^2}{g(x)}\tilde{Q}_1(g(x), 0)\eta + \frac{(g_x)^2}{g(x)^2}\tilde{Q}_2(g(x), 0) \\ &= s_x^2 \left( \frac{1}{4}\eta^2 + \frac{\beta(\eta)\eta}{s} + \frac{\gamma}{s^2} \right) - C^{-1} \left( \frac{g_{xx}}{g_x} C_x - C_{xx} \right). \end{aligned}$$

Further (1.18) entails

$$(1.76) \quad \left. \frac{\tilde{Q}_0(\tilde{x}, 0)}{\tilde{x}} \right|_{\tilde{x}=g(x)} = \frac{1}{4} \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \Big|_{\tilde{x}=g(x)} = \frac{1}{4} g_x(x)^{-2}.$$

Hence we may rewrite (1.75) as

$$(1.77) \quad \begin{aligned} & \frac{\tilde{Q}_0(g(x), 0)}{g(x)}\eta^2 + \frac{\tilde{Q}_1(g(x), 0)}{g(x)}\eta + \frac{\tilde{Q}_2(g(x), 0)}{g(x)^2} \\ &= g_x^{-2} s_x^2 \left( \frac{1}{4}\eta^2 + \frac{\beta(\eta)\eta}{s} + \frac{\gamma}{s^2} \right) - D(x, \eta) \end{aligned}$$

where

$$(1.78) \quad D(x, \eta) = g_x(x)^{-2} C(x, \eta)^{-1} \left( \frac{g_{xx}(x)}{g_x(x)} C_x(x, \eta) - C_{xx}(x, \eta) \right).$$

Thus our task is to find the series  $s(x, \eta)$  that satisfies (1.77), and we want to find the required series in terms of  $x^{(0)}(\tilde{x}, \eta)$  constructed in the



proof of Theorem 1.1, by somehow relating (1.77) with (1.6). In order to relate (1.77) with (1.6), we substitute  $x = x_0^{(0)}(\tilde{x})$  into (1.77) so that the relation is described in terms of the  $\tilde{x}$ -variable. To facilitate the description of (1.77) in  $\tilde{x}$ -coordinate, we introduce

$$(1.79) \quad \tilde{s}(\tilde{x}, \eta) = s(x_0^{(0)}(\tilde{x}), \eta)$$

and

$$(1.80) \quad \tilde{C}(\tilde{x}, \eta) = C(x_0^{(0)}(\tilde{x}), \eta).$$

Then we find

$$(1.81) \quad \frac{d\tilde{s}}{d\tilde{x}} = \left( \frac{ds}{dx} \Big|_{x=x_0^{(0)}(\tilde{x})} \right) \frac{dx_0^{(0)}}{d\tilde{x}} = \left( \frac{ds}{dx} \Big|_{x=x_0^{(0)}(\tilde{x})} \right) \left( \left( \frac{dg}{dx} \right)^{-1} \Big|_{x=x_0^{(0)}(\tilde{x})} \right),$$

and hence by (1.73) with  $\Gamma = 1$

$$(1.82) \quad \tilde{C}(\tilde{x}, \eta) = \left( \frac{d\tilde{s}}{d\tilde{x}} \right)^{-1/2}.$$

On the other hand it follows from the definition (1.80) of  $\tilde{C}(\tilde{x}, \eta)$  that

$$(1.83) \quad C(x, \eta) = \tilde{C}(g(x), \eta),$$

$$(1.84) \quad C_x(x, \eta) = \left( \frac{d\tilde{C}}{d\tilde{x}} \Big|_{\tilde{x}=g(x)} \right) \frac{dg}{dx},$$

$$(1.85) \quad C_{xx}(x, \eta) = \left( \frac{d^2\tilde{C}}{d\tilde{x}^2} \Big|_{\tilde{x}=g(x)} \right) \left( \frac{dg}{dx} \right)^2 + \left( \frac{d\tilde{C}}{d\tilde{x}} \Big|_{\tilde{x}=g(x)} \right) \frac{d^2g}{dx^2}.$$

Thus the substitution of (1.84) and (1.85) into (1.78) shows

$$(1.86) \quad \begin{aligned} D(x, \eta) &= g_x^{-2} C(x, \eta)^{-1} \left( -\frac{d^2\tilde{C}}{d\tilde{x}^2} \Big|_{\tilde{x}=g(x)} \right) g_x^2 \\ &= -C(x, \eta)^{-1} \left( \frac{d^2\tilde{C}}{d\tilde{x}^2} \Big|_{\tilde{x}=g(x)} \right). \end{aligned}$$

We now use (1.82) to compute  $\tilde{C}_{\tilde{x}\tilde{x}} (= d^2\tilde{C}/d\tilde{x}^2)$ :

$$(1.87) \quad \frac{d^2\tilde{C}}{d\tilde{x}^2} = -\frac{1}{2} \left( \frac{d\tilde{s}}{d\tilde{x}} \right)^{-1/2} \left( \frac{\tilde{s}_{\tilde{x}\tilde{x}\tilde{x}}}{\tilde{s}_{\tilde{x}}} - \frac{3}{2} \left( \frac{\tilde{s}_{\tilde{x}\tilde{x}}}{\tilde{s}_{\tilde{x}}} \right)^2 \right).$$

Then the substitution of  $x = x_0^{(0)}(\tilde{x})$  into (1.86) entails  
(1.88)

$$D(x_0^{(0)}(\tilde{x}), \eta) = \frac{1}{2} \tilde{C}(\tilde{x}, \eta)^{-1} \left( \frac{d\tilde{s}}{d\tilde{x}} \right)^{-1/2} \left( \frac{\tilde{s}_{\tilde{x}\tilde{x}\tilde{x}}}{\tilde{s}_{\tilde{x}}} - \frac{3}{2} \left( \frac{\tilde{s}_{\tilde{x}\tilde{x}}}{\tilde{s}_{\tilde{x}}} \right)^2 \right) = \frac{1}{2} \{\tilde{s}; \tilde{x}\}.$$

Now we substitute  $x = x_0^{(0)}(\tilde{x})$  into (1.77) and use (1.81) and (1.88) to obtain

$$(1.89) \quad \frac{\tilde{Q}_0(\tilde{x}, 0)}{\tilde{x}} \eta^2 + \frac{\tilde{Q}_1(\tilde{x}, 0)}{\tilde{x}} \eta + \frac{\tilde{Q}_2(\tilde{x}, 0)}{\tilde{x}^2} \\ = \left( \frac{d\tilde{s}}{d\tilde{x}} \right)^2 \left( \frac{1}{4} \eta^2 + \frac{\beta(\eta)\eta}{\tilde{s}(\tilde{x}, \eta)} + \frac{\gamma}{\tilde{s}(\tilde{x}, \eta)^2} \right) - \frac{1}{2} \{\tilde{s}; \tilde{x}\}.$$

Comparing (1.89) with (1.6) we find by (1.65) and (1.66) that the series  $x^{(0)}(\tilde{x}, \eta)$  constructed in the proof of Theorem 1.1 gives us the series  $\tilde{s}(\tilde{x}, \eta)$  that satisfies (1.89). Furthermore the growth order condition (B.4) in Appendix B guarantees that  $\tilde{s}(\tilde{x}, \eta)$  is the symbol of a microdifferential operator of order 0. Therefore we obtain the required symbol  $s(x, \eta)$  by setting

$$(1.90) \quad s(x, \eta) = \tilde{s}(g(x), \eta).$$

Note that the top degree part of  $s(x, \eta)$ , i.e.,  $s_0(x)$  is, by its definition,  $x_0^{(0)}(g(x)) = x$ . Hence the series  $s$  given by (1.90) has the form (1.71). Hence  $r(x, \eta)$  is the symbol of a microdifferential operator of order  $-1$ . Furthermore the fact that  $s_0(x) = x$  together with (1.73) and (1.74) entails that the highest degree in  $\eta$  parts, i.e., degree 0 parts of  $C$  and  $C^*$  are both  $(g_x)^{1/2}$ , which never vanishes on a sufficiently small neighborhood  $\omega_0$  of the origin. This implies that  $C$  and  $C^*$  are invertible on  $\Omega_0$ , and hence  $\mathcal{X} = CE$  and  $\mathcal{Y} = C^*E$  are also invertible there. Since

$$(1.91) \quad \sigma(L\mathcal{X}) = \sigma(\mathcal{Y}M)$$

holds on  $\Omega_0^*$  by the way of constructing  $\mathcal{X}$  and  $\mathcal{Y}$ , we find

$$(1.92) \quad L\mathcal{X} = \mathcal{Y}M$$

on  $\Omega_0^*$ . This completes the proof of the theorem.  $\square$

*Remark 1.2.* As is evident from the above proof of Theorem 1.6, Theorem 1.6 may be understood as a Borel-transformed version of Theorem 1.3. Actually it follows from (1.59), (1.81) and (1.73) with  $\Gamma$  being 1 that, if we write down the Borel transform of  $(dx^{(0)}(\tilde{x}, \eta)/d\tilde{x})^{-1/2} \psi(x^{(0)}(\tilde{x}, \eta), \eta)$  in  $(x, y)$ -coordinate (not in  $(\tilde{x}, y)$ -coordinate) for a WKB solution of (1.33), we then find  $\mathcal{X}\psi_B(x, y)$  for the operator  $\mathcal{X}$  in Theorem 1.6.

In stating Theorem 1.6 we have considered the relation (1.55) only on  $\Omega_0^*$ . This is just because operators  $L$  and  $M$  contain singularities at  $x = 0$ . As is clear from the above construction, operators  $\mathcal{X}$  and  $\mathcal{Y}$  are well-defined on  $\Omega_0$ . Furthermore, as we will show in Appendix C, Proposition C.1 and Theorem B.1 in Appendix B entail Theorem 1.7 below. In stating the theorem, we let  $U$  (resp.,  $S_j$  ( $j = 1, 2, \dots, N$ )) denote an open set (resp., an analytic hypersurface) given by the following:

$$(1.93) \quad U = \{(x, y) \in \mathbb{C}^2; |x|, |y| < \delta\}$$

and

$$(1.94) \quad S_j = \{(x, y) \in U; y = s_j(x)\},$$

where  $\delta$  is a sufficiently small positive number. We also define

$$(1.95) \quad U^* = U - \left( \{(x, y) \in U; x = 0\} \cup \left( \bigcup_{j=1}^N S_j \right) \right).$$

**Theorem 1.7.** *Let  $\mathcal{X}$  be the microdifferential operator given by (1.59). Then its action upon a multi-valued analytic function  $\varphi(x, y)$  defined on  $U^*$  is represented as an integro-differential operator of the form*

$$(1.96) \quad \mathcal{X}\varphi(x, y) = \int_{y_0}^y K(x, y - y', \partial/\partial x) \varphi(x, y') dy',$$

where  $K(x, y, \partial/\partial x)$  is a differential operator of infinite order that is defined on  $\{(x, y) \in \mathbb{C}^2; |x| < C \text{ and } |y| < C' \text{ for some positive constants } C \text{ and } C'\}$ , and  $y_0$  is a constant that fixes the action of  $(\partial/\partial y)^{-1}$  as an integral operator. (See Figure 1.1 below.) The operator  $\mathcal{Y}$  given by (1.60) also enjoys a similar expression.

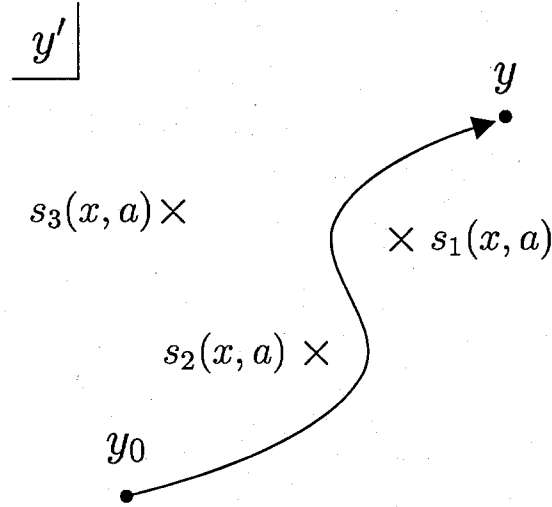


Figure 1.1.

*Remark 1.3.* When the operand  $\varphi$  is a Borel transformed WKB solution of a particular (i.e.,  $\alpha_0^{(0)} = 0$ )  $\infty$ -Whittaker equation, the relevant singular points are only  $y = s_1(x) = x/2$  and  $y = s_2(x) = -x/2$  ([Ko3]); that is, no fixed singularities are observed in this case. (See [KT2, Future Directions and Problems] for the notion and importance of fixed singularities (versus movable ones like the pair  $(s_1(x), s_2(x))$ ) as above.) On the other hand, the power of the expression (1.96) is most manifest when we study the structure of a Borel transformed WKB solution near its fixed singular points, as we will do in Section 5. Hence we do not discuss the action of operators upon Borel transformed WKB solutions of an MPPT equation with  $a = 0$  any more. One more reason to avoid here the further discussion of WKB solutions of an MPPT equation with  $a = 0$ , i.e., a ghost equation, is that we have not yet been able to find a universal and canonical way (like that to be

used in Theorem 2.2 in the next section) of normalizing WKB solutions applicable to all ghost equations. This is mainly due to the existence of infinitely many simple poles in  $S_{\text{odd}}$ , as is shown in the proof of Proposition 1.5 based on Corollary 1.4, and it stands in total contrast to the situation of MPPT equation with  $a \neq 0$ , which we will discuss in Section 2 and Section 5.

*Remark 1.4.* In this section we have analyzed the phenomena which are observed through the confluence of a simple pole and a simple turning point. It is also an interesting problem to study a situation where a double pole and a turning point merge. (Cf. [KKT1], where we study the confluence of a double pole and a simple turning point.)

## 2 Construction of the transformation to the canonical form, II. — the case where $a \neq 0$

The purpose of this section is to find a canonical form of an MPPT equation, i.e., a Schrödinger equation obtained by the addition of a term  $aq(x, a)/x$  to the potential of the ghost equation; to begin with we present the following

**Theorem 2.1.** *Let  $\tilde{Q}_j(\tilde{x}, a)$  ( $j = 0, 1, 2$ ) be holomorphic functions defined on a neighborhood of  $(\tilde{x}, a) = (0, 0)$ , and suppose that*

$$(2.1) \quad \tilde{Q}_0(0, a) \neq 0 \text{ if } a \neq 0,$$

and

$$(2.2) \quad \tilde{Q}_0(\tilde{x}, 0) = c_0^{(0)}\tilde{x} + O(\tilde{x}^2) \text{ holds with } c_0^{(0)} \text{ being a constant different from } 0.$$

Then there exist an open neighborhood  $U$  of  $\tilde{x} = 0$ , an open neighborhood  $V$  of  $a = 0$ , holomorphic functions  $x_k^{(j)}(\tilde{x})$  ( $j, k \geq 0$ ) defined on  $U$  and constants  $\alpha_k^{(j)}$  for which the following conditions (2.3) ~ (2.8) are satisfied:

$$(2.3) \quad \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right) (0) \neq 0,$$

$$(2.4) \quad x_k^{(j)}(0) = 0 \text{ for every } j \text{ and } k,$$

$$(2.5) \quad x_k(\tilde{x}, a) = \sum_{j \geq 0} x_k^{(j)}(\tilde{x}) a^j \quad \text{is holomorphic on } U \times V,$$

$$(2.6) \quad \alpha_k(a) = \sum_{j \geq 0} \alpha_k^{(j)} a^j \quad \text{is holomorphic on } V,$$

$$(2.7) \quad x(\tilde{x}, a, \eta) = \sum_{k \geq 0} x_k(\tilde{x}, a) \eta^{-k} \quad \text{and} \\ \alpha(a, \eta) = \sum_{k \geq 0} \alpha_k(a) \eta^{-k} \quad \text{are Borel transformable series,}$$

$$(2.8)$$

$$\begin{aligned} & \tilde{x}^{-1} \tilde{Q}_0(\tilde{x}, a) + \eta^{-1} \tilde{x}^{-1} \tilde{Q}_1(\tilde{x}, a) + \eta^{-2} \tilde{x}^{-2} \tilde{Q}_2(\tilde{x}, a) \\ &= \left( \frac{\partial x(\tilde{x}, a, \eta)}{\partial \tilde{x}} \right)^2 \left( \frac{1}{4} + \frac{\alpha(a, \eta)}{x(\tilde{x}, a, \eta)} + \eta^{-2} \frac{\tilde{Q}_2(0, a)}{x(\tilde{x}, a, \eta)^2} \right) - \frac{1}{2} \eta^{-2} \{x; \tilde{x}\}. \end{aligned}$$

In this section we only describe how to construct  $x_k^{(j)}(\tilde{x})$  and  $\alpha_k^{(j)}$  so that they formally satisfy (2.8); (2.5), (2.6) and (2.7) are proved in Appendix B (Theorem B.1).

The construction of  $\{x_k^{(j)}\}$  and  $\{\alpha_k^{(j)}\}$  makes use of the perturbation in powers of  $a$ , starting with  $x^{(0)}(\tilde{x}, \eta)$  and  $\alpha^{(0)}(\eta)$  constructed in the preceding section. We introduce  $z(\tilde{x}, a, \eta)$  given by

$$(2.9) \quad \tilde{x}^{-1} x(\tilde{x}, a, \eta)$$

to find (2.10) below in parallel with (1.15):

$$(2.10)$$

$$\begin{aligned} & \tilde{Q}_0(\tilde{x}, a) + \eta^{-1} \tilde{Q}_1(\tilde{x}, a) = \left( \frac{dx}{d\tilde{x}} \right)^2 \left( \frac{\tilde{x}}{4} + \frac{\alpha(a, \eta)}{z} \right) \\ & + \eta^{-2} \left( -\tilde{R}_2(\tilde{x}, a) + 2\tilde{Q}_2(0, a) \frac{z\tilde{x}}{z} + \tilde{Q}_2(0, a) \tilde{x} \left( \frac{z\tilde{x}}{z} \right)^2 - \frac{1}{2} \tilde{x} \{x; \tilde{x}\} \right), \end{aligned}$$

where

$$(2.11) \quad \tilde{R}_2(\tilde{x}, a) = (\tilde{Q}_2(\tilde{x}, a) - \tilde{Q}_2(0, a)) / \tilde{x}.$$

As (1.16) shows,  $(z^{(0)})^{-1}$  is a well-defined (formal) series in  $\eta^{-1}$  thanks to (1.21); hence  $z^{-1}$  is a well-defined formal power series of  $a$ :

$$(2.12) \quad \begin{aligned} z^{-1} &= (z^{(0)} + az^{(1)} + a^2z^{(2)} + \dots)^{-1} \\ &= (z^{(0)})^{-1} \left( 1 - a \left( \frac{z^{(1)}}{z^{(0)}} + a \frac{z^{(2)}}{z^{(0)}} + \dots \right) \right. \\ &\quad \left. + a^2 \left( \frac{z^{(1)}}{z^{(0)}} + a \frac{z^{(2)}}{z^{(0)}} + \dots \right)^2 + \dots \right). \end{aligned}$$

Thus, if we let  $R$  denote the coefficient of  $\eta^{-2}$  in the right-hand side of (2.10), we find it can be formally expanded as a power series of  $a$ :

$$(2.13) \quad R = R^{(0)} + aR^{(1)} + a^2R^{(2)} + \dots,$$

where

$$(2.14) \quad R^{(N)} \text{ is free from } a \text{ and expressed in terms of } z^{(j_0)}, z_{\tilde{x}}^{(j_1)}, z_{\tilde{x}\tilde{x}}^{(j_2)}, z_{\tilde{x}\tilde{x}\tilde{x}}^{(j_3)} \text{ } (0 \leq j_0, j_1, j_2, j_3 \leq N) \text{ and } \tilde{x};$$

furthermore (2.14) entails

$$(2.15) \quad \text{the coefficient } R_l^{(N)} \text{ of } \eta^{-l} \text{ in } R^{(N)} \text{ is expressed in terms of } \tilde{x} \text{ and } z_k^{(j)} \text{ and its derivatives with } 0 \leq j \leq N \text{ and } 0 \leq k \leq l-2.$$

Here  $z_k^{(j)}$  stands for the coefficient of  $\eta^{-k}$  of  $z^{(j)}$ .

Theorem 1.1 shows that  $x^{(0)}$  and  $z^{(0)} = \tilde{x}^{-1}x^{(0)}$  satisfy (2.10) with  $a = 0$ . The comparison of coefficients of  $a^1$  in (2.10) leads to

$$(2.16) \quad \begin{aligned} &\frac{\partial}{\partial a} (\tilde{Q}_0(\tilde{x}, a) + \eta^{-1}\tilde{Q}_1(\tilde{x}, a)) \Big|_{a=0} \\ &= \frac{\tilde{x}}{2} (x_{\tilde{x}}^{(0)} x_{\tilde{x}}^{(1)}) + \frac{2\alpha^{(0)}}{z^{(0)}} (x_{\tilde{x}}^{(0)} x_{\tilde{x}}^{(1)}) + (x_{\tilde{x}}^{(0)})^2 \frac{\alpha^{(1)}}{z^{(0)}} \\ &\quad - (x_{\tilde{x}}^{(0)})^2 \frac{\alpha^{(0)} z^{(1)}}{z^{(0)2}} + \eta^{-2} R^{(1)}. \end{aligned}$$

In what follows we let  $\tilde{Q}_k^{(j)}(\tilde{x})$  ( $k = 0, 1$ ) denote the following:

$$(2.17) \quad \frac{1}{j!} \frac{\partial^j}{\partial a^j} \tilde{Q}_k(\tilde{x}, a) \Big|_{a=0}.$$

Let us first pick up every coefficient of  $\eta^0$  in (2.16), including some terms which actually vanish:

$$(2.16.0) \quad \tilde{Q}_0^{(1)}(\tilde{x}) = \frac{\tilde{x}}{2} \left( \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_0^{(1)}}{d\tilde{x}} \right) + \frac{2\alpha_0^{(0)}}{z_0^{(0)}} \left( \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_0^{(1)}}{d\tilde{x}} \right) \\ + \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \frac{\alpha_0^{(1)}}{z_0^{(0)}} - \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \frac{\alpha_0^{(0)} z_0^{(1)}}{z_0^{(0)2}}.$$

In the right-hand side of (2.16.0) the second term and the fourth term vanish because  $\alpha_0^{(0)}$  vanishes by (1.19). Hence, by setting  $\tilde{x} = 0$  in (2.16.0), we obtain

$$(2.18) \quad \tilde{Q}_0^{(1)}(0) = \alpha_0^{(1)} z_0^{(0)}(0).$$

Choosing  $\alpha_0^{(1)}$  as above, we find a holomorphic function  $h(\tilde{x})$  that satisfies

$$(2.19) \quad \tilde{Q}_0^{(1)}(0) - \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \frac{\alpha_0^{(1)}}{z_0^{(0)}} = \tilde{x} h(\tilde{x}).$$

Hence, by dividing (2.16.0) by  $\tilde{x}$ , we arrive at

$$(2.20) \quad \frac{1}{2} \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_0^{(1)}}{d\tilde{x}} = h(\tilde{x}).$$

Then we solve (2.20) with the initial condition

$$(2.21) \quad x_0^{(1)}(0) = 0.$$

Thus we find a solution  $x_0^{(1)}$  such that  $z_0^{(1)} = \tilde{x}^{-1} x_0^{(1)}$  is holomorphic near  $\tilde{x} = 0$  and that satisfies (2.16.0).

Next we collect terms of degree  $-1$  in  $\eta$  in (2.16); this time we dispose of terms containing  $\alpha_0^{(0)}$  as a factor. Then we find

$$(2.16.1)$$

$$\tilde{Q}_1^{(1)}(\tilde{x})$$



$$\begin{aligned}
&= \frac{\tilde{x}}{2} \left( \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_1^{(1)}}{d\tilde{x}} + \frac{dx_1^{(0)}}{d\tilde{x}} \frac{dx_0^{(1)}}{d\tilde{x}} \right) + 2 \left( \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_0^{(1)}}{d\tilde{x}} \right) \frac{\alpha_1^{(0)}}{z_0^{(0)}} \\
&+ 2 \left( \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_1^{(0)}}{d\tilde{x}} \right) \frac{\alpha_0^{(1)}}{z_0^{(0)}} + \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \left( \frac{\alpha_1^{(1)}}{z_0^{(0)}} - \frac{\alpha_0^{(1)} z_1^{(0)}}{z_0^{(0)2}} \right) \\
&- \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \frac{\alpha_1^{(0)} z_0^{(1)}}{z_0^{(0)2}} \\
&= \left[ \frac{\tilde{x}}{2} \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_1^{(1)}}{d\tilde{x}} + \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \frac{\alpha_1^{(1)}}{z_0^{(0)}} \right] \\
&+ \left[ \frac{\tilde{x}}{2} \frac{dx_1^{(0)}}{d\tilde{x}} \frac{dx_0^{(1)}}{d\tilde{x}} + 2 \left( \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_0^{(1)}}{d\tilde{x}} \right) \frac{\alpha_1^{(0)}}{z_0^{(0)}} + 2 \left( \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_1^{(0)}}{d\tilde{x}} \right) \frac{\alpha_0^{(1)}}{z_0^{(0)}} \right. \\
&\quad \left. - \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \frac{\alpha_0^{(1)} z_1^{(0)}}{z_0^{(0)2}} - \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \frac{\alpha_1^{(0)} z_0^{(1)}}{z_0^{(0)2}} \right].
\end{aligned}$$

Hence (2.16.1) evaluated at  $\tilde{x} = 0$  reads as follows:

$$\begin{aligned}
(2.22) \quad &\tilde{Q}_1^{(1)}(0) \\
&= z_0^{(0)}(0) \alpha_1^{(1)} + 2z_0^{(1)}(0) \alpha_1^{(0)} + 2z_1^{(0)}(0) \alpha_0^{(1)} - \alpha_0^{(1)} z_1^{(0)}(0) - \alpha_1^{(0)} z_0^{(1)}(0) \\
&= z_0^{(0)}(0) \alpha_1^{(1)} + z_0^{(1)}(0) \alpha_1^{(0)} + z_1^{(0)}(0) \alpha_0^{(1)}.
\end{aligned}$$

Since all terms in (2.22) are, except for  $z_0^{(0)}(0) \alpha_1^{(1)}$ , values of functions which have already been fixed, (2.22) fixes the constant  $\alpha_1^{(1)}$ . Furthermore this choice of  $\alpha_1^{(1)}$  enables us to divide (2.16.1) by  $\tilde{x}$  to find a differential equation of the form

$$(2.23) \quad \frac{dx_1^{(1)}(\tilde{x})}{d\tilde{x}} = f(\tilde{x})$$

for a holomorphic function  $f(\tilde{x})$  defined near the origin. We then solve (2.23) with the initial condition

$$(2.24) \quad x_1^{(1)}(0) = 0$$

to obtain the required  $x_1^{(1)}(\tilde{x})$ . The treatment of terms of  $\eta^{-l}$  in (2.16) can be done in a similar way; we first find

$$(2.16.l) \quad (l \geq 2)$$

$$\begin{aligned} 0 = & \frac{\tilde{x}}{2} \left( \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_l^{(1)}}{d\tilde{x}} + F_l \right) + \left( \frac{2\alpha_0^{(0)}}{z_0^{(0)}} \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_l^{(1)}}{d\tilde{x}} + G_l \right) \\ & + \left( \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \frac{\alpha_l^{(1)}}{z_0^{(0)}} + H_l \right) - \left( \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \frac{\alpha_0^{(0)} z_l^{(1)}}{(z_0^{(0)})^2} + K_l \right) + R_l^{(1)}, \end{aligned}$$

where  $F_l$  etc. are respectively collections of terms of degree  $l$  in  $\eta^{-1}$  that originate from  $(x_{\tilde{x}}^{(0)} x_{\tilde{x}}^{(1)})$  etc. and that have been already fixed (like  $(dx_j^{(0)}/d\tilde{x}) (dx_k^{(1)}/d\tilde{x})$  ( $j+k=l$ ,  $0 \leq k \leq l-1$ ). In the above, in order to manifest the origin of  $G_l$  and  $K_l$  we have included terms which are actually 0, i.e., terms multiplied by  $\alpha_0^{(0)}$ . Thus (2.16.l) assumes the following form:

$$(2.25) \quad \frac{\tilde{x}}{2} \left( \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_l^{(1)}}{d\tilde{x}} \right) + \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \frac{\alpha_l^{(1)}}{z_0^{(0)}} + L_l = 0,$$

where  $L_l$  is a sum of terms which have already been fixed. Thus we should, and really do, choose

$$(2.26) \quad \alpha_l^{(1)} = - \left( \frac{1}{z_0^{(0)}} L_l \right) \Big|_{\tilde{x}=0}.$$

Then dividing (2.25) by  $\tilde{x}$  we obtain

$$(2.27) \quad \left( \frac{1}{2} \frac{dx_0^{(0)}}{d\tilde{x}} \right) \frac{dx_l^{(1)}}{d\tilde{x}} = h(\tilde{x})$$

with a holomorphic function  $h$  near the origin. Hence we can solve (2.27) with the initial condition  $x_l^{(1)}(0) = 0$ . Then the resulting function  $x_l^{(1)}$  together with the constant  $\alpha_l^{(1)}$  satisfies (2.16.l).

It is now evident that we can construct  $\{\alpha_k^{(j)}, x_k^{(j)}\}$  for any  $(j, k)$  by the same procedure. Actually the comparison of the coefficients of  $a^N$  gives us an equation  $(E_N)$ , and the computation of the coefficients of  $\eta^{-l}$  in  $(E_N)$  presents the equation  $(E_N, l)$  to be resolved. In the equation  $(E_N, l)$ ,  $\{x_k^{(j)}, z_k^{(j)}, \alpha_k^{(j)}\}$  are regarded to be known objects if

(i)  $j \leq N - 1$

or

(ii)  $j = N, k \leq l - 1$ .

The concrete form of  $(E_N, l)$  is as follows;

$$(2.28) \quad 0 = \frac{\tilde{x}}{2} \frac{dx_0^{(0)}}{d\tilde{x}} \frac{dx_l^{(N)}}{d\tilde{x}} + \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \frac{\alpha_l^{(N)}}{z_0^{(0)}} + (\text{known functions}).$$

Here we note that  $-\tilde{Q}_l^{(N)}$  is included among known functions when  $l$  is 0 or 1. Thus we first fix  $\alpha_l^{(N)}$  so that the equation (2.28) is divisible by  $\tilde{x}$ , and then the equation for  $x_l^{(N)}$  obtained by the division by  $\tilde{x}$  assumes the normal form. Thus we can solve the equation with the initial condition  $x_l^{(N)}(0) = 0$ . Thus we can construct  $x(\tilde{x}, a, \eta) = \sum_{j,k \geq 0} x_k^{(j)}(\tilde{x}) a^j \eta^{-k}$

and  $\alpha(a, \eta) = \sum_{j,k \geq 0} \alpha_k^{(j)} a^j \eta^{-k}$  that satisfy (2.8). The convergence of these

series in  $a$  and their Borel transformability concerning  $\eta$  are assured by Theorem B.1 in Appendix B.

□

*Remark 2.1.* (i) It is worth emphasizing that the growth order properties of  $\{x_k^{(j)}, \alpha_k^{(j)}\}$  as  $j$  tends to  $\infty$  and those as  $k$  tends to  $\infty$  are substantially different despite the fact that the construction of  $\{x_k^{(j)}, \alpha_k^{(j)}\}$  can be done in a symmetric way with respect to indexes  $j$  and  $k$ ; the equation for  $x_l^{(N)}$  can be found by first writing down the equation  $(\mathcal{E}_l)$  through the comparison of the coefficients of  $\eta^{-l}$  under the assumption that all coefficients of  $\eta^{-l'}$  ( $l' \leq l - 1$ ) are known and then finding out the required equation by the comparison of the coefficients of  $a^N$  in  $(\mathcal{E}_l)$  under the assumption that all the coefficients of  $a^{N'}$  ( $N' \leq N - 1$ ) in  $(\mathcal{E}_l)$  are known. The asymmetry of the growth order is tied up with the estimation of higher order derivatives contained in the seemingly

ancillary term  $\eta^{-2}\tilde{x}\{x;\tilde{x}\}/2$  in (2.10). (See Remark B.2 in Appendix B.)

(ii) It is also noteworthy that the convergence property (2.5) (with  $k = 0$ ) automatically entails the following geometric result: it follows from (2.3) and (2.8) that the solution  $\tilde{x} = \tilde{x}_0(a)$  of the equation

$$(2.29) \quad x_0(\tilde{x}, a) + 4\alpha_0(a) = 0,$$

whose existence is guaranteed again by (2.3) for  $|a|$  sufficiently small, satisfies

$$(2.30) \quad \tilde{Q}_0(\tilde{x}_0(a), a) = 0.$$

Otherwise stated, the function  $x = x_0(\tilde{x}, a)$  maps the simple turning point of the given MPPT equation to that of the  $\infty$ -Whittaker equation. Note that it should be difficult to image such a picture only by tracing the algebraic construction of  $x(\tilde{x}, a, \eta)$  given above.

In parallel with the reasoning in Section 1, Theorem 2.1 gives us several results on the structure of WKB solutions of a generic (i.e.,  $a \neq 0$ ) MPPT equation. Among other things, we first note Theorem 2.2 below. To obtain Theorem 2.2 we make essential use of the simple turning point  $\tilde{x} = \tilde{x}_0(a)$ ; it is known ([AKT2, Proposition 1.6]) that  $\tilde{S}_{\text{odd}}$ , the odd part of a solution  $\tilde{S}$  of the associated Riccati equation, has singularities of square-root type near a simple turning point  $\tilde{x} = t$  in general. Hence the integral

$$(2.31) \quad \int_t^{\tilde{x}} \tilde{S}_{\text{odd}} d\tilde{x}$$

is well-defined ([KT2, (2.24)]), and we use this integral to define a WKB solution  $\psi_{\pm}$  of an MPPT equation that is normalized at the simple turning point in question, that is,

$$(2.32) \quad \tilde{\psi}_{\pm}(\tilde{x}, a, \eta) = \frac{1}{\sqrt{\tilde{S}_{\text{odd}}}} \exp \left( \pm \int_{\tilde{x}_0(a)}^{\tilde{x}} \tilde{S}_{\text{odd}}(\tilde{x}, a, \eta) d\tilde{x} \right).$$

As is shown in [KT2, Section 2], we can deduce Theorem 2.2 below from Theorem 2.1 using the above normalization of WKB solutions.

**Theorem 2.2.** *Let  $\tilde{\psi}_+(\tilde{x}, a, \eta)$  be a WKB solution of an MPPT equation (2.33) below, and suppose that it is normalized at its simple turning point as above.*

$$(2.33) \quad \left( \frac{d^2}{d\tilde{x}^2} - \eta^2 \tilde{Q}(\tilde{x}, a, \eta) \right) \tilde{\psi}(\tilde{x}, a, \eta) = 0 \quad (a \neq 0),$$

where

$$(2.34) \quad \tilde{Q} = \frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}} + \eta^{-1} \frac{\tilde{Q}_1(\tilde{x}, a)}{\tilde{x}} + \eta^{-2} \frac{\tilde{Q}_2(\tilde{x}, a)}{\tilde{x}^2}$$

satisfies (2.1) and (2.2). Then, for a sufficiently small  $a$  ( $\neq 0$ ), we can find a WKB solution  $\psi_+(x, \eta; \alpha(a, \eta))$  of the  $\infty$ -Whittaker equation

$$(2.35) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{1}{4} + \frac{\alpha(a, \eta)}{x} + \eta^{-2} \frac{\tilde{Q}_2(0, a)}{x^2} \right) \right) \psi(x, \eta; \alpha(a, \eta)) = 0$$

that is also normalized at its simple turning point  $x = -4\alpha_0(a)$  so that it satisfies the following relation:

$$(2.36) \quad \tilde{\psi}_+(\tilde{x}, a, \eta) = \left( \frac{\partial x(\tilde{x}, a, \eta)}{\partial \tilde{x}} \right)^{-1/2} \psi_+(x(\tilde{x}, a, \eta), \eta; \alpha(a, \eta)),$$

where  $x(\tilde{x}, a, \eta)$  and  $\alpha(a, \eta)$  are the series constructed in Theorem 2.1.

The proof of Theorem 2.2 is essentially the same as that of Corollary 2.18 in [KT2], and we omit it here. We call the attention of the reader to the fact that normalization of the WKB solution  $\tilde{\psi}(\tilde{x}, \eta)$  is not fixed in the corresponding result in Section 1, i.e., Theorem 1.3.

As there is no problem related to the normalization concerning solutions of the Riccati equation, we can obtain the results similar to Theorem 1.2 and Corollary 1.4 by using the series  $x(\tilde{x}, a, \eta)$  and  $\alpha(a, \eta)$  constructed in Theorem 2.1. For example we obtain the following Theorem 2.3 as a counterpart of Corollary 1.4.

**Theorem 2.3.** *Let  $S$  and  $\tilde{S}$  respectively be a solution of*

$$(2.37) \quad S^2 + \frac{dS}{dx} = \eta^2 \left( \frac{1}{4} + \frac{\alpha(a, \eta)}{x} + \eta^{-2} \frac{\tilde{Q}_2(0, a)}{x^2} \right)$$

and

$$(2.38) \quad \tilde{S}^2 + \frac{d\tilde{S}}{d\tilde{x}} = \eta^2 \tilde{Q}(\tilde{x}, a, \eta),$$

and suppose that

$$(2.39) \quad \arg \tilde{S}_{-1}(\tilde{x}, a) = \arg \left( \frac{dx_0(\tilde{x}, a)}{d\tilde{x}} S_{-1}(x_0(\tilde{x}, a), \alpha_0(a)) \right)$$

holds. Then they satisfy

$$(2.40) \quad \tilde{S}_{\text{odd}}(\tilde{x}, a, \eta) = \left( \frac{dx(\tilde{x}, a, \eta)}{d\tilde{x}} \right) S_{\text{odd}}(x(\tilde{x}, a, \eta), \alpha(a, \eta), \eta).$$

We refer the reader to [KT2, Section 2] for the proof.

Now we note the following important

**Lemma 2.4.** *Let  $S$  be a solution of (2.37) whose top degree part  $S_{-1}(x, \alpha_0)$  is chosen so that it is positive for positive  $x$  and  $\alpha_0$ . Then we find*

$$(2.41) \quad \oint_{\Gamma(\alpha_0)} S_{\text{odd}}(x, \alpha(a, \eta), \eta) dx = 2\pi i \alpha(a, \eta) \eta,$$

where  $\Gamma(\alpha_0)$  designates a closed curve in the cut plane shown in Figure 2.1 below.

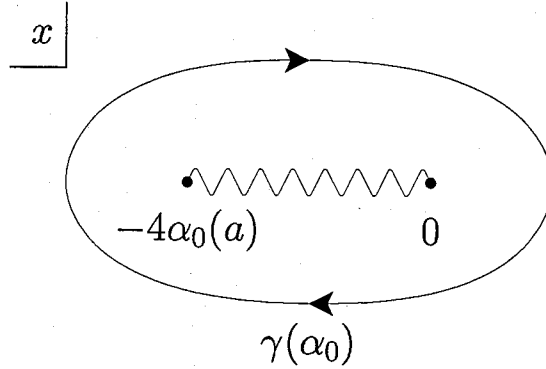


Figure 2.1.

*Proof.* By a straightforward computation we find

$$(2.42) \quad S_{-1}^{(\pm)} = \pm \frac{1}{2} \sqrt{\frac{x + 4\alpha_0}{x}},$$

$$(2.43) \quad S_0^{(\pm)} = \frac{\alpha_0}{x(x + 4\alpha_0)} \pm \frac{\alpha_1}{\sqrt{x}\sqrt{x + 4\alpha_0}}.$$

Then we can readily find the concrete form of  $S_l^{(\pm)}$  ( $l \geq 1$ ) by the induction on  $l$ :

$$(2.44) \quad S_l^{(\pm)} = \sum c_{p,q}^{(\pm)}(l) x^{-\frac{p}{2}} (x + 4\alpha_0)^{-\frac{q}{2}},$$

where  $c_{p,q}^{(\pm)}(l)$  are constants,  $p$  and  $q$  are integers that satisfy

$$(2.45) \quad p + q = 2m, \quad m = l + 1, l, \dots, 1.$$

Furthermore we see that the surviving constant  $c_{p,q}^{(\pm)}(l)$  with  $p + q = 2$  is only for  $p = q = 1$  and that

$$(2.46) \quad c_{1,1}^{(\pm)}(l) = \alpha_{l+1}.$$

By computing the residue at  $\infty$  of  $x^{-p/2}(x + 4\alpha_0)^{-q/2}$ , we find

$$(2.47) \quad \oint_{\Gamma(\alpha_0)} \sqrt{\frac{x + 4\alpha_0}{x}} dx = 4\pi i \alpha_0,$$

$$(2.48) \quad \oint_{\Gamma(\alpha_0)} \frac{dx}{\sqrt{x(x + 4\alpha_0)}} = 2\pi i$$

and

$$(2.49) \quad \oint_{\Gamma(\alpha_0)} \frac{dx}{x^{p/2}(x + 4\alpha_0)^{q/2}} = 0 \quad \text{if } p + q = 2m \geq 4.$$

Therefore (2.43), (2.44) and (2.46) imply

$$(2.50) \quad \oint_{\Gamma(\alpha_0)} S_{\text{odd}} dx = 2\pi i \alpha(\eta) \eta.$$

□

Combining Theorem 2.3 and Lemma 2.4 we obtain the following

**Proposition 2.5.** *Let  $\tilde{S}$  be a solution of the Riccati equation (2.38) that is associated with a generic MPPT equation. Then with an appropriate choice of the branch of  $\tilde{S}_{-1}$ , we find*

$$(2.51) \quad \oint_{\tilde{\Gamma}(a)} \tilde{S}_{\text{odd}}(\tilde{x}, a, \eta) d\tilde{x} = 2\pi i \alpha(a, \eta) \eta,$$

where  $\tilde{\Gamma}(a)$  designates a closed curve in the cut plane shown in Figure 2.2.

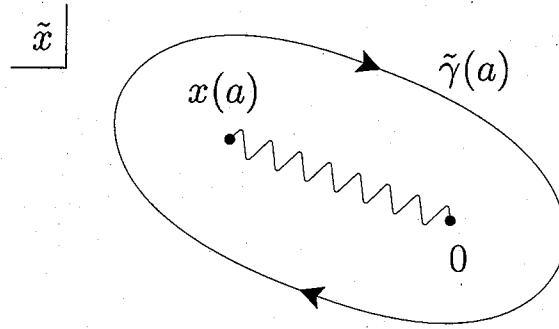


Figure 2.2.

In view of the logical structure of the discussions in Section 1, one naturally expects that some intertwining microdifferential operators between a generic MPPT operator and an  $\infty$ -Whittaker operator may be constructed with the help of the series  $x(\tilde{x}, a, \eta)$  and  $\alpha(a, \eta)$  constructed in Theorem 2.1. This expectation can be readily validated if we introduce a holomorphic function  $g(x, a)$ , instead of  $g(x)$  given in (1.48), which satisfies

$$(2.52) \quad x = x_0(g(x, a), a), \quad \tilde{x} = g(x_0(\tilde{x}, a), a)$$

on a neighborhood of  $(x, a) = (0, 0)$ . The unique existence of such a holomorphic function is guaranteed by (2.3), and hence we find

$$(2.53) \quad g(x, 0) = g(x).$$



The proof of Theorems 2.6 and 2.7 below are essentially the same as that of Theorems 1.6 and 1.7. Here we only repeat the definitions of relevant operators for the convenience of the reader. First  $L$  designates a Borel transformed generic MPPT operator expressed in  $(x, a, y)$ -coordinate and then multiplied by  $(\partial g / \partial x)^2$ . That is,

$$(2.54) \quad L = \frac{\partial^2}{\partial x^2} - \left( \frac{\partial^2 g / \partial x^2}{\partial g / \partial x} \right) \frac{\partial}{\partial x} - \left( \frac{\partial g}{\partial x} \right)^2 \tilde{Q} \left( g(x, a), a, \frac{\partial}{\partial y} \right).$$

In parallel with (1.52) we designate by  $M$  the Borel transformed  $\infty$ -Whittaker equation, that is,

$$(2.55) \quad \frac{\partial^2}{\partial x^2} - \left( \frac{1}{4} + \frac{\alpha(a, \partial / \partial y)}{x} \right) \frac{\partial^2}{\partial y^2} - \frac{\tilde{Q}_2(0, a)}{x^2}.$$

Using the series  $x(\tilde{x}, a, \eta) = \sum_{k \geq 0} x_k(\tilde{x}, a) \eta^{-k}$  constructed in Theorem 2.1, we define another series  $r(x, a, \eta)$  by

$$(2.56) \quad \sum_{k \geq 1} x_k(g(x, a), a) \eta^{-k}.$$

Then, using the same reasoning as in the proof of Theorems 1.6, we obtain Theorem 2.6 below with the help of Theorem B.1 in Appendix B.

**Theorem 2.6.** *There exist invertible microdifferential operators  $\mathcal{X}$  and  $\mathcal{Y}$  with a holomorphic parameter  $a$  that satisfy*

$$(2.57) \quad L\mathcal{X} = \mathcal{Y}M$$

*near  $(x, a) = (0, 0)$  with the exception of  $x\eta = 0$ . The concrete form of operators  $\mathcal{X}$  and  $\mathcal{Y}$  are as follows:*

$$(2.58) \quad \mathcal{X} =: \left( \frac{\partial g}{\partial x} \right)^{1/2} \left( 1 + \frac{\partial r}{\partial x} \right)^{-1/2} \exp(r(x, a, \eta)\xi) :,$$

$$(2.59) \quad \mathcal{Y} =: \left( \frac{\partial g}{\partial x} \right)^{1/2} \left( 1 + \frac{\partial r}{\partial x} \right)^{3/2} \exp(r(x, a, \eta)\xi) :.$$

*Remark 2.2.* In parallel with Remark 1.2, we see from (2.56) and (2.58) that Theorem 2.6 is a Borel-transformed version of Theorem 2.2;  $\mathcal{X}\psi_{+,B}$  is the Borel transform of  $(\partial x(\tilde{x}, a, \eta)/\partial \tilde{x})^{-1/2} \psi_+(x(\tilde{x}, a, \eta), \eta; \alpha(\alpha, \eta))$  written down in  $(x, y)$ -coordinate (not in  $(\tilde{x}, y)$ -coordinate), where  $\psi_+$  is a WKB solution of the  $\infty$ -Whittaker equation (2.35).

Furthermore Theorem B.1 together with Proposition C.1 entails the following

**Theorem 2.7.** *The action of the microdifferential operator  $\mathcal{X}$  upon the Borel transformed WKB solution  $\psi_{+,B}$  of the  $\infty$ -Whittaker equation is expressed as an integro-differential operator of the following form:*

$$(2.60) \quad \mathcal{X}\psi_{+,B} = \int_{y_0}^y K(x, a, y - y', \partial/\partial x) \psi_{+,B}(x, a, y') dy',$$

where  $K(x, a, y, \partial/\partial x)$  is a differential operator of infinite order that is defined on  $\{(x, a, y) \in \mathbb{C}^3; (x, a) \in \omega \text{ for an open neighborhood } \omega \text{ of the origin and } |y| < C \text{ for some positive constant } C\}$ , and  $y_0$  is a constant that fixes the action of  $(\partial/\partial y)^{-1}$  as an integral operator.

*Remark 2.3.* Since  $\alpha_0(a)$  tends to 0 as  $a$  tends to 0, Theorem B.1 guarantees that we can choose  $\omega$  to be of the form  $\omega_0 \times D$ , where

$$(2.61) \quad D = \{a \in \mathbb{C}; |a| < \delta \text{ for some positive constant } \delta\},$$

and

$$(2.62) \quad \omega_0 \text{ is a simply connected open set in } \mathbb{C} \text{ that contains the origin and the simple turning point of the } \infty\text{-Whittaker equation, i.e., } x = -4\alpha_0(a), \text{ for every } a \text{ in } D.$$

Then the integral operator in the right-hand side of (2.60) acts on any multi-valued analytic function defined on  $\omega_0 \times D \times \{y \in \mathbb{C}; |y - y_0| < C\}$ .

### 3 Analytic properties of WKB solutions of the Whittaker equation with a large parameter

In order to analyze WKB solutions of the  $\infty$ -Whittaker equation, which plays a central role in subsequent sections as the canonical form of an MPPT equation for  $a \neq 0$ , we first recall several basic facts about

WKB solutions of the Whittaker equation with a large parameter  $\eta$ , i.e., the equation:

$$(3.1) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{1}{4} + \frac{\alpha}{x} + \eta^{-2} \frac{\gamma(\gamma+1)}{x^2} \right) \right) \psi = 0,$$

where  $\alpha (\neq 0)$  and  $\gamma$  are complex numbers. We refer the reader to [KoT] for the details. As [KoT] has recently found, the Voros coefficient  $\phi(\alpha, \gamma; \eta)$  for (3.1) can be explicitly expressed in terms of the Bernoulli numbers and its Borel transform  $\phi_B(\alpha, \gamma; y)$  is concretely written down by elementary functions. Here the Voros coefficient means, by definition,

$$(3.2) \quad \int_{-4\alpha}^{\infty} (S_{\text{odd}} - \eta S_{-1}) dx,$$

where  $S_{\text{odd}}$  designates the odd part of a solution  $S$  of the Riccati equation associated with (3.1), that is,

$$(3.3) \quad S^2 + \frac{dS}{dx} = \eta^2 \left( \frac{1}{4} + \frac{\alpha}{x} + \eta^{-2} \frac{\gamma(\gamma+1)}{x^2} \right).$$

As we see in Theorem 3.1 below, the concrete form of  $\phi_B(\alpha, \gamma; y)$  enables us to find the singularity structure of Borel transformed WKB solution of (3.1) through the relation

$$(3.4) \quad \psi_+(x, \eta) = (\exp(\phi(\alpha, \gamma; \eta))) \psi_+^{(\infty)}(x, \eta),$$

where  $\psi_+(x, \eta)$  (resp.,  $\psi_+^{(\infty)}(x, \eta)$ ) designates the WKB solution of (3.1) that is normalized at the simple turning point  $x = -4\alpha$  (resp., at infinity); that is,

$$(3.5) \quad \psi_+(x, \eta) = \frac{1}{\sqrt{S_{\text{odd}}}} \exp \left( \int_{-4\alpha}^x S_{\text{odd}} dx \right)$$

and

$$(3.6) \quad \psi_+^{(\infty)}(x, \eta) = \frac{1}{\sqrt{S_{\text{odd}}}} \exp \left( \int_{-4\alpha}^x \eta S_{-1} dx + \int_{\infty}^x (S_{\text{odd}} - \eta S_{-1}) dx \right).$$

An important property of  $\psi_+^{(\infty)}(x, \eta)$  is that it is Borel summable on the condition that

$$(3.7) \quad \text{the path of integration from } \infty \text{ to } x \text{ in the right-hand side of (3.6) never touches a Stokes curve of (3.1).}$$

See [KoT] for the proof of the Borel summability of  $\psi_+^{(\infty)}(x, \eta)$ . See also [DDP1] and [DP] for the corresponding result for the Weber equation. Thus (3.4) implies that the computation of the alien derivative of  $\psi_+(x, \eta)$  is reduced to that of  $\exp \phi(\alpha, \gamma; \eta)$ . In order to compute the latter one we first recall the concrete form of  $\phi_B(\alpha, \gamma; y)$  and then employ the alien calculus ([P2], [Sa]) to obtain the required result.

Now, the result in [KoT] tells us the following:

$$(3.8) \quad \begin{aligned} \phi_B(\alpha, \gamma; y) &= \frac{1}{2y} \left( \frac{\exp(y/\alpha) + 1}{\exp(y/\alpha) - 1} \right) \cosh\left(\frac{\gamma y}{\alpha}\right) - \frac{\alpha}{y^2} + \frac{1}{2y} \sinh\left(\frac{\gamma y}{\alpha}\right). \end{aligned}$$

A straightforward computation shows that

$$(3.9) \quad \phi_B(\alpha, \gamma; y) = \frac{1}{2\alpha} \left( \frac{1}{6} + \gamma + \gamma^2 \right) + O(y) \quad \text{near } y = 0$$

and that

$$(3.10) \quad \begin{aligned} \phi_B(\alpha, \gamma; y) &= \left( \frac{\exp(2m\pi i \gamma) + \exp(-2m\pi i \gamma)}{4m\pi i} \right) \frac{1}{y - 2m\pi i \alpha} + O(1) \\ &\quad \text{near } y = 2m\pi i \alpha \quad (m : \text{a non-zero integer}). \end{aligned}$$

Thus  $\phi_B(\alpha, \gamma; y)$  is seen to be a single-valued analytic function with simple poles located at  $y = 2m\pi i \alpha$  ( $m \neq 0$ ). The computation of the alien derivative  $\Delta \phi$  of such a series, i.e., a series whose Borel transform is single-valued and only with simple poles, is exceptionally simple;

$$(3.11) \quad \Delta \phi = \sum_{m \geq 1} \Delta_{y=2m\pi i \alpha} \phi$$

with

$$(3.12) \quad \Delta_{y=2m\pi i \alpha} \phi = \frac{\exp(2m\pi i \gamma) + \exp(-2m\pi i \gamma)}{2m}.$$

(See [P1] and [Sa].) Hence, by using the alien calculus, we find

$$(3.13) \quad \Delta_{y=2m\pi i \alpha}(\exp \phi) = \frac{\exp(2m\pi i \gamma) + \exp(-2m\pi i \gamma)}{2m} \exp \phi.$$

(See [P1], [CNP] and [Sa].) For the convenience of the description of several formulae below we introduce

$$(3.14) \quad y_+(x) = \int_{-4\alpha}^x S_{-1} dx = \int_{-4\alpha}^x \sqrt{\frac{x+4\alpha}{4x}} dx.$$

Then, on the condition that (3.7) is satisfied, we find

$$(3.15) \quad \Delta(\exp(-y_+(x)\eta)\psi_+^{(\infty)}(x, \eta)) = 0.$$

Hence we conclude that

$$\begin{aligned} (3.16) \quad & \Delta_{y=-y_+(x)+2m\pi i\alpha}(\exp(-y_+(x)\eta)\psi_+(x, \eta)) \\ &= \Delta_{y=-y_+(x)+2m\pi i\alpha}(\exp(-y_+(x)\eta)\exp(\phi(\alpha, \gamma; \eta))\psi_+^{(\infty)}(x, \eta)) \\ &= \frac{\exp(2m\pi i\gamma) + \exp(-2m\pi i\gamma)}{2m} \\ & \quad \times (\exp(-y_+(x)\eta)\exp(\phi(\alpha, \gamma; \eta))\psi_+^{(\infty)}(x, \eta)) \\ &= \frac{\exp(2m\pi i\gamma) + \exp(-2m\pi i\gamma)}{2m} (\exp(-y_+(x)\eta)\psi_+(x, \eta)) \end{aligned}$$

holds if  $x$  is chosen so that the condition (3.7) may be satisfied.

Summing up the obtained results, we find the following

**Theorem 3.1.** *Let  $\psi_+(x, \eta)$  denote the WKB solution of the Whittaker equation that is normalized at the simple turning point  $x = -4\alpha$  as in (3.5). Then its Borel transform  $\psi_{+,B}(x, y)$  is singular at*

$$(3.17) \quad y = -y_+(x) + 2m\pi i\alpha \quad (m = 0, \pm 1, \pm 2, \dots),$$

where  $y_+(x)$  is the function given by (3.14), and its alien derivative there, i.e.,  $\Delta_{y=-y_+(x)+2m\pi i\alpha}\psi_+(x, \eta)$  satisfies the relation (3.18) below for  $x$  that can be connected with a point at infinity by a path that is contained in the interior of a Stokes region of the Whittaker equation.

$$\begin{aligned} (3.18) \quad & (\Delta_{y=-y_+(x)+2m\pi i\alpha}\psi_+)_B(x, y) \\ &= \frac{\exp(2m\pi i\gamma) + \exp(-2m\pi i\gamma)}{2m} \psi_{+,B}(x, y - 2m\pi i\alpha). \end{aligned}$$

## 4 Structure of WKB solutions of the $\infty$ -Whittaker equation

As Theorems 2.1, 2.2 and 2.7 show, the WKB-theoretic canonical form of an MPPT equation for  $a \neq 0$  is the  $\infty$ -Whittaker equation

$$(4.1) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{1}{4} + \frac{\alpha(a, \eta)}{x} + \eta^{-2} \frac{c(a)}{x^2} \right) \right) \tilde{\psi}(x, \eta; \alpha(a, \eta), c(a)) = 0,$$

where  $\alpha(a, \eta)$  satisfies the condition (B.3) and  $c(a)$  is  $\tilde{Q}_2(0, a)$ . Hence the study of singularity structure of Borel transformed WKB solutions of an MPPT equation for  $a \neq 0$  is reduced to the study of the corresponding objects of the  $\infty$ -Whittaker equation. Thus the analysis of the  $\infty$ -Whittaker equation is our next target, and by relating (4.1) with the Whittaker equation

$$(4.2) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{1}{4} + \frac{\alpha}{x} + \eta^{-2} \frac{c}{x^2} \right) \right) \psi(x, \eta; \alpha, c) = 0$$

we achieve the target. A crucial idea in achieving it is the use of microdifferential operators, which becomes possible thanks to the estimate (B.3) of  $\{\alpha_k^{(j)}\}$ . (See also (B.32.k.j).)

In what follows, to avoid technical complexities, we assume the following condition:

$$(4.3) \quad \left( \frac{\partial \tilde{Q}_0}{\partial a} \right) (0, 0) \neq 0.$$

This is a natural strengthening of the assumption (2.1); actually by using the Taylor expansion of  $\tilde{Q}_0(\tilde{x}, a)$ , one immediately sees that the assumption (4.3) together with (2.2) entails (2.1). It is also clear from (2.18) that (4.3) entails

$$(4.4) \quad \alpha_0^{(1)} \neq 0,$$

and hence we find by using (2.6)

$$(4.5) \quad \left. \frac{d\alpha_0(a)}{da} \right|_{a=0} \neq 0.$$

Therefore we may employ  $\alpha_0$  as an independent variable in substitution for  $a$ ; thus we regard  $\alpha_j(a)$  ( $j \geq 1$ ) as functions of  $\alpha_0$  in what follows.

Now, in order to relate the Borel transformed WKB solution  $\psi_B$  of the Whittaker equation (3.1) and the Borel transformed WKB solution  $\tilde{\psi}_B$  of the  $\infty$ -Whittaker equation, we rewrite a WKB solution  $\tilde{\psi}(x, \eta; \alpha(\alpha_0, \eta), c(\alpha_0))$  of (4.1) in the following manner:

$$(4.6) \quad \tilde{\psi}(x, \eta; \alpha(\alpha_0, \eta), c(\alpha_0)) = \left( \sum_{n \geq 0} \frac{(\alpha_1 \eta^{-1} + \alpha_2 \eta^{-2} + \dots)^n}{n!} \frac{\partial^n}{\partial \alpha_0^n} \psi(x, \eta; \alpha_0, c) \right) \Big|_{c=c(\alpha_0)},$$

where  $\psi(x, \eta; \alpha_0, c)$  designates a WKB solution of (4.2) with

$$(4.7) \quad \alpha = \alpha_0.$$

Then the estimate (B.3) that  $\alpha_k$ 's satisfy enables us to apply the Borel transformation to (4.6); we then find

$$(4.8) \quad \tilde{\psi}_B(x, y) = \left( \mathcal{A}\left(\alpha_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial \alpha_0}\right) \psi_B(x, y; \alpha_0, c) \right) \Big|_{c=c(\alpha_0)},$$

where

$$(4.9) \quad \mathcal{A}\left(\alpha_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial \alpha_0}\right) = \sum_{n \geq 0} \frac{(\alpha_1 (\partial/\partial y)^{-1} + \alpha_2 (\partial/\partial y)^{-2} + \dots)^n}{n!} \frac{\partial^n}{\partial \alpha_0^n}$$

is a well-defined microdifferential operator on

$$(4.10) \quad \{(y, \alpha_0; \eta, \theta) \in T^*\mathbb{C}^2; |\alpha_0| < \delta_0, \eta \neq 0\}$$

for some positive constant  $\delta_0$ . In what follows we identify  $\eta$  and  $\theta$  respectively with the symbol  $\sigma(\partial/\partial y)$  and the symbol  $\sigma(\partial/\partial \alpha_0)$ ; using these symbols we may write

$$(4.11) \quad \mathcal{A} =: \sum_{n \geq 0} \frac{(\alpha_1 \eta^{-1} + \alpha_2 \eta^{-2} + \dots)^n \theta^n}{n!} :.$$

In parallel with the above treatment of Borel transformed WKB solutions with the use of a microdifferential operator relevant to the parameter  $\alpha$ , the Borel transform  $V_B(y)$  of the exponential of the Voros coefficient of the  $\infty$ -Whittaker equation can be expressed in terms of

the corresponding function of the Whittaker equation in the following manner:

$$(4.12) \quad V_B(y) = (\mathcal{A}(\alpha_0, \partial/\partial y, \partial/\partial \alpha_0) ((\exp \phi(\alpha_0, c, \eta))_B)) \Big|_{c=c(\alpha_0)}.$$

*Remark 4.1.* Although the target variable is  $\alpha_0$ , not  $x$ , we can use the same reasoning as in Section 2 to see the concrete expression of the operator  $\mathcal{A}$  as an integro-differential operator; the right-hand side of (4.8) and (4.12) should be understood as a multi-valued analytic function acted upon by an integro-differential operator determined by the microdifferential operator  $\mathcal{A}$ . While the estimate (B.3) guarantees the existence of a common domain of definition of the operator as  $a$  tends to 0, the quantity  $\alpha_0(a)$  tends to 0 as  $a$  tends to 0. On the other hand (3.17) means that a fixed singular point of  $\psi_{+,B}(x, y)$  (“fixed” with respect to  $y = -y_+(x)$ ) is located at  $y = -y_+(x) + 2m\pi i\alpha$ . Thus each individual fixed singular point of  $\tilde{\psi}_{+,B}(x, y)$  is contained, for sufficiently small  $a$ , in the domain of definition of the integro-differential operator in question. Hence, in what follows, we do not worry about the existence of a sufficiently large domain of definition of the integro-differential operator; if necessary, we assume that  $a$  (or, equivalently  $\alpha_0$ ) is sufficiently close to 0.

Using the results obtained in the preceding section for the Whittaker equation we obtain the following

**Theorem 4.1.** *Let  $\tilde{\psi}_+(x, \eta)$  and  $\phi(\alpha(a), \gamma(a); \eta)$  respectively denote*

$$(4.13) \quad \frac{1}{\sqrt{\tilde{S}_{\text{odd}}}} \exp \left( \int_{-4\alpha_0(a)}^x \tilde{S}_{\text{odd}} dx \right)$$

and

$$(4.14) \quad \int_{-4\alpha_0(a)}^{\infty} (\tilde{S}_{\text{odd}} - \eta \tilde{S}_{-1}) dx,$$

where  $\tilde{S}_{\text{odd}}$  designates the odd part of a solution  $\tilde{S}$  of the following Riccati equation

$$(4.15) \quad \tilde{S}^2 + \frac{d\tilde{S}}{dx} = \eta^2 \left( \frac{1}{4} + \frac{\alpha(a)}{x} + \eta^{-2} \frac{\gamma(a)^2 + \gamma(a)}{x^2} \right)$$



with

$$(4.16) \quad \gamma(a)^2 + \gamma(a) = c(a).$$

Then the Borel transform  $\tilde{\psi}_{+,B}(x, y)$  of  $\tilde{\psi}_+(x, \eta)$  and the Borel transform  $V_B$  of the exponentiated Voros coefficient  $V = \exp(\phi(\alpha(a), \gamma(a); \eta))$  satisfy the following relations:

$$(4.17) \quad \begin{aligned} & (\Delta_{y=-y_+(x)+2m\pi i\alpha_0} \tilde{\psi}_+)_B(x, y) \\ &= \frac{\exp(2m\pi i\gamma(\alpha_0)) + \exp(-2m\pi i\gamma(\alpha_0))}{2m} \\ & \times : \exp(-2m\pi i(\alpha_1 + \alpha_2\eta^{-1} + \cdots)) : \tilde{\psi}_{+,B}(x, y - 2m\pi i\alpha_0), \end{aligned}$$

$$(4.18) \quad \begin{aligned} & (\Delta_{y=2m\pi i\alpha_0} V)_B(y) \\ &= \frac{\exp(2m\pi i\gamma(\alpha_0)) + \exp(-2m\pi i\gamma(\alpha_0))}{2m} \\ & \times : \exp(-2m\pi i(\alpha_1 + \alpha_2\eta^{-1} + \cdots)) : V_B(y - 2m\pi i\alpha_0), \end{aligned}$$

where  $m = 1, 2, 3, \dots$ , and  $y_+(x)$  denotes

$$(4.19) \quad \int_{-4\alpha_0}^x \sqrt{\frac{x + 4\alpha_0}{4x}} dx.$$

*Proof.* For the notational convenience let  $\mathcal{B}^{-1}\rho$  denote the inverse Borel transform of  $\rho$ . (This is just to avoid the use of the sign  $\Delta\rho$  when  $\rho$  is the Borel transform of a formal series  $\chi$ , although  $\Delta\rho$  is sometimes used to mean  $\Delta\chi$  in references in alien calculus.) Then it follows from (4.8) and the definition of the alien derivative that we obtain

$$(4.20) \quad \begin{aligned} & (\Delta_{y=-y_+(x)+2m\pi i\alpha_0} \tilde{\psi}_+)_B(x, y) \\ &= \left( \Delta_{y=-y_+(x)+2m\pi i\alpha_0} \mathcal{B}^{-1} \left( \mathcal{A}(\alpha_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial \alpha_0}) \psi_{+,B}(x, y; \alpha_0, c) \right) \right)_B(x, y) \Big|_{c=c(\alpha_0)} \\ &= \left( \mathcal{A}(\alpha_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial \alpha_0}) \left( (\Delta_{y=-y_+(x)+2m\pi i\alpha_0} \tilde{\psi}_+)_B(x, y, \alpha_0, c) \right) (x, y) \right) \Big|_{c=c(\alpha_0)}. \end{aligned}$$

Then it follows from Theorem 3.1 that the rightmost term of (4.20) coincides with

$$(4.21) \quad \left( \mathcal{A}(\alpha_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial \alpha_0}) \left[ \frac{\exp(2m\pi i \gamma) + \exp(-2m\pi i \gamma)}{2m} \right. \right. \\ \left. \left. \times \psi_{+,B}(x, y - 2m\pi i \alpha_0; \alpha_0, c) \right] \right) \Big|_{c=c(\alpha_0)}.$$

To relate this function with  $\tilde{\psi}_{+,B}(x, y - 2m\pi i \alpha_0)$  we use the technique of [AKT4]; we introduce the following coordinate transformation from  $(y, \alpha_0)$  to  $(\tilde{y}, \tilde{\alpha}_0)$ :

$$(4.22) \quad \begin{cases} \tilde{y} = y - 2m\pi i \alpha_0 \\ \tilde{\alpha}_0 = \alpha_0. \end{cases}$$

Correspondingly  $\tilde{\eta} = \sigma(\partial/\partial \tilde{y})$  and  $\tilde{\theta} = \sigma(\partial/\partial \tilde{\alpha}_0)$  are related with  $\eta$  and  $\theta$  in the following manner:

$$(4.23) \quad \begin{cases} \eta = \tilde{\eta} \\ \theta = -2m\pi i \tilde{\eta} + \tilde{\theta}. \end{cases}$$

Using  $(\tilde{y}, \tilde{\alpha}_0)$ -variable, we then find

$$(4.24) \quad \left( \mathcal{A}(\alpha_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial \alpha_0}) \psi_{+,B}(x, y - 2m\pi i \alpha_0; \alpha_0, c) \right) \Big|_{c=c(\alpha_0)} \\ = \left( : \sum_{n \geq 0} \frac{(\alpha_1 \tilde{\eta}^{-1} + \alpha_2 \tilde{\eta}^{-2} + \dots)^n (\tilde{\theta} - 2m\pi i \tilde{\eta})^n}{n!} : \right. \\ \left. \times \psi_{+,B}(x, \tilde{y}; \tilde{\alpha}_0, c) \right) \Big|_{c=c(\tilde{\alpha}_0)} \\ = \left( : \sum_{n \geq 0} \frac{1}{n!} (\alpha_1 \tilde{\eta}^{-1} + \alpha_2 \tilde{\eta}^{-2} + \dots)^n \sum_{\substack{k+l=n \\ k, l \geq 0}} \frac{n!}{k!l!} \tilde{\theta}^k (-2m\pi i \tilde{\eta})^l : \right. \\ \left. \times \psi_{+,B}(x, \tilde{y}; \tilde{\alpha}_0, c) \right) \Big|_{c=c(\tilde{\alpha}_0)}$$

$$\begin{aligned}
&= \left( : \sum_{l \geq 0} \frac{1}{l!} (-2m\pi i(\alpha_1 + \alpha_2 \tilde{\eta}^{-1} + \dots)) \right)^l : \\
&\times : \sum_{k \geq 0} \frac{1}{k!} (\alpha_1 \tilde{\eta}^{-1} + \alpha_2 \tilde{\eta}^{-2} + \dots)^k \tilde{\theta}^k : \psi_{+,B}(x, \tilde{y}; \tilde{\alpha}_0, c) \Big|_{c=c(\tilde{\alpha}_0)} \\
&= \left( : \exp(-2m\pi i(\alpha_1 + \alpha_2 \tilde{\eta}^{-1} + \dots)) : \right. \\
&\quad \left. \times \mathcal{A}(\tilde{\alpha}_0, \frac{\partial}{\partial \tilde{y}}, \frac{\partial}{\partial \tilde{\alpha}_0}) \psi_{+,B}(x, \tilde{y}; \tilde{\alpha}_0, c) \right) \Big|_{c=c(\tilde{\alpha}_0)} \\
&=: \exp(-2m\pi i(\alpha_1 + \alpha_2 \eta^{-1} + \dots)) : \tilde{\psi}_{+,B}(x, y - 2m\pi i \alpha_0).
\end{aligned}$$

Combining (4.20), (4.21) and (4.24), we obtain (4.17). The proof of (4.18) can be given in exactly the same manner.  $\square$

## 5 Analytic properties of Borel transformed WKB solutions of an MPPT equation for $a \neq 0$

In the preceding section we have seen that the Borel transform  $\psi_B$  of a WKB solution of the  $\infty$ -Whittaker equation

$$(5.1) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{1}{4} + \frac{\alpha(a, \eta)}{x} + \eta^{-2} \frac{c(a)}{x^2} \right) \right) \psi(x, \eta; \alpha(a, \eta), c(a)) = 0$$

can be represented in the form

$$(5.2) \quad (\mathcal{A}(\alpha_0, \partial/\partial y, \partial/\partial \alpha_0) \psi_{0,B}(x, y; \alpha_0, c)) \Big|_{c=c(\alpha_0)},$$

where  $\mathcal{A}$  is a microdifferential operator and  $\psi_{0,B}$  is a Borel transformed WKB solution  $\psi_0$  of the Whittaker equation

$$(5.3) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{1}{4} + \frac{\alpha_0}{x} + \eta^{-2} \frac{c}{x^2} \right) \right) \psi_0(x, \eta; \alpha_0, c) = 0,$$

where  $\alpha_0$  and  $c$  are complex numbers. We note that we have changed the notation  $(\tilde{\psi}, \psi)$  used in Section 4 to  $(\psi, \psi_0)$  for the convenience of the presentation in this section. On the other hand, Theorem 2.2 shows

that the study of a WKB solution  $\tilde{\psi}_+(\tilde{x}, a, \eta)$  of an MPPT equation for  $a \neq 0$  can be reduced to that of a WKB solution  $\psi_+$  of the  $\infty$ -Whittaker equation in that they are related as in (5.4) below with the infinite series  $x(\tilde{x}, a, \eta)$  and  $\alpha(a, \eta)$  constructed in Theorem 2.1:

$$(5.4) \quad \tilde{\psi}_+(\tilde{x}, a, \eta) = \left( \frac{\partial x(\tilde{x}, a, \eta)}{\partial \tilde{x}} \right)^{-1/2} \psi_+(x(\tilde{x}, a, \eta), \eta; \alpha(a, \eta), \tilde{Q}_2(0, a)).$$

Furthermore, as is noted in Remark 2.2, the growth order condition (B.4) that  $\{x_k(\tilde{x}, a)\}_{k \geq 0}$  satisfies has enabled us to rewrite (5.4) as the following microdifferential relation between  $\tilde{\psi}_{+,B}$  and  $\psi_{+,B}$ :

$$(5.5) \quad \tilde{\psi}_{+,B}(x, a, y) = \mathcal{X} \psi_{+,B}(x, y),$$

where

$$(5.6) \quad \mathcal{X} =: \left( \frac{\partial g}{\partial x}(x, a) \right)^{1/2} \left( 1 + \frac{\partial r}{\partial x} \right)^{-1/2} \exp(r(x, a, \eta)\xi) :$$

with the notations in Section 2. (See (2.58).) In view of the concrete expression (2.60) of  $\mathcal{X}$  as an integro-differential operator, we find by Theorem 4.1 that the singularities of  $\tilde{\psi}_{+,B}(x, a, y)$  are confined to

$$(5.7) \quad y = -y_+(x, a) + 2m\pi i \alpha_0(a) \quad (m = 0, \pm 1, \pm 2, \dots)$$

in a sufficiently small neighborhood of the origin  $(x, a, y) = (0, 0, 0)$ , where

$$(5.8) \quad y_+(x, a) = \int_{-4\alpha_0(a)}^x \sqrt{\frac{x + 4\alpha_0(a)}{4x}} dx.$$

Then it follows from the comparison of degree 0 part of (2.8) that the corresponding point is expressed in  $(\tilde{x}, a, y)$ -coordinate as follows:

$$(5.9) \quad y = -y_+(\tilde{x}, a) + 2m\pi i \alpha_0(a)$$

where

$$(5.10) \quad y_+(\tilde{x}, a) = \int_{\tilde{x}_0(a)}^{\tilde{x}} \sqrt{\frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}}} d\tilde{x}$$

with  $\tilde{x}_0(a)$  in (2.30) (i.e., the simple turning point of the MPPT equation in question.) Since the alien derivative of  $\psi_{+,B}$  at the point is given by (4.17), the application of the operator  $\mathcal{X}$  entails the following

**Theorem 5.1.** *Let  $\tilde{\psi}_+(\tilde{x}, a, \eta)$  be a WKB solution of a generic (i.e.,  $a \neq 0$ ) MPPT equation that is normalized as in (2.32). Then for each positive integer  $m$  the following relation (5.11) holds for sufficiently small  $a(\neq 0)$ :*

(5.11)

$$\begin{aligned} & \left( \Delta_{y=-y_+(\tilde{x}, a)+2m\pi i\alpha_0(a)} \tilde{\psi}_+ \right)_B(\tilde{x}, a, y) \\ &= \frac{\exp(2m\pi i\gamma(a)) + \exp(-2m\pi i\gamma(a))}{2m} \end{aligned}$$

$$\times : \exp(-2m\pi i(\alpha_1(a) + \alpha_2(a)\eta^{-1} + \dots)) : \tilde{\psi}_{+,B}(\tilde{x}, a, y - 2m\pi i\alpha_0(a))$$

where

$$(5.12) \quad y_+(\tilde{x}, a) = \int_{\tilde{x}_0(a)}^{\tilde{x}} \sqrt{\frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}}} d\tilde{x},$$

$$(5.13) \quad \gamma(a)^2 + \gamma(a) = \tilde{Q}_2(0, a)$$

and

$$(5.14) \quad \alpha_j(a) = \frac{1}{2\pi i} \oint_{\tilde{\Gamma}(a)} \tilde{S}_{j-1}(\tilde{x}, a) d\tilde{x}$$

with  $\tilde{\Gamma}(a)$  being the closed curve in Figure 2.2 and with  $\tilde{S}_k$  designating the degree  $k$  part of  $\tilde{S}_{\text{odd}}$ , the odd part of  $\tilde{S}$  that satisfies

$$(5.15) \quad \tilde{S}^2 + \frac{d\tilde{S}}{d\tilde{x}} = \eta^2 \tilde{Q}(\tilde{x}, a).$$

## A Convergence of the top order part of the transformation which brings an MPPT equation to its canonical form

In Appendix A and Appendix B, we give the estimates of the transformation

$$(A.1) \quad x(\tilde{x}, a, \eta) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} x_k^{(j)}(\tilde{x}) a^j \eta^{-k}$$

that appears in Section 2, which brings an MPPT equation

$$(A.2) \quad \left( \frac{d^2}{d\tilde{x}^2} - \eta^2 \left( \frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}} + \eta^{-1} \frac{\tilde{Q}_1(\tilde{x}, a)}{\tilde{x}} + \eta^{-2} \frac{\tilde{Q}_2(\tilde{x}, a)}{\tilde{x}^2} \right) \right) \tilde{\psi}(\tilde{x}, \eta) = 0$$

to its canonical form

$$(A.3) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{1}{4} + \frac{\alpha(a, \eta)}{x} + \eta^{-2} \frac{\gamma(a)}{x^2} \right) \right) \psi(x, \eta) = 0$$

with

$$(A.4) \quad \alpha(a, \eta) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \alpha_k^{(j)} a^j \eta^{-k}.$$

Here we assume that  $\tilde{Q}_j (j = 0, 1, 2)$  are holomorphic in a neighborhood of  $(\tilde{x}, a) = (0, 0)$  and satisfy

$$(A.5) \quad \tilde{Q}_0(0, 0) = 0,$$

$$(A.6) \quad \frac{\partial \tilde{Q}_0}{\partial \tilde{x}}(0, 0) \neq 0,$$

$$(A.7) \quad \gamma(a) = \tilde{Q}_2(0, a).$$

We also obtain the estimates of  $\alpha(a, \eta)$  in the course of the estimation of  $x(\tilde{x}, a, \eta)$ .

The series  $x(\tilde{x}, a, \eta)$  and  $\alpha(a, \eta)$  are constructed so that they satisfy (2.8), that is,

$$(A.8) \quad \begin{aligned} & \frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}} + \eta^{-1} \frac{\tilde{Q}_1(\tilde{x}, a)}{\tilde{x}} + \eta^{-2} \frac{\tilde{Q}_2(\tilde{x}, a)}{\tilde{x}^2} \\ &= \left( \frac{\partial x}{\partial \tilde{x}} \right)^2 \left( \frac{1}{4} + \frac{\alpha(a, \eta)}{x} + \eta^{-2} \frac{\gamma(a)}{x^2} \right) - \frac{1}{2} \eta^{-2} \{x; \tilde{x}\}. \end{aligned}$$

For simplicity, we use the following notations: For multi-indices  $\tilde{\kappa} = (\kappa_1, \dots, \kappa_\mu)$  and  $\tilde{\lambda} = (\lambda_1, \dots, \lambda_\mu)$  in  $\mathbb{N}_0^\mu$  with  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ , we define

$$(A.9) \quad |\tilde{\lambda}|_\mu := \sum_{j=1}^{\mu} \lambda_j,$$

$$(A.10) \quad \tilde{\lambda}! := \prod_{j=1}^{\mu} \lambda_j!,$$

$$(A.11) \quad C(\tilde{\lambda}) := \prod_{j=1}^{\mu} C(\lambda_j), \quad C(\lambda_j) := \frac{3}{2\pi^2(\lambda_j + 1)^2}.$$

For  $(\lambda_j, \kappa_j)$ -dependent ( $j = 1, 2, \dots, \mu$ ) quantities  $\rho_{\kappa_j}^{\lambda_j}$  and  $\sigma_{\kappa_j}$  we also use the following notations:

$$(A.12) \quad \rho_{\tilde{\kappa}}^{\tilde{\lambda}} := \prod_{j=1}^{\mu} \rho_{\kappa_j}^{\lambda_j},$$

$$(A.13) \quad \sum_{|\tilde{\kappa}|_{\mu}=l}^* \sigma_{\tilde{\kappa}} := \begin{cases} 1 & \text{for } \mu = 0, \\ \sum_{|\tilde{\kappa}|_{\mu}=l, j=1}^{\mu} \prod_{\kappa_j \geq 1} \sigma_{\kappa_j} & \text{for } \mu \geq 1. \end{cases}$$

In what follows  $x_k^{(j)}$  or functions related to it such as  $dx_k^{(j)}/d\tilde{x}$  etc. typically stands for  $\rho_k^j$ . We also use the notation  $\sum_{|\tilde{\lambda}|_{\mu}=l}^* \rho_{\tilde{\kappa}}^{\tilde{\lambda}}$  to mean

imposing the constraint on  $\lambda_j$  exactly in the same way as in (A.13). We denote the supremum of a function  $f(\tilde{x})$  on  $\{\tilde{x} \in \mathbb{C}; |\tilde{x}| \leq r\}$  by

$$(A.14) \quad \|f\|_{[r]} := \sup_{|\tilde{x}| \leq r} |f(\tilde{x})|.$$

As in Section 2, we introduce  $z(\tilde{x}, a, \eta)$  given by

$$(A.15) \quad z(\tilde{x}, a, \eta) := \tilde{x}^{-1} x(\tilde{x}, a, \eta).$$

The purpose of Appendix A is to confirm (2.5) and (2.6) for  $k = 0$ , that is, to prove Proposition A.1 below. As we will see in Appendix B the convergence of the series  $x_0(\tilde{x}, a)$  and  $\alpha_0(a)$  plays a central role in our subsequent discussions.

**Proposition A.1.** *Let*

$$(A.16) \quad x_0(\tilde{x}, a) = \sum_{j=0}^{\infty} x_0^{(j)}(\tilde{x}) a^j \quad \text{and} \quad \alpha_0(a) = \sum_{j=0}^{\infty} \alpha_0^{(j)} a^j$$

be the top order part (with respect to  $\eta^{-1}$ ) of the transformation and the coefficient of the canonical form constructed in Section 2 respectively. Then,  $x_0(\tilde{x}, a)$  and  $\alpha_0(a)$  converge in a neighborhood of  $(\tilde{x}, a) = (0, 0)$ .

*Proof.* To begin with, we briefly recall how to construct  $x_0^{(j)}$  and  $\alpha_0^{(j)}$ .

Comparing the coefficients of  $\eta^0$  in (A.8), we have

$$(A.17) \quad \frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}} = \left( \frac{\partial x_0}{\partial \tilde{x}} \right)^2 \left( \frac{1}{4} + \frac{\alpha_0(a)}{x_0} \right).$$

Further, by comparing the coefficients of  $a^0$  in (A.17), we find

$$(A.18) \quad \tilde{Q}_0^{(0)}(\tilde{x}) = \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^2 \left( \frac{\tilde{x}}{4} + \frac{\alpha_0^{(0)}}{z_0^{(0)}} \right),$$

where  $\tilde{Q}_k^{(j)}$  denotes the Taylor coefficient (with respect to  $a$ ) of  $\tilde{Q}_k$  at  $a = 0$  (cf. (2.17)). Our choice of  $x_0^{(0)}$  and  $\alpha_0^{(0)}$  are as follows:

$$(A.19) \quad \alpha_0^{(0)} = 0, \quad x_0^{(0)}(\tilde{x}) = \int_0^{\tilde{x}} 2\sqrt{\frac{\tilde{Q}_0^{(0)}(y)}{y}} dy.$$

It follows from (A.6) that  $x_0^{(0)}$  thus chosen is holomorphic in a neighborhood of 0 and satisfies

$$(A.20) \quad x_0^{(0)}(0) = 0,$$

$$(A.21) \quad \frac{dx_0^{(0)}}{d\tilde{x}}(0) \neq 0.$$

By a similar procedure, we determine  $x_0^{(j)}$  and  $\alpha_0^{(j)}$  successively in the following way: first comparing the coefficients of  $a^j$  in (A.17), we have

$$(A.22) \quad \begin{aligned} \tilde{Q}_0^{(j)}(\tilde{x}) = & \sum_{j_1+j_2+j_3=m} \frac{dx_0^{(j_1)}}{d\tilde{x}} \frac{dx_0^{(j_2)}}{d\tilde{x}} \\ & \times \left( \delta_{0,j_3} \frac{\tilde{x}}{4} + \sum_{j'_1+j'_2=j_3} \frac{\alpha_0^{(j'_1)}}{z_0^{(0)}} \sum_{\mu=\min\{1,j'_2\}}^{j'_2} \sum_{|\tilde{\lambda}|_{\mu}=j'_2}^* \frac{(-1)^\mu z_0^{(\tilde{\lambda})}}{(z_0^{(0)})^\mu} \right). \end{aligned}$$



Here, and in what follows,  $\delta_{p,q}$  designates Kronecker's delta (i.e.,  $= 1$  for  $p = q$  and  $= 0$  if  $p \neq q$ ). By multiplying (A.22) by  $-2z_0^{(0)} \left( dx_0^{(0)} / d\tilde{x} \right)^{-2}$  and taking  $w = x_0^{(0)}(\tilde{x})$  as a new independent variable, we can rewrite (A.22) as follows:

$$(A.23) \quad w \frac{d}{dw} x_0^{(j)} + 2\alpha_0^{(j)} = 2\Phi^{(j)}(w).$$

Here the explicit form of  $\Phi^{(j)}(w)$  is given by the following:

$$(A.24) \quad \begin{aligned} \Phi^{(j)}(w) := & - \sum_{\substack{j_1+j_2+j_3=j \\ 1 \leq j_3 \leq j-1}} \frac{dx_0^{(j_1)}}{dw} \frac{dx_0^{(j_2)}}{dw} \sum_{j'_1+j'_2=j_3} \alpha_0^{(j'_1)} \\ & \times \sum_{j'_2}^* \sum_{\mu=\min\{1,j'_2\}}^* \frac{(-1)^\mu z_0^{(\tilde{\lambda})}}{\left(z_0^{(0)}\right)^\mu} \\ & - \sum_{j'_1+j'_2=m}^* \alpha_0^{(j'_1)} \sum_{j'_2}^* \sum_{\mu=\min\{1,j'_2\}}^* \frac{(-1)^\mu z_0^{(\tilde{\lambda})}}{\left(z_0^{(0)}\right)^\mu} \\ & - \frac{w}{4} \sum_{j_1+j_2=m}^* \frac{dx_0^{(j_1)}}{dw} \frac{dx_0^{(j_2)}}{dw} + z_0^{(0)} \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^{-2} \tilde{Q}_0^{(j)}(w). \end{aligned}$$

We then define  $\alpha_0^{(j)}$  by

$$(A.25) \quad \alpha_0^{(j)} := \Phi^{(j)}(0).$$

With this choice of  $\alpha_0^{(j)}$  we solve (A.23) to obtain

$$(A.26) \quad x_0^{(j)}(w) = 2 \int_0^w \frac{\Phi^{(j)}(\tilde{w}) - \alpha_0^{(j)}}{\tilde{w}} d\tilde{w}.$$

In view of the definition of  $\alpha_0^{(j)}$ , we find that  $x_0^{(j)}(w)$  is holomorphic in some neighborhood of  $\{w \in \mathbb{C}; |w| \leq r\}$  for some  $r > 0$ .

To verify the convergence of the series  $x_0(\tilde{x}, a)$  and  $\alpha_0(a)$ , we use the majorant series method; that is, we construct a majorant series

$A(a) = \sum_{j \geq 0} A^{(j)} a^j$  of  $x_0(\tilde{x}, a)$  and  $\alpha_0(a)$ . Hence our task is to find a sequence  $\{A^{(j)}\}_{j \geq 0}$  of complex numbers so that they satisfy the following relation (A.27.j) for every  $j \geq 0$ :

$$(A.27.j) \quad \begin{cases} |\alpha_0^{(j)}| & \leq \frac{A^{(j)}}{4}, \\ \|x_0^{(j)}\|_{[r]} & \leq A^{(j)}, \\ \left\| \frac{dx_0^{(j)}}{dw} \right\|_{[r]}, \|z_0^{(j)}\|_{[r]} & \leq \frac{A^{(j)}}{r}. \end{cases}$$

To begin with, we choose  $A^{(0)}$  and  $A^{(1)}$  so that they respectively satisfy (A.27.0) and (A.27.1). To define  $A^{(j)}$  ( $j \geq 2$ ) we introduce an auxiliary constant  $C$  so that the following relations may be satisfied:

$$(A.28) \quad \|\tilde{Q}_0^{(j)}\|_{[r]} \leq C^{j+1},$$

$$(A.29) \quad \left\| \left( \frac{dx_0^{(0)}}{dw} \right)^{-1} \right\|_{[r]}, \left\| \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^{-1} \right\|_{[r]}, \left\| (z_0^{(0)})^{-1} \right\|_{[r]} \leq C.$$

Since  $\tilde{Q}_0(w, a)$  is holomorphic at  $(w, a) = (0, 0)$  and  $(dx_0^{(0)}/d\tilde{x})(0)$ ,  $z_0^{(0)}(0) \neq 0$ , we can find such a constant  $C$  by taking  $r(> 0)$  sufficiently small. Using this constant  $C$  we recursively define  $A^{(j)}$  by the following:

(A.30)

$$\begin{aligned} A^{(j)} := & \sum_{\substack{j_1+j_2+j_3=j \\ 1 \leq j_3 \leq j-1}} \frac{A^{(j_1)} A^{(j_2)}}{r^2} \sum_{\substack{j'_1+j'_2=j_3 \\ j'_1 \geq 1}} A^{(j'_1)} \sum_{\mu=\min\{1, j'_2\}}^{j'_2} \sum_{|\tilde{\lambda}|_{\mu}=j'_2}^* \left( \frac{C}{r} \right)^{\mu} A^{(\tilde{\lambda})} \\ & + \sum_{j'_1+j'_2=j}^* A^{(j'_1)} \sum_{\mu=\min\{1, j'_2\}}^{j'_2} \sum_{|\tilde{\lambda}|_{\mu}=j'_2}^* \left( \frac{C}{r} \right)^{\mu} A^{(\tilde{\lambda})} \\ & + \sum_{j_1+j_2=j}^* \frac{A^{(j_1)} A^{(j_2)}}{r} + 4C^{j+3} \frac{A^{(0)}}{r}. \end{aligned}$$

By using the induction on  $j$  we prove that  $A^{(j)}$  satisfies (A.27.j).

Let us now suppose  $A^{(j)}$  satisfies (A.27.j) for  $0 \leq j \leq m-1$ . Then by using (A.24), (A.28), (A.29) and (A.30) we find

(A.31)

$$\begin{aligned}
\|\Phi^{(m)}\|_{[r]} &\leq \sum_{\substack{j_1+j_2+j_3=m \\ 1 \leq j_3 \leq m-1}} \left\| \frac{dx_0^{(j_1)}}{dw} \right\|_{[r]} \left\| \frac{dx_0^{(j_2)}}{dw} \right\|_{[r]} \sum_{j'_1+j'_2=j_3} |\alpha_0^{(j'_1)}| \\
&\quad \times \sum_{j'_2} \sum_{\mu=\min\{1, j'_2\}}^* \left\| z_0^{(\tilde{\lambda})} \right\|_{[r]} \left\| \left( z_0^{(0)} \right)^{-1} \right\|_{[r]}^\mu \\
&\quad + \sum_{j'_1+j'_2=m}^* |\alpha_0^{(j'_1)}| \sum_{j'_2} \sum_{\mu=\min\{1, j'_2\}}^* \left\| z_0^{(\tilde{\lambda})} \right\|_{[r]} \left\| \left( z_0^{(0)} \right)^{-1} \right\|_{[r]}^\mu \\
&\quad + \frac{|w|}{4} \sum_{j_1+j_2=m}^* \left\| \frac{dx_0^{(j_1)}}{dw} \right\|_{[r]} \left\| \frac{dx_0^{(j_2)}}{dw} \right\|_{[r]} \\
&\quad + \left\| z_0^{(0)} \right\|_{[r]} \left\| \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^{-1} \right\|_{[r]}^2 \left\| \tilde{Q}_0^{(m)}(w) \right\|_{[r]} \\
&\leq \sum_{\substack{j_1+j_2+j_3=m \\ 1 \leq j_3 \leq m-1}} \frac{A^{(j_1)} A^{(j_2)}}{r^2} \sum_{\substack{j'_1+j'_2=j_3 \\ j'_1 \geq 1}} \frac{A^{(j'_1)}}{4} \sum_{j'_2} \sum_{\mu=\min\{1, j'_2\}}^* \left( \frac{C}{r} \right)^\mu A^{(\tilde{\lambda})} \\
&\quad + \sum_{j'_1+j'_2=m}^* \frac{A^{(j'_1)}}{4} \sum_{j'_2} \sum_{\mu=\min\{1, j'_2\}}^* \left( \frac{C}{r} \right)^\mu A^{(\tilde{\lambda})} \\
&\quad + \frac{r}{4} \sum_{j_1+j_2=m}^* \frac{A^{(j_1)} A^{(j_2)}}{r^2} + C^{m+3} \frac{A^{(0)}}{r} \\
&= \frac{A^{(m)}}{4}.
\end{aligned}$$

To deduce (A.27.m) from (A.31) we use the following

**Lemma A.2.** *Let  $v(w)$  be a holomorphic function on  $D_r = \{w; |w| \leq r\}$ . We consider the following differential equation for  $u(w)$ :*

$$(A.32) \quad w \frac{du}{dw}(w) + 2\alpha = 2v(w),$$

*where  $\alpha$  is a constant. Then there exist a constant  $\alpha$  and a holomorphic function  $u(w)$  on  $D_r$  that vanishes at  $w = 0$  so that (A.32) and the following inequalities are satisfied:*

$$(A.33) \quad |\alpha| \leq \|v\|_{[r]},$$

$$(A.34) \quad \|u\|_{[r]} \leq 4 \|v\|_{[r]},$$

$$(A.35) \quad \left\| \frac{du}{dw} \right\|_{[r]}, \left\| \frac{u}{w} \right\|_{[r]} \leq \frac{4}{r} \|v\|_{[r]}.$$

*Proof.* By setting  $w$  to be 0 in (A.32) we find

$$(A.36) \quad \alpha = v(0),$$

and then we define

$$(A.37) \quad u(w) = 2 \int_0^w \frac{v(\tilde{w}) - \alpha}{\tilde{w}} d\tilde{w}.$$

Then we easily see that  $u(w)$  is a holomorphic solution of (A.32) on  $D_r$  that vanishes at  $w = 0$ . For this choice of  $\alpha$  and  $u(w)$ , (A.33) is clearly satisfied, and the first inequality of (A.35) is an immediate consequence of the Schwarz lemma, because  $u(w)$  satisfies (A.38) below as a solution of (A.32).

$$(A.38) \quad \left\| w \frac{du}{dw} \right\|_{[r]} \leq 2 \|v\|_{[r]} + 2|\alpha| \leq 4 \|v\|_{[r]}.$$

Since  $u(0) = 0$ , we also find the following:

$$(A.39) \quad \begin{aligned} \|u\|_{[r]} &\leq \left\| \int_0^w \frac{du}{dw}(\tilde{w}) d\tilde{w} \right\|_{[r]} \\ &\leq r \left\| \frac{du}{dw} \right\|_{[r]} \end{aligned}$$

$$\leq 4 \|v\|_{[r]}.$$

We thus obtain the second inequality of (A.35) by using the Schwarz lemma again. □

By applying Lemma A.2 to  $\alpha_0^{(m)}$  and  $x_0^{(m)}$ , we obtain (A.27.m). Thus the induction proceeds. This means that we have confirmed that

$$(A.40) \quad A(a) := \sum_{j \geq 0} A^{(j)} a^j$$

is a majorant series of  $\alpha_0(a)$  and  $x_0(\tilde{x}, a)$ . Hence what we should show is the convergence of the series (A.40). The required convergence follows from the implicit function theorem by the following reasoning: first, by comparing the coefficients of  $a^j$ , we observe that  $A(a)$  satisfies the following equation:

$$(A.41) \quad \begin{aligned} A = & A^{(0)} + A^{(1)}a + \frac{1}{r^2}(A^2 - (A^{(0)})^2)(A - A^{(0)}) \left( \frac{1}{1 - (A - A^{(0)})C/r} \right) \\ & + (A - A^{(0)}) \left( \frac{1}{1 - (A - A^{(0)})C/r} - 1 \right) \\ & + \frac{1}{r}(A - A^{(0)})^2 + 4C^3 \frac{A^{(0)}}{r} \frac{(Ca)^2}{1 - Ca}. \end{aligned}$$

Therefore if we define  $\Xi(a, A)$  by

$$(A.42) \quad \begin{aligned} \Xi(a, A) := & (A - A^{(0)} - A^{(1)}a) \\ & - \frac{1}{r^2}(A^2 - (A^{(0)})^2)(A - A^{(0)}) \left( \frac{1}{1 - (A - A^{(0)})C/r} \right) \\ & - (A - A^{(0)}) \left( \frac{1}{1 - (A - A^{(0)})C/r} - 1 \right) \\ & - \frac{1}{r}(A - A^{(0)})^2 - 4C^3 \frac{A^{(0)}}{r} \frac{(Ca)^2}{1 - Ca}, \end{aligned}$$

then we find that  $A(a)$  is a solution of  $\Xi(a, A) = 0$ . Since  $\Xi$  is holomorphic in a neighborhood of  $(a, A) = (0, A^{(0)})$  and satisfies

$$(A.43) \quad \Xi(0, A^{(0)}) = 0, \quad \left( \frac{\partial \Xi}{\partial A} \right) (0, A^{(0)}) = 1 \neq 0,$$

it follows from the implicit function theorem that  $\Xi(a, A) = 0$  has a unique holomorphic solution satisfying  $A(0) = A^{(0)}$  near  $(a, A) = (0, A^{(0)})$ . Hence  $A(a)$  is convergent. This implies the convergence of the series  $\alpha_0(a)$  and  $x_0(\tilde{x}, a)$ . □

## B Estimation of the transformation which brings an MPPT equation to its canonical form

The purpose of this subsection is to prove (2.5), (2.6) and (2.7), that is, to prove the following

**Theorem B.1.** *Let*

$$(B.1) \quad x(\tilde{x}, a, \eta) = \sum_{k=0}^{\infty} x_k(\tilde{x}, a) \eta^{-k}$$

*be the transformation that brings an MPPT equation (2.33) to the canonical form (2.35) with*

$$(B.2) \quad \alpha(a, \eta) = \sum_{k=0}^{\infty} \alpha_k(a) \eta^{-k}.$$

*Then,  $x$  and  $\alpha$  satisfy the following conditions for some positive constants  $r_0$  and  $A_0$ :*

(i)  $x_k$  and  $\alpha_k$  ( $k = 0, 1, 2, \dots$ ) are holomorphic respectively on  $\{(\tilde{x}, a); |\tilde{x}| \leq r_0, |a| \leq r_0\}$  and  $\{a; |a| \leq r_0\}$ .

