論文の内容の要旨

Rationality of the moduli spaces of 2-elementary $K3$ surfaces

(対合付き $K3$ 曲面のモジュライの有理性)

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複素 $K3$ 曲面 X とその上の周期に非自明に作用する対合 ι の組 (X,ι) を 2-elementary $K3$ 曲面と呼ぶ。Nikulin の分類によってそのような組は位相的には７５種類あることが知られており、各位相型は適当な自然数の三つ組 (r,a,δ) によってラベルづけられる。位相型 (r,a,δ) を一つ固定すればそれに属する 2-elementary $K3$ 曲面たちの同型類はある代数多様体 $\mathcal{M}_{r,a,\delta}$（モジュライ空間）によって自然にパラメトライズされる。周期写像の理論により、$\mathcal{M}_{r,a,\delta}$ は適当な IV 型対称領域の算術商から因子を除いた補集合として構成することができる。本論文の主題はこのモジュライ多様体 $\mathcal{M}_{r,a,\delta}$ の双有理型を調べることである。具体的には次の結果を証明する。

定理 0.1. 以下の８つの (r,a,δ) を除きモジュライ空間 $\mathcal{M}_{r,a,\delta}$ は有理的、すなわち射影空間と双有理同値である*:*
$(1,1,1), (2,2,0), (10,10,1), (r,22-r,1), 11 \leq r \leq 15$。

こうしてほとんどの $\mathcal{M}_{r,a,\delta}$ は双有理変換を除けば最も単純な代数多様体であることがわかった。別の言葉で言い換えれば、それらの算術商 $\mathcal{M}_{r,a,\delta}$ に対する保型函数体は $\mathbb{C}$ 上純超越的である。

いくつかの $\mathcal{M}_{r,a,\delta}$ は従来有理的であると知られていた。$\mathcal{M}_{10,10,0}$ と $\mathcal{M}_{10,2,0}$ は金銅によって有理性が証明され、$\mathcal{M}_{5,5,1}$ は Shepherd-Barron の研究によって実質的に有理性が示されていた。$\mathcal{M}_{10,10,0}$ は特に Enriques 曲面のモジュライであり、$\mathcal{M}_{5,5,1}$ は種数６の曲線のモジュライと自然に双有理同値である。本研究はそれらの研究をモデルとしており、それらの先行結果が実はより一般的な現象の中に位置づけられることを示している。残りの８つの $\mathcal{M}_{r,a,\delta}$ が有理的か否かはまだ明らかではない。有理性より少し弱い性質だが、それらが単有理的であることはわかっている。

さて、何かあるモジュライ空間 $\mathcal{M}$ が有理的であることを証明しようとする時には、代数群の作用の問題（広い意味での不変式論）に持ち込むことが一つの標準的手法である。具体的には、(1) $\mathcal{M}$ のメンバーの構成方法を考案することでパラメータ空間 U を定義する。U には自然な変換群 G が作用している。(2) 商多様体 U/G から $\mathcal{M}$ への自然な写像を調べる。もしも (1) で考案した構成法が標準的なものであったならばこれは双有理同型になる。(3) その時 U への G 作用を解析して U/G の有理性を示す、という手順を踏む。同型類をパラメトライズする $\mathcal{M}$ よりも G 作用の無駄を許して具体的な記述方法を与える U の方から解析を始めるのである。もちろん U の構成は $\mathcal{M}$ のメンバーの個性に完全に依存しており、また (3) の解析はデリケートさを伴うことが多い。その結果、一般にモジュライの有理性問題というのは個別解析の傾向が強く、本研究も例外にもれない。１つ１つ $\mathcal{M}_{r,a,\delta}$ の有理性を証明していくのである。とはいえほとんどの $\mathcal{M}_{r,a,\delta}$ に対して共通した構成のスキームを採るのでそれを以下説明しよう。

与えられた (r,a,δ) に対して、適当な Hirzebruch 曲面もしくは射影平面 Y の上のある種の決められたタイプの特異性と既約分解を持つ $-2K_Y$ 曲線の族を考え、そのパラメータ空間を U とする。Y の自己同型群を G とする。各 $B \in U$ に対して B で分岐する Y の２重被覆をとってその特異点を解消すれば $\mathcal{M}_{r,a,\delta}$ のメンバーが得られる。(より正確には、(r,a,δ) にヒットするように予め特異性等を規定しておいた。) この構成によって周期写像 $\mathcal{P} : U/G \to \mathcal{M}_{r,a,\delta}$ が定まるのだが、この写像は必ずしも双有理同型になるとは限らない。こうした構成法 U は幾つも考えられるので、そこで、いろいろな U を試してみてその中から $\mathcal{P}$ が次数 1 になるものがあるか探し当てることをする。本論文の 4.3 節ではこの種の周期写像の次数を系統的に計算するレシピを提示した。実際の証明は長くなるもののこのレシピに従えば次数計算は容易である。そうして実験の末 $\mathcal{P}$ が次数 1 になるような (U,G) が見つかれば上で説明したように G 作用の解析に帰着する。

この構成を少し角度を変えて説明しよう。(Y,B) というのはそれから作られる 2-elementary $K3$ 曲面の商曲面と分岐曲線をブローダウンしたものである。この時 $\mathcal{P}$ の次数というのは与えられたタイプの (Y,B) たちにブローダウンするやり方が何通りあるかを数えており、それが 1 通りしかないようなタイプを探して $\mathcal{M}_{r,a,\delta}$ の研究に利用するのである。それが見つかれば、商と分岐そのものよりもそれを（1 通りしかない）ブローダウンによって単純な曲面上の特異な曲線に転換した方が解析がしやすい。というのは、２つあったモジュライ要素が１つに統合されるし、よくわかっている群 G の作用の問題に帰着するからである。

これが基本的な議論のスキームだが、次数 1 の周期写像が見つからずこの枠組みからはみ出す場合も幾つかある。その主要なものは $r = a \geq 3$ と $r + a = 20, r \leq 14$ の２系列である。前者は del Pezzo 曲面の標準的な幾何

を用いて解析される。後者の系列の研究が実の所本論文の到達点である。次数計算のレシピを応用して、del Pezzo 曲面の幾何がより意外な形で見出された。

最後に、筆者を育ててくれた指導教官の吉川謙一先生と宮岡洋一先生に深い感謝の意を表したい。修士課程の始めに吉川先生が $K3$ 曲面という豊かな題材を紹介してくれたことが、筆者が数学の道を進む契機となった。本研究に取り組んだのも吉川先生の示唆に基づいている。吉川先生の転出に伴って、宮岡先生が博士課程の途中から指導教官を引き受けてくれ、筆者の成長を促しつつ様々な相談事に親身になってアドバイスをくれた。この２人の先生に出会えたことは誠に幸運なことであったと思う。

Rationality of the moduli spaces of 2-elementary $K3$ surfaces

（対合付き$K3$曲面のモジュライの有理性）

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RATIONALITY OF THE MODULI SPACES OF 2-ELEMENTARY $K3$ SURFACES

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Abstract. $K3$ surfaces with non-symplectic involution are classified by open sets of seventy-five arithmetic quotients of type IV. We prove that sixty-seven of those moduli spaces are rational.

1. Introduction

$K3$ surfaces with non-symplectic involution are basic objects in the study of $K3$ surfaces. They connect $K3$ surfaces with rational surfaces, Enriques surfaces, and low genus curves. In this article we address the rationality problem for the moduli spaces of $K3$ surfaces with non-symplectic involution.

