論文の内容の要旨

論文題目: Spherical functions associated to the principal series representations of $SL(3,\mathbf{R})$ and higher rank Epstein zeta functions
($SL(3,\mathbf{R})$ の主系列表現に付随する球関数, 及び高階 Epstein ゼータ関数について)

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次の 4 本の論文を合わせたものを以て博士論文とする:
I. Shintani functions on $SL(3,\mathbf{R})$ ($SL(3,\mathbf{R})$ の新谷関数)
II. The matrix coefficients with minimal K-types of the spherical and non-spherical principal series representations of $SL(3,\mathbf{R})$ ($SL(3,\mathbf{R})$ の主系列表現の行列係数)
III. A second limit formula for higher rank twisted Epstein zeta functions and some applications (高階 Epstein ゼータ関数の第 2 極限公式とその応用)
IV. Fourth moment of the Epstein zeta functions (Epstein ゼータ関数の 4 乗平均).
大まかに分けると, I と II は $SL(3,\mathbf{R})$ の主系列表現に付随する球関数についての考察であり, III と IV は一般のランクの正定値行列に付随する Epstein ゼータ関数についての考察である. それぞれの論文の概要について順に説明しよう.

まず, 論文 I では $SL(3,\mathbf{R})$ の主系列表現に付随する新谷関数について研究した. 新谷関数は 1970 年代に新谷卓郎氏により p 進体上の GL_n の"Whittaker 関数"として導入された. 彼は群作用に関するある不変性を持つような $GL(n,k)$(k は $\mathbf{Q}_p$ の有限次拡大体) 上の関数を定義し, その存在と一意性を示した. この結果は後に村瀬篤・菅野孝史の両氏によって非アルキメデスな体上のより一般的なケースに拡張され, 新谷関数による L-関数の積分公式や重複度 1 定理が得られた. 一方で, アルキメデス的な体からなる群上の新谷関数の研究は比較的最近になされ, 例えば $GL(2,\mathbf{R})$, $GL(2,\mathbf{C})$ (平野幹), $SU(1,1)$, $U(n,1)$ (都築正男), $Sp(2,\mathbf{R})$(森山知則) などの群上の新谷関数がこれまでのところ研究されている. 私は $G=SL(3,\mathbf{R})$ 上の新谷関数のうち, 特に主系列表現の像として特徴付けられるものについて研究した. この場合の新谷関

数は,

$$K = SO(3),$$

$$H = \left\{ \left(\begin{array}{cc} H_1 & 0 \\ 0 & h_2 \end{array} \right) \in G \,\middle|\, (H_1, h_2) \in GL(2,\mathbf{R}) \times GL(1,\mathbf{R}) \right\},$$

(τ, V_τ) を K の有限次元表現, (η, V_η) を H のユニタリ表現とするとき, 滑らかなベクトル値関数 $F : G \to V_\eta \otimes V_\tau$ であって, $F(hgk) = (\eta(h) \otimes \tau(k^{-1}))F(g)$ $((h,g,k) \in H \times G \times K)$ を満たすものである. 論文 I では η は H のユニタリ指標に限定し, K の表現としては主系列表現の minimal K-type を取った. 新谷関数を調べる手法は, Whittaker 関数の場合と同様であり, Casimir 方程式と Dirac-Schmid 方程式という 2 つの方程式 (この場合は常微分方程式になる) を構成し, それらを解くことによる. 論文 I では主系列表現が spherical なときと non-spherical なときの両方の場合において, 微分方程式を調べることで新谷関数の明示公式を与え, 関数空間の次元が 1 以下になることを示した.

次に論文 II の概要を説明しよう. この論文では, $SL(3,\mathbf{R})$ の主系列表現 (spherical な場合と non-spherical な場合の両方) の行列係数について研究した. この行列係数は $SL(3,\mathbf{R})$ 上の関数であるが, 先と同じ記号で $G = SL(3,\mathbf{R})$, $K = SO(3)$ とするとき, K の有限次元表現 (τ_L, V_L), (τ_R, V_R) があり, 滑らかな関数 $\phi : G \to V_L \otimes V_R$ であって $\phi(k_L g k_R^{-1}) = (\tau_L(k_L) \otimes \tau_R(k_R))\phi(g)$ $(k_L, k_R \in K,\ g \in G)$ を満たすようなものと同一視できる. 論文 II では, K の表現 τ_R, τ_L が主系列表現 π の minimal K-type になる場合を扱った. すなわち, π が spherical な場合は τ_R, τ_L として K の自明な表現 $\mathbf{1}$ を取り, π が non-spherical な場合は τ_R, τ_L として 3 次元の tautorogical 表現 τ_2 を取った. 球関数を調べるための道具は論文 I と同じであり, Casimir 方程式と Dirac-Schmid 方程式 (gradient 方程式) を構成してそれらを解くことによるが, 今回は 2 変数の偏微分方程式となる. 論文の主結果は spherical 表現, non-spherical 表現の場合でそれぞれ (a) 6 つの特性根に対応する級数解の表示を得, (b) 行列係数をそれら 6 つの級数解の線型結合で表したこと, である. (b) の線型結合に現れる係数は c-関数とよばれる. 半単純 Lie 群の class one 主系列表現の行列係数における c-関数の導出は古典的な結果として知られているが, 論文 II で行ったような non-spherical な場合の c-関数の計算はこれまでに例が無く, 新しい結果であると言える.

次に, 論文 III について説明しよう. この論文では, $n \times n$ 正定値対称行列 Q と $\mathbf{u} \in \mathbf{R}^n$ に対し, $\mathrm{Re}(s) > \frac{n}{2}$ において

$$\zeta_n(s, \mathbf{u}, Q) = \sum_{\mathbf{u} \in \mathbf{Z}^n \setminus \{0\}} e^{2\pi i {}^t\mathbf{m} \cdot \mathbf{u}} ({}^t\mathbf{m} Q \mathbf{m})^{-s}$$

で定義される Epstein ゼータ関数 $\zeta_n(s, \mathbf{u}, Q)$ を扱った. Riemann ゼータ関数と同様, $\zeta_n(s, \mathbf{u}, Q)$ は全複素平面に有理型に解析接続され, $\mathbf{u} \in \mathbf{Z}^n$ なら $s = \frac{n}{2}$ に 1 位の極を持ち, $\mathbf{u} \notin \mathbf{Z}^n$ ならば全平面で正則になる. 前者の場合に $s = \frac{n}{2}$

での Laurent 展開の定数項を与えるものを Kronecker の (第 1) 極限公式と言い, 後者の場合に $s=\frac{n}{2}$ での値を与えるものを Kronecker の第 2 極限公式と言う. 論文 III では主に後者の場合, 即ち $\mathbf{u}\in\mathbf{R}^n\backslash\mathbf{Z}^n$ の場合を考えた. $n=2$ のときは Kronecker による古典的な結果として知られている. $n=3$ の場合は 1980 年代に Efrat により研究されており, 彼はその結果を利用しある関数空間に作用する Dirac 作用素とラプラシアンの行列式の解釈を与えた. 論文 III は Efrat の結果の一般の $n\geq 2$ への拡張にあたる. 主結果として, $\zeta_n(\frac{n}{2},\mathbf{u},Q)$ の値を, 階数の 1 つ低い $\zeta_{n-1}(\frac{n}{2},\mathbf{u}',Q')$ ($\mathbf{u}'\in\mathbf{R}^{n-1}$, Q' は $(n-1)\times(n-1)$ 正定値対称行列) と, Dedekind の η-関数の一般化にあたる保型性を持つ関数で表した. また, 計算の途中経過に現れる式を利用し, $\zeta_n(s,\mathbf{u},Q)$ の K-Bessel 展開 (Chowla-Selberg の公式の類似) を得た. この公式は, Epstein ゼータ関数の実軸上の零点の存在を調べる場合などに有益であると期待される. 更に Efrat の結果の一般化として, 第 2 極限公式を利用してある保型関数の空間に作用するラプラシアンの行列式の解釈を与えた.

最後に, 論文 IV の概略を説明しよう. 論文 IV では, $n\geq 4$ とし, $n\times n$ 正定値行列 Q に付随する Epstein ゼータ関数 $\zeta(s;Q)$ (論文 III の $\zeta_n(s,\mathbf{u},Q)$ で $\mathbf{u}=\mathbf{0}$ としたもの) の $\mathrm{Re}(s)=\frac{n-1}{2}$ における 4 乗モーメント, 即ち, 積分

$$I(T;Q):=\int_0^T\left|\zeta\left(\frac{n-1}{2}+it;Q\right)\right|^4dt$$

の T に関するオーダーの上限を調べた. Riemann ゼータ関数に関しては,

$$I_k(T):=\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}dt$$

と置くとき, 古典的な結果として $I_1(T)\sim T\mathrm{log}T$ (Hardy-Littlewood, 1918 年), $I_2(T)\sim\frac{1}{2\pi^2}T\mathrm{log}T$ (Ingham, 1926 年) となることが知られている. 一般には $I_k(T)\sim c_kT(\mathrm{log}T)^{k^2}$ ($\exists\, c_k$:定数) となると予想されているが, $k=0,1,2$ の場合を除きまだ証明されていない. このようなモーメントを計算する際の有力な道具として, 近似関数等式がある. 上の $I_1(T)$, $I_2(T)$ はそれぞれ $\zeta(s)$, $\zeta(s)^2$ の近似関数等式を利用して計算されたものである. $k\geq 3$ の場合に同様の結果が出せない最大の理由は, $k\geq 3$ のとき $\zeta(s)^k$ の近似関数等式が積分の評価において十分には役立たないという点にある. Dirichlet L-関数をはじめ多くの L-関数でモーメントの研究がなされているが, 同じ理由により高次のモーメントに関しては予想に見合う結果は得られていない. さて, Epstein ゼータ関数の話に戻ると, $\zeta(s;Q)$ の近似関数等式は存在するが, 関数等式の Γ-因子の関係からこれは $\zeta(s)^2$ の近似関数等式に似た形になる. 従って $\zeta(s;Q)$ の 2 乗モーメントは $I_2(T)$ の計算と同様にして求めることができ, 既に結果が知られている. ところが 4 乗モーメントだと $\zeta(s)^4$ の近似関数等式に似たものを用いることになるが, 上述したとおりこれは上手く機能しない. 従って別の方法を考える必要がある. 論文 IV の基本的なアイデアは次の通りである. ま

ず Epstein ゼータ関数の Dirichlet 係数から作られるテータ関数を考えると, これは重さ $\frac{n}{2}$ のモジュラー形式になり, Eisenstein 級数と cusp form の和に分解する. 従って Epstein ゼータ関数は Eisenstein 級数と cusp form のそれぞれに付随する L-関数の和に分解する. cusp form の Fourier 係数は比較的小さくなることから, cusp form に付随する L-関数の積分は $O(T)$ と評価できる. 一方で Eisenstein 級数に付随する L-関数は, Hecke, Malyshev, Siegel らの結果を利用すると, Dirichlet L-関数を含む有限ないし無限和で書くことができ, Dirichlet L-関数の 4 乗モーメントの結果を利用して積分が $O(T(\log T)^4)$ と評価できることが分かる. 従って Epstein ゼータ関数の $\mathrm{Re}(s) = \frac{n-1}{2}$ 上の 4 乗モーメントは $O(T(\log T)^4)$ と評価できる. Riemann ゼータ関数をはじめ, これまで多くの L-関数でモーメントが研究されてきたが, 上からの評価が $T^{1+\epsilon}$ ($\forall\epsilon > 0$) 程度になるものは（私が知る限り）全て近似関数等式が有効に働く場合である. 論文 IV はそうでないケースにおいて相応の上限を得たという点において意義があると考えられる.

Spherical functions associated to the principal series representations of $SL(3, \mathbf{R})$ and higher rank Epstein zeta functions
($SL(3, \mathbf{R})$の主系列表現に付随する球関数, 及び高階Epsteinゼータ関数について)

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Spherical functions associated to the principal series representations of $SL(3,\mathbf{R})$ and higher rank Epstein zeta functions

Keiju Sono

序. 論文の概要

当博士論文は, 次の2つの問題を大きな主題とする:
第1部: $SL(3,\mathbf{R})$ の spherical 及び non-spherical 主系列表現に付随する球関数の研究 (行列係数と新谷関数)
第2部: Epstein ゼータ関数の解析的研究 (Kronecker の第2極限公式の一般化と, 4乗平均の評価).

$SL(n,\mathbf{R})$ あるいは $GL(n,\mathbf{R})$ 上の保型形式の研究は $n=2$ のときは20世紀前半に Hecke によって発展させられた. $n\geq 3$ のときは散発的な試みがあったものの, $n=2$ のときの Hecke 理論を Iwasawa-Tate 風にアデール群上の言葉で書き直す Jacquet と Langlands の試みの成功によって大きな動機付けを得た. Shalika の重複度1定理の $GL(n)$ への一般化によって, Jacquet, Piatetski-Shapiro, Shalika らは $GL(n)$ の保型的 L-関数の研究に大きな進歩をもたらした. しかしながら, まだ基本的なレベルでも多くの問題が残されている. 例えば局所的な問題に限っても zeta 因子の計算は無限素点の場合ですら明示的には計算されていない. また, 大域的な L-関数についても, 解析接続や関数等式の他, 素数定理の類似の証明などの限られた話題に集中し, より広い解析的整数論の道具立てを用いることには, $n=2$ 以外はほとんど取り組まれていない. 当論文では, まず第1部において上に述べた無限素点における局所的な問題に対する考察を行い, 次に第2部において Epstein ゼータ関数についての既存の結果のより包括的な一般化及び拡張を行った. これはより難しい cusp 形式に対する研究のための先行的な試みのつもりである.

より細かく分けると, 論文は次の4つの論文を合わせたものである:
I. Shintani functions on $SL(3,\mathbf{R})$ ($SL(3,\mathbf{R})$ の新谷関数)
II. The matrix coefficients with minimal K-types of the spherical and non-spherical principal series representations of $SL(3,\mathbf{R})$ ($SL(3,\mathbf{R})$

の主系列表現の行列係数)
III. A second limit formula for higher rank twisted Epstein zeta functions and some applications (高階 Epstein ゼータ関数の第 2 極限公式とその応用)
IV. Fourth moment of the Epstein zeta functions (Epstein ゼータ関数の 4 乗平均).
I と II が第 1 部, すなわち $SL(3,\mathbf{R})$ の主系列表現に付随する球関数についての考察である. III と IV は第 2 部であり, 一般のランクの正定値行列に付随する Epstein ゼータ関数についての考察である. それぞれの論文の概要について順に説明しよう.

まず論文 I に関して説明する. 新谷関数は 1970 年代に新谷卓郎氏によって $GL(n,k)$(k は $\mathbf{Q}_p$ の有限次拡大体) 上の "Whittaker 関数" として導入された. 彼は定義した関数の明示公式を得, 更に関数の一意性を示した. 後に, non-archimedean の場合のより一般的な研究が村瀬篤・菅野孝史の両氏によってなされ, 新谷関数を用いた L-関数の積分公式や有限素点における重複度 1 定理などが得られた.

一方で無限素点の群上の新谷関数は比較的最近になって研究された. 例を挙げると, $GL(2,\mathbf{R})$, $GL(2,\mathbf{C})$(平野幹), $SU(1,1)$, $U(n,1)$(都築正男), $Sp(2,\mathbf{R})$(森山知則) などの群上の新谷関数がこれまでに研究されている. 研究方法は, 新谷関数の満たす微分方程式系を構成し, それらを解くことによる. 多くの場合, 解は Gauss の超幾何関数により記述される. また, 解空間の次元を調べて新谷関数の空間の次元を決定することも重要である.

論文 I では $G = SL(3,\mathbf{R})$ とし, G の主系列表現に付随する新谷関数を研究した. G の部分群

$$H = \left\{ \begin{pmatrix} H_1 & 0 \\ 0 & h_2 \end{pmatrix} \in G \,\middle|\, (H_1, h_2) \in GL(2,\mathbf{R}) \times GL(1,\mathbf{R}) \right\},$$

及び極大コンパクト部分群 $K = SO(3)$ を取る. π を G の任意の既約ユニタリ表現, (η, V_η) を H の既約ユニタリ表現とし, $C_\eta^\infty(H\backslash G)$ を滑らかな関数 $F : G \to V_\eta$ であって $F(hg) = \eta(h)F(g)$ $((h,g) \in H \times G)$ を満たすものからなる空間とする. $I_{\eta,\pi} := \mathrm{Hom}_{(\mathfrak{g}_\mathbf{C},K)}(\pi, C_\eta^\infty(H\backslash G))$ とし, その minimal K-type (τ, V_τ) への制限

$$I_{\eta,\tau} \to \mathrm{Hom}_K(\tau, C_\eta^\infty(H\backslash G)) \cong C_{\eta,\tau^*}^\infty(H\backslash G/K)$$

を考える. ここで (τ^*, V_{τ^*}) は (τ, V_τ) の随伴表現, $C_{\eta,\tau^*}^\infty(H\backslash G/K)$ は滑らかな関数 $F : G \to V_\eta \otimes V_{\tau^*}$ であって, $F(hgk) = (\eta(h) \otimes \tau^*(k^{-1}))F(g)$ $((n,g,k) \in H \times G \times K)$ を満たすようなものからなる空間である. 上の写像の像に属する関数を新谷関数という. 論文 I では π は G の既約ユニタリ主系列表現, η は H のユニタリ指標の場合を考えた.

論文 I の第 4 節では, spherical (class one) 主系列表現に付随する

新谷関数を調べる. この場合表現の minimal K-type は 1 次元の自明な表現になる. このとき, 新谷関数は Casimir 方程式という $Z(\mathfrak{g})$ の元の作用から得られる微分方程式を調べることで把握できる. 定理 4.8 において, 新谷関数の存在する必要条件及び存在した場合の明示公式を与え, 更に新谷関数の空間の次元が 1 以下になること (重複度 1 定理) を示した.

一方で第 5 節では, non-spherical な主系列表現に付随する新谷関数を調べた. このとき minimal K-type は K の 3 次元の表現になる. この場合は, Casimir 方程式と gradient 方程式の 2 種類の微分方程式を構成する. これらの方程式を調べることで, 定理 5.7 において non-spherical な主系列表現の場合で上記と同様の結果を得た.

応用として, 村瀬・菅野の両氏が行ったような局所ゼータ積分の計算における利用などが期待される. また, 論文 I の結果は対称空間上の調和解析の観点からそれ自体興味深いと思われる.

論文 II では, $SL(3,\mathbf{R})$ の主系列表現の行列係数を扱う. 上と同様, $G = SL(3,\mathbf{R})$, $K = SO(3)$ とするとき, G の主系列表現 π の行列係数は K のある有限次元表現 (τ_L, V_L), (τ_R, V_R) があって, 球関数の空間

$$C^{\infty}_{\tau_L,\tau_R}(K\backslash G/K)$$
$$:= \{\phi : G \to V_L \otimes V_R | \phi(k_L g k_R^{-1}) = (\tau_L(k_L) \otimes \tau_R(k_R))\phi(g) (k_L, k_R \in K, g \in G)\}$$

を満たすようなものの元と同一視できる. 論文 II では, π が spherical な主系列表現の場合は $\tau_L = \tau_R = \mathbf{1}$ (K の 1 次元の自明な表現) とし, π が non-spherical な主系列表現の場合は $\tau_L = \tau_R = \tau_2$ (K の 3 次元 tautological 表現) とした. すなわち, 行列係数のベクトルとして π の minimal K-type の元を取った.

上の球関数を調べる手段は論文 I の新谷関数の場合と同様であり, spherical な主系列表現の場合は Casimir 方程式, non-spherical な主系列表現の場合は Casimir 方程式と gradient 方程式を構成し, それらを組み合わせて解くことによる. 新谷関数とは違い, この場合は 2 変数の偏微分方程式となる. いずれの場合も, 6 つの特性根に対応する 6 つの級数解が得られる. 級数解の係数は非常に複雑であり, かつ後の計算には利用しないので第 7 節の Appendix にまとめた.

論文 II の主定理は定理 5.6 と定理 6.6 であり, それぞれ spherical 主系列表現と non-spherical 主系列表現に対し, その行列係数を得られた級数解の線型結合で表すというものである. 半単純 Lie 群の spherical な主系列表現の行列係数をベキ級数解の線型結合で表すということは古典的な結果として知られており, 線型結合に現れる係数は c-関数と呼ばれ, 一般に Γ-関数の積で表わされることが知られている. 一方で non-spherical な主系列表現の場合はこのような c-関数の導出はほとんどなされていなかった. その理由として, G. Shiffmann が spherical な表現の場合に行ったような議論が non-spherical な表現の場合は上手く帰納的に機能しないことなどが挙げられる. 論文 II では, 定理 6.6 において, この non-spherical 主系列表現の行列係数

に対し, モノドロミー・データを調べ, 微分方程式系の解が正則になるという条件を利用して, c-関数を導出することに成功した.

以上が第1部の $SL(3,\mathbf{R})$ の主系列表現に付随する球関数の研究についてである. 続いて論文 III について説明しよう. 20 世紀初頭, P. Epstein は $n\times n$ の正定値対称行列 Q, $u,v\in\mathbf{R}^n$ に対し, ゼータ関数 $\zeta_n(s,u,v,Q)$ を $\mathrm{Re}(s)>\frac{n}{2}$ において

$$\zeta_n(s,u,v,Q)=\sum_{m\in\mathbf{Z}^n,m+x\neq 0}e^{2\pi i^t m\cdot u}Q[m+v]^{-s}$$

で定義した. ここで $Q[x]:={}^t xQx$ は Q に付随する 2 次形式である. Riemann ゼータ関数と同様, $\zeta_n(s,u,v,Q)$ は全 s-平面に有理型に解析接続され, 関数等式

$$\pi^{-s}\Gamma(s)\zeta_n(s,u,v,Q)=e^{-2\pi i^t u\cdot v}|Q|^{-\frac{1}{2}}\pi^{-(\frac{n}{2}-s)}\Gamma\left(\frac{n}{2}-s\right)\zeta_n\left(\frac{n}{2}-s,v,-u,Q^{-1}\right)$$

を満たす. ここで, $|Q|:=\det Q$. $\zeta_n(s,u,v,Q)$ は $u\in\mathbf{Z}^n$ なら $s=\frac{n}{2}$ に 1 位の極を持ち, そうでないなら全平面で正則となる. 更に, Epstein は $\zeta_n(s;Q):=\zeta_n(s.0,0,Q)$ に対するいわゆる Kronecker の極限公式を得た. これは, $\zeta_n(s;Q)$ の $s=\frac{n}{2}$ の周りでの Laurent 展開の定数項を与える公式である.

Epstein ゼータ関数 (あるいは Eisenstein 級数) に関するもう 1 つの重要な公式として, Chowla-Selberg の公式がある. 今, $a,c>0$, $b\in\mathbf{R}$, $d:=b^2-ac<0$ とする. このとき,

$$Z(s)=\frac{1}{2}\sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}}(am^2+bmn+cn^2)^{-s}\quad(\mathrm{Re}(s)>1)$$

で定義される Epstein ゼータ関数 $Z(s)$ は, $\forall s\in\mathbf{C}$ に対し次の等式を満たす:

$$Z(s)=a^{-s}\zeta(2s)+a^{-s}\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\left(\frac{\sqrt{|d|}}{2a}\right)^{1-2s}\zeta(2s-1)+R_Q(s),$$

$R_Q(s)$ は K-Bessel 関数によって記述される無限級数である. 上の Chowla-Selberg の公式には多くの応用があるが, 特にパラメータ a, b, c が変動するときの $Z(s)$ の零点を調べる際に有用である.

論文 III の第 2 節では, 一般の $n\geq 2$ に対し,

$$\zeta_n(s,u,0,Q)=\sum_{m\in\mathbf{Z}^n\setminus\{0\}}e^{2\pi i^t m\cdot u}Q[m]^{-s}\quad\left(\mathrm{Re}(s)>\frac{n}{2}\right)$$

で定義される Epstein ゼータ関数を調べた. 上述した通り, $u\notin\mathbf{Z}^n$ ならば $\zeta_n(s,u,0,Q)$ は正則であり, $\zeta_n(\frac{n}{2},u,0,Q)$ を表す公式は第 2 極

限公式と呼ばれる．$n=2$ のときは Kronecker による古典的な結果として知られており，これを特に Kronecker の第 2 極限公式という．1980 年代，Efrat は $n=3$ の場合に Q に岩澤分解により座標を入れ，$\zeta_3(\frac{3}{2},u,0,Q)$ $(u\notin \mathbf{Z}^3)$ の公式を得た．更に彼はその公式を利用し，$\mathbf{R}^3$ 上のある asymmetric な保型関数の空間に作用するラプラシアンと Dirac 作用素の行列式の解釈を与えた．

論文 III の大部分は上の Efrat の結果の一般の n への拡張にあたる．第 2 節では，まず $n\times n$ 正定値対称行列 Q に岩澤分解を利用して座標を入れ，それによって Epstein ゼータ関数を書き換え，$u\notin \mathbf{Z}^n$ の場合に $\zeta_n(\frac{n}{2},u,0,Q)$ を表す公式を得た (定理 2.1)．その系として，無限積で定義された Dedekind の η-関数の一般化に相当するある種の保型性を持った関数が得られる (系 2.2)．　更に，計算過程の式を利用し，$\zeta_n(s,u,0,Q)$ $(u\in \mathbf{Z}^n$ の場合も含む) の K-Bessel 展開の公式，すなわち Chowla-Selberg の公式の類似を得た (定理 2.3)．この公式は，$n=2$ の場合に Bateman と Grosswald が行ったように Epstein ゼータ関数の実軸上の零点の有無を調べる等の利用が期待される．

第 3 節では，Efrat の結果の一般化として，定理 2.1 で得た $\zeta_n(s,u,0,Q)$ の第 2 極限公式を利用し，$\mathbf{R}^n$ 上の asymmetric な保型関数の空間に作用するラプラシアンの行列式の解釈を与えた．すなわち，通常意味では発散する固有値の積を Epstein ゼータを介して正規積として計算した．

最後に，論文 IV について説明しよう．論文 IV では，$n\geq 4$ とし，$n\times n$ 正定値行列 Q であって 2 次形式が $\mathbf{Z}^n$ 上整数に値を取るようなものに付随する Epstein ゼータ関数 $\zeta(s;Q)$ (上の $\zeta_n(s,u,v,Q)$ で $u=v=0$ としたもの) の，$\mathrm{Re}(s)=\frac{n-1}{2}$ 上のモーメントについて調べ，積分

$$I(T;Q):=\int_0^T\left|\zeta\left(\frac{n-1}{2}+it;Q\right)\right|^4dt$$

の上からの評価を与えた．

Riemann ゼータ関数や他の L-関数のモーメントは Hardy-Littlewood の時代から 100 年以上に渡って研究されている．Riemann ゼータ関数についての有名な予想として，$\zeta(\frac{1}{2}+it)$ の t に関するオーダー評価に纏わる Lindelöf 予想があるが，これは

$$I_k(T)=\int_1^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}dt$$

と置くとき，任意の正整数 k に対し $I_k(T)=O(T^{1+\epsilon})$ $(\forall\epsilon>0)$ となることと同値である．この $I_k(T)$ を $\zeta(s)$ の $\mathrm{Re}(s)=\frac{1}{2}$ における $2k$ 乗モーメントと呼ぶ (本来ならこれを T で割ったものをそう呼ぶべきであるが，便宜上ここでは $I_k(T)$ をモーメントと呼ぶことにする)．1918 年，Hardy と Littlewood は 2 乗モーメントに関して $I_1(T)\sim T\log T$ となることを示した．更に 1926 年，Ingham は 4 乗モーメントを研

究し, $I_2(T) \sim \frac{1}{2\pi^2}T(\log T)^4$ となることを示した. 一般には, $\forall k \geq 0$ に対して $I_k(T) \sim c_k T(\log T)^{k^2}$ ($\exists\, c_k$:定数) となると予想されているが, Ingham の結果から 1 世紀近く経つにも関わらずこれは $k = 0, 1, 2$ の場合を除き依然として未解決である (条件付きの結果として, Riemann 予想を仮定すると $T(\log T)^{k^2} \ll I_k(T) \ll T(\log T)^{k^2+\epsilon}$ となることが Soundararajan により最近証明された).

L-関数のモーメントを計算する上での有効な道具として, 近似関数等式が挙げられる. 上の $I_1(T)$, $I_2(T)$ の漸近公式は, 各々$\zeta(s)$, $\zeta(s)^2$ の近似関数等式を用いて計算されたものである. $k > 2$ のときに $I_k(T)$ の計算が上手くいかない理由も近似関数等式にある. すなわち, k が 3 以上の整数の場合の $\zeta(s)^k$ の近似関数等式は既に存在するが, これを用いて計算すると main term の評価が小さくならず (より正確には, オーダーの大きい部分の寄与は打ち消し合うと期待されるが, それが示せない), 予想通りの $I_k(T)$ の漸近公式には至らない. Dirichlet L-関数などについても同様にモーメントの研究がなされているが, その辺りの事情は同じであり, やはり高次のモーメントに関して予想に見合う結果は得られていない.

さて, Epstein ゼータ関数の話に戻ると, $\zeta(s;Q)$ の近似関数等式は Chandrasekharan と Narashimhan による一般的な定理の系として得られるが, これは関数等式の Γ-因子の関係から $\zeta(s)^2$ の近似関数等式に相当する. また, モーメントの観点からは, Riemann ゼータ関数における critical line $\mathrm{Re}(s) = \frac{1}{2}$ は Epstein ゼータ関数における $\mathrm{Re}(s) = \frac{n-1}{2}$ に相当する. 従って Ingham による $I_2(T)$ の計算と同様にして, $\zeta(s;Q)$ の $\mathrm{Re}(s) = \frac{n-1}{2}$ における 2 乗モーメントは計算することができ, Müller, Fomenko らによって一般的な結果が得られている.

一方で Epstein ゼータ関数の 4 乗モーメント $I(T;Q)$ は Riemann ゼータ関数の 8 乗モーメントに相当し, 上に述べた理由から近似関数等式の活用は期待出来ず, 計算には新しい手法が必要となる. 論文 IV の基本的なアイデアは次の通りである. $n \geq 4$, Q を $n \times n$ 正定値対称行列であって, $\mathbf{x} \in \mathbf{Z}^n \backslash \{\mathbf{0}\}$ に対し $Q[\mathbf{x}] \in \mathbf{N}$ となるものとする. $k \in \mathbf{Z}_{\geq 0}$ に対し, $r_Q(k)$ を $\mathbf{x} \in \mathbf{Z}^n$ であって $Q[\mathbf{x}] = k$ を満たすものの数とすると, Epstein ゼータ関数 $\zeta(s;Q)$ は $\mathrm{Re}(s) > \frac{n}{2}$ において

$$\zeta(s;Q) = \sum_{k=1}^{\infty} \frac{r_Q(k)}{k^s}$$

と表わされる. 対応するテータ関数

$$\theta(z;Q) = \sum_{k=0}^{\infty} r_Q(k) e^{2\pi i z}$$

を考えると, これは重さ $\frac{n}{2}$ のモジュラー形式になり, Eisenstein 級数

と cusp form の和に分解する. 従って, Epstein ゼータ関数 $\zeta(s;Q)$ は Eisenstein 級数に付随する L-関数と cusp form に付随する L-関数の和に分解するので, これら 2 つの L-関数のモーメントを評価すれば良い. まず, cusp form の Fourier 係数は $r_Q(k)$ に比べて小さくなることより, cusp form に付随する L-関数の $\mathrm{Re}(s)=\frac{n-1}{2}$ におけるモーメントは $O(T)$ と評価できることが古典的な解析的整数論の議論により分かる (補題 2.1). 一方で Eisenstein 級数に付随する L-関数の方は, $n\geq 4$, 偶数のときは Hecke, $n\geq 7$, 奇数のときは Malyshev, $n=5$ のときは Siegel の結果を用いると, Dirichlet L-関数からなる有限ないし無限和で書けることが分かる. $n\geq 7$, 奇数の場合は Fomenko によって得られた結果であり, 彼はこれを用いて Epstein ゼータ関数のオーダーの評価を得た. この表示において Montgomery らによる Dirichlet L-関数の 4 乗モーメントの結果上手く適用すると, 4 乗モーメントが上から $O(T(\log T)^4)$ と評価できることが示せる. 従って, $\zeta(s;Q)$ の 4 乗モーメント $I(T;Q)$ も上から $O(T(\log T)^4)$ と評価できる (定理 2.5, 2.6). 自然な予想として, Q のみに依存する定数 C があって, $I(T;Q)\sim CT(\log T)^4$ となると思われる. モーメントの下からの評価に関しては今後の課題である.

定理 2.5, 2.6 の 1 つの応用として, 論文 IV の第 3 節では約数問題を扱った. 正の整数 l(この論文では $l\geq 4$) に対し, $\mathrm{Re}(s)>\frac{n}{2}$ において

$$\zeta(s;Q)^l=\sum_{k=1}^{\infty}\frac{r_Q^{(l)}(k)}{k^s}$$

と表わしたとき, 和 $\sum_{k\leq x}r_Q^{(l)}(k)$ の $x\to\infty$ のときのオーダー (正確には, 下で述べる誤差項のオーダー) を評価する問題を約数問題という. この和に関して, 次の漸近式が成り立つことが知られている:

$$\sum_{k\leq x}r_Q^{(l)}(k)=M_l^{(n)}(x)+\Delta_l^{(n)}(x)\quad(x\to\infty).$$

ここで, $M_l^{(n)}(x)$ は main term であり, 次数 $l-1$ の計算可能な多項式 P_l を用いて $x^{\frac{n}{2}}P_l(\log x)$ と書ける. $\Delta_l^{(n)}(x)$ は誤差項であり, 一般には $o(x^{\frac{n}{2}})$ となることが知られている. この $\Delta_l^{(n)}(x)$ の評価に関しては, Q が一般の場合は Sankaranarayanan により, 特殊な Q の場合 (Eisenstein 級数に付随する L-関数が Riemann ゼータ関数で表せる場合) には Lü らにより研究されているが, 第 3 節では, 4 乗平均に関する結果と Fomenko のオーダー評価を組み合わせることにより, 一般の Q(2 次形式が整数値になることは仮定するが) の場合において $\Delta_l^{(n)}(x)$ の現在知られている中で最も良い評価を得た (定理 3.2).

論文 IV の最大のポイントは近似関数等式が上手く機能しない L-関数においてモーメントの (おそらく最善の) 評価を得たという点で

ある. 私が知る限り, これまでモーメントがオーダーの意味で正確に計算された例は全て近似関数等式を利用したものである. 論文IVはそうでない例において, 上からの評価のみであるが期待される評価を得たという点において, 相応の意義があると思われる.

最後に, 当博士論文は指導教官の織田孝幸教授, 宮崎直氏, 論文のレフェリーを担当して頂いた先生方をはじめ, 多くの方々の協力とアドバイスによって成り立ったものである. この場を借りて感謝の意を申し上げたい.

I. Shintani functions on $SL(3,\mathbf{R})$

ABSTRACT: We investigate the Shintani functions attached to the spherical and non-spherical principal series representations of $SL(3,\mathbf{R})$. We give the explicit formulas of the radial part of Shintani functions and evaluate the dimension of the space of Shintani functions.

1 Introduction

Shintani function is originally introduced by T.Shintani for p-adic linear group $GL(n,k)$, where k is a finite extension of the p-adic field $\mathbf{Q}_p$ [11]. He defined some "Whittaker functions" on $GL(n,k)$ and obtained the explicit formulas of them. Moreover, he proved the uniqueness of his function. Later, more detail study of Shintani functions for $GL(n)$ was done by A.Murase and T.Sugano [10] (see also [9]). They obtained new kinds of integral formulas for the L-functions in terms of the global Shintani functions, and proved the multiplicity one theorem of the local one at the finite primes.

On the other hand, the multiplicity and explicit formulas of the archimedean Shintani functions were more recently investigated by some mathematicians. For example, M.Hirano studied the Shintani functions on $GL(2,\mathbf{R})$ [3] and $GL(2,\mathbf{C})$ [4], M.Tsuzuki on $SU(1,1)$ [13] and $U(n,1)$ [14], T.Moriyama on $Sp(2,\mathbf{R})$ [7], [8]. They constructed the differential equations satisfied by the radial part of the Shintani functions and obtained the explicit formulas by solving them. Most of them are expressed by some linear combinations of the Gaussian hypergeometric functions. Moreover, the dimensions of the spaces of Shintani functions are obtained, which are sometimes bigger than 1.

In this paper, we investigate the Shintani functions on $G = SL(3,\mathbf{R})$, attached to the principal series representations of G. Now we explain the definition of the Shintani functions on G. We take

$$H = \left\{ \begin{pmatrix} H_1 & 0 \\ 0 & h_2 \end{pmatrix} \in G \,\middle|\, (H_1,h_2) \in GL(2,\mathbf{R}) \times GL(1,\mathbf{R}) \right\}$$

as a subgroup of G and take $K = SO(3)$ as a maximal compact subgroup of G. Let π be an arbitrary irreducible unitary representation of G and (η, V_η) be an irreducible unitary representation of H, and let $C_\eta^\infty(H\backslash G)$ be the space of smooth functions $F : G \to V_\eta$ satisfying $F(hg) = \eta(h)F(g)$ ($(h,g) \in H \times G$). We consider the intertwining space $I_{\eta,\pi} = \mathrm{Hom}_{(\mathfrak{g}_\mathbf{C},K)}(\pi, C_\eta^\infty(H\backslash G))$ and its restriction

$$I_{\eta,\pi} \to \mathrm{Hom}_K(\tau, C_\eta^\infty(H\backslash G)) \cong C_{\eta,\tau^*}(H\backslash G/K)$$

to the minimal K-type (τ, V_τ) of π, where (τ^*, V_{τ^*}) is the contragredient representation of (τ, V_τ) and $C_{\eta,\tau^*}^\infty(H\backslash G/K)$ is the space of smooth functions $F : G \to V_\eta \otimes V_{\tau^*}$ satisfying $F(hgk) = (\eta(h) \otimes \tau^*(k^{-1}))F(g)$ for $(h,g,k) \in H \times G \times K$.

The function which belongs to the image of above map is called the Shintani function. In this paper, we assume that π is the irreducible unitary principal series representation of G and η is the unitary character of H. The study of Shintani functions for the general unitary representation η of H is a further problem.

In section 4, we investigate the Shintani functions attached to the spherical (or class one) principal series representations. These representations have unique K-fixed vector and hence the minimal K-type is one dimensional. In this case, the explicit formulas of Shintani functions are obtained by solving two Casimir equations which are characterized by the action of the center of universal enveloping algebra. We also obtain the necessary condition of the existence of non-zero Shintani functions and prove that the dimension of the space of Shintani functions is equal or less than 1 (Theorem 4.8).

On the other hand, in section 5, we investigate the Shintani functions attached to the non-spherical principal series representations, whose minimal K-type is three dimensional representation of K. In this case, we construct two kinds of differential equations. One is the Casimir equation we used in section 4, and the other is the gradient equation. The key point is as follows. We have three different non-spherical principal series with the same infinitesimal characters $Z(\mathfrak{g}) \to \mathbf{C}$. We cannot distinguish them only by the elements of $Z(\mathfrak{g})$. This is the reason we need the gradient operator which has distinct eigenvalues for different non-spherical principal series. By combining these equations, we obtain the explicit formulas of the Shintani functions, the necessary condition of the existence of non-zero Shintani functions, and prove that the dimension of the space of Shintani functions is equal or less than 1 (Theorem 5.7).

As an application of the results of this paper, our explicit formulas for Shintani functions will be useful to compute the local zeta integral in the theory of Murase and Sugano ([9], [10]). (See also [15], in the case of $U(n,1)$.) Furthermore, the auther thinks these results are interesting themselves in view of the harmonic analysis on homogeneous spaces.

The auther would like to express his gratitude to Professor T.Oda for his constant encouragement and many advices. He also thanks to Professor M.Hirano, T.Ishii, T.Moriyama, M.Tsuzuki and T.Miyazaki for helpful discussion.

2 Preliminaries

2.1 Groups and algebras

Let G be the real reductive Lie group $SL(3,\mathbf{R})$ and $\mathfrak{g} = \mathfrak{sl}(3,\mathbf{R})$ be its Lie algebra. The Cartan involution $\theta : G \to G$ is defined by $\theta(g) = ({}^t g)^{-1}\,(g \in G)$, and its differential $d\theta : \mathfrak{g} \to \mathfrak{g}$ is given by $d\theta(X) = -{}^t X\,(X \in \mathfrak{g})$, where ^{t} means the transposition of matrices. Then the fixed subgroup of θ in G is equal to $K = SO(3)$, which is the maximal compact subgroup of G. Next, we define another involutive automorphism σ of G by $\sigma(g) = JgJ\,(g \in G)$, where $J = \mathrm{diag}(-1,-1,1)$. Its differential $d\sigma : \mathfrak{g} \to \mathfrak{g}$ is given by $d\sigma(X) = JXJ\,(X \in \mathfrak{g})$.

The fixed subgroup H of σ in G is isomorphic to $GL(2, \mathbf{R})$, i.e.

$$H = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \in G \right\} \cong GL(2, \mathbf{R}).$$

We define +1 and -1 eigenspaces of $d\theta$, $d\sigma$ in $\mathfrak{g}$ by

$$\mathfrak{k} = \{X \in \mathfrak{g} | \, d\theta(X) = X\}, \quad \mathfrak{p} = \{X \in \mathfrak{g} | \, d\theta(X) = -X\},$$
$$\mathfrak{h} = \{X \in \mathfrak{g} | \, d\sigma(X) = X\}, \quad \mathfrak{q} = \{X \in \mathfrak{g} | \, d\sigma(X) = -X\}.$$

Then $\mathfrak{k}$, $\mathfrak{h}$ are the Lie algebras of K, H, respectively. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$. Let $E_{ij} \in M(3, \mathbf{R})$ be the matrix whose (i,j)-component is 1 and the other components are 0 $(1 \leq i, j \leq 3)$. For $1 \leq i < j \leq 3$, we put $K_{ij} = E_{ij} - E_{ji}$, $X_{ij} = E_{ij} + E_{ji}$, $H_{ij} = E_{ii} - E_{jj}$. Then we have

$$\mathfrak{k} = \bigoplus_{i<j} \mathbf{R} K_{ij}, \quad \mathfrak{p} = \bigoplus_{i<j} \mathbf{R} X_{ij} \oplus \mathbf{R} H_{12} \oplus \mathbf{R} H_{23}.$$

Next, we take

$$Y_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

as a basis of $\mathfrak{h}$. We have $\mathfrak{p} \cap \mathfrak{q} = \mathbf{R} X_{13} \oplus \mathbf{R} X_{23}$, and we take $\mathfrak{a} = \mathbf{R} X_{13}$ as a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. We define a subgroup A of G by

$$A = \exp(\mathfrak{a}) = \left\{ a_t := \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \middle| \, t \in \mathbf{R} \right\}.$$

Then G has a decomposition $G = HAK$.
For a Lie algebra $\mathfrak{l}$, we denote its complexification by $\mathfrak{l}_{\mathbf{C}}$, that is, $\mathfrak{l}_{\mathbf{C}} = \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$.