(ii) *the following inequalities hold for  $k = 1, 2, \dots$ :*

$$(B.3) \quad \sup_{|a| \leq r_0} |\alpha_k(a)| \leq k! A_0^k,$$

$$(B.4) \quad \sup_{|\tilde{x}|, |a| \leq r_0} |x_k(\tilde{x}, a)| \leq k! A_0^k,$$

$$(B.5) \quad \sup_{|\tilde{x}|, |a| \leq r_0} \left| \frac{\partial x_k}{\partial \tilde{x}}(\tilde{x}, a) \right| \leq k! A_0^k.$$

In order to prove Theorem B.1, we use the following lemmas frequently:

**Lemma B.2.** *For  $l, \mu \in \mathbb{N} = \{1, 2, 3, \dots\}$  with  $\mu \leq l$ , the following inequality holds:*

$$(B.6) \quad \sum_{|\tilde{\lambda}|_\mu = l}^* \tilde{\lambda}! \leq 4^{\mu-1} (l - \mu + 1)!.$$

*Proof.* We shall verify (B.6) by induction on  $\mu \geq 1$ . For the case  $\mu = 1$ , (B.6) is trivial. For  $\mu = 2$ , we have

$$(B.7) \quad \begin{aligned} \sum_{|\tilde{\lambda}|_2 = l}^* \lambda_1! \lambda_2! &= (l-1)! \left( 2 + \sum_{\substack{\lambda_1 + \lambda_2 = l \\ \lambda_1, \lambda_2 \geq 2}} \frac{\lambda_1! \lambda_2!}{(l-1)!} \right) \\ &= (l-1)! \left( 2 + \frac{2}{l-1} \sum_{\lambda=2}^{l-2} \frac{\lambda(\lambda-1) \cdots 3}{(l-2)(l-3) \cdots (l-\lambda+1)} \right) \\ &\leq 2(l-1)! \left( 1 + \frac{l-2}{l-1} \right) \\ &\leq 4(l-1)!. \end{aligned}$$

If we assume that (B.6) holds for  $\mu - 1 \geq 1$ , then we obtain

$$(B.8) \quad \begin{aligned} \sum_{|\tilde{\lambda}|_\mu = l}^* \tilde{\lambda}! &= \sum_{\substack{l' + \lambda_\mu = l \\ l' \geq \mu-1, \lambda_\mu \geq 1}} \lambda_\mu! \sum_{\lambda_1 + \dots + \lambda_{\mu-1} = l'}^* \lambda_1! \cdots \lambda_{\mu-1}! \\ &\leq \sum_{\substack{l' + \lambda_\mu = l \\ l' \geq \mu-1, \lambda_\mu \geq 1}} 4^{\mu-2} (l' - \mu + 2)! \lambda_\mu! \end{aligned}$$

$$\begin{aligned}
&= 4^{\mu-2} \sum_{\substack{l'+\lambda_\mu=l-\mu+2 \\ l' \geq 1, \lambda_\mu \geq 1}} l'! \lambda_\mu! \\
&\leq 4^{\mu-1} (l - \mu + 1)!.
\end{aligned}$$

□

**Lemma B.3.** For  $\tilde{\lambda} = (\lambda_1, \dots, \lambda_\mu) \in \mathbb{N}_0^\mu$  with  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ , the following inequality holds for  $C(\tilde{\lambda})$  given by (A.11):

$$(B.9) \quad \sum_{|\tilde{\lambda}|_\mu=l} C(\tilde{\lambda}) \leq C(l).$$

*Proof.* We first prove (B.9) for the case  $\mu = 2$ :

$$(B.10) \quad \left(\frac{3}{2\pi^2}\right)^2 \sum_{\lambda_1+\lambda_2=l} \frac{1}{(\lambda_1+1)^2} \frac{1}{(\lambda_2+1)^2} \leq \frac{3}{2\pi^2} \frac{1}{(l+1)^2}.$$

Since

$$(B.11) \quad \sum_{\lambda=0}^{\infty} \frac{1}{(\lambda+1)^2} = \frac{\pi^2}{6},$$

we have

$$\begin{aligned}
(B.12) \quad &\sum_{\lambda_1+\lambda_2=l} \frac{(l+2)^2}{(\lambda_1+1)^2(\lambda_2+1)^2} \\
&= \sum_{\lambda_1+\lambda_2=l} \left( \frac{1}{\lambda_1+1} + \frac{1}{\lambda_2+1} \right)^2 \\
&= \sum_{\lambda_1=0}^l \frac{1}{(\lambda_1+1)^2} + \sum_{\lambda_1+\lambda_2=l} \frac{2}{(\lambda_1+1)(\lambda_2+1)} + \sum_{\lambda_2=0}^l \frac{1}{(\lambda_2+1)^2} \\
&\leq 2 \sum_{\lambda=0}^{\infty} \frac{1}{(\lambda+1)^2} + 2 \left( \sum_{\lambda_1=0}^l \frac{1}{(\lambda_1+1)^2} \right)^{1/2} \left( \sum_{\lambda_2=0}^l \frac{1}{(\lambda_2+1)^2} \right)^{1/2} \\
&\leq 4 \sum_{\lambda=0}^{\infty} \frac{1}{(\lambda+1)^2} = \frac{2\pi^2}{3}.
\end{aligned}$$



Then (B.10) immediately follows from this. Since (B.9) is trivial for the case  $\mu = 1$ , we obtain (B.9) for  $\mu \geq 2$  by the successive use of (B.10).  $\square$

*Proof of Theorem B.1.* We rewrite (A.8) as (2.10), that is,

(B.13)

$$\begin{aligned} & \tilde{Q}_0(\tilde{x}, a) + \eta^{-1} \tilde{Q}_1(\tilde{x}, a) \\ &= \left( \frac{dx}{d\tilde{x}} \right)^2 \left( \frac{\tilde{x}}{4} + \frac{\alpha(a, \eta)}{z} \right) \\ &+ \eta^{-2} \left( -\tilde{R}_2(\tilde{x}, a) + 2 \frac{dz}{d\tilde{x}} \frac{\gamma(a)}{z} + \tilde{x} \left( \frac{dz}{d\tilde{x}} \right)^2 \frac{\gamma(a)}{z^2} \right) - \frac{1}{2} \eta^{-2} \{x; \tilde{x}\} \tilde{x}. \end{aligned}$$

Here  $\tilde{R}_2(\tilde{x}, a)$  is the function given by (2.11), that is,

$$(B.14) \quad \tilde{R}_2(\tilde{x}, a) = \frac{\tilde{Q}_2(\tilde{x}, a) - \gamma(a)}{\tilde{x}}.$$

The choice (A.7) of  $\gamma(a)$  guarantees that  $\tilde{R}_2(\tilde{x}, a)$  is holomorphic in a neighborhood of  $(\tilde{x}, a) = (0, 0)$ . By comparing the coefficients of  $\eta^{-k}$  ( $k \geq 1$ ), we obtain

(B.15)

$$\begin{aligned} & \delta_{k,1} \tilde{Q}_1(\tilde{x}, a) \\ &= \sum_{k_1+k_2+k_3=k} \frac{dx_{k_1}}{d\tilde{x}} \frac{dx_{k_2}}{d\tilde{x}} \sum_{k'_1+k'_2=k_3} \frac{\alpha_{k'_1}}{z_0} \sum_{\nu=\min\{1,k'_2\}}^{k'_2} \sum_{\tilde{\kappa}|\nu=k'_2}^* \frac{(-1)^\nu z_{\tilde{\kappa}}}{z_0^\nu} \\ &+ \frac{\tilde{x}}{2} \frac{dx_0}{d\tilde{x}} \frac{dx_k}{d\tilde{x}} + \frac{\tilde{x}}{4} \sum_{k_1+k_2=k}^* \frac{dx_{k_1}}{d\tilde{x}} \frac{dx_{k_2}}{d\tilde{x}} - \delta_{k,2} \tilde{R}_2(\tilde{x}, a) \\ &+ 2\gamma(a) \sum_{k_1+k_2=k-2} \frac{dz_{k_1}}{d\tilde{x}} \frac{1}{z_0} \sum_{\nu=\min\{1,k_2\}}^{k_2} \sum_{\tilde{\kappa}|\nu=k_2}^* \frac{(-1)^\nu z_{\tilde{\kappa}}}{z_0^\nu} \\ &+ \tilde{x}\gamma(a) \sum_{k_1+k_2+k_3=k-2} \frac{dz_{k_1}}{d\tilde{x}} \frac{dz_{k_2}}{d\tilde{x}} \frac{1}{z_0^2} \sum_{\nu=\min\{1,k_3\}}^{k_3} \sum_{\tilde{\kappa}|\nu=k_3}^* (-1)^\nu (\nu+1) \frac{z_{\tilde{\kappa}}}{z_0^\nu} \end{aligned}$$

$$\begin{aligned}
& -\frac{\tilde{x}}{2} \sum_{k_1+k_2=k-2} \frac{d^3 x_{k_1}}{d\tilde{x}^3} \left( \frac{dx_0}{d\tilde{x}} \right)^{-1} \sum_{\nu=\min\{1,k_2\}}^{k_2} \sum_{|\tilde{\kappa}|_\nu=k_2}^* (-1)^\nu \frac{dx_{\tilde{\kappa}}}{d\tilde{x}} \left( \frac{dx_0}{d\tilde{x}} \right)^{-\nu} \\
& + \frac{3}{4} \tilde{x} \sum_{k_1+k_2+k_3=k-2} \frac{d^2 x_{k_1}}{d\tilde{x}^2} \frac{d^2 x_{k_2}}{d\tilde{x}^2} \left( \frac{dx_0}{d\tilde{x}} \right)^{-2} \\
& \times \sum_{\nu=\min\{1,k_3\}}^{k_3} \sum_{|\tilde{\kappa}|_\nu=k_3}^* (-1)^\nu (\nu+1) \frac{dx_{\tilde{\kappa}}}{d\tilde{x}} \left( \frac{dx_0}{d\tilde{x}} \right)^{-\nu}.
\end{aligned}$$

Further, by comparing the coefficients of  $a^j$  in (B.15) and taking  $w = x_0^{(0)}(\tilde{x})$  as a new independent variable, we have

$$(B.16) \quad w \frac{dx_k^{(j)}}{dw} + 2\alpha_k^{(j)} = 2 \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^{-2} z_0^{(0)} \Phi_k^{(j)},$$

where  $\Phi_k^{(j)}$  is

$$(B.17) \quad \Phi_k^{(j)} = \Phi_{k,1}^{(j)} + \Phi_{k,2}^{(j)} + \Phi_{k,3}^{(j)}$$

and  $\Phi_{k,i}^{(j)}$  ( $i = 1, 2, 3$ ) are defined as follows:

$$\begin{aligned}
(B.18) \quad \Phi_{k,1}^{(j)} = & -2 \sum_{k_1+k_2=k-2} \sum_{j_1+j_2+j_3+j_4=j} \gamma^{(j_1)} \frac{dz_{k_1}^{(j_2)}}{d\tilde{x}} \\
& \times \sum_{\nu=\min\{1,k_2\}}^{k_2} (-1)^\nu (z_0^{-\nu-1})^{(j_3)} \sum_{|\tilde{\kappa}|_\nu=k_2}^* \sum_{|\tilde{\lambda}|_\nu=j_4} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \\
& - \tilde{x} \sum_{k_1+k_2+k_3=k-2} \sum_{j_1+j_2+j_3+j_4+j_5=j} \gamma^{(j_1)} \frac{dz_{k_1}^{(j_2)}}{d\tilde{x}} \frac{dz_{k_2}^{(j_3)}}{d\tilde{x}} \\
& \times \sum_{\nu=\min\{1,k_3\}}^{k_3} (-1)^\nu (\nu+1) (z_0^{-\nu-2})^{(j_4)} \sum_{|\tilde{\kappa}|_\nu=k_3}^* \sum_{|\tilde{\lambda}|_\nu=j_5} z_{\tilde{\kappa}}^{(\tilde{\lambda})}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\tilde{x}}{2} \sum_{k_1+k_2=k-2} \sum_{j_1+j_2+j_3=j} \frac{d^3 x_{k_1}^{(j_1)}}{d\tilde{x}^3} \\
& \times \sum_{\nu=\min\{1,k_2\}}^{k_2} (-1)^\nu \left( \left( \frac{dx_0}{d\tilde{x}} \right)^{-\nu-1} \right)^{(j_2)} \sum_{|\tilde{\kappa}|_\nu=k_2}^* \sum_{|\tilde{\lambda}|_\nu=j_3} \frac{dx_{\tilde{\kappa}}^{(\tilde{\lambda})}}{d\tilde{x}} \\
& - \frac{3}{4} \tilde{x} \sum_{k_1+k_2+k_3=k-2} \sum_{j_1+j_2+j_3+j_4=j} \frac{d^2 x_{k_1}^{(j_1)}}{d\tilde{x}^2} \frac{d^2 x_{k_2}^{(j_2)}}{d\tilde{x}^2} \\
& \times \sum_{\nu=\min\{1,k_3\}}^{k_3} (-1)^\nu (\nu+1) \left( \left( \frac{dx_0}{d\tilde{x}} \right)^{-\nu-2} \right)^{(j_3)} \\
& \times \sum_{|\tilde{\kappa}|_\nu=k_3}^* \sum_{|\tilde{\lambda}|_\nu=j_4} \frac{dx_{\tilde{\kappa}}^{(\tilde{\lambda})}}{d\tilde{x}} \\
& + \delta_{k,2} \tilde{R}_2^{(j)}(w),
\end{aligned}$$

(B.19)

$$\begin{aligned}
\Phi_{k,2}^{(j)} &= \delta_{k,1} \tilde{Q}_1^{(j)}(w) \\
& - \frac{\tilde{x}}{4} \sum_{k_1+k_2=k}^* \sum_{j_1+j_2=j} \frac{dx_{k_1}^{(j_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(j_2)}}{d\tilde{x}} \\
& - \sum_{\substack{k_1+k_2+k_3=k \\ 1 \leq k_3 \leq k-1}} \sum_{l_1+l_2+l_3=j} \frac{dx_{k_1}^{(l_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(l_2)}}{d\tilde{x}} \sum_{k'_1+k'_2=k_3} \sum_{l'_1+l'_2+l'_3=l_3} \alpha_{k'_1}^{(l'_1)} \\
& \times \sum_{\nu=\min\{1,k'_2\}}^{k'_2} (-1)^\nu (z_0^{-\nu-1})^{(l'_2)} \sum_{|\tilde{\kappa}|_\nu=k'_2}^* \sum_{|\tilde{\lambda}|_\nu=l'_3} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \\
& - \sum_{k_1+k_2=k}^* \sum_{j_1+j_2+j_3+j_4=j} \frac{dx_0^{(j_1)}}{d\tilde{x}} \frac{dx_0^{(j_2)}}{d\tilde{x}} \alpha_{k_1}^{(j_3)}
\end{aligned}$$

$$\times \sum_{\nu=1}^{k_2} \sum_{j'_1+j'_2=j_4} (-1)^\nu (z_0^{-\nu-1})^{(j'_1)} \sum_{|\tilde{\kappa}|_\nu=k_2}^* \sum_{|\tilde{\lambda}|_\nu=j'_2} z_{\tilde{\kappa}}^{(\tilde{\lambda})},$$

(B.20)

$$\begin{aligned} \Phi_{k,3}^{(j)} = & - \sum_{\substack{j_1+j_2+j_3+j_4=j \\ j_3 \leq j-1}} \frac{dx_0^{(j_1)}}{d\tilde{x}} \frac{dx_0^{(j_2)}}{d\tilde{x}} \alpha_k^{(j_3)} (z_0^{-1})^{(j_4)} \\ & - \frac{\tilde{x}}{2} \sum_{\substack{j_1+j_2=j \\ j_2 \leq j-1}} \frac{dx_0^{(j_1)}}{d\tilde{x}} \frac{dx_k^{(j_2)}}{d\tilde{x}} \\ & - \sum_{k_1+k_2=k} \sum_{\substack{j_1+j_2+j_3+j_4=j \\ 1 \leq j_3}} \frac{dx_{k_1}^{(j_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(j_2)}}{d\tilde{x}} \alpha_0^{(j_3)} (z_0^{-1})^{(j_4)} \\ & - \sum_{\substack{j_1+j_2+j_3+j_4=j \\ 1 \leq j_3}} \frac{dx_0^{(j_1)}}{d\tilde{x}} \frac{dx_0^{(j_2)}}{d\tilde{x}} \alpha_0^{(j_3)} \\ & \times \sum_{\nu=1}^k \sum_{j'_1+j'_2=j_4} (-1)^\nu (z_0^{-\nu-1})^{(j'_1)} \sum_{|\tilde{\kappa}|_\nu=k}^* \sum_{|\tilde{\lambda}|_\nu=j'_2} z_{\tilde{\kappa}}^{(\tilde{\lambda})}. \end{aligned}$$

Here we denote the coefficients of  $a^j$  of  $z_0^{-\nu}$  and  $(dx_0/d\tilde{x})^{-\nu}$  respectively by  $(z_0^{-\nu})^{(j)}$  and  $((dx_0/d\tilde{x})^{-\nu})^{(j)}$ .

The above decomposition of  $\Phi_k^{(j)}$  into three parts  $\Phi_{k,i}^{(j)}$  ( $i = 1, 2, 3$ ) is made so that we may dominate each term in  $\Phi_{k,i}^{(j)}$  by constants of the uniform form

$$(B.21) \quad c_i M_k^{(j)},$$

where  $c_i$  and  $M_k^{(j)}$  are described with the notations to be given later in the following manner:

$$(B.22) \quad c_1 = \delta_0/A,$$

$$(B.23) \quad c_2 = \delta_0,$$

$$(B.24) \quad c_3 = B/C,$$

$$(B.25) \quad M_k^{(j)} = k!(A\varepsilon^{-1})^k C(j) C^j \delta_0 M.$$

We also note that  $\Phi_{1,1}^{(j)}$  is regarded to be 0 as a convention. As we discussed in the proof of Theorem 2.1,  $\alpha_k^{(j)}$  and  $x_k^{(j)}$  are determined by

$$(B.26) \quad \alpha_k^{(j)} = \left( z_0^{(0)}(0) \right)^{-1} \Phi_k^{(j)}(0)$$

$$(B.27) \quad x_k^{(j)} = \int_0^w \frac{2}{\tilde{w}} \left( \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^{-2} z_0^{(0)} \Phi_k^{(j)}(\tilde{w}) - \alpha_k^{(j)} \right) d\tilde{w}.$$

We now estimate the growth order of  $x_k^{(j)}$  and  $\alpha_k^{(j)}$  as  $j$  and  $k$  tend to infinity, by using the induction on the double index  $(j, k)$  appropriately ordered. Since we proved in Appendix A that  $\sum_{j \geq 0} x_0^{(j)}(\tilde{x}) a^j$  and  $\sum_{j \geq 0} \alpha_0^{(j)} a^j$  are convergent near the origin, we can find constants  $C_0, B$  and  $\rho$  so that the following relations (B.28)  $\sim$  (B.31) hold:

(B.28)

$$\|x_0^{(0)}\|_{[r]}, \|z_0^{(0)}\|_{[r]}, \left\| \frac{dx_0^{(0)}}{d\tilde{x}} \right\|_{[r]}, \left\| \left( \frac{dx_0^{(0)}}{d\tilde{x}} \right)^{-1} \right\|_{[r]}, \left\| (z_0^{(0)})^{-1} \right\|_{[r]} \leq C_0 C(0),$$

(B.29)

$$\|\tilde{x}(w)\|_{[r]}, \left\| \left( \frac{d\tilde{x}}{dw} \right)^{-1} \right\|_{[r]}, \sup_{|w| \leq r, |a| \leq \rho} \left| \left( \frac{dx_0}{d\tilde{x}} \right)^{-1} \right|, \sup_{|w| \leq r, |a| \leq \rho} |(z_0)^{-1}|, \sup_{|a| \leq \rho} |\alpha_0| \leq C_0,$$

(B.30)

$$\|z_0^{(j)}\|_{[r]}, \left\| \frac{dx_0^{(j)}}{d\tilde{x}} \right\|_{[r]}, |\alpha_0^{(j)}|, \|\tilde{Q}_1^{(j)}\|_{[r]}, \|\tilde{R}_2^{(j)}\|_{[r]}, |\gamma^{(j)}| \leq C_0 C(j) B^j,$$

(B.31)

$$\left\| \left( \left( \frac{dx_0}{d\tilde{x}} \right)^{-\nu} \right)^{(j)} \right\|_{[r]}, \|(z_0^{-\nu})^{(j)}\|_{[r]} \leq C_0^\nu C(j) B^j.$$

We now try to show that the following dominance relation (B.32.k.j) ( $k \geq 1, j \geq 0$ ) holds for some constants  $A, C$  and  $\delta_0$  which satisfy (B.33)

and (B.34) below:

(B.32.k.j)

$$\left\| x_k^{(j)} \right\|_{[r-\varepsilon]}, \left\| z_k^{(j)} \right\|_{[r-\varepsilon]}, \left\| \frac{dx_k^{(j)}}{dw} \right\|_{[r-\varepsilon]}, |\alpha_k^{(j)}| \leq k! (A\varepsilon^{-1})^k C(j) C^j \delta_0$$

for any  $\varepsilon$  that satisfies (B.35) below:

$$(B.33) \quad 1 < \sqrt{A} \delta_0, \quad 0 < \delta_0 \ll 1,$$

$$(B.34) \quad 0 < B \ll C,$$

$$(B.35) \quad 0 < \varepsilon < r/3.$$

We note that (B.32.1.0) is validated by (B.33) if we choose  $A$  sufficiently large.

Now we will confirm (B.32.k.j) for every  $(k, j)$  ( $k \geq 1, j \geq 0$ ) by using the following induction procedure:

[I] We first confirm (B.32.n.m) by assuming that (B.32.n'.m') ( $0 \leq m', 1 \leq n' \leq n-1$ ) and (B.32.n.m') ( $0 \leq m' \leq m-1$ ) are all validated,

and then

[II] we confirm (B.32.n.0) by assuming that (B.32.n'.0) ( $0 \leq n' \leq n-1$ ) are validated.

As we know (B.32.1.0) is valid for a sufficiently large  $A$ , these confirmations suffice for our purpose. To attain this goal we first note that application of Lemma A.2 to (B.16) entails the following relations:

$$(B.36) \quad \left\| x_k^{(j)} \right\|_{[r-\varepsilon]} \leq 4(C_0 C(0))^3 \left\| \Phi_k^{(j)} \right\|_{[r-\varepsilon]},$$

$$(B.37) \quad \left\| \frac{dx_k^{(j)}}{dw} \right\|_{[r-\varepsilon]}, \left\| z_k^{(j)} \right\|_{[r-\varepsilon]} \leq \frac{4}{r-\varepsilon} (C_0 C(0))^3 \left\| \Phi_k^{(j)} \right\|_{[r-\varepsilon]}.$$

From (B.26) we also find

$$(B.38) \quad |\alpha_k^{(j)}| \leq C_0 C(0) \left\| \Phi_k^{(j)} \right\|_{[r-\varepsilon]}.$$

Thus it suffices for us to estimate  $\Phi_k^{(j)}$  under the appropriate induction hypothesis.

Let us first consider the case [I]; we assume that (B.32. $n'.m'$ ) ( $0 \leq m', 1 \leq n' \leq n-1$ ) and (B.32. $n.m'$ ) ( $0 \leq m' \leq m-1$ ) have been validated, and we try to prove the following estimates:

$$(B.39.i) \quad \left\| \Phi_{n,i}^{(m)} \right\|_{[r-\varepsilon]} \leq c_i M_n^{(m)}$$

for  $i = 1, 2, 3$ . Here  $c_i$  and  $M_n^{(m)}$  are given by (B.22)  $\sim$  (B.25) with  $M$  in (B.25) being a constant independent of  $n, m, \delta_0, C, A$ .

Before embarking on the estimation we note the following

**Lemma B.4.** *Suppose that (B.32. $k.j$ ) holds. Then we find*

$$(B.40) \quad \left\| \frac{d^2 x_k^{(j)}}{dw^2} \right\|_{[r-\varepsilon]}, \left\| \frac{dz_k^{(j)}}{dw} \right\|_{[r-\varepsilon]} \leq e(k+1)! A^k \varepsilon^{-k-1} C(j) C^j \delta_0,$$

$$(B.41) \quad \left\| \frac{d^3 x_k^{(j)}}{dw^3} \right\|_{[r-\varepsilon]} \leq e^2(k+2)! A^k \varepsilon^{-k-2} C(j) C^j \delta_0.$$

*Proof.* Let  $\tilde{\varepsilon}$  denote  $k\varepsilon/(k+1)$ . Then (B.32. $k.j$ ) entails

$$(B.42) \quad \begin{aligned} \sup_{|w| \leq r-\tilde{\varepsilon}} |z_k^{(j)}(w)| &\leq k! A^k \tilde{\varepsilon}^{-k} C(j) C^j \delta_0 \\ &= k! A^k \left(1 + \frac{1}{k}\right)^k \varepsilon^{-k} C(j) C^j \delta_0 \\ &\leq ek! A^k \varepsilon^{-k} C(j) C^j \delta_0, \end{aligned}$$

where  $e = \exp(1)$ . On the other hand, Cauchy's formula tells us

$$(B.43) \quad \frac{dz_k^{(j)}(w)}{dw} = \frac{1}{2\pi\sqrt{-1}} \int_{|\tilde{w}-w|=(k+1)^{-1}\varepsilon} \frac{z_k^{(j)}(\tilde{w})}{(\tilde{w}-w)^2} d\tilde{w}.$$

In view of the definition of  $\tilde{\varepsilon}$  we find  $\tilde{w}$  that appears in the above contour integral satisfies the following (B.44) for  $w$  with  $|w| \leq r - \varepsilon$ :

$$(B.44) \quad \begin{aligned} |\tilde{w}| &\leq |\tilde{w} - w| + |w| \\ &\leq (k+1)^{-1}\varepsilon + r - \varepsilon. \end{aligned}$$

$$= r - \tilde{\varepsilon}.$$

Hence (B.42) shows (B.40) for  $dz_k^{(j)}/dw$ . The estimation of  $d^2x_k^{(j)}/dw^2$  and  $d^3x_k^{(j)}/dw^3$  can be done in exactly the same manner.  $\square$

*Remark B.1.* For a holomorphic function  $f(\tilde{x})$  of  $\tilde{x}$  and a change of variables  $\tilde{x} = \tilde{x}(w)$ , the following relations hold for the differentiation of  $f(\tilde{x})$  with respect to the two variables  $\tilde{x}$  and  $w$ :

$$(B.45) \quad \frac{df}{d\tilde{x}}(\tilde{x}(w)) = \left( \frac{d\tilde{x}(w)}{dw} \right)^{-1} \frac{d}{dw} f(\tilde{x}(w)),$$

$$(B.46) \quad \frac{d^2f}{d\tilde{x}^2}(\tilde{x}(w)) = \left( \frac{d\tilde{x}(w)}{dw} \right)^{-2} \frac{d^2}{dw^2} f(\tilde{x}(w)) \\ + \frac{1}{2} \frac{d}{dw} \left( \frac{d\tilde{x}(w)}{dw} \right)^{-2} \frac{d}{dw} f(\tilde{x}(w)),$$

$$(B.47) \quad \frac{d^3f}{d\tilde{x}^3}(\tilde{x}(w)) = \left( \frac{d\tilde{x}(w)}{dw} \right)^{-3} \frac{d^3}{dw^3} f(\tilde{x}(w)) \\ + \frac{d}{dw} \left( \frac{d\tilde{x}(w)}{dw} \right)^{-3} \frac{d^2}{dw^2} f(\tilde{x}(w)) \\ + \frac{1}{2} \left( \frac{d\tilde{x}(w)}{dw} \right)^{-1} \frac{d^2}{dw^2} \left( \frac{d\tilde{x}(w)}{dw} \right)^{-2} \frac{d}{dw} f(\tilde{x}(w)).$$

Since  $(d\tilde{x}/dw)^{-1}$  satisfy (B.29) we obtain the following estimate from Cauchy's inequality:

$$(B.48) \quad \left\| \frac{d^k}{dw^k} \left( \frac{d\tilde{x}(w)}{dw} \right)^{-l} \right\|_{[r-\varepsilon]} \leq k! \varepsilon^{-k} \left\| \left( \frac{d\tilde{x}(w)}{dw} \right)^{-l} \right\|_{[r]} \\ \leq k! \varepsilon^{-k} C_0^l.$$

Using the relations (B.45)  $\sim$  (B.47) and the estimate (B.48) we obtain the following inequalities:

$$(B.49) \quad \left\| \frac{df}{d\tilde{x}}(\tilde{x}(w)) \right\|_{[r-\varepsilon]} \leq C_0 \left\| \frac{d}{dw} f(\tilde{x}(w)) \right\|_{[r-\varepsilon]},$$



$$(B.50) \quad \left\| \frac{d^2 f}{d\tilde{x}^2}(\tilde{x}(w)) \right\|_{[r-\varepsilon]} \leq C_0^2 \left\| \frac{d^2}{dw^2} f(\tilde{x}(w)) \right\|_{[r-\varepsilon]} + \frac{\varepsilon^{-1}}{2} C_0^2 \left\| \frac{d}{dw} f(\tilde{x}(w)) \right\|_{[r-\varepsilon]},$$

$$(B.51) \quad \left\| \frac{d^3 f}{d\tilde{x}^3}(\tilde{x}(w)) \right\|_{[r-\varepsilon]} \leq C_0^3 \left\| \frac{d^3}{dw^3} f(\tilde{x}(w)) \right\|_{[r-\varepsilon]} + \varepsilon^{-1} C_0^3 \left\| \frac{d^2}{dw^2} f(\tilde{x}(w)) \right\|_{[r-\varepsilon]} + \varepsilon^{-2} C_0^3 \left\| \frac{d}{dw} f(\tilde{x}(w)) \right\|_{[r-\varepsilon]}.$$

Then the following estimates immediately follow from the above inequalities (B.49) ~ (B.51) and Lemma B.4 for  $k \geq 1$ :

$$(B.52) \quad \left\| \frac{dz_k^{(j)}}{d\tilde{x}} \right\|_{[r-\varepsilon]} \leq C_0 e(k+1)! A^k \varepsilon^{-k-1} C(j) C^j \delta_0,$$

$$(B.53) \quad \left\| \frac{d^l x_k^{(j)}}{d\tilde{x}^l} \right\|_{[r-\varepsilon]} \leq l C_0^l e^{l-1} (k+l-1)! A^k \varepsilon^{-k-l+1} C(j) C^j \delta_0 \quad (l = 1, 2, 3).$$

For  $k = 0$ , we have the following estimates from (B.30) by the same discussion of Lemma B.4:

$$(B.54) \quad \left\| \frac{dz_0^{(j)}}{d\tilde{x}} \right\|_{[r-\varepsilon]} \leq e \varepsilon^{-1} C(j) B^j C_0^2,$$

$$(B.55) \quad \left\| \frac{d^l x_0^{(j)}}{d\tilde{x}^l} \right\|_{[r-\varepsilon]} \leq l e^{l-1} (l-1)! \varepsilon^{-l+1} C(j) B^j C_0^{l+1} \quad (l = 1, 2, 3).$$

*Remark B.2.* Lemma B.4 explains the background reason of the asymmetry of the estimate of  $|x_k^{(j)}|$  with respect to  $j$  and  $k$ ; we dominate  $|x_k^{(j)}|$  by  $C^{j+1}$  as  $j$  tends to infinity, whereas we include a much worse

factor  $k!$  to control their behavior as  $k$  tends to infinity. As the estimate (B.64) below shows, the seemingly innocent term

$$(B.56) \quad -\frac{\tilde{x}}{2} \frac{d^3 x_{k-2}}{d\tilde{x}^3} \left( \frac{dx_0}{d\tilde{x}} \right)^{-1}$$

in (B.15) forces us to introduce the  $k!$ -factor for making the induction reasoning run smoothly. This observation indicates that the singular perturbative character of the problem in question originates mainly from the Schwarzian derivative multiplied by  $\eta^{-2}$  in (B.13).

Now we begin the estimation of  $\Phi_{n,i}^{(m)}$  ( $i = 1, 2, 3$ ).

1) The estimation of  $\Phi_{n,1}^{(m)}$ .

First we estimate  $\Phi_{n,1}^{(m)}$ . The background of the expected form (B.39.1) is as follows: we observe that the sum of suffixes in each term that are relevant to  $\eta^{-1}$ , that is, the sum of  $k_p$ 's, is  $n - 2$ . Hence by using (B.32.k.j) we will encounter the factor  $A^{n-2}$  in the resulting estimate. Then (B.33) may be used to rewrite it as follows:

$$(B.57) \quad A^{n-2} = A^n A^{-2} < A^n A^{-1} \delta_0^2.$$

Thus we expect the extra factor  $A^{-1}$  in our estimation. Let us concretely check whether this argument really goes well. We shall estimate the first term of (B.18) for  $n \geq 2$ : By using (B.30), (B.31), induction hypothesis (B.32), (B.52) and (B.54) we have the following estimate:

$$(B.58) \quad \left\| 2 \sum_{k_1+k_2=n-2} \sum_{l_1+l_2+l_3+l_4=m} \gamma^{(l_1)} \frac{dz_{k_1}^{(l_2)}}{d\tilde{x}} \right. \\ \times \sum_{\nu=\min\{1,k_2\}}^{k_2} (-1)^\nu (z_0^{-\nu-1})^{(l_3)} \sum_{|\tilde{\kappa}|_\nu=k_2}^* \sum_{|\tilde{\lambda}|_\nu=l_4} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \left. \right\|_{[r-\epsilon]} \\ \leq 2 \sum_{k_1+k_2=n-2} \sum_{l_1+l_2+l_3+l_4=m} C_0^3 C(l_1) B^{l_1} e(k_1+1)! A^{k_1} \varepsilon^{-k_1-1} C(l_2) C^{l_2} \\ \times \sum_{\nu=\min\{1,k_2\}}^{k_2} C_0^{\nu+1} C(l_3) B^{l_3} \sum_{|\tilde{\kappa}|_\nu=k_2}^* \sum_{|\tilde{\lambda}|_\nu=l_4} \tilde{\kappa}! (A\varepsilon^{-1})^{k_2} C(\tilde{\lambda}) C^{l_4} \delta_0^\nu.$$

Here we applied (B.52) to  $dz_{k_1}^{(l_2)}/d\tilde{x}$  for  $k_1 \geq 1$  with replacing  $\delta_0$  of (B.52) by  $C_0$  in order to estimate  $dz_{k_1}^{(l_2)}/d\tilde{x}$  ( $k_1 \geq 1$ ) and  $dz_0^{(l_2)}/d\tilde{x}$  in the same form. Further, by applying Lemma B.3 to the summation on  $l_1, \dots, l_4$  and  $\tilde{\lambda}$  and also by using (B.34), we find

(B.59)

$$\begin{aligned}
& 2 \sum_{k_1+k_2=n-2} \sum_{l_1+l_2+l_3+l_4=m} C_0^3 C(l_1) B^{l_1} e(k_1+1)! A^{k_1} \varepsilon^{-k_1-1} C(l_2) C^{l_2} \\
& \times \sum_{\nu=\min\{1, k_2\}}^{k_2} C_0^{\nu+1} C(l_3) B^{l_3} \sum_{|\tilde{\kappa}|_\nu=k_2}^* \sum_{|\tilde{\lambda}|_\nu=l_4} \tilde{\kappa}! (A\varepsilon^{-1})^{k_2} C(\tilde{\lambda}) C^{l_4} \delta_0^\nu \\
& \leq 2eC_0^4 C(m) C^m \varepsilon^{-n+1} A^{n-2} \sum_{k_1+k_2=n-2} (k_1+1)! \sum_{\nu=\min\{1, k_2\}}^{k_2} (C_0 \delta_0)^\nu \sum_{|\tilde{\kappa}|_\nu=k_2}^* \tilde{\kappa}!.
\end{aligned}$$

Then we obtain the following estimation from Lemma B.2:

(B.60)

$$\begin{aligned}
& 2eC_0^4 C(m) C^m \varepsilon^{-n+1} A^{n-2} \sum_{k_1+k_2=n-2} (k_1+1)! \sum_{\nu=\min\{1, k_2\}}^{k_2} (C_0 \delta_0)^\nu \sum_{|\tilde{\kappa}|_\nu=k_2}^* \tilde{\kappa}! \\
& \leq 2eC_0^4 C(m) C^m \varepsilon^{-n+1} A^{n-2} \\
& \times \left( (n-1)! + \sum_{\substack{k_1+k_2=n-2 \\ 1 \leq k_2}} (k_1+1)! k_2! \sum_{\nu=1}^{k_2} (C_0 \delta_0)^\nu 4^{\nu-1} \frac{(k_2 - \nu + 1)!}{k_2!} \right) \\
& \leq 2eC_0^4 C(m) C^m (A\varepsilon^{-1})^n \varepsilon A^{-2} \\
& \times \left( (n-1)! + \sum_{k'_1+k_2=n-1}^* k'_1! k_2! C_0 \delta_0 \sum_{\nu=1}^{\infty} (4C_0 \delta_0)^{\nu-1} \frac{1}{\nu!} \right) \\
& \leq 2eC_0^4 C(m) C^m (A\varepsilon^{-1})^n \varepsilon A^{-2} \\
& \times \left( (n-1)! + C_0 \delta_0 e^{4C_0 \delta_0} \sum_{k'_1+k_2=n-1}^* k'_1! k_2! \right)
\end{aligned}$$

$$\leq 2eC_0^4 C(m)C^m(A\varepsilon^{-1})^n \varepsilon A^{-2} ((n-1)! + 4C_0\delta_0 e^{4C_0\delta_0} (n-2)!).$$

Consequently, since we can assume that  $\delta_0$  is sufficiently small as

$$(B.61) \quad C_0\delta_0 e^{4C_0\delta_0} < 1,$$

we obtain the following inequality from (B.57):

$$(B.62) \quad \left\| 2 \sum_{k_1+k_2=n-2} \sum_{l_1+l_2+l_3+l_4=m} \gamma^{(l_1)} \frac{dz_{k_1}^{(l_2)}}{d\tilde{x}} \right. \\ \times \sum_{\nu=\min\{1,k_2\}}^{k_2} (-1)^\nu (z_0^{-\nu-1})^{(l_3)} \sum_{|\tilde{\kappa}|_\nu=k_2}^* \sum_{|\tilde{\lambda}|_\nu=l_4} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \left. \right\|_{[r-\varepsilon]} \\ \leq n!(A\varepsilon^{-1})^n C(m)C^m\delta_0^2 A^{-1} 2eC_0^4 \varepsilon \left( \frac{1}{n} + \frac{4}{n(n-1)} \right).$$

We find that similar estimates hold for other terms:

$$(B.63) \quad \left\| \tilde{x} \sum_{k_1+k_2+k_3=n-2} \sum_{l_1+l_2+l_3+l_4+l_5=m} \gamma^{(l_1)} \frac{dz_{k_1}^{(l_2)}}{d\tilde{x}} \frac{dz_{k_2}^{(l_3)}}{d\tilde{x}} \right. \\ \times \sum_{\nu=\min\{1,k_3\}}^{k_3} (-1)^\nu (\nu+1) (z_0^{-\nu-2})^{(l_4)} \sum_{|\tilde{\kappa}|_\nu=k_3}^* \sum_{|\tilde{\lambda}|_\nu=l_5} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \left. \right\|_{[r-\varepsilon]} \\ \leq \sum_{k_1+k_2+k_3=n-2} \sum_{l_1+l_2+l_3+l_4+l_5=m} C_0^6 C(l_1)C(l_2)C(l_3) A^{k_1+k_2} B^{l_1} C^{l_2+l_3} \\ \times e^2 (k_1+1)!(k_2+1)! \varepsilon^{-k_1-k_2-2} \\ \times \sum_{\nu=\min\{1,k_3\}}^{k_3} (\nu+1) C_0^{\nu+2} C(l_4) B^{l_4} \sum_{|\tilde{\kappa}|_\nu=k_3}^* \sum_{|\tilde{\lambda}|_\nu=l_5} \tilde{\kappa}! (A\varepsilon^{-1})^{k_3} C(\tilde{\lambda}) C^{l_5} \delta_0^\nu \\ \leq e^2 C_0^8 A^{-2} (A\varepsilon^{-1})^n C(m)C^m \left( \sum_{k'_1+k'_2=n}^* k'_1! k'_2! \right)$$

$$\begin{aligned}
& + \sum_{k'_1+k'_2+k_3=n}^* k'_1!k'_2!k_3! \sum_{\nu=1}^{k_3} (\nu+1)(C_0\delta_0)^\nu 4^{\nu-1} \frac{(k_3-\nu+1)!}{k_3!} \Big) \\
& \leq e^2 C_0^8 A^{-2} (A\varepsilon^{-1})^n C(m) C^m \\
& \quad \times \left( 4(n-1)! + 16(n-2)! C_0 \delta_0 \sum_{\nu=1}^{\infty} (4C_0 \delta_0)^{\nu-1} \frac{2}{(\nu-1)!} \right) \\
& \leq n! (A\varepsilon^{-1})^n C(m) C^m \delta_0^2 A^{-1} \left( \frac{4}{n} + \frac{32}{n(n-1)} \right) C_0^8 e^2,
\end{aligned}$$

(B.64)

$$\begin{aligned}
& \left\| \frac{\tilde{x}}{2} \sum_{k_1+k_2=n-2} \sum_{l_1+l_2+l_3=m} \frac{d^3 x_{k_1}^{(l_1)}}{d\tilde{x}^3} \right. \\
& \quad \times \sum_{\nu=\min\{1,k_2\}}^{k_2} (-1)^\nu \left( \left( \frac{dx_0}{d\tilde{x}} \right)^{-\nu-1} \right)^{(l_2)} \sum_{|\tilde{\kappa}|_\nu=k_2}^* \sum_{|\tilde{\lambda}|_\nu=l_3} \frac{dx_{\tilde{\kappa}}^{(\tilde{\lambda})}}{d\tilde{x}} \Big\|_{[r-\varepsilon]} \\
& \leq \sum_{k_1+k_2=n-2} \sum_{l_1+l_2+l_3=m} \frac{3}{2} C_0^4 C(l_1) A^{k_1} C^{l_1} e^2 (k_1+2)! \varepsilon^{-k_1-2} \\
& \quad \times \sum_{\nu=\min\{1,k_2\}}^{k_2} C_0^{\nu+1} C(l_2) B^{l_2} \sum_{|\tilde{\kappa}|_\nu=k_2}^* \sum_{|\tilde{\lambda}|_\nu=l_3} C_0^\nu \tilde{\kappa}! (A\varepsilon^{-1})^{k_2} C(\tilde{\lambda}) C^{l_3} \delta_0^\nu \\
& \leq \frac{3}{2} e^2 C_0^5 C(m) C^m (A\varepsilon^{-1})^n A^{-2} \\
& \quad \times \left( n! + \sum_{k'_1+k_2=n}^* k'_1!k_2! C_0^2 \delta_0 \sum_{\nu=1}^{\infty} (4C_0^2 \delta_0)^{\nu-1} \frac{1}{\nu!} \right) \\
& \leq \frac{3}{2} e^2 C_0^5 C(m) C^m (A\varepsilon^{-1})^n A^{-2} \left( n! + 4C_0^2 \delta_0 e^{4C_0^2 \delta_0} (n-1)! \right) \\
& \leq n! (A\varepsilon^{-1})^n C(m) C^m \delta_0^2 A^{-1} \left( 1 + \frac{4}{n} \right) \frac{3C_0^5 e^2}{2},
\end{aligned}$$

(B.65)

$$\begin{aligned}
& \left\| \frac{3}{4} \tilde{x} \sum_{k_1+k_2+k_3=n-2} \sum_{l_1+l_2+l_3+l_4=m} \frac{d^2 x_{k_1}^{(l_1)}}{d\tilde{x}^2} \frac{d^2 x_{k_2}^{(l_2)}}{d\tilde{x}^2} \right. \\
& \quad \times \sum_{\nu=\min\{1, k_3\}}^{k_3} (-1)^\nu (\nu+1) \left( \left( \frac{dx_0}{d\tilde{x}} \right)^{-\nu-2} \right)^{(l_3)} \sum_{|\tilde{\kappa}|_\nu=k_3}^* \sum_{|\tilde{\lambda}|_\nu=l_4} \frac{dx_{\tilde{\kappa}}^{(\tilde{\lambda})}}{d\tilde{x}} \left. \right\|_{[r-\varepsilon]} \\
& \leq \sum_{k_1+k_2+k_3=n-2} \sum_{l_1+l_2+l_3+l_4=m} 3C_0^5 C(l_1) C(l_2) A^{k_1+k_2} C^{l_1+l_2} \\
& \quad \times e^2 (k_1+1)! (k_2+1)! \varepsilon^{-k_1-k_2-2} \sum_{\nu=\min\{1, k_3\}}^{k_3} (\nu+1) C_0^{\nu+2} C(l_3) B^{l_3} \\
& \quad \times \sum_{|\tilde{\kappa}|_\nu=k_3}^* \sum_{|\tilde{\lambda}|_\nu=l_4} C_0^\nu \tilde{\kappa}! (A\varepsilon^{-1})^{k_3} C(\tilde{\lambda}) C^{l_4} \delta_0^\nu \\
& \leq 3e^2 C_0^7 A^{-2} (A\varepsilon^{-1})^n C(m) C^m \\
& \quad \times \left( \sum_{k'_1+k'_2=n}^* k'_1! k'_2! + 2C_0^2 \delta_0 e^{4C_0^2 \delta_0} \sum_{k'_1+k'_2+k_3=n}^* k'_1! k'_2! k_3! \right) \\
& \leq n! (A\varepsilon^{-1})^n C(m) C^m \delta_0^2 A^{-1} 3 \left( \frac{4}{n} + \frac{32}{n(n-1)} \right) C_0^7 e^2,
\end{aligned}$$

(B.66)

$$\left\| \delta_{n,2} \tilde{R}_2^{(m)}(z) \right\|_{[r-\varepsilon]} \leq \delta_{n,2} (A\varepsilon^{-1})^2 C(m) C^m \delta_0^2 A^{-1} C_0.$$

In the estimation of (B.64) and (B.65), we assumed that  $\delta_0$  is sufficiently small as

$$(B.67) \quad C_0^2 \delta_0 e^{4C_0^2 \delta_0} < 1.$$

Since  $n \geq 2$  and  $A^{-1}, \varepsilon < 1$ , we obtain (B.39.1).

The worst estimate in the above appears in (B.64) since no factor that weakens  $n!$  is contained. This is the reason why (B.3)  $\sim$  (B.5) must contain the factor  $k!$ .

2) The estimation of  $\Phi_{n,2}^{(m)}$ .

The appearance of the extra factor  $\delta_0$  in the estimate

$$(B.68) \quad \left\| \delta_{n,1} \tilde{Q}_1^{(m)}(z) \right\|_{[r-\varepsilon]} \leq A\varepsilon^{-1} C(m) C^m \delta_0^2 C_0 \varepsilon$$

is an immediate consequence of the assumption (B.33). To obtain this extra factor in the estimation of other terms of  $\Phi_{n,2}^{(m)}$ , we note each term in the summation contains two factors each of whose suffix  $k$  is greater than or equal to 1. It then follows from the induction hypothesis that we find the extra  $\delta_0$  factor. Let us confirm the estimation of the most complicated term in  $\Phi_{n,2}^{(m)}$ . Since  $x_k^{(j)}, \alpha_k^{(j)}$  ( $k \geq 1$ ) and  $x_0^{(j)}, \alpha_0^{(j)}$  respectively satisfy different type of estimation (B.32. $k.j$ ) and (B.30), we have to separate its summand depending on its suffix. However the procedure of its estimation is essentially the same with that of (B.58).

(B.69)

$$\begin{aligned} & \left\| \sum_{\substack{k_1+k_2+k_3=n \\ 1 \leq k_3 \leq n-1}} \sum_{l_1+l_2+l_3=m} \frac{dx_{k_1}^{(l_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(l_2)}}{d\tilde{x}} \sum_{k'_1+k'_2=k_3} \sum_{l'_1+l'_2+l'_3=l_3} \alpha_{k'_1}^{(l'_1)} \right. \\ & \quad \times \sum_{\nu=\min\{1,k'_2\}}^{k'_2} (-1)^\nu (z_0^{-\nu-1})^{(l'_2)} \sum_{|\tilde{\kappa}|_\nu=k'_2}^* \sum_{|\tilde{\lambda}|_\nu=l'_3} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \left. \right\|_{[r-\varepsilon]} \\ &= \left\| \sum_{l_1+l_2+l_3=m} \left( \sum_{k_1+k_2+k_3=n}^* \frac{dx_{k_1}^{(l_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(l_2)}}{d\tilde{x}} + 2 \sum_{k+k_3=n}^* \frac{dx_0^{(l_1)}}{d\tilde{x}} \frac{dx_k^{(l_2)}}{d\tilde{x}} \right) \right. \\ & \quad \times \sum_{l'_1+l'_2+l'_3=l_3} \left( \alpha_0^{(l'_1)} \sum_{\nu=1}^{k_3} \sum_{|\tilde{\kappa}|_\nu=k_3}^* + \sum_{\substack{k'_1+k'_2=k_3 \\ 1 \leq k'_1}} \alpha_{k'_1}^{(l'_1)} \sum_{\nu=\min\{1,k'_2\}}^{k'_2} \sum_{|\tilde{\kappa}|_\nu=k'_2}^* \right) \\ & \quad \times (-1)^\nu (z_0^{-\nu-1})^{(l'_2)} \sum_{|\tilde{\lambda}|_\nu=l'_3} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \left. \right\|_{[r-\varepsilon]} \\ &\leq \sum_{l_1+l_2+l_3=m} C_0^2 C(l_1) C(l_2) \left( \sum_{k_1+k_2+k_3=n}^* C^{l_1+l_2} k_1! k_2! (A\varepsilon^{-1})^{k_1+k_2} \delta_0^2 \right. \end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{k+k_3=n}^* B^{l_1} C^{l_2} k! (A\varepsilon^{-1})^k \delta_0 \Big) \\
& \times \sum_{l'_1+l'_2+l'_3=l_3} \left( C_0 C(l'_1) B^{l'_1} \sum_{\nu=1}^{k_3} \sum_{|\tilde{\kappa}|_\nu=k_3}^* \right. \\
& \quad \left. + \sum_{\substack{k'_1+k'_2=k_3 \\ 1 \leq k'_1}} k'_1! (A\varepsilon^{-1})^{k'_1} C(l'_1) C^{l'_1} \delta_0 \sum_{\nu=\min\{1,k'_2\}}^{k'_2} \sum_{|\tilde{\kappa}|_\nu=k'_2}^* \right) \\
& \quad \times C_0^{\nu+1} C(l'_2) B^{l'_2} \sum_{|\tilde{\lambda}|_\nu=l'_3} \tilde{\kappa}! (A\varepsilon^{-1})^{|\tilde{\kappa}|_\nu} C(\tilde{\lambda}) C^{l'_3} \delta_0^\nu \\
& \leq (A\varepsilon^{-1})^n C(m) C^m C_0^2 \left( \sum_{k_1+k_2+k_3=n}^* k_1! k_2! \delta_0^2 + 2 \sum_{k+k_3=n}^* k! \delta_0 \right) \\
& \quad \times \left( C_0 \sum_{\nu=1}^{k_3} \sum_{|\tilde{\kappa}|_\nu=k_3}^* C_0^{\nu+1} \tilde{\kappa}! \delta_0^\nu + \sum_{\substack{k'_1+k'_2=k_3 \\ 1 \leq k'_1}} k'_1! \delta_0 \sum_{\nu=\min\{1,k'_2\}}^{k'_2} \sum_{|\tilde{\kappa}|_\nu=k'_2}^* C_0^{\nu+1} \tilde{\kappa}! \delta_0^\nu \right) \\
& \leq (A\varepsilon^{-1})^n C(m) C^m C_0^2 \left( \sum_{k_1+k_2+k_3=n}^* k_1! k_2! \delta_0^2 + 2 \sum_{k+k_3=n}^* k! \delta_0 \right) \\
& \quad \times \left( C_0^3 k_3! \delta_0 \sum_{\nu=1}^{\infty} \frac{(4C_0 \delta_0)^{\nu-1}}{\nu!} + k_3! C_0 \delta_0 \right. \\
& \quad \left. + \sum_{k'_1+k'_2=k_3}^* k'_1! k'_2! (C_0 \delta_0)^2 \sum_{\nu=1}^{\infty} \frac{(4C_0 \delta_0)^{\nu-1}}{\nu!} \right) \\
& \leq n! (A\varepsilon^{-1})^n C(m) C^m \delta_0^2 C_0^3 \left( \frac{16\delta_0}{n(n-1)} + \frac{8}{n} \right) ((C_0^2 + 4C_0 \delta_0) e^{4C_0 \delta_0} + 1).
\end{aligned}$$



Similarly we can estimate the other terms as follows:

(B.70)

$$\begin{aligned}
& \left\| \frac{\tilde{x}}{4} \sum_{k_1+k_2=n}^* \sum_{l_1+l_2=m} \frac{dx_{k_1}^{(l_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(l_2)}}{d\tilde{x}} \right\|_{[r-\varepsilon]} \\
& \leq \frac{C_0}{4} \sum_{k_1+k_2=n}^* \sum_{l_1+l_2=m} k_1!k_2!(A\varepsilon^{-1})^{k_1+k_2} C(l_1)C(l_2)C^{l_1+l_2} \delta_0^2 C_0^2 \\
& \leq n!(A\varepsilon^{-1})^n C(m) C^m \frac{\delta_0^2 C_0^3}{n},
\end{aligned}$$

(B.71)

$$\begin{aligned}
& \left\| \sum_{k_1+k_2=n}^* \sum_{l_1+l_2+l_3+l_4=m} \frac{dx_0^{(l_1)}}{d\tilde{x}} \frac{dx_0^{(l_2)}}{d\tilde{x}} \alpha_{k_1}^{(l_3)} \right. \\
& \quad \times \sum_{\nu=1}^{k_2} \sum_{l'_1+l'_2=l_4} (-1)^\nu (z_0^{-\nu-1})^{(l'_1)} \sum_{|\tilde{\kappa}|_\nu=k_2}^* \sum_{|\tilde{\lambda}|_\nu=l'_2} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \left. \right\|_{[r-\varepsilon]} \\
& \leq \sum_{k_1+k_2=n}^* \sum_{l_1+l_2+l_3+l_4=m} C_0^2 C(l_1)C(l_2)C(l_3) B^{l_1+l_2} k_1! (A\varepsilon^{-1})^{k_1} C^{l_3} \delta_0 \\
& \quad \times \sum_{\nu=1}^{k_2} \sum_{l'_1+l'_2=l_4} C_0^{\nu+1} C(l'_1) B^{l'_1} \sum_{|\tilde{\kappa}|_\nu=k_2}^* \sum_{|\tilde{\lambda}|_\nu=l'_2} \tilde{\kappa}! C(\tilde{\lambda}) (A\varepsilon^{-1})^{k_2} C^{l'_2} \delta_0^\nu \\
& \leq n!(A\varepsilon^{-1})^n C(m) C^m \delta_0^2 \frac{4C_0^4 e^{4C_0\delta_0}}{n}.
\end{aligned}$$

Therefore we obtain (B.39.2).

3) The estimation of  $\Phi_{n,3}^{(m)}$ .

To find the extra factor  $BC^{-1}$  in the estimate of each term in  $\Phi_{n,3}^{(m)}$ , we first note that the constant  $B$  is dominated by the inverse of the radius of convergence of  $z_0, \alpha_0$ , etc. (cf. (B.30)) and that the constant  $C$  is relevant to the radius of convergence of  $z_m, \alpha_m$ , etc. Hence we obtain this factor thanks to the fact that each term in the summation in  $\Phi_{n,3}^{(m)}$  contains a factor that originates from the coefficient of  $\eta^0 a^j$  ( $j \geq 1$ );

for example, we find

$$\begin{aligned}
& \text{(B.72)} \\
& \left\| \sum_{\substack{l_1+l_2+l_3+l_4=m \\ l_3 \leq m-1}} \frac{dx_0^{(l_1)}}{d\tilde{x}} \frac{dx_0^{(l_2)}}{d\tilde{x}} \alpha_n^{(l_3)} (z_0^{-1})^{(l_4)} \right\|_{[r-\varepsilon]} \\
& \leq \sum_{\substack{l_1+l_2+l_3+l_4=m \\ l_3 \leq m-1}} C_0^3 C(l_1) C(l_2) C(l_3) C(l_4) B^{l_1+l_2+l_4} C^{l_3} n! (A\varepsilon^{-1})^n \delta_0 \\
& \leq n! (A\varepsilon^{-1})^n C(m) C^m \delta_0 \frac{B}{C} C_0^3,
\end{aligned}$$

because  $l_1 + l_2 + l_4 = m - l_3 \geq 1$  holds by the constraint of the range of indexes which is due to the fact that  $\alpha_n^{(m)}$  is excluded in the summation. Similarly we find

$$\begin{aligned}
& \text{(B.73)} \\
& \left\| \sum_{k_1+k_2=n} \sum_{\substack{l_1+l_2+l_3+l_4=m \\ 1 \leq l_3}} \frac{dx_{k_1}^{(l_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(l_2)}}{d\tilde{x}} \alpha_0^{(l_3)} (z_0^{-1})^{(l_4)} \right\|_{[r-\varepsilon]} \\
& = \left\| \sum_{\substack{l_1+l_2+l_3+l_4=m \\ 1 \leq l_3}} \left( \sum_{k_1+k_2=n}^* \frac{dx_{k_1}^{(l_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(l_2)}}{d\tilde{x}} + 2 \frac{dx_0^{(l_1)}}{d\tilde{x}} \frac{dx_n^{(l_2)}}{d\tilde{x}} \right) \alpha_0^{(l_3)} (z_0^{-1})^{(l_4)} \right\|_{[r-\varepsilon]} \\
& \leq \sum_{\substack{l_1+l_2+l_3+l_4=m \\ 1 \leq l_3}} C_0^4 C(l_1) C(l_2) C(l_3) C(l_4) (A\varepsilon^{-1})^n \\
& \quad \times \left( \sum_{k_1+k_2=n}^* C^{l_1+l_2} k_1! k_2! \delta_0^2 + 2 B^{l_1} C^{l_2} n! \delta_0 \right) B^{l_3+l_4} \\
& \leq n! (A\varepsilon^{-1})^n C(m) C^m \delta_0 \frac{B}{C} C_0^4 \left( \frac{4\delta_0}{n} + 2 \right).
\end{aligned}$$

This time the condition that  $l_3 \geq 1$  is due to the fact that  $\alpha_0^{(0)}$  vanishes.

By the same reasoning we also find

$$\begin{aligned}
(B.74) \quad & \left\| \frac{\tilde{x}}{2} \sum_{\substack{l_1+l_2=m \\ l_2 \leq m-1}} \frac{dx_0^{(l_1)}}{d\tilde{x}} \frac{dx_n^{(l_2)}}{d\tilde{x}} \right\|_{[r-\varepsilon]} \\
& \leq \frac{C_0}{2} \sum_{\substack{l_1+l_2=m \\ l_2 \leq m-1}} C_0 C(l_1) B^{l_1} C_0 n! (A\varepsilon^{-1})^n C(l_2) C^{l_2} \delta_0 \\
& \leq n! (A\varepsilon^{-1})^n C(m) C^m \delta_0 \frac{B}{C} \frac{C_0^3}{2},
\end{aligned}$$

$$\begin{aligned}
(B.75) \quad & \left\| \sum_{\substack{l_1+l_2+l_3+l_4=m \\ 1 \leq l_3}} \frac{dx_0^{(l_1)}}{d\tilde{x}} \frac{dx_0^{(l_2)}}{d\tilde{x}} \alpha_0^{(l_3)} \right. \\
& \times \sum_{\nu=1}^n \sum_{l'_1+l'_2=l_4} (-1)^\nu (z_0^{-\nu-1})^{(l'_1)} \sum_{|\tilde{\kappa}|_\nu=n}^* \sum_{|\tilde{\lambda}|_\nu=l'_2} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \left. \right\|_{[r-\varepsilon]} \\
& \leq \sum_{\substack{l_1+l_2+l_3+l_4=m \\ 1 \leq l_3}} C_0^3 C(l_1) C(l_2) C(l_3) B^{l_1+l_2+l_3} \\
& \times \sum_{\nu=1}^n \sum_{l'_1+l'_2=l_4} C_0^{\nu+1} C(l'_1) B^{l'_1} \sum_{|\tilde{\kappa}|_\nu=n}^* \sum_{|\tilde{\lambda}|_\nu=l'_2} \tilde{\kappa}! C(\tilde{\lambda}) (A\varepsilon^{-1})^n C^{l'_2} \delta_0^\nu \\
& \leq n! (A\varepsilon^{-1})^n C(m) C^m \delta_0 \frac{B}{C} C_0^5 e^{4C_0 \delta_0}.
\end{aligned}$$

Hence we obtain (B.39.3).

In conclusion,  $\Phi_n^{(m)}$  satisfies the following inequality:

$$(B.76) \quad \left\| \Phi_n^{(m)} \right\|_{[r-\varepsilon]} \leq n! (A\varepsilon^{-1})^n C(m) C^m \delta_0 \left( \frac{\delta_0}{A} + \delta_0 + \frac{B}{C} \right) M.$$

By taking  $\delta_0$  sufficiently small at first and then,  $A$  and  $C$  sufficiently large, we can assume that the following holds:

$$(B.77) \quad 6r^{-1} (C_0 C(0))^3 M \left( \frac{\delta_0}{A} + \delta_0 + \frac{B}{C} \right) < 1.$$

Since  $0 < \varepsilon < r/3$ , from (B.36)  $\sim$  (B.38), (B.76) and (B.77), we obtain (B.32.k.j). Thus the induction proceeds in the case [I], and it remains to consider the case [II]; we are to confirm (B.32.n.0) under the assumption (B.32.k.0) ( $1 \leq k \leq n-1$ ). But, we can readily confirm this fact by the same estimation as in the case [I]. Actually  $\Phi_{n,3}^{(0)}$  vanishes in this case, and the estimation is easier than before. Therefore we obtain (B.32.k.j) for every  $k \geq 1$  and  $j \geq 0$ . Then by fixing  $\varepsilon > 0$  and taking  $r_0$  and  $A_0$  in Theorem B.1 as  $\min\{r - \varepsilon, C^{-1}\}$  and  $A\varepsilon^{-1}$ , respectively, we obtain Theorem B.1.

□

## C Representation of the action of $\mathcal{X}$ as an integro-differential operator

Using the results obtained in Appendix B we now study how the microdifferential operator  $\mathcal{X}$  constructed in Theorem 1.6 and Theorem 2.6 acts upon multi-valued analytic functions. Although the situation where this operator appears is different from the situation where its counterpart (also denoted by  $\mathcal{X}$ ) appeared in [AKT4], their structures are essentially the same; the reasoning in [AKT4, Appendix C] applies to our case almost word for word. But, in order to make this paper self-contained, we describe the core part of the argument in this appendix. As the following reasoning indicates, the operator  $\mathcal{X}$  constructed in Theorem 1.6 and that in Theorem 2.6 can be dealt with in exactly the same manner. In what follows we discuss the operator  $\mathcal{X}$  constructed in Theorem 2.6 for the sake of definiteness. It then follows from (2.58) that it has the following form:

$$(C.1) \quad \mathcal{X} =: \left( \frac{\partial g}{\partial x} \right)^{1/2} \left( 1 + \frac{\partial r}{\partial x} \right)^{-1/2} \exp(r(x, a, \eta)\xi) :,$$

where

$$(C.2) \quad r = r(x, a, \eta) = \sum_{k \geq 1} r_k(x, a) \eta^{-k}$$

$$(C.3) \quad r_k = x_k(g(x, a), a)$$

and  $g(x, a)$  is the inverse function of  $x = x_0(\tilde{x}, a)$  given in (2.52), that is,

$$(C.4) \quad x = x_0(g(x, a), a).$$

Here  $x_k (k \geq 0)$  is the function given in (2.5) and  $\xi$  stands for the symbol  $\sigma(\partial/\partial x)$  of the differential operator  $\partial/\partial x$ . For the sake of convenience, we introduce  $r_k^\dagger(x)$  by

$$(C.5) \quad \left(\frac{\partial g}{\partial x}\right)^{-1} \left(1 + \frac{\partial r}{\partial x}\right) = \sum_{k=0}^{\infty} r_k^\dagger(x, a) \eta^{-k}.$$

Then the coefficients  $\{h_k\}_{k \geq 0}$  and  $\{f_{l,k}\}_{1 \leq l \leq k}$  in the expansion (C.6) and (C.7) below can be explicitly expressed in terms of  $\{r_k\}$  and  $\{r_k^\dagger\}$  as undermentioned in (C.8) and (C.9):

$$(C.6) \quad \left(\frac{\partial g}{\partial x}\right)^{1/2} \left(1 + \frac{\partial r}{\partial x}\right)^{-1/2} = \sum_{k=0}^{\infty} h_k(x, a) \eta^{-k},$$

$$(C.7) \quad \exp(r(x, a, \eta)\xi) = 1 + \sum_{1 \leq l \leq k} \eta^{-k} \xi^l f_{l,k}(x, a),$$

$$(C.8) \quad \begin{cases} h_0 = (r_0^\dagger)^{1/2}, \\ h_k = (r_0^\dagger)^{1/2} \sum_{l=1}^k \frac{(-1)^l \Gamma(l + \frac{1}{2})}{l! \Gamma(\frac{1}{2})} \sum_{|\tilde{\lambda}|_l=k}^* \frac{r_{\tilde{\lambda}}^\dagger}{(r_0^\dagger)^l} \quad (k \geq 1), \end{cases}$$

and

$$(C.9) \quad f_{l,k} = \frac{1}{l!} \sum_{|\tilde{\lambda}|_l=k}^* r_{\tilde{\lambda}}.$$

Hence it follows from the definition (C.1) of  $\mathcal{X}$  that its total symbol  $\sigma(\mathcal{X})$  is written down as follows:

$$(C.10) \quad \sum_{k=0}^{\infty} \eta^{-k} \left( h_k + \sum_{k'=1}^k \sum_{l=1}^{k'} \xi^l h_{k-k'} f_{l,k'} \right).$$

As the parameter  $a$  does not play an important role in the following discussion, we omit to write  $a$  for the sake of simplicity.

Since  $r_k$  and  $r_k^\dagger$  are respectively given by (C.3) and (C.5), Theorem B.1 and its proof tell us that there exist a neighborhood  $\omega_1$  of  $(x, a) = (0, 0)$  and a constant  $C_0 > 0$  such that

$$(C.11) \quad \sup_{\omega_1} |r_k| \leq k! C_0^k \quad (k = 1, 2, \dots),$$

$$(C.12) \quad \sup_{\omega_1} |r_k^\dagger| \leq k! C_0^k \quad (k = 1, 2, \dots),$$

and

$$(C.13) \quad \max \left\{ \sup_{\omega_1} |r_0^\dagger|, \sup_{\omega_1} |(r_0^\dagger)^{-1}| \right\} \leq C_0.$$

Then it follows from Lemma B.2 that the following holds:

$$(C.14) \quad \begin{aligned} \sup_{\omega_1} |h_k| &\leq C_0^{1/2} \sum_{l=1}^k \frac{\Gamma(l + \frac{1}{2})}{l! \Gamma(\frac{1}{2})} \sum_{|\bar{\lambda}|_l=k}^* \bar{\lambda}! C_0^{k+l} \\ &\leq C_0^{k+1/2} \sum_{l=1}^k 4^{l-1} (k-l+1)! C_0^l \\ &\leq C_0^{3/2} k! C_0^k \sum_{l=1}^k \frac{4^{l-1} C_0^{l-1}}{(l-1)!} \\ &\leq C_0^{3/2} e^{4C_0} k! C_0^k \end{aligned}$$

for  $k \geq 1$  and

$$(C.15) \quad \sup_{\omega_1} |f_{l,k}| \leq \frac{(k-l+1)!}{l!} 4^{l-1} C_0^k \quad (1 \leq l \leq k).$$

Using these estimates together with Proposition C.1 below we obtain Theorem 2.7. Although the following Proposition C.1 is the same as Proposition C.1 in [AKT4], we include it here for the convenience of the reader.

**Proposition C.1.** *For a domain  $U$  in  $\mathbb{C}_x$ , let  $\Omega$  denote*

$$(C.16) \quad \Omega = \{(x, y; \xi, \eta) \in T^*(U \times \mathbb{C}_y); \eta \neq 0\},$$

and let  $P = P(x, \partial/\partial x, \partial/\partial y)$  be a microdifferential operator of order 0 on  $\Omega$  with the total symbol

$$(C.17) \quad \sigma(P) = \sum_{k=0}^{\infty} P_k(x, \eta^{-1}\xi) \eta^{-k}.$$

Here, we assume that each  $P_k(x, \zeta)$  is an entire function of  $\zeta$  and that the following growth order condition should hold: There exists a constant  $C_0 > 0$  so that, for any compact subset  $K$  of  $U \times \mathbb{C}_\zeta$ , we can find another constant  $M_K$  satisfying

$$(C.18) \quad \sup_{(x, \zeta) \in K} |P_k(x, \zeta)| \leq M_K k! C_0^k$$

for  $k = 0, 1, 2, \dots$ . Then, the action of  $P$  upon a (multi-valued) analytic function  $\phi(x, y)$  is represented in the following form:

$$(C.19) \quad P\phi(x, y) = \int_{y_0}^y K(x, y - y', d/dx) \phi(x, y') dy',$$

where  $K(x, y, d/dx)$  is a differential operator of infinite order that is defined on  $\{(x, y); x \in U \text{ and } |y| < 1/C_0\}$  and  $y_0$  is an arbitrarily chosen point that fixes the action of  $(\partial/\partial y)^{-1}$  as an integral operator.

Although we omit the proof of Proposition C.1 and refer the reader to [AKT4] for it, we describe below how the differential operator  $K$  is expressed in terms of  $P_k$ : Let  $a_{l,k}(x)$  denote the coefficient of  $\zeta^l$  in the Taylor expansion of  $P_k$ , i.e.,

$$(C.20) \quad P_k(x, \zeta) = \sum_{l=0}^{\infty} a_{l,k}(x) \zeta^l.$$

Then we find

$$(C.21) \quad \begin{aligned} P\phi(x, y) &= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} : \eta^{-k-l} a_{l,k}(x) : \left( \frac{\partial}{\partial x} \right)^l \phi(x, y) \\ &= \int_{y_0}^y \left( \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} a_{l,k}(x) \frac{(y - y')^{k+l-1}}{(k+l-1)!} \left( \frac{\partial}{\partial x} \right)^l \right) \phi(x, y') dy' \end{aligned}$$

for some reference point  $y_0$  that fixes the action of  $\eta^{-k-l}$  upon  $\phi(x, y)$ . Hence the operator  $K$  should have the form

$$(C.22) \quad \sum_{l=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{l,k}(x) \frac{y^{k+l-1}}{(k+l-1)!} \right) \left( \frac{\partial}{\partial x} \right)^l$$

and our task is to show that

$$(C.23) \quad c_l(x, y) = \sum_{k=0}^{\infty} a_{l,k}(x) \frac{y^{k+l-1}}{(k+l-1)!}$$

enjoys the following property:

(C.24) For any compact subset  $K'$  of  $U$ , any constant  $r$  that is smaller than  $C_0^{-1}$  and any positive constant  $\varepsilon$ , there exists a constant  $M$  for which

$$\sup_{x \in K', |y| \leq r} |c_l(x, y)| \leq M \frac{\varepsilon^l}{(l-1)!}$$

holds for  $l = 1, 2, \dots$ .

This fact can be confirmed by the assumption (C.18) (cf. [AKT4]).

In order to apply Proposition C.1 to the microdifferential operator  $\mathcal{X}$  in question, we rewrite the total symbol (C.10) of  $\mathcal{X}$  in the following manner:

$$(C.25) \quad \begin{aligned} & \left( \sum_{j=0}^{\infty} h_j \eta^{-j} \right) \left( 1 + \sum_{1 \leq l \leq k} f_{l,k} \eta^{-k} \xi^l \right) \\ &= \left( \sum_{j=0}^{\infty} h_j \eta^{-j} \right) \left( 1 + \sum_{k=0}^{\infty} \eta^{-k} \sum_{l=1}^{\infty} f_{l,l+k} (\eta \xi)^{-l} \right) \\ &= \sum_{j=0}^{\infty} h_j \eta^{-j} + \sum_{j,k=0}^{\infty} \eta^{-(j+k)} h_j \sum_{l=1}^{\infty} f_{l,l+k} (\eta^{-1} \xi)^l \\ &= \sum_{m=0}^{\infty} \eta^{-m} \left[ h_m + \sum_{l=1}^{\infty} \left( \sum_{j+k=m} h_j f_{l,l+k} \right) (\eta^{-1} \xi)^l \right]. \end{aligned}$$



Thus, if we define  $P_m(x, \zeta)$  by

$$(C.26) \quad P_m(x, \zeta) = h_m + \sum_{l=1}^{\infty} \left( \sum_{j+k=m} h_j f_{l,l+k} \right) \zeta^l,$$

we find that the total symbol of  $\mathcal{X}$  has the form (C.17). Then (C.14) and (C.15) entail the following:

$$(C.27) \quad |P_m| \leq |h_m| + \sum_{l=1}^{\infty} \left( \sum_{\substack{j+k=m, \\ j,k \geq 0}} |h_j f_{l,l+k}| \right) |\zeta|^l \\ \leq C_0^{3/2} e^{4C_0} m! C_0^m \\ + \sum_{l=1}^{\infty} \left( \sum_{j+k=m} C_0^{3/2} e^{4C_0} \frac{j!(k+1)!}{l!} 4^{l-1} C_0^{j+k+l} \right) |\zeta|^l.$$

Then the application of Lemma B.2 shows that this is further dominated in the following way:

$$(C.28) \quad C_0^{3/2} e^{4C_0} C_0^m \left[ m! + \sum_{l=1}^{\infty} \frac{4^{l-1} C_0^l |\zeta|^l}{l!} \left( \sum_{\substack{j+\tilde{k}=m+1, \\ j \geq 0, \tilde{k} \geq 1}} j! \tilde{k}! \right) \right] \\ \leq C_0^{3/2} e^{4C_0} C_0^m \left[ m! + \frac{1}{4} \sum_{l=1}^{\infty} \frac{(4C_0 |\zeta|)^l}{l!} ((m+1)! + 4m!) \right] \\ \leq C_0^{3/2} e^{4C_0} C_0^m (m+1)! \left( 1 + \frac{5}{4} \sum_{l=1}^{\infty} \frac{(4C_0 |\zeta|)^l}{l!} \right) \\ = C_0^{3/2} e^{4C_0} \left( 1 + \frac{5}{4} (e^{4C_0 |\zeta|} - 1) \right) (m+1)! C_0^m.$$

Therefore  $P_m(x, \zeta)$  given by (C.26) is an entire function of  $\zeta$  and it satisfies the growth order condition (C.18). Hence Proposition C.1 entails that the operator  $\mathcal{X}$  is represented as in (C.19) with a differential operator  $K$  of infinite order. This completes the proof of Theorem 2.7.

Part III

**Exact WKB analysis of a  
Schrödinger equation with a  
merging triplet of two simple poles  
and one turning point  
— its relevance to the Mathieu  
equation and the Legendre  
equation**

## 0 Introduction

The primary aim of this paper is to study the analytic structure of the Borel transform of a WKB solution  $\psi$  of the Schrödinger equation

$$(0.1) \quad \left( \frac{d^2}{dx^2} - \eta^2 Q \right) \psi = 0 \quad (\eta : \text{a large parameter})$$

when the potential  $Q$  contains two simple poles. As a simple pole and a simple turning point give similar effects on the analytic structure of Borel transformed WKB solutions ([Ko1] and [Ko2]), the above problem is, in its setting, a natural counterpart of the problems discussed in [AKT4] and [KKKoT1], where  $Q$  contains two simple turning points (in [AKT4]) and one simple pole and one simple turning point (in [KKKoT1]). But we need much deeper insight into the structure of the Schrödinger equation in question this time. The difficulty becomes clearly visible if we consider  $Q_a$  below as the simplest example of such a potential;

$$(0.2) \quad Q_a = \frac{1}{a^2 - x^2} \quad (a : \text{a parameter}).$$

For this potential  $Q_a$  we find the following relation

$$(0.3) \quad \int_{-a}^a \sqrt{Q_a} dx = \pi,$$

and this indicates that the distance between two singular points of the Borel transformed WKB solutions whose relative location is independent of  $x$  (the so-called “fixed singularities” (cf. [DP], [KT2, p.112], [V])) does not diminish when two simple poles in the potential (i.e.,  $x = \pm a$ ) coalesce into the origin. In the situation studied in [AKT4] and [KKKoT1], integrals corresponding to (0.3) tend to 0 as the relevant turning points (with a simple pole being regarded as a turning point) coalesce, and this fact played a key role in the semi-global study of the problem in [AKT4] and [KKKoT1]. To overcome this difficulty we first generalize our target class of Schrödinger operators so that each operator in the class contains in its potential  $Q$  two simple poles and one simple turning point which merge as a parameter  $a$  contained in  $Q$  tends to 0. The addition of a simple turning point abates the geometric rigidity which we observed above when two and only two simple

poles are relevant. For the sake of brevity and clarity we call such an operator an M2P1T operator, an operator with merging two poles and one turning point. We note that an MTP operator (resp., an MPPT operator) in [AKT4] (resp., [KKKoT1]) may be called an M2T operator (resp., an M1P1T operator) if we follow this form of wording. By way of parenthesis we recall that “P” in MTP is the abbreviation of “point” and that “PP” in MPPT is that of “pair of a pole and”; that is, “MTP” means “merging turning points”, whereas “MPPT” means “merging pair of a simple pole and a simple turning point”. Now, as we will show in Section 1 that a WKB-theoretic canonical form (in the sense of [KT2, Chap.2]) of an M2P1T equation is a Mathieu equation with a large parameter  $\eta$ :

$$(0.4) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{aA(a, \eta) + xB(a, \eta)}{x^2 - a^2} + \eta^{-2} \left( \frac{\gamma_+(a)}{(x-a)^2} + \frac{\gamma_-(a)}{(x+a)^2} \right) \right) \right) \psi = 0,$$

where

$$(0.5) \quad A(a, \eta) = \sum_{j,k} A_k^{(j)} a^j \eta^{-k} \text{ and } B(a, \eta) = \sum_{j,k} B_k^{(j)} a^j \eta^{-k} \text{ with } A_k^{(j)} \text{ and } B_k^{(j)} \text{ satisfying appropriate growth order conditions (cf. Proposition 1.2.1),}$$

$$(0.6) \quad A_0^{(0)2} \neq B_0^{(0)2}, \quad A_0^{(0)} B_0^{(0)} \neq 0,$$

and

$$(0.7) \quad \gamma_{\pm}(a) \text{ are holomorphic near } a = 0.$$

The appearance of infinite series  $A$  and  $B$  connotes the necessity of employing microdifferential operators whose symbols (in the sense of microlocal analysis (e.g. [K<sup>3</sup>])) are  $A(a, \eta)$  and  $B(a, \eta)$  in our analysis (Section 5), and the growth order conditions on  $A_k^{(j)}$  and  $B_k^{(j)}$  are intended to guarantee the existence of such microdifferential operators. When we want to emphasize the infinite series character of the constants contained in the Mathieu equation, we call it the  $\infty$ -Mathieu equation. This WKB-theoretic reduction of an M2P1T operator to the  $\infty$ -Mathieu is interesting in its own right, as this is the first example where three turning points (with a simple pole being counted as a turning point) are simultaneously analyzed. But the Mathieu equation is

notoriously hard to analyze. Hence to attain our original purpose, that is, to study the analytic structure of Borel transformed WKB solutions near their fixed singularities relevant to the simple poles at  $x = \pm a$ , we further try to separate out the simple turning point of the Mathieu equation from the simple poles so that we may make use of the results of Koike ([Ko4]) for the Legendre equation. In order to put this idea into practice we further introduce another parameter  $\rho$  into an M2P1T operator so that the geometric situation required in Section 2 may be realized. In a word, the role of the parameter  $\rho$  in Definition 1.1 is designed to visualize the situation where two simple poles coalesce into the origin with a simple turning point being kept away from the origin; such a situation is realized by letting  $\rho$  tend to 0 with keeping  $\rho/a$  being a non-zero constant.

The main results in this article were announced in [KKT1].

#### Acknowledgment.

We sincerely thank Professor T. Koike for providing us with his draft on the Voros coefficients of the Legendre equation.

## 1 Reduction of an M2P1T equation to the Mathieu equation

The purpose of this section is to construct a WKB-theoretic transformation that brings an M2P1T equation to its canonical form, i.e., the  $\infty$ -Mathieu equation with a large parameter  $\eta$ . As our reasoning is highly intricate, we divide it into several steps to facilitate the understanding of the reader. To begin with let us present the precise definition of an M2P1T operator, i.e., a Schrödinger operator that contains a triplet of two simple poles and one simple turning point which merge as the parameter  $a$  tends to 0: Let  $U$  (resp.,  $V$  and  $O$ ) be a sufficiently small open neighborhood of the origin  $\{t = 0\}$  (resp.,  $\{a = 0\}$  and  $\{\rho = 0\}$ ) and let  $f(t, a, \rho)$  be a holomorphic function that has the following form on  $U \times V \times O$ :

$$(1.1) \quad f(t, a, \rho) = t\rho g(t, \rho) + \sum_{j \geq 1} a^j f^{(j)}(t, \rho)$$

with

$$(1.2) \quad g(t, \rho) \text{ and } f^{(j)}(t, \rho) \text{ being holomorphic on } U \times O,$$

$$(1.3) \quad g(0, \rho) = 1,$$

$$(1.4) \quad f^{(1)}(0, 0) \neq 0,$$

$$(1.5) \quad \rho^2 \neq f^{(1)}(0, \rho)^2 \text{ for } \rho \text{ in } O.$$

In what follows we use symbols  $f^{(0)}(t, \rho)$  and  $\tilde{f}^{(0)}(t, \rho)$  respectively to denote  $t\rho g(t, \rho)$  and  $\rho g(t, \rho)$ .

**Definition 1.1.** Let  $f(t, a, \rho)$  be as above, let  $g_{\pm}(t)$  be holomorphic functions on  $U$  and let  $Q$  denote the following potential

$$(1.6) \quad \frac{f(t, a, \rho)}{t^2 - a^2} + \eta^{-2} \left( \frac{g_+(t)}{(t - a)^2} + \frac{g_-(t)}{(t + a)^2} \right) \quad (\eta : \text{a large parameter}).$$

Then the Schödinger operator

$$(1.7) \quad \frac{d^2}{dt^2} - \eta^2 Q(t, a, \rho)$$

is called an M2P1T operator.

*Remark 1.1.* It follows from (1.3) and the implicit function theorem that the Schödinger operator (1.7) has a simple turning point for  $a \neq 0$  in  $V$  if  $V$  is sufficiently small, on the condition that  $\rho$  is different from 0.

*Remark 1.2.* To see how and why the numerator  $f$  in the potential  $Q$  abates the rigidity of the potential  $Q_a$  in (0.2) we note the following obvious relation:

$$(1.8) \quad \frac{t\tilde{f}^{(0)} + af^{(1)}}{t^2 - a^2} = \frac{\tilde{f}^{(0)} + f^{(1)}}{2(t - a)} + \frac{\tilde{f}^{(0)} - f^{(1)}}{2(t + a)}.$$

Then the condition (1.5) implies in this situation that the numerators in the right-hand side of (1.8) are different from 0 when evaluated at  $t = 0$ . Thus two simple poles cross in an additive manner as  $a$  passes through 0.

### 1.1 Formal construction of the transformation that brings an M2P1T equation to the Mathieu equation

Supposing

$$(1.1.1) \quad \rho \neq 0$$

and

$$(1.1.2) \quad \rho^2 \neq f^{(1)}(0, \rho)^2,$$

we first construct the formal series

$$(1.1.3) \quad x = x(t, a, \rho; \eta) = \sum_{j,k \geq 0} x_{2k}^{(j)}(t, \rho) a^j \eta^{-2k},$$

$$(1.1.4) \quad A = A(a, \rho; \eta) = \sum_{j,k \geq 0} A_{2k}^{(j)}(\rho) a^j \eta^{-2k}$$

and

$$(1.1.5) \quad B = B(a, \rho; \eta) = \sum_{j,k \geq 0} B_{2k}^{(j)}(\rho) a^j \eta^{-2k}$$

so that they satisfy

$$(1.1.6) \quad \begin{aligned} & Q(t, a, \rho; \eta) \\ &= \left( \frac{\partial x}{\partial t} \right)^2 \left( \frac{aA + xB}{x^2 - a^2} + \eta^{-2} \left( \frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2} \right) \right) \\ & \quad - \frac{1}{2} \eta^{-2} \{x; t\}, \end{aligned}$$

where  $\{x; t\}$  designates the Schwarzian derivative, i.e.,

$$(1.1.7) \quad \{x; t\} = \frac{\partial^3 x / \partial t^3}{\partial x / \partial t} - \frac{3}{2} \left( \frac{\partial^2 x / \partial t^2}{\partial x / \partial t} \right)^2.$$

It is known (e.g. [KT2, Chap.2]) that appropriate growth order conditions on  $\{x_{2k}^{(j)}, A_{2k}^{(j)}, B_{2k}^{(j)}\}$  enables these series to relate Borel transformed WKB solutions of an M2P1T equation and those of its canonical form, i.e., the  $\infty$ -Mathieu equation. The growth order conditions will be studied later in Section 1.2.

### 1.1.1 Construction of $\{A_0^{(j)}, B_0^{(j)}, x_0^{(j)}\}$ — the first few terms

Comparing the coefficients of  $\eta^0$  in (1.1.6) we find

$$(1.1.1.1) \quad \frac{f(t, a, \rho)}{t^2 - a^2} = \left( \frac{\partial x_0}{\partial t} \right)^2 \frac{aA_0 + x_0B_0}{x_0^2 - a^2},$$

where

$$(1.1.1.2) \quad x_0(t, a, \rho) = \sum_{j \geq 0} x_0^{(j)}(t, \rho) a^j,$$

$$(1.1.1.3) \quad A_0(a, \rho) = \sum_{j \geq 0} A_0^{(j)}(\rho) a^j,$$

$$(1.1.1.4) \quad B_0(a, \rho) = \sum_{j \geq 0} B_0^{(j)}(\rho) a^j.$$

By multiplying (1.1.1.1) by  $(t^2 - a^2)(x_0^2 - a^2)$ , we are to find  $(A_0, B_0, x_0)$  so that they satisfy

$$(1.1.1.5) \quad \left( \sum_{j \geq 0} f^{(j)}(t, \rho) a^j \right) \left( \left( \sum_{j \geq 0} x_0^{(j)}(t, \rho) a^j \right)^2 - a^2 \right) \\ = (t^2 - a^2) \left( \sum_{j \geq 0} \frac{\partial x_0^{(j)}}{\partial t} a^j \right)^2 \left( \sum_{j \geq 0} A_0^{(j)}(t, \rho) a^{j+1} \right. \\ \left. + \left( \sum_{j \geq 0} x_0^{(j)}(t, \rho) a^j \right) \left( \sum_{j \geq 0} B_0^{(j)}(\rho) a^j \right) \right).$$

Comparing the coefficients of like powers of  $a$ , we find

$$(1.1.1.5.p) \quad (= [5.p])$$

$$- f^{(p-2)} + \sum_{j+k+l=p} x_0^{(j)} x_0^{(k)} f^{(l)} \\ = t^2 \left( \sum_{j+k+l=p} \frac{\partial x_0^{(j)}}{\partial t} \frac{\partial x_0^{(k)}}{\partial t} A_0^{(l-1)} + \sum_{j+k+l+m=p} \frac{\partial x_0^{(j)}}{\partial t} \frac{\partial x_0^{(k)}}{\partial t} x_0^{(l)} B_0^{(m)} \right)$$



$$- \left( \sum_{j+k+l=p-2} \frac{\partial x_0^{(j)}}{\partial t} \frac{\partial x_0^{(k)}}{\partial t} A_0^{(l-1)} + \sum_{j+k+l+m=p-2} \frac{\partial x_0^{(j)}}{\partial t} \frac{\partial x_0^{(k)}}{\partial t} x_0^{(l)} B_0^{(m)} \right).$$

In what follows we use the symbol  $[5.p]$  to denote (1.1.1.5.p) for the brevity of the notation. We also note that terms whose indices do not meet the requirements should be ignored in  $[5.p]$ ; e.g. for  $p = 1$ ,  $f^{(p-2)}$ ,

$\sum_{j+k+l=p-2} x_0^{(j)'} x_0^{(k)'} A_0^{(l-1)}$  and  $\sum_{j+k+l+m=p-2} x_0^{(j)'} x_0^{(k)'} x_0^{(l)} B_0^{(m)}$  are absent in  $[5,p]$  ( $=[5.1]$ ). Here and also in the following,  $x'$  designates  $\partial x / \partial t$ . With these conventions we find

$$[5.0] \quad t x_0^{(0)2} \tilde{f}^{(0)} = t^2 x_0^{(0)'} x_0^{(0)} B_0^{(0)}.$$

Dividing this by  $t x_0^{(0)}$ , we find

$$[5.0]' \quad x_0^{(0)} \tilde{f}^{(0)} = t x_0^{(0)'} B_0^{(0)}.$$

Hence we find

$$(1.1.1.6) \quad x_0^{(0)}(t, \rho) = \frac{1}{4B_0^{(0)}} \left( \int_0^t \frac{\sqrt{\tilde{f}^{(0)}(t, \rho)}}{\sqrt{t}} dt \right)^2.$$

Here we assume that  $B_0^{(0)}$  can be chosen to be different from 0; we will see later (cf. (1.1.1.22) below) that this is automatically satisfied thanks to the assumption (1.1.1). We note that (1.1.1.6) together with (1.2) and (1.3) entails the existence of holomorphic function  $\tilde{x}_0^{(0)}(t, \rho)$  that satisfies

$$(1.1.1.7) \quad x_0^{(0)}(t, \rho) = t \tilde{x}_0^{(0)}(t, \rho)$$

with

$$(1.1.1.8) \quad \tilde{x}_0^{(0)}(0, \rho) = \frac{\rho}{B_0^{(0)}}.$$

Although  $x_0^{(0)}$  depends on  $B_0^{(0)}$  at this stage,  $B_0^{(0)}$  will be eventually fixed. Hence we do not make the dependence of  $x_0^{(0)}$  on  $B_0^{(0)}$  explicit in the above notation. The remark of this sort applies to  $x_0^{(p)}$  to be studied below. Next we study

$$[5.1] \quad 2x_0^{(0)} x_0^{(1)} f^{(0)} + x_0^{(0)2} f^{(1)}$$

$$= t^2(x_0^{(0)'} A_0^{(0)} + 2x_0^{(0)'} x_0^{(1)'} x_0^{(0)} B_0^{(0)} + x_0^{(0)'} x_0^{(1)} B_0^{(0)} + x_0^{(0)'} x_0^{(0)} B_0^{(1)}).$$

It then follows from (1.1.1.7) that the left-hand side of [5.1] has the form

$$(1.1.1.9) \quad t^2(2\tilde{x}_0^{(0)} x_0^{(1)} \tilde{f}^{(0)} + \tilde{x}_0^{(0)2} f^{(1)}).$$

Thus we are to solve

$$\begin{aligned} [5.1]' \quad & 2x_0^{(0)'} x_0^{(1)'} x_0^{(0)} B_0^{(0)} + x_0^{(0)'} x_0^{(1)} B_0^{(0)} - 2\tilde{x}_0^{(0)} x_0^{(1)} \tilde{f}^{(0)} \\ & = -x_0^{(0)'} A_0^{(0)} - x_0^{(0)'} x_0^{(0)} B_0^{(1)} + \tilde{x}_0^{(0)2} f^{(1)}. \end{aligned}$$

In view of (1.1.1.7) and (1.1.1.8) we now introduce a new variable

$$(1.1.1.10) \quad s = x_0^{(0)}(t, \rho);$$

in what follows we use the symbol  $\dot{x}(s, \rho)$  to designate  $dx/ds$ . Dividing [5.1]' by  $x_0^{(0)'}^2$  and rewriting the equation in  $s$ -variable, we use [5.0]' to find

$$\begin{aligned} [5.1]'' \quad & 2B_0^{(0)} s \frac{dx_0^{(1)}(s, \rho)}{ds} - B_0^{(0)} x_0^{(1)}(s, \rho) \\ & = -A_0^{(0)} - sB_0^{(1)} + \left[ (x_0^{(0)'})^{-2} \tilde{x}_0^{(0)2} f^{(1)} \right] (t(s, \rho), \rho), \end{aligned}$$

where  $t(s, \rho)$  designates the inverse function of  $s = x_0^{(0)}(t, \rho)$ . Then we find that [5.1]'' is a differential equation with a regular singularity at  $s = 0$  with the characteristic index  $1/2$ . Hence it has a holomorphic solution  $x_0^{(1)}(s, \rho)$  near  $s = 0$  for any  $A_0^{(0)}$  and  $B_0^{(0)}$ , which are arbitrary constants at this stage. Furthermore we find

$$(1.1.1.11) \quad x_0^{(1)}(0, \rho) = \frac{1}{B_0^{(0)}} (A_0^{(0)} - f^{(1)}(0, \rho))$$

and

$$(1.1.1.12) \quad \dot{x}_0^{(1)}(0, \rho) = \frac{1}{B_0^{(0)}} (-B_0^{(1)} + Z_0^{-1}(z'(0, \rho)f^{(1)}(0, \rho) + f^{(1)'}(0, \rho))),$$

where

$$(1.1.1.13) \quad Z_0 = x_0^{(0)'}(0, \rho) \left( = \tilde{x}_0^{(0)}(0, \rho) = \frac{\rho}{B_0^{(0)}} \right)$$

and

$$(1.1.1.14) \quad z(t, \rho) = (x_0^{(0)'}(t, \rho))^{-2} \tilde{x}_0^{(0)}(t, \rho)^2.$$

We next consider

$$\begin{aligned} [5.2] \quad & -f^{(0)} + (2x_0^{(2)}x_0^{(0)} + x_0^{(1)2})f^{(0)} + 2x_0^{(1)}x_0^{(0)}f^{(1)} + x_0^{(0)2}f^{(2)} \\ & = t^2 \left[ x_0^{(0)2}A_0^{(1)} + 2x_0^{(0)'}x_0^{(1)'}A_0^{(0)} + x_0^{(0)2}x_0^{(2)}B_0^{(0)} \right. \\ & \quad + 2x_0^{(0)'}x_0^{(1)'}x_0^{(1)}B_0^{(0)} + x_0^{(0)2}x_0^{(1)}B_0^{(1)} \\ & \quad + (2x_0^{(2)'}x_0^{(0)'} + x_0^{(1)2})x_0^{(0)}B_0^{(0)} + 2x_0^{(1)'}x_0^{(0)'}x_0^{(0)}B_0^{(1)} \\ & \quad \left. + x_0^{(0)2}x_0^{(0)}B_0^{(2)} \right] - x_0^{(0)2}x_0^{(0)}B_0^{(0)}. \end{aligned}$$

Here we observe a new feature which we did not encounter in the study of [5.p] ( $p = 0, 1$ ): [5.2] is not divisible by  $t^2$  as it stands. Thus the existence of a holomorphic solution  $x_0^{(2)}(t, \rho)$  near  $t = 0$  requires that the following function  $\mathcal{B}^{(1)}(t, \rho)$  given by (1.1.1.15) should vanish at  $t = 0$ . Note that  $t\mathcal{B}^{(1)}(t, \rho)$  is the sum of terms in [5.2] which contain the factor  $t^1$  only, at least explicitly.

$$(1.1.1.15) \quad \mathcal{B}^{(1)}(t, \rho) = \tilde{f}^{(0)} - x_0^{(1)2}\tilde{f}^{(0)} - 2\tilde{x}_0^{(0)}x_0^{(1)}f^{(1)} - x_0^{(0)2}\tilde{x}_0^{(0)}B_0^{(0)}.$$

Substituting (1.3), (1.1.1.8) and (1.1.1.11) into  $\mathcal{B}^{(1)}(t, \rho)$ , we find

$$\begin{aligned} (1.1.1.16) \quad & \mathcal{B}^{(1)}(0, \rho) = \rho - \rho(B_0^{(0)})^{-2}(A_0^{(0)} - f^{(1)}(0, \rho))^2 \\ & \quad - 2\rho(B_0^{(0)})^{-1}(B_0^{(0)})^{-1}(A_0^{(0)} - f^{(1)}(0, \rho))f^{(1)}(0, \rho) \\ & \quad - (\rho(B_0^{(0)})^{-1})^3 B_0^{(0)} \end{aligned}$$

$$\begin{aligned}
&= \rho(B_0^{(0)})^{-2} \left( B_0^{(0)2} - (A_0^{(0)} - f^{(1)}(0, \rho))^2 \right. \\
&\quad \left. - 2(A_0^{(0)} - f^{(1)}(0, \rho))f^{(1)}(0, \rho) - \rho^2 \right) \\
&= \rho(B_0^{(0)})^{-2} \left( B_0^{(0)2} - A_0^{(0)2} + f^{(1)}(0, \rho)^2 - \rho^2 \right).
\end{aligned}$$

In view of the assumption (1.1.1) we thus require

$$(1.1.1.17) \quad B_0^{(0)2} - A_0^{(0)2} + f^{(1)}(0, \rho)^2 - \rho^2 = 0.$$

Assuming (1.1.1.17), we can divide [5.2] by  $t^2 x_0^{(0)'}(t, \rho)^2$  to find

$$\begin{aligned}
[5.2]' \quad & B_0^{(0)} \left( 2s \frac{d}{ds} x_0^{(2)}(s, \rho) + x_0^{(2)}(s, \rho) \right) \\
& - 2(x_0^{(0)'}(t, \rho))^{-2} \tilde{x}_0^{(0)}(t, \rho) \tilde{f}^{(0)}(t, \rho) x_0^{(2)}(t, \rho) \\
& = -A_0^{(1)} - B_0^{(2)} s - 2\dot{x}_0^{(1)}(s, \rho) A_0^{(0)} \\
& \quad - 2\dot{x}_0^{(1)}(s, \rho) x_0^{(1)}(s, \rho) B_0^{(0)} - x_0^{(1)}(s, \rho) B_0^{(1)} \\
& \quad - \dot{x}_0^{(1)}(s, \rho)^2 s B_0^{(0)} - 2\dot{x}_0^{(1)}(s, \rho) s B_0^{(1)} + z(t, \rho) f^{(2)}(t, \rho) \\
& \quad - t^{-1} (x_0^{(0)'}(t))^{-2} (\mathcal{B}^{(1)}(t, \rho) - \mathcal{B}^{(1)}(0, \rho)).
\end{aligned}$$

Here we note one universal (i.e., common to every  $p$ ) phenomenon, which was also observed for  $p = 1$ : [5.0]' entails that the left-hand side of [5.2]' is equal to

$$(1.1.1.18) \quad B_0^{(0)} \left( 2s \frac{d}{ds} - 1 \right) x_0^{(2)}(s, \rho).$$

This considerably facilitates the computation of  $x_0^{(2)}(0, \rho)$  and  $\dot{x}_0^{(2)}(0, \rho)$ , which are needed in our reasoning. But we postpone their actual computation until the stage where  $(A_0^{(0)}, B_0^{(0)})$  is fixed;  $(A_0^{(0)}, B_0^{(0)})$  will be fixed without knowing the explicit form of  $x_0^{(2)}(0, \rho)$  and  $\dot{x}_0^{(2)}(0, \rho)$ , whereas their explicit form becomes substantially simplified when  $(A_0^{(0)}, B_0^{(0)})$  is fixed. Here we only note that  $(\partial \mathcal{B}^{(1)} / \partial t)(0, \rho)$  etc. should

be taken into account in the computation of  $x_0^{(2)}(0, \rho)$  etc. To fix  $(A_0^{(0)}, B_0^{(0)})$ , we consider next stage, i.e., [5.3].

$$\begin{aligned}
[5.3] \quad & -f^{(1)} + \sum_{j+k+l=3} x_0^{(j)} x_0^{(k)} f^{(l)} \\
& = t^2 \left( \sum_{j+k+l=2} x_0^{(j)'} x_0^{(k)'} A_0^{(l)} + \sum_{j+k+l+m=3} x_0^{(j)'} x_0^{(k)'} x_0^{(l)} B_0^{(m)} \right) \\
& \quad - \left[ x_0^{(0)'/2} A_0^{(0)} + x_0^{(0)'/2} x_0^{(0)} B_0^{(1)} + \left( x_0^{(0)'/2} x_0^{(1)} + 2x_0^{(0)'} x_0^{(1)'} x_0^{(0)} \right) B_0^{(0)} \right].
\end{aligned}$$

For the existence of a holomorphic solution  $x_0^{(3)}(t, \rho)$  of [5.3] near  $t = 0$ , we clearly need the coincidence of the value of the left-hand side at  $t = 0$  and that of the right-hand side. Although one immediately notices another condition is necessary for the existence of  $x_0^{(3)}(t, \rho)$ , we first concentrate our attention on this coincidence. Then it follows from (1.1.1.8) and (1.1.1.11) that we have

$$\begin{aligned}
(1.1.1.19) \quad & f^{(1)}(0, \rho) \left[ -1 + \left( \frac{1}{B_0^{(0)}} (A_0^{(0)} - f^{(1)}(0, \rho)) \right)^2 \right] \\
& = - \left( \frac{\rho}{B_0^{(0)}} \right)^2 \left[ A_0^{(0)} + A_0^{(0)} - f^{(1)}(0, \rho) \right].
\end{aligned}$$

Then the substitution of

$$(1.1.1.17') \quad A_0^{(0)2} - B_0^{(0)2} = f^{(1)}(0, \rho)^2 - \rho^2$$

into (1.1.1.19) entails

$$\begin{aligned}
(1.1.1.20) \quad & 0 = f^{(1)}(0, \rho) (f^{(1)}(0, \rho)^2 - \rho^2 - 2A_0^{(0)} f^{(1)}(0, \rho) \\
& \quad + f^{(1)}(0, \rho)^2) + \rho^2 (2A_0^{(0)} - f^{(1)}(0, \rho)) \\
& = 2f^{(1)}(0, \rho)^2 (f^{(1)}(0, \rho) - A_0^{(0)}) - 2\rho^2 (f^{(1)}(0, \rho) - A_0^{(0)}) \\
& = 2(f^{(1)}(0, \rho)^2 - \rho^2) (f^{(1)}(0, \rho) - A_0^{(0)}).
\end{aligned}$$

Thus the assumption (1.1.2) implies

$$(1.1.1.21) \quad A_0^{(0)} = f^{(1)}(0, \rho).$$

Substituting (1.1.1.21) into (1.1.1.17) we obtain

$$(1.1.1.22) \quad B_0^{(0)2} = \rho^2,$$

that is,

$$(1.1.1.22') \quad B_0^{(0)} = \pm \rho.$$

These results lead to the following important assertions: First (1.1.1.22) together with (1.1.1.13) implies

$$(1.1.1.23) \quad x_0^{(0)'}(0, \rho) = \pm 1,$$

and second, a still more important result follows from (1.1.1.11) and (1.1.1.21):

$$(1.1.1.24) \quad x_0^{(1)}(0, \rho) = 0!$$

This result will repeatedly play a decisively important role in our subsequent reasoning.