To be more precise, let X be a complex $K3$ surface with an involution ι. When ι acts nontrivially on $H^0(K_X)$, ι is called *non-symplectic*, and the pair (X, ι) is called a *2-elementary* $K3$ *surface*. By Nikulin [30], the deformation type of (X, ι) is determined by its *main invariant* (r, a, δ), a triplet of integers associated to the lattice of ι-invariant cycles. He classified all main invariants of 2-elementary $K3$ surface, which turned out to be seventy-five in number (see Figure 1). For each main invariant (r, a, δ), let L_- be an even lattice of signature $(2, 20 - r)$ whose discriminant form is 2-elementary of length a and parity δ. Yoshikawa [35], [36] showed that the moduli space $\mathcal{M}_{r,a,\delta}$ of 2-elementary $K3$ surfaces of type (r, a, δ) is the complement of a Heegner divisor in the arithmetic quotient defined by the orthogonal group $O(L_-)$ of L_-. In particular, $\mathcal{M}_{r,a,\delta}$ is irreducible of dimension $20 - r$.

We shall prove the following.

Theorem 1.1. *The moduli space $\mathcal{M}_{r,a,\delta}$ of 2-elementary $K3$ surfaces of type (r, a, δ) is rational, possibly except when $(r, a, \delta) = (1, 1, 1), (2, 2, 0), (10, 10, 1)$ and when $r + a = 22, 11 \leq r \leq 15$.*

In other terms, the arithmetic quotients defined by $O(L_-)$ for the 2-elementary primitive sublattices L_- of the $K3$ lattice are mostly rational. We will later on comment on the excluded eight $\mathcal{M}_{r,a,\delta}$.

A few $\mathcal{M}_{r,a,\delta}$ have been known to be rational: $\mathcal{M}_{10,10,0}$ is the moduli of Enriques surfaces and is rational by Kondō [18]; he also proved that $\mathcal{M}_{10,2,0}$ is rational by identifying it with the moduli of certain trigonal curves [18]; $\mathcal{M}_{5,5,1}$ is birational

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to the moduli of genus six curves (see [34], [3]), and so is rational by [34]. Now Theorem 1.1 shows that these results are not of sporadic nature.

In [20] we proved that all $\mathcal{M}_{r,a,\delta}$ are unirational. The approach there was to use some isogenies between $\mathcal{M}_{r,a,\delta}$ to reduce the problem to fewer main invariants. In fact, in [20] we studied certain covers of $\mathcal{M}_{r,a,\delta}$ rather than $\mathcal{M}_{r,a,\delta}$ themselves. But the rationality problem is much more subtle, requiring individual treatment and delicate analysis. Thus in this article we study the spaces $\mathcal{M}_{r,a,\delta}$ one by one.

While our approach is case-by-case, the schemes of proofs are mostly common, which we shall try to explain here. We first find a *birational* period map

$$(1.1) \qquad \mathcal{P} : U/G \dashrightarrow \mathcal{M}_{r,a,\delta}$$

from the quotient space U/G of a parameter space U by an algebraic group G. Then we prove that U/G is rational as a problem in invariant theory (cf. [12], [31]). The space U parametrizes (possibly singular) $-2K_Y$-curves B on some smooth rational surfaces Y, and the map $\mathcal{P}$ is defined by taking the desingularizations of the double covers of Y branched along B. Thus our approach for 2-elementary $K3$ surfaces is essentially from quotient surfaces and branch curves.

The fact that $\mathcal{P}$ is birational means that our construction is "canonical" for the generic member of $\mathcal{M}_{r,a,\delta}$. The verifications of this property mostly fall into the following two patterns: (α) when the curves B are smooth, so that the desingularization process does not take place, it is just a consequence of the equivalence between (variety, involution) and (quotient, branch). This simple approach is taken when $r = a$ (Y are del Pezzo) and when $(r, a, \delta) = (10, 8, 1)$. ($\beta$) For many other (r, a, δ), the surfaces Y are $\mathbb{P}^2$ or Hirzebruch surfaces $\mathbb{F}_n$, the curves B have simple singularities of prescribed type, and the group G is $\mathrm{Aut}(Y)$. For this type of period map we can calculate its degree systematically. A recipe of such calculation is presented in §4.3. By finding a period map of degree 1 of this type, our rationality problem is reduced to invariant theory over $\mathbb{P}^2$ or $\mathbb{F}_n$.

It should be noted, however, that the existence of birational period map as above, especially of type (β), is a priori not clear. Indeed, when $r + a = 20$, $6 \leq a \leq 9$, $\delta = 1$, we partially give up the above approach and study a *non-birational* period map $U/G \dashrightarrow \mathcal{M}_{r,a,\delta}$ of type (β). Using the recipe in §4.3, we show that it is a quotient map by a Weyl group, and then analyze that action to derive the rationality. The proof for these cases would be most advanced in this article. On the other hand, when $r + a \leq 18$, some of our birational period maps seem related to Nakayama's minimal model process for log del Pezzo surfaces of index 2 [28], which may explain their birationality.

Recall that for most (r, a, δ) the fixed locus X^{ι} of an $(X, \iota) \in \mathcal{M}_{r,a,\delta}$ is the union of a genus $g = 11 - 2^{-1}(r + a)$ curve C^g and some other (-2)-curves ([30]). As an application of our birational period maps, we will determine the generic structure of the fixed curve map

$$(1.2) \qquad F : \mathcal{M}_{r,a,\delta} \to \mathcal{M}_g, \qquad (X, \iota) \mapsto C^g,$$

when $g \geq 3$. The point is that F is the composition of the inverse period map $\mathcal{P}^{-1} \colon \mathcal{M}_{r,a,\delta} \dashrightarrow U/G$ with the map $U/G \to \mathcal{M}_g$ that associates to a $-2K_Y$-curve

B its component of maximal genus. We will find that the latter map identifies U/G with a natural fibration over a sublocus of $\mathcal{M}_g$ defined in terms of special line bundles and points. Thus, via F, $\mathcal{M}_{r,a,\delta}$ is in a nice relation to $\mathcal{M}_g$. This generalizes the descriptions of $\mathcal{M}_{10,2,0}$ and $\mathcal{M}_{5,5,1}$ referred above. Our period maps will be useful in the study of 2-elementary $K3$ surfaces.

The rationality problem for the remaining eight $\mathcal{M}_{r,a,\delta}$ is open. $\mathcal{M}_{1,1,1}$ and $\mathcal{M}_{2,2,0}$ are respectively birational to the moduli of plane sextics and of bidegree $(4,4)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$. Hence the rationality problem for these two may be purely invariant-theoretic. Other six are related to point sets in $\mathbb{P}^2$, as explored in [20]. $\mathcal{M}_{11,11,1}$ is the moduli of classical Coble surfaces. $\mathcal{M}_{10,10,1}$ is a fibration over the moduli of Halphen surfaces of index 2. $\mathcal{M}_{14,8,1}$ and $\mathcal{M}_{15,7,1}$ are finite quotients of configuration spaces of seven points in $\mathbb{P}^2$. Similarly, $\mathcal{M}_{12,10,1}$ and $\mathcal{M}_{13,9,1}$ are related to configuration spaces of eight points in $\mathbb{P}^2$.

