論文題目: Entire curves in projective algebraic varieties
(射影代数多様体内の正則曲線について)

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Contents

Preface

The purpose of this thesis is to study properties of an entire curve in a complex projective algebraic variety, i.e., a holomorphic map from $\mathbb{C}$ to a complex projective algebraic variety.

In Chapter 1, we recall some notation and basic results of jet bundles, logarithmic jet bundles, Demailly-Semple jet bundles, Nevanlinna theory and Kobayashi hyperbolicity.

In Chapter 2, we prove the Nevanlinna second main theorem for some families of non-linear hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ using a meromorphic partial projective connection.

Now we state our main theorems in Chapter 2 precisely. Let $s_0, \ldots, s_n$ be homogeneous polynomials of degree d in $\mathbb{C}[X_0, \ldots, X_n]$ such that

$$\det\left(\frac{\partial s_j}{\partial X_k}\right)_{0\leq j,k\leq n} \not\equiv 0.$$

Then we construct the meromorphic connection $\widetilde{\nabla} = d + \widetilde{\Gamma}$ on $\mathbb{C}^{n+1}$ defined by

$$\sum_{0\leq\lambda\leq n} \frac{\partial s_\kappa}{\partial X_\lambda}\, \widetilde{\Gamma}^{\lambda}_{i,j} = \frac{\partial^2 s_\kappa}{\partial X_i \partial X_j}.$$

This meromorphic connection induces the meromorphic partial projective connection ∇ on $\mathbb{P}^n(\mathbb{C})$ (see Section 2.2).

Theorem 0.0.1. *Let σ_k, $k = 1, \ldots, q$ be elements of a linear system $|\{s_0, \ldots, s_n\}|$ such that $\sum_{1\leq k\leq q} \sigma_k$ is an effective reduced divisor only with simple normal crossings.*

Assume that $X_0^{d-l_0}|s_0, \ldots, X_n^{d-l_n}|s_n$ for non-negative integers $l_j \leq d$, $j = 0, \ldots, n$. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a non-constant holomorphic map whose *image is neither contained in the support of an element of the linear system* $|\{s_0, \ldots, s_n\}|$ *nor contained in the polar locus of* ∇. *Then we have*

$$\left(q - \frac{n+1}{d} - \frac{1}{2d}(n-1)n(n+1+l_0+\cdots+l_n)\right) T_f(r, dH) \leq \sum_{1\leq i\leq q} N_n(r, f^*\sigma_i) + S_f(r),$$

where H *is a hyperplane bundle on* $\mathbb{P}^n(\mathbb{C})$, *and* $S_f(r) = O(\log^+ T_f(r) + \log^+ r)\|$. *Here "* $\|$ *" means that the inequality holds for all* $r \in (0, +\infty)$ *possibly except for subset with finite Lebesgue measure.*

We also investigate the second main theorem for singular hypersurfaces by pulling back meromorphic partial projective connections. As an application, we show the second main theorem for hypersurfaces in $\mathbb{P}^2(\mathbb{C})$ which is not in general position.

Let $s_0, s_1, s_2 \in \mathbb{C}[X_0, X_1, X_2]$ be homogeneous polynomials of degree d such that $\det(\partial s_j/\partial X_k)_{0\leq j,k\leq 2} \not\equiv 0$, and $X_0^{d-l_0}|s_0, X_1^{d-l_1}|s_1, X_2^{d-l_2}|s_2$ for $0 \leq l_0, l_1, l_2 \leq d$. Let $\sigma_0, \ldots, \sigma_q$ be elements of linear system $|\{s_0, s_1, s_2\}|$ such that σ_j is a non-singular divisor in $\mathbb{P}^2(\mathbb{C})$. Assume that σ_j intersects σ_k transversally for all $1 \leq j \neq k \leq q$. Let $x_1, \ldots, x_p$ be points of $\mathbb{P}^2(\mathbb{C})$ such that $\sum_{i=1}^q(\sigma_i)$ is an effective reduced divisor only with simple normal crossings in $\mathbb{P}^2(\mathbb{C}) \setminus \{x_1, \ldots, x_p\}$. Let $\pi : \widetilde{\mathbb{P}}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ be the blowing-up at $\{x_1, \ldots, x_p\}$, and let $E = \sum_{i=1}^p E_i$ be the exceptional divisor of π, where E_i is irreducible and $\pi(E_i) = x_i$. Let $\widetilde{\sigma}_i$ be the proper transform of σ_i under the blowing-up π.

Theorem 0.0.2. (a) *Let* H *be the hyperplane bundle on* $\mathbb{P}^2(\mathbb{C})$. *Let* $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ *be a holomorphic map such that* $f(\mathbb{C})$ *is neither contained in the support of elements of* $|\{s_0, s_1, s_2\}|$ *nor in* $\{\det(\partial s_j/\partial X_k) = 0\}$. *Let* $\widetilde{f} : \mathbb{C} \to \widetilde{\mathbb{P}}^2(\mathbb{C})$ *be the lift of* f.

(i) *When* $d = 1$, *we have*

$$\sum_{i=1}^q T_{\widetilde{f}}(r, [\widetilde{\sigma}_i]) + \sum_{i=1}^p T_{\widetilde{f}}(r, [E_i]) - 3T_f(r, H) \leq \sum_{i=1}^q N_2(r, \widetilde{f}^*\widetilde{\sigma}) + \sum_{i=1}^p N(r, \widetilde{f}^* E_i) + S_f(r). \quad (1)$$

(ii) *When* $d \geq 2$, *we have*

$$\sum_{i=1}^{q} T_{\widetilde{f}}(r,[\widetilde{\sigma}_i]) + \sum_{i=1}^{p} T_{\widetilde{f}}(r,[E_i]) - (6+l_0+l_1+l_2)T_f(r,H)$$
$$\leq \sum_{i=1}^{q} N_2(r,\widetilde{f^*\widetilde{\sigma}}) + \sum_{i=1}^{p} N(r,\widetilde{f}^*E_i) + S_f(r). \qquad (2)$$

(b) *Furthermore, we assume that* $\sigma_1,\dots,\sigma_q$ *are in* m*-subgeneral position. Let* $H_1,H_2,H_3 \subset \mathbb{P}^2(\mathbb{C})$ *be hyperplanes in general position which do not pass through* $\{x_1,\dots,x_p\}$.

(i) *When* $d=1$, *we have*

$$(q-3)T_f(r,H) \leq \sum_{i=1}^{q} N_2(r,\widetilde{f^*\widetilde{\sigma}_i}) + m\sum_{i=1}^{p} N(r,\widetilde{f}^*E_i) + \frac{m-1}{2}\sum_{i=1}^{3} N_2(r,f^*H_i) + S_f(r). \qquad (3)$$

(ii) *When* $d \geq 2$, *we have*

$$\left(q-\frac{6+l_0+l_1+l_2}{d}\right)T_f(r,H) \leq \sum_{i=1}^{p} N_2(r,\widetilde{f^*\widetilde{\sigma}_i}) + m\sum_{i=1}^{p} N(r,\widetilde{f}^*E_i) + \frac{m-1}{2}\sum_{i=1}^{3} N_2(r,f^*H_i) + S_f(r). \quad (4)$$

In Chapter 3, we deal with entire curves in the product space of the Riemann spheres. We prove the second main theorem for algebraic divisors in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which are compactification of one-dimensional subtori in $\mathbb{C}^* \times \mathbb{C}^*$.

Now we state our main theorem in Chapter 3. Let $[X_0 : X_1]$ and $[Y_0 : Y_1]$ be the homogeneous coordinates in the first and second factors of the product space of the $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let m', n', m'', n'' be positive integers. We define the effective divisors D', D'' on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ by the polynomials $X_0^{m'}Y_0^{n'} - X_1^{m'}Y_1^{n'}$, $X_0^{m''}Y_1^{n''} - X_1^{m''}Y_0^{n''}$. We prove the second main theorem for divisors D' and D''. Let $H_{1,0}$, $H_{1,1}$, $H_{2,0}$ and $H_{2,1}$ be the hyperplanes in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which are defined by the monomials X_0, X_1, Y_0 and Y_1. Put $Z_0 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Then there exists the sequence of the blowing-up

$$\begin{aligned} \pi_{1,0} &: Z_1 \to Z_0, \\ \pi_{2,1} &: Z_2 \to Z_1, \\ &\vdots \\ \pi_{k,k-1} &: Z_k \to Z_{k-1}. \end{aligned}$$

which satisfies the following condition $(*)$:

Put $\pi_{j,i} = \pi_{i+1,i} \circ \cdots \circ \pi_{j,j-1}$ for $i < j$. Let $\widetilde{D}'$, $\widetilde{D}''$ and $\widetilde{H}_{i,j}$, $1 \leq i \leq 2, 0 \leq j \leq 1$ be the proper transform of D', D'', and $H_{i,j}$ under $\pi_{k,0}$. Let E_i, $1 \leq i \leq k$ be the exceptional divisor of the blowing-up $\pi_{i,i-1}$, and let $\widetilde{E}_i$ be the proper transform of E_i under $\pi_{k,i}$. Then

$(*)$ $\widetilde{D}' + \widetilde{D}'' + \sum_{i=1}^{2} \sum_{j=0}^{1} \widetilde{H}_{i,j} + \sum_{i=1}^{k} \widetilde{E}_i$ is an effective reduced divisor only with simple normal crossings in Z_k.

Theorem 0.0.3. *Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map. Let $\widetilde{f} : \mathbb{C} \to Z_k$ be the lift of f. Assume that*

$$f(\mathbb{C}) \not\subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \quad | \quad C_0 X_0^{r_1} Y_0^{r_2} - C_1 X_1^{r_1} Y_1^{r_2} = 0\},$$

for all $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ and all $(C_0, C_1) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$, and assume that there exist no holomorphic functions g_1, g_2 on $\mathbb{C}$ and no $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$f = (\exp g_1, \exp g_2),$$

$$a g_1 + b g_2 = (\text{constant}),$$

on $\mathbb{C}$. Then it follows that

$$T_{\widetilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \leq N_2(r, \widetilde{f}^* \widetilde{D}') + N_2(r, \widetilde{f}^* \widetilde{D}'') + 2 \sum_{i=1}^{2} \sum_{j=0}^{1} N_1(r, \widetilde{f}^* \widetilde{H}_{i,j}) + 2 \sum_{i=1}^{k} N_1(r, \widetilde{f}^* \widetilde{E}_i) + S_f(r).$$

In Chapter 4, our main result is a characterization of an algebraic divisor on an algebraic torus whose complement is Kobayashi hyperbolically imbedded into a toric projective variety. Before stating our main theorem in Chapter 4, we give necessary definitions.

We fix a free module $N = \mathbb{Z}^r$ of rank r over the ring $\mathbb{Z}$ of rational integers. Let $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) = \mathbb{Z}^r$ be the dual $\mathbb{Z}$-module of N. Let

$$\langle \, , \, \rangle : M \times N \to \mathbb{Z}$$

be the canonical $\mathbb{Z}$-bilinear pairing. Let $T_N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = (\mathbb{C}^*)^r$ be the r-dimensional algebraic torus. Let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Let A be a finite subset of M. Define

$$\mathcal{L}_A := \{a - b \in M_{\mathbb{R}} \mid a, b \in A\}.$$

Let V_A be an $\mathbb{R}$-vector subspace of $M_{\mathbb{R}}$ generated by all elements in $\mathcal{L}_A$. Define

$$\mathcal{H}_A := \{H \subset V_A \,|\, \text{hyperplane of } V_A \text{ generated by elements in } \mathcal{L}_A\},$$

where a hyperplane of V_A is an $\mathbb{R}$-vector subspace of codimension one in V_A.

Let P be an integral convex polytope in $M_{\mathbb{R}}$ such that $\dim P = r$. Here the dimension of a convex polytope P is the dimension of a subspace of $M_{\mathbb{R}}$ which is generated by $\{a - b \,|\, a, b \in P\}$. Then there exists the toric projective variety X associated to P (see [Od] Chap. 2), and there exists the imbedding $i : T_N \to X$.

Theorem 0.0.4. *Let S be a finite subset of M such that $S \subset P$. Assume the following conditions for all positive dimensional faces τ of P:*

(i) $\tau \cap S \neq \emptyset$, and the dimension of the convex hull of $\tau \cap S$ is equal to the dimension of τ.

(ii) Let $H \in \mathcal{H}_{\tau\cap S}$, and let $\phi_H : V_{\tau\cap S} \to V_{\tau\cap S}/H$ be the canonical morphism. Let $x \in \tau \cap S$. Then $\sharp(\phi_H(\tau \cap S - x)) \geq \dim \tau + 1$ for all $H \in \mathcal{H}_{\tau\cap S}$, where $\sharp(\phi_H(\tau \cap S - x))$ is the number of the elements in

$$\{\phi_H(y - x) \in V_{\tau\cap S}/H \,|\, y \in \tau \cap S\}$$

(note that this condition is independent of a choice of x in $\tau \cap S$).

Then $T_N \setminus \operatorname{supp} D$ is Kobayashi hyperbolically imbedded into X for a general divisor D of the linear system $|\{z_1^{i_1} z_2^{i_2} \cdots z_r^{i_r}\}_{(i_1, i_2, \ldots, i_r) \in S}|$ in T_N.

As a corollary, we prove the following: the complement of the union of $n+1$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position and a general hypersurface of degree n in $\mathbb{P}^n(\mathbb{C})$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^n(\mathbb{C})$.

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1
Preliminaries

In this chapter, we collect some definitions and well-known results of jet bundles, Nevanlinna theory and Kobayashi hyperbolicity. In particular we explain the Nevanlinna second main theorem and the Kobayashi conjecture.

1.1 Jet bundles, logarithmic jet bundles and Demailly-Semple jet bundles

A lot of properties of entire curves are obtained from the structure of the jet bundle. In this section, we introduce jet bundles, logarithmic jet bundles and Demailly-Semple jet bundles.

1.1.1 Jet bundles

Let X be an n-dimensional complex manifold, and let x be a point of X. Let $f : (\mathbb{C}, 0) \to (X, x)$ be a germ of a holomorphic map from a neighborhood of $0 \in \mathbb{C}$ to X such that $f(0) = x$. We denote by $H(\mathbb{C}, X)_x$ the set of all germs of those holomorphic mappings. Take a holomorphic local coordinate system $(z_1, \cdots, z_n)$ around x, and put $f_i = z_i \circ f$, $g_i = z_i \circ g$ for $f, g \in H(\mathbb{C}, X)_x$. Then we write $f \overset{k}{\sim} g$ if

$$\frac{d^j}{dz^j} f_i(0) = \frac{d^j}{dz^j} g_i(0) \quad \text{for } 0 \leq j \leq k, 1 \leq i \leq n.$$

This equivalence relation does not depend on the choice of a holomorphic local coordinate system. Let $j_{k,x}(f)$ denote the equivalence class of $f \in H(\mathbb{C}, X)_x$ and set

$$J_k(X)_x = H(\mathbb{C}, X)_x / \overset{k}{\sim} .$$

Define

$$J_k(X) = \bigcup_{x \in X} J_k(X)_x.$$

Let $p : J_k(X) \to X$ be the natural projection. Then $J_k(X)$ naturally carries a structure of a complex manifold, and the triple $(J_k(X), p, X)$ forms a holomorphic fiber bundle over X. This holomorphic fiber bundle is called the k-jet bundle over X. Let $f : U \to X$ be a holomorphic map from an open set U in $\mathbb{C}$ to X. Then there exists the holomorphic map $j_k(f) : U \to J_k(X)$ such that $j_k(f)(z) = j_{k,f(z)}(f)$, $z \in U$. We call $j_k(f)$ the lifting of order k of f. A holomorphic (resp. meromorphic) functional on $J_k(X)$ which is a polynomial on every fiber is called *holomorphic (resp. meromorphic) k-jet differential* on X. We denote the sheaf of germs of *holomorphic k-jet differentials* by $E_k^{GG}\Omega_X$, and define the sheaf of germs of *holomorphic k-jet differentials* of degree m by

$$\begin{aligned} E_{k,m}^{GG}\Omega_{X,x} &= \{Q \in E_k^{GG}\Omega_{X,x} \mid Q(j_k(f(az))) = a^m Q(j_k(f(z))) \\ &\quad \text{for all } a \in \mathbb{C} \text{ and for all germs of holomorphic map} \\ &\quad f : (\mathbb{C}, 0) \to (X, x)\}. \end{aligned}$$

There exists the canonical derivative $d : E_{k,m}^{GG}\Omega_{X,x} \to E_{k+1,m+1}^{GG}\Omega_{X,x}$ which is defined by the following way. Let $Q \in E_{k,m}^{GG}\Omega_{X,x}$, and let $f : (\mathbb{C}, 0) \to (X, x)$ be a germ of a holomorphic map. Then

$$dQ(j_{k+1}(f))(z) = \frac{dQ(j_k(f))}{dz}.$$

1.1.2 Logarithmic jet bundles

In this subsection, we recall some basic setup of logarithmic jet bundles due to Noguchi [Nog2]. Let D be a simple normal crossing divisor on X. Let Ω be the cotangent bundle of X. The logarithmic jet bundle is a generalization of logarithmic cotangent bundle $\Omega_X(\log D)$. Let V be an affine open subset of X, and let $X_1, \ldots, X_n$ be a local coordinate system around every point of V such that $X_1 \cdots X_i = 0$ defines D. Then we define

$$\begin{aligned} J_k(V; \log D) &= V \times \operatorname{Spec} \mathbb{C}\left[\frac{dX_1}{X_1}, \ldots, \frac{dX_i}{X_i}, dX_{i+1}, \ldots, dX_n, \ldots, \right. \\ &\qquad \left. \frac{d^k X_1}{X_1}, \ldots, \frac{d^k X_i}{X_i}, d^k X_{i+1}, \ldots, d^k X_n\right]. \end{aligned}$$

Let $\{V_i\}_i$ be an affine open covering of X. By glueing $\{J_k(V_i; \log D)\}_i$, We obtain the logarithmic jet bundle $J_k(X; \log D)$. A holomorphic (resp. meromorphic) functional on $J_k(X; \log D)$ which is a polynomial on every fiber is called a *holomorphic (resp. meromorphic) logarithmic k-jet differential* on X.

1.1.3 Demailly-Semple jet bundles

We recall the construction of the Demailly-Semple jet bundle due to [De]. Let $\mathbb{G}_k$ be the group of germs of k-jets biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \psi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \qquad a_1 \in \mathbb{C}^*, a_j \in \mathbb{C}, j \geq 2,$$

in which the composition law is taken modulo terms t^j of degree $j > k$. Then $\mathbb{G}_k$ acts on $J_k X$ by $(\psi, j_{k,x}(f)) \mapsto j_{k,x}(f \circ \psi)$ where f is a germ of a holomorphic map $f : (\mathbb{C}, 0) \to (X, x)$. Let $J_k X^{\mathrm{reg}}$ be the bundle of regular k-jets of maps $f : (\mathbb{C}, 0) \to (X, x)$, that is , jets f such that $f'(0) \neq 0$. The Demailly-Semple jet bundle is a compactification of $J_k X^{\mathrm{reg}}/\mathbb{G}_k$.

Let V be a subbundle of T_X. We call the pair (X, V) as the *directed manifold.* We define a sequence of directed manifolds (X_k, V_k) as follows. Let $(X_0, V_0) = (X, V)$. We define $X_k = P(V_{k-1})$ by the projectivized bundle of lines in the vector bundle V_{k-1}. Let $x \in X_{k-1}$, and let $v \in V_{k-1}$. Then we define V_k at any point $(x, [v]) \in X_k$ by

$$V_{k,(x,[v])} = \{\xi \in T_{X_k,(x,[v])} \mid (\pi_{k,k-1})_* \xi \in \mathbb{C} \cdot v\},$$

where $\pi_{k,k-1} : X_k \to X_{k-1}$ is the projection map. We denote by $\mathcal{O}_{X_k}(-1)$ the tautological line bundle. Let $f : (\mathbb{C}, 0) \to X$ be a germ of a holomorphic map. Then f lifts to $f_{[k]} : (\mathbb{C}, 0) \to X_k$ which is tangent to V_k. This lift induces the injection $J_k X^{\mathrm{reg}}/\mathbb{G}_k \to X_k$ and the image of this injection is Zariski dense in X_k. Moreover, the derivative $f'_{[k-1]}$ gives the value of $f^*_{[k]}\mathcal{O}_k(-1)$. We assume that $V = \Omega_X$. We define that

$$\begin{aligned} E_{k,m}\Omega_X &= \{Q \in E^{GG}_{k,m}\Omega_X \mid Q(f \circ \psi) = \psi'^m Q(f) \circ \psi, \\ &\qquad \text{for every germ } f \in J_k X \text{ and every germ } \psi \in \mathbb{G}_k\}, \end{aligned}$$

i.e., $E_{k,m}\Omega_X$ is set of germs of $E^{GG}_{k,m}\Omega_X$ which are invariant under arbitrary changes of parametrization. Let $\pi_{k,0} : X_k \to X$ be the projection map. Then the following theorem follows.

Theorem 1.1.1 ([De]). *The injection $J_k X^{\mathrm{reg}}/\mathbb{G}_k \to X_k$ induces the isomorphism*

$$E_{k,m}\Omega_X \simeq \pi_{k,0*}\mathcal{O}_{X_k}(m).$$

1.1.4 Vanishing theorem and degeneracy of entire curves

A degeneracy of entire curves in X follows from the next theorem.

Theorem 1.1.2 ([GG], [De], [SY]). *Let A be an ample line bundle on X. Assume that there exists a non-zero section*

$$Q \in H^0(X, E_{k,m}^{GG}\Omega_X \otimes A^{-1})$$

for $k, m > 0$. Then $Q(j_k(f)) \equiv 0$.

By Theorem 1.1.1, $H^0(X, E_{k,m}\Omega_X \otimes A^{-1}) \simeq H^0(X, \pi_{k,0*} \otimes A^{-1})$ and this isomorphism sometimes enables us to calculate the dimension of $H^0(X, E_{k,m}\Omega_X \otimes A^{-1})$.

1.2 Nevanlinna theory

Nevanlinna theory is an intersection theory of transcendental curves and effective divisors. We recall some basic definitions and well-known results in Nevanlinna theory.

1.2.1 Notation

We introduce some functions which play an important role in the Nevanlinna theory. Let E be an effective divisor on $\mathbb{C}$. We write $E = \sum m_j P_j$, where $\{P_j\}$ is a set of discrete points in $\mathbb{C}$ and m_j are positive integers. Put $n_k(r, E) = \sum_{|P_j|<r} \min\{k, m_j\}$ where $k > 0$ or $+\infty$. We define the counting function of E by

$$N_k(r, E) = \int_1^r \frac{n_k(t, E)}{t} dt.$$

Let X be a complex projective algebraic manifold, and let D be a divisor on X. Let $L = [D]$ be the holomorphic line bundle on X which is defined by the divisor D, and let $\operatorname{supp} D$ be the support of D. Let σ be a holomorphic section of L such that the zero divisor of σ is D. Let $f : \mathbb{C} \to X$ be a non-constant holomorphic map. We define the proximity function of D by

$$m_f(r, D) = \int_0^{2\pi} \log \frac{1}{\|\sigma(f(re^{i\theta}))\|} \frac{d\theta}{2\pi} + O(1),$$

where $\|\cdot\|$ is a Hermitian metric in L. Let $R(L, \|\cdot\|)$ be the curvature form of the metrized line bundle $(L, \|\cdot\|)$ representing the first Chern class. Then we define the characteristic function of L by

$$T_f(r, L) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^* R(L, \|\cdot\|) + O(1),$$

where $\Delta(t) = \{z \in \mathbb{C} \,|\, |z| < t\}$. We set $T_f(r) = T_f(r, L)$ if L is an ample line bundle on X.

1.2.2 Nevanlinna first and second main theorems

In Nevanlinna theory, there exists two fundamental theorems, i.e., Nevanlinna first and second main theorems. The first main theorem explains the relation between the characteristic function, counting function and proximity function.

Theorem 1.2.1 (First Main Theorem). *Let X be a complex projective manifold, and let L be a holomorphic line bundle on X. Let D be an effective divisor on X such that $[D] = L$. Let $f : \mathbb{C} \to X$ be a non-constant entire curve such that $f(\mathbb{C}) \not\subset \operatorname{supp} D$. Then it follows that*

$$T_f(r, L) = N(r, f^*D) + m_f(r, D) + O(1).$$

The first main theorem implies that

$$T_f(r, L) \geq N(r, f^*D) + O(1).$$

To study entire curves, the first main theorem is not sufficient. The second main theorem is an inequality which estimates the characteristic functions from above using a counting function. The following lemma is important to prove the second main theorem.

Lemma 1.2.2 (Lemma on logarithmic derivative). *Let f be a non-constant meromorphic function on $\mathbb{C}$. Then it follows that*

$$\int_0^{2\pi} \log^+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \leq S_f(r),$$

where $\log^+ r = \max\{0, \log r\}$, and $S_f(r) = O(\log^+ T_f(r) + \log^+ r)||$. Here "$||$" means that the inequality holds for all $r \in (0, +\infty)$ possibly except for a Borel subset with finite Lebesgue measure.

The cases in which the second main theorem is proved are not so many. The following second main theorem of H. Cartan is fundamental.

Theorem 1.2.3 ([Ca]). *Let $H_1, H_2, \ldots, H_q$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map which is not linearly degenerate, i.e., there exists no hyperplane in $\mathbb{P}^n(\mathbb{C})$ which contains $f(\mathbb{C})$. Let H be a hyperplane bundle on $\mathbb{P}^n(\mathbb{C})$. Then it follows that*

$$(q - n - 1)T_f(r, H) \leq \sum_{i=1}^{q} N_n(r, f^*H_i) + S_f(r).$$

Cartan's second main theorem implies that a holomorphic map $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ which omits $(n+2)$ hyperplanes in general position linearly degenerates, i.e., $f(\mathbb{C})$ is contained in a hyperplane of $\mathbb{P}^n(\mathbb{C})$. After Cartan's proof came out, Y.-T. Siu [Si1] and J. Noguchi [Nog5] reproved Cartan's second main theorem using totally geodesic connections. Let ∇ be a C^∞-connection in TX. Define the Wronskian operator by

$$W(\nabla, f) = f' \wedge \nabla_{f'} f' \wedge \cdots \wedge \nabla_{f'}^{n-1} f', \quad f : (\mathbb{C}, 0) \to \mathbb{P}^n(\mathbb{C}).$$

Then the following theorem holds.

Theorem 1.2.4 ([Nog5]). *Let $f : \mathbb{C} \to X$ be a ∇-nondegenerate holomorphic curve and let $D = \sum D_i$ be an effective reduced divisor only with simple normal crossings. Assume*

(i) $\log |W(\nabla, f)|$ *is subharmonic;*

(ii) every D_i is ∇-totally geodesic.

Then we have

$$T_f(r, L(D)) + T_f(r, K_X) \leq \sum N_n(r, f^* D_i) + S_f(r).$$

Let ∇ be the Fubini-Study metric connection on $\mathbb{P}^n(\mathbb{C})$. Then hyperplanes in $\mathbb{P}^n(\mathbb{C})$ are totally geodesic with respect to ∇. In [Nog5], Noguchi proved that $W(\nabla, f)$ is holomorphic, and obtained Cartan's second main theorem. K. Yamanoi also proved Cartan's second main theorem by showing the algebro-geometric version of Nevanlinna's lemma on logarithmic derivative (ANLD) in [Ya].

We give another new case in which the second main theorem was proved. In Noguchi, Winkelmann and Yamanoi [NWY1], [NWY3], the second main theorem for holomorphic map f from $\mathbb{C}$ to a semi-Abelian variety A was proved;

Theorem 1.2.5 ([NWY3]). *Let $f : \mathbb{C} \to A$ be a holomorphic map which is not algebraically degenerate, i.e., $f(\mathbb{C})$ is Zariski dense in A. There exists the compactification of A such that $\overline{A}$ is smooth, equivalent with respect to the A-action, independent of f, and it follows that*

$$T_f(r, [\overline{D}]) \leq N_1(r, f^* D) + \varepsilon T_f(r, [\overline{D}])||_\varepsilon,$$

for arbitrary $\varepsilon > 0$, where $\overline{D}$ is the closure of D in $\overline{A}$.