Before proceeding further, we show how these results are used in the explicit computation of  $x_0^{(2)}(0, \rho)$ , which will later become necessary to compute  $(A_0^{(1)}, B_0^{(1)})$ . First, in order to see the explicit form of  $[t^{-1}(x_0^{(0)'})^{-2} (\mathcal{B}^{(1)}(t, \rho) - \mathcal{B}^{(1)}(0, \rho))]$  evaluated at  $t = 0$ , we calculate  $(\partial \mathcal{B}^{(1)} / \partial t)(0, \rho)$ :

$$(1.1.1.25) \quad \frac{\partial \mathcal{B}^{(1)}}{\partial t}(0, \rho) = \rho g'(0, \rho) - 2Z_0 f^{(1)}(0, \rho) x_0^{(1)'}(0, \rho) \\ - 2B_0^{(0)} x_0^{(0)''}(0, \rho) - B_0^{(0)} \tilde{x}_0^{(0)'}(0, \rho).$$

In obtaining this result we used (1.1.1.24) at several spots. Replacing  $f^{(1)}(0, \rho)$  by  $A_0^{(0)}$ , we encounter one remarkable cancellation of terms containing  $A_0^{(0)}$  in  $[5.2]'$  evaluated at  $s = 0$ :

$$(1.1.1.26) \quad -2\dot{x}_0^{(1)}(0, \rho) A_0^{(0)} + 2Z_0^{-2} (Z_0^2 A_0^{(0)} \dot{x}_0^{(1)}(0, \rho)) = 0.$$

Cancellation of this sort will play a crucially important role in the construction of  $(x_0^{(p)}, A_0^{(p)}, B_0^{(p)})$  and their estimation in the subsequent sections. Using (1.1.1.24) again, we thus find

$$(1.1.1.27) \quad B_0^{(0)} x_0^{(2)}(0, \rho) = A_0^{(1)} - f^{(2)}(0, \rho) + \chi_0^{(0)} B_0^{(0)},$$

where  $\chi_0^{(0)}$  is a constant fixed by  $g(t, \rho)$  (and  $Z_0 = \pm 1$ ). Here we notice no  $B_0^{(1)}$ -dependent terms remain in the right-hand side of (1.1.1.27).

Now let us return to the study of [5.3]. To find the conditions that guarantee the existence of holomorphic  $x_0^{(3)}(t, \rho)$ , let us introduce the following functions  $\mathcal{B}$ ,  $\mathcal{B}^{(0)}$ ,  $\mathcal{B}^{(1)}$  and  $\mathcal{B}^{(2)}$ :

$$(1.1.1.28) \quad \begin{aligned} \mathcal{B}(t, \rho) = & -f^{(1)}(t, \rho) + \sum_{j+k+l=3} x_0^{(j)} x_0^{(k)} f^{(l)} \\ & + x_0^{(0)'} x_0^{(0)} A_0^{(0)} + x_0^{(0)'} x_0^{(0)} B_0^{(1)} \\ & + (x_0^{(0)'} x_0^{(1)} + 2x_0^{(0)'} x_0^{(1)'} x_0^{(0)}) B_0^{(0)}, \end{aligned}$$

$$(1.1.1.29) \quad \mathcal{B}^{(0)} = -f^{(1)} + x_0^{(0)'} x_0^{(0)} A_0^{(0)},$$

$$(1.1.1.30) \quad \begin{aligned} \mathcal{B}^{(1)} = & 2\tilde{x}_0^{(0)} x_0^{(2)} f^{(1)} + x_0^{(0)'} \tilde{x}_0^{(0)} B_0^{(1)} \\ & + (x_0^{(0)'} \tilde{x}_0^{(1)} + 2x_0^{(0)'} x_0^{(1)'} \tilde{x}_0^{(0)}) B_0^{(0)}, \end{aligned}$$

where

$$(1.1.1.31) \quad \tilde{x}_0^{(1)}(t, \rho) = x_0^{(1)}(t, \rho)/t,$$

and

$$(1.1.1.32) \quad \begin{aligned} \mathcal{B}^{(2)} = & 2(\tilde{x}_0^{(0)} x_0^{(3)} + \tilde{x}_0^{(1)} x_0^{(2)}) \tilde{f}^{(0)} + \tilde{x}_0^{(1)2} f^{(1)} \\ & + 2\tilde{x}_0^{(0)} \tilde{x}_0^{(1)} f^{(2)} + \tilde{x}_0^{(0)2} f^{(3)}. \end{aligned}$$

It is obvious that we have

$$(1.1.1.33) \quad \mathcal{B} = \mathcal{B}^{(0)} + t\mathcal{B}^{(1)} + t^2\mathcal{B}^{(2)}.$$

One immediately notices that  $\mathcal{B}^{(0)}(0, \rho) = 0$  is equivalent to (1.1.1.19) and that “another condition” needed for the existence of holomorphic  $x_0^{(3)}(t, \rho)$  is given by

$$(1.1.1.34) \quad \frac{\partial \mathcal{B}^{(0)}}{\partial t}(0, \rho) + \mathcal{B}^{(1)}(0, \rho) = 0.$$

Thus we obtain

$$(1.1.1.35) \quad \begin{aligned} & 2Z_0 A_0^{(0)} x_0^{(2)}(0, \rho) + Z_0 B_0^{(1)} \\ & + (\tilde{x}_0^{(1)}(0, \rho) + 2x_0^{(1)'}(0, \rho)) B_0^{(0)} \\ & - \frac{\partial f^{(1)}}{\partial t}(0, \rho) + 2A_0^{(0)} Z_0 x_0^{(0)''}(0, \rho) = 0 \end{aligned}$$

with the help of (1.1.1.21), (1.1.1.23) and (1.1.1.24). We now substitute (1.1.1.12) and (1.1.1.27) into (1.1.1.35) to find

$$(1.1.1.36) \quad \begin{aligned} & 2Z_0 \frac{A_0^{(0)}}{B_0^{(0)}} (A_0^{(1)} - f^{(2)}(0, \rho) + \chi_0^{(0)} B_0^{(0)}) \\ & - 2Z_0 B_0^{(1)} + 3z'(0, \rho) A_0^{(0)} + 2f^{(1)'}(0, \rho) \\ & + 2Z_0 x_0^{(0)''}(0, \rho) A_0^{(0)} = 0. \end{aligned}$$

Dividing (1.1.1.36) by  $Z_0 (= \pm 1)$ , we find

$$(1.1.1.37) \quad \begin{aligned} & 2 \frac{A_0^{(0)}}{B_0^{(0)}} A_0^{(1)} - 2B_0^{(1)} \\ & = 2 \frac{A_0^{(0)}}{B_0^{(0)}} f^{(2)}(0, \rho) - 2A_0^{(0)} \chi_0^{(0)} - 3Z_0^{-1} z'(0, \rho) A_0^{(0)} \\ & - 2Z_0^{-1} f^{(1)'}(0, \rho) - 2x_0^{(0)''}(0, \rho) A_0^{(0)}. \end{aligned}$$

Thus “another condition” for the existence of holomorphic  $x_0^{(3)}(t, \rho)$  gives a constraint on  $(A_0^{(1)}, B_0^{(1)})$ .



Now assumptions (1.1.1.19) and (1.1.1.34) enable us to divide [5.3] by  $t^2(x_0^{(0)'})^2$  to obtain

[5.3]'

$$\begin{aligned}
& B_0^{(0)} \left( 2s \frac{d}{ds} + 1 \right) x_0^{(3)}(s, \rho) \\
&= -A_0^{(2)} - sB_0^{(3)} - \left[ \sum_{\substack{j+k+l=2 \\ l \leq 1}} \dot{x}_0^{(j)} \dot{x}_0^{(k)} A_0^{(l)} + \sum_{\substack{j+k+l+m=3 \\ j,k,l,m \leq 2}} \dot{x}_0^{(j)} \dot{x}_0^{(k)} x_0^{(l)} B_0^{(m)} \right] \\
&+ A_0^{(0)} + sB_0^{(1)} + (x_0^{(1)} + 2s\dot{x}_0^{(1)}) B_0^{(0)} \\
&+ \left[ \sum_{n=0}^3 \phi_n(t, \rho) \right] \Big|_{t=t(s, \rho)}
\end{aligned}$$

where

$$(1.1.1.38) \quad \phi_0 = 2(x_0^{(0)'})^{-2} \tilde{x}_0^{(0)} \tilde{f}^{(0)} x_0^{(3)},$$

$$\begin{aligned}
(1.1.1.39) \quad \phi_1 = & (x_0^{(0)'})^{-2} (2\tilde{x}_0^{(1)} x_0^{(2)} \tilde{f}^{(0)} + \tilde{x}_0^{(1)2} f^{(3)} \\
& + 2\tilde{x}_0^{(0)} \tilde{x}_0^{(1)} f^{(1)} + \tilde{x}_0^{(0)2} f^{(3)})
\end{aligned}$$

$$(1.1.1.40) \quad \phi_2 = 2(x_0^{(0)'})^{-2} (\tilde{x}_0^{(0)} f^{(1)}) x_0^{(2)'}$$

$$\begin{aligned}
(1.1.1.41) \quad \phi_3 = & t^{-2} (x_0^{(0)'})^{-2} [\mathcal{B}^{(0)}(t, \rho) + t\mathcal{B}^{(1)}(t, \rho) - \mathcal{B}^{(0)}(0, \rho) \\
& - t(\mathcal{B}^{(1)}(0, \rho) + (\partial\mathcal{B}^{(0)}/\partial t)(0, \rho)) - t^2(x_0^{(0)'})^2 \phi_2].
\end{aligned}$$

Here we have separated out  $\phi_0$  (resp.,  $\phi_2$ ) from  $\phi_1$  (resp.,  $\phi_3$ ) to call the attention of the reader to the peculiar roles  $\phi_0$  and  $\phi_2$  play in our computation, as has already been noticed when  $p = 2$ : First, [5.0]' entails  $\phi_0$  coincides with  $2B_0^{(0)} x_0^{(3)}(t, \rho)$ ; second, we observe  $\phi_2(t(0, \rho))$  coincides with  $2\dot{x}_0^{(2)}(0, \rho) A_0^{(0)}$ .

It is clear that [5.3]' has a holomorphic solution  $x_0^{(3)}(t, \rho)$  near  $t = 0$  for any  $A_0^{(2)}$  and  $B_0^{(3)}$ , on the condition that  $(A_0^{(0)}, B_0^{(0)})$  satisfies (1.1.1.21) and (1.1.1.22) and that  $(A_0^{(1)}, B_0^{(1)})$  obeys the constraint (1.1.1.37).

Using this holomorphic solution  $x_0^{(3)}(t, \rho)$ , we can write down [5.4]:

$$\begin{aligned}
[5.4] \quad & -f^{(2)} + \sum_{j+k+l=4} x_0^{(j)} x_0^{(k)} f^{(l)} \\
& = t^2 \left( \sum_{j+k+l=3} x_0^{(j)'} x_0^{(k)'} A_0^{(l)} + \sum_{j+k+l+m=4} x_0^{(j)'} x_0^{(k)'} x_0^{(l)} B_0^{(m)} \right) \\
& \quad - \left( \sum_{j+k+l=1} x_0^{(j)'} x_0^{(k)'} A_0^{(l)} + \sum_{j+k+l+m=2} x_0^{(j)'} x_0^{(k)'} x_0^{(l)} B_0^{(m)} \right).
\end{aligned}$$

Assuming  $x_0^{(4)}(t, \rho)$  is holomorphic near  $t = 0$ , we set  $t = 0$  in [5.4] to obtain

$$\begin{aligned}
(1.1.1.42) \quad & -f^{(2)}(0, \rho) + x_0^{(0)'}(0, \rho) A_0^{(1)} \\
& + 2x_0^{(0)'}(0, \rho) x_0^{(1)'}(0, \rho) A_0^{(0)} + x_0^{(0)'}(0, \rho) x_0^{(2)}(0, \rho) B_0^{(0)} \\
& = 0.
\end{aligned}$$

Here we have used (1.1.1.7) and (1.1.1.24) to guarantee that there is no contribution from the sum  $\sum_{j+k+l=4} x_0^{(j)} x_0^{(k)} f^{(l)}$  and  $\left( \sum_{j+k+l+m=2} x_0^{(j)'} x_0^{(k)'} x_0^{(l)} B_0^{(m)} \right) - x_0^{(0)'}(0, \rho) x_0^{(2)}(0, \rho) B_0^{(0)}$ . Substituting (1.1.1.12) and (1.1.1.27) into (1.1.1.42), we find

$$\begin{aligned}
(1.1.1.43) \quad & -f^{(2)}(0, \rho) + A_0^{(1)} + 2A_0^{(0)}(B_0^{(0)})^{-1} \left( -B_0^{(1)} + Z_0^{-1}(z'(0, \rho) A_0^{(0)} \right. \\
& \quad \left. + f^{(1)'}(0, \rho)) \right) + A_0^{(1)} - f^{(2)}(0, \rho) + \chi_0^{(0)} B_0^{(0)} \\
& = 0.
\end{aligned}$$

Then the assumption (1.1.2) enables us to solve (1.1.1.37) and (1.1.1.43);  $(A_0^{(1)}, B_0^{(1)})$  is fixed in terms of  $f^{(2)}(0, \rho)$ ,  $A_0^{(0)}$ ,  $B_0^{(0)}$ ,  $z'(0, \rho)$ ,  $f^{(1)'}(0, \rho)$ ,  $\chi_0^{(0)}$ ,  $Z_0$  and  $x_0^{(0)''}(0, \rho)$ . We next calculate the coefficient of  $t^1$  in [5.4] to find a constraint on  $(A_0^{(2)}, B_0^{(2)})$  which guarantees the existence of

holomorphic  $x_0^{(4)}(t, \rho)$  near  $t = 0$ . In principle, what we are to do now is to repeat this procedure to find  $(x_0^{(p)}(t, \rho), A_0^{(p)}, B_0^{(p)})$  for every  $p$  and then to estimate them. But the computation becomes more and more complicated as  $p$  increases; hence we first describe the core feature of the induction process in Section 1.1.2 and then brush it up in Section 1.1.3, so that the estimation may become smoothly performed with the refined version.

### 1.1.2 Description of the dependence of $\{x_0^{(p)}(t, \rho)\}_{p \geq 0}$ upon $\{A_0^{(q)}, B_0^{(q)}\}_{q \geq 0}$

As the concrete computation in the preceding subsection indicates, one constraint is placed on  $(A_0^{(q)}, B_0^{(q)})$  for the existence of holomorphic  $x_0^{(q+2)}(t, \rho)$  and another constraint on  $(A_0^{(q)}, B_0^{(q)})$  is added for the existence of holomorphic  $x_0^{(q+3)}(t, \rho)$ ; these two conditions combined will fix  $(A_0^{(q)}, B_0^{(q)})$ . In order to confirm that this process runs smoothly by the assumption (1.1.2), we like to know the concrete structure of  $x_0^{(p)}(t, \rho)$ , or at least its “principal part”. For this purpose let us first prepare some notations related to [5.p] ( $= (1.1.1.5.p)$ ).

**Definition 1.1.2.1.** Assume  $p \geq 4$ . Then  $\mathcal{B}[p] = \mathcal{B}[p](t, \rho)$ ,  $\mathcal{B}[p]^{(0)}$ ,  $\mathcal{B}[p]^{(1)}$  and  $\mathcal{B}[p]^{(2)}$  are respectively defined by the following:

$$\begin{aligned}
 (1.1.2.1) \quad \mathcal{B}[p] = & \sum_{i+j+k=p-3} \frac{\partial x_0^{(i)}}{\partial t}(t, \rho) \frac{\partial x_0^{(j)}}{\partial t}(t, \rho) A_0^{(k)} \\
 & + \sum_{i+j+k+l=p-2} x_0^{(k)}(t, \rho) \frac{\partial x_0^{(i)}}{\partial t}(t, \rho) \frac{\partial x_0^{(j)}}{\partial t}(t, \rho) B_0^{(l)} \\
 & + \sum_{i+j+k=p} x_0^{(i)}(t, \rho) x_0^{(j)}(t, \rho) f^{(k)}(t, \rho) - f^{(p-2)}(t, \rho),
 \end{aligned}$$

$$(1.1.2.2) \quad \mathcal{B}[p]^{(0)} = \sum_{i+j+k=p-3} \frac{\partial x_0^{(i)}}{\partial t}(t, \rho) \frac{\partial x_0^{(j)}}{\partial t}(t, \rho) A_0^{(k)}$$

$$\begin{aligned}
& + \sum_{\substack{i+j+k+l=p-2 \\ k \geq 2}} x_0^{(k)}(t, \rho) \frac{\partial x_0^{(i)}}{\partial t}(t, \rho) \frac{\partial x_0^{(j)}}{\partial t}(t, \rho) B_0^{(l)} \\
& + \sum_{\substack{i+j+k=p \\ i, j \geq 2, k \geq 1}} x_0^{(i)}(t, \rho) x_0^{(j)}(t, \rho) f^{(k)}(t, \rho) - f^{(p-2)}(t, \rho)
\end{aligned}$$

$$\begin{aligned}
(1.1.2.3) \quad \mathcal{B}[p]^{(1)} &= \tilde{x}_0^{(0)}(t, \rho) \left( \sum_{i+j+l=p-2} \frac{\partial x_0^{(i)}}{\partial t}(t, \rho) \frac{\partial x_0^{(j)}}{\partial t}(t, \rho) B_0^{(l)} \right) \\
&+ \tilde{x}_0^{(1)}(t, \rho) \left( \sum_{i+j+l=p-3} \frac{\partial x_0^{(i)}}{\partial t}(t, \rho) \frac{\partial x_0^{(j)}}{\partial t}(t, \rho) B_0^{(l)} \right) \\
&+ 2\tilde{x}_0^{(0)}(t, \rho) \left( \sum_{\substack{j+k=p \\ j \geq 2, k \geq 1}} x_0^{(j)}(t, \rho) f^{(k)}(t, \rho) \right) \\
&+ 2\tilde{x}_0^{(1)}(t, \rho) \left( \sum_{\substack{j+k=p-1 \\ j \geq 2, k \geq 1}} x_0^{(j)}(t, \rho) f^{(k)}(t, \rho) \right) \\
&+ \sum_{\substack{i+j=p \\ i, j \geq 2}} x_0^{(i)}(t, \rho) x_0^{(j)}(t, \rho) \tilde{f}^{(0)}(t, \rho),
\end{aligned}$$

$$\begin{aligned}
(1.1.2.4) \quad \mathcal{B}[p]^{(2)} &= \sum_{\substack{i+j+k=p \\ i, j=0,1; k \geq 1}} \tilde{x}_0^{(i)}(t, \rho) \tilde{x}_0^{(j)}(t, \rho) f^{(k)}(t, \rho) \\
&+ 2 \left( \tilde{x}_0^{(0)}(t, \rho) x_0^{(p)}(t, \rho) + \tilde{x}_0^{(1)}(t, \rho) x_0^{(p-1)}(t, \rho) \right) \tilde{f}^{(0)}(t, \rho).
\end{aligned}$$

*Remark 1.1.2.1.* In parallel with (1.1.1.33) we have

$$(1.1.2.5) \quad \mathcal{B}[p] = \mathcal{B}[p]^{(0)} + t\mathcal{B}[p]^{(1)} + t^2\mathcal{B}[p]^{(2)}.$$

To rewrite [5.p] more concretely we further introduce the following symbols. First, we let  $E^{(p)}$  denote

$$(1.1.2.6) \quad \sum_{i+j+k=p-1} \frac{\partial x_0^{(i)}}{\partial t}(t, \rho) \frac{\partial x_0^{(j)}}{\partial t}(t, \rho) A_0^{(k)}$$

$$+ \sum_{i+j+k+l=p} \frac{\partial x_0^{(i)}}{\partial t}(t, \rho) \frac{\partial x_0^{(j)}}{\partial t}(t, \rho) x_0^{(k)}(t, \rho) B_0^{(l)}.$$

Second, we define  $t$ -independent functions  $C_0^{(p)}$  and  $D_0^{(p)}$  by the following:

$$(1.1.2.7) \quad C_0^{(p)}(\rho) = \mathcal{B}[p]^{(0)}(0, \rho),$$

$$(1.1.2.8) \quad D_0^{(p)}(\rho) = \mathcal{B}[p]^{(1)}(0, \rho) + \frac{\partial \mathcal{B}[p]^{(0)}}{\partial t}(0, \rho).$$

Using  $C_0^{(p)}$  and  $D_0^{(p)}$ , we define  $\mathcal{F}^{(p)}$  by

$$(1.1.2.9) \quad \mathcal{B}[p]^{(0)}(t, \rho) + t\mathcal{B}[p]^{(1)}(t, \rho) - (C_0^{(p)} + tD_0^{(p)}).$$

It is then clear that  $\mathcal{F}^{(p)}$  is divisible by  $t^2$  and we use the symbol  $\mathcal{E}^{(p)}$  to denote

$$(1.1.2.10) \quad t^{-2}\mathcal{F}^{(p)}.$$

Having in mind the results in Section 1.1.1, we plan to fix constants  $(A_0^{(q)}, B_0^{(q)})$  by equations

$$(1.1.2.11) \quad C_0^{(q+3)} = 0 \quad \text{and} \quad D_0^{(q+2)} = 0,$$

and construct  $x_0^{(p)}$  by solving

$$[5.p] \quad E^{(p)} - \mathcal{E}^{(p)} - \mathcal{B}[p]^{(2)} = 0.$$

As is observed in the preceding subsection, we can rewrite [5.p] using the variable

$$(1.1.2.12) \quad s = x_0^{(0)}(t, \rho)$$

and its inverse function  $t(s, \rho)$  as follows:

$$[5.p]' \quad B_0^{(0)} \left( 2s \frac{d}{ds} - 1 \right) x_0^{(p)}(s, \rho)$$

$$\begin{aligned}
&= -A_0^{(p-1)} - B_0^{(p)} s - \sum_{\substack{i+j+k=p-1 \\ k \leq p-2}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} A_0^{(k)} - \sum_{\substack{i+j+k+l=p \\ i,j,k,l \leq p-1}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} x_0^{(k)} B_0^{(l)} \\
&+ \left[ (x_0^{(0)'}(t, \rho))^{-2} \left( \mathcal{E}^{(p)} + 2\tilde{x}_0^{(1)}(t, \rho) \tilde{f}^{(0)}(t, \rho) x_0^{(p-1)}(t, \rho) \right. \right. \\
&\left. \left. + \sum_{\substack{i+j+k=p \\ i,j=0,1; k \geq 1}} \tilde{x}_0^{(i)}(t, \rho) \tilde{x}_0^{(j)}(t, \rho) f^{(k)}(t, \rho) \right) \right] \Big|_{t=t(s, \rho)},
\end{aligned}$$

where  $\mathcal{E}^{(p)}$  denotes the sum of functions given by (1.1.2.10).

*Remark 1.1.2.2.* In parallel with (1.1.1.40) we note that the value at  $s=0$  of  $2\dot{x}_0^{(0)} \dot{x}_0^{(p-1)} A_0^{(0)}$  coincides with that of  $\left[ 2(x_0^{(0)'})^{-2} \tilde{x}_0^{(0)} f^{(1)} x_0^{(p-1)'} \right] \Big|_{t=t(s, \rho)}$ ,

which originates from  $\partial \mathcal{B}[p]^{(1)} / \partial t$ ; through the Taylor expansion this term appears among the terms of  $\mathcal{E}^{(p)}$  evaluated at  $s=0$ . A similar relation holds between  $2\dot{x}_0^{(1)} \dot{x}_0^{(p-2)} A_0^{(0)}$  and  $\left[ 2(x_0^{(0)'})^{-2} \tilde{x}_0^{(1)} f^{(1)} x_0^{(p-2)'} \right] \Big|_{t=t(s, \rho)}$ , which also originates from  $\partial \mathcal{B}[p]^{(1)} / \partial t$ . Furthermore the value at  $s=0$  of

$$\sum_{\substack{i+j=p-1 \\ i,j \geq 2}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} A_0^{(0)}$$

is also coincident with that of

$$\left[ (x_0^{(0)'})^{-2} \sum_{\substack{i+j=p-1 \\ i,j \geq 2}} x_0^{(i)'} x_0^{(j)'} f^{(1)} \right] \Big|_{t=t(s, \rho)},$$

which is a part of the coefficient of  $t^2$  in the Taylor expansion of  $\mathcal{B}[p]^{(0)}(t, \rho)$ . These coincidences will play important roles in our subsequent reasoning.

In order to facilitate pointing out the core part of our reasoning, we further prepare the following

**Definition 1.1.2.2.** Let  $(\vec{A}_0[p], \vec{B}_0[p'])$  stand for  $(A_0^{(0)}, A_0^{(1)}, \dots, A_0^{(p)}, B_0^{(0)}, B_0^{(1)}, \dots, B_0^{(p')})$  and let  $X = X(\vec{A}_0[p], \vec{B}_0[p'])$  and  $Y = Y(\vec{A}_0[p], \vec{B}_0[p'])$  be their functions. If  $X-Y$  depends only on  $(\vec{A}_0[q-1], \vec{B}_0[q-1])$  for some  $q$ , then we say

$$(1.1.2.13) \quad X \equiv_{(q)} Y.$$

*Remark 1.1.2.3.* In the above definition we concentrate our attention on the dependence on  $(\vec{A}_0[p], \vec{B}_0[p'])$  which are newly introduced to our discussions as parameters in the canonical form of an M2P1T operator. Hence the influence of the quantities contained in the starting operator such as  $f^{(k)}(0, \rho)$  are taken into account only through their effects on  $(\vec{A}_0[p], \vec{B}_0[p'])$ .

As a preparation for Proposition 1.1.2.1 below, we present the following Lemma 1.1.2.1, where we suppose  $p \geq 4$  for the sake of the uniformity of expression. (Cf. Remark 1.1.2.4 below.)

**Lemma 1.1.2.1.** (i)  $C_0^{(p+1)}(\rho)$  has the following structure:

$$(1.1.2.14) \quad C_0^{(p+1)}(\rho) = \left[ (x_0^{(0)'})^2 A_0^{(p-2)} \right. \\ \left. + 2x_0^{(0)'} x_0^{(p-2)'} A_0^{(0)} + (x_0^{(0)'})^2 x_0^{(p-1)} B_0^{(0)} \right] \Big|_{t=0} \\ + \mathcal{C}^{(p+1)} \left( x_0^{(i)'} (i \leq p-3), x_0^{(j)} (j \leq p-2), \right. \\ \left. A_0^{(k)} (k \leq p-3), B_0^{(l)} (l \leq p-3) \right) \Big|_{t=0},$$

where

$$(1.1.2.15) \quad \mathcal{C}^{(p+1)} = \sum_{\substack{i+j+k=p-2 \\ i,j,k \leq p-3}} x_0^{(i)'} x_0^{(j)'} A_0^{(k)} \\ + \sum_{\substack{i+j+k+l=p-1 \\ 2 \leq k \leq p-2}} x_0^{(i)'} x_0^{(j)'} x_0^{(k)} B_0^{(l)} \\ + \sum_{\substack{i+j+k=p+1 \\ i,j \geq 2; k \geq 1}} x_0^{(i)} x_0^{(j)} f^{(k)} - f^{(p-1)}.$$

(ii)  $D_0^{(p)}(\rho)$  has the following structure:

$$(1.1.2.16) \quad D_0^{(p)}(\rho) = \left[ 2\tilde{x}_0^{(0)} x_0^{(0)'} x_0^{(p-2)'} B_0^{(0)} \right. \\ \left. + \tilde{x}_0^{(0)} x_0^{(0)'}{}^2 B_0^{(p-2)} + 2\tilde{x}_0^{(0)} f^{(1)} x_0^{(p-1)} \right]$$

$$\begin{aligned}
& + (x_0^{(0)'})^2 x_0^{(p-2)'} B_0^{(0)} \Big|_{t=0} \\
& + \mathcal{D}^{(p)} \left( x_0^{(i)'} , x_0^{(i')''} (i, i' \leq p-3), x_0^{(j)} (j \leq p-2), \right. \\
& \left. A_0^{(k)} (k \leq p-3), B_0^{(l)} (l \leq p-3) \right) \Big|_{t=0},
\end{aligned}$$

where

$$\begin{aligned}
(1.1.2.17) \quad \mathcal{D}^{(p)} = & \tilde{x}_0^{(0)} \left( \sum_{\substack{i+j+l=p-2 \\ i,j,l \leq p-3}} x_0^{(i)'} x_0^{(j)'} B_0^{(l)} \right) + \tilde{x}_0^{(1)} \left( \sum_{i+j+l=p-3} x_0^{(i)'} x_0^{(j)'} B_0^{(l)} \right) \\
& + 2\tilde{x}_0^{(0)} \left( \sum_{\substack{j+k=p \\ j,k \geq 2}} x_0^{(j)} f^{(k)} \right) + 2\tilde{x}_0^{(1)} \left( \sum_{\substack{j+k=p-1 \\ j \geq 2, k \geq 1}} x_0^{(j)} f^{(k)} \right) \\
& + \sum_{\substack{i+j=p \\ i,j \geq 2}} x_0^{(i)} x_0^{(j)} \tilde{f}^{(0)} + 2 \sum_{i+j+k=p-3} x_0^{(i)''} x_0^{(j)'} A_0^{(k)} \\
& + \sum_{\substack{i+j+k+l=p-2 \\ 2 \leq k \leq p-3}} x_0^{(k)'} x_0^{(i)'} x_0^{(j)'} B_0^{(l)} + 2 \sum_{\substack{i+j+k+l=p-2 \\ k \geq 2}} x_0^{(k)} x_0^{(i)''} x_0^{(j)'} B_0^{(l)} \\
& + 2 \sum_{\substack{i+j+k=p \\ i,j \geq 2, k \geq 1}} x_0^{(i)'} x_0^{(j)} f^{(k)} + \sum_{\substack{i+j+k=p \\ i,j \geq 2, k \geq 1}} x_0^{(i)} x_0^{(j)} f^{(k)'} - f^{(p-2)'}.
\end{aligned}$$

*Remark 1.1.2.4.* In our later reasoning we will basically use the pair of equations  $C_0^{(p+1)} = 0$  and  $D_0^{(p)} = 0$  to fix  $(A_0^{(p-2)}, B_0^{(p-2)})$ . Hence for the convenience of the future reference we have listed the concrete form of  $C_0^{(p+1)}$  and  $D_0^{(p)}$ , not  $C_0^{(p)}$  and  $D_0^{(p)}$ . We also note that  $\mathcal{C}^{(p+1)}$  and  $\mathcal{D}^{(p)}$  will turn out to be “non-principal parts” in the computation in what follows, in the sense that only  $(A_0^{(q)}, B_0^{(q)})$  ( $q \leq p-3$ ) are relevant to these parts. (See Remark 1.1.2.5 after Proposition 1.1.2.1 below.) From the experience in the previous subsection one might find the following term in the “principal parts” of  $D_0^{(p)}$

$$(1.1.2.18) \quad (x_0^{(0)'})^2 x_0^{(p-2)'} B_0^{(0)}$$



to be somewhat unexpected. As a matter of fact this term originates from

$$(1.1.2.19) \quad \frac{\partial}{\partial t} \left( \sum_{\substack{i+j+k+l=p-2 \\ k \geq 2}} x_0^{(k)} x_0^{(i)'} x_0^{(j)'} B_0^{(l)} \right),$$

and hence

$$(1.1.2.20) \quad p - 2 \geq 2, \text{ i.e., } p \geq 4$$

is required for the appearance of this term. This is the reason why we did not encounter this term when  $p = 3$ . Thus for the sake of the uniformity of presentation we assume  $p \geq 4$  in Proposition 1.1.2.1 below. At the same time we note that the term

$$(1.1.2.21) \quad \tilde{x}_0^{(1)} \left( \sum_{i+j+l=p-3} x_0^{(i)'} x_0^{(j)'} B_0^{(l)} \right) \Big|_{t=0}$$

in the “non-principal part”  $\mathcal{D}^{(p)}$  coincides with (1.1.2.18) evaluated at  $t = 0$  when  $p = 3$ . Since  $x_0^{(1)'}(0, \rho) = \tilde{x}_0^{(1)}(0, \rho)$ , the term (1.1.2.21) had better been regarded as one of the principal terms when  $p = 3$ . This coincidence of terms peculiar to  $p = 3$  explains why the “principal part” of (1.1.1.35) assumes the same form as that claimed in  $\mathcal{A}_0(p)$  (vi) ( $p \geq 4$ ) in Proposition 1.1.2.1 below; this fact might, at first, look somewhat puzzling in view of the absence of (1.1.2.18) in the “principal part” of (1.1.1.35).

Using these notations we now state the following

**Proposition 1.1.2.1.** *Let  $x_0^{(p)}(s, \rho)$  be a solution of the equation  $[5.p]'$  (listed below (1.1.2.12)) with subsidiary conditions*

$$(1.1.2.22) \quad C_0^{(p)} = D_0^{(p)} = 0.$$

*Then the following set  $\mathcal{A}_0(p)$  of assertions ( $\mathcal{A}_0(p)$ (i),  $\mathcal{A}_0(p)$ (ii),  $\dots$ ,  $\mathcal{A}_0(p)$ (vi)) is valid for every  $p \geq 4$ .*

$$\mathcal{A}_0(p) : \begin{cases} \mathcal{A}_0(p)(i) : & x_0^{(p)}(s, \rho) \text{ is holomorphic near } s = 0, \\ \mathcal{A}_0(p)(ii) : & x_0^{(p)}(s, \rho) \text{ depends on } (\vec{A}_0[p-1], \vec{B}_0[p]) \\ & = (A_0^{(0)}, A_0^{(1)}, \dots, A_0^{(p-1)}, B_0^{(0)}, B_0^{(1)}, \dots, B_0^{(p)}), \\ \mathcal{A}_0(p)(iii) : & x_0^{(p)}(0, \rho) \equiv_{(p-1)} A_0^{(p-1)} / B_0^{(0)}, \\ \mathcal{A}_0(p)(iv) : & \frac{dx_0^{(p)}}{ds}(0, \rho) \equiv_{(p)} -B_0^{(p)} / B_0^{(0)}, \\ \mathcal{A}_0(p)(v) : & C_0^{(p)} \equiv_{(p-3)} 2A_0^{(p-3)} - 2\frac{A_0^{(0)}}{B_0^{(0)}} B_0^{(p-3)}, \\ \mathcal{A}_0(p)(vi) : & D_0^{(p)} \equiv_{(p-2)} 2\frac{Z_0 A_0^{(0)}}{B_0^{(0)}} A_0^{(p-2)} - 2Z_0 B_0^{(p-2)} \end{cases}$$

*Remark 1.1.2.5.* The validity of  $\mathcal{A}_0(p)(v)$  and  $\mathcal{A}_0(p)(vi)$  justifies calling  $\mathcal{C}^{(p)}$  and  $\mathcal{D}^{(p)}$  “non-principal parts”.

*Proof of Proposition 1.1.2.1.* [I] Let us first confirm  $\mathcal{A}_0(4)$ . As the argument for this case serves as a good specimen of the reasoning for the general case, we give it in a detailed manner. To begin with we summarize the results obtained in the precedent subsection. First, we know (i) the explicit form of the equation that  $x_0^{(0)}(t, \rho)$  satisfies (cf. [5.0]'), (i') the concrete form of  $x_0^{(0)}(t, \rho)$  and (ii)  $x_0^{(0)}(0, \rho)$  and  $x_0^{(0)'}(0, \rho)$  (cf. (1.1.1.7), (1.1.1.8) and (1.1.1.23)); second, we know (i) the concrete form of the equation that  $x_0^{(1)}(s, \rho)$  satisfies (cf. [5.1]') and (ii)  $x_0^{(1)}(0, \rho)$  and  $\dot{x}_0^{(1)}(0, \rho)$  (cf. (1.1.1.11), (1.1.1.12) and (1.1.1.24)); third, we know (i) the explicit form of the equation that  $x_0^{(2)}(s, \rho)$  satisfies (cf. [5.2]' and (1.1.1.18)) and (ii)  $x_0^{(2)}(0, \rho)$  (cf. (1.1.1.27)), and fourthly we present the explicit form of the equation that  $x_0^{(3)}(s, \rho)$  satisfies (cf. [5.3]'). These results, among other things, guarantee the validity of  $\mathcal{A}_0(q)(ii)$  ( $q \leq 3$ ). At the same time we notice that we have so far fixed  $(A_0^{(1)}, B_0^{(1)})$  (cf. (1.1.1.37) and (1.1.1.43)) to guarantee the holomorphy of  $x_0^{(q)}(s, \rho)$  ( $q \leq 3$ ) near  $s = 0$ . One important observation to be made is that holomorphic  $x_0^{(3)}(s, \rho)$  exists for arbitrary constants  $(A_0^{(2)}, B_0^{(2)}, B_0^{(3)})$

at this stage; any constraints have not yet been imposed upon these constants on which  $x_0^{(3)}(s, \rho)$  depends.

Now, to find a holomorphic solution  $x_0^{(4)}$  of [5.4]' we are to suppose  $C_0^{(4)} = D_0^{(4)} = 0$ . To find the explicit constraints on the parameters  $(A_0^{(1)}, B_0^{(1)})$  and others, we want to have concrete expressions of  $C_0^{(4)}$  and  $D_0^{(4)}$  which enable us to see their implications. The explicit computation of all terms in  $C_0^{(p)}$  and  $D_0^{(p)}$  is a laborious task, but the filtration with respect to  $p$  we are using facilitates our computation substantially. For example, the thorough computation of  $x_0^{(2)}(0, \rho)$  is considerably more arduous than that of  $x_0^{(2)}(0, \rho)$ , but the confirmation of  $\mathcal{A}_0(2)(iv)$  is a rather straightforward task; in the right-hand side of [5.2]' all terms except for  $-A_0^{(1)} - B_0^{(2)}s$  are expressed in terms of

$$(1.1.2.23) \quad \{x_0^{(q)}(q = 0, 1) \text{ and their derivatives, } f^{(0)}, f^{(1)} \text{ and } f^{(2)}, \\ A_0^{(0)}, B_0^{(0)} \text{ and } B_0^{(1)}\},$$

and hence, thanks to  $\mathcal{A}_0(q)(ii)$  ( $q = 0, 1$ ), we may ignore them in confirming  $\mathcal{A}_0(2)(iv)$ . Similarly the confirmation of  $\mathcal{A}_0(3)(iii)$ , which we need in confirming  $\mathcal{A}_0(4)(vi)$ , is not difficult, if we note the following fact (C):

(C) If we set  $s = 0$  in the right-hand side of [5.3]', the remaining terms are free from  $B_0^{(2)}$ .

Clearly  $\mathcal{A}_0(3)(iii)$  follows from (C), and the fact (C) is a consequence of the following two facts (C.i) and (C.ii):

(C.i)  $-2\dot{x}_0^{(2)}(0, \rho)A_0^{(0)}$  and  $\phi_2(0, \rho)$  cancel out (cf. Remark 1.1.2.2),

$$(C.ii) \quad \left[ \sum_{\substack{j+k+l+m=3 \\ j, k, l, m \leq 2}} \dot{x}_0^{(j)} \dot{x}_0^{(k)} x_0^{(l)} B_0^{(m)} \right] \Big|_{s=0} \\ = \left[ \sum_{\substack{j+k+l+m=3 \\ l=2}} \dot{x}_0^{(j)} \dot{x}_0^{(k)} x_0^{(l)} B_0^{(m)} \right] \Big|_{s=0}$$

$$= \left[ \sum_{j+k+m=1} \dot{x}_0^{(j)} \dot{x}_0^{(k)} B_0^{(0)-1} (A_0^{(1)} - f^{(2)}(0, \rho) + \chi_0^{(0)} B_0^{(0)}) B_0^{(m)} \right] \Big|_{s=0},$$

which follows from (1.1.1.24) and (1.1.1.27). Since these terms are the only terms in the right-hand side of [5.3]' that may contain  $B_0^{(2)}$ , (C.i) and (C.ii) entail (C). The disappearance of  $B_0^{(p-1)}$  in the right-hand side of [5.p]' is a universal phenomenon, as we will see below.

Using  $\mathcal{A}_0(2)(iv)$  and  $\mathcal{A}_0(3)(iii)$ , which we have just confirmed, together with the results obtained in the preceding subsection, we can now confirm  $\mathcal{A}_0(4)(v)$  and  $\mathcal{A}_0(4)(vi)$ . Let us first compute  $C_0^{(4)}$ . Then it follows from (1.1.2.14), (1.1.1.12) and (1.1.1.27) that

$$\begin{aligned} (1.1.2.24) \quad C_0^{(4)}(\rho) &\equiv_{(1)} A_0^{(1)} + 2\dot{x}_0^{(1)}(0, \rho) A_0^{(0)} + x_0^{(2)}(0, \rho) B_0^{(0)} \\ &\equiv_{(1)} 2A_0^{(1)} - 2\frac{A_0^{(0)}}{B_0^{(0)}} B_0^{(1)}. \end{aligned}$$

This confirms  $\mathcal{A}_0(4)(v)$ . To compute  $D_0^{(4)}$  we apply  $\mathcal{A}_0(2)(iv)$  and  $\mathcal{A}_0(3)(iii)$  together with  $\mathcal{A}_0(q)(ii)$  ( $q \leq 2$ ) to (1.1.2.16) to find

$$\begin{aligned} (1.1.2.25) \quad D_0^{(4)}(\rho) &\equiv_{(2)} 2Z_0 \left( -\frac{B_0^{(2)}}{B_0^{(0)}} \right) B_0^{(0)} + Z_0 B_0^{(2)} \\ &\quad + 2Z_0 A_0^{(0)} \left( \frac{A_0^{(2)}}{B_0^{(0)}} \right) + Z_0 \left( -\frac{B_0^{(2)}}{B_0^{(0)}} \right) B_0^{(0)} \\ &= 2\frac{Z_0 A_0^{(0)}}{B_0^{(0)}} A_0^{(2)} - 2Z_0 B_0^{(2)}. \end{aligned}$$

This validates  $\mathcal{A}_0(4)(vi)$ .

These concrete expressions of the “top parts” of  $C_0^{(4)}$  and  $D_0^{(4)}$  tell us how the subsidiary conditions given by (1.1.2.22) (with  $p = 4$ ) put new constraints on  $(A_0^{(1)}, B_0^{(1)}, A_0^{(2)}, B_0^{(2)})$ . Now we know  $(A_0^{(1)}, B_0^{(1)})$  obeys the constraint (1.1.1.37) which may be summarized, in our current context, as follows:

$$(1.1.2.26) \quad 2\frac{A_0^{(0)}}{B_0^{(0)}} A_0^{(1)} - 2B_0^{(1)} = \text{given data.}$$

Considering this equation simultaneously with  $C_0^{(4)}(\rho) = 0$ , we find that these two constraints are consistent, i.e., admit a simultaneous (unique) solution  $(A_0^{(1)}, B_0^{(1)})$  thanks to our assumption (1.1.2) (supplemented with (1.1.1.21) and (1.1.1.22)). Thus  $\mathcal{A}_0(4)(i)$  is valid, and then  $\mathcal{A}_0(4)(ii)$ ,  $\mathcal{A}_0(4)(iii)$  and  $\mathcal{A}_0(4)(iv)$  can be readily confirmed. In order to make our argument as concrete as possible, let us write down [5.4]' explicitly:

$$\begin{aligned}
[5.4]' \quad & B_0^{(0)} \left( 2s \frac{d}{ds} - 1 \right) x_0^{(4)}(s, \rho) \\
&= -A_0^{(3)} - B_0^{(4)} s - \sum_{\substack{i+j+k=3 \\ k \leq 2}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} A_0^{(k)} - \sum_{\substack{i+j+k+l=4 \\ i,j,k,l \leq 3}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} x_0^{(k)} B_0^{(l)} \\
&+ \left[ (x_0^{(0)'}(t, \rho))^{-2} \left( \mathcal{E}^{(4)} + 2\tilde{x}_0^{(1)}(t, \rho) \tilde{f}^{(0)}(t, \rho) x_0^{(3)}(t, \rho) \right. \right. \\
&\left. \left. + \sum_{\substack{i+j+k=4 \\ i,j=0,1 \ k \geq 1}} \tilde{x}_0^{(i)}(t, \rho) \tilde{x}_0^{(j)}(t, \rho) f^{(k)}(t, \rho) \right) \right] \Big|_{t=t(s, \rho)},
\end{aligned}$$

where

$$\begin{aligned}
(1.1.2.27) \quad \mathcal{E}^{(4)} = & t^{-2} \left[ \sum_{i+j+k=1} x_0^{(i)'}(t, \rho) x_0^{(j)'}(t, \rho) A_0^{(k)} \right. \\
& + x_0^{(2)}(t, \rho) (x_0^{(0)'}(t, \rho))^2 B_0^{(0)} - f^{(2)} \\
& + t\tilde{x}_0^{(0)}(t, \rho) \left( \sum_{i+j+l=2} x_0^{(i)'}(t, \rho) x_0^{(j)'}(t, \rho) B_0^{(l)} \right) \\
& + t\tilde{x}_0^{(1)}(t, \rho) \left( \sum_{i+j+l=1} x_0^{(i)'}(t, \rho) x_0^{(j)'}(t, \rho) B_0^{(l)} \right) \\
& + 2t\tilde{x}_0^{(0)}(t, \rho) \left( \sum_{\substack{j+k=4 \\ j \geq 2, k \geq 1}} x_0^{(j)}(t, \rho) f^{(k)}(t, \rho) \right) \\
& + 2t\tilde{x}_0^{(1)}(t, \rho) x_0^{(2)}(t, \rho) f^{(1)}(t, \rho) \\
& \left. + (x_0^{(2)}(t, \rho))^2 f^{(0)}(t, \rho) - (C_0^{(4)} + tD_0^{(4)}) \right].
\end{aligned}$$

Since  $x_0^{(4)}(s, \rho)$  is a (unique) holomorphic solution of [5.4]', which has a regular singularity at  $s = 0$  with its characteristic index  $1/2$ , it suffices to examine the structure of each term in the right-hand side of [5.4]' to find how  $x_0^{(4)}(s, \rho)$  depends on the parameters. Since we have validated  $\mathcal{A}_0(q)(ii)$  ( $q \leq 3$ ), the explicit form of the right-hand side of [5.4]' entails that  $x_0^{(4)}(s, \rho)$  depends on  $(\vec{A}_0[3], \vec{B}_0[4])$ . This means that  $\mathcal{A}_0(4)(ii)$  is confirmed. To validate  $\mathcal{A}_0(4)(iii)$ , we need  $\mathcal{A}_0(3)(iv)$ , which we have not yet checked; but its confirmation is a straightforward one, because all terms except for  $-B_0^{(3)}s$  in the right-hand side of [5.3]' are free from  $B_0^{(3)}$  (and  $A_0^{(3)}$ ). Exactly in parallel with the confirmation of  $\mathcal{A}_0(3)(iii)$ , we then use the cancellation of  $-2\tilde{x}_0^{(3)}(0, \rho)A_0^{(0)}$  and  $2\tilde{x}_0^{(0)}(0, \rho)f^{(1)}(0, \rho)x_0^{(3)'}(0, \rho)$  (cf. Remark 1.1.2.2) and the relation

$$(1.1.2.28) \quad \left[ \sum_{\substack{i+j+k+l=4 \\ i,j,k,l \leq 3}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} x_0^{(k)} B_0^{(l)} \right] \Big|_{s=0} \\ = \left[ \sum_{\substack{i+j+k+l=4 \\ k=2,3}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} x_0^{(k)} B_0^{(l)} \right] \Big|_{s=0},$$

which follows from (1.1.1.24). Here we clearly observe that the right-hand side of (1.1.2.28) is free from  $B_0^{(3)}$ . Then by checking indices of all terms in the right-hand side of [5.4]' (including terms in  $\mathcal{E}^{(4)}$ ) we use  $\mathcal{A}_0(q)(ii)$  ( $q \leq 3$ ) to conclude that the right-hand side of [5.4]' evaluated at  $s = 0$  is independent of  $B_0^{(3)}$ . Thus we have confirmed  $\mathcal{A}_0(4)(iii)$ . The confirmation of  $\mathcal{A}_0(4)(iv)$  is a straightforward one, because all terms except for  $-B_0^{(4)}s$  in the right-hand side of [5.4]' are free from  $B_0^{(4)}$  (and  $A_0^{(4)}$ , which has not yet come into our discussion).

Thus we have confirmed  $(\mathcal{A}_0(4)(i), \mathcal{A}_0(4)(ii), \dots, \mathcal{A}_0(4)(vi))$ . In the course of the confirmation new parameters  $(A_0^{(3)}, B_0^{(4)})$  came into our discussion, whereas  $(A_0^{(1)}, B_0^{(1)})$  was fixed and one constraint  $D_0^{(4)} = 0$  was imposed on  $(A_0^{(2)}, B_0^{(2)})$ . Thus our reasoning enters the next stage with free parameters  $(A_0^{(3)}, B_0^{(3)}, B_0^{(4)})$ , neither free nor fixed parameters  $(A_0^{(2)}, B_0^{(2)})$  (i.e., constants controlled by  $D_0^{(4)} = 0$ ) and fixed constants  $(A_0^{(q)}, B_0^{(q)})$  ( $q = 0, 1$ ).

[II] Let us now suppose that  $\mathcal{A}_0(p)$  ( $4 \leq p \leq q$ ) has been validated and show that  $\mathcal{A}_0(q+1)$  is valid. To begin with, we note that in part [I] of

this proof we have confirmed the following statements (S1), (S2) and (S3) besides our real target  $\mathcal{A}_0(4)$ .

(S1)  $\mathcal{A}_0(p)(i)$  and  $\mathcal{A}_0(p)(ii)$  are valid for  $0 \leq p \leq 3$  (with the conventional understanding that  $A_0^{(-1)} = 0$ ).

(S2)  $\mathcal{A}_0(p)(iii)$  is valid for  $p = 2, 3$ , and  $\mathcal{A}_0(p)(iv)$  is valid for  $1 \leq p \leq 3$ .

(S2)  $(A_0^{(1)}, B_0^{(1)})$  is fixed.

We also note that  $(A_0^{(0)}, B_0^{(0)})$  has been fixed in Section 1.1.1.

It then follows from (S1) that the right-hand side of  $[5.q+1]'$  depends on  $(\vec{A}_0[q], \vec{B}_0[q+1])$ . On the other hand, the conditions  $C_0^{(q+1)} = D_0^{(q+1)} = 0$  guarantee the unique existence of holomorphic solution  $x_0^{(q+1)}(s, \rho)$  of  $[5.q+1]'$ . Hence  $\mathcal{A}_0(q+1)(i)$  and  $\mathcal{A}_0(q+1)(ii)$  are valid on the condition that  $C_0^{(q+1)} = D_0^{(q+1)} = 0$  are consistent with previously imposed constraints on  $(\vec{A}_0[q], \vec{B}_0[q+1])$ . In parallel with the reasoning in part [I] it suffices to confirm  $\mathcal{A}_0(q+1)(v)$  and  $\mathcal{A}_0(q+1)(vi)$ ;  $\mathcal{A}_0(q+1)(v)$  combined with  $\mathcal{A}_0(q+1)(vi)$  shows the existence of constants  $(A_0^{(q-2)}, B_0^{(q-2)})$  that satisfy  $D_0^{(q)} = C_0^{(q+1)} = 0$ , with the help of the assumption (1.1.2). Parenthetically  $\mathcal{A}_0(q+1)(vi)$  describes the constraint upon  $(A_0^{(q-1)}, B_0^{(q-1)})$ , which will be used to fix them at the next stage. On the other hand, the confirmation of  $\mathcal{A}_0(q+1)(v)$  and  $\mathcal{A}_0(q+1)(vi)$  is readily done by

( $\alpha$ ) applying  $\mathcal{A}_0(p)(ii)$  ( $0 \leq p \leq q$ ),  $\mathcal{A}_0(p)(iii)$  ( $2 \leq p \leq q$ ) and  $\mathcal{A}_0(p)(iv)$  ( $1 \leq p \leq q-1$ ) to (1.1.2.14), and

( $\beta$ ) applying  $\mathcal{A}_0(p)(ii)$  ( $0 \leq p \leq q$ ),  $\mathcal{A}_0(p)(iii)$  ( $2 \leq p \leq q$ ) and  $\mathcal{A}_0(p)(iv)$  ( $1 \leq p \leq q-1$ ) to (1.1.2.16).

By way of parenthesis the counterpart of  $\mathcal{A}_0(p)(iii)$  ( $p = 0, 1$ ) (resp.,  $\mathcal{A}_0(0)(iv)$ ) is given by  $x_0^{(0)}(0, \rho) = x_0^{(1)}(0, \rho) = 0$  (resp.,  $\dot{x}_0^{(0)}(0, \rho) = 1$ ), which are used in the above confirmation.

Thus what remains to be confirmed is  $(\mathcal{A}_0(q+1)(iii), \mathcal{A}_0(q+1)(iv))$ . Using the explicit form of  $[5.q+1]'$  together with  $\mathcal{A}_0(p)(ii)$  ( $0 \leq p \leq q$ ), we immediately find  $\mathcal{A}_0(q+1)(iv)$ . To validate  $\mathcal{A}_0(q+1)(iii)$ , we use the setoff between  $-2\dot{x}_0^{(q)}(0, \rho)$   $A_0^{(0)}$  and  $2\tilde{x}_0^{(0)}(0, \rho)$   $f^{(1)}(0, \rho)$   $x_0^{(q)'}(0, \rho)$  together with the relation

$$(1.1.2.29) \quad \left[ \sum_{\substack{i+j+k+l=q+1 \\ i,j,k,l \leq q}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} x_0^{(k)} B_0^{(l)} \right] \Big|_{s=0}$$

$$= \left[ \sum_{\substack{i+j+k+l=q+1 \\ 2 \leq k \leq q}} x_0^{(i)} x_0^{(j)} x_0^{(k)} B_0^{(l)} \right] \Big|_{s=0}.$$

Thus exactly the same reasoning used to confirm  $\mathcal{A}_0(4)(iii)$  shows that  $\mathcal{A}_0(q+1)(iii)$  is valid. Thus we have confirmed  $(\mathcal{A}_0(q+1)(i), \mathcal{A}_0(q+1)(ii), \dots, \mathcal{A}_0(q+1)(vi))$ , and hence the induction proceeds.

□

### 1.1.3 Formal construction of $\{x_n^{(p)}, A_n^{(p)}, B_n^{(p)}\}_{p,n \geq 0}$ — the case where $g_{\pm}(t) = 0$

Although the reasoning in the previous subsection is natural and instructive, the setting employed there is somewhat clumsy, particularly when we want to estimate the growth order of  $\{x_0^{(p)}, A_0^{(p)}, B_0^{(p)}\}_{p \geq 0}$ . The primary purpose of this subsection is to present a more refined induction procedure for the construction of  $\{x_0^{(p)}, A_0^{(p)}, B_0^{(p)}\}_{p \geq 0}$ . We later (in Proposition 1.1.3.2) confirm that the procedure works for the construction of  $\{x_n^{(p)}, A_n^{(p)}, B_n^{(p)}\}_{p,n \geq 0}$  which are used to transform an M2P1T equation to its canonical form; for the sake of simplicity of reasoning we assume  $g_{\pm}(t) = 0$  in this subsection. In what follows,  $x_0^{(0)}(s, \rho)$  denotes the holomorphic function given by (1.1.1.6) and  $x_0^{(1)}(s, \rho)$  is the holomorphic solution of [5.1]'' satisfying the condition (1.1.1.24), that is,

$$(1.1.3.1) \quad x_0^{(1)}(0, \rho) = 0.$$

The constants  $A_0^{(0)}$  and  $B_0^{(0)}$  are those satisfying (1.1.1.21) and (1.1.1.22), respectively and  $(A_0^{(1)}, B_0^{(1)})$  designates a common solution of (1.1.1.37) and (1.1.1.43): In this subsection we conventionally understand that the relations  $C_0^{(3)}(A_0^{(0)}, B_0^{(0)}) = D_0^{(2)}(A_0^{(0)}, B_0^{(0)}) = 0$  and  $C_0^{(4)}(A_0^{(1)}, B_0^{(1)}) = D_0^{(3)}(A_0^{(1)}, B_0^{(1)}) = 0$  respectively mean the relations that  $(A_0^{(0)}, B_0^{(0)})$  and  $(A_0^{(1)}, B_0^{(1)})$  satisfy. We also understand  $C_0^{(2)}(A_0^{(-1)}, B_0^{(-1)}) = 0$  to be an empty condition, which is a reflection of the fact that [5.2] is free from the constant term. By way of parenthesis we note that  $D_0^{(3)}(A_0^{(1)}, B_0^{(1)}) = 0$  is well-defined (i.e., without any extra convention) as is given by (1.1.1.37) despite the seeming ambiguity in separating



out its “principal part” (cf. Remark 1.1.2.4). Similarly  $C_0^{(p+1)}$  with  $p = 3$  given by (1.1.2.14) is coincident with (1.1.1.43).

In order to present the refined induction procedure we prepare some notations and auxiliary results. We use the symbol  $\mathfrak{A}_0(p)$  to mean the assertion that a triplet of data  $T_0^{(r)} = \{x_0^{(r)}(s, \rho), A_0^{(r)}, B_0^{(r)}\}$  is given for  $0 \leq r \leq p$  so that they satisfy the following conditions:

(1.1.3.2.r)  $x_0^{(r)}(s, \rho)$  is a holomorphic solution of  $[5.r]'$  (to be found below (1.1.2.12)) near  $s = 0$ ,

(1.1.3.3.r)  $x_0^{(r)}(s, \rho)$  depends on  $(\vec{A}_0[r-1], \vec{B}_0[r]) = (A_0^{(0)}, A_0^{(1)}, \dots, A_0^{(r-1)}, B_0^{(0)}, B_0^{(1)}, \dots, B_0^{(r)})$ ,

(1.1.3.4.r)  $C_0^{(r+3)}(\rho)$  and  $D_0^{(r+2)}(\rho)$  depend on  $(\vec{A}_0[r], \vec{B}_0[r])$ , and  $(\vec{A}_0[r], \vec{B}_0[r])$  satisfies the relations  $C_0^{(r+3)}(\rho) = D_0^{(r+2)}(\rho) = 0$ ,

$$(1.1.3.5.r) \quad C_0^{(r+3)}(\rho) \equiv_{(r)} 2A_0^{(r)} - 2\frac{A_0^{(0)}}{B_0^{(0)}}B_0^{(r)},$$

$$(1.1.3.6.r) \quad D_0^{(r+2)}(\rho) \equiv_{(r)} 2Z_0A_0^{(r)}\frac{A_0^{(0)}}{B_0^{(0)}} - 2Z_0B_0^{(r)}.$$

We will show later in Proposition 1.1.3.1 that  $\mathfrak{A}_0(p)$  entails  $\mathfrak{A}_0(p+1)$ .

*Remark 1.1.3.1.* A main difference of the contents of  $\mathfrak{A}_0(p)$  and  $\mathfrak{A}_0(p)$  is that  $\mathfrak{A}_0(p)$  refers to the structure of  $C_0^{(r+3)}(\rho)$  and  $D_0^{(r+2)}(\rho)$  for  $r \leq p-1$ ; in view of Lemma 1.1.2.1 one might be puzzled with the appearance of  $x_0^{(p+1)}(0, \rho)$  in the expression of  $C_0^{(p+3)}(\rho)$  and  $D_0^{(p+2)}(\rho)$ . As Lemma 1.1.3.3 and Lemma 1.1.3.4 below show,  $x_0^{(p+1)}(0, \rho)$  can be written down in terms of  $\{T_0^{(r)}\}_{0 \leq r \leq p}$  and  $x_0^{(p+1)}(0, \rho) - A_0^{(p)}/B_0^{(0)}$  is free from  $A_0^{(p)}$  and  $B_0^{(p)}$ . These facts are implicitly woven into conditions (1.1.3.4.r), (1.1.3.5.r) and (1.1.3.6.r). The reader will find the mechanism in the proof of Proposition 1.1.3.1, where conditions (1.1.3.5.r) and (1.1.3.6.r) are confirmed for  $r = p+1$ .

In proving Lemma 1.1.3.1 ~ Lemma 1.1.3.4 below we assume that  $\mathfrak{A}_0(p)$  ( $p \geq 1$ ) has been validated.

**Lemma 1.1.3.1.** *The right-hand side of  $[5.p + 1]'$  ( $p \geq 1$ ) has the following form:*

$$(1.1.3.7) \quad -A_0^{(p)} - B_0^{(p+1)}s + B_0^{(0)}R_0^{(p+1)}(s, \rho),$$

where  $B_0^{(p+1)}$  is a complex number and

$$(1.1.3.8) \quad \begin{aligned} B_0^{(0)}R_0^{(p+1)}(s, \rho) = & - \sum_{\substack{i+j+k=p \\ k \leq p-1}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} A_0^{(k)} - \sum_{\substack{i+j+k+l=p+1 \\ i,j,k,l \leq p}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} x_0^{(k)} B_0^{(l)} \\ & + \left[ (x_0^{(0)'}(t, \rho))^{-2} t^{-2} \left( \sum_{i+j+k=p-2} x_0^{(i)'} x_0^{(j)'} A_0^{(k)} \right. \right. \\ & + \sum_{i+j+k+l=p-1} x_0^{(i)'} x_0^{(j)'} x_0^{(k)} B_0^{(l)} + \sum_{\substack{i+j+k=p+1 \\ k \geq 1}} x_0^{(i)} x_0^{(j)} f^{(k)} \\ & \left. \left. + \sum_{\substack{i+j=p+1 \\ i,j \geq 1}} x_0^{(i)} x_0^{(j)} f^{(0)} - f^{(p-1)} \right) \right] \Big|_{t=t(s, \rho)}. \end{aligned}$$

*Remark 1.1.3.2.* The factor  $B_0^{(0)}$  in front of  $R_0^{(p+1)}(s, \rho)$  is a rather conventional one; it will turn out to be notationally convenient when we estimate the growth order of  $x_0^{(p)}(s, \rho)$  etc. with an emphasis on their  $\rho$ -dependence. Recall that  $B_0^{(0)} = \pm \rho$  holds by (1.1.1.22').

*Proof of Lemma 1.1.3.1.* Since  $C_0^{(p+1)}(\rho) = D_0^{(p+1)}(\rho) = 0$  holds by the assumption, we can read off the above result immediately from (1.1.1.5. $p+1$ ) in view of the definition of  $[5.p+1]'$ . We only note that we have shifted

$$(1.1.3.9) \quad 2(x_0^{(0)'}(t, \rho))^{-2} \tilde{x}_0^{(0)}(t, \rho) x_0^{(p+1)}(t, \rho) \tilde{f}^{(0)}(t, \rho) \Big|_{t=t(s, \rho)}$$

to the left-hand side of  $[5.p+1]'$ ; we have left

$$(1.1.3.10) \quad \left[ (x_0^{(0)'}(t, \rho))^{-2} t^{-2} \left( \sum_{\substack{i+j=p+1 \\ i,j \geq 1}} x_0^{(i)} x_0^{(j)} f^{(0)} \right) \right] \Big|_{t=t(s, \rho)}$$

in  $B_0^{(0)} R_0^{(p+1)}(s, \rho)$  despite the fact that a term similar to (1.1.3.9), i.e.,

$$(1.1.3.11) \quad 2(x_0^{(0)'})^{-2} \tilde{x}_0^{(1)} x_0^{(p)} \tilde{f}^{(0)} \Big|_{t=t(s, \rho)},$$

is contained in the sum (1.1.3.10); this non-uniformity of treatment is just due to the convention that the left-hand of  $[5.p + 1]'$  should contain only the (at this level) unknown function  $x_0^{(p+1)}(s, \rho)$  and that its right-hand side should consist of given data.

□

**Lemma 1.1.3.2.** *The function  $R_0^{(p+1)}(s, \rho)$  is determined by  $\{T_0^{(r)}\}_{0 \leq r \leq p}$ , and it is free from  $A_0^{(p)}$ .*

*Proof.* This is an immediate consequence of the concrete expression (1.1.3.8) of  $R_0^{(p+1)}(s, \rho)$ .

□

**Lemma 1.1.3.3.** (i) *For an arbitrary complex number  $B_0^{(p+1)}$  we find a unique holomorphic solution  $x_0^{(p+1)}(s, \rho)$  near  $s = 0$  of the following equation  $[5.p + 1]'$ :*

$$(1.1.3.12) \quad (= [5.p + 1]') \\ B_0^{(0)} \left( 2s \frac{d}{ds} - 1 \right) x_0^{(p+1)}(s, \rho) = -A_0^{(p)} - B_0^{(p+1)} s + B_0^{(0)} R_0^{(p+1)}(s, \rho).$$

(ii) *The solution  $x_0^{(p+1)}(s, \rho)$  depends on  $(\vec{A}_0[p], \vec{B}_0[p + 1])$ .*

(iii) *For the above solution  $x_0^{(p+1)}(s, \rho)$  we find*

$$(1.1.3.13) \quad B_0^{(0)} x_0^{(p+1)}(0, \rho) = A_0^{(p)} - B_0^{(0)} R_0^{(p+1)}(0, \rho)$$

and

$$(1.1.3.14) \quad B_0^{(0)} \dot{x}_0^{(p+1)}(0, \rho) = -B_0^{(p+1)} + B_0^{(0)} \dot{R}_0^{(p+1)}(0, \rho).$$

*Proof.* (i) Since  $C_0^{(p+1)}(\rho) = D_0^{(p+1)}(\rho) = 0$  holds by the assumption, and since  $B_0^{(0)}$  is different from 0 by the assumption (1.1.1) together with the relation (1.1.1.22'), the unique existence of a holomorphic solution of  $[5.p+1]'$  is evident.

(ii) This immediately follows from Lemma 1.1.3.2 (on the condition that  $\mathfrak{A}_0(p)$  is valid).

(iii) By setting  $s = 0$  in (1.1.3.12), we readily obtain (1.1.3.13). By first differentiating both sides of (1.1.3.12) and then setting  $s = 0$ , we obtain (1.1.3.14).

□

*Remark 1.1.3.3.* It is clear that relations similar to (1.1.3.13) and (1.1.3.14) hold for any holomorphic solution  $x_0^{(q)}(s, \rho)$  of the following equation

$$(1.1.3.15) \quad B_0^{(0)} \left( 2s \frac{d}{ds} - 1 \right) x_0^{(q)}(s, \rho) = -A_0^{(q-1)} - B_0^{(q)} s + B_0^{(0)} R_0^{(q)}(s, \rho),$$

where  $A_0^{(q-1)}$  and  $B_0^{(q)}$  are complex numbers and  $R_0^{(q)}(s, \rho)$  is holomorphic near  $s = 0$ ; that is, we have

$$(1.1.3.16) \quad B_0^{(0)} x_0^{(q)}(0, \rho) = A_0^{(q-1)} - B_0^{(0)} R_0^{(q)}(0, \rho)$$

and

$$(1.1.3.17) \quad B_0^{(0)} \dot{x}_0^{(q)}(0, \rho) = -B_0^{(q)} + B_0^{(0)} \dot{R}_0^{(q)}(0, \rho).$$

**Lemma 1.1.3.4.** *The value  $B_0^{(0)} R_0^{(p+1)}(0, \rho)$  is free from  $B_0^{(p)}$ .*

*Proof.* When  $p = 0$ ,  $[5,1]''$  together with (1.1.1.21) entails that  $B_0^{(0)} R_0^{(1)}(0, \rho)$  coincides with  $A_0^{(0)}$ ; thus it is free from  $B_0^{(0)}$ . Hence we assume  $p \geq 1$  in the discussion below. It then follows from (1.1.3.3.r) that  $x_0^{(r)}(s, \rho)$  ( $0 \leq r \leq p-1$ ) is free from  $B_0^{(p)}$ . Hence the terms in  $R_0^{(p+1)}(s, \rho)$  whose relevance we have to check are those containing  $B_0^{(p)}$ ,  $x_0^{(p)}$  or  $\dot{x}_0^{(p)}$ . Furthermore, (1.1.3.16) with  $q = p$  guarantees that it

suffices to concentrate our attention on terms containing  $B_0^{(p)}$  or  $\dot{x}_0^{(p)}$ . Thus the terms to be checked are the following:

$$(1.1.3.18) \quad - \left( \sum_{i+j+k=1} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) x_0^{(k)}(0, \rho) \right) B_0^{(p)},$$

$$(1.1.3.19) \quad - \left( \sum_{\substack{i+j=p+1 \\ i, j \leq p}} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \right) x_0^{(0)}(0, \rho) B_0^{(0)} \\ - \left( \sum_{i+j=p} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \right) \left( \sum_{k+l=1} x_0^{(k)}(0, \rho) B_0^{(l)} \right),$$

$$(1.1.3.20) \quad -2\dot{x}_0^{(0)}(0, \rho) \dot{x}_0^{(p)}(0, \rho) A_0^{(0)}$$

and terms in the coefficients of the Taylor expansion of

$$(1.1.3.21) \quad \left[ (x_0^{(0)'})^{-2} t^{-2} (2x_0^{(0)} x_0^{(p)} f^{(1)} + 2x_0^{(1)} x_0^{(p)} f^{(0)}) \right] \Big|_{t=t(s, \rho)}.$$

Here we encounter a situation essentially the same as that observed in the fact (C) used for the confirmation of  $\mathcal{A}_0(3)(iii)$  in the proof of Proposition 1.1.2.1. First the important relation (1.1.3.1) together with (1.1.1.7), i.e.,  $x_0^{(0)}(0, \rho) = 0$ , entails the vanishing of each term in the sum (1.1.3.18) and the sum (1.1.3.19); this reasoning corresponds to (C.ii). Second, (1.1.3.20) is cancelled out by the term

$$(1.1.3.22) \quad 2(x_0^{(0)'})^{-2} \tilde{x}_0^{(0)} x_0^{(p)'} f^{(1)} \Big|_{t=t(0, \rho)} = 2\dot{x}_0^{(p)}(0, \rho) f^{(1)}(0, \rho) = 2\dot{x}_0^{(p)}(0, \rho) A_0^{(0)},$$

which originates from

$$(1.1.3.23) \quad \left[ (x_0^{(0)'})^{-2} t^{-2} (2x_0^{(0)} x_0^{(p)} f^{(1)}) \right] \Big|_{t=t(s, \rho)}.$$

This fact corresponds to (C.i). We note that the contribution from

$$(1.1.3.24) \quad \left[ (x_0^{(0)'})^{-2} t^{-2} (2x_0^{(1)} x_0^{(p)} f^{(0)}) \right] \Big|_{t=t(s, \rho)}$$

is

$$(1.1.3.25) \quad 2(x_0^{(0)'}(0, \rho))^{-2} \tilde{x}_0^{(1)}(0, \rho) \tilde{f}^{(0)}(0, \rho) x_0^{(p)}(0, \rho);$$

thus this part is irrelevant to  $B_0^{(p)}$ . This completes the proof of Lemma 1.1.3.4.

□

So far we have constructed a holomorphic solution  $x_0^{(p+1)}(s, \rho)$  of  $[5.p+1]'$  by using the data given in  $\mathfrak{A}_0(p)$  together with a newly added arbitrary complex number  $B_0^{(p+1)}$ . Since

$$(1.1.3.26) \quad C_0^{(p+3)}(\rho) = D_0^{(p+2)}(\rho) = 0$$

is contained in the assertion  $\mathfrak{A}_0(p)$ , the equation  $[5.p+2]'$  is given by (1.1.3.27)

$$B_0^{(0)} \left( 2s \frac{d}{ds} - 1 \right) x_0^{(p+2)}(s, \rho) = -A_0^{(p+1)} - B_0^{(p+2)} s + B_0^{(0)} R_0^{(p+2)}(s, \rho),$$

where  $A_0^{(p+1)}$  and  $B_0^{(p+2)}$  are newly added arbitrary complex numbers and  $R_0^{(p+2)}(s, \rho)$  is given by replacing  $p$  with  $p+1$  in (1.1.3.8). Note that  $x_0^{(p+1)}(s, \rho)$  and  $B_0^{(p+1)}$  are available at this stage. Furthermore, by using exactly the same reasoning as in the proof of Lemma 1.1.3.4, we find

$$(1.1.3.28) \quad R_0^{(p+2)}(0, \rho) \text{ is free from } (A_0^{(p+1)} \text{ and } B_0^{(p+1)}).$$

For the sake of the completeness of the reasoning we note that no condition on  $A_0^{(p)}$  and  $B_0^{(p)}$  are used in the proof of Lemma 1.1.3.4.

We are now ready to prove the following

**Proposition 1.1.3.1.** *The assertion  $\mathfrak{A}_0(p)$  is valid for every  $p \geq 1$ .*

*Proof.* As we have confirmed the validity of  $\mathfrak{A}_0(1)$  in previous subsections, it suffices to validate  $\mathfrak{A}_0(p+1)$  supposing that  $\mathfrak{A}_0(p)$  is valid. (It is possible to start the induction from  $p=0$ , but to avoid the use of conventional interpretation of the symbol such as  $D_0^{(2)}$  we have started from  $p=1$ .) As we have seen above, we have constructed  $x_0^{(p+1)}(s, \rho)$  that satisfied (1.1.3.2. $p+1$ ) and (1.1.3.3. $p+1$ ) by incorporating an a priori arbitrary complex number  $B_0^{(p+1)}$  with the given data. Furthermore the condition (1.1.3.26) contained in  $\mathfrak{A}_0(p)$  enables us to find the equation (1.1.3.27) for  $x_0^{(p+2)}(s, \rho)$ , where  $A_0^{(p+1)}$  and  $B_0^{(p+2)}$  are a priori arbitrary complex numbers and  $B_0^{(p+1)}$  and  $x_0^{(p+1)}(s, \rho)$  are used to define  $R_0^{(p+2)}(s, \rho)$ . Thus what we have to do for confirming  $\mathfrak{A}_0(p+1)$  is to

show (1.1.3.5. $p+1$ ) and (1.1.3.6. $p+1$ ) and to prove that  $(A_0^{(p+1)}, B_0^{(p+1)})$  can be chosen so that

$$(1.1.3.29) \quad C_0^{(p+4)}(\rho) = D_0^{(p+3)}(\rho) = 0$$

may be satisfied. Meanwhile, once we confirm (1.1.3.5. $p+1$ ) and (1.1.3.6. $p+1$ ), we can readily solve (1.1.3.29) to fix  $(A_0^{(p+1)}, B_0^{(p+1)})$  thanks to the assumption (1.1.2) combined with (1.1.1.21) and (1.1.1.22). To confirm (1.1.3.5. $p+1$ ) and (1.1.3.6. $p+1$ ) we substitute (1.1.3.14) and (1.1.3.16) with  $q = p+2$  into (1.1.2.14) and (1.1.2.16). Then the required results follow from (1.1.3.3. $r$ ) ( $r \leq p+1$ ) together with (1.1.3.13). As the reasoning is the same for (1.1.3.5. $p+1$ ) and (1.1.3.6. $p+1$ ), we show the reasoning for  $C_0^{(p+4)}(\rho)$ . By substituting (1.1.3.14) and (1.1.3.16) (with  $q = p+2$ ) into (1.1.2.14) we find the following:

$$(1.1.3.30) \quad C_0^{(p+4)}(\rho) = A_0^{(p+1)} + 2 \frac{A_0^{(0)}}{B_0^{(0)}} (-B_0^{(p+1)} + B_0^{(0)} \dot{R}_0^{(p+1)}(0, \rho)) \\ + (A_0^{(p+1)} - B_0^{(0)} R_0^{(p+2)}(0, \rho)) \\ + C^{(p+4)}(x_0^{(i)'} (i \leq p), x_0^{(j)} (j \leq p+1), \\ A_0^{(k)} (k \leq p), B_0^{(l)} (l \leq p)) \Big|_{t=0}.$$

Then it follows from (1.1.3.28) and the structure of  $C^{(p+4)}|_{t=0}$  supplemented by (1.1.3.13) that

$$(1.1.3.31) \quad 2A_0^{(0)} \dot{R}_0^{(p+1)}(0, \rho) - B_0^{(0)} R_0^{(p+2)}(0, \rho) + C_0^{(p+4)} \Big|_{t=0}$$

is free from  $A_0^{(p+1)}$  and  $B_0^{(p+1)}$ ; it depends on  $(\vec{A}_0[p], \vec{B}_0[p])$  by (1.1.3.3. $r$ ) ( $r \leq p$ ). Thus we find

$$(1.1.3.5.p+1) \quad C_0^{(p+4)} \equiv_{(p+1)} 2A_0^{(p+1)} - 2 \frac{A_0^{(0)}}{B_0^{(0)}} B_0^{(p+1)}.$$

As we have noted earlier, we can readily find  $(A_0^{(p+1)}, B_0^{(p+1)})$  that annihilates  $C_0^{(p+4)}(\rho)$  and  $D_0^{(p+3)}(\rho)$  by their expressions (1.1.3.5. $p+1$ )

and (1.1.3.6. $p + 1$ ). Thus we obtain the required triplet  $T_0^{(p+1)} = \{x_0^{(p+1)}(s, \rho), A_0^{(p+1)}, B_0^{(p+1)}\}$ . Therefore  $\mathfrak{A}_0(p + 1)$  is validated, and the induction proceeds.

□

Next we study how the construction of triplets  $T_l^{(r)} = \{x_l^{(r)}(s, \rho), A_l^{(r)}, B_l^{(r)}\}$  ( $l, r \geq 0$ ) are done. In what follows we use the symbol

$$(1.1.3.32) \quad \{x; t\}_n^{(p)}$$

to denote the coefficient of  $a^p \eta^{-n}$  of the expansion of  $\{x; t\}$ , that is,

$$(1.1.3.33) \quad \{x; t\} = \sum_{p, n \geq 0} \{x; t\}_n^{(p)} a^p \eta^{-n}.$$

We eventually need more explicit description of  $\{x; t\}$  in terms of the derivatives of  $x_l^{(r)}$ , but it suffices to use this simplified symbol for the time being.

First we note (1.1.6) with  $g_{\pm} = 0$  entails

$$(1.1.3.34) \quad (x^2 - a^2)f = (t^2 - a^2) \left( \frac{\partial x}{\partial t} \right)^2 (aA + xB) - \frac{1}{2} \eta^{-2} (t^2 - a^2) (x^2 - a^2) \{x; t\}.$$

Since

$$(1.1.3.35) \quad x_{2\nu+1}(t, a, \rho) = A_{2\nu+1}(a, \rho) = B_{2\nu+1}(a, \rho) = 0 \quad (\nu = 0, 1, 2, \dots)$$

holds by Proposition A.1 in Appendix A, we then find the following relation (1.1.3.36) for  $n \geq 1$  by the comparison of the coefficients of  $\eta^{-2n}$  of (1.1.3.34) :

$$(1.1.3.36) \quad \left( \sum_{i+j=n} x_{2i} x_{2j} \right) f$$



$$\begin{aligned}
&= (t^2 - a^2) \left( \sum_{i+j+k=n} x'_{2i} x'_{2j} a A_{2k} + \sum_{i+j+k+l=n} x'_{2i} x'_{2j} x_{2k} B_{2l} \right) \\
&\quad - \frac{1}{2} (t^2 - a^2) \sum_{\substack{i+j+k=n-1 \\ r \geq 0}} x_{2i} x_{2j} \{x; t\}_{2k}^{(r)} a^r \\
&\quad + \frac{1}{2} (t^2 - a^2) a^2 \left( \sum_{r \geq 0} \{x; t\}_{2(n-1)}^{(r)} a^r \right).
\end{aligned}$$

Expanding (1.1.3.36) in powers of  $a$  and comparing the coefficients of  $a^p$ , we obtain

$$\begin{aligned}
(1.1.3.37) \quad & \sum_{\substack{q+r+u=p \\ i+j=n}} x_{2i}^{(q)} x_{2j}^{(r)} f^{(u)} \\
&= t^2 \left[ \sum_{\substack{q+r+u=p-1 \\ i+j+k=n}} x_{2i}^{(q)'} x_{2j}^{(r)'} A_{2k}^{(u)} + \sum_{\substack{q+r+u+v=p \\ i+j+k+l=n}} x_{2i}^{(q)'} x_{2j}^{(r)'} x_{2k}^{(u)} B_{2l}^{(v)} \right. \\
&\quad \left. - \frac{1}{2} \sum_{\substack{q+r+u=p \\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x; t\}_{2k}^{(u)} + \frac{1}{2} \{x; t\}_{2(n-1)}^{(p-2)} \right] \\
&\quad - \left[ \sum_{\substack{q+r+u=p-3 \\ i+j+k=n}} x_{2i}^{(q)'} x_{2j}^{(r)'} A_{2k}^{(u)} + \sum_{\substack{q+r+u+v=p-2 \\ i+j+k+l=n}} x_{2i}^{(q)'} x_{2j}^{(r)'} x_{2k}^{(u)} B_{2l}^{(v)} \right. \\
&\quad \left. - \frac{1}{2} \sum_{\substack{q+r+u=p-2 \\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x; t\}_{2k}^{(u)} + \frac{1}{2} \{x; t\}_{2(n-1)}^{(p-4)} \right].
\end{aligned}$$

Let us now define  $\Phi_{2n}^{(p)}$  and  $\Psi_{2n}^{(p)}$  by the following:

$$\begin{aligned}
(1.1.3.38) \quad \Phi_{2n}^{(p)} &= \sum_{\substack{q+r+u=p-3 \\ i+j+k=n}} x_{2i}^{(q)'} x_{2j}^{(r)'} A_{2k}^{(u)} + \sum_{\substack{q+r+u+v=p-2 \\ i+j+k+l=n}} x_{2i}^{(q)'} x_{2j}^{(r)'} x_{2k}^{(u)} B_{2l}^{(v)} \\
&\quad + \left( \sum_{\substack{q+r+u=p \\ i+j=n}} x_{2i}^{(q)} x_{2j}^{(r)} f^{(u)} - 2x_0^{(0)} x_{2n}^{(p)} f^{(0)} \right)
\end{aligned}$$

$$-\frac{1}{2} \sum_{\substack{q+r+u=p-2 \\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x; t\}_{2k}^{(u)} + \frac{1}{2} \{x; t\}_{2(n-1)}^{(p-4)},$$

(1.1.3.39)

$$\begin{aligned} \Psi_{2n}^{(p)} = & \sum_{\substack{q+r+u=p-1 \\ i+j+k=n}} x_{2i}^{(q)'} x_{2j}^{(r)'} A_{2k}^{(u)} + \sum_{\substack{q+r+u+v=p \\ i+j+k+l=n}} x_{2i}^{(q)'} x_{2j}^{(r)'} x_{2k}^{(u)} B_{2l}^{(v)} \\ & - 2\tilde{x}_0^{(0)} \tilde{f}^{(0)} x_{2n}^{(p)} - \frac{1}{2} \sum_{\substack{q+r+u=p \\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x; t\}_{2k}^{(u)} + \frac{1}{2} \{x; t\}_{2(n-1)}^{(p-2)}. \end{aligned}$$

*Remark 1.1.3.4.* The separation of terms into  $\Phi_{2n}^{(p)}$  and  $\Psi_{2n}^{(p)}$  is somewhat loosely done to make the expression simpler in view of our experience in Section 1.1.1. Some terms which evidently contain the factor  $t^2$  remain in  $\Phi_{2n}^{(p)}$ ; a typical example is  $\sum_{\substack{i+j=n \\ i,j \leq n-1}} x_{2i}^{(0)} x_{2j}^{(p)} f^{(0)}$ . Since leaving these

terms in  $\Phi_{2n}^{(p)}$  does not cause any problems in our induction procedure described below, we have not paid much attention to this point. The term  $2x_0^{(0)} x_{2n}^{(p)} f^{(0)}$  plays an exceptional role in our reasoning, and we have separated it from  $\Phi_{2n}^{(p)}$  and put  $-2t^{-2} x_0^{(0)} f^{(0)} x_{2n}^{(p)}$  into  $\Psi_{2n}^{(p)}$ .

Thus we are to determine  $T_l^{(r)} = \{x_l^{(r)}, A_l^{(r)}, B_l^{(r)}\}$  ( $r, l \geq 0$ ) so that they satisfy

$$(1.1.3.40) \quad \Phi_{2n}^{(p)} - t^2 \Psi_{2n}^{(p)} = 0$$

for every  $p, n \geq 0$ . Using the variable

$$(1.1.3.41) \quad s = x_0^{(0)}(t, \rho),$$

we can rewrite (1.1.3.40) as

$$(1.1.3.42) \quad \left(2s \frac{d}{ds} - 1\right) x_{2n}^{(p)} = -\frac{A_{2n}^{(p-1)}}{B_0^{(0)}} - \frac{B_{2n}^{(p)}}{B_0^{(0)}} s + R_{2n}^{(p)}(s, \rho),$$

where

$$(1.1.3.43) \quad R_{2n}^{(p)}(s, \rho)$$

$$= - \sum_{\substack{q+r+u=p-1 \\ i+j+k=n \\ (u,k) \neq (p-1,n)}} \dot{x}_{2i}^{(q)} \dot{x}_{2j}^{(r)} \frac{A_{2k}^{(u)}}{B_0^{(0)}} \quad (\alpha.i)$$

$$- \sum_{\substack{q+r+u+v=p \\ i+j+k+l=n}}^* \dot{x}_{2i}^{(q)} \dot{x}_{2j}^{(r)} x_{2k}^{(u)} \frac{B_{2l}^{(v)}}{B_0^{(0)}} \quad (\alpha.ii)$$

$$+ t^{-2} \sum_{\substack{q+r+u=p-3 \\ i+j+k=n}} \dot{x}_{2i}^{(q)} \dot{x}_{2j}^{(r)} \frac{A_{2k}^{(u)}}{B_0^{(0)}} \quad (\alpha.iii)$$

$$+ t^{-2} \sum_{\substack{q+r+u+v=p-2 \\ i+j+k+l=n}} \dot{x}_{2i}^{(q)} \dot{x}_{2j}^{(r)} x_{2k}^{(u)} \frac{B_{2l}^{(v)}}{B_0^{(0)}} \quad (\alpha.iv)$$

$$+ \frac{t^{-2}}{B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 \sum_{\substack{q+r+u=p \\ i+j=n \\ i,j \leq n-1}} x_{2i}^{(q)} x_{2j}^{(r)} f^{(u)} \quad (\alpha.v)$$

$$+ \frac{2t^{-2}}{B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 \sum_{\substack{q+r+u=p \\ q \leq p-1}} x_{2n}^{(q)} x_0^{(r)} f^{(u)} \quad (\alpha.vi)$$

$$- \frac{t^{-2}}{2B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 \sum_{\substack{q+r+u=p-2 \\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x; t\}_{2k}^{(u)} \quad (\alpha.vii)$$

$$+ \frac{t^{-2}}{2B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 \{x; t\}_{2(n-1)}^{(p-4)} \quad (\alpha.viii)$$

$$+ \frac{1}{2B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 \sum_{\substack{q+r+u=p \\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x; t\}_{2k}^{(u)} \quad (\alpha.ix)$$

$$-\frac{1}{2B_0^{(0)}}\left(\frac{dt}{ds}\right)^2\{x;t\}_{2(n-1)}^{(p-2)} \quad (\alpha.x)$$

with  $\sum^*$  in  $(\alpha.ii)$  meaning the following:

$$(1.1.3.44) \quad \sum_{\substack{q+r+u+v=p \\ i+j+k+l=n}}^* = \sum_{\substack{q+r+u+v=p \\ i+j+k+l=n \\ (q,i),(r,j),(u,k),(v,l) \neq (p,n)}}^*$$

Here the formula number  $(\alpha.l)$  is put to each sum for the later reference. We note that, as is usual,

$$(1.1.3.45) \quad \frac{2t^{-2}}{B_0^{(0)}}\left(\frac{dt}{ds}\right)^2 x_{2n}^{(p)} x_0^{(0)} f^{(0)}$$

has been shifted to the left-hand side of (1.1.3.42) thanks to [5.0]'; this is the reason why we encounter somewhat puzzling sums  $(\alpha.v)$  and  $(\alpha.vi)$ . Our task is to show a generalization of Proposition 1.1.3.1 that is applicable to  $T_l^{(r)} = \{x_l^{(r)}, A_l^{(r)}, B_l^{(r)}\}$  ( $l \geq 0$ ). In order to see how we can, and really do, adjust the constants contained in  $R_{2n}^{(p)}$  to find a holomorphic solution  $x_{2n}^{(p)}(s, \rho)$  of (1.1.3.42) near  $s = 0$ , we first show a generalization of Proposition 1.1.2.1. To present the generalization we prepare some notations.

**Definition 1.1.3.1.** (i) The infinite vector  $(x_l^{(0)}, x_l^{(1)}, \dots, x_l^{(r)}, \dots)$  (resp.,  $(A_l^{(0)}, A_l^{(1)}, \dots, A_l^{(r)}, \dots)$  and  $(B_l^{(0)}, B_l^{(1)}, \dots, B_l^{(r)}, \dots)$ ) is denoted by  $\vec{x}_l[\infty]$  (resp.,  $\vec{A}_l[\infty]$  and  $\vec{B}_l[\infty]$ ).

(ii)  $\vec{x}_n[p]$  (resp.,  $\vec{A}_n[p]$  and  $\vec{B}_n[p]$ ) stands for  $(\vec{x}_0[\infty], \vec{x}_1[\infty], \dots, \vec{x}_{n-1}[\infty], x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(p)})$  (resp.,  $(\vec{A}_0[\infty], \vec{A}_1[\infty], \dots, \vec{A}_{n-1}[\infty], A_n^{(0)}, A_n^{(1)}, \dots, A_n^{(p)})$  and  $(\vec{B}_0[\infty], \vec{B}_1[\infty], \dots, \vec{B}_{n-1}[\infty], B_n^{(0)}, B_n^{(1)}, \dots, B_n^{(p)})$ ).

(iii) We say  $\vec{x}_l[\infty]$  is holomorphic near  $s = 0$  (or  $t = 0$ ) if there exists a neighborhood  $U$  (resp.,  $O$ ) of  $\{s \in \mathbb{C}; s = 0\}$  (resp.,  $\{\rho \in \mathbb{C}; \rho = 0\}$ ) for which  $x_l^{(r)}(s, \rho)$  is holomorphic on  $U \times (O - \{0\})$  for every  $r \geq 0$ .

(iv) We say  $\vec{x}_n[p]$  is holomorphic near  $s = 0$  (or  $t = 0$ ) if there exists a neighborhood  $U$  (resp.,  $O$ ) of  $\{s \in \mathbb{C}; s = 0\}$  (resp.,  $\{\rho \in \mathbb{C}; \rho = 0\}$ ) for which the following holds:

(iv.a)  $x_l^{(r)}(s, \rho)$  is holomorphic on  $U \times (O - \{0\})$  for  $0 \leq l \leq n-1$   
and  $r \geq 0$ ,

and

(iv.b)  $x_n^{(r)}(s, \rho)$  is holomorphic on  $U \times (O - \{0\})$  for  $0 \leq r \leq p$ .

(v) Let  $\mathcal{X} = \mathcal{X}(\vec{A}_n[p], \vec{B}_n[p'])$  and  $\mathcal{Y} = \mathcal{Y}(\vec{A}_n[p], \vec{B}_n[p'])$  be functions of  $\vec{A}_n[p]$  and  $\vec{B}_n[p']$ . If  $\mathcal{X} - \mathcal{Y}$  depends only on  $(\vec{A}_n[q-1], \vec{B}_n[q-1])$ , then we say

$$(1.1.3.46) \quad \mathcal{X} \underset{(n,q)}{\equiv} \mathcal{Y}.$$

If there is no fear of confusion, we abbreviate it as

$$(1.1.3.47) \quad \mathcal{X} \underset{(q)}{\equiv} \mathcal{Y}.$$

*Remark 1.1.3.5.* As a convention we understand

$$(1.1.3.48) \quad \begin{aligned} & (\vec{A}_n[-1], \vec{B}_n[-1]) \\ &= (\vec{A}_0[\infty], \vec{A}_1[\infty], \dots, \vec{A}_{n-1}[\infty], \vec{B}_0[\infty], \vec{B}_1[\infty], \dots, \vec{B}_{n-1}[\infty]). \end{aligned}$$

Although the following Lemma 1.1.3.5 is an immediate consequence of (1.1.3.38) (together with (1.1.1.23)), it plays an important role in finding the concrete description of the conditions which guarantee the existence of a holomorphic solution  $x_{2n}^{(p)}(s, \rho)$  of (1.1.3.42) with  $p > r$ . (Cf. Proposition 1.1.3.2 and Proposition 1.1.3.3 below.)

**Lemma 1.1.3.5.** *If  $\vec{x}_{2n}[r]$  is holomorphic near  $s = 0$ , then  $\Phi_{2n}^{(r)}(t, \rho)$  is holomorphic near  $t = 0$ .*

As mentioned in the above, this is an immediate consequence of the definition of  $\Phi_{2n}^{(r)}$ . The importance of Lemma 1.1.3.5 consists in the fact that the holomorphy of  $\Phi_{2n}^{(r+1)}(t, \rho)$  near  $t = 0$  is needed to describe the conditions which guarantee the existence of a holomorphic solution

$x_{2n}^{(r+1)}(t, \rho)$  of (1.1.3.42) with  $p = r + 1$  on the condition that  $\vec{x}_{2n}[r]$  is holomorphic. In what follows we let  $[E; r, l]$  designate the following equation:

$$[E; r, l] \quad \left(2s \frac{d}{ds} - 1\right) x_l^{(r)}(s, \rho) = -\frac{A_l^{(r-1)}}{B_0^{(0)}} - \frac{B_l^{(r)}}{B_0^{(0)}} s + R_l^{(r)}(s, \rho),$$

where

$$(1.1.3.49) \quad A_l^{(r-1)} \text{ and } B_l^{(r)} \text{ are complex numbers,}$$

$$(1.1.3.50) \quad A_l^{(-1)} = 0,$$

$$(1.1.3.51) \quad A_{2\nu+1}^{(r-1)} = B_{2\nu+1}^{(r)} = R_{2\nu+1}^{(r)} = 0 \text{ for } r, \nu = 0, 1, 2, \dots,$$

$$(1.1.3.52) \quad R_{2n}^{(p)} \text{ is given by (1.1.3.43).}$$

*Remark 1.1.3.6.* In our subsequent discussion, we arrange our reasoning so that each quantity in the definition of  $R_{2n}^{(p)}$  has been given by preceding arguments.