This article is organized as follows. §2 and §3 are preliminaries on invariant theory. In §4, after reviewing basic theory of 2-elementary $K3$ surfaces, we explain how to calculate the degrees of period maps of certain type. The proof of Theorem 1.1 begins with §5, where we treat the case $r = a$ using del Pezzo surfaces. The case $r > a$ is divided according to the genus g of main fixed curves. This division policy comes from our observation on the relation of $\mathcal{M}_{r,a,\delta}$ to $\mathcal{M}_g$. In §6 we treat the case $g \geq 7$ where trigonal curves of fixed Maroni invariant are central. Section $7 \leq n \leq 11$ is for the case $g = 13 - n$. In §12 we treat the case $g = 1$, $11 \leq r \leq 14$, $\delta = 1$, where del Pezzo surfaces appear again in more nontrivial way. We study in §13 the rest cases with $g = 1$, and in §14 the case $g = 0$.

2. Rationality of quotient variety

Let U be an irreducible variety acted on by an algebraic group G. Throughout this article we shall denote by U/G a *rational quotient*, namely a variety whose function field is isomorphic to the invariant subfield $\mathbb{C}(U)^G$ of the function field $\mathbb{C}(U)$ of U. As $\mathbb{C}(U)^G$ is finitely generated over $\mathbb{C}$, a rational quotient always exists (cf. [31]) and is unique up to birational equivalence. The inclusion $\mathbb{C}(U)^G \subset \mathbb{C}(U)$ induces a quotient map $\pi\colon U \dashrightarrow U/G$. Every G-invariant rational map $\varphi\colon U \to V$ is factorized as $\varphi = \psi \circ \pi$ by a unique rational map $\psi\colon U/G \dashrightarrow V$. We are interested in the problem when U/G is rational. We recall some results and techniques following [12] and [31].

Theorem 2.1 (Miyata [25])**.** *If G is a connected solvable group and U is a linear representation of G, the quotient U/G is rational.*

Theorem 2.2 (Katsylo, Bogomolov [15] [5])**.** *If U is a linear representation of $G = \mathrm{SL}_2 \times (\mathbb{C}^\times)^k$, the quotient U/G is rational.*

Theorem 2.2 is for the most part a consequence of Katsylo's theorem in [15]. The exception in [15] was settled by [5]. See also [31] Theorem 2.12.

The G-action on U is *almost transitive* if U contains an open orbit. The following is a special case of the slice method ([14], [12]).

Proposition 2.3 (slice method). *Let $f : U \to V$ be a G-equivariant morphism with almost transitive G-action on V. If $G_v \subset G$ is the stabilizer group of a general point $v \in V$, then U/G is birational to $f^{-1}(v)/G_v$.*

Proof. Indeed, the fiber $f^{-1}(v)$ is a slice of U in the sense of [12] with the stabilizer G_v. □

The G-action on U is *almost free* if a general point of U has trivial stabilizer.

Proposition 2.4 (no-name method). *If $E \to U$ is a G-linearized vector bundle with almost free G-action on U, then the induced map $E/G \dashrightarrow U/G$ is birationally equivalent to a vector bundle over U/G.*

When G is reductive, the no-name lemma is a consequence of the descent theory for principal G-bundle. A proof for general G is given in [9] Lemma 4.4. We will use the no-name method also in the following form.

Proposition 2.5. *Let $E \to U$ be a G-linearized vector bundle and let $G_0 = \{g \in G, g|_U = id\}$. Suppose that (i) $\overline{G} = G/G_0$ acts almost freely on U, (ii) G_0 acts on E by a scalar multiplication $\alpha : G_0 \to \mathbb{C}^\times$, and (iii) there exists a G-linearized line bundle $L \to U$ on which G_0 acts by α. Then $\mathbb{P}E/G$ is birational to $\mathbb{P}^N \times (U/G)$.*

Proof. Apply Proposition 2.4 to the $\overline{G}$-linearized vector bundle $E \otimes L^{-1}$. We have a canonical identification $\mathbb{P}(E \otimes L^{-1}) = \mathbb{P}E$. Notice that the quotient map $E \otimes L^{-1} \dashrightarrow (E \otimes L^{-1})/\overline{G}$ is linear on the fibers (cf. [9]). □

3. Automorphism action on Hirzebruch surfaces

We prepare miscellaneous results concerning automorphism action on Hirzebruch surfaces. For $n \geq 0$ let $\mathcal{E}_n$ be the vector bundle $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$ over $\mathbb{P}^1$. The projectivization $\mathbb{F}_n = \mathbb{P}\mathcal{E}_n$ is a Hirzebruch surface. Our convention is that a point of $\mathbb{F}_n$ represents a line in a fiber of $\mathcal{E}_n$. Then the section $\mathbb{P}\mathcal{O}_{\mathbb{P}^1}(n) \subset \mathbb{F}_n$ is a $(-n)$-curve which we denote by Σ. Let $\pi : \mathbb{F}_n \to \mathbb{P}^1$ be the natural projection. If F is a π-fiber, we shall denote the line bundle $\mathcal{O}_{\mathbb{F}_n}(a(\Sigma + nF) + bF)$ by $L_{a,b}$. The Picard group $\mathrm{Pic}(\mathbb{F}_n)$ consists of the bundles $L_{a,b}$, $a, b \in \mathbb{Z}$. For example, the section $\mathbb{P}\mathcal{O}_{\mathbb{P}^1} \subset \mathbb{F}_n$ belongs to $|L_{1,0}|$; the fiber F belongs to $|L_{0,1}|$; the canonical bundle $K_{\mathbb{F}_n}$ is isomorphic to $L_{-2,n-2}$. We have $(L_{a,b}.F) = a$ and $(L_{a,b}.\Sigma) = b$.

Except for §3.3, we assume $n > 0$ in this section. Under this assumption, the action of $\mathrm{Aut}(\mathbb{F}_n)$ preserves π and Σ. Consequently, we have the exact sequence

$$1 \to R \to \mathrm{Aut}(\mathbb{F}_n) \to \mathrm{Aut}(\Sigma) \to 1 \tag{3.1}$$

where $R = \mathrm{Aut}(\mathcal{E}_n)/\mathbb{C}^\times$. Via the given splitting $\mathcal{E}_n = \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$, we may identify

$$R = \left\{ g_{\alpha,s} = \begin{pmatrix} \alpha & s \\ 0 & 1 \end{pmatrix}, \quad \alpha \in \mathbb{C}^\times, s \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(n)) \right\}. \tag{3.2}$$

Thus we have $R \simeq \mathbb{C}^\times \ltimes H^0(\mathcal{O}_{\mathbb{P}^1}(n))$. When n is even, $\mathcal{E}_n$ admits a PGL_2-linearization via that of $\mathcal{O}_{\mathbb{P}^1}(n)$, so that the sequence (3.1) splits.

3.1. **Linearization of line bundles.** Let $\widetilde{G} = \mathrm{SL}_2 \ltimes R$ where SL_2 acts on the component $H^0(\mathcal{O}_{\mathbb{P}^1}(n))$ of R in the natural way. Via the identification (3.2) and the SL_2-linearization of $\mathcal{O}_{\mathbb{P}^1}(n)$, the bundle $\mathcal{E}_n$ is $\widetilde{G}$-linearized. This induces a surjective homomorphism $\widetilde{G} \to \mathrm{Aut}(\mathbb{F}_n)$, whose kernel is $\mathbb{Z}/2\mathbb{Z}$ generated by $\rho = (-1, g_{-1,0})$ when n is odd, and by $\sigma = (-1, g_{1,0})$ when n is even.