2.2 The principal series representations

As a representation of G, we take the principal series representation defined as follows. Let P_0 be a minimal parabolic subgroup of G given by the upper triangular matrices in G, and $P_0 = M A_{P_0} N$ be the Langlands decomposition of P_0 with

$$A_{P_0} = \{\text{diag}(a_1, a_2, a_3) | a_i > 0, a_1 a_2 a_3 = 1\}$$

$$M = K \cap \{\text{diagonals in}\, G\} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z},$$

and

$$N = \left\{ \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \in G \middle| x_1, x_2, x_3 \in \mathbf{R} \right\}.$$

To define a principal series representation with respect to the minimal parabolic subgroup P_0 of G, we firstly fix a character σ of M and a linear form $\nu \in \mathfrak{a}_{P_0}^* \otimes_{\mathbf{R}} \mathbf{C} = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_{P_0}, \mathbf{C})$, where $\mathfrak{a}_{P_0}$ is the Lie algebra of A_{P_0}. We write

$$\nu(\mathrm{diag}(t_1, t_2, t_3)) = \nu_1 t_1 + \nu_2 t_2$$

for $\mathrm{diag}(t_1, t_2, t_3) \in \mathfrak{a}_{P_0}$. Then we can define a representation $\sigma \otimes a^{\nu}$ of MA_{P_0}, and extend this to P_0 by the identification $P_0/N \simeq MA_{P_0}$, taking the trivial representation $\mathbf{1}_N$ as the representation of N. Then the induced representation

$$\pi_{\sigma,\nu} := C^{\infty}\mathrm{Ind}_{P_0}^{G}(\sigma \otimes a^{\nu+\rho} \otimes 1_N)$$

is called the principal series representation of G. Here ρ is the half sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$ given by $a^{\rho} = a_1^2 a_2$, for $a = \mathrm{diag}(a_1, a_2, a_3) \in A_{P_0}$. Concretely, the representation space is given by

$$C_{(M,\sigma)}^{\infty}(K) := \{f \in C^{\infty}(K) | f(mk) = \sigma(m)f(k), m \in M, k \in K\}$$

and the action of G is defined by

$$(\pi_{\sigma,\nu}(x)f)(k) = a(kx)^{\nu+\rho} f(\kappa(kx))\, (x \in G, k \in K).$$

Here, for $g \in G$, $g = n(g)a(g)\kappa(g)$ $(n(g) \in N, a(g) \in A_{P_0}, \kappa(g) \in K)$ is the Iwasawa decomposition. Throughout this paper, we assume that the representation $\pi_{\sigma,\nu}$ is irreducible. Moreover, we assume that ν_1, ν_2 are the elements of $\sqrt{-1}\mathbf{R}$. Then this representation becomes unitary.
Next, we define characters σ_j $(j = 0, 1, 2, 3)$ of M as follows. The group M consisting of four elements is a finite abelian group of (2,2)-type, and its elements except for the unity are given by

$$m_1 = \mathrm{diag}(1, -1, -1), m_2 = \mathrm{diag}(-1, 1, -1), m_3 = \mathrm{diag}(-1, -1, 1).$$

The set $\hat{M}$ consists of 4 characters $\{\sigma_j | j = 0, 1, 2, 3\}$, where σ_0 is the trivial character of M, and $\sigma_1, \sigma_2, \sigma_3$ are defined by the following table:

	m_1	m_2	m_3
σ_1	1	-1	-1
σ_2	-1	1	-1
σ_3	-1	-1	1

The following proposition ([6], Proposition 1.1) gives the correspondence between the character σ of M and the minimal K-type of the principal series representation $\pi_{\sigma,\nu}$ of G:

Proposition 2.1. 1) *If σ is the trivial character of M, the representation $\pi_{\sigma,\nu}$ is spherical or class one. That is, it has a unique K-invariant vecter in $H_{\sigma,\nu}$.*
2)*If σ is not trivial, the minimal K-type of the restriction $\pi_{\sigma,\nu}|_K$ to K is a 3-dimensional representation of K, which is isomorphic to the unique standard one (τ_2, V_2). The multiplicity of this minimal K-type is one:*

$$\dim_{\mathbf{C}}\mathrm{Hom}_K(\tau_2, H_{\sigma,\nu}) = 1.$$

3 The space of Shintani functions

3.1 The definition of Shintani functions

As a representation of H, we take the unitary character $\eta = \eta_{s,k} : H \to \mathbf{C}^\times$ ($s \in \sqrt{-1}\mathbf{R}$, $k \in \{0,1\}$) defined by

$$\eta\left(\left(\begin{array}{cc|c} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ \hline 0 & 0 & h_1 \end{array}\right)\right) = \det(H_1)^k|\det(H_1)|^{s-k},\ H_1 = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \in GL(2,\mathbf{R}). \tag{3.1}$$

Let $\eta = \eta_{s,k}$ be the unitary character of H defined as above. We consider the induced representation $C^\infty\mathrm{Ind}_H^G(\eta)$ with the representation space

$$C_\eta^\infty(H\backslash G) = \{F \in C^\infty(G) | F(hg) = \eta(h)F(g), \forall h \in H, \forall g \in G\}.$$

G acts on this space by right translation. Let $\pi_{\sigma,\nu}$ be the principal series representation of G. We consider the intertwining space

$$I_{\eta,\pi_{\sigma,\nu}} := \mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\pi_{\sigma,\nu}, C_\eta^\infty(H\backslash G)).$$

We denote its image by $S_{\eta,\pi_{\sigma,\nu}}$, that is,

$$S_{\eta,\pi_{\sigma,\nu}} := \bigcup_{T\in I_{\eta,\pi_{\sigma,\nu}}} \mathrm{Image}(T).$$

We call the element of $S_{\eta,\pi_{\sigma,\nu}}$ the Shintani function of type $(\eta, \pi_{\sigma,\nu})$. Let (τ, V_τ) be the K-type of the principal series representation $\pi_{\sigma,\nu}$ and let $\iota : \tau \to \pi_{\sigma,\nu}$ be the K-embedding of τ and ι^* be the pullback via ι. Then the map

$$\iota^* : I_{\eta,\pi_{\sigma,\nu}} \to \mathrm{Hom}_K(\tau, C_\eta^\infty(H\backslash G)) \cong C_{\eta,\tau^*}^\infty(H\backslash G/K)$$

gives the restriction of $T \in S_{\eta,\pi_{\sigma,\nu}}$ to τ, where τ^* is the contragredient representation of τ and the space $C_{\eta,\tau^*}^\infty(H\backslash G/K)$ is defined by

$$C_{\eta,\tau^*}^\infty(H\backslash G/K) = \{F : G \to V_{\tau^*} | F(hgk^{-1}) = \eta(h)\tau^*(k)F(g), \forall(h,g,k) \in H\times G\times K\}.$$

We denote the image of $I_{\eta,\pi_{\sigma,\nu}}$ in $C_{\eta,\tau^*}^\infty(H\backslash G/K)$ by $C_{\eta,\tau^*}^\infty(H\backslash G/K)_{\pi_{\sigma,\nu}}$, and the element of this space is called the Shintani function of type $(\pi_{\sigma,\nu}, \eta, \tau)$.

3.2 The A-radial part

Because G has the decomposition $G = HAK$, the element of $C^{\infty}_{\eta,\tau}(H\backslash G/K)$ is characterized by its restriction to A. We denote the centralizer and the normalizer of A in $K \cap H$ by $Z_{K\cap H}(A)$, $N_{K\cap H}(A)$ respectively. It is easy to verify that $K \cap H$, $Z_{K\cap H}(A)$ and $N_{K\cap H}(A)$ are given as follows.

Lemma 3.1. *We have*

$$K\cap H = \left\{ \left(\begin{array}{cc|c} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \middle| \, \theta \in \mathbf{R} \right\} \bigsqcup \left\{ \left(\begin{array}{cc|c} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ \hline 0 & 0 & -1 \end{array} \right) \middle| \, \theta \in \mathbf{R} \right\},$$

$$Z_{K\cap H}(A) = \left\{ I, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\},$$

$$N_{K\cap H}(A) = \left\{ I, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

Here, I is the unit element of $GL(3,\mathbf{R})$.

Let $w_0 := \mathrm{diag}(-1,-1,1)$. Then $w_0 Z_{K\cap H}(A)$ is the unique non trivial element of $W = N_{K\cap H}(A)/Z_{K\cap H}(A)$. Let $\eta : H \to \mathbf{C}$ be the unitary character of H and (τ, V_τ) be the finite dimensional representation of K. We denote by $C^{\infty}_{W}(A;\eta,\tau)$ the space of smooth functions $F : A \to V_\tau$ satisfying the following conditions:

$$\begin{aligned} &(1)\, \eta(m)\tau(m)F(a) = F(a) \quad (\forall m \in Z_{K\cap H}(A), \forall a \in A) \\ &(2)\, \eta(w_0)\tau(w_0)F(a) = F(a^{-1}) \quad (\forall a \in A) \\ &(3)\, \eta(l)\tau(l)F(1) = F(1) \quad (\forall l \in K \cap H) \end{aligned} \tag{3.2}$$

The following lemma is proved by Flensted-Jensen ([2], Theorem 4.1).

Lemma 3.2. *The restriction to A gives the following isomorphism:*

$$C^{\infty}_{\eta,\tau}(H\backslash G/K) \cong C^{\infty}_{W}(A;\eta,\tau).$$

Through this isomorphism, we denote the image of $C^{\infty}_{\eta,\tau}(H\backslash G/K)_{\pi_{\sigma,\nu}}$ in $C^{\infty}_{W}(A;\eta,\tau)$ by $C^{\infty}_{W}(A;\eta,\tau)_{\pi_{\sigma,\nu}}$. This is our target space in this paper. The following two lemmas are obvious:

Lemma 3.3. *For $t \neq 0$, we have*

$$\mathfrak{g} = \mathrm{Ad}(a_t^{-1})\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

Lemma 3.4. *Let $F \in C^{\infty}_{\eta,\tau}(H\backslash G/K)$. For $X \in \mathfrak{k}$, $Y \in \mathfrak{h}$, $Z \in \mathfrak{a}$, we have*

$$R((\mathrm{Ad}(a_t^{-1})Y)ZX)F(a_t) = \eta(Y)\tau(-X)(Zf)(a_t).$$

4 Shintani functions attached to the spherical principal series representations

Throughout this section, as a character of M, we take the trivial character $\sigma = \sigma_0$. Then the principal series representation $\pi_{\sigma_0,\nu}$ is the spherical or class one principal series representation whose minimal K-type is the one dimensional trivial representation $(\mathbf{1}, V_1)$ of K, which occurs of multiplicity one in $\pi_{\sigma_0,\nu}|_K$ (Proposition 2.1).

4.1 The Capelli elements

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. $Z(\mathfrak{g})$ has two independent generators, and they are obtained as the Capelli elements because $\mathfrak{g} = \mathfrak{sl}_3$ is of type A_2 (see [5]). For $i = 1, 2, 3$, we put

$$E'_{ii} = E_{ii} - \frac{1}{3}\left(\sum_{k=1}^{3} E_{kk}\right).$$

The following proposition gives the explicit description of the independent generators of $Z(\mathfrak{g})$ (see [6]).

Proposition 4.1. *The independent generators $\{Cp_2, Cp_3\}$ of $Z(\mathfrak{g})$ are given as follows:*

$$\begin{aligned} Cp_2 =& (E'_{11} - 1)E'_{22} + E'_{22}(E'_{33} + 1) + (E'_{11} - 1)(E'_{33} + 1) \\ & - E_{23}E_{32} - E_{13}E_{31} - E_{12}E_{21} \\ Cp_3 =& (E'_{11} - 1)E'_{22}(E'_{33} + 1) + E_{12}E_{23}E_{31} + E_{1,3}E_{21}E_{32} \\ & - (E'_{11} - 1)E_{23}E_{32} - E_{13}E'_{22}E_{31} - E_{12}E_{21}(E'_{33} + 1). \end{aligned}$$

Since Cp_2, Cp_3 are the elements of $Z(\mathfrak{g})$, they act on $\pi_{\sigma_0,\nu}$ as the scalar operators. And since the space of Shintani functions is the image of the $(\mathfrak{g}_{\mathbf{C}}, K)$-homomorphism of $\pi_{\sigma_0,\nu}$, they act on the space of Shintani functions as the same scalar operators respectively.

4.2 Eigenvalues of Cp_2, Cp_3

In order to construct the partial differential equations satisfied by spherical functions attached to the spherical principal series, we have to compute the eigenvalues of the actions of the Capelli elements Cp_2, Cp_3. For the spherical principal series representation , $\sigma = \sigma_0$ is the trivial character of M. Let f_0 be the generator of the minimal K-type in $H_{\sigma_0,\nu}$ normalized such that $f_0|K \equiv 1$. The actions of Cp_2, Cp_3 on f_0 are computed in [6], and the result is as follows:

Proposition 4.2. *The Capelli elements Cp_2, Cp_3 act on f_0 by scalar multiples, and the eigenvalues are given as follows:*

$$Cp_2 f_0 = -\frac{1}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2)f_0,$$

$$Cp_3 f_0 = -\frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2)f_0.$$

4.3 Construction of the Casimir equations

Next, we compute the actions of Cp_2, Cp_3 on $F(a_t) \in C^\infty_{\eta,1}(H\backslash G/K)|_A$. Here, $\eta = \eta_{s,k}$ be the unitary character of H.

Lemma 4.3. *For $F(a_t) \in C^\infty_{\eta,\tau}(H\backslash G/K)|_A$, we have*

$$R(\mathrm{Ad}(a_t^{-1})Y_i)F(a_t) = sF(a_t) \quad (i = 1, 4)$$

$$R(\mathrm{Ad}(a_t^{-1})Y_i)F(a_t) = 0 \quad (i = 2, 3)$$

$$R(X_{13})F(a_t) = \frac{dF}{dt}(a_t).$$

Proof. By definition, we have

$$\begin{aligned} R(\mathrm{Ad}(a_t^{-1})Y_i)F(a_t) &= \frac{d}{du}F(a_t \exp(u\mathrm{Ad}(a_t^{-1})Y_i))|_{u=0} \\ &= \frac{d}{du}F(\exp(uY_i)a_t)|_{u=0} \\ &= \frac{d}{du}\eta(\exp(uY_i))|_{u=0}F(a_t). \end{aligned}$$

Since $\exp(uY_1) = \mathrm{diag}(e^u, 1, e^{-u})$, we have $\eta((\exp(uY_1)) = e^{ku} \cdot e^{(s-k)u} = e^{su}$. Therefore, we have

$$R(\mathrm{Ad}(a_t^{-1})Y_1)F(a_t) = sF(a_t).$$

Next, since

$$\exp(uY_2) = \begin{pmatrix} \cosh u & \sinh u & 0 \\ \sinh u & \cosh u & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\eta(\exp(uY_2)) = 1^k \cdot |1|^{s-k} = 1$. Therefore, we have

$$R(\mathrm{Ad}(a_t^{-1})Y_2)F(a_t) = 0.$$

The computations of the actions of $\mathrm{Ad}(a_t^{-1})(Y_i)$ $(i = 3, 4)$ are similar. Finally, since $\exp(uX_{13}) = a_u$, we have

$$\begin{aligned} R(X_{13})F(a_t) &= \frac{d}{du}F(a_{t+u})|_{u=0} \\ &= \frac{dF}{dt}(a_t). \end{aligned}$$

□

By simple computations of matrices, we have the following expressions of the elements in $M(3,\mathbf{R})$.

Lemma 4.4.

$$E'_{11} = \frac{\cosh^2 t + 1}{3\cosh(2t)}\mathrm{Ad}(a_t^{-1})Y_1 - \frac{1}{3}\mathrm{Ad}(a_t^{-1})Y_4 - \frac{1}{2}\tanh(2t)K_{13},$$

$$E'_{22} = \frac{1}{3}(-H_{12} + H_{23}),$$

$$E'_{33} = \frac{\sinh^2 t - 1}{3\cosh(2t)}\mathrm{Ad}(a_t^{-1})Y_1 - \frac{1}{3}\mathrm{Ad}(a_t^{-1})Y_4 + \frac{1}{2}\tanh(2t)K_{13},$$

$$E_{23} = \frac{1}{2\sinh t}\mathrm{Ad}(a_t^{-1})Y_3 + \frac{1}{2\tanh t}K_{12} + \frac{1}{2}K_{23},$$

$$E_{13} = \frac{1}{2}X_{13} + \frac{1}{2}K_{13},$$

$$E_{12} = \frac{1}{2\cosh t}\mathrm{Ad}(a_t^{-1})Y_2 - \frac{1}{2}\tanh t K_{23} + \frac{1}{2}K_{12}.$$

To make use of Lemma 3.4, we have to rewrite Cp_2, Cp_3 in the form of linear combinations of the elements in $(\mathrm{Ad}(a_t^{-1})\mathfrak{h})\mathfrak{a}\mathfrak{k}$. To do this, we use the following formulas which can be obtained by direct computation.

Lemma 4.5. *We have*

$$[K_{13}, \mathrm{Ad}(a_t^{-1})Y_1] = -2\cosh(2t)X_{13}, \quad [K_{13}, \mathrm{Ad}(a_t^{-1})Y_4] = -\cosh(2t)X_{13},$$

$$[K_{12}, \mathrm{Ad}(a_t^{-1})Y_3] = \sinh t X_{13},$$

$$[K_{23}, \mathrm{Ad}(a_t^{-1})Y_3] = -\frac{2\sinh^3 t}{\cosh(2t)}\mathrm{Ad}(a_t^{-1})Y_1 + 2\sinh t\mathrm{Ad}(a_t^{-1})Y_4 + \frac{\cosh t}{\cosh(2t)}K_{13},$$

$$[K_{13}, X_{13}] = \frac{2}{\cosh(2t)}\mathrm{Ad}(a_t^{-1})Y_1 - 2\tanh(2t)K_{13},$$

$$[K_{23}, \mathrm{Ad}(a_t^{-1})Y_2] = -\cosh t X_{13},$$

$$[K_{12}, \mathrm{Ad}(a_t^{-1})Y_2] = \frac{2\cosh^3 t}{\cosh(2t)}\mathrm{Ad}(a_t^{-1})Y_1 - 2\cosh t\mathrm{Ad}(a_t^{-1})Y_4 - \frac{\sinh t}{\cosh(2t)}K_{13},$$

$$[K_{12}, X_{13}] = -\frac{1}{\sinh t}\mathrm{Ad}(a_t^{-1})Y_3 - \frac{1}{\tanh t}K_{12},$$

$$[K_{23}, X_{13}] = \frac{1}{\cosh t}\mathrm{Ad}(a_t^{-1})Y_2 - \tanh t K_{23},$$

$$[X_{13}, \mathrm{Ad}(a_t^{-1})Y_2] = -\tanh t\mathrm{Ad}(a_t^{-1})Y_2 - \frac{1}{\cosh t}K_{23},$$

$$[K_{13}, \mathrm{Ad}(a_t^{-1})Y_2] = -\frac{1}{\tanh t}\mathrm{Ad}(a_t^{-1})Y_3 - \frac{\cosh(2t)}{\sinh t}K_{12},$$

$$[K_{13}, \mathrm{Ad}(a_t^{-1})Y_3] = \mathrm{tanh}t\mathrm{Ad}(a_t^{-1})Y_2 - \frac{\cosh(2t)}{\cosh t}K_{23},$$

$$[X_{13}, \mathrm{Ad}(a_t^{-1})Y_1] = -2\tanh(2t)\mathrm{Ad}(a_t^{-1})Y_1 - \frac{2}{\cosh(2t)}K_{13},$$

$$[X_{13}, \mathrm{Ad}(a_t^{-1})Y_4] = -\tanh(2t)\mathrm{Ad}(a_t^{-1})Y_1 - \frac{1}{\cosh(2t)}K_{13},$$

$$[K_{12}, \mathrm{Ad}(a_t^{-1})Y_1] = -\frac{\cosh(2t)}{\cosh t}\mathrm{Ad}(a_t^{-1})Y_2 - \mathrm{tanh}tK_{23},$$

$$[K_{12}, \mathrm{Ad}(a_t^{-1})Y_4] = \frac{1-\sinh^2 t}{\cosh t}\mathrm{Ad}(a_t^{-1})Y_2 - 2\mathrm{tanh}tK_{23},$$

$$[K_{23}, \mathrm{Ad}(a_t^{-1})Y_1] = -\frac{\cosh(2t)}{\sinh t}\mathrm{Ad}(a_t^{-1})Y_3 - \frac{1}{\tanh t}K_{12},$$

$$[K_{23}, \mathrm{Ad}(a_t^{-1})Y_4] = -\frac{1+\cosh^2 t}{\sinh t}\mathrm{Ad}(a_t^{-1})Y_3 - \frac{2}{\tanh t}K_{12}.$$

Here, $[X, Y] := XY - YX$ *is the Lie bracket on* $\mathfrak{g}$.

By using Lemma 4.5, we can rewrite Cp_2, Cp_3 as we wished. Now, since $\forall F(a_t) \in C^\infty_{\eta,1}(H\backslash G/K)|_A$ is annihilated by the action of $U(\mathfrak{g})\mathfrak{k}$ and $\mathrm{Ad}(a_t^{-1})Y_2U(\mathfrak{g})$, $\mathrm{Ad}(a_t^{-1})Y_3U(\mathfrak{g})$, and the actions of $\mathrm{Ad}(a_t^{-1})Y_1$ and $\mathrm{Ad}(a_t^{-1})Y_4$ on F are the same (the multiplication by s), we may regard Cp_2, Cp_3 as the elements in $U(\mathfrak{g})(\mathrm{mod}\,\mathfrak{P})$ where $\mathfrak{P}$ is the subalgebra of $U(\mathfrak{g})$ defined by

$$\mathfrak{P} = (\mathrm{Ad}(a_t^{-1})Y_1 - \mathrm{Ad}(a_t^{-1})Y_4)U(\mathfrak{g}) + \mathrm{Ad}(a_t^{-1})Y_2U(\mathfrak{g}) + \mathrm{Ad}(a_t^{-1})Y_3U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}.$$

Lemma 4.6. *We have the congruences*

$$\begin{aligned} Cp_2 \equiv &-\frac{1}{4}X_{13}^2 - \frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)X_{13} \\ &+ \left(\frac{1}{4}\tanh^2(2t) - \frac{1}{3}\right)(\mathrm{Ad}(a_t^{-1})Y_1)^2 - 1 \quad (\mathrm{mod}\,\mathfrak{P}), \\ Cp_3 \equiv &-\frac{1}{12}(\mathrm{Ad}(a_t^{-1})Y_1)X_{13}^2 - \frac{1}{6}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)(\mathrm{Ad}(a_t^{-1})Y_1)X_{13} \\ &+ \left(-\frac{1}{27} + \frac{1}{12}\tanh^2(2t)\right)(\mathrm{Ad}(a_t^{-1})Y_1)^3 - \frac{1}{3}(\mathrm{Ad}(a_t^{-1})Y_1) \quad (\mathrm{mod}\,\mathfrak{P}). \end{aligned}$$

By combining Proposition 4.2, Lemma 4.3 and Lemma 4.6, we have the following two differential equations.

Theorem 4.7. *The Shintani function $F(a_t) \in C^{\infty}_{\eta,1}(H\backslash G/K)_{\pi_{\sigma_0,\nu}}|_A$ satisfies the following equations.*
1. The differential equation obtained from the action of Cp_2:

$$\begin{aligned}&-\frac{1}{4}\frac{d^2F}{dt^2}(a_t) - \frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)\frac{dF}{dt}(a_t)\\ &+\left\{\left(-\frac{1}{3} + \frac{1}{4}\tanh^2(2t)\right)s^2 - 1\right\}F(a_t) = \lambda_2 F(a_t).\end{aligned} \tag{4.1}$$

2. The differential equation obtained from the action of Cp_3:

$$\begin{aligned}&-\frac{1}{12}s\frac{d^2F}{dt^2}(a_t) - \frac{1}{6}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)s\frac{dF}{dt}(a_t)\\ &+\left\{\left(-\frac{2}{27} + \frac{1}{12}\tanh^2(2t)\right)s^3 - \frac{1}{3}s\right\}F(a_t) = \lambda_3 F(a_t).\end{aligned} \tag{4.2}$$

Here,

$$\lambda_2 = -\frac{1}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2),$$
$$\lambda_3 = -\frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2)$$

are the eigenvalues of the Capelli elements on principal series representations.

(4.2)-(4.1)$\times\frac{1}{3}s$ gives

$$\left\{\frac{1}{27}s^3 + \frac{1}{3}\lambda_2 s - \lambda_3\right\}F(a_t) = 0.$$

Therefore, if $F(a_t)$ is not identically zero, we have

$$s^3 - 3(\nu_1^2 - \nu_1\nu_2 + \nu_2^2)s + (2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2) = 0.$$

By solving this equation, we have

$$s = 2\nu_1 - \nu_2,\ 2\nu_2 - \nu_1,\ -\nu_1 - \nu_2.$$

Therefore, one of the necessary conditions of the existence of non-trivial Shintani functions is that the parameter s is one of the above three values. Now, we assume that s satisfies this condition. We put $x = \tanh(2t)$, $\tilde{F}(x) := F(a_t)$ in (4.1). Then we have

$$\begin{aligned}&-4x(1-x)^2\frac{d^2\tilde{F}}{dx^2}(x) - 4(1-x)^2\frac{d\tilde{F}}{dx}(x)\\ &+\left\{(-\frac{1}{3} + \frac{1}{4}x)s^2 - 1 - \lambda_2\right\}\tilde{F}(x) = 0.\end{aligned}$$

Next, we put $\tilde{F}(x) = (1-x)^{\mu} G_0(x)$ $(\mu \in \mathbf{C})$. Then G_0 satisfies

$$
\begin{aligned}
&-4x(1-x)^{\mu+2}\frac{d^2G_0}{dx^2}(x) + (-4+(8\mu+4)x)(1-x)^{\mu+1}\frac{dG_0}{dx}(x) \\
&+\left\{-4\mu(\mu-1)x + 4\mu - 4\mu x + (-\frac{1}{3}+\frac{1}{4}x)s^2 - 1 - \lambda_2\right\}(1-x)^{\mu}G_0(x) = 0.
\end{aligned}
\tag{4.3}
$$

We want to divide the left hand side of (4.3) by $(1-x)^{\mu+1}$. To do this, we take $\mu \in \mathbf{C}$ so that μ satisfies

$$-4\mu^2 + 4\mu - \frac{1}{12}s^2 - 1 - \lambda_2 = 0.$$

The value of $\mu \in \mathbf{C}$ is as follows:
1) If $s = 2\nu_1 - \nu_2$, $\mu = \frac{2\pm\nu_2}{4}$. So we take $\mu = \frac{2+\nu_2}{4}$.
2) If $s = 2\nu_2 - \nu_1$, $\mu = \frac{2\pm\nu_1}{4}$. So we take $\mu = \frac{2+\nu_1}{4}$.
3) If $s = -\nu_1 - \nu_2$, $\mu = \frac{2\pm(\nu_1-\nu_2)}{4}$. So we take $\mu = \frac{2+\nu_1-\nu_2}{4}$.
For this μ, the left hand side of (4.3) is divided by $(1-x)^{\mu+1}$, and the equation becomes

$$
\begin{aligned}
&x(x-1)\frac{d^2G_0}{dx^2}(x) + (-1+(1+2\mu)x)\frac{dG_0}{dx} \\
&+(\mu^2 - \frac{1}{16}s^2)G_0(x) = 0.
\end{aligned}
\tag{4.4}
$$

This is the Gaussian hypergeometric differential equation. Note that the Shintani function $F(a_t)$ on A is regular at the origin ($\Leftrightarrow x = 0$). Therefore, $G_0(x) = (1-x)^{-\mu}\tilde{F}(x)$ is also regular at $x = 0$. (4.4) has just one solution which is regular around $x = 0$ (up to constant multiples), and it is given by

$$G_0(x) = {}_2F_1(\alpha, \beta; 1; x)$$

where ${}_2F_1$ is the Gaussian hypergeometric function and $\alpha, \beta \in \mathbf{C}$ are defined by

$$1 + \alpha + \beta = 1 + 2\mu$$

$$\alpha\beta = \mu^2 - \frac{1}{16}s^2.$$

Explicitly, by solving these equations, α, β are given as follows:
1) In case of $s = 2\nu_1 - \nu_2$, $(\alpha, \beta) = (\frac{\nu_1+1}{2}, \frac{-\nu_1+\nu_2+1}{2})$.
2) In case of $s = 2\nu_2 - \nu_1$, $(\alpha, \beta) = (\frac{\nu_2+1}{2}, \frac{-\nu_2+\nu_1+1}{2})$.
3) In case of $s = -\nu_1 - \nu_2$, $(\alpha, \beta) = (\frac{\nu_1+1}{2}, \frac{-\nu_2+1}{2})$.

Finally, we consider the three conditions in (3.2). The condition (1) is equivalent to $(-1)^k F(a_t) = F(a_t)$. The condition (2) always holds. The condition (3) is equivalent to $(-1)^k F(1) = F(1)$, which holds if the condition (1) is satisfied. Summing up, we have the following theorem.

Theorem 4.8. *Let $\eta = \eta_{s,k}$ be the unitary character of H defined by (3.1). Then the necessary condition of the existence of the non-trivial elements in $C^\infty_{\eta,1}(H\backslash G/K)_{\pi_{\sigma_0,\nu}}$ is that*

$$k = 0 \quad \text{and} \quad s = 2\nu_1 - \nu_2, \quad 2\nu_2 - \nu_1, \quad -\nu_1 - \nu_2.$$

Suppose that this condition is satisfied and non-trivial Shintani functions exist. If we put $x = \tanh^2(2t)$, $F(a_t) = \tilde{F}(x) \in C^\infty_{\eta,1}(H\backslash G/K)_{\pi_{\sigma_0,\nu}}|_A$ is given as follows (up to constant multiples):

1) *In case of $s = 2\nu_1 - \nu_2$, we have*

$$\tilde{F}(x) = (1-x)^{\frac{2+\nu_2}{4}} {}_2F_1\left(\frac{\nu_1+1}{2}, \frac{-\nu_1+\nu_2+1}{2}; 1; x\right).$$

2) *In case of $s = 2\nu_2 - \nu_1$, we have*

$$\tilde{F}(x) = (1-x)^{\frac{2+\nu_1}{4}} {}_2F_1\left(\frac{\nu_2+1}{2}, \frac{-\nu_2+\nu_1+1}{2}; 1; x\right).$$

3) *In case of $s = -\nu_1 - \nu_2$, we have*

$$\tilde{F}(x) = (1-x)^{\frac{2+\nu_1-\nu_2}{4}} {}_2F_1\left(\frac{\nu_1+1}{2}, \frac{-\nu_2+1}{2}; 1; x\right).$$

Especially, we have

$$\dim C^\infty_{\eta,1}(H\backslash G/K)_{\pi_{\sigma_0,\nu}} \leq 1.$$

5 Shintani functions attached to the non-spherical principal series representations

In this section, as a character of M, we take a non-trivial character $\sigma = \sigma_i$ $(i = 1, 2, 3)$. Then the minimal K-type of $\pi_{\sigma_i,\nu}$ is the three dimensional representation of K which is isomorphic to the tautological representation $\tau_2 : K = SO(3,\mathbf{R}) \hookrightarrow GL(3,\mathbf{R})$ which occurs of multiplicity one in $\pi_{\sigma_i,\nu}|_K$. We take τ_2^* instead of τ_2 as a minimal K-type of $\pi_{\sigma_i,\nu}$. The representation space of τ_2 is denoted by V_{τ_2} $(= \mathbf{R}^3)$, and we take $s_1 = {}^t(1,0,0)$, $s_2 = {}^t(0,1,0)$, $s_3 = {}^t(0,0,1)$ as a basis of V_{τ_2}. Let $\Psi \in C^\infty_{\eta,\tau_2}(H\backslash G/K)_{\pi_{\sigma_i,\nu}}$ be the Shintani function. Then Ψ is expressed by

$$\Psi(g) = F_0(g)s_1 + G_0(g)s_2 + H_0(g)s_3.$$

Ψ is characterized by its restriction to A. To investigate $\Psi|_A$, we construct two kinds of differential equations. One is the Casimir equation of degree two and the other is the gradient equation (or the Dirac-Schmidt equation).

5.1 The Casimir equation

Firstly, we construct the Casimir equation of degree two. Since the Capelli element Cp_2 acts on the representation space of the principal series representation $\pi_{\sigma_i,\nu}$ as a scalar operator (λ_2-multiple), and the space of Shintani functions $C^\infty_{\eta,\tau_2}(H\backslash G/K)_{\pi_{\sigma_i,\nu}}$ is the image of $(\mathfrak{g}_{\mathbf{C}},K)$-homomorphism of $\pi_{\sigma_i,\nu}$, Cp_2 acts on this space as the same scalar operator. Since $\forall\Psi(a_t)\in C^\infty_{\eta,\tau_2}(H\backslash G/K)|_A$ is annihilated by the action of $\mathrm{Ad}(a_t^{-1})Y_2U(\mathfrak{g})$, $\mathrm{Ad}(a_t^{-1})Y_3U(\mathfrak{g})$, and the actions of $\mathrm{Ad}(a_t^{-1})Y_1$ and $\mathrm{Ad}(a_t^{-1})Y_4$ on F are the same (the multiplication by s), we may regard Cp_2 as the element in $U(\mathfrak{g})(\mathrm{mod}\,\mathfrak{P}')$, where $\mathfrak{P}'$ is a subalgebra of $U(\mathfrak{g})$ defined by

$$\mathfrak{P}'=(\mathrm{Ad}(a_t^{-1})Y_1-\mathrm{Ad}(a_t^{-1})Y_4)U(\mathfrak{g})+\mathrm{Ad}(a_t^{-1})Y_2U(\mathfrak{g})+\mathrm{Ad}(a_t^{-1})Y_3U(\mathfrak{g}).$$

By using Lemma 4.4 and Lemma 4.5, we can rewrite Cp_2 in Proposition 4.1 as follows.

Lemma 5.1. *We have*

$$\begin{aligned}Cp_2\equiv&-\frac{1}{4}X_{13}^2-\frac{1}{2}\left(\tanh(2t)+\frac{1}{\tanh(2t)}\right)X_{13}\\&+\left(\frac{1}{4}\tanh^2(2t)-\frac{1}{3}\right)(\mathrm{Ad}(a_t^{-1})Y_1)^2-1\\&+\frac{\tanh(2t)}{2\cosh(2t)}\mathrm{Ad}(a_t^{-1})Y_1K_{13}+\left(-\frac{1}{4}\tanh^2(2t)+\frac{1}{4}\right)K_{13}^2\\&+\left(-\frac{1}{4}\tanh^2t+\frac{1}{4}\right)K_{12}^2+\left(\frac{1}{4}-\frac{1}{4}\tanh^2t\right)K_{23}^2\\&(\mathrm{mod}\quad(\mathrm{Ad}(a_t^{-1})Y_1-\mathrm{Ad}(a_t^{-1})Y_4)U(\mathfrak{g})+\mathrm{Ad}(a_t^{-1})Y_2U(\mathfrak{g})+\mathrm{Ad}(a_t^{-1})Y_3U(\mathfrak{g})).\end{aligned}$$

By using this Lemma, the action of Cp_2 on $\Psi(a_t)=F_0(a_t)s_1+G_0(a_t)s_2+H_0(a_t)s_3\in C^\infty_{\eta,\tau_2}(H\backslash G/K)|_A$ can be computed easily. We have

$$Cp_2\Psi(a_t)=F'(a_t)s_1+G'(a_t)s_2+H'(a_t)s_3,$$

where

$$\begin{aligned}F'(a_t)=&-\frac{1}{4}\frac{d^2F_0}{dt^2}(a_t)-\frac{1}{2}\left(\tanh(2t)+\frac{1}{\tanh(2t)}\right)\frac{dF_0}{dt}(a_t)\\&+\left\{\left(\frac{1}{4}\tanh^2(2t)-\frac{1}{3}\right)s^2+\frac{1}{4}\tanh^2(2t)+\frac{1}{4\tanh^2t}-\frac{3}{2}\right\}F_0(a_t)\\&-\frac{\tanh(2t)}{2\cosh(2t)}sH_0(a_t),\end{aligned}$$

$$\begin{aligned}G'(a_t)=&-\frac{1}{4}\frac{d^2G_0}{dt^2}(a_t)-\frac{1}{2}\left(\tanh(2t)+\frac{1}{\tanh(2t)}\right)\frac{dG_0}{dt}(a_t)\\&+\left\{\left(\frac{1}{4}\tanh^2t-\frac{1}{3}\right)s^2+\frac{1}{4}\tanh^2t+\frac{1}{4\tanh^2t}-\frac{3}{2}\right\}G_0(a_t),\end{aligned}$$

$$
\begin{aligned}
H^{'}(a_t) =& -\frac{1}{4}\frac{d^2H_0}{dt^2}(a_t) - \frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)\frac{dH_0}{dt}(a_t) \\
&+ \left\{\left(\frac{1}{4}\tanh^2(2t) - \frac{1}{3}\right)s^2 + \frac{1}{4}\tanh^2(2t) + \frac{1}{4}\tanh^2 t - \frac{3}{2}\right\}H_0(a_t) \\
&+ \frac{\tanh(2t)}{2\cosh(2t)}sF_0(a_t).
\end{aligned}
$$

Since $\Psi(a_t)$ satisfies $Cp_2\Psi(a_t) = \lambda_2\Psi(a_t)$, we have the following three differential equations:

Theorem 5.2. *For $\Psi(a_t) = F_0(a_t)s_1 + G_0(a_t)s_2 + H_0(a_t)s_3 \in C^\infty_{\eta,\tau_2}(H\backslash G/K)_{\pi_{\sigma_i,\nu}}|_A$, the functions F_0, G_0, H_0 satisfy the following equations.*

$$
\begin{aligned}
&-\frac{1}{4}\frac{d^2F_0}{dt^2}(a_t) - \frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)\frac{dF_0}{dt}(a_t) \\
&+ \left\{\left(\frac{1}{4}\tanh^2(2t) - \frac{1}{3}\right)s^2 + \frac{1}{4}\tanh^2(2t) + \frac{1}{4\tanh^2 t} - \frac{3}{2}\right\}F_0(a_t) \\
&- \frac{\tanh(2t)}{2\cosh(2t)}sH_0(a_t) = \lambda_2 F_0(a_t)
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&-\frac{1}{4}\frac{d^2G_0}{dt^2}(a_t) - \frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)\frac{dG_0}{dt}(a_t) \\
&+ \left\{\left(\frac{1}{4}\tanh^2 t - \frac{1}{3}\right)s^2 + \frac{1}{4}\tanh^2 t + \frac{1}{4\tanh^2 t} - \frac{3}{2}\right\}G_0(a_t) \\
&= \lambda_2 G_0(a_t)
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
&-\frac{1}{4}\frac{d^2H_0}{dt^2}(a_t) - \frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)\frac{dH_0}{dt}(a_t) \\
&+ \left\{\left(\frac{1}{4}\tanh^2(2t) - \frac{1}{3}\right)s^2 + \frac{1}{4}\tanh^2(2t) + \frac{1}{4}\tanh^2 t - \frac{3}{2}\right\}H_0(a_t) \\
&+ \frac{\tanh(2t)}{2\cosh(2t)}sF_0(a_t) = \lambda_2 H_0(a_t).
\end{aligned}
\tag{5.3}
$$

5.2 The gradient equation

For the spherical function $\Psi(g) \in C^\infty_{\eta,\tau_2}(H\backslash G/K)$, we define the right gradient operator ∇^R as follows:

Definition 5.3. For the orthonormal basis $\{X_i\}_{i=1}^5$ of $\mathfrak{p}$, the right gradient operator ∇^R is defined by

$$
\nabla^R\Psi(g) := \sum_{i=1}^{5} R(X_i)\Psi \otimes X_i^*.
$$

Here, X_i^* is the dual basis of X_i with respect to the inner product $(X, Y) \in \mathfrak{p} \times \mathfrak{p} \to \mathrm{Tr}(XY) \in \mathbf{C}$.

The set $\{H_{12}, H_{23}, X_{12}, X_{23}, X_{13}\}$ becomes the orthonormal basis of $\mathfrak{p}$, and $\{\frac{1}{3}(2H_{12}+H_{23}), \frac{1}{3}(H_{12}+2H_{23}), \frac{1}{2}X_{12}, \frac{1}{2}X_{23}, \frac{1}{2}X_{13}\}$ is its dual basis. Therefore, the gradient operator ∇^R is explicitly given by

$$\begin{aligned}\nabla^R\Psi(g) =& \frac{1}{3}R(H_{12})\Psi \otimes (2H_{12} + H_{23}) + \frac{1}{3}R(H_{23})\Psi \otimes (H_{12} + 2H_{23}) \\ &+ \frac{1}{2}\sum_{i<j} R(X_{ij})\Psi \otimes X_{ij}.\end{aligned}$$

We rewrite this by using the basis of $\mathfrak{p}_{\mathbf{C}}$.

Claim 1. We define five elements w_i $(0 \le i \le 4)$ in $\mathfrak{p}_{\mathbf{C}}$ by

$$w_0 := -2(H_{23} - \sqrt{-1}X_{23}),\ w_4 := -2(H_{23} + \sqrt{-1}X_{23}),$$

$$w_2 := \frac{2}{3}(2H_{12} + H_{23}),$$

$$w_1 := X_{13} + \sqrt{-1}X_{12},\ w_3 := -X_{13} + \sqrt{-1}X_{12}.$$

Then $\{w_i | 0 \le i \le 4\}$ becomes the basis of $\mathfrak{p}_{\mathbf{C}}$.

With this basis, the gradient operator ∇^R is rewritten as

$$\begin{aligned}\nabla^R\Psi =& \frac{1}{16}R(w_4)\Psi \otimes w_0 + \frac{1}{16}R(w_0)\Psi \otimes w_4 - \frac{1}{4}R(w_3)\Psi \otimes w_1 \\ &- \frac{1}{4}R(w_1)\Psi \otimes w_3 + \frac{3}{8}R(w_2)\Psi \otimes w_2 \\ =& \frac{1}{4}\left\{\frac{1}{4}R(w_4)\Psi \otimes w_0 + \frac{1}{4}R(w_0)\Psi \otimes w_4 - R(w_3)\Psi \otimes w_1\right. \\ &\left. -R(w_1)\Psi \otimes w_3 + \frac{3}{2}R(w_2)\Psi \otimes w_2\right\}.\end{aligned}$$

The Lie algebra $\mathfrak{p}_{\mathbf{C}}$ becomes the representation space of the adjoint action of K. We denote this representation by (τ_4, W_4). By the Clebsh-Gordan theorem, $\tau_2 \otimes \tau_4$ has the irreducible decomposition

$$\tau_2 \otimes \tau_4 \cong \tau_2 \oplus \tau_4 \oplus \tau_6.$$

Here, each τ_n is the $(n+1)$-dimensional irreducible representation of K. In this decomposition, the projector of K-modules

$$pr_2 : \tau_2 \otimes \tau_4 \twoheadrightarrow \tau_2,\ s_i \otimes w_j \mapsto pr_2(s_i \otimes w_j)$$

is described as in the following table:

	w_0	w_1	w_2	w_3	w_4
s_1	0	$-\frac{1}{4}(s_3+\sqrt{-1}s_2)$	$-\frac{1}{3}s_1$	$\frac{1}{4}(s_3-\sqrt{-1}s_2)$	0
s_2	$\frac{1}{2}(s_2-\sqrt{-1}s_3)$	$-\frac{\sqrt{-1}}{4}s_1$	$\frac{1}{6}s_2$	$-\frac{\sqrt{-1}}{4}s_1$	$\frac{1}{2}(s_2+\sqrt{-1}s_3)$
s_3	$-\frac{1}{2}(s_3+\sqrt{-1}s_2)$	$-\frac{1}{4}s_1$	$\frac{1}{6}s_3$	$\frac{1}{4}s_1$	$\frac{1}{2}(-s_3+\sqrt{-1}s_2)$

$\nabla^R\Psi$ is a $\tau_2\otimes(\tau_2\otimes\mathfrak{p}_{\mathbf{C}})$-valued function. Then, by mapping $s_i\otimes w_k$ to $pr_2(s_i\otimes w_k)$, we have a K-homomorphism

$$\tilde{pr}_2\circ\nabla^R : C^\infty_{\eta,\tau_2}(H\backslash G/K)\to C^\infty_{\eta,\tau_2}(H\backslash G/K).$$

Since the minimal K-type τ_2^* occurs of multiplicity one, $\tilde{pr}_2\circ\nabla^R$ is a map of constant multiple. To compute the action of the gradient operator $\tilde{pr}_2\circ\nabla^R$ on the space of the Shintani functions $C^\infty_{\eta,\tau_2}(H\backslash G/K)_{\pi_{\sigma_i,\nu}}$, we have to decompose w_i $(i=0,1,2,3,4)$ along the decomposition $\mathfrak{g}_{\mathbf{C}}=\mathrm{Ad}(a_t^{-1})\mathfrak{h}_{\mathbf{C}}\oplus\mathfrak{a}_{\mathbf{C}}\oplus\mathfrak{k}_{\mathbf{C}}$.