Let us begin our discussion by showing the following

**Lemma 1.1.3.6.** *Suppose that constants  $(\vec{A}_l[\infty], \vec{B}_l[\infty])$  ( $l = 0, 1, \dots, 2n-1$ ) and holomorphic (near  $s = 0$ )  $\vec{x}_l[\infty]$  ( $l = 0, 1, \dots, 2n-1$ ) are given with  $x_l^{(r)}$  satisfying  $[E; r, l]$ . Suppose further*

$$(1.1.3.53) \quad x_l^{(0)}(0, \rho) = 0 \quad (l = 0, 1, \dots, 2n-1).$$

*Then there exists a holomorphic (in  $s$ ) solution  $x_{2n}^{(r)}(s, \rho)$  of  $[E; r, 2n]$  for  $r = 0, 1$  for any  $(A_{2n}^{(0)}, B_{2n}^{(0)}, B_{2n}^{(1)})$ . Furthermore they satisfy the following:*

$$(1.1.3.54) \quad x_{2n}^{(0)}(0, \rho) = 0,$$

$$(1.1.3.55) \quad \dot{x}_{2n}^{(0)}(0, \rho) \underset{(2n;0)}{\equiv} -\frac{B_{2n}^{(0)}}{B_0^{(0)}},$$

$$(1.1.3.56) \quad x_{2n}^{(1)}(0, \rho) \underset{(2n;0)}{\equiv} \frac{A_{2n}^{(0)}}{B_0^{(0)}},$$

$$(1.1.3.57) \quad \dot{x}_{2n}^{(1)}(0, \rho) \underset{(2n;1)}{\equiv} -\frac{B_{2n}^{(1)}}{B_0^{(0)}}.$$

*Proof.* We first show the existence of holomorphic  $x_{2n}^{(0)}(s, \rho)$  and confirm its properties (1.1.3.54) and (1.1.3.55). Checking each term in (1.1.3.43), we readily find that the possible singularity of  $R_{2n}^{(0)}$  arises from the sum  $(\alpha.v)$ . On the other hand, (1.1.3.53) and the definition of  $f^{(0)}$  entail

$$(1.1.3.58) \quad \sum_{\substack{q+r+u=0 \\ i+j=n \\ i,j \leq n-1}} x_{2i}^{(q)} x_{2j}^{(r)} f^{(u)} = \left( \sum_{\substack{i+j=n \\ i,j \leq n-1}} x_{2i}^{(0)} x_{2j}^{(0)} \right) f^{(0)} = O(t^3).$$

Hence the contribution from  $(\alpha.v)$  is holomorphic near  $t = 0$ . Therefore  $[E; 0, 2n]$  has a (unique) holomorphic solution  $x_{2n}^{(0)}(s, \rho)$  for any complex number  $B_{2n}^{(0)}$ . Furthermore the contribution from  $(\alpha.v)$  depends only on  $(\vec{A}_0[\infty], \vec{A}_1[\infty], \dots, \vec{A}_{2n-1}[\infty], \vec{B}_0[\infty], \vec{B}_1[\infty], \dots, \vec{B}_{2n-1}[\infty])$ , and it vanishes at  $t = 0$ . On the other hand it follows from (1.1.3.44) that each term in  $(\alpha.ii)$  with  $p = 0$  contains a factor  $x_{2k}^{(0)}$  with  $k \leq n - 1$ . Hence the value of  $(\alpha.ii)$  at  $s = 0$  is 0. Clearly  $(\alpha.ix)$  with  $p = 0$  also vanishes at  $s = 0$ . Thus we obtain (1.1.3.54). Since  $(\alpha.ii)$  also depends only on  $(\vec{A}_0[\infty], \vec{A}_1[\infty], \dots, \vec{A}_{2n-1}[\infty], \vec{B}_0[\infty], \vec{B}_1[\infty], \dots, \vec{B}_{2n-1}[\infty])$ ,  $R_{2n}^{(0)}(s, \rho)$  depends only on these parameters. Therefore we find

$$(1.1.3.59) \quad \left( 2s \frac{d}{ds} - 1 \right) x_{2n}^{(0)}(s, \rho) + \frac{B_{2n}^{(0)}}{B_0^{(0)}} s \underset{(2n;0)}{\equiv} 0,$$

and, in particular, we obtain (1.1.3.55).

We next investigate the structure of  $R_{2n}^{(1)}$ . The contribution from  $(\alpha.v)$  with  $p = 1$  is:

$$(1.1.3.60) \quad \left( \sum_{\substack{q+r=1 \\ i+j=n \\ i,j \leq n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \right) f^{(0)} + \left( \sum_{\substack{i+j=n \\ i,j \leq n-1}} x_{2i}^{(0)} x_{2j}^{(0)} \right) f^{(1)}.$$

Then it follows from (1.1.3.53) that

$$(1.1.3.61) \quad \sum_{\substack{q+r=1 \\ i+j=n \\ i,j \leq n-1}} x_{2i}^{(q)} x_{2j}^{(r)} = O(t),$$

$$(1.1.3.62) \quad \sum_{\substack{i+j=n \\ i,j \leq n-1}} x_{2i}^{(0)} x_{2j}^{(0)} = O(t^2).$$

Hence the contribution from  $(\alpha.v)$  with  $p = 1$  is holomorphic near  $t = 0$ . Similarly the contribution from  $(\alpha.vi)$  with  $p = 1$  is holomorphic near  $t = 0$ , because

$$(1.1.3.63) \quad x_{2n}^{(0)} (x_0^{(1)} f^{(0)} + x_0^{(0)} f^{(1)}) = O(t^2)$$

by (1.1.3.54). Other terms in  $R_{2n}^{(1)}$  are evidently holomorphic near  $s = 0$ , and hence  $[E; 1, 2n]$  has a holomorphic solution  $x_{2n}^{(1)}(s, \rho)$  near  $s = 0$  for any complex numbers  $A_{2n}^{(0)}$ ,  $B_{2n}^{(0)}$  and  $B_{2n}^{(1)}$ . To confirm its property (1.1.3.56) we next show

$$(1.1.3.64) \quad R_{2n}^{(1)}(0, \rho) \text{ is free from } B_{2n}^{(0)}.$$

The proof of this fact is basically the same as that of Proposition 1.1.2.1; in parallel with the cancellation (C.i),

$$(1.1.3.65) \quad \frac{2}{B_0^{(0)}} \dot{x}_{2n}^{(0)}(0, \rho) \dot{x}_0^{(0)}(0, \rho) f^{(1)}(0, \rho),$$

which originates from the Taylor expansion of the second term in (1.1.3.63), is cancelled out by the term

$$(1.1.3.66) \quad -2\dot{x}_0^{(0)} \dot{x}_{2n}^{(0)} \frac{A_0^{(0)}}{B_0^{(0)}} \Big|_{s=0}$$

in the sum  $(\alpha.i)$  evaluated at  $s = 0$ , whereas, in parallel with (C.ii), the term which contains  $\dot{x}_{2n}^{(0)}$  and  $B_{2n}^{(0)}$  in the sum  $(\alpha.ii)$  evaluated at  $s = 0$ , that is,

$$(1.1.3.67) \quad -\left(2\dot{x}_{2n}^{(0)} \dot{x}_0^{(1)} x_0^{(0)} + 2\dot{x}_{2n}^{(0)} \dot{x}_0^{(0)} x_0^{(1)} + \sum_{q+r+u=1} \dot{x}_0^{(q)} \dot{x}_0^{(r)} x_0^{(u)} \frac{B_{2n}^{(0)}}{B_0^{(0)}}\right) \Big|_{s=0}$$



is equal to

$$(1.1.3.68) \quad - \left( 2\dot{x}_{2n}^{(0)}\dot{x}_0^{(1)}x_0^{(0)} + 2\dot{x}_{2n}^{(0)}\dot{x}_0^{(0)}x_0^{(1)} + (x_0^{(1)} + 2\dot{x}_0^{(1)}x_0^{(1)}) \frac{B_{2n}^{(0)}}{B_0^{(0)}} \right) \Big|_{s=0},$$

which vanishes by (1.1.1.7) and (1.1.1.24). Thus the evaluation of  $[E; 1, 2n]$  at  $s = 0$  entails

$$(1.1.3.69) \quad x_{2n}^{(1)}(0, \rho) \underset{(2n,0)}{\equiv} \frac{A_{2n}^{(0)}}{B_0^{(0)}},$$

as is required. Since  $R_{2n}^{(1)}(s, \rho)$  is clearly free from  $B_{2n}^{(1)}$  (and  $A_{2n}^{(1)}$ , which has not yet appeared in our discussion), the relation (1.1.3.57) is an immediate consequence of  $[E; 1, 2n]$ . □

An important fact which lies behind the existence of holomorphic  $x_{2n}^{(r)}(s, \rho)$  with  $r = 0, 1$  is the validity of the following:

$$(1.1.3.70) \quad \Phi_{2n}^{(0)} \Big|_{t=0} = \frac{d\Phi_{2n}^{(0)}}{dt} \Big|_{t=0} = \Phi_{2n}^{(1)} \Big|_{t=0} = \frac{d\Phi_{2n}^{(1)}}{dt} \Big|_{t=0} = 0.$$

In passing we note

$$(1.1.3.71) \quad \Phi_{2n}^{(2)} \Big|_{t=0} = 0$$

also follows from (1.1.3.53) and (1.1.3.54), although we cannot expect

$$(1.1.3.72) \quad \frac{d\Phi_{2n}^{(2)}}{dt} \Big|_{t=0} = 0$$

in general. Actually as we will see below (1.1.3.72) gives a constraint on  $A_{2n}^{(0)}$  and  $B_{2n}^{(0)}$ , which are free parameters in Lemma 1.1.3.6. Now, in parallel with Proposition 1.1.2.1 we find the following

**Proposition 1.1.3.2.** *Let us suppose the same conditions as in Lemma 1.1.3.6, that is, the existence of constants  $(\vec{A}_l[\infty], \vec{B}_l[\infty])$  ( $l = 0, 1, \dots, 2n - 1$ ) and holomorphic  $\vec{x}_l[\infty]$  ( $l = 0, 1, \dots, 2n - 1$ ) that satisfies (1.1.3.53). Then the following set  $\mathcal{A}_{2n}(p)$  of assertions ( $\mathcal{A}_{2n}(p)(i)$ ,  $\mathcal{A}_{2n}(p)(ii)$ ,  $\dots$ ,  $\mathcal{A}_{2n}(p)(vi)$ ) is valid for every  $p \geq 0$  with the proviso*

that  $\mathcal{A}_{2n}(p)(v)$  ( $p = 0, 1, 2$ ) and  $\mathcal{A}_{2n}(p)(vi)$  ( $p = 0, 1$ ) are void statements (i.e., trivially correct statements in the sense that both sides are 0 under the convention

$$(1.1.3.73) \quad A_{2n}^{(q)} = B_{2n}^{(q')} = 0 \text{ for } q, q' = -3, -2 \text{ and } q' = -1,$$

which supplements (1.1.3.50).)

$$\mathcal{A}_{2n}(p) : \left\{ \begin{array}{ll} \mathcal{A}_{2n}(p)(i) : & \text{We can find constraints on parameters } (A_{2n}^{(p-2)}, B_{2n}^{(p-2)}, A_{2n}^{(p-3)}, B_{2n}^{(p-3)}) \text{ which are consistent with the constraints on } (\vec{A}_{2n}[p-3], \vec{B}_{2n}[p-3]) \text{ that have been given in previous stages (i.e., in } \mathcal{A}_{2n}(p')(i) \text{ (} 0 \leq p' \leq p-1 \text{)), so that a solution } x_{2n}^{(p)}(s, \rho) \text{ of } [E; p, 2n] \text{ is holomorphic in } s, \\ \mathcal{A}_{2n}(p)(ii) : & \text{The solution } x_{2n}^{(p)}(s, \rho) \text{ found in } \mathcal{A}_{2n}(p)(i) \text{ depends on } (\vec{A}_{2n}[p-1], \vec{B}_{2n}[p]), \\ \mathcal{A}_{2n}(p)(iii) : & x_{2n}^{(p)}(0, \rho)_{(2n; p-1)} \equiv \frac{A_{2n}^{(p-1)}}{B_0^{(0)}}, \\ \mathcal{A}_{2n}(p)(iv) : & \dot{x}_{2n}^{(p)}(0, \rho)_{(2n; p)} \equiv -\frac{B_{2n}^{(p)}}{B_0^{(0)}}, \\ \mathcal{A}_{2n}(p)(v) : & \Phi_{2n}^{(p)}|_{t=0}{}_{(2n; p-3)} \equiv A_{2n}^{(p-3)} - 2\frac{A_0^{(0)}}{B_0^{(0)}}B_{2n}^{(p-3)}, \\ \mathcal{A}_{2n}(p)(vi) : & \frac{d\Phi_{2n}^{(p)}}{dt}|_{t=0}{}_{(2n; p-2)} \equiv 2Z_0\frac{A_0^{(0)}}{B_0^{(0)}}A_{2n}^{(p-2)} - 2Z_0B_{2n}^{(p-2)}. \end{array} \right.$$

*Proof.* With the convention (1.1.3.73) we find by Lemma 1.1.3.6 and (1.1.3.70) that  $\mathcal{A}_{2n}(0)$  and  $\mathcal{A}_{2n}(1)$  are valid. To make the induction run smoothly we confirm  $\mathcal{A}_{2n}(2)$  separately, although one may build it in the induction procedure. We first note that  $\mathcal{A}_{2n}(2)(vi)$  follows from

$\mathcal{A}_{2n}(1)(iii)$  and  $\mathcal{A}_{2n}(0)(iv)$  through the explicit computation of each term in  $d\Phi_{2n}^{(2)}/dt|_{t=0}$ . In the computation we repeatedly use (1.1.1.24); for example,  $2f^{(0)'}x_{2n}^{(1)}x_0^{(1)}|_{t=0}$ , which may depend on  $A_{2n}^{(0)}$  through  $x_{2n}^{(1)}|_{t=0}$ , actually vanishes thanks to the vanishing factor  $x_0^{(1)}|_{t=0}$ , and so on. Then, as the constraint on  $(A_{2n}^{(0)}, B_{2n}^{(0)})$  required in  $\mathcal{A}_{2n}(2)(i)$ , we employ

$$(1.1.3.74) \quad \frac{d\Phi_{2n}^{(2)}}{dt}\Big|_{t=0} = 0;$$

the confirmed assertion  $\mathcal{A}_{2n}(2)(vi)$  guarantees that this gives a linear relation of  $(A_{2n}^{(0)}, B_{2n}^{(0)})$  whose coefficients are determined by  $(\vec{A}_{2n}[-1], \vec{B}_{2n}[-1])$  (in the notation of (1.1.3.48)). It is clear from the definition of  $R_{2n}^{(2)}$  that (1.1.3.74) together with (1.1.3.71) entails the holomorphy of  $R_{2n}^{(2)}(s, \rho)$  near  $s = 0$  and hence the existence of a holomorphic solution  $x_{2n}^{(2)}(s, \rho)$  of  $[E; 2, 2n]$ . Thus we have validated  $\mathcal{A}_{2n}(2)(i)$ . The assertion  $\mathcal{A}_{2n}(2)(ii)$  then immediately follows from the definition of the equation  $[E; 2, 2n]$ . To confirm  $\mathcal{A}_{2n}(2)(iii)$  it suffices to show that  $R_{2n}^{(2)}(0, \rho)$  is free from  $B_{2n}^{(1)}$ . This fact can be verified by a reasoning similar to the proof of Lemma 1.1.3.4; the terms we have to examine are the following:

$$(1.1.3.75) \quad -2\dot{x}_{2n}^{(1)}(0, \rho)\dot{x}_0^{(0)}(0, \rho)A_0^{(0)}/B_0^{(0)},$$

$$(1.1.3.76) \quad -\left(\sum_{q+r+u=1} \dot{x}_0^{(q)}(0, \rho)\dot{x}_0^{(r)}(0, \rho)x_0^{(u)}(0, \rho)\right)B_{2n}^{(1)}/B_0^{(0)},$$

$$(1.1.3.77) \quad -2\dot{x}_{2n}^{(1)}(0, \rho)\left(\sum_{r+u+v=1} \dot{x}_0^{(r)}(0, \rho)x_0^{(u)}(0, \rho)B_0^{(v)}/B_0^{(0)}\right)$$

and

$$(1.1.3.78) \quad 2\dot{x}_{2n}^{(1)}(0, \rho)\dot{x}_0^{(0)}(0, \rho)f^{(1)}(0, \rho)/B_0^{(0)},$$

which originates from the Taylor expansion of

$$(1.1.3.79) \quad 2\sum_{r+u=1} x_{2n}^{(1)}x_0^{(r)}f^{(u)}/B_0^{(0)}.$$

Then, as we have often observed (1.1.3.75) and (1.1.3.78) sum up to 0, and (1.1.3.76) and (1.1.3.77) vanish by (1.1.1.24) together with (1.1.1.7). Thus we have validated  $\mathcal{A}_{2n}(2)(iii)$ . The confirmation of  $\mathcal{A}_{2n}(2)(iv)$  is trivial, as  $R_{2n}^{(2)}(s, \rho)$  does not contain  $B_{2n}^{(2)}$ . Summing up, we have confirmed  $\mathcal{A}_{2n}(2)$ . Let us now begin the induction argument. Suppose that  $\mathcal{A}_{2n}(p)$  is valid for  $0 \leq p \leq p_0 - 1$  with  $p_0 \geq 3$ . Then, as is in the confirmation of  $\mathcal{A}_{2n}(2)$ , we see that  $\mathcal{A}_{2n}(p_0)(v)$  follows from  $\mathcal{A}_{2n}(p_0 - 2)(iii)$  and  $\mathcal{A}_{2n}(p_0 - 3)(iv)$ , and that  $\mathcal{A}_{2n}(p_0)(vi)$  follows from  $\mathcal{A}_{2n}(p_0 - 1)(iii)$  and  $\mathcal{A}_{2n}(p_0 - 2)(iv)$ . In order to guarantee the existence of a holomorphic solution  $x_{2n}^{(p_0)}(s, \rho)$  of  $[E; p_0, 2n]$ , we require

$$(1.1.3.80) \quad \Phi_{2n}^{(p_0)}|_{t=0} = 0$$

and

$$(1.1.3.81) \quad \frac{d\Phi_{2n}^{(p_0)}}{dt}|_{t=0} = 0.$$

The condition (1.1.3.81) gives a linear constraint on  $(A_{2n}^{(p_0-2)}, B_{2n}^{(p_0-2)})$  whose coefficients are described by  $(\vec{A}_{2n}[p_0 - 3], \vec{B}_{2n}[p_0 - 3])$ , whereas (1.1.3.80) supplemented by  $\mathcal{A}_{2n}(p_0)(v)$ , together with the constraint on  $(A_{2n}^{(p_0-3)}, B_{2n}^{(p_0-3)})$  given in the preceding stage, i.e.,

$$(1.1.3.82) \quad \frac{d\Phi_{2n}^{(p_0-1)}}{dt}|_{t=0} = 0,$$

fixes  $(A_{2n}^{(p_0-3)}, B_{2n}^{(p_0-3)})$  in terms of  $(\vec{A}_{2n}[p_0 - 4], \vec{B}_{2n}[p_0 - 4])$ . Here we have used the assumption (1.1.2) together with (1.1.1.21) and (1.1.1.22). Then the validity of  $\mathcal{A}_{2n}(p_0)(i)$  and  $\mathcal{A}_{2n}(p_0)(ii)$  is obvious. The confirmation of  $\mathcal{A}_{2n}(p_0)(iii)$  requires the validation of the fact that  $R_{2n}^{(p_0)}(0, \rho)$  is free from  $B_{2n}^{(p_0-1)}$ ; this validation can be done by exactly the same reasoning used when  $p_0 = 2$ . Thus we have confirmed  $\mathcal{A}_{2n}(p_0)(iii)$ . The validation of  $\mathcal{A}_{2n}(p_0)(iv)$  is trivial, as  $R_{2n}^{(p_0)}(s, \rho)$  is free from  $B_{2n}^{(p_0)}$ . Hence the induction proceeds, and  $\mathcal{A}_{2n}(p)$  is seen to be valid for every  $p \geq 0$ .

□

*Remark 1.1.3.7.* As is clear from the above proof, “constraints on parameters  $(A_{2n}^{(p-2)}, B_{2n}^{(p-2)}, A_{2n}^{(p-3)}, B_{2n}^{(p-3)})$ ” to be found in  $\mathcal{A}_{2n}(p)(i)$  are

(1.1.3.80) and (1.1.3.81). These conditions turn out to be consistent with previously imposed constraints on  $(\vec{A}_{2n}[p-3], \vec{B}_{2n}[p-3])$  by  $\mathcal{A}_{2n}(p)(v)$  and  $\mathcal{A}_{2n}(p)(vi)$ , and hence we have avoided the explicit statement of the conditions in  $\mathcal{A}_{2n}(p)(i)$ .

In Proposition 1.1.3.2 indices of the fixed quantity at the stage  $\mathcal{A}_{2n}(p)$  are not uniform;  $x_{2n}^{(p)}$  is fixed with free parameters  $(A_{2n}^{(p-1)}, B_{2n}^{(p-1)}, B_{2n}^{(p)})$  and parameters  $(A_{2n}^{(p-2)}, B_{2n}^{(p-2)})$  constrained by (1.1.3.81), whereas  $(\vec{A}_{2n}[p-3], \vec{B}_{2n}[p-3])$  is fixed. Hence we rearrange the setting so that  $T_l^{(r)} = \{x_l^{(r)}, A_l^{(r)}, B_l^{(r)}\}$  ( $l, r \geq 0$ ), following the way in which Proposition 1.1.3.1 is stated. In what follows we assume the same conditions as in Lemma 1.1.3.6, that is, the existence of constants  $(\vec{A}_l[\infty], \vec{B}_l[\infty])$  ( $l = 0, 1, \dots, 2n-1$ ) and holomorphic  $\vec{x}_l[\infty]$  ( $l = 0, 1, \dots, 2n-1$ ) that satisfies (1.1.3.53). Under this assumption we use the symbol  $\mathfrak{A}_{2n}(p-1)$  to mean the assertion that a triplet of data  $T_{2n}^{(r)} = \{x_{2n}^{(r)}(s, \rho), A_{2n}^{(r)}, B_{2n}^{(r)}\}$  is given for  $0 \leq r \leq p-1$  so that they satisfy the following conditions:

$$(1.1.3.83.r) \quad x_{2n}^{(r)}(s, \rho) \text{ is a holomorphic solution of } [E; r, 2n] \text{ near } s=0,$$

$$(1.1.3.84.r) \quad x_{2n}^{(r)}(s, \rho) \text{ depends on } (\vec{A}_{2n}[r-1], \vec{B}_{2n}[r]),$$

$$(1.1.3.85.r) \quad x_{2n}^{(0)}(0, \rho) = 0,$$

$$(1.1.3.86.r) \quad \Phi_{2n}^{(r+3)}|_{t=0} \text{ and } \frac{d\Phi_{2n}^{(r+2)}}{dt}|_{t=0} \text{ depend on } (\vec{A}_{2n}[r], \vec{B}_{2n}[r]),$$

$$\text{and } (\vec{A}_{2n}[r], \vec{B}_{2n}[r]) \text{ satisfies } \Phi_{2n}^{(r+3)}|_{t=0} = \frac{d\Phi_{2n}^{(r+2)}}{dt}|_{t=0} = 0,$$

$$(1.1.3.87.r) \quad \Phi_{2n}^{(r+3)}|_{t=0} \equiv_{(2n;r)} 2A_{2n}^{(r)} - 2\frac{A_0^{(0)}}{B_0^{(0)}}B_{2n}^{(r)},$$

$$(1.1.3.88.r) \quad \left. \frac{d\Phi_{2n}^{(r+2)}}{dt} \right|_{t=0} \underset{(2n;r)}{\equiv} 2Z_0 \frac{A_0^{(0)}}{B_0^{(0)}} A_{2n}^{(r)} - 2Z_0 B_{2n}^{(r)}.$$

**Proposition 1.1.3.3.** *The assertion  $\mathfrak{A}_{2n}(p)$  is valid for every  $p \geq 0$ .*

As the proof is essentially the same as that of Proposition 1.1.3.1, we describe its core part only. In the course of the proof of Proposition 1.1.3.2 we have seen that  $\mathfrak{A}_{2n}(p)$  ( $p = 0, 1, 2$ ) are valid. Let us suppose that  $\mathfrak{A}_{2n}(p)$  is valid for  $0 \leq p \leq p_0 - 1$  with  $p_0 \geq 3$ , and we want to confirm  $\mathfrak{A}_{2n}(p_0)$ . By adding an arbitrary complex number  $B_{2n}^{(p_0)}$  to the given data  $T_{2n}^{(r)}$  ( $r \leq p_0 - 1$ ) we can define the equation  $[E; p_0, 2n]$ . It then follows from (1.1.3.86. $(p_0 - 3)$ ) and (1.1.3.86. $(p_0 - 2)$ ) that

$$(1.1.3.89) \quad \Phi_{2n}^{(p_0)}|_{t=0} = \left. \frac{d\Phi_{2n}^{(p_0)}}{dt} \right|_{t=0} = 0$$

holds. Hence  $R_{2n}^{(p_0)}$  is holomorphic near  $s = 0$ . Then (1.1.3.83. $p_0$ ) and (1.1.3.84. $p_0$ ) are immediate consequences of  $[E; p_0, 2n]$ . As the confirmation of (1.1.3.87. $p_0$ ) and (1.1.3.88. $p_0$ ) requires the description of  $x_{2n}^{(p_0+1)}(0, \rho)$ , we further consider  $[E; p_0 + 1, 2n]$ ; we add arbitrary constants  $A_{2n}^{(p_0)}$  and  $B_{2n}^{(p_0+1)}$  to the given data to write down  $[E; p_0 + 1, 2n]$  with the aid of  $x_{2n}^{(p_0)}(s, \rho)$  we have just constructed. Then (1.1.3.86. $(p_0 - 2)$ ) and (1.1.3.86. $(p_0 - 1)$ ) guarantee

$$(1.1.3.90) \quad \Phi_{2n}^{(p_0+1)}|_{t=0} = \left. \frac{d\Phi_{2n}^{(p_0+1)}}{dt} \right|_{t=0} = 0.$$

Hence  $R_{2n}^{(p_0+1)}(s, \rho)$  is holomorphic near  $s = 0$ . Furthermore, by the same reasoning as in the proof of Lemma 1.1.3.4, we can verify that  $R_{2n}^{(p_0+1)}(0, \rho)$  is free from  $B_{2n}^{(p_0)}$ . Then, evaluating  $[E; p_0 + 1, 2n]$  at  $s = 0$ , we find

$$(1.1.3.91) \quad x_{2n}^{(p_0+1)}(0, \rho) \underset{(2n;p_0)}{\equiv} \frac{A_{2n}^{(p_0)}}{B_0^{(0)}} - R_{2n}^{(p_0+1)}(0, \rho).$$

Using this relation together with

$$(1.1.3.92) \quad \dot{x}_{2n}^{(p_0)}(0, \rho) \underset{(2n;p_0)}{\equiv} - \frac{B_{2n}^{(p_0)}}{B_0^{(0)}},$$

we find (1.1.3.87. $p_0$ ) and (1.1.3.88. $p_0$ ). Then we can fix  $(A_{2n}^{(p_0)}, B_{2n}^{(p_0)})$  so that  $(\vec{A}_{2n}[p_0], \vec{B}_{2n}[p_0])$  annihilates  $\Phi_{2n}^{(p_0+3)}|_{t=0}$  and  $d\Phi_{2n}^{(p_0+2)}/dt|_{t=0}$ , as is required in (1.1.3.86. $p_0$ ). We note that no constraint is imposed upon the complex number  $B_{2n}^{(p_0+1)}$  introduced for defining  $[E; p_0 + 1, 2n]$  at this stage. Hence the induction proceeds, completing the proof.

## 1.2 Growth order properties of $T_n^{(p)} = \{x_n^{(p)}, A_n^{(p)}, B_n^{(p)}\}$ ( $p, n \geq 0$ ) — the case where $g_{\pm}(t) = 0$

The purpose of this section is to estimate the growth order properties of  $\{T_n^{(p)}\}_{p,n \geq 0}$  so that the formal transformation of an M2P1T operator to its canonical form (the  $\infty$ -Mathieu equation) may acquire the microlocal analytic meaning, as will be explained later in Section 5. For the sake of simplicity of our reasoning we assume  $g_{\pm}(t) = 0$  in this section. The proof of the corresponding result when  $g_{\pm} \neq 0$  is given in Appendix C. Let us first prepare some notations and elementary inequalities which will be frequently used in our computation.

**Definition 1.2.1.** For  $l$  in  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$  and  $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$  in  $\mathbb{N}_0^n$ , we define

$$(1.2.1) \quad C(l) = \frac{3}{2\pi^2(l+1)^2},$$

$$(1.2.2) \quad C(\vec{\lambda}) = \prod_{j=1}^n C(\lambda_j).$$

An important property they enjoy is described by the following

**Lemma 1.2.1.** When  $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$  ranges over the set of all vectors that satisfy

$$(1.2.3) \quad \lambda_1 + \lambda_2 + \dots + \lambda_n = l,$$

the sum of  $C(\vec{\lambda})$  is dominated by  $C(l)$ , that is,

$$(1.2.4) \quad \sum_{\lambda_1 + \lambda_2 + \dots + \lambda_n = l} C(\vec{\lambda}) \leq C(l).$$

See [KKKoT1, Lemma B.3] for the proof.

**Lemma 1.2.2.** *The following inequality (1.2.5) holds for any positive integers  $l$  and  $n$  satisfying  $l \geq n$ :*

$$(1.2.5) \quad \sum_{\substack{\lambda_1 + \lambda_2 + \dots + \lambda_n = l \\ \lambda_1, \lambda_2, \dots, \lambda_n \geq 1}} \lambda_1! \lambda_2! \dots \lambda_n! \leq 4^{n-1} (l - n + 1)!$$

See [AKT4, Lemma A.4] for the proof.

In what follows we use the symbol  $\|h\|_{[r]}$  for a holomorphic function  $h(s)$  on  $\{s \in \mathbb{C}; |s| \leq r\}$  ( $r > 0$ ) to denote its supremum norm on the disc, that is,

$$(1.2.6) \quad \|h\|_{[r]} = \sup_{|s| \leq r} |h(s)|.$$

Using these symbols we now give the precise statement on the growth order of  $|f^{(j)}(s, \rho)|$ :

There exist positive constants  $\sigma_0, \kappa_0$  and  $L_0$  for which the following inequality (1.2.7) holds for every  $j$  in  $\mathbb{N}_0$  and  $\rho$  in  $\{\rho \in \mathbb{C}; 0 < |\rho| \leq \sigma_0\}$ :

$$(1.2.7) \quad \|f^{(j)}(\cdot, \rho)\|_{[\sigma_0]} \leq \kappa_0 C(j) L_0^j.$$

Here the auxiliary factor  $C(j)$  is intended for the convenience in performing the induction procedure in what follows.

We begin our estimation by studying the growth order property of the triplet  $T_0^{(p)} = \{x_0^{(p)}(s, \rho), A_0^{(p)}, B_0^{(p)}\}$  ( $p \geq 0$ ). For the sake of convenience we introduce the following notations:

$$(1.2.8) \quad \tilde{A}_0^{(p)} \stackrel{\text{def}}{=} A_0^{(p)} / B_0^{(0)} \quad \text{and} \quad \tilde{B}_0^{(p)} \stackrel{\text{def}}{=} B_0^{(p)} / B_0^{(0)},$$

$$(1.2.9) \quad \tilde{A}_0^{(-1)} = 0,$$

$$(1.2.10) \quad z_0^{(p)}(s, \rho) \stackrel{\text{def}}{=} x_0^{(p)}(s, \rho) - \tilde{A}_0^{(p-1)} + \tilde{B}_0^{(p)} s.$$

It then follows from

$$(1.1.3.15') \quad \left(2s \frac{d}{ds} - 1\right) z_0^{(p)}(s, \rho) = R_0^{(p)}(s, \rho),$$



(1.1.3.16) and (1.1.3.17) (with  $q = p$ ) that we find

$$(1.2.11) \quad z_0^{(p)}(0, \rho) = x_0^{(p)}(0, \rho) - \tilde{A}_0^{(p-1)} = -R_0^{(p)}(0, \rho)$$

and

$$(1.2.12) \quad \dot{z}_0^{(p)}(0, \rho) = \dot{x}_0^{(p)}(0, \rho) + \tilde{B}_0^{(p)} = \dot{R}_0^{(p)}(0, \rho).$$

We first prepare the following

**Lemma 1.2.3.** *There exist positive constants  $(r_0, R_0)$  and sufficiently small positive constant  $C_0$  for which the following estimate  $[G; p, 0]$  holds for every  $p \geq 1$  and  $\rho$  in  $\{\rho \in \mathbb{C}; 0 < |\rho| \leq r_0\}$ .*

$$[G; p, 0] \left\{ \begin{array}{ll} (p.i) & |z_0^{(p+1)}(0, \rho)| \leq C_0 C(p) (R_0 |\rho|^{-1})^p \\ (p.ii) & |\dot{z}_0^{(p)}(0, \rho)| \leq C_0 C(p) (R_0 |\rho|^{-1})^p \\ (p.iii) & \|z_0^{(p)}(\cdot, \rho)\|_{[r_0]} \leq C_0 C(p) (R_0 |\rho|^{-1})^p \\ (p.iv) & \|\dot{z}_0^{(p)}(\cdot, \rho)\|_{[r_0]} \leq C_0 C(p) (R_0 |\rho|^{-1})^p \\ (p.v) & |\tilde{A}_0^{(p)}(\rho)| \leq C_0 C(p) (R_0 |\rho|^{-1})^p \\ (p.vi) & |\tilde{B}_0^{(p)}(\rho)| \leq C_0 C(p) (R_0 |\rho|^{-1})^p \end{array} \right.$$

*Remark 1.2.1.* We may assume that  $r_0$  and  $R_0^{-1}$  are sufficiently small, and hence,  $(R_0 |\rho|^{-1})^{-1}$  is also sufficiently small. (In what follows, we consider  $r_0$  and  $R_0^{-1}$  as sufficiently small positive constants.) Therefore it is clear that  $[G; p, 0]$  entails

$$(p.\widetilde{iii}) \quad \|x_0^{(p)}(\cdot, \rho)\|_{[r_0]} \leq (1 + r_0 + (R_0 |\rho|^{-1})^{-1}) C_0 C(p) (R_0 |\rho|^{-1})^p \\ \leq 2C_0 C(p) (R_0 |\rho|^{-1})^p$$

and

$$(p.\widetilde{iv}) \quad \|\dot{x}_0^{(p)}(\cdot, \rho)\|_{[r_0]} \leq 2C_0 C(p) (R_0 |\rho|^{-1})^p.$$

Furthermore these estimates hold for  $p = 1$  by the concrete computation in Section 1.1.3. We also note that, as the form of the estimates  $[G; p, 0]$  for  $p \geq 1$  indicates, we can take  $C_0 > 0$  arbitrarily small by taking  $R_0 > 0$  sufficiently large.

*Proof of Lemma 1.2.3.* Before embarking on the induction, we check the situation concretely when  $p = 0$ . When  $p = 0$ ,  $\tilde{B}_0^{(0)} = 1$  and  $|\tilde{A}_0^{(0)}| = |A_0^{(0)}|/|\rho|$ . Thus (0.v) and (0.vi) are violated. Furthermore  $x_0^{(1)}(0, \rho) = 0$  entails

$$(1.2.13) \quad z_0^{(1)}(0, \rho) = -\tilde{A}_0^{(0)},$$

and hence (0.i) is also violated, whereas (0.ii), (0.iii) and (0.iv) trivially hold as  $z_0^{(0)}(s, \rho) = x_0^{(0)}(s, \rho) - \tilde{B}_0^{(0)}s = 0$  holds. Since the results in Section 1.1.3 confirm  $[G; 1, 0]$ , we assume that  $[G; p, 0]$  is valid for  $1 \leq p \leq p_0 - 1$  and validate  $[G; p_0, 0]$ . As the reasoning is lengthy, we separate it into several parts.

[I] Let us first confirm the most delicate statement  $(p_0.i)$ . As we will see later, the confirmation of  $(p_0.ii)$  can be done in a similar manner (actually simpler because the relevant index is  $p_0$ , not  $p_0 + 1$ ). To begin with we note that Proposition 1.1.3.1 guarantees that  $T_0^{(p)}$  exists for every  $p \geq 0$  and that it annihilates  $\Phi_0^{(p)}|_{t=0}$  and  $d\Phi_0^{(p)}/dt|_{t=0}$  (cf. (1.1.3.38)) for every  $p$ . Hence  $R_0^{(p)}(s, \rho)$  given by (1.1.3.8) is holomorphic in  $s$  if taken as a whole, though each individual term in the sum may be singular at  $s = 0$ . Therefore we find

$$(1.2.14) \quad \begin{aligned} R_0^{(p_0+1)}(0, \rho) &= \frac{1}{2\pi i} \int_{|s|=r_0} R_0^{(p_0+1)}(s, \rho) \frac{ds}{s} \\ &= \frac{1}{2\pi i} \oint R_0^{(p_0+1)}(s, \rho) \frac{ds}{s}. \end{aligned}$$

In order to clarify our reasoning we label the terms in  $R_0^{(p_0+1)}$  as follows:

$$(1.2.15)$$

$$\begin{aligned} &R_0^{(p_0+1)}(s, \rho) \\ &= - \sum_{\substack{i+j+k=p_0 \\ k \leq p_0-1}} \dot{x}_0^{(i)}(s, \rho) \dot{x}_0^{(j)}(s, \rho) \tilde{A}_0^{(k)} \end{aligned} \quad (\beta.i)$$

$$- \sum_{\substack{i+j+k+l=p_0+1 \\ i,j,k,l \leq p_0}} \dot{x}_0^{(i)}(s, \rho) \dot{x}_0^{(j)}(s, \rho) x_0^{(k)}(s, \rho) \tilde{B}_0^{(l)} \quad (\beta.ii)$$

$$+ t^{-2} \left( \sum_{i+j+k=p_0-2} \dot{x}_0^{(i)}(s, \rho) \dot{x}_0^{(j)}(s, \rho) \tilde{A}_0^{(k)} \right) \quad (\beta.iii)$$

$$+ t^{-2} \left( \sum_{i+j+k+l=p_0-1} \dot{x}_0^{(i)}(s, \rho) \dot{x}_0^{(j)}(s, \rho) x_0^{(k)}(s, \rho) \tilde{B}_0^{(l)} \right) \quad (\beta.iv)$$

$$+ \left( \frac{dt}{ds} \right)^2 \frac{t^{-2}}{B_0^{(0)}} \left( \sum_{\substack{i+j+k=p_0+1 \\ k \geq 2}} x_0^{(i)}(s, \rho) x_0^{(j)}(s, \rho) f^{(k)}(t(s, \rho), \rho) \right) \quad (\beta.v)$$

$$+ \left( \frac{dt}{ds} \right)^2 \frac{t^{-2}}{B_0^{(0)}} \left( \sum_{i+j=p_0} x_0^{(i)}(s, \rho) x_0^{(j)}(s, \rho) f^{(1)}(t(s, \rho), \rho) \right) \quad (\beta.vi)$$

$$+ \left( \frac{dt}{ds} \right)^2 \frac{t^{-1}}{B_0^{(0)}} \tilde{f}^{(0)}(t(s, \rho), \rho) \left( \sum_{\substack{i+j=p_0+1 \\ i, j \geq 1}} x_0^{(i)}(s, \rho) x_0^{(j)}(s, \rho) \right) \quad (\beta.vii)$$

$$- \left( \frac{dt}{ds} \right)^2 \frac{t^{-2}}{B_0^{(0)}} f^{(p_0-1)}(t(s, \rho), \rho). \quad (\beta.viii)$$

In what follows we use the symbol  $(\beta.j)$  ( $j = i, ii, \dots, viii$ ) to denote the sum labeled by the symbol; for example, we denote Cauchy's integral of the second sum in  $R_0^{(p_0+1)}$  as follows:

$$(1.2.16) \quad \frac{1}{2\pi i} \oint (\beta.ii) \frac{ds}{s}.$$

Since  $(\beta.ii)$  is holomorphic near  $s = 0$ , this is equal to

$$(1.2.17) \quad - \sum_{\substack{i+j+k+l=p_0+1 \\ i, j, k, l \leq p_0}} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) x_0^{(k)}(0, \rho) \tilde{B}_0^{(l)}.$$

In using the induction hypothesis we have to take extra care in dealing with  $\dot{x}_0^{(0)}$ ,  $x_0^{(0)}$  and  $\tilde{B}_0^{(0)}$ , and we also use

$$(1.2.18) \quad x_0^{(1)}(0, \rho) = 0$$

as an excellent substitute of  $(p.i)$  with  $p = 0$ . Thanks to the constraint on the indices in (1.2.17), at most two indices among  $(i, j, k, l)$  may become 0. Furthermore (1.2.18) implies the vanishing of annoying terms

such as  $\dot{x}_0^{(0)}(0, \rho)^2 x_0^{(1)}(0, \rho) \tilde{B}_0^{(p_0)}$  and  $\dot{x}_0^{(0)}(0, \rho) \dot{x}_0^{(p_0)}(0, \rho) x_0^{(1)}(0, \rho) \tilde{B}_0^{(0)}$ . Among the surviving terms let us consider the estimation of the following terms as an example; this term is one of the terms that give the worst contribution to the estimates of  $(\beta.ii)$ :

$$(1.2.19) \quad \left| \dot{x}_0^{(0)}(0, \rho) \dot{x}_0^{(1)}(0, \rho) x_0^{(p_0)}(0, \rho) \tilde{B}_0^{(0)} \right| \\ \leq 2^2 (C(0))^{-2} C_0^2 C(0) C(1) C(p_0 - 1) C(0) (R_0 |\rho|^{-1})^{p_0}.$$

Since  $\dot{x}_0^{(0)}(0, \rho) = \tilde{B}_0^{(0)} = 1$ , the estimates (1.2.19) follows from  $(p_0 - 1.i)$ ,  $(1.ii)$ ,  $(p_0 - 1.v)$  and  $(1.vi)$ . The unnecessary factor  $(C(0))^{-2} C(0)^2$  is inserted for the convenience of applying Lemma 1.2.1 to the estimation of the constant  $(N.ii)$  used in (1.2.21) below. In this way, we obtain the following estimates from the induction hypothesis and Lemma 1.2.1:

$$(1.2.20) \quad \left| - \sum_{\substack{i+j+k+l=p_0+1 \\ i,j,k,l \leq p_0}} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) x_0^{(k)}(0, \rho) \tilde{B}_0^{(l)} \right| \\ \leq (2^2 (C(0))^{-2} + 2^3 (C(0))^{-1} C_0 + 2^4 C_0^2) C(p_0) (R_0 |\rho|^{-1})^{p_0}.$$

(Actually it suffices to use  $(2^2 (C(0))^{-2} + 2^3 (C(0))^{-1} C_0 + 2^4 C_0^2) C_0$  as the extra factor due to the vanishing of  $x_0^{(0)}(0, \rho)$ .) Hence we obtain

$$(1.2.21) \quad \left| \frac{1}{2\pi i} \oint (\beta.ii) \frac{ds}{s} \right| \leq N(ii) C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0},$$

where

$$(1.2.22) \quad N(ii) = (2^2 (C(0))^{-2} + 2^3 (C(0))^{-1} C_0 + 2^4 C_0^2) C_0.$$

It is clear that  $N(ii)$  has the form  $\gamma C_0$  with a constant  $\gamma$  that is uniformly bounded for  $C_0 \leq 1$ . Otherwise stated, we can choose a sufficiently small constant  $N(ii)$  that is independent of  $p_0$  by choosing  $C_0$  sufficiently small. The choice of  $N(ii)$  is made in accordance with the number of sums used to compute  $z_0^{(p_0+1)}(0, \rho)$ , that is, 7 at this stage, although we need to make it smaller to sum up around 20 kinds of such sums in computing  $(\tilde{A}_0^{(p_0)}, \tilde{B}_0^{(p_0)})$ . This is the reason why we keep an extra constant  $N_0$  in (1.2.56) below. Thus, logically speaking, we should fix  $N(j)$  at the very end of the proof of this lemma. The important point is that we can choose them independent of  $p_0$ ,

Since the domination of Cauchy's integral of  $(\beta.i)$  requires some delicate treatment as we will see below, we next study the contribution from  $(\beta.j)$  ( $j = \text{iii}, \text{iv}, \text{v}$ ). As these terms may contain singularities at  $s = 0$  through the factor  $t^{-2}$ , we estimate the contour integral for  $r_0 \neq 0$ . When  $p_0 = 2$ ,  $(\beta.\text{iii})$  reduces to  $t^{-2} \tilde{A}_0^{(0)}$ , and hence we find

$$(1.2.23) \quad \frac{1}{2\pi i} \oint (\beta.\text{iii}; p_0 = 2) \frac{ds}{s} = \frac{1}{2\pi i} \int_{|s|=r_0} \frac{s^2}{t^2} \tilde{A}_0^{(0)} \frac{ds}{s^3} \stackrel{\text{def}}{=} \tilde{A}_0^{(0)} I(r).$$

Therefore we have the following relation (1.2.24) for  $R_0 \geq 1$ :

$$(1.2.24) \quad \left| \frac{1}{2\pi i} \int_{|s|=r_0} (\beta.\text{iii}; p_0 = 2) \frac{ds}{s} \right| \leq N_0(\text{iii}) C_0 C(2) (R_0 |\rho|^{-1})^2,$$

where  $N_0(\text{iii})$  is a constant which has the form

$$(1.2.25) \quad \gamma R_0^{-1}$$

with  $\gamma$  being given by

$$(1.2.26) \quad (C_0 C(2))^{-1} |A_0^{(0)}| |I(r)| (R_0 |\rho|^{-1})^{-1}.$$

When  $p_0 \geq 3$ , the induction hypothesis entails the following:

$$\begin{aligned} (1.2.27) \quad & \left| \frac{1}{2\pi i} \oint (\beta.\text{iii}) \frac{ds}{s} \right| \\ &= \left| \frac{1}{2\pi i} \int_{|s|=r_0} \frac{s^2}{t^2} \left[ \tilde{A}_0^{(0)} \left( 2\dot{x}_0^{(p_0-2)}(s, \rho) + \sum_{\substack{i+j=p_0-2 \\ i,j \geq 1}} \dot{x}_0^{(i)}(s, \rho) \dot{x}_0^{(j)}(s, \rho) \right) \right. \right. \\ & \quad + \sum_{1 \leq k \leq p_0-3} \tilde{A}_0^{(k)} \left( 2\dot{x}_0^{(p_0-2-k)}(s, \rho) + \sum_{\substack{i+j=p_0-2-k \\ i,j \geq 1}} \dot{x}_0^{(i)}(s, \rho) \dot{x}_0^{(j)}(s, \rho) \right) \\ & \quad \left. \left. + \tilde{A}_0^{(p_0-2)} \right] \frac{ds}{s^3} \right| \\ &\leq |I(r)| \left( 4(C(0))^{-1} |\rho|^{-1} |A_0^{(0)}| (1 + C_0) + 4C_0(1 + C_0) + 1 \right) \\ &\quad \times C_0 C(p_0-2) (R_0 |\rho|^{-1})^{p_0-2} \end{aligned}$$

$$\leq N(\text{iii})C_0C(p_0)(R_0|\rho|^{-1})^{p_0},$$

where

(1.2.28)

$$N(\text{iii}) = 4|I(r)| \left( 4(C(0))^{-1}|A_0^{(0)}|(1+C_0) + 4|\rho|C_0(1+C_0) + |\rho| \right) |\rho|R_0^{-2}.$$

Here the factor 4 dominates  $C(p_0-2)/C(p_0)$  for  $p_0 \geq 3$ . The estimation of the integral of  $(\beta.j)$  ( $j = \text{iv}, \text{v}$ ) can be done in a similar manner, and we find

$$(1.2.29) \quad \left| \oint (\beta.\text{iv}) \frac{ds}{s} \right| \leq N(\text{iv})C_0C(p_0-1)(R_0|\rho|^{-1})^{p_0}$$

and

$$(1.2.30) \quad \left| \oint (\beta.\text{v}) \frac{ds}{s} \right| \leq N(\text{v})C_0C(p_0-1)(R_0|\rho|^{-1})^{p_0},$$

where

$$(1.2.31) \quad N(\text{iv}), N(\text{v}) = \gamma(R_0|\rho|^{-1})^{-1}$$

with a uniformly bounded constant  $\gamma$  for  $R_0|\rho|^{-1} \gg 1$ . The domination of the integral of  $(\beta.\text{viii})$  is trivial: by (1.2.7) we have

$$(1.2.32) \quad \left| \oint (\beta.\text{viii}) \frac{ds}{s} \right| \leq N(\text{viii})C_0C(p_0-1)(R_0|\rho|^{-1})^{p_0}$$

with

$$(1.2.33) \quad N(\text{viii}) = \gamma(R_0|\rho|^{-1})^{-1}.$$

Thus what remain to be examined are  $(\beta.\text{i})$ ,  $(\beta.\text{vi})$  and  $(\beta.\text{vii})$ . Interestingly enough, their estimation is closely related to the fact C observed below (1.1.2.23) in the proof of Proposition 1.1.2.1.

We first study  $(\beta.\text{vii})$ . By the Taylor expansion we find

$$(1.2.34) \quad \sum_{\substack{i+j=p_0+1 \\ i,j \geq 1}} x_0^{(i)}(s, \rho) x_0^{(j)}(s, \rho) \\ = \sum_{\substack{i+j=p_0+1 \\ i,j \geq 1}} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho)$$

$$+ 2 \sum_{\substack{i+j=p_0+1 \\ i,j \geq 1}} x_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) s + O(s^2).$$

Since  $\tilde{f}^{(0)} = \rho g(t, \rho)$  with  $g(0, \rho) = 1$ , the substitution of (1.2.34) into the integral in the left-hand side of (1.2.35) below entails the following:

$$\begin{aligned} (1.2.35) \quad & \left| \frac{1}{2\pi i} \oint \left( \frac{dt}{ds} \right)^2 \frac{t^{-1}}{B_0^{(0)}} \tilde{f}^{(0)} \left( \sum_{\substack{i+j=p_0+1 \\ i,j \geq 1}} x_0^{(i)} x_0^{(j)} \right) \frac{ds}{s} \right| \\ &= \left| \frac{1}{2\pi i} \oint \left( \frac{dt}{ds} \right)^2 \left( \frac{s}{t} \right) Z_0 g(t, \rho) \left\{ \sum_{\substack{i+j=p_0+1 \\ i,j \geq 1}} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) \right. \right. \\ &\quad \left. \left. + 2s \left( \sum_{\substack{i+j=p_0+1 \\ i,j \geq 1}} x_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \right) + O(s^2) \right\} \frac{ds}{s^2} \right|, \end{aligned}$$

where  $Z_0 = \pm 1$  (cf. (1.1.1.13) and (1.1.1.22)). Clearly there is no contribution to the resulting integral from the third term in the braces (i.e.,  $O(s^2)$ ), whereas  $[G; p, 0]$  ( $p \leq p_0 - 1$ ) is effectively used to estimate the contribution from the first sum and that from the second one. Let us now recall

$$(1.2.36) \quad x_0^{(1)}(0, \rho) = 0.$$

Hence the indices  $(i, j)$  in the first sum and the index  $i$  in the second sum may be assumed to be equal to or greater than 2. Hence the induction hypothesis entails

$$(1.2.37) \quad \left| \frac{1}{2\pi i} \oint (\beta.vii) \frac{ds}{s} \right| \leq N(vii) C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0},$$

where

$$(1.2.38) \quad N(vii) = \gamma C_0 ((R_0 |\rho|^{-1})^{-1} + 2)$$

with  $\gamma$  being a uniformly bounded constant for  $C_0 \leq 1$ .

We next study the contribution from  $(\beta.i)$  and  $(\beta.vi)$ . At first one might be puzzled by the term

$$(1.2.39) \quad - \sum_{i+j=p_0} \dot{x}_0^{(i)}(s, \rho) \dot{x}_0^{(j)}(s, \rho) \tilde{A}_0^{(0)},$$

which contains

$$(1.2.40) \quad -2\dot{x}_0^{(p_0)} \tilde{A}_0^{(0)}.$$

Fortunately the contribution of this term is cancelled by the contribution from the coefficient of  $s^2$  in the Taylor expansion of

$$(1.2.41) \quad \left(\frac{dt}{ds}\right)^2 \frac{t^{-2}}{B_0^{(0)}} \left( \sum_{i+j=p_0} x_0^{(i)}(s, \rho) x_0^{(j)}(s, \rho) \right) f^{(1)}(t(s, \rho), \rho)$$

after the contour integration  $\oint ds/s$ , as we see below. By expanding

$$(1.2.42) \quad \sum_{i+j=p_0} x_0^{(i)}(s, \rho) x_0^{(j)}(s, \rho)$$

in powers of  $s$  as

$$(1.2.43) \quad \begin{aligned} & \sum_{i+j=p_0} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) \\ & + 2s \sum_{i+j=p_0} x_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \\ & + s^2 \left\{ \sum_{i+j=p_0} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \right. \\ & \quad \left. + \sum_{i+j=p_0} x_0^{(i)}(0, \rho) \ddot{x}_0^{(j)}(0, \rho) \right\} \\ & + O(s^3), \end{aligned}$$

we find

$$(1.2.44) \quad \frac{1}{2\pi i} \oint (\beta.vi) \frac{ds}{s} = I_0 + I_1,$$

where

$$(1.2.45) \quad I_0 = \frac{1}{2\pi i} \frac{1}{B_0^{(0)}} \oint \frac{s^2}{t^2} \left(\frac{dt}{ds}\right)^2 \left( \sum_{i+j=p_0} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \right) f^{(1)}(t, \rho) \frac{ds}{s}$$



and

$$\begin{aligned}
(1.2.46) \quad I_1 = & \frac{1}{2\pi i} \frac{1}{B_0^{(0)}} \oint \frac{s^2}{t^2} \left( \frac{dt}{ds} \right)^2 \left\{ \sum_{\substack{i+j=p_0 \\ i,j \geq 2}} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) \right. \\
& + 2s \left( \sum_{\substack{i+j=p_0 \\ i \geq 2}} x_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \right) \\
& + s^2 \left( \sum_{\substack{i+j=p_0 \\ i \geq 2}} x_0^{(i)}(0, \rho) \ddot{x}_0^{(j)}(0, \rho) \right) \\
& \left. + O(s^3) \right\} f^{(1)}(t, \rho) \frac{ds}{s^3}.
\end{aligned}$$

On the other hand (1.1.1.21) and (1.1.1.23) entail

$$(1.2.47) \quad I_0 = \frac{1}{B_0^{(0)}} \left( \sum_{i+j=p_0} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \right) A_0^{(0)}.$$

Hence the puzzling part (1.2.39) of the contribution from  $(\beta.i)$  is cancelled out by  $I_0$  ! Therefore

$$(1.2.48) \quad \frac{1}{2\pi i} \oint \{(\beta.i) + (\beta.vi)\} \frac{ds}{s} = I_2 + I_1,$$

where

$$(1.2.49) \quad I_2 = \frac{-1}{2\pi i} \oint \left( \sum_{\substack{i+j+k=p_0 \\ 1 \leq k \leq p_0-1}} \dot{x}_0^{(i)}(s, \rho) \dot{x}_0^{(j)}(s, \rho) \tilde{A}_0^{(k)} \right) \frac{ds}{s}.$$

Since

$$\begin{aligned}
(1.2.50) \quad & \sum_{\substack{i+j+k=p_0 \\ 1 \leq k \leq p_0-1}} \dot{x}_0^{(i)}(s, \rho) \dot{x}_0^{(j)}(s, \rho) \tilde{A}_0^{(k)} \\
& = 2\dot{x}_0^{(0)}(s, \rho) \left( \sum_{\substack{j+k=p_0 \\ j \geq 1, p_0-1 \geq k \geq 1}} \dot{x}_0^{(j)}(s, \rho) \tilde{A}_0^{(k)} \right) \\
& \quad + \sum_{\substack{i+j+k=p_0 \\ i, j \geq 1, p_0-1 \geq k \geq 1}} \dot{x}_0^{(i)}(s, \rho) \dot{x}_0^{(j)}(s, \rho) \tilde{A}_0^{(k)}
\end{aligned}$$

holds, the induction hypothesis entails the existence of a constant  $N(i)$  which satisfy the following:

$$(1.2.51) \quad |I_2| \leq N(i)C_0C(p_0)(R_0|\rho|^{-1})^{p_0},$$

$$(1.2.52) \quad N(i) = 4C_0(2 + C_0).$$

To estimate  $I_1$ , we note that  $\ddot{x}_0^{(0)}(s, \rho) = 0$  and

$$(1.2.53) \quad |\ddot{x}_0^{(j)}(0, \rho)| = \left| \frac{1}{2\pi i} \oint \dot{x}_0^{(j)}(s, \rho) \frac{ds}{s^2} \right| \\ \leq 2r_0^{-2}C_0C(p_0)(R_0|\rho|^{-1})^{p_0}$$

holds for  $j \geq 1$ . Using these facts, we find

$$(1.2.54) \quad |I_1| \leq N(vi)C_0C(p_0)(R_0|\rho|^{-1})^{p_0}$$

with

$$(1.2.55) \quad N(vi) = \gamma \left( C_0(R_0)^{-2}|\rho| + R_0^{-1} + r_0^{-1}C_0R_0^{-1} \right),$$

where  $\gamma$  is a constant originating from innocent factors in the integrand (i.e., irrelevant to  $C_0, R_0$  and  $|\rho|^{-1}$ ).

Summing up the estimates of the contributions from  $\beta(j)$  ( $j = i, ii, \dots, viii$ ) we find that  $[G; p, 0]$  ( $1 \leq p \leq p_0 - 1$ ) entails

$$(1.2.56) \quad |z_0^{(p_0+1)}(0, \rho)| \leq N_0C_0C(p_0)(R_0|\rho|^{-1})^{p_0},$$

where  $N_0$  is a constant which is independent of  $p_0$  and can be chosen as small as we want if we choose  $C_0$  and  $R_0^{-1}$  sufficiently small. Note that each  $N(j)$  found in the above contains a factor  $C_0$  or  $R_0^{-1}$  or their sum. We also note that the estimate (1.2.56) validates, in particular,  $(p_0.i)$ .

Let us next confirm  $(p_0.ii)$ . In view of (1.2.12) we start with (1.2.57) below, whose counterpart in the confirmation of  $(p_0.i)$  is (1.2.14).

$$(1.2.57) \quad \dot{z}_0^{(p_0)}(0, \rho) = \dot{R}_0^{(p_0)}(0, \rho) = \frac{1}{2\pi i} \oint R_0^{(p_0)}(s, \rho) \frac{ds}{s^2}.$$

An important difference between (1.2.14) and (1.2.57) is the following point: the index in question in (1.2.14) was  $(p_0 + 1)$ , whereas the

corresponding index is  $p_0$  in (1.2.57). Thus the domination is easier this time. Actually, as we will note below, even the estimation of the contribution from  $(\beta.i)$  (cf. (1.2.40) and (1.2.60) below) does not require the subtle reasoning related to the fact C observed in the proof of Proposition 1.1.2.1. Hence we avoid the detailed reasoning and content ourselves with locating the points which need some special attention. In what follows we let  $I(j)$  ( $j = i, ii, \dots, viii$ ) denote

$$(1.2.58) \quad \frac{1}{2\pi i} \oint [(\beta.j) \text{ with the index } (p_0 + 1) \text{ being replaced by } p_0] \frac{ds}{s^2}.$$

(i) Concerning the estimation of  $I(i)$ : Since we have

$$(1.2.59) \quad \begin{aligned} & \sum_{\substack{i+j+k=p_0-1 \\ k \leq p_0-2}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} \tilde{A}_0^{(k)} \\ &= \tilde{A}_0^{(0)} \left( 2\dot{x}_0^{(p_0-1)} + \sum_{\substack{i+j=p_0-1 \\ i, j \geq 1}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} \right) \\ &+ \sum_{1 \leq k \leq p_0-2} \tilde{A}_0^{(k)} \left( 2\dot{x}_0^{(p_0-1-k)} + \sum_{\substack{i+j=p_0-1-k \\ i, j \geq 1}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} \right), \end{aligned}$$

we find that the most troublesome term may be

$$(1.2.60) \quad 2\tilde{A}_0^{(0)} \dot{x}_0^{(p_0-1)}.$$

However, even the contribution from this term is dominated by

$$(1.2.61) \quad \begin{aligned} & |A_0^{(0)}| |\rho|^{-1} 2C_0 C(p_0 - 1) (R_0 |\rho|^{-1})^{p_0-1} \\ &= 2|A_0^{(0)}| \frac{C(p_0 - 1)}{C(p_0)} R_0^{-1} C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0}. \end{aligned}$$

Hence we can readily find

$$(1.2.62) \quad |I(i)| \leq N(i) C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0}$$

with a constant  $N(i)$  that can be chosen arbitrarily small independently of  $p_0$  by choosing  $R_0^{-1}$  sufficiently small. Making a contrast to the earlier

estimation of  $\oint(\beta.i)ds/s$  with the index  $(p_0 + 1)$ , the estimation  $I(i)$  does not require the cancellation among terms in  $(\beta.i)$  and  $(\beta.vi)$ .

(ii) Concerning the estimation of  $I(ii)$ : The integral  $I(ii)$  cannot enjoy such a simple form as (1.2.17), because the double pole  $s^{-2}$  is contained in the integrand. Still, the restriction on indices  $(i, j, k, l)$  again guarantees that at most two of them are allowed to be 0. Hence the induction hypothesis entails

$$(1.2.63) \quad |I(ii)| \leq N(ii)C_0C(p_0)(R_0|\rho|^{-1})^{p_0}$$

for a constant  $N(ii)$  that contains a factor  $C_0$  like the constant  $N(ii)$  in (1.2.22).

(iii) Concerning the estimation of  $I(iii)$ : Since we have (for  $p_0 \geq 4$ )

$$(1.2.64) \quad \begin{aligned} I(iii) &= \frac{1}{2\pi i} \oint \frac{s^2}{t^2} \left( \sum_{i+j+k=p_0-3} \dot{x}_0^{(i)} \dot{x}_0^{(j)} \tilde{A}_0^{(k)} \right) \frac{ds}{s^4} \\ &= \frac{1}{2\pi i} \oint \frac{s^2}{t^2} \left( \tilde{A}_0^{(0)} \left( 2\dot{x}_0^{(p_0-3)} + \sum_{\substack{i+j=p_0-3 \\ i,j \geq 1}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} \right) \right. \\ &\quad \left. + \sum_{1 \leq k \leq p_0-4} \tilde{A}_0^{(k)} \left( 2\dot{x}_0^{(p_0-3-k)} + \sum_{\substack{i+j=p_0-3-k \\ i,j \geq 1}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} \right) + \tilde{A}_0^{(p_0-3)} \right) \frac{ds}{s^4}, \end{aligned}$$

we use the induction hypothesis to find

$$(1.2.65) \quad |I(iii)| \leq N(iii)C_0C(p_0)(R_0|\rho|^{-1})^{p_0-2}$$

with a constant  $N(iii)$  that can be chosen arbitrarily small independently of  $p_0$  by choosing  $C_0$  and  $R_0^{-1}$  sufficiently small.

(iv), (v) The estimation of  $I(iv)$  and  $I(v)$  can be done in the same way as in the estimation of  $I(iii)$ .

(vi) Concerning the estimation of  $I(vi)$ : By using the Taylor expansion of

$$(1.2.66) \quad \sum_{i+j=p_0-1} x_0^{(i)}(s, \rho) x_0^{(j)}(s, \rho)$$

in  $s$ , we can readily confirm

$$(1.2.67) \quad |I(\text{vi})| \leq N(\text{vi})C_0C(p_0)(R_0|\rho|^{-1})^{p_0}$$

with a constant  $N(\text{vi})$  that can be chosen arbitrarily small independently of  $p_0$  by choosing  $C_0$  and  $R_0^{-1}$  sufficiently small. We note that the estimation is uniformly done including the part corresponding to  $I_0$  given by (1.2.45), that is,

$$(1.2.68) \quad \frac{1}{2\pi i} \frac{1}{B_0^{(0)}} \oint \frac{s^2}{t^2} \left( \frac{dt}{ds} \right)^2 \left( \sum_{i+j=p_0-1} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \right) f^{(1)}(t, \rho) \frac{ds}{s^2},$$

just because the required exponent in the right-hand side of (1.2.67) is  $p_0$ , not  $p_0 - 1$ .

(vii) The estimation of  $I(\text{vii})$  can be done similarly as that of  $I(\text{vi})$  with the help of the Taylor expansion of  $x_0^{(i)}(s, \rho)$  in  $s$ .

(viii) The required estimation of  $I(\text{viii})$  is attained by choosing  $R_0$  sufficiently large compared with  $\kappa_0$  and  $L_0$  in (1.2.7).

Summing up the observations (i), (ii),  $\dots$ , (viii) we find that the validity  $[G; p, 0]$  ( $1 \leq p \leq p_0 - 1$ ) implies that

$$(1.2.69) \quad |\dot{z}_0^{(p_0)}(0, \rho)| \leq N_1 C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0}$$

holds for any given small constant  $N_1$  if we choose  $C_0$  and  $R_0^{-1}$  sufficiently small. In particular we have thus confirmed  $(p_0.\text{ii})$ .

[II] Using the results in part [I], together with the induction hypothesis, we next confirm  $(p_0.\text{v})$  and  $(p_0.\text{vi})$ . For this purpose let us write down the conditions  $\Phi_0^{(p_0+3)}|_{t=0}$  and  $d\Phi_0^{(p_0+2)}/dt|_{t=0}$  using  $s$ -variable. For the sake of notational simplicity, in what follows, we keep some  $t$ -derivatives as they are; they are denoted as  $x_0^{(k)'}$  etc. as usual.

$$(1.2.70) \quad \left( \frac{ds}{dt} \right)^{-2} \Phi_0^{(p_0+3)} \Big|_{s=0} \\ = \left[ \sum_{i+j+k=p_0} \dot{x}_0^{(i)} \dot{x}_0^{(j)} A_0^{(k)} \right]$$

$$\begin{aligned}
& + \left( \frac{ds}{dt} \right)^{-2} \left( \sum_{i+j+k=p_0+3} x_0^{(i)} x_0^{(j)} f^{(k)} - 2x_0^{(0)} x_0^{(p_0+3)} f^{(0)} \right) \\
& + \sum_{i+j+k+l=p_0+1} \dot{x}_0^{(i)} \dot{x}_0^{(j)} x_0^{(k)} B_0^{(l)} \Big|_{s=0} \\
& = A_0^{(p_0)} + 2\dot{x}_0^{(p_0)}(0, \rho) A_0^{(0)} \\
& + \sum_{\substack{i+j+k=p_0 \\ i,j,k \leq p_0-1}} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) A_0^{(k)} \\
& + \left( \sum_{\substack{i+j=p_0+2 \\ i,j \geq 2}} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) \right) f^{(1)}(0, \rho) \\
& + \sum_{\substack{i+j+k=p_0+3 \\ i,j,k \geq 2}} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) f^{(k)}(0, \rho) \\
& + x_0^{(p_0+1)}(0, \rho) B_0^{(0)} \\
& + \sum_{\substack{i+j+k+l=p_0+1 \\ 2 \leq k \leq p_0}} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) x_0^{(k)}(0, \rho) B_0^{(l)} \\
& = A_0^{(p_0)} + 2(\dot{z}_0^{(p_0)}(0, \rho) - \tilde{B}_0^{(p_0)}) A_0^{(0)} \\
& + (z_0^{(p_0+1)}(0, \rho) + \tilde{A}_0^{(p_0)}) B_0^{(0)} \\
& + \sum_{\substack{i+j+k=p_0 \\ i,j,k \leq p_0-1}} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) A_0^{(k)} \\
& + \left( \sum_{\substack{i+j=p_0+2 \\ i,j \geq 2}} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) \right) A_0^{(0)} \\
& + \sum_{\substack{i+j+k=p_0+3 \\ i,j,k \geq 2}} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) f^{(k)}(0, \rho)
\end{aligned}$$

$$+ \sum_{\substack{i+j+k+l=p_0+1 \\ 2 \leq k \leq p_0}} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) x_0^{(k)}(0, \rho) B_0^{(l)},$$

(1.2.71)

$$\begin{aligned} & \left( \frac{ds}{dt} \right)^{-2} \frac{d\Phi_0^{(p_0+2)}}{dt} \Big|_{s=0} \\ &= \left[ 2 \left( \frac{ds}{dt} \right)^{-1} \sum_{i+j+k=p_0-1} x_0^{(i)''} \dot{x}_0^{(j)} A_0^{(k)} \right. \\ & \quad + \left( \frac{ds}{dt} \right)^{-2} \left( \sum_{i+j+k=p_0+2} x_0^{(i)} x_0^{(j)} f^{(k)'} - 2x_0^{(0)} x_0^{(p_0+2)} f^{(0)'} \right) \\ & \quad + 2 \left( \frac{ds}{dt} \right)^{-1} \left( \sum_{i+j+k=p_0+2} \dot{x}_0^{(i)} \dot{x}_0^{(j)} f^{(k)} - \left( \frac{d}{ds} (x_0^{(0)} x_0^{(p_0+2)}) \right) f^{(0)} \right) \\ & \quad + 2 \left( \frac{ds}{dt} \right)^{-1} \left( \sum_{i+j+k+l=p_0} x_0^{(i)''} \dot{x}_0^{(j)} x_0^{(k)} B_0^{(l)} \right) \\ & \quad \left. + \sum_{i+j+k+l=p_0} \dot{x}_0^{(i)} \dot{x}_0^{(j)} x_0^{(k)'} B_0^{(l)} \right] \Big|_{s=0} \\ &= 2Z_0 \sum_{i+j+k=p_0-1} x_0^{(i)''}(0, \rho) \dot{x}_0^{(j)}(0, \rho) A_0^{(k)} \\ & \quad + \left( \sum_{\substack{i+j=p_0+2 \\ i, j \geq 2}} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) \right) \rho \\ & \quad + \sum_{\substack{i+j+k=p_0+2 \\ i, j \geq 2, k \geq 1}} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) f^{(k)'}(0, \rho) \\ & \quad + 2Z_0 x_0^{(p_0+1)}(0, \rho) A_0^{(0)} \\ & \quad + 2Z_0 \sum_{\substack{i+j=p_0+1 \\ 2 \leq j \leq p_0}} \dot{x}_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) A_0^{(0)} \end{aligned}$$

$$\begin{aligned}
& + 2Z_0 \sum_{\substack{i+j+k=p_0+2 \\ j,k \geq 2}} \dot{x}_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) f^{(k)}(0, \rho) \\
& + 2Z_0 x_0^{(0)''}(0, \rho) x_0^{(p_0)}(0, \rho) B_0^{(0)} \\
& + 2Z_0 \sum_{\substack{i+j+k+l=p_0 \\ 2 \leq k \leq p_0-1}} x_0^{(i)''}(0, \rho) \dot{x}_0^{(j)}(0, \rho) x_0^{(k)}(0, \rho) B_0^{(l)} \\
& + 3Z_0 \dot{x}_0^{(p_0)}(0, \rho) B_0^{(0)} + Z_0 B_0^{(p_0)} \\
& + Z_0 \sum_{\substack{i+j+k+l=p_0 \\ i,j,k,l \leq p_0-1}} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \dot{x}_0^{(k)}(0, \rho) B_0^{(l)} \\
& = 2Z_0 \left( z_0^{(p_0+1)}(0, \rho) + \tilde{A}_0^{(p_0)} \right) A_0^{(0)} \\
& + 3Z_0 \left( \dot{z}_0^{(p_0)}(0, \rho) - \tilde{B}_0^{(p_0)} \right) B_0^{(0)} + Z_0 B_0^{(p_0)} \\
& + 2Z_0 \sum_{i+j+k=p_0-1} x_0^{(i)''}(0, \rho) \dot{x}_0^{(j)}(0, \rho) A_0^{(k)} \\
& + \rho \left( \sum_{\substack{i+j=p_0+2 \\ i,j \geq 2}} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) \right) \\
& + \sum_{\substack{i+j+k=p_0+2 \\ i,j \geq 2, k \geq 1}} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) f^{(k)'}(0, \rho) \\
& + 2Z_0 \sum_{\substack{i+j=p_0+1 \\ 2 \leq j \leq p_0}} \dot{x}_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) A_0^{(0)} \\
& + 2Z_0 \sum_{\substack{i+j+k=p_0+2 \\ j,k \geq 2}} \dot{x}_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) f^{(k)}(0, \rho) \\
& + 2Z_0 x_0^{(0)''}(0, \rho) x_0^{(p_0)}(0, \rho) B_0^{(0)}
\end{aligned}$$



$$\begin{aligned}
& + 2Z_0 \sum_{\substack{i+j+k+l=p_0 \\ 2 \leq k \leq p_0-1}} x_0^{(i)''}(0, \rho) \dot{x}_0^{(j)}(0, \rho) x_0^{(k)}(0, \rho) B_0^{(l)} \\
& + Z_0 \sum_{\substack{i+j+k+l=p_0 \\ i,j,k,l \leq p_0-1}} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \dot{x}_0^{(k)}(0, \rho) B_0^{(l)}.
\end{aligned}$$

In the above computation we have repeatedly used

$$(1.2.72) \quad x_0^{(0)}(s, \rho) = s \quad (\text{cf. (1.1.1.10)}),$$

$$(1.2.73) \quad x_0^{(1)}(0, \rho) = 0 \quad (\text{cf. (1.1.1.24)}),$$

$$(1.2.74) \quad f^{(0)}(t, \rho) = t\rho g(t, \rho) \text{ with } g(0, \rho) = 1 \text{ (cf. (1.1) and (1.3))},$$

$$(1.2.75) \quad Z_0 = x_0^{(0)'}(0, \rho) = \pm 1 \quad (\text{cf. (1.1.1.13) and (1.1.1.23)}),$$

$$(1.2.76) \quad f^{(1)}(0, \rho) = A_0^{(0)} \quad (\text{cf. (1.1.1.21)}),$$

$$(1.2.77) \quad B_0^{(0)} = Z_0^{-1}\rho \quad (\text{cf. (1.1.1.13)}),$$

$$(1.2.78) \quad x_0^{(p_0+1)}(0, \rho) = z_0^{(p_0+1)}(0, \rho) + \tilde{A}_0^{(p_0)} \quad (\text{cf. (1.2.11)})$$

and

$$(1.2.79) \quad \dot{x}_0^{(p_0)}(0, \rho) = \dot{z}_0^{(p_0)}(0, \rho) - \tilde{B}_0^{(p_0)} \quad (\text{cf. (1.2.12)}),$$

and we have separated out  $(A_0^{(p_0)}, B_0^{(p_0)})$  from other terms. Here we have used Lemma 1.1.3.4 together with (1.1.3.3.r) that  $T_0^{(r)}$  satisfies.

Thus we have found the following relations which determine  $(\tilde{A}_0^{(p_0)}, \tilde{B}_0^{(p_0)})$ :

$$\begin{aligned}
(1.2.70') \quad & -2 \left( B_0^{(0)} \tilde{A}_0^{(p_0)} - A_0^{(0)} \tilde{B}_0^{(p_0)} \right) \\
& = -2A_0^{(p_0)} + 2 \frac{A_0^{(0)}}{B_0^{(0)}} B_0^{(p_0)}
\end{aligned}$$

$$= 2\dot{z}_0^{(p_0)}(0, \rho)A_0^{(0)} \quad (\gamma.i)$$

$$+ z_0^{(p_0+1)}(0, \rho)B_0^{(0)} \quad (\gamma.ii)$$

$$+ \sum_{\substack{i+j+k=p_0 \\ i,j,k \leq p_0-1}} \dot{x}_0^{(i)}(0, \rho)\dot{x}_0^{(j)}(0, \rho)B_0^{(0)}\tilde{A}_0^{(k)} \quad (\gamma.iii)$$

$$+ \sum_{\substack{i+j=p_0+2 \\ i,j \geq 2}} x_0^{(i)}(0, \rho)x_0^{(j)}(0, \rho)B_0^{(0)}\tilde{A}_0^{(0)} \quad (\gamma.iv)$$

$$+ \sum_{\substack{i+j+k=p_0+3 \\ i,j,k \geq 2}} x_0^{(i)}(0, \rho)x_0^{(j)}(0, \rho)f^{(k)}(0, \rho) \quad (\gamma.v)$$

$$+ \sum_{\substack{i+j+k+l=p_0+1 \\ 2 \leq k \leq p_0}} \dot{x}_0^{(i)}(0, \rho)\dot{x}_0^{(j)}(0, \rho)x_0^{(k)}(0, \rho)B_0^{(0)}\tilde{B}_0^{(l)} \quad (\gamma.vi)$$

$$\stackrel{\text{def}}{=} \Gamma_0^{(p_0)}$$

$$(1.2.71') \quad -2\left(A_0^{(0)}\tilde{A}_0^{(p_0)} - B_0^{(0)}\tilde{B}_0^{(p_0)}\right)$$

$$= -2\frac{A_0^{(0)}}{B_0^{(0)}}A_0^{(p_0)} + 2B_0^{(p_0)}$$

$$= 2z_0^{(p_0+1)}(0, \rho)A_0^{(0)} \quad (\delta.i)$$

$$+ 3\dot{z}_0^{(p_0)}(0, \rho)B_0^{(0)} \quad (\delta.ii)$$

$$+ 2 \sum_{i+j+k=p_0-1} x_0^{(i)''}(0, \rho)\dot{x}_0^{(j)}(0, \rho)B_0^{(0)}\tilde{A}_0^{(k)} \quad (\delta.iii)$$

$$+ Z_0\rho\left(\sum_{\substack{i+j=p_0+2 \\ i,j \geq 2}} x_0^{(i)}(0, \rho)x_0^{(j)}(0, \rho)\right) \quad (\delta.iv)$$

$$+ Z_0 \sum_{\substack{i+j+k=p_0+2 \\ i,j \geq 2, k \geq 1}} x_0^{(i)}(0, \rho)x_0^{(j)}(0, \rho)f^{(k)'}(0, \rho) \quad (\delta.v)$$

$$+ 2 \sum_{\substack{i+j=p_0+1 \\ 2 \leq j \leq p_0}} \dot{x}_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) A_0^{(0)} \quad (\delta.vi)$$

$$+ 2 \sum_{\substack{i+j+k=p_0+2 \\ j, k \geq 2}} \dot{x}_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) f^{(k)}(0, \rho) \quad (\delta.vii)$$

$$+ 2 x_0^{(0)''}(0, \rho) x_0^{(p_0)}(0, \rho) B_0^{(0)} \quad (\delta.viii)$$

$$+ 2 \sum_{\substack{i+j+k+l=p_0 \\ 2 \leq k \leq p_0-1}} x_0^{(i)''}(0, \rho) \dot{x}_0^{(j)}(0, \rho) x_0^{(k)}(0, \rho) B_0^{(0)} \tilde{B}_0^{(l)} \quad (\delta.ix)$$

$$+ \sum_{\substack{i+j+k+l=p_0 \\ i, j, k, l \leq p_0-1}} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \dot{x}_0^{(k)}(0, \rho) B_0^{(0)} \tilde{B}_0^{(l)} \quad (\delta.x)$$

$$\stackrel{\text{def}}{=} \Delta_0^{(p_0)}.$$

Then, by using the assumption (1.1.2) together with (1.2.75), (1.2.76) and (1.2.77), we obtain the following relation (1.2.80) from (1.2.70') and (1.2.71'):

$$(1.2.80) \quad \begin{pmatrix} \tilde{A}_0^{(p_0)} \\ \tilde{B}_0^{(p_0)} \end{pmatrix} = \left( -\frac{1}{2} \right) \left( B_0^{(0)2} - A_0^{(0)2} \right)^{-1} \begin{pmatrix} B_0^{(0)} \Gamma_0^{(p_0)} - A_0^{(0)} \Delta_0^{(p_0)} \\ A_0^{(0)} \Gamma_0^{(p_0)} - B_0^{(0)} \Delta_0^{(p_0)} \end{pmatrix}.$$

Hence it suffices to dominate each of terms  $(\gamma.j)$  ( $j = i, ii, \dots, vi$ ) and  $(\delta.j)$  ( $j = i, ii, \dots, x$ ) by a constant of the form

$$(1.2.81) \quad N C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0}$$

with a constant  $N$  which can be chosen sufficiently small and independent of  $p_0$  by letting  $C_0$  and  $R_0^{-1}$  sufficiently small. As we have already confirmed the estimate of this sort for  $(\gamma.i)$ ,  $(\gamma.ii)$ ,  $(\delta.i)$  and  $(\delta.ii)$  it is enough to examine other terms. The reasoning is basically the same as that used in part [I]. For example we find the following estimate (1.2.82) for the sum  $(\gamma.iv)$ , which one may think to be the most troublesome one in view of the range of indices:

$$(1.2.82) \quad |(\gamma.iv)|$$

$$\begin{aligned}
&= \left| \sum_{\substack{i'+j'=p_0 \\ i',j' \geq 1}} x_0^{(i'+1)}(0, \rho) x_0^{(j'+1)}(0, \rho) A_0^{(0)} \right| \\
&= \left| \sum_{\substack{i'+j'=p_0 \\ i',j' \geq 1}} (z_0^{(i'+1)}(0, \rho) + \tilde{A}_0^{(i')})(z_0^{(j'+1)}(0, \rho) + \tilde{A}_0^{(j')}) A_0^{(0)} \right| \\
&\leq 4C_0^2 C(p_0) (R_0 |\rho|^{-1})^{p_0} |A_0^{(0)}|.
\end{aligned}$$

Therefore we find

$$(1.2.83) \quad |(\gamma.\text{iv})| \leq N(\gamma.\text{iv}) C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0}$$

with

$$(1.2.84) \quad N(\gamma.\text{iv}) = 4 |A_0^{(0)}| C_0.$$

The same technique that makes full use of the estimate of  $|z_0^{(p_0+1)}(0, \rho)|$  also applies to  $(\gamma.\text{v})$ ,  $(\gamma.\text{vi})$ ,  $(\delta.\text{iv})$ ,  $(\delta.\text{v})$ ,  $(\delta.\text{vi})$ ,  $(\delta.\text{vii})$ ,  $(\delta.\text{viii})$  and  $(\delta.\text{ix})$ , whereas the rest of terms, i.e.,  $(\gamma.\text{iii})$ ,  $(\delta.\text{iii})$  and  $(\delta.\text{x})$  are rather easy to handle. For example we readily find

$$(1.2.85) \quad |(\delta.\text{x})| \leq N(\delta.\text{x}) C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0}$$

with

$$(1.2.86) \quad N(\delta.\text{x}) = 4 \left( (C(0))^{-2} + 2(C(0))^{-1} C_0 + 4C_0^2 \right) |\rho| C_0.$$

Thus the induction hypothesis together with (1.2.80) entails that

$$(1.2.87) \quad |\tilde{A}_0^{(p_0)}| \leq N_2 C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0}$$

and

$$(1.2.88) \quad |\tilde{B}_0^{(p_0)}| \leq N_2 C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0}$$

hold, where  $N_2$  is a constant which is independent of  $p_0$  and can be chosen as small as we want if we choose  $C_0$  and  $R_0^{-1}$  sufficiently small. In particular we have thus confirmed  $(p_0.\text{v})$  and  $(p_0.\text{vi})$ .

[III] Next we validate  $(p_0.\text{iii})$  and  $(p_0.\text{iv})$ . We first note that, by the same reasoning with the estimation (1.2.69) of  $\dot{R}_0^{(p_0)}(0, \rho)$  (cf. (1.2.57)), we find

$$(1.2.89) \quad \|R_0^{(p_0)}(\cdot, \rho)\|_{r_0} \leq N_2 C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0}$$

holds, where  $N_2$  is a constant which is independent of  $p_0$  and can be chosen as small as we want if we choose  $C_0$  and  $R_0^{-1}$  sufficiently small. ( Since  $R_0^{(p_0)}$  is holomorphic at  $s = 0$ , the estimates (1.2.89) directly follows from the maximum modulus principle and the induction hypothesis.) Then, to obtain  $(p_0.iii)$ , we use the following integral representation (1.2.91) of the holomorphic solution  $x_0^{(p_0)}(s, \rho)$  of the equation (1.2.90).

(1.2.90) ( $= [E; p_0, 0]$  )

$$\left(2s \frac{d}{ds} - 1\right) x_0^{(p_0)}(s, \rho) = -\tilde{A}_0^{(p_0-1)} - \tilde{B}_0^{(p_0)} s + R_0^{(p_0)}(s, \rho),$$

(1.2.91)

$$x_0^{(p_0)}(s, \rho) = x_0^{(p_0)}(0, \rho) + \frac{s^{1/2}}{2} \int_0^s u^{-3/2} \left( R_0^{(p_0)}(u, \rho) - \tilde{A}_0^{(p_0-1)} - \tilde{B}_0^{(p_0)} u + x_0^{(p_0)}(0, \rho) \right) du.$$

Here we note that the integrand of the integral in the right-hand side of (1.2.91) is integrable near  $u = 0$ , because (1.2.11) entails that it has the form

$$(1.2.92) \quad u^{-1/2} \left( (R_0^{(p_0)}(u, \rho) - R_0^{(p_0)}(0, \rho)) u^{-1} - \tilde{B}_0^{(p_0)} \right).$$

Therefore, combining the results in part [I], [II] and (1.2.89), we obtain the following estimates:

$$(1.2.93) \quad \|x_0^{(p_0)}(\cdot, \rho)\|_{r_0} \leq N_2 C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0},$$

where  $N_2$  is a sufficiently small constant. Then  $(p_0.iii)$  immediately follows from (1.2.87), (1.2.88) and (1.2.93). Further, to obtain  $(p_0.iv)$ , we rewrite (1.2.90) as follows:

$$(1.2.94) \quad \dot{x}_0^{(p_0)}(s, \rho) = \frac{1}{2s} \left( x_0^{(p_0)}(s, \rho) - \tilde{A}_0^{(p_0-1)} - \tilde{B}_0^{(p_0)} s + R_0^{(p_0)}(s, \rho) \right).$$

Then the following estimates follow from the maximum modulus principle:

$$(1.2.95) \quad \|\dot{x}_0^{(p_0)}(\cdot, \rho)\|_{r_0} \leq N_2 C_0 C(p_0) (R_0 |\rho|^{-1})^{p_0},$$

where  $N_2$  is a sufficiently small constant. Thus (p<sub>0</sub>.iv) follows from (1.2.88) and (1.2.93).

Summing up the results in part [I], [II] and [III], we conclude that the induction proceeds. This completes the proof of Lemma 1.2.3.  $\square$

We now embark on the proof of Proposition 1.2.1 below. In order to facilitate the concrete expression of the Taylor expansion of the Schwarzian derivative  $\{x; t\}$  we prepare the following notations.

**Definition 1.2.2.** (i) For multi-indices  $\vec{\kappa} = (\kappa_1, \kappa_2, \dots, \kappa_\mu)$  and  $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_\mu)$  in  $\mathbb{N}_0^\mu$ , we define

$$(1.2.96) \quad |\vec{\lambda}|_\mu = \sum_{j=1}^{\mu} \lambda_j,$$

$$(1.2.97) \quad \vec{\lambda}! = \prod_{j=1}^{\mu} \lambda_j!.$$

(ii) For  $(\vec{\lambda}, \vec{\kappa})$ -dependent quantities  $X_{\kappa_j}^{(\lambda_j)}$  (such as  $dx_{\kappa_j}^{(\lambda_j)}/dt$ ) we define

$$(1.2.98) \quad X_{\vec{\kappa}}^{(\vec{\lambda})} = \prod_{j=1}^{\mu} X_{\kappa_j}^{(\lambda_j)}$$

and

$$(1.2.99) \quad \sum_{|\vec{\kappa}|_\mu=k}^* \sum_{|\vec{\lambda}|_\mu=l} X_{\vec{\kappa}}^{(\vec{\lambda})} = \begin{cases} 1 & \text{for } \mu = 0 \\ \sum_{\substack{|\vec{\kappa}|_\mu=k \\ \kappa_j \geq 1}} \sum_{|\vec{\lambda}|_\mu=l} \prod_{j=1}^{\mu} X_{\kappa_j}^{(\lambda_j)} & \text{for } \mu \geq 1. \end{cases}$$

For the notational convenience we also introduce the following

**Definition 1.2.3.** We define  $\tilde{A}_{2n}^{(p)}$  and  $\tilde{B}_{2n}^{(p)}$  by the following:

$$(1.2.100) \quad \tilde{A}_{2n}^{(p)} = A_{2n}^{(p)} / B_0^{(0)}, \quad \tilde{B}_{2n}^{(p)} = B_{2n}^{(p)} / B_0^{(0)},$$

$$(1.2.101) \quad \tilde{A}_{2n}^{(-1)} = 0.$$

**Proposition 1.2.1.** *There exist positive constants  $(r_0, R, A)$  and a sufficiently small constant  $N_0$  for which the following estimate  $[G; p, 2n]$  holds for every  $p \geq 0$ , every  $n \geq 1$ , every  $\rho$  in  $\{\rho \in \mathbb{C}; 0 < \rho \leq r_0\}$  and any positive constant  $\varepsilon$  that is smaller than  $r_0/3$ :*

$$[G; p, 2n] = \begin{cases} (p, 2n)(i) & |x_{2n}^{(p+1)}(0, \rho)| \leq N_0 C(p) (R|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n, \\ (p, 2n)(ii) & |\tilde{A}_{2n}^{(p)}(\rho)| \leq N_0 C(p) (R|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n, \\ (p, 2n)(iii) & |\tilde{B}_{2n}^{(p)}(\rho)| \leq N_0 C(p) (R|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n, \\ (p, 2n)(iv) & \|x_{2n}^{(p)}(\cdot, \rho)\|_{[r_0-\varepsilon]} \leq N_0 C(p) (R|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n, \\ (p, 2n)(v) & \|\dot{x}_{2n}^{(p)}(\cdot, \rho)\|_{[r_0-\varepsilon]} \leq N_0 C(p) (R|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n. \end{cases}$$

In what follows, for the simplicity of the notation, we use the symbol  $\|h\|_{[r]}$  to denote  $\|h(\cdot, \rho)\|_{[r]}$  even when a holomorphic function  $h(s, \rho)$  contains  $\rho$  as an auxiliary variables other than  $s$ .

*Remark 1.2.2.* We note that, as the form of the estimates  $[G; p, 2n]$  for  $n \geq 1$  indicates, we can take  $N_0 > 0$  arbitrarily small by taking  $A > 0$  sufficiently large.

*Remark 1.2.3.* As we will see in the proof below, the order of  $|\rho|$  relevant to  $n$  in  $[G; p, 2n]$  is inductively determined by the contribution from  $(\alpha.ix)$  in (1.1.3.43). (Cf. (1.2.179) and (1.2.180).)

*Remark 1.2.4.* In view of Remark 1.2.1, we see that  $[G; p, 2n]$  with  $n = 0$  coincides with  $[G; p, 0]$  in Lemma 1.2.3.

*Proof.* Aside from the treatment of terms originating from the Schwarzian derivative, the flow of the reasoning is basically the same as that in the proof of Lemma 1.2.3. As the proof is lengthy, we separate it into four parts, part [I]  $\sim$  part [IV]. Before beginning the proof we note that the term in the left-hand side of each  $(p, 2n)(j)$  ( $j = (i), (ii), \dots, (v)$ ) with  $(p, n) = (0, 1)$  vanishes. This fact is implicitly confirmed in what follows, but, in view of its interest, we give a detailed proof in Appendix B.

[I] Let us first study how to dominate the contribution from  $\{x; t\}_{2(n-1)}^{(p)}$ . Using the Taylor expansion we find

(1.2.102)

$$\begin{aligned}
& \{x; t\}_{2(n-1)}^{(p)} \\
&= \sum_{\substack{k_1+k_2=n-1 \\ l_1+l_2+l_3=p}} \frac{d^3 x_{2k_1}^{(l_1)}}{dt^3} \sum_{\nu=\min\{1, k_2\}}^{k_2} (-1)^\nu \left( \left( \frac{dx_0}{dt} \right)^{-\nu-1} \right)^{(l_2)} \sum_{|\vec{k}|_\nu=k_2}^* \sum_{|\vec{l}|_\nu=l_3} \frac{dx_{2\vec{k}}^{(\vec{l})}}{dt} \\
&- \frac{3}{2} \sum_{\substack{k_1+k_2+k_3=n-1 \\ l_1+l_2+l_3+l_4=p}} \frac{d^2 x_{2k_1}^{(l_1)}}{dt^2} \frac{d^2 x_{2k_2}^{(l_2)}}{dt^2} \sum_{\nu=\min\{1, k_3\}}^{k_3} (-1)^\nu (\nu+1) \left( \left( \frac{dx_0}{dt} \right)^{-\nu-2} \right)^{(l_3)} \\
&\times \sum_{|\vec{k}|_\nu=k_3}^* \sum_{|\vec{l}|_\nu=l_4} \frac{dx_{2\vec{k}}^{(\vec{l})}}{dt},
\end{aligned}$$

where we use the symbol  $\left( \left( \frac{dx_0}{dt} \right)^{-\nu-1} \right)^{(l_2)}$  (resp.,  $\left( \left( \frac{dx_0}{dt} \right)^{-\nu-2} \right)^{(l_3)}$ ) to mean the coefficient of  $a^{l_2}$  (resp.,  $a^{l_3}$ ) of the Taylor expansion of  $(dx_0/dt)^{-\nu-1}$  (resp.,  $(dx_0/dt)^{-\nu-2}$ ) in powers of  $a$ . To dominate them we prepare the following

**Lemma 1.2.4.** *Let  $x_0(s, a, \rho)$  denote*

$$(1.2.103) \quad \sum_{p \geq 0} x_0^{(p)}(s, \rho) a^p.$$

*Then Lemma (1.2.3) entails the existence of some positive constants  $r_0, M_0$  and  $R$  for which the following inequality holds:*

$$(1.2.104) \quad \left\| \left( \left( \frac{dx_0}{dt} \right)^{-\nu} \right)^{(l)} \right\|_{[r_0]} \leq M_0^\nu C(l) (R|\rho|^{-1})^l.$$

*Proof.* Since we may assume that

$$(1.2.105) \quad \frac{dx_0^{(0)}}{dt}(s, \rho) = \left( \frac{dt}{ds} \right)^{-1} \neq 0$$

holds on  $\{s; |s| \leq r_0\}$ , it follows from the estimates (p.iii) of  $\dot{x}_0^{(p)}$  in Remark 1.2.1 that there exist some positive constant  $\tilde{M}_0$  for which



$(dx_0/dt)^{-1}$  is holomorphic on  $\Omega = \{(s, a, \rho); |s| \leq r_0, 2R_0|a| \leq |\rho|\}$  and

$$(1.2.106) \quad \sup_{\Omega} \left| \frac{dx_0}{dt} \right|^{-1} \leq \tilde{M}_0$$

holds. Hence we find

$$(1.2.107) \quad \sup_{\Omega} \left| \left( \frac{dx_0}{dt} \right)^{-\nu} \right| \leq \tilde{M}_0^{\nu}.$$

This then implies

$$(1.2.108) \quad \left\| \left( \left( \frac{dx_0}{dt} \right)^{-\nu} \right)^{(l)} \right\|_{[r_0]} \leq \tilde{M}_0^{\nu} (2R_0|\rho|^{-1})^l.$$

On the other hand it immediately follows from the definition (1.2.1) of  $C(l)$  that

$$(1.2.109) \quad \frac{3}{2\pi^2} 2^{-l-2} \leq C(l)$$

holds for every  $l$  in  $\mathbb{N}_0$ . Therefore we obtain

$$(1.2.110) \quad \left\| \left( \left( \frac{dx_0}{dt} \right)^{-\nu} \right)^{(l)} \right\|_{[r_0]} \leq M_0^{\nu} C(l) (R|\rho|^{-1})^l$$

by setting

$$(1.2.111) \quad M_0 = \frac{8}{3} \pi^2 \tilde{M}_0 \quad \text{and} \quad R = 4R_0.$$

This completes the proof of Lemma 1.2.4. □

We now resume the proof of Proposition 1.2.1. Let us begin our reasoning by dominating the first sum in the right-hand side of (1.2.102), namely

$$(1.2.112) \quad S_{2(n-1)}^{(p)} \stackrel{\text{def}}{=} \sum_{\substack{k_1+k_2=n-1 \\ l_1+l_2+l_3=p}} \frac{d^3 x_{2k_1}^{(l_1)}}{dt^3} \sum_{\nu=\min\{1, k_2\}}^{k_2} (-1)^{\nu} \left( \left( \frac{dx_0}{dt} \right)^{-\nu-1} \right)^{(l_2)}$$

$$\times \sum_{|\vec{\kappa}|_\nu = k_2}^* \sum_{|\vec{\lambda}|_\nu = l_3} \frac{dx_{2\vec{\kappa}}^{(\vec{\lambda})}}{dt}.$$

We first note that (p.iii) Remark 1.2.1 and Cauchy's integral formula applied to  $dx_0^{(l)}/dt$  entail

$$(1.2.113) \quad \left\| \frac{d^2 x_0^{(l)}}{dt^2} \right\|_{[r_0-\varepsilon]} \leq M_0 C(l) (R|\rho|^{-1})^l \varepsilon^{-1},$$

$$(1.2.114) \quad \left\| \frac{d^3 x_0^{(l)}}{dt^3} \right\|_{[r_0-\varepsilon]} \leq 2! M_0 C(l) (R|\rho|^{-1})^l \varepsilon^{-2}$$

for  $l \geq 0$  and some positive constant  $M_0$ . Indeed, (1.2.113) and (1.2.114) follow from (1.2.105) and the following relations for the differentiation of a holomorphic function  $f(s)$  with respect to the two variables  $t$  and  $s$ :

$$(1.2.115) \quad \frac{d^2 f}{dt^2}(s) = \left( \frac{dt(s)}{ds} \right)^{-2} \frac{d^2}{ds^2} f(s) + \frac{1}{2} \frac{d}{ds} \left( \frac{dt(s)}{ds} \right)^{-2} \frac{d}{ds} f(s),$$

$$(1.2.116) \quad \begin{aligned} \frac{d^3 f}{dt^3}(s) = & \left( \frac{dt(s)}{ds} \right)^{-3} \frac{d^3}{ds^3} f(s) + \frac{d}{ds} \left( \frac{dt(s)}{ds} \right)^{-3} \frac{d^2}{ds^2} f(s) \\ & + \frac{1}{2} \left( \frac{dt(s)}{ds} \right)^{-1} \frac{d^2}{ds^2} \left( \frac{dt(s)}{ds} \right)^{-2} \frac{d}{ds} f(s). \end{aligned}$$

*Remark 1.2.5.* Since we can take the constant  $C_0$  in (p.iii) in Remark 1.2.1 for  $p \geq 1$  arbitrarily small by taking  $R_0$  sufficiently large, we can take  $M_0$  in (1.2.113) and (1.2.114) for  $l \geq 1$  also arbitrarily small. However this fact does not hold for  $l = 0$ . Fortunately, our reasoning below does not require  $M_0$  to be arbitrarily small. Hence, for the simplicity of presentation, we use the estimates (1.2.113) and (1.2.114) in the form that is applicable to both cases  $l = 0$  and  $l \geq 1$ , that is, we only assert the existence of some positive constant  $M_0$  there.

Further, the following lemma follows from the induction hypothesis:

**Lemma 1.2.5.** *For each  $(l, k)$  ( $l \geq 0, k \geq 1$ ),  $[G; l, 2k](v)$  entails the following:*

$$(1.2.117) \quad \left\| \frac{d^2 x_{2k}^{(l)}}{ds^2} \right\|_{[r-\varepsilon]} \leq e^2 N_0 C(l) (R|\rho|^{-1})^l (2k+1)! \varepsilon^{-2k-1} (A|\rho|^{-1})^k,$$

$$(1.2.118) \quad \left\| \frac{d^3 x_{2k}^{(l)}}{ds^3} \right\|_{[r-\varepsilon]} \leq e^2 N_0 C(l) (R|\rho|^{-1})^l (2k+2)! \varepsilon^{-2k-2} (A|\rho|^{-1})^k,$$

where  $e = 2.718 \dots$ .