Lemma 3.1. *Every line bundle on $\mathbb{F}_n$ is $\widetilde{G}$-linearized.*

Proof. The SL_2-linearization of $\mathcal{O}_{\mathbb{P}^1}(1)$ induces a $\widetilde{G}$-linearization of $\pi^*\mathcal{O}_{\mathbb{P}^1}(1) = L_{0,1}$. Moreover, $\mathcal{O}_{\mathbb{F}_n}(\Sigma)$ is the dual of the tautological bundle over $\mathbb{F}_n$, which is the blow-up of $\mathcal{E}_n$ along the zero section. Hence $\mathcal{O}_{\mathbb{F}_n}(\Sigma) = L_{1,-n}$ is $\widetilde{G}$-linearized. □

Proposition 3.2. *When n is odd, every line bundle on $\mathbb{F}_n$ is $\mathrm{Aut}(\mathbb{F}_n)$-linearized. When n is even, the bundle $L_{a,b}$ admits an $\mathrm{Aut}(\mathbb{F}_n)$-linearization if b is even.*

Proof. In the $\widetilde{G}$-linearization of $\mathcal{E}_n$, the element σ (resp. ρ) acts trivially when n is even (resp. odd). Hence $\mathcal{O}_{\mathbb{F}_n}(\Sigma)$ is always $\mathrm{Aut}(\mathbb{F}_n)$-linearized. Also $L_{0,-2} = \pi^*K_{\mathbb{P}^1}$ is $\mathrm{Aut}(\mathbb{F}_n)$-linearized via the PGL_2-linearization of $K_{\mathbb{P}^1}$. This proves the assertion for even n. The homomorphism $\widetilde{G} \to \mathbb{C}^\times$ defined by $(\gamma, g_{\alpha,s}) \mapsto \alpha$ induces a 1-dimensional representation V of $\widetilde{G}$ on which ρ acts by -1. Hence the bundle $L_{0,1} \simeq L_{0,1} \otimes V$ is $\mathrm{Aut}(\mathbb{F}_n)$-linearized for odd n. □

3.2. **Some linear systems.** We study the $\mathrm{Aut}(\mathbb{F}_n)$-action on the linear systems $|L_{0,1}|$, $|L_{1,0}|$, $|L_{1,1}|$, and $|L_{2,0}|$. Recall that we are assuming $n > 0$.

Proposition 3.3. *The group $\mathrm{Aut}(\mathbb{F}_n)$ acts on the linear system $|L_{0,1}|$ transitively, and the stabilizer of a point of $|L_{0,1}|$ is connected and solvable.*

Proof. We have a canonical identification $|L_{0,1}| \simeq \Sigma$ given by $F \mapsto F \cap \Sigma$. The sequence (3.1) restricted to the stabilizer G_p of a $p \in \Sigma$ gives the exact sequence

$$1 \to R \to G_p \to \mathrm{Aut}(\Sigma, p) \to 1,$$

where $\mathrm{Aut}(\Sigma, p)$ is the stabilizer of p in $\mathrm{Aut}(\Sigma)$. Since both R and $\mathrm{Aut}(\Sigma, p) \simeq \mathbb{C}^\times \ltimes \mathbb{C}$ are connected and solvable, so is G_p. □

Every smooth divisor H in $|L_{1,0}|$ is a section of π disjoint from Σ. Hence it corresponds to another splitting $\mathcal{E}_n \simeq \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$ for which the image of $\mathbb{P}\mathcal{O}_{\mathbb{P}^1}$ is H.

Proposition 3.4. *The group $\mathrm{Aut}(\mathbb{F}_n)$ acts transitively on the open set $U \subset |L_{1,0}|$ of smooth divisors. The sequence (3.1) restricted to the stabilizer $G_H \subset \mathrm{Aut}(\mathbb{F}_n)$ of an $H \in U$ gives the exact sequence*

$$1 \to \mathbb{C}^\times \to G_H \to \mathrm{Aut}(\Sigma) \to 1. \tag{3.3}$$

Proof. In fact, the subgroup $R \subset \mathrm{Aut}(\mathbb{F}_n)$ acts transitively on U because any two splittings $\mathcal{E}_n \simeq \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$ are $\mathrm{Aut}(\mathcal{E}_n)$-equivalent. When H corresponds to the original splitting of $\mathcal{E}_n$, we have $G_H \cap R = \{g_{\alpha,0}, \alpha \in \mathbb{C}^\times\}$. The homomorphism $G_H \to \mathrm{Aut}(\Sigma)$ is surjective thanks to the SL_2-linearization of $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$. □

Proposition 3.5. *The group* $\mathrm{Aut}(\mathbb{F}_n)$ *acts on* $|L_{1,1}|$ *almost transitively with the stabilizer of a general point being connected and solvable.*

Proof. We first show that the subgroup $R \subset \mathrm{Aut}(\mathbb{F}_n)$ acts on $|L_{1,1}|$ almost freely. Indeed, if an element $g \in R$ preserves (and hence fixes) a smooth $D \in |L_{1,1}|$, then for a general $H \in |L_{1,0}|$ the $n+1$ points $H \cap D$ are fixed by g. Since $(L_{1,0}.L_{1,0}) = n$, then g fixes H and so must be trivial. The group R preserves the sub linear system $\mathbb{P}V_p \subset |L_{1,1}|$ of curves passing each $p \in \Sigma$. Since $\dim R = \dim \mathbb{P}V_p = n+2$, the R-action on $\mathbb{P}V_p$ is also almost transitive. Thus $\mathrm{Aut}(\mathbb{F}_n)$ acts on $|L_{1,1}|$ almost transitively. Let $G \subset \mathrm{Aut}(\mathbb{F}_n)$ be the stabilizer of a general $D \in |L_{1,1}|$. Since $G \cap R = \{\mathrm{id}\}$, the natural homomorphism $G \to \mathrm{Aut}(\Sigma)$ is injective. Its image is contained in the stabilizer $\mathrm{Aut}(\Sigma, p)$ of the point $p = D \cap \Sigma$. Then the embedding $G \to \mathrm{Aut}(\Sigma, p)$ is open because $\dim G = 2$. Therefore G is connected and solvable. □

Every smooth divisor in $|L_{2,0}|$ is a hyperelliptic curve of genus $n-1$ whose g^1_2 is given by the restriction of π.

Proposition 3.6. *If* $U \subset |L_{2,0}|$ *is the open set of smooth divisors, a geometric quotient* $U/\mathrm{Aut}(\mathbb{F}_n)$ *exists and is isomorphic to the moduli space* $\mathcal{H}_{n-1}$ *of hyperelliptic curves of genus* $n-1$.