Lemma 5.4. *We have*

$$\begin{aligned}
w_0 =&\frac{2\mathrm{sinh}^2t}{\mathrm{cosh}(2t)}\mathrm{Ad}(a_t^{-1})Y_1-2\mathrm{Ad}(a_t^{-1})Y_4+\tanh(2t)K_{13}\\
&+\frac{2\sqrt{-1}}{\mathrm{sinh}t}\mathrm{Ad}(a_t^{-1})Y_3+\frac{2\sqrt{-1}}{\mathrm{tanh}t}K_{12},\\
w_4 =&\frac{2\mathrm{sinh}^2t}{\mathrm{cosh}(2t)}\mathrm{Ad}(a_t^{-1})Y_1-2\mathrm{Ad}(a_t^{-1})Y_4+\tanh(2t)K_{13}\\
&-\frac{2\sqrt{-1}}{\mathrm{sinh}t}\mathrm{Ad}(a_t^{-1})Y_3-\frac{2\sqrt{-1}}{\mathrm{tanh}t}K_{12},\\
w_2 =&\frac{2(\mathrm{cosh}^2t+1)}{3\mathrm{cosh}(2t)}\mathrm{Ad}(a_t^{-1})Y_1-\frac{2}{3}\mathrm{Ad}(a_t^{-1})Y_4\\
&-\tanh(2t)K_{13},\\
w_1 =&X_{13}+\frac{\sqrt{-1}}{\mathrm{cosh}t}\mathrm{Ad}(a_t^{-1})Y_2-\sqrt{-1}\mathrm{tanh}tK_{23},\\
w_3 =&-X_{13}+\frac{\sqrt{-1}}{\mathrm{cosh}t}\mathrm{Ad}(a_t^{-1})Y_2-\sqrt{-1}\mathrm{tanh}tK_{23}.
\end{aligned}$$

By using Lemma 3.4 and Lemma 4.3, and the table of projections, we can compute the action of the gradient operator. For $\Psi(a_t)=F_0(a_t)s_1+G_0(a_t)s_2+H_0(a_t)s_3\in C^\infty_{\eta,\tau_2}(H\backslash G/K)|_A$, we have

$$\tilde{pr}_2\circ\nabla^R\Psi(a_t)=F''(a_t)s_1+G''(a_t)s_2+H''(a_t)s_3,$$

where

$$\begin{aligned}
F''(a_t)=&-\left(\frac{1}{2\mathrm{cosh}(2t)}-\frac{1}{6}\right)sF_0(a_t)\\
&-\frac{1}{2}\frac{dH_0}{dt}(a_t)-\frac{1}{2}\left(\tanh(2t)+\mathrm{tanh}t\right)H_0(a_t),
\end{aligned}$$

$$G''(a_t) = -\frac{1}{3}sG_0(a_t),$$

$$\begin{aligned} H''(a_t) = &\left(\frac{1}{2\cosh(2t)} + \frac{1}{6}\right) sH_0(a_t) \\ &- \frac{1}{2}\frac{dF_0}{dt}(a_t) - \frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh t}\right) F_0(a_t). \end{aligned}$$

On the other hand, the eigenvalue of the gradient operator on the spherical functions of the principal series representation $\pi_{\sigma_i,\nu}$ depends on the choice of σ_i, denoted by λ_{σ_i} $(i = 1, 2, 3)$. These values are computed in [6] and they are as follows:

$$\lambda_{\sigma_1} = -\frac{1}{3}(2\nu_1 - \nu_2), \quad \lambda_{\sigma_2} = -\frac{1}{3}(2\nu_2 - \nu_1), \quad \lambda_{\sigma_3} = \frac{1}{3}(\nu_1 + \nu_2). \tag{5.4}$$

Therefore, since $\Psi(a_t) = F_0(a_t)s_1+G_0(a_t)s_2+H_0(a_t)s_3 \in C^\infty_{\eta,\tau_2}(H\backslash G/K)_{\pi_{\sigma_i,\nu}}|_A$ satisfies $\tilde{pr}_2 \circ \nabla^R\Psi(a_t) = \lambda_{\sigma_i}\Psi(a_t)$, we have the following three differential equations:

Theorem 5.5. *For* $\Psi(a_t) = F_0(a_t)s_1+G_0(a_t)s_2+H_0(a_t)s_3 \in C^\infty_{\eta,\tau_2}(H\backslash G/K)_{\pi_{\sigma_i,\nu}}|_A$, *we have*

$$\begin{aligned} &-\left(\frac{1}{2\cosh(2t)} - \frac{1}{6}\right) sF_0(a_t) - \frac{1}{2}\frac{dH_0}{dt}(a_t) \\ &-\frac{1}{2}\left(\tanh(2t) + \tanh t\right) H_0(a_t) = \lambda_{\sigma_i}F_0(a_t) \end{aligned} \tag{5.5}$$

$$-\frac{1}{3}sG_0(a_t) = \lambda_{\sigma_i}G_0(a_t) \tag{5.6}$$

$$\begin{aligned} &\left(\frac{1}{2\cosh(2t)} + \frac{1}{6}\right) sH_0(a_t) - \frac{1}{2}\frac{dF_0}{dt}(a_t) \\ &-\frac{1}{2}\left(\tanh(2t) + \frac{1}{\tanh t}\right) F_0(a_t) = \lambda_{\sigma_i}H_0(a_t). \end{aligned} \tag{5.7}$$

We consider the case of $\sigma = \sigma_1$. We have $\lambda_{\sigma_1} = -\frac{1}{3}(2\nu_1 - \nu_2)$. By (5.6), if $s \neq -3\lambda_{\sigma_1} = 2\nu_1 - \nu_2$, we have $G_0(a_t) \equiv 0$. Suppose that $s = 2\nu_1 - \nu_2$. We put $x = \tanh^2(2t)$, $G_0(a_t) = G_1(x)$ in (5.6). Then the equation becomes

$$\begin{aligned} &-4x^2(1-x)^2\frac{d^2G_1}{dx^2}(x) - 4x(1-x)^2\frac{dG_1}{dx}(x) \\ &+\left\{\left(\frac{1}{4}x^2 - \frac{1}{3}x\right)s^2 - (2+\lambda_2)x + 1\right\} G_1(x) = 0. \end{aligned} \tag{5.8}$$

We put $G_1(x) = x^\alpha(1-x)^\beta \tilde{G}_1(x)\,(\alpha,\beta \in \mathbf{C})$. Then the equation becomes

$$\begin{aligned}
&-4x^2(1-x)^2\frac{d^2\tilde{G}_1}{dx^2}(x)\\
&+4x(1-x)\{-2\alpha-1+(2\alpha+2\beta+1)x\}\frac{d\tilde{G}_1}{dx}(x)\\
&+\{-4\alpha(\alpha-1)(1-x)^2-4\beta(\beta-1)x^2+8\alpha\beta x(1-x)\\
&-4\alpha(1-x)^2+4\beta x(1-x)\\
&+(\frac{1}{4}x^2-\frac{1}{3}x)s^2-(2+\lambda_2)x+1\}\tilde{G}_1(x)=0.
\end{aligned} \tag{5.9}$$

We want to divide the left hand side of (5.9) by $x(1-x)$. For this purpose, $\alpha, \beta \in \mathbf{C}$ must satisfy

$$-4\alpha^2+1=0,$$
$$-4\beta(\beta-1)-\frac{1}{12}s^2-\lambda_2-1=0.$$

The solutions are $\alpha = \pm\frac{1}{2}, \beta = \frac{2\pm\nu_2}{4}$. We choose $\alpha = \frac{1}{2}, \beta = \frac{2+\nu_2}{4}$. Then the left hand side of (5.9) can be divided by $x(1-x)$ and the equation becomes

$$\begin{aligned}
&x(x-1)\frac{d^2\tilde{G}_1}{dx^2}(x)+\left\{-2+\left(3+\frac{\nu_2}{2}\right)x\right\}\frac{d\tilde{G}_1}{dx}(x)\\
&+\frac{1}{4}(-\nu_1^2+\nu_1\nu_2+2\nu_2+4)\tilde{G}_1(x)=0.
\end{aligned} \tag{5.10}$$

This is a Gaussian hypergeometric differential equation, and its regular solution is given by

$$\tilde{G}_1(x) = {}_2F_1\left(\frac{1}{2}\nu_1+1, \frac{1}{2}\nu_2-\frac{1}{2}\nu_1+1; 2; x\right)$$

up to constant multiples. (The other solutions are not regular around $x=0$, since they contain $\log x$). Therefore, we have

$$G_0(a_t) = G_0(x) = Cx^{\frac{1}{2}}(1-x)^{\frac{2+\nu_2}{4}}\,{}_2F_1\left(\frac{1}{2}\nu_1+1, \frac{1}{2}\nu_2-\frac{1}{2}\nu_1+1; 2; x\right)$$

where C is a constant number and $x = \tanh^2(2t)$. Next , we consider the equations satisfied by F_0 and H_0. From (5.5) and (5.7), we have

$$\begin{aligned}
\frac{dF_0}{dt}(a_t) = &\left(\frac{1}{\cosh(2t)}+\frac{1}{3}\right)sH_0(a_t)\\
&-\left(\tanh(2t)+\frac{1}{\tanh t}\right)F_0(a_t)\\
&-2\lambda_{\sigma_1}H_0(a_t),
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
\frac{dH_0}{dt}(a_t) = &-\left(\frac{1}{\cosh(2t)}-\frac{1}{3}\right)sF_0(a_t)\\
&-(\tanh(2t)+\tanh t)\,H_0(a_t)\\
&-2\lambda_{\sigma_1}F_0(a_t).
\end{aligned} \tag{5.12}$$

By differentiating both sides of (5.11), (5.12) by t, we have

$$\begin{aligned}\frac{d^2F_0}{dt^2}(a_t) = &\Big\{3\tanh^2(2t) + \frac{2}{\tanh^2 t} + \frac{2\tanh(2t)}{\tanh t} - 3\\ &-\frac{4}{3}s\lambda_{\sigma_1} - \left(\frac{1}{\cosh^2(2t)} - \frac{1}{9}\right)s^2 + 4\lambda_{\sigma_1}^2\Big\} F_0(a_t)\\ &+\Big\{-\left(\frac{4\tanh(2t)}{\cosh(2t)} + \frac{2}{3}\tanh(2t) + \frac{2}{3\tanh(2t)} + \frac{2}{\sinh(2t)}\right)s\\ &+4\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)\lambda_{\sigma_1}\Big\} H_0(a_t),\end{aligned} \tag{5.13}$$

$$\begin{aligned}\frac{d^2H_0}{dt^2}(a_t) = &\Big\{3\tanh^2(2t) + 2\tanh^2 t + 2\tanh t\tanh(2t) - 3\\ &-\frac{4}{3}s\lambda_{\sigma_1} - \left(\frac{1}{\cosh^2(2t)} - \frac{1}{9}\right)s^2 + 4\lambda_{\sigma_1}^2\Big\} H_0(a_t)\\ &+\Big\{\left(\frac{4\tanh(2t)}{\cosh(2t)} - \frac{2}{3}\tanh(2t) - \frac{2}{3\tanh(2t)} + \frac{2}{\sinh(2t)}\right)s\\ &+4\left(\tanh(2t) + \frac{1}{\tanh(2t)}\right)\lambda_{\sigma_1}\Big\} F_0(a_t).\end{aligned} \tag{5.14}$$

By inserting (5.11), (5.12), (5.13), (5.14) into the Casimir equation (5.1), (5.3) to eliminate the differential terms, we have

$$\left(\frac{1}{3}s\lambda_{\sigma_1} - \lambda_{\sigma_1}^2 - \frac{1}{9}s^2 - \lambda_2\right)F_0(a_t) = 0,$$

$$\left(\frac{1}{3}s\lambda_{\sigma_1} - \lambda_{\sigma_1}^2 - \frac{1}{9}s^2 - \lambda_2\right)H_0(a_t) = 0.$$

Therefore, if the parameter $s \in \mathbf{C}$ satisfies $\frac{1}{3}s\lambda_{\sigma_1} - \lambda_{\sigma_1}^2 - \frac{1}{9}s^2 - \lambda_2 \neq 0$, then $F_0(a_t), H_0(a_t) \equiv 0$. Suppose that $\frac{1}{3}s\lambda_{\sigma_1} - \lambda_{\sigma_1}^2 - \frac{1}{9}s^2 - \lambda_2 = 0$. Since $\lambda_{\sigma_1} = -\frac{1}{3}(2\nu_1 - \nu_2)$, $\lambda_2 = -\frac{1}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2)$, we have

$$s^2 + (2\nu_1 - \nu_2)s + (\nu_1^2 - \nu_1\nu_2 - 2\nu_2^2) = 0.$$

Therefore, we have

$$s = -\nu_1 - \nu_2,\ -\nu_1 + 2\nu_2. \tag{5.15}$$

That is, (5.15) is the necessary condition of the existence of non-trivial $F_0(a_t), H_0(a_t)$. We put

$$F_0(a_t) = \cosh^{-\frac{1}{2}}(2t)(\sinh t)^{-1}\tilde{F}(t),$$
$$H_0(a_t) = \cosh^{-\frac{1}{2}}(2t)(\cosh t)^{-1}\tilde{H}(t)$$

and insert these into (5.5), (5.7). Then we have

$$-\left(\frac{1}{2\cosh(2t)} - \frac{1}{6}\right)s\frac{1}{\tanh t}\tilde{F}(t) - \frac{1}{2}\frac{d\tilde{H}}{dt}(t) = \lambda_{\sigma_1}\frac{1}{\tanh t}\tilde{F}(t),$$

$$\left(\frac{1}{2\cosh(2t)}+\frac{1}{6}\right) s\ \tanh t\tilde{H}(t)-\frac{1}{2}\frac{d\tilde{F}}{dt}(t)=\lambda_{\sigma_1}\tanh t\tilde{H}(t).$$

Next, we put $\tanh^2 t = u$, $\tilde{F}(t) = \tilde{F}_1(u)$, $\tilde{H}(t) = \tilde{H}_1(u)$. Then the above equations become

$$-\left(\frac{1-u}{2(1+u)}-\frac{1}{6}\right) s\tilde{F}_1(u)-u(1-u)\frac{d\tilde{H}_1}{du}(u)=\lambda_{\sigma_1}\tilde{F}_1(u), \tag{5.16}$$

$$-(1-u)\frac{d\tilde{F}_1}{du}(u)+\left(\frac{1-u}{2(1+u)}+\frac{1}{6}\right) s\tilde{H}_1(u)=\lambda_{\sigma_1}\tilde{H}_1(u). \tag{5.17}$$

For a while, we consider the case of $s=0$, that is, $\eta=\eta_{0,k}$ is a signature $\mathrm{sgn}_{(k)}$ of H where

$$\mathrm{sgn}_{(k)}\left(\left(\begin{array}{ccc} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & h_1 \end{array}\right)\right)=\det(H_1)^k|\det(H_1)|^{-k}, H_1=\left(\begin{array}{cc} h_{11} & h_{12} \\ h_{21} & h_{22} \end{array}\right). \tag{5.18}$$

Then, by combining (5.16), (5.17), we have

$$u(1-u)^2\lambda_{\sigma_1}\frac{d^2\tilde{F}_1}{du^2}(u)-u(1-u)\lambda_{\sigma_1}\frac{d\tilde{F}_1}{du}(u)-\lambda_{\sigma_1}^3\tilde{F}_1(u)=0.$$

Suppose that $\lambda_{\sigma_1}\neq 0$. Then the equation becomes

$$u(1-u)^2\frac{d^2\tilde{F}_1}{du^2}(u)-u(1-u)\frac{d\tilde{F}_1}{du}(u)-\lambda_{\sigma_1}^2\tilde{F}_1(u)=0.$$

We put $\tilde{F}_1(u)=(1-u)^{\lambda_{\sigma_1}}F_2(u)$. Then $F_2(u)$ satisfies

$$u(1-u)\frac{d^2F_2}{du^2}(u)-(2\lambda_{\sigma_1}+1)u\frac{dF_2}{du}(u)-\lambda_{\sigma_1}^2F_2(u)=0. \tag{5.19}$$

The equation (5.19) is the Gaussian hypergeometric differential equation. Now, since $F_2(u) = (1-u)^{-\lambda_{\sigma_1}}\cosh^{\frac{1}{2}}(2t)\sinh tF_0(a_t)$ and $F_0(a_t)$ is regular at $a_t = 1(\Leftrightarrow t=0 \Leftrightarrow u=0)$, $F_2(u)$ must be regular at $u=0$. The regular solution of (5.19) is given by

$$F_2(u)=u{}_2F_1(\lambda_{\sigma_1}+1,\lambda_{\sigma_1}+1;2;u)$$

up to constant multiples. Therefore, we have

$$\tilde{F}_1(u)=C\cdot u(1-u)^{\lambda_{\sigma_1}}{}_2F_1(\lambda_{\sigma_1}+1,\lambda_{\sigma_1}+1;2;u)\quad (C:\text{constant}).$$

Similarly, we have

$$\tilde{H}_1(u)=C'\cdot(1-u)^{\lambda_{\sigma_1}}{}_2F_1(\lambda_{\sigma_1}+1,\lambda_{\sigma_1};1;u)\quad (C':\text{constant}).$$

We want to find the relation between C and C'. By expanding $\tilde{F}_1(u)$ and $\tilde{H}_1(u)$ around $u = 0$, we have

$$\tilde{F}_1(u) = C \cdot \left(u + \frac{\lambda_{\sigma_1^2} + 1}{2} u^2 + u^3 P_1(u) \right),$$

$$\tilde{H}_1(u) = C' \cdot \left(1 + \lambda_{\sigma_1}^2 u + u^2 P_2(u) \right),$$

where $P_1(u)$ and $P_2(u)$ are the analytic functions around $u = 0$. By inserting these into the equation (5.16), we have

$$-3u(1-u^2)C' \left(\lambda_{\sigma_1}^2 + uP_3(u) \right) = 3(1+u)\lambda_{\sigma_1} C \left(u + u^2 P_4(u) \right),$$

where $P_3(u)$ and $P_4(u)$ are also the analytic functions around $u = 0$. By compairing the coefficients of u of both sides, we have

$$C = -\lambda_{\sigma_1} C'.$$

Summing up, $F_0(a_t)$ and $H_0(a_t)$ are given by

$$F_0(a_t) = -C'\lambda_{\sigma_1} \cosh^{-\frac{1}{2}}(2t)(\sinh t)^{-1} u(1-u)^{\lambda_{\sigma_1}} {}_2F_1(\lambda_{\sigma_1}+1, \lambda_{\sigma_1}+1; 2; u),$$

$$H_0(a_t) = C' \cosh^{-\frac{1}{2}}(2t)(\cosh t)^{-1}(1-u)^{\lambda_{\sigma_1}} {}_2F_1(\lambda_{\sigma_1}+1, \lambda_{\sigma_1}; 1; u).$$

(C': some constant, $u = \tanh^2 t$). In our computation, we assumed that $\lambda_{\sigma_1} \neq 0$, but the result above holds without this assumption.
Next, we consider the conditions (1), (2), (3) in (3.2). For $\Psi = {}^t(F_0, G_0, H_0) \in C^\infty_{\mathrm{sgn}_{(k)}, \tau_2}(H \backslash G / K)$, the condition (1) is equivalent to

$$(-1)^k \cdot {}^t(-F_0(a_t), G_0(a_t), -H_0(a_t)) = {}^t(F_0(a_t), G_0(a_t), H_0(a_t)).$$

This is equivalent to

$$\begin{aligned} k = 0 &\Rightarrow F_0 \equiv 0, H_0 \equiv 0, \\ k = 1 &\Rightarrow G_0 \equiv 0. \end{aligned} \tag{5.20}$$

The condition (2) is equivalent to

$${}^t(-F_0(a_t), -G_0(a_t), H_0(a_t)) = {}^t(F_0(a_t), G_0(a_t), H_0(a_t)).$$

The solutions we have always satisfy this condition. The condition (3) is equivalent to

$${}^t(\cos\theta F_0(1) + \sin\theta G_0(1), -\sin\theta F_0(1) + \cos\theta G_0(1), H_0(1)) = {}^t(F_0(1), G_0(1), H_0(1)), \tag{5.21}$$

$$\begin{aligned} &{}^t(\cos\theta F_0(1) + \sin\theta G_0(1), \sin\theta F_0(1) - \cos\theta G_0(1), -H_0(1)) \\ &= (-1)^k \cdot {}^t(F_0(1), G_0(1), H_0(1)). \end{aligned} \tag{5.22}$$

($\forall \theta \in \mathbf{R}$). Since $F_0(1) = G_0(1) = 0$, (5.21), (5.22) are equivalent to

$$k = 0 \Rightarrow H_0 \equiv 0.$$

But this condition holds if the condition (5.20) is satisfied. We have obtained a result about the Shintani functions attached to the non-spherical principal series representation $\pi_{\sigma_1,\nu}$. Note that since the transform $\nu_1 \mapsto \nu_2, \nu_2 \mapsto \nu_1$ does not change the eigenvalue of Casimir operator λ_2 and change the eigenvalue of gradient operator λ_{σ_1} to λ_{σ_2}, this transform gives the result in case of $\sigma = \sigma_2$. Similarly, the transform $\nu_1 \mapsto -\nu_1, \nu_2 \mapsto -\nu_1 + \nu_2$ gives the result in case of $\sigma = \sigma_3$. Summing up these results, we have the following theorem.

Theorem 5.6. *Let $\eta = \mathrm{sgn}_{(k)}$ ($k \in \{0,1\}$) be a signature of H defined by (5.18) and τ_2 be a three dimensional tautological representation of K , and let $\Psi = {}^t(F_0, G_0, H_0) \in C^\infty_{\mathrm{sgn}_{(k)},\tau_2}(H\backslash G/K)_{\pi_{\sigma_1,\nu}}$ be a Shintani function corresponding to the non-spherical principal series representation $\pi_{\sigma_1,\nu}$ of G. Then the restriction of Ψ to A is given as follows:*

1) *In case of $k = 0$,*

a) If $2\nu_1 - \nu_2 \neq 0$, $\Psi \equiv 0$.

b) If $2\nu_1 - \nu_2 = 0$, we have

$$\begin{pmatrix} F_0(a_t) \\ G_0(a_t) \\ H_0(a_t) \end{pmatrix} = C \cdot \begin{pmatrix} 0 \\ x^{\frac{1}{2}}(1-x)^{\frac{2+\nu_2}{4}} {}_2F_1\left(\frac{1}{2}\nu_1 + 1, \frac{1}{2}\nu_2 - \frac{1}{2}\nu_1 + 1, ; 2; x\right) \\ 0 \end{pmatrix}.$$

$$(x = \tanh^2(2t), C : \textit{some constant}).$$

2) *In case of $k = 1$,*

a) If $(-\nu_1 - \nu_2)(-\nu_1 + 2\nu_2) \neq 0$, $\Psi \equiv 0$.

b) If $(-\nu_1 - \nu_2)(-\nu_1 + 2\nu_2) = 0$, we have

$$\begin{pmatrix} F_0(a_t) \\ G_0(a_t) \\ H_0(a_t) \end{pmatrix} = C \cdot \begin{pmatrix} -\lambda_{\sigma_1}\cosh^{-\frac{1}{2}}(2t)(\sinh t)^{-1}u(1-u)^{\lambda_{\sigma_1}} {}_2F_1(\lambda_{\sigma_1}+1, \lambda_{\sigma_1}+1; 2; u) \\ 0 \\ \cosh^{-\frac{1}{2}}(2t)(\cosh t)^{-1}(1-u)^{\lambda_{\sigma_1}} {}_2F_1(\lambda_{\sigma_1}+1, \lambda_{\sigma_1}; 1; u) \end{pmatrix}.$$

$$(u = \tanh^2 t, C : \textit{some constant}).$$

Especially, in any cases, we have

$$\dim C^\infty_{\mathrm{sgn}_{(k)},\tau_2}(H\backslash G/K)_{\pi_{\sigma_1,\nu}} \leq 1.$$

The transform $\nu_1 \mapsto \nu_2, \nu_2 \mapsto \nu_1$ gives the result in case of $\sigma = \sigma_2$ and the transform $\nu_1 \mapsto -\nu_1, \nu_2 \mapsto -\nu_1 + \nu_2$ gives the result in case of $\sigma = \sigma_3$.

Next, we compute the Shintani functions $\Psi \in C^\infty_{\eta,\tau_2}(H\backslash G/K)_{\pi_{\sigma_1,\nu}}$ for general unitary character $\eta = \eta_{s,k}$ under the assumption that $1, \nu_1, \nu_2$ are linearly independent over $\mathbf{Q}$. We have already known that the necessary condition of the existence of non zero Ψ is that the parameter s is one of $2\nu_1 - \nu_2$, $2\nu_2 - \nu_1$, or $-\nu_1 - \nu_2$, and we have already solved the differential equations in case of $s = 2\nu_1 - \nu_2$. Hereafter, we suppose that the parameter s is either $2\nu_2 - \nu_1$ or

$-\nu_1-\nu_2$. We put $w=\frac{1-u}{1+u}$, $\tilde{F}_2(w)=\tilde{F}_1(u)$, $\tilde{H}_2(w)=\tilde{H}_1(u)$ in (5.16), (5.17). Then we have

$$-\left(\frac{w}{2}-\frac{1}{6}\right)s\tilde{F}_2(w)+w(1-w)\frac{d\tilde{H}_2}{dw}(w)=\lambda_{\sigma_1}\tilde{F}_2(w), \tag{5.23}$$

$$w(1+w)\frac{d\tilde{F}_2}{dw}(w)+\left(\frac{w}{2}+\frac{1}{6}\right)s\tilde{H}_2(w)=\lambda_{\sigma_1}\tilde{H}_2(w). \tag{5.24}$$

We put

$$\tilde{F}_2(w)=\sum_{n=0}^{\infty}a_nw^{n+\alpha}, \tilde{H}_2(w)=\sum_{n=0}^{\infty}c_nw^{n+\gamma}$$

($\alpha,\gamma\in\mathbf{C}$, $a_0,c_0\neq 0$) in (5.23), (5.24) and compute the power series solutions. By inserting these series into (5.23), (5.24), we have

$$\frac{s}{6}a_0w^{\alpha}+w^{\alpha+1}Q_1(w)+\gamma c_0w^{\gamma}+w^{\gamma+1}Q_2(w)=\lambda_{\sigma_1}a_0w^{\alpha}+w^{\alpha+1}Q_3(w), \tag{5.25}$$

$$\alpha a_0w^{\alpha}+w^{\alpha+1}Q_4(w)+\frac{s}{6}c_0w^{\gamma}+w^{\gamma+1}Q_5(w)=\lambda_{\sigma_1}c_0w^{\gamma}+w^{\gamma+1}Q_6(w), \tag{5.26}$$

where $Q_i(w)$ $(i=1,\cdots,6)$ are the analytic functions around $w=0$. By compairing the lowest terms in power series, easily we have $\alpha=\gamma$ (in this argument, we use the fact that $1,\nu_1,\nu_2$ are linearly independent over $\mathbf{Q}$ carefully). Therefore, from (5.25), (5.26), we have

$$\left(\frac{s}{6}-\lambda_{\sigma_1}\right)+\gamma c_0=0,$$

$$\alpha a_0+\left(\frac{s}{6}-\lambda_{\sigma_1}\right)c_0=0.$$

Since $(a_0,c_0)\neq(0,0)$, we have

$$\left(\frac{s}{6}-\lambda_{\sigma_1}\right)^2-\alpha\gamma=0.$$

By combining this and $\alpha=\gamma$, we have

$$\alpha=\gamma=\pm\left(\frac{s}{6}-\lambda_{\sigma_1}\right).$$

Hereafter, we put $A=A(\nu_1,\nu_2)=\frac{s}{6}-\lambda_{\sigma_1}$. Then $\tilde{F}_2(w)$ (resp. $\tilde{H}_2(w)$) are expressed by the linear combination of some power series $\sum_{n=0}^{\infty}a_nw^{n+A}$, $\sum_{n=0}^{\infty}a_n'w^{n-A}$ (resp. $\sum_{n=0}^{\infty}c_nw^{n+A}$, $\sum_{n=0}^{\infty}c_n'w^{n-A}$). That is, there exist common constants C_+, C_- such that

$$\tilde{F}_2(w)=C_+\sum_{n=0}^{\infty}a_nw^{n+A}+C_-\sum_{n=0}^{\infty}a_n'w^{n-A},$$

$$\tilde{H}_2(w) = C_+ \sum_{n=0}^{\infty} c_n w^{n+A} + C_- \sum_{n=0}^{\infty} c_n' w^{n-A}.$$

By inserting

$$\tilde{F}_2(w) = \sum_{n=0}^{\infty} a_n w^{n+A}, \tilde{H}_2(w) = \sum_{n=0}^{\infty} c_n w^{n+A}$$

into the equations (5.23), (5.24) and picking up the coefficients of w^{n+A}, we have the following recurrence relations:

$$-\frac{s}{2}a_{n-1} + Aa_n - (n+A-1)c_{n-1} + (n+A)c_n = 0, \tag{5.27}$$

$$(n+A-1)a_{n-1} + (n+A)a_n + \frac{s}{2}c_{n-1} + Ac_n = 0. \tag{5.28}$$

for all $n \geq 0$. Here, we assume that $a_l = c_l = 0$ if $l < 0$. From (5.27), (5.28), easily we have $c_n = (-1)^{n+1}a_n$ for all $n \geq 0$ by induction. Therefore, by inserting $c_n = (-1)^{n+1}a_n$, $c_{n-1} = (-1)^n a_{n-1}$ into the equation (5.27), we have

$$\left(A + (n+A)(-1)^{n+1}\right) a_n = \left(\frac{s}{2} + (n+A-1)(-1)^n\right) a_{n-1}.$$

Thus we have

$$a_n = \left\{ \prod_{k=1}^{n} \frac{\frac{s}{2} + (n+A-1)(-1)^k}{A + (k+A)(-1)^{k+1}} \right\} a_0, \tag{5.29}$$

$$c_n = (-1)^{n+1} \left\{ \prod_{k=1}^{n} \frac{\frac{s}{2} + (n+A-1)(-1)^k}{A + (k+A)(-1)^{k+1}} \right\} a_0. \tag{5.30}$$

Similarly, if the characteristic roots are $\alpha = \gamma = -A$, by inserting

$$\tilde{F}_2(w) = \sum_{n=0}^{\infty} a_n' w^{n-A}, \tilde{H}_2(w) = \sum_{n=0}^{\infty} c_n' w^{n-A}$$

into the equations (5.23), (5.24), we have

$$a_n' = \left\{ \prod_{k=1}^{n} \frac{\frac{s}{2} + (n-A-1)(-1)^k}{A + (k-A)(-1)^{k+1}} \right\} a_0, \tag{5.31}$$

$$c_n' = (-1)^n \left\{ \prod_{k=1}^{n} \frac{\frac{s}{2} + (n-A-1)(-1)^k}{A + (k-A)(-1)^{k+1}} \right\} a_0. \tag{5.32}$$

$(\forall n \geq 1)$. From (5.29), a_{2n} and a_{2n-1} are expressed by

$$a_{2n} = \frac{\left(\frac{s}{4} + \frac{A}{2} + \frac{1}{2}\right)_n \left(-\frac{s}{4} + \frac{A}{2}\right)_n}{n! \left(A + \frac{1}{2}\right)_n} a_0,$$

$$a_{2n-1} = -\frac{\left(\frac{s}{4} + \frac{A}{2} + \frac{1}{2}\right)_{n-1} \left(-\frac{s}{4} + \frac{A}{2}\right)_n}{(n-1)! \left(A + \frac{1}{2}\right)_n} a_0.$$

Here, for $\alpha \in \mathbf{C}$, $n \in \mathbf{Z}_{>0}$, we define $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$ and $(\alpha)_0 := 1$. Therefore, if we normalize $a_0 = 1$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{2n} w^{2n} =& \sum_{n=0}^{\infty} \frac{\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}\right)_n \left(-\frac{s}{4}+\frac{A}{2}\right)_n}{n!\left(A+\frac{1}{2}\right)_n} w^{2n} \\ =& {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right),\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} a_{2n-1} w^{2n-1} =& -\frac{-\frac{s}{4}+\frac{A}{2}}{A+\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}\right)_{n-1} \left(-\frac{s}{4}+\frac{A}{2}+1\right)_{n-1}}{(n-1)!\left(A+\frac{3}{2}\right)_{n-1}} w^{2n-2} \cdot w \\ =& \frac{\frac{s}{4}-\frac{A}{2}}{A+\frac{1}{2}} w \sum_{n=0}^{\infty} \frac{\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}\right)_n \left(-\frac{s}{4}+\frac{A}{2}+1\right)_n}{n!\left(A+\frac{3}{2}\right)_n} w^{2n} \\ =& \frac{\frac{s}{4}-\frac{A}{2}}{A+\frac{1}{2}} w\, {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right).\end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_n w^{n+A} =& w^A \left\{ {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right) \right. \\ & \left. + \frac{\frac{s}{4}-\frac{A}{2}}{A+\frac{1}{2}} w\, {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right) \right\}.\end{aligned} \tag{5.33}$$

And since $c_n = (-1)^{n+1} a_n$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} c_n w^{n+A} =& w^A \left\{ -{}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right) \right. \\ & \left. + \frac{\frac{s}{4}-\frac{A}{2}}{A+\frac{1}{2}} w\, {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right) \right\}.\end{aligned} \tag{5.34}$$

Similarly, by using

$$a'_{2n} = \frac{\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}\right)_n \left(\frac{s}{4}-\frac{A}{2}\right)_n}{n!\left(-A+\frac{1}{2}\right)_n} a'_0,$$

$$a'_{2n-1} = -\frac{\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}\right)_{n-1} \left(\frac{s}{4}-\frac{A}{2}\right)_n}{(n-1)!\left(-A+\frac{1}{2}\right)_n} a'_0$$

and $c'_n = (-1)^n a'_n$, if we normalize $a'_0 = 1$, we have

$$\sum_{n=0}^{\infty} a_n' w^{n-A} = w^{-A}\left\{ {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}; -A+\frac{1}{2}; w^2\right) \right.$$
$$\left. - \frac{\frac{s}{4}-\frac{A}{2}}{-A+\frac{1}{2}} w\, {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}+1; -A+\frac{3}{2}; w^2\right)\right\}, \tag{5.35}$$

$$\sum_{n=0}^{\infty} c_n' w^{n-A} = w^{-A}\left\{ {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}; -A+\frac{1}{2}; w^2\right) \right.$$
$$\left. + \frac{\frac{s}{4}-\frac{A}{2}}{-A+\frac{1}{2}} w\, {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}+1; -A+\frac{3}{2}; w^2\right)\right\}. \tag{5.36}$$

Therefore, $\tilde{F}_2(w)$ and $\tilde{H}_2(w)$ are expressed as follows:

$$\begin{aligned}\tilde{F}_2(w) = {} & C_+ w^{A}\left\{ {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right)\right. \\ & \left. + \frac{\frac{s}{4}-\frac{A}{2}}{A+\frac{1}{2}} w\, {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right)\right\} \\ & + C_- w^{-A}\left\{ {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}; -A+\frac{1}{2}; w^2\right)\right. \\ & \left. - \frac{\frac{s}{4}-\frac{A}{2}}{-A+\frac{1}{2}} w\, {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}+1; -A+\frac{3}{2}; w^2\right)\right\},\end{aligned} \tag{5.37}$$

$$\begin{aligned}\tilde{H}_2(w) = {} & C_+ w^{A}\left\{ -{}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right)\right. \\ & \left. + \frac{\frac{s}{4}-\frac{A}{2}}{A+\frac{1}{2}} w\, {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right)\right\} \\ & + C_- w^{-A}\left\{ {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}; -A+\frac{1}{2}; w^2\right)\right. \\ & \left. + \frac{\frac{s}{4}-\frac{A}{2}}{-A+\frac{1}{2}} w\, {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}+1; -A+\frac{3}{2}; w^2\right)\right\}.\end{aligned} \tag{5.38}$$

We want the relation between C_+ and C_-. We can find the relation by using the regularity of the Shintani function and the asymptotic formula of Gaussian hypergeometric function ${}_2F_1$. Now, since $F_0(a_t)$ and $H_0(a_t)$ are regular around $t = 0\,(\Leftrightarrow w = 1)$, and since

$$\tilde{F}(t) = \cosh^{\frac{1}{2}}(2t)\sinh t F_0(a_t),$$

$$\tilde{H}(t) = \cosh^{\frac{1}{2}}(2t)\cosh t H_0(a_t),$$

$\tilde{F}_2(w) = \tilde{F}(t)$ and $\tilde{H}_2(w) = \tilde{H}(t)$ must be regular around $w = 1$ and $\tilde{F}_2(w)$ must satisfy $\tilde{F}_2(1) = 0$. Since all hypergeometric functions appearing in the right hand sides of (5.37), (5.38) are in the form of ${}_2F_1(a, b; a+b; w^2)$, to investigate the behavior of the right hand sides of (5.37), (5.38) around $w = 1$, we use the following asymptotic formula:

Formula 1. We have

$${}_2F_1(a, b; a+b; z) = -\frac{\Gamma(a+b)(\log(1-z) + \psi(a) + \psi(b) + 2\gamma)}{\Gamma(a)\Gamma(b)}(1 + O(z-1))$$
$$(z \to 1).$$

Here, γ is the Euler constant and ψ is defined by

$$\psi(z) = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z-1}\right) - \gamma.$$

We apply this formula to the right hand sides of (5.37), (5.38). Firstly, the coefficients of $\log(1-w^2)$ of $\tilde{F}_2(w)$ equals

$$\begin{aligned}
&C_+\left\{-\frac{\Gamma(A+\frac{1}{2})}{\Gamma(\frac{s}{4}+\frac{A}{2}+\frac{1}{2})\Gamma(-\frac{s}{4}+\frac{A}{2})} - \frac{\frac{s}{4}-\frac{A}{2}}{A+\frac{1}{2}}\frac{\Gamma(A+\frac{3}{2})}{\Gamma(\frac{s}{4}+\frac{A}{2}+\frac{1}{2})\Gamma(-\frac{s}{4}+\frac{A}{2}+1)}\right\}\\
&+C_-\left\{-\frac{\Gamma(-A+\frac{1}{2})}{\Gamma(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2})\Gamma(\frac{s}{4}-\frac{A}{2})} + \frac{\frac{s}{4}-\frac{A}{2}}{-A+\frac{1}{2}}\frac{\Gamma(-A+\frac{3}{2})}{\Gamma(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2})\Gamma(\frac{s}{4}-\frac{A}{2}+1)}\right\}\\
&= C_+\cdot 0 + C_-\cdot 0 = 0.
\end{aligned}$$

Therefore, $\tilde{F}_2(w)$ is regular around $w = 1$ regardless of the values of C_+, C_-. Next, the coefficients of $\log(1-w^2)$ of $\tilde{H}_2(w)$ equals

$$\begin{aligned}
&C_+\left\{\frac{\Gamma(A+\frac{1}{2})}{\Gamma(\frac{s}{4}+\frac{A}{2}+\frac{1}{2})\Gamma(-\frac{s}{4}+\frac{A}{2})} - \frac{\frac{s}{4}-\frac{A}{2}}{A+\frac{1}{2}}\frac{\Gamma(A+\frac{3}{2})}{\Gamma(\frac{s}{4}+\frac{A}{2}+\frac{1}{2})\Gamma(-\frac{s}{4}+\frac{A}{2}+1)}\right\}\\
&+C_-\left\{-\frac{\Gamma(-A+\frac{1}{2})}{\Gamma(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2})\Gamma(\frac{s}{4}-\frac{A}{2})} - \frac{\frac{s}{4}-\frac{A}{2}}{-A+\frac{1}{2}}\frac{\Gamma(-A+\frac{3}{2})}{\Gamma(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2})\Gamma(\frac{s}{4}-\frac{A}{2}+1)}\right\}\\
&= \frac{2\Gamma(A+\frac{1}{2})}{\Gamma(\frac{s}{4}+\frac{A}{2}+\frac{1}{2})\Gamma(-\frac{s}{4}+\frac{A}{2})}C_+ - \frac{2\Gamma(-A+\frac{1}{2})}{\Gamma(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2})\Gamma(\frac{s}{4}-\frac{A}{2})}C_-.
\end{aligned}$$

Since this coefficient must be 0, we have

$$C_+ = \frac{\Gamma(-A+\frac{1}{2})}{\Gamma(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2})\Gamma(\frac{s}{4}-\frac{A}{2})},$$

$$C_- = \frac{\Gamma(A+\frac{1}{2})}{\Gamma(\frac{s}{4}+\frac{A}{2}+\frac{1}{2})\Gamma(-\frac{s}{4}+\frac{A}{2})}$$

up to (the same) constant multiples. Note that we can easily verify that $\tilde{F}_2(1) = 0$ for these C_+, C_- by using formulas $\Gamma(z+1) = z\Gamma(z)$, $\psi(z+1) = \psi(z) + \frac{1}{z}$. Now, we have completely determined $\tilde{F}_2(w)$, $\tilde{H}_2(w)$. We have

$$\begin{aligned}
\tilde{F}_2(w) = & \frac{\Gamma(-A+\frac{1}{2})}{\Gamma(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2})\Gamma(\frac{s}{4}-\frac{A}{2})} w^A \left\{ {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right) \right. \\
& \left. + \frac{\frac{s}{4}-\frac{A}{2}}{A+\frac{1}{2}} w {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right)\right\} \\
& + \frac{\Gamma(A+\frac{1}{2})}{\Gamma(\frac{s}{4}+\frac{A}{2}+\frac{1}{2})\Gamma(-\frac{s}{4}+\frac{A}{2})} w^{-A} \left\{ {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}; -A+\frac{1}{2}; w^2\right) \right. \\
& \left. - \frac{\frac{s}{4}-\frac{A}{2}}{-A+\frac{1}{2}} w {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}+1; -A+\frac{3}{2}; w^2\right)\right\},
\end{aligned} \tag{5.39}$$

$$\begin{aligned}
\tilde{H}_2(w) = & \frac{\Gamma(-A+\frac{1}{2})}{\Gamma(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2})\Gamma(\frac{s}{4}-\frac{A}{2})} w^A \left\{ -{}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}; A+\frac{1}{2}; w^2\right) \right. \\
& \left. + \frac{\frac{s}{4}-\frac{A}{2}}{A+\frac{1}{2}} w {}_2F_1\left(\frac{s}{4}+\frac{A}{2}+\frac{1}{2}, -\frac{s}{4}+\frac{A}{2}+1; A+\frac{3}{2}; w^2\right)\right\} \\
& + \frac{\Gamma(A+\frac{1}{2})}{\Gamma(\frac{s}{4}+\frac{A}{2}+\frac{1}{2})\Gamma(-\frac{s}{4}+\frac{A}{2})} w^{-A} \left\{ {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}; -A+\frac{1}{2}; w^2\right) \right. \\
& \left. + \frac{\frac{s}{4}-\frac{A}{2}}{-A+\frac{1}{2}} w {}_2F_1\left(-\frac{s}{4}-\frac{A}{2}+\frac{1}{2}, \frac{s}{4}-\frac{A}{2}+1; -A+\frac{3}{2}; w^2\right)\right\}
\end{aligned} \tag{5.40}$$

up to (the same) constant multiples. Here, $A = A(\nu_1, \nu_2) = \frac{s}{6} - \lambda_{\sigma_1}$, $\lambda_{\sigma_1} = -\frac{1}{3}(2\nu_1 - \nu_2)$, $w = \frac{1-u}{1+u} = \frac{1-\tanh^2 t}{1+\tanh^2 t}$, $\tilde{F}_2(w) = \tilde{F}(t) = \cosh^{\frac{1}{2}}(2t)\sinh t F_0(a_t)$, $\tilde{H}_2(w) = \tilde{H}(t) = \cosh^{\frac{1}{2}}(2t)\cosh t H_0(a_t)$. By the same argument we have done when η is the signature, we can easily verify that the conditions (1), (2), (3) in (3.2) are equivalent to the condition (5.20). We have already computed $F_0(a_t)$, $G_0(a_t)$ and $H_0(a_t)$ in any cases. Summing up, we obtain the following theorem:

Theorem 5.7. *Assume that $1, \nu_1, \nu_2$ are linearly independent over* $\mathbf{Q}$. *Let $\eta = \eta_{s,k}$ be a unitary character of H defined by (3.1) and $\sigma = \sigma_1$ be the character of M. Then the necessary condition of the existence of the non-trivial Shintani functions attached to the non-spherical principal series representation $\pi_{\sigma_1,\nu}$ is that*

$$k = 0 \quad \text{and} \quad s = 2\nu_1 - \nu_2$$

or

$$k = 1 \quad \text{and} \quad s = -\nu_1 - \nu_2 \quad \text{or} \quad -\nu_1 + 2\nu_2.$$

That is, if the condition above is not satisfied, we have

$$C^{\infty}_{\eta,\tau_2}(H\backslash G/K)_{\pi_{\sigma_1,\nu}} = \{0\}.$$

Let $\Psi(a_t) = {}^t(F_0(a_t), G_0(a_t), H_0(a_t)) \in C^{\infty}_{\eta,\tau_2}(H\backslash G/K)_{\pi_{\sigma_1,\nu}}|_A$ *and suppose that the condition above is satisfied. Then*
1) *If* $k = 0$ *and* $s = 2\nu_1 - \nu_2$, *we have*

$$\begin{pmatrix} F_0(a_t) \\ G_0(a_t) \\ H_0(a_t) \end{pmatrix} = C \cdot \begin{pmatrix} 0 \\ x^{\frac{1}{2}}(1-x)^{\frac{2+\nu_2}{4}} {}_2F_1\left(\frac{1}{2}\nu_1 + 1, \frac{1}{2}\nu_2 - \frac{1}{2}\nu_1 + 1, ; 2; x\right) \\ 0 \end{pmatrix}.$$

$$(x = \tanh^2(2t), C : \textit{some constant}).$$

2) *If* $k = 1$ *and* $s = -\nu_1 - \nu_2$ *or* $-\nu_1 + 2\nu_2$, *we have*

$$\begin{pmatrix} F_0(a_t) \\ G_0(a_t) \\ H_0(a_t) \end{pmatrix} = C \cdot \begin{pmatrix} \cosh^{-\frac{1}{2}}(2t)\sinh^{-1}t\tilde{F}_2(w) \\ 0 \\ \cosh^{-\frac{1}{2}}(2t)\cosh^{-1}t\tilde{H}_2(w) \end{pmatrix}.$$

where C *is some constant and* $\tilde{F}_2(w)$ *and* $\tilde{H}_2(w)$ *are the functions given by* (5.39), (5.40). *Especially, in any cases, we have*

$$\dim C^{\infty}_{\eta,\tau_2}(H\backslash G/K)_{\pi_{\sigma_1,\nu}} \leq 1.$$

The transform $\nu_1 \mapsto \nu_2, \nu_2 \mapsto \nu_1$ *gives the result in case of* $\sigma = \sigma_2$ *and the transform* $\nu_1 \mapsto -\nu_1, \nu_2 \mapsto -\nu_1 + \nu_2$ *gives the result in case of* $\sigma = \sigma_3$.

Remark 5.8. By using the relation $1 - x = w^2$ and the formulas of the hypergeometric function

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} {}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\alpha)\Gamma(\beta - \alpha)}{\Gamma(\gamma - \alpha)}(1-z)^{-\alpha} {}_2F_1(\alpha, \gamma - \beta; \alpha - \beta + 1; \frac{1}{1-z})$$
$$+ \frac{\Gamma(\beta)\Gamma(\alpha - \beta)}{\Gamma(\gamma - \beta)}(1-z)^{-\beta} {}_2F_1(\beta, \gamma - \alpha; \beta - \alpha + 1; \frac{1}{1-z}),$$

$$\begin{aligned} {}_2F_1(\alpha, \beta, \gamma; z) &= (1-z)^{-\alpha} {}_2F_1(\alpha, \gamma - \beta; \gamma; \frac{z}{z-1}) \\ &= (1-z)^{\gamma - \alpha - \beta} {}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma; z), \end{aligned}$$

we can rewrite $\tilde{F}_2(w)$, $\tilde{H}_2(w)$ in Theorem 5.7 as functions in $x = \tanh^2(2t)$. We put $\tilde{F}_2(w) = F_3(x)$, $\tilde{H}_2(w) = H_3(x)$. Then we have

$$
\begin{aligned}
F_3(x) = \left(-\frac{s}{4}+\frac{A}{2}\right)&\left\{-(1-x)^{\frac{A}{2}}{}_2F_1\left(\frac{s}{4}+\frac{A+1}{2},-\frac{s}{4}+\frac{A}{2};1;x\right)\right.\\
&\left.+(1-x)^{\frac{A+1}{2}}{}_2F_1\left(\frac{s}{4}+\frac{A+1}{2},-\frac{s}{4}+\frac{A}{2}+1;1;x\right)\right\},\\
H_3(x) = \left(-\frac{s}{4}+\frac{A}{2}\right)&\left\{(1-x)^{\frac{A}{2}}{}_2F_1\left(\frac{s}{4}+\frac{A+1}{2},-\frac{s}{4}+\frac{A}{2};1;x\right)\right.\\
&\left.+(1-x)^{\frac{A+1}{2}}{}_2F_1\left(\frac{s}{4}+\frac{A+1}{2},-\frac{s}{4}+\frac{A}{2}+1;1;x\right)\right\}.
\end{aligned}
$$

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II. The matrix coefficients with minimal K-types of the spherical and non-spherical principal series representations of $SL(3,\mathbf{R})$

ABSTRACT: We compute the holonomic system of rank 6 for the radial part of the matrix coefficients of spherical and non-spherical principal series representations of $SL(3,\mathbf{R})$. We obtain six power series solutions corresponding to the set of six characteristic roots, and express the matrix coefficients by linear combinations of these power series. Among others, the c-functions of non-spherical principal series are obtained.

1 Introduction

It is a classical result to have the matrix coefficient of the spherical (or class one) principal series of a semisimple real Lie group as a linear combination of asymptotic power series solutions [7]. Among others there appear c-functions as the coefficients of this linear combination, which are obtained as certain explicit products of the gamma factors. The functions of this kind were firstly computed explicitly by G. Schiffmann [7] only for the case of class one representations (see also chap.9 of Warner [8]). It is generally believed that we have a similar formula for non-spherical case. But in spite of this folklore, we find few references even for small Lie groups. One of the reasons is that the inductive argument used in [7] does not work for non-spherical case.