There exist a lot of applications of this theorem, for example, see [NWY2] and [CN].

We would like to mention M. McQuillan's work in [Mcl]. Let X be a complex projective algebraic manifold, and let E be a vector bundle over X. We define $\mathbb{P}(E) = \mathbf{Proj} \oplus_{d\geq 0} S^d E$. This algebraic variety comes with a tautological line sheaf $\mathcal{O}(1)$.

Theorem 1.2.6 (McQuillan's tautological inequality). *Let D be an effective simple normal crossing divisor on X. Let $f : \mathbb{C} \to X$ be a non-constant holomorphic map such that $f(\mathbb{C}) \not\subset D$. Let $f' : \mathbb{C} \to \mathbb{P}(\Omega_X(\log D)))$ be the canonical lifting of f. Then it follows that*

$$T_{f'}(r, \mathcal{O}(1)) \leq N_1(r, D) + S_f(r).$$

Theorem 1.2.6 implies the lemma on logarithmic derivative if the dimension of the target space is one. McQuillan used Theorem 1.2.6 to prove the second main theorem for a parabolic leaf of a singular foliation on a surface of general type. McQuillan also obtained the following theorem by combining the second main theorem for parabolic leaf and F. Bogomolov's theorem [Bo].

Theorem 1.2.7 ([Mcl]). *If X is a surface of general type with $c_1^2 > c_2$, then all entire curves of X are algebraically degenerate.*

1.3 Kobayashi hyperbolicity and Kobayashi hyperbolic imbeddings

In this section, we recall some basic definitions and results in Kobayashi hyperbolicity.

1.3.1 Definition

Let $\Delta(1)$ be the unit disk in the complex plane $\mathbb{C}$. Let ρ be the Poincaré distance on $\Delta(1)$. Then

$$\rho(z, w) = \log \frac{1 + |\lambda|}{1 - |\lambda|}, \quad \lambda = \frac{z - w}{1 - \bar{w}z}$$

for $z, w \in \Delta(1)$. Let X be a complex space and let p, q be points of X. A chain of holomorphic disks from p to q is a chain of points $p = p_0, p_1, \dots, p_k = q$ of X, pairs of points $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ in $\Delta(1)$, and holomorphic maps $f_1, \dots, f_k$ from $\Delta(1)$ to X such that

$$f(a_i) = p_{i-1}, \quad f(b_i) = p_i, \quad \text{for } 1 \leq i \leq k.$$

Let α be such a chain. Define its length $l(\alpha)$ by

$$l(\alpha) = \sum_{i=1}^{k} \rho(a_i, b_i),$$

and define the Kobayashi pseudodistance d_X between p and q by

$$d_X(p, q) = \inf_{\alpha} l(\alpha).$$

Definition 1.3.1. The complex space X is Kobayashi hyperbolic if the pseudodistance d_X is a distance.

Example 1.3.2.

(i) The Kobayashi pseudodistance on $\mathbb{C}$ is identically zero.

(ii) The Kobayashi pseudodistance on $\Delta(1)$ is equal to the Poincaré distance.

(iii) Let $X = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. The unit disk $\Delta(1)$ is the universal covering space of X. Because a complex manifold is Kobayashi hyperbolic if and only if its universal covering space is Kobayashi hyperbolic, X is Kobayashi hyperbolic(small Picard theorem).

Let X, Y be complex spaces, and let $f : X \to Y$ be a holomoprhic map. By the definition of the Kobayashi pseudodistance, it follows that

$$d_Y(f(x), f(y)) \leq d_X(x, y)$$

for $x, y \in X$. Hence there exists no non-constant holomorphic map from $\mathbb{C}$ to a Kobayashi hyperbolic complex space. If the Kobayashi hyperbolic complex space is compact, the converse is also true.

Theorem 1.3.3 ([Br]). *A compact complex space X is Kobayashi hyperbolic if and only if there exist no non-constant entire curves in X.*

Now we define the Kobayashi hyperbolic imbedding. Let X be a complex space, and let Y be a locally closed complex subspace of X such that the closure $\overline{Y}$ is compact in X.

Definition 1.3.4. Y is Kobayashi hyperbolically imbedded into X if, for every pair of distinct point $p, q \in \overline{Y}$, there exist a neighborhood U, V of p, q in X such that

$$d_Y(U \cap Y, V \cap Y) > 0.$$

Next theorem is an analogy of Theorem 1.3.3.

Theorem 1.3.5 ([Za1], [Za2]). *Let X be a compact complex manifold, and let $D_i, i = 1, \dots, k$ be smooth divisors in X such that $D = \sum_{i=1}^k D_i$ has only simple normal crossing. Then $X \setminus D$ is Kobayashi hyperbolically imbedded into X if and only if the following conditions are satisfied:*

(i) There exist no non-constant entire curves in $X \setminus D$.

(ii) Let I, J be a partition of $\{1, \dots, k\}$. Then there exist no non-constant entire curves in $\bigcap_{i \in I} D_i \setminus \bigcup_{j \in J} D_j$.

The following theorem is a generalization of the big Picard theorem.

Theorem 1.3.6 ([Ki]). *Let X be an m-dimensional complex manifold and let A be a closed complex subspace of X consisting of hypersurfaces with normal crossing singularities. Let Z be a complex space and Y be a locally closed complex subspace of Z. If Y is Kobayashi hyperbolically imbedded into Z, then every holomorphic map $h : X \setminus A \to Y$ extends to a holomorphic map $\tilde{h} : X \to Z$.*

Kobayashi hyperbolic imbedding is also closely related to a boundedness property of a family of holomorphic mappings (see, e.g., [Ko2] Chap. 6, [Nog3]).

1.3.2 Kobayashi conjecture

It is a famous conjecture proposed by S. Kobayashi that

(a) a generic hypersurface of degree $d \geq 2n - 1$ in $\mathbb{P}^n(\mathbb{C})$ is Kobayashi hyperbolic;

(b) the complement of a generic hypersurface of degree $d \geq 2n+1$ in $\mathbb{P}^n(\mathbb{C})$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^n(\mathbb{C})$.

If the hypersurface is a union of hyperplanes in $\mathbb{P}^n(\mathbb{C})$, Conjecture (b) holds;

Theorem 1.3.7 ([Fu]). *Let $H_1, \dots, H_q$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position and $q \geq 2n + 1$. Then $\mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^q H_i$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^n(\mathbb{C})$.*

In [De], Demailly calculated the dimension of $H^0(X_k, \mathcal{O}_{X_k}(m) \otimes A^{-1})$ using the Rieman-Roch formula and the vanishing theorem due to Bogomolov. By Theorem 1.1.2 and McQuillan's result [Mc1], hyperbolicity of high degree generic hypersurfaces in $\mathbb{P}^3(\mathbb{C})$ was proved.

Theorem 1.3.8 ([Mc2], [DE], [Pa]). *A very generic hypersurfaces in* $\mathbb{P}^3(\mathbb{C})$ *of degree* $d \geq 18$ *is Kobayashi hyperbolic.*

The above theorem was extended to a logarithmic case, and the following theorem was obtained.

Theorem 1.3.9 ([DL], [Ro]). *The complement of a very generic hypersurfaces of degree* $d \geq 14$ *in* $\mathbb{P}^2(\mathbb{C})$ *is Kobayashi hyperbolically imbedded into* $\mathbb{P}^2(\mathbb{C})$.

It is an important problem to construct explicit families of hyperbolic hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. In [MN], K. Masuda and J. Noguchi constructed Kobayashi hyperbolic hypersurfaces in an arbitrary dimensional complex projective space. Using a stability of the Kobayashi hyperbolicity and deforming a union of hyperplanes in $\mathbb{P}^2(\mathbb{C})$, M. Zaidenberg [Za3] showed that there exists hypersurfaces of degree five in $\mathbb{P}^2(\mathbb{C})$ such that the complement is Kobayashi hyperbolically imbedded in $\mathbb{P}^2(\mathbb{C})$. J. Duval [Du] constructed the hypersurfaces of degree six in $\mathbb{P}^3(\mathbb{C})$ which are Kobayashi hyperbolic. Kobayashi hyperbolic hypersurfaces of degree five in $\mathbb{P}^3(\mathbb{C})$ are not found yet.

2 The second main theorem for hypersurfaces in the projective space

2.1 Introduction

In the Nevanlinna theory, it has been a fundamental problem to prove the Second Main Theorem for a holomorphic map from $\mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$. In 1933, H. Cartan [Ca] proved the Second Main Theorem for hyperplanes in general position. The case of non-linear hypersurfaces had been studied by many authors. For examples, J. Noguchi [Nog1], B. Shiffman [Sh2], Eremenko and Sodin [ES], Y. -T. Siu [Si2], and M. Ru [Ru]. In these results, the degree of hypersurfaces does not appear in the defect relation. In A. Biancofiore [Bi], the degree of hypersurfaces concerns the defect relation for special holomorphic mappings. In this paper, the degree of hypersurfaces appears in our defect relation.

To prove the Second Main Theorem, we use meromorphic partial projective connection which is defined in J.-P. Demailly [De]. We recall the definition and some basic properties of a meromorphic partial projective connection in §2.2. A reference of this section is §11 of J.-P. Demailly [De]. In the Nevanlinna theory, the idea of using meromorphic connection is due to Y. -T. Siu [Si1]. Later, by using meromorphic connection, A. Nadel [Na] constructed Kobayashi hyperbolic hypersurfaces in $\mathbb{P}^3(\mathbb{C})$. J. El Goul [El] also constructed Kobayashi hyperbolic hypersurfaces in $\mathbb{P}^3(\mathbb{C})$ by simplifying Nadel's method. J.-P. Demailly [De] developed a new general concept called meromorphic partial projective connections. However, the Nevanlinna theory was not used in A. Nadel [Na], J. El Goul [El], J.-P. Demailly [De], or J.-P. Demailly and J. El Goul [DE]. These papers mainly dealt with holomorphic curves into non-linear hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ by using a negative curvature method.

In §2.6, we prove the Nevanlinna Second Main Theorem for singular hypersurfaces by using the pull back of a meromorphic partial projective connection. The Second Main Theorem for singular divisors was dealt with in B. Shiffman [Sh1]. In B. Shiffman [Sh1], a singular divisor is reduced to smooth one by resolving the singularity. In this paper, we also resolve the singularity of divisors. By the same method, we show the Second Main Theorem for hypersurfaces in m-subgeneral position ($m \geq 2$) in $\mathbb{P}^2(\mathbb{C})$ such that any two hypersurfaces intersect transversally. We say that hypersurfaces are in m-subgeneral position if the intersection of any $m+1$ hypersurfaces is empty. In the case where hypersurfaces are hyperplanes, E. I. Nochka proved the Second Main Theorem in [Noc]. The approach that we employ is different from Nochka's one (see Theorem 0.0.2).

2.2 Meromorphic partial projective connections and totally geodesic hypersurfaces

In this section, we recall some definitions and properties of meromorphic partial projective connections and totally geodesic hypersurfaces. A reference for this section is J.-P. Demailly [De], §11.

Let X be an n-dimensional complex projective algebraic manifold. Let $\{U_j\}_{1\leq j\leq N}$ be an affine open covering of X.

Definition 2.2.1. A meromorphic partial projective connection ∇ relative to an affine open covering $\{U_j\}_{1\leq j\leq N}$ of X is a collection of meromorphic connections ∇_j on U_j, satisfying

$$\nabla_j - \nabla_k = \alpha_{jk} \otimes \mathrm{Id}_{T_X} + \mathrm{Id}_{T_X} \otimes \beta_{jk} \quad \text{on } U_j \cap U_k,$$

for all $1 \leq j, k \leq N$, where α_{jk}, β_{jk} are meromorphic one-forms on $U_j \cap U_k$. We write $\nabla = \{(\nabla_j, U_j)\}_{1\leq j\leq N}$.

Let S_j be the smallest subvariety of X such that ∇_j is a holomorphic connection on $U_j \setminus S_j \cap U_j$. We set $\mathrm{supp}\,(\nabla)_\infty = \bigcup_{1\leq j\leq N} S_j$ and call it the polar locus of ∇.

Example 2.2.2. Let $\{U_j\}_{0\leq j\leq n}$ be an affine open covering of $\mathbb{P}^n(\mathbb{C})$ such that $U_j = \{[X_0 : \cdots : X_n] \in \mathbb{P}^n(\mathbb{C}) | X_j \neq 0\}$. There is the canonical isomorphism $U_j \simeq \mathbb{C}^n$ and we take flat connections d_j on U_j. Then $\{(d_j, U_j)\}_{0\leq j\leq n}$ is a partial projective connection on $\mathbb{P}^n(\mathbb{C})$. □

Let f be a holomorphic map from $\mathbb{C}$ to X, and let ∇ be a meromorphic partial projective connection relative to an affine covering $\{U_j\}$ of X. Assume that $f(\mathbb{C})$ is not contained in the polar locus of ∇. We write

$$\nabla_j^{(m)} = \overbrace{\nabla_j \circ \cdots \circ \nabla_j}^{m-\text{times}}.$$

By the definition of a meromorphic partial projective connection, we have

$$\begin{aligned} & f'(z) \wedge \nabla_{j\, f'(z)} f'(z) \wedge \cdots \wedge \nabla_{j\, f'(z)}^{(n-1)} f'(z) \\ = \; & f'(z) \wedge \nabla_{k\, f'(z)} f'(z) \wedge \cdots \wedge \nabla_{k\, f'(z)}^{(n-1)} f'(z) \in \bigwedge^{n} TX_{f(z)}, \end{aligned}$$

for $f(z) \in U_j \cap U_k \setminus \operatorname{supp}(\nabla)_\infty$.

Definition 2.2.3. The *Wronskian* $W_\nabla(f)$ of f relative to a meromorphic partial projective connection ∇ is defined by

$$W_\nabla(f)(z) = f'(z) \wedge \nabla_{f'(z)} f'(z) \wedge \cdots \wedge \nabla_{f'(z)}^{(n-1)} f'(z) \in \bigwedge^{n} TX_{f(z)},$$

where ∇ is ∇_j for $f(z) \in U_j \setminus \operatorname{supp}(\nabla)_\infty$.

Let $\mathbb{P}^n(\mathbb{C})$ be an n-dimensional complex projective space, and let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n(\mathbb{C})$ be the canonical projection. Put $U_j = \{[X_0 : \cdots : X_n] \in \mathbb{P}^n(\mathbb{C}) \mid X_j \neq 0\}$ where $[X_0 : \ldots : X_n]$ is a homogeneous coordinate system of $\mathbb{P}^n(\mathbb{C})$. In U_j, we take a local coordinate system

$$\left(\frac{X_0}{X_j}, \ldots, \frac{X_{j-1}}{X_j}, \frac{X_{j+1}}{X_j}, \ldots, \frac{X_n}{X_j} \right).$$

Let $\eta_j : U_j \to \mathbb{C}^{n+1} \setminus \{0\}$ be a holomorphic map such that

$$\eta_j \left(\frac{X_0}{X_j}, \ldots, \frac{X_{j-1}}{X_j}, \frac{X_{j+1}}{X_j}, \ldots, \frac{X_n}{X_j} \right) = \left(\frac{X_0}{X_j}, \ldots, \overset{\text{j-th}}{\overset{\downarrow}{1}}, \ldots, \frac{X_n}{X_j} \right).$$

Then $\pi \circ \eta_j = \mathrm{Id}_{U_j}$. A meromorphic connection $\widetilde{\nabla}$ on $\mathbb{C}^{n+1}$ induces a meromorphic connection ∇_j on U_j by

$$\nabla_j = \pi_*(\eta_j^* \widetilde{\nabla}),$$

where $\eta_j^* \widetilde{\nabla}$ is the induced connection on $\eta_j^* T_{\mathbb{C}^{n+1}}$ and $\pi_* : T_{\mathbb{C}^{n+1}} \to \pi^* T_{\mathbb{P}^n(\mathbb{C})}$ is the projection.

The following lemma is Corollary 11.10. of J.-P. Demailly [De]:

Lemma 2.2.4. *Let $\widetilde{\nabla} = d + \widetilde{\Gamma}$ be a meromorphic connection on $\mathbb{C}^{n+1}$, and let $\varepsilon = \sum z_j \partial/\partial z_j$ be the Euler vector field on $\mathbb{C}^{n+1}$. Then $\{(\pi_*(\eta_j^* \widetilde{\nabla}), U_j)\}_{0 \le j \le n}$ is a meromorphic partial projective connection on $\mathbb{P}^n(\mathbb{C})$ provided that*

(i) the Christoffel symbols $\widetilde{\Gamma}^{\lambda}_{j\mu}$ of $\widetilde{\nabla}$ are homogeneous rational functions of degree -1,

(ii) on every intersection $(\pi^{-1}U_i) \cap (\pi^{-1}U_j)$ $(i \ne j)$ there are meromorphic functions α, β and meromorphic 1-forms γ, ξ on $\mathbb{C}^{n+1} \setminus \{0\}$ such that

$$\widetilde{\Gamma}(\varepsilon, v) = \alpha v + \gamma(v)\varepsilon, \quad \widetilde{\Gamma}(\omega, \varepsilon) = \beta \omega + \xi(\omega)\varepsilon,$$

for all vector fields v, ω.

Proof. See the proof of Lemma 11.8. of J.-P. Demailly [De]. □

Let D be a reduced effective divisor of an n-dimensional complex projective algebraic manifold X, and let ∇ be a meromorphic connection. Take the holomorphic function s on an open set $U \subset X$ such that $D|_U = (s)$, and take a local coordinate system $(z_1, \ldots, z_n)$ on U.

We define that D is totally geodesic with respect to ∇ on U if there exist meromorphic one-forms $a = \sum_{1 \le j \le n} a_j dz_j$, $b = \sum_{1 \le j \le n} b_j dz_j$ and a meromorphic two-form $c = \sum_{1 \le j, \mu \le n} c_{j\mu} dz_j \otimes dz_\mu$ such that no polar locus of a_j, b_j, or $c_{j\mu}$ $(1 \le j, \mu \le n)$ contains $\operatorname{supp} D|_U$, and

$$\nabla^*(ds) = d^2 s - ds \circ \Gamma = a \otimes ds + ds \otimes b + sc$$

in U, where ∇^* is the induced connection on T^*_X.

The following Lemma 2.2.5 was obtained by J.-P. Demailly [De].

Lemma 2.2.5. *Assume that D is totally geodesic with respect to ∇ on U, i.e., there exist meromorphic one-forms a, b and meromorphic two-form c on U such that*

$$\nabla^*(ds) = a \otimes ds + ds \otimes b + sc.$$

Let β be a holomorphic function on U such that βa, βb, βc are holomorphic forms. Let V be a domain in $\mathbb{C}$. Let $f : V \to U$ be a holomorphic map. Then we have

$$\begin{aligned} \frac{d^k(s \circ f)}{dz^k}(z) =& \gamma_k(z)(s \circ f)(z) + \sum_{0 \le l \le k-2} \gamma_{l,k}(z)(ds \cdot \nabla^{(l)}_{f'} f')(z) \\ &+ (ds \cdot \nabla^{(k-1)}_{f'} f')(z), \qquad z \in V \end{aligned}$$

for $k \in \mathbb{N}$. Here γ_k and $\gamma_{l,k}$ are meromorphic functions on V such that $\beta^{k-1}(f)\gamma_k$ and $\beta^{k-l-1}(f)\gamma_{l,k}$ are holomorphic functions on V.

Proof. The lemma holds for $k = 1$, because

$$\frac{ds \circ f}{dz} = ds \cdot f'.$$

We shall prove the lemma by induction over the order k. Hence we assume that the lemma has already been proved for $k-1$. Then we have

$$\begin{aligned}
&\frac{d}{dz}\left(\frac{d^{k-1}(s\circ f)}{dz^{k-1}}\right)(z) \\
&= \frac{d}{dz}\Bigg(\gamma_{k-1}(z)(s\circ f)(z) + \sum_{0\le l\le k-3} \gamma_{l,k-1}(z)(ds\cdot \nabla_{f'}^{(l)} f')(z) \\
&\quad + (ds\cdot \nabla_{f'}^{(k-2)} f')(z)\Bigg) \\
&= \frac{d\gamma_{k-1}}{dz}(z)(s\circ f)(z) + \gamma_{k-1}(z)\frac{d(s\circ f)}{dz}(z) \\
&\quad + \sum_{0\le l\le k-3}\left(\frac{d\gamma_{l,k-1}}{dz}(z)(ds\cdot \nabla_{f'}^{(l)} f')(z) + \gamma_{l,k-1}(z)\frac{d}{dz}(ds\cdot \nabla_{f'}^{(l)} f')(z)\right) \\
&\quad + \frac{d}{dz}(ds\cdot \nabla_{f'}^{(k-2)} f')(z). \qquad (2.1)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{d}{dz}(ds\cdot \nabla_{f'}^{(l)} f')(z) \\
&= (ds\cdot \nabla_{f'}^{(l+1)} f')(z) + (\nabla_{f'}^{*}(ds)\cdot(\nabla_{f'}^{(l)} f'))(z) \\
&= (ds\cdot \nabla_{f'}^{(l+1)} f')(z) + (a(f)\cdot f')(z)(ds\cdot \nabla_{f'}^{(l)} f')(z) \\
&\quad + (ds\cdot f')(z)(b(f)\cdot(\nabla_{f'}^{(l)} f'))(z) + (s\circ f)(z)(c(f)\cdot(f',\nabla_{f'}^{(l)} f'))(z) \qquad (2.2)
\end{aligned}$$

By (2.1) and (2.2), the lemma holds. □

Let $\nabla = \{(\nabla_j, U_j)\}_{1\le j\le N}$ be a meromorphic partial projective connection relative to an affine covering $\{U_j\}_{1\le j\le N}$ of X, and let s_j be a holomorphic function on U_j such that $D|_{U_j} = (s_j)$. By the definition of the meromorphic partial projective connection,

$$\nabla_j - \nabla_k = \alpha_{jk} \otimes \mathrm{Id}_{T_X} + \mathrm{Id}_{T_X} \otimes \beta_{jk},$$

on $U_j \cap U_k$ with meromorphic one-forms α_{jk} and β_{jk}. Then we have

$$(\nabla_j^* - \nabla_k^*)\, ds_j = -ds_j \otimes \alpha_{jk} - \beta_{jk} \otimes ds_j.$$

This implies that D is totally geodesic with respect to ∇_j on $U_j \cap U_k$ if D is totally geodesic with respect to ∇_k and $\operatorname{supp} D|_{U_j\cap U_k}$ is not contained in the polar loci of α_{jk}, β_{jk}.

Definition 2.2.6. Let ∇ be a meromorphic partial projective connection relative to an affine open covering $\{U_j\}$ of X. Let D be an effective divisor on X such that $\operatorname{supp} D|_{U_j} \not\subset \operatorname{supp}(\nabla)_\infty$. Then D is said to be totally geodesic with respect to ∇ if $D|_{U_j}$ is totally geodesic with respect to ∇_j on U_j for all j.

Let $s_0, \dots, s_n$ be homogeneous polynomials of $\mathbb{C}[X_0, \dots, X_n]$ such that $\deg(s_0) = \cdots = \deg(s_n) = d$ and $\det(\partial s_j/\partial X_k)_{0\le j,k\le n} \not\equiv 0$. We define a meromorphic connection $\widetilde{\nabla} = d + \widetilde{\Gamma}$ on $\mathbb{C}^{n+1}$ by

$$\sum_{0\le\lambda\le n} \frac{\partial s_\kappa}{\partial X_\lambda} \widetilde{\Gamma}^\lambda_{i\,j} = \frac{\partial^2 s_\kappa}{\partial X_i \partial X_j},$$

for $0 \le i, j \le n$. Then $\widetilde{\nabla}^* ds_j \equiv 0$ for all $0 \le j \le n$. Let $U_j = \{[X_0 : \cdots : X_n] \in \mathbb{P}^n(\mathbb{C}) \mid X_j \neq 0\}$ be an affine open subset of $\mathbb{P}^n(\mathbb{C})$. Let $\eta_j : U_j \to \mathbb{C}^{n+1} \setminus \{0\}$ be the canonical section of the $\mathbb{C}^*$-fiber bundle $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ such that

$$\eta_j([X_0 : \cdots : X_n]) = \left(\frac{X_0}{X_j}, \cdots, \overset{\text{j-th}}{\overset{\downarrow}{1}}, \cdots, \frac{X_n}{X_j}\right).$$

Then meromorphic connection $\widetilde{\nabla}$ induces a meromorphic partial projective connection $\nabla = \{(\pi_*(\eta_j^*\widetilde{\nabla}), U_j)\}$ on $\mathbb{P}^n(\mathbb{C})$ by Lemma 2.2.4 (see §11 of J.-P. Demailly [De]). A reduced divisor s of the linear system $|\{s_0, \dots, s_n\}|$ is totally geodesic with respect to ∇ if $\operatorname{supp}(s)$ is not contained in $\operatorname{supp}(\nabla)_\infty$.

Remark 2.2.7. Let $(z_0, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$ be a local coordinate system on U_j such that $z_k = X_k/X_j$. Put $\nabla_j = \pi_*(\eta_j^*\widetilde{\nabla}) = d + \left(\Gamma^\lambda_{i\,\mu}\right)$ where $\Gamma^\lambda_{i\,\mu}$ is a Christoffel symbols with respect to this coordinate system. Then one can check that

$$\Gamma^\lambda_{i\,\mu} = \eta_j^*\widetilde{\Gamma}^\lambda_{i\,\mu} - z_\lambda \eta_j^*\widetilde{\Gamma}^j_{i\,\mu},$$
$$\nabla^* d(\eta_j^* s_\kappa) = \deg(s_\kappa)\, \eta_j^* s_\kappa \sum_{i,\mu} \eta_j^* \widetilde{\Gamma}^j_{i\,\mu} dz_i dz_\mu$$

on U_j.

The following lemma was obtained by J.-P. Demailly [De].

Lemma 2.2.8. *Let $\nabla = \{(\nabla_i, U_i)\}_{0\le i\le n}$ be the meromorphic partial projective connection on $\mathbb{P}^n(\mathbb{C})$ constructed as above. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a non-constant holomorphic map such that $f(\mathbb{C})$ is not contained in* $\operatorname{supp}(\nabla)_\infty$. *Then*

$$W_\nabla(f) = f' \wedge \nabla_{f'} f' \wedge \cdots \wedge \nabla_{f'}^{(n-1)} f' \equiv 0$$

if and only if $f(\mathbb{C})$ is contained in a support of an element of a linear system $|\{s_0, \ldots, s_n\}|$.

Proof. Assume that $W_\nabla(f) \equiv 0$. Let z be a point of $\mathbb{C}$ such that $f(z)$ is not contained in supp $(\nabla)_\infty$. There exists j such that $f(z) \in U_j$. We can take a non-trivial solution $(\alpha_0, \cdots, \alpha_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ which sutisfies

$$\begin{aligned}
&\alpha_0\ \eta_j^* s_0(f(z)) + \cdots + \alpha_n\ \eta_j^* s_n(f(z)) = 0, \\
&\alpha_0\ ((\eta_j^* ds_0) \cdot f')(z) + \cdots + \alpha_n\ ((\eta_j^* ds_n) \cdot f')(z) = 0, \\
&\alpha_0\ ((\eta_j^* ds_0) \cdot \nabla_{f'}^{(l)} f')(z) + \cdots + \alpha_n\ ((\eta_j^* ds_n) \cdot \nabla_{f'}^{(l)} f')(z) = 0
\end{aligned}$$

for $1 \leq l \leq n-1$ where $\nabla = \nabla_j$. Let $s = \alpha_0 s_0 + \cdots + \alpha_n s_n$ be an element of the linear system $|\{s_0, \ldots, s_n\}|$. It follows that

$$s(f(z)) = 0, \qquad (ds \cdot f')(z) = 0, \qquad (ds \cdot \nabla_{f'}^{(l)} f')(z) = 0 \tag{2.3}$$

for all $1 \leq l \leq n-1$. Then there exist holomorphic functions $a_0, \cdots, a_{k-1}$ for $k \leq n-1$ on a neighborhood of z such that

$$\nabla_{f'}^{(k)} f' = a_0 f' + a_1 \nabla_{f'} f' + \cdots + a_{k-1} \nabla_{f'}^{(k-1)} f' \tag{2.4}$$

on that neighborhood. By (2.3) and (2.4), we have

$$(ds \cdot \nabla_{f'}^{(l)} f')(z) = 0$$

for all $l \in \mathbb{N}$. Because supp (s) is totally geodesic with respect to ∇, we have

$$\frac{d^l (s \circ f)}{dz^l}(z) = 0$$

for all $l \in \mathbb{N}$ by Lemma 2.2.5. Therefore $s \circ f \equiv 0$. Then the image of f is contained in a support of (s).