*Proof.* Let  $\tilde{\varepsilon}$  denote  $k\varepsilon/(k+1)$ . Then  $[G; l, 2k](v)$  entails

$$(1.2.119) \quad \begin{aligned} & \sup_{|s| \leq r-\tilde{\varepsilon}} |\dot{x}_{2k}^{(l)}(s)| \\ & \leq N_0 C(l) (R|\rho|^{-1})^l (2k)! \tilde{\varepsilon}^{-2k} (A|\rho|^{-1})^k \\ & = N_0 C(l) (R|\rho|^{-1})^l (2k)! \left(1 + \frac{1}{k}\right)^{2k} \varepsilon^{-2k} (A|\rho|^{-1})^k \\ & \leq e^2 N_0 C(l) (R|\rho|^{-1})^l (2k)! \varepsilon^{-2k} (A|\rho|^{-1})^k. \end{aligned}$$

To derive (1.2.117) and (1.2.118), we use (1.2.119) together with the following representation of  $d^j x_{2k}^{(l)}/ds^j$  ( $j = 2, 3$ ):

$$(1.2.120) \quad \frac{d^j x_{2k}^{(l)}}{ds^j} = \frac{(j-1)!}{2\pi\sqrt{-1}} \int_{|\tilde{s}-s|=(k+1)^{-1}\varepsilon} \frac{\dot{x}_{2k}^{(l)}(\tilde{s})}{(\tilde{s}-s)^{1+j}} d\tilde{s}.$$

Since

$$(1.2.121) \quad \begin{aligned} |\tilde{s}| & \leq |\tilde{s}-s| + |s| \\ & \leq (k+1)^{-1}\varepsilon + r - \varepsilon \\ & = r - \tilde{\varepsilon} \end{aligned}$$

holds for  $s$  in  $\{s; |s| \leq r - \varepsilon\}$  and  $\tilde{s}$  on the above contour, we obtain (1.2.117) and (1.2.118). □

We note that Lemma 1.2.5 together with (1.2.115) and (1.2.116) implies the following inequalities (1.2.122) and (1.2.123) for some positive constant  $M_0$ :

$$(1.2.122) \quad \left\| \frac{d^2 x_{2k}^{(l)}}{dt^2} \right\|_{[r_0-\varepsilon]} \leq M_0 N_0 C(l) (2k+1)! (R|\rho|^{-1})^l \varepsilon^{-2k-1} (A|\rho|^{-1})^k,$$

$$(1.2.123) \quad \left\| \frac{d^3 x_{2k}^{(l)}}{dt^3} \right\|_{[r_0-\varepsilon]} \leq M_0 N_0 C(l) (2k+2)! (R|\rho|^{-1})^l \varepsilon^{-2k-2} (A|\rho|^{-1})^k$$

Let us again return to the proof of Proposition 1.2.1. First we observe that (1.2.114) and Lemma 1.2.3 (via Lemma 1.2.4) entail the following:

$$(1.2.124) \quad \|S_0^{(p)}\|_{[r_0-\varepsilon]} = \left\| \sum_{l_1+l_2=p} \frac{d^3 x_0^{(l_1)}}{dt^3} \left( \left( \frac{dx_0}{dt} \right)^{-1} \right)^{(l_2)} \right\|_{[r_0-\varepsilon]} \\ \leq 2M_0^2 C(p) (R|\rho|^{-1})^p \varepsilon^{-2}.$$

To dominate  $S_{2(n-1)}^{(p)}$  for  $n \geq 2$  we assume that  $[G; l, 2k]$  for  $0 \leq l \leq p$  and  $1 \leq k \leq n-1$ ; this assumption is a part of the induction hypothesis to be employed in parts [II], [III] and [V]. We then perform its estimation by separating the situation into the following three cases: (i)  $k_1 = 0$ , (ii)  $k_2 = 0$  and (iii)  $k_1, k_2 \neq 0$ .

(i)  $k_1 = 0$  : In this case, applying Lemma 1.2.1 and Lemma 1.2.2, we find

$$(1.2.125) \quad \left\| \sum_{l_1+l_2+l_3=p} \frac{d^3 x_0^{(l_1)}}{dt^3} \sum_{\nu=1}^{n-1} (-1)^\nu \left( \left( \frac{dx_0}{dt} \right)^{-\nu-1} \right)^{(l_2)} \sum_{|\vec{k}|_\nu=n-1}^* \sum_{|\vec{l}|_\nu=l_3} \frac{dx_{2\vec{k}}^{(\vec{l})}}{dt} \right\|_{[r_0-\varepsilon]} \\ \leq \sum_{l_1+l_2+l_3=p} 2M_0^2 C(l_1) (R|\rho|^{-1})^{l_1} \varepsilon^{-2} \left( \sum_{\nu=1}^{n-1} M_0^\nu C(l_2) (R|\rho|^{-1})^{l_2} \right) \\ \times 4^{-1} (4M_0 N_0)^\nu C(l_3) (2(n-1) - \nu + 1)! \\ \times (R|\rho|^{-1})^{l_3} \varepsilon^{-2(n-1)} (A|\rho|^{-1})^{n-1} \\ \leq 2M_0^4 N_0 C(p) (R|\rho|^{-1})^p (2(n-1))! \varepsilon^{-2n} \\ \times (A|\rho|^{-1})^{n-1} \sum_{\nu=1}^{n-1} \frac{(4M_0^2 N_0)^{\nu-1}}{(\nu-1)!} \\ \leq 2e^{4M_0^2 N_0} M_0^4 N_0 C(p) (R|\rho|^{-1})^p (2(n-1))! \varepsilon^{-2n} (A|\rho|^{-1})^{n-1}.$$

Here  $M_0$  is taken so that

$$(1.2.126) \quad \sup_{|t| \leq r_0} \left| \frac{ds}{dt} \right| \leq M_0$$

holds.

(ii)  $k_2 = 0$  : In this case we find

$$(1.2.127) \quad \left\| \sum_{l_1+l_2=p} \frac{d^3 x_{2(n-1)}^{(l_1)}}{dt^3} \left( \left( \frac{dx_0}{dt} \right)^{-1} \right)^{(l_2)} \right\|_{[r_0-\varepsilon]} \\ \leq M_0^2 N_0 C(p) (2n)! (R|\rho|^{-1})^p \varepsilon^{-2n} (A|\rho|^{-1})^{n-1}.$$

(iii)  $k_1, k_2 \geq 1$  : We first observe

$$(1.2.128) \quad \left\| \sum_{\nu=\min\{1, k_2\}}^{k_2} (-1)^\nu \left( \left( \frac{dx_0}{dt} \right)^{-\nu-1} \right)^{(l_2)} \sum_{|\vec{\kappa}|_\nu=k_2}^* \sum_{|\vec{\lambda}|_\nu=l_3} \frac{dx_{2\vec{\kappa}}^{(\vec{\lambda})}}{dt} \right\|_{[r_0-\varepsilon]} \\ \leq \sum_{\nu=1}^{k_2} M_0^{\nu+1} C(l_2) (R|\rho|^{-1})^{l_2} (M_0 N_0)^\nu C(l_3) 4^{\nu-1} (2k_2 - \nu + 1)! \\ \times (R|\rho|^{-1})^{l_3} \varepsilon^{-2k_2} (A|\rho|^{-1})^{k_2} \\ \leq M_0^3 e^{4M_0^2 N_0} N_0 C(l_2) C(l_3) (R|\rho|^{-1})^{l_2+l_3} (2k_2)! \varepsilon^{-2k_2} (A|\rho|^{-1})^{k_2}.$$

Hence we obtain (1.2.129) below by (1.2.123) and (1.2.128):

$$(1.2.129) \quad \left\| \sum_{\substack{k_1+k_2=n-1 \\ l_1+l_2+l_3=p \\ k_1, k_2 \geq 1}} \frac{d^3 x_{2k_1}^{(l_1)}}{dt^3} \sum_{\nu=\min\{1, k_2\}}^{k_2} (-1)^\nu \left( \left( \frac{dx_0}{dt} \right)^{-\nu-1} \right)^{(l_2)} \sum_{|\vec{\kappa}|_\nu=k_2}^* \sum_{|\vec{\lambda}|_\nu=l_3} \frac{dx_{2\vec{\kappa}}^{(\vec{\lambda})}}{dt} \right\|_{[r_0-\varepsilon]} \\ \leq M_0^4 e^{4M_0^2 N_0} N_0^2 C(p) (|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^{n-1}.$$

Thus the following estimate (1.2.130) follows from (1.2.125), (1.2.127) and (1.2.129) for some positive constant  $M$  that is independent of  $N_0, C_0, R$  and  $A$ :

$$(1.2.130) \quad \|S_{2(n-1)}^{(p)}\|_{[r_0-\varepsilon]} \leq M N_0 C(p) (R|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^{n-1}.$$

The reasoning given so far equally applies to the second sum in the right-hand side of (1.2.102), i.e.,

$$(1.2.131) \quad -\frac{3}{2} \sum_{\substack{k_1+k_2+k_3=n-1 \\ l_1+l_2+l_3+l_4=p}} \frac{d^2 x_{2k_1}^{(l_1)}}{dt^2} \frac{d^2 x_{2k_2}^{(l_2)}}{dt^2} \\ \times \sum_{\nu=\min\{1, k_3\}}^{k_3} (-1)^\nu (\nu+1) \left( \left( \frac{dx_0}{dt} \right)^{-\nu-2} \right)^{(l_3)} \sum_{|\vec{k}|_\nu=k_3}^* \sum_{|\vec{l}|_\nu=l_4} \frac{dx_{2\vec{k}}^{(\vec{l})}}{dt}.$$

Summing up, we have found

$$(1.2.132) \quad \|\{x; t\}_0^{(p)}\|_{[r_0-\varepsilon]} \leq 2! MC(p) (R|\rho|^{-1})^p \varepsilon^{-2},$$

by Lemma 1.2.3 via Lemma 1.2.4, and we have also confirmed, for  $n \geq 2$ ,

$$(1.2.133) \quad \|\{x; t\}_{2(n-1)}^{(p)}\|_{[r_0-\varepsilon]} \leq MN_0 C(p) (R|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^{n-1}$$

for some positive constant  $M$  that is independent of  $N_0, C_0, R$  and  $A$  by assuming the validity of  $[G; l, 2k]$  for  $0 \leq l \leq p$  and  $1 \leq k \leq n-1$ , besides Lemma 1.2.4.

Making use of these results, we now show that the validity of  $[G; q, 2k]$  ( $q$ : arbitrary,  $1 \leq k \leq n-1$ ) together with the validity of  $[G; r, 2n]$  ( $r \leq p_0 - 1$ ) entails  $[G; p_0, 2n]$ . In what follows we call these assumptions as the induction hypothesis for short. It is clear that the induction hypothesis is stronger than the assumptions we have used to confirm (1.2.133) with  $p = p_0$ . We also remark that the validity of  $[G; 0, 2n]$  is guaranteed by the same reasoning if we assume the validity of  $[G; q, 2k]$  ( $q$ : arbitrary,  $1 \leq k \leq n-1$ ) besides Lemma 1.2.3 (i.e., the validity of  $[G; q, 0]$  for  $q \geq 1$ ). Parenthetically we note that it suffices to use only  $[G; q, 2k]$  ( $1 \leq k \leq n-1$ ) with  $q \leq p_0$  to validate  $[G; p_0, 2n]$ ; the situation is the same when  $p_0 = 0$ .

[II] Let us first dominate  $x_{2n}^{(p_0+1)}(0, \rho) - \tilde{A}_{2n}^{(p_0)}$  on the above induction hypothesis. The reasoning used for the domination is basically the same as that used in the proof of Lemma 1.2.3 reinforced with the results in part [I], which are applied to the estimation of terms  $(\alpha, j)$  ( $j = \text{vii}, \text{viii}, \text{ix}, \text{x}$ ) in (1.1.3.43). Hence in what follows we focus our

attention on the points which require some special care, and we will try to avoid routine repetitions. As in the proof of Lemma 1.2.3 we use the concrete expression (1.1.3.43) of  $R_{2n}^{(p_0+1)}(s, \rho)$  to dominate

$$(1.2.134) \quad -x_{2n}^{(p_0+1)}(0, \rho) + \tilde{A}_{2n}^{(p_0)} = R_{2n}^{(p_0+1)}(0, \rho) \\ = \frac{1}{2\pi i} \int_{|s|=r_0-\varepsilon} R_{2n}^{(p_0+1)}(s, \rho) \frac{ds}{s} = \frac{1}{2\pi i} \oint R_{2n}^{(p_0+1)} \frac{ds}{s}.$$

As in (1.2.16) we also use the notation

$$(1.2.135) \quad \frac{1}{2\pi i} \oint (\alpha.j) \frac{ds}{s}$$

to denote Cauchy's integral of the term labelled by  $(\alpha.j)$  ( $j = i, ii, \dots, x$ ) in (1.1.3.43). In what follows we use the notation introduced in Definition 1.2.3;  $A_{2k}^{(u)}/B_0^{(0)}$  etc. in (1.1.3.43) are respectively denoted by  $\tilde{A}_{2k}^{(u)}$  etc.

To begin with, we note that the contribution from the parts

$$(1.2.136) \quad - \sum_{\substack{q+r=p_0 \\ i+j=n, i,j \leq n-1}} \dot{x}_{2i}^{(q)}(s, \rho) \dot{x}_{2j}^{(r)}(s, \rho) \tilde{A}_0^{(0)}$$

and

$$(1.2.137) \quad -2 \sum_{q+r=p_0} \dot{x}_{2n}^{(q)}(s, \rho) \dot{x}_0^{(r)}(s, \rho) \tilde{A}_0^{(0)}$$

in  $(\alpha.i)$  are cancelled out respectively by the worst (in estimating) part of the contribution from  $(\alpha.v)$  with  $u = 1$  and by that from  $(\alpha.vi)$  with  $u = 1$ , that is,

$$(1.2.138) \quad \frac{1}{2\pi i} \oint \frac{f^{(1)}(t, \rho)}{B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 \frac{s^2}{t^2} \left( \sum_{\substack{q+r=p_0 \\ i+j=n, i,j \leq n-1}} \dot{x}_{2i}^{(q)}(0, \rho) \dot{x}_{2j}^{(r)}(0, \rho) \right) \frac{ds}{s}$$

and

$$(1.2.139) \quad \frac{2}{2\pi i} \oint \frac{f^{(1)}(t, \rho)}{B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 \frac{s^2}{t^2} \left( \sum_{q+r=p_0} \dot{x}_{2n}^{(q)}(0, \rho) \dot{x}_0^{(r)}(0, \rho) \right) \frac{ds}{s}.$$

The mechanism of the cancellation is the same for both parts; first we consider the Taylor expansion  $x_{2i}^{(q)}(s, \rho)x_{2j}^{(r)}(s, \rho)$  and pick up the coefficient of  $s^2$  and then we use

$$(1.2.140) \quad f^{(1)}(t, \rho) \left( \frac{dt}{ds} \right)^2 \frac{s^2}{t^2} \Big|_{s=0} = A_0^{(0)}.$$

Once (1.2.138) is set aside, other contributions from  $(\alpha.v)$  with  $u = 1$ , i.e.,

$$(1.2.141) \quad \sum_{\substack{q+r=p_0 \\ i+j=n, i, j \leq n-1}} \frac{1}{2\pi i} \oint \frac{f^{(1)}(t, \rho)}{B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 \frac{1}{t^2} \left( x_{2i}^{(q)}(0, \rho) x_{2j}^{(r)}(0, \rho) \right. \\ \left. + 2s x_{2i}^{(q)}(0, \rho) \dot{x}_{2j}^{(r)}(0, \rho) + s^2 x_{2i}^{(q)}(0, \rho) \ddot{x}_{2j}^{(r)}(0, \rho) \right) \frac{ds}{s}$$

is seen to be tame. In fact, each integral to be examined contains either  $x_{2i}^{(q)}(0, \rho)$  ( $1 \leq i \leq n-1$ ) in its integrand and hence the integral is dominated by

$$(1.2.142) \quad M|\rho|^{-1} N_0^2 C(p_0) (R|\rho|^{-1})^{p_0-1} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n \\ = MR^{-1} N_0^2 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n,$$

where  $M$  is a constant that originates from the innocent part of the integrand such as

$$(1.2.143) \quad \left( \left( \frac{dt}{ds} \right)^2 \frac{1}{t^2} f^{(1)}(t, \rho) \right) \frac{ds}{s}.$$

If we set aside (1.2.139), we use the same reasoning to find the contribution from  $(\alpha.vi)$  with  $u = 1$  is dominated by

$$(1.2.144) \quad MR^{-1} N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n.$$

Because of the constraint on the indices

$$(1.2.145) \quad q + r + u = p_0 + 1,$$

contributions from  $(\alpha.vi)$  with  $u \geq 2$  and  $(\alpha.v)$  with  $u \geq 2$  are dominated by similar constants, whereas contributions from  $(\alpha.vi)$  with  $u = 0$  and  $(\alpha.v)$  with  $u = 0$  require some special care. To fix the



notation we discuss the contribution from  $(\alpha.vi)$  with  $u = 0$ ; the contribution from  $(\alpha.v)$  with  $u = 0$  is handled in the same manner. We first note that  $f^{(0)}$  has the form  $\rho t g(t, \rho)$  with  $g(0, \rho) = 1$ . Hence we find

$$\begin{aligned}
 (1.2.146) \quad & \frac{1}{2\pi i} \oint \frac{2t^{-2}}{B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 f^{(0)} \left( \sum_{\substack{q+r=p_0+1 \\ q \leq p_0}} x_{2n}^{(q)} x_0^{(r)} \right) \frac{ds}{s} \\
 &= \frac{1}{2\pi i} \oint 2Z_0 \frac{s}{t} g(t, \rho) \left( \frac{dt}{ds} \right)^2 \left( \sum_{\substack{q+r=p_0+1 \\ q \leq p_0}} x_{2n}^{(q)} x_0^{(r)} \right) \frac{ds}{s^2}.
 \end{aligned}$$

Thus we observe that the annoying factor  $1/B_0^{(0)}$  has disappeared and that it suffices to study the Taylor expansion (in powers of  $s$ ) of  $\sum x_{2n}^{(q)} x_0^{(r)}$  up to the degree 1 part; each term in the Taylor expansion to be estimated contains  $x_{2n}^{(q)}(0, \rho)$  or  $x_0^{(r)}(0, \rho)$  as its factor. Since  $x_0^{(0)}$  does not appear in the sum,  $[G; p, 0]$  ( $p \geq 1$ ) and the induction hypothesis guarantee that each contribution is dominated by

$$(1.2.147) \quad MC_0 N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n$$

with some positive constant  $M$  that is independent of  $C_0, N_0, R$  and  $A$ . (In what follows,  $M$  stands for such a constant.)

Returning to the estimation of  $(\alpha.i)$ , we find by  $[G; p, 0]$  ( $p \geq 1$ ) and the induction hypothesis that each term in  $(\alpha.i)$  except for (1.2.137) and (1.2.136) is dominated by a constant of the form

$$(1.2.148) \quad M(C_0 + N_0) N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n.$$

Next let us study the contribution from  $(\alpha.ii)$ . This term is basically handled by the application of the induction hypothesis. Since  $\tilde{x}_0^{(0)}(0, \rho)$  and  $\tilde{B}_0^{(0)}$  are not covered by  $[G; p, 0]$  ( $p \geq 1$ ), we have to pay attention to them. However, all the terms in  $(\alpha.ii)$  contain two factors; one of them has a suffix  $(2k_1, (q_1))$  with  $k_1 \geq 1$  and the other has a suffix  $(2k_2, (q_2))$  with  $q_2 \geq 1$ . Thus we can dominate the contribution from  $(\alpha.ii)$  by

$$(1.2.149) \quad MC_0 N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n.$$

The succeeding target is ( $\alpha$ .iii). In view of the structure of the induction hypotheses, we rewrite

(1.2.150)

$$\begin{aligned}
& \sum_{\substack{q+r+u=p_0-2 \\ i+j+k=n}} \dot{x}_{2i}^{(q)} \dot{x}_{2j}^{(r)} \tilde{A}_{2k}^{(u)} \\
&= \tilde{A}_0^{(0)} \left( 2\dot{x}_0^{(0)} \dot{x}_{2n}^{(p_0-2)} + \sum_{\substack{q+r=p_0-2 \\ i+j=n \\ (q,i),(r,j) \neq (0,0)}} \dot{x}_{2i}^{(q)} \dot{x}_{2j}^{(r)} \right) + \sum_{\substack{q+r+u=p_0-2 \\ i+j+k=n \\ (u,k) \neq (0,0)}} \tilde{A}_{2k}^{(u)} \dot{x}_{2i}^{(q)} \dot{x}_{2j}^{(r)} \\
&= 2\tilde{A}_0^{(0)} \dot{x}_0^{(0)} \dot{x}_{2n}^{(p_0-2)} + \tilde{A}_0^{(0)} \left( \sum_{\substack{q+r=p_0-2 \\ i+j=n \\ (q,i),(r,j) \neq (0,0)}} \dot{x}_{2i}^{(q)} \dot{x}_{2j}^{(r)} \right) + \tilde{A}_{2n}^{(p_0-2)} \dot{x}_0^{(0)2} \\
&\quad + 2 \sum_{\substack{r+u=p_0-2 \\ j+k=n \\ (u,k),(r,j) \neq (0,0)}} \tilde{A}_{2k}^{(u)} \dot{x}_0^{(0)} \dot{x}_{2j}^{(r)} + \sum_{\substack{q+r+u=p_0-2 \\ i+j+k=n \\ (u,k),(q,i),(r,j) \neq (0,0)}} \tilde{A}_{2k}^{(u)} \dot{x}_{2i}^{(q)} \dot{x}_{2j}^{(r)}.
\end{aligned}$$

Thus the worst contribution from ( $\alpha$ .iii) is dominated as follows:

$$\begin{aligned}
(1.2.151) \quad & \left| \frac{1}{2\pi i} \oint \frac{s^2}{t^2} (2\tilde{A}_0^{(0)} \dot{x}_{2n}^{(p_0-2)}) \frac{ds}{s^3} \right| \\
& \leq M|\rho|^{-1} N_0 C(p_0) (R|\rho|^{-1})^{p_0-2} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n \\
& \leq MR^{-1} N_0 C(p_0) (R|\rho|^{-1})^{p_0-1} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n.
\end{aligned}$$

Parenthetically we note that the contribution from  $\tilde{A}_{2n}^{(p_0-2)} \dot{x}_0^{(0)2}$  ( $= \tilde{A}_{2n}^{(p_0-2)}$ ) is weaker than (1.2.151) by the factor  $|\rho|$ .

In parallel with the study of ( $\alpha$ .iii) we can readily find that the worst contribution from ( $\alpha$ .iv) is

$$(1.2.152) \quad \left| \frac{1}{2\pi i} \oint \frac{s^2}{t^2} (\dot{x}_0^{(0)2} x_{2n}^{(p_0-1)} \tilde{B}_0^{(0)}) \frac{ds}{s^3} \right|,$$

which is dominated by

$$(1.2.153) \quad MN_0 C(p_0) (R|\rho|^{-1})^{p_0-1} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n.$$

The domination of contributions from  $(\alpha.vii) \sim (\alpha.x)$  can be done in a similar manner. Since the domination of contributions from  $(\alpha.viii)$  and  $(\alpha.x)$  are straightforward, we concentrate our attention on  $(\alpha.vii)$  and  $(\alpha.ix)$ . Among the contributions from  $(\alpha.vii)$  the worst ones are

$$(1.2.154) \quad \left| \frac{1}{2\pi i} \oint \frac{t^{-2}}{2B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 x_0^{(0)2} \{x; t\}_{2(n-1)}^{(p_0-1)} \frac{ds}{s} \right|$$

and

$$(1.2.155) \quad \left| \frac{1}{2\pi i} \oint \frac{t^{-2}}{B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 (x_0^{(0)} x_{2(n-1)}^{(p_0-1)} \{x; t\}_0^{(0)}) \frac{ds}{s} \right|,$$

which are respectively dominated by

$$(1.2.156) \quad |\rho|^{-1} M N_0 C(p_0) (R|\rho|^{-1})^{p_0-1} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^{n-1} \\ = M R^{-1} N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^{n-1}$$

and

$$(1.2.157) \quad M R^{-1} N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^{n-1}.$$

Concerning the contributions of  $(\alpha.ix)$  we discuss the case  $n = 1$  and the case  $n \geq 2$  separately. When  $n = 1$ ,  $(\alpha.ix)$  evaluated at  $s = 0$  is given by

$$(1.2.158) \quad \frac{1}{2B_0^{(0)}} \sum_{\substack{q+r+u=p_0+1 \\ q, r \geq 2}} x_0^{(q)}(0, \rho) x_0^{(r)}(0, \rho) \{x; t\}_0^{(u)} \Big|_{s=0},$$

which is dominated by

$$(1.2.159) \quad |\rho|^{-1} M C_0^2 C(p_0) (R|\rho|^{-1})^{p_0-1} \varepsilon^{-2} \\ = M C_0^2 (N_0 R A)^{-1} N_0 C(p_0) (R|\rho|^{-1})^{p_0} \varepsilon^{-2} A.$$

When  $n \geq 2$ , it follows from the results in part [I] together with the induction hypotheses that the sum  $(\alpha.ix)$  evaluated at  $s = 0$  is dominated by

$$(1.2.160) \quad M C_0 N_0 |\rho|^{-1} C(p_0) (R|\rho|^{-1})^{p_0-1} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^{n-1}$$

$$= MC_0 R^{-1} N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^{n-1}.$$

Summing up the results obtained in this part, we find

$$(1.2.161) \quad \begin{aligned} |x_{2n}^{(p_0+1)}(0, \rho) - \tilde{A}_{2n}^{(p_0)}| &= |R_{2n}^{(p_0+1)}(0, \rho)| \\ &\leq N_2 N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n, \end{aligned}$$

where

$$(1.2.162) \quad N_2 = M(C_0 + N_0 + R^{-1} + C_0(N_0 R A)^{-1}).$$

By using the same reasoning as above, we also find

$$(1.2.163) \quad \begin{aligned} |\dot{x}_{2n}^{(p_0)}(0, \rho) + \tilde{B}_{2n}^{(p_0)}| &= |\dot{R}_{2n}^{(p_0)}(0, \rho)| \\ &\leq N_2 N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n. \end{aligned}$$

Actually the domination is easier than the confirmation of (1.2.161), because this time we do not need to seek for the cancellation of annoying terms such as (1.2.136), (1.2.137), (1.2.138) and (1.2.139). Hence we omit the proof of (1.2.163).

*Remark 1.2.6.* By taking  $C_0$  and  $N_0$  sufficiently small and then letting  $R$  and  $A$  sufficiently large, we may consider the factor  $N_2$  is sufficiently small. Here we note that the factor  $A$  is not used essentially in the estimation in part [II] (and also part [III] below), that is, we can obtain (1.2.161) and (1.2.163) with  $N_2$  sufficiently small from the induction hypothesis without taking  $A$  sufficiently large. The factor  $A$  plays an essential role in part [IV] to make the constant  $M(N_0 A)^{-1} N_0$  (resp.,  $M A^{-1} N_0$ ) in (1.2.179) (resp., (1.2.180)) sufficiently small. Parenthetically we also note that this stage of the reasoning is not an appropriate place to detect the proper order of  $|\rho|$  relevant to  $n$ ; for example the order in question is 0 in (1.2.160), whereas it is  $-1$  in the estimate (1.2.179) of the corresponding term in part [IV]. Since (1.2.179) is a consequence of Lemma 1.2.3 as we will see later, that is the spot where we find the appropriate order.

[III] Using the results in part [II] we dominate  $\tilde{A}_{2n}^{(p_0)}$  and  $\tilde{B}_{2n}^{(p_0)}$  by the induction on  $p_0$ . The reasoning is basically the same as the reasoning in

part [II] of the proof of Lemma 1.2.3 except for the estimation of terms involving the effect of the Schwarzian derivative. By concretely writing down the conditions  $(ds/dt)^{-2}\Phi_{2n}^{(p_0+3)}|_{s=0} = (ds/dt)^{-2}(d\Phi_{2n}^{(p_0+2)}/dt)|_{s=0} = 0$ , we obtain the following relations which determine  $(\tilde{A}_{2n}^{(p_0)}, \tilde{B}_{2n}^{(p_0)})$ , where  $z_{2n}^{(p_0)}(s, \rho)$  stands for  $x_{2n}^{(p_0)}(s, \rho) - \tilde{A}_{2n}^{(p_0-1)} + \tilde{B}_{2n}^{(p_0)}s$ . (Cf. (1.2.70), (1.2.70'), (1.2.71) and (1.2.71').)

(1.2.164)

$$- 2 \left( B_0^{(0)} \tilde{A}_{2n}^{(p_0)} - A_0^{(0)} \tilde{B}_{2n}^{(p_0)} \right) = 2 \dot{z}_{2n}^{(p_0)}(0, \rho) A_0^{(0)} \quad (\tilde{\gamma}.i)$$

$$+ z_{2n}^{(p_0+1)}(0, \rho) B_0^{(0)} \quad (\tilde{\gamma}.ii)$$

$$+ \sum_{\substack{q+r+u=p_0 \\ i+j+k=n \\ (q,i),(r,j),(u,k) \neq (p_0,n)}} \dot{x}_{2i}^{(q)}(0, \rho) \dot{x}_{2j}^{(r)}(0, \rho) B_0^{(0)} \tilde{A}_{2k}^{(u)} \quad (\tilde{\gamma}.iii)$$

$$+ 2 \sum_{\substack{q+r+u=p_0+3 \\ q \geq 2, r, u \geq 1}} x_0^{(q)}(0, \rho) x_{2n}^{(r)}(0, \rho) f^{(u)}(0, \rho) \quad (\tilde{\gamma}.iv)$$

$$+ \sum_{\substack{q+r+u=p_0+3, q, r, u \geq 1 \\ i+j=n, i, j \geq 1}} x_{2i}^{(q)}(0, \rho) x_{2j}^{(r)}(0, \rho) f^{(u)}(0, \rho) \quad (\tilde{\gamma}.v)$$

$$+ \sum_{\substack{q+r+u+v=p_0+1 \\ i+j+k+l=n \\ (u,k) \neq (p_0+1,n)}} \dot{x}_{2i}^{(q)}(0, \rho) \dot{x}_{2j}^{(r)}(0, \rho) x_{2k}^{(u)}(0, \rho) B_0^{(0)} \tilde{B}_{2l}^{(v)} \quad (\tilde{\gamma}.vi)$$

$$+ \frac{1}{2} \{x; t\}_{2(n-1)}^{(p_0-1)}|_{s=0} \quad (\tilde{\gamma}.vii)$$

$$- \frac{1}{2} \sum_{\substack{q+r+v=p_0+1 \\ i+j+k=n-1}} x_{2i}^{(q)}(0, \rho) x_{2j}^{(r)}(0, \rho) \{x; t\}_{2k}^{(v)}|_{s=0} \quad (\tilde{\gamma}.viii)$$

$$\stackrel{\text{def}}{=} \Gamma_{2n}^{(p_0)},$$

(1.2.165)

$$- 2 \left( A_0^{(0)} \tilde{A}_{2n}^{(p_0)} - B_0^{(0)} \tilde{B}_{2n}^{(p_0)} \right) \\ = 2A_0^{(0)} z_{2n}^{(p_0+1)}(0, \rho) \quad (\tilde{\delta}.i)$$

$$+ 3B_0^{(0)} z_{2n}^{(p_0)}(0, \rho) \quad (\tilde{\delta}.ii)$$

$$+ 2 \sum_{\substack{q+r+u=p_0-1 \\ i+j+k=n}} x_{2i}^{(q)''} \dot{x}_{2j}^{(r)} A_{2k}^{(u)} \Big|_{s=0} \quad (\tilde{\delta}.iii)$$

$$+ \sum_{\substack{q+r=p_0+2 \\ i+j=n}} x_{2i}^{(q)}(0, \rho) x_{2j}^{(r)}(0, \rho) \left( f^{(0)'} \Big|_{t=0} \right) \quad (\tilde{\delta}.iv)$$

$$+ \sum_{\substack{q+r+u=p_0+2, u \geq 1 \\ i+j=n}} x_{2i}^{(q)}(0, \rho) x_{2j}^{(r)}(0, \rho) \left( f^{(u)'} \Big|_{t=0} \right) \quad (\tilde{\delta}.v)$$

$$+ \sum_{\substack{q+r=p_0+1 \\ i+j=n, (r,j) \neq (p_0+1, n)}} \dot{x}_{2i}^{(q)}(0, \rho) x_{2j}^{(r)}(0, \rho) f^{(1)}(0, \rho) \quad (\tilde{\delta}.vi)$$

$$+ \sum_{\substack{q+r+u=p_0+2, u \geq 2 \\ i+j=n}} \dot{x}_{2i}^{(q)}(0, \rho) x_{2j}^{(r)}(0, \rho) f^{(u)}(0, \rho) \quad (\tilde{\delta}.vii)$$

$$+ 2 \sum_{\substack{q+r+u+v=p_0 \\ i+j+k+l=n}} \left( x_{2i}^{(q)''} \dot{x}_{2j}^{(r)} \right) \Big|_{t=0} x_{2k}^{(u)}(0, \rho) B_{2l}^{(v)} \quad (\tilde{\delta}.viii)$$

$$+ \sum_{\substack{q+r+u+v=p_0 \\ i+j+k+l=n \\ (q,i), (r,j), (u,k), (v,l) \neq (p_0, n)}} \dot{x}_{2i}^{(q)}(0, \rho) \dot{x}_{2j}^{(r)}(0, \rho) \dot{x}_{2k}^{(u)}(0, \rho) B_{2l}^{(v)} \quad (\tilde{\delta}.ix)$$

$$+ \frac{1}{2} \left( \frac{d}{dt} \{x; t\}_{2(n-1)}^{(p_0-2)} \right) \Big|_{t=0} \quad (\tilde{\delta}.x)$$

$$- \frac{1}{2} \left( \frac{d}{dt} \sum_{\substack{q+r+u=p_0 \\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x; t\}_{2k}^{(u)} \right) \Big|_{t=0} \quad (\tilde{\delta}.xi)$$

$$\stackrel{\text{def}}{=} \Delta_{2n}^{(p_0)}.$$

Thus, as in part [II] of the proof of Lemma 1.2.3, it suffices to confirm that

$$(1.2.166) \quad |\Gamma_{2n}^{(p_0)}|, |\Delta_{2n}^{(p_0)}| \leq N_3 N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n$$

holds, where  $N_3$  is a sufficiently small constant given by (1.2.162).

Using the induction hypothesis, we readily find that  $|(\tilde{\gamma}.j)|$  ( $j = \text{i, ii, iii}$ ) is dominated by a constant of the form

$$(1.2.167) \quad N_3 N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n$$

with

$$(1.2.168) \quad N_3 = M N_2 \quad \text{for } (\tilde{\gamma}.\text{i})$$

$$(1.2.169) \quad N_3 = |\rho| N_2 \quad \text{for } (\tilde{\gamma}.\text{ii})$$

$$(1.2.170) \quad N_3 = M C_0 \quad \text{for } (\tilde{\gamma}.\text{iii}).$$

In view of the wideness of the range of indices we are to pay some attention to  $(\tilde{\gamma}.\text{iv})$  with  $u = 1$ . This term is seen to be dominated by a constant of the form (1.2.167) with (1.2.170) if we set

$$(1.2.171) \quad \tilde{q} = q - 1, \quad \tilde{r} = r - 1$$

and use  $[G; \tilde{q}, 0]$  ( $(\tilde{q}.\text{i})$  and  $(\tilde{q}.\text{v})$ ) and  $[G; \tilde{r}, 2n]$  ( $(\tilde{r}, 2n)(\text{i})$ ). Parenthetically, here we observe that  $\tilde{q}, \tilde{r} \leq p_0$ , as we have noted before beginning the discussion of part [II]; this is consistent with our delicate way of constructing  $x_{2n}^{(p)}(s, \rho)$ . (Cf. Proposition 1.1.3.3.) The same reasoning also applies to  $(\tilde{\gamma}.\text{v})$  and  $(\tilde{\gamma}.\text{vi})$ . We find they are dominated by a constant of the form (1.2.167) with

$$(1.2.172) \quad N_3 = M N_0$$

and (1.2.170) respectively. It immediately follows from (1.2.132) and (1.2.133) that  $|(\tilde{\gamma}.\text{vii})|$  is dominated by a constant of the form (1.2.167) with

$$(1.2.173) \quad N_3 = M |\rho|^2 (N_0 R A)^{-1} \quad \text{for } n = 1$$

and

$$(1.2.174) \quad N_3 = M|\rho|^2(RA)^{-1} \quad \text{for } n \geq 2.$$

To dominate  $(\tilde{\gamma}.viii)$  we use (1.2.132) and (1.2.133) together with the technique employed in dominating  $|(\tilde{\gamma}.iv)|$ . Then we find  $(\tilde{\gamma}.viii)$  satisfies the estimates of the form (1.2.167) with

$$(1.2.175) \quad N_3 = M|\rho|^2 C_0(RA)^{-1}.$$

Thus we have seen that  $|\Gamma_{2n}^{(p_0)}|$  satisfies (1.2.166). The domination of  $|\Delta_{2n}^{(p_0)}|$  can be done in the same manner. We only note that, using Cauchy's inequality, the domination of  $x_{2i}^{(q)''}$  in  $(\tilde{\delta}.iii)$  and the differentiated terms  $(\tilde{\delta}.x)$  and  $(\tilde{\delta}.xi)$  can be done without any trouble, because their value are considered at  $s = 0$ ; the order of  $\varepsilon$  is not affected by differentiation as in Lemma 1.2.5. Thus by rewriting (1.2.164) and (1.2.165) in the form of (1.2.80) we conclude that  $|\tilde{A}_{2n}^{(p_0)}|$  and  $|\tilde{B}_{2n}^{(p_0)}|$  are dominated by

$$(1.2.176) \quad N_2 N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n,$$

where  $N_2$  is a sufficiently small constant of the form (1.2.162).

[IV] Finally let us dominate  $\|x_{2n}^{(p_0)}\|_{[r_0-\varepsilon]}$  and  $\|\dot{x}_{2n}^{(p_0)}\|_{[r_0-\varepsilon]}$ . We first note that, by a straightforward calculation, we find

$$(1.2.177) \quad \|R_{2n}^{(p_0)}\|_{[r_0-\varepsilon]} \leq N_4 N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n$$

with

$$(1.2.178) \quad N_4 = M(C_0 + N_0 + R^{-1} + (N_0 A)^{-1}).$$

(We can not expect the cancellation of terms in  $R_{2n}^{(p_0)}(u, \rho)$  which is similar to that observed between (1.2.136) and (1.2.138). However, without the cancellation, we can still confirm (1.2.177), although  $N_4$  contains a term  $(N_0 A)^{-1}$ ; to make this term small we take  $A$  sufficiently large.) Here we only mention the estimation of  $(\alpha.ix)$ , whose contribution determines the order of  $|\rho|$  relevant to  $n$  in  $[G; p, 2n]$ . It follows from (1.2.132) and (1.2.133) that

$$(1.2.179) \quad \left\| \frac{1}{2B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 \sum_{q+r+u=p_0} x_0^{(q)} x_0^{(r)} \{x; t\}_0^{(u)} \right\|_{[r_0-\varepsilon]}$$



$$\begin{aligned}
&\leq M|\rho|^{-1}C(p_0)(R|\rho|^{-1})^{p_0}2!\varepsilon^{-2} \\
&\leq M(N_0A)^{-1}N_0C(p_0)(R|\rho|^{-1})^{p_0}2!\varepsilon^{-2}A|\rho|^{-1}
\end{aligned}$$

for  $n = 1$  and

$$\begin{aligned}
(1.2.180) \quad &\left\| \frac{1}{2B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 \sum_{\substack{q+r+u=p_0 \\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x; t\}_{2k}^{(u)} \right\|_{[r_0-\varepsilon]} \\
&\leq M|\rho|^{-1}N_0C(p_0)(R|\rho|^{-1})^{p_0}(2n)!\varepsilon^{-2n}(A|\rho|^{-1})^{n-1} \\
&\leq MA^{-1}N_0C(p_0)(R|\rho|^{-1})^{p_0}(2n)!\varepsilon^{-2n}(A|\rho|^{-1})^n
\end{aligned}$$

for  $n \geq 2$ .

Then the domination of  $\|x_{2n}^{(p_0)}\|_{[r_0-\varepsilon]}$  and  $\|\dot{x}_{2n}^{(p_0)}\|_{[r_0-\varepsilon]}$  can be readily done by the same reasoning as part [III] in the proof of Lemma 1.2.3. Thus the induction proceeds. This completes the proof of Proposition 1.2.1.

□

### 1.3 Correspondence between a WKB solution of an M2P1T equation and that of the Mathieu equation

The purpose of this section is to show how we can relate a WKB solution of an M2P1T equation to an appropriate WKB solution of an  $\infty$ -Mathieu equation. To begin with we summarize the results in Section 1.1, Section 1.2 and Appendix C in the form of Theorem 1.3.1 below. To avoid the notational confusions which we will later explain in Remark 1.3.1, we now assume

$$(1.3.1) \quad B_0^{(0)} = \rho.$$

**Theorem 1.3.1.** *Let  $Q(t, a, \rho)$  be a potential of an M2P1T operator given in Definition 1.1. Then there exist positive constants  $r$  and  $R_0$ , and holomorphic functions*

$$(1.3.2) \quad A_{2n}(a, \rho) = \sum_{j=0}^{\infty} A_{2n}^{(j)}(\rho) a^j,$$

$$(1.3.3) \quad B_{2n}(a, \rho) = \sum_{j=0}^{\infty} B_{2n}^{(j)}(\rho) a^j$$

and

$$(1.3.4) \quad x_{2n}(t, a, \rho) = \sum_{j=0}^{\infty} x_{2n}^{(j)}(t, \rho) a^j$$

( $n \geq 0$ ) on

$$(1.3.5) \quad E_{r, R_0}^1 = \{(t, a, \rho) \in \mathbb{C}^3 : |t| \leq r, 0 < |\rho| \leq r, R_0|a| \leq |\rho|\}$$

for which the following conditions are satisfied there:

$$(1.3.6) \quad A(a, \rho, \eta), B(a, \rho, \eta) \text{ and } x(t, a, \rho, \eta) \text{ satisfy (1.1.6),}$$

$$(1.3.7) \quad A_0(0, \rho) = f^{(1)}(0, \rho),$$

$$(1.3.8) \quad B_0(0, \rho) = \rho,$$

$$(1.3.9) \quad \frac{\partial x_0}{\partial t}(0, 0, \rho) = 1,$$

$$(1.3.10) \quad \text{the function } x_0(t, a, \rho) \text{ of } t \text{ is injective for each fixed } a \text{ and } \rho \\ \text{on } E_{r, R_0}^1,$$

$$(1.3.11) \quad x_0(t, a, \rho)|_{t=\pm a} = \pm a.$$

Furthermore there exists a positive constant  $R_1$  for which the following estimates hold for  $n \geq 1$ :

$$(1.3.12) \quad |A_{2n}(a, \rho)| \leq |\rho|(2n)!R_1^n|\rho|^{-n},$$

$$(1.3.13) \quad |B_{2n}(a, \rho)| \leq |\rho|(2n)!R_1^n|\rho|^{-n},$$

$$(1.3.14) \quad |x_{2n}(t, a, \rho)| \leq (2n)!R_1^n|\rho|^{-n},$$

$$(1.3.15) \quad \left| \frac{dx_{2n}}{dt}(t, a, \rho) \right| \leq (2n)!R_1^n|\rho|^{-n}.$$

*Remark 1.3.1.* Using this occasion we make a correction in our announcement paper [KKT1, (1.105) and (1.106)]; the exponent of  $|\rho|$  should be  $-n$ , not  $-n+1$ . We note that the exponent of  $|\rho|$  in [KKT1, (1.103) and (1.104)] should be kept intact, i.e.,  $-n+1$ .

*Proof.* It suffices to show (1.3.10) and (1.3.11), as the other relations have been explicitly stated in Section 1.1 and Section 1.2. In what follows, by taking  $r$  sufficiently small, we assume

$$(1.3.16) \quad f(\pm a, a, \rho) \neq 0,$$

which the assumptions (1.3), (1.4) and (1.5) guarantee. Since  $A_0$ ,  $B_0$  and  $x_0$  satisfy

$$(1.3.17) \quad (x_0^2 - a^2)f = (t^2 - a^2)(x_0')^2(aA_0 + x_0B_0),$$

by letting  $t = \pm a$  in (1.3.17), we find that

$$(1.3.18) \quad x_0^2(\pm a, a, \rho) = a^2$$

holds. Since  $x_0^{(j)}(0, \rho) = 0$  ( $j = 0, 1$ ), it follows from (1.1.1.13) that

$$(1.3.19) \quad \begin{aligned} \frac{x_0(\pm a, a, \rho)}{a} &= \frac{x_0^{(0)}(\pm a, \rho)}{a} + x_0^{(1)}(\pm a, \rho) + a \sum_{j=2}^{\infty} x_0^{(j)}(\pm a, \rho) a^{j-2} \\ &\xrightarrow{a \rightarrow 0} \pm \frac{\partial x_0^{(0)}}{\partial t}(0, \rho) = \pm \frac{\rho}{B_0^{(0)}}. \end{aligned}$$

Hence (1.3.1) and (1.3.18) entail (1.3.11).

To confirm (1.3.10) we use  $s = x_0^{(0)}(t, \rho)$  as a coordinate. Take  $r_1$  and  $\varepsilon$  be sufficiently small so that  $x_0(s, a, \rho)$  is holomorphic on

$$(1.3.20) \quad \tilde{E}_{r_1+2\varepsilon, R_0}^1 = \{(s, a, \rho) \in \mathbb{C}^3 : |s| \leq r_1 + 2\varepsilon, 0 < |\rho| \leq r_1, R_0|a| \leq |\rho|\}.$$

Then, by taking  $R_0$  sufficiently large, we can assume that

$$(1.3.21) \quad |x_0(s, a, \rho) - s| < \varepsilon$$

holds on  $\tilde{E}_{r_1+2\varepsilon, R_0}^1$ . Therefore, for any  $\hat{s}$  in  $x_0(\tilde{E}_{r_1, R_0}^1)$ , we find  $|s - \hat{s}| > \varepsilon$  holds on  $\{s \in \mathbb{C} : |s| = r_1 + 2\varepsilon\}$ . Appealing to Rouché's theorem, we

find that  $x_0(s, a, \rho)$  is injective on  $\{s \in \mathbb{C} : |s| \leq r_1\}$ . By taking  $r$  so that  $x_0^{(0)}(t, \rho)$  is injective and satisfies  $|x_0^{(0)}(t, \rho)| \leq r_1$  on  $E_{r, R_0}^1$ , we obtain (1.3.10).  $\square$

*Remark 1.3.2.* When  $B_0^{(0)} = -\rho$ , some minor adjustments of signs etc. are needed at several points in Theorem 1.3.1. For the sake of the reader's convenience, we list up the formulas that require the adjustments below; each formula is appropriately modified and endowed with a new label obtained by adding ' to the original number of formulas. In accordance with the adjustments, (1.1.6) is also changed to

$$(1.1.6') \quad Q(t, a, \rho; \eta) \\ = \left( \frac{\partial x}{\partial t} \right)^2 \left( \frac{aA + xB}{x^2 - a^2} + \eta^{-2} \left( \frac{g_+(-a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2} \right) \right) \\ - \frac{1}{2} \eta^{-2} \{x; t\}.$$

$$(1.3.3') \quad A(a, \rho, \eta), B(a, \rho, \eta) \text{ and } x(t, a, \rho, \eta) \text{ satisfy (1.1.6').}$$

$$(1.3.5') \quad B_0(0, \rho) = -\rho.$$

$$(1.3.6') \quad \frac{\partial x_0}{\partial t}(0, 0, \rho) = -1.$$

$$(1.3.8') \quad x_0(t, a, \rho)|_{t=\pm a} = \mp a.$$

As is shown in [KT2], Theorem 1.3.1 entails the following

**Theorem 1.3.2.** *Let  $\hat{S}$  and  $\tilde{S}$  be a solution of*

$$(1.3.22) \quad \hat{S}^2 + \frac{\partial \hat{S}}{\partial t} = \eta^2 Q(t, a, \rho, \eta)$$

and

$$(1.3.23) \quad \tilde{S}^2 + \frac{\partial \tilde{S}}{\partial x} = \eta^2 \left( \frac{aA + xB}{x^2 - a^2} + \eta^{-2} \left( \frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2} \right) \right)$$

respectively, and suppose that

(1.3.24)

$$\arg \hat{S}_{-1}(t, a, \rho) = \arg \left( \frac{\partial x_0}{\partial t} S_{-1}(x_0(t, a, \rho), a, A_0(a, \rho), B_0(a, \rho)) \right)$$

holds. Then they satisfy

(1.3.25)  $\hat{S}_{\text{odd}}(t, a, \rho, \eta)$

$$= \left( \frac{\partial x}{\partial t} \right) \tilde{S}_{\text{odd}}(x(t, a, \rho, \eta), a, A(a, \rho, \eta), B(a, \rho, \eta), \eta),$$

where  $\hat{S}_{\text{odd}}$  and  $\tilde{S}_{\text{odd}}$  respectively be the odd part of  $\hat{S}$  and  $\tilde{S}$ .

We also have the following theorem (cf. [AKT1]):

**Theorem 1.3.3.** Let  $\hat{\psi}_{\pm}(t, a, \rho, \eta)$  be WKB solutions of a generic (i.e.,  $a\rho \neq 0$ ) M2P1T equation (1.7) that are normalized at a simple pole  $t = a$  as

$$(1.3.26) \quad \hat{\psi}_{\pm}(t, a, \rho, \eta) = \frac{1}{\sqrt{\hat{S}_{\text{odd}}}} \exp \left( \pm \int_a^t \hat{S}_{\text{odd}} dt \right),$$

and let  $\tilde{\psi}_{\pm}(x, a, A, B, \eta)$  denote WKB solutions of the Mathieu equation

$$(1.3.27) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{aA + xB}{x^2 - a^2} + \eta^{-2} \left( \frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2} \right) \right) \right) \tilde{\psi} = 0$$

which are normalized at a simple pole  $x = a$  as

$$(1.3.28) \quad \tilde{\psi}_{\pm}(x, a, A, B, \eta) = \frac{1}{\sqrt{\tilde{S}_{\text{odd}}}} \exp \left( \pm \int_a^x \tilde{S}_{\text{odd}} dx \right).$$

Then  $\hat{\psi}_{\pm}$  and  $\tilde{\psi}_{\pm}$  satisfy the following relation on the set  $E_{r, R_0}^1$  given by (1.3.5):

(1.3.29)  $\hat{\psi}_{\pm}(t, a, \rho, \eta)$

$$= \left( \frac{\partial x}{\partial t} \right)^{-1/2} \tilde{\psi}_{\pm}(x(t, a, \rho, \eta), a, A(a, \rho, \eta), B(a, \rho, \eta), \eta),$$

where  $x(t, a, \rho, \eta)$ ,  $A(a, \rho, \eta)$  and  $B(a, \rho, \eta)$  are the series given in Theorem 1.3.1.

## 2 Reduction of the Mathieu equation to the Legendre equation near its simple poles

The main purpose of this section is to construct a transformation that brings the Mathieu equation

$$(2.1) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{aA + xB}{x^2 - a^2} + \eta^{-2} \left( \frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2} \right) \right) \right) \tilde{\psi} = 0$$

with genuine constants  $A(\neq 0)$  and  $B$  to the following Legendre equation

$$(2.2) \quad \left( \frac{d^2}{dz^2} - \eta^2 \left( \frac{a\Lambda^2}{z^2 - a^2} + \eta^{-1} \frac{\sqrt{a}\Lambda}{z^2 - a^2} + \eta^{-2} \frac{az\nu + a^2(\mu^2 - 1)}{(z^2 - a^2)^2} \right) \right) \phi = 0$$

on a neighborhood of the line segment connecting two simple poles at  $x = \pm a$ .

We note that introducing the large parameter  $\eta$  as in (2.2) to the classical Legendre equation is a natural one from the WKB-theoretic viewpoint; an elementary evidence for the naturalness is given by the fact that the WKB solutions  $\psi_{\pm}$  of (2.2) with  $\nu = 0$  and  $\mu^2 = 1/4$  is expressed in a closed form, i.e.,

$$(2.3) \quad \left( \eta\sqrt{a}\Lambda + \frac{1}{2} \right)^{-1/2} (z^2 - a^2)^{1/4} \left( \frac{z + \sqrt{z^2 - a^2}}{a} \right)^{\pm(\eta\sqrt{a}\Lambda + 1/2)},$$

which forms a counterpart of the interesting formula for  $P_{\sqrt{a}\Lambda}^{\pm 1/2}$  and  $Q_{\sqrt{a}\Lambda}^{\pm 1/2}$  ([Er, vol. I, p.150])). This naturalness seems to have enabled Koike ([Ko4]) to find the explicit form of the Voros coefficient for (2.2), of which we will make essential use in Section 4. However, there is one technical problem with the equation (2.2); it contains a term with degree 1 in  $\eta$ . Although the appearance of degree 1 (or, more generally, an odd degree part) in  $\eta$  is natural from the viewpoint of the general theory of simple-pole type operators (cf. e.g. [KKoT]), it is somewhat unhandy in this paper; the equations we are dealing with in this paper contain only even degree terms in  $\eta$ . Hence, as an auxiliary equation we consider the following equation:

$$(2.4) \quad \left( \frac{d^2}{dz^2} - \eta^2 \left( \frac{a\Gamma}{z^2 - a^2} + \eta^{-2} \left( \frac{g_+(a)}{(z-a)^2} + \frac{g_-(-a)}{(z+a)^2} \right) \right) \right) \psi = 0,$$

which can be smoothly related with (2.1). We will show the WKB-theoretic equivalence of (2.2) and (2.4) later in Proposition 2.1. Thus our first task is to construct the transformation series

$$(2.5) \quad z(x, a, A, B, \eta) = \sum_{n=0}^{\infty} z_{2n}(x, a, A, B) \eta^{-2n}$$

and

$$(2.6) \quad \Gamma(a, A, B, \eta) = \sum_{n=0}^{\infty} \Gamma_{2n}(a, A, B) \eta^{-2n}$$

so that they satisfy

$$(2.7) \quad \begin{aligned} & \frac{aA + xB}{x^2 - a^2} + \eta^{-2} \left( \frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2} \right) \\ &= \left( \frac{\partial z}{\partial x} \right)^2 \left( \frac{a\Gamma}{z^2 - a^2} + \eta^{-2} \left( \frac{g_+(a)}{(z-a)^2} + \frac{g_-(-a)}{(z+a)^2} \right) \right) \\ & \quad - \frac{1}{2} \eta^{-2} \{z; x\}. \end{aligned}$$

In order to attain the required reduction of the Mathieu equation to the Legendre equation near its simple poles, we need several delicate properties of the series including their domains of definition and estimates. Hence the precise target is to prove the following

**Theorem 2.1.** *There exist holomorphic functions  $z_{2n}(x, a, A, B)$  and  $\Gamma_{2n}(a, A, B)$  on*

$$(2.8) \quad E_{r_1, r_2}^2 = \{(x, a, A, B) \in \mathbb{C}^4 : |x| < r_1|a|, a \neq 0, A \neq 0, |B| < r_2|A|\}$$

for some constants  $r_1 > 1$  and  $r_2 > 0$  such that  $z(x, a, A, B, \eta)$  and  $\Gamma(a, A, B, \eta)$  respectively given by (2.5) and (2.6) satisfy (2.7) and the following conditions there:

$$(2.9) \quad \text{the function } z_0(x, a, A, B) \text{ of } x \text{ is injective on } D_{r_1|a|} = \{x \in \mathbb{C} : |x| < r_1|a|\} \text{ for fixed } a, A \text{ and } B,$$

$$(2.10) \quad z_0(\pm a, a, A, B) = \pm a,$$

$$(2.11) \quad \frac{\partial z_0}{\partial x}(x, a, A, B) \neq 0.$$

Furthermore they satisfy the following estimates: for any  $h > 0$  we can take sufficiently small  $\delta > 0$  so that

$$(2.12) \quad |z_{2n}(x, a, A, B)| \leq (2n)!h^n |aA|^{-n},$$

$$(2.13) \quad |\Gamma_{2n}(a, A, B)| \leq (2n)!h^n |aA|^{-n}$$

hold on  $E_{r_1, \delta}^2$  for  $n \geq 1$ .

In order to explain the geometric meaning of Theorem 2.1, we give some remarks before beginning its proof.

*Remark 2.1.* Since the two simple poles of (2.1) are contained in  $D_{r_1|a|}$ , Theorem 2.1 guarantees that the reduction of (2.1) to (2.4) is successful on a full neighborhood of the line segment joining these two poles. On the other hand, Theorem 2.1 does not say anything about the simple turning point of (2.1).

*Remark 2.2.* Two simple poles at  $x = \pm a$  and the simple turning point at  $x = -aA/B$  all merge at the origin when  $a$  tends to 0. But, by taking  $B/A$  sufficiently small, we can regard that the turning point is sufficiently far away from the two simple poles in the scale of  $a$ .

*Proof of Theorem 2.1.* Let  $\tilde{x}, \tilde{z}, \tilde{B}$  and  $\tilde{\eta}$  be

$$(2.14) \quad \tilde{x} = x/a,$$

$$(2.15) \quad \tilde{z} = z/a,$$

$$(2.16) \quad \tilde{B} = B/A,$$

$$(2.17) \quad \tilde{\eta} = \sqrt{aA}\eta,$$

then (2.1) is rewritten as follows:

$$(2.18) \quad \left( \frac{d^2}{d\tilde{x}^2} - \tilde{\eta}^2 \left( \frac{1 + \tilde{x}\tilde{B}}{\tilde{x}^2 - 1} + \tilde{\eta}^{-2} \left( \frac{g_+(a)}{(\tilde{x} - 1)^2} + \frac{g_-(-a)}{(\tilde{x} + 1)^2} \right) \right) \right) \tilde{\psi} = 0.$$

Hence if we construct

$$(2.19) \quad \tilde{z}(\tilde{x}, \tilde{B}, \tilde{\eta}) = \sum_{n=0}^{\infty} \tilde{z}_{2n}(\tilde{x}, \tilde{B}) \tilde{\eta}^{-2n}$$



and

$$(2.20) \quad \tilde{\Gamma}(\tilde{B}, \tilde{\eta}) = \sum_{n=0}^{\infty} \tilde{\Gamma}_{2n}(\tilde{B}) \tilde{\eta}^{-2n}$$

so that they satisfy

$$(2.21) \quad \begin{aligned} & \frac{1 + \tilde{x}\tilde{B}}{\tilde{x}^2 - 1} + \tilde{\eta}^{-2} \left( \frac{g_+(a)}{(\tilde{x} - 1)^2} + \frac{g_-(-a)}{(\tilde{x} + 1)^2} \right) \\ &= \left( \frac{\partial \tilde{z}}{\partial \tilde{x}} \right)^2 \left( \frac{\tilde{\Gamma}}{\tilde{z}^2 - 1} + \tilde{\eta}^{-2} \left( \frac{g_+(a)}{(\tilde{z} - 1)^2} + \frac{g_-(-a)}{(\tilde{z} + 1)^2} \right) \right) \\ & \quad - \frac{1}{2} \tilde{\eta}^{-2} \{ \tilde{z}; \tilde{x} \}, \end{aligned}$$

then we find

$$(2.22) \quad z(x, a, A, B, \eta) = a\tilde{z}(x/a, B/A, \sqrt{aA}\eta)$$

and

$$(2.23) \quad \Gamma(a, A, B, \eta) = A\tilde{\Gamma}(B/A, \sqrt{aA}\eta)$$

satisfy (2.7).

Therefore it suffices to show the following properties of  $\tilde{z}$  and  $\tilde{\Gamma}$ :

$$(2.24) \quad \begin{aligned} & \tilde{z}_{2n}(\tilde{x}, \tilde{B}) \text{ and } \tilde{\Gamma}_{2n}(\tilde{B}) \text{ are holomorphic on} \\ & \tilde{E}_{r_1, r_2}^2 = \left\{ (\tilde{x}, \tilde{B}) \in \mathbb{C}^2 : |\tilde{x}| \leq r_1, |\tilde{B}| \leq r_2 \right\} \\ & \text{for some positive constants } r_1 > 1 \text{ and } r_2 > 0, \end{aligned}$$

$$(2.25) \quad \begin{aligned} & \text{the function } \tilde{z}_0(\tilde{x}, \tilde{B}) \text{ of } \tilde{x} \text{ is injective on } D_{r_1} = \{ \tilde{x} \in \mathbb{C} : \\ & |\tilde{x}| \leq r_1 \} \text{ for fixed } \tilde{B} \text{ with } |\tilde{B}| \leq r_2, \end{aligned}$$

$$(2.26) \quad \tilde{z}_0(\pm 1, \tilde{B}) = \pm 1,$$

$$(2.27) \quad \frac{\partial \tilde{z}_0}{\partial \tilde{x}}(\tilde{x}, \tilde{B}) \neq 0 \quad \text{on } D_{r_1}$$

and they satisfy the following estimates: for any  $h > 0$  we can take sufficiently small  $\delta > 0$  so that

$$(2.28) \quad |\tilde{\Gamma}_{2n}(\tilde{B})| \leq (2n)!h^n,$$

$$(2.29) \quad |\tilde{z}_{2n}(\tilde{x}, \tilde{B})| \leq (2n)!h^n$$

hold on  $\tilde{E}_{r_1, \delta}^2$  for  $n \geq 1$ .

We first show (2.25), (2.26) and (2.27). Comparing the coefficients of  $\tilde{\eta}^0$  of (2.21), we find that  $\tilde{z}_0$  and  $\tilde{\Gamma}_0$  satisfy

$$(2.30) \quad \frac{1 + \tilde{x}\tilde{B}}{1 - \tilde{x}^2} = \left( \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \right)^2 \frac{\tilde{\Gamma}_0}{1 - \tilde{z}_0^2}.$$

Therefore we take  $\tilde{z}_0$  and  $\tilde{\Gamma}_0$  as follows:

$$(2.31) \quad \tilde{z}_0(\tilde{x}, \tilde{B}) = \cos \left( \frac{1}{\sqrt{\tilde{\Gamma}_0}} \int_1^{\tilde{x}} \sqrt{\frac{1 + \tilde{x}\tilde{B}}{1 - \tilde{x}^2}} d\tilde{x} \right),$$

$$(2.32) \quad \sqrt{\tilde{\Gamma}_0(\tilde{B})} = \frac{-1}{\pi} \int_1^{-1} \sqrt{\frac{1 + \tilde{x}\tilde{B}}{1 - \tilde{x}^2}} d\tilde{x}.$$

From (2.31) and (2.32), we immediately find that

$$(2.33) \quad \tilde{z}_0(\tilde{x}, 0) = \tilde{x},$$

$$(2.34) \quad \sqrt{\tilde{\Gamma}_0(0)} = 1$$

and (2.26) hold. Let  $r_1 > 1$  be a constant. Then, for any positive constant  $\varepsilon$ , we can take  $\delta > 0$  so that  $\tilde{z}_0(\tilde{x}, \tilde{B})$  and  $\tilde{\Gamma}_0(\tilde{B})$  are holomorphic on  $\tilde{E}_{r_1+2\varepsilon, \delta}^2 = \{(\tilde{x}, \tilde{B}) \in \mathbb{C}^2 : |\tilde{x}| \leq r_1 + 2\varepsilon, |\tilde{B}| \leq \delta\}$  and satisfy the following estimates there:

$$(2.35) \quad \max \left\{ |\tilde{z}_0(\tilde{x}, \tilde{B}) - \tilde{x}|, \left| \frac{\partial \tilde{z}_0}{\partial \tilde{x}} - 1 \right|, |\tilde{\Gamma}_0(\tilde{B}) - 1| \right\} < \varepsilon.$$

Therefore, for any  $y \in \tilde{z}_0(\tilde{E}_{r_1, \delta}^2)$ , we find  $|\tilde{x} - y| > \varepsilon$  holds on  $\{\tilde{x} \in \mathbb{C} : |\tilde{x}| = r_1 + 2\varepsilon\}$  and hence, appealing to Rouché's theorem, we find that (2.25) holds for  $r_2 < \delta$ . Note that (2.27) follows also from (2.35).

Next we show (2.24). Comparing the coefficients of  $\tilde{\eta}^{-2n}$  ( $n \geq 1$ ) of (2.21), we obtain the following relations for  $(\tilde{z}_{2n}, \tilde{\Gamma}_{2n})$  ( $n \geq 1$ ):

$$(2.36) \quad \frac{2\tilde{\Gamma}_0}{\tilde{z}_0^2 - 1} \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \frac{\partial \tilde{z}_{2n}}{\partial \tilde{x}} - \left( \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \right)^2 \frac{2\tilde{z}_0 \tilde{\Gamma}_0}{(\tilde{z}_0^2 - 1)^2} \tilde{z}_{2n} + \left( \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \right)^2 \frac{\tilde{\Gamma}_{2n}}{\tilde{z}_0^2 - 1} = \tilde{\Phi}_{2n},$$

where  $\tilde{\Phi}_{2n}$  ( $n \geq 1$ ) is a sum of terms that are determined by  $(\tilde{z}_{2k}, \tilde{\Gamma}_{2k})$  ( $0 \leq k \leq n-1$ ). Multiplying both sides of (2.36) by  $(\tilde{z}_0^2 - 1)/(2\tilde{\Gamma}_0 \partial \tilde{z}_0 / \partial \tilde{x})$ , we can rewrite (2.36) as follows:

$$(2.37) \quad \frac{\partial \tilde{z}_{2n}}{\partial \tilde{x}} - \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \frac{\tilde{z}_0}{\tilde{z}_0^2 - 1} \tilde{z}_{2n} + \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \frac{\tilde{\Gamma}_{2n}}{2\tilde{\Gamma}_0} = \Phi_{2n}.$$

Now, we give the concrete form of  $\Phi_{2n}$  ( $n \geq 1$ ). The concrete form of  $\Phi_2$  is rather simple:

$$(2.38) \quad \Phi_2 = \frac{\tilde{z}_0^2 - 1}{2\tilde{\Gamma}_0} \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-1} \left\{ \frac{1}{2} \{ \tilde{z}_0, \tilde{x} \} + \left( \frac{g_+(a)}{(\tilde{x} - 1)^2} + \frac{g_-(-a)}{(\tilde{x} + 1)^2} \right) - \left( \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \right)^2 \left( \frac{g_+(a)}{(\tilde{z}_0 - 1)^2} + \frac{g_-(-a)}{(\tilde{z}_0 + 1)^2} \right) \right\}.$$

Then, to simplify the expression of  $\Phi_{2n}$  ( $n \geq 2$ ) and also the discussion given below, we introduce  $y_{2k}(\tilde{x}, \tilde{B})$  ( $k = 0, 1, \dots$ ) by

$$(2.39) \quad y_0(\tilde{x}, \tilde{B}) = \frac{\tilde{z}_0^2(\tilde{x}, \tilde{B}) - 1}{\tilde{x}^2 - 1},$$

$$(2.40) \quad y_{2k}(\tilde{x}, \tilde{B}) = \frac{1}{\tilde{x}^2 - 1} \sum_{l=0}^k \tilde{z}_{2l}(\tilde{x}, \tilde{B}) \tilde{z}_{2(k-l)}(\tilde{x}, \tilde{B}) \quad (k \geq 1).$$

We immediately see that they satisfy the following relation:

$$(2.41) \quad (\tilde{x}^2 - 1) \sum_{n=0}^{\infty} \tilde{\eta}^{-2n} y_{2n}(\tilde{x}, \tilde{B}) = \tilde{z}^2(\tilde{x}, \tilde{B}, \tilde{\eta}) - 1.$$

Further we use the following notation: for a multi-index  $\vec{\kappa} = (\kappa_1, \kappa_2, \dots, \kappa_\mu)$  in  $\mathbb{N}_0^\mu$  and for  $\kappa_j$ -dependent ( $j = 1, 2, \dots, \mu$ ) quantities  $X_{\kappa_j}$ , we define

$$(2.42) \quad |\vec{\kappa}|_\mu = \sum_{j=1}^{\mu} \kappa_j,$$

$$(2.43) \quad \sum_{|\vec{\kappa}|_\mu=k}^* X_{\kappa_1} \cdots X_{\kappa_\mu} = \begin{cases} 1 & \text{for } \mu = 0 \\ \sum_{\substack{|\vec{\kappa}|_\mu=k \\ \kappa_j \geq 1}} X_{\kappa_1} \cdots X_{\kappa_\mu} & \text{for } \mu \geq 1. \end{cases}$$

Using these notations, we can describe the concrete form of  $\Phi_{2n}$  ( $n \geq 2$ ) as follows:

$$(2.44) \quad \Phi_{2n} = \Phi_{2n}^{(1)} + \Phi_{2n}^{(2)} + \Phi_{2n}^{(3)},$$

where  $\Phi_{2n}^{(1)}$ ,  $\Phi_{2n}^{(2)}$  and  $\Phi_{2n}^{(3)}$  are

$$(2.45) \quad \begin{aligned} \Phi_{2n}^{(1)} = & -\frac{1}{2} \frac{d\tilde{z}_0}{d\tilde{x}} \sum_{\mu=2}^n \sum_{|\vec{\kappa}|_\mu=n}^* (-1)^\mu \frac{y_{2\kappa_1} \cdots y_{2\kappa_\mu}}{y_0^\mu} \\ & - \frac{1}{2\tilde{\Gamma}_0} \sum_{\substack{k_1+\dots+k_4=n \\ k_1, \dots, k_4 \leq n-1}} \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-1} \frac{d\tilde{z}_{2k_1}}{d\tilde{x}} \frac{d\tilde{z}_{2k_2}}{d\tilde{x}} \tilde{\Gamma}_{2k_3} \\ & \times \sum_{\mu=\min\{1, k_4\}}^{k_4} \sum_{|\vec{\kappa}|_\mu=k_4}^* (-1)^\mu \frac{y_{2\kappa_1} \cdots y_{2\kappa_\mu}}{y_0^\mu} \\ & + \frac{1}{2} \frac{1}{\tilde{z}_0^2 - 1} \frac{d\tilde{z}_0}{d\tilde{x}} \sum_{k_1+k_2=n}^* \tilde{z}_{2k_1} \tilde{z}_{2k_2}, \\ (2.46) \quad \Phi_{2n}^{(2)} = & \frac{\tilde{z}_0^2 - 1}{4\tilde{\Gamma}_0} \sum_{k_1+k_2=n-1} \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-2} \frac{d^3 \tilde{z}_{2k_1}}{d\tilde{x}^3} \\ & \times \sum_{\mu=\min\{1, k_2\}}^{k_2} \sum_{|\vec{\kappa}|_\mu=k_2}^* (-1)^\mu \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-\mu} \frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}} \cdots \frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}} \end{aligned}$$

$$\begin{aligned}
& -\frac{3(\tilde{z}_0^2-1)}{8\tilde{\Gamma}_0} \sum_{k_1+k_2+k_3=n-1} \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-3} \frac{d^2\tilde{z}_{2k_1}}{d\tilde{x}^2} \frac{d^2\tilde{z}_{2k_2}}{d\tilde{x}^2} \\
& \times \sum_{\mu=\min\{1,k_3\}}^{k_3} \sum_{|\vec{\kappa}|_\mu=k_3}^* (-1)^\mu (\mu+1) \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu} \frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}} \cdots \frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}, \\
(2.47) \quad \Phi_{2n}^{(3)} &= -\frac{\tilde{z}_0^2-1}{2\tilde{\Gamma}_0} \sum_{k_1+k_2+k_3=n-1} \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1} \frac{d\tilde{z}_{2k_1}}{d\tilde{x}} \frac{d\tilde{z}_{2k_2}}{d\tilde{x}} \frac{g_+(a)}{(\tilde{z}_0-1)^2} \\
& \times \sum_{\mu=\min\{1,k_3\}}^{k_3} \sum_{|\vec{\kappa}|_\mu=k_3}^* (-1)^\mu (\mu+1) \frac{\tilde{z}_{2\kappa_1} \cdots \tilde{z}_{2\kappa_\mu}}{(\tilde{z}_0-1)^\mu} \\
& -\frac{\tilde{z}_0^2-1}{2\tilde{\Gamma}_0} \sum_{k_1+k_2+k_3=n-1} \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1} \frac{d\tilde{z}_{2k_1}}{d\tilde{x}} \frac{d\tilde{z}_{2k_2}}{d\tilde{x}} \frac{g_-(-a)}{(\tilde{z}_0+1)^2} \\
& \times \sum_{\mu=\min\{1,k_3\}}^{k_3} \sum_{|\vec{\kappa}|_\mu=k_3}^* (-1)^\mu (\mu+1) \frac{\tilde{z}_{2\kappa_1} \cdots \tilde{z}_{2\kappa_\mu}}{(\tilde{z}_0+1)^\mu}.
\end{aligned}$$

Then we recursively determine  $(\tilde{z}_{2n}(\tilde{x}, \tilde{B}), \tilde{\Gamma}_{2n}(\tilde{B}))$  ( $n \geq 1$ ) as follows:

$$(2.48) \quad \tilde{\Gamma}_{2n}(\tilde{B}) = \frac{-2\tilde{\Gamma}_0}{\pi} \int_1^{-1} (1-\tilde{z}_0^2)^{-1/2} \Phi_{2n}(\tilde{x}, \tilde{B}) d\tilde{x},$$

$$\begin{aligned}
(2.49) \quad \tilde{z}_{2n}(\tilde{x}, \tilde{B}) &= (1-\tilde{z}_0^2)^{1/2} \int_1^{\tilde{x}} (1-\tilde{z}_0^2)^{-1/2} \left( \Phi_{2n}(\tilde{x}, \tilde{B}) - \frac{\tilde{\Gamma}_{2n}}{2\tilde{\Gamma}_0} \frac{d\tilde{z}_0}{d\tilde{x}} \right) d\tilde{x} \\
&= (1-\tilde{z}_0^2)^{1/2} \int_{-1}^{\tilde{x}} (1-\tilde{z}_0^2)^{-1/2} \left( \Phi_{2n}(\tilde{x}, \tilde{B}) - \frac{\tilde{\Gamma}_{2n}}{2\tilde{\Gamma}_0} \frac{d\tilde{z}_0}{d\tilde{x}} \right) d\tilde{x}.
\end{aligned}$$

Now we inductively confirm that  $(\tilde{z}_{2n}(\tilde{x}, \tilde{B}), \tilde{\Gamma}_{2n}(\tilde{B}))$  ( $n \geq 1$ ) satisfy (2.24), (2.37) and

$$(2.50) \quad \tilde{z}_{2n}(\pm 1, \tilde{B}) = 0.$$

We first confirm that  $(\tilde{z}_2(\tilde{x}, \tilde{B}), \tilde{\Gamma}_2(\tilde{B}))$  satisfies (2.24) and (2.50). From (2.27) we immediately see that  $\{\tilde{z}_0; \tilde{x}\}$  is holomorphic on  $\tilde{E}_{r_1, r_2}^2$ . Fur-

thermore, using (2.25) and (2.26), we find that

$$\begin{aligned}
(2.51) \quad & (\tilde{z}_0^2 - 1) \left( \frac{g_+(a)}{(\tilde{x} - 1)^2} - \left( \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \right)^2 \frac{g_+(a)}{(\tilde{z}_0 - 1)^2} \right) \\
&= g_+(a) \frac{\tilde{z}_0 + 1}{\tilde{z}_0 - 1} \left( \left( \frac{\tilde{z}_0 - 1}{\tilde{x} - 1} \right)^2 - \left( \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \right)^2 \right)
\end{aligned}$$

is holomorphic at  $\tilde{x} = 1$  and hence on  $\tilde{E}_{r_1, r_2}^2$ . By the same reasoning, the counterpart of (2.51) in  $\Phi_2$ , i.e.,

$$\begin{aligned}
(2.52) \quad & (\tilde{z}_0^2 - 1) \left( \frac{g_-(a)}{(\tilde{x} + 1)^2} - \left( \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \right)^2 \frac{g_-(a)}{(\tilde{z}_0 + 1)^2} \right) \\
&= g_-(a) \frac{\tilde{z}_0 - 1}{\tilde{z}_0 + 1} \left( \left( \frac{\tilde{z}_0 + 1}{\tilde{x} + 1} \right)^2 - \left( \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \right)^2 \right)
\end{aligned}$$

is also holomorphic on  $\tilde{E}_{r_1, r_2}^2$ . In conclusion  $\Phi_2$  is holomorphic on  $\tilde{E}_{r_1, r_2}^2$ . It is clear from the representation (2.48) (resp. (2.49)) of  $\tilde{\Gamma}_2$  (resp.  $\tilde{z}_2$ ) that they are holomorphic and satisfy (2.37) and (2.50) on  $\tilde{E}_{r_1, r_2}^2$ .

Next we confirm that  $(\tilde{z}_{2n}, \tilde{\Gamma}_{2n})$  satisfies (2.24), (2.37) and (2.50) under the assumption that  $(\tilde{z}_{2k}, \tilde{\Gamma}_{2k})$  ( $1 \leq k \leq n-1$ ) satisfy these properties. By the same reasoning as the case of  $n=1$ , it suffices to show that  $\Phi_{2n}$  is holomorphic on  $\tilde{E}_{r_1, r_2}^2$ . We first note that, since  $\tilde{z}_0$  satisfies (2.25) and (2.26),  $y_0$  is holomorphic and satisfies

$$(2.53) \quad y_0(\tilde{x}, \tilde{B}) \neq 0 \quad \text{on} \quad \tilde{E}_{r_1, r_2}^2.$$

Further the holomorphy of  $y_{2k}$  ( $1 \leq k \leq n-1$ ) follows from the induction hypothesis (2.50). Then the holomorphy of  $\Phi_{2n}^{(1)}$  and  $\Phi_{2n}^{(2)}$  immediately follows from the induction hypothesis also. On the other hand the seeming poles at  $\tilde{x} = \pm a$  that appear in (2.47) are cancelled out thanks to Lemma C.1 in Appendix C, and hence  $\Phi_{2n}^{(3)}$  is holomorphic on  $\tilde{E}_{r_1, r_2}^2$ . (Indeed, we can apply Lemma C.1 with  $w_0 = \tilde{z}_0 \pm 1$  and  $w_k = \tilde{z}_{2k}$  ( $k = 1, 2, \dots$ ) in this case.) Thus, we find that  $\Phi_{2n}$  is holomorphic on  $\tilde{E}_{r_1, r_2}^2$ . Then the induction proceeds, and hence we obtain (2.24) and (2.50) for  $n \geq 1$ .

Now we embark on the proof of the estimates (2.28) and (2.29). Let  $N$  be an arbitrarily large natural number. In order to derive these estimates, we introduce a new variable  $\zeta$  given by

$$(2.54) \quad \zeta = \exp \left[ \frac{1}{N} \log \left( \frac{\tilde{x}}{N} \right) \right]$$

and we consider a holomorphic function  $g(\tilde{x})$  on  $D_N = \{\tilde{x} \in \mathbb{C} : |\tilde{x}| \leq N\}$  as a holomorphic function  $g(N\zeta^N)$  on  $\{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$ .

*Remark 2.3.* As we will see below, to obtain (2.28) and (2.29) for arbitrarily small  $h$ , we will let  $N$  sufficiently large so that (2.65) holds. Then  $D_N$  becomes larger and larger as  $N$  increases. Still, the same reasoning as in the proof of (2.24), (2.25) and (2.27) guarantee that,

$$(2.55) \quad \text{for arbitrary large } N, \text{ we can take } \delta > 0 \text{ sufficiently small so that } \tilde{z}_{2n}(\tilde{x}, \tilde{B}) \text{ } (n = 0, 1, 2, \dots) \text{ are holomorphic on } \tilde{E}_{N,\delta}^2.$$

In what follows, we use the following notation: for a holomorphic function  $f(\zeta)$  on  $\{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$ , we define  $\|f\|_{\{\varepsilon\}}$  by

$$(2.56) \quad \|f\|_{\{\varepsilon\}} := \sup_{|\zeta| \leq 1-\varepsilon} |f(\zeta)|$$

for  $0 < \varepsilon < 1$ . Then our task is to show the following

**Lemma 2.1.** *There exist positive constants  $C_0(< 1)$  and  $C_1$  such that, for arbitrarily large natural number  $N$ , we can take a sufficiently small positive constant  $\delta$  (depending on  $N$ ) so that the following estimates hold for  $|\tilde{B}| \leq \delta$  and  $0 < \varepsilon \leq (2N)^{-1} \log N$ : for  $1 \leq k \leq N-1$ ,*

$$(2.57) \quad |\tilde{\Gamma}_{2k}| \leq C_0 N^{k-N} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k,$$

$$(2.58) \quad \|\tilde{z}_{2k}\|_{\{\varepsilon\}} \leq C_0 N^{k+1-N} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k,$$

$$(2.59) \quad \left\| \frac{d\tilde{z}_{2k}}{d\tilde{x}} \right\|_{\{\varepsilon\}} \leq C_0 N^{k-N} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k,$$

$$(2.60) \quad \left\| \frac{y_{2k}}{y_0} \right\|_{\{\varepsilon\}} \leq C_0 N^{k-N} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k,$$

and for  $k \geq N$ ,

$$(2.61) \quad |\tilde{\Gamma}_{2k}| \leq C_0 N^{-1} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k,$$

$$(2.62) \quad \|\tilde{z}_{2k}\|_{\{\varepsilon\}} \leq C_0(\varepsilon N)^{-2k}(2k)!(C_1 \log N)^k,$$

$$(2.63) \quad \left\| \frac{d\tilde{z}_{2k}}{d\tilde{x}} \right\|_{\{\varepsilon\}} \leq C_0 N^{-1}(\varepsilon N)^{-2k}(2k)!(C_1 \log N)^k,$$

$$(2.64) \quad \left\| \frac{y_{2k}}{y_0} \right\|_{\{\varepsilon\}} \leq C_0 N^{-1}(\varepsilon N)^{-2k}(2k)!(C_1 \log N)^k.$$

As the proof of Lemma 2.1 is delicate and lengthy, we describe its role in our whole reasoning before proving it; the proof of Lemma 2.1 will be given after we explain its role. Now a crucial point is that the estimate (2.28) (resp. (2.29)) we want to prove follows from (2.57) and (2.61) (resp. (2.58) and (2.62)). This can be confirmed in the following manner: Let  $h > 0$  be an arbitrarily small number. Then we take  $N$  so that it satisfies

$$(2.65) \quad \frac{4C_1}{\log N} < h.$$

By taking  $\varepsilon = (2N)^{-1} \log N$ , we obtain the following estimates from (2.57) and (2.61) for  $n \geq 1$ :

$$(2.66) \quad |\tilde{\Gamma}_{2n}(\tilde{B})| \leq C_0 N^{-1}(2n)! \left( \frac{4C_1}{\log N} \right)^n$$

for  $|\tilde{B}| \leq \delta$ , where  $\delta$  is a positive constant appearing in Lemma 2.1. By the same way, we obtain the following estimates from (2.58) and (2.62):

$$(2.67) \quad |\tilde{z}_{2n}(\tilde{x}, \tilde{B})| = |\tilde{z}_{2n}(N\zeta^N, \tilde{B})| \leq C_0(2n)! \left( \frac{4C_1}{\log N} \right)^n$$

for  $|\zeta| \leq 1 - (2N)^{-1} \log N$  and  $|\tilde{B}| \leq \delta$ . Here we note that, for sufficiently large  $N$ ,

$$(2.68) \quad |\tilde{x}| \geq N^{1/2}/2 \text{ holds when } |\zeta| = 1 - (2N)^{-1} \log N.$$

Indeed, (2.68) follows from the relation  $\tilde{x} = N\zeta^N$  and the following inequality:

$$(2.69) \quad N \left( 1 - \frac{\log N}{2N} \right)^N \geq \frac{1}{2} N \exp \left[ -\frac{1}{2} \log N \right] = \frac{1}{2} N^{1/2}$$



holds for sufficiently large  $N$ . Thus we can assume that (2.67) holds for  $|\tilde{x}| \leq N^{1/2}/2$ . Hence, by taking  $N$  so that it satisfies  $r_1 \leq N^{1/2}/2$  and (2.65), we obtain (2.28) and (2.29). In conclusion, we obtain Theorem 2.1. Thus the proof of Theorem 2.1 will be completed if we verify Lemma 2.1.

*Proof of Lemma 2.1.* To begin with we confirm that (2.57)  $\sim$  (2.60) hold for  $k = 1$ . We first show that  $\Phi_2$  satisfies the following estimates: there exists a positive constant  $\tilde{C}_0$  such that, for an arbitrary positive constant  $p > 1$ , we can take a positive constant  $\delta$  so that

$$(2.70) \quad \sup_{|\tilde{x}| \leq N} |\Phi_2(\tilde{x}, \tilde{B})| \leq \tilde{C}_0 N^{-p+1}$$

holds for  $|\tilde{B}| \leq \delta$ .

*Remark 2.4.* As (2.33) and (2.34) indicate, we readily find

$$(2.71) \quad \Phi_{2n}(\tilde{x}, 0) = 0 \quad \text{for } n \geq 1.$$

Therefore it is natural to expect that (2.70) holds by taking  $\delta$  sufficiently small depending on  $N$  and  $p$ .

Indeed, by taking  $\delta > 0$  sufficiently small, we may assume that  $\tilde{\Gamma}_0$ ,  $y_0$  and  $d\tilde{z}_0/d\tilde{x}$  are holomorphic on  $\tilde{E}_{N+N^p, \delta}^2$ . Furthermore, since  $\tilde{\Gamma}_0(0) = y_0(\tilde{x}, 0) = d\tilde{z}_0/d\tilde{x}(\tilde{x}, 0) = 1$ , by letting  $\delta > 0$  sufficiently small again, we may also assume that

$$(2.72) \quad \sup_{\substack{|\tilde{x}| \leq N+N^p \\ |\tilde{B}| \leq \delta}} \left\{ |(\tilde{\Gamma}_0)^{\pm 1}|, |(y_0)^{\pm 1}|, \left| \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{\pm 1} \right| \right\} \leq 2$$

holds. We fix  $\tilde{B}$  in the disc  $\{\tilde{B} : |\tilde{B}| \leq \delta\}$ . Then we obtain the following estimates for  $j = 1, 2, \dots$  from Cauchy's inequality:

$$(2.73) \quad \sup_{|\tilde{x}| \leq N} \left| \frac{d^j \tilde{z}_0}{d\tilde{x}^j} \right| \leq 2(j-1)! N^{-(j-1)p}.$$

Therefore, from (2.73) we obtain

$$(2.74) \quad |\{\tilde{z}_0; \tilde{x}\}| \leq \left| \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-1} \frac{d^3 \tilde{z}_0}{d\tilde{x}^3} \right| + \frac{3}{2} \left| \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-2} \left( \frac{d^2 \tilde{z}_0}{d\tilde{x}^2} \right)^2 \right|$$

$$\leq 32N^{-2p}$$

on  $D_N$ . In what follows, we fix  $p$  at  $N+2$  and take  $\delta$  sufficiently small so that (2.70) holds.

Next we derive the estimates of (2.51) on  $D_N$ . Since (2.51) is holomorphic on  $D_N$ , appealing to the maximum modulus principle, it suffices to estimate (2.51) on the boundary  $\partial D_N$  of  $D_N$ . Further, since  $g_{\pm}(x)$  is holomorphic at the origin, we can assume that

$$(2.75) \quad |g_{\pm}(\pm a)| \leq \tilde{C}_1$$

holds for some positive constant  $\tilde{C}_1$ . From the representation

$$(2.76) \quad \tilde{z}_0(\tilde{x}) = 1 + (\tilde{x} - 1) \frac{\partial \tilde{z}_0}{\partial \tilde{x}} - \int_1^{\tilde{x}} (\tilde{x} - 1) \frac{\partial^2 \tilde{z}_0}{\partial \tilde{x}^2} d\tilde{x}$$

of  $\tilde{z}_0(\tilde{x})$ , we obtain

$$(2.77) \quad \left( \frac{\tilde{z}_0 - 1}{\tilde{x} - 1} \right)^2 - \left( \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \right)^2 = - \frac{2}{\tilde{x} - 1} \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \int_1^{\tilde{x}} (\tilde{x} - 1) \frac{\partial^2 \tilde{z}_0}{\partial \tilde{x}^2} d\tilde{x} \\ + \left( \frac{1}{\tilde{x} - 1} \int_1^{\tilde{x}} (\tilde{x} - 1) \frac{\partial^2 \tilde{z}_0}{\partial \tilde{x}^2} d\tilde{x} \right)^2.$$

Here we note that it follows from (2.73) that the following estimates hold on  $\partial D_N$ :

$$(2.78) \quad \left| \frac{1}{\tilde{x} - 1} \int_1^{\tilde{x}} (\tilde{x} - 1) \frac{\partial^2 \tilde{z}_0}{\partial \tilde{x}^2} d\tilde{x} \right| \leq \frac{2(N+1)^2}{N-1} N^{-p}.$$

Further, by taking  $\delta$  sufficiently small, we may assume that (2.35) holds with  $\varepsilon = 1/2$  on  $D_N$ , and hence,

$$(2.79) \quad N - \frac{3}{2} \leq |\tilde{z}_0 \pm 1| \leq N + \frac{3}{2}$$

holds on  $\partial D_N$ . Then, combining (2.75), (2.78) and (2.79), we obtain the following estimates of (2.51):

$$(2.80) \quad \sup_{|\tilde{x}|=N} \left| g_+(a) \frac{\tilde{z}_0 + 1}{\tilde{z}_0 - 1} \left( \left( \frac{\tilde{z}_0 - 1}{\tilde{x} - 1} \right)^2 - \left( \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \right)^2 \right) \right|$$

$$\begin{aligned}
&\leq \tilde{C}_1 \frac{N+3/2}{N-3/2} \left( \frac{8(N+1)^2}{N-1} N^{-p} + \left( \frac{2(N+1)^2}{N-1} N^{-p} \right)^2 \right) \\
&\leq 320 \tilde{C}_1 N^{-p+1}.
\end{aligned}$$

By the same reasoning, we obtain the same estimates with (2.80) for (2.52). Therefore, combining (2.72), (2.74), (2.79) and (2.80), we obtain (2.70).

Now we derive (2.57)  $\sim$  (2.60) from (2.70) for  $k = 1$ . Since  $0 < \varepsilon \leq (2N)^{-1} \log N$ , (2.57) for  $k = 1$  immediately follows from the representation (2.48) of  $\tilde{\Gamma}_2$ , (2.70) for  $C_1 \geq (C_0)^{-1} \tilde{C}_0$  as follows:

$$\begin{aligned}
(2.81) \quad |\tilde{\Gamma}_2(\tilde{B})| &\leq 4 \left| \frac{\tilde{\Gamma}_0}{\pi} \right| \sup_{|\tilde{x}| \leq N} |\Phi_2| \int_1^{-1} |1 - \tilde{z}_0^2|^{-1/2} \left| \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-1} \right| |d\tilde{z}_0| \\
&\leq 8 \tilde{C}_0 N^{-p+1} \\
&\leq 8 \tilde{C}_0 N^{-p+1} (\varepsilon N)^{-2} 2^{-2} (\log N)^2 \\
&\leq C_0 N^{-p+2} (\varepsilon N)^{-2} 2! (C_0)^{-1} \tilde{C}_0 \log N.
\end{aligned}$$

Next we consider the estimates of  $\tilde{z}_2$ . Since  $\tilde{z}_2$  is holomorphic on  $D_N$ , it suffices to estimate it for  $\tilde{x} \in \partial D_N$ . We obtain the following estimates from (2.49) and (2.70) for  $\tilde{x} \in \partial D_N \cap \{\operatorname{Re} \tilde{x} \geq 0\}$ :

$$\begin{aligned}
(2.82) \quad |\tilde{z}_2(\tilde{x}, \tilde{B})| &\leq |1 - \tilde{z}_0^2|^{1/2} \left( 2 \sup_{|\tilde{x}| \leq N} |\Phi_2| + \frac{|\tilde{\Gamma}_2|}{2|\tilde{\Gamma}_0|} \right) \int_1^{\tilde{z}_0(\tilde{x})} |1 - \tilde{z}_0^2|^{-1/2} |d\tilde{z}_0| \\
&\leq 5(2N+3) \tilde{C}_0 N^{-p+1} \tilde{C}_2 \log N \\
&\leq 20 \tilde{C}_0 \tilde{C}_2 N^{-p+2} \log N \\
&\leq C_0 N^{-p+4} (\varepsilon N)^{-2} 2! (C_0)^{-1} 5 \tilde{C}_0 \tilde{C}_2 \log N,
\end{aligned}$$

where the integration path is taken as a straight line segment that connects 1 and  $\tilde{x}$ ; thus this choice of the integration path together with the assumption on  $\tilde{x}$  enables us to dominate the multivalued integral

in the following manner:

$$(2.83) \quad \sup_{\substack{\pm \operatorname{Re} \tilde{x} \geq 0 \\ |\tilde{x}| \leq N}} \int_{\pm 1}^{\tilde{z}_0(\tilde{x})} |1 - \tilde{z}_0^2|^{-1/2} |d\tilde{z}_0| \leq \tilde{C}_2 \log N,$$

where  $\tilde{C}_2$  is a positive constant that is independent of  $N$ . Using the second representation of (2.49), we find that  $\tilde{z}_2$  satisfies the same estimates with (2.82) for  $\tilde{x} \in \partial D_N \cap \{-\operatorname{Re} \tilde{x} \geq 0\}$ . Therefore (2.82) holds on  $D_N$ . Then, since  $|N\zeta^N| \leq N$  for  $|\zeta| \leq 1 - \varepsilon$ , by taking  $C_1 \geq (C_0)^{-1} 5\tilde{C}_0\tilde{C}_2$ , we immediately have (2.58) for  $k = 1$ . Further, from (2.37), (2.70), (2.81) and (2.82), we obtain the following estimates on  $D_N$ :

$$(2.84) \quad \begin{aligned} \left| \frac{\partial \tilde{z}_2}{\partial \tilde{x}} \right| &\leq \left| \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \frac{\tilde{z}_0}{\tilde{z}_0^2 - 1} \tilde{z}_2 \right| + \left| \frac{\partial \tilde{z}_0}{\partial \tilde{x}} \frac{\tilde{\Gamma}_2}{2\tilde{\Gamma}_0} \right| + |\Phi_2| \\ &\leq \frac{N + 1/2}{(N - 3/2)^2} 40\tilde{C}_0\tilde{C}_2 N^{-p+2} \log N + 16\tilde{C}_0 N^{-p+1} + \tilde{C}_0 N^{-p+1} \\ &\leq \tilde{C}_0 (320\tilde{C}_2 \log N + 17) N^{-p+1} \\ &\leq C_0 N^{-p+3} (\varepsilon N)^{-2} 2! (2C_0)^{-1} \tilde{C}_0 (320\tilde{C}_2 + 17) \log N. \end{aligned}$$

Therefore, by taking  $C_1 \geq (2C_0)^{-1} \tilde{C}_0 (320\tilde{C}_2 + 17)$ , we obtain (2.59) for  $k = 1$ . Finally, from (2.40) and (2.82), we obtain the following estimates on  $D_N$ :

$$(2.85) \quad \begin{aligned} |y_2| &\leq \left| \frac{2\tilde{z}_0}{\tilde{x}^2 - 1} \right| |\tilde{z}_2| \\ &\leq \frac{N + 1/2}{(N - 1)^2} 40\tilde{C}_0\tilde{C}_2 N^{-p+2} \log N \\ &\leq C_0 N^{-p+3} (\varepsilon N)^{-2} 2! (C_0)^{-1} \tilde{C}_0 160\tilde{C}_2 \log N. \end{aligned}$$

Hence we obtain (2.60) for  $k = 1$  with  $C_1 \geq (C_0)^{-1} \tilde{C}_0 160\tilde{C}_2$ . In conclusion, we obtain (2.57)  $\sim$  (2.60) for  $k = 1$ . Here we remark that, from the discussion above,

(2.86) we can take  $C_0 > 0$  arbitrarily small by taking  $C_1$  sufficiently large.