In this paper, we investigate the matrix coefficients of the spherical and non-spherical principal series representations of $G = SL(3,\mathbf{R})$. Let us introduce the contents of this paper. We take $K = SO(3,\mathbf{R})$ as a maximal compact subgroup of G. Given a principal series representation (π, H_π) of G, the matrix coefficients of π are regarded as the elements of the space of the spherical functions defined by

$$C^\infty_{\eta,\tau}(K\backslash G/K)$$
$$:= \{\phi : G \to V_\eta \otimes V_\tau \,|\, \phi(k_L g k_R^{-1}) = (\eta(k_L) \otimes \tau(k_R))\,\phi(g), k_L, k_R \in K, g \in G\}$$

with some finite dimensional representations (η, V_η), (τ, V_τ) of K. We consider the following two cases. If π is the spherical principal series representation, then we take $\eta = \tau = \mathbf{1}$, the trivial representation of K, and if π is the non-spherical principal series representation, we take $\eta = \tau = \tau_2$, the three dimensional tautological representation of K. In other words, we restrict the vectors of the matrix coefficients to the elements of the minimal K-type of π.

The basic method is to construct and investigate the differential equations satisfied by the spherical functions we mentioned above. Let $\mathfrak{g}$ be the Lie algebra of G and $U(\mathfrak{g})$ its universal enveloping algebra. The spherical function attached to the class one principal series is completely determined by the actions of the Capelli elements Cp_2, Cp_3, which are the generators of the center of $U(\mathfrak{g})$ together with the regularity at the identity of G. This part is more or less a classical fact. Meanwhile, to investigate the spherical functions attached to the non-spherical principal series representations, we construct the following two kinds of differential equations: (a) the differential equations characterized by the action of the Casimir element of degree two (this is also one of the generators of the center of $U(\mathfrak{g})$), (b) the differential equations characterized by the action of the gradient operator (or the Dirac-Schmid operator). We have three different non-spherical principal series with the same infinitesimal characters $Z(\mathfrak{g}) \to \mathbf{C}$. We cannot distinguish them only by the elements of $Z(\mathfrak{g})$. This is the reason we need the gradient operator which has distinct eigenvalues for different non-spherical principal series. This part seems to be new. In both cases, we obtain six power series solutions corresponding to the six characteristic roots.

The main theorems in this paper are Theorem 5.6 and Theorem 6.6, which give the exact power series expansions of the matrix coefficients of the spherical and non-spherical principal series representations (Since the explicit forms of the coefficients of the power series solutions are quite complicated, they are introduced in the "Appendix", section 7 of this paper). We express the matrix coefficients by the linear combinations of the power series solutions. The method we use is classical (for example, introduced in [3]). We investigate a part of the monodromy data of our holonomic system to have the unique solutions invariant under the fundamental group of the regular part of the split Cartan subgroup in $SL(3, \mathbf{R})$.

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2 Preliminaries

2.1 Notation

Let $G = SL(3, \mathbf{R})$ and fix $K = SO(3, \mathbf{R})$ as a maximal compact subgroup of G, and set $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{sl}(3, \mathbf{R}), \mathfrak{k} = \mathrm{Lie}(K) = \mathfrak{so}(3)$. Put

$$A := \left\{ \mathrm{diag}(a_1, a_2, a_3) \in G \;\middle|\; \prod_{i=1}^{3} a_i = 1, a_i \in \mathbf{R}_{>0} \right\}$$

and set $\mathfrak{a} = \mathrm{Lie}(A)$. The Cartan involution $\theta : G \to G$ is defined by $g \mapsto ({}^t g)^{-1}$ ($g \in G$), and its Lie algebra version is $\theta : \mathfrak{g} \to \mathfrak{g}, X \mapsto -{}^t X$. Then

$$K = G^{\theta} = \{ g \in G \,|\, \theta(g) = g \}$$

and

$$\mathfrak{k} = \mathfrak{g}^{\theta} = \{ X \in \mathfrak{g} \,|\, \theta(X) = X \}.$$

Put

$$\mathfrak{p} = \mathfrak{g}^{-\theta} = \{X \in \mathfrak{g} \,|\, \theta(X) = -X\}.$$

Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, called the Cartan decomposition. Let $E_{i,j}$ $(1 \le i,j \le 3)$ be the matrix unit with 1 at the (i,j)-th entry and 0 at other entries. Put

$$H_{i,j} := E_{i,i} - E_{j,j} \in \mathfrak{a}\, (i \neq j).$$

Put $X_{i,j} = E_{i,j} + E_{j,i}\, (i \neq j) \in \mathfrak{p}$ and $K_{i,j} = E_{i,j} - E_{j,i}\, (i \neq j) \in \mathfrak{k}$.

2.2 The principal series representations

Let P_0 be a minimal parabolic subgroup of G given by the upper triangular matrices in G, and $P_0 = MAN$ be the Langlands decomposition of P_0 with $M = K \cap \{\text{diagonals in}\, G\}$, and

$$N = \left\{ \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \in G \,\middle|\, x_i \in \mathbf{R}, i = 1,2,3 \right\}.$$

To define a principal series representation with respect to the minimal parabolic subgroup P_0 of G, we firstly fix a character σ of M and a linear form $\nu \in \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C} = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$. We write $\nu(\mathrm{diag}(t_1,t_2,t_3)) = \nu_1 t_1 + \nu_2 t_2$. Then we can define a representation $\sigma \otimes a^{\nu}$ of MA, and extend this to P_0 by the identification $P_0/N \simeq MA$. Then we set

$$\pi_{\sigma,\nu} = C^{\infty}\mathrm{Ind}_{P_0}^{G}(\sigma \otimes a^{\nu+\rho} \otimes 1_N).$$

Here ρ is the half sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$ given by $a^{\rho} = a_1^2 a_2$, for $a = \mathrm{diag}(a_1, a_2, a_3) \in A$. The representation space is

$$C^{\infty}_{(M,\sigma)}(K) = \{f \in C^{\infty}(K) | f(mk) = \sigma(m) f(k), m \in M, k \in K\}$$

and the action of G is defined by

$$(\pi(x)f)(k) = a(kx)^{\nu+\rho} f(\kappa(kx))\, (x \in G, k \in K).$$

Here, for $g \in G$, $g = n(g)a(g)\kappa(g)$ $(n(g) \in N, a(g) \in A, \kappa(g) \in K)$ is the Iwasawa decomposition. Next, we define characters σ_j $(j = 0,1,2,3)$ of M as follows. The group M consisting of four elements is a finite abelian group of (2,2)-type, and its elements except for the unity are given by

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since M is commutative, all irreducible unitary representations of M are 1-dimensional. For any $\sigma \in \widehat{M}$, we have $\sigma^2 = 1$. Therefore, the set $\widehat{M}$ consisting of 4 characters $\{\sigma_j | j = 0,1,2,3\}$, where each σ_j, except for the trivial character σ_0, is specified by the following table of values at the elements m_i $(i = 1,2,3)$.

	m_1	m_2	m_3
σ_1	1	-1	-1
σ_2	-1	1	-1
σ_3	-1	-1	1

The correspondence of a character of M and the minimal K-type of $\pi_{\sigma,\nu}$ is as follows ([6]).

Proposition 2.1. 1) *If σ is the trivial character of M, the representation $\pi_{\sigma,\nu}$ is spherical or class one. That is, it has a unique K-invariant vecter in $H_{\sigma,\nu}$.*
2) *If σ is not trivial, the minimal K-type of the restriction $\pi_{\sigma,\nu}|_K$ to K is a 3-dimensional representation of K, which is isomorphic to the unique standard one (τ_2, V_2). In this case, we call $\pi_{\sigma,\nu}$ the non-spherical principal series representation. The multiplicity of this minimal K-type is one:*

$$\dim_{\mathbf{C}}\mathrm{Hom}_K(\tau_2, H_{\sigma,\nu}) = 1.$$

2.3 The definition of spherical functions

Let (π, H_π) be the principal series representation of $G = SL(3, \mathbf{R})$. We want to study the matrix coefficient

$$\Phi_{w,v} : G \to \mathbf{C}, \quad g \mapsto \langle w, \pi(g)v\rangle \quad (w \in H_\pi^*, v \in H_\pi).$$

Let (τ_L, V_L) be a K-type of H_π^* and (τ_R, V_R) be a K-type of H_π. And let $\iota : \tau_L \boxtimes \tau_R \to \pi^* \boxtimes \pi$ be a $K \times K$ embedding. The bilinear form $(w, v) \mapsto \Phi_{w,v}$ is the element of $\mathrm{Hom}_{G\times G}(H_\pi^* \otimes H_\pi, C^\infty(G))$. We define a homomorphism $\mathrm{Hom}_{G\times G}(H_\pi^* \otimes H_\pi, C^\infty(G)) \to \mathrm{Hom}_{K\times K}(V_L \otimes V_R, C^\infty(G))$ by $\Phi \mapsto \Phi \circ \iota$. The space $\mathrm{Hom}_{K\times K}(V_L \otimes V_R, C^\infty(G))$ is identified with a space
$C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K)$
$:= \{F : G \to V_L^* \otimes V_R^* | F(k_1 g k_2) = (\tau_L^* \boxtimes \tau_R^*)(k_1, k_2^{-1})F(g),\ k_1, k_2 \in K, g \in G\}$
by the correspondence

$$\langle F_\phi(g), v_1 \otimes v_2\rangle = \phi(v_1 \otimes v_2)(g) \quad (\forall(v_1, v_2) \in V_L \times V_R)$$

for $\phi \in \mathrm{Hom}_{K\times K}(V_L \otimes V_R, C^\infty(G))$, $F_\phi \in C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K)$. Hence the matrix coefficient

$$\phi(v_1 \otimes v_2)(g) = \langle v_1, \pi(g)v_2\rangle$$

with $v_1 \in V_L$, $v_2 \in V_R$ is determined by $F_\phi \in C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K)$. Elements of $C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K)$ are called spherical functions. In particular, the element of $C^\infty_{\tau_L^*,\tau_R^*}(K\backslash G/K)$ which is realized as the image of the bilinear form $(w, v) \mapsto \Phi_{w,v}$ $(w \in H_\pi^*, v \in H_\pi)$ is called the spherical function attached to the principal series representation π. Because of the Cartan double coset decomposition $G = KAK$, spherical functions are determined by their restriction to A.

3 The double coset Cartan decomposition

Because G has the double coset decomposition $G = KAK$, we consider the decomposition of the standard elements in $\mathfrak{p}$ with respect to the double coset decomposition:

$$\mathfrak{g} = \mathrm{Ad}(a^{-1})\mathfrak{k} + \mathfrak{a} + \mathfrak{k}.$$

Here $a \in A$ is a regular element in A. For $x \in \mathbf{R}_{>0}$, put $sh(x) = \frac{1}{2}(x - \frac{1}{x})$, $ch(x) = \frac{1}{2}(x + \frac{1}{x})$. We have the following decomposition:

Lemma 3.1. *We have*

$$X_{i,j} = -\frac{1}{sh(\frac{a_i}{a_j})}\mathrm{Ad}(a^{-1})K_{i,j} + 0 + \frac{ch(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_j})}K_{i,j} \quad ;$$

$$H_{i,j} = 0 + H_{i,j} + 0$$

with respect to the decomposition $\mathfrak{g} = \mathrm{Ad}(a^{-1})\mathfrak{k} + \mathfrak{a} + \mathfrak{k}$.

4 The $(\mathfrak{g}, K)$-modules of principal series representations

4.1 The Capelli elements

The center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ has two independent generators, and they are obtained as Capelli elements because $\mathfrak{g} = \mathfrak{sl}_3$ is of type A_2 (see [2]). For $i = 1, 2, 3$, we put

$$E'_{i,i} = E_{i,i} - \frac{1}{3}\left(\sum_{k=1}^{3} E_{k,k}\right).$$

The following proposition ([6], Proposition 3.1) gives the explicit description of the independent generators of $Z(\mathfrak{g})$.

Proposition 4.1. *The independent generators* $\{Cp_2, Cp_3\}$ *of* $Z(\mathfrak{g})$ *are given as follows:*

$$\begin{aligned} Cp_2 =& (E'_{1,1} - 1)E'_{2,2} + E'_{2,2}(E'_{3,3} + 1) + (E'_{1,1} - 1)(E'_{3,3} + 1) \\ & - E_{2,3}E_{3,2} - E_{1,3}E_{3,1} - E_{1,2}E_{2,1} \\ Cp_3 =& (E'_{1,1} - 1)E'_{2,2}(E'_{3,3} + 1) + E_{1,2}E_{2,3}E_{3,1} + E_{1,3}E_{2,1}E_{3,2} \\ & - (E'_{1,1} - 1)E_{2,3}E_{3,2} - E_{1,3}E'_{2,2}E_{3,1} - E_{1,2}E_{2,1}(E'_{3,3} + 1). \end{aligned}$$

4.2 Reduction of Capelli elements

To compute the actions of Capelli elements on spherical functions attached to the spherical principal series, we may regard the above two elements as elements in $Z(\mathfrak{g})\,(\mathrm{mod} U(\mathfrak{g})\,\mathfrak{k})$, because these functions are annihilated by the right action of $\mathfrak{k}$. Cp_2, Cp_3 $(\mathrm{mod} U(\mathfrak{g})\mathfrak{k})$ are given in [6]. They are as follows:

Lemma 4.2. *The Capelli elements Cp_2,Cp_3 satisfy the following congruences:*

$$\begin{aligned}
Cp_2 \equiv & (E_{1,1}' - 1)E_{2,2}' + E_{2,2}'(E_{3,3}' + 1) + (E_{1,1}' - 1)(E_{3,3}' + 1) \\
& - E_{2,3}^2 - E_{1,3}^2 - E_{1,2}^2 \quad (\mathrm{mod}\, U(\mathfrak{g})\mathfrak{k}), \\
Cp_3 \equiv & (E_{1,1}' - 1)E_{2,2}'(E_{3,3}' + 1) + E_{1,2}E_{2,3}E_{1,3} + E_{1,3}E_{1,2}E_{2,3} - E_{1,3}^2 \\
& - E_{2,3}^2(E_{1,1}' - 1) - E_{1,3}^2 E_{2,2}' - E_{1,2}^2(E_{3,3}' + 1) \\
& (\mathrm{mod}\, U(\mathfrak{g})\mathfrak{k}).
\end{aligned}$$

4.3 Eigenvalues of Cp_2, Cp_3

In order to construct the partial differential equations satisfied by spherical functions attached to the spherical principal series, we have to compute the eigenvalues of the actions of the Capelli elements Cp_2, Cp_3. For the spherical principal series, $\sigma = \sigma_0$ is the trivial character of M. Let f_0 be the generator of the minimal K-type in $H_{\sigma_0,\nu}$ normalized such that $f_0|K \equiv 1$. The actions of Cp_2, Cp_3 on f_0 are computed in [6], and the result is as follows:

Proposition 4.3. *The Capelli elements Cp_2, Cp_3 act on f_0 by scalar multiples, and the eigenvalues are given as follows:*

$$Cp_2 f_0 = S_2\left(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(-\nu_1 + 2\nu_2), -\frac{1}{3}(\nu_1 + \nu_2)\right) f_0,$$

$$Cp_3 f_0 = S_3\left(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(-\nu_1 + 2\nu_2), -\frac{1}{3}(\nu_1 + \nu_2)\right) f_0.$$

Here, $S_2(a, b, c) = ab + bc + ca$, $S_3(a, b, c) = abc$.

5 The matrix coefficient of the spherical principal series representation

5.1 Construction of the differential equations

We put

$$y_1 = y_1(a) := a_1/a_2,\ y_2 = y_2(a) := a_2/a_3$$

for $a = \mathrm{diag}(a_1, a_2, a_3) \in A$. By definition of the action of Lie algebra, we have the following formula.

Lemma 5.1. *For $f(y_1, y_2) = f(a) \in C^\infty(A)$, we have*

$$H_{1,2}f = \left(2y_1\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial y_2}\right)f,\ H_{2,3}f = \left(-y_1\frac{\partial}{\partial y_1} + 2y_2\frac{\partial}{\partial y_2}\right)f.$$

Now we want to construct the partial differential equations satisfied by spherical functions attached to the spherical principal series representation $\pi_{\sigma_0,\nu}$. We define differential operators ∂_1, ∂_2 by

$$\partial_i := y_i\frac{\partial}{\partial y_i} \quad (i = 1, 2).$$

By direct computations, we have the following two lemmas.

Lemma 5.2. *For $1 \le i, j \le 3$ such that $i \neq j$, we have*

$$[K_{i,j}, \mathrm{Ad}(a^{-1})K_{i,j}] = -2sh\left(\frac{a_i}{a_j}\right)H_{i,j}.$$

Lemma 5.3. *For $i, j, k \in \{1, 2, 3\}$ such that $i \neq j, j \neq k, k \neq i$, we have*

$$[K_{i,j}, \mathrm{Ad}(a^{-1})K_{j,k}] = \frac{sh(\frac{a_j}{a_k})}{sh(\frac{a_i}{a_k})}\mathrm{Ad}(a^{-1})K_{i,k} + \frac{sh(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_k})}K_{i,k},$$

$$[K_{i,j}, \mathrm{Ad}(a^{-1})K_{k,i}] = \frac{sh(\frac{a_i}{a_k})}{sh(\frac{a_j}{a_k})}\mathrm{Ad}(a^{-1})K_{j,k} - \frac{sh(\frac{a_i}{a_j})}{sh(\frac{a_j}{a_k})}K_{j,k}.$$

By combining Lemma 5.1, Lemma 5.2, and Lemma 5.3, the actions of Cp_2, Cp_3 in Lemma 4.2 on f_0 are obtained by direct computations. The eigenvalues are obtained in Proposition 4.3. Therefore, we can construct two differential equations characterized by the actions of Capelli elements Cp_2, Cp_3.
Let us compute the action of Cp_2 on the bi- K-invariant spherical function $F(y_1, y_2) \in C^\infty(K\backslash G/K)|_A$. Firstly, since we have

$$E'_{1,1} = \frac{2}{3}H_{1,2} + \frac{1}{3}H_{2,3},$$

$$E'_{2,2} = -\frac{1}{3}H_{1,2} + \frac{1}{3}H_{2,3},$$

$$E'_{3,3} = -\frac{1}{3}H_{1,2} - \frac{2}{3}H_{2,3},$$

the actions of $E'_{1,1}$, $E'_{2,2}$, $E'_{3,3}$ on F are given by ∂_1, $-\partial_1 + \partial_2$, $-\partial_2$ respectively. Therefore, the action of $(E'_{1,1} - 1)E'_{2,2} + E'_{2,2}(E'_{3,3} + 1) + (E'_{1,1} - 1)(E'_{3,3} + 1)$ on F is given by

$$\begin{aligned}&(\partial_1 - 1)(-\partial_1 + \partial_2) + (-\partial_1 + \partial_2)(-\partial_2 + 1) + (\partial_1 - 1)(-\partial_2 + 1)\\ &= -\partial_1^2 + \partial_1\partial_2 - \partial_2^2 + \partial_1 + \partial_2 - 1.\end{aligned}$$

Next, since

$$\begin{aligned} E_{2,3} &= \frac{1}{2}(K_{2,3} + X_{2,3}) \\ &= \frac{1}{2}\left\{ -\frac{1}{sh(\frac{a_2}{a_3})}\mathrm{Ad}(a^{-1})K_{2,3} + \left(\frac{ch(\frac{a_2}{a_3})}{sh(\frac{a_2}{a_3})} + 1 \right) K_{2,3} \right\} \end{aligned}$$

and F is annihilated by the action of $\mathrm{Ad}(a^{-1})\mathfrak{k}U(\mathfrak{g})$ and $U(\mathfrak{g})\mathfrak{k}$, the action of $E_{2,3}^2$ on F is equivalent to that of

$$-\frac{1}{4sh(\frac{a_2}{a_3})}\left(\frac{ch(\frac{a_2}{a_3})}{sh(\frac{a_2}{a_3})} + 1 \right) K_{2,3}\mathrm{Ad}(a^{-1})K_{2,3}.$$

By using Lemma 5.2, this equals

$$-\frac{1}{4sh(\frac{a_2}{a_3})}\left(\frac{ch(\frac{a_2}{a_3})}{sh(\frac{a_2}{a_3})} + 1 \right) \cdot \left\{ -2sh\left(\frac{a_2}{a_3}\right) H_{2,3} \right\} = \frac{1}{2}\left(\frac{ch(\frac{a_2}{a_3})}{sh(\frac{a_2}{a_3})} + 1 \right) H_{2,3}.$$

Since $\frac{ch(\frac{a_2}{a_3})}{sh(\frac{a_2}{a_3})} = \frac{y_2^2+1}{y_2^2-1}$ and the action of $H_{2,3}$ is given by $-\partial_1 + 2\partial_2$, we conclude that

$$(E_{2,3}^2 F)(y_1, y_2) = \frac{1}{2}\left(\frac{y_2^2+1}{y_2^2-1} + 1 \right)(-\partial_1 + 2\partial_2)F(y_1, y_2).$$

Similarly, we have

$$(E_{1,3}^2 F)(y_1, y_2) = \frac{1}{2}\left(\frac{y_1^2 y_2^2+1}{y_1^2 y_2^2-1} + 1 \right)(\partial_1 + \partial_2)F(y_1, y_2),$$

$$(E_{1,2}^2 F)(y_1, y_2) = \frac{1}{2}\left(\frac{y_1^2+1}{y_1^2-1} + 1 \right)(2\partial_1 - \partial_2)F(y_1, y_2).$$

From Proposition 4.3, the eigenvalue $\chi_{C_{P_2}}$ is given by

$$\chi_{C_{P_2}} = -\frac{1}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2).$$

By combining these results, we have following differential equation:

$$\begin{aligned} &(-\partial_1^2 + \partial_1\partial_2 - \partial_2^2 + \partial_1 + \partial_2 - 1)F(y_1, y_2) \\ &- \frac{1}{2}\left(-\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2 y_2^2+1}{y_1^2 y_2^2-1} + 2\frac{y_1^2+1}{y_1^2-1} + 2 \right)\partial_1 F(y_1, y_2) \\ &- \frac{1}{2}\left(2\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2 y_2^2+1}{y_1^2 y_2^2-1} - \frac{y_1^2+1}{y_1^2-1} + 2 \right)\partial_2 F(y_1, y_2) \\ &= -\frac{1}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2)F(y_1, y_2). \end{aligned}$$

By multiplying both sides by -2, we have

$$\begin{aligned}
&2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)F(y_1, y_2) \\
&+ \left(-\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} + 2\frac{y_1^2+1}{y_1^2-1} \right) \partial_1 F(y_1, y_2) \\
&+ \left(2\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} - \frac{y_1^2+1}{y_1^2-1} \right) \partial_2 F(y_1, y_2) \\
&+ \left\{ -\frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2 \right\} F(y_1, y_2) = 0.
\end{aligned}$$

The construction of the differential equation characterized by the action of Cp_3 is more complicated, but the way is similar. We omit this computation. After all, we have the following two differential equations.

Theorem 5.4. *Let $F \in C^\infty_{1,1}(K\backslash G/K)$ be a spherical function attached to the spherical principal series representation $\pi_{\sigma_0,\nu}$. Then its restriction to A: $F|_A = F(y_1, y_2)$ satisfies two partial differential equations:*

$$\begin{aligned}
&2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)F \\
&+ \left(-\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} + 2\frac{y_1^2+1}{y_1^2-1} \right) \partial_1 F \\
&+ \left(2\frac{y_2^2+1}{y_2^2-1} + \frac{y_1^2y_2^2+1}{y_1^2y_2^2-1} - \frac{y_1^2+1}{y_1^2-1} \right) \partial_2 F \\
&+ \left\{ -\frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2 \right\} F = 0,
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
&\partial_1^2\partial_2 F - \partial_1\partial_2^2 F + \left(-1 + \frac{y_2^2}{y_2^2-1} + \frac{y_1^2y_2^2}{y_1^2y_2^2-1} \right) \partial_1^2 F \\
&+ \left(-\frac{2y_2^2}{y_2^2-1} + \frac{2y_1^2}{y_1^2-1} \right) \partial_1\partial_2 F + \left(1 - \frac{y_1^2}{y_1^2-1} - \frac{y_1^2y_2^2}{y_1^2y_2^2-1} \right) \partial_2^2 F \\
&+ \left(\frac{1}{2} - \frac{y_1^2y_2^2}{(y_2^2-1)(y_1^2y_2^2-1)} + \frac{3y_1^2y_2^2}{(y_1^2-1)(y_2^2-1)} + \frac{y_1^2y_2^2}{(y_1^2-1)(y_1^2y_2^2-1)} \right. \\
&\left. - \frac{y_1^2y_2^2+1}{2(y_1^2y_2^2-1)} - \frac{y_2^2}{y_2^2-1} - \frac{2y_1^2}{y_1^2-1} \right) \partial_1 F \\
&+ \left(-\frac{3}{2} - \frac{y_1^2y_2^2}{(y_2^2-1)(y_1^2y_2^2-1)} - \frac{3y_1^2y_2^2}{(y_1^2-1)(y_2^2-1)} + \frac{y_1^2y_2^2}{(y_1^2-1)(y_1^2y_2^2-1)} \right. \\
&\left. - \frac{y_1^2y_2^2+1}{2(y_1^2y_2^2-1)} + \frac{2y_2^2}{y_2^2-1} + \frac{y_1^2}{y_1^2-1} \right) \partial_2 F \\
&+ \frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2)F = 0.
\end{aligned} \tag{5.2}$$

5.2 The expansion of the matrix coefficients in terms of the power series around $y_1 = y_2 = 0$

For the spherical function $F \in C^{\infty}_{1,1}(K\backslash G/K)$ above, we want to find its series expansion at the origin $y_1 = 0$, $y_2 = 0$ by solving (5.1) and (5.2). Firstly, we put

$$F(y_1, y_2) = \sum_{n,m=0}^{\infty} a_{n,m} y_1^{n+\mu_1} y_2^{m+\mu_2} \quad (a_{0,0} \neq 0). \tag{5.3}$$

The first task is to compute the characteristic roots (μ_1, μ_2). By substituting (5.3) for F into the equation (5.1), and picking up the coefficient of $y_1^{n+\mu_1} y_2^{m+\mu_2}$, we have the following equation satisfied by $\{a_{n,m}\}$:

$$\begin{aligned}
&\{2(n^{'}-4)^2 - 2(n^{'}-4)(m^{'}-4) + 2(m^{'}-4)^2 \\
&+ 2(n^{'}-4) + 2(m^{'}-4) + \lambda\} a_{n-4,m-4} \\
&+ \{-2(n^{'}-4)^2 + 2(n^{'}-4)(m^{'}-2) - 2(m^{'}-2)^2 \\
&- 4(n^{'}-4) + 2(m^{'}-2) - \lambda\} a_{n-4,m-2} \\
&+ \{-2(n^{'}-2)^2 + 2(n^{'}-2)(m^{'}-4) - 2(m^{'}-4)^2 \\
&+ 2(n^{'}-2) - 4(m^{'}-4) - \lambda\} a_{n-2,m-4} \\
&+ \{2(n^{'}-2)^2 - 2(n^{'}-2)m^{'} + 2m^{'2} + 2(n^{'}-2) - 4m^{'} + \lambda\} a_{n-2,m} \\
&+ \{2n^{'2} - 2n^{'}(m^{'}-2) + 2(m^{'}-2)^2 - 4n^{'} + 2(m^{'}-2) + \lambda\} a_{n,m-2} \\
&+ (-2n^{'2} + 2n^{'}m^{'} - 2m^{'2} + 2n^{'} + 2m^{'} - \lambda) a_{n,m} = 0.
\end{aligned} \tag{5.4}$$

Here, $\lambda := -\frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2$, $n^{'} := n + \mu_1$, $m^{'} = m + \mu_2$, and $a_{i,j} = 0$ if $i < 0$ or $j < 0$.

Proposition 5.5. *The characteristic roots (μ_1, μ_2) take following six values:*

$$\begin{aligned}
&(\mu_1, \mu_2) \\
&= \left(\frac{1}{3}(2\nu_1 - \nu_2) + 1, -\frac{1}{3}(2\nu_2 - \nu_1) + 1\right), \left(\frac{1}{3}(2\nu_2 - \nu_1) + 1, -\frac{1}{3}(2\nu_1 - \nu_2) + 1\right), \\
&\left(\frac{1}{3}(2\nu_2 - \nu_1) + 1, \frac{1}{3}(\nu_1 + \nu_2) + 1\right), \left(-\frac{1}{3}(\nu_1 + \nu_2) + 1, -\frac{1}{3}(2\nu_2 - \nu_1) + 1\right), \\
&\left(\frac{1}{3}(2\nu_1 - \nu_2) + 1, \frac{1}{3}(\nu_1 + \nu_2) + 1\right), \left(-\frac{1}{3}(\nu_1 + \nu_2) + 1, -\frac{1}{3}(2\nu_1 - \nu_2) + 1\right).
\end{aligned} \tag{5.5}$$

Proof. Because $a_{i,j} = 0$ (if $i < 0$ or $j < 0$), and $a_{0,0} \neq 0$, by putting $n = m = 0$ in (5.4), we have

$$-2(\mu_1^2 - \mu_1\mu_2 + \mu_2^2 - \mu_1 - \mu_2) - \lambda = 0.$$

This equation is equivalent to

$$(\mu_1 - 1)^2 - (\mu_1 - 1)(\mu_2 - 1) + (\mu_2 - 1)^2 = \frac{1}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2). \tag{5.6}$$

Next, by computing the recurrence equation given by equation (5.2), and substituting $n = m = 0$ in the coefficient of $a_{n,m}$, we have

$$\mu_1^2\mu_2 - \mu_1\mu_2^2 - \mu_1^2 + \mu_2^2 + \mu_1 - \mu_2 = -\frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2).$$

This equation is equivalent to

$$\{(\mu_1 - 1) - (\mu_2 - 1)\}(\mu_1 - 1)(\mu_2 - 1) = -\frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2). \quad (5.7)$$

By combining (5.6) and (5.7), we have the result. □

(Since the explicit forms of these power series are quite complicated, and we do not use them at all in the computation of c-functions, we leave them to the " Appendix " ; section 7 of this paper.)

Now we have known that the equations (5.1) and (5.2) have six power series solutions corresponding to the six characteristic roots in Proposition 5.5. And the coefficients of each power series satisfy the recurrence relation (5.4). For a characteristic root (α, β), we express the power seires solution corresponding to (α, β) by $\psi_{\alpha,\beta}$. We assume that the constant term of $\psi_{\alpha,\beta}$ is 1. By Proposition 5.5, β takes following three values:

$$\beta_1 = \frac{1}{3}(\nu_1 + \nu_2) + 1, \beta_2 = -\frac{1}{3}(2\nu_1 - \nu_2) + 1, \beta_3 = -\frac{1}{3}(2\nu_2 - \nu_1) + 1.$$

And for each β_i, we have two power series solutions. Therefore, we can write matrix coefficient F by

$$F(y_1, y_2) = \sum_{i=1}^{3} c_i a_i(y_1, y_2) y_2^{\beta_i}.$$

Here, c_i $(i = 1, 2, 3)$ are some constants and $a_i(y_1, y_2)$ $(i = 1, 2, 3)$ are some analytic functions around $y_2 = 0$. By substituting $a_i(y_1, y_2)y_2^{\beta_i}$ into the equation (5.1), we have

$$\begin{aligned}
&y_2^{\beta_i}\Big\{2(\partial_1^2 a_i(y_1, y_2) - \partial_1\partial_2 a_i(y_1, y_2) - \beta_i\partial_1 a_i(y_1, y_2) \\
&+ \partial_2^2 a_i(y_1, y_2) + 2\beta_i\partial_2 a_i(y_1, y_2) + \beta_i^2 a_i(y_1, y_2)) \\
&+ \left(-\frac{y_2^2 + 1}{y_2^2 - 1} + \frac{y_1^2y_2^2 + 1}{y_1^2y_2^2 - 1} + 2\frac{y_1^2 + 1}{y_1^2 - 1}\right)\partial_1 a_i(y_1, y_2) \\
&+ \left(2\frac{y_2^2 + 1}{y_2^2 - 1} + \frac{y_1^2y_2^2 + 1}{y_1^2y_2^2 - 1} - \frac{y_1^2 + 1}{y_1^2 - 1}\right)(\partial_2 a_i(y_1, y_2) + \beta_i a_i(y_1, y_2)) \\
&+ \left(-\frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2\right) a_i(y_1, y_2)\Big\} = 0.
\end{aligned}$$

Dividing both sides by $y_2^{\beta_i}$ and taking the limit $y_2 \to 0$, then we obtain

$$\begin{aligned}&2\partial_1^2 a_i(y_1,0) + \left(2\frac{y_1^2+1}{y_1^2-1} - 2\beta_i\right)\partial_1 a_i(y_1,0)\\&+\left(2\beta_i^2 - \frac{y_1^2+1}{y_1^2-1}\beta_i - 3\beta_i - \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2\right) a_i(y_1,0) = 0.\end{aligned} \tag{5.8}$$

We put $y_1^2 = u$, $f_i(u) = a_i(y_1,0)$. Then the equation (5.8) becomes

$$\begin{aligned}&8u^2\frac{d^2 f_i}{du^2} + \left(4\frac{u+1}{u-1} - 4\beta_i + 8\right) u\frac{df_i}{du}\\&+\left(2\beta_i^2 - \frac{u+1}{u-1}\beta_i - 3\beta_i - \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2\right) f_i = 0.\end{aligned} \tag{5.9}$$

Next, we put $f_i(u) = u^x g_i(u)$ $(x \in \mathbf{C})$ and substitute this into (5.9). Then we have

$$\begin{aligned}&8u^2\frac{d^2 g_i}{du^2} + \left(4\frac{u+1}{u-1} - 4\beta_i + 8 + 16x\right) u\frac{dg_i}{du}\\&+\left(8x^2 + (4-4\beta_i)x + 2\beta_i^2 - 4\beta_i - \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2)\right.\\&\left.+2 + \frac{8x - 2\beta_i}{u-1}\right) g_i = 0.\end{aligned} \tag{5.10}$$

Now, we choose x_i satisfying

$$8x_i^2 + (4-4\beta_i)x_i + 2\beta_i^2 - 4\beta_i - \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) + 2 = 0$$

and substitute $x = x_i$ into (5.10). Then we have

$$8u^2\frac{d^2 g_i}{du^2} + \left(4\frac{u+1}{u-1} - 4\beta_i + 8 + 16x_i\right) u\frac{dg_i}{du} + \frac{8x_i - 2\beta_i}{u-1} g_i = 0. \tag{5.11}$$

Finally, we put $u = \frac{1}{\zeta}$ and substitute this into (5.11). Then we have

$$\begin{aligned}&\zeta(\zeta-1)\frac{d^2 g_i}{d\zeta^2} + \left(-\frac{1}{2}(\beta_i - 4x_i + 1) + \frac{1}{2}(\beta_i - 4x_i + 3)\zeta\right)\frac{dg_i}{d\zeta}\\&+(-x_i + \frac{\beta_i}{4})g_i = 0.\end{aligned} \tag{5.12}$$

(5.12) is a Gaussian hypergeometric differential equation, and if we define p_i, q_i as the complex numbers satisfying

$$1 + p_i + q_i = \frac{1}{2}(\beta_i - 4x_i + 3),$$

$$p_i q_i = -x_i + \frac{\beta_i}{4}$$

and define r_i by

$$r_i = \frac{1}{2}(\beta_i - 4x_i + 1),$$

then the solution is expressed by

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & p_i & ;\zeta \\ 1-r_i & r_i - p_i - q_i & q_i & \end{matrix}\right\} = P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & p_i & ;1-\zeta \\ r_i - p_i - q_i & 1-r_i & q_i & \end{matrix}\right\}.$$

Here, $P\{\quad\}$ denotes Riemann's P-function. The regular solution is,

$$\begin{aligned} g_i(y_1) &= {}_2F_1\left(p_i, q_i; 1 - r_i + p_i + q_i; 1 - \zeta\right) \\ &= {}_2F_1\left(p_i, q_i; 1 - r_i + p_i + q_i; 1 - \frac{1}{y_1^2}\right). \end{aligned}$$

(See [9]). Since ${}_2F_1$ satisfies a formula ([5])

$$\begin{aligned} {}_2F_1(a,b;c;z) =& (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}{}_2F_1\left(a, c-b; 1+a-b; \frac{1}{1-z}\right) \\ &+ (1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}{}_2F_1\left(b, c-a; 1+b-a; \frac{1}{1-z}\right), \end{aligned} \tag{5.13}$$

we have

$$\begin{aligned} g_i(y_1) =& y_1^{2p_i}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(q_i-p_i)}{\Gamma(1-r_i+q_i)\Gamma(q_i)}{}_2F_1(p_i, 1-r_i+p_i; 1+p_i-q_i; y_1^2) \\ &+ y_1^{2q_i}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(p_i-q_i)}{\Gamma(1-r_i+p_i)\Gamma(p_i)}{}_2F_1(q_i, 1-r_i+q_i; 1+q_i-p_i; y_1^2). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &a_i(y_1, 0) \\ &= u^{x_i}g_i(y_1) = y_1^{2x_i}g_i(y_1) \\ &= y_1^{2(p_i+x_i)}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(q_i-p_i)}{\Gamma(1-r_i+q_i)\Gamma(q_i)}{}_2F_1(p_i, 1-r_i+p_i; 1+p_i-q_i; y_1^2) \\ &+ y_1^{2(q_i+x_i)}\frac{\Gamma(1-r_i+p_i+q_i)\Gamma(p_i-q_i)}{\Gamma(1-r_i+p_i)\Gamma(p_i)}{}_2F_1(q_i, 1-r_i+q_i; 1+q_i-p_i; y_1^2). \end{aligned} \tag{5.14}$$

Next, for $i = 1, 2, 3$, we compute (x_i, p_i, q_i, r_i). (Although the equation for x_i has two different solutions in general, the result doesn't depend on the choice of x_i.)

A) In case of $\beta_i = \beta_1 = \frac{1}{3}(\nu_1 + \nu_2) + 1$, $x_1 = \frac{1}{3}\nu_1 - \frac{1}{6}\nu_2$, $p_1 = -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2}$, $q_1 = \frac{1}{2}$, $r_1 = -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1$.

B) In case of $\beta_i = \beta_2 = -\frac{1}{3}(2\nu_1 - \nu_2) + 1$, $x_2 = -\frac{1}{6}\nu_1 + \frac{1}{3}\nu_2$, $p_2 = -\frac{1}{2}\nu_2 + \frac{1}{2}$, $q_2 = \frac{1}{2}$, $r_2 = -\frac{1}{2}\nu_2 + 1$.

C) In case of $\beta_i = \beta_3 = -\frac{1}{3}(2\nu_2 - \nu_1) + 1$, $x_3 = \frac{1}{3}\nu_1 - \frac{1}{6}\nu_2$, $p_3 = -\frac{1}{2}\nu_1 + \frac{1}{2}$, $q_3 = \frac{1}{2}$, $r_3 = -\frac{1}{2}\nu_1 + 1$.

By substituting these results into equation (5.14), we have

$$
\begin{aligned}
&a_1(y_1, 0) \\
&= y_1^{\frac{1}{3}(2\nu_2-\nu_1)+1} \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})} {}_2F_1\left(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1; y_1^2\right) \\
&+ y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})} {}_2F_1\left(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1; y_1^2\right),
\end{aligned}
$$

$$
\begin{aligned}
&a_2(y_1, 0) \\
&= y_1^{-\frac{1}{3}(\nu_1+\nu_2)+1} \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})} {}_2F_1\left(-\frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_2 + 1; y_1^2\right) \\
&+ y_1^{\frac{1}{3}(2\nu_2-\nu_1)+1} \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})} {}_2F_1\left(\frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\nu_2 + 1; y_1^2\right),
\end{aligned}
$$

$$
\begin{aligned}
&a_3(y_1, 0) \\
&= y_1^{-\frac{1}{3}(\nu_1+\nu_2)+1} \frac{\Gamma(\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})} {}_2F_1\left(-\frac{1}{2}\nu_1 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_1 + 1; y_1^2\right) \\
&+ y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} \frac{\Gamma(-\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})} {}_2F_1\left(\frac{1}{2}\nu_1 + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\nu_1 + 1; y_1^2\right).
\end{aligned}
$$

Therefore, by comparing the leading terms, we have

$$
\begin{aligned}
&\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1}(y_1, y_2) \\
&= y_1^{\frac{1}{3}(2\nu_2-\nu_1)+1} y_2^{\frac{1}{3}(\nu_1+\nu_2)+1} {}_2F_1\left(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1; y_1^2\right) \\
&+ (\text{higher order terms with respect to } y_2),
\end{aligned}
$$

$$
\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1}(y_1, y_2)
$$

$$= y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} y_2^{\frac{1}{3}(\nu_1+\nu_2)+1} {}_2F_1\left(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1; y_1^2\right)$$
$$+ \text{(higher order terms with respect to } y_2),$$

$$\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1}(y_1, y_2)$$
$$= y_1^{-\frac{1}{3}(\nu_1+\nu_2)+1} y_2^{-\frac{1}{3}(2\nu_1-\nu_2)+1} {}_2F_1\left(-\frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_2 + 1; y_1^2\right)$$
$$+ \text{(higher order terms with respect to } y_2),$$

$$\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1}(y_1, y_2)$$
$$= y_1^{\frac{1}{3}(2\nu_2-\nu_1)+1} y_2^{-\frac{1}{3}(2\nu_1-\nu_2)+1} {}_2F_1\left(\frac{1}{2}\nu_2 + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\nu_2 + 1; y_1^2\right)$$
$$+ \text{(higher order terms with respect to } y_2),$$

$$\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1}(y_1, y_2)$$
$$= y_1^{-\frac{1}{3}(\nu_1+\nu_2)+1} y_2^{-\frac{1}{3}(2\nu_2-\nu_1)+1} {}_2F_1\left(-\frac{1}{2}\nu_1 + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}\nu_1 + 1; y_1^2\right)$$
$$+ \text{(higher order terms with respect to } y_2),$$

$$\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1}(y_1, y_2)$$
$$= y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1} y_2^{-\frac{1}{3}(2\nu_2-\nu_1)+1} {}_2F_1\left(\frac{1}{2}\nu_1 + \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\nu_1 + 1; y_1^2\right)$$
$$+ \text{(higher order terms with respect to } y_2),$$

and

$$\begin{aligned} F = c_1 &\left\{ \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right. \\ &\left. + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\} \\ + c_2 &\left\{ \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\ &\left. + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right\} \\ + c_3 &\left\{ \frac{\Gamma(\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right. \\ &\left. + \frac{\Gamma(-\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right\}. \end{aligned} \tag{5.15}$$

The next work is to determine the values of c_1, c_2, c_3. To do this, we apply the same method to y_2-part. That is, for $\alpha_1 = -\frac{1}{3}(\nu_1+\nu_2)+1, \alpha_2 = \frac{1}{3}(2\nu_1 - \nu_2)+1, \alpha_3 = \frac{1}{3}(2\nu_2-\nu_1)+1$, we can write

$$F(y_1, y_2) = \sum_{i=1}^{3} d_i b_i(y_1, y_2) y_1^{\alpha_i}$$

and by investigating the differential equations satisfied by $b_i(0, y_2)$ $(i = 1, 2, 3)$ and comparing the leading terms with respect to y_2, we have

$$\begin{aligned} F = d_1 &\left\{ \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\ &\left. + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right\} \\ + d_2 &\left\{ \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right. \\ &\left. + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\} \\ + d_3 &\left\{ \frac{\Gamma(\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\ &\left. + \frac{\Gamma(-\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\}. \end{aligned} \tag{5.16}$$

By comparing the coefficients of $\psi_{\alpha,\beta}$ in the equation (5.15) and (5.16), we can determine c_i, d_i $(i = 1, 2, 3)$ up to constant multiples. In particular,

$$c_1 = \frac{\Gamma(-\frac{1}{2}\nu_1)\Gamma(-\frac{1}{2}\nu_2)}{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})}, \quad c_2 = \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1)}{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})},$$

$$c_3 = \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_2)}{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})}.$$

For the power series $\sum_{n,m=0}^{\infty} a_{n,m} y_1^{n+\mu_1} y_2^{m+\mu_2}$ $(a_{0,0} \neq 0)$, we call $a_{0,0}$ the first term of this power series. We completely determined the six coefficients appearing in the linear combination of power series. Summing up, we have the following theorem:

Theorem 5.6. *Let F be a spherical function attached to the spherical principal series representation $\pi_{\sigma_0,\nu}$, and $\psi_{\alpha,\beta}$ be the power series solution around $y_1 = y_2 = 0$ corresponding to the characteristic root (α, β) whose first term is equal to 1. Then we have*

$$F = \frac{\Gamma(-\frac{1}{2}\nu_1)\Gamma(-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1}$$

$$
\begin{aligned}
&+\frac{\Gamma(-\frac{1}{2}\nu_1)\Gamma(-\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\\
&+\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1)\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\\
&+\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1)\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\\
&+\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\\
&+\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}.
\end{aligned}
$$

6 The matrix coefficients of the non-spherical principal seires representations

In this section, we investigate the matrix coefficients of the non-spherical principal series representations whose minimal K-type is the 3-dimensional tautological representation of K. Let $\tau_2 : K = SO(3) \hookrightarrow GL(3, \mathbf{R})$ be the tautological representation. Then we say that

$$\{s_1 = {}^t(1,0,0), s_2 = {}^t(0,1,0), s_3 = {}^t(0,0,1)\}$$

is the natural basis of this representation τ_2. We consider a spherical function $\Psi \in C^\infty_{\tau_2,\tau_2}(K\backslash G/K)$ attached to the non-spherical principal series representation $\pi_{\sigma_i,\nu}$ $(i = 1,2,3)$. Ψ can be written in terms of the basis $\{s_i | i = 1,2,3\}$:

$$\Psi(g) = \sum_{i=1}^{3}\sum_{j=1}^{3} d_{ij}(g) s_i^L \otimes s_j^R.$$

Note that Ψ satisfies

$$\Psi(k_1 g k_2^{-1}) = \sum_{i=1}^{3}\sum_{j=1}^{3} d_{ij}(g)(\tau_2(k_1)s_i^L) \otimes (\tau_2(k_2)s_j^R)$$

for $k_1, k_2 \in K$, $g \in G$.