Conversely, assume that s is an element of the linear system $|\{s_0, \ldots, s_n\}|$ such that $s \circ f \equiv 0$. Then, by Lemma 2.2.5,

$$(ds \cdot f')(z) = 0, \qquad (ds \cdot \nabla_{f'}^{(l)} f')(z) = 0$$

for all $l \in \mathbb{N}$. So f', $\nabla_{f'}^{(l)} f'$ are elements of a kernel of ds. Because the dimension of Ker(ds) is less than $n-1$, we have $W_\nabla(f) \equiv 0$. □

2.3 Proof of Theorem 0.0.1

To prove the Second Main Theorem, we need Borel's lemma.

Lemma 2.3.1. *Let $h(r) > 0$ be a monotone increasing function in $r \geq 1$. Then, for arbitrary $\delta > 0$, we have*

$$\frac{dh(r)}{dr} \leq (h(r))^{1+\delta}\|.$$

Proof. See Noguchi-Ochiai [NO], Chapter V, §5. □

Let X be an n-dimensional complex projective algebraic manifold, and let σ_i $(1 \leq i \leq q)$ be a holomorphic section of the holomorphic line bundle L_i on X. Let $\nabla = \{(\nabla_j, U_j)\}_{1\leq j\leq N}$ be a meromorphic partial projective connection relative to an affine covering $\{U_j\}_{1\leq j\leq N}$ of X. Let β be a holomorphic section of the holomorphic line bundle L on X such that $\beta\nabla_j$ is holomorphic on U_j for all $1 \leq j \leq N$.

Lemma 2.3.2. *Assume that (σ_i) is smooth and $\sum_{1\leq i\leq q}(\sigma_i)$ is a simple normal crossing divisor of X. Assume that* $\operatorname{supp}(\sigma_j)$ *is not contained in* $\operatorname{supp}(\nabla)_\infty$ *and (σ_j) is totally geodesic with respect to ∇ for $1 \leq j \leq q$. Let $f : \mathbb{C} \to X$ be a holomorphic map such that $f(\mathbb{C})$ is not contained in* $\operatorname{supp}(\nabla)_\infty$ *and the Wronskian $W_\nabla(f) \not\equiv 0$. Then we have*

$$\int_{|z|=r} \log^+ \frac{\|W_\nabla(f)(z)\|_{\bigwedge^n TX}\|\beta(f(z))\|_L^{n(n-1)/2}}{\prod_{i=1}^q \|\sigma_i(f(z))\|_{L_j}} \frac{d\theta}{2\pi} \leq S_f(r),$$

where $S_f(r) = O(\log^+ r + \log^+ T_f(r))\|$.

Proof. Take an open covering $\{V_j\}_{1\leq j\leq N}$ such that $V_j \Subset U_j$ (i.e., topological closure $\overline{V}_j$ is contained in U_j and $\overline{V}_j$ is compact), and take a partition of unity $\{\phi_j\}_{1\leq j\leq N}$ subordinate to the covering $\{V_j\}_{1\leq j\leq N}$. Take holomorphic functions $z_1, \ldots, z_n$ on U_j such that $dz_1, \ldots, dz_n$ are linearly independent and

$$U_j \cap \bigcup_{i=1}^q \operatorname{supp}(\sigma_i) = \{w \in U_j |\, z_1(w)\cdots z_p(w) = 0\},$$

for some p, $0 \leq p \leq n$. We put $f_l = z_l \circ f$, $(\nabla_{f'}^{(k)} f')_l = dz_l \cdot \nabla_{f'}^{(k)} f'$. Then we have

$$\phi_j(f)\log^+\frac{\|f'\wedge\nabla_{f'}f'\wedge\cdots\wedge\nabla_{f'}{}^{(n-1)}f'\|_{\bigwedge^n TX}\|\beta(f)\|_L^{n(n-1)/2}}{\prod_{i=1}^q\|\sigma_i(f)\|_{L_j}}$$

$$= \phi_j(f)\log^+\Bigg(\varphi_j(f)\|\beta(f)\|_L^{n(n-1)/2}$$

$$\times\left|\begin{matrix} \frac{f_1'}{f_1} & \cdots & \frac{f_p'}{f_p} & f_{p+1}' & \cdots & f_n' \\ \frac{((\nabla_j)_{f'}f')_1}{f_1} & \cdots & \frac{((\nabla_j)_{f'}f')_p}{f_p} & ((\nabla_j)_{f'}f')_{p+1} & \cdots & ((\nabla_j)_{f'}f')_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{((\nabla_j)_{f'}^{(n-1)}f')_1}{f_1} & \cdots & \frac{((\nabla_j)_{f'}^{(n-1)}f')_p}{f_p} & ((\nabla_j)_{f'}^{(n-1)}f')_{p+1} & \cdots & ((\nabla_j)_{f'}^{(n-1)}f')_n \end{matrix}\right|\Bigg)$$

on $f^{-1}(U_j)$ where φ_j is a C^∞-function on U_j. By Lemma 2.2.5,

$$((\nabla_j)_{f'}^{(l)}f')_i(z) = \sum_{0\le k\le l+1} a_{i,l,k}(z)\frac{d^k f_i}{dz^k}(z),$$

for $1\le i\le p$. Here $a_{i,l,k}$ are meromorphic functions on $f^{-1}(U_j)$ such that $a_{i,l,k}(z)(\beta\circ f(z))^l$ is a holomorphic function. Hence it follows that

$$\int_{|z|=r}\phi_j(f)\log^+\frac{\|f'\wedge\nabla_{f'}f'\wedge\cdots\wedge\nabla_{f'}{}^{(n-1)}f'\|_{\bigwedge^n TX}\|\beta(f)\|_L^{n(n-1)/2}}{\prod_{i=1}^q\|\sigma_i(f)\|_{L_j}}\frac{d\theta}{2\pi}$$
$$\le\int_{|z|=r}\Phi(f(z))\frac{d\theta}{2\pi}+\sum_{1\le k\le p}\sum_{1\le l\le n}\int_{|z|=r}\log^+\frac{|f_k^{(l)}(z)|}{|f_k(z)|}\frac{d\theta}{2\pi}$$
$$+\sum_{p+1\le k\le n}\sum_{1\le l\le n}\int_{|z|=r}\Psi(f(z))\log^+|f_k^{(l)}(z)|\frac{d\theta}{2\pi},$$

where Φ,Ψ is a bounded C^∞-function on X. By using the lemma on logarithmic derivative, it follows that

$$\int_{|z|=r}\log^+\frac{|f_k^{(l)}(z)|}{|f_k(z)|}\frac{d\theta}{2\pi}\le S_f(r),$$

$$\int_{|z|=r}\Psi(f(z))\log^+|f_k^{(l)}(z)|\frac{d\theta}{2\pi}$$
$$\le\int_{|z|=r}\log^+\frac{|f_k^{(l)}(z)|}{|f_k'(z)|}\frac{d\theta}{2\pi}+\int_{|z|=r}\Psi(f(z))\log^+|f_k'(z)|\frac{d\theta}{2\pi}$$
$$\le\int_{|z|=r}\Psi(f(z))\log^+|f_k'(z)|\frac{d\theta}{2\pi}+S_f(r).$$

It follows that

$$\int_{|z|=r} \Psi(f(z)) \log^+ |f_k'(z)| \frac{d\theta}{2\pi} = \frac{1}{2} \int_{|z|=r} \Psi(f(z)) \log^+ |f_k'(z)|^2 \frac{d\theta}{2\pi}$$
$$\leq \frac{1}{2} \int_{|z|=r} \log^+ \|f'(z)\|_{TX}^2 \frac{d\theta}{2\pi} + O(1),$$

where $\|\cdot\|_{TX}$ is a hermitian metric of TX.

By Lemma 2.3.1 and the concavity of log, we have that for $\delta > 0$,

$$\begin{aligned}
&\frac{1}{2} \int_{|z|=r} \log^+ \|f'(z)\|_{TX}^2 \frac{d\theta}{2\pi} \\
&\leq \frac{1}{2} \int_{|z|=r} \log\{\|f'(z)\|_{TX}^2 + 1\} \frac{d\theta}{2\pi} \\
&\leq \frac{1}{2} \log \left(1 + \int_{|z|=r} \|f'(z)\|_{TX}^2 \frac{d\theta}{2\pi}\right) + O(1) \\
&\leq \frac{1}{2} \log \left(1 + \frac{1}{2\pi r} \frac{d}{dr} \int_{|z|\leq r} \|f'(z)\|_{TX}^2 \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}\right) + O(1) \\
&\leq \frac{1}{2} \log \left(1 + \frac{1}{2\pi r} \left(\int_{|z|\leq r} \|f'(z)\|_{TX}^2 \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}\right)^{1+\delta}\right) + O(1)\| \\
&= \frac{1}{2} \log \left(1 + \frac{r^\delta}{2\pi} \left(\frac{d}{dr} \int_1^r \frac{dt}{t} \int_{|z|\leq r} \|f'(z)\|_{TX}^2 \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}\right)^{1+\delta}\right) + O(1)\| \\
&\leq \frac{1}{2} \log \left(1 + \frac{r^\delta}{2\pi} \left(\int_1^r \frac{dt}{t} \int_{|z|\leq r} \|f'(z)\|_{TX}^2 \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}\right)^{(1+\delta)^2}\right) + O(1)\| \\
&\leq S_f(r).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\int_{|z|=r} \log^+ \frac{\|W_\nabla(f)(z)\|_{\wedge^n TX} \|\beta(f(z))\|_L^{n(n-1)/2}}{\prod_{i=1}^q \|\sigma_i(f(z))\|_{L_i}} \frac{d\theta}{2\pi} \\
&= \sum_j \int_{|z|=r} \phi_j(f(z)) \log^+ \frac{\|W_\nabla(f)(z)\|_{\wedge^n TX} \|\beta(f(z))\|_L^{n(n-1)/2}}{\prod_{i=1}^q \|\sigma_i(f(z))\|_{L_i}} \frac{d\theta}{2\pi} \\
&\leq S_f(r).
\end{aligned}$$

□

Theorem 2.3.3. *Let K_X be the canonical line bundle of X. Under the hypothesis of Lemma 2.3.2, we have*

$$\sum_{1\leq i\leq q} T_f(r, L_i) + T_f(r, K_X) - \frac{1}{2} n(n-1) T_f(r, L) \leq \sum_{1\leq i\leq q} N_n(r, f^*(\sigma_i)) + S_f(r).$$

Proof. We denote by $\mathrm{ord}_z\,(\sigma_j \circ f)$ the order of zero of $\sigma_j \circ f$ at the point of $z \in \mathbb{C}$, and we denote by $\mathrm{ord}_z\,\beta(f)^{n(n-1)/2}W_\nabla(f)$ the order of zero of $\beta(f)^{n(n-1)/2}W_\nabla(f)$ at the point $z \in \mathbb{C}$. If $\mathrm{ord}_z\,(\sigma_j \circ f) \geq n+1$ for $z \in \mathbb{C}$, then

$$\mathrm{ord}_z\,\beta(f)^{n(n-1)/2}W_\nabla(f) \geq \mathrm{ord}_z(\sigma_j \circ f) - n,$$

by Lemma 2.2.5. So we have, by the Nevanlinna First Main Theorem,

$$\sum_j T_f(r, L_j) - \sum_j N_n(r, f^*(\sigma_j)) - T_f(r, \bigwedge^n T_X) - \frac{1}{2}n(n-1)T_f(r, L)$$
$$\leq \int_{|z|=r} \log \frac{\|W_\nabla(f)(z)\|_{\bigwedge^n TX} \|\beta(f(z))\|_L^{n(n-1)/2}}{\prod_{i=1}^q \|\sigma_i(f(z))\|_{L_j}} \frac{d\theta}{2\pi}.$$

From Lemma 2.3.2 the theorem follows. □

Proof of Theorem 0.0.1. We construct the meromorphic partial projective connection

$$\nabla = \{(\nabla_j, U_j)\}_{0 \leq j \leq n},$$

on $\mathbb{P}^n(\mathbb{C})$ as in §2. By Cramer's rule, the degree of the pole divisor of each ∇_j is less than or equal to $l_0 + \cdots + l_n + n + 1$. The Main Theorem follows from Theorem 2.3.3 and $K_{\mathbb{P}^n} = -(n+1)H$. □

Now we show two typical corollaries. Define the defect

$$\delta_f((\sigma_j)) = 1 - \limsup_{r\to\infty} \frac{N(r, f^*\sigma_j)}{T_f(r, [D_j])}.$$

Corollary 2.3.4. (Defect Relation) *Under the hypothesis of Theorem 0.0.1, we have*

$$\sum_{1\leq j\leq q} \delta_f((\sigma_j)) \leq \frac{n+1}{d} + \frac{1}{2d}(n-1)n(l_0 + \cdots + l_n + n + 1)$$

Proof. This is deduced from Theorem 0.0.1 and the same arguments in Noguchi-Ochiai [NO], Chapter V, §5. □

Remark 2.3.5. When $q = 1$ and

$$\frac{n+1}{d} + \frac{1}{2d}(n-1)n(l_0 + \cdots + l_n + n + 1) < 1,$$

the holomorphic map omitting the hypersurface is algebraically degenerate by Corollary 2.3.4.

Corollary 2.3.6. (Ramification Theorem) *Assume that*

$$f^*\sigma_j \geq \mu_j \operatorname{supp}(f^*\sigma_j)$$

for some positive integers μ_j, $1 \leq j \leq q$. *Under the hypothesis of Theorem 0.0.1, we have*

$$\sum_{1 \leq j \leq q} \left(1 - \frac{n}{\mu_j}\right) \leq \frac{n+1}{d} + \frac{1}{2d}(n-1)n(l_0 + \cdots + l_n + n + 1).$$

Proof. This is deduced from Theorem 0.0.1 and the same arguments in Noguchi-Ochiai [NO], Chapter V, §5. □

Example 2.3.7. Put $s_0 = X_0^d, \ldots, s_n = X_n^d \in \mathbb{C}[X_0, \ldots, X_n]$. Let $\sigma_1, \ldots, \sigma_q \in |\{s_0, \ldots, s_n\}|$ be smooth Fermat hypersurfaces such that the divisor $\sigma_1 + \cdots + \sigma_q$ is of simple normal crossing. By Theorem 0.0.1, we have

$$\left(q - \frac{n+1}{d} - \frac{1}{2d}(n-1)n(n+1)\right) T_f(r, dH) \leq \sum_{1 \leq i \leq q} N_n(r, f^*(\sigma_i)) + S_f(r).$$

□

Example 2.3.8. Put $s_0 = X_0^d, \ldots, s_{n-1} = X_{n-1}^d, s_n = X_n^{d-1}(\varepsilon_0 X_0 + \cdots + \varepsilon_n X_n) \in \mathbb{C}[X_0, \ldots, X_n]$, $\varepsilon_0, \ldots, \varepsilon_n \in \mathbb{R}$. Let $\sigma = s_0 + \cdots + s_n$. Assume that the hypersurface defined by σ in $\mathbb{P}^n(\mathbb{C})$ is smooth. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map such that the image of f is Zariski dense in $\mathbb{P}^n(\mathbb{C})$. If

$$\frac{(n+1)(n^2 - n + 1)}{d} < 1,$$

the image of f intersects the hypersurface defined by σ. □

2.4 Restriction of the meromorphic partial projective connection

Let $s_0, \ldots, s_{n+p}$ be homogeneous polynomials in $\mathbb{C}[X_0, \cdots, X_{n+p}]$ such that

$$\det \left(\frac{\partial s_j}{\partial X_k}\right)_{0 \leq j,k \leq n+q} \not\equiv 0.$$

Let $X \subset \mathbb{P}^{n+p}(\mathbb{C})$ be a smooth n-dimensional complete intersection for some hypersurfaces associated to elements of linear system $|s_0, \ldots, s_{n+p}|$. Then we construct the meromorphic partial projective connection ∇ associated to $\{s_0, \dots s_{n+p}\}$ on $\mathbb{P}^{n+p}(\mathbb{C})$ as in §2. We may assume that ∇ is a meromorphic partial projective connection on X because elements of $|s_0, \dots, s_{n+p}|$ is totally geodesic with respect to ∇.

For $\alpha = (\alpha_0, \dots, \alpha_{n+p}) \in \mathbb{C}^{n+p+1}$ we put

$$s_\alpha = \alpha_0 s_0 + \cdots + \alpha_{n+p} s_{n+p}.$$

We denote the hypersurface in $\mathbb{P}^{n+p}(\mathbb{C})$ corresponding to s_α by Y_α.

The next lemma is due to Theorem 11.19. of J.-P. Demailly [De].

Lemma 2.4.1. *Let*

$$Z = Y_{\alpha^1} \cap \cdots \cap Y_{\alpha^p} \subset \mathbb{P}^{n+p}(\mathbb{C})$$

be a smooth n-dimensional complete intersection, for linearly independent elements $\alpha^1, \dots, \alpha^p \in \mathbb{C}^{n+p+1}$ such that $ds_{\alpha^1} \wedge \cdots \wedge ds_{\alpha^p}$ does not vanish along Z. Assume that Z is not contained in $\{\det(\partial s_j/\partial X_k)_{0 \le j,k \le n+q} = 0\}$. Let $f : \mathbb{C} \to Z$ be a non-constant holomorphic map. Assume that $f(\mathbb{C})$ is not contained in $\{\det(\partial s_j/\partial X_k)_{0 \le j,k \le n+q} = 0\}$ nor contained in a hypersurface Y_α which satisfies $Z \not\subset Y_\alpha$. Then we have

$$W_\nabla(f) = f' \wedge \nabla_{f'} f' \wedge \cdots \wedge \nabla_{f'}^{(n-1)} f' \not\equiv 0.$$

Proof. See the proof of the Theorem 11.19. of J.-P. Demailly [De] □

Let $i : Z \to \mathbb{P}^{n+p}(\mathbb{C})$ be the inclusion map from Z to $\mathbb{P}^{n+p}(\mathbb{C})$, and let $H_Z = i^* H$ be the pull back of the hyperplane bundle on $\mathbb{P}^{n+p}(\mathbb{C})$.

Theorem 2.4.2. *Let $f : \mathbb{C} \to Z$ be a holomorphic map such that $f(\mathbb{C})$ is not contained in $\{\det(\partial s_j/\partial X_k)_{0 \le j,k \le n+q} = 0\}$ nor contained in a hypersurface Y_α which satisfies $Z \not\subset Y_\alpha$. Assume that $X_0^{d-l_0} | s_0, \dots, X_{n+p}^{d-l_{n+p}} | s_{n+p}$ for $0 \le l_j \le d$. Let $\sigma_1, \dots, \sigma_q$ be elements of a linear system $|\{s_j\}_{j=0,\dots,n+p}|$ such that $(i^*\sigma_j)$ is smooth and $\sum_{1 \le j \le q} i^*(\sigma_j)$ is a simply normal crossing divisor on Z. Then we have*

$$\left(q + p - \frac{n+p+1}{d} - \frac{1}{2d} n(n-1)(n+1+l_0+\cdots+l_{n+p})\right) T_f(r, dH_Z) \le \sum_{1 \le j \le q} N_n(r, f^*\sigma_j) + S_f(r).$$

Proof. Because the canonical line bundle $K_Z = (pd - n - p - 1)H_Z$, Theorem 2.3.3 implies the statement. □

In paricular, if $q = 0$ and

$$p - \frac{n+p+1}{d} - \frac{1}{2d} n(n-1)(n+1+l_0+\cdots+l_{n+p}) > 0,$$

then f is algebraically degenerate (cf. Theorem 11.19 of J.-P. Demailly [De]).

2.5 Pull back of the meromorphic partial projective connection

Let X and $\widetilde{X}$ be n-dimensional complex projective algebraic manifolds. Let $\pi : \widetilde{X} \to X$ be a surjective holomorphic map. Then there exists a proper subvariety S of X such that $\widetilde{X} \setminus \pi^{-1}(S)$ and $X \setminus S$ are locally biholomorphic. Let $\nabla = \{(\nabla_j, U_j)\}_{1 \le j \le N}$ be a meromorphic partial projective connection on X relative to an affine open covering $\{U_j\}_{1\le j\le N}$ on X. We shall now construct the meromorphic partial projective connection $\widetilde{\nabla} = \{(\widetilde{\nabla}_j, \pi^{-1}U_j)\}_{1\le j\le N}$ on $\widetilde{X}$. Let $p \in \pi^{-1}U_k \subset \widetilde{X}$. Let $u, v \in \Gamma(V, T_X)$ be local holomorphic vector fields on a small neighborhood V of p. Then $V \setminus \pi^{-1}(S)$ is locally biholomorphic with $\pi(V) \setminus S$. We define

$$(\widetilde{\nabla}_k)_u v|_{V \setminus \pi^{-1}(S)} = (\pi_*|_{V\setminus\pi^{-1}S})^{-1} \, (\nabla_k)_{\pi_* u} \pi_* v|_{V\setminus\pi^{-1}S},$$

on $V \setminus \pi^{-1}S$. Then, the meromorphic vector field $(\widetilde{\nabla}_k)_u v|_{V\setminus\pi^{-1}(S)}$ on $V\setminus\pi^{-1}S$ is uniquely extended to the meromorphic vector field on V. In this way, we define the meromorphic connection $\widetilde{\nabla}_k$ on $\pi^{-1}(U_k)$. Let α_{ij} and β_{ij} be meromorphic one-forms on $U_i \cap U_j$ such that

$$\nabla_i - \nabla_j = \alpha_{ij} \otimes \mathrm{Id}_{TX} + \mathrm{Id}_{TX} \otimes \beta_{ij}.$$

Then we have

$$\widetilde{\nabla}_i - \widetilde{\nabla}_j = \pi^*\alpha_{ij} \otimes \mathrm{Id}_{T\widetilde{X}} + \mathrm{Id}_{T\widetilde{X}} \otimes \pi^*\beta_{ij}.$$

So $\widetilde{\nabla} = \{(\widetilde{\nabla}_j, \pi^{-1}U_j)\}_{1\le j\le N}$ is a meromorphic partial projective connection on $\widetilde{X}$ relative to an affine open covering $\{\pi^{-1}U_j\}_{1\le j\le N}$ of $\widetilde{X}$.

Assume that π is the blowing-up of X at a point of X. Let D be a reduced effective divisor in X such that $\operatorname{supp} D$ is not contained in $\operatorname{supp}(\nabla)_\infty$. Take a holomorphic function s_j on U_j such that $D|_{U_j} = (s_j)$.

Lemma 2.5.1. *Assume that D is totally geodesic with respect to ∇, and the strict transform of D under π is smooth. Then the strict transform of D is totally geodesic with respect to the meromorphic partial projective connection $\widetilde{\nabla} = \{(\widetilde{\nabla}_j, \pi^{-1}(U_j))\}_{1\le j\le N}$ on $\widetilde{X}$.*

Proof. There exist meromorphic one-forms a_j, b_j and meromorphic two-form c_j on U_j such that no polar locus of a_j, b_j, c_j does not contain $\operatorname{supp} D|_{U_j}$, and a_j, b_j, c_j satisfy

$$\nabla_j{}^* ds_j = a_j \otimes ds_j + ds_j \otimes b_j + s_j c_j,$$

for all $1 \leq j \leq N$. So we have

$$\widetilde{\nabla}_j^* d\pi^* s_j = \pi^* a_j \otimes d\pi^* s_j + d\pi^* s_j \otimes \pi^* b_j + \pi^* s_j \pi^* c_j.$$

Let E be an exceptional divisor of π. Let $\widetilde{D}$ be a strict transform of D under the blowing-up π. Then, $\operatorname{supp} \widetilde{D}$ is not contained in $\operatorname{supp}(\widetilde{\nabla})_\infty$. We may assume that there exists a holomorphic function e on an affine open subset in $\pi^{-1}(U_j)$ such that $(e) = E|_V$. Then we have

$$\widetilde{D}|_V = \left(\frac{\pi^* s_j}{e^k}\right),$$

for some non-negative integer k. On V, it follows that

$$\begin{aligned}
d\left(\frac{\pi^* s_j}{e^k}\right) &= \frac{d\pi^* s_j}{e^k} - k\,\frac{\pi^* s_j}{e^k}\frac{de}{e}, \\
\widetilde{\nabla}_j^*\left(\frac{d\pi^* s_j}{e^k}\right) &= d\left(\frac{1}{e^k}\right) \otimes d\pi^* s_j + \frac{1}{e^k}\widetilde{\nabla}_j^* d\pi^* s_j \\
&= -k\frac{de}{e} \otimes \frac{d\pi^* s_j}{e^k} \\
&\quad + \frac{1}{e^k}(\pi^* a_j \otimes d\pi^* s_j + d\pi^* s_j \otimes \pi^* b_j + \pi^* s_j\, \pi^* c_j) \\
&= \left(\pi^* a_j - k\frac{de}{e}\right) \otimes \frac{d\pi^* s_j}{e^k} + \frac{d\pi^* s_j}{e^k} \otimes \pi^* b_j + \frac{\pi^* s_j}{e^k}\pi^* c_j, \\
\widetilde{\nabla}_j^*\left(-k\frac{\pi^* s_j}{e^k}\frac{de}{e}\right) &= -k\,d\left(\frac{\pi^* s_j}{e^k}\right) \otimes \frac{de}{e} - k\frac{\pi^* s_j}{e^k}\widetilde{\nabla}_j^*\left(\frac{de}{e}\right).
\end{aligned}$$

So we have

$$\begin{aligned}
\widetilde{\nabla}_j^* d\left(\frac{\pi^* s_j}{e^k}\right) &= \widetilde{\nabla}_j^*\left(\frac{d\pi^* s_j}{e^k}\right) + \widetilde{\nabla}_j^*\left(-k\frac{\pi^* s_j}{e^k}\frac{de}{e}\right) \\
&= \left(\pi^* a_j - k\frac{de}{e}\right) \otimes \frac{d\pi^* s_j}{e^k} + \frac{d\pi^* s_j}{e^k} \otimes \pi^* b_j - k\,d\left(\frac{\pi^* s_j}{e^k}\right) \otimes \frac{de}{e} \\
&\quad + \frac{\pi^* s_j}{e^k}\left(\pi^* c_j - k\widetilde{\nabla}_j^*\frac{de}{e}\right) \\
&= \left(\pi^* a_j - k\frac{de}{e}\right) \otimes \left(d\left(\frac{\pi^* s_j}{e^k}\right) + k\,\frac{\pi^* s_j}{e^k}\frac{de}{e}\right) \\
&\quad + \left(d\left(\frac{\pi^* s_j}{e^k}\right) + k\,\frac{\pi^* s_j}{e^k}\frac{de}{e}\right) \otimes \pi^* b_j - k\,d\left(\frac{\pi^* s_j}{e^k}\right) \otimes \frac{de}{e} \\
&\quad + \frac{\pi^* s_j}{e^k}\left(\pi^* c_j - k\widetilde{\nabla}_j^*\frac{de}{e}\right) \\
&= \widetilde{a}_j \otimes d\left(\frac{\pi^* s_j}{e^k}\right) + d\left(\frac{\pi^* s_j}{e^k}\right) \otimes \widetilde{b}_j + \frac{\pi^* s_j}{e^k}\widetilde{c}_j,
\end{aligned}$$

where

$$\widetilde{a}_j = \pi^* a_j - k\frac{de}{e}, \qquad \widetilde{b}_j = \pi^* b_j - k\frac{de}{e},$$

$$\widetilde{c}_j = k\pi^* a_j \otimes \frac{de}{e} + k\frac{de}{e} \otimes \pi^* b_j + \pi^* c_j - k^2 \frac{de}{e} \otimes \frac{de}{e} - k\,\widetilde{\nabla}_j^* \frac{de}{e}.$$

□

2.6 The second main theorem for singular divisors

Let X be an n-dimensional complex algebraic projective manifold, and let π be the blowing-up of X at the point p of X. Let U be an affine open neighborhood of p, and let ∇ be a meromorphic connection on U. Let $z_1, \ldots, z_n$ be holomorphic functions on U such that $dz_1, \ldots, dz_n$ are linearly independent on U and $p = \{x \in U \mid z_1(x) = \cdots = z_n(x) = 0\}$. Then,

$$\pi^{-1}(U) \simeq \{(x, [y_1 : \cdots : y_n]) \in U \times \mathbb{P}^{n-1}(\mathbb{C}) \mid z_i(x) y_j = z_j(x) y_i \quad \text{for all } i, j\}.$$

Let $\widetilde{U}_k = \{(x, [y_1 : \cdots : y_n]) \in \pi^{-1}(U) \mid y_k \neq 0\}$ be an affine open set of $\widetilde{X}$. Define a holomorphic function $u_i = y_i / y_k$ on $\widetilde{U}_k$. Then

$$du_1, \ldots, du_{k-1}, dz_k, du_{k+1}, \ldots, du_n,$$

are linearly independent on $\widetilde{U}_k$, and $E|_{\widetilde{U}_k} = (z_k)$ where E is the exceptional divisor of π. We shall now show that $W_{\widetilde{\nabla}}$ has only logarithmic poles on the exceptional divisor. Here $\widetilde{\nabla}$ is the pull back of ∇.