Proof. Recall that $\mathcal{H}_{n-1}$ is a normal irreducible variety. We have a natural $\mathrm{Aut}(\mathbb{F}_n)$-invariant morphism $\psi\colon U \to \mathcal{H}_{n-1}$. In order to show that ψ is surjective, let C be a hyperelliptic curve of genus $n-1$ and $\pi\colon C \to \mathbb{P}^1$ be its g^1_2. The curve C is naturally embedded in $\mathbb{P}(\pi_*\mathcal{O}_C)^\vee$ over $\mathbb{P}^1$ by the evaluation map. Let $\pi_*\mathcal{O}_C = \mathcal{L}_+ \oplus \mathcal{L}_-$ be the decomposition with respect to the hyperelliptic involution ι, where ι acts on $\mathcal{L}_\pm$ by ± 1. It is clear that $\mathcal{L}_+ \simeq \mathcal{O}_{\mathbb{P}^1}$, and a cohomology calculation shows that $\mathcal{L}_- \simeq \mathcal{O}_{\mathbb{P}^1}(-n)$. Thus $\mathbb{P}(\pi_*\mathcal{O}_C)^\vee$ is isomorphic to $\mathbb{F}_n$. Then C belongs to $|L_{2,b}|$ for some $b \geq 0$, and the genus formula shows that $b = 0$. Therefore ψ is surjective. Conversely, for a $C \in U$ the natural embedding $C \subset \mathbb{P}(\pi_*\mathcal{O}_C)^\vee$ extends to an isomorphism $\mathbb{F}_n \to \mathbb{P}(\pi_*\mathcal{O}_C)^\vee$ (e.g., by the evaluation at C via the canonical identifications $\mathbb{P}(\pi_*\mathcal{O}_C)^\vee \simeq \mathbb{P}(\pi_*\mathcal{O}_C)$ and $\mathbb{F}_n \simeq \mathbb{P}\mathcal{E}_n^\vee$). This implies that the ψ-fibers are $\mathrm{Aut}(\mathbb{F}_n)$-orbits. Now our assertion follows from [27] Proposition 0.2. □

By the proof, the hyperelliptic involution of a smooth $C \in |L_{2,0}|$ uniquely extends to an involution ι_C of $\mathbb{F}_n$. Concretely, for each π-fiber F consider the involution of F which fixes the point $F \cap \Sigma$ and exchanges the two points $F \cap C$ (or fixes $F \cap C$ when F is tangent to C). This defines ι_C.

Corollary 3.7. *When* $n \geq 3$, *the stabilizer in* $\mathrm{Aut}(\mathbb{F}_n)$ *of a general* $C \in |L_{2,0}|$ *is* $\langle \iota_C \rangle$.

Proof. By Proposition 3.6 C has no automorphism other than its hyperelliptic involution. Any automorphism of $\mathbb{F}_n$ acting trivially on C must be trivial because it fixes three points of general π-fibers. □

3.3. **Trigonal curves.** *We allow* $n \geq 0$ *in this subsection.* It is known that trigonal curves are naturally related to Hirzebruch surfaces. We recall here some basic facts (cf. [2], [22]). A canonically embedded trigonal curve $C \subset \mathbb{P}^{g-1}$ of genus $g \geq 5$ is contained in a unique rational normal scroll, that is, the image of a Hirzebruch surface $\mathbb{F}_n$ by a linear system $|L_{1,m}|$. The scroll is swept out by the lines spanned by the fibers of the trigonal map $C \to \mathbb{P}^1$ (which is unique), and may also be cut out by the quadrics containing C. The number n is the *scroll invariant* of C, and the number m is the *Maroni invariant* of C. If we regard C as a curve on $\mathbb{F}_n$, the system $|L_{1,m}|$ gives the canonical system of C, and $|L_{0,1}|$ gives the g^1_3 of C. Then C belongs to $|L_{3,b}|$ for $b = m - n + 2$. The genus formula derives the relation $g = 3n + 2b - 2$. Conversely, for a smooth curve C on $\mathbb{F}_n$ with $C \in |L_{3,b}|$, the linear system $|L_{1,m}|$ with $m = b + n - 2$ is identified with $|K_C|$ by restriction, and the curve $\phi_{L_{1,m}}(C)$ in $|L_{1,m}|^\vee$ is a canonical model of C which is contained in the scroll $\phi_{L_{1,m}}(\mathbb{F}_n)$.

For $g \geq 5$ we denote by $\mathcal{T}_{g,n} \subset \mathcal{M}_g$ the locus of trigonal curves of scroll invariant n in the moduli space of genus g curves. In some literature, $\mathcal{T}_{g,n}$ is called a *Maroni locus*. General trigonal curves have scroll invariant 0 or 1 depending on whether g is even or odd. The above facts infer the following.

Proposition 3.8. *For $g \geq 5$ and $n \geq 0$ the space $\mathcal{T}_{g,n}$ is naturally birational to a rational quotient $|L_{3,b}|/G$, where $2b = g - 3n + 2$ and G is the identity component of* $\mathrm{Aut}(\mathbb{F}_n)$.

3.4. **A coordinate system.** The Hirzebruch surface $\mathbb{F}_n$ has a natural coordinate system. Let $[X, Y]$ be the homogeneous coordinate of $\mathbb{P}^1$. Let $V_0 = \{Y \neq 0\}$ and $V_1 = \{X \neq 0\}$ be open sets of $\mathbb{P}^1$. We fix the original splitting $\mathcal{E}_n = \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$ and denote $H = \mathbb{P}\mathcal{O}_{\mathbb{P}^1} \subset \mathbb{F}_n$. An open covering $\{U_i\}_{i=1}^4$ of $\mathbb{F}_n$ is defined by

$$U_1 = \pi^{-1}(V_0) - H, \qquad U_2 = \pi^{-1}(V_1) - H,$$
$$U_3 = \pi^{-1}(V_0) - \Sigma, \qquad U_4 = \pi^{-1}(V_1) - \Sigma.$$

Let $\mathbf{1} \in H^0(\mathcal{O}_{\mathbb{P}^1})$ be the section given by the constant function 1. Let $s_0, s_1 \in H^0(\mathcal{O}_{\mathbb{P}^1}(n))$ be the sections given by $s_0 = Y^n$ and $s_1 = X^n$ respectively. Note that $s_1 = (Y^{-1}X)^n s_0$ where we regard $Y^{-1}X \in H^0(\mathcal{O}_{V_0 \cap V_1})$. We shall use $(\mathbf{1}, s_i)$ as a local frame of $\mathcal{E}_n$ over V_i. Then we have an isomorphism $U_1 \to \mathbb{C}^2$ (resp. $U_3 \to \mathbb{C}^2$) given by

$$([X, Y],\ \mathbb{C}(a\mathbf{1} + bs_0)) \mapsto (Y^{-1}X, b^{-1}a) \qquad (\text{resp.}\ \mapsto (Y^{-1}X, a^{-1}b)),$$

and an isomorphism $U_2 \to \mathbb{C}^2$ (resp. $U_4 \to \mathbb{C}^2$) given by

$$([X, Y],\ \mathbb{C}(a\mathbf{1} + cs_1)) \mapsto (X^{-1}Y, c^{-1}a) \qquad (\text{resp.}\ \mapsto (X^{-1}Y, a^{-1}c)).$$

Thus the open sets $U_i \simeq \mathbb{C}^2$ have coordinates (x_i, y_i) glued by

$$x_1 = x_3 = x_2^{-1} = x_4^{-1},$$
$$y_3 = y_1^{-1}, \quad y_4 = y_2^{-1}, \quad y_2 = x_1^n y_1, \quad y_4 = x_3^{-n} y_3.$$

The $(-n)$-curve Σ is defined by the equation $y_1 = y_2 = 0$.

Let us describe some of the $\mathrm{Aut}(\mathbb{F}_n)$-action on $\mathbb{F}_n$ in terms of these coordinates. The elements $g_{\alpha,s} \in R$ leave U_3 invariant. If $s \in H^0(\mathcal{O}_{\mathbb{P}^1}(n))$ is written as $s = \sum_{i=0}^n \lambda_i X^i Y^{n-i}$, then $g_{\alpha,s}$ acts by

$$g_{\alpha,s} : U_3 \ni (x_3, y_3) \mapsto (x_3, \alpha y_3 + \sum_{i=0}^{n} \lambda_i x_3^i) \in U_3. \tag{3.4}$$

The group GL_2 acts on $\mathbb{F}_n$ via the GL_2-linearization of $\mathcal{E}_n$. Then the elements $h_\beta = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$ and $j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of GL_2 act respectively by

$$h_\beta : U_3 \ni (x_3, y_3) \mapsto (\beta x_3, y_3) \in U_3, \tag{3.5}$$

$$j : U_3 \ni (x_3, y_3) \mapsto (x_3, y_3) \in U_4. \tag{3.6}$$

In (3.6), the second (x_3, y_3) is the coordinate in U_4. This action is also regarded as the rational transformation $U_3 \ni (x_3, y_3) \mapsto (x_3^{-1}, x_3^{-n} y_3) \in U_3$.