Lemma 6.1. *For $a \in A$, we have $d_{ij}(a) = 0$ if $i \neq j$.*

Proof. A subgroup M of G is defined by

$$
\begin{aligned}
M = Z_K(A) &= \{k \in K | ak = ka \quad (\forall a \in A)\}\\
&= \{\mathrm{diag}(\epsilon_1,\epsilon_2,\epsilon_3) | \epsilon_i \in \{\pm 1\}, \epsilon_1\epsilon_2\epsilon_3 = 1\}.
\end{aligned}
$$

Then for $m \in M$, $a \in A$, we have

$$\tau_L(m)\Psi(a) = \Psi(ma) = \Psi(am) = \tau_R(m^{-1})\Psi(a).$$

Therefore, for example, for $m_3 = \text{diag}(-1,-1,1) \in M$, we have

$$\tau_2(m_3) \otimes \Psi(a) = 1 \otimes \tau_2(m_3^{-1})\Psi(a).$$

Hence

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} (d_{ij}(a)) = (d_{ij}(a)) \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

From this, we have $d_{13}(a) = d_{31}(a) = d_{23}(a) = d_{32}(a) = 0$. Similarly, the actions of the other elements of M show that $d_{ij}(a) = 0$ if $i \neq j$. □

6.1 The action of Casimir operator

We use the same coordinate $y_1 = \frac{a_1}{a_2}, y_2 = \frac{a_2}{a_3}$ ($a = \text{diag}(a_1, a_2, a_3) \in A$) as in section 5. We compute the action of the Casimir element, which is an element of $Z(\mathfrak{g})$. The Casimir operator C of $SL(3, \mathbf{R})$ is decomposed into two parts with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$:

$$C = C(\mathfrak{p}) + C(\mathfrak{k}).$$

Here,

$$C(\mathfrak{p}) = \frac{2}{3}(H_{1,2}^2 + H_{1,2}H_{2,3} + H_{2,3}^2) + \frac{1}{2}\sum_{i<j} X_{i,j}^2,$$

$$C(\mathfrak{k}) = -\frac{1}{2}\sum_{i<j} K_{i,j}^2.$$

Firstly, we consider the action of $C(\mathfrak{p})$.

$$\left(\text{The action of } \frac{2}{3}(H_{1,2}^2 + H_{1,2}H_{2,3} + H_{2,3}^2)\right)$$
$$= \frac{2}{3}\{(2\partial_1 - \partial_2)^2 + (2\partial_1 - \partial_2)(-\partial_1 + 2\partial_2) + (-\partial_1 + 2\partial_2)^2\}$$
$$= 2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2).$$

Next,

$$X_{i,j}^2 = \left\{ -\frac{1}{sh(\frac{a_i}{a_j})}\text{Ad}(a^{-1})K_{i,j} + \frac{ch(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_j})}K_{i,j} \right\}^2$$

$$= \frac{1}{sh(\frac{a_i}{a_j})^2}(\mathrm{Ad}(a^{-1})K_{i,j})^2 - 2\frac{ch(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_j})^2}\mathrm{Ad}(a^{-1})K_{i,j}\cdot K_{i,j}$$
$$+ \frac{ch(\frac{a_i}{a_j})^2}{sh(\frac{a_i}{a_j})^2}K_{i,j}^2 - \frac{ch(\frac{a_i}{a_j})}{sh(\frac{a_i}{a_j})^2}[K_{i,j}, \mathrm{Ad}(a^{-1})K_{i,j}].$$

The bracket product above is given by Lemma 5.2. Therefore, we have

$$\begin{aligned}
\frac{1}{2}\sum_{i<j} X_{i,j}^2 &= \frac{1}{2}\frac{1}{sh(y_1)^2}(\mathrm{Ad}(a^{-1})K_{1,2})^2 - \frac{ch(y_1)}{sh(y_1)^2}\mathrm{Ad}(a^{-1})K_{1,2}\cdot K_{1,2} \\
&+ \frac{1}{2}\frac{ch(y_1)^2}{sh(y_1)^2}K_{1,2}^2 + \frac{ch(y_1)}{sh(y_1)}H_{1,2} \\
&+ \frac{1}{2}\frac{1}{sh(y_1y_2)^2}(\mathrm{Ad}(a^{-1})K_{1,3})^2 - \frac{ch(y_1y_2)}{sh(y_1y_2)^2}\mathrm{Ad}(a^{-1})K_{1,3}\cdot K_{1,3} \\
&+ \frac{1}{2}\frac{ch(y_1y_2)^2}{sh(y_1y_2)^2}K_{1,3}^2 + \frac{ch(y_1y_2)}{sh(y_1y_2)}H_{1,3} \\
&+ \frac{1}{2}\frac{1}{sh(y_2)^2}(\mathrm{Ad}(a^{-1})K_{2,3})^2 - \frac{ch(y_2)}{sh(y_2)^2}\mathrm{Ad}(a^{-1})K_{2,3}\cdot K_{2,3} \\
&+ \frac{1}{2}\frac{ch(y_2)^2}{sh(y_2)^2}K_{2,3}^2 + \frac{ch(y_2)}{sh(y_2)}H_{2,3}.
\end{aligned}$$

The actions of $(\mathrm{Ad}(a^{-1})K_{i,j})^2$, $(\mathrm{Ad}(a^{-1})K_{i,j})K_{i,j}$, $K_{i,j}^2$ on $\Psi(g) = \sum_i \sum_j d_{i,j}(g) s_i^L \otimes s_j^R$ are given by

$$\begin{aligned}
(\mathrm{Ad}(a^{-1})K_{i,j})^2\Psi(a) &= -d_{ii}(a)s_{ii}^{LR} - d_{jj}(a)s_{jj}^{LR}, \\
(\mathrm{Ad}(a^{-1})K_{i,j})K_{i,j}\Psi(a) &= d_{jj}(a)s_{ii}^{LR} + d_{ii}(a)s_{jj}^{LR}, \\
K_{i,j}^2\Psi(a) &= -d_{ii}(a)s_{ii}^{LR} - d_{jj}(a)s_{jj}^{LR}
\end{aligned}$$

on A. Here, we put $s_{ij}^{LR} := s_i^L \otimes s_j^R$. Therefore, we have

$$\begin{aligned}
&C(\mathfrak{p})\Psi(a) \\
&= 2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)\Psi(a) \\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1\Psi(a) \\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2\Psi(a) \\
&- \frac{1}{2}\frac{1}{sh(y_1)^2}\{d_{11}(a)s_{11}^{LR} + d_{22}(a)s_{22}^{LR}\} - \frac{1}{2}\frac{1}{sh(y_1y_2)^2}\{d_{11}(a)s_{11}^{LR} + d_{33}(a)s_{33}^{LR}\} \\
&- \frac{1}{2}\frac{1}{sh(y_2)^2}\{d_{22}(a)s_{22}^{LR} + d_{33}(a)s_{33}^{LR}\}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{ch(y_1)}{sh(y_1)^2}\{d_{22}(a)s_{11}^{LR}+d_{11}(a)s_{22}^{LR}\}+\frac{ch(y_1y_2)}{sh(y_1y_2)^2}\{d_{33}(a)s_{11}^{LR}+d_{11}(a)s_{33}^{LR}\}\\
&+\frac{ch(y_2)}{sh(y_2)^2}\{d_{33}(a)s_{22}^{LR}+d_{22}(a)s_{33}^{LR}\}\\
&-\frac{1}{2}\frac{ch(y_1)^2}{sh(y_1)^2}\{d_{11}(a)s_{11}^{LR}+d_{22}(a)s_{22}^{LR}\}-\frac{1}{2}\frac{ch(y_1y_2)^2}{sh(y_1y_2)^2}\{d_{11}(a)s_{11}^{LR}+d_{33}(a)s_{33}^{LR}\}\\
&-\frac{1}{2}\frac{ch(y_2)^2}{sh(y_2)^2}\{d_{22}(a)s_{22}^{LR}+d_{33}(a)s_{33}^{LR}\}.
\end{aligned}
$$

Next, the action of $C(\mathfrak{k})=-\frac{1}{2}\sum_{i<j}K_{i,j}^2$ is given as follows:

$$
\begin{aligned}
&C(\mathfrak{k})\Psi(a)\\
&=\frac{1}{2}\{d_{11}(a)s_{11}^{LR}+d_{22}(a)s_{22}^{LR}\}+\frac{1}{2}\{d_{22}(a)s_{22}^{LR}+d_{33}(a)s_{33}^{LR}\}\\
&+\frac{1}{2}\{d_{11}(a)s_{11}^{LR}+d_{33}(a)s_{33}^{LR}\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(C\Psi)(a)\\
&=2(\partial_1^2-\partial_1\partial_2+\partial_2^2)\Psi(a)\\
&+\left(2\frac{ch(y_1)}{sh(y_1)}+\frac{ch(y_1y_2)}{sh(y_1y_2)}-\frac{ch(y_2)}{sh(y_2)}\right)\partial_1\Psi(a)\\
&+\left(-\frac{ch(y_1)}{sh(y_1)}+\frac{ch(y_1y_2)}{sh(y_1y_2)}+2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2\Psi(a)\\
&-\frac{1}{sh(y_1)^2}\{d_{11}(a)s_{11}^{LR}+d_{22}(a)s_{22}^{LR}\}-\frac{1}{sh(y_1y_2)^2}\{d_{11}(a)s_{11}^{LR}+d_{33}(a)s_{33}^{LR}\}\\
&-\frac{1}{sh(y_2)^2}\{d_{22}(a)s_{22}^{LR}+d_{33}(a)s_{33}^{LR}\}\\
&+\frac{ch(y_1)}{sh(y_1)^2}\{d_{22}(a)s_{11}^{LR}+d_{11}(a)s_{22}^{LR}\}+\frac{ch(y_1y_2)}{sh(y_1y_2)^2}\{d_{33}(a)s_{11}^{LR}+d_{11}(a)s_{33}^{LR}\}\\
&+\frac{ch(y_2)}{sh(y_2)^2}\{d_{33}(a)s_{22}^{LR}+d_{22}(a)s_{33}^{LR}\}.
\end{aligned}
$$

The next step is to compute the eigenvalue λ of the Casimir operator C. We compute the action on $f\in H_{\pi_{\sigma_i,\nu}}$ such that $f(e)=1$. Firstly, $\frac{2}{3}(H_{1,2}^2+H_{1,2}H_{2,3}+H_{2,3}^2)$ acts on f by scalar multiplication. Its scalar is given by

$$
\frac{2}{3}\{(\nu_1-\nu_2+1)^2+(\nu_1-\nu_2+1)(\nu_2+1)+(\nu_2+1)^2\}
$$

$$= \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2 + 3\nu_1 + 3).$$

Next,

$$\frac{1}{2}\sum_{i<j} X_{i,j}^2 - \frac{1}{2}\sum_{i<j} K_{i,j}^2 = \frac{1}{2}\sum_{i<j}(E_{i,j} + E_{j,i})^2 - \frac{1}{2}\sum_{i<j}(E_{i,j} - E_{j,i})^2$$
$$= \sum_{i<j}(E_{i,j}E_{j,i} + E_{j,i}E_{i,j}).$$

Since $Xf(e) = 0$ for $X \in \mathfrak{n}$, the action of $E_{i,j}E_{j,i}$ is 0. On the other hand, since

$$[E_{i,j}, E_{j,i}] = H_{i,j},$$

we have

$$E_{j,i}E_{i,j} = E_{i,j}E_{j,i} - H_{i,j}.$$

Thus

$$\left(\frac{1}{2}\sum_{i<j} X_{i,j}^2 - \frac{1}{2}\sum_{i<j} K_{i,j}^2\right) f$$
$$= -\left(\sum_{i<j} H_{i,j}\right) f$$
$$= -2H_{1,3}f = -2(\nu_1 + 2)f.$$

Therefore, we have

$$\lambda = \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2 + 3\nu_1 + 3) - 2(\nu_1 + 2)$$
$$= \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) - 2.$$

For $\Psi(g) = \sum_{i=1}^3 \sum_{j=1}^3 d_{ij}(g) s_i^L \otimes s_j^R \in C^\infty_{\tau_2,\tau_2}(K\backslash G/K)$, we put

$$d_{11}(a) = F(a) = F(y_1, y_2),$$
$$d_{22}(a) = G(a) = G(y_1, y_2),$$
$$d_{33}(a) = H(a) = H(y_1, y_2).$$

Then, by comparing the coefficients of s_{ii}^{LR} in both sides of the equation $C\Psi = \lambda\Psi$, we have the following theorem:

Theorem 6.2. *Let $\Psi = {}^t(F, G, H)$ be a spherical function attached to the non-spherical principal series representation $\pi_{\sigma_i,\nu}$ $(i = 1, 2, 3)$ restricted to A. Then, F, G, H satisfy the following differential equations:*

$$\begin{aligned}
&2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)F(y_1, y_2) \\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1 F(y_1, y_2) \\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2 F(y_1, y_2) \\
&- \left(\frac{1}{sh(y_1)^2} + \frac{1}{sh(y_1y_2)^2}\right)F(y_1, y_2) \\
&+ \frac{ch(y_1)}{sh(y_1)^2}G(y_1, y_2) + \frac{ch(y_1y_2)}{sh(y_1y_2)^2}H(y_1, y_2) \\
&= \lambda F(y_1, y_2),
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
&2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)G(y_1, y_2) \\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1 G(y_1, y_2) \\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2 G(y_1, y_2) \\
&- \left(\frac{1}{sh(y_1)^2} + \frac{1}{sh(y_2)^2}\right)G(y_1, y_2) \\
&+ \frac{ch(y_1)}{sh(y_1)^2}F(y_1, y_2) + \frac{ch(y_2)}{sh(y_2)^2}H(y_1, y_2) \\
&= \lambda G(y_1, y_2),
\end{aligned} \tag{6.2}$$

$$\begin{aligned}
&2(\partial_1^2 - \partial_1\partial_2 + \partial_2^2)H(y_1, y_2) \\
&+ \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1 H(y_1, y_2) \\
&+ \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2 H(y_1, y_2) \\
&- \left(\frac{1}{sh(y_2)^2} + \frac{1}{sh(y_1y_2)^2}\right)H(y_1, y_2) \\
&+ \frac{ch(y_1y_2)}{sh(y_1y_2)^2}F(y_1, y_2) + \frac{ch(y_2)}{sh(y_2)^2}G(y_1, y_2) \\
&= \lambda H(y_1, y_2).
\end{aligned} \tag{6.3}$$

Here, $\lambda = \frac{2}{3}(\nu_1^2 - \nu_1\nu_2 + \nu_2^2) - 2.$

6.2 The gradient operator

For a spherical function $\Psi(g) \in C^\infty_{\tau_2,\tau_2}(K\backslash G/K)$ attached to the non-spherical principal series representation $\pi_{\sigma_i,\nu}$ $(i = 1, 2, 3)$, we define the right gradient operator ∇^R as follows:

Definition 6.3. For the orthonormal basis $\{X_i\}_{i=1}^5$ of $\mathfrak{p}$, the right gradient operator ∇^R is defined by

$$\nabla^R\Psi := \sum_{i=1}^{5} R_{X_i}\Psi \otimes X_i^*.$$

Here, X_i^* is the dual basis of X_i with respect to the inner product $(X,Y) \in \mathfrak{p}\times\mathfrak{p} \to \mathrm{Tr}(XY) \in \mathbf{C}$.

If we take $\{H_{1,2}, H_{2,3}, X_{1,2}, X_{2,3}, X_{1,3}\}$ as a basis of $\mathfrak{p}$, its dual basis is $\{\frac{1}{3}(2H_{1,2}+H_{2,3}), \frac{1}{3}(H_{1,2}+2H_{2,3}), \frac{1}{2}X_{1,2}, \frac{1}{2}X_{2,3}, \frac{1}{2}X_{1,3}\}$. Therefore,

$$\begin{aligned}\nabla^R\Psi =& \frac{1}{3}R_{H_{1,2}}\Psi \otimes (2H_{1,2}+H_{2,3}) + \frac{1}{3}R_{H_{2,3}}\Psi \otimes (H_{1,2}+2H_{2,3}) \\ &+ \frac{1}{2}\sum_{i<j} R_{X_{i,j}}\Psi \otimes X_{i,j}.\end{aligned}$$

Claim 2. We define $\{w_i | 0 \le i \le 4\} \subset \mathfrak{p}_{\mathbf{C}} = \mathfrak{p}\otimes_{\mathbf{R}} \mathbf{C}$ by

$$\begin{aligned} w_0 &:= -2(H_{2,3} - \sqrt{-1}X_{2,3}) \\ w_4 &:= -2(H_{2,3} + \sqrt{-1}X_{2,3}) \\ w_2 &:= \frac{2}{3}(2H_{1,2}+H_{2,3}) \\ w_1 &:= X_{1,3} + \sqrt{-1}X_{1,2} \\ w_3 &:= -X_{1,3} + \sqrt{-1}X_{1,2}. \end{aligned}$$

Then $\{w_i | 0 \le i \le 4\}$ becomes a basis of $\mathfrak{p}_{\mathbf{C}}$.

With this basis, the gradient operator ∇^R is written as

$$\begin{aligned}\nabla^R\Psi =& \frac{1}{16}R_{w_4}\Psi\otimes w_0 + \frac{1}{16}R_{w_0}\Psi\otimes w_4 - \frac{1}{4}R_{w_3}\Psi\otimes w_1 \\ &- \frac{1}{4}R_{w_1}\Psi\otimes w_3 + \frac{3}{8}R_{w_2}\Psi\otimes w_2 \\ =& \frac{1}{4}\left(\frac{1}{4}R_{w_4}\Psi\otimes w_0 + \frac{1}{4}R_{w_0}\Psi\otimes w_4 - R_{w_3}\Psi\otimes w_1\right. \\ &\left. -R_{w_1}\Psi\otimes w_3 + \frac{3}{2}R_{w_2}\Psi\otimes w_2\right).\end{aligned}$$

K acts on $\mathfrak{p}_{\mathbf{C}}$ by adjoint action. We denote this representation by (τ_4, W_4). By the Clebsh-Gordan theorem, $\tau_2 \otimes \tau_4$ has the irreducible decomposition

$$\tau_2\otimes\tau_4 \cong \tau_2 \oplus \tau_4 \oplus \tau_6.$$

Here, each τ_n is a $(n+1)$-dimensional irreducible representation of K. In this decomposition, the projector of K-modules

$$pr_2 : \tau_2 \otimes \tau_4 \to \tau_2$$

is described as in the following table:

Table 1: Table of $pr_2(s_j \otimes w_k)$

	w_0	w_1	w_2	w_3	w_4
s_1	0	$-\frac{1}{4}(s_3+\sqrt{-1}s_2)$	$-\frac{1}{3}s_1$	$\frac{1}{4}(s_3-\sqrt{-1}s_2)$	0
s_2	$\frac{1}{2}(s_2-\sqrt{-1}s_3)$	$-\frac{\sqrt{-1}}{4}s_1$	$\frac{1}{6}s_2$	$-\frac{\sqrt{-1}}{4}s_1$	$\frac{1}{2}(s_2+\sqrt{-1}s_3)$
s_3	$-\frac{1}{2}(s_3+\sqrt{-1}s_2)$	$-\frac{1}{4}s_1$	$\frac{1}{6}s_3$	$\frac{1}{4}s_1$	$\frac{1}{2}(-s_3+\sqrt{-1}s_2)$

$\nabla^R\Psi$ is a $(\tau_2 \otimes \tau_2) \otimes \mathfrak{p}_{\mathbf{C}}$-valued function. Then, by mapping $s_i^L \otimes s_j^R \otimes w_k$ to $s_i^L \otimes s_j^R w_k$ (here, $s_j^R w_k := pr_2(s_j^R \otimes w_k)$) , we have a K-homomorphism

$$\tilde{pr}_2 \circ \nabla^R : C^\infty_{\tau_2,\tau_2}(K\backslash G/K) \to C^\infty_{\tau_2,\tau_2}(K\backslash G/K).$$

Since the minimal K-type τ_2 is of multiplicity one, $\tilde{pr}_2 \circ \nabla^R$ is a map of scalar multiplication.
We compute $4\tilde{pr}_2(\nabla^R\Psi)(a)$ for $\Psi(g) = \sum_i \sum_j d_{ij}(g) s_i^L \otimes s_j^R$, $a \in A$.

1)

$$\begin{aligned}
\frac{1}{4}\tilde{pr}_2(R_{w_4}\Psi \otimes w_0)(a) &= \frac{1}{4}\tilde{pr}_2(R_{-2(H_{2,3}+\sqrt{-1}X_{2,3})}\Psi \otimes w_0)(a) \\
&= -\frac{1}{2}\tilde{pr}_2(R_{H_{2,3}}\Psi \otimes w_0)(a) - \frac{\sqrt{-1}}{2}\tilde{pr}_2(R_{X_{2,3}}\Psi \otimes w_0)(a)
\end{aligned}$$

Firstly,

$$\begin{aligned}
&-\frac{1}{2}\tilde{pr}_2(R_{H_{2,3}}\Psi \otimes w_0)(a) \\
&= -\frac{1}{2}(-\partial_1 + 2\partial_2)\sum_{i=1}^{3} d_{ii}(a) s_i^L \otimes s_i^R w_0 \\
&= -\frac{1}{4}(-\partial_1 + 2\partial_2) d_{22}(a) s_{22}^{LR} + \frac{\sqrt{-1}}{4}(-\partial_1 + 2\partial_2) d_{22}(a) s_{23}^{LR} \\
&+ \frac{\sqrt{-1}}{4}(-\partial_1 + 2\partial_2) d_{33}(a) s_{32}^{LR} + \frac{1}{4}(-\partial_1 + 2\partial_2) d_{33}(a) s_{33}^{LR}.
\end{aligned}$$

Next, since

$$X_{2,3} = -\frac{1}{sh(y_2)}\mathrm{Ad}(a^{-1})K_{2,3} + \frac{ch(y_2)}{sh(y_2)}K_{2,3},$$

we have

$$-\frac{\sqrt{-1}}{2}\tilde{pr}_2(R_{X_{2,3}}\Psi \otimes w_0)(a)$$

$$= -\frac{\sqrt{-1}}{2}\tilde{pr}_2\left(-\frac{1}{sh(y_2)}R_{\mathrm{Ad}(a^{-1})K_{2,3}}\Psi\otimes w_0 + \frac{ch(y_2)}{sh(y_2)}R_{K_{2,3}}\Psi\otimes w_0\right)(a)$$
$$= \left(\frac{1}{4sh(y_2)}d_{33}(a) - \frac{ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{22}^{LR}$$
$$+ \left(-\frac{\sqrt{-1}}{4sh(y_2)}d_{33}(a) + \frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{23}^{LR}$$
$$+ \left(-\frac{\sqrt{-1}}{4sh(y_2)}d_{22}(a) + \frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{32}^{LR}$$
$$+ \left(-\frac{1}{4sh(y_2)}d_{22}(a) + \frac{ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{33}^{LR}.$$

Therefore, we have

$$\frac{1}{4}\tilde{pr}_2(R_{w_4}\Psi\otimes w_0)(a)$$
$$= -\frac{1}{4}(-\partial_1 + 2\partial_2)d_{22}(a)s_{22}^{LR} + \frac{\sqrt{-1}}{4}(-\partial_1 + 2\partial_2)d_{22}(a)s_{23}^{LR}$$
$$+ \frac{\sqrt{-1}}{4}(-\partial_1 + 2\partial_2)d_{33}(a)s_{32}^{LR} + \frac{1}{4}(-\partial_1 + 2\partial_2)d_{33}(a)s_{33}^{LR}$$
$$+ \left(\frac{1}{4sh(y_2)}d_{33}(a) - \frac{ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{22}^{LR}$$
$$+ \left(-\frac{\sqrt{-1}}{4sh(y_2)}d_{33}(a) + \frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{23}^{LR}$$
$$+ \left(-\frac{\sqrt{-1}}{4sh(y_2)}d_{22}(a) + \frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{32}^{LR}$$
$$+ \left(-\frac{1}{4sh(y_2)}d_{22}(a) + \frac{ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{33}^{LR}.$$

2) Similarly, we have

$$\frac{1}{4}\tilde{pr}_2(R_{w_0}\Psi\otimes w_4)(a)$$
$$= -\frac{1}{4}(-\partial_1 + 2\partial_2)d_{22}(a)s_{22}^{LR} - \frac{\sqrt{-1}}{4}(-\partial_1 + 2\partial_2)d_{22}(a)s_{23}^{LR}$$
$$- \frac{\sqrt{-1}}{4}(-\partial_1 + 2\partial_2)d_{33}(a)s_{32}^{LR} + \frac{1}{4}(-\partial_1 + 2\partial_2)d_{33}(a)s_{33}^{LR}$$
$$+ \left(\frac{1}{4sh(y_2)}d_{33}(a) - \frac{ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{22}^{LR}$$
$$+ \left(\frac{\sqrt{-1}}{4sh(y_2)}d_{33}(a) - \frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{22}(a)\right)s_{23}^{LR}$$

$$+\left(\frac{\sqrt{-1}}{4sh(y_2)}d_{22}(a)-\frac{\sqrt{-1}ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{32}^{LR}$$
$$+\left(-\frac{1}{4sh(y_2)}d_{22}(a)+\frac{ch(y_2)}{4sh(y_2)}d_{33}(a)\right)s_{33}^{LR}.$$

3)

$$\begin{aligned}&-\tilde{pr}_2(R_{w_3}\Psi\otimes w_1)(a)\\&=\tilde{pr}_2(R_{X_{1,3}}\Psi\otimes w_1)(a)-\sqrt{-1}\tilde{pr}_2(R_{X_{1,2}}\otimes w_1)(a).\end{aligned}$$

Firstly,

$$\begin{aligned}&\tilde{pr}_2(R_{X_{1,3}}\Psi\otimes w_1)(a)\\&=-\frac{1}{sh(y_1y_2)}R_{\mathrm{Ad}(a^{-1})K_{1,3}}\left(\sum_{i=1}^{3}d_{ii}(a)s_i^L\otimes s_i^R\right)w_1\\&+\frac{ch(y_1y_2)}{sh(y_1y_2)}R_{K_{1,3}}\left(\sum_{i=1}^{3}d_{ii}(a)s_i^L\otimes s_i^R\right)w_1\\&=-\frac{1}{sh(y_1y_2)}(-d_{11}(a)s_3^L\otimes s_1^Rw_1+d_{33}(a)s_1^L\otimes s_3^Rw_1)\\&-\frac{ch(y_1y_2)}{sh(y_1y_2)}(-d_{11}(a)s_1^L\otimes s_3^Rw_1+d_{33}(a)s_3^L\otimes s_1^Rw_1)\\&=\left(\frac{1}{4sh(y_1y_2)}d_{33}(a)-\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{11}(a)\right)s_{11}^{LR}\\&+\left(-\frac{\sqrt{-1}}{4sh(y_1y_2)}d_{11}(a)+\frac{\sqrt{-1}ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s_{32}^{LR}\\&+\left(-\frac{1}{4sh(y_1y_2)}d_{11}(a)+\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s_{33}^{LR}.\end{aligned}$$

Next,

$$\begin{aligned}&-\sqrt{-1}\tilde{pr}_2(R_{X_{1,2}}\Psi\otimes w_1)(a)\\&=\frac{\sqrt{-1}}{sh(y_1)}R_{\mathrm{Ad}(a^{-1})K_{1,2}}\left(\sum_{i=1}^{3}d_{ii}(a)s_i^L\otimes s_i^R\right)w_1\\&-\frac{\sqrt{-1}ch(y_1)}{sh(y_1)}R_{K_{1,2}}\left(\sum_{i=1}^{3}d_{ii}(a)s_i^L\otimes s_i^R\right)w_1\\&=\frac{\sqrt{-1}}{sh(y_1)}(-d_{11}(a)s_2^L\otimes s_1^Rw_1+d_{22}(a)s_1^L\otimes s_2^Rw_1)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\sqrt{-1}ch(y_1)}{sh(y_1)}(-d_{11}(a)s_1^L \otimes s_2^R w_1 + d_{22}(a)s_2^L \otimes s_1^R w_1) \\
&= \left(\frac{1}{4sh(y_1)}d_{22}(a) - \frac{ch(y_1)}{4sh(y_1)}d_{11}(a)\right) s_{11}^{LR} \\
&+ \left(\frac{\sqrt{-1}}{4sh(y_1)}d_{11}(a) - \frac{\sqrt{-1}ch(y_1)}{4sh(y_1)}d_{22}(a)\right) s_{23}^{LR} \\
&+ \left(-\frac{1}{4sh(y_1)}d_{11}(a) + \frac{ch(y_1)}{4sh(y_1)}d_{22}(a)\right) s_{22}^{LR}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&- \tilde{pr}_2(R_{w_3}\Psi \otimes w_1)(a) \\
&= \left(\frac{1}{4sh(y_1)}d_{22}(a) + \frac{1}{4sh(y_1y_2)}d_{33}(a)\right. \\
&\left. - \frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{11}(a) - \frac{ch(y_1)}{4sh(y_1)}d_{11}(a)\right) s_{11}^{LR} \\
&+ \left(\frac{\sqrt{-1}}{4sh(y_1)}d_{11}(a) - \frac{\sqrt{-1}ch(y_1)}{4sh(y_1)}d_{22}(a)\right) s_{23}^{LR} \\
&+ \left(-\frac{\sqrt{-1}}{4sh(y_1y_2)}d_{11}(a) + \frac{\sqrt{-1}ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right) s_{32}^{LR} \\
&+ \left(-\frac{1}{4sh(y_1)}d_{11}(a) + \frac{ch(y_1)}{4sh(y_1)}d_{22}(a)\right) s_{22}^{LR} \\
&+ \left(-\frac{1}{4sh(y_1y_2)}d_{11}(a) + \frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right) s_{33}^{LR}.
\end{aligned}
$$

4) Similarly, we have

$$
\begin{aligned}
&- \tilde{pr}_2(R_{w_1}\Psi \otimes w_3)(a) \\
&= \left(\frac{1}{4sh(y_1y_2)}d_{33}(a) - \frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{11}(a)\right. \\
&\left. + \frac{1}{4sh(y_1)}d_{22}(a) - \frac{ch(y_1)}{4sh(y_1)}d_{11}(a)\right) s_{11}^{LR} \\
&+ \left(-\frac{\sqrt{-1}}{4sh(y_1)}d_{11}(a) + \frac{\sqrt{-1}ch(y_1)}{4sh(y_1)}d_{22}(a)\right) s_{23}^{LR} \\
&+ \left(\frac{\sqrt{-1}}{4sh(y_1y_2)}d_{11}(a) - \frac{\sqrt{-1}ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right) s_{32}^{LR} \\
&+ \left(-\frac{1}{4sh(y_1)}d_{11}(a) + \frac{ch(y_1)}{4sh(y_1)}d_{22}(a)\right) s_{22}^{LR}
\end{aligned}
$$

$$+\left(-\frac{1}{4sh(y_1y_2)}d_{11}(a)+\frac{ch(y_1y_2)}{4sh(y_1y_2)}d_{33}(a)\right)s_{33}^{LR}.$$

5) Finally,

$$\begin{aligned}
&\frac{3}{2}\tilde{pr}_2(R_{w_2}\Psi\otimes w_2)(a)\\
&=\frac{3}{2}\tilde{pr}_2(R_{\frac{2}{3}(2H_{1,2}+H_{2,3})}\Psi\otimes w_2)(a)\\
&=R_{2H_{1,2}+H_{2,3}}\left(\sum_{i=1}^{3}d_{ii}(a)s_i^L\otimes s_i^R\right)w_2\\
&=-\partial_1 d_{11}(a)s_{11}^{LR}+\frac{1}{2}\partial_1 d_{22}(a)s_{22}^{LR}+\frac{1}{2}\partial_1 d_{33}(a)s_{33}^{LR}.
\end{aligned}$$

By combining these results, we have

$$\begin{aligned}
&4\tilde{pr}_2(\nabla_R\Psi)(a)\\
&=\left\{-\left(\partial_1+\frac{ch(y_1)}{2sh(y_1)}+\frac{ch(y_1y_2)}{2sh(y_1y_2)}\right)d_{11}(a)\right.\\
&\left.+\frac{1}{2sh(y_1)}d_{22}(a)+\frac{1}{2sh(y_1y_2)}d_{33}(a)\right\}s_{11}^{LR}\\
&+\left\{-\frac{1}{2sh(y_1)}d_{11}(a)+\left(\partial_1-\partial_2+\frac{ch(y_1)}{2sh(y_1)}-\frac{ch(y_2)}{2sh(y_2)}\right)d_{22}(a)\right.\\
&\left.+\frac{1}{2sh(y_2)}d_{33}(a)\right\}s_{22}^{LR}\\
&+\left\{-\frac{1}{2sh(y_1y_2)}d_{11}(a)-\frac{1}{2sh(y_2)}d_{22}(a)\right.\\
&\left.+\left(\partial_2+\frac{ch(y_2)}{2sh(y_2)}+\frac{ch(y_1y_2)}{2sh(y_1y_2)}\right)d_{33}(a)\right\}s_{33}^{LR}.
\end{aligned}$$

This equals $\lambda_i\sum_{j=1}^{3}d_{jj}(a)s_{jj}^{LR}$, where λ_i $(i=1,2,3)$ are some constants depending on the choice of $\sigma=\sigma_i$ of the principal series representation. These eigenvalues λ_i $(i=1,2,3)$ are computed in [6], and they are $\lambda_1=-\frac{1}{3}(2\nu_1-\nu_2)$, $\lambda_2=\frac{1}{3}(\nu_1-2\nu_2)$, $\lambda_3=\frac{1}{3}(\nu_1+\nu_2)$. Summing up, we have the following result:

Theorem 6.4. *Let* $\Psi(g)=\sum_{i=1}^{3}\sum_{j=1}^{3}d_{ij}(g)s_i^L\otimes s_j^R\in C^\infty_{\tau_2,\tau_2}(K\backslash G/K)$ *be a spherical function attached to the non-spherical principal series representation* $\pi_{\sigma_i,\nu}$ $(i=1,2,3)$. *Put*

$$d_{11}(a)=F(y_1,y_2),d_{22}(a)=G(y_1,y_2),d_{33}(a)=H(y_1,y_2).$$

Then F, G, H satisfy the following differential equations:

$$
\begin{aligned}
&-\left(\partial_1+\frac{ch(y_1)}{2sh(y_1)}+\frac{ch(y_1y_2)}{2sh(y_1y_2)}\right)F(y_1,y_2)+\frac{1}{2sh(y_1)}G(y_1,y_2)\\
&+\frac{1}{2sh(y_1y_2)}H(y_1,y_2)=\lambda_iF(y_1,y_2)
\end{aligned}
\tag{6.4}
$$

$$
\begin{aligned}
&-\frac{1}{2sh(y_1)}F(y_1,y_2)+\left(\partial_1-\partial_2+\frac{ch(y_1)}{2sh(y_1)}-\frac{ch(y_2)}{2sh(y_2)}\right)G(y_1,y_2)\\
&+\frac{1}{2sh(y_2)}H(y_1,y_2)=\lambda_iG(y_1,y_2)
\end{aligned}
\tag{6.5}
$$

$$
\begin{aligned}
&-\frac{1}{2sh(y_1y_2)}F(y_1,y_2)-\frac{1}{2sh(y_2)}G(y_1,y_2)\\
&+\left(\partial_2+\frac{ch(y_2)}{2sh(y_2)}+\frac{ch(y_1y_2)}{2sh(y_1y_2)}\right)H(y_1,y_2)=\lambda_iH(y_1,y_2).
\end{aligned}
\tag{6.6}
$$

Here, $\lambda_1=-\frac{1}{3}(2\nu_1-\nu_2)$, $\lambda_2=\frac{1}{3}(\nu_1-2\nu_2)$, $\lambda_3=\frac{1}{3}(\nu_1+\nu_2)$.

6.3 The expansion of the matrix coefficients in terms of the power series around $y_1=y_2=0$

We put

$$
F(y_1,y_2)=\sum_{n,m=0}^{\infty}a_{n,m}y_1^{n+\mu_1}y_2^{m+\mu_2},
$$

$$
G(y_1,y_2)=\sum_{n,m=0}^{\infty}b_{n,m}y_1^{n+\mu_1}y_2^{m+\mu_2},
$$

$$
H(y_1,y_2)=\sum_{n,m=0}^{\infty}c_{n,m}y_1^{n+\mu_1}y_2^{m+\mu_2}.
$$

$((a_{0,0},b_{0,0},c_{0,0})\neq(0,0,0))$. We want to find the recurrence relations between $(a_{n,m})$, $(b_{n,m})$, $(c_{n,m})$ and the values of characteristic roots (μ_1,μ_2). Hereafter, we assume that $1,\nu_1,\nu_2$ are linearly independent over $\mathbf{Q}$. By inserting these power series into the Casimir equation (6.1), (6.2), (6.3) and the gradient equation (6.4), (6.5), (6.6) and picking up the coefficient of $y_1^{n+\mu_1}y_2^{m+\mu_2}$, we have the following recurrence relations:

$$
\begin{aligned}
&\{2(n'^2 - n'm' + m'^2 - n' - m') - \lambda\}a_{n,m} \\
&+ 2\sum_{k=1}^{\infty}(-2n' + m' + 2k)a_{n-2k,m} + 2\sum_{k=1}^{\infty}(n' - 2m' + 4k)a_{n,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(-n' - m' + 2k)a_{n-2k,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(2k-1)b_{n-2k+1,m} + 2\sum_{k=1}^{\infty}(2k-1)c_{n-2k+1,m-2k+1} = 0,
\end{aligned} \tag{6.7}
$$

$$
\begin{aligned}
&\{2(n'^2 - n'm' + m'^2 - n' - m') - \lambda\}b_{n,m} \\
&+ 2\sum_{k=1}^{\infty}(-2n' + m' + 2k)b_{n-2k,m} + 2\sum_{k=1}^{\infty}(n' - 2m' + 2k)b_{n,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(-n' - m' + 4k)b_{n-2k,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(2k-1)a_{n-2k+1,m} + 2\sum_{k=1}^{\infty}(2k-1)c_{n,m-2k+1} = 0,
\end{aligned} \tag{6.8}
$$

$$
\begin{aligned}
&\{2(n'^2 - n'm' + m'^2 - n' - m') - \lambda\}c_{n,m} \\
&+ 2\sum_{k=1}^{\infty}(-2n' + m' + 4k)c_{n-2k,m} + 2\sum_{k=1}^{\infty}(n' - 2m' + 2k)c_{n,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(-n' - m' + 2k)c_{n-2k,m-2k} \\
&+ 2\sum_{k=1}^{\infty}(2k-1)a_{n-2k+1,m-2k+1} + 2\sum_{k=1}^{\infty}(2k-1)b_{n,m-2k+1} = 0,
\end{aligned} \tag{6.9}
$$

$$
\begin{aligned}
&(-n' + 1 - \lambda_i)a_{n,m} + \sum_{k=1}^{\infty} a_{n-2k,m} + \sum_{k=1}^{\infty} a_{n-2k,m-2k} \\
&- \sum_{k=1}^{\infty} b_{n-2k+1,m} - \sum_{k=1}^{\infty} c_{n-2k+1,m-2k+1} = 0,
\end{aligned} \tag{6.10}
$$

$$
\begin{aligned}
&(n' - m' - \lambda_i)b_{n,m} - \sum_{k=1}^{\infty} b_{n-2k,m} + \sum_{k=1}^{\infty} b_{n,m-2k} \\
&+ \sum_{k=1}^{\infty} a_{n-2k+1,m} - \sum_{k=1}^{\infty} c_{n,m-2k+1} = 0,
\end{aligned} \tag{6.11}
$$

$$
\begin{aligned}
&(m' - 1 - \lambda_i)c_{n,m} - \sum_{k=1}^{\infty} c_{n,m-2k} - \sum_{k=1}^{\infty} c_{n-2k,m-2k} \\
&+ \sum_{k=1}^{\infty} a_{n-2k+1,m-2k+1} + \sum_{k=1}^{\infty} b_{n,m-2k+1} = 0.
\end{aligned} \tag{6.12}
$$

Here, $n' = n+\mu_1$, $m' = m+\mu_2$, λ is the eigenvalue of the Casimir operator given in Theorem 6.2, and λ_i $(i = 1, 2, 3)$ are the eigenvalues of the gradient operator given in Theorem 6.4. In these identities, we assume that $a_{i,j}, b_{i,j}, c_{i,j} = 0$ if $i < 0$ or $j < 0$. Note that in this computation, we used the power series expansions

$$
\frac{ch(y)}{sh(y)} = -1 - 2\sum_{k=1}^{\infty} y^{2k}, \; \frac{1}{sh(y)} = -2\sum_{k=1}^{\infty} y^{2k-1},
$$

$$
\frac{1}{sh(y)^2} = 4\sum_{k=1}^{\infty} k y^{2k}, \; \frac{ch(y)}{sh(y)^2} = 2\sum_{k=1}^{\infty}(2k-1)y^{2k-1}
$$

$(|y| < 1)$. By inserting $n = m = 0$ into (6.7), (6.8), (6.9), we have

$$
2(\mu_1^2 - \mu_1\mu_2 + \mu_2^2 - \mu_1 - \mu_2) - \lambda = 0. \tag{6.13}
$$

Moreover, by inserting $n = m = 0$ into (6.10), (6.11), (6.12), we have

$$
(-\mu_1 + 1 - \lambda_i)a_{0,0} = 0
$$

$$
(\mu_1 - \mu_2 - \lambda_i)b_{0,0} = 0
$$

$$
(\mu_2 - 1 - \lambda_i)c_{0,0} = 0.
$$

Since $(a_{0,0}, b_{0,0}, c_{0,0}) \neq (0, 0, 0)$, at least one of $-\mu_1 + 1 - \lambda_i$, $\mu_1 - \mu_2 - \lambda_i$, $\mu_2 - 1 - \lambda_i$ is 0. By combining this with the equation (6.13), we can compute the values of (μ_1, μ_2). (Because of the assumption of the linearly independence of $1, \nu_1, \nu_2$, we know that just one of $-\mu_1 + 1 - \lambda_i$, $\mu_1 - \mu_2 - \lambda_i$, $\mu_2 - 1 - \lambda_i$ is 0, and the other two are not 0.)

Proposition 6.5. 1) *In case of* $\sigma = \sigma_1$, $\lambda_i = \lambda_1 = -\frac{1}{3}(2\nu_1 - \nu_2)$.
a) If $-\mu_1 + 1 - \lambda_1 = 0$,

$$
(\mu_1, \mu_2) = \left(\frac{2\nu_1 - \nu_2 + 3}{3}, \frac{\nu_1 - 2\nu_2 + 3}{3}\right), \left(\frac{2\nu_1 - \nu_2 + 3}{3}, \frac{\nu_1 + \nu_2 + 3}{3}\right)
$$

and $a_{0,0} \neq 0, b_{0,0} = 0, c_{0,0} = 0$.
b) If $\mu_1 - \mu_2 - \lambda_1 = 0$,

$$
(\mu_1, \mu_2) = \left(\frac{2\nu_2 - \nu_1 + 3}{3}, \frac{\nu_1 + \nu_2 + 3}{3}\right), \left(\frac{-\nu_1 - \nu_2 + 3}{3}, \frac{\nu_1 - 2\nu_2 + 3}{3}\right)
$$

and $a_{0,0} = 0, b_{0,0} \neq 0, c_{0,0} = 0$.
c) If $\mu_2 - 1 - \lambda_1 = 0$,

$$(\mu_1, \mu_2) = \left(\frac{2\nu_2 - \nu_1 + 3}{3}, \frac{\nu_2 - 2\nu_1 + 3}{3}\right), \left(\frac{-\nu_1 - \nu_2 + 3}{3}, \frac{\nu_2 - 2\nu_1 + 3}{3}\right)$$

and $a_{0,0} = 0, b_{0,0} = 0, c_{0,0} \neq 0$.
2) *The change of notation* $\lambda_1 \mapsto \lambda_2$, $\nu_1 \mapsto \nu_2$, $\nu_2 \mapsto \nu_1$ *in* 1) *gives the values of* (μ_1, μ_2) *in case of* $\sigma = \sigma_2$, *and the change of notation* $\lambda_1 \mapsto \lambda_3$, $\nu_1 \mapsto -\nu_1$, $\nu_2 \mapsto -\nu_1 + \nu_2$ *in* 1) *gives the values of* (μ_1, μ_2) *in case of* $\sigma = \sigma_3$.