We may assume "$k = 1$" without loss of generality.

We have

$$\pi_* \left(\frac{\partial}{\partial z_1} \frac{\partial}{\partial u_2} \cdots \frac{\partial}{\partial u_n} \right) = \left(\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_n} \right) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ u_2 & z_1 & & \\ \vdots & & \ddots & \text{\Large 0} \\ u_n & \text{\Large 0} & & z_1 \end{pmatrix}.$$

We denote the above Jacobian matrix by A. Then we have

$$A^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -u_2/z_1 & 1/z_1 & & \\ \vdots & & \ddots & \mathbf{0} \\ -u_n/z_1 & \mathbf{0} & & 1/z_1 \end{pmatrix}.$$

Put $\nabla = d + \Gamma$ where $\Gamma = (\Gamma^\lambda{}_\mu)_{1\le\lambda,\mu\le n}$ is a connection form with respect to the local frame $\partial/\partial z_1, \dots, \partial/\partial z_n$, i.e., $\Gamma^\lambda{}_\mu = \sum_{j=1}^n \Gamma^\lambda_{i,\mu} dz_i$. Let $\widetilde{\Gamma} = (\widetilde{\Gamma}^\lambda{}_\mu)_{1\le\lambda,\mu\le n}$ be the connection form of the meromorphic connection $\widetilde{\nabla}$ on $\widetilde{U}_1$ with respect to the local frame $\partial/\partial z_1, \partial/\partial u_2, \dots, \partial/\partial u_n$. Then we have $\widetilde{\Gamma} = A^{-1}dA + A^{-1}\pi^*\Gamma A$. Since

$$\frac{d\pi^* z_1}{z_1} = \frac{dz_1}{z_1}, \qquad \frac{d\pi^* z_j}{z_1} = du_j + u_j\frac{dz_1}{z_1},$$

it follows that

$$\frac{\pi^*\Gamma^\lambda{}_\mu}{z_1} = \phi_{1,\lambda,\mu}\frac{dz_1}{z_1} + \sum_{j=2}^n \phi_{j,\lambda,\mu} du_j$$

where $\phi_{j,\lambda,\mu}$ is a meromorphic function on $\widetilde{U}_1$. Let β be a holomorphic function on U such that $\beta\nabla$ is holomorphic on U. It follows that $\pi^*\beta\,\phi_{1,\lambda,\mu}$ is holomorphic function. So $\pi^*\beta\, A^{-1}\pi^*\Gamma A$ has only logarithmic poles on the exceptional divisor.

It follows that

$$\begin{aligned} A^{-1}dA &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -u_2/z_1 & 1/z_1 & & \\ \vdots & & \ddots & \mathbf{0} \\ -u_n/z_1 & \mathbf{0} & & 1/z_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ du_2 & dz_1 & & \\ \vdots & & \ddots & \mathbf{0} \\ du_n & \mathbf{0} & & dz_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ (du_2)/z_1 & (dz_1)/z_1 & & \\ \vdots & & \ddots & \mathbf{0} \\ (du_n)/z_1 & \mathbf{0} & & (dz_1)/z_1 \end{pmatrix}. \end{aligned}$$

We define the meromorphic connection $\widetilde{\nabla}_1$ on $\widetilde{U}_1$ by

$$\widetilde{\nabla}_1 = \widetilde{\nabla} - \frac{dz_1}{z_1} \otimes \mathrm{Id}_{T\widetilde{X}} - \mathrm{Id}_{T\widetilde{X}} \otimes \frac{dz_1}{z_1}.$$

Then $\{(\widetilde{\nabla}, \widetilde{U}_1), (\widetilde{\nabla}_1, \widetilde{U}_1)\}$ is a meromorphic partial projective connection on $\widetilde{U}_1$, so $W_{\widetilde{\nabla}} = W_{\widetilde{\nabla}_1}$. One sees that

$$\begin{aligned}
\widetilde{\nabla}_1 &= d + A^{-1}\pi^*\Gamma A + A^{-1}dA - \frac{dz_1}{z_1} \otimes \mathrm{Id}_{T\widetilde{X}} - \mathrm{Id}_{T\widetilde{X}} \otimes \frac{dz_1}{z_1} \\
&= d + A^{-1}\pi^*\Gamma A + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ (du_2)/z_1 & (dz_1)/z_1 & & \\ \vdots & & \ddots & \mathbf{0} \\ (du_n)/z_1 & \mathbf{0} & & (dz_1)/z_1 \end{pmatrix} \\
&\quad - \frac{dz_1}{z_1} \begin{pmatrix} 1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1 \end{pmatrix} - \begin{pmatrix} (dz_1)/z_1 & 0 & \cdots & 0 \\ (du_2)/z_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (du_n)/z_1 & 0 & \cdots & 0 \end{pmatrix} \\
&= d + A^{-1}\pi^*\Gamma A - \begin{pmatrix} (2dz_1)/z_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{0} & \\ 0 & & & \end{pmatrix}.
\end{aligned}$$

Therefore $\pi^*\beta\,\widetilde{\nabla}_1$ has only logarithmic poles on the exceptional divisor. In the same way as above, we can construct the meromorphic connection $\widetilde{\nabla}_k$ on $\widetilde{U}_k$ for every k, $1 \le k \le n$. Then we obtain the meromorphic partial projective connection $\{(\widetilde{\nabla}_i, \widetilde{U}_i)\}_{1\le i\le n}$ on $\pi^{-1}(U)$ such that each $\pi^*\beta\,\widetilde{\nabla}_i$ has only logarithmic poles on the exceptional divisor.

Let X be an n-dimensional complex algebraic projective manifold, and let S be a reduced effective divisor on X such that the singular locus of S is $\{x_1, \ldots, x_p\} \subset X$ where x_i is a point of X. Let $\pi : \widetilde{X} \to X$ be the blowing-up at $\{x_1, \ldots, x_p\}$. Let E_i, $1 \le i \le p$ be irreducible divisors of $\widetilde{X}$ such that $E = \bigcup_{1\le i\le p} E_i$ is an exceptional divisor of π and $\pi(E_i) = x_i$. Let

$\nabla = \{(\nabla_j, U_j)\}_{1\le j\le N}$ be a meromorphic partial projective connection on X relative to an affine open covering $\{U_j\}_{1\le j\le N}$ of X. Assume that S is not contained in the polar locus of ∇, and S is totally geodesic with respect to ∇. Let $\beta \in \Gamma(X, L)$ be a holomorphic section of a line bundle L on X such that $\beta\nabla_j$ is holomorphic for every j.

Lemma 2.6.1. *Let $f : \mathbb{C} \to X$ be a non-constant holomorphic map such that $f(\mathbb{C})$ is not contained in the polar locus of ∇ and $W_\nabla(f) \not\equiv 0$. Let $\widetilde{f} : \mathbb{C} \to \widetilde{X}$ denote the lift of f. Assume that the proper transform $\widetilde{S} \subset \widetilde{X}$ of S is non-singular, and $\widetilde{S}$ intersects E transversally. Let $\widetilde{\sigma}$ be a holomorphic section of a holomorphic line bundle of $\widetilde{X}$ such that $(\widetilde{\sigma}) = \widetilde{S}$. Then it follows that*

$$\int_{|z|=r} \log^+ \frac{\|W_{\widetilde{\nabla}}(\widetilde{f})(z)\|_{\wedge^n T\widetilde{X}} \|\beta(f)(z)\|_L^{n(n-1)/2}}{\|\widetilde{\sigma}(\widetilde{f})(z)\|_{[\widetilde{S}]}} \frac{d\theta}{2\pi} \le S_f(r).$$

Proof. We may assume that each U_i has at most one singular point of S. Let $I = \{1, 2, \cdots, N\}$, and let I' and I'' be subsets of I such that

$$I' = \{i \in I \,|\, U_i \cap \{x_1, \cdots, x_p\} \neq \emptyset\},$$

$$I'' = I \setminus I' = \{i \in I \,|\, U_i \cap \{x_1, \cdots, x_p\} = \emptyset\}.$$

Let $\widetilde{\nabla} = \{(\pi^*\nabla_i, \pi^{-1}(U_i))\}_{1\le i\le N}$ be the pull back of ∇. If $j \in I'$, there exists one $x_\nu \in U_j$. By the above argument, we can construct a meromorphic partial projective connection $\{(\widetilde{\nabla}_{j,k}, \widetilde{U}_{j,k})\}_{1\le k\le n}$ on $\pi^{-1}(U_j)$ such that each $\pi^*\beta\widetilde{\nabla}_{j,k}$ has only logarithmic poles on E_ν. Here $\{\widetilde{U}_{j,k}\}_{1\le k\le n}$ is the affine open covering of $\pi^{-1}(U_j)$. We define an affine open covering $\{\Omega_j\}_{j=1,\ldots N'}$ of $\widetilde{X}$ and a meromorphic partial projective connection $\{(\widehat{\nabla}_j, \Omega_j)\}_{1\le j\le N'}$ on $\widetilde{X}$ by

$$\{(\widetilde{\nabla}_{j,k}, \widetilde{U}_{j,k})\}_{j\in I', 1\le k\le n} \cup \{(\widetilde{\nabla}_j, \pi^{-1}U_j)\}_{j\in I''},$$

where $(\widehat{\nabla}_j, \Omega_j)$ is equal to $(\widetilde{\nabla}_{j,k}, \widetilde{U}_{j,k})$, $j \in I'$, or equal to $(\widetilde{\nabla}_j, \pi^{-1}U_j)$, $j \in I''$. Then it follows that the Wronskian of $\widetilde{\nabla}$ is equal to the Wronskian of $\widehat{\nabla}$. Take an open covering $\{V_j\}_{1\le j\le N'}$ of $\widetilde{X}$ such that $V_j \Subset \Omega_j$, and take a partition of unity $\{\phi_j\}_{1\le j\le N'}$ subordinate to the open covering $\{V_j\}_{1\le j\le N'}$. If $\Omega_i = \widetilde{U}_{j,k}$ for some $j \in I', 1 \le k \le n$, then $\widetilde{S}$ intersects exceptional divisor E transversally in Ω_i. So we can take holomorphic functions $X_1, \cdots, X_n$ in Ω_i such that $(X_1) = E|_{\Omega_i}$ and $(X_2) = \widetilde{S}|_{\Omega_i}$, and $dX_1, \cdots, dX_n$ are linearly independent. We trivialize the n-jet bundle of $\widetilde{X}$ on Ω_i by

$$X_1^{(1)}, \cdots, X_n^{(1)}, X_1^{(2)} \cdots, X_n^{(2)}, \cdots, X_n^{(n)},$$

where $dX_l = X_l^{(1)}$ and $dX_l^{(k)} = X_l^{(k+1)}$. There exists a meromorphic n-jet differential ω on Ω_i such that

$$W_{\widetilde{\nabla}_i} = \omega \frac{\partial}{\partial X_1} \wedge \cdots \wedge \frac{\partial}{\partial X_n}.$$

Let $\widetilde{\beta}_i$ be a holomorphic function in Ω_i such that $(\widetilde{\beta}_i) = \pi^*\beta|_{\Omega_i}$. From Lemma 2.2.5 it follows that

$$\widetilde{\beta}_i^{\,n(n-1)/2} \frac{\omega}{X_2}$$

is logarithmic n-jet differential along the divisor $(X_1) + (X_2)$ (cf. Noguchi [Nog1]). So it follows that

$$\int_{|z|=r} \phi_i(\widetilde{f}) \log^+ \frac{\|W_{\widetilde{\nabla}}(\widetilde{f})(z)\|_{\bigwedge^n T\widetilde{X}} \|\beta(f)(z)\|_L^{n(n-1)/2}}{\|\widetilde{\sigma}(\widetilde{f})(z)\|_{[\widetilde{S}]}} \frac{d\theta}{2\pi} \leq S_{\widetilde{f}}(r),$$

by the same arguments as that in the proof of Lemma 2.3.2. If $\Omega_i = \pi^{-1}(U_j)$ for $j \in \Gamma$", the above inequality also holds. Because $\pi : \widetilde{X} \to X$ is a bimeromorphic map, we have $S_f(r) = S_{\widetilde{f}}(r)$. So we complete the proof. □

Theorem 2.6.2. *Under the hypothesis of Lemma 2.6.1, we have*

$$\begin{aligned} & T_{\widetilde{f}}(r, [\widetilde{S}]) + T_f(r, K_X) + (n-1)\sum_{i=1}^{p} T_{\widetilde{f}}(r, [E_i]) \\ \leq \; & N_n(r, \widetilde{f}^*\widetilde{S}) + \frac{1}{2}n(n-1)T_f(r, L) + \frac{1}{2}n(n-1)\sum_{i=1}^{p} N(r, \widetilde{f}^*E_i) + S_f(r). \end{aligned}$$

Proof. Let $e_j \in \Gamma(X, [E_j])$ be a holomorphic section of E_j such that $(e_j) = E_j$. By Lemma 2.6.1, it follows that

$$\begin{aligned} & \int_{|z|=r} \log^+ \frac{\|W_{\widetilde{\nabla}}(\widetilde{f})\|_{\bigwedge^n T\widetilde{X}} \|\beta(f)\|_L^{n(n-1)/2} \prod_{i=1}^{p} \|e_i(\widetilde{f})\|_{E_i}^{n(n-1)/2}}{\|\widetilde{\sigma}(\widetilde{f})\|_{[\widetilde{S}]}} \frac{d\theta}{2\pi} \\ & \leq -\frac{1}{2}n(n-1)\sum_{i=1}^{p} m_{\widetilde{f}}(r, E_i) + S_f(r). \end{aligned}$$

Then we have

$$\begin{aligned} & T_{\widetilde{f}}(r, [\widetilde{S}]) + T_{\widetilde{f}}(r, K_{\widetilde{X}}) - \frac{1}{2}n(n-1)\sum_{i=1}^{p} T_{\widetilde{f}}(r, [E_i]) - \frac{1}{2}n(n-1)T_f(r, L) \\ & \leq N_n(r, \widetilde{f}^*\widetilde{S}) - \frac{1}{2}n(n-1)\sum_{i=1}^{p} m_{\widetilde{f}}(r, E_i) + S_f(r). \end{aligned}$$

Since $N(r, \widetilde{f}^*E_i) = T_{\widetilde{f}}(r, [E_i]) - m_{\widetilde{f}}(r, E_i)$ and $K_{\widetilde{X}} = K_X + (n-1)\sum_{i=1}^{p} E_i$, we complete the proof. □

Example 2.6.3. Let

$$S = \{X_0^{d-2}(\varepsilon_1 X_1^2 + \varepsilon_2 X_2^2) + X_1^d + X_2^d = 0\},$$

$$\varepsilon_1 \neq 0,\ \varepsilon_2 \neq 0,\ |\varepsilon_1| \neq |\varepsilon_2|.$$

Then S is smooth except the point $[1:0:0]$. Let

$$\pi : \widetilde{\mathbb{P}}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$$

be the blowing-up at $[1:0:0] \in \mathbb{P}^2(\mathbb{C})$ and $\widetilde{S}$ be the proper transformation of S. One can check that $\widetilde{S}$ is non-singular, and $\widetilde{S}$ intersects transversally the exceptional divisor E of π. Let $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ be a holomorphic map and $\widetilde{f} : \mathbb{C} \to \widetilde{\mathbb{P}}^2(\mathbb{C})$ be the lift of f. Put $s_0 = X_0^{d-2}(\varepsilon_1 X_1^2 + \varepsilon_2 X_2^2)$, $s_1 = X_1^d$, $s_2 = X_2^d$. Then we construct a meromorphic partial projective connection ∇ on $\mathbb{P}^2(\mathbb{C})$ as in §2. Then the degree of the pole of ∇ is five. By Theorem 2.6.2, we have

$$T_{\widetilde{f}}(r, [\widetilde{S}]) + T_{\widetilde{f}}(r, [E]) - 8T_f(r, H) \leq N_2(r, \widetilde{f}^*\widetilde{S}) + N(r, \widetilde{f}^*E) + S_f(r),$$

where H is a hyperplane bundle of $\mathbb{P}^2(\mathbb{C})$. Because $\pi^*S = \widetilde{S} + 2E$, and $T_f(r, H) = T_{\widetilde{f}}(r, \pi^*H) \geq T_{\widetilde{f}}(r, E)$, we have

$$\left(1 - \frac{9}{d}\right) T_f(r, H) \leq N_2(r, \widetilde{f}^*\widetilde{S}) + N(r, \widetilde{f}^*E) + S_f(r).$$

□

Now we prove Theorem 0.0.2.

proof of Theorem 0.0.2. We construct the meromorphic partial projective connection ∇ on $\mathbb{P}^2(\mathbb{C})$ as in §2. Then $\sigma_1, \ldots, \sigma_q$ are totally geodesic with respect to ∇. When $d = 1$, the pole degree of ∇ is 0. When $d \geq 2$, the pole degree of ∇ is $3 + l_0 + l_1 + l_2$. Because σ_i intersects σ_j transversally for all $1 \leq i \neq j \leq q$, the divisor $\widetilde{\sigma}_1 + \cdots + \widetilde{\sigma}_q + E$ is simple normal crossing, and the divisor $\widetilde{\sigma}_1 + \cdots + \widetilde{\sigma}_q$ is smooth. (1) and (2) follow by the same arguments as that in the proof of Theorem 2.6.2.

If $\sigma_1, \ldots, \sigma_q$ are in m-subgeneral position. We have

$$\sum_{i=1}^{q} \pi^*\sigma_i \leq \sum_{i=1}^{q} \widetilde{\sigma}_i + m \sum_{i=1}^{p} E_i,$$

on $\widetilde{\mathbb{P}}^2(\mathbb{C})$. It follows that

$$\sum_{i=1}^{q} T_f(r,[\sigma_i]) \leq \sum_{i=1}^{q} T_{\tilde{f}}(r,[\widetilde{\sigma}_i]) + m\sum_{i=1}^{p} T_{\tilde{f}}(r,[E_i]). \tag{2.5}$$

Take hyperplanes $\{L_i', L_i''\}_{1\leq i\leq p}$ in $\mathbb{P}^2(\mathbb{C})$ such that L_i' and L_i'' pass through x_i, and the divisor $H_1 + H_2 + H_3 + \sum_{i=1}^{p}(L_i' + L_i'')$ is in general position. Then, by Cartan's Second Main Theorem (cf. [Ca]) we have

$$\begin{aligned}&\sum_{i=1}^{3} T_f(r,[H_i]) + \sum_{i=1}^{p} T_f(r,[L_i'] + [L_i'']) - 3T_f(r,H)\\ \leq\ &\sum_{i=1}^{3} N_2(r,H_i) + \sum_{i=1}^{p} N_2(r, f^*(L_i' + L_i'')) + S_f(r).\end{aligned}$$

Let $\widetilde{L}_i'$ and $\widetilde{L}_i''$ be proper transforms of L_i', L_i'' under the blowing-up π. Since

$$T_f(r, L_i' + L_i'') = T_{\tilde{f}}(r, \widetilde{L}_i' + \widetilde{L}_i'') + 2T_{\tilde{f}}(r, E_i),$$

and

$$N(r, f^*(L_i' + L_i'')) = N(r, \tilde{f}^*(\widetilde{L}_i' + \widetilde{L}_i'') + 2N(r, \tilde{f}^*E_i),$$

we have

$$\begin{aligned}&2\sum_{i=1}^{p} T_{\tilde{f}}(r, E_i) + \sum_{i=1}^{p} T_{\tilde{f}}(r, \widetilde{L}_i' + \widetilde{L}_i'')\\ \leq\ &2\sum_{i=1}^{p} N(r, \tilde{f}^*E_i) + \sum_{i=1}^{p} N(r, \tilde{f}^*(\widetilde{L}_i' + \widetilde{L}_i'')) + \sum_{i=1}^{3} N_2(r, f^*H_i) + S_f(r).\end{aligned}$$

By the First Main Theorem,

$$T_{\tilde{f}}(r, [\widetilde{L}_i'] + [\widetilde{L}_i'']) \geq N(r, \tilde{f}^*(\widetilde{L}_i' + \widetilde{L}_i'')).$$

Therefore we have

$$\sum_{i=1}^{p} T_{\tilde{f}}(r, E_i) \leq \sum_{i=1}^{p} N(r, \tilde{f}^*E_i) + \frac{1}{2}\sum_{i=1}^{3} N_2(r, f^*H_i) + S_f(r). \tag{2.6}$$

Then we deduce (3) and (4) from (1), (2), (2.5), and (2.6). □

3 Holomorphic curves into the product space of the Riemann spheres

3.1 Introduction

The purpose of this Chapter is to prove the second main theorem for a holomorphic map from the complex plane $\mathbb{C}$ to the product space of the one-dimensional projective spaces $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$.

The second main theorem for hypersurfaces in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is first obtained in Noguchi [Nog5]. In Section 5 of Noguchi [Nog5], there are some additional conditions for maps from $\mathbb{C}$ to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. In our paper, we do not assume these conditions.

In Siu [Si1], a meromorphic connection is used to prove the second main theorem. Because $\mathbb{C}^* \times \mathbb{C}^*$ is a Lie group, there exists the canonical connection on $\mathbb{C}^* \times \mathbb{C}^*$. We extend this connection to the meromorphic connection ∇ on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. This meromorphic connction ∇ is used in Section 5 of Noguchi [Nog5]. We also use ∇ to prove Theorem 0.0.3. This connection does not "vary" under the blowing-up, and plays an important role in our arguments.

Let $i : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the inclusion map where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then Z_k is the compactification of the semi-Abelian variety $\mathbb{C}^* \times \mathbb{C}^*$. If holomorphic map $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ does not intersect $H_{1,0}, H_{1,1}, H_{2,0}, H_{2,1}$, then f is a holomorphic map from $\mathbb{C}$ to the semi-Abelian variety $\mathbb{C}^* \times \mathbb{C}^*$. In Noguchi, Winkelmann and Yamanoi [NWY1], [NWY3], the second main theorem for holomorphic map f from $\mathbb{C}$ to a semi-Abelian variety A with D is proved, where D is an effective reduced divisor on A. If $A = \mathbb{C}^* \times \mathbb{C}^*$, and $D = D' + D''$ in Theorem 1.2.5, Theorem 0.0.3 deals with the holomorphic curves into $\overline{A}$.

3.2 Meromorphic connection and blowing-up

Let D', D'' be divisors on $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ defined by the polynomials $X_0^{m'}Y_0^{n'}-X_1^{m'}Y_1^{n'}$, $X_0^{m''}Y_1^{n''}-X_1^{m''}Y_0^{n''}$. Let $i:\mathbb{C}^*\times\mathbb{C}^*\to\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ be the inclusion map. Then $\operatorname{supp} i^*D'$ is a subgroup of $\mathbb{C}^*\times\mathbb{C}^*$. Therefore there exists the canonical connection ∇ on $\mathbb{C}^*\times\mathbb{C}^*$ such that $\operatorname{supp} i^*D'$ is totally geodesic with respect to ∇. This connection is extended to the meromorphic connection on $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$. We also denote this extended connection by ∇. Let $U_{i,j}=\{([X_0:X_1],[Y_0:Y_1])\in\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})|X_i\neq 0, Y_j\neq 0\}$, $0\leq i,j\leq 1$. Take the canonical local coordinate system (z,w) on $U_{i,j}\simeq\mathbb{C}\times\mathbb{C}$. Then, the meromorphic connection ∇ is written by

$$d+\begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{dw}{w}\end{pmatrix},$$

on $U_{i,j}$. It is easy to see that $\operatorname{supp} i^*D''$ is also totally geodesic with respect to ∇.

The universal covering space of $\mathbb{C}^*\times\mathbb{C}^*$ is $\mathbb{C}\times\mathbb{C}$. The connection on $\mathbb{C}\times\mathbb{C}$ which is induced by ∇ is the flat connection d on $\mathbb{C}\times\mathbb{C}$. Let $f:\mathbb{C}\to\mathbb{C}^*\times\mathbb{C}^*$ be a non-constant holomorphic map, and let $F:\mathbb{C}\to\mathbb{C}\times\mathbb{C}$ be the lift of f. Then $f'\wedge\nabla_{f'}f'\equiv 0$ if and only if F is a translation of a linear map.

Lemma 3.2.1. *Let $f:\mathbb{C}\to\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map such that $f(\mathbb{C})$ is not contained in the support of $H_{i,j}$, $i=1,2$, $j=0,1$. Then f satisfies*

$$f'\wedge\nabla_{f'}f'\equiv 0,$$

if and only if f satisfies the following condition (i) or (ii):
(i)

$$f(\mathbb{C})\subset\{([X_0:X_1],[Y_0:Y_1])\in\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})\,|\,X_0^{r_1}Y_0^{r_2}-CX_1^{r_1}Y_1^{r_2}=0\},$$

for some $(r_1,r_2)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$ and some $C\in\mathbb{C}\setminus\{0\}$.

(ii)
There exist holomorphic functions g_1,g_2 on $\mathbb{C}$ and $(a,b)\in\mathbb{C}\times\mathbb{C}\setminus\{(0,0)\}$ such that

$$f=(\exp g_1,\exp g_2):\mathbb{C}\to\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C}),$$
$$ag_1+bg_2=(\text{constant}),$$

on $\mathbb{C}$.

Proof. Without loss of generality, we may assume that $f(0) \in \mathbb{C}^* \times \mathbb{C}^*$. The holomorphic map

$$\left(\exp(2\pi\sqrt{-1}\cdot), \exp(2\pi\sqrt{-1}\cdot)\right) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*,$$

is the universal covering of $\mathbb{C}^* \times \mathbb{C}^*$. The induced connection on the covering space $\mathbb{C} \times \mathbb{C}$ by ∇ is the flat connection d. We put $f = (f_1, f_2)$ where f_1 and f_2 are meromorphic functions on $\mathbb{C}$. Let

$$h_i = \frac{1}{2\pi\sqrt{-1}} \log f_i, \qquad i = 1,\, 2.$$

Assume that $f' \wedge \nabla_{f'} f' \equiv 0$. Then there exists a meromorphic function h on $\mathbb{C}$ such that

$$\begin{pmatrix} h_1''(z) \\ h_2''(z) \end{pmatrix} = h(z) \begin{pmatrix} h_1'(z) \\ h_2'(z) \end{pmatrix},$$

on $\mathbb{C}$.