We describe the linear systems $|L_{a,b}|$ in terms of these coordinates. We may assume $a, b \geq 0$, which is equivalent to the condition that $|L_{a,b}|$ has no fixed component. If $C \in |L_{a,b}|$, then $C \cap U_1$ is a curve on U_1 and thus defined by the equation $F(x_1, y_1) = 0$ for a polynomial $F(x_1, y_1)$.

Proposition 3.9. *For $a, b \geq 0$ the linear system $|L_{a,b}|$ is identified by restriction to U_1 with the projectivization of the vector space $\{\sum_{i=0}^{a} f_i(x_1) y_1^i,\ \deg f_i \leq b + in\}$ of polynomials of x_1, y_1.*

Proof. Let $F(x_1, y_1) = 0$ be an equation of $C|_{U_1}$ for a general $C \in |L_{a,b}|$. Expanding $F(x_1, y_1) = \sum_{i=0}^{d} f_i(x_1) y_1^i$, we have $d = (C.F) = a$ and $\deg f_0 = (C.\Sigma) = b$. By substitution, the curve $C|_{U_2}$ on U_2 is defined by $x_2^k F(x_2^{-1}, x_2^n y_2) = 0$ for some k. Putting $y_2 = 0$, we know that $k = b$. Since $x_2^b F(x_2^{-1}, x_2^n y_2)$ should be a polynomial of x_2, y_2, we must have $\deg f_i \leq b + in$. Then the equality $h^0(L_{a,b}) = \chi(L_{a,b}) = \frac{1}{2}(a+1)(an + 2b + 2)$ concludes the proof. □

A defining polynomial $\sum_{i=0}^{a} f_i(x_1) y_1^i$ in U_1 for a curve $C \in |L_{a,b}|$ is transformed into $\sum_{i=0}^{a} f_i(x_2^{-1}) x_2^{b+in} y_2^i$ in U_2, into $\sum_{i=0}^{a} f_i(x_3) y_3^{a-i}$ in U_3, and into $\sum_{i=0}^{a} f_i(x_4^{-1}) x_4^{b+in} y_4^{a-i}$ in U_4.

4. 2-ELEMENTARY $K3$ SURFACE

4.1. **2-elementary $K3$ surface.** We review basic theory of 2-elementary $K3$ surfaces following [1] and [35]. Let (X, ι) be a 2-elementary $K3$ surface, namely X is a complex $K3$ surface and ι is a non-symplectic involution on X. The presence of ι implies that X is algebraic. We denote by $L_\pm(X, \iota) \subset H^2(X, \mathbb{Z})$ the lattice of cohomology classes l with $\iota^* l = \pm l$. Then $L_+(X, \iota)$ is the orthogonal complement of $L_-(X, \iota)$ and contained in the Néron-Severi lattice NS_X. If r is the rank of $L_+(X, \iota)$, then $L_+(X, \iota)$ and $L_-(X, \iota)$ have signature $(1, r-1)$ and $(2, 20-r)$ respectively. Let $L_\pm(X, \iota)^\vee$ be the dual lattice of $L_\pm(X, \iota)$. The discriminant form of $L_\pm(X, \iota)$ is the finite quadratic form $(D_{L_\pm}, q_\pm)$ where $D_{L_\pm} = L_\pm(X, \iota)^\vee / L_\pm(X, \iota)$ and $q_\pm : D_{L_\pm} \to \mathbb{Q}/2\mathbb{Z}$ is induced by the quadratic form on $L_\pm(X, \iota)^\vee$. We have a canonical isometry $(D_{L_+}, q_+) \simeq (D_{L_-}, -q_-)$. The abelian group D_{L_+} is 2-elementary, namely $D_{L_+} \simeq (\mathbb{Z}/2\mathbb{Z})^a$ for some $a \geq 0$. The parity δ of q_+ is defined by $\delta = 0$ if $q_+(D_{L_+}) \subset \mathbb{Z}$, and $\delta = 1$ otherwise. The triplet (r, a, δ) is called the *main invariant* of the lattice $L_+(X, \iota)$, and also of the 2-elementary $K3$ surface (X, ι). By [29], the isometry class of $L_\pm(X, \iota)$ is uniquely determined by (r, a, δ).

Proposition 4.1 (Nikulin [30]). *Let $X^\iota \subset X$ be the fixed locus of ι.*

(i) *If $(r, a, \delta) = (10, 10, 0)$, then $X^\iota = \emptyset$.*

(ii) *If $(r, a, \delta) = (10, 8, 0)$, then X^ι is a union of two elliptic curves.*

(iii) *In other cases, X^ι is decomposed as $X^\iota = C^g \sqcup E_1 \sqcup \cdots \sqcup E_k$ such that C^g is a genus g curve and $E_1, \cdots, E_k$ are (-2)-curves with*

$$g = 11 - \frac{r+a}{2}, \qquad k = \frac{r-a}{2}. \tag{4.1}$$

One has $\delta = 0$ if and only if the class of X^ι is divisible by 2 in $L_+(X, \iota)$.

Theorem 4.2 (Nikulin [30]). *The deformation type of a 2-elementary K3 surface (X, ι) is determined by its main invariant (r, a, δ). All possible main invariants of 2-elementary K3 surfaces are shown on the following Figure 1 (which is identical to the table in* [1] *page 31).*

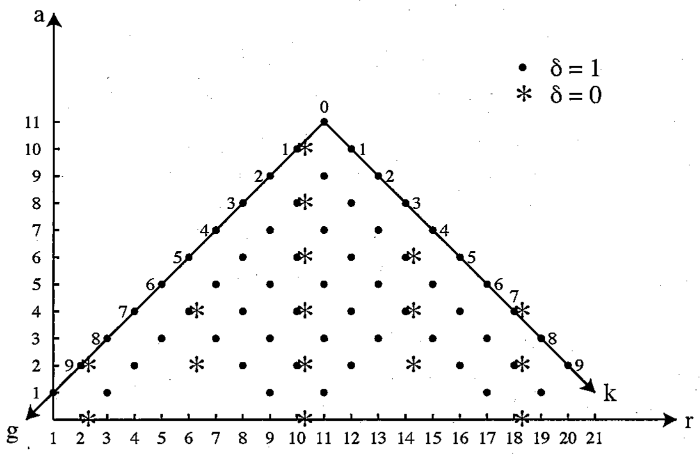


FIGURE 1. Distribution of main invariants (r, a, δ)

A moduli space of 2-elementary $K3$ surfaces of type (r, a, δ) is constructed as follows. Let L_- be an even lattice of signature $(2, 20-r)$ whose discriminant form is 2-elementary of length a and parity δ. The orthogonal group $\mathrm{O}(L_-)$ of L_- acts on the domain

$$\Omega_{L_-} = \{\mathbb{C}\omega \in \mathbb{P}(L_- \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0\}.$$

The quotient space $\mathcal{F}(\mathrm{O}(L_-)) = \mathrm{O}(L_-)\backslash\Omega_{L_-}$ turns out to be an irreducible, normal, quasi-projective variety of dimension $20 - r$. The complex analytic divisor $\sum \delta^\perp \subset \Omega_{L_-}$, where δ are (-2)-vectors in L_-, is the inverse image of an algebraic divisor $H \subset \mathcal{F}(\mathrm{O}(L_-))$. We set

$$\mathcal{M}_{r,a,\delta} = \mathcal{F}(\mathrm{O}(L_-)) - H.$$

For a 2-elementary $K3$ surface (X, ι) of type (r, a, δ) we have an isometry $\Phi\colon L_-(X, \iota) \to L_-$. Then $\Phi(H^{2,0}(X))$ is contained in Ω_{L_-}. The period of (X, ι) is defined by

$$\mathcal{P}(X, \iota) = [\Phi(H^{2,0}(X))] \in \mathcal{M}_{r,a,\delta},$$

which is independent of the choice of Φ.