Because of the same reasons we mentioned in section 5, we leave the explicit formulas for the power series to the " Appendix ", section 7 of this paper. Now, we have known that there exsists six power series corresponding to the six characteristic roots given in Proposition 6.5, and the coefficients of power series satisfy the recurrence relations from (6.7) to (6.12). Firstly, we take $\sigma = \sigma_1$ for the character of M. Let $\Psi = {}^t(F, G, H)$ be the matrix coefficient with K-type of three dimensional tautological representation, and $\psi_{\alpha,\beta} = {}^t(f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta})$ be the power series solution around $y_1 = y_2 = 0$ corresponding to the characteristic root (α, β) whose first term is ${}^t(1,0,0)$ or ${}^t(0,1,0)$ or ${}^t(0,0,1)$.
Since α is one of the values of μ_1 in Proposition 6.5, α takes following three values:

$$\alpha_1 = -\frac{1}{3}(\nu_1 + \nu_2) + 1, \alpha_2 = \frac{1}{3}(2\nu_1 - \nu_2) + 1, \alpha_3 = \frac{1}{3}(2\nu_2 - \nu_1) + 1.$$

Therefore, we can write

$$F(y_1, y_2) = \sum_{i=1}^{3} d_i b_i^{(1)}(y_1, y_2) y_1^{\alpha_i}$$
$$G(y_1, y_2) = \sum_{i=1}^{3} d_i b_i^{(2)}(y_1, y_2) y_1^{\alpha_i}$$
$$H(y_1, y_2) = \sum_{i=1}^{3} d_i b_i^{(3)}(y_1, y_2) y_1^{\alpha_i}.$$

Here, $d_i (i = 1, 2, 3)$ are some constants and $b_i^{(j)}(y_1, y_2)$ are analytic functions for $0 < y_2 < 1$ and $0 < y_1 \ll 1$. By inserting $F = b_2^{(1)}(y_1, y_2) y_1^{\alpha_2}, G = b_2^{(2)}(y_1, y_2) y_1^{\alpha_2}, H = b_2^{(3)}(y_1, y_2) y_1^{\alpha_2}$ into the equation (6.1), we have

$$\begin{aligned} y_1^{\alpha_2}\Big\{ & 2(\partial_1^2 b_2^{(1)}(y_1, y_2) + 2\alpha_2 \partial_1 b_2^{(1)}(y_1, y_2) + \alpha_2^2 b_2^{(1)}(y_1, y_2) \\ & - \partial_1 \partial_2 b_2^{(1)}(y_1, y_2) - \alpha_2 \partial_2 b_2^{(1)}(y_1, y_2) + \partial_2^2 b_2^{(1)}(y_1, y_2)) \\ & + \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1 y_2)}{sh(y_1 y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)(\alpha_2 b_2^{(1)}(y_1, y_2) + \partial_1 b_2^{(1)}(y_1, y_2)) \end{aligned}$$

$$
\begin{aligned}
&+\left(-\frac{ch(y_1)}{sh(y_1)}+\frac{ch(y_1y_2)}{sh(y_1y_2)}+2\frac{ch(y_2)}{sh(y_2)}\right)\partial_2 b_2^{(1)}(y_1,y_2)\\
&-\left(\frac{1}{sh(y_1)^2}+\frac{1}{sh(y_1y_2)^2}\right)b_2^{(1)}(y_1,y_2)\\
&+\frac{ch(y_1)}{sh(y_1)^2}b_2^{(2)}(y_1,y_2)+\frac{ch(y_1y_2)}{sh(y_1y_2)^2}b_2^{(3)}(y_1,y_2)\\
&-\lambda b_2^{(1)}(y_1,y_2)\Big\}=0.
\end{aligned}
$$

By dividing both sides by $y_1^{\alpha_2}$ and taking the limit $y_1 \to 0$, we have

$$
\begin{aligned}
2\partial_2^2 b_2^{(1)}(0,y_2)&+\left(2\frac{y_2^2+1}{y_2^2-1}-2\alpha_2\right)\partial_2 b_2^{(1)}(0,y_2)\\
&+\left(2\alpha_2^2-\frac{y_2^2+1}{y_2^2-1}\alpha_2-3\alpha_2-\lambda\right)b_2^{(1)}(0,y_2)=0.
\end{aligned}
$$

(Note that because of Proposition 6.5, since $a_{0,0}\neq 0$ if $\alpha=\alpha_2$, $b_2^{(1)}(0,y_2)$ is not identically 0. Hereafter the same statement holds for all the functions we compute.) This equation is the same type of equation as (5.8). By solving this, we have

$$
\begin{aligned}
b_2^{(1)}(0,y_2)=&y_2^{-\frac{1}{3}(2\nu_2-\nu_1)+1}\frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}{}_2F_1\left(-\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};-\frac{1}{2}\nu_2+1;y_2^2\right)\\
&+y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}\frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}{}_2F_1\left(\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};\frac{1}{2}\nu_2+1;y_2^2\right)
\end{aligned}
$$

and by comparing the leading terms, we have

$$
\begin{aligned}
&f_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}(y_1,y_2)\\
&=y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}y_2^{-\frac{1}{3}(2\nu_2-\nu_1)+1}{}_2F_1\left(-\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};-\frac{1}{2}\nu_2+1;y_2^2\right)\\
&+(\text{higher order terms with respect to } y_1),
\end{aligned}
$$

$$
\begin{aligned}
&f_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}(y_1,y_2)\\
&=y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}{}_2F_1\left(\frac{1}{2}\nu_2+\frac{1}{2},\frac{1}{2};\frac{1}{2}\nu_2+1;y_2^2\right)\\
&+(\text{higher order terms with respect to } y_1)
\end{aligned}
$$

and

$$\Psi = d_2\left\{\frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1} + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\right\} + (\text{linear combination of the other four solutions}). \tag{6.14}$$

Next, since β is one of the values of μ_2 in Proposition 6.5, β takes following three values:

$$\beta_1 = \frac{1}{3}(\nu_1+\nu_2)+1, \beta_2 = -\frac{1}{3}(2\nu_1-\nu_2)+1, \beta_3 = -\frac{1}{3}(2\nu_2-\nu_1)+1.$$

Therefore, we can write

$$F(y_1,y_2) = \sum_{i=1}^{3} c_i a_i^{(1)}(y_1,y_2)y_2^{\beta_i},$$
$$G(y_1,y_2) = \sum_{i=1}^{3} c_i a_i^{(2)}(y_1,y_2)y_2^{\beta_i},$$
$$H(y_1,y_2) = \sum_{i=1}^{3} c_i a_i^{(3)}(y_1,y_2)y_2^{\beta_i}.$$

By inserting $F = a_2^{(1)}(y_1,y_2)y_2^{\beta_2}$, $G = a_2^{(2)}(y_1,y_2)y_2^{\beta_2}$, $H = a_2^{(3)}(y_1,y_2)y_2^{\beta_2}$ into the equation (6.3) and applying the same method, we have

$$\Psi = c_2\left\{\frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1} + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2+\frac{1}{2})}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,-\frac{1}{3}(2\nu_1-\nu_2)+1}\right\} + (\text{linear combination of the other four solutions}). \tag{6.15}$$

Next, we take $i = 1$ or 3. By inserting $F = a_i^{(1)}(y_1,y_2)y_2^{\beta_i}, G = a_i^{(2)}(y_1,y_2)y_2^{\beta_i}, H = a_i^{(3)}(y_1,y_2)y_2^{\beta_i}$ into the equation (6.1), we have

$$y_2^{\beta_i}\Big\{2(\partial_1^2 a_i^{(1)}(y_1,y_2) - \partial_1\partial_2 a_i^{(1)}(y_1,y_2) - \beta_i\partial_1 a_i^{(1)}(y_1,y_2) + \partial_2^2 a_i^{(1)}(y_1,y_2) + 2\beta_i a_i^{(1)}(y_1,y_2) + \beta_i^2 a_i^{(1)}(y_1,y_2)) + \left(2\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} - \frac{ch(y_2)}{sh(y_2)}\right)\partial_1 a_i^{(1)}(y_1,y_2) + \left(-\frac{ch(y_1)}{sh(y_1)} + \frac{ch(y_1y_2)}{sh(y_1y_2)} + 2\frac{ch(y_2)}{sh(y_2)}\right)(\beta_i a_i^{(1)}(y_1,y_2) + \partial_2 a_i^{(1)}(y_1,y_2))$$

$$
\begin{aligned}
&- \left(\frac{1}{sh(y_1)^2} + \frac{1}{sh(y_1y_2)^2}\right) a_i^{(1)}(y_1, y_2) \\
&+ \frac{ch(y_1)}{sh(y_1)^2} a_i^{(2)}(y_1, y_2) + \frac{ch(y_1y_2)}{sh(y_1y_2)^2} a_i^{(3)}(y_1, y_2) \\
&- \lambda a_i^{(1)}(y_1, y_2)\Big\} = 0.
\end{aligned}
$$

By dividing both sides by $y_2^{\beta_i}$ and taking the limit $y_2 \to 0$, we have

$$
\begin{aligned}
&2\partial_1^2 a_i^{(1)}(y_1, 0) + \left(2\frac{y_1^2+1}{y_1^2-1} - 2\beta_i\right) \partial_1 a_i^{(1)}(y_1, 0) \\
&+ \left(2\beta_i^2 - \frac{y_1^2+1}{y_1^2-1}\beta_i - 3\beta_i - \frac{4y_1^2}{(y_1^2-1)^2} - \lambda\right) a_i^{(1)}(y_1, 0) \\
&+ \frac{2(y_1^3+y_1)}{(y_1^2-1)^2} a_i^{(2)}(y_1, 0) = 0.
\end{aligned}
\tag{6.16}
$$

Next, by inserting $F = a_i^{(1)}(y_1, y_2)y_2^{\beta_i}, G = a_i^{(2)}(y_1, y_2)y_2^{\beta_i}, H = a_i^{(3)}(y_1, y_2)y_2^{\beta_i}$ into the equation (6.4), we have

$$
\begin{aligned}
&y_2^{\beta_i}\Big\{ - \partial_1 a_i^{(1)}(y_1, y_2) - \frac{ch(y_1)}{2sh(y_1)} a_i^{(1)}(y_1, y_2) \\
&- \frac{ch(y_1y_2)}{2sh(y_1y_2)} a_i^{(1)}(y_1, y_2) + \frac{1}{2sh(y_1)} a_i^{(2)}(y_1, y_2) \\
&+ \frac{1}{2sh(y_1y_2)} a_i^{(3)}(y_1, y_2) - \lambda_1 a_i^{(1)}(y_1, y_2)\Big\} = 0.
\end{aligned}
$$

By dividing both sides by $y_2^{\beta_i}$ and taking the limit $y_2 \to 0$, we have

$$
\frac{y_1}{y_1^2-1} a_i^{(2)}(y_1, 0) = \partial_1 a_i^{(1)}(y_1, 0) + \left(\frac{y_1^2+1}{2(y_1^2-1)} + \lambda_1 - \frac{1}{2}\right) a_i^{(1)}(y_1, 0). \tag{6.17}
$$

By combining equations (6.16) and (6.17) to eliminate $a_i^{(2)}(y_1, 0)$, we have

$$
\begin{aligned}
&2\partial_1^2 a_i^{(1)}(y_1, 0) + \left(4\frac{y_1^2+1}{y_1^2-1} - 2\beta_i\right) \partial_1 a_i^{(1)}(y_1, 0) \\
&+ \left(\frac{y_1^2+1}{y_1^2-1}(2\lambda_1 - \beta_i - 1) + 2\beta_i^2 - 3\beta_i - \lambda + 1\right) a_i^{(1)}(y_1, 0) = 0.
\end{aligned}
$$

We put $y_1^2 = u$, $f_i(u) := a_i^{(1)}(y_1, 0)$ and define a differential operator $\tilde{\partial}_1$ by $\tilde{\partial}_1 := u\frac{d}{du}$. Then the equation becomes

$$
8\tilde{\partial}_1{}^2 f_i(u) + \left(8 - 4\beta_i + \frac{16}{u-1}\right) \tilde{\partial}_1 f_i(u)
$$

$$+\left(2\beta_i^2-4\beta_i+2\lambda_1-\lambda+\frac{4\lambda_1-2\beta_i-2}{u-1}\right)f_i(u)=0.$$

Next, we put $f_i(u)=u^x g_i(u)$ $(x\in\mathbf{C})$. Then the equation becomes

$$8\tilde{\partial_1}^2 g_i(u)+\left(8-4\beta_i+\frac{16}{u-1}+16x\right)\tilde{\partial_1}g_i(u)$$
$$+\left(8x^2+(8-4\beta_i)x+2\beta_i^2-4\beta_i+2\lambda_1-\lambda+\frac{16x+4\lambda_1-2\beta_i-2}{u-1}\right)g_i(u)=0.$$

We take $x=x_i$ as the number satisfying

$$8x_i^2+(8-4\beta_i)x_i+2\beta_i^2-4\beta_i+2\lambda_1-\lambda=0.$$

Then we have

$$8u^2\frac{d^2g_i}{du^2}+\left(16-4\beta_i+16x_i+\frac{16}{u-1}\right)u\frac{dg_i}{du}$$
$$+\frac{16x_i+4\lambda_1-2\beta_i-2}{u-1}g_i(u)=0.$$

Finally, we put $u=\frac{1}{\zeta}$. Then the equation becomes

$$\begin{aligned}&\zeta(\zeta-1)\frac{d^2g_i}{d\zeta^2}\\&+\left(\left(\frac{1}{2}\beta_i-2x_i+2\right)\zeta-\frac{1}{2}\beta_i+2x_i\right)\frac{dg_i}{d\zeta}\\&+\left(-2x_i-\frac{1}{2}\lambda_1+\frac{1}{4}\beta_i+\frac{1}{4}\right)g_i=0.\end{aligned}\tag{6.18}$$

(6.18) is a Gaussian hypergeometric differential equation, and if we define p_i,q_i by complex numbers satisfying

$$1+p_i+q_i=\frac{1}{2}\beta_i-2x_i+2$$
$$p_iq_i=-2x_i-\frac{1}{2}\lambda_1+\frac{1}{4}\beta_i+\frac{1}{4}$$

and r_i by

$$r_i=\frac{1}{2}\beta_i-2x_i,$$

then the general solution is expressed by

$$P\left\{\begin{matrix}0&1&\infty&\\0&0&p_i&;\zeta\\1-r_i&r_i-p_i-q_i&q_i&\end{matrix}\right\}=P\left\{\begin{matrix}0&1&\infty&\\0&0&p_i&;1-\zeta\\r_i-p_i-q_i&1-r_i&q_i&\end{matrix}\right\}.$$

The regular solution is given by

$$g_i(y_1)={}_2F_1\left(p_i,q_i;1-r_i+p_i+q_i;1-\zeta\right)$$

$$= {}_2F_1\left(p_i, q_i; 1 - r_i + p_i + q_i; 1 - \frac{1}{y_1^2}\right).$$

Since ${}_2F_1$ satisfies a formula (5.13), we have

$$\begin{aligned} g_i(y_1) =& y_1^{2p_i} \frac{\Gamma(1 - r_i + p_i + q_i)\Gamma(q_i - p_i)}{\Gamma(1 - r_i + q_i)\Gamma(q_i)} {}_2F_1(p_i, 1 - r_i + p_i; 1 + p_i - q_i; y_1^2) \\ &+ y_1^{2q_i} \frac{\Gamma(1 - r_i + p_i + q_i)\Gamma(p_i - q_i)}{\Gamma(1 - r_i + p_i)\Gamma(p_i)} {}_2F_1(q_i, 1 - r_i + q_i; 1 + q_i - p_i; y_1^2). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &a_i^{(1)}(y_1, 0) = u^{x_i} g_i(y_1) = y_1^{2x_i} g_i(y_1) \\ &= y_1^{2(p_i + x_i)} \frac{\Gamma(1 - r_i + p_i + q_i)\Gamma(q_i - p_i)}{\Gamma(1 - r_i + q_i)\Gamma(q_i)} {}_2F_1(p_i, 1 - r_i + p_i; 1 + p_i - q_i; y_1^2) \\ &+ y_1^{2(q_i + x_i)} \frac{\Gamma(1 - r_i + p_i + q_i)\Gamma(p_i - q_i)}{\Gamma(1 - r_i + p_i)\Gamma(p_i)} {}_2F_1(q_i, 1 - r_i + q_i; 1 + q_i - p_i; y_1^2) \end{aligned} \tag{6.19}$$

for $i = 1, 3$. The values (x_i, p_i, q_i, r_i) $(i = 1, 3)$ are given as follows:
A) When $i = 1$, we have

$$\begin{aligned} x_1 &= \frac{1}{3}\nu_1 - \frac{1}{6}\nu_2, \quad p_1 = -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1, \\ q_1 &= \frac{1}{2}, \quad r_1 = -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2}. \end{aligned}$$

B) When $i = 3$, we have

$$\begin{aligned} x_3 &= \frac{1}{3}\nu_1 - \frac{1}{6}\nu_2, \quad p_3 = -\frac{1}{2}\nu_1 + 1, \\ q_3 &= \frac{1}{2}, \quad r_3 = -\frac{1}{2}\nu_1 + \frac{1}{2}. \end{aligned}$$

By inserting these results into (6.19), we have

$$\begin{aligned} &a_1^{(1)}(y_1, 0) \\ &= y_1^{\frac{1}{3}(2\nu_2 - \nu_1) + 2} \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 - \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)} {}_2F_1\left(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1, \frac{3}{2}; -\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{3}{2}; y_1^2\right) \\ &+ y_1^{\frac{1}{3}(2\nu_1 - \nu_2) + 1} \frac{2\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)} {}_2F_1\left(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1, \frac{1}{2}; \frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2}; y_1^2\right), \\ &a_3^{(1)}(y_1, 0) \\ &= y_1^{-\frac{1}{3}(\nu_1 + \nu_2) + 2} \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + 1)} {}_2F_1\left(-\frac{1}{2}\nu_1 + 1, \frac{3}{2}; -\frac{1}{2}\nu_1 + \frac{3}{2}; y_1^2\right) \end{aligned}$$

$$+y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}\frac{2\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)}{}_2F_1\left(\frac{1}{2}\nu_1+1,\frac{1}{2};\frac{1}{2}\nu_1+\frac{1}{2};y_1^2\right).$$

Here, we used $\Gamma(\frac{1}{2})=\sqrt{\pi}$, $\Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{2}$. From the equation of $a_1^{(1)}(y_1,0)$, we have

$$\begin{aligned}&f_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}(y_1,y_2)\\&=y_1^{\frac{1}{3}(2\nu_1-\nu_2)+1}y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}{}_2F_1\left(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1,\frac{1}{2};\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2};y_1^2\right)\\&+(\text{higher order terms with respect to } y_2),\end{aligned}$$

$$\begin{aligned}&f_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}(y_1,y_2)\\&=y_1^{\frac{1}{3}(2\nu_2-\nu_1)+2}y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}{}_2F_1\left(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1,\frac{3}{2};-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{3}{2};y_1^2\right)\\&+(\text{higher order terms with respect to } y_2).\end{aligned}$$

And since the coefficient of $y_1^{\frac{1}{3}(2\nu_2-\nu_1)+2}y_2^{\frac{1}{3}(\nu_1+\nu_2)+1}$ of $f_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}$ in $\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}$ is $\frac{1}{\nu_1-\nu_2-1}$, if the coefficient of $\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}$ is $\frac{2\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}$, the coefficient of $\Psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}$ is given by

$$\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2-\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\times(\nu_1-\nu_2-1)=\frac{2\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}.$$

Therefore, we have

$$\begin{aligned}\Psi=&c_1\left\{\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2}\nu_2+1)}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\right.\\&\left.+\frac{\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1-\frac{1}{2}\nu_2+1)}\psi_{\frac{1}{3}(2\nu_2-\nu_1)+1,\frac{1}{3}(\nu_1+\nu_2)+1}\right\}\\&+(\text{linear combination of the other four solutions}).\end{aligned}\tag{6.20}$$

Similarly, from the equation of $a_3^{(1)}(y_1,0)$, we have

$$\begin{aligned}\Psi=&c_3\left\{\frac{\Gamma(-\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1+1)}\psi_{\frac{1}{3}(2\nu_1-\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\right.\\&\left.+\frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1+1)}\psi_{-\frac{1}{3}(\nu_1+\nu_2)+1,-\frac{1}{3}(2\nu_2-\nu_1)+1}\right\}\\&+(\text{linear combination of the other four solutions}).\end{aligned}\tag{6.21}$$

Next, we insert $F=b_i^{(1)}(y_1,y_2)y_1^{\alpha_i}$, $G=b_i^{(2)}(y_1,y_2)y_1^{\alpha_i}$, $H=b_i^{(3)}(y_1,y_2)y_1^{\alpha_i}$ $(i=1,3)$ into the equation (6.3), (6.6). By applying the same method as we used above to $b_i^{(j)}(y_1, y_2)$ $(i = 1, 3, j = 1, 2, 3)$ (eliminate $b_i^{(2)}(0, y_2)$ and construct the differential equation with respect to $b_i^{(3)}(0, y_2)$), we have

$$\begin{aligned}\Psi =& d_1 \left\{ \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\ & \left. + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right\} \\ & + (\text{linear combination of the other four solutions})\end{aligned} \tag{6.22}$$

and

$$\begin{aligned}\Psi =& d_3 \left\{ \frac{\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + 1)} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\ & \left. + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + 1)} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\} \\ & + (\text{linear combination of the other four solutions}).\end{aligned} \tag{6.23}$$

Now we have six equations with respect to the matrix coefficient Ψ (i.e. (6.20), (6.15), (6.21), (6.22), (6.14), (6.23)). By combining these equations, we obtain two different expressions of Ψ. That is,

$$\begin{aligned}\Psi =& c_1 \left\{ \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right. \\ & \left. + \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\} \\ & + c_2 \left\{ \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\ & \left. + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right\} \\ & + c_3 \left\{ \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + 1)} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right. \\ & \left. + \frac{\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + 1)} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right\} \\ =& d_1 \left\{ \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \right. \\ & \left. + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right\} \\ & + d_2 \left\{ \frac{\Gamma(\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \right. \\ & \left. + \frac{\Gamma(-\frac{1}{2}\nu_2)}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\}\end{aligned}$$

$$+ d_3 \left\{ \frac{\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + 1)} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + 1)} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \right\}.$$

By comparing the coefficients of $\psi_{\alpha,\beta}$, we have

$$c_1 = \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2)}{\Gamma(-\frac{1}{2}\nu_1 + 1)\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})}, \quad c_2 = \frac{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})}{\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)\Gamma(\frac{1}{2}\nu_1 + 1)},$$

$$c_3 = \frac{\Gamma(\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)}$$

up to (the same) constant multiples. Thus we obtained the expression of Ψ in case of $\sigma = \sigma_1$. Note that since the transform $\nu_1 \mapsto \nu_2, \nu_2 \mapsto \nu_1$ does not change the eigenvalue of Casimir operator λ and change the eigenvalue of gradient operator λ_1 to λ_2, this transform gives the expression of Ψ in case of $\sigma = \sigma_2$. Similarly, the transform $\nu_1 \mapsto -\nu_1, \nu_2 \mapsto -\nu_1 + \nu_2$ gives the expression in case of $\sigma = \sigma_3$. Trerefore, we obtained the following theorem.

Theorem 6.6. *Let $\Psi = {}^t(F, G, H) \in C^\infty_{\tau_2,\tau_2}(K\backslash G/K)|_A$ be a spherical function attached to the non-spherical principal series $\pi_{\sigma_1,\nu}$ whose minimal K-type is three dimensional tautological representation τ_2, and $\psi_{\alpha,\beta} = {}^t(f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta})$ be the power series solution around $y_1 = y_2 = 0$ corresponding to the characteristic root (α, β) whose first term is ${}^t(1,0,0)$ or ${}^t(0,1,0)$ or ${}^t(0,0,1)$. Then we have*

$$\begin{aligned}
\Psi = & \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + 1)\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \\
& + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + 1)\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, \frac{1}{3}(\nu_1+\nu_2)+1} \\
& + \frac{\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + 1)\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \\
& + \frac{\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_2)\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + 1)\Gamma(-\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_1 - \frac{1}{2}\nu_2 + 1)} \psi_{\frac{1}{3}(2\nu_2-\nu_1)+1, -\frac{1}{3}(2\nu_1-\nu_2)+1} \\
& + \frac{\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(-\frac{1}{2}\nu_1 + 1)\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)} \psi_{\frac{1}{3}(2\nu_1-\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1} \\
& + \frac{\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2})\Gamma(\frac{1}{2}\nu_2)\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{1}{2}\nu_1 + 1)\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2})\Gamma(-\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 1)} \psi_{-\frac{1}{3}(\nu_1+\nu_2)+1, -\frac{1}{3}(2\nu_2-\nu_1)+1}.
\end{aligned} \tag{6.24}$$

The transform $\nu_1 \mapsto \nu_2, \nu_2 \mapsto \nu_1$ in (6.24) gives the expression of Ψ in case of $\sigma = \sigma_2$ and the transform $\nu_1 \mapsto -\nu_1, \nu_2 \mapsto -\nu_1 + \nu_2$ in (6.24) gives the expression of Ψ in case of $\sigma = \sigma_3$.

7 Appendix

We give the explicit formulas of the coefficients of power series solutions of both spherical and non-spherical case under certain assumptions without proof. The spherical functions corresponding to the matrix coefficients with minimal K-types are expressed by the linear combinations of these power series (Theorem 5.6, Theorem 6.6). There is a similar result in [6], in the case of Whittaker functions.

7.1 In case of spherical principal series representation

Firstly, we consider the matrix coefficient attached to the spherical principal series. Let $F(y_1, y_2)$ be the matrix coefficient of the spherical principal series representation restricted to A. Next, we put

$$F(y_1, y_2) = sh(y_1)^{-\frac{1}{2}} sh(y_2)^{-\frac{1}{2}} sh(y_1 y_2)^{-\frac{1}{2}} G(y_1, y_2) \quad (0 < y_1, y_2 < 1)$$

and compute the power series of G at the origin $y_1 = y_2 = 0$. We put

$$G(y_1, y_2) = \sum_{n,m=0}^{\infty} \tilde{a}_{n,m} y_1^{n+\tilde{\mu}_1} y_2^{m+\tilde{\mu}_2} \quad (\tilde{a}_{0,0} \neq 0). \tag{7.1}$$

The characteristic roots take six values:

$$(\tilde{\mu}_1, \tilde{\mu}_2) = (\mu_1 - 1, \mu_2 - 1)$$

for the six values (μ_1, μ_2) given in Proposition 5.5. We put

$$p(n, m) = n^2 - nm + m^2 + (2\tilde{\mu}_1 - \tilde{\mu}_2)n + (2\tilde{\mu}_2 - \tilde{\mu}_1)m.$$

Assume that $p(n, m) \neq 0$ if $(n, m) \neq (0, 0)$. Let $\mathbf{P}_{n,m}$ be the family of all sets $\{p(2n_k, 2m_k), \cdots, p(2n_0, 2m_0)\}$ such that

$$n_k = n, m_k = m, n_0 = m_0 = 0$$

and

$$(n_{i+1}, m_{i+1}) = (n_i + l_i, m_i) \, or \, (n_i, m_i + l_i) \, or \, (n_i + l_i, m_i + l_i)$$
$$(\exists l_i \in \mathbf{Z}_{>0}), \quad (i = 0, \cdots, k - 1).$$

(Here, k depends on each set). For $\{p(2n_k, 2m_k), \cdots, p(2n_0, 2m_0)\} \in \mathbf{P}_{n,m}$ and $0 \leq i \leq k - 1$, we define $d_i \in \mathbf{Z}$ by $d_i = -l_i$. And we put

$$C_{(n_1, \cdots n_k; m_1, \cdots m_k)} = \prod_{i=0}^{k-1} d_i. \tag{7.2}$$

Then, $\tilde{a}_{n,m} = 0$ if n or m is odd, and

$$\tilde{a}_{2n,2m} = \sum_{\{p(2n_k,2m_k),\cdots,p(2n_0,2m_0)\} \in \mathbf{P}_{n,m}} \frac{C_{(n_1, \cdots n_k; m_1, \cdots m_k)}}{p(2n_k, 2m_k) \cdots p(2n_1, 2m_1)} \tilde{a}_{0,0}. \tag{7.3}$$

for $(n, m) \neq (0, 0)$.

7.2 In case of non-spherical principal series representations

Next, we give the explicit formulas for the power series of the matrix coefficients of the non-spherical principal series. We give the power series solution of the equations obtained in Theorem 6.3 and Theorem 6.5. Firstly, we modify F, G, H by

$$\tilde{F}(y_1, y_2) = sh(y_1)^{\frac{1}{2}} sh(y_2)^{\frac{1}{2}} sh(y_1 y_2)^{\frac{1}{2}} F(y_1, y_2)$$
$$\tilde{G}(y_1, y_2) = sh(y_1)^{\frac{1}{2}} sh(y_2)^{\frac{1}{2}} sh(y_1 y_2)^{\frac{1}{2}} G(y_1, y_2)$$
$$\tilde{H}(y_1, y_2) = sh(y_1)^{\frac{1}{2}} sh(y_2)^{\frac{1}{2}} sh(y_1 y_2)^{\frac{1}{2}} H(y_1, y_2).$$

We put

$$\tilde{F}(y_1, y_2) = \sum_{n,m=0}^{\infty} \tilde{a}_{n,m} y_1^{n+\tilde{\mu}_1} y_2^{m+\tilde{\mu}_2} \tag{7.4}$$

$$\tilde{G}(y_1, y_2) = \sum_{n,m=0}^{\infty} \tilde{b}_{n,m} y_1^{n+\tilde{\mu}_1} y_2^{m+\tilde{\mu}_2} \tag{7.5}$$

$$\tilde{H}(y_1, y_2) = \sum_{n,m=0}^{\infty} \tilde{c}_{n,m} y_1^{n+\tilde{\mu}_1} y_2^{m+\tilde{\mu}_2} \tag{7.6}$$

$$(\tilde{a}_{0,0}, \tilde{b}_{0,0}, \tilde{c}_{0,0}) \neq (0, 0, 0).$$

Now, we put

$$p(n,m) = q(n,m) = r(n,m) = n^2 - nm + m^2 + (2\tilde{\mu}_1 - \tilde{\mu}_2)n + (2\tilde{\mu}_2 - \tilde{\mu}_1)m.$$

(Though $p(n,m)$, $q(n,m)$, $r(n,m)$ are the same polynomials, we use the different symbols. By doing so, the expressions of the coefficients $\tilde{a}_{n,m}$, $\tilde{b}_{n,m}$, $\tilde{c}_{n,m}$ become a little easier.) The characteristic roots take six values:

$$(\tilde{\mu}_1, \tilde{\mu}_2) = (\mu_1 - 1, \mu_2 - 1)$$

for the six values (μ_1, μ_2) given in Proposition 6.5.
The following formula gives the explicit expressions of the coefficients $(\tilde{a}_{n,m})$, $(\tilde{b}_{n,m})$, $(\tilde{c}_{n,m})$. Let $\mathbf{P}_{n,m}$ be the family of all sets $\{\alpha_k(n_k, m_k), \cdots, \alpha_0(n_0, m_0)\}$ satisfying the following rules:

A) $\alpha_i = p$ or q or r $(i = 0, \cdots, k)$, $(n_k, m_k) = (n, m)$, $(n_0, m_0) = (0, 0)$,

B) The relations of $\alpha_i(n_i, m_i)$ and $\alpha_{i-1}(n_{i-1}, m_{i-1})$ are as follows. And for each correspondence, we associate one number (the number after ;).

In case of $\alpha_i(n_i, m_i) = p(n_i, m_i)$,

$$\alpha_{i-1}(n_{i-1}, m_{i-1}) = \begin{cases} p(n_i - 2l_i, m_i)\,; l_i \quad \text{or} \\ p(n_i, m_i - 2l_i)\,; -l_i \quad \text{or} \\ p(n_i - 2l_i, m_i - 2l_i)\,; l_i \quad \text{or} \\ q(n_i - 2l_i + 1, m_i)\,; -(2l_i - 1) \quad \text{or} \\ r(n_i - 2l_i + 1, m_i - 2l_i + 1)\,; -(2l_i - 1). \end{cases} \tag{7.7}$$

In case of $\alpha_i(n_i, m_i) = q(n_i, m_i)$,

$$\alpha_{i-1}(n_{i-1}, m_{i-1}) = \begin{cases} q(n_i - 2l_i, m_i)\,; l_i \quad \text{or} \\ q(n_i, m_i - 2l_i);\, l_i \quad \text{or} \\ q(n_i - 2l_i, m_i - 2l_i)\,; -l_i \quad \text{or} \\ p(n_i - 2l_i + 1, m_i)\,; -(2l_i - 1) \quad \text{or} \\ r(n_i, m_i - 2l_i + 1)\,; -(2l_i - 1). \end{cases} \tag{7.8}$$

In case of $\alpha_i(n_i, m_i) = r(n_i, m_i)$,

$$\alpha_{i-1}(n_{i-1}, m_{i-1}) = \begin{cases} r(n_i - 2l_i, m_i)\,; -l_i \quad \text{or} \\ r(n_i, m_i - 2l_i);\, l_i \quad \text{or} \\ r(n_i - 2l_i, m_i - 2l_i)\,; l_i \quad \text{or} \\ p(n_i - 2l_i + 1, m_i - 2l_i + 1)\,; -(2l_i - 1) \quad \text{or} \\ q(n_i, m_i - 2l_i + 1)\,; -(2l_i - 1). \end{cases} \tag{7.9}$$

For each i, we denote the number after ; of each correspondence by d_i. We put

$$(\delta^a_{n,m}, \delta^b_{n,m}, \delta^c_{n,m}) = \begin{cases} (\tilde{a}_{0,0}, \tilde{b}_{0,0}, \tilde{c}_{0,0}) & (n; \text{even}, m; \text{even}) \\ (\tilde{a}_{0,0}, \tilde{c}_{0,0}, \tilde{b}_{0,0}) & (n; \text{even}, m; \text{odd}) \\ (\tilde{b}_{0,0}, \tilde{a}_{0,0}, \tilde{c}_{0,0}) & (n; \text{odd}, m; \text{even}) \\ (\tilde{c}_{0,0}, \tilde{b}_{0,0}, \tilde{a}_{0,0}) & (n; \text{odd}, m; \text{odd}) \end{cases}$$

And we put
$\mathbf{P}^p_{n,m} := \mathbf{P}_{n,m} \cap \{\alpha_k = p\}$, $\mathbf{P}^q_{n,m} := \mathbf{P}_{n,m} \cap \{\alpha_k = q\}$, $\mathbf{P}^r_{n,m} := \mathbf{P}_{n,m} \cap \{\alpha_k = r\}$.
Then we have

$$\tilde{a}_{n,m} = \sum_{\{\alpha_k(n_k,m_k),\cdots,\alpha_0(n_0,m_0)\} \in \mathbf{P}^p_{n,m}} \frac{\left(\prod_{i=1}^k d_i\right) \delta^a_{n,m}}{\alpha_k(n_k, m_k) \cdots \alpha_1(n_1, m_1)}, \tag{7.10}$$

$$\tilde{b}_{n,m} = \sum_{\{\alpha_k(n_k,m_k),\cdots,\alpha_0(n_0,m_0)\} \in \mathbf{P}^q_{n,m}} \frac{\left(\prod_{i=1}^k d_i\right) \delta^b_{n,m}}{\alpha_k(n_k, m_k) \cdots \alpha_1(n_1, m_1)}, \tag{7.11}$$

$$\tilde{c}_{n,m} = \sum_{\{\alpha_k(n_k,m_k),\cdots,\alpha_0(n_0,m_0)\} \in \mathbf{P}^r_{n,m}} \frac{\left(\prod_{i=1}^k d_i\right) \delta^c_{n,m}}{\alpha_k(n_k, m_k) \cdots \alpha_1(n_1, m_1)}. \tag{7.12}$$

for $(n,m) \neq (0,0)$. Here, $\tilde{a}_{n,m} = 0$ (resp. $\tilde{b}_{n,m} = 0, \tilde{c}_{n,m} = 0$) if $\mathbf{P}^p_{n,m} = \emptyset$ (resp. $\mathbf{P}^q_{n,m} = \emptyset$, $\mathbf{P}^r_{n,m} = \emptyset$).

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III. A second limit formula for higher rank twisted Epstein zeta functions and some applications

ABSTRACT: We give the second limit formula and an analogue of the Chowla-Selberg formula for the twisted Epstein zeta functions of rank $n \geq 2$. As an application, we compute the determinant of the Euclidean Laplacian on the space of asymmetrically automorphic functions on $\mathbf{R}^n$ by using our second limit formula.

1 Introduction

Let Q be a positive definite symmetric $n \times n$ matrix, $u, v \in \mathbf{R}^n$, and let s be a complex variable with $\mathrm{Re}(s) > n/2$. Epstein [7] considered a kind of zeta function defined by

$$\zeta_n(s, u, v, Q) = \sum_{m \in \mathbf{Z}^n, m+v \neq 0} e^{2\pi i {}^t m \cdot u} Q[m+v]^{-s}$$

where $Q[x] := {}^t x Q x$ for $x \in \mathbf{R}^n$. This function admits the analytic continuation to the whole s-plane, and satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta_n(s, u, v, Q) = e^{-2\pi i {}^t u \cdot v}|Q|^{-\frac{1}{2}}\pi^{-(\frac{n}{2}-s)}\Gamma\left(\frac{n}{2}-s\right)\zeta_n\left(\frac{n}{2}-s, v, -u, Q^{-1}\right)$$

where $|Q| := \det(Q)$. Further, he obtained so called the Kronecker limit formula for $\zeta_n(s; Q) := \zeta_n(s, 0, 0, Q)$. This is the computation of the constant term of the Laurent expansion of $\zeta_n(s; Q)$ around $s = n/2$ (see also Terras [12]). There are many generalizations of the Epstein zeta function. For example, Siegel [11] defined a generalized Epstein zeta function by

$$\zeta(s, u, v, Q, P) = \sum_{m \in \mathbf{Z}^n, m+v \neq 0} e^{2\pi i {}^t m \cdot u} \frac{P(m+v)}{Q[m+v]^{s+\frac{g}{2}}}$$

for $\mathrm{Re}(s) > n/2$. Here, $u, v \in \mathbf{R}^n$ are column vectors, Q is a positive definite symmetric $n \times n$ matrix, and $P(x) = P(x_1, \cdots, x_n)$ is a homogeneous polynomial of degree g satisfying

$$\sum_{1 \leq i,j \leq n} q^*_{ij} \frac{\partial^2 P(x)}{\partial x_i \partial x_j} = 0$$

where $(q^*_{ij}) = Q^{-1}$. He proved that this function admits the analytic continuation to the whole s-plane and satisfies some functional equation. He introduced many examples of applications of the Kronecker limit formula to algebraic number theory, mainly on the quadratic fields.

On the other hand, Chowla and Selberg [4] obtained another important formula called the Chowla-Selberg formula. Let a and c be positive numbers and b be real. Assume that $d = b^2 - 4ac < 0$. Then the Epstein zeta function defined by

$$Z(s) = \frac{1}{2} \sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} (am^2 + bmn + cn^2)^{-s} \quad (\mathrm{Re}(s) > 1)$$

satisfies the following identity:

$$Z(s) = a^{-s}\zeta(2s) + a^{-s}\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\left(\frac{\sqrt{|d|}}{2a}\right)^{1-2s}\zeta(2s-1) + R_Q(s),$$

$$R_Q(s) = \frac{4a^{-s}}{\pi^{-s}\Gamma(s)}\left(\frac{\sqrt{|d|}}{2a}\right)^{-s+\frac{1}{2}}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\left(\sum_{d|n} d^{1-2s}\right) K_{s-\frac{1}{2}}\left(\frac{\pi n\sqrt{|d|}}{a}\right)\cos\left(\frac{n\pi b}{a}\right).$$

for any $s \in \mathbf{C}$, where $K_\nu(z)$ is the K-Bessel function. There are a lot of applications of this formula in number theory, for example, to investigate the distribution of zeros of $Z(s)$.

In Section 2, we consider the Epstein zeta function defined by

$$\zeta_n(s,u,0,Q) = \sum_{m\in\mathbf{Z}^n\setminus\{0\}} e^{2\pi i\,{}^t m\cdot u} Q[m]^{-s} \quad \left(\mathrm{Re}(s) > \frac{n}{2}\right)$$

for general $n \geq 2$. It is known that if $u \notin \mathbf{Z}^n$, $\zeta_n(s,u,0,Q)$ is entire and the identity which expresses $\zeta_n(n/2,u,0,Q)$ is known as the second limit formula. The case of $n = 2$ is classical, called the second Kronecker limit formula. In [6], Efrat obtained the second limit formula for $\zeta_3(s,u,0,Q)$ and applied this formula to compute the determinant of the Dirac operators and Laplacians. We generalize his method to general $n \geq 2$ case, and obtain the second limit formula for $\zeta_n(s,u,0,Q)$, where u is the element of $\mathbf{R}^n\setminus\mathbf{Z}^n$ (Theorem 2.1). As a corollary, we obtain a certain generalized Dedekind η-function which has some modular properties (Corollary 2.2). Further, we give the K-Bessel expansion of $\zeta_n(s,u,0,Q)$ with $u \in \mathbf{R}^n$ (Theorem 2.3), which is an analogue of the Chowla-Selberg formula.

In Section 3, we compute the determinant of Laplacian on the space of asymmetrically automorphic functions by using our second limit formula, which is also a generalization of Efrat's result. This corresponds to the bosonic and fermionic string theory, as explained in [1].

The auther wants to express his gratitude to the referee for his valuable advices.

2 Twisted Epstein zeta functions

2.1 Definition of the twisted Epstein zeta functions

Let $n \geq 2$ be an integer and $Q \in GL(n,\mathbf{R})$ be a positive definite symmetric matrix. Let $u, v \in \mathbf{R}^n$. In this paper, any vector in $\mathbf{R}^n$ is regarded as a column vector in principle. For $a \in \mathbf{R}^n$, we write $Q[a] := {}^t a Q a$ where ${}^t a$ is the transpose of a. Then the Epstein zeta function $\zeta_n(s,u,v,Q)$ is defined by

$$\zeta_n(s,u,v,Q) = \sum_{m \in \mathbf{Z}^n, m+v \neq 0} e^{2\pi i {}^t m \cdot u} Q[m+v]^{-s} \quad \left(\mathrm{Re}(s) > \frac{n}{2}\right). \tag{2.1}$$

It is known that $\zeta_n(s,u,v,Q)$ admits an analytic continuation to the whole plane and has a simple pole at $s = n/2$ if $u \in \mathbf{Z}^n$ and is entire if $u \notin \mathbf{Z}^n$. In this paper, we call $\zeta_n(s,u,v,Q)$ the twisted Epstein zeta function if $u \notin \mathbf{Z}^n$. $\zeta_n(s,u,v,Q)$ satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta_n(s,u,v,Q) = e^{-2\pi i {}^t u \cdot v} |Q|^{-\frac{1}{2}} \pi^{-(\frac{n}{2}-s)} \Gamma\left(\frac{n}{2}-s\right) \zeta_n\left(\frac{n}{2}-s, v, -u, Q^{-1}\right) \tag{2.2}$$

where $|Q| := \det(Q)$. Throughout this paper we assume that $u \notin \mathbf{Z}^n, v = 0$ except for the subsection 2.4 of Section 2.

2.2 Formulation of the twisted Epstein zeta functions

Let $Z(\mathbf{R})$ be the center of $GL(n,\mathbf{R})$ which is isomorphic to $\mathbf{R}^\times$ and $H_n = GL(n,\mathbf{R})/O(n)Z(\mathbf{R})$ be the upper half plane of degree n. By Iwasawa decomposition, the element $\tau \in H_n$ is uniquely expressed by

$$\tau = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & y_1 y_2 \cdots y_{n-2} x_{1,2} & y_1 y_2 \cdots y_{n-3} x_{1,3} & \cdots & x_{1,n} \\ 0 & y_1 y_2 \cdots y_{n-2} & y_1 y_2 \cdots y_{n-3} x_{2,3} & \cdots & x_{2,n} \\ 0 & 0 & y_1 y_2 \cdots y_{n-3} & \cdots & x_{3,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & x_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2.3}$$

with $x_{i,j} \in \mathbf{R}\,(1 \le i < j \le n)$, $y_i > 0\,(i = 1,2,\cdots n-1)$. Let $Q \in GL(n,\mathbf{R})$ be the positive definite symmetric matrix. Then there exists a unique $\tau \in H_n$ above such that

$$Q = |Q|^{\frac{1}{n}} (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{-\frac{2}{n}} \tau \cdot {}^t\tau. \tag{2.4}$$

Since the i-th component of ${}^t\tau \cdot m$ is $y_1 y_2 \cdots y_{n-i}(x_{1,i}m_1 + x_{2,i}m_2 + \cdots + x_{i-1,i}m_{i-1} + m_i)$ (if $i = n$, $y_1 y_2 \cdots y_{n-i} := 1$), we have

$$\begin{aligned} Q[m] =& |Q|^{\frac{1}{n}} (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{-\frac{2}{n}} ({}^t m \cdot \tau)({}^t\tau \cdot m) \\ =& |Q|^{\frac{1}{n}} (y_1^{n-1} y_2^{n-2} \cdots y_{n-1})^{-\frac{2}{n}} \\ & \cdot \sum_{i=1}^{n} y_1^2 y_2^2 \cdots y_{n-i}^2 (x_{1,i}m_1 + x_{2,i}m_2 + \cdots + x_{i-1,i}m_{i-1} + m_i)^2. \end{aligned}$$

Therefore, we have

$$\zeta_n(s,u,0,Q)$$
$$= |Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}$$
$$\cdot \sum_{m\in\mathbf{Z}^n\setminus\{0\}} e^{2\pi i^t m\cdot u}\left\{\sum_{i=1}^{n} y_1^2y_2^2\cdots y_{n-i}^2(x_{1,i}m_1+x_{2,i}m_2+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\right\}^{-s} \tag{2.5}$$

for $\mathrm{Re}(s) > n/2$.

2.3 Second limit formula for $\zeta_n(s,u,0,Q)$

In [11], Siegel asked whether one can obtain the second limit formula for $\zeta_n(s,u,v,Q)$, i.e, evaluate this function at $s=n/2$ when $u\notin\mathbf{Z}^n$, in analogy with the $n=2$ situation which is called the Kronecker limit formula. In [6], Efrat answered this question in case of $n=3$ and gave the second limit formula for $\zeta_3(s,u,0,Q)$. The following theorem is the answer in case of general $n\geq 2$ and $v=0$.