This means that

$$h_i'(z) = h_i'(0) \exp H(z), \qquad i = 1,\, 2,$$

in a simple connected neighborhood U of $0 \in \mathbb{C}$. Here

$$H(z) = \int_0^z h(t)dt.$$

If $h_i'(0) = 0$, it follows that h_i is a constant function. Hence $(h_1'(0), h_2'(0)) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$. It holds that

$$h_i(z) = h_i'(0) \int_0^z \exp H(t)dt + h_i(0), \qquad i = 1,\, 2.$$

It follows that

$$h_2'(0)h_1(z) - h_1'(0)h_2(z) = h_2'(0)h_1(0) - h_1'(0)h_2(0).$$

Conversely, assume that there exists $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$ah_1(z) + bh_2(z) = (\text{constant}),$$

on $\mathbb{C}$. Then $ah_1'(z) + bh_2'(z) = 0, ah_1''(z) + bh_2''(z) = 0$. Therefore it follows that $f' \wedge \nabla_{f'} f' \equiv 0$.

Therefore $f' \wedge \nabla_{f'} f' \equiv 0$ if and only if there exists $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$a \log f_1(z) + b \log f_2(z) = (\text{constant}),$$

on $\mathbb{C}$.

Assume that

$$a \log f_1(z) + b \log f_2(z) = c, \tag{3.1}$$

for some $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$, $c \in \mathbb{C}$. Without loss of generality, we may assume that $a = 1$. For $x \in \mathbb{C}$, we put

$$f_i(z) = (z - x)^{r_i} h_i(z), \qquad i = 1, 2,$$

where $r_i \in \mathbb{Z}$, $h_i(z)$ is a holomorphic function on an open neighborhood of x such that $h_i(x) \neq 0$. Then, by (3.1), we have $r_1 + br_2 = 0$. When $r_2 \neq 0$ for some $x \in \mathbb{C}$, it follows that

$$\log(f_1(z))^{r_2} + \log(f_2(z))^{-r_1} = r_2 c.$$

Then it holds that the meromorphic function $(f_1(z))^{r_2}(f_2(z))^{-r_1}$ is a constant function on $\mathbb{C}$. This means that

$$f(\mathbb{C}) \subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid X_1^{r_2} Y_0^{r_1} - C X_0^{r_2} Y_1^{r_1} = 0\},$$

for $r_1 \in \mathbb{Z}$, $r_2 \in \mathbb{Z} \setminus \{0\}$, $C \in \mathbb{C}$. When $r_2 = 0$ for all $x \in \mathbb{C}$, we have $r_1 = 0$ for all $x \in \mathbb{C}$. This means that there exist holomorphic functions g_1, g_2 on $\mathbb{C}$ such that $f_i = \exp g_i$, $i = 1, 2$. Then $g_1 + bg_2 = c$.

Conversely, if f satisfies the condition (i) or (ii). It is easy to see that there exists $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$a \log f_1(z) + b \log f_2(z) = (\text{constant}),$$

on $\mathbb{C}$. □

Remark 3.2.2. The condition of (b) in Lemma 3.2.1 does not mean the algebraical degeneracy of $f(\mathbb{C})$. For example, take

$$f(z) = (\exp z, \exp \sqrt{-1} z) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}).$$

The divisor

$$D' + D'' + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j},$$

is not simple normal crossing at $\{([0:1],[0:1]), ([0:1],[1:0]), ([1:0],[0:1]), ([1:0],[1:0])\} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Put $Z_0 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $\pi_{1,0} : Z_1 \to Z_0$ be the blowing-up of Z_0 at the center $\{([0:1],[0:1]), ([0:1],[1:0]), ([1:0],[0:1]), ([1:0],[1:0])\}$. Let $D'_1, D''_1, H_{i,j,1}$ be the strict transform of $D', D'', H_{i,j}$ under $\pi_{1,0}$, and let E_1 be the exceptional divisor of $\pi_{1,0}$.

If the divisor

$$D'_1 + D''_1 + \sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,1} + E_1,$$

is not simple normal crossing in Z_1, we blow up Z_1 at the points where that divisor is not simple normal crossing. We repeat this process for several times. We put the l-th blowing-up $\pi_{l,l-1} : Z_l \to Z_{l-1}$. Let E_l be the exceptional divisor of $\pi_{l,l-1}$. We define

$$\pi_{j,i} = \pi_{i+1,i} \circ \pi_{i+2,i+1} \circ \cdots \circ \pi_{j,j-1},$$

for $i \leq j$ (we define $\pi_{i,i} = \mathrm{Id}$). Let D'_l, D''_l, $H_{i,j,l}$ be the strict transform of D', D'', $H_{i,j}$ under $\pi_{l,0}$, and let $E_{i,l}$, $1 \leq i \leq l$, be the strict transform of E_i under $\pi_{l,i}$.

Then there exists a positive integer k such that

$$D'_k + D''_k + \sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,k} + \sum_{i=1}^{k} E_{i,k},$$

is simple normal crossing. We put $\widetilde{D}' = D'_k$, $\widetilde{D}'' = D''_k$, $\widetilde{H}_{i,j} = H_{i,j,k}$, and $\widetilde{E}_i = E_{i,k}$.

Example 3.2.3. Let D', D'' be the divisor which are defined by the polynomials

$$X_0^2Y_0 - X_1^2Y_1, \qquad X_0^3Y_1^2 - X_1^3Y_0^2.$$

Let $\pi_{1,0} : Z_1 \to Z_0$ be the blowing-up as above. Then $D'_1+D''_1+\sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,1}+ E_1$ is not simple normal crossing at four points in Z_1. Then $\pi_{2,1} : Z_2 \to Z_1$ is the blowing-up at these four points. We see that $D'_2+D''_2+\sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,2}+ \sum_{i=1}^{2} E_{i,2}$ is not simple normal crossing at two points in Z_2. Then $\pi_{3,2} : Z_3 \to Z_2$ is the blowing-up at these two points. Then $D'_2+D''_2+\sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,2}+ \sum_{i=1}^{2} E_{i,2}$ is normal crossing.

Let E'_i and E''_i be irreducible components of E_i such that $\pi_{i,0}(\operatorname{supp} E'_i) \subset \operatorname{supp} D'$ and $\pi_{i,0}(\operatorname{supp} E''_i) \subset \operatorname{supp} D''$. Then $E_1 = E'_1 + E''_1$, $E_2 = E'_2 + E''_2$ and $E_3 = E''_3$. Let $\widetilde{E}'_i$ and $\widetilde{E}''_i$ be the proper transform of E'_i and E''_i. Then it follows that

$$\pi_{3,0}^* D' = \widetilde{D}' + \widetilde{E}'_1 + 2\widetilde{E}'_2,$$

and

$$\pi_{3,0}^* D' = \widetilde{D}'' + 2\widetilde{E}''_1 + 3\widetilde{E}''_2 + 6\widetilde{E}''_3.$$

Lemma 3.2.4. *There exist affine open coverings $\{U^l_s\}_{1\le s\le N_l}$ of Z_l, for $0\le l\le k$, such that every U^l_s satisfies the following five conditions:*
(i)

$$U^l_s \simeq \mathbb{C}\times\mathbb{C}.$$

Take the canonical local coordinate system (z,w) of U^l_s.
(ii)

$$\sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,l}|_{U^l_s} + \sum_{1\le i\le l} E_{i,l}|_{U^l_s} = (z)+(w),$$

on U^l_s.
(iii)

$$D'_l|_{U^l_s} = (z^{p'}-w^{q'}) \quad (\text{or} \quad (1-z^{p'}w^{q'}) \quad \text{respectively}),$$

on U^l_s, where p' and q' are non-negative integers (p', q' may depend on l and s).
(iv)

$$D''_l|_{U^l_s} = (1-z^{p''}w^{q''}) \quad (\text{or} \quad (z^{p''}-w^{q''}) \quad \text{respectively}),$$

on U^l_s, where p'' and q'' are non-negative integers (p'', q'' may depend on l and s).
(v)

$$\pi^*_{l,0}\nabla|_{U^l_s} = d + \begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{dw}{w} \end{pmatrix},$$

on U^l_s.

Proof. We take affine open coverings $\{U^l_s\}_{1\le s\le N_l}$ by induction over l. For $l=0$, we put $\{U^0_s\}_{1\le s\le 4} = \{U_{i,j}\}_{0\le i,j\le 1}$. Here $U_{i,j} = \{[X_0:X_1],[Y_0:Y_1]\in\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})|X_i\ne 0, Y_j\ne 0\}$. Then $\{U^0_s\}_{1\le s\le 4}$ satisfies above five conditions. Assume that we take the affine open covering $\{U^{l-1}_s\}_{1\le s\le N_{l-1}}$ of Z_{l-1} for $l\le k$ which satisfies the above five conditions. Let $U^{l-1}_t\in\{U^{l-1}_s\}_{1\le s\le N_{l-1}}$. Take the canonical local coordinate system (z,w) of $U^{l-1}_t\simeq\mathbb{C}\times\mathbb{C}$.

If $D'_{l-1}|_{U^{l-1}_t} = (z^{p'}-w^{q'})$ for some positive integers p', q'. Then

$$D''_{l-1}|_{U^{l-1}_t} = (1-z^{p''}w^{q''}),$$

for some non-negative integers p'', q''. The divisor

$$D'_{l-1}+D''_{l-1}+\sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,l-1} + \sum_{1\le i\le l-1} E_{i,l-1},$$

in Z_{l-1}, is not normal crossing at $(0,0) \in U_t^{l-1}$. Then $(0,0)$ is contained in the center of the blowing-up $\pi_{l,l-1}$. We have

$$\pi_{l,l-1}^{-1}(U_t^{l-1}) = \{((z,w),[W_0 : W_1]) \in U_t^{l-1} \times \mathbb{P}^1(\mathbb{C}) \mid zW_1 = wW_0\}.$$

Let $V_i = \{((z,w),[W_0 : W_1]) \in \pi_{l,l-1}^{-1}(U_t^{l-1}) \mid W_i \neq 0\}$, $i = 0,1$. Then $\{V_0, V_1\}$ are affine open covering of $\pi_{l,l-1}^{-1}(U_t^{l-1})$. We show that affine open sets V_0 and V_1 satisfy the five conditions of lemma. Let $u = W_1/W_0$ be the holomorphic function on V_0. Then (z,u) is the local coordinate system of V_0. It is easy to verify that V_0 satisfies (i), (ii), (iii) and (iv). Since

$$\pi_{l,l-1}^* z = z, \quad \pi_{l,l-1}^* w = zu,$$

we have

$$\pi_{l,l-1*}\begin{pmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial u} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial w} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}.$$

Let Γ be the connection form of $\pi_{l,l-1}^* \nabla|_{V_0}$ with respect to the frame $\partial/\partial z, \partial/\partial u$. Then it follows that

$$\Gamma = \begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}^{-1} d\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}^{-1} \pi_{l,l-1}^* \begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{dw}{w} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{u}{z} & \frac{1}{z} \end{pmatrix},$$

we have

$$\begin{aligned}
\Gamma &= \begin{pmatrix} 0 & 0 \\ \frac{du}{z} & \frac{dz}{z} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -\frac{u}{z} & \frac{1}{z} \end{pmatrix} \begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{dz}{z} - \frac{du}{u} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ \frac{du}{z} & \frac{dz}{z} \end{pmatrix} + \begin{pmatrix} -\frac{dz}{z} & 0 \\ \frac{u}{z}\frac{dz}{z} & -\frac{1}{z}\left(\frac{dz}{z} + \frac{du}{u}\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ \frac{du}{z} & \frac{dz}{z} \end{pmatrix} + \begin{pmatrix} -\frac{dz}{z} & 0 \\ \frac{u}{z}\frac{dz}{z} - \frac{u}{z}\left(\frac{dz}{z} + \frac{du}{u}\right) & -\frac{dz}{z} - \frac{du}{u} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{du}{u} \end{pmatrix}.
\end{aligned}$$

Hence V_0 satisfies (v). In the same way, we can show that V_1 satisfies the conditions of the lemma.

If

$$D'_{l-1}|_{U_t^{l-1}} = (1 - z^{p'} w^{q'}), \qquad D''_{l-1}|_{U_t^{l-1}} = (z^{p''} - w^{q''}),$$

for some non-negative integers p', q' and some positive integers p'', q''. In the same way as above, we can take the affine open sets in $\pi_{l,l-1}^{-1}(U_t^{l-1})$ which satisfy the five conditions of the lemma.

If

$$D'_{l-1}|_{U_t^{l-1}} = (1 - z^{p'}w^{q'}), \qquad D''_{l-1}|_{U_t^{l-1}} = (1 - z^{p''}w^{q''}),$$

for some non-negative integers p', q', p'', q''. Then $U_t^{l-1} \simeq \pi_{l,l-1}^{-1}(U_t^{l-1})$ because U_t^{l-1} does not contain the center of the blowing-up $\pi_{l,l-1}$. By the assumption of induction, the affine open subset $\pi_{l,l-1}^{-1}(U_t^{l-1})$ satisfies the five conditions of the lemma.

This completes the proof. □

3.3 Proof of the bigness of $\widetilde{D}' + \widetilde{D}''$

In this section, we show that $\widetilde{D}' + \widetilde{D}''$ is big in Z_k. We note that there exists the proof of the bigness for more general cases in Proposition 3.9. of [NWY3].

To prove the bigness of the line bundle $\widetilde{D}' + \widetilde{D}''$, it is sufficient to show the following lemma (cf. Theorem 2.2.16 of R. Lazarsfeld [La]).

Lemma 3.3.1. *The divisor* $\widetilde{D}' + \widetilde{D}''$ *is nef and* $(\widetilde{D}' + \widetilde{D}'')^2 > 0$.

Proof. Because

$$(\widetilde{D}' + \widetilde{D}'')^2 = (\widetilde{D}')^2 + 2\widetilde{D}' \cdot \widetilde{D}'' + (\widetilde{D}'')^2,$$

it is enough to show that $(\widetilde{D}')^2 = (\widetilde{D}'')^2 = 0$ and $\widetilde{D}'$ and $\widetilde{D}''$ are nef. Without loss of generality, we may assume that $m' \leq n'$. Let E'_1 be the reduced divisor on Z_1 such that

$$\pi_{1,0}^* D' = D'_1 + m' E'_1.$$

Let F' be the divisor on Z_k such that

$$\pi_{k,1}^* D'_1 = \widetilde{D}' + F'.$$

It follows that

$$\begin{aligned}
(\widetilde{D}')^2 &= (\pi_{k,0}^* D' - F' - m'\pi_{k,1}^* E'_1)^2 \\
&= (\pi_{k,0}^* D')^2 + (F')^2 + m'^2(\pi_{k,1}^* E'_1)^2 - 2\pi_{k,0}^* D' \cdot F' \\
&\quad + 2m'F' \cdot \pi_{k,1}^* E'_1 - 2m'\pi_{k,1}^* E'_1 \cdot \pi_{k,0}^* D' \\
&= 2m'n' + (F')^2 - 2m'^2 - 2(\widetilde{D}' + F' + m'\pi_{k,1}^* E'_1) \cdot F' \\
&\quad + 2m'F' \cdot \pi_{k,1}^* E'_1 - 2m'\pi_{k,1}^* E'_1 \cdot (\pi_{k,1}^* D'_1 + m'\pi_{k,1}^* E'_1) \\
&= 2m'n' - 2m'^2 - (F')^2 - 2\widetilde{D}' \cdot F' - 2m' D'_1 \cdot E'_1 - 2m'^2 (E'_1)^2.
\end{aligned}$$

Because $D'_1 \cdot E'_1 = 2m'$, we have

$$(\widetilde{D}')^2 = 2m'n' - 2m'^2 - (F')^2 - 2\widetilde{D}' \cdot F'. \tag{3.2}$$

If $m' = n'$, then $\widetilde{D}' = D'_1$, $F' = 0$ and we have $(\widetilde{D}')^2 = 0$.

Now we prove $(\widetilde{D}')^2 = 0$ by the induction over the positive integer $m'+n'$. Let E'_i, $i = 2, \cdots k$ be reduced effective divisors on Z_i such that

$$\mathrm{supp}\,(\pi^*_{i,i-1}D'_{i-1} - D'_i) = \mathrm{supp}\,E'_i.$$

Let $\widetilde{E}'_i$ be the strict transform of E'_i under $\pi_{k,i}$. There exist non-negative integers $a_2, a_3, \cdots, a_k$ such that

$$F = a_2\widetilde{E}'_2 + a_3\widetilde{E}'_3 + \cdots + a_k\widetilde{E}'_k.$$

Now we take another divisor A' on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is defined by the polynomial

$$X_0^{m'}Y_0^{n'-m'} - X_1^{m'}Y_1^{n'-m'}.$$

There is, as in Section 3, the sequence of the blowing-up

$$\begin{aligned} \sigma_{1,0} \ &: \ W_1 \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}), \\ &\ \ \vdots \\ \sigma_{k-1,k-2} \ &: \ W_{k-1} \to W_{k-2}, \end{aligned}$$

such that the following condition $(**)$ satisfies:

Let S be the reduced divisor such that

$$\mathrm{supp}\left(\sigma^*_{k-1,0}\left(A' + \sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j}\right)\right) = \mathrm{supp}\,S,$$

where $\sigma_{k-1,1} = \sigma_{1,0} \circ \cdots \circ \sigma_{k-1,k-2}$. Then

$(**)$ $\quad S$ is normal crossing in W_{k-1}.

Let B'_i be the exceptional divisor of $\sigma_{i,i-1}$, and let $\widetilde{B}'_i$ be the strict transform of B'_i under $\sigma_{i+1,i} \circ \cdots \circ \sigma_{k-1,k-2}$. Let $\widetilde{A}'$ be the strict transform of A' under $\sigma_{1,0} \circ \cdots \circ \sigma_{k-1,k-2}$. It follows that

$$(\sigma_{1,0} \circ \cdots \circ \sigma_{k-1,k-2})^*A' = \widetilde{A}' + a_2\widetilde{B}'_1 + a_3\widetilde{B}'_2 + \cdots + a_k\widetilde{B}'_{k-1},$$

and

$$\widetilde{E}'_i \cdot \widetilde{E}'_j = \widetilde{B}'_{i-1} \cdot \widetilde{B}'_{j-1}, \qquad \widetilde{D}' \cdot \widetilde{E}'_i = \widetilde{A}' \cdot \widetilde{B}'_{i-1},$$

for all $2 \leq i, j \leq k$. Put $G' = a_2\widetilde{B}'_1 + a_3\widetilde{B}'_2 + \cdots + a_k\widetilde{B}'_{k-1}$. We have

$$(F')^2 = (G')^2, \qquad \widetilde{D}' \cdot F' = \widetilde{A}' \cdot G'. \tag{3.3}$$

It follows that

$$\begin{aligned}(\widetilde{A}')^2 &= (\sigma^*_{k-1,0}A' - G')^2 \\ &= 2m'(n'-m') - 2\sigma^*_{k-1,0}A' \cdot G' + (G')^2 \\ &= 2m'(n'-m') - 2(\widetilde{A}' + G') \cdot G' + (G')^2 \\ &= 2m'(n'-m') - (G')^2 - 2\widetilde{A}' \cdot G'.\end{aligned}$$

By the assumption of the induction, we have

$$(G')^2 + 2\widetilde{A}' \cdot G' - 2m'(n'-m') = 0. \tag{3.4}$$

. By (3.2), (3.3) and (3.4), it follows that

$$(\widetilde{D}')^2 = 2m'n' - 2m'^2 - (F')^2 - 2\widetilde{D}' \cdot F' = 2m'(n'-m') - (G')^2 - 2\widetilde{A}' \cdot G' = 0.$$

Then we complete the induction. By the same way, we can show that $(\widetilde{D}'')^2 = 0$.

Now we show that $\widetilde{D}'$ is nef. Let $m' = dp$, $n' = dq$, where d is the greatest common divisor of m' and n'. Then it follows that

$$X_0^{m'}Y_0^{n'} - X_1^{m'}Y_1^{n'} = \prod_{i=0}^{d-1}\left(X_0^pY_0^q - (\varepsilon_d)^i X_1^pY_1^q\right),$$

where $\varepsilon_d = \exp((2\pi\sqrt{-1})/d)$. Let C_i be the irreducible divisor on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is defined by the polynomial $X_0^pY_0^q - (\varepsilon_d)^iX_1^pY_1^q$, and let $\widetilde{C}_i$ be the strict transform of C_i under $\pi_{k,0}$. By the above arguments, we have $(\widetilde{C}_0)^2 = 0$. Because $\widetilde{C}_0$ and $\widetilde{C}_i$, $1 \leq i \leq d-1$, are linearly equivalent, we have

$$\widetilde{C}_i \cdot \widetilde{D}' = (\widetilde{C}_i)^2 = (\widetilde{C}_0)^2 = 0.$$

Therefore $\widetilde{D}'$ is nef. By the same way, we can show that $\widetilde{D}''$ is nef. □

3.4 Proof of Theorem 0.0.3

Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the holomorphic map, let $\widetilde{f} : \mathbb{C} \to Z_k$ be the lift of f, and let $\widetilde{\nabla} = \pi^*_{k,0}\nabla$.

$$
\begin{array}{ccccccccc}
Z_0 & \xleftarrow{\pi_{1,0}} & Z_1 & \xleftarrow{\pi_{2,1}} & Z_2 & \xleftarrow{\pi_{3,2}} & \cdots & \xleftarrow{\pi_{k,k-1}} & Z_k \\
{\scriptstyle f}\uparrow & & & & & & & \nearrow {\scriptstyle \widetilde{f}} & \\
\mathbb{C} & & & & & & & &
\end{array}
$$

Let $\widetilde{\sigma}' \in \Gamma(Z_k, [\widetilde{D}'])$, $\widetilde{\sigma}'' \in \Gamma(Z_k, [\widetilde{D}''])$, $\widetilde{h}_{i,j} \in \Gamma(Z_k, [\widetilde{H}_{i,j}])$, $\widetilde{e}_i \in \Gamma(Z_k, \widetilde{E}_i)$ be the holomorphic section such that

$$(\widetilde{\sigma}') = \widetilde{D}', \qquad (\widetilde{\sigma}'') = \widetilde{D}'', \qquad (\widetilde{h}_{i,j}) = \widetilde{H}_{i,j}, \qquad (\widetilde{e}_i) = \widetilde{E}_i.$$

Lemma 3.4.1. *Assume that*

$$f(\mathbb{C}) \not\subset \operatorname{supp}\left(D' + D'' + \sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j}\right),$$

and assume that

$$f' \wedge \nabla_{f'} f' \not\equiv 0.$$

Then it follows that

$$\int_{|z|=r} \log^+ \frac{\|\widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'} \widetilde{f}'(z)\|_{\bigwedge^2 TZ_k}}{\|\widetilde{\sigma}'(\widetilde{f})\|_{[\widetilde{D}']} \|\widetilde{\sigma}''(\widetilde{f})\|_{[\widetilde{D}'']} \prod_{i=1}^{2}\prod_{j=0}^{1} \|\widetilde{h}_{i,j}(\widetilde{f})\|_{[\widetilde{H}_{i,j}]} \prod_{i=1}^{k} \|\widetilde{e}_i(\widetilde{f})\|_{[\widetilde{E}_i]}} \frac{d\theta}{2\pi}$$
$$= S_f(r).$$

Proof. For the convenience of the notation, we assume that D' and D'' are irreducible. Put

$$A = \widetilde{D}' + \widetilde{D}'' + \sum_{i=1}^{2}\sum_{j=0}^{1} \widetilde{H}_{i,j} + \sum_{i=1}^{k} \widetilde{E}_i,$$

and put

$$\xi(z) = \frac{\|\widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'} \widetilde{f}'(z)\|_{\bigwedge^2 TZ_k}}{\|\widetilde{\sigma}'(\widetilde{f})\|_{[\widetilde{D}']} \|\widetilde{\sigma}''(\widetilde{f})\|_{[\widetilde{D}'']} \prod_{i=1}^{2}\prod_{j=0}^{1} \|\widetilde{h}_{i,j}(\widetilde{f})\|_{[\widetilde{H}_{i,j}]} \prod_{i=1}^{k} \|\widetilde{e}_i(\widetilde{f})\|_{[\widetilde{E}_i]}}.$$

Note that A is simple normal crossing in Z_k.

Let

$$x \in \bigcup_{i=1}^{2}\bigcup_{j=0}^{1} \operatorname{supp} \widetilde{H}_{i,j} \cap \bigcup_{i=1}^{k} \operatorname{supp} \widetilde{E}_l.$$

By Lemma 3.2.4, there exists an affine open neighborhood U_x of x and local coordinate system z_x, w_x on U_x which satisfies the five conditions of Lemma 3.2.4. We put

$$V_x = U_x \setminus \operatorname{supp}(\widetilde{D}' + \widetilde{D}''),$$

and put

$$\widetilde{f}_1 = z_x \circ \widetilde{f}, \qquad \widetilde{f}_2 = w_x \circ \widetilde{f},$$

on $\widetilde{f}^{-1}(V_x)$. It follows that

$$\widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'} \widetilde{f}' = \left(\widetilde{f}_1' \widetilde{f}_2'' - \widetilde{f}_1'' \widetilde{f}_2' + \widetilde{f}_1' \widetilde{f}_2' \frac{\widetilde{f}_1'}{\widetilde{f}_1} - \widetilde{f}_1' \widetilde{f}_2' \frac{\widetilde{f}_2'}{\widetilde{f}_2} \right) \frac{\partial}{\partial z_x} \wedge \frac{\partial}{\partial w_x}.$$

on $\widetilde{f}^{-1}(V_x)$. Then it follows that

$$\xi(z) = \left(\frac{\widetilde{f}_1'}{\widetilde{f}_1} \frac{\widetilde{f}_2''}{\widetilde{f}_2} - \frac{\widetilde{f}_1''}{\widetilde{f}_1} \frac{\widetilde{f}_2'}{\widetilde{f}_2} + \left(\frac{\widetilde{f}_1'}{\widetilde{f}_1} \right)^2 \frac{\widetilde{f}_2'}{\widetilde{f}_2} - \frac{\widetilde{f}_1'}{\widetilde{f}_1} \left(\frac{\widetilde{f}_2'}{\widetilde{f}_2} \right)^2 \right) \Phi_x(f(z)), \tag{3.5}$$

on $\widetilde{f}^{-1}(V_x)$, where Φ_x is a smooth function on V_x.

Let

$$x \in \operatorname{supp} \widetilde{D}' \cap \left(\bigcup_{i=1}^{2} \bigcup_{j=0}^{1} \operatorname{supp} \widetilde{H}_{i,j} \cup \bigcup_{i=1}^{k} \operatorname{supp} \widetilde{E}_i \right)$$

$$\left(\text{or} \quad x \in \operatorname{supp} \widetilde{D}'' \cap \left(\bigcup_{i=1}^{2} \bigcup_{j=0}^{1} \operatorname{supp} \widetilde{H}_{i,j} \cup \bigcup_{i=1}^{k} \operatorname{supp} \widetilde{E}_i \right), \text{ respectively} \right)$$

By Lemma 3.2.4, there exists an affine open neighborhood U_x of x and local coordinate system z_x, w_x on U_x which satisfies the five condition of Lemma 3.2.4. Because D' and D'' are irreducible, it follows that

$$D'|_{U_x} = (z_x - 1) \qquad (\text{or } D''|_{U_x} = (z_x - 1), \text{ respectively}),$$

and $z_x(x) = 1, w_x(x) = 0$. We take $z_x' = z_x - 1$. Let V_x be an affine open subset of U_x such that

$$A|_{V_x} = (z_x') + (w_x),$$

and

$$\nabla|_{V_x} = d + \begin{pmatrix} -(dz_x')/(z_x' + 1) & 0 \\ 0 & -(dw_x)/w_x \end{pmatrix}.$$

We note that $z_x'(x) + 1 \neq 0$ on $\widetilde{f}^{-1}(V_x)$. We put

$$\widetilde{f}_1 = z_x' \circ \widetilde{f}, \qquad \widetilde{f}_2 = w_x \circ \widetilde{f},$$

on $\widetilde{f}^{-1}(V_x)$. It follows that

$$\xi(z) = \left(\frac{\widetilde{f}_1'}{\widetilde{f}_1} \frac{\widetilde{f}_2''}{\widetilde{f}_2} - \frac{\widetilde{f}_1''}{\widetilde{f}_1} \frac{\widetilde{f}_2'}{\widetilde{f}_2} - \frac{\widetilde{f}_1'}{\widetilde{f}_1} \left(\frac{\widetilde{f}_2'}{\widetilde{f}_2} \right)^2 \right) \Phi_x(f(z)) \tag{3.6}$$
$$+ \widetilde{f}_1' \frac{\widetilde{f}_1'}{\widetilde{f}_1} \frac{\widetilde{f}_2'}{\widetilde{f}_2} \Psi_x(f(z)),$$

on $\widetilde{f}^{-1}(V_x)$, where Φ_x and Ψ_x are smooth functions on V_x.