Theorem 4.3 (Yoshikawa [35], [36]). *The variety $\mathcal{M}_{r,a,\delta}$ is a moduli space of 2-elementary K3 surfaces of type (r, a, δ) in the sense that, for a family $(\mathfrak{X} \to U, \iota)$ of such 2-elementary K3 surfaces the period map $U \to \mathcal{M}_{r,a,\delta}$, $u \mapsto \mathcal{P}(\mathfrak{X}_u, \iota_u)$, is a morphism of varieties, and via the period mapping the points of $\mathcal{M}_{r,a,\delta}$ bijectively correspond to the isomorphism classes of such 2-elementary K3 surfaces.*

Let $\mathcal{M}_g$ be the moduli of genus g curves. When $(r, a, \delta) \neq (10, 10, 0), (10, 8, 0)$, setting $g = 11 - 2^{-1}(r + a)$, we have the *fixed curve map*

$$F : \mathcal{M}_{r,a,\delta} \to \mathcal{M}_g, \quad (X, \iota) \mapsto C^g, \tag{4.2}$$

where C^g is the genus g component of X^ι. In this article we will determine the generic structure of F in terms of $\mathcal{M}_g$ for $g \geq 3$. For example, one will find that

- F is generically injective for $r \leq 5$ and for $8 \leq r \leq 12$, $a \leq 2$.
- The members of $F(\mathcal{M}_{r,a,\delta})$ have Clifford index ≤ 2. If $k > 0$ in addition, they have Clifford index ≤ 1.
- When $r = 2 + 4n$, we have $\delta = 0$ if and only if the generic member of $F(\mathcal{M}_{r,a,\delta})$ possesses a theta characteristic of projective dimension $3 - n$.

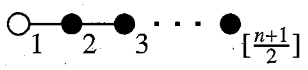
4.2. **DPN pair.** We shall explain a generalized double cover construction of 2-elementary $K3$ surfaces. Recall from [1] that a *DPN pair* is a pair (Y, B) of a smooth rational surface Y and a bi-anticanonical curve $B \in |-2K_Y|$ with only A-D-E singularities. When B is smooth, (Y, B) is called a *right DPN pair*. 2-elementary $K3$ surfaces (X, ι) with $X^\iota \neq \emptyset$ are in canonical correspondence with right DPN pairs: for such an (X, ι) the quotient $Y = X/\iota$ is a smooth rational surface, and the branch curve B of the quotient map $X \to Y$ is a $-2K_Y$-curve isomorphic to X^ι. Conversely, for a right DPN pair (Y, B) the double cover $f \colon X \to Y$ branched along B is a $K3$ surface, with the covering transformation being a non-symplectic involution. From B one knows the invariant (r, a) of X via Proposition 4.1. Also one has $r = \rho(Y)$. The lattice $L_+(X, \iota)$ is generated by the sublattice f^*NS_Y and the classes of components of X^ι (cf. [20]). By [1], if $B = \sum_{i=0}^k B_i$ is the irreducible decomposition of B, then (X, ι) has parity $\delta = 0$ if and only if $\sum_{i=0}^k (-1)^{n_i} B_i \in 4NS_Y$ for some $n_i \in \{0, 1\}$.

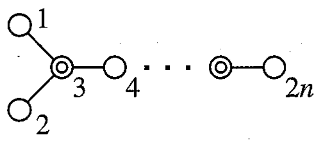
Let (Y, B) be a DPN pair. A *right resolution* of (Y, B) is a triplet (Y', B', π) such that (Y', B') is a right DPN pair and $\pi \colon Y' \to Y$ is a birational morphism with $\pi(B') = B$. When Y is obvious from the context, we also call it a right resolution of B. A right resolution exists and is unique up to isomorphism. It may be constructed explicitly as follows ([1]). Let

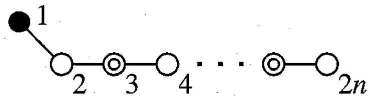
$$\cdots \xrightarrow{\pi_{i+1}} (Y_i, B_i) \xrightarrow{\pi_i} (Y_{i-1}, B_{i-1}) \xrightarrow{\pi_{i-1}} \cdots \xrightarrow{\pi_1} (Y_0, B_0) = (Y, B) \tag{4.3}$$

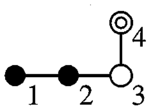
be the blow-ups defined inductively by $\pi_{i+1} : Y_{i+1} \to Y_i$ being the blow-up at the singular points of B_i, and $B_{i+1} = \widetilde{B_i} + \sum_p E_p$ where $\widetilde{B_i}$ is the strict transform of B_i and E_p are the (-1)-curves over the triple points p of B_i. Each (Y_i, B_i) is also a DPN pair. This process will terminate and we finally obtain a right DPN pair

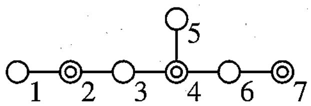
$(Y', B') = (Y_N, B_N)$. Let p be a singular point of B. According to the type of singularity, the dual graph of the curves on Y' contracted to p is as follows.

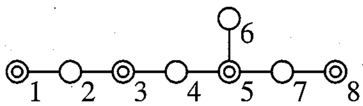
A_n


D_{2n}


D_{2n+1}


E_6


E_7


E_8


Here black vertices represent (-2)-curves, white vertices represent (-1)-curves, and double circles represent (-4)-curves. The (-4)-curves are components of B', while the (-2)-curves are disjoint from B'. The (-1)-curves intersect with B' transversely at two points unless p is A_{2n}-type; when p is an A_{2n}-point, the (-1)-curve is tangent to B' at one point. The labeling for the vertices will be used later. Note that identification of the dual graph of the curves with the above abstract graph is not unique when p is D_{2n}-type. In that case, such an identification is obtained after one distinguish the three branches (resp. two tangential branches) of B at p when $n = 2$ (resp. $n > 2$).

Let (Y, B) be a DPN pair with a right resolution (Y', B', π). Taking the double cover of Y' branched over B', we associate a 2-elementary $K3$ surface (X, ι) to (Y, B). The composition $X \to Y$ of the quotient map $X \to Y'$ and the blow-down π is called the *right covering map* for (Y, B). Note that (X, ι) is also the minimal desingularization of the double cover of Y branched over B. By the above dual graphs, the invariant (r, a) of (X, ι) is calculated in terms of (Y, B) as follows. Let a_n (resp. d_n, e_n) be the number of singularities of B of type A_n (resp. D_n, E_n). Then $r = \rho(Y')$ is given by

$$\rho(Y) + \sum_{l \geq 1} l\,(a_{2l-1} + a_{2l}) + \sum_{m \geq 2} 2m(d_{2m} + d_{2m+1}) + 4e_6 + 7e_7 + 8e_8. \tag{4.4}$$

If k_0 is the number of components of B, the number $k+1$ of components of $X^\iota \simeq B'$ is given by

$$k_0 + \sum_{m\geq 2}(m-1)(d_{2m} + d_{2m+1}) + e_6 + 3e_7 + 4e_8. \tag{4.5}$$

The genus g of the main component of X^ι is the maximal geometric genus of components of B. For the parity δ we will later use the following criteria.