Theorem 2.1. *We define $A_1, A_2, \cdots, A_{n-1}$ inductively by*

$$A_1 = -u_n,$$

$$A_{k+1} = -u_{n-k} - \sum_{i=1}^{k}(A_i + m_{n+1-i})x_{n-k,n+1-i} \quad (k\geq 1).$$

For Q in (2.4) with τ in (2.3), we put

$$f(Q,u,m_2,\cdots,m_n) = 2\pi i u_1 + 2\pi i\sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}$$
$$- 2\pi y_1y_2\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}}.$$

For $u = {}^t(u_1,u_2,\cdots,u_n)\in\mathbf{R}^n$, we put $u' = {}^t(u_2,u_3,\cdots,u_n)\in\mathbf{R}^{n-1}$. For $\tau\in H_n$ in (2.3), we define $\tau'\in H_{n-1}$ by

$$\tau' = \begin{pmatrix} y_1y_2\cdots y_{n-2} & y_1y_2\cdots y_{n-3}x_{2,3} & y_1y_2\cdots y_{n-4}x_{2,4} & \cdots & x_{2,n} \\ 0 & y_1y_2\cdots y_{n-3} & y_1y_2\cdots y_{n-4}x_{3,4} & \cdots & x_{3,n} \\ 0 & 0 & y_1y_2\cdots y_{n-4} & \cdots & x_{4,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & x_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2.6}$$

For $Q\in GL(n,\mathbf{R})$ in (2.4), we define $Q'\in GL(n-1,\mathbf{R})$ by

$$Q' = (y_1^{n-2}y_2^{n-3}\cdots y_{n-2})^{-\frac{2}{n-1}}\tau'\cdot{}^t\tau'. \tag{2.7}$$

Then we have

$$\zeta_n\left(\frac{n}{2},u,0,Q\right)=|Q|^{-\frac{1}{2}}y_1^{\frac{1}{n-1}}y_2^{\frac{2}{n-1}}\cdots y_{n-1}^{\frac{n-1}{n-1}}\zeta_{n-1}\left(\frac{n}{2},u',0,Q'\right)$$
$$-2\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)^{-1}|Q|^{-\frac{1}{2}}\log\prod_{(m_2,\cdots,m_n)\in\mathbf{Z}^{n-1}}|1-e^{f(Q,u,m_2,\cdots,m_n)}|. \tag{2.8}$$

Proof. We have

$$\pi^{-s}\Gamma(s)\zeta_n(s,u,0,Q)$$
$$=\pi^{-s}\Gamma(s)|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}$$
$$\cdot\sum_{m\in\mathbf{Z}^n\setminus\{0\}}e^{2\pi i{}^tm\cdot u}\left\{\sum_{i=1}^{n}y_1^2y_2^2\cdots y_{n-i}^2(x_{1,i}m_1+x_{2,i}m_2+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\right\}^{-s}. \tag{2.9}$$

For $m={}^t(m_1,m_2,\cdots,m_n)\in\mathbf{Z}^n$, we put $m'={}^t(m_2,m_3,\cdots,m_n)\in\mathbf{Z}^{n-1}$. Firstly, the part of $m_1=0$ terms in (2.9) is equal to

$$\pi^{-s}\Gamma(s)|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}$$
$$\cdot\sum_{m'\in\mathbf{Z}^{n-1}\setminus\{0\}}e^{2\pi i{}^tm'\cdot u'}\left\{\sum_{i=2}^{n}y_1^2y_2^2\cdots y_{n-i}^2(x_{2,i}m_2+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\right\}^{-s}$$
$$=\pi^{-s}\Gamma(s)|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}$$
$$\cdot\sum_{m'\in\mathbf{Z}^{n-1}\setminus\{0\}}e^{2\pi i{}^tm'\cdot u'}\left\{\sum_{i=1}^{n-1}y_1^2y_2^2\cdots y_{(n-1)-i}^2(x_{2,i+1}m_2+\cdots+x_{i,i+1}m_i+m_{i+1})^2\right\}^{-s}$$
$$=\pi^{-s}\Gamma(s)|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}(y_1^{n-2}y_2^{n-3}\cdots y_{n-2})^{-\frac{2s}{n-1}}\zeta_{n-1}(s,u',0,Q'). \tag{2.10}$$

Next, we compute the part of $m_1\neq 0$ terms in (2.9). Since

$$\frac{\pi^{-s}\Gamma(s)}{\alpha^s}=\int_0^\infty e^{-\pi\alpha t}t^s\frac{dt}{t}\quad(\alpha>0),$$

The part of $m_1\neq 0$ terms in (2.9) is expressed by

$$|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}$$
$$\cdot\sum_{m_1\in\mathbf{Z}\setminus\{0\},m'\in\mathbf{Z}^{n-1}}e^{2\pi i(m_1u_1+\cdots m_nu_n)}$$
$$\cdot\int_0^\infty e^{-\pi t\{\sum_{i=1}^n y_1^2y_2^2\cdots y_{n-i}^2(x_{1,i}m_1+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\}}t^s\frac{dt}{t}. \tag{2.11}$$

We transform the summation of (2.11) by applying the Poisson summation formula

$$\sum_{n\in\mathbf{Z}}e^{-\pi t(n+\alpha)^2}=\frac{1}{\sqrt{t}}\sum_{n\in\mathbf{Z}}e^{2\pi in\alpha-\frac{\pi n^2}{t}} \tag{2.12}$$

to the summations in $m_n, m_{n-1}, \cdots, m_2$. Firstly, we pick up the summation in m_n. Since

$$\begin{aligned}
&2\pi i u_n m_n \\
&- \pi t(x_{1,n}m_1 + \cdots + x_{n-1,n}m_{n-1} + m_n)^2 \\
&= -\pi t\left(m_n + x_{1,n}m_1 + \cdots + x_{n-1,n}m_{n-1} - \frac{iu_n}{t}\right)^2 \\
&- 2\pi i u_n(x_{1,n}m_1 + \cdots + x_{n-1,n}m_{n-1}) \\
&- \frac{\pi u_n^2}{t},
\end{aligned}$$

by applying the Poisson summation formula to the summation in m_n in (2.11), we have

$$\begin{aligned}
&\sum_{m_n \in \mathbf{Z}} e^{2\pi i u_n m_n - \pi t(x_{1,n}m_1 + \cdots + x_{n-1,n}m_{n-1} + m_n)^2} \\
&= e^{-2\pi i u_n(x_{1,n}m_1 + \cdots + x_{n-1,n}m_{n-1}) - \frac{\pi u_n^2}{t}} \\
&\cdot \sum_{m_n \in \mathbf{Z}} e^{-\pi t\left(m_n + x_{1,n}m_1 + \cdots + x_{n-1,n}m_{n-1} - \frac{iu_n}{t}\right)^2} \\
&= e^{-2\pi i u_n(x_{1,n}m_1 + \cdots + x_{n-1,n}m_{n-1}) - \frac{\pi u_n^2}{t}} \\
&\cdot \frac{1}{\sqrt{t}} \sum_{m_n \in \mathbf{Z}} e^{2\pi i(x_{1,n}m_1 + \cdots + x_{n-1,n}m_{n-1} - \frac{iu_n}{t})m_n - \frac{\pi m_n^2}{t}}.
\end{aligned}$$

Therefore, by rewriting the summation in m_n, $m_1 \neq 0$ part (2.11) is rewritten as follows:

$$\begin{aligned}
&|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}} \\
&\times \sum_{m_1 \in \mathbf{Z}\setminus\{0\}, m' \in \mathbf{Z}^{n-1}} e^{2\pi i(m_1u_1 + \cdots + m_{n-1}u_{n-1}) - 2\pi i u_n(x_{1,n}m_1 + \cdots + x_{n-1.n}m_{n-1})} \\
&\quad \cdot e^{2\pi i(x_{1,n}m_1 + \cdots + x_{n-1,n}m_{n-1})m_n} \\
&\quad \cdot \int_0^\infty e^{-\pi t\{\sum_{i=1}^{n-1} y_1^2\cdots y_{n-i}^2(x_{1,i}m_1 + \cdots + x_{i-1,i}m_{i-1} + m_i)^2\}} \\
&\quad \cdot e^{-\frac{\pi}{t}(-u_n + m_n)^2} t^{s-\frac{1}{2}} \frac{dt}{t}.
\end{aligned} \tag{2.13}$$

Similarly, by applying the Poisson summation formula to the summations in $m_{n-1}, m_{n-2}, \cdots, m_{n-(k-1)}$ $(1 \leq k \leq n-1)$, we have the following identity.

$$\begin{aligned}
&|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}} \\
&\cdot \sum_{m_1 \in \mathbf{Z}\setminus\{0\}, m' \in \mathbf{Z}^{n-1}} e^{2\pi i(m_1u_1 + \cdots m_nu_n)}
\end{aligned}$$

$$\cdot \int_0^\infty e^{-\pi t\{\sum_{i=1}^n y_1^2 y_2^2 \cdots y_{n-i}^2 (x_{1,i}m_1+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\}} t^s \frac{dt}{t}$$

$$\begin{aligned}
=&|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}} y_1^{-(k-1)} y_2^{-(k-2)} \cdots y_{k-1}^{-1}\\
\times & \sum_{m_1\in\mathbf{Z}\setminus\{0\}, m'\in\mathbf{Z}^{n-1}} e^{2\pi i(m_1u_1+\cdots+m_{n-k}u_{n-k})}\\
&\cdot e^{2\pi i(m_n+A_1)(x_{1,n}m_1+x_{2,n}m_2+\cdots+x_{n-k,n}m_{n-k})}\\
&\cdot e^{2\pi i(m_{n-1}+A_2)(x_{1,n-1}m_1+x_{2,n-1}m_2+\cdots+x_{n-k,n-1}m_{n-k})}\\
&\vdots\\
&\cdot e^{2\pi i(m_{n+1-k}+A_k)(x_{1,n-k+1}m_1+x_{2,n-k+1}m_2+\cdots+x_{n-k,n-k+1}m_{n-k})}\\
&\cdot \int_0^\infty e^{-\pi t\{\sum_{i=1}^{n-k} y_1^2\cdots y_{n-i}^2(x_{1,i}m_1+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\}}\\
&\cdot e^{-\frac{\pi}{ty_1^2\cdots y_{k-1}^2}(m_{n+1-k}+A_k)^2-\frac{\pi}{ty_1^2\cdots y_{k-2}^2}(m_{n+2-k}+A_{k-1})^2-\cdots-\frac{\pi}{t}(m_n+A_1)^2}\\
&\cdot t^{s-\frac{k}{2}}\frac{dt}{t} \quad (1\le k\le n-1).
\end{aligned} \tag{2.14}$$

We prove (2.14) by induction with respect to k. We have already known that (2.14) holds if $k=1$. Suppose that (2.14) holds for $k\in\mathbf{Z}$ such that $1\le k\le n-2$. By applying the Poisson summation formula (2.12), the summation in m_{n-k} is

$$\begin{aligned}
&\sum_{m_{n-k}\in\mathbf{Z}} e^{2\pi i u_{n-k}m_{n-k}+2\pi i(m_n+A_1)x_{n-k,n}m_{n-k}+\cdots+2\pi i(m_{n+1-k}+A_k)x_{n-k,n-k+1}m_{n-k}}\\
&\cdot e^{-\pi t y_1^2\cdots y_k^2(x_{1,n-k}m_1+\cdots+x_{n-k-1,n-k}m_{n-k-1}+m_{n-k})^2}\\
&=e^{2\pi i A_{k+1}(x_{1,n-k}m_1+\cdots+x_{n-k-1,n-k}m_{n-k-1})-\frac{\pi A_{k+1}^2}{ty_1^2\cdots y_k^2}}\\
&\cdot \sum_{m_{n-k}\in\mathbf{Z}} e^{-\pi t y_1^2\cdots y_k^2\left(m_{n-k}+x_{1,n-k}m_1+\cdots+x_{n-k-1,n-k}m_{n-k-1}+\frac{iA_{k+1}}{ty_1^2\cdots y_k^2}\right)^2}\\
&=e^{2\pi i A_{k+1}(x_{1,n-k}m_1+\cdots+x_{n-k-1,n-k}m_{n-k-1})-\frac{\pi A_{k+1}^2}{ty_1^2\cdots y_k^2}}\\
&\cdot\frac{1}{\sqrt{t}y_1\cdots y_k}\sum_{m_{n-k}\in\mathbf{Z}} e^{2\pi i(x_{1,n-k}m_1+\cdots+x_{n-k-1,n-k}m_{n-k-1}+\frac{iA_{k+1}}{ty_1^2\cdots y_k^2})m_{n-k}-\frac{\pi m_{n-k}^2}{ty_1^2\cdots y_k^2}}.
\end{aligned}$$

Therefore, by rewriting the summation in m_{n-k}, the right hand side of (2.14) becomes

$$\begin{aligned}
&|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}} y_1^{-k} y_2^{-(k-1)} \cdots y_k^{-1}\\
\times & \sum_{m_1\in\mathbf{Z}\setminus\{0\}, m'\in\mathbf{Z}^{n-1}} e^{2\pi i(m_1u_1+\cdots+m_{n-k-1}u_{n-k-1})}
\end{aligned}$$

$$\cdot\, e^{2\pi i(m_n+A_1)(x_{1,n}m_1+x_{2,n}m_2+\cdots+x_{n-k-1,n}m_{n-k-1})}$$
$$\cdot\, e^{2\pi i(m_{n-1}+A_2)(x_{1,n-1}m_1+x_{2,n-1}m_2+\cdots+x_{n-k-1,n-1}m_{n-k-1})}$$
$$\vdots$$
$$\cdot\, e^{2\pi i(m_{n+1-k}+A_k)(x_{1,n-k+1}m_1+x_{2,n-k+1}m_2+\cdots+x_{n-k-1,n-k+1}m_{n-k-1})}$$
$$\cdot\, e^{2\pi i(m_{n-k}+A_{k+1})(x_{1,n-k}m_1+x_{2,n-k}m_2+\cdots+x_{n-k-1,n-k}m_{n-k-1})}$$
$$\cdot \int_0^\infty e^{-\pi t\{\sum_{i=1}^{n-k-1} y_1^2\cdots y_{n-i}^2(x_{1,i}m_1+\cdots+x_{i-1,i}m_{i-1}+m_i)^2\}}$$
$$\cdot\, e^{-\frac{\pi}{ty_1^2\cdots y_k^2}(m_{n-k}+A_{k+1})^2}$$
$$\cdot\, e^{-\frac{\pi}{ty_1^2\cdots y_{k-1}^2}(m_{n+1-k}+A_k)^2-\frac{\pi}{ty_1^2\cdots y_{k-2}^2}(m_{n+2-k}+A_{k-1})^2-\cdots-\frac{\pi}{t}(m_n+A_1)^2}\, t^{s-\frac{k+1}{2}}\frac{dt}{t}.$$

This is what we changed k to $k+1$ in the right hand side of (2.14). Therefore, (2.14) holds for any $k \in \mathbf{Z}$ such that $1 \le k \le n-1$. By inserting $k = n-1$ in the identity (2.14), (2.11) is equal to

$$\begin{aligned} &|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1} \\ &\times \sum_{m_1\neq 0, m'\in\mathbf{Z}^{n-1}} e^{2\pi i m_1u_1+2\pi i m_1\sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}} \\ &\quad\cdot \int_0^\infty e^{-\pi t y_1^2\cdots y_{n-1}^2m_1^2-\frac{\pi}{ty_1^2\cdots y_{n-2}^2}(m_2+A_{n-1})^2-\cdots-\frac{\pi}{t}(m_n+A_1)^2}\, t^{s-\frac{n-1}{2}}\frac{dt}{t}. \end{aligned} \tag{2.15}$$

For $a, b > 0$, K-Bessel function $K_s(a,b)$ is defined by

$$K_s(a,b) = \int_0^\infty e^{-(a^2t+\frac{b^2}{t})}t^s\frac{dt}{t}. \tag{2.16}$$

The value at $s = 1/2$ is given by

$$K_{\frac{1}{2}}(a,b) = \frac{\sqrt{\pi}}{a}e^{-2ab}.$$

Since the integral in (2.15) is expressed by $K_{s-\frac{n-1}{2}}(a,b)$, where

$$a = \sqrt{\pi}|m_1|y_1\cdots y_{n-1},$$

$$b = \sqrt{\pi}\left(\frac{1}{y_1^2\cdots y_{n-2}^2}(m_2+A_{n-1})^2 + \frac{1}{y_1^2\cdots y_{n-3}^2}(m_3+A_{n-2})^2+\cdots+(m_n+A_1)^2\right)^{\frac{1}{2}},$$

we have

$$\begin{aligned} &(\text{The integral in (2.15)})|_{s=\frac{n}{2}} \\ &= \frac{1}{|m_1|y_1\cdots y_{n-1}}e^{-2\pi|m_1|y_1\cdots y_{n-1}\left(\frac{1}{y_1^2\cdots y_{n-2}^2}(m_2+A_{n-1})^2+\cdots+(m_n+A_1)^2\right)^{\frac{1}{2}}}. \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&(2.15)|_{s=\frac{n}{2}} \\
&= |Q|^{-\frac{1}{2}} \sum_{m_1\in\mathbf{Z}\backslash\{0\}, m'\in\mathbf{Z}^{n-1}} e^{2\pi i m_1 u_1 + 2\pi i m_1 \sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}} \\
&\qquad \cdot \frac{1}{|m_1|} e^{-2\pi|m_1|y_1\cdots y_{n-1}\left(\frac{1}{y_1^2\cdots y_{n-2}^2}(m_2+A_{n-1})^2+\cdots+(m_n+A_1)^2\right)^{\frac{1}{2}}} \qquad (2.17)\\
&= |Q|^{-\frac{1}{2}} \sum_{m'\in\mathbf{Z}^{n-1}} 2\mathrm{Re} \sum_{m_1=1}^{\infty} \frac{1}{m_1} e^{m_1 f(Q,u,m_2,\cdots,m_n)} \\
&= -2|Q|^{-\frac{1}{2}}\log \prod_{m'\in\mathbf{Z}^{n-1}} |1-e^{f(Q,u,m_2,\cdots,m_n)}|.
\end{aligned}
$$

In the above computation, we used the identity

$$
\mathrm{Re}\sum_{n=1}^{\infty}\frac{z^n}{n} = -\log|1-z|.
$$

By combining $m_1 = 0$ part (2.10) and $m_1 \neq 0$ part (2.17), we obtain the second limit formula (2.8).

□

We fix $u \in \mathbf{Z}^n\backslash\{0\}$. We define a function $\eta_n(\tau, u)$ of $\tau \in H_n$ by

$$
\begin{aligned}
\eta_n(\tau,u) =& \exp\left(-\frac{1}{2}\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right) y_1^{\frac{1}{n-1}} y_2^{\frac{2}{n-1}}\cdots y_{n-1}^{\frac{n-1}{n-1}}\zeta_{n-1}\left(\frac{n}{2},u',0,Q'\right)\right) \\
&\times \prod_{m'\in\mathbf{Z}^{n-1}}(1-e^{f(Q,u,m_2,\cdots,m_n)}).
\end{aligned}
\qquad (2.18)
$$

Since $\zeta_{n-1}\left(\frac{n}{2},u',0,Q'\right)$ and $f(Q,u,m_2,\cdots,m_n)$ may be regarded as functions of $\tau \in H_n$ (not functions of Q), this is well-defined. This is one of the generalizations of the Dedekind η-function

$$
\eta(z) = e^{\frac{\pi i z}{12}}\prod_{m=1}^{\infty}(1-e^{2\pi i m z}) \quad (z = x+iy,\ y>0).
$$

Some modular properties of $|\eta(z)|$ are obtained from the original Kronecker limit formula. In the same way, we can obtain some properties of $|\eta_n(\tau,u)|$ by using our second limit formula. From the second limit formula (2.8), we have

$$
-\frac{1}{2}\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)|Q|^{\frac{1}{2}}\zeta_n\left(\frac{n}{2},u,0,Q\right) = \log|\eta_n(\tau,u)|. \qquad (2.19)
$$

Let

$$
\iota : GL(n,\mathbf{R}) \to GL(n,\mathbf{R})/O(n)\cdot Z(\mathbf{R}) = H_n
$$

be the canonical projection and define the action of $GL(n,\mathbf{R})$ on H_n by

$$\tau \mapsto g\circ\tau := \iota(g\tau) \quad (g\in GL(n,\mathbf{R}), \tau\in H_n).$$

Note that this is a group action. By the definition of Epstein zeta function, $\zeta_n(s,u,0,Q)$ satisfies

$$\zeta_n(s,u,0,gQ^tg) = \zeta_n(s,g^{-1}u,0,Q)$$

for $\forall g\in \mathrm{GL}(n,\mathbf{Z})$. By combining this and the relation (2.19), as a corollary of Theorem 2.1, we obtain the following formula for $|\eta_n(\tau,u)|$:

Corollary 2.2. *The function* $|\eta_n(\tau,u)|$ *satisfies*

$$|\eta_n(g\circ\tau,u)| = |\eta_n(\tau,g^{-1}u)| \quad (\forall g\in \mathrm{GL}(n,\mathbf{Z})). \tag{2.20}$$

2.4 An analogue of Chowla-Selberg formula

We return to the identity (2.15) in the proof of the Theorem 2.1. From the definition of $A_1,\cdots,A_{n-1}$, we can easily verify that the condition $m_2+A_{n-1} = m_3+A_{n-3} = \cdots = m_n+A_1 = 0$ is equivalent to $u' = (u_2,\cdots,u_n)\in\mathbf{Z}^{n-1}$ and $m_2=u_2,\cdots,m_n=u_n$.
Firstly, we consider the case of $u'\notin\mathbf{Z}^{n-1}$. The integral in (2.15) is expressed by

$$K_{s-\frac{n-1}{2}}\left(\sqrt{\pi}|m_1|y_1\cdots y_{n-1}, \sqrt{\pi}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}}\right)$$

where $K_s(a,b)$ is the K-Bessel function defined by (2.16). Since $u'\notin\mathbf{Z}^{n-1}$, for any $(m_2,\cdots,m_n)\in\mathbf{Z}^{n-1}$, we have $\sum_{j=1}^{n-1}(m_{n+1-j}+A_j)^2/y_1^2\cdots y_{j-1}^2>0$. For $c>0$, we define another K-Bessel function $K_s(c)$ by

$$K_s(c) = \int_0^\infty e^{-\frac{c}{2}(t+\frac{1}{t})}t^s\frac{dt}{t}. \tag{2.21}$$

The relation between $K_s(c)$ and $K_s(a,b)$ is given by

$$K_s(a,b) = \left(\frac{b}{a}\right)^s K_s(2ab) \quad (a,b>0). \tag{2.22}$$

Therefore, the integral in (2.15) is expressed by

$$\begin{aligned}&\left\{\frac{1}{|m_1|y_1\cdots y_{n-1}}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}}\right\}^{s-\frac{n-1}{2}}\\ &\cdot K_{s-\frac{n-1}{2}}\left(2\pi|m_1|y_1\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}}\right).\end{aligned} \tag{2.23}$$

Thus the right hand side of (2.15) is expressed by

$$|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}(y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1})(y_1y_2\cdots y_{n-1})^{-s+\frac{n-1}{2}}$$
$$\times \sum_{m_1\neq 0, m'\in\mathbf{Z}^{n-1}} e^{2\pi i m_1 u_1+2\pi i m_1\sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}}$$
$$\cdot |m_1|^{-s+\frac{n-1}{2}}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}(s-\frac{n-1}{2})}$$
$$\cdot K_{s-\frac{n-1}{2}}\left(2\pi|m_1|y_1\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}}\right) \tag{2.24}$$

when $\mathrm{Re}(s)>n/2$. Since this K-Bessel expansion converges absolutely for any $s\in\mathbf{C}$, this becomes the entire function. Therefore, by combining $m_1=0$ terms given by (2.10) and $m_1\neq 0$ terms given by (2.24), we have the following formula.

$$\pi^{-s}\Gamma(s)\zeta_n(s,u,0,Q)$$
$$=\pi^{-s}\Gamma(s)|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}(y_1^{n-2}y_2^{n-3}\cdots y_{n-2})^{-\frac{2s}{n-1}}\zeta_{n-1}(s,u',0,Q')$$
$$+|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}(y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1})(y_1y_2\cdots y_{n-1})^{-s+\frac{n-1}{2}}$$
$$\times \sum_{m_1\neq 0, m'\in\mathbf{Z}^{n-1}} e^{2\pi i m_1 u_1+2\pi i m_1\sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}}$$
$$\cdot |m_1|^{-s+\frac{n-1}{2}}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}(s-\frac{n-1}{2})}$$
$$\cdot K_{s-\frac{n-1}{2}}\left(2\pi|m_1|y_1\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}}\right). \tag{2.25}$$

By the theory of analytic continuation, the right hand side of (2.25) represents $\pi^{-s}\Gamma(s)\zeta_n(s,u,0,Q)$ over the whole s-plane.

Next, we consider the case of $u'=(u_2,\cdots,u_n)\in\mathbf{Z}^{n-1}$. The computation of $m'=(m_2,\cdots,m_n)\neq(u_2,\cdots,u_n)$ part is the same, that is, this part is expressed by the series of K-Bessel functions which converges absolutely for general $s\in\mathbf{C}$ and becomes the entire function. The computation of $(m_2,\cdots,m_n)=(u_2,\cdots,u_n)$ part in the right hand side of (2.15) is given by

$$|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}(y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1})$$
$$\cdot \sum_{m_1\neq 0} e^{2\pi i m_1 u_1}\cdot \int_0^\infty e^{-\pi t y_1^2\cdots y_{n-1}^2 m_1^2} t^{s-\frac{n-1}{2}}\frac{dt}{t}$$
$$= 2\pi^{-(s-\frac{n-1}{2})}\Gamma\left(s-\frac{n-1}{2}\right)|Q|^{-\frac{s}{n}}$$
$$\cdot (y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}(y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1})(y_1y_2\cdots y_{n-1})^{-2s+n-1}$$
$$\cdot \sum_{m_1=1}^{\infty}\frac{\cos(2\pi m_1 u_1)}{m_1^{2s-(n-1)}}. \tag{2.26}$$

Now, the function $\sum_{m_1=1}^{\infty}\cos(2\pi m_1 u_1)/m_1^{2s-(n-1)}$ $(\mathrm{Re}(s)>n/2)$ in (2.26) is expressed by $\{\mathrm{Li}_{2s-(n-1)}(e^{2\pi i u_1})+\mathrm{Li}_{2s-(n-1)}(e^{-2\pi i u_1})\}/2$, where $\mathrm{Li}_s(z)$ is the polylogarithm originally defined by

$$\mathrm{Li}_s(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^s}$$

for $\mathrm{Re}(s)>1$ and $|z|\leq 1$. It is known that as a function of s, this function has the analytic continuation to the whole s-plane. Therefore, we have the following formula:

$$\pi^{-s}\Gamma(s)\zeta_n(s,u,0,Q)$$
$$=\pi^{-s}\Gamma(s)|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}(y_1^{n-2}y_2^{n-3}\cdots y_{n-2})^{-\frac{2s}{n-1}}\zeta_{n-1}(s,u',0,Q')$$
$$+\pi^{-(s-\frac{n-1}{2})}\Gamma\left(s-\frac{n-1}{2}\right)|Q|^{-\frac{s}{n}}$$
$$\cdot (y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}(y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1})(y_1y_2\cdots y_{n-1})^{-2s+n-1}$$
$$\cdot \left\{\mathrm{Li}_{2s-(n-1)}(e^{2\pi i u_1})+\mathrm{Li}_{2s-(n-1)}(e^{-2\pi i u_1})\right\}$$

$$+|Q|^{-\frac{s}{n}}(y_1^{n-1}y_2^{n-2}\cdots y_{n-1})^{\frac{2s}{n}}(y_1^{-(n-2)}y_2^{-(n-3)}\cdots y_{n-2}^{-1})(y_1y_2\cdots y_{n-1})^{-s+\frac{n-1}{2}}$$
$$\times \sum_{m_1\neq 0, m'\in\mathbf{Z}^{n-1}\setminus\{u'\}} e^{2\pi i m_1 u_1+2\pi i m_1\sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}}$$
$$\cdot |m_1|^{-s+\frac{n-1}{2}}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}(s-\frac{n-1}{2})}$$
$$\cdot K_{s-\frac{n-1}{2}}\left(2\pi|m_1|y_1\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}}\right). \tag{2.27}$$

for any $s\in\mathbf{C}$. Summing up, we have the following theorem.

Theorem 2.3. *The twisted Epstein zeta function $\zeta_n(s,u,0,Q)$ satisfies the following identity.*
1) *If* $u' = (u_2,\cdots,u_n) \notin \mathbf{Z}^{n-1}$, *we have*

$$\begin{aligned}
&\zeta_n(s,u,0,Q) \\
&= |Q|^{-\frac{s}{n}}\zeta_{n-1}(s,u',0,Q')\prod_{i=1}^{n-1} y_i^{\frac{2is}{n(n-1)}} \\
&+\pi^s\Gamma(s)^{-1}|Q|^{-\frac{s}{n}}\prod_{i=1}^{n-1} y_i^{\frac{n-2i}{n}s-\frac{n-2i-1}{2}} \\
&\times \sum_{m_1\neq 0, m'\in\mathbf{Z}^{n-1}} e^{2\pi i m_1 u_1 + 2\pi i m_1 \sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}} \\
&\quad\cdot |m_1|^{-s+\frac{n-1}{2}}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}(s-\frac{n-1}{2})} \\
&\quad\cdot K_{s-\frac{n-1}{2}}\left(2\pi|m_1|y_1\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}}\right).
\end{aligned} \tag{2.28}$$

2) *If* $u' = (u_2,\cdots,u_n) \in \mathbf{Z}^{n-1}$, *we have*

$$\begin{aligned}
&\zeta_n(s,u,0;Q) \\
&= |Q|^{-\frac{s}{n}}\zeta_{n-1}(s,u',0,Q')\prod_{i=1}^{n-1} y_i^{\frac{2is}{n(n-1)}} \\
&+\pi^{\frac{n-1}{2}}\Gamma(s)^{-1}\Gamma\left(s-\frac{n-1}{2}\right)|Q|^{-\frac{s}{n}} \\
&\cdot\left\{\mathrm{Li}_{2s-(n-1)}(e^{2\pi i u_1})+\mathrm{Li}_{2s-(n-1)}(e^{-2\pi i u_1})\right\}\prod_{i=1}^{n-1} y_i^{-\frac{2is}{n}+i} \\
&+\pi^s\Gamma(s)^{-1}|Q|^{-\frac{s}{n}}\prod_{i=1}^{n-1} y_i^{\frac{n-2i}{n}s-\frac{n-2i-1}{2}} \\
&\times \sum_{m_1\neq 0, m'\in\mathbf{Z}^{n-1}\setminus\{u'\}} e^{2\pi i m_1 u_1 + 2\pi i m_1 \sum_{j=1}^{n-1}(m_{n+1-j}+A_j)x_{1,n+1-j}} \\
&\quad\cdot |m_1|^{-s+\frac{n-1}{2}}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}(s-\frac{n-1}{2})} \\
&\quad\cdot K_{s-\frac{n-1}{2}}\left(2\pi|m_1|y_1\cdots y_{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{n+1-j}+A_j)^2\right)^{\frac{1}{2}}\right).
\end{aligned} \tag{2.29}$$

Theorem 2.3 gives the relations between $\zeta_n(s,u,0,Q)$ and $\zeta_{n-1}(s,u',0,Q')$. Assume that $u=(u_1,\cdots,u_n)=(0,\cdots,0)$. In this case, the polylogarithms above become the Riemann zeta function. By using (2.29) inductively, we can express $\zeta_n(s;Q)=\zeta_n(s,0,0,Q)$ by Riemann zeta function and K-Bessel series. To do this, we need some notations.

For $\tau\in H_n$ in (2.3), we define $\tau^{(j)}\in H_{n-j}$ by

$$\tau^{(j)}=\begin{pmatrix} y_1y_2\cdots y_{n-1-j} & y_1y_2\cdots y_{n-2-j}x_{j+1,j+2} & \cdots & x_{j+1,n} \\ 0 & y_1y_2\cdots y_{n-2-j} & \cdots & x_{j+2,n} \\ 0 & 0 & \cdots & x_{j+3,n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \ddots & x_{n-1,n} \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

(that is, we remove first j column vectors and j row vectors from τ) ($j=0,1,\cdots,n-2$) and $\tau^{(n-1)}:=(1)$. For Q in (2.4), we define $(n-j)\times(n-j)$ positive definite symmetric matrix $Q^{(j)}$ by

$$Q^{(j)}=(y_1^{n-j-1}y_2^{n-j-2}\cdots y_{n-1-j})^{-\frac{2}{n-j}}\tau^{(j)t}\tau^{(j)}.$$

For $m=(m_1,\cdots,m_n)\in\mathbf{Z}^n$, we define $m^{(j)}\in\mathbf{Z}^{n-j}$ by $m^{(j)}=(m_{j+1},\cdots,m_n)$ ($j=1,\cdots,n-1$). For $\tau\in H_l$ in (2.3) (we replace n by l), we define $A_1^{(l)},A_2^{(l)},\cdots,A_{l-1}^{(l)}$ inductively by

$$A_1^{(l)}=0,\quad A_{k+1}^{(l)}=-\sum_{i=1}^{k}(A_i^{(l)}+m_{l+1-i})x_{l-k,l+1-i}\quad(k\ge 1)$$

($l=2,3,\cdots,n$). For $s\in\mathbf{C}$, $m_1,\cdots,m_l\in\mathbf{Z}$, $\tau\in H_l$ in (2.3), we define an entire function $K_l(s,m_1,\cdots,m_l,\tau)$ by

$$\begin{aligned} &K_l(s,m_1,\cdots,m_l,\tau)\\ &=e^{2\pi i m_1\sum_{j=1}^{l-1}(m_{l+1-j}+A_j^{(l)})x_{1,l+1-j}}\\ &\quad\cdot|m_1|^{-s+\frac{l-1}{2}}\left(\sum_{j=1}^{l-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{l+1-j}+A_j^{(l)})^2\right)^{\frac{1}{2}(s-\frac{l-1}{2})}\\ &\quad\cdot K_{s-\frac{l-1}{2}}\left(2\pi|m_1|y_1\cdots y_{l-1}\left(\sum_{j=1}^{l-1}\frac{1}{y_1^2\cdots y_{j-1}^2}(m_{l+1-j}+A_j^{(l)})^2\right)^{\frac{1}{2}}\right)\end{aligned}$$

($l=2,3,\cdots,n$). Finally, for $s\in\mathbf{C}$, $y_1,\cdots,y_{l-1}>0$, we define three functions $a_l(s,y_1,\cdots,y_{l-1})$, $b_l(s,y_1,\cdots,y_{l-1})$, $c_l(s,y_1,\cdots,y_{l-1})$ by

$$a_l(s,y_1,\cdots,y_{l-1})=\prod_{i=1}^{l-1}y_i^{\frac{2is}{l(l-1)}},\quad b_l(s,y_1,\cdots,y_{l-1})=\prod_{i=1}^{l-1}y_i^{-\frac{2is}{l}+i},$$

$$c_l(s,y_1,\cdots,y_{l-1})=\prod_{i=1}^{l-1} y_i^{\frac{l-2i}{l}s-\frac{l-2i-1}{2}}.$$

Then the identity (2.26) becomes

$$\begin{aligned}
&|Q|^{\frac{s}{n}}\zeta_n(s;Q)\\
=&a_n(s,y_1,\cdots,y_{n-1})\zeta_{n-1}(s;Q^{(1)})\\
&+2\pi^{\frac{n-1}{2}}\Gamma(s)^{-1}\Gamma\left(s-\frac{n-1}{2}\right)b_n(s,y_1,\cdots,y_{n-1})\zeta(2s-(n-1))\\
&+\pi^s\Gamma(s)^{-1}c_n(s,y_1,\cdots,y_{n-1})\sum_{m_1\neq 0,m^{(1)}\in\mathbf{Z}^{n-1}\setminus\{0\}}K_n(s,m_1,\cdots,m_n,\tau).
\end{aligned}$$

Note that $\zeta_{n-1}(s;Q^{(1)})$ also satisfies a similar identity. By using this formula inductively, we obtain the following identity:

$$\begin{aligned}
&|Q|^{\frac{s}{n}}\zeta_n(s;Q)\\
=&\left\{\prod_{i=1}^{k}a_{n-(i-1)}(s,y_1,\cdots,y_{n-i})\right\}\zeta_{n-k}(s;Q^{(k)})\\
&+2\sum_{j=1}^{k}\pi^{\frac{n-j}{2}}\Gamma(s)^{-1}\Gamma\left(s-\frac{n-j}{2}\right)\left\{\prod_{i=1}^{j-1}a_{n-(i-1)}(s,y_1,\cdots,y_{n-i})\right\}\\
&\cdot b_{n-(j-1)}(s,y_1,\cdots,y_{n-j})\zeta(2s-(n-j))\\
&+\pi^s\Gamma(s)^{-1}\sum_{j=1}^{k}\left\{\prod_{i=1}^{j-1}a_{n-(i-1)}(s,y_1,\cdots,y_{n-i})\right\}\\
&\cdot c_{n-(j-1)}(s,y_1,\cdots,y_{n-j})\sum_{m_j\neq 0,m^{(j)}\in\mathbf{Z}^{n-j}\setminus\{0\}}K_{n-(j-1)}(s,m_j,\cdots,m_n,\tau^{(j-1)}).
\end{aligned} \tag{2.30}$$

$(1\le k\le n-1)$. In particular, since $\zeta_1(s;Q^{(n-1)})=2\zeta(2s)$, we have the following expansion formula by letting $k=n-1$.

Corollary 2.4. *The Epstein zeta function $\zeta_n(s;Q)=\zeta_n(s,0,0,Q)$ satisfies the following identity:*

$$\begin{aligned}
&|Q|^{\frac{s}{n}}\zeta_n(s;Q)\\
=&2\left\{\prod_{i=1}^{n-1}a_{n-(i-1)}(s,y_1,\cdots,y_{n-i})\right\}\zeta(2s)\\
&+2\sum_{j=1}^{n-1}\pi^{\frac{n-j}{2}}\Gamma(s)^{-1}\Gamma\left(s-\frac{n-j}{2}\right)\left\{\prod_{i=1}^{j-1}a_{n-(i-1)}(s,y_1,\cdots,y_{n-i})\right\}\\
&\cdot b_{n-(j-1)}(s,y_1,\cdots,y_{n-j})\zeta(2s-(n-j))
\end{aligned}$$

$$+\pi^s\Gamma(s)^{-1}\sum_{j=1}^{n-1}\left\{\prod_{i=1}^{j-1} a_{n-(i-1)}(s,y_1,\cdots,y_{n-i})\right\}$$

$$\cdot c_{n-(j-1)}(s,y_1,\cdots,y_{n-j}) \sum_{m_j\neq 0, m^{(j)}\in \mathbf{Z}^{n-j}\backslash\{0\}} K_{n-(j-1)}(s,m_j,\cdots,m_n,\tau^{(j-1)}). \tag{2.31}$$

Remark 2.5. The auther thinks this formula is useful to investigate the real zeros of $\zeta_n(s;Q)$, since the term of K-Bessel series becomes sufficiently small when $y_1,\cdots,y_{n-1}>0$ are sufficiently large. For example, Bateman and Grosswald [3] established a sufficient condition for the existence of real zeros of Epstein zeta functions in case of $n=2$ by using the Chowla-Selberg formula.

3 Determinant of Laplacian

Let Δ be a positive self-adjoint elliptic operator on L^2 functions of some closed connected Riemannian manifold with non-zero eigenvalues $0<\lambda_1\leq\lambda_2\leq\lambda_3\leq\cdots$. (The eigenvalues 0 are excluded if they exist). For such Δ, we associate the Dirichlet series

$$\zeta_\Delta(s)=\sum_{i=1}^{\infty}\frac{1}{\lambda_i^s}.$$

It is shown in [9] that this series converges when $\mathrm{Re}(s)$ is sufficiently large, and has a meromorphic continuation to the entire plane, which is regular at $s=0$. Formally, we have

$$-\log\prod_{i=1}^{\infty}\lambda_i=\frac{d}{ds}\zeta_\Delta(s)|_{s=0}.$$

Therefore, we define the determinant of Δ by

$$\det(\Delta)=e^{-\zeta'_\Delta(0)}.$$

We compute the determinant of the Euclidean Laplacian Δ on some functional space dependent on $u\in\mathbf{R}^n\backslash\mathbf{Z}^n$ by using the second limit formula we obtained. Let Γ be a lattice in $\mathbf{R}^n$ with basis $a_1,\cdots,a_n\in\mathbf{R}^n$ and $T^n=\mathbf{R}^n/\Gamma$ be a n-dimensional torus. We put $M=(a_1\cdots a_n)\in GL(n,\mathbf{R})$. For $u=(u_1,\cdots,u_n)\in\mathbf{R}^n\backslash\mathbf{Z}^n$, we define a space of asymmetrically automorphic functions on $\mathbf{R}^n$ by

$$S^u(T^n)=\{f:\mathbf{R}^n\to\mathbf{C},\ \text{smooth}\ | f(x+a_j)=e^{2\pi i u_j}f(x),\ j=1,\cdots,n\}.$$

For any $m\in\mathbf{Z}^n$, we define $\alpha_m\in\mathbf{R}^n$ by $\alpha_m={}^tM^{-1}\cdot(m+u)$. Then the set $\{e^{2\pi i\,{}^t\alpha_m\cdot x}|m\in\mathbf{Z}^n\}$ forms a basis of $S^u(T^n)$. Moreover, these $e^{2\pi i\,{}^t\alpha_m\cdot x}$ are the eigenfunctions of the Laplacian

$$\Delta:=-\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)$$

since

$$\begin{aligned}\Delta e^{2\pi i {}^t\alpha_m \cdot x} =& 4\pi^2\ {}^t\alpha_m \cdot \alpha_m e^{2\pi i {}^t\alpha_m \cdot x} \\ =& 4\pi^2 ({}^tM \cdot M)^{-1}[m+u] e^{2\pi i {}^t\alpha_m \cdot x}.\end{aligned}$$

Therefore, if we put $Q = {}^tM \cdot M$ and define the zeta function attached to Δ by

$$\zeta_\Delta(s) := \sum_{m \in \mathbf{Z}^n} Q^{-1}[m+u]^{-s} = \zeta_n(s, 0, u, Q^{-1}),$$

the determinant of Laplacian Δ on $S^u(T^n)$ is given by $e^{-\zeta_\Delta'(0)}$. (We normalized Q by $1/4\pi^2$, which does not change the value $\zeta_\Delta'(0)$). From the functional equation (2.2), we have

$$\pi^{-s}\Gamma(s)\zeta_n(s, 0, u, Q^{-1}) = |Q|^{\frac{1}{2}} \pi^{-(\frac{n}{2}-s)} \Gamma\left(\frac{n}{2} - s\right) \zeta_n\left(\frac{n}{2} - s, u, 0, Q\right). \quad (3.1)$$

Since the right hand side of (3.1) is regular at $s = 0$, the left hand side of (3.1) is also regular at $s = 0$. But since $\Gamma(s)$ has a simple pole at $s = 0$ ($\Gamma(s) = 1/s + O(1)$), the function $\zeta_n(s, 0, u, Q^{-1})$ must vanish at $s = 0$. (This is the reason why the value $\zeta_\Delta'(0)$ is not changed by the normalization of Q). In other words, $\zeta_n(s, 0, u, Q^{-1})$ has a Taylor expansion

$$\zeta_n(s, 0, u, Q^{-1}) = \zeta_n'(0, 0, u, Q^{-1})s + O(s^2) \quad (3.2)$$

around $s = 0$. Therefore, by using the functional equation (3.1) and the second limit formula (2.8), we have

$$\begin{aligned}\zeta_\Delta'(0) &= \lim_{s \to 0} \frac{1}{s} \zeta_n(s, 0, u, Q^{-1}) \\ &= \lim_{s \to 0} \pi^{-s}\Gamma(s)\zeta_n(s, 0, u, Q^{-1}) \\ &= |Q|^{\frac{1}{2}} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \zeta_n\left(\frac{n}{2}, u, 0, Q\right) \\ &= \pi^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) y_1^{\frac{1}{n-1}} y_2^{\frac{2}{n-1}} \cdots y_{n-1}^{\frac{n-1}{n-1}} \zeta_{n-1}\left(\frac{n}{2}, u', 0, Q'\right) \\ &\quad - 2\log \prod_{m' \in \mathbf{Z}^{n-1}} |1 - e^{f(Q, u, m_2, \cdots, m_n)}|.\end{aligned}$$

Therefore, we have the following result.

Theorem 3.1. *The determinant of Laplacian* $\Delta = -\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)$ *on* $S^u(T^n)$ *is given by*

$$\begin{aligned}\det(\Delta) =& \exp\left\{-\pi^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) y_1^{\frac{1}{n-1}} y_2^{\frac{2}{n-1}} \cdots y_{n-1}^{\frac{n-1}{n-1}} \zeta_{n-1}\left(\frac{n}{2}, u', 0, Q'\right)\right\} \\ &\cdot \prod_{m' \in \mathbf{Z}^{n-1}} |1 - e^{f(Q, u, m_2, \cdots, m_n)}|^2.\end{aligned} \quad (3.3)$$

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IV. Fourth moment of the Epstein zeta functions

ABSTRACT: We study the fourth moment of the Epstein zeta function $\zeta(s;Q)$ associated to the $n \times n$ positive definite symmetric matrix Q ($n \geq 4$) on the line $\mathrm{Re}(s) = \frac{n-1}{2}$. We prove that the integral $\int_0^T |\zeta(\frac{n-1}{2}+it;Q)|^4 dt$ is evaluated by $O(T(\log T)^4)$ if Q satisfies some conditions. As an application, we consider the divisor problem with respect to the coefficients of the Dirichlet series of Epstein zeta functions. Certain estimates for the error term of the summation of the Dirichlet coefficients are obtained by combining our results and Fomenko's estimates for $\zeta(\frac{n-1}{2}+it;Q)$.

1 Introduction

Moments of the Riemann zeta function and other L-functions have been studied for about one hundred years, from the age of Hardy and Littlewood. In 1918, Hardy and Littlewood ([2]) proved that

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt \sim T\log T, \tag{1.1}$$

and in 1926, Ingham ([8]) showed that

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \sim \frac{1}{2\pi^2}T(\log T)^4. \tag{1.2}$$

It is conjectured that

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \sim C_k T(\log T)^{k^2} \tag{1.3}$$

holds for $k \geq 0$ with some positive constant number C_k, but this has not been proved except for the cases $k = 0, 1, 2$ (it is known that the asymptotic formula (1.3) holds for all real numbers $k \in [0, 2]$ under the assumption of Riemann hypothesis).

In this paper, we deal with the Epstein zeta function $\zeta(s;Q)$, where Q is a $n \times n$ positive definite symmetric matrix ($n \geq 4$) which gives an integer-valued quadratic form. We consider the fourth moment of $\zeta(s;Q)$ on the line $\mathrm{Re}(s) = \frac{n-1}{2}$, and prove that the integral $\int_0^T |\zeta(\frac{n-1}{2}+it;Q)|^4 dt$ is evaluated by $O(T(\log T)^4)$ when $T \to \infty$ (Theorem 2.5, Theorem 2.6). Further, we apply our results to the divisor problem for the coefficients of the Dirichlet series of $\zeta(s;Q)$, and obtain certain estimates for the error terms of them.