Let $x \in \operatorname{supp} \widetilde{D}' \cap \operatorname{supp} \widetilde{D}''$. There exists an affine open neighborhood V_x of x and holomorphic functions z_x, w_x on V_x such that

$$\widetilde{D}'|_{V_x} = (z_x), \qquad \widetilde{D}''|_{V_x} = (w_x),$$
$$A|_{V_x} = (z_x) + (w_x),$$

on V_x. It follows that dz_x and dw_x are linearly independent on V_x. We put

$$\widetilde{f}_1 = z_x \circ \widetilde{f}, \qquad \widetilde{f}_2 = w_x \circ \widetilde{f}.$$

By Lemma 2.2.5, there exist holomorphic functions g_0, g_1, h_0, h_1 on V_x such that

$$dz_x \cdot \nabla_{\widetilde{f}'} \widetilde{f}'(\gamma) = g_0(\widetilde{f}(\gamma))\widetilde{f}_1(\gamma) + g_1(\widetilde{f}(\gamma))\widetilde{f}_1'(\gamma) + \widetilde{f}_1''(\gamma),$$

for all $\gamma \in \widetilde{f}^{-1}(V_x)$, and

$$dw_x \cdot \nabla_{\widetilde{f}'} \widetilde{f}'(\gamma) = h_0(\widetilde{f}(\gamma))\widetilde{f}_2(\gamma) + h_1(\widetilde{f}(\gamma))\widetilde{f}_2'(\gamma) + \widetilde{f}_2''(\gamma),$$

for all $\gamma \in \widetilde{f}^{-1}(V_x)$. It follows that

$$\widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'} \widetilde{f}'$$
$$= \left[\widetilde{f}_1' \left(h_0(\widetilde{f})\widetilde{f}_2 + h_1(\widetilde{f})\widetilde{f}_2' + \widetilde{f}_2'' \right) - \widetilde{f}_2' \left(g_0(\widetilde{f})\widetilde{f}_1 + g_1(\widetilde{f})\widetilde{f}_1' + \widetilde{f}_2'' \right) \right] \frac{\partial}{\partial z_x} \wedge \frac{\partial}{\partial w_x}.$$

Then it follows that

$$\xi(z) = \Phi_{x,1}(\widetilde{f}) \frac{\widetilde{f}_1'}{\widetilde{f}_1} + \Phi_{x,2}(\widetilde{f}) \frac{\widetilde{f}_2'}{\widetilde{f}_2} \tag{3.7}$$
$$+ \Phi_{x,3}(\widetilde{f}) \frac{\widetilde{f}_1'}{\widetilde{f}_1} \frac{\widetilde{f}_2'}{\widetilde{f}_2} + \Phi_{x,4}(\widetilde{f}) \frac{\widetilde{f}_1'}{\widetilde{f}_1} \frac{\widetilde{f}_2''}{\widetilde{f}_2} + \Phi_{x,5}(\widetilde{f}) \frac{\widetilde{f}_1''}{\widetilde{f}_1} \frac{\widetilde{f}_2'}{\widetilde{f}_2},$$

on $\widetilde{f}^{-1}(V_x)$, where $\Phi_{x,1}, \ldots, \Phi_{x,5}$ are smooth functions on V_x.

Let $R = \{x \in Z_k |\ x$ is contained in two irreducible components of A $\}$. Note that R is a finite subset of Z_k. For $x \in R$, we take an affine open subset V_x and holomorphic functions z_x, w_x as above arguments. Then $\{V_x\}_{x\in R}$ is an open covering of Z_k. We take an open covering $\{V'_x\}_{x\in R}$ of Z_k such that $V'_x \subset V_x$ and V'_x is relatively compact in V_x. We take a partition of unity $\{\phi_x\}_{x\in R}$ which is subordinate to the covering $\{V'_x\}_{x\in R}$. Fix $x \in R$. Let $\widetilde{f_1} = z_x \circ \widetilde{f}$, $\widetilde{f_2} = w_x \circ \widetilde{f}$ be a holomorphic function on $\widetilde{f}^{-1}(V_x)$. Then $\widetilde{f_1}$ and $\widetilde{f_2}$ are extended to meromorphic functions on $\mathbb{C}$. By (3.5), (3.6) and (3.7), we have

$$\begin{aligned}
&\int_{|z|=r} \phi_i(\widetilde{f}(z)) \log^+ \xi(z) \frac{d\theta}{2\pi} \\
&\leq \int_{|z|=r} \Gamma(\widetilde{f}(z)) \frac{d\theta}{2\pi} + 4\sum_{i=1}^{2} \int_{|z|=r} \log^+ \frac{|\widetilde{f}_i'(z)|}{|\widetilde{f}_i(z)|} \frac{d\theta}{2\pi} \\
&+ \sum_{i=1}^{2} \int_{|z|=r} \log^+ \frac{|\widetilde{f}_i''(z)|}{|\widetilde{f}_i(z)|} \frac{d\theta}{2\pi} + \int_{|z|=r} \log^+ \phi_i(\widetilde{f}(z))|\widetilde{f}_1'(z)| \frac{d\theta}{2\pi},
\end{aligned}$$

where Γ is a bounded smooth function on Z_k. By using the lemma on logarithmic derivative, it follows that

$$\int_{|\gamma|=r} \log^+ \frac{|\widetilde{f}_i'(\gamma)|}{|\widetilde{f}_i(\gamma)|} \frac{d\theta}{2\pi} \leq S_{\widetilde{f}}(r).$$

It holds that

$$\begin{aligned}
\int_{|z|=r} \log^+ \phi_i(\widetilde{f}(z))|\widetilde{f}_1'(z)| \frac{d\theta}{2\pi} &= \frac{1}{2} \int_{|z|=r} \log^+ \phi_i(\widetilde{f}(z))^2 |\widetilde{f}_1'(z)|^2 \frac{d\theta}{2\pi} \\
&\leq \frac{1}{2} \int_{|z|=r} \log^+ \|\widetilde{f}'(z)\|^2_{TZ_k} \frac{d\theta}{2\pi} + O(1),
\end{aligned}$$

where $\|\cdot\|_{TZ_k}$ is a hermitian metric of TZ_k. By the lemma on logarithmic derivative and the concavity of log, we have that

$$
\begin{aligned}
&\frac{1}{2}\int_{|z|=r} \log^+ \|\widetilde{f}'(z)\|^2_{TZ_k}\frac{d\theta}{2\pi} \\
&\leq \frac{1}{2}\int_{|z|=r} \log\{\|\widetilde{f}'(z)\|^2_{TZ_k}+1\}\frac{d\theta}{2\pi} \\
&\leq \frac{1}{2}\log\left(1+\int_{|z|=r} \|\widetilde{f}'(z)\|^2_{TZ_k}\frac{d\theta}{2\pi}\right)+O(1) \\
&\leq \frac{1}{2}\log\left(1+\frac{1}{2\pi r}\frac{d}{dr}\int_{|z|\leq r} \|\widetilde{f}'(z)\|^2_{TZ_k}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)+O(1) \\
&\leq \frac{1}{2}\log\left(1+\frac{1}{2\pi r}\left(\int_{|z|\leq r} \|\widetilde{f}'(z)\|^2_{TZ_k}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)^{1+\delta}\right)+O(1)\| \\
&= \frac{1}{2}\log\left(1+\frac{r^\delta}{2\pi}\left(\frac{d}{dr}\int_1^r \frac{dt}{t}\int_{|z|\leq r} \|\widetilde{f}'(z)\|^2_{TZ_k}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)^{1+\delta}\right)+O(1)\| \\
&\leq \frac{1}{2}\log\left(1+\frac{r^\delta}{2\pi}\left(\int_1^r \frac{dt}{t}\int_{|z|\leq r} \|\widetilde{f}'(z)\|^2_{TZ_k}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)^{(1+\delta)^2}\right)+O(1)\| \\
&\leq S_f(r),
\end{aligned}
$$

where δ is any positive number.

Because $\sum_{x\in R}\phi_x(\widetilde{f})=1$ on $\mathbb{C}$, it follows that

$$
\int_{|z|=r}\log^+\xi(z)\frac{d\theta}{2\pi}=\sum_{x\in R}\int_{|z|=r}\phi_x(\widetilde{f}(z))\log^+\xi(z)\frac{d\theta}{2\pi}\leq S_f(r).
$$

□

The following lemma is useful.

Lemma 3.4.2. *It follows that*

$$
\sum_{i=1}^{2}\sum_{j=0}^{1}\pi^*_{k,0}H_{i,j}=\sum_{i=1}^{2}\sum_{j=0}^{1}\widetilde{H}_{i,j}+\sum_{i=1}^{k}\pi^*_{k,i}E_i+\sum_{i=1}^{k}\widetilde{E}_i.
$$

Proof. Let the divisor $H_{i,j,l}$ on Z_l be the strict transform of $H_{i,j}$ under $\pi_{l,0}$, and let $E_{i,l}$, $i\leq l$, be the strict transform of E_i under $\pi_{l,i}$, where $E_{l,l}=E_l$.

We show

$$
\sum_{i=1}^{2}\sum_{j=0}^{1}\pi^*_{l,0}H_{i,j}=\sum_{i=1}^{2}\sum_{j=0}^{1}H_{i,j,l}+\sum_{i=1}^{l}\pi^*_{l,i}E_i+\sum_{i=1}^{l}E_{i,l},
$$

by induction over l. If $l = 1$, we have

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{1,0}^{*}H_{i,j} = \sum_{i=1}^{2}\sum_{j=0}^{1}H_{i,j,1} + 2E_i.$$

Therefore the statement of the induction holds for $l = 1$. Assume that the statement holds for $l-1$, $1 < l \leq k$. Let C_i $i = 1, 2, \ldots, r$ be irreducible divisor on Z_{l-1} such that

$$\operatorname{supp}\left(\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l-1,0}^{*}H_{i,j}\right) = \bigcup_{i=1}^{r}\operatorname{supp} C_i.$$

There exist positive integers $a_1, a_2, \ldots, a_r$ such that

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l-1,0}^{*}H_{i,j} = \sum_{i=1}^{r}a_iC_i.$$

By the assumption of the induction, we have

$$\sum_{i=1}^{l}\pi_{l,i}^{*}E_i = \sum_{i=1}^{r}(a_i - 1)C_i.$$

Let $x \in Z_{l-1}$ be one of the points of the center of $\pi_{l,l-1}$, and let F_l be the irreducible component of E_l such that $\pi_{l,l-1}(\operatorname{supp} F_l) = x$. Assume that $x \in \operatorname{supp} C_p \cap \operatorname{supp} C_q$ for $1 \leq p < q \leq r$. Then the coefficients of F_l in $\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l,0}^{*}H_{i,j}$ is $a_p + a_q$, and the coefficient of F_l in $\sum_{i=1}^{l-1}\pi_{l,i}^{*}E_i$ is $a_p + a_q - 2$. Therefore we have

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l,0}^{*}H_{i,j} - \sum_{i=1}^{l}\pi_{l,i}^{*}E_i = \sum_{i=1}^{2}\sum_{j=0}^{1}H_{i,j,l} + \sum_{i=1}E_{i,l}.$$

This complete the induction, and the lemma follows. □

Proof Theorem 0.0.3. We put $W_{\widetilde{\nabla}}(\widetilde{f}) = \widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'}\widetilde{f}'$. We denote by $\operatorname{ord}_z g$ the order of zero of g at z, where g is a holomorphic section of a line bundle on a neighborhood of z. By (3.5), (3.6) and (3.7) in Lemma 3.4.1, it follows that

$$\begin{aligned}
&\operatorname{ord}_z\left(\widetilde{\sigma}'(\widetilde{f})\widetilde{\sigma}''(\widetilde{f})\prod_{i=1}^{2}\prod_{j=0}^{1}\widetilde{h}_{i,j}(\widetilde{f})\prod_{i=1}^{k}\widetilde{e}_i(\widetilde{f})\right) - \operatorname{ord}_z\left(W_{\widetilde{\nabla}}(\widetilde{f})\right)\\
&\leq \min\{\operatorname{ord}_z\widetilde{\sigma}'(\widetilde{f}), 2\} + \min\{\operatorname{ord}_z\widetilde{\sigma}''(\widetilde{f}), 2\}\\
&+ 2\sum_{i=1}^{2}\sum_{j=0}^{1}\min\{\operatorname{ord}_z\widetilde{h}_{i,j}(\widetilde{f}), 1\} + 2\sum_{i=1}^{k}\min\{\operatorname{ord}_z\widetilde{e}_i(\widetilde{f}), 1\}.
\end{aligned}$$

Therefore it follows that

$$T_{\widetilde{f}}(r, K_{Z_k}) + T_{\widetilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) + \sum_{i=1}^{2}\sum_{j=0}^{1} T_{\widetilde{f}}(r, \widetilde{H}_{i,j}) + \sum_{i=1}^{k} T_{\widetilde{f}}(r, \widetilde{E}_i) \quad (3.8)$$

$$\leq N_2(r, \widetilde{f}^*\widetilde{D}') + N_2(r, \widetilde{f}^*\widetilde{D}'') + 2\sum_{i=1}^{2}\sum_{j=0}^{1} N_1(r, \widetilde{f}^*\widetilde{H}_{i,j})$$
$$+ 2\sum_{1\leq i\leq k} N_1(r, \widetilde{f}^*\widetilde{E}_i) + S_f(r),$$

where K_{Z_k} is the canonical line bundle of Z_k. The canonical line bundle of Z_k is equal to

$$\pi_{k,0}^* K_{\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})} + \pi_{k,1}^* E_1 + \pi_{k,2}^* E_2 + \cdots + E_k.$$

By Lemma 3.4.2, it follows that

$$\begin{aligned} -T_f(r, K_{\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})}) &= T_f(r, \mathcal{O}(2,2)) = \sum_{i=1}^{2}\sum_{j=0}^{1} T_{\widetilde{f}}(r, \pi_{k,0}^* H_{i,j}) \quad (3.9) \\ &= \sum_{i=1}^{2}\sum_{j=0}^{1} T_{\widetilde{f}}(r, \widetilde{H}_{i,j}) + \sum_{i=1}^{k} T_{\widetilde{f}}(r, \pi_{k,i}^* E_i) \\ &\quad + \sum_{i=1}^{k} T_{\widetilde{f}}(r, \widetilde{E}_i) \end{aligned}$$

By (3.8), (3.9), it follows that

$$T_{\widetilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \leq N_2(r, \widetilde{f}^*\widetilde{D}') + N_2(r, \widetilde{f}^*\widetilde{D}'')$$
$$+ 2\sum_{i=1}^{2}\sum_{j=0}^{1} N_1(r, \widetilde{f}^*\widetilde{H}_{i,j}) + 2\sum_{i=1}^{k} N_1(r, \widetilde{f}^*\widetilde{E}_i) + S_f(r).$$

By Lemma 3.2.1 and Lemma 3.3.1, Theorem 0.0.3 follows. □

Corollary 3.4.3. *Let $f : \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be a non-constant map. Assume that*

$$f(\mathbb{C}) \not\subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid C_0 X_0^{r_1} Y_0^{r_2} - C_1 X_1^{r_1} Y_1^{r_2} = 0\},$$

for all $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ and all $(C_0, C_1) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$, and assume that there exists no $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$a \log f_1 + b \log f_2 = (\text{constant}),$$

on $\mathbb{C}$. Then it follows that

$$T_{\widetilde{f}}(r,[\widetilde{D}]) \leq N_2(r,f^*D') + N_2(r,f^*D'') + S_f(r).$$

Proof. Because $N_2(r,\widetilde{f}^*\widetilde{H}_{i,j}) = 0$ and $N_2(r,\widetilde{f}^*\widetilde{E}_i) = 0$, we have the corollary. □

Example 3.4.4. Let D', D'' be the divisor which are defined by the polynomials

$$X_0Y_0 - X_1Y_1, \qquad X_0Y_1 - X_1Y_0.$$

Then

$$D_1' + D_1'' + \sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,1} + E_1,$$

is normal crossing in Z_1. Therefore $\widetilde{D}' = D_1', \widetilde{D}'' = D_1''$. Let $E_{(0,0)}, E_{(0,\infty)}, E_{(\infty,0)}, E_{(\infty,\infty)}$ be irreducible components of E_1 such that

$$\pi_{1,0}(\operatorname{supp} E_{(0,0)}) = ([0:1],[0:1]), \qquad \pi_{1,0}(\operatorname{supp} E_{(0,\infty)}) = ([0:1],[1:0]),$$

$$\pi_{1,0}(\operatorname{supp} E_{(\infty,0)}) = ([1:0],[0:1]), \qquad \pi_{1,0}(\operatorname{supp} E_{(\infty,\infty)}) = ([1:0],[1:0]).$$

Let $f = (f_1, f_2) : \mathbb{C} \to Z_0$ be a non-constant holomorphic map, and let $\widetilde{f} : \mathbb{C} \to Z_1$ be the lift of f. It follows that

$$T_{\widetilde{f}}(r,[\widetilde{D}']) = T_{\widetilde{f}}(r,[\pi_{1,0}^*D']) - T_{\widetilde{f}}(r,[E_{(0,\infty)}]) - T_{\widetilde{f}}(r,[E_{(\infty,0)}]),$$

and

$$T_{\widetilde{f}}(r,[\pi_{1,0}^*D']) = T_f(r,\mathcal{O}(1,1)) = T(r,f_1) + T(r,f_2),$$

where

$$T(r,f_i) = \int_{|z|=r} \log^+|f_i|\frac{d\theta}{2\pi} + N(r,(f_i)_\infty),$$

for $i = 1,2$. By the first main theorem, we have

$$T_{\widetilde{f}}(r,E_{(0,\infty)}) = N(r,\widetilde{f}^*E_{(0,\infty)}) + m_{\widetilde{f}}(r,E_{(0,\infty)}),$$

$$T_{\widetilde{f}}(r,E_{(\infty,0)}) = N(r,\widetilde{f}^*E_{(\infty,0)}) + m_{\widetilde{f}}(r,E_{(\infty,0)}).$$

It holds that

$$m_{\widetilde{f}}(r,E_{(0,\infty)}) = \int_{|z|=r} \log^+ \frac{1}{\sqrt{|f_1|^2 + |f_2^{-1}|^2}}\frac{d\theta}{2\pi},$$

and

$$m_{\widetilde{f}}(r,E_{(\infty,0)}) = \int_{|z|=r} \log^+ \frac{1}{\sqrt{|f_1^{-1}|^2 + |f_2|^2}}\frac{d\theta}{2\pi}.$$

By these equations, we have

$$T_{\widetilde{f}}(r,\widetilde{D}') = N(r,(f_1)_\infty) + N(r,(f_2)_\infty) - N(r,\widetilde{f}^*E_{(0,\infty)}) - N(r,\widetilde{f}^*E_{(\infty,0)})$$
$$+ \int_{|z|=r} \left(\log^+|f_1| + \log^+|f_2|\right)\frac{d\theta}{2\pi}$$
$$- \int_{|z|=r} \left(\log^+\frac{1}{\sqrt{|f_1|^2+|f_2^{-1}|^2}} + \log^+\frac{1}{\sqrt{|f_1^{-1}|^2+|f_2|^2}}\right)\frac{d\theta}{2\pi}$$

Let $f_1 = P(z), f_2 = \exp z$, where $P(z)$ is a polynomial of degree p on $\mathbb{C}$. Then $T(r,f_1) = p\log r + O(1)$, and $T(r,f_2) = |r| + O(1)$. Because

$$\log^+\frac{1}{\sqrt{|f_1|^2+|f_2^{-1}|^2}} \leq \log^+\frac{1}{|f_1|},$$

it follows that

$$m_{\widetilde{f}}(r,E_{(0,\infty)}) \leq T(r,f_1^{-1}) = T(r,f_1) + O(1) = p\log|r| + O(1).$$

Therefore we have

$$m_{\widetilde{f}}(r,E_{(0,\infty)}) = o(r).$$

By the same arguments, we have

$$m_{\widetilde{f}}(r,E_{(\infty,0)}) = o(r).$$

Then it holds that

$$T_{\widetilde{f}}(r,\widetilde{D}') = r + o(r).$$

Let D' and D'' be divisors on $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ which are defined by the polynomials

$$X_0^mY_0^n - X_1^mY_1^n, \qquad X_0^nY_1^m - X_1^nY_0^m.$$

(i,e,. $m = m' = n''$ and $n = n' = m''$.) We have the following theorem.

Theorem 3.4.5. *Let $f:\mathbb{C}\to\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map. Let $\widetilde{f}:\mathbb{C}\to Z_k$ be the lift of f. Assume that*

$$f(\mathbb{C}) \not\subset \{([X_0:X_1],[Y_0:Y_1])\in\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})\,|\,C_0X_0^{r_1}Y_0^{r_2} - C_1X_1^{r_1}Y_1^{r_2} = 0\},$$

for all $(r_1,r_2)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$ and all $(C_0,C_1)\in\mathbb{C}\times\mathbb{C}\setminus\{(0,0)\}$, and assume that there exist no holomorphic functions g_1,g_2 on $\mathbb{C}$ and no $(a,b)\in\mathbb{C}\times\mathbb{C}\setminus\{(0,0)\}$ such that

$$f = (\exp g_1, \exp g_2):\mathbb{C}\to\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C}),$$

$$ag_1 + bg_2 = \text{(constant)},$$

on $\mathbb{C}$. Then it follows that

$$\left(1 - \frac{4}{m+n}\right) T_{\tilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \leq N_2(r, \tilde{f}^*\widetilde{D}') + N_2(r, \tilde{f}^*\widetilde{D}'') + S_f(r).$$

Proof. Let $a_1 = \min\{m, n\}$. It follows that

$$\pi_{1,0}^*(D' + D'') = D_1' + D_1'' + a_1 E_1,$$

on Z_1, where D_1' and D_1'' are proper transform of D' and D'' under $\pi_{1,0}$. Let $a_2 = \min\{\max\{m, n\} - a_1, a_1\} \leq a_1$. It follows that

$$\pi_{2,0}^*(D' + D'') = D_2' + D_2'' + a_2 E_2 + a_1 \pi_{2,1}^* E_1,$$

on Z_2, where D_2' and D_2'' are proper transform of D' and D'' under $\pi_{2,0}$. Repeating this process, there exist positive integers $a_3 \cdots, a_k$ such that

$$\pi_{k,0}^*(D' + D'') = \widetilde{D}' + \widetilde{D}'' + \sum_{i=1}^{k} a_i \pi_{k,i}^* E_i.$$

Without loss of generality, we may assume that $m \leq n$. Then it holds that $m \geq a_1 \geq a_2 \geq \cdots \geq a_k$. It follows that

$$T_{\tilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \geq T_{\tilde{f}}(r, \pi_{k,0}^* \mathcal{O}(m+n, m+n)) - m \sum_{i=1}^{k} T_{\tilde{f}}(r, \pi_{k,i}^* E_i).$$

By Lemma 3.4.2, we have

$$T_{\tilde{f}}(r, \pi_{k,0}^* \mathcal{O}(2,2)) = \sum_{i=1}^{2} \sum_{j=0}^{1} T_{\tilde{f}}(r, \widetilde{H}_{i,j}) + \sum_{i=1}^{k} T_{\tilde{f}}(r, \pi_{k,i}^* E_i) + \sum_{i=1}^{k} T_{\tilde{f}}(r, \widetilde{E}_i)$$

Then we have

$$\begin{aligned} &T_{\tilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \\ &\geq \frac{m+n}{2} \left(T_{\tilde{f}}(r, \pi_{k,0}^* \mathcal{O}(2,2)) - \sum_{i=1}^{k} T_{\tilde{f}}(r, \pi_{k,i}^* E_i) \right) + \left(\frac{m+n}{2} - m \right) \sum_{i=1}^{k} T_{\tilde{f}}(r, \pi_{k,i}^* E_i) \\ &\geq \frac{m+n}{2} \left(\sum_{i=1}^{2} \sum_{j=0}^{1} T_{\tilde{f}}(r, \widetilde{H}_{i,j}) + \sum_{i=1}^{k} T_{\tilde{f}}(r, \widetilde{E}_i) \right). \end{aligned}$$

By Theorem 0.0.3, it follows that

$$T_{\tilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \leq N_2(r, \tilde{f}^*\widetilde{D}') + N_2(r, \tilde{f}^*\widetilde{D}'')) + \frac{4}{m+n} T_{\tilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) + S_f(r).$$

Then the theorem follows. □

Corollary 3.4.6. *Assume the hypothesis of Theorem 3.4.5, and assume that*

$$f(\mathbb{C}) \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \setminus \operatorname{supp}(D' + D'').$$

If $m + n \geq 5$, *then it follows that* $f(\mathbb{C}) \subset \operatorname{supp} H_{i,j}$ *for* $i = 1, 2$ *and* $j = 0, 1$.

Proof. Assume that $f(\mathbb{C})$ is not contained in the support of $\sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j}$. By Theorem 3.4.5, f satisfies the following condition (i) or condition (ii):

(i) $\quad f(\mathbb{C}) \subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid X_0^{r_1} Y_0^{r_2} - C_1 X_1^{r_1} Y_1^{r_2} = 0\}$,

for some $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ and some $C_1 \in \mathbb{C} \setminus \{(0)\}$.

(ii) There exist holomorphic functions g_1, g_2 on $\mathbb{C}$ and $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$f = (\exp g_1, \exp g_2) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}),$$

$$ag_1 + bg_2 = (\text{constant}),$$

on $\mathbb{C}$.

If f satisfies condition (i), without loss of generality, we may assume that $r_1 > 0, r_2 \geq 0$. Assume that $r_2 > 0$. Let R be an irreducible component of $\{X_0^{r_1} Y_0^{r_2} - C X_1^{r_1} Y_1^{r_2} = 0\}$. Then $([0 : 1], [1 : 0]), ([1 : 0], [0 : 1]) \in \operatorname{supp} R \cap \operatorname{supp} D'$, and $\operatorname{supp} R \cap \operatorname{supp} D''$ contains at least one point which is not $([0 : 1], [1 : 0])$ nor $([1 : 0], [0 : 1])$. Therefore the holomorphic map

$$f : \mathbb{C} \to \operatorname{supp} R \setminus \operatorname{supp}(D' + D'')$$

is a constant map.
Assume that $r_2 = 0$. We have

$$f(\mathbb{C}) \subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}(\mathbb{C}) \times \mathbb{P}(\mathbb{C}) \mid X_0^{r_1} - C X_1^{r_1} = 0\}.$$

Let S be an irreducible component of $\{X_0^{r_1} - C X_1^{r_1} = 0\}$. Because $m + n \geq 5$, m or n is more than 2, it follows that $\operatorname{supp} S \cap \operatorname{supp} D'$ or $\operatorname{supp} S \cap \operatorname{supp} D''$ contains at least three points. Then f is a constant map.