Next we show (2.57)  $\sim$  (2.60) for  $k = n$  ( $2 \leq n \leq N - 1$ ) under the assumption that these estimates hold for  $1 \leq k \leq n - 1$ . We first confirm the following estimates:

$$(2.87) \quad \|\Phi_{2n}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0^2 N^{n-N} (\varepsilon N)^{-2n} (2n)! C_1^n (\log N)^{n-1},$$

where  $\tilde{C}_0$  is a positive constant that is independent of  $n$ ,  $N$  and  $\varepsilon$ . Let us consider the first term of  $\Phi_{2n}^{(1)}$ . From Lemma 1.2.2 and the induction hypothesis, we obtain the following estimates:

$$(2.88) \quad \begin{aligned} & \left\| \frac{1}{2} \frac{d\tilde{z}_0}{d\tilde{x}} \sum_{\mu=2}^n \sum_{|\vec{\kappa}|_\mu=n}^* (-1)^\mu \frac{y_{2\kappa_1} \cdots y_{2\kappa_\mu}}{y_0^\mu} \right\|_{\{\varepsilon\}} \\ & \leq \frac{1}{2} \left\| \frac{d\tilde{z}_0}{d\tilde{x}} \right\|_{\{\varepsilon\}} \sum_{\mu=2}^n \sum_{|\vec{\kappa}|_\mu=n}^* \left\| \frac{y_{2\kappa_1}}{y_0} \right\|_{\{\varepsilon\}} \cdots \left\| \frac{y_{2\kappa_\mu}}{y_0} \right\|_{\{\varepsilon\}} \\ & \leq \sum_{\mu=2}^n \sum_{|\vec{\kappa}|_\mu=n}^* C_0^\mu N^{n-\mu N} (\varepsilon N)^{-2n} (2\kappa_1)! \cdots (2\kappa_\mu)! (C_1 \log N)^n \\ & \leq \sum_{\mu=2}^n (4C_0)^\mu (2n - \mu + 1)! N^{n-\mu N} (\varepsilon N)^{-2n} (C_1 \log N)^n \\ & \leq N^n (\varepsilon N)^{-2n} (2n - 1)! (C_1 \log N)^n \sum_{\mu=2}^n \frac{(4C_0 N^{-N})^\mu}{(\mu - 2)!} \\ & \leq 16e^{4C_0} C_0^2 N^{n-2N} (\varepsilon N)^{-2n} (2n - 1)! (C_1 \log N)^n. \end{aligned}$$

Here we note that the same reasoning as in (2.88) entails the following estimates for  $1 \leq k \leq n - 1$ :

$$(2.89) \quad \begin{aligned} & \left\| \sum_{\mu=1}^k \sum_{|\vec{\kappa}|_\mu=k}^* (-1)^\mu \frac{y_{2\kappa_1} \cdots y_{2\kappa_\mu}}{y_0^\mu} \right\|_{\{\varepsilon\}} \\ & \leq 4e^{4C_0} C_0 N^{k-N} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k. \end{aligned}$$

Next we consider the second term of  $\Phi_{2n}^{(1)}$ . Since at least two of  $k_j$ 's are non-zero, the factor  $C_0 N^{-N}$  appears at least twice in the estimation of the term. For example, the following part of the term with  $k_2 = k_3 = 0$

is one of the essential terms in the estimation:

$$(2.90) \quad \left\| \frac{1}{2} \sum_{\substack{k_1+k_4=n \\ 1 \leq k_1, k_4 \leq n-1}} \frac{d\tilde{z}_{2k_1}}{d\tilde{x}} \sum_{\mu=1}^{k_4} \sum_{|\tilde{\kappa}|_\mu=k_4}^* (-1)^\mu \frac{y_{2\kappa_1} \cdots y_{2\kappa_\mu}}{y_0^\mu} \right\|_{\{\varepsilon\}} \\ \leq 8e^{4C_0} C_0^2 N^{n-2N} (\varepsilon N)^{-2n} (2n-1)! (C_1 \log N)^n.$$

On the other hand, when at least three of  $k_1, \dots, k_4$  are non-zero, the factor  $C_0 N^{-N}$  appears at least three times. Then, since  $C_0 N^{-N} \ll 1$ , we obtain better estimates than (2.90) for these terms. Therefore the second term of  $\Phi_{2n}^{(1)}$  satisfies the following estimates for some positive constant  $\tilde{C}_0$ :

$$(2.91) \quad \left\| \frac{1}{2\tilde{\Gamma}_0} \sum_{\substack{k_1+\dots+k_4=n \\ k_1, \dots, k_4 \leq n-1}} \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-1} \frac{d\tilde{z}_{2k_1}}{d\tilde{x}} \frac{d\tilde{z}_{2k_2}}{d\tilde{x}} \tilde{\Gamma}_{2k_3} \right. \\ \times \left. \sum_{\mu=\min\{1, k_4\}}^{k_4} \sum_{|\tilde{\kappa}|_\mu=k_4}^* (-1)^\mu \frac{y_{2\kappa_1} \cdots y_{2\kappa_\mu}}{y_0^\mu} \right\|_{\{\varepsilon\}} \\ \leq \tilde{C}_0 C_0^2 N^{n-2N} (\varepsilon N)^{-2n} (2n-1)! (C_1 \log N)^n.$$

Finally, since  $|\tilde{z}_0^2 - 1| \geq (|\tilde{x}| - 3/2)^2 \geq N/8$  holds for  $|\zeta| = 1 - \varepsilon$  ( $0 < \varepsilon \leq (2N)^{-1} \log N$ ) (cf. (2.68) and (2.79)), the estimates of the third term of  $\Phi_{2n}^{(1)}$  follows from the maximum modulus principle and the induction hypothesis as follows:

$$(2.92) \quad \left\| \frac{1}{2} \frac{1}{\tilde{z}_0^2 - 1} \frac{d\tilde{z}_0}{d\tilde{x}} \sum_{k_1+k_2=n}^* \tilde{z}_{2k_1} \tilde{z}_{2k_2} \right\|_{\{\varepsilon\}} \\ \leq 8C_0^2 N^{n+1-2N} (\varepsilon N)^{-2n} (2n-1)! (C_1 \log N)^n.$$

We thus obtain the following estimates of  $\Phi_{2n}^{(1)}$  from (2.88), (2.91) and (2.92) for some positive constant  $\tilde{C}_0$ :

$$(2.93) \quad \|\Phi_{2n}^{(1)}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0^2 N^{n+1-2N} (\varepsilon N)^{-2n} (2n-1)! (C_1 \log N)^n \\ \leq \tilde{C}_0 C_0^2 N^{n-N} (\varepsilon N)^{-2n} (2n-1)! C_1^m (\log N)^{n-1}$$

Now we consider the estimation of  $\Phi_{2n}^{(2)}$ . We first show the following

**Lemma 2.2.** Let  $d\tilde{z}_{2k}/d\tilde{x}$  satisfy (2.59) for  $0 < \varepsilon \leq (2N)^{-1} \log N$ . Then the following inequalities hold:

(2.94)

$$\left\| \frac{d^2 \tilde{z}_{2k}}{d\tilde{x}^2} \right\|_{\{\varepsilon\}} \leq e^2 (1 - \varepsilon)^{-N} C_0 N^{k-1-N} (\varepsilon N)^{-2k-1} (2k+1)! (C_1 \log N)^k,$$

(2.95)

$$\begin{aligned} \left\| \frac{d^3 \tilde{z}_{2k}}{d\tilde{x}^3} \right\|_{\{\varepsilon\}} &\leq e^2 (1 - \varepsilon)^{-2N} C_0 N^{k-2-N} \\ &\quad \times \left( 1 + \frac{\log N}{2k+2} \right) (\varepsilon N)^{-2k-2} (2k+2)! (C_1 \log N)^k. \end{aligned}$$

*Proof.* Appealing to the maximum modulus principle, it is enough to show (2.94) and (2.95) for  $|\zeta| = 1 - \varepsilon$ . We first note the following relation:

$$(2.96) \quad \frac{d^2 \tilde{z}_{2k}}{d\tilde{x}^2} = \frac{\zeta^{-N+1}}{N^2} \frac{d}{d\zeta} \frac{d\tilde{z}_{2k}}{d\tilde{x}},$$

$$(2.97) \quad \frac{d^3 \tilde{z}_{2k}}{d\tilde{x}^3} = \left( \frac{\zeta^{-2N+2}}{N^4} \frac{d^2}{d\zeta^2} + \frac{-N+1}{N^4} \zeta^{-2N+1} \frac{d}{d\zeta} \right) \frac{d\tilde{z}_{2k}}{d\tilde{x}}.$$

We use the following representation:

$$(2.98) \quad \frac{d^j}{d\zeta^j} \frac{d\tilde{z}_{2k}}{d\tilde{x}} = \frac{j!}{2\pi\sqrt{-1}} \int_{|\tilde{\zeta}-\zeta|=(k+1)^{-1}\varepsilon} \frac{d\tilde{z}_{2k}}{d\tilde{x}} \frac{d\tilde{\zeta}}{(\tilde{\zeta}-\zeta)^{j+1}}.$$

We immediately find that the integral path of (2.98) is contained in  $|\tilde{\zeta}| \leq 1 - \tilde{\varepsilon}$  with  $\tilde{\varepsilon} = k\varepsilon/(k+1)$ . Since

$$\begin{aligned} (2.99) \quad \left\| \frac{d\tilde{z}_{2k}}{d\tilde{x}} \right\|_{\{\tilde{\varepsilon}\}} &\leq C_0 N^{k-N} (\tilde{\varepsilon} N)^{-2k} (2k)! (C_1 \log N)^k \\ &= C_0 N^{k-N} \left( 1 + \frac{1}{k} \right)^{2k} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k \\ &\leq e^2 C_0 N^{k-N} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k \end{aligned}$$

follows from (2.59), we obtain the following estimates from (2.98):

(2.100)

$$\left\| \frac{d^j}{d\zeta^j} \frac{d\tilde{z}_{2k}}{d\tilde{x}} \right\|_{\{\varepsilon\}} \leq j! (k+1)^j \varepsilon^{-j} e^2 C_0 N^{k-N} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k.$$

Then, from (2.100), we obtain the estimates of (2.96) and (2.97) as follows:

$$(2.101) \quad \left\| \frac{\zeta^{-N+1}}{N^2} \frac{d}{d\zeta} \frac{d\tilde{z}_{2k}}{d\tilde{x}} \right\|_{\{\varepsilon\}} \leq e^2(1-\varepsilon)^{-N} C_0 N^{k-1-N} (\varepsilon N)^{-2k-1} (2k+1)! (C_1 \log N)^k,$$

$$(2.102) \quad \left\| \left( \frac{\zeta^{-2N+2}}{N^4} \frac{d^2}{d\zeta^2} + \frac{-N+1}{N^4} \zeta^{-2N+1} \frac{d}{d\zeta} \right) \frac{d\tilde{z}_{2k}}{d\tilde{x}} \right\|_{\{\varepsilon\}} \leq e^2(1-\varepsilon)^{-2N} C_0 N^{k-2-N} (\varepsilon N)^{-2k-2} (2k+2)! (C_1 \log N)^k + 2e^2(1-\varepsilon)^{-2N} C_0 N^{k-2-N} (\varepsilon N)^{-2k-1} (2k+1)! (C_1 \log N)^k.$$

Since  $\varepsilon N \leq 2^{-1} \log N$ , (2.94) and (2.95) immediately follow from (2.101) and (2.102).  $\square$

We return to the estimation of  $\Phi_{2n}^{(2)}$ . Let us consider the first term of  $\Phi_{2n}^{(2)}$ . By the same reasoning as the estimation of (2.89), the following holds for  $k \geq 1$ :

$$(2.103) \quad \left\| \sum_{\mu=1}^k \sum_{|\tilde{\kappa}|_{\mu}=k}^* (-1)^{\mu} \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-\mu} \frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}} \cdots \frac{d\tilde{z}_{2\kappa_{\mu}}}{d\tilde{x}} \right\|_{\{\varepsilon\}} \leq 8e^{8C_0} C_0 N^{k-N} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k.$$

By the discussion similar to the estimation of (2.91), we find that the terms with  $k_1 = 0$  or  $k_2 = 0$  are essential in the estimation. In particular, since (2.73) holds, we see that the following term with  $k_2 = 0$  is the worst contribution:

$$(2.104) \quad \left\| \frac{\tilde{z}_0^2 - 1}{4\tilde{\Gamma}_0} \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-2} \frac{d^3 \tilde{z}_{2(n-1)}}{d\tilde{x}^3} \right\|_{\{\varepsilon\}} \leq 2(N^2(1-\varepsilon)^{2N} + 3)e^2(1-\varepsilon)^{-2N} C_0 N^{n-3-N} \times \left( 1 + \frac{\log N}{2n} \right) (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1}$$



$$\leq e^2 C_0 N^{n-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1}.$$

Therefore, having (2.73) in mind, we obtain the following estimates for some positive constant  $\tilde{C}_0$ :

$$(2.105) \quad \left\| \frac{\tilde{z}_0^2 - 1}{4\tilde{\Gamma}_0} \sum_{k_1+k_2=n-1} \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-2} \frac{d^3 \tilde{z}_{2k_1}}{d\tilde{x}^3} \right. \\ \times \sum_{\mu=\min\{1,k_2\}}^{k_2} \sum_{|\vec{\kappa}|_\mu=k_2}^* (-1)^\mu \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-\mu} \frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}} \cdots \frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}} \left. \right\|_{\{\varepsilon\}} \\ \leq \tilde{C}_0 C_0 N^{n-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1}.$$

By the same reasoning, we find that the following estimates for the second term of  $\Phi_{2n}^{(2)}$  follows from (2.94) and Lemma 1.2.2:

$$(2.106) \quad \left\| \frac{3(\tilde{z}_0^2 - 1)}{8\tilde{\Gamma}_0} \sum_{k_1+k_2+k_3=n-1} \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-3} \frac{d^2 \tilde{z}_{2k_1}}{d\tilde{x}^2} \frac{d^2 \tilde{z}_{2k_2}}{d\tilde{x}^2} \right. \\ \times \sum_{\mu=\min\{1,k_3\}}^{k_3} \sum_{|\vec{\kappa}|_\mu=k_3}^* (-1)^\mu (\mu+1) \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-\mu} \frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}} \cdots \frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}} \left. \right\|_{\{\varepsilon\}} \\ \leq \tilde{C}_0 C_0 N^{n-1-2N} (\varepsilon N)^{-2n} (2n-1)! (C_1 \log N)^{n-1}.$$

Thus we see that the following estimates hold for some positive constant  $\tilde{C}_0$ :

$$(2.107) \quad \|\Phi_{2n}^{(2)}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0 N^{n-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1}.$$

Finally we consider the estimation of  $\Phi_{2n}^{(3)}$ . Let us consider the first term of  $\Phi_{2n}^{(3)}$ . Since it is holomorphic on  $|\zeta| < 1$ , it suffices to estimate it for  $|\zeta| = 1 - \varepsilon$ . We first note that, since  $|\tilde{z}_0 - 1| \geq 4^{-1}\sqrt{N}$  holds on  $|\zeta| = 1 - \varepsilon$ , we find the following estimates:

$$(2.108) \quad \left\| \sum_{\mu=1}^k \sum_{|\vec{\kappa}|_\mu=k}^* (-1)^\mu (\mu+1) \frac{\tilde{z}_{2\kappa_1} \cdots \tilde{z}_{2\kappa_\mu}}{(\tilde{z}_0 - 1)^\mu} \right\|_{\{\varepsilon\}} \\ \leq 32e^{32C_0} C_0 N^{k-N+1/2} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k.$$

By the same discussion as the estimation of (2.91), we find that the terms with one of  $k_j$ 's being  $n - 1$  are essential in the estimation. In particular, since  $32e^{32C_0}N^{-1/2} \ll 1$ , comparison of (2.108) and (2.59) entails that the worst is the term with  $k_3 = n - 1$ , which can be estimated as follows:

$$\begin{aligned}
(2.109) \quad & \left\| \frac{\tilde{z}_0 + 1}{\tilde{z}_0 - 1} \frac{d\tilde{z}_0}{d\tilde{x}} \frac{g_+(a)}{2\tilde{\Gamma}_0} \sum_{\mu=1}^{n-1} \sum_{|\tilde{\kappa}|_\mu=n-1}^* (-1)^\mu (\mu + 1) \frac{\tilde{z}_{2\kappa_1} \cdots \tilde{z}_{2\kappa_\mu}}{(\tilde{z}_0 - 1)^\mu} \right\|_{\{\varepsilon\}} \\
& \leq \frac{N(1 - \varepsilon)^N + 3/2}{N(1 - \varepsilon)^N - 3/2} 4^3 \tilde{C}_1 e^{32C_0} C_0 \\
& \quad \times N^{n-N-1/2} (\varepsilon N)^{-2(n-1)} (2n - 2)! (C_1 \log N)^{n-1} \\
& \leq 4^3 \tilde{C}_1 e^{32C_0} C_0 N^{n-N-1/2} \frac{(\log N)^2}{n^2} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1} \\
& \leq 4^5 \tilde{C}_1 e^{32C_0} C_0 N^{n-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1},
\end{aligned}$$

where  $\tilde{C}_1$  is a positive constant that satisfies (2.75). In this way, we can obtain the estimates of the first term of  $\Phi_{2n}^{(3)}$ . On the other hand, we immediately find that the second term also satisfies the same estimates with the first term. Therefore we find that the following estimates hold for  $\Phi_{2n}^{(3)}$  with a positive constant  $\tilde{C}_0$ :

$$(2.110) \quad \|\Phi_{2n}^{(3)}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0 N^{n-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1}.$$

By taking  $C_1^{-1} \leq C_0$  and summing up (2.93), (2.107) and (2.110), we obtain (2.87).

Now we confirm (2.57)  $\sim$  (2.60) for  $k = n$ . We first note that the following estimates follow from (2.48) and (2.87):

$$\begin{aligned}
(2.111) \quad & |\tilde{\Gamma}_{2n}(\tilde{B})| \leq \frac{4|\tilde{\Gamma}_0|}{\pi} \int_1^{-1} |1 - \tilde{z}_0^2|^{-1/2} |d\tilde{z}_0| \|\Phi_{2n}\|_{\{\varepsilon\}} \\
& \leq 8\tilde{C}_0 C_0^2 N^{n-N} (\varepsilon N)^{-2n} (2n)! C_1^n (\log N)^{n-1}.
\end{aligned}$$

Then, by taking  $C_0$  sufficiently small so that they satisfy  $8\tilde{C}_0 C_0 < 1$ , we obtain (2.57). Next, from (2.49), (2.87) and (2.111), we obtain the

following estimates on  $\{|\tilde{x}| = N(1 - \varepsilon)^N\} \cap \{\operatorname{Re} \tilde{x} \geq 0\}$ :

(2.112)

$$\begin{aligned} |\tilde{z}_{2n}(\tilde{x}, \tilde{B})| &\leq |1 - \tilde{z}_0^2|^{1/2} \int_1^{\tilde{x}} |1 - \tilde{z}_0^2|^{-1/2} |d\tilde{z}_0| \left( 2\|\Phi_{2n}\|_{\{\varepsilon\}} + \frac{|\tilde{\Gamma}_{2n}|}{2|\tilde{\Gamma}_0|} \right) \\ &\leq 20(1 - \varepsilon)^N \tilde{C}_0 \tilde{C}_2 C_0^2 N^{n+1-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^n, \end{aligned}$$

where  $\tilde{C}_2$  is a positive constant appearing in (2.83). By the same discussion, we find from the second representation of (2.49) that (2.112) also holds on  $\{|\tilde{x}| = N(1 - \varepsilon)^N\} \cap \{-\operatorname{Re} \tilde{x} \geq 0\}$ . Since  $\tilde{z}_{2n}$  is holomorphic on  $\{|\tilde{x}| \leq N(1 - \varepsilon)^N\}$ , we see that (2.112) holds there. Hence, by taking  $C_0$  so that  $20\tilde{C}_0\tilde{C}_2C_0 < 1$  holds, we obtain (2.58) for  $k = n$ . Then, using the relation (2.37), we obtain the following estimates from (2.87), (2.111) and (2.112):

(2.113)

$$\begin{aligned} \left\| \frac{\partial \tilde{z}_{2n}}{\partial \tilde{x}} \right\|_{\{\varepsilon\}} &\leq 2 \frac{N(1 - \varepsilon)^N + 1/2}{(N(1 - \varepsilon)^N - 3/2)^2} \|\tilde{z}_{2n}\|_{\{\varepsilon\}} + 2|\tilde{\Gamma}_{2n}| + \|\Phi_{2n}\|_{\{\varepsilon\}} \\ &\leq (320\tilde{C}_0\tilde{C}_2 + 9\tilde{C}_0)C_0^2 N^{n-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^n. \end{aligned}$$

Therefore, by taking  $C_0$  so that  $(320\tilde{C}_0\tilde{C}_2 + 9\tilde{C}_0)C_0 < 1$  holds, we find that  $\tilde{z}_{2n}$  satisfies (2.59). Furthermore, by the maximum modulus principle, we obtain the following estimates from (2.40), (2.58), (2.69) and (2.112):

(2.114)

$$\begin{aligned} &\|y_{2n}\|_{\{\varepsilon\}} \\ &\leq \frac{1}{N^2(1 - \varepsilon)^{2N} - 1} \left( 2\|\tilde{z}_0\|_{\{\varepsilon\}} \|\tilde{z}_{2n}\|_{\{\varepsilon\}} + \sum_{k=1}^{n-1} \|\tilde{z}_{2k}\|_{\{\varepsilon\}} \|\tilde{z}_{2(n-k)}\|_{\{\varepsilon\}} \right) \\ &\leq \frac{4}{N^2(1 - \varepsilon)^{2N}} \left( 80N(1 - \varepsilon)^{2N} \tilde{C}_0 \tilde{C}_2 + 2n^{-1} N^{1-N} \right) \\ &\quad \times C_0^2 N^{n+1-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^n \\ &\leq 4(80\tilde{C}_0\tilde{C}_2 + 1)C_0^2 N^{n-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^n. \end{aligned}$$

Therefore, by taking  $C_0$  so that it satisfies  $4(80\tilde{C}_0\tilde{C}_2 + 1)C_0 < 1$ , we obtain (2.60) for  $k = n$ . Thus the induction proceeds and we obtain (2.57)  $\sim$  (2.60) for  $1 \leq k \leq N - 1$ .

Now, we confirm (2.61)  $\sim$  (2.64) for  $k \geq N$ . We first remark that, from (2.57)  $\sim$  (2.60), we find that (2.61)  $\sim$  (2.64) also hold for  $1 \leq k \leq N - 1$ . Hence we show (2.61)  $\sim$  (2.64) for  $k = n$  ( $n \geq N$ ) under the assumption that these estimates hold for  $1 \leq k \leq n - 1$ . By the same discussion with the derivation of (2.57)  $\sim$  (2.60) from (2.87), it suffices to show the following estimates:

$$(2.115) \quad \|\Phi_{2n}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0^2 N^{-1} (\varepsilon N)^{-2n} (2n)! C_1^n (\log N)^{n-1},$$

where  $\tilde{C}_0$  is some positive constant. We first confirm the following estimates:

$$(2.116) \quad \|\Phi_{2n}^{(1)}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0^2 N^{-1} (\varepsilon N)^{-2n} (2n - 1)! (C_1 \log N)^n.$$

Then, since  $\log N \leq n$ , we find that  $\Phi_{2n}^{(1)}$  satisfies (2.115). As in the derivation of (2.93), the following term is essential in the estimation of  $\Phi_{2n}^{(1)}$ :

$$(2.117) \quad \left\| \frac{1}{2} \frac{1}{\tilde{z}_0^2 - 1} \frac{d\tilde{z}_0}{d\tilde{x}} \sum_{k_1+k_2=n}^* \tilde{z}_{2k_1} \tilde{z}_{2k_2} \right\|_{\{\varepsilon\}} \\ \leq 8N^{-1} C_0^2 (\varepsilon N)^{-2n} (C_1 \log N)^n \sum_{k_1+k_2=n}^* (2k_1)! (2k_2)! \\ \leq 32C_0^2 N^{-1} (\varepsilon N)^{-2n} (2n - 1)! (C_1 \log N)^n.$$

Here we used the fact that  $|\tilde{z}_0^2 - 1| \geq N/8$  holds for  $|\zeta| = 1 - \varepsilon$ . In this way, we can show that the first and the second term of  $\Phi_{2n}^{(1)}$  also satisfy (2.116) by the same discussion with the estimation of (2.88) and (2.91).

Next, we show the following estimates:

$$(2.118) \quad \|\Phi_{2n}^{(2)}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0 N^{-1} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1}.$$

We first note that, by the same discussion with the proof of Lemma 2.2, we obtain the following estimates for  $k = 1, 2, \dots$  from (2.63):

$$(2.119) \quad \left\| \frac{d^2 \tilde{z}_{2k}}{d\tilde{x}^2} \right\|_{\{\varepsilon\}} \leq e^2 (1 - \varepsilon)^{-N} C_0 N^{-2} (\varepsilon N)^{-2k-1} (2k + 1)! (C_1 \log N)^k,$$

(2.120)

$$\left\| \frac{d^3 \tilde{z}_{2k}}{d\tilde{x}^3} \right\|_{\{\varepsilon\}} \leq e^2(1-\varepsilon)^{-2N} C_0 N^{-3} \times \left( 1 + \frac{\log N}{2k+2} \right) (\varepsilon N)^{-2k-2} (2k+2)! (C_1 \log N)^k.$$

Let us consider the first term of  $\Phi_{2n}^{(2)}$ , which is essential in the estimation of  $\Phi_{2n}^{(2)}$ . Since  $\log N \leq n$ , we find the following estimates from (2.120):

(2.121)

$$\left\| \frac{d^3 \tilde{z}_{2(n-1)}}{d\tilde{x}^3} \right\|_{\{\varepsilon\}} \leq \frac{3e^2}{2} (1-\varepsilon)^{-2N} C_0 N^{-3} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1}.$$

And, since  $\log N \leq N$ , we find the following estimates from (2.120) for  $1 \leq k \leq n-2$ :

(2.122)

$$\left\| \frac{d^3 \tilde{z}_{2(k-1)}}{d\tilde{x}^3} \right\|_{\{\varepsilon\}} \leq e^2(1-\varepsilon)^{-2N} C_0 N^{-2} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^{k-1}.$$

Then, by the same reasoning with the estimates of (2.105), we obtain the following estimates for the first term of  $\Phi_{2n}^{(2)}$ :

(2.123)

$$\begin{aligned} & \left\| \frac{\tilde{z}_0^2 - 1}{4\tilde{\Gamma}_0} \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-2} \left\{ \frac{d^3 \tilde{z}_{2(n-1)}}{d\tilde{x}^3} \right. \right. \\ & + \sum_{k_1+k_2=n-1}^* \frac{d^3 \tilde{z}_{2k_1}}{d\tilde{x}^3} \sum_{\mu=\min\{1,k_2\}}^{k_2} \sum_{|\tilde{\kappa}|_\mu=k_2}^* (-1)^\mu \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-\mu} \frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}} \dots \frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}} \\ & \left. + \frac{d^3 \tilde{z}_0}{d\tilde{x}^3} \sum_{\mu=1}^{n-1} \sum_{|\tilde{\kappa}|_\mu=k_2}^* (-1)^\mu \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-\mu} \frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}} \dots \frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}} \right\} \right\|_{\{\varepsilon\}} \\ & \leq 8N^2(1-\varepsilon)^{2N} \{ e^2(1-\varepsilon)^{-2N} C_0 N^{-3} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1} \\ & + 8e^{8C_0+2} (1-\varepsilon)^{-2N} C_0^2 N^{-3} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1} \\ & + 8e^{8C_0} C_0 N^{-1-p} (\varepsilon N)^{-2(n-1)} (2n-2)! (C_1 \log N)^{n-1} \} \end{aligned}$$

$$\leq 8C_0N^{-1}(\varepsilon N)^{-2n}(2n)!(C_1 \log N)^{n-1} \\ \times \{e^2 + 8e^{8C_0+2}C_0 + e^{8C_0}N^{-p+2}n^{-2}(\log N)^2\}.$$

Since  $p \geq N+2$ , we immediately find that the first term of  $\Phi_{2n}^{(2)}$  satisfies (2.118). In the same way, from (2.119), we can show the following estimates:

$$(2.124) \quad \left\| \frac{3(\tilde{z}_0^2 - 1)}{8\tilde{\Gamma}_0} \sum_{k_1+k_2+k_3=n-1} \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-3} \frac{d^2\tilde{z}_{2k_1}}{d\tilde{x}^2} \frac{d^2\tilde{z}_{2k_2}}{d\tilde{x}^2} \right. \\ \times \sum_{\mu=\min\{1,k_3\}}^{k_3} \sum_{|\tilde{\kappa}|_\mu=k_3}^* (-1)^\mu (\mu+1) \left( \frac{d\tilde{z}_0}{d\tilde{x}} \right)^{-\mu} \frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}} \dots \frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}} \left. \right\|_{\{\varepsilon\}} \\ \leq \tilde{C}_0 C_0 N^{-2}(\varepsilon N)^{-2n}(2n-1)!(C_1 \log N)^{n-1}.$$

Hence, from (2.123) and (2.124), we obtain (2.118).

Finally, by the same discussion with the estimation of (2.109), we obtain the following estimates:

$$(2.125) \quad \|\Phi_{2n}^{(3)}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0 N^{-3/2}(\varepsilon N)^{-2n+2}(2n-2)!(C_1 \log N)^{n-1}.$$

Then, since  $N^{1/2} \leq n$  and  $(\varepsilon N)^2 \leq n$ , we find that  $\Phi_{2n}^{(3)}$  satisfies (2.115).

Summing up, we have confirmed (2.61) ~ (2.64) for  $k = n$ . Thus the induction proceeds. This completes the proof of Lemma 2.1, completing the proof of Theorem 2.1. □

As is shown in [KT2], we can deduce the following Theorem 2.2 from Theorem 2.1:

**Theorem 2.2.** *Let  $\tilde{S}$  and  $S$  respectively be a solution of*

$$(2.126) \quad \tilde{S}^2 + \frac{\partial \tilde{S}}{\partial x} = \eta^2 \left( \frac{aA + xB}{x^2 - a^2} + \eta^{-2} \left( \frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2} \right) \right)$$

and

$$(2.127) \quad S^2 + \frac{\partial S}{\partial z} = \eta^2 \left( \frac{a\Gamma}{z^2 - a^2} + \eta^{-2} \left( \frac{g_+(a)}{(z-a)^2} + \frac{g_-(-a)}{(z+a)^2} \right) \right),$$

and suppose that

(2.128)

$$\arg \tilde{S}_{-1}(x, a, A, B) = \arg \left( \frac{\partial z_0}{\partial x} S_{-1}(z_0(x, a, A, B), a, \Gamma_0(a, A, B)) \right)$$

holds. Then they satisfy

(2.129)

$$\begin{aligned} \tilde{S}_{\text{odd}}(x, a, A, B, \eta) \\ = \left( \frac{\partial z}{\partial x} \right) S_{\text{odd}}(z(x, a, A, B, \eta), a, \Gamma(a, A, B, \eta), \eta) \end{aligned}$$

on  $E_{r_1, r_2}^2$ , where  $\tilde{S}_{\text{odd}}$  and  $S_{\text{odd}}$  respectively denote the odd part of  $\tilde{S}$  and  $S$ .

We also have the following

**Theorem 2.3.** Let  $\tilde{\psi}_{\pm}(x, a, A, B, \eta)$  be WKB solutions of the generic (i.e.,  $a \neq 0$ ) Mathieu equation (2.1) that are normalized at a simple pole  $x = a$  as

$$(2.130) \quad \tilde{\psi}_{\pm}(x, a, A, B, \eta) = \frac{1}{\sqrt{\tilde{S}_{\text{odd}}}} \exp \left( \pm \int_a^x \tilde{S}_{\text{odd}} dx \right),$$

and  $\psi_{\pm}(z, a, \Gamma, \eta)$  be WKB solutions of the Legendre equation (2.4) that is normalized at a simple pole  $z = a$  as

$$(2.131) \quad \psi_{\pm}(z, a, \Gamma, \eta) = \frac{1}{\sqrt{S_{\text{odd}}}} \exp \left( \pm \int_a^z S_{\text{odd}} dz \right).$$

Then they satisfy the following relation (2.132) on an open set  $E_{r_1, r_2}^2$  given by (2.8):

(2.132)

$$\begin{aligned} \tilde{\psi}_{\pm}(x, a, A, B, \eta) \\ = \left( \frac{\partial z}{\partial x} \right)^{-1/2} \psi_{\pm}(z(x, a, A, B, \eta), a, \Gamma(a, A, B, \eta), \eta), \end{aligned}$$

where  $z(x, a, A, B, \eta)$  and  $\Gamma(a, A, B, \eta)$  are the series constructed in Theorem 2.1.

We have so far discussed how WKB solutions of (2.1) are related to WKB solutions of (2.4). But we need in Section 3 the Legendre equation in the form (2.2). Here we discuss how WKB solutions of (2.4) and those of (2.2) are related; as we will see below the relation can be found in a straightforward manner. For the sake of simplicity of description we consider the situation when the parameter  $\Gamma$  in (2.4) is a genuine constant; this restriction does not cause any problems in our later discussion, as appropriate use of microdifferential operators will enable us to relate (2.4) with  $\Gamma$  being a genuine constant and (2.4) with  $\Gamma$  being infinite series. (See Proposition 4.3.) To relate (2.4) and (2.2) we define an infinite series

$$(2.133) \quad \Lambda(a, \Gamma, \eta) = \sum_{n=0}^{\infty} \Lambda_n(a, \Gamma) \eta^{-n}$$

and functions  $\mu(a)$  and  $\nu(a)$  of  $a$  by

$$(2.134) \quad \Lambda = \sqrt{\Gamma + (\sqrt{a}\eta)^{-2} \left( g_+(a) + g_-(-a) + \frac{1}{4} \right)} - \frac{(\sqrt{a}\eta)^{-1}}{2},$$

$$(2.135) \quad \mu = \sqrt{1 + 2(g_+(a) + g_-(-a))},$$

$$(2.136) \quad \nu = 2(g_+(a) - g_-(-a)).$$

Since  $\Lambda(a, \Gamma, \eta)$  satisfy

$$(2.137) \quad a\Gamma = a\Lambda^2 + \eta^{-1}\sqrt{a}\Lambda - \eta^{-2}(g_+(a) + g_-(-a)),$$

we immediately obtain (2.4) from (2.2) by choosing  $\Lambda, \mu$  and  $\nu$  in (2.2) respectively by (2.134), (2.135) and (2.136). Therefore we find the following

**Proposition 2.1.** *Let  $T_{\text{odd}}(z, a, \Lambda, \mu, \nu, \eta)$  and  $\phi_{\pm}(z, a, \Lambda, \mu, \nu, \eta)$  respectively be the odd part of the solution of the Riccati equation*

$$(2.138) \quad T^2 + \frac{\partial T}{\partial z} = \eta^2 \left( \frac{a\Lambda^2}{z^2 - a^2} + \eta^{-1} \frac{\sqrt{a}\Lambda}{z^2 - a^2} + \eta^{-2} \frac{az\nu + a^2(\mu^2 - 1)}{(z^2 - a^2)^2} \right)$$



and WKB solutions of (2.2) that are normalized at a simple pole  $z = a$  as

$$(2.139) \quad \phi_{\pm}(z, a, \Lambda, \mu, \nu, \eta) = \frac{1}{\sqrt{T_{\text{odd}}}} \exp \left( \pm \int_a^z T_{\text{odd}} dz \right).$$

Then the following relations hold:

$$(2.140) \quad S_{\text{odd}}(z, a, \Gamma, \eta) = T_{\text{odd}}(z, a, \Lambda(a, \Gamma, \eta), \mu(a), \nu(a), \eta),$$

$$(2.141) \quad \psi_{\pm}(z, a, \Gamma, \eta) = \phi_{\pm}(z, a, \Lambda(a, \Gamma, \eta), \mu(a), \nu(a), \eta),$$

where the infinite series  $\Lambda(a, \Gamma, \eta)$  and the functions  $\mu(a)$  and  $\nu(a)$  are those given by (2.134), (2.135) and (2.136) respectively.

*Remark 2.5.* Since  $\Lambda(a, \Gamma, \eta)$  given by (2.134) is a convergent power series in  $\eta$ ,  $\Lambda_n(a, \Gamma)$  ( $n \geq 1$ ) satisfy the following estimates: There exists a positive constant  $C$  such that

$$(2.142) \quad |\Lambda_n(a, \Gamma)| \leq \sqrt{|\Gamma|} \left( \frac{C}{\sqrt{|a\Gamma|}} \right)^n$$

holds for  $a\Gamma \neq 0$  and  $n \geq 1$ .

### 3 Analytic properties of Borel transformed WKB solutions of the Legendre equation with a large parameter

The main purpose of this section is to present analytic properties of Borel transformed WKB solutions of (2.2) with genuine constants  $a, \Lambda, \mu$  and  $\nu$ . To begin with, we show the following important

**Proposition 3.1.** *Let  $T_{\text{odd}}(z, a, \Lambda, \eta)$  be the odd part of the solution of (2.138) whose top degree part  $T_{-1}(z, a, \Lambda)$  is chosen so that it is positive for positive  $a$ ,  $z(> a)$  and  $\Lambda$ . Then we have*

$$(3.1) \quad \oint_{\gamma} T_{\text{odd}}(z, a, \Lambda, \eta) dz = 2\pi i \sqrt{a} \Lambda \eta + \pi i,$$

where  $\gamma$  is a closed curve that encircles two simple poles  $z = \pm a$  counterclockwise.

*Proof.* Let

$$(3.2) \quad T^{(\pm)}(z, a, \Lambda, \eta) = \sum_{n=-1}^{\infty} T_n^{(\pm)}(z, a, \Lambda) \eta^{-n}$$

be the solutions of (2.138) whose top degree parts  $T_{-1}^{(\pm)}(z, a, \Lambda)$  are respectively given by

$$(3.3) \quad T_{-1}^{(\pm)}(z, a, \Lambda) = \pm \sqrt{\frac{a\Lambda^2}{z^2 - a^2}}.$$

Then  $T_0^{(\pm)}$  and  $T_1^{(\pm)}$  are respectively given by

$$(3.4) \quad T_0^{(\pm)} = \frac{1}{2} \frac{z}{z^2 - a^2} \pm \frac{1}{2} \frac{1}{\sqrt{z^2 - a^2}}$$

and

$$(3.5) \quad T_1^{(\pm)} = \pm \frac{4a\Lambda z + a^2(4\mu^2 - 1)}{8\sqrt{a\Lambda}(z^2 - a^2)^{3/2}}.$$

Further we can inductively confirm that  $T_n^{(\pm)}$  ( $n \geq 2$ ) have the following form:

$$(3.6) \quad \begin{aligned} T_n^{(\pm)} = & \sum_{2 \leq p \leq n+2} c_{p,n}^{(\pm)} (z^2 - a^2)^{-p/2} \\ & + \sum_{3 \leq p \leq n+2} d_{p,n}^{(\pm)} z (z^2 - a^2)^{-p/2}, \end{aligned}$$

where  $c_{p,n}^{(\pm)}$  and  $d_{p,n}^{(\pm)}$  are constants. Hence, by noting that

$$(3.7) \quad \oint_{\gamma} \frac{dz}{\sqrt{z^2 - a^2}} = 2\pi i$$

and

$$(3.8) \quad \oint_{\gamma} T_n^{(\pm)} dz = 0$$

hold for  $n \geq 1$ , we immediately obtain (3.1). □

Now we consider the Voros coefficient

$$(3.9) \quad V(a, \Lambda, \eta) = \sum_{n=1}^{\infty} V_n \eta^{-n}$$

of (2.2), which is, by definition, given by

$$(3.10) \quad \int_a^{\infty} \left( T_{\text{odd}} - \eta T_{-1} - \frac{1}{2z} \right) dz$$

(cf. [DP], [AKT4]). Let  $\phi_{\pm}^{(\infty)}$  be WKB solutions of (2.2) that are normalized at infinity as

$$(3.11) \quad \phi_{\pm}^{(\infty)} = \frac{z^{\pm 1/2}}{\sqrt{T_{\text{odd}}}} e^{\pm \eta y_+} \exp \left[ \pm \int_{\infty}^z \left( T_{\text{odd}} - \eta T_{-1} - \frac{1}{2z} \right) dz \right],$$

where

$$(3.12) \quad y_+(z, a, \Lambda) = \int_a^z \sqrt{\frac{a\Lambda^2}{z^2 - a^2}} dz.$$

Then WKB solutions (2.139) of (2.2) that are normalized at  $z = a$  as (2.139) are written by  $V$  and  $\phi_{\pm}^{(\infty)}$  as follows:

$$(3.13) \quad \phi_{\pm} = a^{\mp 1/2} \exp(\pm V) \phi_{\pm}^{(\infty)}.$$

An important property of  $\phi_{\pm}^{(\infty)}$  is that they are Borel summable when

$$(3.14) \quad \text{the path of integration of (3.11) from } \infty \text{ to } z \text{ can be deformed so that it does not intersect Stokes curves of (2.2).}$$

See [KoS2] for the proof of the Borel summability of  $\phi_{\pm}^{(\infty)}$ . Hence the representation (3.13) of  $\phi_{\pm}$  entails that the calculation of the alien derivative of  $\phi_{\pm}$  is reduced to that of  $V$ . Fortunately the explicit form of  $V$  has been given by T. Koike ([Ko4]) as follows:

$$(3.15) \quad V_n = \frac{1}{n(n+1)(\sqrt{a}\Lambda)^n} \times \left[ B_{n+1} + \sum_{\substack{k+2l=n+1 \\ k, l \geq 0}} \frac{(n+1)!}{k!(2l)!} B_k \left\{ \left( \frac{1}{2} \right)^{2l} - \theta_+^{2l} - \theta_-^{2l} \right\} \right]$$

for  $n \geq 1$ , where  $B_n$  ( $n = 0, 1, 2, \dots$ ) are Bernoulli numbers defined by

$$(3.16) \quad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n$$

and

$$(3.17) \quad \theta_{\pm}(\mu, \nu) = \sqrt{\frac{\mu^2 \pm \sqrt{\mu^4 - \nu^2}}{2}}.$$

In [Ko4] the derivation of (3.15) is done in a parallel way to the computation of the Voros coefficient of the Weber equation and the Whittaker equation. See [SS] and [T] (resp., [KoT]) for the computation of the Voros coefficient of the Weber equation (resp., the Whittaker equation). Hence the Borel transform  $V_B(a, \Lambda, y)$  of  $V$  is concretely given by

$$(3.18) \quad V_B = \frac{1}{y(\exp(y/\sqrt{a}\Lambda) - 1)} \times \left\{ 1 + \cosh\left(\frac{y}{2\sqrt{a}\Lambda}\right) - \cosh\left(\frac{\theta_+ y}{\sqrt{a}\Lambda}\right) - \cosh\left(\frac{\theta_- y}{\sqrt{a}\Lambda}\right) \right\}.$$

It immediately follows from (3.18) that  $V_B$  behaves as

$$(3.19) \quad V_B = \frac{1}{2\sqrt{a}\Lambda} \left( \frac{1}{4} - (\theta_+^2 + \theta_-^2) \right) + O(y)$$

near  $y = 0$  and

$$(3.20) \quad V_B = \frac{1 + (-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{2m\pi i(y - 2m\pi i\sqrt{a}\Lambda)} + O(1)$$

near  $y = 2m\pi i\sqrt{a}\Lambda$  for  $m \in \mathbb{Z} \setminus \{0\}$ . Therefore  $V_B$  is singular at  $y = 2m\pi i\sqrt{a}\Lambda$  ( $m \in \mathbb{Z} \setminus \{0\}$ ) and it has simple poles there.

Now let us compute the alien derivative

$$(3.21) \quad \Delta V = \sum_{m \geq 1} \Delta_{y=2m\pi i\sqrt{a}\Lambda} V$$

of the Voros coefficient  $V$  by using the alien calculus initiated by [Ec] and developed by [P2], [DP] and [Sa]. Since  $V_B$  is single-valued and only has simple pole singularities,  $\Delta_{y=2m\pi i\sqrt{a}\Lambda} V$  is given by the residue of  $V_B$  at  $y = 2m\pi i\sqrt{a}\Lambda$ , i.e.,

$$(3.22) \quad \Delta_{y=2m\pi i\sqrt{a}\Lambda} V = \frac{1 + (-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{m}.$$

Then, by employing the alien calculus, we find

$$(3.23) \quad \begin{aligned} \Delta_{y=2m\pi i\sqrt{a}\Lambda} \exp(\pm V) \\ = \pm \frac{1 + (-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{m} \exp(\pm V). \end{aligned}$$

Noting the fact that

$$(3.24) \quad \Delta_{y=2m\pi i\sqrt{a}\Lambda} \left( e^{\mp\eta y_+} a^{\mp 1/2} \phi_{\pm}^{(\infty)} \right) = 0$$

hold for  $m \geq 1$  under the condition (3.14), we find (3.13) entails the following relations when (3.14) is satisfied:

$$(3.25) \quad \begin{aligned} \Delta_{y=2m\pi i\sqrt{a}\Lambda} (e^{\mp\eta y_+} \phi_{\pm}) \\ = \Delta_{y=2m\pi i\sqrt{a}\Lambda} \left( e^{\mp\eta y_+} a^{\mp 1/2} \exp(\pm V) \phi_{\pm}^{(\infty)} \right) \\ = e^{\mp\eta y_+} a^{\mp 1/2} \phi_{\pm}^{(\infty)} \Delta_{y=2m\pi i\sqrt{a}\Lambda} (\exp(\pm V)) \\ = \pm \frac{1 + (-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{m} \\ \quad \times e^{\mp\eta y_+} a^{\mp 1/2} \phi_{\pm}^{(\infty)} \exp(\pm V) \\ = \pm \frac{1 + (-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{m} e^{\mp\eta y_+} \phi_{\pm}. \end{aligned}$$

Summing up all these, we obtain the following

**Theorem 3.1.** *Let  $\phi_{\pm}(z, \eta)$  denote the WKB solutions of the Legendre equation (2.2) that are normalized at a simple pole  $z = a$  as in (2.139). Then their Borel transform  $\phi_{\pm,B}(z, y)$  are singular at*

$$(3.26) \quad y = \mp y_+(z) + 2m\pi i\sqrt{a}\Lambda \quad (m = 0, \pm 1, \pm 2, \dots),$$

where  $y_+(z)$  is the function given by (3.12), and its alien derivative there satisfies the following relation (3.27) for  $z$  that can be connected with  $z = \infty$  by a path that is contained in the interior of a Stokes region of the Legendre equation.

(3.27)

$$(\Delta_{y=\mp y_+ + 2m\pi i\sqrt{a}\Lambda}\phi_{\pm})_B(z, y) = \pm \Xi_m(\mu, \nu)\phi_{\pm, B}(z, y - 2m\pi i\sqrt{a}\Lambda),$$

where

(3.28)

$$\Xi_m(\mu, \nu) = \frac{1}{m} \left\{ 1 + (-1)^m - \cosh \left( 2\pi i m \sqrt{\frac{\mu^2 + \sqrt{\mu^4 - \nu^2}}{2}} \right) - \cosh \left( 2\pi i m \sqrt{\frac{\mu^2 - \sqrt{\mu^4 - \nu^2}}{2}} \right) \right\}.$$

#### 4 Analytic properties of Borel transformed WKB solutions of the Mathieu equation — properties relevant to simple poles

The principal aim of this section is to deduce analytic properties of Borel transformed WKB solutions of the Mathieu equation (2.1) for  $a \neq 0$  and  $A \neq 0$  that are relevant to its two simple poles from those of the Legendre equation (2.2) through the transformation obtained in Section 2. To begin with, we show a result corresponding to Proposition 3.1 for the Mathieu equation. First, combining Proposition 2.1 and Proposition 3.1 we immediately find

$$(4.1) \quad \oint_{\gamma} S_{\text{odd}}(z, a, \Gamma, \eta) dz = 2\pi i \sqrt{a}\Lambda(a, \Gamma, \eta)\eta + \pi i,$$

where  $\gamma$  is the path given in Proposition 3.1. Therefore Proposition 4.1 below follows from Theorem 2.2.

**Proposition 4.1.** *Let  $\tilde{S}_{\text{odd}}(x, a, A, B, \eta)$  be the odd part of the solution of (2.126) whose top degree part  $\tilde{S}_{-1}(x, a, A, B)$  is chosen so that it*

satisfies (2.128). Then we have

$$(4.2) \quad \oint_{\gamma} \tilde{S}_{\text{odd}}(x, a, A, B, \eta) dx = 2\pi i \sqrt{a} \Lambda(a, \Gamma(a, A, B, \eta), \eta) \eta + \pi i,$$

where the infinite series  $\Lambda(a, \Gamma, \eta)$  and  $\Gamma(a, A, B, \eta)$  are those given in Proposition 2.1 and Theorem 2.2 respectively and  $\gamma$  is a closed curve that encircles two simple poles counterclockwise.

Let us now employ the relation (2.141) between  $\phi_{\pm}$  and  $\psi_{\pm}$  to deduce analytic properties of  $\psi_{\pm, B}$  from those of  $\phi_{\pm, B}$ . Here we make full use of microlocal analysis, which has been made possible by the estimation (2.142) that  $\Lambda_n$  satisfies. The concrete procedure is as follows: first, by the Taylor expansion, the right-hand side of (2.141), can be written as

$$(4.3) \quad \sum_{k=0}^{\infty} \frac{\tilde{\Lambda}^k(a, \Gamma, \eta)}{k!} \frac{\partial^k}{\partial \Lambda^k} \phi_{\pm}(z, a, \Lambda_0(a, \Gamma), \mu(a), \nu(a), \eta),$$

where

$$(4.4) \quad \tilde{\Lambda}(a, \Gamma, \eta) = \Lambda(a, \Gamma, \eta) - \Lambda_0(a, \Gamma).$$

Then, taking into account the estimates (2.142) of  $\Lambda_n$ , we can rewrite (4.3) in the form of an action of a microdifferential operator

$$(4.5) \quad \mathcal{L} =: \exp \left( \tilde{\Lambda} \theta_{\Lambda} \right) :$$

upon  $\phi_{\pm, B}$  through the Borel transformation. Here  $: \cdot :$  designates the normal ordered product (cf. [A1]) and  $\theta_{\Lambda}$  is the symbol of  $\partial_{\Lambda}$ , i.e.,  $: \theta_{\Lambda} := \partial_{\Lambda}$ . More concretely, we can write the action of  $\mathcal{L}$  as an action of an integro-differential operator so that (4.3) can be rewritten as follows:

**Proposition 4.2.** *Suppose that the constants  $a \neq 0$  and  $\Lambda$  in (2.2) are different from 0. Let  $\phi_{\pm, B}$  (resp.,  $\psi_{\pm, B}$ ) be the Borel transformed WKB solutions of (2.2) (resp., (2.4)) and suppose that they are both normalized at a simple pole  $z = a$ . Then they satisfy the following relation:*

$$(4.6) \quad \psi_{\pm, B}(z, a, \Gamma, y)$$

$$= \int_{\mp y_+}^y K_\Lambda(a, \Gamma, y - y', \partial_\Lambda) \phi_{\pm, B}(z, a, \Lambda, \mu(a), \nu(a), y') dy' \Big|_{\Lambda = \Lambda_0(a, \Gamma)},$$

where  $K_\Lambda(a, \Gamma, y, \partial_\Lambda)$  is a differential operator of infinite order that is defined on  $\{(\Lambda, y) \in \mathbb{C}^2\}$ , which analytically depends on  $a$  and  $\Gamma$  with the exception  $a\Gamma = 0$ , and

$$(4.7) \quad y_+(z, a, \Gamma) = \int_a^z \sqrt{\frac{a\Gamma}{z^2 - a^2}} dz,$$

$$(4.8) \quad \Lambda_0(a, \Gamma) = \sqrt{\Gamma}.$$

Here  $\mu(a)$  and  $\nu(a)$  are functions that are respectively given by (2.135) and (2.136).

See [K] and [SKK] for the notion of differential operators of infinite order.

*Remark 4.1.* The differential operator  $K_\Lambda$  is locally defined for  $a, \Gamma \neq 0$ . However, as (2.134) implies,  $K_\Lambda$  is multivalued on  $\{(a, \Gamma, \Lambda, y) \in \mathbb{C}^4 : a, \Gamma \neq 0\}$ .

*Remark 4.2.* It immediately follows from (4.7) and (4.8) that

$$(4.9) \quad y_+(z, a, \Gamma) = \int_a^z \sqrt{\frac{a\Lambda_0^2(a, \Gamma)}{z^2 - a^2}} dz.$$

Therefore, comparing (3.12) and (4.9), we find that  $y_+$  is preserved by a change of parameters from  $(\Lambda, \mu, \nu)$  to  $(\Gamma, g_+(a), g_-(-a))$ .

Combining Theorem 3.1 and Proposition 4.2, we obtain the following

**Lemma 4.1.** *Let  $\psi_\pm(z, a, \Gamma, \eta)$  denote the WKB solutions of the Legendre equation (2.4) that are normalized at a simple pole  $z = a$  as in (2.131). Then their Borel transform  $\psi_{\pm, B}(z, a, \Gamma, y)$  are singular at*

$$(4.10) \quad y = \mp y_+(z, a, \Gamma) + 2m\pi i \sqrt{a\Gamma} \quad (m = 0, \pm 1, \pm 2, \dots),$$

where  $y_+(z)$  is the function given by (4.7). Furthermore their alien derivatives there satisfy the following relation (4.11) on the condition



that  $z$  can be connected with  $z = \infty$  by a path that is contained in the interior of a Stokes region of the Legendre equation (2.4):

(4.11)

$$\left( \Delta_{y=\mp y_+ + 2m\pi i\sqrt{a}\Gamma} \psi_{\pm} \right)_B (z, a, \Gamma, y)$$

$$= \pm \Xi_m(\mu, \nu) \left( \exp(-2m\pi i\sqrt{a}\tilde{\Lambda}\eta) \psi_{\pm} \right)_B (z, a, \Gamma, y - 2m\pi i\sqrt{a}\Gamma),$$

where  $\mu = \mu(a)$  and  $\nu = \nu(a)$  are functions that are given by (2.135) and (2.136) respectively and  $\tilde{\Lambda}(a, \Gamma, \eta)$  is a formal power series given by (2.134) and (4.4).

*Proof.* From the representation (4.6) of  $\psi_{\pm, B}$  and the definition of the alien derivative, we find

(4.12)

$$\left( \Delta_{y=2m\pi i\sqrt{a}\Gamma} e^{\mp\eta y_+} \psi_{\pm} \right)_B (z, a, \Gamma, y)$$

$$= \mathcal{L}_{2m\pi i\sqrt{a}\Lambda} \left( \Delta_{y=2m\pi i\sqrt{a}\Lambda} \phi_{\pm}^{(0)} \right)_B (z, a, \Lambda, y) \Big|_{\Lambda=\Lambda_0(a, \Gamma)}$$

holds, where  $\mathcal{L}_{y_0}$  is the integro-differential operator obtained by taking  $y = y_0$  as the end point of integration instead of  $y = \mp y_+$  in (4.6) and  $\phi_{\pm}^{(0)} = e^{\mp\eta y_+} \phi_{\pm}$ . Therefore it follows from Theorem 3.1 that the right hand side of (4.12) is equal to

(4.13)

$$\pm \Xi_m(\mu(a), \nu(a)) \mathcal{L}_{2m\pi i\sqrt{a}\Lambda} \left( \phi_{\pm, B}^{(0)}(z, a, \Lambda, y - 2m\pi i\sqrt{a}\Lambda) \right) \Big|_{\Lambda=\Lambda_0(a, \Gamma)}.$$

Let us introduce the following coordinate transformation from  $(y, \Lambda)$  to  $(y', \Lambda')$ :

(4.14)

$$\begin{cases} y' = y - 2m\pi i\sqrt{a}\Lambda \\ \Lambda' = \Lambda. \end{cases}$$

We now prepare the following general lemma:

**Lemma 4.2.** *Let  $F : (y, \Lambda_1, \dots, \Lambda_p) \rightarrow (y', \Lambda'_1, \dots, \Lambda'_p)$  be a coordinate transformation given by*

(4.15)

$$\begin{cases} y' = y + f(\Lambda_1, \dots, \Lambda_p) \\ \Lambda'_1 = \Lambda_1 \\ \vdots \\ \Lambda'_p = \Lambda_p, \end{cases}$$

where  $f(\Lambda_1, \dots, \Lambda_p)$  is a holomorphic function of  $\Lambda = (\Lambda_1, \dots, \Lambda_p) \in \mathbb{C}^p$  at  $\Lambda = \overset{\circ}{\Lambda}$ . Let  $\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_p$  be symbols of microdifferential operators of the following form:

$$(4.16) \quad \tilde{\Lambda}_j(\Lambda_1, \dots, \Lambda_p, \eta) = \sum_{n=1}^{\infty} \eta^{-n} \Lambda_{j,n}(\Lambda_1, \dots, \Lambda_p) \quad (j = 1, \dots, p).$$

Then the following relation holds:

$$(4.17) \quad : \exp(\tilde{\Lambda}(\Lambda, \eta) \cdot \theta_{\Lambda}) : \\ =: \exp[\eta'(f(\Lambda' + \tilde{\Lambda}) - f(\Lambda'))] :: \exp(\tilde{\Lambda}(\Lambda', \eta') \cdot \theta_{\Lambda'}) :,$$

where  $\eta' = \sigma(\partial/\partial y')$ ,  $\theta_{\Lambda_1} = \sigma(\partial/\partial \Lambda_1), \dots, \theta_{\Lambda} = (\theta_{\Lambda_1}, \dots, \theta_{\Lambda_p})$ , etc., and  $\cdot$  is the inner product.

*Proof.* Let  $P(\Lambda, \theta_{\Lambda}, \eta)$  denote the symbol of the left-hand side of (4.17), i.e.,

$$(4.18) \quad P(\Lambda, \theta_{\Lambda}, \eta) = \exp(\tilde{\Lambda}(\Lambda, \eta) \cdot \theta_{\Lambda}).$$

We first note the following equality:

$$(4.19) \quad : P(\Lambda, \theta_{\Lambda}, \eta) :=: P'(\Lambda', \theta_{\Lambda'}, \eta') :,$$

where  $P'$  is given by

$$(4.20) \quad \begin{aligned} & P'(\Lambda', \theta_{\Lambda'}, \eta') \\ &= \exp[-F(y, \Lambda) \cdot (\eta', \theta_{\Lambda'})] \exp(\partial_{\hat{\eta}} \cdot \partial_{\hat{y}} + \partial_{\hat{\theta}_{\Lambda}} \cdot \partial_{\hat{\Lambda}}) \\ & \quad \times P(\Lambda, \hat{\theta}_{\Lambda}, \hat{\eta}) \exp[F(\hat{y}, \hat{\Lambda}) \cdot (\eta', \theta_{\Lambda'})] \Big|_{\substack{(y, \Lambda) = (\hat{y}, \hat{\Lambda}) = F^{-1}(y', \Lambda') \\ \hat{\theta}_{\Lambda} = \hat{\eta} = 0}} \\ &= \exp(\partial_{\hat{\eta}} \cdot \partial_{\hat{y}} + \partial_{\hat{\theta}_{\Lambda}} \cdot \partial_{\hat{\Lambda}}) P(\Lambda, \hat{\theta}_{\Lambda}, \hat{\eta}) \\ & \quad \times \exp[(F(y + \hat{y}, \Lambda + \hat{\Lambda}) - F(y, \Lambda)) \cdot (\eta', \theta_{\Lambda'})] \Big|_{\substack{(y, \Lambda) = F^{-1}(y', \Lambda') \\ \hat{\Lambda} = \hat{y} = \hat{\theta}_{\Lambda} = \hat{\eta} = 0}} \end{aligned}$$

(Cf. [SKK, Chapter 2, Theorem 1.5.5]. See also the proof of [AKY, Proposition 1.2.13].) Since

$$(4.21) \quad (F(y + \hat{y}, \Lambda + \hat{\Lambda}) - F(y, \Lambda)) \cdot (\eta', \theta_{\Lambda'})$$

$$= \hat{y}\eta' + (f(\Lambda + \hat{\Lambda}) - f(\Lambda))\eta' + \hat{\Lambda} \cdot \theta_{\Lambda'}$$

and

$$(4.22) \quad e^{-\hat{z}\zeta} \exp(\partial_{\hat{\zeta}} \cdot \partial_z) e^{\hat{z}\zeta} f(\hat{\zeta}) = f(\hat{\zeta} + \zeta)$$

holds for a holomorphic function  $f(\zeta)$ , we find

$$(4.23) \quad \begin{aligned} P'(\Lambda', \theta_{\Lambda'}, \eta') &= \exp(\partial_{\hat{\theta}_{\Lambda}} \cdot \partial_{\hat{\Lambda}}) P(\Lambda, \hat{\theta}_{\Lambda}, \hat{\eta} + \eta') \\ &\quad \times \exp \left[ (f(\Lambda + \hat{\Lambda}) - f(\Lambda))\eta' + \hat{\Lambda} \cdot \theta_{\Lambda'} \right] \Big|_{\substack{(y, \Lambda) = F^{-1}(y', \Lambda') \\ \hat{\Lambda} = \hat{y} = \hat{\theta}_{\Lambda} = \hat{\eta} = 0}} \\ &= \exp(\partial_{\hat{\theta}_{\Lambda}} \cdot \partial_{\hat{\Lambda}}) \exp(\tilde{\Lambda}_1(\Lambda, \eta') \hat{\theta}_{\Lambda_1}) \cdots \exp(\tilde{\Lambda}_p(\Lambda, \eta') \hat{\theta}_{\Lambda_p}) \\ &\quad \times \exp \left[ (f(\Lambda + \hat{\Lambda}) - f(\Lambda))\eta' + \hat{\Lambda} \cdot \theta_{\Lambda'} \right] \Big|_{\substack{\Lambda = \Lambda' \\ \hat{\Lambda} = \hat{\theta}_{\Lambda} = 0}} \\ &= \exp \left[ \eta' (f(\Lambda' + \tilde{\Lambda}) - f(\Lambda')) + \tilde{\Lambda} \cdot \theta_{\Lambda'} \right]. \end{aligned}$$

Thus we obtain (4.17) from (4.19). □

We resume the proof of Lemma 4.1. It follows from (4.17) that

$$(4.24) \quad \begin{aligned} \mathcal{L} &=: \exp(\tilde{\Lambda}(a, \Gamma, \eta)\theta_{\Lambda}) : \\ &=: \exp(-2m\pi i \sqrt{a} \eta' \tilde{\Lambda}(a, \Gamma, \eta')) :: \exp(\tilde{\Lambda}(a, \Gamma, \eta')\theta_{\Lambda'}) : . \end{aligned}$$

Therefore we find

$$(4.25) \quad \begin{aligned} &\mathcal{L}_{2m\pi i \sqrt{a} \Lambda}(\phi_{\pm, B}^{(0)}(z, a, \Lambda, y - 2m\pi i \sqrt{a} \Lambda)) \\ &=: \exp(-2m\pi i \sqrt{a}(\Lambda_1 + \Lambda_2 \eta'^{-1} + \cdots)) : (\mathcal{L}_0 \phi_{\pm, B}^{(0)})(z, a, \Lambda', y'), \end{aligned}$$

where the action of  $:\eta'^{-1}:$  is fixed by taking  $y' = 0$  as the end point of integration. Here we note that, from (4.6) and (4.8), we obtain

$$(4.26) \quad \begin{aligned} &(\mathcal{L}_0 \phi_{\pm, B}^{(0)})(z, a, \Lambda, y - 2m\pi i \sqrt{a} \Lambda) \Big|_{\Lambda = \Lambda_0(a, \Gamma)} \\ &= (e^{\mp \eta y \pm} \psi_{\pm})_B(z, a, \Gamma, y - 2m\pi i \sqrt{a} \Gamma). \end{aligned}$$

Then (4.11) follows from (4.12), (4.13), (4.25) and (4.26). □

From (4.1) and (4.8), we find that (4.11) can be rewritten as follows:

$$(4.27) \quad \left( \Delta_{y=\mp y_+ + 2m\pi i\sqrt{a\Gamma}} \psi_{\pm} \right)_B(z, a, \Gamma, y) \\ = \pm(-1)^m \Xi_m(\mu, \nu) \left( \exp(-m \oint_{\gamma} S_{\text{odd}} dx) \psi_{\pm} \right)_B(z, a, \Gamma, y).$$

Now, we will study the singularity structure of Borel transformed WKB solutions of the Mathieu equation (2.1) using the transformation obtained in Theorem 2.1. To begin with, to simplify the notation, we restate the estimates (2.12) and (2.13) in the following form: there exists

$$(4.28) \quad \text{a continuous increasing function } h : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0} \text{ that satisfies} \\ h(\delta) \rightarrow 0 \text{ when } \delta \rightarrow 0$$

such that  $z_{2n}$  and  $\Gamma_{2n}$  ( $n \geq 1$ ) given in Theorem 2.1 satisfy the following estimates on  $E_{r_1, \delta}^2$  for  $0 < \delta < r_2$ :

$$(4.29) \quad |z_{2n}(x, a, A, B)| \leq (2n)! h^n(\delta) |aA|^{-n},$$

$$(4.30) \quad |\Gamma_{2n}(a, A, B)| \leq (2n)! h^n(\delta) |aA|^{-n}.$$

Let us consider the following  $\infty$ -Legendre equation:

$$(4.31) \quad \left( \frac{d^2}{dz^2} - \eta^2 \left( \frac{a\Gamma(a, A, B, \eta)}{z^2 - a^2} + \eta^{-2} \left( \frac{g_+(a)}{(z-a)^2} + \frac{g_-(-a)}{(z+a)^2} \right) \right) \right) \psi^{\dagger} = 0.$$

We immediately see that WKB solutions  $\psi_{\pm}^{\dagger}(z, a, A, B, \eta)$  of (4.31) that are normalized at its simple pole  $z = a$  are given by

$$(4.32) \quad \psi_{\pm}^{\dagger}(z, a, A, B, \eta) = \psi_{\pm}(z, a, \Gamma(a, A, B, \eta), \eta).$$

Similarly to the relation between  $\phi_{\pm, B}$  and  $\psi_{\pm, B}$  discussed in Proposition 4.2, by applying the Taylor expansion and the Borel transformation successively to (4.32), we can relate the Borel transform of  $\psi_{\pm}^{\dagger}$  with that of  $\psi_{\pm}$  through the action of a microdifferential operator defined by

$$(4.33) \quad \mathcal{G} =: \exp(\tilde{\Gamma}\theta_{\Gamma}) :,$$

where

$$(4.34) \quad \tilde{\Gamma}(a, A, B, \eta) = \Gamma(a, A, B, \eta) - \Gamma_0(a, A, B)$$

and  $\theta_\Gamma$  is the symbol of  $\partial_\Gamma$ . To be more specific, we find the following thanks to (4.30):

**Proposition 4.3.** *Let  $\psi_{\pm, B}$  (resp.  $\psi_{\pm, B}^\dagger$ ) be the Borel transformed WKB solutions of (2.4) (resp. (4.31)) for  $a \neq 0$  (resp.  $A \neq 0$ ) that are normalized at a simple pole  $z = a$ . Then  $\psi_{\pm, B}$  and  $\psi_{\pm, B}^\dagger$  satisfy the following relation:*

$$(4.35) \quad \psi_{\pm, B}^\dagger(z, a, A, B, y) = \int_{\mp y_+}^y K_\Gamma(a, A, B, y - y', \partial_\Gamma) \psi_{\pm, B}(z, a, \Gamma, y') dy' \Big|_{\Gamma=\Gamma_0(a, A, B)},$$

where  $K_\Gamma(a, A, B, y, \partial_\Gamma)$  is a differential operator of infinite order that is defined on

$$(4.36) \quad \{(a, A, B, \Gamma, y) \in \mathbb{C}^5 : a, A \neq 0, |B/A| < r_2, h(|B/A|)|y| < \sqrt{|aA|}\},$$

$$(4.37) \quad y_+(z, a, A, B) = \int_a^z \sqrt{\frac{a\Gamma_0(a, A, B)}{z^2 - a^2}} dz,$$

and

$$(4.38) \quad \sqrt{\Gamma_0(a, A, B)} = \frac{1}{2\pi i \sqrt{a}} \int_\gamma \sqrt{\frac{aA + xB}{x^2 - a^2}} dx.$$

In view of Lemma 4.1, we expect that  $\psi_{\pm, B}^\dagger$  have singularities at  $y = \mp y_+ + 2m\pi i \sqrt{a\Gamma_0}$  ( $m = 0, \pm 1, \pm 2, \dots$ ). This is the case if the representation (4.35) holds there, that is, they actually have the singularities there that correspond to those of  $\psi_{\pm, B}$ . Let us confirm this fact when these singularities are contained in the domain of definition of the integro-differential operator given in Proposition 4.3. We first note that  $\Gamma_0$  is independent of  $a$ . Indeed, by taking  $\tilde{x} = x/a$  as a new variable, we obtain

$$(4.39) \quad \sqrt{\Gamma_0(a, A, B)} = \frac{1}{2\pi i} \int_\gamma \sqrt{\frac{A + \tilde{x}B}{\tilde{x}^2 - 1}} d\tilde{x}.$$

Therefore, by taking  $r_2$  sufficiently small, we can assume that

$$(4.40) \quad \frac{1}{2}|A| < |\Gamma_0(a, A, B)| < 2|A|$$

holds on  $\{|B| < r_2|A|\}$ . Hence, if  $m \in \mathbb{Z}$ ,  $A$  and  $B$  satisfy

$$(4.41) \quad 2\sqrt{2}|m|\pi h(|B/A|) < 1,$$

the  $m$ -th singular point is in the domain of definition of the integro-differential operator. For each  $m \in \mathbb{Z}$ , this condition is satisfied by taking  $|B/A|$  sufficiently small. Further, through the representation (4.35), we can derive from Lemma 4.1 the following

**Lemma 4.3.** *Let  $\psi_{\pm}^{\dagger}(z, a, A, B, \eta)$  denote the WKB solutions of the  $\infty$ -Legendre equation (4.31) that are normalized at a simple pole  $z = a$ . Then, when (4.41) holds, its Borel transform  $\psi_{\pm, B}^{\dagger}(z, a, A, B, y)$  is singular at*

$$(4.42) \quad y = \mp y_+(z, a, A, B) + 2m\pi i \sqrt{a\Gamma_0(a, A, B)}$$

and its alien derivative there satisfies

$$(4.43) \quad \left( \Delta_{y=\mp y_+ + 2m\pi i \sqrt{a\Gamma_0}} \psi_{\pm}^{\dagger} \right)_B(z, a, A, B, y) \\ = \pm(-1)^m \Xi_m(\mu, \nu) \left( \exp(-m \oint_{\gamma} S_{\text{odd}}^{\dagger} dx) \psi_{\pm}^{\dagger} \right)_B(z, a, A, B, y),$$

where  $\mu = \mu(a)$  and  $\nu = \nu(a)$  are functions that are given by (2.135) and (2.136) respectively and  $S_{\text{odd}}^{\dagger}$  is the odd part of the solutions of the Riccati equation associated with (4.31).

*Proof.* As in the proof of Lemma 4.1, it suffices to show

$$(4.44) \quad \mathcal{G}_{2m\pi i \sqrt{a\Gamma}} \left( \left( \exp(-2m\pi i \sqrt{a\tilde{\Lambda}}\eta) \psi_{\pm}^{(0)} \right)_B(z, a, \Gamma, y - 2m\pi i \sqrt{a\Gamma}) \right) \Big|_{\Gamma=\Gamma_0} \\ = \left( \exp(-m \oint_{\gamma} S_{\text{odd}}^{\dagger} dx) e^{\mp \eta y_+} \psi_{\pm}^{\dagger} \right)_B(z, a, A, B, y),$$

where  $\mathcal{G}_{y_0}$  is the integro-differential operator obtained by taking  $y = y_0$  as the end point of integration instead of  $y = \mp y_+$  in (4.35) and  $\psi_{\pm}^{(0)} =$

$e^{\mp\eta y+\psi_{\pm}}$ . Let us introduce the following coordinate transformation from  $(y, \Gamma)$  to  $(y', \Gamma')$ :

$$(4.45) \quad \begin{cases} y' = y - 2m\pi i\sqrt{a\Gamma} \\ \Gamma' = \Gamma. \end{cases}$$

Then, from Lemma 4.2, we obtain

$$(4.46) \quad \begin{aligned} \mathcal{G} &=: \exp(\tilde{\Gamma}(a, A, B, \eta)\theta_{\Gamma}) : \\ &=: \exp[-2m\pi i\sqrt{a}\eta'(\sqrt{\Gamma'+\tilde{\Gamma}}-\sqrt{\Gamma'})] :: \exp(\tilde{\Gamma}(a, A, B, \eta')\theta_{\Gamma'}) :. \end{aligned}$$

Therefore we find

$$(4.47) \quad \begin{aligned} &\mathcal{G}_{2m\pi i\sqrt{a\Gamma}}((\exp(-2m\pi i\sqrt{a}\tilde{\Lambda}\eta)\psi_{\pm}^{(0)})_B(z, a, \Gamma, y - 2m\pi i\sqrt{a\Gamma})) \\ &=: \exp[-2m\pi i\sqrt{a}\eta'(\sqrt{\Gamma'+\tilde{\Gamma}}-\sqrt{\Gamma'})] : \\ &\quad \times : \exp(\tilde{\Gamma}(a, A, B, \eta')\theta_{\Gamma'}) : (\exp(-2m\pi i\sqrt{a}\tilde{\Lambda}\eta')\psi_{\pm}^{(0)})_B(z, a, \Gamma', y') \\ &=: \exp[-2m\pi i\sqrt{a}\eta'(\sqrt{\Gamma'+\tilde{\Gamma}}-\sqrt{\Gamma'})] : \\ &\quad \times (\exp(-2m\pi i\sqrt{a}\eta'\tilde{\Lambda}(a, \Gamma'+\tilde{\Gamma}, \eta'))\psi_{\pm}^{(0)}(z, a, \Gamma'+\tilde{\Gamma}, \eta'))_B \end{aligned}$$

where the action of  $:\eta'^{-1}:$  is fixed by taking  $y' = 0$  as the end point of integration. From (4.8) and (4.32), we find that, by replacing  $\Gamma$  with  $\Gamma_0(a, A, B)$ , the rightmost term of (4.47) equals to

$$(4.48) \quad \begin{aligned} &: \exp[-2m\pi i\sqrt{a}\eta(\Lambda(a, \Gamma(a, A, B, \eta), \eta) - \sqrt{\Gamma_0(a, A, B)})] : \\ &\quad \times (e^{\mp\eta y+\psi_{\pm}^{\dagger}})_B(z, a, A, B, y - 2m\pi i\sqrt{a\Gamma_0}). \end{aligned}$$

Then (4.43) follows from the following equality:

$$(4.49) \quad \oint_{\gamma} S_{\text{odd}}^{\dagger}(z, a, A, B, \eta)dz = 2\pi i\sqrt{a}\Lambda(a, \Gamma(a, A, B, \eta), \eta)\eta + \pi i.$$

□

Now, we derive the singularity structure of Borel transformed WKB solutions of the Mathieu equation (2.1) from Lemma 4.3. We first remark that the Mathieu equation has two simple poles and one simple turning point. On the other hand, the ( $\infty$ -)Legendre equation has only two simple poles. Therefore, if we want to relate the Mathieu equation with the Legendre equation, in other words, if we want to focus our attention on the two simple poles of the Mathieu equation, we have to remove the effect of the simple turning point. This can be attained by controlling the merging velocity of the turning point, that is,  $|A/B|$ . Indeed, since the turning point is located at  $x = -aA/B$ , it is distant enough from the poles located at  $x = \pm a$  if  $|A/B|$  is large. The existence of the function  $h(\delta)$  that satisfies (4.28)  $\sim$  (4.30) enables us to ignore the effect of the simple turning point and to derive the structure of Borel transformed WKB solutions of the Mathieu equation at the (fixed) singularities related only to the two simple poles from that of the Legendre equation as is discussed below (especially in Theorem 4.2).

Let  $\tilde{\psi}_{\pm}$  be WKB solutions of the Mathieu equation (2.1). Then, from (2.132), we obtain the following relation:

$$(4.50) \quad \tilde{\psi}_{\pm}(x, a, A, B, \eta) = \left(\frac{\partial z}{\partial x}\right)^{-1/2} \psi_{\pm}^{\dagger}(z(x, a, A, B, \eta), a, A, B, \eta).$$

For the simplicity of discussion, we take  $z_0(x, a, A, B)$  as a new coordinate variable instead of  $x$ . This is guaranteed by Theorem 2.1. Let  $M$  and  $L_{\infty}$  respectively be the Borel transformed Mathieu operator expressed in  $(z_0, a, A, B, y)$ -coordinate and the Borel transformed  $\infty$ -Legendre operator, i.e.,

$$(4.51) \quad M = \left(\frac{\partial x}{\partial z_0}\right)^{-2} \frac{\partial^2}{\partial z_0^2} - \frac{\partial^2 x}{\partial z_0^2} \left(\frac{\partial x}{\partial z_0}\right)^{-3} \frac{\partial}{\partial z_0} \\ - \frac{aA + xB}{x^2 - a^2} \frac{\partial^2}{\partial y^2} - \frac{g_+(a)}{(x - a)^2} - \frac{g_-(-a)}{(x + a)^2},$$

$$(4.52) \quad L_{\infty} = \frac{\partial^2}{\partial z_0^2} - \frac{a\Gamma(a, A, B, \partial/\partial y)}{z_0^2 - a^2} \frac{\partial^2}{\partial y^2} - \frac{g_+(a)}{(z_0 - a)^2} - \frac{g_-(-a)}{(z_0 + a)^2}.$$

Then, we find the following



**Theorem 4.1.** *There exist invertible microdifferential operators  $\mathcal{Z}$  and  $\mathcal{W}$  that satisfy*

$$(4.53) \quad M\mathcal{Z} = \mathcal{W}L_\infty$$

on

$$(4.54) \quad \{(z_0, y, a, A, B; \zeta_0, \eta) \in T^*\mathbb{C}_{z_0} \times \dot{T}^*\mathbb{C}_y \times \mathbb{C}^3 :$$

$$|z_0| < r_1|a|, a \neq 0, A \neq 0, |B| < r_2|A|\}$$

for some positive constants  $r_1$  and  $r_2$  with the exception of  $z_0^2 - a^2 = 0$ . The concrete form of  $\mathcal{Z}$  and  $\mathcal{W}$  are as follows:

$$(4.55) \quad \mathcal{Z} =: \left(\frac{\partial x}{\partial z_0}\right)^{1/2} \left(1 + \frac{\partial \tilde{z}}{\partial z_0}\right)^{-1/2} \exp(\tilde{z}(z_0, a, A, B, \eta)\zeta_0) :,$$

$$(4.56) \quad \mathcal{W} =: \left(\frac{\partial x}{\partial z_0}\right)^{-3/2} \left(1 + \frac{\partial \tilde{z}}{\partial z_0}\right)^{3/2} \exp(\tilde{z}(z_0, a, A, B, \eta)\zeta_0) :,$$

where

$$(4.57) \quad \tilde{z}(z_0, a, A, B, \eta) = z(x(z_0, a, A, B), a, A, B, \eta) - z_0.$$

Theorem 4.1 follows from the following proposition (cf. [AY]):

**Proposition 4.4.** *Let  $x(t)$  be a holomorphic change of variables at the origin from  $\mathbb{C}_t$  to  $\mathbb{C}_x$  satisfying*

$$(4.58) \quad x(0) = 0 \text{ and } \frac{dx}{dt}(0) \neq 0$$

and suppose that the following microdifferential operators  $\mathcal{P}$  and  $\mathcal{Q}$  are given:

$$(4.59) \quad \mathcal{P} = \frac{\partial^2}{\partial t^2} - p\left(t, \frac{\partial}{\partial y}\right) \frac{\partial^2}{\partial y^2},$$

$$(4.60) \quad \mathcal{Q} = \frac{\partial^2}{\partial x^2} - q\left(x, \frac{\partial}{\partial y}\right) \frac{\partial^2}{\partial y^2},$$

where  $p$  (resp.,  $q$ ) are microdifferential operators of order 0 defined near  $t = 0$  (resp.,  $x = 0$ ) except for  $\eta = 0$ . Furthermore let  $r(x, \eta)$  be the

symbol of a microdifferential operator of order  $-1$  and suppose that the total symbols  $p(t, \eta) := \sigma(p(t, \partial/\partial y))$ ,  $q(x, \eta) := \sigma(q(x, \partial/\partial y))$  and  $z(x, \eta) = x + r(x, \eta)$  satisfy the following relation:

$$(4.61) \quad p(t, \eta) = \left( \frac{dz(x(t), \eta)}{dt} \right)^2 q(z(x(t), \eta), \eta) - \frac{1}{2} \eta^{-2} \{z(x(t), \eta); t\}.$$

Then the following relation holds:

$$(4.62) \quad \mathcal{P}\mathcal{X} = \mathcal{Y}\mathcal{Q},$$

where  $\mathcal{X}$  and  $\mathcal{Y}$  are microdifferential operators defined by

$$(4.63) \quad \mathcal{X} =: \left( \frac{dz}{dt} \right)^{-1/2} \exp(r(x(t), \eta)\xi) :,$$

$$(4.64) \quad \mathcal{Y} =: \left( \frac{dz}{dt} \right)^{3/2} \exp(r(x, \eta)\xi) :$$

and  $\xi = \sigma(\partial/\partial x)$ .

*Proof.* Let  $P(x, \xi, \eta)$ ,  $Q(x, \xi, \eta)$ ,  $X(x, \xi, \eta)$  and  $Y(x, \xi, \eta)$  respectively be total symbols of  $\mathcal{P}$ ,  $\mathcal{Q}$ ,  $\mathcal{X}$  and  $\mathcal{Y}$  in  $(x, y)$ -coordinate. For example,  $P(x, \xi, \eta)$  and  $Q(x, \xi, \eta)$  are respectively given by

$$(4.65) \quad P(x, \xi, \eta) = \left( \frac{dt}{dx} \right)^{-2} \xi^2 - \left( \frac{dt}{dx} \right)^{-3} \frac{d^2 t}{dx^2} \xi - \eta^2 p(t(x), \eta)$$

and

$$(4.66) \quad Q(x, \xi, \eta) = \xi^2 - \eta^2 q(x, \eta),$$

where  $t(x)$  is the inverse function of  $x(t)$ . Then it suffices to show

$$(4.67) \quad P \circ X(x, \xi, \eta) = Y \circ Q(x, \xi, \eta),$$

where the composition  $\circ$  is defined by

$$(4.68) \quad P \circ X(x, \xi, \eta) = \exp(\partial_\xi \partial_{\hat{x}}) P(x, \hat{\xi}, \eta) X(\hat{x}, \xi, \eta) \Big|_{\substack{\hat{x}=x \\ \hat{\xi}=\xi}}.$$

(Cf. [A2, Proposition 2.5].) We first note that  $P(x, \xi, \eta)$  is expressed in terms of the total symbol

$$(4.69) \quad \tilde{P}(t, \tau, \eta) = \tau^2 - \eta^2 p(t, \eta)$$

of  $\mathcal{P}$  in  $(t, y)$ -coordinate, where  $\tau = \sigma(\partial/\partial t)$ , as follows:

$$(4.70) \quad P(x, \xi, \eta) = e^{-x\xi} \exp(\partial_{\hat{\tau}} \partial_{\hat{t}}) \tilde{P}(t, \hat{\tau}, \eta) e^{x(\hat{t})\xi} \Big|_{\substack{\hat{t}=t(x) \\ \hat{\tau}=0}}.$$

Combining (4.68) and (4.70), we find

$$(4.71) \quad \begin{aligned} P \circ X(x, \xi, \eta) &= \exp(\partial_{\hat{\tau}} \partial_{\hat{t}}) \tilde{P}(t, \hat{\tau}, \eta) \exp(\partial_{\hat{\xi}} \partial_{\hat{x}}) e^{(x(\hat{t})-x)\hat{\xi}} X(\hat{x}, \xi, \eta) \Big|_{\substack{\hat{t}=t(x), \hat{\tau}=0 \\ \hat{x}=x, \hat{\xi}=\xi}} \\ &= \exp(\partial_{\hat{\tau}} \partial_{\hat{t}}) \tilde{P}(t, \hat{\tau}, \eta) e^{(x(\hat{t})-x)\xi} X(x(\hat{t}), \xi, \eta) \Big|_{\substack{\hat{t}=t(x) \\ \hat{\tau}=0}}. \end{aligned}$$

Therefore it follows from the concrete form of  $\tilde{P}(t, \tau, \eta)$  and (4.71) that

$$(4.72) \quad \begin{aligned} P \circ X(x, \xi, \eta) &= -\eta^2 p(t(x), \eta) X(x, \xi, \eta) \\ &\quad + \frac{\partial^2}{\partial \hat{t}^2} \left( e^{(x(\hat{t})-x)\xi} X(x(\hat{t}), \xi, \eta) \right) \Big|_{\hat{t}=t(x)}. \end{aligned}$$

On the other hand, since  $Y$  satisfies  $\partial_{\xi}^k Y = r^k(x, \eta) Y$ , we find

$$(4.73) \quad \begin{aligned} Y \circ Q(x, \xi, \eta) &= Y(x, \xi, \eta) Q(x, \xi, \eta) - \eta^2 \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial \xi^k} Y(x, \xi, \eta) \frac{\partial^k}{\partial x^k} q(x, \eta) \\ &= Y(x, \xi, \eta) Q(x, \xi, \eta) + \eta^2 Y(x, \xi, \eta) q(x, \eta) \\ &\quad - \eta^2 Y(x, \xi, \eta) q(x + r(x, \eta), \eta) \\ &= Y(x, \xi, \eta) \xi^2 - \eta^2 Y(x, \xi, \eta) q(z(x, \eta), \eta). \end{aligned}$$

Then, since  $Y = (dz/dt)^2 X$ , it follows from (4.61) and (4.73) that

$$(4.74) \quad \begin{aligned} Y \circ Q(x, \xi, \eta) &= Y(x, \xi, \eta) \xi^2 - \eta^2 X(x, \xi, \eta) p(t(x), \eta) \\ &\quad - \frac{1}{2} \{z; t\} X(x, \xi, \eta). \end{aligned}$$

Thus, comparing (4.72) and (4.74), we find that (4.67) immediately follows if the following relation is confirmed:

$$(4.75) \quad \frac{\partial^2}{\partial \hat{t}^2} \left( e^{(x(\hat{t})-x)\xi} X(x(\hat{t}), \xi, \eta) \right) \Big|_{\hat{t}=t(x)} = Y(x, \xi, \eta) \xi^2 - \frac{1}{2} \{z; t\} X(x, \xi, \eta).$$

Since the left hand side of (4.75) is equal to

$$(4.76) \quad e^{-x\xi} \frac{\partial^2}{\partial \hat{t}^2} \left( \left( \frac{dz(x(\hat{t}), \eta)}{d\hat{t}} \right)^{-1/2} \exp(z(x(\hat{t}), \eta)\xi) \right) \Big|_{\hat{t}=t(x)} \\ = \exp(r(x, \eta)\xi) \frac{\partial^2}{\partial t^2} \left( \frac{dz}{dt} \right)^{-1/2} + \left( \frac{dz}{dt} \right)^{3/2} \xi^2 \exp(r(x, \eta)\xi),$$

we find that (4.75) is an immediate consequence of

$$(4.77) \quad \{z; t\} = -2 \left( \frac{dz}{dt} \right)^{1/2} \frac{d^2}{dt^2} \left( \frac{dz}{dt} \right)^{-1/2}.$$

This completes the proof. □

*Remark 4.3.* In the situation of Theorem 4.1,  $\mathcal{P}$  and  $\mathcal{Q}$  correspond to  $M$  and  $L_\infty$  respectively.

In view of (4.29), we obtain the following

**Proposition 4.5.** *Let  $\psi_{\pm, B}$  and  $\tilde{\psi}_{\pm, B}$  respectively be the Borel transformed WKB solutions of (2.4) and (2.1) for  $a \neq 0$  and  $A \neq 0$  that are normalized at their simple poles as (2.131) and (2.130). Then they satisfy the following relation:*

$$(4.78) \quad \tilde{\psi}_{\pm, B}(z_0, a, A, B, y) \\ = \int_{\mp y_+}^y K_z(z_0, a, A, B, y - y', \partial_{z_0}) \psi_{\pm, B}(z_0, a, A, B, y') dy',$$

where  $K_z(z_0, a, A, B, y, \partial_{z_0})$  is a differential operator of infinite order that is defined on

$$(4.79) \quad \tilde{E}_{r_1, h}^2 = \{(z_0, a, A, B, y) \in \mathbb{C}^5 : a, A \neq 0, |x| < r_1|a|, |B/A| < r_2,$$

$$h(|B/A|)|y| < \sqrt{|aA|}\}$$

with some positive constants  $r_1 > 1$  and  $r_2 > 0$  and

$$(4.80) \quad y_+(z_0, a, A, B) = \int_a^{z_0} \sqrt{\frac{a\Gamma_0(a, A, B)}{z_0^2 - a^2}} dz_0.$$

In conclusion, by employing similar discussions to Lemma 4.3 and Proposition 4.1, we obtain

**Theorem 4.2.** *Let  $\tilde{\psi}_{\pm}(x, a, A, B, \eta)$  be WKB solutions of the Mathieu equation (2.1) with  $a \neq 0$  and  $A \neq 0$  that is normalized at a simple pole  $x = a$ . Then, for each integer  $m$  we can take some positive constant  $\delta$  so that the following holds when  $|B/A| < \delta$  is satisfied. The Borel transform  $\tilde{\psi}_{\pm, B}(x, a, A, B, y)$  of  $\tilde{\psi}_{\pm}(x, a, A, B, \eta)$  is singular at*

$$(4.81) \quad y = \mp y_+(x, a, A, B) + 2m\pi i \sqrt{a\Gamma_0(a, A, B)}$$

and its alien derivative there satisfies

$$(4.82) \quad \left( \Delta_{y=\mp y_+ + 2m\pi i \sqrt{a\Gamma_0}} \tilde{\psi}_{\pm} \right)_B(x, a, A, B, y) \\ = \pm (-1)^m \Xi_m(\mu, \nu) \left( \exp(-m \oint_{\gamma} \tilde{S}_{\text{odd}} dx) \tilde{\psi}_{\pm} \right)_B(x, a, A, B, y),$$

where

$$(4.83) \quad \Xi_m(\mu, \nu) = \frac{1}{m} \left\{ 1 + (-1)^m - \cosh \left( 2\pi i m \sqrt{\frac{\mu^2 + \sqrt{\mu^4 - \nu^2}}{2}} \right) \right. \\ \left. - \cosh \left( 2\pi i m \sqrt{\frac{\mu^2 - \sqrt{\mu^4 - \nu^2}}{2}} \right) \right\},$$

$$(4.84) \quad \mu = \mu(a) = \sqrt{1 + 2(g_+(a) + g_-(-a))},$$

$$(4.85) \quad \nu = \nu(a) = 2(g_+(a) - g_-(-a)),$$

$$(4.86) \quad y_+(x, a, A, B) = \int_a^x \sqrt{\frac{aA + xB}{x^2 - a^2}} dx$$

and

$$(4.87) \quad \sqrt{\Gamma_0(a, A, B)} = \frac{1}{2\pi i \sqrt{a}} \int_{\gamma} \sqrt{\frac{aA + xB}{x^2 - a^2}} dx.$$

Here  $\gamma$  is a closed curve that encircles two simple poles counterclockwise.

*Remark 4.4.* In Theorem 4.2, the positive constant  $\delta$  should be taken so small that (4.41) is satisfied for  $|B/A| < \delta$  for an arbitrarily given  $m \in \mathbb{Z}$ .

*Remark 4.5.* In  $(x, a, A, B)$ -coordinate,  $y_+(x, a, A, B)$  is given by (4.86). However, since

$$(4.88) \quad z_0(x, a, A, B) = a \cos \left( \frac{1}{\sqrt{a\Gamma_0}} \int_a^x \sqrt{\frac{aA + xB}{x^2 - a^2}} dx \right)$$

satisfies

$$(4.89) \quad \frac{aA + xB}{x^2 - a^2} = \left( \frac{\partial z_0}{\partial x} \right)^2 \frac{a\Gamma_0}{z_0^2 - a^2},$$

we find (4.80) is equivalent to (4.86).

## 5 Analytic properties of Borel transformed WKB solutions of an M2P1T equation

In this section, we study WKB theoretic structure of an M2P1T equation

$$(5.1) \quad \left( \frac{d^2}{dt^2} - \eta^2 Q(t, a, \rho, \eta) \right) \hat{\psi} = 0,$$

where the potential  $Q(t, a, \rho)$  is given in Definition 1.1. We constructed transformation series  $x(t, a, \rho, \eta)$ ,  $A(a, \rho, \eta)$  and  $B(a, \rho, \eta)$  in Section 1 that give equivalence between an M2P1T equation and the following  $\infty$ -Mathieu equation:

$$(5.2) \quad \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{aA(a, \rho, \eta) + xB(a, \rho, \eta)}{x^2 - a^2} \right) \right) \psi = 0$$

$$+ \eta^{-2} \left( \frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2} \right) \tilde{\psi}^\dagger = 0.$$

As the following discussion shows, (5.2) behaves as the WKB theoretic canonical form of an M2P1T equation.

Let  $\hat{\psi}_\pm$  and  $\tilde{\psi}_\pm^\dagger$  respectively be WKB solutions of (5.1) and (5.2) that are normalized at their simple poles  $t = a$  and  $x = a$ . Then, from Theorem 1.3.3, we find the following relation holds:

$$(5.3) \quad \hat{\psi}_\pm(t, a, \rho, \eta) = \left( \frac{\partial x}{\partial t} \right)^{-1/2} \tilde{\psi}_\pm^\dagger(x(t, a, \rho, \eta), a, \rho, \eta).$$

For the simplicity of discussion, we take  $x_0(t, a, \rho)$  as a new coordinate variable instead of  $t$ . This is guaranteed by Theorem 1.3.1. Let  $N$  and  $M_\infty$  respectively be the Borel transformed M2P1T operator expressed in  $(x_0, a, \rho, y)$ -coordinate and the Borel transformed  $\infty$ -Mathieu operator, i.e.,

$$(5.4) \quad N = \left( \frac{\partial t}{\partial x_0} \right)^{-2} \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2 t}{\partial x_0^2} \left( \frac{\partial t}{\partial x_0} \right)^{-3} \frac{\partial}{\partial x_0} - Q(t, a, \rho, \partial/\partial y) \frac{\partial^2}{\partial y^2}$$

$$(5.5) \quad M_\infty = \frac{\partial^2}{\partial x_0^2} - \frac{aA(a, \rho, \partial/\partial y) + x_0B(a, \rho, \partial/\partial y)}{x_0^2 - a^2} \frac{\partial^2}{\partial y^2} \\ - \frac{g_+(a)}{(x_0 - a)^2} - \frac{g_-(-a)}{(x_0 + a)^2}.$$

Then, from Theorem 1.3.1 and Proposition 4.4, we obtain the following

**Theorem 5.1.** *There exist invertible microdifferential operators  $\mathcal{X}$  and  $\mathcal{Y}$  that satisfy*

$$(5.6) \quad N\mathcal{X} = \mathcal{Y}M_\infty$$

on

$$(5.7) \quad \{(x_0, y, a, \rho; \xi_0, \eta) \in T^*\mathbb{C}_{x_0} \times T^*\mathbb{C}_y \times \mathbb{C}^2 :$$

$$|x_0| < r, 0 < |\rho| < r, R_0|a| < |\rho|\}$$

for some positive constants  $r$  and  $R_0$  with the exception of  $x_0^2 - a^2 = 0$ . The concrete form of  $\mathcal{X}$  and  $\mathcal{Y}$  are as follows:

$$(5.8) \quad \mathcal{Z} =: \left( \frac{\partial t}{\partial x_0} \right)^{1/2} \left( 1 + \frac{\partial \tilde{x}}{\partial x_0} \right)^{-1/2} \exp(\tilde{x}(x_0, a, \rho, \eta)\xi_0) :,$$

$$(5.9) \quad \mathcal{W} =: \left( \frac{\partial t}{\partial x_0} \right)^{-3/2} \left( 1 + \frac{\partial \tilde{x}}{\partial x_0} \right)^{3/2} \exp(\tilde{x}(x_0, a, \rho, \eta) \xi_0) :,$$

where

$$(5.10) \quad \tilde{x}(x_0, a, \rho, \eta) = x(t(x_0, a, \rho), a, \rho, \eta) - x_0.$$

For the correspondence of Borel transformed WKB solutions, we have the following

**Proposition 5.1.** *Let  $\hat{\psi}_{\pm, B}$  and  $\tilde{\psi}_{\pm, B}^\dagger$  respectively be Borel transformed WKB solutions of a generic M2P1T equation (i.e.  $a, \rho \neq 0$ ) and the  $\infty$ -Mathieu equation that are normalized at their simple poles  $t = a$  and  $x = a$ . Then they satisfy the following relation:*

$$(5.11) \quad \hat{\psi}_{\pm, B}(x_0, a, \rho, y) = \int_{\mp y_+}^y K_x(x_0, a, \rho, y - y', \partial_{x_0}) \tilde{\psi}_{\pm, B}^\dagger(x_0, a, \rho, y') dy',$$

where  $K_x(x_0, a, \rho, y, \partial_{x_0})$  is a differential operator of infinite order that is defined on

$$(5.12) \quad \tilde{E}_{r, R_0, R_1}^1 = \{(x_0, a, \rho, y) \in \mathbb{C}^4 : |x_0| < r, 0 < |\rho| < r, \\ R_0|a| < |\rho|, R_1|y| < \sqrt{|\rho|}\},$$

and

$$(5.13) \quad y_+(x_0, a, \rho) = \int_a^{x_0} \sqrt{\frac{aA(a, \rho) + x_0B(a, \rho)}{x_0^2 - a^2}} dx_0.$$

Thus, the analysis of the singularity structure of Borel transformed WKB solutions of an M2P1T equation should be reduced to that of the  $\infty$ -Mathieu equation. However the complete singularity structure of Borel transformed WKB solutions of the ( $\infty$ -)Mathieu equation is too complicated to be analyzed directly. Fortunately, as the discussion in Section 4 shows, the singularity structure of Borel transformed WKB solutions of the Mathieu equation that is relevant to its two simple poles is now clarified. Using this knowledge for the Mathieu equation, we discuss the singularity structure of Borel transformed WKB solutions of an M2P1T equation in what follows.



We first relate the  $\infty$ -Mathieu equation with the Mathieu equation. To this end we use the following relation:

$$(5.14) \quad \tilde{\psi}_{\pm}^{\dagger}(x, a, \rho, \eta) = \tilde{\psi}_{\pm}(x, a, A(a, \rho, \eta), B(a, \rho, \eta), \eta).$$

Applying the Borel transformation to (5.14), we can relate the Borel transform  $\tilde{\psi}_{\pm, B}^{\dagger}$  of  $\tilde{\psi}_{\pm}^{\dagger}$  with  $\tilde{\psi}_{\pm, B}$  through the action of a microdifferential operator

$$(5.15) \quad \mathcal{AB} =: \exp(\tilde{A}\theta_A + \tilde{B}\theta_B) :,$$

where

$$(5.16) \quad \tilde{A}(a, \rho, \eta) = A(a, \rho, \eta) - A_0(a, \rho),$$

$$(5.17) \quad \tilde{B}(a, \rho, \eta) = B(a, \rho, \eta) - B_0(a, \rho)$$

and  $\theta_A$  (resp.  $\theta_B$ ) is the symbol of  $\partial_A$  (resp.  $\partial_B$ ). Thanks to the estimates (1.3.12) and (1.3.13), we obtain the following

**Proposition 5.2.** *Let  $\tilde{\psi}_{\pm, B}^{\dagger}$  and  $\tilde{\psi}_{\pm, B}$  respectively be Borel transformed WKB solutions of the  $\infty$ -Mathieu equation and the Mathieu equation that are normalized at their simple poles  $x = a$ . Then they satisfy the following relation:*

$$(5.18) \quad \begin{aligned} & \tilde{\psi}_{\pm, B}^{\dagger}(x, a, \rho, y) \\ &= \int_{\mp y+}^y K_{A, B}(a, \rho, y - y', \partial_A, \partial_B) \tilde{\psi}_{\pm, B}(x, a, A, B, y') dy' \Big|_{\substack{A=A_0(a, \rho) \\ B=B_0(a, \rho)}}, \end{aligned}$$

where  $K_{A, B}(a, \rho, y - y', \partial_A, \partial_B)$  is a differential operator of infinite order that is defined on

$$(5.19) \quad \{(a, \rho, A, B, y) \in \mathbb{C}^5 : 0 < |\rho| < r, R_0|a| < |\rho|, R_1|y| < \sqrt{|\rho|}\}$$

with some positive constants  $r, R_0$  and  $R_1$  and

$$(5.20) \quad y_+(x, a, \rho) = \int_a^x \sqrt{\frac{aA(a, \rho) + xB(a, \rho)}{x^2 - a^2}} dx.$$

Now we study the singular structure of  $\tilde{\psi}_{\pm,B}^\dagger$  using Theorem 4.2. Let us focus on the  $m$ -th singular point of  $\tilde{\psi}_{\pm,B}$  located at (4.81). Evidently, from (4.41), the following condition should be satisfied:

$$(5.21) \quad 2\sqrt{2}|m|\pi h(|B_0(a, \rho)/A_0(a, \rho)|) < 1,$$

where  $h(\delta)$  is a function that satisfies (4.28)  $\sim$  (4.30). Since  $A_0(0, 0) = f^{(1)}(0, 0) \neq 0$  and  $B_0(0, \rho) = \rho$ , Lemma 1.2.3 tells us that, by taking  $R_0$  sufficiently large, we can assume that  $A_0(a, \rho)$  and  $B_0(a, \rho)$  satisfy

$$(5.22) \quad \frac{1}{2}|f^{(1)}(0, 0)| \leq |A_0(a, \rho)| \leq \frac{3}{2}|f^{(1)}(0, 0)|,$$

$$(5.23) \quad \frac{1}{2}|\rho| \leq |B_0(a, \rho)| \leq \frac{3}{2}|\rho|$$

on  $\{R_0|a| < |\rho|\}$ . Since  $h(\delta)$  is an increasing function, we find that (5.21) follows from

$$(5.24) \quad 2\sqrt{2}|m|\pi h(3|\rho|/|f^{(1)}(0, 0)|) < 1.$$

Therefore, by taking  $\rho$  sufficiently small with keeping the relation  $R_0|a| < |\rho|$ , we can make  $|B_0(a, \rho)/A_0(a, \rho)|$  arbitrary small so that (5.24) holds. On the other hand, when

$$(5.25) \quad 2|m|\pi R_1 \sqrt{|a\Gamma_0(a, A_0(a, \rho), B_0(a, \rho))|} < \sqrt{|\rho|}$$

is satisfied, the  $m$ -th singular point is contained in the domain of definition of the integro-differential operator in (5.18). Hence, in view of (4.40) and (5.22), it suffices to take  $a$  sufficiently small relative to  $\rho$  so that

$$(5.26) \quad 2\sqrt{3}|m|\pi R_1 \sqrt{|f^{(1)}(0, 0)|} \sqrt{|a|} < \sqrt{|\rho|}$$

holds. Then, using Theorem 4.2 and Proposition 5.2, we can show the following

**Lemma 5.1.** *Let  $\tilde{\psi}_\pm^\dagger$  denote the WKB solutions of the  $\infty$ -Mathieu equation (5.2) that are normalized at a simple pole  $x = a$ . Then, when (5.24) and (5.26) hold, its Borel transform  $\tilde{\psi}_{\pm,B}^\dagger(x, a, A, B, y)$  is singular at*

$$(5.27) \quad y = \mp y_+(x, a, \rho) + mp(a, \rho)$$

and its alien derivative there satisfies

$$(5.28) \quad \left( \Delta_{y=\mp y_+ + mp} \tilde{\psi}_{\pm}^{\dagger} \right)_B (x, a, \rho, y) \\ = \pm (-1)^m \Xi_m(\mu, \nu) \left( \exp(-m \oint_{\gamma} \tilde{S}_{\text{odd}}^{\dagger} dx) \tilde{\psi}_{\pm}^{\dagger} \right)_B (x, a, \rho, y),$$

where

$$(5.29) \quad p(a, \rho) = \int_{\gamma} \sqrt{\frac{f(t, a, \rho)}{t^2 - a^2}} dt$$

and  $\Xi_m(\mu, \nu)$ ,  $\mu(a)$  and  $\nu(a)$  are functions that are given by (4.83), (4.84) and (4.85) respectively.