Let us introduce the basic idea of this paper. For a $n \times n$ positive definite symmetric matrix Q, the quadratic form associated to Q is defined by $Q[\mathrm{x}] =$

${}^t\mathbf{x}Q\mathbf{x}$ for $\mathbf{x} \in \mathbf{R}^n$. We assume that $Q[\mathbf{x}] \in \mathbf{N}$ for any $\mathbf{x} \in \mathbf{Z}^n \backslash \{\mathbf{0}\}$. For $k \in \mathbf{Z}_{\geq 0}$, we define $r_Q(k)$ by the number of $\mathbf{x} \in \mathbf{Z}^n$ which satisfies $Q[\mathbf{x}] = k$. Then the Epstein zeta function $\zeta(s; Q)$ is expressed by

$$\zeta(s; Q) = \sum_{k=1}^{\infty} \frac{r_Q(k)}{k^s} \tag{1.4}$$

for $\mathrm{Re}(s) > \frac{n}{2}$. It is well-known that the corresponding theta series

$$\theta(z; Q) = \sum_{k=0}^{\infty} r_Q(k) e^{2\pi i k z}$$

becomes a modular form of weight $\frac{n}{2}$, which decomposes into the summation of Eisenstein series and cusp form. Therefore, $\zeta(s; Q)$ decomposes into the summation of the L-function associated to the Eisenstein series and the L-function associated to the cusp form. Hence to establish the upper bound for the moments of $\zeta(s; Q)$, it suffices to evaluate the moments of these two L-functions. We can easily prove that the fourth moment of the L-function associated to the cusp form on the line $\mathrm{Re}(s) = \frac{n-1}{2}$ is evaluated by $O(T)$ (Lemma 2.1), and our main problem is to evaluate the moment of L-function associated to the Eisenstein series. For this purpose, we use the classical theories due to Hecke ([7]), Malyshev ([13]), and Siegel ([16]). By using their theorems, we prove that the L-function associated to the Eisenstein series is expressed by some series consisting of the Dirichlet L-functions and thus we can use the theory of the moments of the Dirichlet L-functions. Our method is similar to that of Fomenko, who obtained some order estimates for $\zeta(s; Q)$ on the line $\mathrm{Re}(s) = \frac{n-1}{2}$ (see [1]).

As the easiest example, we take $Q = I_4$, the 4×4 unit matrix. Then the Epstein zeta function $\zeta(s; I_4)$ is expressed by

$$\zeta(s; I_4) = 8(1 - 2^{2-2s})\zeta(s)\zeta(s-1). \tag{1.5}$$

Since the factor $(1 - 2^{2-2s})\zeta(s)$ is bounded on the line $\mathrm{Re}(s) = \frac{3}{2}$, the fourth moment $\int_0^T |\zeta(\frac{3}{2} + it; I_4)|^4 dt$ is evaluated by $O(T(\log T)^4)$, by using Ingham's asymptotic formula (1.2). Of course, the general case is more complicated, but the underlying idea is similar.

In section 3, we apply our results to the divisor problem. For a positive integer l (in this paper, we assume that $l \geq 4$), we write

$$\zeta(s; Q)^l = \sum_{k=1}^{\infty} \frac{r_Q^{(l)}(k)}{k^s} \quad \left(\mathrm{Re}(s) > \frac{n}{2}\right).$$

Evaluating the magnitude of the summation $\sum_{k \leq x} r_Q^{(l)}(k)$ $(x \to \infty)$ is called the divisor problem. It is well-known that the following asymptotic formula holds (see [11], [15]):

$$\sum_{k \leq x} r_Q^{(l)}(k) = M_l^{(n)}(x) + \Delta_l^{(n)}(x) \quad (x \to \infty),$$

where $M_l^{(n)}(x)$ is the main term, expressed by $x^{\frac{n}{2}}P_l(\log x)$ with some polynomial P_l of degree $l-1$, and $\Delta_l^{(n)}(x)$ is the error term which becomes $o(x^{\frac{n}{2}})$. In [15], Sankaranarayanan showed that the error term $\Delta_l^{(n)}(x)$ is evaluated by $O(x^{\frac{n}{2}-\frac{1}{l}+\epsilon})$ for $n \geq 3, l \geq 2$ by using the order estimate for $\zeta(\frac{n-1}{2}+it;Q)$ and the fact that the mean-square $\int_0^T |\zeta(\frac{n-1}{2}+it;Q)|^2 dt$ is evaluated by $O(T^{1+\epsilon})$ ($\forall\epsilon > 0$) (this fact is proved in [10]). Note that the order estimate for $\zeta(\frac{n-1}{2}+it;Q)$ above relies on Stirling's formula for the gamma function and the Phragmén-Lindelöf principle, which is weaker than Fomenko's ones. Further, Lü ([11], [12]) obtained more sharp estimates for $\Delta_l^{(n)}(x)$ for special kinds of Q, whose the "Eisenstein part" of the associated Epstein zeta functions are expressed by using the Riemann zeta function, like (1.5). His method relies on the sophisticated theories for the Riemann zeta function, for example, the order estimate for $\zeta(s)$ on the critical line, and the estimate for the twelfth moment of $\zeta(s)$ by Heath-Brown ([6]). We investigate the divisor problem for general Q ($n \times n$ positive definite symmetric matrix ($n \geq 4$) which gives an integer valued quadratic form). Instead of the theories for the Riemann zeta function above, we use Fomenko's estimates for the order of $\zeta(\frac{n-1}{2}+it;Q)$ ([1]), and our theorems for the fourth moment of $\zeta(s;Q)$. Certain estimates for $\Delta_l^{(n)}(x)$ are obtained (Theorem 3.2), which are a little better than Sankaranarayanan's one.

2 Fourth moment of the Epstein zeta functions

2.1 Notation and some basic results

Let n be a positive integer and Q be a $n \times n$ positive definite symmetric matrix. The Epstein zeta function associated to Q is defined by

$$\zeta(s;Q) = \sum_{\mathbf{x}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}} Q[\mathbf{x}]^{-s} \quad \left(\mathrm{Re}(s) > \frac{n}{2}\right)$$

where $Q[\mathbf{x}] := {}^t\mathbf{x}Q\mathbf{x}$. This function has the meromorphic continuation to the whole s-plane and satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta(s;Q) = (\det Q)^{-\frac{1}{2}}\pi^{s-\frac{n}{2}}\Gamma\left(\frac{n}{2}-s\right)\zeta\left(\frac{n}{2}-s;Q^{-1}\right). \qquad (2.1)$$

$\zeta(s;Q)$ is holomorphic everywhere except for a simple pole at $s=\frac{n}{2}$ with residue $\pi^{\frac{n}{2}}/(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})$. Throughout this paper, we assume that $Q[\mathbf{x}] \in \mathbf{N}$ for any $\mathbf{x} \in \mathbf{Z}^n\setminus\{\mathbf{0}\}$. Let $r_Q(k)$ be the number of $\mathbf{x} \in \mathbf{Z}^n$ which satisfies $Q[\mathbf{x}] = k$. Then $\zeta(s;Q)$ has the following Dirichlet series expansion in $\mathrm{Re}(s) > \frac{n}{2}$:

$$\zeta(s;Q) = \sum_{k=1}^{\infty}\frac{r_Q(k)}{k^s}.$$

Hereafter, we assume that $n \geq 4$. We consider the theta series corresponding to $\zeta(s;Q)$ defined by

$$\theta(z;Q) = \sum_{k=0}^{\infty} r_Q(k) e^{2\pi ikz}.$$

It is well-known that $\theta(z;Q)$ is decomposed into the summation of Eisenstein series and cusp form:

$$\theta(z;Q) = E(z) + S(z) \tag{2.2}$$

where

$$E(z) = \sum_{k=0}^{\infty} e(k) e^{2\pi ikz}$$

is the Eisenstein series and

$$S(z) = \sum_{k=1}^{\infty} s(k) e^{2\pi ikz}$$

is the cusp form. Moreover, it is known that the coefficient $s(k)$ of $S(z)$ is evaluated by

$$s(k) \ll k^{\frac{n}{4}-\frac{1}{2}+\epsilon} \tag{2.3}$$

if n is even, and

$$s(k) \ll k^{\frac{n}{4}-\frac{1}{4}+\epsilon} \tag{2.4}$$

if n is odd, where ϵ is always an arbitrary positive number in this paper.

2.2 The fourth moment of $\zeta(s;Q)$ on the line $\mathrm{Re}(s) = \frac{n-1}{2}$

We want to establish some upper bound for the magnitude of the integral

$$\int_0^T \left|\zeta\left(\frac{n-1}{2} + it; Q\right)\right|^4 dt$$

when $T \to \infty$. Since $\zeta(s;Q)$ is expressed by

$$\zeta(s;Q) = \hat{E}(s) + \hat{S}(s) \tag{2.5}$$

where $\hat{E}(s)$ and $\hat{S}(s)$ are defined by

$$\hat{E}(s) = \sum_{k=1}^{\infty} \frac{e(k)}{k^s}, \quad \hat{S}(s) = \sum_{k=1}^{\infty} \frac{s(k)}{k^s}$$

in $\mathrm{Re}(s) > \frac{n}{2}$ and analytically continued to the whole s-plane, it is enough to evaluate the two integrals

$$\int_0^T \left|\hat{E}\left(\frac{n-1}{2} + it\right)\right|^4 dt, \quad \int_0^T \left|\hat{S}\left(\frac{n-1}{2} + it\right)\right|^4 dt.$$

Firstly, the following lemma gives an upper bound for the integral $\int_0^T |\hat{S}(\frac{n-1}{2} + it)|^4 dt$.

Lemma 2.1. *When $T \to \infty$, we have*

$$\int_0^T \left| \hat{S}\left(\frac{n-1}{2} + it\right)\right|^4 dt = O(T). \tag{2.6}$$

Proof. If $n \geq 6$, the estimates (2.3), (2.4) yield the series $\sum_{k=1}^{\infty} \frac{s(k)}{k^s}$ converges absolutely on the line $\mathrm{Re}(s) = \frac{n-1}{2}$. So the estimate (2.6) trivially holds. In case of $n = 4$ or 5, the line $\mathrm{Re}(s) = \frac{n-1}{2}$ is so to say the "absolute convergence line", that is, it is assured that the Dirichlet series for $\hat{S}(s)$ converges absolutely for $\mathrm{Re}(s) > \frac{n-1}{2}$ by the estimates (2.3), (2.4), but not on $\mathrm{Re}(s) = \frac{n-1}{2}$. To prove (2.6), we use a classical method. Assume that

$$s(k) = O(k^{\alpha+\epsilon})$$

holds. Then the Dirichlet series

$$\hat{S}(s) = \sum_{k=1}^{\infty} \frac{s(k)}{k^s} \tag{2.7}$$

converges absolutely for $\mathrm{Re}(s) > \alpha + 1$. If $n = 4$, $\alpha = \frac{1}{2}$ and if $n = 5$, $\alpha = 1$. Note that the L-function $\hat{S}(s)$ associated to the cusp form is expressed by the linear combination of some Epstein zeta functions (see [16]), for any fixed σ, the estimate $\hat{S}(\sigma + it) \ll |t|^{\mu(\sigma)}$ $(|t| \to \infty)$ holds for some $\mu(\sigma) < \infty$. Put

$$\hat{S}(s)^2 = \sum_{k=1}^{\infty} \frac{s^{(2)}(k)}{k^s} \quad (\mathrm{Re}(s) > \alpha + 1).$$

By using the Mellin inversion formula

$$e^{-x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) x^{-s} ds \quad (c > 0, x > 0),$$

for $\delta > 0$,

$$\sum_{k=1}^{\infty} \frac{s^{(2)}(k)}{k^s} e^{-\delta k} = \frac{1}{2\pi i} \int_{\alpha+2-i\infty}^{\alpha+2+i\infty} \Gamma(z - s) \hat{S}(z)^2 \delta^{s-z} dz \tag{2.8}$$

holds when $\mathrm{Re}(s) < \alpha+2$. We take a real number β which satisfies $\alpha < \beta < \alpha+1$, and assume that s satisfies

$$\max\left\{\alpha + \frac{1}{2}, \beta\right\} < \mathrm{Re}(s) < \beta + 1.$$

We move the path of integral to the line $\mathrm{Re}(s) = \beta$. The pole of integrant between the lines $\mathrm{Re}(s) = \alpha + 2$, $\mathrm{Re}(s) = \beta$ is $z = s$, and the residue is $\hat{S}(s)^2$. Therefore, by Cauchy's residue theorem, we have

$$\begin{aligned} \hat{S}(s)^2 = &\sum_{k=1}^{\infty} \frac{s^{(2)}(k)}{k^s} e^{-\delta k} \\ &- \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(z - s) \hat{S}(z)^2 \delta^{s-z} dz. \end{aligned} \tag{2.9}$$

We put

$$S_1 = \sum_{k=1}^{\infty} \frac{s^{(2)}(k)}{k^s} e^{-\delta k},$$

$$S_2 = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \Gamma(z-s)\hat{S}(z)^2 \delta^{s-z} dz.$$

Firstly,

$$\begin{aligned} \int_0^T |S_1|^2 dt &= \int_0^T \left(\sum_{k=1}^{\infty} \frac{s^{(2)}(k)}{k^{\sigma+it}} e^{-\delta k} \right) \left(\sum_{l=1}^{\infty} \frac{s^{(2)}(l)}{l^{\sigma-it}} e^{-\delta l} \right) dt \\ &= T \sum_{k=1}^{\infty} \frac{s^{(2)}(k)^2}{k^{2\sigma}} e^{-2\delta k} \\ &+ O\left(\sum\sum_{l<k} \frac{s^{(2)}(k)s^{(2)}(l)e^{-(k+l)\delta}}{(kl)^{\sigma} \log(\frac{k}{l})} \right). \end{aligned} \tag{2.10}$$

Since $s^{(2)}(k)$ is evaluated by

$$s^{(2)}(k) = O(k^{\alpha+\epsilon}),$$

the second term of the right hand side of (2.10) is evaluated by $O(\delta^{2\sigma-2\alpha-2-\epsilon})$ when $\delta \to +0$ (see [17], p117). Therefore,

$$\int_0^T |S_1|^2 dt = T \sum_{k=1}^{\infty} \frac{s^{(2)}(k)^2}{k^{2\sigma}} e^{-2\delta k} + O(\delta^{2\sigma-2\alpha-2-\epsilon}). \tag{2.11}$$

Note that the series $\sum_{k=1}^{\infty} \frac{s^{(2)}(k)^2}{k^{2\sigma}}$ converges for $\sigma > \alpha + \frac{1}{2}$. Next, by using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} S_2 &\ll \delta^{\sigma-\beta} \int_{\beta-i\infty}^{\beta+i\infty} |\Gamma(z-s)||\hat{S}(z)|^2 dz \\ &\ll \delta^{\sigma-\beta} \left(\int_{-\infty}^{\infty} |\Gamma(\beta+iv-s)| dv \int_{-\infty}^{\infty} |\Gamma(\beta+iv-s)||\hat{S}(\beta+iv)|^4 dv \right)^{\frac{1}{2}} \\ &\ll \delta^{\sigma-\beta} \left(\int_{-\infty}^{\infty} |\Gamma(\beta+iv-s)||\hat{S}(\beta+iv)|^4 dv \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\int_0^T |S_2|^2 dt \ll \delta^{2\sigma-2\beta} \int_0^T \int_{-\infty}^{\infty} |\Gamma(\beta+iv-s)||\hat{S}(\beta+iv)|^4 dv dt. \tag{2.12}$$

The contribution of the integral in $|v| \geq 2T$ is evaluated by

$$\ll \delta^{2\sigma-2\beta} \int_0^T \int_{|v|\geq 2T} e^{-C_1|v-t|} |v|^{C_2} dv dt$$

$$\ll \delta^{2\sigma-2\beta}\int_0^T e^{-C_3T}dt \ll \delta^{2\sigma-2\beta}.$$

Here, C_1, C_2, C_3 are some positive constants. The remaining part is evaluated by

$$\begin{aligned}
&\ll \delta^{2\sigma-2\beta}\int_{-2T}^{2T}|\hat{S}(\beta+iv)|^4\int_0^T|\Gamma(\beta+iv-s)|dtdv \\
&\ll \delta^{2\sigma-2\beta}\int_{-2T}^{2T}|\hat{S}(\beta+iv)|^4dv \\
&\ll \delta^{2\sigma-2\beta}T^{4\mu(\beta)+1}.
\end{aligned}$$

Therefore,

$$\int_0^T|S_2|^2dt \ll \delta^{2\sigma-2\beta}T^{4\mu(\beta)+1}. \tag{2.13}$$

By combining (2.11) and (2.13), we have

$$\begin{aligned}
\int_0^T|\hat{S}(s)|^4dt &\ll \int_0^T|S_1|^2dt+\int_0^T|S_2|^2dt \\
&\ll O(T)+O(\delta^{2\sigma-2\alpha-2-\epsilon})+O(\delta^{2\sigma-2\beta}T^{4\mu(\beta)+1}).
\end{aligned} \tag{2.14}$$

We put

$$\delta = T^{-\frac{4\mu(\beta)+1}{2\alpha-2\beta+2}}.$$

Then (2.14) becomes

$$\int_0^T|\hat{S}(s)|^4dt \ll O(T)+O(T^{\frac{(\alpha+1-\sigma)(4\mu(\beta)+1)}{\alpha-\beta+1}+\epsilon}). \tag{2.15}$$

The second term of the right hand side of (2.15) becomes $o(T)$ if $\sigma = \mathrm{Re}(s)$ satisfies

$$\sigma > \alpha+1-\frac{\alpha-\beta+1}{4\mu(\beta)+1}.$$

In particular, the left hand side of (2.15) becomes $O(T)$ for $\sigma = \alpha+1$.

□

Our main problem is to evaluate the integral $\int_0^T|\hat{E}(\frac{n-1}{2}+it)|^4dt$. To do this, we use the relations between the L-function associated to the Eisenstein series and the Dirichlet L-functions. Before stating our main theorems, we prepare three lemmas. The first lemma is the estimate for the fourth moment of Dirichlet L-functions (for example, see [5]):

Lemma 2.2. *When $T \to \infty$, we have*

$$\sum_{\chi(\mathrm{mod}q)}\int_0^T\left|L\left(\frac{1}{2}+it,\chi\right)\right|^4dt \ll qT(\mathrm{log}qT)^4 \tag{2.16}$$

where $\sum_{\chi(\mathrm{mod}q)}$ denotes the summation over all Dirichlet characters modulo q.

The second lemma is the estimates for the order of Dirichlet L-functions on the critical line by Heath-Brown ([3], [4]):

Lemma 2.3. *Let $L(s,\chi)$ be a Dirichlet L-function associated to a Dirichlet character modulo q. Then, when $t \to \infty$, the following estimates hold:*

$$L\left(\frac{1}{2}+it,\chi\right) \ll q^{\frac{1}{2}}t^{\frac{1}{6}}\log(qt), \tag{2.17}$$

$$L\left(\frac{1}{2}+it,\chi\right) \ll (qt)^{\frac{3}{16}+\epsilon}. \tag{2.18}$$

The third lemma is a simple inequality, used to evaluate the fourth moment of $\hat{E}(s)$ in case of n is odd and $n \geq 7$.

Lemma 2.4. *For $x_1, \cdots, x_m \geq 0$, we have*

$$x_1^{\frac{1}{4}} + \cdots + x_m^{\frac{1}{4}} \leq m^{\frac{3}{4}}(x_1 + \cdots + x_m)^{\frac{1}{4}}. \tag{2.19}$$

Proof. The inequality (2.19) is equivalent to

$$\frac{x_1^{\frac{1}{4}} + \cdots + x_m^{\frac{1}{4}}}{m} \leq \left(\frac{x_1 + \cdots + x_m}{m}\right)^{\frac{1}{4}}$$

which directly follows from the convexity of the function $f(x) = x^{\frac{1}{4}}$. □

Theorem 2.5. *Assume that n is even and $n \geq 4$, or n is odd and $n \geq 7$. Then, when $T \to \infty$, the following estimate holds:*

$$\int_0^T \left|\zeta\left(\frac{n-1}{2}+it;Q\right)\right|^4 dt = O(T(\log T)^4). \tag{2.20}$$

Proof. Firstly, we assume that n is even and $n \geq 4$. Then, the Eisenstein series $E(z)$ is a modular form of weight $\frac{n}{2}$ and level N, where N is a positive integer such that NA^{-1} becomes an integral matrix for $A = 2Q$ (see [9]). According to Hecke ([7], Theorem 44), the series $\hat{E}(s)$ is expressed by some linear combination of the form

$$(t_1t_2)^{-s}L(s,\chi_1)L\left(s-\frac{n}{2}+1,\chi_2\right)$$

where t_1, t_2 are positive divisors of level N and χ_1, χ_2 are Dirichlet characters modulo $\frac{N}{t_1}$, $\frac{N}{t_2}$, respectively. We write

$$\hat{E}(s) = \sum_{l=1}^{L} c_l(t_{1,l}t_{2,l})^{-s}L(s,\chi_{1,l})L\left(s-\frac{n}{2}+1,\chi_{2,l}\right).$$

Then

$$\begin{aligned}\int_0^T \left|\hat{E}\left(\frac{n-1}{2}+it\right)\right|^4 dt &\ll \sum_{l=1}^{L}\int_0^T \left|L\left(\frac{n-1}{2}+it,\chi_{1,l}\right)L\left(\frac{1}{2}+it,\chi_{2,l}\right)\right|^4 dt \\ &\ll \sum_{l=1}^{L}\int_0^T \left|L\left(\frac{1}{2}+it,\chi_{2,l}\right)\right|^4 dt\end{aligned} \tag{2.21}$$

since $L\left(\frac{n-1}{2}+it,\chi_{1,l}\right)$ is bounded with respect to t. Moreover, the estimate (2.16) yields each integral $\int_0^T |L(\frac{1}{2}+it,\chi_{2,l})|^4 dt$ is evaluated by $O(T(\log T)^4)$. Therefore, the statement of theorem is proved in this case.

Next, we assume that n is odd and $n \geq 7$. In this case, the Fourier coefficient of the Eisenstein series $E(z)$ has the following expression (see [13]):

$$e(k) = \frac{\pi^{\frac{n}{2}}}{(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})} k^{\frac{n}{2}-1} H(Q;k)$$

where

$$H(Q;k) = \sum_{q=1}^{\infty} \left\{ \sum_{h(\bmod q)}{}' q^{-n} S(hQ;q) e^{-2\pi i \frac{kh}{q}} \right\}$$

is a singular series, $\sum'$ means the summation over a reduced residue system, and

$$S(Q;q) = \sum_{x_1,\cdots,x_n=0}^{q-1} e^{\frac{2\pi i Q(x_1,\cdots,x_n)}{q}}$$

is a Gaussian sum. Therefore, the associated Dirichlet series is given by

$$\hat{E}(s) = \frac{\pi^{\frac{n}{2}}}{(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})} \sum_{k=1}^{\infty} \frac{1}{k^{s-\frac{n}{2}+1}} \sum_{q=1}^{\infty} \sum_{h(\bmod q)}{}' q^{-n} S(hQ;q) e^{-2\pi i \frac{kh}{q}}$$

for $\mathrm{Re}(s) > \frac{n}{2}$. Let $(k,q)=d$, $k=k_1 d$, $q=q_1 d$, $(k_1,q_1)=1$ and $k_1 = k_2 q_1 + l$, $(q_1,l)=1$. Then the right hand side becomes

$$\frac{\pi^{\frac{n}{2}}}{(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})} \sum_{d=1}^{\infty} \frac{1}{d^{s-\frac{n}{2}+1}} \sum_{q_1=1}^{\infty} \sum_{h(\bmod q_1 d)}{}' (q_1 d)^{-n} S(hQ;q_1 d)$$
$$\cdot \sum_{l(\bmod q_1)}{}' e^{-\frac{2\pi i h l}{q_1}} \sum_{k_1 \equiv l (\bmod q_1)} \frac{1}{k_1^{s-\frac{n}{2}+1}}.$$

By using the well-known identity

$$\sum_{\chi(\bmod q_1)} \overline{\chi}(l)\chi(k) = \begin{cases} \phi(q_1) & (k \equiv l \pmod{q_1}) \\ 0 & (\text{otherwise}), \end{cases}$$

we have

$$\sum_{k_1 \equiv l(\bmod q_1)} \frac{1}{k_1^{s-\frac{n}{2}+1}} = \frac{1}{\phi(q_1)} \sum_{\chi(\bmod q_1)} \overline{\chi}(l) \sum_{k_1=1}^{\infty} \frac{\chi(k_1)}{k_1^{s-\frac{n}{2}+1}}$$
$$= \frac{1}{\phi(q_1)} \sum_{\chi(\bmod q_1)} \overline{\chi}(l) L\left(s-\frac{n}{2}+1,\chi\right)$$

for $\mathrm{Re}(s) > \frac{n}{2}$. Therefore,

$$\begin{aligned}\hat{E}(s) =& \frac{\pi^{\frac{n}{2}}}{(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})}\sum_{d=1}^{\infty}\frac{1}{d^{s-\frac{n}{2}+1}}\sum_{q_1=1}^{\infty}\sideset{}{'}\sum_{h(\mathrm{mod} q_1 d)}\frac{S(hQ;q_1 d)}{(q_1 d)^n} \\ &\cdot \sideset{}{'}\sum_{l(\mathrm{mod} q_1)} e^{-\frac{2\pi i h l}{q_1}}\frac{1}{\phi(q_1)}\sum_{\chi(\mathrm{mod} q_1)}\overline{\chi}(l)L\left(s-\frac{n}{2}+1,\chi\right)\end{aligned} \tag{2.22}$$

holds for $\mathrm{Re}(s) > \frac{n}{2}$. It is known that the following estimate holds (see [13]):

$$S(hQ;q) \ll q^{\frac{n}{2}}.$$

Therefore, the absolute value of the right hand side of (2.22) is estimated by

$$\begin{aligned}&\ll \sum_{d=1}^{\infty}\frac{1}{d^{\sigma-\frac{n}{2}+1}}\sum_{q_1=1}^{\infty}\phi(q_1 d)\frac{(q_1 d)^{\frac{n}{2}}}{(q_1 d)^n}\cdot\phi(q_1)\frac{1}{\phi(q_1)}\sum_{\chi(\mathrm{mod} q_1)}\left|L\left(s-\frac{n}{2}+1,\chi\right)\right| \\ &\ll \sum_{d=1}^{\infty}\frac{1}{d^{\sigma}}\sum_{q_1=1}^{\infty}\frac{1}{q_1^{\frac{n}{2}-1}}\sum_{\chi(\mathrm{mod} q_1)}\left|L\left(s-\frac{n}{2}+1,\chi\right)\right|.\end{aligned} \tag{2.23}$$

The estimate (2.18) yields the right hand side of (2.23) converges on the line $\mathrm{Re}(s) = \frac{n-1}{2}$, hence $\hat{E}(s)$ is continued analytically to some domain containing the line $\mathrm{Re}(s) = \frac{n-1}{2}$ by (2.22) and the estimate

$$\left|\hat{E}\left(\frac{n-1}{2}+it\right)\right| \ll \sum_{q_1=1}^{\infty}\frac{1}{q_1^{\frac{n}{2}-1}}\sum_{\chi(\mathrm{mod} q_1)}\left|L\left(\frac{1}{2}+it,\chi\right)\right| \tag{2.24}$$

holds. By applying Minkowski's inequality to (2.24), we have

$$\begin{aligned}&\left(\int_0^T\left|\hat{E}\left(\frac{n-1}{2}+it\right)\right|^4 dt\right)^{\frac{1}{4}} \\ &\quad\ll \sum_{q_1=1}^{\infty}\frac{1}{q_1^{\frac{n}{2}-1}}\sum_{\chi(\mathrm{mod} q_1)}\left(\int_0^T\left|L\left(\frac{1}{2}+it,\chi\right)\right|^4 dt\right)^{\frac{1}{4}}\end{aligned} \tag{2.25}$$

By applying the inequality (2.19) to the summation in $\chi(\mathrm{mod} q_1)$ and using the estimate (2.16), the right hand side of (2.25) is evaluated by

$$\begin{aligned}&\le \sum_{q_1=1}^{\infty}\frac{1}{q_1^{\frac{n}{2}-1}}\phi(q_1)^{\frac{3}{4}}\left(\sum_{\chi(\mathrm{mod} q_1)}\int_0^T\left|L\left(\frac{1}{2}+it,\chi\right)\right|^4 dt\right)^{\frac{1}{4}} \\ &\ll \sum_{q_1=1}^{\infty}\frac{1}{q_1^{\frac{n}{2}-1}}q_1^{\frac{3}{4}}(q_1 T(\log q_1 T)^4)^{\frac{1}{4}}\end{aligned}$$

$$\ll \left(\sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-2-\epsilon}} \right) T^{\frac{1}{4}} \log T.$$

The series $\sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-2-\epsilon}}$ converges when $n > 6$. Therefore, the estimate

$$\left(\int_0^T \left| \hat{E}\left(\frac{n-1}{2} + it \right) \right|^4 dt \right)^{\frac{1}{4}} \ll T^{\frac{1}{4}} \log T$$

holds when $n \geq 7$, hence the statement of theorem is proved. □

Next, we consider the case of $n = 5$. In this case, we cannot use the method we used in the proof of Theorem 2.5, since the right hand side of (2.23) may not converge on the line $\mathrm{Re}(s) = 2$ if $n = 5$. We use another formula proved by Siegel ([16]) under some additional conditions.

Theorem 2.6. *Let Q be a 5×5 positive definite symmetric integral matrix which satisfies* $\det Q = 1$. *Then, when $T \to \infty$, we have*

$$\int_0^T |\zeta(2 + it; Q)|^4 dt = O(T(\log T)^4). \tag{2.26}$$

Proof. Assume that Q satisfies the conditions of theorem. In this case, Siegel showed that $\hat{E}(s)$ has the following expression (see [16], Theorem 12):

$$\hat{E}(s) = 2\pi^s \frac{\Gamma(\frac{5}{2} - s)}{\Gamma(\frac{5}{2})} \left\{ \psi(s) + \psi\left(\frac{5}{2} - s \right) \right\} \tag{2.27}$$

for $1 < \mathrm{Re}(s) < \frac{3}{2}$, where

$$\begin{aligned} \psi(s) =& 2^{s-\frac{5}{2}} \left\{ \cos\frac{\pi}{4}(2s-5) \sum_{a,b\ b\equiv 1 (\mathrm{mod} 4)} \chi_b(a) a^{s-\frac{5}{2}} b^{-s} \right. \\ & \left. + \cos\frac{\pi}{4}(2s+5) \sum_{a,b\ b\equiv 3 (\mathrm{mod} 4)} \chi_b(a) a^{s-\frac{5}{2}} b^{-s} \right\} \end{aligned} \tag{2.28}$$

and $\chi_b(a) = (\frac{a}{b})$ denoting the Legendre-Jacobi symbol. For fixed b, we have

$$\sum_a \chi_b(a) a^{s-\frac{5}{2}} = L\left(\frac{5}{2} - s, \chi_b \right)$$

for $\mathrm{Re}(s) < \frac{3}{2}$. Therefore,

$$\sum_{a,b\ b\equiv j (\mathrm{mod} 4)} \chi_b(a) a^{s-\frac{5}{2}} b^{-s} = \sum_{b\equiv j (\mathrm{mod} 4)} b^{-s} L\left(\frac{5}{2} - s, \chi_b \right) \tag{2.29}$$

(j=1,3) holds for $\mathrm{Re}(s) < \frac{3}{2}$. By using the estimate (2.17), the series of the right hand side of (2.29) converges absolutely on $\mathrm{Re}(s) = 2$, so the left hand side of (2.29) can be continued analytically to some domain containing the line $\mathrm{Re}(s) = 2$ by (2.29). Therefore, $\psi(s)$ can be continued analytically to some domain containing the line $\mathrm{Re}(s) = 2$ by

$$\begin{aligned}\psi(s) =& 2^{s-\frac{5}{2}} \left\{ \cos\frac{\pi}{4}(2s-5) \sum_{b\equiv 1(\mathrm{mod}4)} b^{-s} L\left(\frac{5}{2}-s, \chi_b\right) \right. \\ & \left. +\cos\frac{\pi}{4}(2s+5) \sum_{b\equiv 3(\mathrm{mod}4)} b^{-s} L\left(\frac{5}{2}-s, \chi_b\right) \right\}.\end{aligned} \tag{2.30}$$

On the other hand, for fixed a,

$$\begin{aligned}&\sum_{b,\ b\equiv j(\mathrm{mod}4)} \chi_b(a) b^{-s} \\ &\qquad = \frac{1}{\phi(4)} \sum_{\chi(\mathrm{mod}4)} \overline{\chi}(j) \sum_{b=1}^{\infty} \chi(b)\chi_b(a) b^{-s} \\ &\qquad = \frac{1}{\phi(4)} \sum_{\chi(\mathrm{mod}4)} \overline{\chi}(j) L(s, \tilde{\chi}_{a,\chi})\end{aligned}$$

(j=1,3) holds for $\mathrm{Re}(s) > 1$, where

$$\tilde{\chi}_{a,\chi}(b) = \chi(b)\chi_b(a) = \chi(b)\left(\frac{a}{b}\right). \tag{2.31}$$

Note that $\tilde{\chi}_{a,\chi}$ becomes a Dirichlet character modulo $4a$. Summing up, we have proved that the identity

$$\begin{aligned}\psi(s) = \frac{2^{s-\frac{5}{2}}}{\phi(4)} & \left\{ \cos\frac{\pi}{4}(2s-5) \sum_{a=1}^{\infty} a^{s-\frac{5}{2}} \sum_{\chi(\mathrm{mod}4)} \overline{\chi}(1) L(s, \tilde{\chi}_{a,\chi}) \right. \\ & \left. +\cos\frac{\pi}{4}(2s+5) \sum_{a=1}^{\infty} a^{s-\frac{5}{2}} \sum_{\chi(\mathrm{mod}4)} \overline{\chi}(3) L(s, \tilde{\chi}_{a,\chi}) \right\}\end{aligned} \tag{2.32}$$

holds for $1 < \mathrm{Re}(s) < \frac{3}{2}$, where $\tilde{\chi}_{a,\chi}$ is a Dirichlet character modulo $4a$. By using Heath-Brown's estimate (2.17) again, the right hand side of (2.32) converges absolutely at $s = \frac{1}{2} + it$, so $\psi(s)$ can be continued analytically to some domain containing the line $\mathrm{Re}(s) = \frac{1}{2}$ by (2.32). Therefore, by combining these results, the L-function $\hat{E}(s)$ has the following expression on the line $\mathrm{Re}(s) = 2$:

$\hat{E}(2+it)$

$$= 2^{\frac{1}{2}+it}\pi^{2+it}\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{5}{2})}\left\{\cos\frac{\pi}{4}(-1+2it)\sum_{b\equiv 1(\mathrm{mod}4)} b^{-2-it}L\left(\frac{1}{2}-it,\chi_b\right)\right.$$

$$\left. +\cos\frac{\pi}{4}(9+2it)\sum_{b\equiv 3(\mathrm{mod}4)} b^{-2-it}L\left(\frac{1}{2}-it,\chi_b\right)\right\}$$

$$+ 2^{-2-it}\pi^{2+it}\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{5}{2})}\left\{\cos\frac{\pi}{4}(-4-2it)\sum_{a=1}^{\infty}a^{-2-it}\sum_{\chi(\mathrm{mod}4)}\overline{\chi}(1)L\left(\frac{1}{2}-it,\tilde{\chi}_{a,\chi}\right)\right.$$

$$\left. +\cos\frac{\pi}{4}(6-2it)\sum_{a=1}^{\infty}a^{-2-it}\sum_{\chi(\mathrm{mod}4)}\overline{\chi}(3)L\left(\frac{1}{2}-it,\tilde{\chi}_{a,\chi}\right)\right\}. \tag{2.33}$$

Note that $\Gamma(\frac{1}{2}-it)\cos\frac{\pi}{4}(\cdot \pm 2it)$ (4 terms) are bounded when $t \to \infty$. By applying Minkowski's inequality, we have

$$\left(\int_0^T |\hat{E}(2+it)|^4 dt\right)^{\frac{1}{4}}$$
$$\ll \sum_{b\equiv 1(\mathrm{mod}4)} b^{-2}\left(\int_0^T \left|L\left(\frac{1}{2}-it,\chi_b\right)\right|^4 dt\right)^{\frac{1}{4}}$$
$$+ \sum_{b\equiv 3(\mathrm{mod}4)} b^{-2}\left(\int_0^T \left|L\left(\frac{1}{2}-it,\chi_b\right)\right|^4 dt\right)^{\frac{1}{4}}$$
$$+\sum_{a=1}^{\infty}a^{-2}\left(\int_0^T \left|L\left(\frac{1}{2}-it,\tilde{\chi}_{a,\chi}\right)\right|^4 dt\right)^{\frac{1}{4}}$$
$$\ll \sum_{b\equiv 1,3(\mathrm{mod}4)} b^{-2}(bT(\mathrm{log}bT)^4)^{\frac{1}{4}} + \sum_{a=1}^{\infty}a^{-2}(aT(\mathrm{log}aT)^4)^{\frac{1}{4}}$$
$$\ll T^{\frac{1}{4}}\mathrm{log}T.$$

Therefore, since

$$\int_0^T |\hat{E}(2+it)|^4 dt \ll T(\mathrm{log}T)^4,$$

we obtain the estimate (2.26). □

Remark 2.7. Assume that Q satisfies the conditions of Theorem 2.6. By applying Heath-Brown's estimate (2.17) to the identity (2.33), the estimate

$$|\hat{E}(2+it)| \ll |t|^{\frac{1}{6}} \quad (|t| \to \infty)$$

holds. Since the Dirichlet series associated to the cusp form satisfies

$$|\hat{S}(2+it)| = O(|t|^{\epsilon}) \quad (|t| \to \infty),$$

the Epstein zeta function associated to Q satisfies

$$|\zeta(2+it;Q)| \ll |t|^{\frac{1}{6}} \quad (|t| \to \infty). \tag{2.34}$$

3 Application to the divisor problem

In this section, we evaluate the summation of the coefficients of Dirichlet series of $\zeta(s;Q)^l$, where l is a positive integer equal or bigger than 4. For $l \in \mathbf{N}$, write

$$\zeta(s;Q)^l = \sum_{k=1}^{\infty} \frac{r_Q^{(l)}(k)}{k^s}$$

for $\mathrm{Re}(s) > \frac{n}{2}$. It is known that the following asypmtotic formula holds (see [11], [15]):

$$\sum_{k \le x} r_Q^{(l)}(k) = M_l^{(n)}(x) + \Delta_l^{(n)}(x) \quad (x \to \infty),$$

where $M_l^{(n)}(x)$ is called the main term, expressed by $x^{\frac{n}{2}} P_l(\log x)$ with a polymonial P_l of degree $l-1$, and $\Delta_l^{(n)}(x)$ is called the error term which becomes $o(x^{\frac{n}{2}})$ in general. Our main problem is to evaluate the error term $\Delta_l^{(n)}(x)$ as small as possible. The following lemma is due to Fomenko ([1]), which plays a fundamental role in the proof of our theorem:

Lemma 3.1. *Let Q be a $n \times n$ positive definite matrix which defines an integer valued quadratic form. Then the estimate*

$$\left|\zeta\left(\frac{n-1}{2} + it; Q\right)\right| \ll |t|^{\alpha_n + \epsilon} \quad (|t| \to \infty) \tag{3.1}$$

holds, where

$$\alpha_n = \begin{cases} \dfrac{9}{56} & (n \ge 4, \mathrm{even}), \\ \dfrac{1}{6} & (n \ge 9, \mathrm{odd}), \\ \dfrac{3}{16} & (n = 7). \end{cases} \tag{3.2}$$

By combining this lemma and our theorems in Section 2, we obtain the following estimates for the error term $\Delta_l^{(n)}(x)$:

Theorem 3.2. *Let Q be a $n \times n$ positive definite symmetrix matrix which satisfies $Q[\mathbf{x}] \in \mathbf{N}$ for any $\mathbf{x} \in \mathbf{Z}^n \backslash \{\mathbf{0}\}$. If $n = 5$, we assume that Q is integral* *and* $\det Q = 1$. *Then, for* $l \geq 4$, *the following asymptotic formula holds when* $x \to \infty$:

$$\sum_{k\leq x} r_Q^{(l)}(k) = M_l^{(n)}(x) + \begin{cases} O\left(x^{\frac{n}{2}-\frac{28}{9l+20}+\epsilon}\right) & (n \geq 4, \text{even}), \\ O\left(x^{\frac{n}{2}-\frac{3}{l+2}+\epsilon}\right) & (n \geq 9, \text{odd or } n=5), \\ O\left(x^{\frac{n}{2}-\frac{8}{3l+4}+\epsilon}\right) & (n=7). \end{cases} \tag{3.3}$$

Here, $M_l^{(n)}(x)$ *is expressed by* $x^{\frac{n}{2}} P_l(\log x)$ *with a polymonial* P_l *of degree* $l-1$.

Proof. We start from the Perron's formula (see [15])

$$\sum_{k\leq x} r_Q^{(l)}(k) = \frac{1}{2\pi i}\int_{\frac{n}{2}+\epsilon-iT}^{\frac{n}{2}+\epsilon+iT} \zeta(s;Q)^l \frac{x^s}{s} ds + O\left(\frac{x^{\frac{n}{2}+\epsilon}}{T}\right) + O(x^\epsilon).$$

We move the path of integral to the parallel segment with $\mathrm{Re}(s) = \frac{n-1}{2}$. Then,

$$\begin{aligned}\sum_{k\leq x} r_Q^{(l)}(k) = \frac{1}{2\pi i}&\left\{\int_{\frac{n-1}{2}-iT}^{\frac{n-1}{2}+iT} + \int_{\frac{n-1}{2}+iT}^{\frac{n}{2}+\epsilon+iT} + \int_{\frac{n}{2}+\epsilon-iT}^{\frac{n-1}{2}-iT}\right\} \zeta(s;Q)^l \frac{x^s}{s} ds \\ &+ \mathrm{Res}\left[\zeta(s;Q)^l \frac{x^s}{s}, s = \frac{n}{2}\right] + O\left(\frac{x^{\frac{n}{2}+\epsilon}}{T}\right) + O(x^\epsilon).\end{aligned} \tag{3.4}$$

We put

$$I_1 := \frac{1}{2\pi i}\int_{\frac{n-1}{2}-iT}^{\frac{n-1}{2}+iT} \zeta(s;Q)^l \frac{x^s}{s} ds,$$

$$I_2 := \frac{1}{2\pi i}\int_{\frac{n-1}{2}+iT}^{\frac{n}{2}+\epsilon+iT} \zeta(s;Q)^l \frac{x^s}{s} ds,$$

$$I_3 := \frac{1}{2\pi i}\int_{\frac{n}{2}+\epsilon-iT}^{\frac{n-1}{2}-iT} \zeta(s;Q)^l \frac{x^s}{s} ds,$$

$$M_l^{(n)}(x) := \mathrm{Res}\left[\zeta(s;Q)^l \frac{x^s}{s}, s = \frac{n}{2}\right].$$

Firstly, since $\zeta(s;Q)^l \frac{x^s}{s}$ has a pole of order l at $s = \frac{n}{2}$, $M_l^{(n)}(x)$ is expressed by

$$M_l^{(n)}(x) = x^{\frac{n}{2}} P_l(\log x) \tag{3.5}$$

with some polynomial P_l of degree $l-1$. Next, we evaluate three integrals I_i $(i = 1, 2, 3)$. Assume that the estimate

$$\left|\zeta\left(\frac{n-1}{2} + it; Q\right)\right| \ll |t|^{\alpha_n+\epsilon} \quad (|t| \to \infty)$$

holds. Then, by using Theorem 2.5 or Theorem 2.6, $|I_1|$ is evaluated by

$$\begin{aligned}
|I_1| &\ll x^{\frac{n-1}{2}} \int_{-T}^{T} \left|\zeta\left(\frac{n-1}{2}+it;Q\right)\right|^{l-4} \left|\zeta\left(\frac{n-1}{2}+it;Q\right)\right|^{4} \frac{dt}{t} \\
&\ll x^{\frac{n-1}{2}} T^{(l-4)\alpha_n+\epsilon} \int_{-T}^{T} \left|\zeta\left(\frac{n-1}{2}+it;Q\right)\right|^{4} \frac{dt}{t} \\
&\ll x^{\frac{n-1}{2}} T^{(l-4)\alpha_n+\epsilon}.
\end{aligned} \tag{3.6}$$

By Phragmén-Lindelöf principle,

$$|\zeta(\sigma \pm iT;Q)| \ll T^{-2\alpha_n(\sigma-\frac{n}{2})+\epsilon}$$

holds for $\frac{n-1}{2} \leq \sigma \leq \frac{n}{2}$. Therefore, $|I_2|+|I_3|$ is evaluated by

$$\begin{aligned}
|I_2|+|I_3| &\ll \int_{\frac{n-1}{2}}^{\frac{n}{2}+\epsilon} T^{-2l\alpha_n(\sigma-\frac{n}{2})+\epsilon} \frac{x^\sigma}{T} d\sigma \\
&= \int_{\frac{n-1}{2}}^{\frac{n}{2}+\epsilon} \left(\frac{x}{T^{2l\alpha_n}}\right)^{\sigma} T^{nl\alpha_n-1+\epsilon} d\sigma \\
&\ll x^{\frac{n-1}{2}} T^{l\alpha_n-1+\epsilon} + x^{\frac{n}{2}+\epsilon} T^{-1+\epsilon}.
\end{aligned} \tag{3.7}$$

By inserting (3.5), (3.6), (3.7) into (3.4), we have

$$\begin{aligned}
\sum_{k\leq x} r_Q^{(l)}(k) =& M_l^{(n)}(x) + O(x^{\frac{n-1}{2}} T^{(l-4)\alpha_n+\epsilon}) + O(x^{\frac{n-1}{2}} T^{l\alpha_n-1+\epsilon}) \\
&+ O(x^{\frac{n}{2}+\epsilon} T^{-1+\epsilon}) + O(x^\epsilon).
\end{aligned} \tag{3.8}$$

We put

$$T = x^{\frac{1}{2(l-4)\alpha_n+2}}.$$

Then (3.8) becomes

$$\sum_{k\leq x} r_Q^{(l)}(k) = M_l^{(n)}(x) + O\left(x^{\frac{n}{2}-\frac{1}{2(l-4)\alpha_n+2}+\epsilon}\right) + O\left(x^{\frac{n}{2}-\frac{1-2\alpha_n}{(l-4)\alpha_n+1}+\epsilon}\right). \tag{3.9}$$

Finally, by insertig the values given in Lemma 3.1 or (2.34) into α_n, we obtain the result. □

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