If f satisfies condition (ii), it is easy to see that f is a constant map. $\square$

Remark 3.4.7. Let $x_{1,0} = ([0 : 1], [1 : 1]), x_{1,1} = ([1 : 0], [1 : 1]), x_{2,0} = ([1 : 1], [0 : 1]), x_{2,1} = ([1 : 1], [1 : 0]) \in Z_0 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $W = Z_0 \setminus \operatorname{supp} D' \cup \operatorname{supp} D''$, and let $W^* = W \setminus \{x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}\}$. By Corollary 3.4.6, there exist no non-constant holomorphic maps from $\mathbb{C}$ to W^*.

Let $i : W^* \to W$ be the inclusion map, and let d_{W^*}, d_W be the Kobayashi pseudo distance of W^*, W (see Noguchi-Ochiai [NO]). By Proposition 1.3.14. of [NO], we have $i^* d_W = d_{W*}$. Therefore W^* is Brody hyperbolic but not Kobayashi hyperbolic.

4 Kobayashi hyperbolic imbeddings into toric varieties

4.1 Introduction

We fix a free module $N = \mathbb{Z}^r$ of rank r over the ring $\mathbb{Z}$ of rational integers. Let $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) = \mathbb{Z}^r$ be the dual $\mathbb{Z}$-module of N. Let

$$\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$$

be the canonical $\mathbb{Z}$-bilinear pairing. Let $T_N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = (\mathbb{C}^*)^r$ be the r-dimensional algebraic torus. Let S be a finite subset of M. Let D be a divisor on T_N which is defined by a Laurent polynomial

$$\sum_{I=(i_1,\dots,i_r)\in S} a_I z_1^{i_1} \cdots z_r^{i_r},$$

where $a_I \in \mathbb{C}^*$.

By the main theorem of [Nog4], every entire curve $f : \mathbb{C} \to T_N \setminus \operatorname{supp} D$ is *algebraically degenerate*, i.e., the image of f is contained in a proper subvariety of T_N. In this paper, we deal with Kobayashi hyperbolicity of $T_N \setminus \operatorname{supp} D$, where $f : \mathbb{C} \to T_N \setminus \operatorname{supp} D$ is *most degenerate* to a constant. Moreover, we give a characterization of D such that $T_N \setminus \operatorname{supp} D$ is Kobayashi hyperbolically imbedded into a toric variety.

Now, we recall some basic facts about Kobayashi hyperbolic imbedding. The concept of Kobayashi hyperbolic imbedding was introduced in Kobayashi [Ko1] to obtain a generalization of the big Picard theorem. The classical big Picard theorem is stated as follows:

If a function f holomorphic on the punctured disk in $\mathbb{C}$ omits $\{0, 1\} \subset \mathbb{C}$, then f can be extended to a meromorphic function on the full disk.

Recall that $\mathbb{C} \setminus \{0, 1\}$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^1(\mathbb{C})$. The following generalization of the big Picard theorem obtained in [Ki]:

Let X be an m-dimensional complex manifold and let A be a closed complex subspace of X consisting of hypersurfaces with normal crossing singularities. Let Z be a complex space and Y be a complex subspace of Z. If Y is Kobayashi hyperbolically imbedded into Z, then every holomorphic map $h : X \setminus A \to Y$ extends to a holomorphic map $\tilde{h} : X \to Z$.

Kobayashi hyperbolic imbedding is also closely related to the structure of a family of holomorphic mappings (see, e.g., [Ko2] Chap. 6, [Nog3]).

It is a famous conjecture proposed by S. Kobayashi that $\mathbb{P}^n(\mathbb{C}) \setminus Y$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^n(\mathbb{C})$ if Y is a generic hypersurface of degree $d \geq 2n+1$. H. Fujimoto proved in [Fu] that Kobayashi conjecture is true if Y is a union of hyperplanes in $\mathbb{P}^n(\mathbb{C})$, i.e., $\mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^{d} H_i$ is Kobayashi hyperbolically imbedded into $\mathbb{P}^n(\mathbb{C})$ if $H_1, \ldots, H_d$ are hyperplanes in general position and $d \geq 2n+1$. As a special case of theorem, we obtain the following:

Corollary 4.1.1. *Let $H_1, \ldots, H_{n+1}$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position, and let Y be a general hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$. If $d \geq n$, then*

$$\mathbb{P}^n(\mathbb{C}) \setminus \left(\bigcup_{i=1}^{n+1} H_i \cup Y \right)$$

is Kobayashi hyperbolically imbedded into $\mathbb{P}^n(\mathbb{C})$.

If an algebraic divisor D on T_N is a union of translations of subtori in T_N, it is much more elementary to prove the existence of a toric variety into which $T_N \setminus \operatorname{supp} D$ is Kobayashi hyperbolically imbedded. Let D_i, $i = 1, \ldots, q$ be an algebraic divisor on T_N which is defined by

$$z_1^{a_{i,1}} \cdots z_r^{a_{i,r}} - c_i = 0,$$

where $(a_{i,1}, \ldots, a_{i,r}) \in M = \mathbb{Z}^r$ and $c_i \in \mathbb{C}^*$ for $i = 1, \ldots, q$. Put $a_i = (a_{i,1}, \ldots, a_{i,r}) \in M$. Assume that $M_{\mathbb{R}}$ is generated by $a_1, \ldots, a_q$, i.e.,

$$M_{\mathbb{R}} = \{k_1 a_1 + \cdots + k_q a_q \in M_{\mathbb{R}} \mid (k_1, \ldots, k_q) \in \mathbb{R}^q\}.$$

Then the following theorem holds.

Theorem 4.1.2. *There exists a toric projective variety X such that $T_N \setminus \operatorname{supp} \left(\sum_{i=1}^{q} D_i\right)$ is Kobayashi hyperbolically imbedded into X.*

4.2 Brody hyperbolicity of $T_N \setminus \operatorname{supp} D$

In this section, we prove the Brody hyperbolicity of $T_N \setminus \operatorname{supp} D$. First, we show the following lemma.

Lemma 4.2.1. *Let $l \in \mathbb{N}$. Let $S_1, \ldots, S_{l+1}$ be subsets of $\mathbb{Z}^l$ such that $\sharp(S_j) < \infty$ for $j = 1, \ldots, l+1$. Let $Q_1(z_1, \ldots, z_l)$, $\ldots$, $Q_{l+1}(z_1, \ldots, z_l)$ be Laurent polynomials of $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_l, z_l^{-1}]$ such that*

$$Q_j(z_1, \ldots, z_l) = \sum_{I=(i_1,\ldots,i_l)\in S_j} a_{j,I} z_1^{i_1} \cdots z_l^{i_l},$$

for $j = 1, \ldots, l+1$. Let $d_j = \sharp(S_j)$, and let $N = \sum_{j=1}^{l+1} d_j$. Then $Q_1, \ldots, Q_{l+1}$ have no common zero point in $(\mathbb{C}^*)^l$ for general $[\ldots : a_{1,I} : \ldots : a_{2,I} : \ldots : a_{l+1,I} : \ldots] \in \mathbb{P}^{N-1}(\mathbb{C})$.

Proof. Let Z be the subvariety in $(\mathbb{C}^*)^l \times \mathbb{P}^{N-1}(\mathbb{C})$ defined by

$$\{((z_1, \ldots, z_l), [\ldots : a_{j,I} : \ldots]) \in (\mathbb{C}^*)^l \times \mathbb{P}^{N-1}(\mathbb{C}) \mid \sum_{I=(i_1,\ldots,i_l)\in S_j} a_{j,I} z_1^{i_1} \cdots z_l^{i_l} = 0 \text{ for } j = 1, \ldots, l+1\}.$$

For $x \in (\mathbb{C}^*)^l$, we denote the fiber of Z over x by Z_x. Then

$$\dim Z_x \leq \sum_{i=1}^{l+1} d_i - (l+1) - 1 = N - l - 2.$$

It follows that $\dim Z \leq (N - l - 2) + l = N - 2$. Let $p : (\mathbb{C}^*)^l \times \mathbb{P}^{N-1}(\mathbb{C}) \to \mathbb{P}^{N-1}(\mathbb{C})$ be the projection. Then $\dim p(Z) \leq N - 2$, and $p(Z)$ is contained in a proper subvariety of $\mathbb{P}^{N-1}(\mathbb{C})$. If $[\ldots : a_{1,I} : \ldots : a_{2,I} : \ldots : a_{l+1,I} : \ldots] \in \mathbb{P}^{N-1}(\mathbb{C})$ is not contained in $p(Z)$, then $Q_1, \ldots, Q_{l+1}$ have no common zero point in $(\mathbb{C}^*)^l$. □

Lemma 4.2.2. *Let S be a finite subset in M. Assume the following condition.*

> *Let $H \in \mathcal{H}_S$, and let $\phi_H : M_{\mathbb{R}} \to M_{\mathbb{R}}/H$ be the canonical morphism. Then $\sharp(\phi(S)) \geq r + 1$ for all $H \in \mathcal{H}_S$, where $\sharp(\phi(S))$ is the number of the elements in $\{\phi(y) \in M_{\mathbb{R}}/H \mid y \in S\}$.*

Then $T_N \setminus \operatorname{supp} D$ and $\operatorname{supp} D$ contain no translation of positive dimensional subtorus in T_N for a general divisor D of the linear system $|\{z_1^{i_1} z_2^{i_2} \cdots z_r^{i_r}\}_{(i_1,i_2,\ldots,i_r)\in S}|$ on T_N.

Proof. Let $H \in \mathcal{H}_S$, and let $(h_1, \ldots, h_r) \in M^r$ be a $\mathbb{Z}$-basis of M such that $h_1, \ldots, h_{r-1}$ generate an $\mathbb{R}$-vector subspace H. We denote $h_i = (h_{i,1}, \ldots, h_{i,r}) \in M$ for $i = 1, \ldots, r$. Let $u_i := z_1^{h_{i,1}} \cdots z_r^{h_{i,r}}$. It follows that $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}] = \mathbb{C}[u_1, u_1^{-1}, \ldots, u_r, u_r^{-1}]$. Let $[\ldots : a_I : \ldots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$, and let

$$\sum_{I=(i_1,\ldots,i_r)\in S} a_I z_1^{i_1} \cdots z_r^{i_r},$$

be a Laurent polynomial in $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}]$. Then there exist non-zero Laurent polynomials $Q_1(u_1, \ldots, u_{r-1}), \ldots, Q_t(u_1, \ldots, u_{r-1})$ in $\mathbb{C}[u_1, u_1^{-1}, \ldots, u_{r-1}, u_{r-1}^{-1}]$ and integers $d_1 < d_2 < \cdots < d_t$ such that

$$\sum_{I=(i_1,\ldots,i_r)\in S} a_I z_1^{i_1} \cdots z_r^{i_r} = \sum_{i=1}^{t} Q_i(u_1, \ldots, u_{r-1}) u_r^{d_i}.$$

By the condition of the lemma, it follows that $t \geq r+1$. Because of Lemma 4.2.1, there exists a proper subvariety Y_H in $\mathbb{P}^{\sharp(S)-1}(\mathbb{C})$ which satisfies the following:

If $[\ldots : a_I : \ldots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$ *is not contained in* Y_H*, then* $Q_{j_1}, Q_{j_2}, \ldots, Q_{j_r}$ *have no common zero point in* $(\mathbb{C}^*)^{r-1}$ *for any* $1 \leq j_1 < j_2 < \ldots < j_r \leq t$.

Since the number of the elements in $\mathcal{H}_S$ is finite, $\bigcup_{H\in\mathcal{H}_S} Y_H$ is a subvariety of $\mathbb{P}^{\sharp(S)-1}(\mathbb{C})$.

Fix $[\ldots : a_I : \ldots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$ which is not contained in $\bigcup_{H\in\mathcal{H}_S} Y_H$. Let D be the divisor of T_N defined by the Laurent polynomial

$$\sum_{I=(i_1,\ldots,i_r)\in S} a_I z_1^{i_1} \cdots z_r^{i_r}.$$

Let Y be a translation of subtorus in T_N such that $1 \leq \dim Y \leq r-1$. Let k be the codimension of Y. There exist primitive elements $b_1 = (b_{1,1}, \ldots, b_{1,r}), \ldots, b_k = (b_{k,1}, \ldots, b_{k,r}) \in M$ and $c_1, \ldots, c_k \in \mathbb{C}^*$ such that

$$S = \{(z_1, \ldots, z_r) \in (\mathbb{C}^*)^r \mid z_1^{b_{j,1}} \cdots z_r^{b_{j,r}} = c_j \text{ for } j = 1, \ldots, k\}.$$

Let W be the subspace in $M_\mathbb{R}$ which is generated by $b_1, \ldots, b_k$. Let W' be the largest subspace of W generated by elements in $\mathcal{L}_S$. Define the canonical morphisms $\phi_W : M_\mathbb{R} \to M_\mathbb{R}/W$, $\phi_{W'} : M_\mathbb{R} \to M_\mathbb{R}/W'$, $\psi : M_\mathbb{R}/W' \to M_\mathbb{R}/W$. By the definition of W', ψ is injective on $\phi_{W'}(S)$. Without loss of generality, we may assume that $b_1, \ldots, b_l$ is a basis of W' where $l \leq k$. There exist $b_{k+1} = (b_{k+1,1}, \ldots, b_{k+1,r}), \ldots, b_r = (b_{r,1}, \ldots, b_{r,r}) \in M$ such that $b_1, \ldots, b_r$ be a basis of M. Put $u_1 = z_1^{b_{1,1}} \cdots z_r^{b_{1,r}}, \dots, u_r = z_1^{b_{r,1}} \cdots z_r^{b_{r,r}}$. There exist the canonical isomorphisms

$$M/(W' \cap M) \simeq \mathbb{Z}b_{l+1} + \cdots + \mathbb{Z}b_r \simeq \mathbb{Z}^{r-l},$$
$$M/(W \cap M) \simeq \mathbb{Z}b_{k+1} + \cdots + \mathbb{Z}b_r \simeq \mathbb{Z}^{r-k},$$

where $\mathbb{Z}b_{l+1} + \cdots + \mathbb{Z}b_r$ (resp. $\mathbb{Z}b_{k+1} + \cdots + \mathbb{Z}b_r$) is the $\mathbb{Z}$-module generated by $b_{l+1}, \dots, b_r$ (resp. $b_{k+1}, \dots, b_r$). Therefore, we may assume that

$$\phi_{W'}(S) \subset \mathbb{Z}b_{l+1} + \cdots + \mathbb{Z}b_r \simeq \mathbb{Z}^{r-l},$$

and

$$\phi_W(S) \subset \mathbb{Z}b_{k+1} + \cdots \mathbb{Z}b_r \simeq \mathbb{Z}^{r-k}.$$

Let $Q_{J'}(u_1, \dots, u_l)$ (resp. $R_J(u_1, \dots, u_k)$) be a Laurent polynomial of $\mathbb{C}[u_1, u_1^{-1}, \dots, u_l, u_l^{-1}]$ (resp. $\mathbb{C}[u_1, u_1^{-1}, \dots, u_k, u_k^{-1}]$) such that

$$\begin{aligned}\sum_{I=(i_1,\dots,i_r)\in S} a_I z_1^{i_1} \cdots z_r^{i_r} &= \sum_{J'=(j'_{l+1},\dots,j'_r)\in\phi_{W'}(S)} Q_{J'}(u_1, \dots, u_l) u_{l+1}^{j'_{l+1}} \cdots u_r^{j'_r} \\ &= \sum_{J=(j_{k+1},\dots,j_r)\in\phi_W(S)} R_J(u_1, \dots, u_k) u_{k+1}^{j_{k+1}} \cdots u_r^{j_r}.\end{aligned}$$

We take $H \in \mathcal{H}_S$ such that $W' \subset H$. Because $[\dots : a_I : \dots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$ is not contained in Y_H, there exist at least two elements in $\{J'\}_{J'\in\phi_{W'}(S)}$ such that $Q_{J'}(c_1, \dots, c_l) \neq 0$. Since ψ is a one-to-one correspondence between $\phi_{W'}(S)$ and $\phi_W(S)$, there exist at least two elements in $\{J\}_{J\in\phi_W(S)}$ such that $R_J(c_1, \dots, c_k) \neq 0$. It follows that

$$D|_Y : \sum_{J=(j_{k+1},\dots,j_r)\in\phi_W(S)} R_J(c_1, \dots, c_k) u_{k+1}^{j_{k+1}} \cdots u_r^{j_r} = 0,$$

since $Y = \{(u_1, \dots, u_r) \in (\mathbb{C}^*)^r \mid u_1 = c_1, \dots, u_k = c_k\}$. Hence $Y \cap \operatorname{supp} D \neq \emptyset$ and $Y \not\subset \operatorname{supp} D$. This completes the proof. □

The following theorem is proved in [Nog4].

Theorem 4.2.3 ([Nog4, Main Theorem, Proposition 1.8]). *Let D be an algebraic effective reduced divisor of a semi-Abelian variety A over the complex number field $\mathbb{C}$ (D may be the zero-divisor). Let $f : \mathbb{C} \to A \setminus \operatorname{supp} D$ be an arbitrary holomorphic mapping. Then the Zariski closure B of the image of f in A is a translate of a semi-Abelian subvariety of A, and $B \cap \operatorname{supp} D = \emptyset$.*

By Lemma 4.2.2 and Theorem 4.2.3, the following theorem holds.

Theorem 4.2.4. *Let S be a finite subset in M. Assume the following condition.*

> *Let $H \in \mathcal{H}_S$, and let $\phi_H : M_{\mathbb{R}} \to M_{\mathbb{R}}/H$ be the canonical morphism. Then $\sharp(\phi_H(S)) \geq r+1$ for all $H \in \mathcal{H}_S$, where $\sharp(\phi_H(S))$ is the number of the elements in $\{\phi_H(y) \in M_{\mathbb{R}}/H \,|\, y \in S\}$.*

Then $T_N \setminus \operatorname{supp} D$ and $\operatorname{supp} D$ have no non-constant holomorphic map from $\mathbb{C}$ for a general divisor D of the linear system $|\{z_1^{i_1} z_2^{i_2} \cdots z_r^{i_r}\}_{(i_1, i_2, \ldots, i_r) \in S}|$ on T_N.

4.3 Proof of Theorem 0.0.4

Let P be an integral convex polytope in $M_{\mathbb{R}}$ such that $\dim P = r$. Let D be an algebraic effective reduced divisor on T_N. There exists the toric projective variety X which is associated to P. We denote the closure of D in X by $\overline{D}$. There exist T_N-invariant irreducible (Weil) divisors $A_1, \ldots, A_k$ in T_N such that $X \setminus \bigcup_{i=1}^k A_i = T_N$.

Lemma 4.3.1. *Assume that the following two conditions are satisfied.*

(a) There exists neither non-constant holomorphic map

$$f : \mathbb{C} \to T_N \setminus \operatorname{supp} D,$$

nor non-constant map

$$f : \mathbb{C} \to \operatorname{supp} D.$$

(b) For any partition of indices $I \cup J = \{1, 2, \ldots, k\}$, there exists neither non-constant holomorphic map

$$f : \mathbb{C} \to \bigcap_{i \in I} A_i \setminus \Big(\bigcup_{j \in J} A_j \cup \operatorname{supp} \overline{D}\Big),$$

nor non-constant holomorphic map

$$f : \mathbb{C} \to \Big(\bigcap_{i \in I} A_i \cap \operatorname{supp} \overline{D}\Big) \setminus \bigcup_{j \in J} A_j.$$

Then $T_N \setminus \operatorname{supp} D$ is Kobayashi hyperbolically imbedded in X.

Proof. Assume $T_N \setminus \operatorname{supp} D$ is not Kobayashi hyperbolically imbedded in X. Then there exists a non-constant holomorphic map $f : \mathbb{C} \to X$ which satisfies the following condition (see Theorem (3.6.5) of Kobayashi [Ko2]):

For any $R > 0$, there exists a sequence of holomorphic maps $f_i : D_R \to T_N \setminus \operatorname{supp} D$ for $i = 1, 2, \ldots$, such that $\{f_i\}_{i=1,2,\ldots}$ converges uniformly on any compact sets in D_R to f. Here $D_R = \{z \in \mathbb{C} \mid |z| < R\}$.

Let Δ be the fan of the toric projective variety X. Assume that $f(z) \in A_i$ for some i and $z \in \mathbb{C}$. There exists an r-dimensional convex cone $\sigma \in \Delta$ such that

$$f(z) \in U_\sigma := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M],$$

where $\sigma = \{m \in M_\mathbb{R} \mid \langle x, m \rangle \geq 0 \text{ for all } x \in \sigma\}$. There exist $h_1, \ldots, h_p \in \mathbb{C}[\sigma^\vee \cap M]$ such that

$$A_i \cap U_\sigma = \{h_1 = 0\} \cap \cdots \cap \{h_p = 0\},$$

and $\{h_j = 0\} \cap T_N = \emptyset$ for all $j = 1, \ldots, p$. Let B be a sufficiently small neighborhood of z. Because $f_j(B)$ is contained in $U_\sigma \cap T_N$ for large j, it follows that $h_l \circ f_j \not\equiv 0$ on B for $l = 1, \ldots, p$ and large j. Then $h_l \circ f \equiv 0$ on B for $l = 1, \ldots, p$ by Hurwitz theorem. It follows that $f(\mathbb{C})$ is contained in A_i. Hence, $f(\mathbb{C}) \cap A_j = \emptyset$ or $f(\mathbb{C}) \subset A_j$ for all $j = 1, \ldots, k$. By the same argument, it follows that $f(\mathbb{C}) \cap \operatorname{supp} \overline{D} = \emptyset$ or $f(\mathbb{C}) \subset \operatorname{supp} \overline{D}$. This contradicts the assumption of the lemma. □

Proof of Theorem 0.0.4. Let $[\ldots : a_I : \ldots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$, and let D be the divisor on T_N defined by the Laurent polynomial

$$\sum_{I=(i_1,\ldots,i_r)\in S} a_I z_1^{i_1} \cdots z_r^{i_r} = 0.$$

We show that X and D satisfy the conditions $(a), (b)$ of Lemma 4.3.1 for general $[\ldots : a_I : \ldots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$. By Theorem 4.2.4, the condition (a) of Lemma 4.3.1 holds for general $[\ldots : a_I : \ldots] \in \mathbb{P}^{\sharp(S)-1}(\mathbb{C})$. Let I, J be a partition of $\{1, 2, \ldots, k\}$. Let Z be an irreducible component of $\bigcap_{i\in I} A_i$. Because there exists the one-to-one correspondence between the faces of P and the T_N-invariant irreducible subvarieties in X (see §2.3 of Oda [Od]), there exists the face τ of P which correspondes to Z. Let l be the dimension of $V_{\tau\cap P}$. Fix a basis $b_1 = (b_{1,1}, \ldots, b_{1,r}), \ldots, b_l = (b_{l,1}, \ldots, b_{l,r}) \in M$ of $\mathbb{Z}$-module $V_{\tau\cap P} \cap M$. Then there is the canonical isomorphism $V_{\tau\cap P} \cap M \simeq \mathbb{Z}^l$. Let $u_1 = z_1^{b_{1,1}} \cdots z_r^{b_{1,r}}, \ldots, u_l = z_1^{b_{l,1}} \cdots z_r^{b_{l,r}}$. Then $Z \setminus \bigcup_{j\in J} A_j$ is biholomorphic to $\operatorname{Spec} \mathbb{C}[u_1, u_1^{-1}, \ldots, u_l, u_l^{-1}]$. Let $x \in \tau \cap S$. It follows that $\tau \cap S - x \in V_{\tau\cap S} \cap M \simeq \mathbb{Z}^r$. Hence $(\overline{D} \setminus \bigcup_{j\in J} A_j)|_Z$ is defined by the Laurent polynomial

$$\sum_{I'=(i_1,\ldots,i_l)\in\tau\cap S-x} c_{I'} u_1^{i_1} \cdots u_l^{i_l},$$

where $c_{I'}$ is equal to some element of $\{a_I\}_{I\in V}$. By the assumption of Theorem 0.0.4 and Theorem 4.2.4, there exists neither non-constant holomorphic map

$$f : \mathbb{C} \to \bigcap_{i\in I} A_i \setminus \Big(\bigcup_{j\in J} A_j \cup \operatorname{supp} \overline{D}\Big),$$

nor non-constant holomorphic map

$$f : \mathbb{C} \to \Big(\bigcap_{i\in I} A_i \cap \operatorname{supp} \overline{D}\Big) \setminus \bigcup_{j\in J} A_j.$$

Hence the condition (b) of Lemma 4.3.1 holds. This completes the proof. □

The following proposition gives examples of P and S which satisfy the conditions of Theorem 0.0.4.

Proposition 4.3.2. *Let S be a finite subset of M, and let P be an integral convex polytope in $M_{\mathbb{R}}$ such that $S \subset P$. Let ϱ be any one-dimensional face of P. If $\sharp(\varrho\cap S) \geq r+1$, then P satisfies the conditions* (i), (ii) *of Theorem 0.0.4.*

Proof. Let τ be a positive dimensional face of P. It is easy to see that τ satisfies the condition (i) of Theorem 0.0.4. Let $H \in \mathcal{H}_{\tau\cap S}$. There exists a one-dimensional face ϱ of τ such that $\varrho - x \not\subset H$ for $x \in \tau \cap S$. Then it follows that

$$\sharp(\phi_H(\tau \cap S - x)) \geq \sharp(\phi_H(\varrho \cap S - x)) \geq r + 1 \geq \dim \tau + 1,$$

where $\phi_H : E_{\tau\cap S} \to E_{\tau\cap S}/H$ is the canonical morphism. This completes the proof. □

Now we prove Corollary 4.1.1. Let $d \geq r$. Let

$$P = \{(x_1, \ldots, x_r) \in M_{\mathbb{R}} \mid \sum_{i=1}^{r} x_i \leq d,\ x_i \geq 0 \text{ for } i = 1, \ldots, r\},$$

and let

$$S = \{(x_1, \ldots, x_r) \in M \mid \sum_{i=1}^{r} x_i \leq d,\ x_i \geq 0 \text{ for } i = 1, \ldots, r\}.$$

Then the toric variety X defined by P is r-dimensional complex projective space $\mathbb{P}^r(\mathbb{C})$, and elements in the linear system $|\{z_1^{i_1} z_2^{i_2} \cdots z_r^{i_r}\}_{(i_1,i_2,\ldots,i_r)\in S}|$ are d-dimensional hypersurfaces of $\mathbb{P}^r(\mathbb{C})$. It is easy to verify that S and P satisfy the assumption of Proposition 4.3.2.

Example 4.3.3. Let $S = \{(0,0),(2,0),(0,2),(1,2),(2,1)\}$. Let D be a divisor on $\mathbb{C}^* \times \mathbb{C}^* = \operatorname{Spec}\mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$ defined by the following polynomial:

$$a_{00} + a_{20}z_1^2 + a_{02}z_2^2 + a_{12}z_1z_2^2 + a_{21}z_1^2z_2,$$

where $[a_{00} : a_{20} : a_{02} : a_{12} : a_{21}]$ is a generic point of $\mathbb{P}^4(\mathbb{C})$. The following cases satisfy the conditions of Theorem 0.0.4.

(1) Take $P = \{(z_1, z_2) \in M_{\mathbb{R}} \mid z_1 + z_2 \leq 3, z_1 \geq 0, z_2 \geq 0\}$. Then X is the two-dimensional complex projective space $\mathbb{P}^2(\mathbb{C})$.

(2) Take $P = \{(z_1, z_2) \in M_{\mathbb{R}} \mid z_1 + z_2 \leq 3, z_1 \geq 0, z_2 \geq 0, z_2 \leq 2\}$. Then X is the Hirzebruch surface F_1.

(3) Take $P = \{(z_1, z_2) \in M_{\mathbb{R}} \mid z_1 \geq 0, z_2 \geq 0, z_1 \leq 2, z_2 \leq 2\}$. Then X is the product space of the one-dimensional projective spaces $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$.