*Proof.* We first note the following relation, which is an immediate consequence of Proposition 4.1:

$$(5.30) \quad \oint_{\gamma} \tilde{S}_{\text{odd}}^{\dagger}(x, a, \rho, \eta) dx \\ = 2\pi i \sqrt{a} \Lambda(a, \Gamma(a, A(a, \rho, \eta), B(a, \rho, \eta), \eta), \eta) \eta + \pi i,$$

where  $\Lambda$  and  $\Gamma$  are formal power series given in Section 2,  $\tilde{S}_{\text{odd}}^{\dagger}$  is the odd part of a solution of the Riccati equation associated with the  $\infty$ -Mathieu equation and  $\gamma$  is a contour that encircles two simple poles of the  $\infty$ -Mathieu equation counterclockwise avoiding its simple turning point. Especially, we find

$$(5.31) \quad p(a, \rho) = 2\pi i \sqrt{a \Gamma_0(a, A_0(a, \rho), B_0(a, \rho))}.$$

Then, in a way parallel to the proof of Lemma 4.3, applying Lemma 4.2 to the symbol of  $\mathcal{AB}$  and using a coordinate transformation  $F : (y, A, B) \rightarrow (y', A', B')$  defined by

$$(5.32) \quad \begin{cases} y' = y - 2m\pi i \sqrt{a \Gamma_0(a, A, B)} \\ A' = A \\ B' = B \end{cases}$$

instead of (4.45), we obtain Lemma 5.1. □

*Remark 5.1.* From (4.40), (5.22) and (5.31), we find

$$(5.33) \quad |p(a, \rho)| = O(\sqrt{|a|})$$

when  $a$  tends to 0.

Now, from Theorem 1.3.2 and (5.30), we obtain the following

**Proposition 5.3.** *Let  $\hat{S}_{\text{odd}}$  be the odd part of a solution of the Riccati equation associated with an M2P1T equation. Then*

$$(5.34) \quad \oint_{\gamma} \hat{S}_{\text{odd}}(t, a, \rho, \eta) dt \\ = 2\pi i \sqrt{a} \Lambda(a, \Gamma(a, A(a, \rho, \eta), B(a, \rho, \eta), \eta), \eta) \eta + \pi i$$

holds, where  $\gamma$  is a contour that encircles two simple poles of the M2P1T equation counterclockwise avoiding its simple turning point.

In conclusion, combining Proposition 5.1, Lemma 5.1 and Proposition 5.3, we obtain

**Theorem 5.2.** *Let  $\hat{\psi}_{\pm}(t, a, \rho, \eta)$  be WKB solutions of a generic (i.e.  $a \neq 0$ ,  $\rho \neq 0$ ) M2P1T equation that is normalized at a simple pole  $t = a$ . Then, for each integer  $m$  we can take some positive constants  $\delta_1$  and  $\delta_2$  so that the following holds when  $|\rho| < \delta_1$  and  $0 < |a| < \delta_2 |\rho|$  are satisfied: The Borel transform  $\hat{\psi}_{\pm, B}(t, a, \rho, y)$  of  $\hat{\psi}_{\pm}(t, a, \rho, \eta)$  is singular at*

$$(5.35) \quad y = \mp y_+(t, a, \rho) + mp(a, \rho)$$

and its alien derivative there satisfies

$$(5.36) \quad \left( \Delta_{y=\mp y_+ + mp} \hat{\psi}_{\pm} \right)_B(t, a, \rho, y) \\ = \pm (-1)^m \Xi_m(\mu, \nu) \left( \exp(-m \oint_{\gamma} \tilde{S}_{\text{odd}} dx) \hat{\psi}_{\pm} \right)_B(t, a, \rho, y),$$

where

$$(5.37) \quad \Xi_m(\mu, \nu) = \frac{1}{m} \left\{ 1 + (-1)^m - \cosh \left( 2\pi i m \sqrt{\frac{\mu^2 + \sqrt{\mu^4 - \nu^2}}{2}} \right) \right\}$$

$$- \cosh \left( 2\pi i m \sqrt{\frac{\mu^2 - \sqrt{\mu^4 - \nu^2}}{2}} \right) \Bigg\},$$

$$(5.38) \quad \mu = \mu(a) = \sqrt{1 + 2(g_+(a) + g_-(-a))},$$

$$(5.39) \quad \nu = \nu(a) = 2(g_+(a) - g_-(-a)),$$

$$(5.40) \quad y_+(t, a, \rho) = \int_a^t \sqrt{\frac{f(t, a, \rho)}{t^2 - a^2}} dt$$

and

$$(5.41) \quad p(a, \rho) = \int_\gamma \sqrt{\frac{f(t, a, \rho)}{t^2 - a^2}} dt.$$

Here  $\gamma$  is a contour that encircles two simple poles of the M2P1T equation counterclockwise avoiding its simple turning point.

*Remark 5.2.* In Theorem 5.2, the positive constants  $\delta_1$  and  $\delta_2$  should be taken so small that (5.24) and (5.26) are satisfied for  $|\rho| < \delta_1$  and  $0 < |a| < \delta_2|\rho|$  for an arbitrarily given  $m \in \mathbb{Z}$ .

*Remark 5.3.* In  $(t, a, \rho)$ -coordinate,  $y_+(t, a, \rho)$  is given by (5.40). However, since  $x_0(t, a, \rho)$  satisfies

$$(5.42) \quad \frac{f(t, a, \rho)}{t^2 - a^2} = \left( \frac{\partial x_0}{\partial t} \right)^2 \frac{aA_0 + x_0B_0}{x_0^2 - a^2}$$

we find (5.40) is equivalent to (5.13).

## A The vanishing of the odd degree (in $\eta^{-1}$ ) part of the transformation $x(t, a, \eta)$

The purpose of this section is to prove Proposition A.1 below. From the logical viewpoint this result should be placed in Section 1.1.3. But in order not to divert the reader's attention from the main stream of the reasoning we separately show this result here. We also note that one can bypass this reasoning by first constructing  $x(t, a, \eta)$  that consists of even degree part and then proving its convergence. We hope the proof of Proposition A.1 will give some insight into the structure of  $x(t, a, \eta)$ . For the sake of simplicity we assume

$$(A.1) \quad g_{\pm} = 0$$

in this section.

**Proposition A.1.** *The transformation  $x$  and constants  $A$  and  $B$  respectively have the form (1.1.3), (1.1.4) and (1.1.5), that is, their odd degree parts in  $\eta^{-1}$  vanish.*

*Proof.* Let us begin our discussion by studying the structure of

$$(A.2) \quad x_1(t, a, \rho) = \sum_{p \geq 0} x_1^{(p)}(t, \rho) a^p$$

and

$$(A.3) \quad A_1(a) = \sum_{p \geq 0} A_1^{(p)} a^p \quad \text{and} \quad B_1(a) = \sum_{p \geq 0} B_1^{(p)} a^p.$$

It then follows from (1.1.6) that we have

$$(A.4) \quad 2x_0x_1f = (t^2 - a^2) [(x'_0)^2(aA_1 + x_0B_1 + x_1B_0) + 2x'_0x'_1(aA_0 + x_0B_0)],$$

where

$$(A.5) \quad x_0 = \sum_{p \geq 0} x_0^{(p)}(t, \rho) a^p$$

and

$$(A.6) \quad A_0 = \sum_{p \geq 0} A_0^{(p)} a^p \quad \text{and} \quad B_0 = \sum_{p \geq 0} B_0^{(p)} a^p.$$

Comparing the coefficients of  $a^0$  in (A.4), we find  
(A.7)

$$2x_0^{(0)}x_1^{(0)}f^{(0)} = t^2[(x_0^{(0)'})^2(x_0^{(0)}B_1^{(0)} + x_1^{(0)}B_0^{(0)}) + 2x_0^{(0)'}x_1^{(0)'}x_0^{(0)}B_0^{(0)}].$$

Then we obtain

(A.8)

$$\begin{aligned} & 2t^2\tilde{x}_0^{(0)}\tilde{f}^{(0)}x_1^{(0)} \\ &= t^2(x_0^{(0)'})^2\left[\left(sB_1^{(0)} + B_0^{(0)}x_1^{(0)}(s, \rho) + 2B_0^{(0)}s\frac{d}{ds}x_1^{(0)}(s, \rho)\right)\right]\Big|_{s=x_0^{(0)}(t, \rho)}. \end{aligned}$$

Dividing both sides of (A.8) by  $t^2(x_0^{(0)'})^2$ , we use [5.0]' divided by  $t$ , i.e.,

$$(A.9) \quad \tilde{x}_0^{(0)}\tilde{f}^{(0)} = (x_0^{(0)'})^2B_0^{(0)}$$

to find

$$(A.10) \quad B_0^{(0)}\left(2s\frac{d}{ds} - 1\right)x_1^{(0)}(s, \rho) = -sB_1^{(0)}.$$

Therefore we obtain

$$(A.11) \quad x_1^{(0)}(s, \rho) = -\frac{B_1^{(0)}}{B_0^{(0)}}s.$$

In particular, we have

$$(A.12) \quad x_1^{(0)}(0, \rho) = 0,$$

$$(A.13) \quad \dot{x}_1^{(0)}(0, \rho) = -\frac{B_1^{(0)}}{B_0^{(0)}}.$$

Similarly comparison of the coefficients of  $a^1$  in (A.4) entails

(A.14)

$$\begin{aligned} & 2(x_0^{(0)}x_1^{(1)} + x_0^{(1)}x_1^{(0)})f^{(0)} + 2x_0^{(0)}x_1^{(0)}f^{(1)} \\ &= t^2[(x_0^{(0)'})^2(A_1^{(0)} + x_0^{(1)}B_1^{(0)} + x_0^{(0)}B_1^{(1)} + x_1^{(0)}B_0^{(1)} + x_1^{(1)}B_0^{(0)})] \end{aligned}$$

$$\begin{aligned}
& + 2x_0^{(0)'} x_0^{(1)'} (x_0^{(0)} B_1^{(0)} + x_1^{(0)} B_0^{(0)}) + 2x_0^{(0)'} x_1^{(0)'} A_0^{(0)} \\
& + 2x_0^{(1)'} x_1^{(0)'} x_0^{(0)} B_0^{(0)} + 2x_0^{(0)'} x_1^{(1)'} x_0^{(0)} B_0^{(0)} \\
& + 2x_0^{(0)'} x_1^{(0)'} x_0^{(1)} B_0^{(0)} + 2x_0^{(0)'} x_1^{(0)'} x_0^{(0)} B_0^{(1)} \Big].
\end{aligned}$$

It then follows from (1.1.1.7), (1.1.1.24) and (A.12) that the left-hand side of (A.14) has the form

$$(A.15) \quad 2t^2 (\tilde{x}_0^{(0)} x_1^{(1)} + t \tilde{x}_0^{(1)} \tilde{x}_1^{(0)}) \tilde{f}^{(0)} + 2t^2 \tilde{x}_0^{(0)} \tilde{x}_1^{(0)} f^{(1)},$$

where

$$(A.16) \quad \tilde{x}_1^{(0)} = t^{-1} x_1^{(0)}.$$

Hence by dividing both sides of (A.14) by  $t^2 (x_0^{(0)'})^2$ , we readily find

$$\begin{aligned}
(A.17) \quad & B_0^{(0)} \left( 2s \frac{d}{ds} - 1 \right) x_1^{(1)}(s, \rho) \\
& = -A_1^{(0)} - s B_1^{(1)} + 2(x_0^{(0)'})^{-2} \tilde{x}_0^{(0)} \tilde{x}_1^{(0)} f^{(1)} - 2\dot{x}_1^{(0)} A_0^{(0)} + V,
\end{aligned}$$

where

$$\begin{aligned}
(A.18) \quad & V = 2(x_0^{(0)'})^{-2} \tilde{x}_0^{(1)} x_1^{(0)} \tilde{f}^{(0)} \Big|_{t=t(s, \rho)} \\
& - (x_0^{(1)} B_1^{(0)} + x_1^{(0)} B_0^{(1)} + 2\dot{x}_0^{(1)} s B_1^{(0)} + 2\dot{x}_0^{(1)} x_1^{(0)} B_0^{(0)} \\
& + 2\dot{x}_0^{(1)} \dot{x}_1^{(0)} s B_0^{(0)} + 2\dot{x}_1^{(0)} x_0^{(1)} B_0^{(0)} + 2\dot{x}_1^{(0)} s B_0^{(1)}).
\end{aligned}$$

Here we note that  $V$  vanishes at  $s = 0$ , and furthermore (1.1.1.21) entails

$$(A.19) \quad 2(x_0^{(0)'})^{-2} \tilde{x}_0^{(0)} \tilde{x}_1^{(0)} f^{(1)} \Big|_{t=0} = 2Z_0^2 \dot{x}_1^{(0)}(0, \rho) A_0^{(0)} = 2A_0^{(0)} \dot{x}_1^{(0)}(0, \rho),$$

where  $Z_0$  stands for  $x_0^{(0)'}(0, \rho) = \pm 1$  (cf. (1.1.1.13) and (1.1.1.23)). Therefore we obtain

$$(A.20) \quad x_1^{(1)}(0, \rho) = \frac{A_1^{(0)}}{B_0^{(0)}}.$$

Next, by comparing the coefficients of  $a^2$  in (A.4), we encounter terms which do not have factor  $t^2$  explicitly, that is,

$$(A.21) \quad 2f^{(1)} x_0^{(0)} x_1^{(1)}$$



in the left-hand side of (A.4) and

$$(A.22) \quad - \left[ (x_0^{(0)'})^2 (x_0^{(0)} B_1^{(0)} + x_1^{(0)} B_0^{(0)}) + 2x_0^{(0)'} x_1^{(0)'} x_0^{(0)} B_0^{(0)} \right]$$

in the right-hand side. It is clear that each term in (A.21) and (A.22) is divisible by  $t^1$ . Hence the existence of a holomorphic solution  $x_1^{(2)}(s, \rho)$  requires

$$(A.23) \quad \left[ 2f^{(1)} \tilde{x}_0^{(0)} x_1^{(1)} + (x_0^{(0)'})^2 (\tilde{x}_0^{(0)} B_1^{(0)} + \tilde{x}_1^{(0)} B_0^{(0)}) + 2x_0^{(0)'} x_1^{(0)'} \tilde{x}_0^{(0)} B_0^{(0)} \right] \Big|_{t=0} = 0.$$

Then by using (A.12), (A.13) and (A.20) we find that

$$(A.24) \quad \begin{aligned} & 2A_0^{(0)} Z_0 \frac{A_1^{(0)}}{B_0^{(0)}} + Z_0 B_1^{(0)} + Z_0 \left( -\frac{B_1^{(0)}}{B_0^{(0)}} \right) B_0^{(0)} + 2Z_0^3 \left( -\frac{B_1^{(0)}}{B_0^{(0)}} \right) B_0^{(0)} \\ & = 2Z_0 \left( \frac{A_0^{(0)}}{B_0^{(0)}} A_1^{(0)} - B_1^{(0)} \right) = 0 \end{aligned}$$

should hold. Similar computation of constant terms in the coefficients of  $a^3$  in (A.4) shows the vanishing of the following sum is required for the existence of  $x_1^{(3)}$ :

$$(A.25) \quad \begin{aligned} & \left[ 2 \left( \sum_{j+k+l=3} x_0^{(j)} x_1^{(k)} f^{(l)} \right) \right. \\ & \quad + (x_0^{(0)'})^2 (A_1^{(0)} + x_0^{(0)} B_1^{(1)} + x_0^{(1)} B_1^{(0)} + x_1^{(0)} B_0^{(1)} + x_1^{(1)} B_0^{(0)}) \\ & \quad + 2x_0^{(0)'} x_0^{(1)'} (x_0^{(0)} B_1^{(0)} + x_1^{(0)} B_0^{(0)}) \\ & \quad + 2x_0^{(0)'} x_1^{(0)'} (A_0^{(0)} + x_0^{(0)} B_0^{(1)} + x_0^{(1)} B_0^{(0)}) \\ & \quad \left. + (2x_0^{(0)'} x_1^{(1)'} + 2x_0^{(1)'} x_1^{(0)'}) x_0^{(0)} B_0^{(0)} \right] \Big|_{t=0} \\ & = A_1^{(0)} + x_1^{(1)}(0, \rho) B_0^{(0)} + 2Z_0^2 \dot{x}_1^{(0)}(0, \rho) A_0^{(0)} \\ & = 2A_1^{(0)} - 2 \frac{A_0^{(0)}}{B_0^{(0)}} B_1^{(0)}. \end{aligned}$$

The vanishing of (A.25) together with (A.24) entails the vanishing of  $(A_1^{(0)}, B_1^{(0)})$  by the assumption (1.1.2) combined with (1.1.1.21) and (1.1.1.22). Then it follows from (A.11) that

$$(A.26) \quad x_1^{(0)}(s, \rho) = 0.$$

Thus we can define

$$(A.27) \quad \hat{x}_1(t, a, \rho) = a^{-1}x_1(t, a, \rho)$$

and

$$(A.28) \quad \hat{A}_1 = a^{-1}A_1 \quad \text{and} \quad \hat{B}_1 = a^{-1}B_1.$$

On the other hand, dividing both sides of (A.4) by  $a$ , we find

$$(A.29) \quad 2x_0\hat{x}_1f = (t^2 - a^2)[(x'_0)^2(a\hat{A}_1 + x_0\hat{B}_1 + \hat{x}_1B_0) + 2x'_0\hat{x}'_1(aA_0 + x_0B_0)].$$

Hence by repeating the reasoning which guaranteed the vanishing of  $(x_1^{(0)}(s, \rho), A_1^{(0)}, B_1^{(0)})$ , we find the vanishing of  $(\hat{x}_1^{(0)}(s, \rho), \hat{A}_1^{(0)}, \hat{B}_1^{(0)}) = (x_1^{(1)}(s, \rho), A_1^{(1)}, B_1^{(1)})$ . By repeating this reasoning we find

$$(A.30) \quad x_1(s, a, \rho) = 0$$

and

$$(A.31) \quad A_1(a, \rho) = B_1(a, \rho) = 0.$$

To prove the required result we use the induction: let us assume

$$(A.32.\nu) \quad x_{2n-1}(s, a, \rho) = 0 \quad \text{and} \quad A_{2n-1}(a, \rho) = B_{2n-1}(a, \rho) = 0$$

hold for  $n \leq \nu$ ,

and show (A.32. $\nu + 1$ ) is valid. First we multiply (1.1.6) (with  $g_{\pm} = 0$ ) by  $(x^2 - a^2)(t^2 - a^2)$  to find

$$(A.33) \quad (x^2 - a^2)f = (x')^2(aA + xB)(t^2 - a^2) - \frac{1}{2}\eta^{-2}(x^2 - a^2)(t^2 - a^2)\{x; t\}.$$

Comparing the coefficients of  $\eta^{-2\nu-1}$  in (A.33) we find

$$(A.34) \quad 2x_0x_{2\nu+1}f = (t^2 - a^2)[(x'_0)^2(aA_{2\nu+1} + x_0B_{2\nu+1} + x_{2\nu+1}B_0)]$$

$$+ 2x'_0 x'_{2\nu+1} (aA_0 + x_0 B_0)].$$

This has the same form as (A.4); only the suffix 1 in (A.4) is replaced by  $\nu + 1$ . Hence the same reasoning used to show  $x_1 = A_1 = B_1 = 0$  applies to (A.34). Then we find (A.32. $\nu + 1$ ) is valid. Therefore the induction proceeds, completing the proof of Proposition A.1.

□

## B The vanishing of $x_{2n}^{(1)}(0, \rho)$ , $\tilde{A}_{2n}^{(0)}$ and $\tilde{B}_{2n}^{(0)}$ for $n \geq 1$ when $g_{\pm}(t) = 0$ .

In Section 1.1 and Section 1.2, the vanishing of  $x_0^{(1)}(0, \rho)$  repeatedly played an important role in our reasoning. Hence it is reasonable for the reader to wonder how is the situation for the higher order terms. The answer is that a similar vanishing is observed if  $g_{\pm}(t) = 0$  but that it does not hold in general when  $g_{\pm}(t) \neq 0$ . Hence we content ourselves with a rather weak statement given in Lemma 1.1.3 so that the reasoning in Section 1.1.3 may be applicable to the case where  $g_{\pm}(t) \neq 0$ . (See the reasoning in Appendix C.) It may be, however, of some interest to see how the actual situation is when  $g_{\pm} = 0$ . Accordingly, we show the following

**Proposition B.1.** *Assume  $g_{\pm}(t) = 0$ . Then we find the following properties for the triplet  $T_{2n}^{(p)} = \{x_{2n}^{(p)}, A_{2n}^{(p)}, B_{2n}^{(p)}\}$  constructed in Section 1.1.3:*

$$(B.1) \quad x_{2n}^{(1)}(0, \rho) = 0 \quad \text{for } n \geq 0,$$

$$(B.2) \quad \dot{x}_{2n}^{(0)}(0, \rho) = 0 \quad \text{for } n \geq 1,$$

$$(B.3) \quad A_{2n}^{(0)} = B_{2n}^{(0)} = 0 \quad \text{for } n \geq 1.$$

*Proof.* Let us first recall

$$(B.4) \quad x_{2j}^{(0)}(0, \rho) = 0 \quad \text{for } j = 0, 1, 2, \dots$$

(Cf. (1.1.3.54).) Then, by using (B.4), we validate by the induction on

$k$  the following statement  $\mathcal{V}(k)$  ( $k \geq 1$ ):

$$(B.5) \quad \mathcal{V}(k) : \begin{cases} (i) & \dot{x}_{2i}^{(0)}(0, \rho) = 0, \quad i = 1, 2, \dots, k, \\ (ii) & x_{2i}^{(1)}(0, \rho) = 0, \quad i = 0, 1, \dots, k, \\ (iii) & A_{2i}^{(0)} = B_{2i}^{(0)} = 0, \quad i = 1, 2, \dots, k. \end{cases}$$

Let us first prove  $\mathcal{V}(1)$ . To begin with, we note

$$(B.6) \quad R_2^{(0)}(s, \rho) = \frac{1}{2B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 s^2 \{x; t\}_0^{(0)};$$

other terms in  $R_2^{(0)}(s, \rho)$  do not exist because of the constraints on the indices. Since  $\tilde{A}_2^{(-1)} = 0$  by the assumption (1.1.3.50), (B.6) entails

$$(B.7) \quad x_2^{(0)}(s, \rho) = -\tilde{B}_2^{(0)}s + O(s^2).$$

We also note that  $z_2^{(0)}(s, \rho)$ , which is, by definition,  $x_2^{(0)}(s, \rho) - \tilde{A}_2^{(-1)} + \tilde{B}_2^{(0)}s$ , satisfies

$$(B.8) \quad \dot{z}_2^{(0)}(0, \rho) = 0.$$

We next show

$$(B.9) \quad R_2^{(1)}(0, \rho) = 0.$$

In what follows (untill (B.18)), we use the symbol  $(\alpha.j)$  ( $j = i, ii, \dots, x$ ) to denote the term labelled by  $(\alpha.j)$  in (1.1.3.43) with  $(p, n) = (1, 1)$ ; for example,  $(\alpha.i)|_{s=0}$  means

$$(B.10) \quad - \sum_{\substack{q+r+u=0 \\ i+j+k=1 \\ (u,k) \neq (0,1)}} \dot{x}_{2i}^{(q)}(0, \rho) \dot{x}_{2j}^{(r)}(0, \rho) \tilde{A}_{2k}^{(u)}.$$

Using this expression together with (B.7), we find

$$(B.11) \quad \begin{aligned} (\alpha.i)|_{s=0} &= -2\dot{x}_2^{(0)}(0, \rho) \dot{x}_0^{(0)}(0, \rho) \tilde{A}_0^{(0)} \\ &= 2\tilde{B}_2^{(0)} \tilde{A}_0^{(0)}. \end{aligned}$$

Seemingly the  $\rho$ -dependence of this term is wilder than that one might expect at this stage. But fortunately, as we will see below (cf. (B.16)), it is cancelled out by  $(\alpha.vi)|_{s=0}$ , which is equal to

$$(B.12) \quad \frac{2t^{-2}}{B_0^{(0)}} t^2 \sum_{r+u=1} x_2^{(0)}(s, \rho) x_0^{(r)}(s, \rho) f^{(u)}(t, \rho) \Big|_{s=0}.$$

Since we have

$$(B.13) \quad x_{2n}^{(0)}(s, \rho) x_0^{(1)}(s, \rho) f^{(0)}(t, \rho) = O(s^3)$$

thanks to the relation

$$(B.14) \quad x_0^{(1)}(0, \rho) = 0,$$

it suffices to study the contribution from  $x_2^{(0)}(s, \rho) x_0^{(0)}(s, \rho) f^{(1)}(t, \rho)$ . Then it follows from (B.7) and the relation  $(ds/dt|_{t=0})^2 = 1$  that

$$(B.15) \quad \begin{aligned} & \frac{2t^{-2}}{B_0^{(0)}} t^2 x_2^{(0)}(s, \rho) x_0^{(0)}(s, \rho) f^{(1)}(t, \rho) \Big|_{s=0} \\ &= \frac{2}{B_0^{(0)}} (-\tilde{B}_2^{(0)}) A_0^{(0)} = -2\tilde{B}_2^{(0)} \tilde{A}_0^{(0)}. \end{aligned}$$

Thus we find

$$(B.16) \quad (\alpha.i)|_{s=0} + (\alpha.vi)|_{s=0} = 0.$$

In view of the constraint on the indices, we find that  $(\alpha.j)$  ( $j = \text{iii, iv, v, vii, viii, x}$ ) contains no term. It is clear that  $(\alpha.ix)|_{s=0}$  vanishes. Thus the remaining term to be studied is only  $(\alpha.ii)|_{s=0}$ ; because of the constraint on the indices, we find either (i)  $q + r + v = 1$  or (ii)  $q + r + v = 0$ . In case (i)  $u$  is 0, and hence  $x_{2k}^{(u)}(0, \rho)$  vanishes by (B.4). In case (ii),  $u$  should be 1 and hence the constraint  $(u, k) \neq (1, 1)$  entails  $k = 0$ , leading to the vanishing of this term by (B.14).

Summing up all these, we thus find

$$(B.17) \quad R_2^{(1)}(0, \rho) = 0.$$

This implies

$$(B.18) \quad z_2^{(1)}(0, \rho) = 0.$$

By using this and (B.8), we next show  $\Gamma_2^{(0)} = \Delta_2^{(0)} = 0$  by examining each term in (1.2.164) and (1.2.165). We use symbols  $(\tilde{\gamma}.j)$  and  $(\tilde{\delta}.k)$  to mean terms labelled by them there.

The vanishing of  $(\tilde{\gamma}.i)$  and  $(\tilde{\gamma}.ii)$  immediately follows from (B.8) and (B.18). For  $(p_0, n) = (0, 1)$ , one of  $(i, j, k)$  in  $(\tilde{\gamma}.iii)$  should be 1, which is forbidden in  $(\tilde{\gamma}.iii)$ . Thus  $(\tilde{\gamma}.iii)$  contains no term.

Concerning  $(\tilde{\gamma}.iv)$ , we first consider the case where  $u = 1$ , i.e.,  $q + r = 2$ . If  $q = 2$ , then  $r = 0$ ; thus  $x_2^{(r)}(0, \rho) = 0$  by (B.4). If  $q = 0$  or 1  $x_0^{(q)}(0, \rho)$  vanishes by (B.4) or (B.14). Thus every term with  $u = 1$  in  $(\tilde{\gamma}.iv)$  is 0. The same reasoning applies to terms with  $u = 2$ . Thus  $(\tilde{\gamma}.iv)$  is 0. It is clear that  $(\tilde{\gamma}.v)$  contains no term when  $n = 1$ . To study  $(\tilde{\gamma}.vi)$ , we first consider the case where  $q + r + v = 1$ . Then  $u = 0$ , and hence  $x_{2k}^{(u)}(0, \rho)$  vanishes by (B.4). When  $q + r + v = 0$ ,  $u = 1$ ; then the constraint  $(u, k) \neq (1, 1)$  forces  $k$  to be 0. Hence  $x_{2k}^{(u)}(0, \rho)$  is 0 by (B.14). Thus  $(\tilde{\gamma}.vi)$  is 0. The term  $(\tilde{\gamma}.vii)$  does not exist for  $(p_0, n) = (0, 1)$ . The vanishing of  $(\tilde{\gamma}.viii)$  is an immediate consequence of (B.4) (with  $j = 0$ ). Thus we have confirmed

$$(B.19) \quad \Gamma_2^{(0)} = 0.$$

We next study  $\Delta_2^{(0)}$ . Again by (B.18) and (B.8) we find that  $(\tilde{\delta}.i)$  and  $(\tilde{\delta}.ii)$  are 0. When  $(p_0, n) = (0, 1)$ ,  $(\tilde{\delta}.iii)$  contains no term. To study  $(\tilde{\delta}.iv)$ , we may assume  $(i, j) = (0, 1)$  without loss of generality. Then  $x_{2i}^{(q)}(0, \rho)$  vanishes for  $q = 0$  or 1 by (B.4) or (B.14), whereas  $x_2^{(r)}(0, \rho)$  vanishes for  $q = 2$  (and hence  $r = 0$ ). Thus  $(\tilde{\delta}.iv)$  is 0 in our case. The vanishing of  $(\tilde{\delta}.v)$  can be confirmed in the same manner. Concerning  $(\tilde{\delta}.vi)$ ,  $x_{2j}^{(r)}(0, \rho) = 0$  for  $r = 0$  by (B.4). On the other hand,  $r = 1$  forces  $j$  to be 0, and hence the vanishing of  $x_{2j}^{(r)}(0, \rho)$  follows from (B.14). Thus  $(\tilde{\delta}.vi)$  is also 0. The vanishing of  $(\tilde{\delta}.vii)$  is clear for  $(p_0, n) = (0, 1)$ . For each term in  $(\tilde{\delta}.viii)$ ,  $u = 0$  in our case, and hence the vanishing of  $(\tilde{\delta}.viii)$  follows from (B.4). When  $(p_0, n) = (0, 1)$ ,  $(\tilde{\delta}.x)$  contains no term because of the constraint on the indices, and  $(\tilde{\delta}.x)$  does not exist. Finally in  $(\tilde{\delta}.xi)$  we find

$$(B.20) \quad x_{2i}^{(q)} x_{2j}^{(r)} = (x_0^{(0)})^2 = s^2.$$

Hence

$$(B.21) \quad \frac{d}{dt} \left( \sum_{\substack{q+r+u=0 \\ i+j+k=0}} x_{2i}^{(q)} x_{2j}^{(r)} \{x; t\}_{2k}^{(u)} \right) \Big|_{t=0} = 0.$$

Thus we have confirmed

$$(B.22) \quad \Delta_2^{(0)} = 0.$$

Therefore (B.19) and (B.22) imply

$$(B.23) \quad A_2^{(0)} = B_2^{(0)} = 0.$$

Then

$$(B.24) \quad \dot{x}_2^{(0)}(0, \rho) = 0$$

follows from (B.7) and (B.23), and

$$(B.25) \quad x_2^{(1)}(0, \rho) = \tilde{A}_2^{(0)} + z_2^{(1)}(0, \rho) = 0$$

follows from (B.18) and (B.23). Thus we have validated  $\mathcal{V}(1)$ .

Let us now validate  $\mathcal{V}(n)$  ( $n \geq 2$ ) by assuming that  $\mathcal{V}(k)$  ( $1 \leq k \leq n-1$ ) have been validated. To validate  $\mathcal{V}(n)$  we first prove

$$(B.26) \quad R_{2n}^{(0)}(s, \rho) = O(s^2).$$

From this point to (B.34),  $(\alpha.j)$  ( $j = \text{i, ii}, \dots, \text{x}$ ) means the term labelled by  $(\alpha.j)$  in (1.1.3.43) with  $(p, n) = (0, n)$ . As  $(\alpha.j)$  ( $j = \text{i, iii, iv, vi, vii, viii, x}$ ) contains no term, we concentrate our attention on other terms.

To study  $(\alpha.\text{ii})$ ,  $p = 0$  implies  $q = r = u = v = 0$ . Then the convention (1.1.3.44) entails

$$(B.27) \quad i, j, k, l \leq n-1.$$

Hence at most two of  $(i, j, k, l)$  are allowed to be 0; otherwise stated, at least two of them are equal to or greater than 1. Therefore it follows from  $\mathcal{V}(n-1)(\text{i}), (\text{iii})$  that

$$(B.28) \quad (\alpha.\text{ii}) = O(s^2)$$

(including the possibility of its vanishing).

Concerning  $(\alpha.v)$  with  $n \geq 2$ , the constraint on the indices entails

$$(B.29) \quad i, j \geq 1.$$

Hence  $\mathcal{V}(n-1)(i)$  implies

$$(B.30) \quad x_{2i}^{(0)} x_{2j}^{(0)} f^{(0)} = O(s^5),$$

that is,

$$(B.31) \quad (\alpha.v) = O(s^3).$$

As to  $(\alpha.ix)$  we divide the situation into two cases: (i)  $k \leq n-2$ , (ii)  $k = n-1$ . In case (i),  $i+j = n-1-k \geq 1$  and hence (B.4) and  $\mathcal{V}(n-1)(i)$  entail

$$(B.32) \quad x_{2i}^{(0)} x_{2j}^{(0)} \{x; t\}_{2k}^{(0)} = O(s^3),$$

whereas in case (ii) we find  $i = j = 0$  and thus

$$(B.33) \quad x_0^{(0)} x_0^{(0)} \{x; t\}_{2(n-1)}^{(0)} = O(s^2).$$

In any event, we obtain

$$(B.34) \quad (\alpha.ix) = O(s^2).$$

Summing up all these, we find

$$(B.35) \quad R_{2n}^{(0)}(s, \rho) = O(s^2).$$

Next we study  $R_{2n}^{(1)}(0, \rho)$ . From this point to (B.44),  $(\alpha.j)$  stands for the term labelled by it in (1.1.3.43) with  $(p, n) = (1, n)$ , where  $n \geq 2$ .

Let us first examine  $(\alpha.i)|_{s=0}$ . It follows from the definition that

$$(B.36) \quad (\alpha.i)|_{s=0} = - \sum_{i+j=n} (\dot{x}_{2i}^{(0)} \dot{x}_{2j}^{(0)} \tilde{A}_0^{(0)}) \Big|_{s=0} - \sum_{\substack{i+j+k=n \\ 1 \leq k \leq n-1}} (\dot{x}_{2i}^{(0)} \dot{x}_{2j}^{(0)} \tilde{A}_{2k}^{(0)}) \Big|_{s=0}.$$

Then the second sum vanishes by  $\mathcal{V}(n-1)(iii)$ . On the other hand, all terms except  $-(\dot{x}_0^{(0)} \dot{x}_{2n}^{(0)} + \dot{x}_{2n}^{(0)} \dot{x}_0^{(0)}) \tilde{A}_0^{(0)} \Big|_{s=0}$  in the first sum vanish by  $\mathcal{V}(n-1)(i)$ . Thus we find

$$(B.37) \quad (\alpha.i)|_{s=0} = -2\dot{x}_{2n}^{(0)}(0, \rho) \tilde{A}_0^{(0)}.$$



As one expects, this term is cancelled out by  $(\alpha.vi)$ ; let us confirm it first, by setting aside the study of other terms. Since (B.4) and  $\mathcal{V}(n-1)(ii)$  guarantee  $x_{2n}^{(0)}x_0^{(1)}f^{(0)} = O(s^3)$ , what we have to worry about in  $(\alpha.vi)$  is the term

$$(B.38) \quad \frac{2t^{-2}}{B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 x_{2n}^{(0)} s f^{(1)}(t, \rho) \Big|_{s=0},$$

which cancels  $(\alpha.i)|_{s=0}$  by (B.4). Let us now return to the study of  $(\alpha.ii)|_{s=0}$ , following the numbering. To study  $(\alpha.ii)|_{s=0}$ , we first note that  $x_{2k}^{(u)}(0, \rho)$  with  $u = 0$  vanishes by (B.4), and hence we suppose  $u = 1$ . But then, the condition  $(u, k) \neq (1, n)$  forces  $k \leq n-1$ . This means that one of  $(i, j, l)$  is equal to or greater than 1. Hence  $\mathcal{V}(n-1)(i), (iii)$  guarantees

$$(B.39) \quad \dot{x}_{2i}^{(0)} \dot{x}_{2j}^{(0)} \tilde{B}_{2l}^{(0)} \Big|_{s=0} = 0.$$

Thus we find

$$(B.40) \quad (\alpha.ii)|_{s=0} = 0.$$

It is clear that  $(\alpha.iii)$  and  $(\alpha.iv)$  contain no term when  $p = 1$ . As to  $(\alpha.v)$  we rewrite

$$(B.41) \quad \sum_{\substack{q+r+u=1 \\ i+j=n \\ i,j \leq n-1}} x_{2i}^{(q)} x_{2j}^{(r)} f^{(u)} = \sum_{\substack{q+r=1 \\ i+j=n \\ i,j \leq n-1}} x_{2i}^{(q)} x_{2j}^{(r)} f^{(0)} + \sum_{\substack{i+j=n \\ i,j \leq n-1}} x_{2i}^{(0)} x_{2j}^{(0)} f^{(1)}.$$

Then, in the first sum either  $q$  or  $r$  is 0 and both  $i$  and  $j$  is equal to or greater than 1. Hence  $\mathcal{V}(n-1)(i), (ii)$  implies

$$(B.42) \quad x_{2i}^{(q)} x_{2j}^{(r)} f^{(0)} = O(s^4).$$

It is also clear from  $\mathcal{V}(n-1)(i)$  that each term in the second sum is of  $O(s^4)$ . Thus we obtain

$$(B.43) \quad (\alpha.v)|_{s=0} = 0.$$

Since  $(\alpha.vii)$ ,  $(\alpha.viii)$  and  $(\alpha.x)$  contain no term and since  $(\alpha.vi)$  has already been examined, what remains to be studied is  $(\alpha.ix)$ . But, either  $q$  or  $r$  is equal to 0 in each term in  $(\alpha.ix)$ . Hence (B.4) guarantees

$$(B.44) \quad (\alpha.ix)|_{s=0} = 0.$$

Thus we have confirmed

$$(B.45) \quad R_{2n}^{(1)}(0, \rho) = 0.$$

We now show

$$(B.46) \quad \Gamma_{2n}^{(0)} = \Delta_{2n}^{(0)} = 0.$$

To begin with we note

$$(B.47) \quad \dot{z}_{2n}^{(0)}(0, \rho) = z_{2n}^{(1)}(0, \rho) = 0$$

follows from (B.35) and (B.45). Then, using the symbol  $(\tilde{\gamma}.i)$  etc. to denote the corresponding term in (1.2.164) and (1.2.165) with  $p_0 = 0$ , we find by (B.47) that

$$(B.48) \quad (\tilde{\gamma}.i) = (\tilde{\gamma}.ii) = 0.$$

Concerning  $(\tilde{\gamma}.iii)$ , we first note

$$(B.49) \quad q = r = u = 0$$

and hence the constraint on the indices entails

$$(B.50) \quad i, j, k \leq n - 1.$$

Therefore at least one of  $(i, j, k)$  is equal to or greater than 1. Then  $\mathcal{V}(n-1)(i)$ , (iii) guarantees

$$(B.51) \quad (\tilde{\gamma}.iii) = 0.$$

It is clear that  $(\tilde{\gamma}.iv)$  contains no term when  $p_0 = 0$ . As to  $(\tilde{\gamma}.v)$  each term in the sum has the form

$$(B.52) \quad x_{2i}^{(1)}(0, \rho) x_{2j}^{(1)}(0, \rho) f^{(1)}(0, \rho)$$

with  $i, j \geq 1$ . Hence  $\mathcal{V}(n-1)(ii)$  implies it vanishes, and thus we have

$$(B.53) \quad (\tilde{\gamma}.v) = 0.$$

Regarding  $(\tilde{\gamma}.vi)$  we divide the situation into two cases: (i)  $u = 1$  and (ii)  $u = 0$ . In case (i),  $(u, k) \neq (1, n)$  entails  $k \leq n - 1$ , and hence at least one of  $(i, j, l)$  is equal to or greater than 1. Therefore  $\mathcal{V}(n-1)(i)$ ,

(iii) implies that the term in question is 0. In case (ii), (B.4) applies to the term. Thus

$$(B.54) \quad (\tilde{\gamma}.vi) = 0.$$

Clearly  $(\tilde{\gamma}.vii)$  does not exist. Concerning the sum  $(\tilde{\gamma}.viii)$  either  $q$  or  $r$  is equal to 0 in each summand and hence (B.4) entails its vanishing. Thus we find

$$(B.55) \quad \Gamma_{2n}^{(0)} = 0.$$

As to  $\Delta_{2n}^{(0)}$ ,  $(\tilde{\delta}.i)$  and  $(\tilde{\delta}.ii)$  vanish by (B.47), and  $(\tilde{\delta}.iii)$  contains no term for  $p_0 = 0$ . Concerning  $(\tilde{\delta}.iv)$ , we rewrite  $\sum_{\substack{q+r=2 \\ i+j=n}} x_{2i}^{(q)}(0, \rho) x_{2j}^{(r)}(0, \rho)$

as follows:

$$(B.56) \quad 2 \sum_{i+j=n} x_{2i}^{(0)}(0, \rho) x_{2j}^{(2)}(0, \rho) + \sum_{i+j=n} x_{2j}^{(1)}(0, \rho) x_{2j}^{(1)}(0, \rho).$$

Then (B.4) implies the vanishing of each term in the first sum. On the other hand, if one of  $(i, j)$  is  $n$  then the other is 0 in each term in the second sum. Hence  $\mathcal{V}(n-1)(ii)$  entails the vanishing of the second sum. Thus we find

$$(B.57) \quad (\tilde{\delta}.iv) = 0.$$

Similarly (B.4) guarantees

$$(B.58) \quad (\tilde{\delta}.v) = 0,$$

and we can readily confirm the vanishing of  $(\tilde{\delta}.vi)$  in the same way as that used for the confirmation of (B.57). The vanishing of  $(\tilde{\delta}.vii)$  and  $(\tilde{\delta}.viii)$  is an immediate consequence of (B.4). Concerning  $(\tilde{\delta}.ix)$  with  $p_0 = 0$ , the constraints on the indices entail

$$(B.59) \quad i, j, k, l \leq n-1,$$

and hence at least two of  $(i, j, k, l)$  are equal to or greater than 1. Hence  $\mathcal{V}(n-1)(i), (iii)$  guarantees that every term in  $(\tilde{\delta}.ix)$  should be 0. As  $(\tilde{\delta}.x)$  does not exist for  $p_0 = 0$ , the last term to be examined for the

confirmation of the vanishing of  $\Delta_{2n}^{(0)}$  is  $(\tilde{\delta}.xi)$ : each term in  $(\tilde{\delta}.x)$  for  $p_0 = 0$  contains the factor  $x_{2i}^{(0)} x_{2j}^{(0)}$ . Thus we find

$$(B.60) \quad (\tilde{\delta}.xi) = 0.$$

Summing up all these we have confirmed

$$(B.61) \quad R_{2n}^{(0)}(s, \rho) = O(s^2),$$

and

$$(B.62) \quad R_{2n}^{(1)}(0, \rho) = 0$$

together with

$$(B.63) \quad \Gamma_{2n}^{(0)} = \Delta_{2n}^{(0)} = 0,$$

which implies

$$(B.64) \quad A_{2n}^{(0)} = B_{2n}^{(0)} = 0.$$

Therefore we find

$$(B.65) \quad \dot{x}_{2n}^{(0)}(0, \rho) = -\tilde{B}_{2n}^{(0)} + \dot{R}_{2n}^{(0)}(0, \rho) = 0,$$

$$(B.66) \quad x_{2n}^{(1)}(0, \rho) = \tilde{A}_{2n}^{(0)} + R_{2n}^{(1)}(0, \rho) = 0.$$

As (B.64), (B.65) and (B.66), together with  $\mathcal{V}(n-1)$ , imply that  $\mathcal{V}(n)$  is validated. Thus the induction proceeds, and the proof of the proposition is completed.

*Remark B.1.* By following the reasoning in Appendix C, one can confirm that, if  $g_{\pm}(t) \neq 0$ ,  $x_2^{(1)}(0, \rho)$ , together with  $(\tilde{A}_2^{(0)}, \tilde{B}_2^{(0)})$ , is different from 0 in general.

□

**C Construction and estimation of the transformation series that brings an M2P1T equation to the Mathieu equation when  $\eta^{-2} \left( \frac{g_+(t)}{(t-a)^2} + \frac{g_-(t)}{(t+a)^2} \right)$  is not 0.**

The purpose of this appendix is to confirm the results in Section 1.1.3 and Section 1.2 without assuming  $g_{\pm}(t) = 0$ . For the sake of definiteness of the description we assume  $B_0^{(0)} = \rho$  (and hence  $x_0^{(0)'}(0, \rho) = 1$ ).

For the sake of computation of terms of the form  $\left( \sum_{l \geq 0} z_l(t) \eta^{-l} \right)^{-p}$  ( $p = 1, 2$ ) we first prepare the following Lemma C.1. The computation of the above series with  $p = 2$  is not used in this appendix but used in Section 1.3. As the reasoning for the case  $p = 2$  is basically the same as that for the case  $p = 1$  we bring them together here.

**Lemma C.1.** *Let  $w_k(t)$  ( $k = 0, 1, 2, \dots, n$ ) be holomorphic functions at  $t = t_0$  and satisfy*

$$(C.1) \quad \frac{dw_0}{dt}(t_0) \neq 0$$

and

$$(C.2) \quad w_k(t_0) = 0 \quad (k = 0, 1, 2, \dots).$$

Then  $f_n(t)$  and  $g_n(t)$  ( $n = 1, 2, \dots$ ) defined by

$$(C.3) \quad f_n(t) = \sum_{k_1+k_2+k_3=n} \frac{dw_{k_1}}{dt} \frac{dw_{k_2}}{dt} \sum_{\mu=\min\{1,k_3\}}^{k_3} \sum_{|\vec{\kappa}|_{\mu}=k_3}^* (-1)^{\mu} \frac{w_{\kappa_1} \cdots w_{\kappa_{\mu}}}{w_0^{\mu}},$$

$$(C.4) \quad g_n(t) = \sum_{k_1+k_2+k_3=n} \frac{dw_{k_1}}{dt} \frac{dw_{k_2}}{dt} \sum_{\mu=\min\{1,k_3\}}^{k_3} \sum_{|\vec{\kappa}|_{\mu}=k_3}^* (-1)^{\mu} (\mu + 1) \frac{w_{\kappa_1} \cdots w_{\kappa_{\mu}}}{w_0^{\mu}}$$

satisfy the following relations:

$$(C.5) \quad f_n(t_0) = \frac{dw_0}{dt} \frac{dw_n}{dt} \Big|_{t=t_0},$$

$$(C.6) \quad g_n(t_0) = 0.$$

In particular,  $w_0^{-1}g_n(t)$  is holomorphic at  $t = t_0$ .

*Proof.* Using the assumption (C.2) we define  $\alpha_k$  by

$$(C.7) \quad \alpha_k = \frac{w_k}{w_0} \Big|_{t=t_0} = \left( \frac{dw_0}{dt} \right)^{-1} \frac{dw_k}{dt} \Big|_{t=t_0}.$$

In order to obtain (C.5), it suffices to show

$$(C.8) \quad \alpha_n = \sum_{k_1+k_2+k_3=n} \alpha_{k_1} \alpha_{k_2} \sum_{\mu=\min\{1,k_3\}}^{k_3} (-1)^\mu \beta_{k_3}^{(\mu)}$$

for  $n \geq 1$ , where  $\beta_k^{(\mu)}$  is a constant defined by

$$(C.9) \quad \beta_k^{(\mu)} = \sum_{|\vec{k}|_\mu=k}^* \alpha_{\kappa_1} \cdots \alpha_{\kappa_\mu}.$$

Since  $\alpha_0 = \beta_0^{(0)} = 1$ ,  $\beta_n^{(k)} = 0$  for  $k \geq n+1$  and

$$(C.10) \quad \sum_{k_1+k_2=n}^* \alpha_{k_1} \beta_{k_2}^{(\mu)} = \beta_n^{(\mu+1)},$$

we find

$$(C.11) \quad \begin{aligned} & \sum_{k_1+k_2+k_3=n} \alpha_{k_1} \alpha_{k_2} \sum_{\mu=\min\{1,k_3\}}^{k_3} (-1)^\mu \beta_{k_3}^{(\mu)} \\ &= 2\alpha_0 \beta_0^{(0)} \alpha_n + \beta_0^{(0)} \sum_{k_1+k_2=n}^* \alpha_{k_1} \alpha_{k_2} + \alpha_0^2 \sum_{\mu=1}^n (-1)^\mu \beta_n^{(\mu)} \\ & \quad + 2\alpha_0 \sum_{k_1+k_2=n}^* \alpha_{k_1} \sum_{\mu=1}^{k_2} (-1)^\mu \beta_{k_2}^{(\mu)} + \sum_{k_1+k_2+k_3=n}^* \alpha_{k_1} \alpha_{k_2} \sum_{\mu=1}^{k_3} (-1)^\mu \beta_{k_3}^{(\mu)} \\ &= 2\alpha_n + \beta_n^{(2)} + \sum_{\mu=1}^n (-1)^\mu \beta_n^{(\mu)} + 2 \sum_{\mu=1}^{n-1} (-1)^\mu \beta_n^{(\mu+1)} + \sum_{\mu=1}^{n-2} (-1)^\mu \beta_n^{(\mu+2)} \\ &= 2\alpha_n - \beta_n^{(1)}. \end{aligned}$$

Since  $\beta_k^{(1)} = \alpha_k$ , we obtain (C.8).

Next, we show

$$(C.12) \quad \sum_{k_1+k_2+k_3=n} \alpha_{k_1} \alpha_{k_2} \sum_{\mu=\min\{1,k_3\}}^{k_3} (-1)^\mu (\mu+1) \beta_{k_3}^{(\mu)} = 0.$$

By using the same result as above, we can rewrite the left-hand side of (C.12) as follows:

$$\begin{aligned} (C.13) \quad & \beta_0^{(0)} \sum_{k_1+k_2=n} \alpha_{k_1} \alpha_{k_2} + \alpha_0^2 \sum_{\mu=1}^n (-1)^\mu (\mu+1) \beta_n^{(\mu)} \\ & + 2\alpha_0 \sum_{k_1+k_2=n}^* \alpha_{k_1} \sum_{\mu=1}^{k_2} (-1)^\mu (\mu+1) \beta_{k_2}^{(\mu)} \\ & + \sum_{k_1+k_2+k_3=n}^* \alpha_{k_1} \alpha_{k_2} \sum_{\mu=1}^{k_3} (-1)^\mu (\mu+1) \beta_{k_3}^{(\mu)} \\ & = \sum_{k_1+k_2=n} \alpha_{k_1} \alpha_{k_2} + \sum_{\mu=1}^n (-1)^\mu (\mu+1) \beta_n^{(\mu)} \\ & \quad + 2 \sum_{\mu=1}^{n-1} (-1)^\mu (\mu+1) \beta_n^{(\mu+1)} + \sum_{\mu=1}^{n-2} (-1)^\mu (\mu+1) \beta_n^{(\mu+2)} \\ & = 2\alpha_n + \beta_n^{(2)} - 2\beta_n^{(1)} - \beta_n^{(2)}. \end{aligned}$$

Since  $\beta_n^{(1)} = \alpha_n$ , we obtain (C.12); thus we have confirmed (C.6).  $\square$

Let us now confirm Proposition 1.1.3.2 together with the estimate  $[G'; p, 2n]$  given in Proposition C.1 below, which is totally the same estimates with  $[G; p, 2n]$  in Proposition 1.2.1, when the lower order term

$$(C.14) \quad \eta^{-2} \left( \frac{g_+(t)}{(t-a)^2} + \frac{g_-(t)}{(t+a)^2} \right)$$

is not assumed to be 0. In what follows we sometimes refer to this lower order term as the additional term so that the background of our reasoning may become apparent. The main reason why we perform

the construction and the estimation simultaneously is that we want to use the analyticity of  $\{x_{2k}(t, a, \rho)\}_{0 \leq k \leq n-1}$  on a sufficiently large set, say on  $E_{r, R_0}^1$  in constructing  $x_{2k}(t, a, \rho)$ ; the analyticity of  $\{x_{2k}\}_{0 \leq k \leq n-1}$  enables us to find Lemma C.2. The relation (C.18) leads us to introduce the auxiliary functions  $\{y_{\pm, 2k}(t, a, \rho)\}_{0 \leq k \leq n-1}$ , which facilitates the manipulation of the singularities at  $t = \pm a$  contained in the additional terms, as we will see below.

**Proposition C.1.** *There exist positive constants  $(r_0, R, A)$  and a sufficiently small constant  $N_0$  for which the following estimate  $[G'; p, 2n]$  holds for every  $p \geq 0$ , every  $n \geq 1$ , every  $\rho$  in  $\{\rho \in \mathbb{C}; 0 < |\rho| \leq r_0\}$  and any positive constant  $\varepsilon$  that is smaller than  $r_0/3$ :*

$$[G'; p, 2n] = \begin{cases} (p, 2n)(i) & |x_{2n}^{(p+1)}(0, \rho)| \leq N_0 C(p) (R|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n, \\ (p, 2n)(ii) & |\tilde{A}_{2n}^{(p)}| \leq N_0 C(p) (R|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n, \\ (p, 2n)(iii) & |\tilde{B}_{2n}^{(p)}| \leq N_0 C(p) (R|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n, \\ (p, 2n)(iv) & \|x_{2n}^{(p)}\|_{[r_0-\varepsilon]} \leq N_0 C(p) (R|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n, \\ (p, 2n)(v) & \|\dot{x}_{2n}^{(p)}\|_{[r_0-\varepsilon]} \leq N_0 C(p) (R|\rho|^{-1})^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n. \end{cases}$$

To confirm Proposition 1.1.3.2 and Proposition C.1 when the potential  $Q$  contains the additional terms, we first note that (1.1.6) requires  $x = \sum_{n \geq 0} x_{2n}(t, a, \rho) \eta^{-2n}$ ,  $A = \sum_{n \geq 0} A_{2n}(a, \rho) \eta^{-2n}$ , and  $B = \sum_{n \geq 0} B_{2n}(a, \rho) \eta^{-2n}$ , should satisfy

(C.15)

$$\begin{aligned} & (x^2 - a^2) \left\{ f + \eta^{-2} \left( \frac{t+a}{t-a} g_+(t) + \frac{t-a}{t+a} g_-(t) \right) \right\} \\ &= (t^2 - a^2) \left( \frac{\partial x}{\partial t} \right)^2 \left\{ aA + xB + \eta^{-2} \left( \frac{x+a}{x-a} g_+(a) + \frac{x-a}{x+a} g_-(-a) \right) \right\} \\ & \quad - \frac{1}{2} \eta^{-2} (t^2 - a^2) (x^2 - a^2) \{x; t\}. \end{aligned}$$

Since the additional terms do not affect the relation that  $A_0(a, \rho)$ ,  $B_0(a, \rho)$  and  $x_0(t, a, \rho)$  should satisfy, Proposition 1.1.2.1 and Lemma 1.2.3 apply to the case where  $Q$  contains the additional terms.



In parallel with (1.1.3.36), the comparison of the coefficients of  $\eta^{-2n}$  ( $n = 1, 2, \dots$ ) of (C.15) leads us to the following relation:

(C.16)

$$\begin{aligned}
& \left( \sum_{k_1+k_2=n} x_{2k_1} x_{2k_2} \right) f \\
& + \left( \sum_{k_1+k_2=n-1} x_{2k_1} x_{2k_2} - \delta_{n,1} a^2 \right) \left( \frac{t+a}{t-a} g_+(t) + \frac{t-a}{t+a} g_-(t) \right) \\
& = (t^2 - a^2) \left( \sum_{k_1+k_2+k_3=n} x'_{2k_1} x'_{2k_2} a A_{2k_3} + \sum_{k_1+\dots+k_4=n} x'_{2k_1} x'_{2k_2} x_{2k_3} B_{2k_4} \right) \\
& + (t^2 - a^2) \sum_{k_1+\dots+k_4=n-1} x'_{2k_1} x'_{2k_2} x_{2k_3} \\
& \quad \times \left( \frac{g_+(a)}{x_0 - a} \sum_{\mu=\min\{1,k_4\}}^{k_4} \sum_{|\vec{\kappa}|_\mu=k_4}^* (-1)^\mu \frac{x_{2\kappa_1} \cdots x_{2\kappa_\mu}}{(x_0 - a)^\mu} \right. \\
& \quad \left. + \frac{g_-(-a)}{x_0 + a} \sum_{\mu=\min\{1,k_4\}}^{k_4} \sum_{|\vec{\kappa}|_\mu=k_4}^* (-1)^\mu \frac{x_{2\kappa_1} \cdots x_{2\kappa_\mu}}{(x_0 + a)^\mu} \right) \\
& + (t^2 - a^2) a \sum_{k_1+k_2+k_3=n-1} x'_{2k_1} x'_{2k_2} \\
& \quad \times \left( \frac{g_+(a)}{x_0 - a} \sum_{\mu=\min\{1,k_3\}}^{k_3} \sum_{|\vec{\kappa}|_\mu=k_3}^* (-1)^\mu \frac{x_{2\kappa_1} \cdots x_{2\kappa_\mu}}{(x_0 - a)^\mu} \right. \\
& \quad \left. - \frac{g_-(-a)}{x_0 + a} \sum_{\mu=\min\{1,k_3\}}^{k_3} \sum_{|\vec{\kappa}|_\mu=k_3}^* (-1)^\mu \frac{x_{2\kappa_1} \cdots x_{2\kappa_\mu}}{(x_0 + a)^\mu} \right) \\
& - \frac{1}{2} (t^2 - a^2) \sum_{k_1+k_2+k_3=n-1} x_{2k_1} x_{2k_2} \{x; t\}_{2k_3} \\
& + \frac{1}{2} (t^2 - a^2) a^2 \{x; t\}_{2(n-1)},
\end{aligned}$$

where  $\delta_{n,1}$  is the Kronecker delta and  $\{x; t\}_{2k}$  is the coefficient of  $\eta^{-2k}$  of  $\{x; t\}$ .

Let us now confirm that  $\mathcal{A}_{2n}(p)$  ( $p \geq 0$ ) hold under the assumption that  $\mathcal{A}_{2k}(p)$  and  $[G''; p, 2k]$  hold for  $0 \leq k \leq n-1$  and  $p \geq 0$ . It follows

from  $[G'; p, 2k]$  ( $p \geq 0$ ) that  $x_{2k}(s, a, \rho)$  is holomorphic on

$$(C.17) \quad \tilde{E}_{r_0, 2R}^1 = \{(s, a, \rho) \in \mathbb{C}^3 : |s| \leq r_0, 0 < |\rho| \leq r_0, |a| \leq (2R)^{-1}|\rho|\}.$$

Using this analyticity we first show the following

**Lemma C.2.**

$$(C.18) \quad x_{2k}(t, a, \rho)|_{t=\pm a} = 0$$

holds for  $1 \leq k \leq n-1$ .

*Proof.* It follows from (C.16) with  $n=1$  that  $x_2$  satisfies the following relation:

$$(C.19) \quad \begin{aligned} & 2x_0x_2f + (x_0^2 - a^2) \left( \frac{t+a}{t-a}g_+(t) + \frac{t-a}{t+a}g_-(t) \right) \\ &= (t^2 - a^2) \left( \sum_{k_1+k_2+k_3=1} x'_{2k_1}x'_{2k_2}aA_{2k_3} + \sum_{k_1+\dots+k_4=1} x'_{2k_1}x'_{2k_2}x_{2k_3}B_{2k_4} \right) \\ &+ (t^2 - a^2)(x'_0)^2 \left( \frac{x_0+a}{x_0-a}g_+(a) + \frac{x_0-a}{x_0+a}g_-(-a) \right) \\ &- \frac{1}{2}(t^2 - a^2)(x_0^2 - a^2)\{x; t\}_0. \end{aligned}$$

Since  $x_0$  satisfies (1.3.11), by setting  $t = \pm a$  in (C.19), we obtain

$$(C.20) \quad 2ax_2(t, a, \rho)f(t, a, \rho)|_{t=\pm a} = 0.$$

Hence (C.18) for  $k=1$  follows from (1.3.16). Next we show (C.18) for  $k=l$  ( $2 \leq l \leq n-1$ ) under the assumption that (C.18) holds for  $1 \leq k \leq l-1$ . By setting  $t = a$  in (C.16) with  $n=l$ , we obtain

$$(C.21) \quad \begin{aligned} & (2ax_{2l}f + 4a^2g_+x'_{2(l-1)})|_{t=a} \\ &= (t^2 - a^2)g_+(a)\frac{x_0+a}{x_0-a} \end{aligned}$$

$$\times \sum_{k_1+k_2+k_4=l-1} x'_{2k_1} x'_{2k_2} \sum_{\mu=\min\{1,k_4\}}^{k_4} \sum_{|\vec{\kappa}|_\mu=k_4}^* (-1)^\mu \frac{x_{2\kappa_1} \cdots x_{2\kappa_\mu}}{(x_0 - a)^\mu} \Big|_{t=a}.$$

Since  $w_0 = x_0 - a$  and  $w_k = x_{2k}$  satisfy (C.1) and (C.2) at  $t_0 = a$ , (C.5) implies that the right-hand side of (C.21) is equal to  $4a^2 g_+ x'_{2(l-1)}|_{t=a}$ . Hence we obtain

$$(C.22) \quad 2ax_{2l}f|_{t=a} = 0.$$

We then see that  $x_{2l}|_{t=a} = 0$ . Using the same reasoning as above, we find  $x_{2l}|_{t=-a} = 0$  holds. Hence we obtain (C.18) for  $k = l$ .  $\square$

Let us now define  $y_{\pm,2k}(t, a, \rho)$  ( $k = 0, 1, 2, \dots$ ) by

$$(C.23) \quad y_{\pm,0} = \frac{x_0 \mp a}{t \mp a}$$

$$(C.24) \quad y_{\pm,2k} = \frac{x_{2k}}{t \mp a}.$$

Then, from Theorem 1.3.1 and (C.18), we find that  $(y_{\pm,0})^{-1}$  and  $y_{\pm,2k}$  ( $k = 0, 1, 2, \dots$ ) are holomorphic on  $\tilde{E}_{r_0,2R}^1$ . We denote the coefficients of  $a^p$  of  $(y_{\pm,0})^{-\mu}$  ( $\mu = 1, 2, \dots$ ) and  $y_{\pm,2k}$  by  $w_{\pm}^{\mu,(p)}$  and  $y_{\pm,2k}^{(p)}$  respectively as follows:

$$(C.25) \quad (y_{\pm,0})^{-\mu}(t, a, \rho) = \sum_{p=0}^{\infty} w_{\pm}^{\mu,(p)}(t, \rho) a^p,$$

$$(C.26) \quad y_{\pm,2k}(t, a, \rho) = \sum_{p=0}^{\infty} y_{\pm,2k}^{(p)}(t, \rho) a^p.$$

We also denote the coefficients of  $t^p$  of  $g_{\pm}$  by  $g_{\pm}^{(p)}$ , i.e.,

$$(C.27) \quad g_{\pm}(t) = \sum_{p=0}^{\infty} g_{\pm}^{(p)} t^p.$$

In parallel with (1.1.3.37), comparison of the coefficients of  $a^p$  in (C.16) leads us to the following relation:

$$(C.28) \quad \sum_{\substack{l_1+l_2+l_3=p \\ k_1+k_2=n}} x_{2k_1}^{(l_1)} x_{2k_2}^{(l_2)} f^{(l_3)} + \mathcal{F}_{2n}^{(p)}$$

$$\begin{aligned}
&= t^2 \left[ \sum_{\substack{l_1+l_2+l_3=p-1 \\ k_1+k_2+k_3=n}} x_{2k_1}^{(l_1)'} x_{2k_2}^{(l_2)'} A_{2k_3}^{(l_3)} + \sum_{\substack{l_1+\dots+l_4=p \\ k_1+\dots+k_4=n}} x_{2k_1}^{(l_1)'} x_{2k_2}^{(l_2)'} x_{2k_3}^{(l_3)} B_{2k_4}^{(l_4)} \right. \\
&\quad - \frac{1}{2} \sum_{\substack{l_1+l_2+l_3=p \\ k_1+k_2+k_3=n-1}} x_{2k_1}^{(l_1)} x_{2k_2}^{(l_2)} \{x; t\}_{2k_3}^{(l_3)} + \frac{1}{2} \{x; t\}_{2(n-1)}^{(p-2)} \Big] \\
&\quad - \left[ \sum_{\substack{l_1+l_2+l_3=p-3 \\ k_1+k_2+k_3=n}} x_{2k_1}^{(l_1)'} x_{2k_2}^{(l_2)'} A_{2k_3}^{(l_3)} + \sum_{\substack{l_1+\dots+l_4=p-2 \\ k_1+\dots+k_4=n}} x_{2k_1}^{(l_1)'} x_{2k_2}^{(l_2)'} x_{2k_3}^{(l_3)} B_{2k_4}^{(l_4)} \right. \\
&\quad \left. - \frac{1}{2} \sum_{\substack{l_1+l_2+l_3=p-2 \\ k_1+k_2+k_3=n-1}} x_{2k_1}^{(l_1)} x_{2k_2}^{(l_2)} \{x; t\}_{2k_3}^{(l_3)} + \frac{1}{2} \{x; t\}_{2(n-1)}^{(p-4)} \right] + \mathcal{G}_{2n}^{(p)},
\end{aligned}$$

where  $\mathcal{F}_{2n}^{(p)}$  and  $\mathcal{G}_{2n}^{(p)}$  are functions that depend only on  $x_{2k}^{(l)}$ ,  $y_{\pm, 2k}^{(l)}$  ( $0 \leq k \leq n-1$ ,  $l \geq 0$ ) and  $g_{\pm}$ . The concrete forms of  $\mathcal{F}_{2n}^{(p)}$  and  $\mathcal{G}_{2n}^{(p)}$  are given as follows:

$$\begin{aligned}
(C.29) \quad \mathcal{F}_2^{(p)} &= t g_+(t) \sum_{l_1+l_2=p} x_0^{(l_1)} y_{+,0}^{(l_2)} + g_+(t) \sum_{l_1+l_2=p-1} x_0^{(l_1)} y_{+,0}^{(l_2)} \\
&\quad + t g_+(t) y_{+,0}^{(p-1)} + g_+(t) y_{+,0}^{(p-2)} \\
&\quad + t g_-(t) \sum_{l_1+l_2=p} x_0^{(l_1)} y_{-,0}^{(l_2)} - g_-(t) \sum_{l_1+l_2=p-1} x_0^{(l_1)} y_{-,0}^{(l_2)} \\
&\quad - t g_-(t) y_{-,0}^{(p-1)} + g_-(t) y_{-,0}^{(p-2)},
\end{aligned}$$

$$\begin{aligned}
(C.30) \quad \mathcal{F}_{2n}^{(p)} &= 2t g_+(t) \sum_{l_1+l_2=p} x_0^{(l_1)} y_{+,2(n-1)}^{(l_2)} \\
&\quad + t g_+(t) \sum_{k_1+k_2=n-1}^* \sum_{l_1+l_2=p} x_{2k_1}^{(l_1)} y_{+,2k_2}^{(l_2)} \\
&\quad + 2g_+(t) \sum_{l_1+l_2=p-1} x_0^{(l_1)} y_{+,2(n-1)}^{(l_2)} \\
&\quad + g_+(t) \sum_{k_1+k_2=n-1}^* \sum_{l_1+l_2=p-1} x_{2k_1}^{(l_1)} y_{+,2k_2}^{(l_2)}
\end{aligned}$$

$$\begin{aligned}
& + 2tg_-(t) \sum_{l_1+l_2=p} x_0^{(l_1)} y_{-,2(n-1)}^{(l_2)} \\
& + tg_-(t) \sum_{k_1+k_2=n-1}^* \sum_{l_1+l_2=p} x_{2k_1}^{(l_1)} y_{-,2k_2}^{(l_2)} \\
& - 2g_-(t) \sum_{l_1+l_2=p-1} x_0^{(l_1)} y_{-,2(n-1)}^{(l_2)} \\
& - g_-(t) \sum_{k_1+k_2=n-1}^* \sum_{l_1+l_2=p-1} x_{2k_1}^{(l_1)} y_{-,2k_2}^{(l_2)}
\end{aligned}$$

for  $n \geq 2$  and

(C.31)

$$\begin{aligned}
\mathcal{G}_{2n}^{(p)} = & \left( t \sum_{\substack{k_1+\dots+k_4=n-1 \\ l_1+\dots+l_6=p}} x_{2k_1}^{(l_1)'} x_{2k_2}^{(l_2)'} x_{2k_3}^{(l_3)} g_+^{(l_4)} \right. \\
& + \sum_{\substack{k_1+\dots+k_4=n-1 \\ l_1+\dots+l_6=p-1}} x_{2k_1}^{(l_1)'} x_{2k_2}^{(l_2)'} x_{2k_3}^{(l_3)} g_+^{(l_4)} \Big) \\
& \times \sum_{\mu=\min\{1,k_4\}}^{k_4} w_+^{\mu+1,(l_5)} \sum_{|\vec{\kappa}|_\mu=k_4}^* \sum_{|\vec{\lambda}|_\mu=l_6} (-1)^\mu y_{+,2\kappa_1}^{(\lambda_{\mu+1})} \dots y_{+,2\kappa_\mu}^{(\lambda_{2\mu})} \\
& + \left( t \sum_{\substack{k_1+\dots+k_4=n-1 \\ l_1+\dots+l_6=p}} x_{2k_1}^{(l_1)'} x_{2k_2}^{(l_2)'} x_{2k_3}^{(l_3)} (-1)^{l_4} g_-^{(l_4)} \right. \\
& - \sum_{\substack{k_1+\dots+k_4=n-1 \\ l_1+\dots+l_6=p-1}} x_{2k_1}^{(l_1)'} x_{2k_2}^{(l_2)'} x_{2k_3}^{(l_3)} (-1)^{l_4} g_-^{(l_4)} \Big) \\
& \times \sum_{\mu=\min\{1,k_4\}}^{k_4} w_-^{\mu+1,(l_5)} \sum_{|\vec{\kappa}|_\mu=k_4}^* \sum_{|\vec{\lambda}|_\mu=l_6} (-1)^\mu y_{-,2\kappa_1}^{(\lambda_{\mu+1})} \dots y_{-,2\kappa_\mu}^{(\lambda_{2\mu})} \\
& + \left( t \sum_{\substack{k_1+\dots+k_4=n-1 \\ l_1+\dots+l_5=p-1}} x_{2k_1}^{(l_1)'} x_{2k_2}^{(l_2)'} g_+^{(l_3)} \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{k_1+\dots+k_4=n-1 \\ l_1+\dots+l_5=p-2}} x_{2k_1}^{(l_1)'} x_{2k_2}^{(l_2)'} g_+^{(l_3)} \\
& \times \sum_{\mu=\min\{1,k_4\}}^{k_4} w_+^{\mu+1,(l_4)} \sum_{|\vec{\kappa}|_\mu=k_4}^* \sum_{|\vec{\lambda}|_\mu=l_5} (-1)^\mu y_{+,2\kappa_1}^{(\lambda_{\mu+1})} \dots y_{+,2\kappa_\mu}^{(\lambda_{2\mu})} \\
& - \left( t \sum_{\substack{k_1+\dots+k_4=n-1 \\ l_1+\dots+l_5=p-1}} x_{2k_1}^{(l_1)'} x_{2k_2}^{(l_2)'} (-1)^{l_3} g_-^{(l_3)} \right. \\
& \quad \left. - \sum_{\substack{k_1+\dots+k_4=n-1 \\ l_1+\dots+l_5=p-2}} x_{2k_1}^{(l_1)'} x_{2k_2}^{(l_2)'} (-1)^{l_3} g_-^{(l_3)} \right) \\
& \times \sum_{\mu=\min\{1,k_4\}}^{k_4} w_-^{\mu+1,(l_4)} \sum_{|\vec{\kappa}|_\mu=k_4}^* \sum_{|\vec{\lambda}|_\mu=l_5} (-1)^\mu y_{-,2\kappa_1}^{(\lambda_{\mu+1})} \dots y_{-,2\kappa_\mu}^{(\lambda_{2\mu})}
\end{aligned}$$

for  $n \geq 1$ .

Now, we define  $\tilde{\Phi}_{2n}^{(p)}$  and  $\tilde{R}_{2n}^{(p)}$  by

$$(C.32) \quad \tilde{\Phi}_{2n}^{(p)} = \Phi_{2n}^{(p)} + \mathcal{F}_{2n}^{(p)} - \mathcal{G}_{2n}^{(p)},$$

$$(C.33) \quad \tilde{R}_{2n}^{(p)} = R_{2n}^{(p)} + \frac{t^{-2}}{B_0^{(0)}} \left( \frac{dt}{ds} \right)^2 (\mathcal{F}_{2n}^{(p)} - \mathcal{G}_{2n}^{(p)}),$$

where  $\Phi_{2n}^{(p)}$  and  $R_{2n}^{(p)}$  are respectively given by (1.1.3.38) and (1.1.3.43). It is evident from (C.28) that, if we want to construct  $\{x, A, B\}$  when  $g_\pm \neq 0$ ,  $\tilde{\Phi}_{2n}^{(p)}$  (resp.,  $\tilde{R}_{2n}^{(p)}$ ) is the required substitute of  $\Phi_{2n}^{(p)}$  (resp.,  $R_{2n}^{(p)}$ ) used in Section 1.1.3. Since  $\mathcal{F}_{2n}^{(p)} \equiv 0$  and  $\mathcal{G}_{2n}^{(p)} \equiv 0$  for any  $p \geq 0$  and  $q \geq 0$ , by the same reasoning as that in Section 1.1.3, we find  $\mathcal{A}_{2n}(p)$  ( $p \geq 0$ ) is also valid in this case.

Next we estimate the constructed series as Proposition C.1 requires. For this purpose we prepare the following

**Lemma C.3.** *The series  $\mathcal{F}_{2n}^{(p)}$  and  $\mathcal{G}_{2n}^{(p)}$  ( $p \geq 0$ ) satisfy the following estimates for some positive constant  $M_0$  under the assumption that*

$[G; p, 0]$  and  $[G'; p, 2k]$  ( $1 \leq k \leq n-1, p \geq 0$ ) hold:

(C.34)

$$|\mathcal{F}_{2n}^{(p+3)}|_{t=0} \leq M_0 A^{-1} N_0 C(p) (R|\rho|^{-1})^p (2n-2)! \varepsilon^{-2n+2} (A|\rho|^{-1})^n,$$

(C.35)

$$|\mathcal{F}_{2n}^{(p+2)'}|_{t=0} \leq M_0 A^{-1} N_0 C(p) (R|\rho|^{-1})^p (2n-2)! \varepsilon^{-2n+2} (A|\rho|^{-1})^n,$$

(C.36)

$$\|\mathcal{F}_{2n}^{(p)}\|_{[r_0-\varepsilon]} \leq M_0 N_0 C(p) (R|\rho|^{-1})^p (2n-2)! \varepsilon^{-2n+2} (A|\rho|^{-1})^{n-1},$$

(C.37)

$$|\mathcal{G}_{2n}^{(p+3)}|_{t=0} \leq M_0 A^{-1} N_0 C(p) (R|\rho|^{-1})^p (2n-2)! \varepsilon^{-2n+2} (A|\rho|^{-1})^n,$$

(C.38)

$$|\mathcal{G}_{2n}^{(p+2)'}|_{t=0} \leq M_0 A^{-1} N_0 C(p) (R|\rho|^{-1})^p (2n-2)! \varepsilon^{-2n+2} (A|\rho|^{-1})^n,$$

(C.39)

$$\|\mathcal{G}_{2n}^{(p)}\|_{[r_0-\varepsilon]} \leq M_0 N_0 C(p) (R|\rho|^{-1})^p (2n-2)! \varepsilon^{-2n+2} (A|\rho|^{-1})^{n-1}.$$

*Proof.* To begin with, we derive the estimates of  $y_{\pm,2k}^{(p)}$  and  $w_{\pm}^{(p)}$  from those of  $x_{\pm,2k}^{(p)}$  as follows:

**Lemma C.4.** *The functions  $y_{\pm,2k}^{(p)}$  and  $w_{\pm}^{\mu,(p)}$  ( $0 \leq k \leq n-1, p \geq 0, \mu \geq 1$ ) satisfy the following estimates for some positive constant  $M$  under the assumption that  $[G; p, 0]$  and  $[G'; p, 2k]$  ( $1 \leq k \leq n-1, p \geq 0$ ) hold:*

$$(C.40) \quad \|y_{\pm,0}^{(p)}\|_{[r_0]} \leq M C(p) (R|\rho|^{-1})^p,$$

$$(C.41) \quad \|w_{\pm}^{\mu,(p)}\|_{[r_0]} \leq M^{\mu} C(p) (R|\rho|^{-1})^p,$$

$$(C.42) \quad \|y_{\pm,2k}^{(p)}\|_{[r_0-\varepsilon]} \leq M N_0 C(p) (R|\rho|^{-1})^p (2k)! \varepsilon^{-2k} (A|\rho|^{-1})^k.$$

*Proof.* Since  $(y_{\pm,0})^{\pm 1}$  are holomorphic on  $\tilde{E}_{r,2R}^1$  and bounded by some positive constant  $M$  there, we find that, by taking  $R$  sufficiently large if necessary,  $y_{\pm,0}^{(p)}$  and  $w_{\pm}^{\mu,(p)}$  ( $p \geq 0, \mu \geq 1$ ) satisfy (C.40) and (C.41). Further, it follows from the definition of  $y_{\pm,2k}$  that  $y_{\pm,2k}^{(p)}$  satisfy the following relation:

$$(C.43) \quad y_{\pm,2k}^{(p)} = t^{-1} (x_{2k}^{(p)} \pm y_{\pm,2k}^{(p-1)}),$$

where we conventionally regard  $y_{\pm,2k}^{(-1)}$  as 0. It is then evident that we can estimate  $\|y_{\pm,2k}^{(p)}\|_{[r_0-\varepsilon]}$  in an inductive manner with the help of  $[G'; p, 2k]$ . Actually, with an appropriate choice of constants  $M$  and  $R$  that is specified below, the maximum modulus principle enables to find the following:

$$\begin{aligned}
(C.44) \quad \|y_{\pm,2k}^{(p)}\|_{[r_0-\varepsilon]} &\leq \frac{M}{2} (\|x_{2k}^{(p)}\|_{[r_0-\varepsilon]} + \|y_{\pm,2k}^{(p-1)}\|_{[r_0-\varepsilon]}) \\
&\leq \frac{M}{2} (1 + MR^{-1}|\rho|C(p-1)C(p)^{-1}) \\
&\quad \times N_0 C(p) (R|\rho|^{-1})^p (2k)! \varepsilon^{-2k} (A|\rho|^{-1})^k \\
&\leq MN_0 C(p) (R|\rho|^{-1})^p (2k)! \varepsilon^{-2k} (A|\rho|^{-1})^k.
\end{aligned}$$

Here we take  $M > 0$  so that

$$(C.45) \quad \sup_{|s|=r_0-\varepsilon} |t^{-1}(s)| \leq M/2$$

holds for  $0 < \varepsilon < r_0/3$  and assume that, by taking  $R$  sufficiently large,

$$(C.46) \quad MR^{-1}|\rho|C(p-1)C(p)^{-1} \leq 1$$

holds. □

*Remark C.1.* As the recursive relation (C.43) for  $y_{\pm,2k}^{(p)}$  ( $k \geq 1$ ) implies, if we write  $y_{\pm,2k}^{(p)}$  in terms of  $x_{2k}^{(p)}$ , it looks as if it had a pole at  $t = 0$  whose order became higher and higher with increasing  $p$ . However (C.18) guarantees that the pole actually does not appear. This is also the case for  $y_{\pm,0}^{(p)}$  and  $w_{\pm}^{\mu,(p)}$ .

Now let us return to the proof of Lemma C.3. Suppose  $g_{\pm}$  is bounded by some positive constant  $M$  as follows:

$$(C.47) \quad \|g_{\pm}\|_{[r_0]} \leq M.$$

Then, for example, the second term of  $\mathcal{F}_2^{(p+3)}$ , i.e.,  $g_+ \sum_{l_1+l_2=p+2} x_0^{(l_1)} y_{+,0}^{(l_2)}$

is estimated as follows for  $t = 0$ :

$$(C.48) \quad \|g_+\|_{[r_0]} \sum_{l_1+l_2=p+2} |x_0^{(l_1)}(0, \rho)| \|y_{+,0}^{(l_2)}\|_{[r_0]}$$



$$\leq M^2 C_0 C(p+2) (R|\rho|^{-1})^{p+1}.$$

In this way, we can readily confirm that the following estimate holds for  $p \geq 0$ :

$$(C.49) \quad |\mathcal{F}_2^{(p+3)}|_{t=0} \leq 4M^2 C_0 C(p+2) (R|\rho|^{-1})^{p+1}.$$

Therefore, by taking  $M_0$  sufficiently large so that  $4M^2 C_0 R \leq M_0 N_0$  holds, we obtain (C.34) for  $n = 1$ . In the same way, we easily find that  $\mathcal{F}_{2n}^{(p+3)}|_{t=0}$  ( $n = 2, 3, \dots$ ) satisfy (C.34). The estimation of  $\mathcal{F}_{2n}^{(p+2)'}|_{t=0}$  required in (C.35) can be also done in a similar manner; by using Cauchy's inequality we can estimate, for example, the derivative of the third term of  $\mathcal{F}_{2n}^{(p+2)}|_{t=0}$  evaluated at  $t = 0$ , i.e.,  $\left(2g_+(t) \sum_{l_1+l_2=p+1} x_0^{(l_1)} y_{+,2(n-1)}^{(l_2)}\right)'|_{t=0}$  as follows:

$$(C.50) \quad \left| \left(2g_+(t) \sum_{l_1+l_2=p+1} x_0^{(l_1)} y_{+,2(n-1)}^{(l_2)}\right)' \right|_{t=0} \\ \leq \frac{2}{r_0 - \varepsilon} \|g_+\|_{[r_0]} \sum_{l_1+l_2=p+1} \|x_0^{(l_1)}\|_{[r_0]} \|y_{+,2(n-1)}^{(l_2)}\|_{[r_0-\varepsilon]} \\ \leq \frac{3M^2 N_0}{r_0} C(p+2) (R|\rho|^{-1})^{p+1} (2n-2)! \varepsilon^{-2n+2} (A|\rho|^{-1})^{n-1}.$$

In this way, we find the estimation (C.35). The estimate (C.36) is an immediate consequence of the induction hypothesis.

Next, we confirm (C.37). Since

$$(C.51) \quad |x_{2k}^{(l)'}(0, \rho)| \leq \frac{M}{r_0 - \varepsilon} \|x_{2k}^{(l)}\|_{[r_0-\varepsilon]}$$

holds for some positive constant  $M$  and since we may assume that  $g_{\pm}^{(l)}$  satisfy

$$(C.52) \quad |g_{\pm}^{(l)}| \leq MC(l) r_0^{-l},$$

the first term of  $\mathcal{G}_{2n}^{(p+3)}|_{t=0}$  can be estimated as follows:

(C.53)

$$\begin{aligned}
& \sum_{\substack{k_1+\dots+k_4=n-1 \\ l_1+\dots+l_6=p+2}} |x_{2k_1}^{(l_1)'}(0, \rho)| |x_{2k_2}^{(l_2)'}(0, \rho)| |x_{2k_3}^{(l_3)}(0, \rho)| |g_+^{(l_4)}| \\
& \times \sum_{\mu=\min\{1, k_4\}}^{k_4} \|w_+^{\mu+1, (l_5)}\|_{[r_0]} \sum_{|\vec{\kappa}|_\mu=k_4}^* \sum_{|\vec{\lambda}|_\mu=l_6} \|y_{+, 2\kappa_1}^{(\lambda_{\mu+1})}\|_{[r_0-\varepsilon]} \cdots \|y_{+, 2\kappa_\mu}^{(\lambda_{2\mu})}\|_{[r_0-\varepsilon]} \\
& \leq \frac{M^3}{(r_0 - \varepsilon)^2} (R|\rho|^{-1})^{p+1} \varepsilon^{-2n+2} (A|\rho|^{-1})^{n-1} \\
& \times \sum_{\substack{k_1+\dots+k_4=n-1 \\ l_1+\dots+l_6=p+2}} C(l_1)C(l_2)C(l_3)C(l_4)(2k_1)!(2k_2)!(2k_3)! \left(\frac{|\rho|}{r_0 R}\right)^{l_4} \\
& \times \sum_{\mu=\min\{1, k_4\}}^{k_4} M^{2\mu+1} N_0^\mu C(l_5)C(l_6) \sum_{|\vec{\kappa}|_\mu=k_4}^* (2\kappa_1)! \cdots (2\kappa_\mu)! \\
& \leq 9r_0^{-2} M^3 e^{4M^2 N_0} C(p+2) (R|\rho|^{-1})^{p+1} (2n-2)! \varepsilon^{-2n+2} (A|\rho|^{-1})^{n-1}.
\end{aligned}$$

Similar estimation is validated for other terms in  $\mathcal{G}_{2n}^{(p+3)}|_{t=0}$ . Hence, by taking  $M_0$  so that  $36r_0^{-2} M^3 e^{4M^2 N_0} R \leq N_0 M_0$  holds, we obtain (C.37). We can confirm (C.38) in a similar manner. The validation of (C.39) is a straightforward task.  $\square$

Finally let us discuss how to deduce  $[G'; p_0, 2n]$  from Lemma C.3. Since the estimates (1.2.161) still holds, we can deduce the following estimates for  $\tilde{R}_{2n}^{(p_0+1)}(0, \rho)$  from (C.34) and (C.37) with  $p = p_0 - 2$ :

$$(C.54) \quad |\tilde{R}_{2n}^{(p_0+1)}(0, \rho)| \leq N_1 N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n,$$

where

$$(C.55) \quad N_1 = M(C_0 + N_0 + R^{-1} + (N_0 A)^{-1})$$

with a positive constant  $M$  that is independent of  $C_0$ ,  $N_0$ ,  $R$  and  $A$ . Since (1.2.163) and (1.2.177) also hold, we obtain the following estimates from (C.36) and (C.39) with  $p = p_0$ :

$$(C.56) \quad |\dot{\tilde{R}}_{2n}^{(p_0)}(0, \rho)| \leq N_1 N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n,$$

$$(C.57) \quad \|\tilde{R}_{2n}^{(p_0)}\|_{[r_0-\varepsilon]} \leq N_1 N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n.$$

Now let us define  $\tilde{\Gamma}_{2n}^{(p)}$  and  $\tilde{\Delta}_{2n}^{(p)}$  by

$$(C.58) \quad \tilde{\Gamma}_{2n}^{(p)} = \Gamma_{2n}^{(p)} + (\mathcal{F}_{2n}^{(p+3)} - \mathcal{G}_{2n}^{(p+3)})|_{t=0},$$

$$(C.59) \quad \tilde{\Delta}_{2n}^{(p)} = \Delta_{2n}^{(p)} + (\mathcal{F}_{2n}^{(p+2)'} - \mathcal{G}_{2n}^{(p+2)'})|_{t=0}.$$

(Here we note that  $\Gamma_{2n}^{(p)}$  and  $\Delta_{2n}^{(p)}$  are obtained from  $\Phi_{2n}^{(p+3)}|_{t=0}$  and  $\Phi_{2n}^{(p+2)'}|_{t=0}$  respectively.) Hence, in view of (C.32), we find that what plays the role of  $\Gamma_{2n}^{(p)}$  (resp.,  $\Delta_{2n}^{(p)}$ ) in this case is  $\tilde{\Gamma}_{2n}^{(p)}$  (resp.,  $\tilde{\Delta}_{2n}^{(p)}$ ). Then, combining (1.2.166), we obtain the following estimates from (C.34), (C.35), (C.37) and (C.38) with  $p = p_0$ :

$$(C.60) \quad |\tilde{\Gamma}_{2n}^{(p_0)}|, |\tilde{\Delta}_{2n}^{(p_0)}| \leq N_1 N_0 C(p_0) (R|\rho|^{-1})^{p_0} (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n.$$

Thus, by the same reasoning with part [III] and [IV] in the proof of Proposition 1.2.1, we find that  $[G'; p, 2n]$  follows from (C.54), (C.56), (C.57) and (C.60). Therefore, the induction proceeds, and hence, we obtain Proposition C.1.

## References

- [A1] T. Aoki: Symbols and formal symbols of pseudodifferential operators, *Advanced Studies in Pure Mathematics*, **4**, Kinokuniya, 1984, pp.181–208.
- [A2] ———: Calcul exponentiel des opérateurs microdifférentiels d'ordre infini. I, *Ann. Inst. Fourier, Grenoble* **33** (1983), 227–250.
- [AKY] T. Aoki, K. Kataoka and S. Yamazaki: *Hyperfunctions, FBI transformations and pseudo-differential operators of infinite order*, (in Japanese) Kyoritsu-Shuppan CO., LTD, 2004.
- [AKT1] T. Aoki, T. Kawai and T. Takei: The Bender-Wu analysis and the Voros theory, *Special Functions*, Springer-Verlag, 1991, pp.1–29.
- [AKT2] ———: New turning points in the exact WKB analysis for higher-order ordinary differential equations, *Analyse algébrique des perturbations singulières, I, Méthodes résurgentes*, Hermann, 1994, pp.69–84.
- [AKT3] ———: WKB analysis of Painlevé transcendents with a large parameter. II. — Multiple-scale analysis of Painlevé transcendents, *Structure of Solutions of Differential Equations*, World Scientific, 1996, pp.1–49.
- [AKT4] ———: The Bender-Wu analysis and the Voros theory. II, *Advanced Studies in Pure Mathematics*, **54**, Math. Soc. Japan, 2009, pp.19–94.
- [AY] T. Aoki and J. Yoshida: Microlocal reduction of ordinary differential operators with a large parameter, *Publ. RIMS, Kyoto Univ.*, **29**(1993), 959–975.
- [BW] C. M. Bender and T. T. Wu: Anharmonic Oscillator, *Phys. Rev.* **184** (1969), 1231–1260.
- [CNP] B. Candelpergher, J.-C. Nosmas and F. Pham: Première pas en calcul étranger, *Ann. Inst. Fourier, Grenoble*, **43**(1993), 201–224.

- [DDP1] E. Delabaere, H. Dillinger and F. Pham: Résurgence de Voros et périodes des courbes hyperelliptiques, *Ann. Inst. Fourier, Grenoble*, **43**(1993), 163–199.
- [DDP2] ———: Exact semiclassical expansions for one-dimensional quantum oscillators, *J. Math. Phys.*, **38**(1997), 6126–6184.
- [DP] E. Delabaere and F. Pham: Resurgent methods in semi-classical asymptotics, *Ann. Inst. Henri Poincaré*, **71**(1999), 1–94.
- [D1] T. M. Dunster: Convergent expansions for linear ordinary differential equations having a simple turning point, with an application to Bessel functions, *Studies in Applied Math*, **107**(2001), 293–323.
- [D2] ———: Convergent expansions for solutions of linear ordinary differential equations having a simple pole, with an application to associated Legendre functions, *Studies in Applied Math*, **113** (2004), 245–270.
- [DLS] T. M. Dunster, D. A. Lutz and R. Schäfke: Convergent Liouville-Green expansions for second-order linear differential equations; with an application to Bessel functions, *Proc. Roy. Soc. Lon, Ser. A*, **440** (1993), 37–54.
- [Ec] J. Ecalle: Les fonctions résurgentes, I, II, III, *Publ. Math. d’Orsay, Univ. Paris-Sud*, 1981 (Tome I, II), 1985 (Tome III).
- [Er] A. Erdélyi: Higher Transcendental Functions, I, II, III, McGraw-Hill, 1955; reprinted in 1981 by Robert E. Krieger Publishing Company, Malabar, Florida.
- [F] M.V.Fedoryuk: Asymptotic Analysis. – Linear Ordinary Differential Equations. Springer-Verlag (1993).
- [KKKoT1] S. Kamimoto, T. Kawai, T. Koike and Y. Takei: On the WKB theoretic structure of the Schrödinger operators with a merging pair of a simple pole and a simple turning point, *Kyoto J. Math.*, **50** (2010), 101–164.

- [KKKoT2] ———: On a Schrödinger equation with a merging pair of a simple pole and a simple turning point — Alien calculus of WKB solutions through microlocal analysis, *Asymptotics in Dynamics, Geometry and PDEs; Generalized Borel Summation*, Publications of the Scuola Normale Superiore, Springer, pp 245–254, (2011).
- [KKT1] S. Kamimoto, T. Kawai and Y. Takei: Microlocal analysis of fixed singularities of WKB solutions of a Schrödinger equation with a merging triplet of two simple poles and a simple turning point, to appear in *The Mathematical Legacy of Leon Ehrenpreis, 1930 - 2010*.
- [KKT2] ———: Exact WKB analysis of a Schrödinger equation with merging triplet of two simple poles and a turning point — its relevance to the Mathieu equation and the Legendre equation, preprint (RIMS-1735), 2011.
- [K<sup>3</sup>] M. Kashiwara, T. Kawai, T. Kimura: *Foundations of Algebraic Analysis*, Princeton University Press, Princeton, 1986.
- [K] T. Kawai: Systems of linear differential equations of infinite order — an aspect of infinite analysis, *Proc. Symp. in Pure Math.*, **49**, Part 1, Amer. Math. Soc., 1989, pp.3–17.
- [KKoT] T. Kawai, T. Koike and Y. Takei: On the exact WKB analysis of higher order simple-pole type operators, *Adv. in Math.*, **228** (2011), 63–96.
- [KT1] T. Kawai and Y. Takei: Secular equations through the exact WKB analysis, *Analyse algébrique des perturbations singulières, I, Méthodes résurgentes*, Hermann, 1994, pp.85–102.
- [KT2] ———: *Algebraic Analysis of Singular Perturbation Theory*, Amer. Math. Soc., 2005.
- [Ko1] T. Koike: On a regular singular point in the exact WKB analysis, *Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear*, Kyoto Univ. Press, 2000, pp.39–54.

- [Ko2] ———: On the exact WKB analysis of second order linear ordinary differential equations with simple poles, Publ. RIMS, Kyoto Univ., **36** (2000), 297–319.
- [Ko3] ———: On “new” turning points associated with regular singular points in the exact WKB analysis, RIMS Kôkyûroku, **1159**, RIMS, 2000, pp.100–110.
- [Ko4] ———: in preparation.
- [KoS1] T. Koike and R. Schäfke: On the Borel summability of WKB solutions of Schrödinger equations with polynomial potentials and its applications, in preparation.
- [KoS2] ———: in preparation.
- [KoT] T. Koike and Y. Takei: On the Voros coefficient for the Whittaker equation with a large parameter — Some progress around Sato’s conjecture in exact WKB analysis, Publ. RIMS, Kyoto Univ., **47** (2011), 375–395.
- [Kr] A. Kramers: Wellenmechanik und halbzahlige Quantisierung, Zeit. f. Physik, **39** (1926), 828–840.
- [P1] F. Pham: Fonctions réurgentes implicites, C. R. Acad. Sci. Paris, Sér. I-Math., **309** (1989), 999–1001.
- [P2] ———: Resurgence, quantized canonical transformations, and multi-instanton expansion, Algebraic Analysis, Vol. II, Academic Press, 1988, pp.699–726.
- [SKK] M. Sato, T. Kawai and M. Kashiwara: Microfunctions and pseudo-differential equations, Lect. Notes in Math., **287**, Springer, 1973, pp.265–529.
- [Sa] D. Sauzin: Resurgent functions and splitting problems, RIMS Kôkyûroku, **1493**, 2006, pp.48–117.
- [SS] H. Shen and H. J. Silverstone: Observations on the JWKB treatment of the quadratic barrier, Algebraic Analysis of Differential Equations, Springer, 2008, pp.307–319.

- [T] Y. Takei: Sato's conjecture for the Weber equation and transformation theory for Schrödinger equations with a merging pair of turning points, *RIMS Kôkyûroku Bessatsu*, **B10** (2008), 205–224.
- [V] A. Voros: The return of the quartic oscillator — The complex WKB method. *Ann. Inst. Henri Poincaré*, **39** (1983), 211–338.
- [W] W. Wasow: Asymptotic expansions for ordinary differential equations, Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1965.



## 参考文献 1

On a Schrödinger equation with a  
merging pair of a simple pole and a  
simple turning point

—Alien calculus of WKB solutions  
through microlocal analysis

神本 晋吾

The purpose of this report is to present the core results of [KKKoT] and [KoT] with emphasis on their background. The object studied in these papers is, in somewhat rough description, a Schrödinger equation

$$(1) \quad \left( \frac{d^2}{dx^2} - \eta^2 Q(x, \eta) \right) \psi(x, \eta) = 0 \quad (\eta : \text{a large parameter})$$

with one simple turning point and with a simple pole in the potential  $Q$ . Now that satisfactory results have been obtained by [AKT2] concerning the WKB theoretic structure of a Schrödinger equation with two simple turning points, it is high time for us to study the above equation in view of the fact that a simple pole in the potential gives the Borel transformed WKB solutions of (1) essentially the same effect as a simple turning point does ([Ko1], [Ko2]).

In studying this problem we have to analyse two (or more) singularities of the Borel transformed WKB solutions whose relative location is fixed (the so-called “fixed singularities” (cf. [DP]; see also [V])). This means that the usual technique (cf. [AKT1], [KT]) of relating Borel transformed WKB solutions through integral operators determined by some microdifferential operators (cf. [SKK], [K<sup>3</sup>], [A]) requires the domain of definition of the relevant operators to be sufficiently large. To circumvent this problem, following the idea in [AKT2], we introduce an auxiliary parameter  $a$  to the potential  $Q$  so that the turning point and the pole in question merge as the parameter  $a$  tends to 0. Interestingly enough, we then naturally encounter the so-called ghost equation (cf. [Ko3], [KKKoT]) at  $a = 0$ , the top degree part  $Q_0(x)$  of whose potential contains neither zeros nor poles. The transformation of a ghost equation to its canonical form is known ([Ko3]; see also [KKKoT; Section 1]), and by perturbing the transformation with respect to the parameter  $a$  we can find the WKB-theoretic canonical operator of an appropriately defined (Definition 1 below) class of

Schrödinger operators with a simple turning point and a simple pole (Theorem 1 below).