(4) Take $P = \{(z_1, z_2) \in M_{\mathbb{R}} \mid z_1 + z_2 \leq 3, z_1 \geq 0, z_2 \geq 0, z_1 \leq 2, z_2 \leq 2\}$. Then X is a one-point blowing-up of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$.

4.4 Proof of Theorem 4.1.2

In this section, we deal with an algebraic divisor on T_N which is a union of translations of subtori. Let D_i, $i = 1, \ldots, q$ be the algebraic divisor on T_N which is defined by

$$z_1^{a_{i,1}} \cdots z_r^{a_{i,r}} - c_i = 0,$$

where $(a_{i,1}, \ldots, a_{i,r}) \in M = \mathbb{Z}^r$ and $c_i \in \mathbb{C}^*$ for $i = 1, \ldots, q$. Put $a_i = (a_{i,1}, \ldots, a_{i,r}) \in M$. Assume that $\mathbb{R}$-vector space $M_{\mathbb{R}}$ is generated by $a_1, \ldots, a_q$, i.e.,

$$M_{\mathbb{R}} = \{k_1a_1 + \cdots + k_qa_q \in M_{\mathbb{R}} \mid (k_1, \ldots, k_q) \in \mathbb{R}^q\}.$$

Let $I = (\delta_1, \ldots, \delta_q) \in \{-1, +1\}^q$ where $\delta_j = -1$ or $(+1)$. Let

$$C_I = \mathbb{R}_{\geq 0}(\delta_1 a_1) + \cdots + \mathbb{R}_{\geq 0}(\delta_q a_q),$$

where

$$\mathbb{R}_{\geq 0}(\delta_1 a_1) + \cdots + \mathbb{R}_{\geq 0}(\delta_q a_q) = \{r_1\delta_1a_1 + \cdots + r_q\delta_qa_q \in M_{\mathbb{R}} \mid r_1 \geq 0, \ldots, r_q \geq 0\}.$$

Then C_I is a convex rational polyhedral cone. We put

$$\Pi = \{C_I \subset M_{\mathbb{R}} \mid I \in \{-1, +1\}^q\},$$

and we put

$$\Pi' = \{C \in \Pi \mid C \text{ is strongly convex}\}.$$

The strong convexity of cone means that it contains no nonzero subspace of $M_{\mathbb{R}}$. Let

$$\Delta(r) = \{C^{\vee} \subset N_{\mathbb{R}} \mid C \in \Pi'\},$$

where

$$C^{\vee} = \{v \in N_{\mathbb{R}} \mid \langle v, m\rangle \geq 0 \text{ for all } m \in C\}.$$

Then $\Delta(r)$ is a finite set of r-dimensional strongly convex rational polyhedral cones in $N_{\mathbb{R}}$. Here the dimension of a cone σ is the dimension of the smallest $\mathbb{R}$-subspace of $N_{\mathbb{R}}$ containing σ. Let Δ be the collection of all faces of cones in $\Delta(r)$, i.e.,

$$\Delta = \{\sigma \subset N_{\mathbb{R}} \mid \text{ there exists } \tau \in \Delta(r) \text{ such that } \sigma \text{ is a face of } \tau\ \}.$$

Because elements in $\Delta(r)$ are strongly convex rational polyhedral cones, Δ is a collection of strongly convex rational polyhedral cones.

Lemma 4.4.1. *The collection Δ is a finite and complete fan in N, i.e., Δ satisfies the following conditions:*

(i) Every face of any $\sigma \in \Delta$ is contained in Δ.

(ii) For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

(iii) Δ is a finite set and the support $|\Delta| = \bigcup_{\sigma\in\Delta} \sigma$ coincides with the entire $N_{\mathbb{R}}$.

Proof. (i) is clear by the definition.

Let $\sigma \in \Delta(r)$, and let $\tau \in \Delta$. We show that $\sigma \cap \tau$ is a face of σ. By the definition, there exists $\sigma' \in \Delta(r)$ and $m \in \sigma'^{\vee}$ such that $\tau = \sigma' \cap \{m\}^{\perp}$, where $\{m\}^{\perp} = \{v \in N_{\mathbb{R}} \mid \langle v, m\rangle = 0\}$. There exist $1 \leq j_1 < \cdots < j_l \leq q$ such that

$$\sigma \cap \sigma' = \sigma \cap \bigcap_{k=1}^{l} \{a_{j_k}\}^{\perp}.$$

It follows that $\sigma \cap \sigma'$ is a face of σ. Because $m \in (\sigma \cap \sigma')^{\vee}$, it follows that $\sigma \cap \tau = (\sigma \cap \sigma') \cap \{m\}^{\perp}$ is a face of $\sigma \cap \sigma'$. Hence $\sigma \cap \tau$ is a face of σ.

Now we show the condition (ii) of the lemma. Let $\tau, \tau' \in \Delta$. There exists $\sigma \in \Delta(r)$ such that τ is a face of σ. Then $\sigma \cap \tau'$ is a face of σ by the above argument. It follows that $\tau \cap \tau' = \tau \cap (\sigma \cap \tau')$ is a face of σ. Hence $\tau \cap \tau'$ is a face of τ. In the same way, $\tau \cap \tau'$ is a face of τ'.

We show the condition (iii) of the lemma. The finiteness of Δ is obvious. For any $v \in N_{\mathbb{R}}$, there exists $(\delta_1, \dots, \delta_q) \in \{-1, +1\}^q$ such that $\langle v, \delta_i a_i \rangle \geq 0$ for $i = 1, \dots, q$, and $C := \{s_1\delta_1 a_1 + \cdots + s_q\delta_q a_q \in M_{\mathbb{R}} \mid s_1 \geq 0, \dots, s_q \geq 0\}$ is strongly convex. Then $\sigma := C^{\vee} \in \Delta$ and $v \in \sigma$. □

Let X be a toric variety associated to the fan Δ. Then X is compact (see Theorem 1.11. of [Od]).

A real valued function $h : |\Delta| \to \mathbb{R}$ is said to be a Δ-linear support function if it is $\mathbb{Z}$-valued on $N \cap |\Delta|$ and is linear on each $\sigma \in \Delta$. Namely, there exists $l_\sigma \in M$ for each $\sigma \in \Delta$ such that $h(n) = \langle l_\sigma, n \rangle$ for $n \in \sigma$ and that $\langle l_\sigma, n \rangle = \langle l_\tau, n \rangle$ holds for $n \in \tau < \sigma$. Here $\tau < \sigma$ means that τ is a face of σ. Assume that, for any $\sigma \in \Delta(r)$ and any $n \in N_{\mathbb{R}}$, we have $\langle l_\sigma, n \rangle \geq h(n)$ with the equality holding if and only if $n \in \sigma$. In this case, h is said to be strictly upper convex with respect to Δ.

Lemma 4.4.2. *X is projective.*

Proof. Define Δ-linear support function h by

$$h(n) = -\sum_{j=1}^{q} |\langle n, a_j \rangle|,$$

for $n \in N_{\mathbb{R}}$. Let $C \in \Pi'$, and let $(\delta_1, \dots, \delta_q) \in \{-1, +1\}^q$ such that $C = \{\mathbb{R}_{\geq 0}\delta_1 a_1 + \cdots + \mathbb{R}_{\geq 0}\delta_q a_q\}$. Then $l_\sigma = -(\delta_1 a_1 + \cdots + \delta_q a_q)$ for $\sigma = C^{\vee}$. Hence $\langle n, l_\sigma \rangle \geq h(n)$ for $n \in N_{\mathbb{R}}$ and the equality holds if and only if $n \in \sigma$. Therefore h is a strictly upper convex with respect to Δ. Then X is a toric projective variety (see Corollary 2.14. of [Od]). □

Let $A_1, \dots, A_k$ be T_N-invariant irreducible (Weil) divisors of X such that $X \setminus \bigcup_{i=1}^{k} A_i = T_N$. Let $\overline{D}_i$ be the closure of D_i in X for $i = 1, \dots, q$. In the same way of the proof of Lemma 4.3.1, the following lemma holds.

Lemma 4.4.3. *Assume that the following two conditions are satisfied.*

(a') There exists no non-constant holomorphic map

$$f : \mathbb{C} \to T_N \setminus \bigcup_{i=1}^{q} \operatorname{supp} D_i.$$

(b') Let $I \subset \{1, \dots, k\}$, $J \subset \{1, \dots, q\}$ such that $I \neq \emptyset$ or $J \neq \emptyset$. Let $I' = \{1, \dots, k\} \setminus I$, $J' = \{1, \dots. q\} \setminus J$. Then there exists no non-constant holomorphic map

$$f : \mathbb{C} \to \left(\bigcap_{j \in I} \operatorname{supp} A_j \cap \bigcap_{j \in J} \operatorname{supp} \overline{D}_j \right) \setminus \left(\bigcup_{j \in I'} \operatorname{supp} A_j \cup \bigcup_{j \in J'} \operatorname{supp} \overline{D}_j \right).$$

Then $T_N \setminus \bigcup_{i=1}^q \operatorname{supp} D_i$ *is Kobayashi hyperbolically imbedded in* X.

Now we prove Theorem 4.1.2

Proof of Theorem 4.1.2. We show that X and $D_1, \ldots, D_q$ satisfy the condition (a'), (b') of Lemma 4.4.3.

Let

$$f : \mathbb{C} \to T_N \setminus \bigcup_{j=1}^{q} \operatorname{supp} D_j,$$

be a holomorphic map. There exist holomorphic functions $g_1, \ldots, g_r$ on $\mathbb{C}$ such that

$$f = (\exp g_1, \ldots, \exp g_r) : \mathbb{C} \to T_N \setminus \bigcup_{i=1}^{q} \operatorname{supp} D_i.$$

It holds that

$$\exp(a_{j,1}g_1 + \cdots + a_{j,r}g_r) - c_j \neq 0,$$

for all $j = 1, \ldots, q$ on $\mathbb{C}$. By the small Picard theorem, $\exp(a_{j,1}g_1 + \cdots + a_{j,r}g_r) - c_j$ is a constant function. Hence $a_{j,1}g_1 + \cdots + a_{j,r}g_r$ is constant. Since $a_1, \ldots, a_q$ generate $\mathbb{R}$-vector space $M_{\mathbb{R}} = \mathbb{R}^r$, it follows that $g_1, \ldots, g_r$ are constant functions. Therefore X and $D_1, \ldots, D_q$ satisfy the condition (a') of Lemma 4.4.3.

Let $I \subset \{1, \ldots, k\}$, $J \subset \{1, \ldots, q\}$ such that $I \neq \emptyset$ or $J \neq \emptyset$. Let $I' = \{1, \ldots, k\} \setminus I$, $J' = \{1, \ldots q\} \setminus J$. Let

$$f : \mathbb{C} \to \left(\bigcap_{j \in I} \operatorname{supp} A_j \cap \bigcap_{j \in J} \operatorname{supp} \overline{D}_j \right) \setminus \left(\bigcup_{j \in I'} \operatorname{supp} A_j \cup \bigcup_{j \in J'} \operatorname{supp} \overline{D}_j \right),$$

be a holomorphic map. It follows that $f(\mathbb{C}) \subset A_i$ (resp. $f(\mathbb{C}) \subset \overline{D}_i$) or $f(\mathbb{C}) \cap A_i = \emptyset$ (resp. $f(\mathbb{C}) \cap \overline{D}_i = \emptyset$) for $i = 1, \ldots, k$ (resp. for $i = 1, \ldots, q$). We show that f is a constant map. There exists an element of $\sigma \in \Delta(r)$ such that $f(\mathbb{C}) \subset U_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. There exist $(\delta_1, \ldots, \delta_q) \in \{-1, +1\}^q$ such that

$$\sigma^\vee = \mathbb{R}_{\geq 0}\delta_1 a_1 + \cdots + \mathbb{R}_{\geq 0}\delta_q a_q.$$

We take primitive elements $b_1 = (b_{1,1}, \ldots, b_{1,r}), \ldots, b_q = (b_{q,1}, \ldots, b_{q,r})$ of M such that $d_i b_i = \delta_i a_i$ where d_i is a positive integer, i.e., $\mathbb{R}a_i \cap M = \mathbb{Z}b_i$ and $\mathbb{R}_{\geq 0}\delta_i a_i = \mathbb{R}_{\geq 0} b_i$. There exist $b_{q+1} = (b_{q+1,1}, \ldots, b_{q+1,r}), \ldots, b_l = (b_{l,1}, \ldots, b_{l,q}) \in M$ such that $\sigma^\vee \cap M = \mathbb{Z}_{\geq 0} b_1 + \cdots + \mathbb{Z}_{\geq 0} b_l$ where l is a positive integer. Let $u_i = z_1^{b_{i,1}} \cdots z_r^{b_{i,r}}$ for $i = 1, \ldots, l$. Then $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[u_1, \ldots, u_l]$. Since $f(\mathbb{C}) \subset A_i$ or $f(\mathbb{C}) \cap A_i = \emptyset$ for $i = 1, \ldots, k$, it follows that $u_i \circ f \equiv 0$ or $u_i \circ f \neq 0$ on $\mathbb{C}$ for $i = 1, \ldots, q$. Since $f(\mathbb{C}) \subset \overline{D}_i$ or $f(\mathbb{C}) \cap \overline{D}_i = \emptyset$ for

$i = 1, \dots, q$, it follows that $u_i^{d_i} \circ f \equiv c_i^{\delta_i}$ or $u_i^{d_i} \circ f \neq c_i^{\delta_i}$ on $\mathbb{C}$ for $i = 1, \dots, q$. By the small Picard theorem, $u_i \circ f$ is a constant function for $i = 1, \dots, q$. Since $b_j \in \mathbb{Q}_{\geq 0} b_1 + \cdots + \mathbb{Q}_{\geq 0} b_q$ for $j > q$, there exist relations such that

$$u_j^{\rho_j} = u_1^{\mu_{j,1}} \cdots u_q^{\mu_{j,q}} \quad \text{for} \quad j > q,$$

where ρ_j is a positive integer and $\mu_{j,i}$ is a non-negative integer. Hence $u_j \circ f$ is constant function for $j > q$, and f is a constant map. Therefore $X, D_1, \dots, D_q$ satisfy the condition (b') of Lemma 4.4.3. $T_N \setminus \operatorname{supp} D$ is Kobayashi hyperbolically imbedded into X by Lemma 4.4.3. □

Corollary 4.4.4. *Let $D(1) := \{x \in \mathbb{C} \mid |x| < 1\}$, and let $D(1)^* := D(1) \setminus \{0\}$. Let f, g be holomorphic functions on $D(1)^*$ such that $f \neq 0, g \neq 0$, $f \neq g$ and $f \neq g^{-1}$ on $D(1)^*$. Then f and g are extended to meromorphic functions on $D(1)$.*

Proof. Let D and D' be the divisors on $(\mathbb{C}^*)^2 = \operatorname{Spec} \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$ defined by $z_1 z_2 - 1 = 0$ and $z_1 z_2^{-1} - 1 = 0$. Then (f, g) is a holomorphic map from $D(1)^*$ to $(\mathbb{C}^*)^2 \setminus \operatorname{supp}(D + D')$. By Theorem 4.1.2, there exists toric projective variety X such that $(\mathbb{C}^*)^2 \setminus \operatorname{supp}(D + D')$ is Kobayashi hyperbolically imbedded into X. By a generalization of the big Picard theorem in [Ki], (f, g) are extended to a holomorphic map $F : D(1) \to X$. Since X and $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ are birational, there exist meromorphic functions $\tilde{f}, \tilde{g}$ on $D(1)$ such that the holomorphic map $(\tilde{f}, \tilde{g}) : D(1) \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is a extension of (f, g). □

Corollary 4.4.4 is the classical big Picard theorem if $g = 1$.

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論文の内容の要旨

論文題目: Entire curves in projective algebraic varieties (射影代数多様体内の正則曲線について)

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1 背景

R. ネヴァンリンナは, 複素平面上の有理型関数の値分布を調べるために 1925 年の論文においてネヴァンリンナ理論を創始した ([Ne]). ネヴァンリンナ理論では第一主要定理と第二主要定理と呼ばれる 2 つの主定理があり, 第一主要定理は特性関数, 個数関数, 接近関数と呼ばれる 3 つの関数の関係を表したもので, 第二主要定理は個数関数の大きさを特性関数の大きさで漸近的に評価したものである. ネヴァンリンナ理論はその後高次元化され, 複素多様体の中の正則曲線 (複素平面から複素多様体への正則な写像) を研究する有力な手段となっている. 第二主要定理が確立されている主な場合として次のようなものがある:

(a) 複素射影空間 $\mathbb{P}^n(\mathbb{C})$ における一般の位置にある超平面に対する第二主要定理 (H. Cartan [Ca]).

(b) 準アーベル多様体とその因子に対する第二主要定理 (Noguchi-Winkelmann-Yamanoi [NWY1], [NWY2]).

(c) 一般型曲面の正則葉層構造に沿った正則曲線に対する第二主要定理 (M. McQuillan [Mc]).

また小林双曲性も, 正則曲線を調べる際には重要な概念である. 小林双曲的多様体は非定値な正則曲線を持たないような複素多様体であり, 小林双曲性については次の小林予想が有名である.

(1) 複素射影空間 $\mathbb{P}^n(\mathbb{C})$ において次数 $d \geq 2n-1$ の一般な超曲面は小林双曲的である.

(2) 複素射影空間 $\mathbb{P}^n(\mathbb{C})$ において次数 $d \geq 2n+1$ の一般な超曲面の補集合は $\mathbb{P}^n(\mathbb{C})$ に小林双曲的に埋め込まれている.

小林予想については, 任意次元で存在が K. Masuda-J. Noguchi [MN] で示された. 小林予想 (1) について $n=3$ のときは J.-P. Demailly, J. El Goul [DE], M. Păun [Pa] などで詳しく研究されて部分的な解決をしている. また小林予想 (2) についても $n=2$ のときは G. Dethloff, S. Lu [DL], E. Rousseau [Ro] により部分的に解決している.

本論文の構成は以下のとおりである. 第一節ではジェット束, ネヴァンリンナ理論や小林双曲性の基本的な定義や性質について述べる. これらは後の節で何度も使われる基本的な概念である. 第二節では J.-P. Demailly [De] で導入された射影的有理型接続を使ってある第二主要定理を証明する. $\mathbb{P}^n(\mathbb{C})$ での特別な超曲面や一般の位置にない超平面の第二主要定理を扱っている. 第三節では有理型接続を使って $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ に対する第二主要定理を証明する. 第四節では代数的トーラスから因子を除いた空間が, あるトーリック多様体に小林双曲的に埋め込まれる十分条件を与える.

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2 複素射影空間の超曲面に対する第二主要定理

H. Cartan [Ca] により複素射影空間 $\mathbb{P}^n(\mathbb{C})$ の一般の位置にある超平面に対する第二主要定理が証明されて以来, 超曲面に対する第二主要定理を証明することが大きな問題となった. 論文の第二節では特殊な超曲面に対する第二主要定理を証明する.

$s_0, \dots, s_n$ を $\mathbb{C}[X_0, \dots, X_n]$ の d 次斉次多項式で, ある正整数 $l_0, \dots, l_n$ に対して

$$X_0^{d-l_0}|s_0, \dots, X_n^{d-l_n}|s_n,$$

$$\det\left(\frac{\partial s_j}{\partial X_k}\right)_{0\le j,k\le n} \not\equiv 0,$$

を満たすとする. このとき $\mathbb{C}^{n+1}$ 上の 有理型接続 $\widetilde{\nabla} = d + \widetilde{\Gamma}$ を

$$\sum_{0\le\lambda\le n} \frac{\partial s_\kappa}{\partial X_\lambda} \widetilde{\Gamma}^\lambda_{i,j} = \frac{\partial^2 s_\kappa}{\partial X_i \partial X_j}$$

で定義する. この接続は $\mathbb{P}^n(\mathbb{C})$ 上に射影的有理型接続 ∇ を誘導する (J.-P. Demailly [De] 参照). 我々は, この ∇ を使って次の第二主要定理を証明する.

定理 2.1 (Theorem 0.0.1) $\sigma_1, \dots, \sigma_q$ を線形系 $|\{s_0, \dots, s_n\}|$ の元で $\sum_{k=1}^q \sigma_k$ は単純正規交叉的とする. H を $\mathbb{P}^n(\mathbb{C})$ の超平面束とする. また $f: \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ を正則曲線で, その像が線形系 $|\{s_0, \dots, s_n\}|$ の元の台や, ∇ の極の集合に入っていないものとする. このとき次が成り立つ.

$$\begin{aligned}&\left(q - \frac{n+1}{d} - \frac{1}{2d}(n-1)n(n+1+l_0+\cdots+l_n)\right) T_f(r, dH)\\ &\le \sum_{1\le i\le q} N_n(r, f^*\sigma_i) + S_f(r),\end{aligned}$$

ここで, $S_f(r)$ は増大度の小さい項を表し, $(0,\infty)$ の測度有限な例外集合の外で次の評価をみたす.

$$S_f(r) = O(\log r + \log T_f(r, H)).$$

また X を射影代数多様体として $\widetilde{X} \to X$ をその固有改変 (proper modification) としたときに X 上の射影的有理型接続を引き戻すことで, $\widetilde{X}$ 上にも射影的有理型接続が誘導される. これによって一般の位置にない超平面に対する次の第二主要定理を得た.

定理 2.2 (Theorem 0.0.2) $S_1, \dots, S_q$ を $\mathbb{P}^2(\mathbb{C})$ の m-準一般の位置にある超平面とする. $\{x_1, \dots, x_p\}$ を $S_1, \dots, S_q$ が単純正規交叉的ではない $\mathbb{P}^2(\mathbb{C})$ の点全体とする. $\widetilde{\mathbb{P}}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ を $\{x_1, \dots, x_p\}$ でブローアップしたものとして, H_1, H_2, H_3 を $\mathbb{P}^2(\mathbb{C})$ の超平面で, $\{x_1, \dots, x_p\}$ を通らないものとする. 正則曲線 $f: \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ を線形非退化なものとする. ここで f が線形非退化とは, f の像を含む $\mathbb{P}^2(\mathbb{C})$ の超平面が存在しないこととする. このとき次が成り立つ.

$$\begin{aligned}(q-3)T_f(r,H) \;&\le\; \sum_{i=1}^q N_2(r, \widetilde{f}^*\widetilde{\sigma}_i) + m\sum_{i=1}^p N(r, \widetilde{f}^*E_i)\\ &\quad + \frac{m-1}{2}\sum_{i=1}^3 N_2(r, f^*H_i) + S_f(r).\end{aligned}$$

一般の位置にない超平面に対する第二主要定理は E. Nochka [Noc] により任意次元で解決されているが上の定理 2.2 は Nochka のものとは異なる.

3　リーマン球面の直積における第二主要定理

第三節ではリーマン球面の直積である $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ 内の正則曲線を扱う．先行する結果として J. Noguchi [Nog] が，ある特別な条件下で得た第二主要定理がある．ここでは，そのような条件を仮定せずに成立することを証明する．$\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ には 2 次元代数トーラス $\mathbb{C}^*\times\mathbb{C}^*$ が含まれているが，その中の部分群である因子のコンパクト化 D', D'' に対する第二主要定理を証明する．ここで

$$D' : z^{m'} - w^{n'} = 0, \quad D'' : z^{m''}w^{n''} - 1 = 0,$$

であり z, w は $\mathbb{C}\times\mathbb{C}\subset\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ の局所座標系とする．代数的トーラス上には平坦な接続 ∇ が存在し，部分群は ∇ に関して全測地的になる．この接続 ∇ を $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ 上に有理型接続として拡張しておく．$Z=\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ とおいて，$\pi:\widetilde{Z}\to Z$ を適当な固有改変とする．すると $\widetilde{Z}$ 上にも有理型接続が誘導されて，それを $\widetilde{\nabla}$ とおく．このとき次の定理が示される．

定理 3.1 (Theorem 0.0.3) $f:\mathbb{C}\to\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ を正則曲線として $\widetilde{f}:\mathbb{C}\to\widetilde{Z}$ をそのリフトとする．また $\widetilde{D}', \widetilde{D}''$ を D', D'' の π による強変換，$E=\sum_{i=1}E_i$ を $\widetilde{\nabla}$ の極の既約分解とする．f の像が $\mathbb{C}^*\times\mathbb{C}^*$ のある真部分群の平行移動の閉包に含まれることはないとすると次が成り立つ．

$$T_{\widetilde{f}}(r,\widetilde{D}) \leq N_2(r,\widetilde{f}^*\widetilde{D}') + N_2(r,\widetilde{f}^*\widetilde{D}'') + 2\sum_{i=1} N_1(r,\widetilde{f}^*E_i) + S_f(r).$$

4　トーリック多様体への小林双曲的な埋め込み

代数的トーラス $(\mathbb{C}^*)^r$ 上の因子 D をローラン多項式

$$\sum_{I=(i_1,\ldots,i_r)} a_I z_1^{i_1}\cdots z_r^{i_r} \in \mathbb{C}[z_1,z_1^{-1},\ldots,z_r,z_r^{-1}]$$

で与えられるものとする．このとき，補空間 $(\mathbb{C}^*)^r\setminus D$ があるトーリック多様体へ小林双曲的に埋め込まれるような D の条件について考察する．以下に定理に必要な定義を述べる．

階数 r の $\mathbb{Z}$ 上の自由加群を $N=\mathbb{Z}^r$ として，$M=\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ とする．係数を $\mathbb{R}$ に拡張したものを $N_{\mathbb{R}}=N\otimes_{\mathbb{Z}}\mathbb{R}$, $M_{\mathbb{R}}=M\otimes_{\mathbb{Z}}\mathbb{R}$ とおく．r 次元代数的トーラスを $T_N=N\otimes_{\mathbb{Z}}\mathbb{C}^*$ とする．M に含まれる有限部分集合 A に対して，

$$\mathcal{L}_A := \{a-b\in M_{\mathbb{R}}\,|\,a,b\in A\}$$

と定義する．V_A を $\mathcal{L}_A$ の元全体で生成される $M_{\mathbb{R}}$ の $\mathbb{R}$ 上線形部分空間として，

$$\mathcal{H}_A := \{H\subset V_A \,|\, \mathcal{L}_A \text{ の元で生成される } V_A \text{ の超平面}\}$$

と定義する．ただしここでいう超平面とは，余次元 1 の $\mathbb{R}$ 上の線形部分空間である．また，P を $M_{\mathbb{R}}$ に含まれる r 次元整凸多面体とする．このとき P に付随した射影的トーリック多様体 X が自然に定義される．

定理 4.1 (Theorem 0.0.4) S を M の中の有限部分集合で $S\subset P$ とする．P に含まれる任意の正次元面 τ に対して次の条件を仮定する:

(i) $\tau\cap S\neq\emptyset$ であり，$\tau\cap S$ の凸包の次元は τ の次元と一致する．

(ii) $H\in\mathcal{H}_{\tau\cap S}$ に対して，$\phi_H : V_{\tau\cap S}\to V_{\tau\cap S}/H$ を自然な準同型とする．x を $\tau\cap S$ のある一点としたとき，$\sharp(\phi_H(\tau\cap S-x))\geq\dim\tau+1$ が任意の $H\in\mathcal{H}_{\tau\cap S}$ に対して成り立つとする．

このとき，線形系 $|\{z_1^{i_1}z_2^{i_2}\cdots z_r^{i_r}\}_{(i_1,i_2,\ldots,i_r)\in S}|$ の一般な元で与えられる T_N の因子 D に対して，$T_N\setminus D$ は X に小林双曲的に埋め込まれている．

また, 主結果の応用として小林予想 (2) に関連した次の系が得られる.

系 4.2 (Corollary 4.1.1) n 次元射影空間 $\mathbb{P}^n(\mathbb{C})$ において 一般の位置にある $(n+1)$ 本の超平面 $H_0, H_1, \ldots, H_n$ をとる. このとき次数 n の一般な超曲面 S に対して, $\mathbb{P}^n(\mathbb{C}) \setminus \left(S \cup \bigcup_{i=0}^n H_i\right)$ は $\mathbb{P}^n(\mathbb{C})$ に小林双曲的に埋め込まれる.

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