A mathematical formulation of the intuitive picture of such an “appropriate” class is given by the following

**Definition 1.** *The Schrödinger equation (1) is called an equation with a merging pair of a simple pole and a simple turning point, or, for short, an MPPT equation if its potential  $Q$  depends also on an auxiliary complex parameter  $a$  and has the following form:*

$$(2) \quad Q = \frac{Q_0(x, a)}{x} + \eta^{-1} \frac{Q_1(x, a)}{x} + \eta^{-2} \frac{Q_2(x, a)}{x^2},$$

where  $Q_j(x, a)$  ( $j = 0, 1, 2$ ) are holomorphic near  $(x, a) = (0, 0)$  and  $Q_0(x, a)$  satisfies the following conditions (3) and (4):

$$(3) \quad \left( \frac{\partial Q_0}{\partial a} \right) (0, 0) \neq 0,$$

$$(4) \quad Q_0(x, 0) = c_0^{(0)} x + O(x^2) \text{ holds with } c_0^{(0)} \text{ being a constant different from } 0.$$

*Remark 1.* In [KKKoT] a slightly weaker condition

$$(3') \quad Q_0(0, a) \neq 0 \text{ if } a \neq 0$$

is imposed instead of (3).

It follows from the above definition that there exists a unique holomorphic function  $x(a)$  near  $a = 0$  that satisfies

$$(5) \quad Q_0(x(a), a) = 0,$$

$$(6) \quad x(a) \neq 0 \text{ if } a \neq 0.$$

Then the assumption (4) guarantees that  $x = x(a)$  ( $a \neq 0, |a| \ll 1$ ) is a simple turning point. Thus the above assumptions visualize our

intuitive picture of the equation. The following Theorem 1 guarantees the appropriateness of the above definition. For the clarity of description we put  $\sim$  to quantities relevant to a general MPPT equation to distinguish them from those of the canonical equation (16).

**Theorem 1.** *Let*

$$(7) \quad \tilde{L}\tilde{\psi} = \left( \frac{d^2}{d\tilde{x}^2} - \eta^2 \tilde{Q}(\tilde{x}, a, \eta) \right) \tilde{\psi}(\tilde{x}, a, \eta) = 0$$

*be an MPPT equation in the sense of Definition 1, that is, the potential  $\tilde{Q}(\tilde{x}, a, \eta)$  is of the form (2) and the conditions (3) and (4) are satisfied. Then there exist an open neighborhood  $U$  of  $\tilde{x} = 0$ , holomorphic functions  $x_k^{(j)}(\tilde{x})$  defined on  $U$  and constants  $\alpha_k^{(j)}$  ( $j, k \geq 0$ ) for which the following conditions (8)  $\sim$  (12) are satisfied:*

$$(8) \quad \frac{dx_0^{(0)}}{d\tilde{x}}(0) \neq 0,$$

$$(9) \quad x_k^{(j)}(0) = 0 \quad \text{for every } j \text{ and } k,$$

$$(10) \quad \alpha_0^{(0)} = 0,$$

$$(11) \quad \sup_{\tilde{x} \in U} |x_k^{(j)}(\tilde{x})|, |\alpha_k^{(j)}| \leq AC_1^j C_2^k k!$$

*with some positive constants  $A$ ,  $C_1$  and  $C_2$ ,*

$$(12)$$

$$\begin{aligned} & \tilde{Q}(\tilde{x}, a, \eta) \\ &= \left( \frac{\partial x(\tilde{x}, a, \eta)}{\partial \tilde{x}} \right)^2 \left( \frac{1}{4} + \frac{\alpha(a, \eta)}{x(\tilde{x}, a, \eta)} + \eta^{-2} \frac{\tilde{Q}_2(0, a)}{x(\tilde{x}, a, \eta)^2} \right) - \frac{1}{2} \eta^{-2} \{x; \tilde{x}\}, \end{aligned}$$

where

$$(13) \quad x(\tilde{x}, a, \eta) = \sum_{k \geq 0} \sum_{j \geq 0} x_k^{(j)}(\tilde{x}) a^j \eta^{-k},$$

$$(14) \quad \alpha(a, \eta) = \sum_{k \geq 0} \sum_{j \geq 0} \alpha_k^{(j)} a^j \eta^{-k}$$

and  $\{x; \tilde{x}\}$  denotes the Schwarzian derivative

$$(15) \quad \frac{d^3 x / d\tilde{x}^3}{dx / d\tilde{x}} - \frac{3}{2} \left( \frac{d^2 x / d\tilde{x}^2}{dx / d\tilde{x}} \right)^2.$$

This theorem combined with the general WKB theory (cf. [KT]) asserts that the WKB theoretically canonical equation of an MPPT equation  $\tilde{L}\tilde{\psi} = 0$  is given by the following

$$(16) \quad M\psi = \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{1}{4} + \frac{\alpha(a, \eta)}{x} + \eta^{-2} \frac{\tilde{Q}_2(0, a)}{x^2} \right) \right) \psi = 0.$$

In parallel with the usage of the name “ $\infty$ -Weber equation” in [AKT2], we call the equation  $M\psi = 0$  an  $\infty$ -Whittaker equation.

An important point is that in the double series  $x(\tilde{x}, a, \eta)$  and  $\alpha(a, \eta)$  in Theorem 1 the growth order property of  $|x_k^{(j)}|$  and  $|\alpha_k^{(j)}|$  as  $j$  tends to  $\infty$  and that as  $k$  tends to  $\infty$  are substantially different despite the fact that their construction is done in a symmetric way with respect to indexes  $j$  and  $k$  (cf. [KKKoT; Remark 2.1]). In particular,

$$(17) \quad x_k(\tilde{x}, a) = \sum_{j \geq 0} x_k^{(j)}(\tilde{x}) a^j$$

and

$$(18) \quad \alpha_k(a) = \sum_{j \geq 0} \alpha_k^{(j)} a^j$$

are holomorphic respectively on  $U \times V$  and on  $V$  for some open neighborhood  $V$  of  $a = 0$ , while  $x(\tilde{x}, a, \eta)$  and  $\alpha(a, \eta)$  are only Borel transformable series in the sense of [KT]. Although the problem is of singular perturbative character, it seems that it is of regular perturbative character in the variable  $a$ . Actually our reasoning indicates that the singular perturbative character originates from the part  $\eta^{-2}(d^3 x_k^{(j)}/d\tilde{x}^3)/(dx_k^{(j)}/d\tilde{x})$  in the defining equation of  $x_k^{(j)}$ , which does not affect much the behavior of  $x_k^{(j)}$  as  $j$  tends to infinity. (See [KKKoT; (B.64)].)

It is readily imagined, and we can really confirm, that the canonical equation  $M\psi = 0$  is further reduced to the following Whittaker equation with a large parameter:

$$(19) \quad M_0\chi = \left( \frac{d^2}{dx^2} - \eta^2 \left( \frac{1}{4} + \frac{\alpha_0}{x} + \eta^{-2} \frac{\gamma(\gamma+1)}{x^2} \right) \right) \chi = 0,$$

where  $\alpha_0$  and  $\gamma$  are complex numbers. Concerning the Whittaker equation with a large parameter for  $\alpha_0 \neq 0$  we know ([KoT]) the following Theorem 2: Let  $\chi_{\pm}(x, \eta)$  be WKB solutions of the Whittaker equation normalized as

$$(20) \quad \chi_{\pm}(x, \eta) = \frac{1}{\sqrt{S_{\text{odd}}}} \exp \left( \pm \int_{-4\alpha_0}^x S_{\text{odd}} dx \right),$$

where  $S_{\text{odd}}$  is the odd part of the formal power series solution  $S = \eta S_{-1}(x) + S_0(x) + \eta^{-1} S_1(x) + \dots$  of the associated Riccati equation (cf. [KKKoT]). Then the following holds.

**Theorem 2.** *Suppose  $\alpha_0 \neq 0$ . Then the Borel transform  $\chi_{+,B}(x, y)$  of  $\chi_+$  has fixed singularities at  $y = -y_+(x) + 2m\pi i\alpha_0$  ( $m = \pm 1, \pm 2, \dots$ ), where*

$$(21) \quad y_+(x) = \int_{-4\alpha_0}^x S_{-1} dx = \int_{-4\alpha_0}^x \sqrt{\frac{x + 4\alpha_0}{4x}} dx$$

and its alien derivative is explicitly given by

$$(22) \quad \begin{aligned} & (\Delta_{y=-y_+(x)+2m\pi i\alpha_0}\chi_+)_B(x, y) \\ &= \frac{\exp(2m\pi i\gamma) + \exp(-2m\pi i\gamma)}{2m} \chi_{+,B}(x, y - 2m\pi i\alpha_0). \end{aligned}$$

Note that the relative location between two singular points  $-y_+(x) + 2m\pi i\alpha_0$  and  $-y_+(x) + 2m'\pi i\alpha_0$  does not vary, that is, their difference  $2(m - m')\pi i\alpha_0$  is a constant independent of  $x$ . The proof of Theorem 2 can be done by using the following expression of the Borel transform of the Voros coefficient  $\phi$ :

$$(23) \quad \begin{aligned} & \phi_B(\alpha_0, \gamma; y) \\ &= \frac{1}{2y} \left( \frac{\exp(y/\alpha_0) + 1}{\exp(y/\alpha_0) - 1} \right) \cosh\left(\frac{\gamma y}{\alpha_0}\right) - \frac{\alpha_0}{y^2} + \frac{1}{2y} \sinh\left(\frac{\gamma y}{\alpha_0}\right), \end{aligned}$$

where the Voros coefficient of the Whittaker equation (19) is defined by

$$(24) \quad \phi(\alpha_0, \gamma; \eta) = \int_{-4\alpha_0}^{\infty} (S_{\text{odd}} - \eta S_{-1}) dx.$$

See [KoT] for the details. Since the concrete computation in alien calculus is normally performed on the Borel plane (cf. [P], [DP]), we have to study the Borel transformed version of Theorem 1. To employ Theorem 2, we assume  $a \neq 0$  in what follows. Thanks to the estimate (11), we have the following Theorem 3 and Theorem 4. To state them we make the following notational preparations: Let  $g(x, a)$  be the inverse function of  $x_0(\tilde{x}, a)$ , i.e., a holomorphic function that satisfies

$$(25) \quad x = x_0(g(x, a), a), \quad \tilde{x} = g(x_0(\tilde{x}, a), a)$$

on a neighborhood of  $(x, a) = (0, 0)$ . Then we consider the Borel transform of  $\tilde{L}$  in  $(x, y, a)$ -variable:

$$(26) \quad \mathcal{L} \stackrel{\text{def}}{=} \left( \frac{\partial g}{\partial x} \right)^2 \times (\text{Borel transform of } \tilde{L})|_{\tilde{x}=g(x,a)}$$

$$= \frac{\partial^2}{\partial x^2} - \left( \frac{\partial^2 g / \partial x^2}{\partial g / \partial x} \right) \frac{\partial}{\partial x} - \left( \frac{\partial g}{\partial x} \right)^2 \tilde{Q}(g(x, a), a, \frac{\partial}{\partial y}).$$

Similarly let  $\mathcal{M}$  (resp.  $\mathcal{M}_0$ ) be the Borel transform of  $M$  (resp.  $M_0$ ):

$$(27) \quad \mathcal{M} = \frac{\partial^2}{\partial x^2} - \left( \frac{1}{4} + \frac{\alpha(a, \partial/\partial y)}{x} \right) \frac{\partial^2}{\partial y^2} - \frac{\tilde{Q}_2(0, a)}{x^2},$$

$$(28) \quad \mathcal{M}_0 = \frac{\partial^2}{\partial x^2} - \left( \frac{1}{4} + \frac{\alpha_0}{x} \right) \frac{\partial^2}{\partial y^2} - \frac{\gamma(\gamma + 1)}{x^2}.$$

**Theorem 3.** *Suppose  $a \neq 0$ . Let  $\omega_0$  be a sufficiently small open neighborhood of  $x = 0$ , and set*

$$(29) \quad \Omega_0 = \{(x, y; \xi, \eta) \in T^* \mathbb{C}_{(x, y)}^2; x \in \omega_0, \eta \neq 0\}.$$

*Then there exist microdifferential operators  $\mathcal{X}$  and  $\mathcal{Y}$  defined on  $\Omega_0$  that satisfy*

$$(30) \quad \mathcal{L}\mathcal{X} = \mathcal{Y}\mathcal{M}$$

*for  $x \neq 0$ . The concrete form of operators  $\mathcal{X}$  and  $\mathcal{Y}$  is as follows:*

$$(31) \quad \mathcal{X} = : \left( \frac{\partial g}{\partial x} \right)^{1/2} \left( 1 + \frac{\partial r}{\partial x} \right)^{-1/2} \exp(r(x, a, \eta)\xi) :,$$

$$(32) \quad \mathcal{Y} = : \left( \frac{\partial g}{\partial x} \right)^{1/2} \left( 1 + \frac{\partial r}{\partial x} \right)^{3/2} \exp(r(x, a, \eta)\xi) :,$$

*where*

$$(33) \quad r(x, a, \eta) = \sum_{k \geq 1} x_k(g(x, a), a) \eta^{-k}$$

*and  $:$  designates the normal ordered product (cf. [A]).*

Theorem 3 implies that the operators  $\mathcal{L}$  and  $\mathcal{M}$  are microlocally equivalent. This fact indicates that the singularities of  $\tilde{\psi}_B(g(x, a), y)$  that satisfies  $\mathcal{L}\tilde{\psi}_B = 0$  and those of  $\psi_B(x, y)$  that satisfies  $\mathcal{M}\psi_B = 0$  are the same. This is really visualized by the following Theorem 4:



**Theorem 4.** *The action of the microdifferential operator  $\mathcal{X}$  upon the Borel transformed WKB solution  $\psi_{+,B}$  of the  $\infty$ -Whittaker equation is expressed as an integro-differential operator of the following form:*

$$(34) \quad \mathcal{X}\psi_{+,B} = \int_{y_0}^y K(x, a, y - y', \partial/\partial x) \psi_{+,B}(x, a, y') dy',$$

where  $K(x, a, y, \partial/\partial x)$  is a differential operator of infinite order that is defined on  $\{(x, a, y) \in \mathbb{C}^3; (x, a) \in \omega \text{ for an open neighborhood } \omega \text{ of the origin and } |y| < C \text{ for some positive constant } C\}$ , and  $y_0$  is a constant that fixes the action of  $(\partial/\partial y)^{-1}$  as an integral operator.

Since a differential operator of infinite order acts on the sheaf of holomorphic functions as a sheaf homomorphism, we can immediately locate the singularities of  $\mathcal{X}\psi_{+,B}$  through the integral representation (34). Another important point in the integral representation (34) is that its domain of definition enjoys the uniformity with respect to the parameter  $a$ , that is, the open neighborhood  $\omega$  is taken to be of the form

$$(35) \quad \{x \in \mathbb{C}; |x| < \delta_1\} \times \{a \in \mathbb{C}; |a| < \delta_2\}$$

for some positive constants  $\delta_1$  and  $\delta_2$ . Note that since  $\alpha_0(a)$  tends to 0 as  $a$  tends to 0 by (10),  $(\delta_1, \delta_2)$  can be chosen so that  $\{|x| < \delta_1\}$  contains  $x = -4\alpha_0(a)$  for every  $a$  in  $\{|a| < \delta_2\}$ . This is the precise meaning of saying “To circumvent the problem (of the existence of a large domain of definition of relevant integral operators)” at the beginning of this report.

In parallel with Theorem 3, we can show that  $\mathcal{M}$  and  $\mathcal{M}_0$  are also microlocally equivalent. For simplicity we employ  $\alpha_0(a)$  as an independent variable in substitution for  $a$  (this substitution of variable is

guaranteed by (3)). Thanks to the estimate (11) we obtain the following

**Theorem 5.** *Let  $\mathcal{A}$  be a microdifferential operator on*

$$(36) \quad \{(\alpha_0, y; \theta, \eta) \in T^*\mathbb{C}^2; |\alpha_0| < \delta_0, \eta \neq 0\}$$

*for some positive constant  $\delta_0$  defined by*

$$(37) \quad \mathcal{A} = : \exp((\alpha_1(\alpha_0)\eta^{-1} + \alpha_2(\alpha_0)\eta^{-2} + \dots)\theta) : .$$

*Here  $\theta$  and  $\eta$  are respectively identified with the symbol  $\sigma(\partial/\partial\alpha_0)$  and the symbol  $\sigma(\partial/\partial y)$ . Then the following holds:*

$$(38) \quad \mathcal{M}\mathcal{A} = (\mathcal{A}\mathcal{M}_0)|_{\gamma(\gamma+1)=\tilde{Q}_2(0,a)}$$

*for  $x \neq 0$ .*

Although the target variable is  $\alpha_0$ , not  $x$ , as is the case for the microdifferential operator  $\mathcal{X}$ , the operator  $\mathcal{A}$  also has a concrete expression as an integro-differential operator stated in Theorem 4. On the other hand, as is indicated in Theorem 2, a fixed singular point of  $\psi_{+,B}(x, y)$  (“fixed” with respect to  $y = -y_+(x)$ ) is located at  $y = -y_+(x) + 2m\pi i\alpha$ . Thus, by the same reasoning for the case of  $\mathcal{X}$ , each individual fixed singular point of  $\tilde{\psi}_{+,B}(x, y)$  is contained, for sufficiently small  $a$ , in the domain of definition of the integro-differential operator  $\mathcal{A}$ .

Summing up all these results, we finally obtain

**Theorem 6.** *Suppose  $a \neq 0$  and let  $\tilde{\psi}_+(\tilde{x}, a, \eta)$  be a WKB solution of an MPPT equation normalized at its turning point  $\tilde{x}_0(a)$  as follows:*

$$(39) \quad \tilde{\psi}_+(x, a, \eta) = \frac{1}{\sqrt{\tilde{S}_{\text{odd}}}} \exp\left(\int_{\tilde{x}_0(a)}^x \tilde{S}_{\text{odd}} dx\right)$$

where  $\tilde{S}_{\text{odd}}$  is the odd part of the formal power series solution  $\tilde{S}$  of the associated Riccati equation. Then for each positive integer  $m$  the following relation (40) holds for sufficiently small  $a$ :

$$\begin{aligned}
(40) \quad & \left( \Delta_{y=-y_+(\tilde{x},a)+2m\pi i\alpha_0(a)} \tilde{\psi}_+ \right)_B(\tilde{x}, a, y) \\
&= \frac{\exp(2m\pi i\gamma(a)) + \exp(-2m\pi i\gamma(a))}{2m} \times \\
& \quad : \exp(-2m\pi i(\alpha_1(a) + \alpha_2(a)\eta^{-1} + \dots)) : \tilde{\psi}_{+,B}(\tilde{x}, a, y - 2m\pi i\alpha_0(a)),
\end{aligned}$$

where

$$(41) \quad y_+(\tilde{x}, a) = \int_{\tilde{x}_0(a)}^{\tilde{x}} \sqrt{\frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}}} d\tilde{x},$$

$$(42) \quad \gamma(a)^2 + \gamma(a) = \tilde{Q}_2(0, a)$$

and

$$(43) \quad \alpha_j(a) = \frac{1}{2\pi i} \oint_{\tilde{\Gamma}(a)} \tilde{S}_{\text{odd},j-1}(\tilde{x}, a) d\tilde{x}$$

with  $\tilde{\Gamma}(a)$  being a closed curve encircling  $\tilde{x}_0(a)$  and the origin as in Figure 1 and with  $\tilde{S}_{\text{odd},k}$  designating the degree  $k$  part of  $\tilde{S}_{\text{odd}}$ .

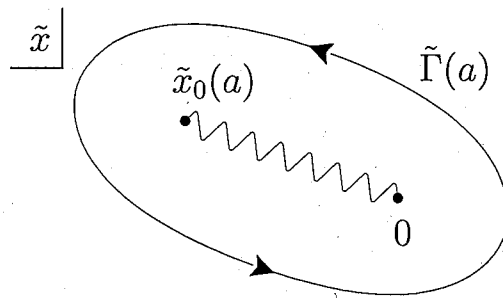


Figure 1.

## References

- [A] T. Aoki: Symbols and formal symbols of pseudodifferential operators, *Advanced Studies in Pure Mathematics*, **4**, Kinokuniya, 1984, pp.181–208.
- [AKT1] T. Aoki, T. Kawai and T. Takei: The Bender-Wu analysis and the Voros theory, *Special Functions*, Springer-Verlag, 1991, pp.1–29.
- [AKT2] ———: The Bender-Wu analysis and the Voros theory. II, *Advanced Studies in Pure Mathematics*, **54**, Math. Soc. Japan, 2009, pp.19–94.
- [DP] E. Delabaere and F. Pham: Resurgent methods in semi-classical asymptotics, *Ann. Inst. Henri Poincaré*, **71**(1999), 1–94.
- [KKKoT] S. Kamimoto, T. Kawai, T. Koike and Y. Takei: On the WKB theoretic structure of the Schrödinger operators with a merging pair of a simple pole and a simple turning point, *Kyoto J. Math.*, **50** (2010), 101–164.
- [K<sup>3</sup>] M. Kashiwara, T. Kawai, T. Kimura: *Foundations of Algebraic Analysis*, Princeton University Press, Princeton, 1986.
- [KT] T. Kawai and Y. Takei: *Algebraic Analysis of Singular Perturbation Theory*, Amer. Math. Soc., 2005.
- [Ko1] T. Koike: On a regular singular point in the exact WKB analysis, *Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear*, Kyoto Univ. Press, 2000, pp.39–54.

- [Ko2] ———: On the exact WKB analysis of second order linear ordinary differential equations with simple poles, Publ. RIMS, Kyoto Univ., **36**(2000), 297–319.
- [Ko3] ———: On “new” turning points associated with regular singular points in the exact WKB analysis, RIMS Kôkyûroku, **1159**, RIMS, 2000, pp.100–110.
- [KoT] T. Koike and Y. Takei: On the Voros coefficient for the Whittaker equation with a large parameter — Some progress around Sato’s conjecture in exact WKB analysis, Publ. RIMS, Kyoto Univ., **47** (2011), 375–395.
- [P] F. Pham: Resurgence, quantized canonical transformations, and multi-instanton expansion, Algebraic Analysis, Vol. II, Academic Press, 1988, pp.699–726.
- [SKK] M. Sato, T. Kawai and M. Kashiwara: Microfunctions and pseudo-differential equations, Lect. Notes in Math., **287**, Springer, 1973, pp.265–529.
- [V] A. Voros: The return of the quartic oscillator — The complex WKB method, Ann. Inst. Henri Poincaré, **39**(1983), 211–338.

## 参考文献 2

Microlocal analysis of fixed  
singularities of WKB solutions  
of a Schrödinger equation  
with a merging triplet of two simple  
poles and a simple turning point

神本 晋吾

## Abstract

We first show that the WKB-theoretic canonical form of an M2P1T (merging two poles and one turning point) Schrödinger equation is given by the algebraic Mathieu equation. We further show that, in analyzing the structure of WKB solutions of a Mathieu equation near fixed singular points relevant to simple poles of the equation, we can focus our attention on the pole part of the equation so that we may reduce it to the Legendre equation. The Borel transformation of WKB-theoretic transformations thus obtained gives rise to microdifferential relations, which lead to the microlocal analysis of the Borel transformed WKB solutions of an M2P1T equation near their fixed singular points. The fully detailed account of the results will be given in [9].

## 0 Introduction

The purpose of this article is to announce the main results of [9] emphasizing the atypical points in its reasoning which cannot be found in earlier papers dealing with seemingly related problems, such as [3] and [8]. As the logical structure of the argument in [9] is intricate, we try to explain the ideas that underlie its formulation of the problem. The target of [9] is the exact WKB analysis of a Schrödinger equation

$$(0.1) \quad \left( \frac{d^2}{dx^2} - \eta^2 Q(x, a) \right) \psi = 0,$$

where  $\eta$  is a large parameter and the potential  $Q$  contains a triplet of two simple poles and one simple turning point that merge as the parameter  $a$  tends to 0. Here “exact WKB analysis” means WKB analysis based on the Borel transformation with respect to the large parameter  $\eta$ ; thus our principal aim is to analyze the singularity structure of the

Borel transformed WKB solution  $\psi_B(x, a, y)$ , which solves the Borel transformed Schrödinger equation

$$(0.2) \quad \left( \frac{\partial^2}{\partial x^2} - Q(x, a) \frac{\partial^2}{\partial y^2} \right) \psi_B(x, a, y) = 0.$$

Hence the exact WKB analysis belongs to the most favorite field of the late Professor Ehrenpreis, Fourier analysis in the complex domain. (Cf. [6]) Our interest in the class of Schrödinger equations with a merging triplet of poles and a turning point originates from our desire to understand the semi-global structure of a Schrödinger equation with two simple poles in its potential. As is now well-known (cf. [12], [13]), a simple pole gives an effect to the Borel transformed WKB solutions that is similar to the effect which a turning point gives. Thus the analysis of the class of Schrödinger equations with two simple poles in their potentials is a natural counterpart of the classes of equations studied in [3] (Schrödinger equations with a merging pair of simple turning points) and in [8] (Schrödinger equations with a merging pair of a simple pole and a simple turning point). One can then easily guess that a WKB-theoretic canonical form of such a Schrödinger equation is the Legendre equation with a large parameter, that is,

$$(0.3) \quad \left( \frac{d^2}{dx^2} - \eta^2 Q_{\text{Leg}}(x, a) \right) \psi = 0,$$

where

$$(0.4) \quad Q_{\text{Leg}} = \frac{\lambda}{x^2 - a^2} + \eta^{-2} \left( \frac{\gamma_+}{(x - a)^2} + \frac{\gamma_-}{(x + a)^2} \right)$$

with  $\gamma_{\pm}$  being complex numbers and with  $\lambda$  being an infinite series in  $\eta^{-1}$  with constant coefficients that satisfies an appropriate growth order condition to be discussed later. To emphasize the fact that  $\lambda$  is not a genuine constant but an infinite series we sometimes call such an equation the  $\infty$ -Legendre equation. Parenthetically we note that, in



what follows, we basically concentrate our attention to the core part of the potential, that is,  $\lambda/(x^2 - a^2)$  by mainly considering the situation where  $\gamma_+$  and  $\gamma_-$  are 0; this limitation is helpful in clarifying the logical structure of our reasoning by avoiding technical complexities. By the way, in the exact WKB analysis, an important subject is the analytic structure of the Borel transformed WKB solutions near their fixed singularities (cf. [11, p.112–p.113]. See also [4], [5] and [17]), that is, singularities located at

(0.5)

$$y = - \int_{\alpha}^x \sqrt{Q(x, a)} dx + 2l \int_{\alpha}^{\tilde{\alpha}} \sqrt{Q(x, a)} dx \quad (l = \pm 1, \pm 2, \dots),$$

where  $\alpha$  and  $\tilde{\alpha}$  are turning points (with a simple pole being regarded as a turning point) of the equation. An important point in [3] and [8] is that the period integral

$$(0.6) \quad 2 \int_{\alpha}^{\tilde{\alpha}} \sqrt{Q(x, a)} dx$$

tends to 0 when we let  $a$  tend to 0; hence by showing that the domain of definition of the transformation operator to the canonical form can be chosen to be independent of  $a$ , we can analyze the analytic structure of the Borel transformed WKB solution near a fixed singularity with  $|l| \gg 1$ . But this time we find

$$(0.7) \quad \int_{-a}^a \frac{dx}{\sqrt{x^2 - a^2}} = \pi i$$

does not change even when  $a$  tends to 0. Thus the strategy in [3] and [8] is not effective in this case. To circumvent the problem we dismantle the potential of its homogeneity and seek for the class of Schrödinger equations which can be transformed to

$$(0.8) \quad \left( \frac{d^2}{dx^2} - \eta^2 \frac{aA + xB}{x^2 - a^2} \right) \psi = 0$$

with  $A$  and  $B$  being infinite series in  $\eta^{-1}$  that are independent of  $x$ , that is, the algebraic  $\infty$ -Mathieu equation (if we follow the usage of the terminology of [7, p.98]), which we call the  $\infty$ -Mathieu equation for short. In view of the explicit form of the potential in (0.8), we imagine that the class which we now try to analyze would consist of Schrödinger equations with two simple poles and one simple turning point. Fortunately this guess turns out to be correct, as is explained in Section 1 below. Thus widening the target class gives a clean result, but the problem is the fact that the Mathieu equation is a notoriously difficult object to analyze. Hence we next contrive to deduce the analytic properties of Borel transformed WKB solutions near the fixed singularities relevant to the pair of simple poles, which was our original target, by “driving off” the simple turning point. This contrivance will be explained in Section 3, but here we note the following geometric fact that explains why we introduce an auxiliary parameter  $\rho$  into our formulation (cf. Definition 1.1 below).

To describe the geometric situation, let  $A_0$  (resp.,  $B_0$ ) denote the degree 0 (in  $\eta$ ) part of  $A$  (resp.,  $B$ ). Then we can confirm

$$(0.9) \quad A_0|_{a=0} \neq 0 \quad (\text{cf. (1.39) and (1.4)})$$

and

$$(0.10) \quad B_0|_{a=0} = Z_0 \rho \quad \text{with} \quad Z_0 = \pm 1 \quad (\text{cf. (1.40)}).$$

Now, keeping  $a/\rho =: \kappa (\neq 0)$  fixed sufficiently small, we let  $\rho$  tend to 0. Then, since the turning point  $t_0$  of (0.8) is given by

$$(0.11) \quad -\frac{aA_0}{B_0} = -\frac{\kappa A_0(0)}{Z_0 + \kappa\beta} + O(\rho)$$

with some constant  $\beta$ , it stays away from 0. On the other hand, the simple poles  $t = \pm a$  tend to 0. Thus one may expect that the singularity structure of Borel transformed WKB solutions near the fixed

singularities relevant to the simple poles can be deduced from that of Borel transformed WKB solutions of the Schrödinger equation whose potential contains two simple poles only, i.e., without a turning point. And this expectation is realized in Section 3. In ending Introduction we note that in deducing the results in the final section (Section 4) from those in Section 1 and Section 3, we make full use of microdifferential relations among objects on the Borel plane which are discussed in Section 2.

## 1 Definition of an M2P1T equation and its reduction to the Mathieu equation

In what follows,  $U$  (resp.,  $V$  and  $O$ ) denotes a sufficiently small open neighborhood of the origin  $\{t \in \mathbb{C}; t = 0\}$  (resp.,  $\{a \in \mathbb{C}; a = 0\}$  and  $\{\rho \in \mathbb{C}; \rho = 0\}$ ) and let  $f(t, a, \rho)$  denote a holomorphic function that has the following form (1.1) on  $U \times V \times O$ :

$$(1.1) \quad f(t, a, \rho) = t\rho g(t, \rho) + \sum_{j \geq 1} a^j f^{(j)}(t, \rho)$$

with

$$(1.2) \quad g(t, \rho) \text{ and } f^{(j)}(t, \rho) \text{ being holomorphic on } U \times O,$$

$$(1.3) \quad g(0, \rho) = 1,$$

$$(1.4) \quad f^{(1)}(0, 0) \neq 0,$$

$$(1.5) \quad \rho^2 \neq (f^{(1)}(0, \rho))^2 \text{ for } \rho \text{ in } O.$$

In what follows we use symbols  $f^{(0)}(t, \rho)$  and  $\tilde{f}^{(0)}(t, \rho)$  respectively to denote  $t\rho g(t, \rho)$  and  $\rho g(t, \rho)$ .

**Definition 1.1.** Let  $f(t, a, \rho)$  be as above, let  $g_{\pm}(t)$  be holomorphic functions on  $U$  and let  $Q$  denote the following potential:

$$(1.6) \quad \frac{f(t, a, \rho)}{t^2 - a^2} + \eta^{-2} \left( \frac{g_+(t)}{(t - a)^2} + \frac{g_-(t)}{(t + a)^2} \right) \quad (\eta : \text{a large parameter}).$$

Then the Schrödinger operator

$$(1.7) \quad \frac{d^2}{dt^2} - \eta^2 Q(t, a, \rho, \eta)$$

is called an *M2P1T*, merging two poles and one turning point, operator.

*Remark 1.1.* For the sake of simplicity we assume the following condition (1.8) in Section 1:

$$(1.8) \quad g_+ = g_- = 0.$$

*Remark 1.2.* It immediately follows from (1.3) that (1.7) for  $\rho \neq 0$  has a simple turning point when  $V$  is chosen sufficiently small.

*Remark 1.3.* It follows from the trivial relation

$$(1.9) \quad \frac{t\tilde{f}^{(0)} + af^{(1)}}{t^2 - a^2} = \frac{\tilde{f}^{(0)} + f^{(1)}}{2(t - a)} + \frac{\tilde{f}^{(0)} - f^{(1)}}{2(t + a)}$$

that we obtain a sum of simple poles at  $a = 0$ , not a double pole. Parenthetically we note that the assumption (1.5) guarantees that their residues are different from 0.

*Remark 1.4.* The reader might wonder why the assumption about the structure of  $\tilde{f}^{(0)}(t, \rho)$  is so restrictive. But, since we want to uniformly deal with the problem for an arbitrarily small parameter  $\rho (\neq 0)$ , some strict restriction on the structure of  $\tilde{f}^{(0)}(t, \rho)$  is inevitable. Actually one will be able to find that the function  $x_0^{(0)}(t, \rho)$  given by (1.27) below cannot be holomorphic on a fixed neighborhood of the origin  $\{t = 0\}$  if we choose, for example,

$$(1.10) \quad \tilde{f}^{(0)}(t, \rho) = t + \rho,$$

although it satisfies

$$(1.11) \quad \tilde{f}^{(0)}(0, \rho) = \rho,$$

the condition we frequently use in our computation.

The purpose of this section is to show that an M2P1T equation is WKB-theoretically transformed to an  $\infty$ -Mathieu equation. We refer the reader to [11, Section 2] for the basic properties of “WKB-theoretic transformations”, but we note their heuristic explanation as follows: in an intuitive description its core is a formal coordinate transformation from  $t$  to  $x = x(t, a, \rho, \eta)$  defined by an infinite series

$$(1.12) \quad x(t, a, \rho, \eta) = \sum_{k \geq 0} x_{2k}(t, a, \rho) \eta^{-2k}$$

which satisfies

$$(1.13) \quad Q(t, a, \rho, \eta) = \left( \frac{\partial x}{\partial t} \right)^2 \left( \frac{aA + xB}{x^2 - a^2} \right) - \frac{1}{2} \eta^{-2} \{x; t\},$$

for some infinite series

$$(1.14) \quad A = \sum_{k \geq 0} A_{2k}(a, \rho) \eta^{-2k}$$

and

$$(1.15) \quad B = \sum_{k \geq 0} B_{2k}(a, \rho) \eta^{-2k},$$

where  $\{x; t\}$  stands for the Schwarzian derivative

$$(1.16) \quad -2 \left( \frac{\partial x}{\partial t} \right)^{1/2} \frac{\partial^2}{\partial t^2} \left( \frac{\partial x}{\partial t} \right)^{-1/2}.$$

In what follows we call the Schrödinger operator

$$(1.17) \quad \left( \frac{d^2}{dx^2} - \eta^2 \frac{aA + xB}{x^2 - a^2} \right)$$

an  $\infty$ -Mathieu operator. Using appropriate growth order conditions that  $x_{2k}(t, a, \rho)$ ,  $A_{2k}(a, \rho)$  and  $B_{2k}(a, \rho)$  satisfy we can construct microdifferential operators  $\mathcal{X}$  and  $\mathcal{Y}$  so that they “intertwine” the Borel transformed M2P1T operator and the Borel transformed  $\infty$ -Mathieu operator; we have (Theorem 2.1)

$$(1.18) \quad N\mathcal{X} = \mathcal{Y}M_\infty,$$

where  $M_\infty$  denotes the Borel transformed  $\infty$ -Mathieu operator and  $N$  denotes the Borel transformed M2P1T operator written in  $(x, y)$ -variable with the effect of the coordinate change appropriately taken into account (cf. (2.4) for the concrete form of  $N$ ). See Section 2 for the explicit description of  $\mathcal{X}$  in terms of the infinite series  $x$ .

In constructing the infinite series  $x$ ,  $A$  and  $B$ , we further expand  $x_{2k}(t, a, \rho)$  etc. in powers of  $a$ ; that is, we will seek for  $x$ ,  $A$  and  $B$  in the form of double series as follows:

$$(1.19) \quad x = \sum_{j,k \geq 0} x_{2k}^{(j)}(t, \rho) a^j \eta^{-2k},$$

$$(1.20) \quad A = \sum_{j,k \geq 0} A_{2k}^{(j)}(\rho) a^j \eta^{-2k},$$

$$(1.21) \quad B = \sum_{j,k \geq 0} B_{2k}^{(j)}(\rho) a^j \eta^{-2k}.$$

Substituting these series into (1.13) and comparing the coefficient of  $\eta^0$  we find

$$(1.22) \quad \frac{f(t, a, \rho)}{t^2 - a^2} = \left( \frac{\partial x_0}{\partial t} \right)^2 \frac{aA_0 + x_0B_0}{x_0^2 - a^2},$$

where

$$(1.23) \quad x_0(t, a, \rho) = \sum_{j \geq 0} x_0^{(j)}(t, \rho) a^j,$$

$$(1.24) \quad A_0(a, \rho) = \sum_{j \geq 0} A_0^{(j)}(\rho) a^j,$$

$$(1.25) \quad B_0(a, \rho) = \sum_{j \geq 0} B_0^{(j)}(\rho) a^j.$$

After multiplying (1.22) by  $(t^2 - a^2)(x_0^2 - a^2)$  we compare the coefficient of  $a^p$  to find

$$(1.26.p) \quad \begin{aligned} & -f^{(p-2)} + \sum_{j+k+l=p} x_0^{(j)} x_0^{(k)} f^{(l)} \\ &= t^2 \left( \sum_{j+k+l=p} \frac{\partial x_0^{(j)}}{\partial t} \frac{\partial x_0^{(k)}}{\partial t} A_0^{(l-1)} + \sum_{j+k+l+m=p} \frac{\partial x_0^{(j)}}{\partial t} \frac{\partial x_0^{(k)}}{\partial t} x_0^{(l)} B_0^{(m)} \right) \\ & - \left( \sum_{j+k+l=p-2} \frac{\partial x_0^{(j)}}{\partial t} \frac{\partial x_0^{(k)}}{\partial t} A_0^{(l-1)} + \sum_{j+k+l+m=p-2} \frac{\partial x_0^{(j)}}{\partial t} \frac{\partial x_0^{(k)}}{\partial t} x_0^{(l)} B_0^{(m)} \right). \end{aligned}$$

In (1.26.p) terms whose indices do not meet the requirements should be ignored, as usual. With this convention (1.26.p) with  $p = 0$  or  $1$  is of a peculiar form. For example, we find

$$(1.26.0) \quad t x_0^{(0)2} \tilde{f}^{(0)} = t^2 x_0^{(0)'} x_0^{(0)} B_0^{(0)}.$$

Here, and in what follows,  $x'$  stands for  $\partial x / \partial t$ . Hence we find

$$(1.27) \quad x_0^{(0)}(t, \rho) = \frac{1}{4B_0^{(0)}} \left( \int_0^t \sqrt{\frac{\tilde{f}^{(0)}(t, \rho)}{t}} dt \right)^2,$$

where  $B_0^{(0)}$  is a non-zero constant to be fixed later. Then it follows from the assumptions (1.2) and (1.3) that there exists a holomorphic function  $\tilde{x}_0^{(0)}(t, \rho)$  that satisfies

$$(1.28) \quad x_0^{(0)}(t, \rho) = t \tilde{x}_0^{(0)}(t, \rho)$$

with

$$(1.29) \quad \tilde{x}_0^{(0)}(0, \rho) = \frac{\rho}{B_0^{(0)}}.$$

Next we consider the case  $p = 1$ . Then, by using (1.28) we find

$$(1.26.1) \quad \begin{aligned} & 2t\tilde{x}_0^{(0)}x_0^{(1)}t\tilde{f}^{(0)} + t^2\tilde{x}_0^{(0)2}f^{(1)} \\ &= t^2 \left( x_0^{(0)'}{}^2 A_0^{(0)} + 2x_0^{(0)'}x_0^{(1)'}x_0^{(0)}B_0^{(0)} + x_0^{(0)'}{}^2 x_0^{(1)}B_0^{(0)} + x_0^{(0)'}{}^2 x_0^{(0)}B_0^{(1)} \right). \end{aligned}$$

Hence it suffices to solve

$$(1.30) \quad \begin{aligned} & 2x_0^{(0)'}x_0^{(1)'}x_0^{(0)}B_0^{(0)} + x_0^{(0)'}{}^2 x_0^{(1)}B_0^{(0)} - 2\tilde{x}_0^{(0)}x_0^{(1)}\tilde{f}^{(0)} \\ &= -x_0^{(0)'}{}^2 A_0^{(0)} - x_0^{(0)'}{}^2 x_0^{(0)}B_0^{(1)} + \tilde{x}_0^{(0)2}f^{(1)}. \end{aligned}$$

Here, and in what follows, we use a new variable  $s$  given by

$$(1.31) \quad s = x_0^{(0)}(t, \rho).$$

Using the symbol  $\dot{x}$  to denote  $dx/ds$ , we then find the following equation (1.32) with the help of (1.27).

$$(1.32) \quad \begin{aligned} & B_0^{(0)} \left( 2s \frac{d}{ds} - 1 \right) x_0^{(1)}(s, \rho) \\ &= -A_0^{(0)} - sB_0^{(1)} + [(x_0^{(0)'})^{-2} \tilde{x}_0^{(0)2} f^{(1)}](t(s, \rho), \rho), \end{aligned}$$

where  $t(s, \rho)$  denotes the inverse function of  $s = x_0^{(0)}(t, \rho)$ . It is clear that (1.32) admits a solution  $x_0^{(1)}(s, \rho)$  that is holomorphic near  $s = 0$  for arbitrary constants  $A_0^{(0)}$  and  $B_0^{(1)}$ , which are to be fixed later. Furthermore we can immediately see

$$(1.33) \quad x_0^{(1)}(0, \rho) = \frac{1}{B_0^{(0)}} (A_0^{(0)} - f^{(1)}(0, \rho)),$$



(1.34)

$$\dot{x}_0^{(1)}(0, \rho) = \frac{1}{B_0^{(0)}} \left( -B_0^{(1)} + Z_0^{-1} (z'(0, \rho) f^{(1)}(0, \rho) + f^{(1)'}(0, \rho)) \right),$$

where

(1.35)

$$Z_0 = x_0^{(0)'}(0, \rho)$$

and

(1.36)

$$z(t, \rho) = (x_0^{(0)'}(t, \rho))^{-2} \tilde{x}_0^{(0)}(t, \rho)^2.$$

For  $p \geq 2$  (1.26.p) assumes the following form:

(1.37.p)

$$C_0^{(p)}(\rho) + D_0^{(p)}(\rho)t + t^2 \mathcal{E}_0^{(p)} = 0,$$

where  $C_0^{(p)}$  and  $D_0^{(p)}$  are free from  $t$  and  $\mathcal{E}_0^{(p)}$  contains in it at least

(1.38)

$$\sum_{j+k+l=p} x_0^{(j)'} x_0^{(k)'} A_0^{(l-1)} + \sum_{j+k+l+m=p} x_0^{(j)'} x_0^{(k)'} x_0^{(l)} B_0^{(m)}.$$

One can readily find that  $C_0^{(2)}$  is absent in (1.37.2) and that  $D_0^{(2)} = 0$  gives a quadratic constraint on  $(A_0^{(0)}, B_0^{(0)})$  (cf. [9, (1.1.1.17)]). Hence, by assuming  $D_0^{(2)} = 0$ , we can solve the equation  $\mathcal{E}_0^{(2)} = 0$  to find  $x_0^{(2)}(t, \rho)$  that is holomorphic near  $t = 0$ . As one of the most exciting points in our computation becomes visible at the next stage, we hasten to study the situation where  $p = 3$ ; we will come back to the explicit computation of  $x_0^{(2)}(t, \rho)$  after the study of the case. For this purpose we assume  $C_0^{(3)} = 0$ . Then a straightforward computation shows that this gives another quadratic constraint on  $(A_0^{(0)}, B_0^{(0)})$  (cf. [9, (1.1.1.19)]). The equations  $D_0^{(2)} = C_0^{(3)} = 0$  lead to

(1.39)

$$A_0^{(0)} = f^{(1)}(0, \rho)$$

and

(1.40)

$$B_0^{(0)2} = \rho^2.$$

Thus it follows from (1.29) and (1.35) that

$$(1.41) \quad Z_0^2 = 1.$$

And, by (1.33) we find the following amazing result:

$$(1.42) \quad x_0^{(1)}(0, \rho) = 0!$$

The relation (1.42) together with (1.41) plays a crucially important role at several points in the reasoning of [9]. As a typical example of such points we show here how (1.41) and (1.42) effect the computation of  $x_0^{(2)}(0, \rho)$ . To begin with, we rewrite (1.26.2) explicitly in  $s (= x_0^{(0)}(t, \rho))$ -variable:

$$(1.26'.2) \quad \begin{aligned} B_0^{(0)} \left( 2s \frac{d}{ds} - 1 \right) x_0^{(2)}(s, \rho) = & -A_0^{(1)} - B_0^{(2)} s \\ & - 2\dot{x}_0^{(1)}(s, \rho) A_0^{(0)} - 2\dot{x}_0^{(1)}(s, \rho) x_0^{(1)}(s, \rho) B_0^{(0)} - 2\dot{x}_0^{(1)}(s, \rho) s B_0^{(1)} \\ & - x_0^{(1)}(s, \rho) B_0^{(1)} - \dot{x}_0^{(1)}(s, \rho)^2 s B_0^{(0)} + (z(t, \rho) f^{(2)}(t, \rho) \\ & - [t^{-1} (x_0^{(0)'}(t, \rho))^{-2} (\mathcal{B}^{(1)}(t, \rho) - \mathcal{B}^{(1)}(0, \rho))]) \Big|_{t=t(s, \rho)}, \end{aligned}$$

where  $z(t, \rho)$  is the function given by (1.36) and

$$(1.43) \quad \mathcal{B}^{(1)}(t, \rho) = \tilde{f}^{(0)} - x_0^{(1)2} \tilde{f}^{(0)} - 2\tilde{x}_0^{(0)} x_0^{(1)} f^{(1)} - x_0^{(0)2} \tilde{x}_0^{(0)} B_0^{(0)}.$$

By way of parenthesis we note that the condition  $D_0^{(2)} = 0$  is given by  $\mathcal{B}^{(1)}(0, \rho) = 0$ . To evaluate the term in the brackets in (1.26'.2) at  $s = 0$  we compute  $\partial \mathcal{B}^{(1)} / \partial t|_{t=0}$  to find

$$(1.44) \quad \rho g'(0, \rho) - 2Z_0 f^{(1)}(0, \rho) x_0^{(1)'}(0, \rho) - 2B_0^{(0)} x_0^{(0)''}(0, \rho) - B_0^{(0)} \tilde{x}_0^{(0)'}(0, \rho).$$

In this computation we have repeatedly used (1.41) and (1.42); for example, we have used (1.42) to claim

$$(1.45) \quad (x_0^{(1)2} \tilde{f}^{(0)})' \Big|_{t=0} = x_0^{(1)} (\tilde{x}_0^{(0)} f^{(1)})' \Big|_{t=0} = 0.$$

Using (1.41), we further notice a remarkable cancellation of terms in the right-hand side of (1.26'.2) when they are evaluated at  $s = 0$ ; it follows from (1.39) that  $-2\dot{x}_0^{(1)}(0, \rho)A_0^{(0)}$  is cancelled by  $-(x_0^{(0)'}(0, \rho))^{-2}(-2Z_0f^{(1)}(0, \rho)x_0^{(1)'}(0, \rho))$  in (1.44), i.e.,

$$(1.46) \quad -2\dot{x}_0^{(1)}(0, \rho)A_0^{(0)} + 2Z_0^{-2}(Z_0^2A_0^{(0)}\dot{x}_0^{(1)}(0, \rho)) = 0.$$

An important implication of (1.46) is that the cancelling terms originally depended on  $B_0^{(1)}$  through  $\dot{x}_0^{(1)}(0, \rho)$  (cf. (1.34)). Furthermore other  $B_0^{(1)}$ -dependent terms in the right-hand side of (1.26'.2), i.e.,

$$(1.47) \quad \begin{aligned} & -2\dot{x}_0^{(1)}(s, \rho)x_0^{(1)}(s, \rho)B_0^{(0)} - 2\dot{x}_0^{(1)}(s, \rho)sB_0^{(1)} \\ & - x_0^{(1)}(s, \rho)B_0^{(1)} - \dot{x}_0^{(1)}(s, \rho)^2sB_0^{(0)} \end{aligned}$$

also vanish when evaluated at  $s = 0$ , thanks to (1.42). It then follows from (1.26'.2) that

$$(1.48) \quad B_0^{(0)}x_0^{(2)}(0, \rho) = A_0^{(1)} - f^{(2)}(0, \rho) + \chi_0^{(0)}B_0^{(0)},$$

where  $\chi_0^{(0)}$  is a constant fixed by  $g(t, \rho)$  (and  $Z_0 = \pm 1$ ). Thus  $x_0^{(2)}(0, \rho)$  is free from  $B_0^{(1)}$ , and this fact, together with the explicit form of  $\dot{x}_0^{(1)}(0, \rho)$  given by (1.34), enables us to explicitly describe  $D_0^{(3)}$  and  $C_0^{(4)}$ . An important point is that these “cancellations and vanishings” occur for every  $p \geq 2$  and that they make the concrete expression of the core parts of  $D_0^{(p+1)}$  and  $C_0^{(p+2)}$  to be “uniform”, as is shown below:

$$(1.49) \quad C_0^{(p+2)} = 2\left(A_0^{(p-1)} - \frac{A_0^{(0)}}{B_0^{(0)}}B_0^{(p-1)}\right) \text{ depends only on } \\ (A_0^{(q)}, B_0^{(q)}) \quad (q \leq p-2) \text{ and given data such as } f^{(q)}(0, \rho) \\ (q \leq p-1),$$

and

$$(1.50) \quad D_0^{(p+1)} - 2Z_0 \left( \frac{A_0^{(0)}}{B_0^{(0)}} A_0^{(p-1)} - B_0^{(p-1)} \right) \text{ depends only on } (A_0^{(q)}, B_0^{(q)}) \quad (q \leq p-2) \text{ and given data.}$$

As is clear from (1.49) and (1.50) we can determine  $(A_0^{(p-1)}, B_0^{(p-1)})$  ( $p \geq 2$ ) recursively by solving linear equations. (The solvability of the equations is guaranteed by the assumption (1.5) together with the explicit computations (1.39) and (1.40) of  $A_0^{(0)}$  and  $B_0^{(0)}$ .) Here we emphasize the importance of the point that the main parts “ $2(A_0^{(p-1)} - A_0^{(0)} B_0^{(p-1)}/B_0^{(0)})$ ” and “ $2Z_0(A_0^{(0)} A_0^{(p-1)}/B_0^{(0)} - B_0^{(p-1)})$ ” are of the same form for every  $p$ . Parenthetically we note that  $C_0^{(p+2)}$  (resp.,  $D_0^{(p+1)}$ ) read off from (1.26.p+2) (resp., (1.26.p+1)) at first contains  $x_0^{(p)}(0, \rho)$  and  $x_0^{(p-1)'}(0, \rho)$ ; their “principal parts”, the parts which may be dependent on  $A_0^{(p-1)}$  and  $B_0^{(p-1)}$ , are at first respectively given as follows (cf. [9, Lemma 1.1.2.1]):

$$(1.51) \quad [(x_0^{(0)'})^2 A_0^{(p-1)} + 2x_0^{(0)'} x_0^{(p-1)'} A_0^{(0)} + x_0^{(0)'} x_0^{(p)} B_0^{(0)}] \Big|_{t=0},$$

$$(1.52) \quad [2\tilde{x}_0^{(0)} x_0^{(0)'} x_0^{(p-1)'} B_0^{(0)} + \tilde{x}_0^{(0)} x_0^{(0)'} x_0^{(p)} B_0^{(p-1)} + 2\tilde{x}_0^{(0)} f^{(1)} x_0^{(p)} + (x_0^{(0)'})^2 x_0^{(p-1)'} B_0^{(0)}] \Big|_{t=0}.$$

Thus the clean and uniform results (1.49) and (1.50) are almost miraculous, and at the same time we believe that, without such uniform expressions, it should be impossible to find conditions that would guarantee the recursive solvability of equations  $C_0^{(p+2)} = D_0^{(p+1)} = 0$ .

Thus a naive way of inductively determining  $(x_0^{(p)}, A_0^{(p)}, B_0^{(p)})$  ( $p \geq 1$ ) is as follows:

In order to find a holomorphic (in  $t$ ) solution  $x_0^{(p)}(t, \rho)$  of (1.26.p) one first requires  $C_0^{(p)} = D_0^{(p)} = 0$ ; then by rewriting (1.26.p) in  $s(=$

$x_0^{(0)}(t, \rho)$ -variable we find

$$(1.26'.p) \quad B_0^{(0)} \left( 2s \frac{d}{ds} - 1 \right) x_0^{(p)}(s, \rho) = -A_0^{(p-1)} - B_0^{(p)} s + B_0^{(0)} R_0^{(p)}(s, \rho),$$

where

(1.53.p)

$$\begin{aligned} B_0^{(0)} R_0^{(p)}(s, \rho) = & - \sum_{\substack{i+j+k=p-1 \\ k \leq p-2}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} A_0^{(k)} - \sum_{\substack{i+j+k+l=p \\ i,j,k,l \leq p-1}} \dot{x}_0^{(i)} \dot{x}_0^{(j)} x_0^{(k)} B_0^{(l)} \\ & + \left[ (x_0^{(0)'}(t, \rho))^{-2} t^{-2} \right. \\ & \times \left( \sum_{i+j+k=p-3} x_0^{(i)'} x_0^{(j)'} A_0^{(k)} + \sum_{i+j+k+l=p-2} x_0^{(i)'} x_0^{(j)'} x_0^{(k)} B_0^{(l)} \right. \\ & \left. \left. + \sum_{\substack{i+j+k=p \\ k \geq 1}} x_0^{(i)} x_0^{(j)} f^{(k)} + \sum_{\substack{i+j=p \\ i,j \geq 1}} x_0^{(i)} x_0^{(j)} f^{(0)} - f^{(p-2)} \right) \right] \Big|_{t=t(s, \rho)}. \end{aligned}$$

It is then clear that (1.26'.p) admits a holomorphic solution  $x_0^{(p)}(s, \rho)$  for any complex numbers  $A_0^{(p-1)}$  and  $B_0^{(p)}$ , as we have assumed  $C_0^{(p)} = D_0^{(p)} = 0$ . On the other hand, if we admit (1.49) and (1.50), the equation  $C_0^{(p)} = 0$  combined with  $D_0^{(p-1)} = 0$ , a relation required in the preceding stage, will fix  $A_0^{(p-3)}$  and  $B_0^{(p-3)}$  (for  $p \geq 4$ ), which have not yet been completely fixed so far. At the same time, the condition  $D_0^{(p)} = 0$  will be used at the next stage to fix  $A_0^{(p-2)}$  and  $B_0^{(p-2)}$ . Thus the reader might find the reasoning to be somewhat clumsy, particularly because of the unevenness of the indices in question. Hence we present here the core of the more refined induction procedure with some comments on its background. We note that the induction scheme we present below is also suited for the growth order estimation of the functions constructed. See [9, Section 1.1.3 and Section 1.2] for the details.

Let us first prepare some notations. We denote a triplet  $\{x_0^{(r)}(s, \rho), A_0^{(r)}, B_0^{(r)}\}$  by  $T_0^{(r)}$  and use the symbol  $\mathfrak{A}_0(p)$  to mean the assertion that  $T_0^{(r)}$  is given for  $0 \leq r \leq p$  so that each of them satisfies the following conditions (1.54.r)  $\sim$  (1.58.r):

- (1.54.r)  $x_0^{(r)}(s, \rho)$  is a holomorphic solution of (1.26'.r) near  $s = 0$ ,
- (1.55.r)  $x_0^{(r)}(s, \rho)$  depends on  $(\vec{A}_0[r-1], \vec{B}_0[r]) \stackrel{\text{def}}{=} (A_0^{(0)}, A_0^{(1)}, \dots, A_0^{(r-1)}, B_0^{(0)}, B_0^{(1)}, \dots, B_0^{(r)})$ ,
- (1.56.r)  $C_0^{(r+3)}$  and  $D_0^{(r+2)}$  depend on  $(\vec{A}_0[r], \vec{B}_0[r])$ , and  $(\vec{A}_0[r], \vec{B}_0[r])$  annihilates them,
- (1.57.r)  $C_0^{(r+3)} - 2(A_0^{(r)} - \frac{A_0^{(0)}}{B_0^{(0)}} B_0^{(r)})$  is independent of  $(A_0^{(r)}, B_0^{(r)})$ ,
- (1.58.r)  $D_0^{(r+2)} - 2Z_0(\frac{A_0^{(0)}}{B_0^{(0)}} A_0^{(r)} - B_0^{(r)})$  is independent of  $(A_0^{(r)}, B_0^{(r)})$ .

Then we obtain

**Proposition 1.1.** *The assertion  $\mathfrak{A}(p)$  is valid for every  $p \geq 1$ .*

The proof of this proposition is done in an inductive manner (cf. [9, Section 1.1.3]). But we imagine that the first reactions to this proposition of the reader might be the following:

[A] Is the claim logically self-contained? For example, the concrete expression (1.51) (resp., (1.52)) of  $C_0^{(p+2)}$  (resp.,  $D_0^{(p+1)}$ ) indicates that we need  $x_0^{(p_0+1)}(0, \rho)$  for the description of  $C_0^{(p_0+3)}$  and  $D_0^{(p_0+2)}$ , but  $\mathfrak{A}_0(p_0)$  refers to  $T_0^{(r)}$  ( $r \leq p_0$ ) only.

[B] Well, this may not be a logical question but a rather psychological one. Still, I wonder why (1.56. $p_0$ ) is valid despite the presence of

$x_0^{(p_0+1)}$  in  $C_0^{(p_0+3)}$ ; in view of (1.55. $p_0 + 1$ ) I think  $\vec{B}_0[r]$  in (1.56. $r$ ) might be  $\vec{B}_0[r + 1]$ .

So let us first dispel potential sources of such uneasiness. Actually both [A] and [B] are reasonable concerns and the core of the proof of Proposition 1.1 is closely related to them. The answer to [A] is rather easy: although  $x_0^{(p_0+1)}(s, \rho)$  is not referred to in  $\mathfrak{A}_0(p_0)$ , the assertion  $\mathfrak{A}_0(p_0)$  trivially entails the vanishing of  $C_0^{(p_0+1)}$  and  $D_0^{(p_0+1)}$  and hence the existence of a holomorphic solution  $x_0^{(p_0+1)}(s, \rho)$  of (1.26'. $p_0 + 1$ ) is guaranteed. Then it follows from (1.26'. $p_0 + 1$ ) that  $x_0^{(p_0+1)}(0, \rho)$  is given by

$$(1.59) \quad x_0^{(p_0+1)}(0, \rho) = (B_0^{(0)})^{-1} A_0^{(p_0)} - R_0^{(p_0+1)}(0, \rho).$$

Thus  $x_0^{(p_0+1)}(0, \rho)$  is described by  $T_0^{(r)}$  ( $r \leq p_0$ ). Note that  $R_0^{(p_0+1)}(s, \rho)$  is determined by  $T_0^{(r)}$  ( $r \leq p_0$ ). (Cf. (1.53. $p$ )) This concrete expression of  $x_0^{(p_0+1)}(0, \rho)$  will also alleviate the anxiety [B]. Still, the reader might wonder:

[B'] How can we proceed with a seemingly rather vague expression like (1.59)? For example, how can we find (1.57. $p_0+1$ ) and (1.58. $p_0+1$ ), which are needed to proceed one step further, that is, to confirm  $\mathfrak{A}_0(p_0 + 1)$  using the data in  $\mathfrak{A}_0(p_0)$ ?

Well, then, we present the core of the proof of Proposition 1.1, which will clarify all these.

*Remark 1.5.* Here we have tried to follow the late Professor Ehrenpreis in his style of lecturing — how do you find it, Professor Ehrenpreis?

To perform the induction procedure, let us suppose that  $\mathfrak{A}_0(p_0)$  is validated. Then, as we noted to see (1.59), we have

$$(1.60) \quad C_0^{(p_0+1)} = D_0^{(p_0+1)} = 0,$$

and hence we can find a holomorphic solution  $x_0^{(p_0+1)}(s, \rho)$  of (1.26'. $p_0+1$ ) for any complex number  $B_0^{(p_0+1)}$ , which meets the requirement (1.54. $p_0+1$ ) and (1.55. $p_0+1$ ). Now, the intriguing part of the proof begins here. Since  $\mathfrak{A}_0(p_0)$  entails

$$(1.61) \quad C_0^{(p_0+2)} = D_0^{(p_0+2)} = 0,$$

we can further find a holomorphic solution  $x_0^{(p_0+2)}(s, \rho)$  of (1.26'. $p_0+2$ ) for any complex numbers  $A_0^{(p_0+1)}$  and  $B_0^{(p_0+2)}$ . To confirm  $\mathfrak{A}_0(p_0+1)$  we do not make full use of  $x_0^{(p_0+2)}(s, \rho)$  but use only  $x_0^{(p_0+2)}(0, \rho)$  for the computation of  $C_0^{(p_0+4)}$  and  $D_0^{(p_0+3)}$ . Since it follows from (1.26'. $p_0+2$ ) that

$$(1.62) \quad B_0^{(0)} x_0^{(p_0+2)}(0, \rho) = A_0^{(p_0+1)} - B_0^{(0)} R_0^{(p_0+2)}(0, \rho),$$

the following Lemma 1.1 is the key to the proof.

**Lemma 1.1.** *Let us suppose  $\mathfrak{A}_0(p_0)$  is validated. Then we find*

$$(1.63) \quad B_0^{(0)} R_0^{(p_0+2)}(0, \rho) \text{ is free from } B_0^{(p_0+1)}.$$

Before giving the proof of this lemma, we note the following three facts: first, once the lemma is proved, the confirmation of  $\mathfrak{A}_0(p_0+1)$  is an easy task as we will note later. Second, although this is a rather obvious comment, the complex number  $B_0^{(p_0+2)}$  introduced to define  $x_0^{(p_0+2)}(s, \rho)$  is actually irrelevant to  $x_0^{(p_0+2)}(0, \rho)$  and has no relevance to the later argument; in validating  $\mathfrak{A}_0(p_0+2)$  we may use another complex number  $\tilde{B}_0^{(p_0+2)}$  to construct  $\tilde{x}_0^{(p_0+2)}(s, \rho)$  needed there, which may be different from  $x_0^{(p_0+2)}(s, \rho)$  constructed above for the auxiliary purpose of finding the constant  $x_0^{(p_0+2)}(0, \rho)$ , which is irrelevant to  $B_0^{(p_0+2)}$ . Third, the cancellation among several terms to be observed in the proof of Lemma 1.1 also plays crucially important roles in the estimation of growth orders of  $T_0^{(p)}$  etc. (See [C1] and [C2] after Remark 1.7).



Now we give

*Proof of Lemma 1.1.* In view of (1.55.r) ( $r \leq p_0$ ) we find that the terms in  $B_0^{(0)} R_0^{(p_0+2)}(0, \rho)$  which may contain  $B_0^{(p_0+1)}$  are those which contain  $x_0^{(p_0+1)}$ ,  $\dot{x}_0^{(p_0+1)}$  and  $B_0^{(p_0+1)}$  itself. Furthermore we note that  $x_0^{(p_0+1)}(0, \rho)$  is seen to be free from  $B_0^{(p_0+1)}$  by (1.59) together with the fact that  $R_0^{(p_0+1)}(s, \rho)$  is determined by  $T_0^{(r)}$  ( $r \leq p_0$ ). Thus we do not worry about  $-\left(\sum_{i+j+k=1} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) B_0^{(k)}\right) x_0^{(p_0+1)}(0, \rho)$  in our computation. Hence it is enough to examine the contribution from the following terms:

$$(1.64) \quad -\left(\sum_{i+j+k=1} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) x_0^{(k)}(0, \rho)\right) B_0^{(p_0+1)},$$

$$(1.65) \quad -\left(\sum_{\substack{i+j=p_0+2 \\ i, j \leq p_0+1}} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho)\right) x_0^{(0)}(0, \rho) B_0^{(0)} \\ -\left(\sum_{i+j=p_0+1} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho)\right) \left(\sum_{k+l=1} x_0^{(k)}(0, \rho) B_0^{(l)}\right),$$

$$(1.66) \quad -2\dot{x}_0^{(0)}(0, \rho) \dot{x}_0^{(p_0+1)}(0, \rho) A_0^{(0)}$$

and

$$(1.67) \quad \text{terms that appear in the coefficients of the Taylor expansion in } s \text{ of } \left[ (x_0^{(0)})^{-2} t^{-2} (2x_0^{(0)} x_0^{(p_0+1)} f^{(1)} + 2x_0^{(1)} x_0^{(p_0+1)} f^{(0)}) \right] \Big|_{t=t(s, \rho)}.$$

Here we observe the following two facts:

$$(1.68) \quad \text{any term that may contain } B_0^{(p_0+1)} \text{ in (1.64) and (1.65) vanishes because of the vanishing of } x_0^{(i)}(0, \rho) \text{ } (i = 0, 1),$$

and

$$(1.69) \quad -2\dot{x}_0^{(0)}(0, \rho)\dot{x}_0^{(p_0+1)}(0, \rho)A_0^{(0)} + 2(x_0^{(0)'})^{-2}\tilde{x}_0^{(0)}x_0^{(p_0+1)'}f^{(1)}\big|_{t=t(0, \rho)} = 0,$$

where the second term in (1.69) is the unique relevant term in (1.67). (Cf. Remark 1.6 below.) It is then evident that (1.68) (resp., (1.69)) is a counterpart of (1.45) (resp., (1.46)), which we encountered in the computation of  $x_0^{(2)}(0, \rho)$ . In any event, (1.68) and (1.69) clearly prove the lemma.

Q.E.D.

*Remark 1.6.* Since  $x_0^{(p_0+1)}(0, \rho)$  is free from  $B_0^{(p_0+1)}$  as noted above,  $B_0^{(p_0+1)}$  is not contained in

$$(1.70) \quad 2(x_0^{(0)'})^{-2}\tilde{x}_0^{(1)}(0, \rho)\tilde{f}^{(0)}(0, \rho)x_0^{(p_0+1)}(0, \rho),$$

despite the fact that (1.70) is resembling to the second term in (1.69) in the sense that (1.70) originates from

$$(1.71) \quad [(x_0^{(0)'})^{-2}t^{-2}(2x_0^{(1)}x_0^{(p_0+1)}f^{(0)})]\big|_{t=t(s, \rho)},$$

which forms the pair to

$$(1.72) \quad [(x_0^{(0)'})^{-2}t^{-2}(2x_0^{(0)}x_0^{(p_0+1)}f^{(1)})]\big|_{t=t(s, \rho)}$$

in (1.67), the term which generates the second term in (1.69).

Now Lemma 1.1 and (1.62) imply

$$(1.73) \quad x_0^{(p_0+2)} - A_0^{(p_0+1)}/B_0^{(0)} \text{ depends on only } (\vec{A}_0[p_0], \vec{B}_0[p_0]).$$

On the other hand (1.26'. $p_0 + 1$ ) entails

$$(1.74) \quad B_0^{(0)}\dot{x}_0^{(p_0+1)}(0, \rho) + B_0^{(p_0+1)} = B_0^{(0)}\dot{R}_0^{(p_0+1)}(0, \rho),$$

which also depends on only  $(\vec{A}_0[p_0], \vec{B}_0[p_0])$ .

Substituting those into (1.51) and (1.52) with  $p = p_0 + 2$ , we can validate (1.57. $p_0 + 1$ ) and (1.58. $p_0 + 1$ ). Then we can readily choose  $(A_0^{(p_0+1)}, B_0^{(p_0+1)})$  so that they satisfy

$$(1.75) \quad C_0^{(p_0+4)} = D_0^{(p_0+3)} = 0.$$

Thus the induction proceeds. This completes the proof of Proposition 1.1.

*Remark 1.7.* As (1.73) and (1.74) show, expressions like (1.59) nicely fit in with our induction scheme. This is the answer to the query [B'], and the important point in the answer is Lemma 1.1.

Thus we have formally constructed  $T_0^{(p)} = \{x_0^{(p)}, A_0^{(p)}, B_0^{(p)}\}$  for every  $p \geq 0$ . We can further confirm (cf. [9, Lemma 1.2.3]) that they actually define a function

$$(1.76) \quad x_0(t, a, \rho) = \sum_{p \geq 0} x_0^{(p)}(t, \rho) a^p,$$

which is holomorphic on

$$(1.77) \quad \{(t, a, \rho) \in \mathbb{C}^3; |t| < r_0, \rho \neq 0, |a|, |\rho| < M_0, |a/\rho| < N_0\}$$

and constants

$$(1.78) \quad A_0(a, \rho) = \sum_{p \geq 0} A_0^{(p)}(\rho) a^p$$

and

$$(1.79) \quad B_0(a, \rho) = \sum_{p \geq 0} B_0^{(p)}(\rho) a^p,$$

which are convergent on

$$(1.80) \quad \{(a, \rho) \in \mathbb{C}^2; \rho \neq 0, |a|, |\rho| < M_0, |a/\rho| < N_0\}$$

for some positive constants  $r_0, M_0$  and  $N_0$ . Although we do not give the details of the proof here, we note the following core facts [C1] and [C2]. Here we use the symbol  $(\sigma.j)$  ( $j = \text{i, ii and iii}$ ) to denote the following sums in  $R_0^{(p_0+1)}(s, \rho)$  (cf. (1.53.p) with  $p = p_0 + 1$ ):

$$(1.81) \quad (\sigma.\text{i}) \stackrel{\text{def}}{=} - \sum_{i+j=p_0} \dot{x}_0^{(i)}(s, \rho) \dot{x}_0^{(j)}(s, \rho) A_0^{(0)} / B_0^{(0)},$$

(cf. the first sum in (1.53.p<sub>0</sub> + 1)),

$$(1.82) \quad (\sigma.\text{ii}) \stackrel{\text{def}}{=} \left[ (x_0^{(0)'}(t, \rho))^{-2} t^{-2} \left( \sum_{i+j=p_0} x_0^{(i)}(s, \rho) x_0^{(j)}(s, \rho) f^{(1)}(t, \rho) / B_0^{(0)} \right) \right] \Big|_{t=t(s, \rho)}$$

(cf. the fifth sum in (1.53.p<sub>0</sub> + 1)),

$$(1.83) \quad (\sigma.\text{iii}) \stackrel{\text{def}}{=} \left[ (x_0^{(0)'}(t, \rho))^{-2} t^{-1} \tilde{f}^{(0)}(t) \left( \sum_{\substack{i+j=p_0+1 \\ i, j \geq 1}} x_0^{(i)}(s, \rho) x_0^{(j)}(s, \rho) / B_0^{(0)} \right) \right] \Big|_{t=t(s, \rho)}$$

(cf. the sixth sum in (1.53.p<sub>0</sub> + 1)).

Now in inductively showing the domination of  $\{x_0^{(p)}, A_0^{(p)}, B_0^{(p)}\}$  which guarantees the domains of convergence (1.77) and (1.80) we at first find that each of these three terms might block the induction reasoning from proceeding. But, fortunately we observe

[C1] What we encounter in the induction process is the estimation of the integral of the form, say,

$$(1.84) \quad I(\text{iii}) = \frac{1}{2\pi i} \oint \frac{(\sigma.\text{iii})}{s} ds;$$

then by the Taylor expansion of

$$(1.85) \quad \sum_{\substack{i+j=p_0+1 \\ i,j \geq 1}} x_0^{(i)}(s, \rho) x_0^{(j)}(s, \rho),$$

we find the following from the relation  $\tilde{f}^{(0)} = \rho g$ :

$$(1.86) \quad |I(\text{iii})| = \left| \frac{1}{2\pi i} \oint \left( \frac{dt}{ds} \right)^2 \left( \frac{s}{t} \right) Z_0 g(t, \rho) \left\{ \sum_{\substack{i+j=p_0+1 \\ i,j \geq 1}} x_0^{(i)}(0, \rho) x_0^{(j)}(0, \rho) \right. \right. \\ \left. \left. + 2s \left( \sum_{\substack{i+j=p_0+1 \\ i,j \geq 1}} x_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \right) + O(s^2) \right\} \frac{ds}{s^2} \right|.$$

Then in order to make the induction reasoning run smoothly we use (1.42); the second sum in the integrand of the right-hand side gives the contribution of the form

$$(1.87) \quad \frac{1}{2\pi i} \oint 2 \left( \sum_{\substack{i+j=p_0+1 \\ i \geq 2, j \geq 1}} x_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \right) \frac{ds}{s}.$$

See [9] for the details which show how this gain in the margin of indices is important in the induction procedures.

**[C2]** The integral

$$(1.88) \quad I(\text{i}) = \frac{1}{2\pi i} \oint \frac{(\sigma, \text{i})}{s} ds$$

is, notably enough, cancelled by the contribution

$$(1.89) \quad I_0 = \frac{1}{2\pi i} \frac{1}{B_0^{(0)}} \oint \frac{s^2}{t^2} \left( \frac{dt}{ds} \right)^2 \left( \sum_{i+j=p_0} \dot{x}_0^{(i)}(0, \rho) \dot{x}_0^{(j)}(0, \rho) \right) f^{(1)}(t, \rho) \frac{ds}{s},$$

which originates from

$$(1.90) \quad I(\text{ii}) = \frac{1}{2\pi i} \oint \frac{(\sigma.\text{ii})}{s} ds,$$

and, furthermore  $I(\text{ii}) - I_0$  is amenable to the induction procedure, as is shown in [9].

We readily find [C1] and [C2] are reasonable counterparts of (1.68) and (1.69) respectively.

Thus we have succeeded in constructing  $\{x_0(t, a, \rho), A_0(a, \rho), B_0(a, \rho)\}$  which satisfies the highest degree (i.e., degree 0) part in  $\eta$  of the required relation (1.13); hence the reasonable approach to the proof of (1.13) is to try to construct the perturbation series  $\{x = \sum_{k \geq 0} x_{2k} \eta^{-2k},$

$A = \sum_{k \geq 0} A_{2k} \eta^{-2k}, B = \sum_{k \geq 0} B_{2k} \eta^{-2k}\}$  so that they satisfy (1.13). As we mentioned earlier we further expand  $\{x_{2k}, A_{2k}, B_{2k}\}$  into the power series of  $a$  (cf. (1.19), (1.20) and (1.21)), and by comparing the coefficients of  $a^p$  in the coefficients of  $\eta^{-2n}$  ( $n \geq 1$ ) of (1.13) multiplied by  $(t^2 - a^2)(x^2 - a^2)$  we obtain

$$(1.91) \quad \sum_{\substack{q+r+u=p \\ i+j=n}} x_{2i}^{(q)} x_{2j}^{(r)} f^{(u)} \\ = t^2 \left[ \sum_{\substack{q+r+u=p-1 \\ i+j+k=n}} x_{2i}^{(q)'} x_{2j}^{(r)'} A_{2k}^{(u)} + \sum_{\substack{q+r+u+v=p \\ i+j+k+l=n}} x_{2i}^{(q)'} x_{2j}^{(r)'} x_{2k}^{(u)} B_{2l}^{(v)} \right. \\ \left. - \frac{1}{2} \sum_{\substack{q+r+u=p \\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x; t\}_{2k}^{(u)} + \frac{1}{2} \{x; t\}_{2(n-1)}^{(p-2)} \right] \\ - \left[ \sum_{\substack{q+r+u=p-3 \\ i+j+k=n}} x_{2i}^{(q)'} x_{2j}^{(r)'} A_{2k}^{(u)} + \sum_{\substack{q+r+u+v=p-2 \\ i+j+k+l=n}} x_{2i}^{(q)'} x_{2j}^{(r)'} x_{2k}^{(u)} B_{2l}^{(v)} \right]$$

$$-\frac{1}{2} \sum_{\substack{q+r+u=p-2 \\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x; t\}_{2k}^{(u)} + \frac{1}{2} \{x; t\}_{2(n-1)}^{(p-4)} \Big],$$

where  $\{x; t\}_{2k}^{(q)}$  designates the coefficient of  $a^q \eta^{-2k}$  of  $\{x; t\}$ , that is,

$$(1.92) \quad \{x; t\} = \sum_{q, k \geq 0} \{x; t\}_{2k}^{(q)} a^q \eta^{-2k}.$$

In view of the resemblance between (1.26.p) and (1.91), one expects that the construction and domination of the triplet  $T_{2n}^{(r)} = \{x_{2n}^{(r)}(s, \rho), A_{2n}^{(r)}(\rho), B_{2n}^{(r)}(\rho)\}$  ( $n \geq 1$ ) may be performed in parallel with the construction and domination of  $T_0^{(r)}$ . And, actually this is really the case. We only note the following facts:

(1.93) in the recursive construction of  $x_{2n}^{(p)}(s, \rho)$  ( $p = 0, 1, 2, \dots$ ) the relation  $x_{2n}^{(0)}(0, \rho) = 0$  plays an important role,

(1.94) assertions similar to [C1] and [C2] (with the appropriate shift of indices) also play important roles,

and

(1.95) in dominating the growth order of  $T_{2n}^{(p)}$  we first dominate  $\{x; t\}_{2(n-1)}^{(p)}$  using the induction hypothesis and then employ the similar argument used in dominating  $T_0^{(p)}$ .

We refer the reader to [9, Section 1.2] for the details. Here we content ourselves by quoting the final result which will be used later.

**Theorem 1.1.** *Let  $Q(t, a, \rho, \eta)$  be a potential of an M2P1T operator given by (1.6). Then there exist positive constants  $r_0, M_0, N_0, R_0$  and holomorphic functions  $A_{2n}(a, \rho), B_{2n}(a, \rho)$  and  $x_{2n}(t, a, \rho)$  ( $n \geq 0$ ) on*

$$(1.96) \quad \{(t, a, \rho) \in \mathbb{C}^3; |t| < r_0, \rho \neq 0, |a|, |\rho| < M_0, |a/\rho| < N_0\}$$

for which the following conditions are satisfied there;

$$(1.97) \quad A(a, \rho, \eta), B(a, \rho, \eta) \text{ and } x(t, a, \rho, \eta) \text{ satisfy (1.13),}$$

$$(1.98) \quad \frac{1}{2}|f^{(1)}(0, 0)| \leq |A_0(a, \rho)| \leq 2|f^{(1)}(0, 0)|,$$

$$(1.99) \quad |B_0(a, \rho)| \leq 2|\rho|,$$

$$(1.100) \quad \frac{\partial x_0}{\partial t}(t, a, \rho) \neq 0,$$

$$(1.101) \quad x_0^2(\pm a, a, \rho) = a^2,$$

$$(1.102) \quad \text{if } t = t_0(a, \rho) \text{ satisfies } f(t_0, a, \rho) = 0 \text{ then } aA_0(a, \rho) + x_0(t_0, a, \rho)B_0(a, \rho) = 0 \text{ holds,}$$

the following estimates hold for  $n \geq 1$ ;

$$(1.103) \quad |A_{2n}(a, \rho)| \leq |\rho|(2n)!R_0^n|\rho|^{-n},$$

$$(1.104) \quad |B_{2n}(a, \rho)| \leq |\rho|(2n)!R_0^n|\rho|^{-n},$$

$$(1.105) \quad |x_{2n}(t, a, \rho)| \leq |\rho|(2n)!R_0^n|\rho|^{-n},$$

$$(1.106) \quad \left| \frac{dx_{2n}}{dt}(t, a, \rho) \right| \leq |\rho|(2n)!R_0^n|\rho|^{-n}.$$

*Remark 1.8.* Although we have presented the results assuming (1.8), the construction and the domination of  $\{x, A, B\}$  can be done without the assumption. In this case the potential of the canonical form of an M2P1T equation is

$$(1.107) \quad \frac{aA + xB}{x^2 - a^2} + \eta^{-2} \left( \frac{g_+(a)}{(x - a)^2} + \frac{g_-(-a)}{(x + a)^2} \right).$$



## 2 Intertwining the Borel transformed Schrödinger operators

As was first observed in [2], the analytic meaning of the formal coordinate transformation becomes most transparent with the help of the Borel transformation. To describe the situation concretely, let us first introduce the inverse function  $h(x, a, \rho)$  of  $x = x_0(t, a, \rho)$ , that is,

$$(2.1) \quad x = x_0(h(x, a, \rho), a, \rho), \quad t = h(x_0(t, a, \rho), a, \rho).$$

Since we formally find

$$(2.2) \quad \psi(x_0 + \eta^{-2}x_2 + \eta^{-4}x_4 + \cdots, \eta) = \sum_{n \geq 0} \frac{1}{n!} \left( \sum_{k \geq 1} x_{2k} \eta^{-2k} \right)^n \frac{\partial^n}{\partial x^n} \psi(x, \eta) \Big|_{x=x_0},$$

its Borel transform has the form

$$(2.3) \quad \left( \sum_{n \geq 0} \frac{1}{n!} \left( \sum_{k \geq 1} x_{2k} (h(x, a, \rho), a, \rho) \left( \frac{\partial}{\partial y} \right)^{-2k} \right)^n \frac{\partial^n}{\partial x^n} \right) \psi_B(x, y) \\ =: \exp \left( \left( \sum_{k \geq 1} x_{2k} (h(x, a, \rho), a, \rho) \eta^{-2k} \right) \xi \right) : \psi_B(x, y).$$

In the right-hand side of (2.3), and also in what follows, we denote by  $\xi$  the symbol of  $\partial/\partial x$  and use the ideograms in the symbol calculus of microdifferential operators; in particular the ideogram  $: \sigma :$  designates the normal ordered product determined by a symbol  $\sigma$ . We note that  $: \sigma :$  makes sense as a microdifferential operator when the formal series  $\sigma$  satisfies some growth order conditions like those we discussed in Theorem 1.1. (Cf. Theorem 2.1 below.) See [1] for the details of the symbol calculus. The relation (2.3) indicates that the structure of Schrödinger equations should be most clearly understood when they are Borel transformed. Actually we find Theorem 2.1 below by making use of the formal series constructed in Section 1.

To state the theorem let us prepare some notations.

Let  $N$  denote the Borel transform of an M2P1T operator written in  $(x, y)$ -coordinate, that is,

$$(2.4) \quad N = \left( \frac{\partial h}{\partial x} \right)^{-2} \frac{\partial^2}{\partial x^2} - \frac{\partial^2 h}{\partial x^2} \left( \frac{\partial h}{\partial x} \right)^{-3} \frac{\partial}{\partial x} - Q \left( h(x, a, \rho), a, \rho, \frac{\partial}{\partial y} \right) \frac{\partial^2}{\partial y^2}.$$

We also denote the Borel transform of the  $\infty$ -Mathieu equation by  $M_\infty$ . Using  $\{x_{2n}\}_{n \geq 0}$  and the function  $h$  in (2.1), we define

$$(2.5) \quad r_{2k} = x_{2k}(h(x, a, \rho), a, \rho)$$

$$(2.6) \quad r = \sum_{k \geq 1} r_{2k} \eta^{-2k},$$

$$(2.7) \quad s = x + r,$$

$$(2.8) \quad \mathcal{X} =: \left( \frac{\partial h}{\partial x} \right)^{1/2} \left( \frac{\partial s}{\partial x} \right)^{-1/2} \exp(r\xi) :,$$

$$(2.9) \quad \mathcal{Y} =: \left( \frac{\partial h}{\partial x} \right)^{-3/2} \left( \frac{\partial s}{\partial x} \right)^{3/2} \exp(r\xi) :.$$

To describe the geometric situation we introduce the following set  $W$  where  $C_0, \delta_0$  and  $\delta_1$  are some positive constants:

$$(2.10) \quad W = \{(a, \rho) \in \mathbb{C}^2; |a| \leq C_0|\rho|, 0 < |\rho| < \delta_0, |a| < \delta_1\}.$$

With these notations we can deduce Theorem 2.1 below from the results in Section 1 by using the same reasoning as in the proof of Theorem 2.6 of [8].

**Theorem 2.1.** *Let  $U$  be a sufficiently small open neighborhood of the closed interval  $[-a, a]$ . Then, for sufficiently small constants*

$C_0, \delta_0$  and  $\delta_1$ , microdifferential operators  $\mathcal{X}$  and  $\mathcal{Y}$  intertwine  $N$  and  $M_\infty$  on  $U \times W_0$  with the exception of  $(x^2 - a^2)\eta = 0$ , that is, we have

$$(2.11) \quad N\mathcal{X} = \mathcal{Y}M_\infty$$

with  $\mathcal{X}$  and  $\mathcal{Y}$  being invertible there.

Although the  $\infty$ -Mathieu equation contains infinite series  $A$  and  $B$ , they satisfy the growth order conditions stated in Theorem 1.1. The growth order conditions enable us to relate, by microdifferential operators, the Borel transformed  $\infty$ -Mathieu operator and the Borel transformed Mathieu operator  $M = M(A, B, c_+, c_-)$ , that is,

$$(2.12) \quad M(A, B, c_+, c_-) = \frac{\partial^2}{\partial x^2} - \frac{aA + xB}{x^2 - a^2} \frac{\partial^2}{\partial y^2} - \frac{c_+}{(x - a)^2} - \frac{c_-}{(x + a)^2}$$

with  $A, B$  and  $c_\pm$  being genuine constants, as the following Theorem 2.2 shows.

**Theorem 2.2.** *There exist microdifferential operators  $\mathcal{A}$  and  $\mathcal{B}$  for which the following relation holds:*

$$(2.13) \quad \mathcal{A}\mathcal{B}M = M_\infty\mathcal{A}\mathcal{B}.$$

The proof is essentially the same as the proof of Theorem 4.1 of [10]; it suffices to define

$$(2.14) \quad \mathcal{A} =: \exp\left(\sum_{k \geq 1} A_{2k} \eta^{-2k}\right) a \alpha_0 :$$

and

$$(2.15) \quad \mathcal{B} =: \exp\left(\sum_{d \geq 1} B_{2k} \eta^{-2k}\right) \beta_0 :,$$

where  $\alpha_0$  (resp.,  $\beta_0$ ) stands for the symbol of  $\partial/\partial(aA_0)$  (resp.,  $\partial/\partial B_0$ ).

These theorems assert that the microlocal structure of Borel transformed WKB solutions of an M2P1T equation coincides with that of a Mathieu equation. By appropriately representing the action of the microdifferential operator in question as an integro-differential operator acting on multi-valued analytic functions, we can deduce informations on the alien derivatives of WKB solutions of an M2P1T equation from those of its canonical equation. To attain this goal, we first show the following

**Theorem 2.3.** *The action of the microdifferential operator  $\mathcal{X}$  (given by (2.8)) upon the Borel-transformed WKB solution  $\psi_{+,B}$  of the  $\infty$ -Mathieu equation is expressed as an integro-differential operator of the form*

$$(2.16) \quad \mathcal{X}\psi_{+,B} = \int_{-y_+}^y K(x, a, \rho, y - y', \partial/\partial x) \psi_{+,B}(x, a, \rho, y') dy',$$

where

$$(2.17) \quad y_+(x, a, \rho) = \int_a^x \sqrt{\frac{aA_0(a, \rho) + xB_0(a, \rho)}{x^2 - a^2}} dx$$

and  $K(x, a, \rho, y, \partial/\partial x)$  is a differential operator of infinite order (in the sense of [15]) which is defined on  $\{(x, a, \rho, y) \in \mathbb{C}^4; (x, a, \rho) \in U \times W, |y| < C|\rho|^{1/2}\}$  for some positive constant  $C$ . Similar expressions are also available for the action of  $\mathcal{A}$  and  $\mathcal{B}$  on the Borel transformed WKB solutions of a Mathieu equation.

### 3 Can we focus our attention on the simple poles of the Mathieu equation?

As we emphasized in Introduction, our original problem was to analyze the singularity structure of Borel transformed WKB solutions near

fixed singularities determined by a pair of simple poles contained in the potential. But the canonical equation of an M2P1T equation, i.e., the Mathieu equation contains a simple turning point besides two simple poles. Unfortunately no effective WKB-theoretic results are known for the Mathieu equation, but T. Koike has succeeded in computing the Voros coefficient for the Legendre equation. (Private communication. See also [14].) Hence, if we can somehow focus our attention on the simple poles of the Mathieu equation, we will be able to make use of the results of Koike. And, actually this expectation is realized in Section 4. The problem is what we mean by saying “focus our attention on the pole part”. The answer is given by Theorem 3.1 below. In what follows,  $Q_L(z, C, \gamma_+, \gamma_-)$  denotes

$$(3.1) \quad \frac{aC}{z^2 - a^2} + \eta^{-2} \left( \frac{\gamma_+}{(z - a)^2} + \frac{\gamma_-}{(z + a)^2} \right),$$

and  $Q_M(x, A, B, c_+, c_-)$  denotes

$$(3.2) \quad \frac{aA + xB}{x^2 - a^2} + \eta^{-2} \left( \frac{c_+}{(x - a)^2} + \frac{c_-}{(x + a)^2} \right).$$

**Theorem 3.1.** *Let  $r_1(> 1)$  and  $r_2$  be positive constants with  $r_2$  sufficiently small and denote by  $\Omega_{r_1, r_2}$  the following set:*

$$(3.3) \quad \{(x, a, A, B) \in \mathbb{C}^4; |x| < r_1|a|, a \neq 0, A \neq 0, |B| < r_2|A|\}.$$

*Then we can construct infinite series*

$$(3.4) \quad z(x, a, A, B, \eta) = \sum_{k \geq 0} z_{2k}(x, a, A, B) \eta^{-2k}$$

*and*

$$(3.5) \quad C(a, A, B, \eta) = \sum_{k \geq 0} C_{2k}(a, A, B) \eta^{-2k}$$

so that they satisfy the following conditions (3.6)  $\sim$  (3.10).

$$(3.6) \quad z_{2k} \text{ and } C_{2k} \text{ are holomorphic on } \Omega_{r_1, r_2},$$

$$(3.7) \quad \text{for each fixed constants } a, A \text{ and } B \text{ the function } z_0(x, a, A, B) \text{ of } x \text{ is injective on } \{x \in \mathbb{C}; |x| < r_1|a|\},$$

$$(3.8) \quad (z_0(\pm a, a, A, B))^2 = a^2,$$

$$(3.9) \quad \frac{\partial z_0}{\partial x}(x, a, A, B) \neq 0 \quad \text{on } \Omega_{r_1, r_2},$$

$$(3.10) \quad \begin{aligned} &Q_M(x, A, B, c_+, c_-) \\ &= \left(\frac{\partial z}{\partial x}\right)^2 Q_L(z(x, a, A, B, \eta), C, c_+, c_-) - \frac{1}{2}\eta^{-2}\{z; x\}. \end{aligned}$$

Further the constructed series  $z$  and  $C$  satisfy the following estimates:

$$(3.11) \quad \text{for any } \varepsilon > 0 \text{ we can find sufficiently small } r_2 \text{ for which}$$

$$(3.11.i) \quad |z_{2k}(x, a, A, B)| \leq (2k)!\varepsilon^k |aA|^{-k}$$

and

$$(3.11.ii) \quad |C_{2k}(a, A, B)| \leq (2k)!\varepsilon^k |aA|^{-k}$$

hold on  $\Omega_{r_1, r_2}$  for every  $k \geq 1$ .

In parallel with the reasoning in Section 2 the relation (3.10) together with the estimates (3.11.i) and (3.11.ii) entails that the Borel transformed Mathieu operator and the Borel transformed Legendre operator are intertwined on  $\Omega_{r_1, r_2}$  by microdifferential operators and that the microdifferential operators enjoy the integral representation

similar to (2.16). The point is that the simple turning point of the Mathieu equation, i.e.,  $-aA/B$ , is necessitated to be outside  $\Omega_{r_1, r_2}$  for sufficiently small  $r_2$ . We refer the reader to [9] for the proof of Theorem 3.1; the formal construction of the series  $z$  and  $C$  is rather straightforward, but their estimation is quite intricate.

As Koike has explicitly written down the Voros coefficient for the Legendre-type equation with a large parameter that has the form

$$(3.12) \quad \left( \frac{d^2}{dz^2} - \eta^2 \left( \frac{a\Lambda^2}{z^2 - a^2} + \eta^{-1} \frac{\sqrt{a}\Lambda}{z^2 - a^2} + \eta^{-2} \frac{az\nu + a^2(\mu^2 - 1)}{(z^2 - a^2)^2} \right) \right) \phi = 0,$$

we prepare Lemma 3.1 below so that we may make use of Koike's results in Section 4.

**Lemma 3.1.** *We can rewrite*

$$(3.13) \quad \left( \frac{d^2}{dz^2} - \eta^2 Q_L(z, C, c_+, c_-) \right) \psi = 0$$

*in the form (3.12) if we choose  $\mu, \nu$  and*

$$(3.14) \quad \Lambda(a, C, \eta) = \sum_{k \geq 0} \Lambda_k(a, C) \eta^{-k},$$

*by*

$$(3.15) \quad \mu^2 = 1 + 2(c_+ + c_-),$$

$$(3.16) \quad \nu = 2(c_+ - c_-),$$

$$(3.17) \quad \Lambda = \sqrt{C - (\sqrt{a}\eta)^{-2} \left( c_+ + c_- - \frac{1}{4} \right)} - \frac{(\sqrt{a}\eta)^{-1}}{2}.$$

The proof is straightforward.

## 4 Singularity structure of the Borel transformed WKB solutions of an M2P1T equation

As stated in Section 3, we can focus our attention on the pole part of the Mathieu equation so that the part may be analyzed with the help of the results for the Legendre equation. Hence by the same reasoning as in [8, Section 5] (cf. [4] and [16] for the basic properties of the alien derivative) we obtain the following

**Theorem 4.1.** *Let  $\tilde{\psi}_+(t, a, \rho, \eta)$  be a WKB solution of a generic (i.e.,  $a \neq 0$ ,  $\rho \neq 0$ ) M2P1T equation that is normalized at a simple pole  $\{t = a\}$ . Then for every positive integer  $l$  we can find positive constants  $\delta_1$  and  $\delta_2$  so that the following relation (4.1) holds, where  $\Delta_{y=-y_+(t,a,\rho)+l\varpi}$  designates the alien derivative at the fixed singularity  $-y_+(t, a, \rho) + l\varpi$  and the suffix  $B$  indicates the Borel transform in the parentheses:*

$$\begin{aligned}
 (4.1) \quad & (\Delta_{y=-y_+(t,a,\rho)+l\varpi} \tilde{\psi}_+)_B(t, a, \rho, y) \\
 &= \frac{(-1)^l}{l} \left\{ 1 + (-1)^l - \cosh \left( 2\pi i l \sqrt{\frac{\mu^2 + \sqrt{\mu^4 - \nu^2}}{2}} \right) \right. \\
 & \quad \left. - \cosh \left( 2\pi i l \sqrt{\frac{\mu^2 - \sqrt{\mu^4 - \nu^2}}{2}} \right) \right\} \\
 & \quad \times \left( \exp \left( -l \oint_{\gamma} \tilde{S}_{\text{odd}} dt \right) \tilde{\psi}_+ \right)_B(t, a, \rho, y),
 \end{aligned}$$

where  $\tilde{S}_{\text{odd}}$  denotes the odd part of the solution  $\tilde{S}$  of the Riccati equation associated with the M2P1T equation and  $\gamma$  is a closed curve that encircles two simple poles counterclockwise, and

$$(4.2) \quad \mu^2 = 1 + 2(g_+(a) + g_-(-a)),$$



$$(4.3) \quad \nu = 2(g_+(a) - g_-(-a)),$$

$$(4.4) \quad y_+(t, a, \rho) = \int_a^t \sqrt{\frac{f(t, a, \rho)}{t^2 - a^2}} dt,$$

$$(4.5) \quad \varpi(a, \rho) = \oint_{\gamma} \sqrt{\frac{f(t, a, \rho)}{t^2 - a^2}} dt.$$

*Remark 4.1.* The highest degree part in  $\eta$  of  $\oint_{\gamma} \tilde{S}_{\text{odd}} dt$  is  $\eta\varpi(a, \rho)$ .

## References

- [1] T. Aoki: Symbols and formal symbols of pseudodifferential operators, *Advanced studies in Pure Math.* **4**, North-Holland, 1984, pp.181–208.
- [2] T. Aoki, T. Kawai and Y. Takei: The Bender-Wu analysis and the Voros theory, *Special Functions*, Springer, 1991, pp.1–29.
- [3] ———: The Bender-Wu analysis and the Voros theory. II, *Advanced Studies in Pure Math.*, **54**, Math. Soc. Japan, 2009, pp.19–94.
- [4] E. Delabaere and F. Pham: Resurgent methods in semi-classical asymptotics, *Ann. Inst. H. Poincaré*, **71** (1999), 1–94.
- [5] J. Ecalle: Les fonctions résurgentes, I, II, III, *Publ. Math. d’Orsay*, Univ. Paris-Sud, 1981 (Tome I, II), 1985 (Tome III).
- [6] L. Ehrenpreis: The Borel transform, *Algebraic Analysis of Differential Equations*, Springer, 2008, pp.119–131.

- [7] A. Erdély: Higher Transcendental Functions, III, McGraw-Hill, 1955; reprinted in 1981 by Robert E. Krieger Publishing Company, Malabar, Florida.
- [8] S. Kamimoto, T. Kawai, T. Koike and Y. Takei: On the WKB theoretic structure of the Schrödinger operators with a merging pair of a simple pole and a simple turning point, *Kyoto J. Math.*, **50** (2010), 101–164.
- [9] S. Kamimoto, T. Kawai and Y. Takei: Exact WKB analysis of a Schrödinger equation with a merging triplet of two simple poles and a one simple turning point — its relevance to the Mathieu equation and the Legendre equation, preprint (**RIMS-1735**), 2011.
- [10] T. Kawai, T. Koike and Y. Takei: On the exact WKB analysis of higher order simple-pole type operators, *Adv. in Math.*, **228** (2011), 63–96.
- [11] T. Kawai and Y. Takei: Algebraic Analysis of Singular Perturbation Theory. Amer. Math. Soc., 2005.
- [12] T. Koike: On a regular singular point in the exact WKB analysis, *Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear*, Kyoto Univ. Press, 2000, pp.39–54.
- [13] ———: On the exact WKB analysis of second order linear ordinary differential equations with simple poles, *Publ. RIMS, Kyoto Univ.*, **36** (2000), 297–319.
- [14] T. Koike and Y. Takei: On the Voros coefficient for the Whittaker equation with a large parameter — Some progress around Sato’s conjecture in exact WKB analysis, *Publ. RIMS, Kyoto Univ.*, **47** (2011), 375–395.

- [15] M. Sato, T. Kawai and M. Kashiwara: Microfunctions and pseudo-differential equations, Lect. Notes in Math., **287**, Springer, 1973, pp.265–529.
- [16] D. Sauzin, Resurgent functions and splitting problems, RIMS Kôkyûroku, **1493**, RIMS, 2006, pp.48–117.
- [17] A. Voros: The return of the quartic oscillator — The complex WKB method. Ann. Inst. Henri Poincaré, **39** (1983), 211 – 338.