Lemma 4.4. *Let (X, ι) and (Y, B) be as above. Then (X, ι) has parity $\delta = 1$ if we have distinct irreducible components B_i of B with either of the following conditions.*

(1) $B_i \cap B_j$ contains a node of B for every $1 \leq i, j \leq 3$.

(2) $B_1 \cap B_2$ contains a node and a D_4-point of B.

(3) $B_1 \cap B_2$ contains a node and a D_{2n}-point of B, in the latter of which B_1 and B_2 share a tangent direction.

(4) B_1 has a node which is also a node of B.

Proof. (1) Let p_i be a node of B contained in $B_j \cap B_k$ with $\{i, j, k\} = \{1, 2, 3\}$. Let $C_i \subset X$ be the (-2)-curve over p_i. It suffices to show that the $\mathbb{Q}$-cycle $D = 2^{-1}\sum_{i=1}^{3} C_i$ belongs to $L_+(X, \iota)^\vee$. If F_i is the component of X^ι over B_i, then $(D.F_i) = 1$ for every $1 \leq i \leq 3$. We have $(D.F) = 0$ for other components F of X^ι. Since $2D$ is the pullback of a divisor on X/ι, D has integral intersection pairing with the pullbacks of divisors on X/ι. This proves the assertion.

(2) Let Y'' be the blow-up of Y at the D_4-point, $E \subset Y''$ the exceptional curve, and $\widetilde{B}_i$ (resp. $\widetilde{B}$) the strict transform of B_i (resp. B) in Y''. Then one may apply the case (1) to the DPN pair $(Y'', \widetilde{B} + E)$ and the components $\widetilde{B}_1, \widetilde{B}_2, E$ of $\widetilde{B} + E$.

(3) Blow-up Y at the D_{2n}-singularity and use the induction on n: the assertion is reduced to the case (2).

(4) If $C \subset X$ is the (-2)-curve over that node, we have $2^{-1}C \in L_+(X, \iota)^\vee$. □

In certain cases, the right covering map $f\colon X \to Y$ may be recovered from a line bundle on X.

Lemma 4.5 (cf. [20]). *Let (X, ι) and (Y, B) be as above and suppose that Y is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_n$ with $n > 0$. Let $L \in \mathrm{Pic}(Y)$ be $\mathcal{O}_{\mathbb{P}^2}(1)$, $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, 1)$, $L_{1,0}$ for respective case. Then the map $f^*\colon |L| \to |f^*L|$ is isomorphic.*

Proof. The bundle f^*L is nef of degree $2(L.L) > 0$ so that $h^0(f^*L) = \chi(f^*L) = 2 + (L.L)$. Hence the assertion follows from the coincidence $h^0(L) = 2 + (L.L)$. □

By this lemma, the morphism $\phi_{f^*L}\colon X \to |f^*L|^\vee$ is identified with the composition $\phi_L \circ f\colon X \to Y \to |L|^\vee$. The morphism ϕ_L is an embedding when $Y = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. If $Y = \mathbb{F}_n$ with $n > 0$, ϕ_L contracts the $(-n)$-curve Σ. Therefore, for $Y \neq \mathbb{F}_1$, one may recover the morphism $f\colon X \to Y$ from the bundle f^*L by desingularizing the surface $\phi_{f^*L}(X) = \phi_L(Y)$. For $Y = \mathbb{F}_1$, $\phi_L\colon \mathbb{F}_1 \to \mathbb{P}^2$ is the blow-down of the (-1)-curve Σ, and one may recover f from f^*L if one could identify the point $\phi_L(\Sigma)$ in $\mathbb{P}^2$.

In the rest of this subsection we prepare some auxiliary results under the following assumption (which is necessary if one wants to obtain general members of $\mathcal{M}_{r,a,\delta}$).

Condition 4.6. The singularities of B are only of type A_1, D_{2n}, E_7, E_8.

For such a singularity p of B, the irreducible curves on Y' contracted to p are only (-4)-curves and (-1)-curves transverse to B'. The reduced preimages of those curves in X are ι-invariant (-2)-curves, and their dual graph is the Dynkin graph of same type as the singularity of p. We denote by $\Lambda_p \subset L_+(X, \iota)$ the generated root lattice. Let $B = \sum_{i=1}^{l} B_i$ be the irreducible decomposition of B, and F_i be the component of X^ι with $f(F_i) = B_i$.

Lemma 4.7. *The lattice $L_+(X, \iota)$ is generated by the sublattice $f^*NS_Y \oplus (\oplus_p \Lambda_p)$, where p are the singularities of B, and the classes of F_i, $1 \leq i \leq l$.*

Proof. Let $f' : X \to Y'$ be the quotient map by ι. By the construction of Λ_p, the lattice $(f')^*NS_{Y'}$ is contained in $f^*NS_Y \oplus (\oplus_p \Lambda_p)$. Also the components of X^ι other than $F_1, \cdots, F_l$ are contracted by f to the triple points of B, so that their classes are contained in $\oplus_p \Lambda_p$. □

We shall construct an ample divisor class on X using the objects in Lemma 4.7. Let p be a triple point of B. Choose an identification of the dual graph of the exceptional curves over p with an abstract graph presented in p.11. Via the labeling given for the latter, we denote by $\{E_{p,i}\}_i$ the (-2)-curves generating Λ_p. Then we define a divisor D_p on X by $D_p = E_{p,1} + E_{p,2} + \sum_{i=4}^{2n} 10^{i-4} E_{p,i}$ when p is D_{2n}-type; $D_p = \sum_{i=3}^{5} 10^{i-3} E_{p,i} + E_{p,6}$ when p is E_7-type; and $D_p = \sum_{i=2}^{6} 10^{i-2} E_{p,i} + E_{p,7}$ when p is E_8-type. This divisor is independent of the choice of an identification of the graphs.

Lemma 4.8. *For an arbitrary ample class $H \in NS_Y$ the divisor class*

$$10^{30} f^*H + 10^{20} \sum_{i=1}^{l} F_i + \sum_p D_p, \tag{4.6}$$

where p are the triple points of B, is an ι-invariant ample class on X.

Proof. The class (4.6) is the pullback of a divisor class L on Y'. Note the bounds $\sum_p \mathrm{rk}(\Lambda_p) \leq 20$ and $l \leq 10$ and apply the Nakai criterion to L. □

By Lemmas 4.7 and 4.8 we have a basis and a polarization of the lattice $L_+(X, \iota)$ defined explicitly in terms of (Y, B).

4.3. **Degree of period map.** Let Y be one of the following rational surfaces:

$$\mathbb{P}^2, \qquad \mathbb{P}^1 \times \mathbb{P}^1, \qquad \mathbb{F}_n \ \ (1 \leq n \leq 4).$$

Suppose that one is given an irreducible, $\mathrm{Aut}(Y)$-invariant locus

$$U \subset |-2K_Y|$$

such that (i) every member $B_u \in U$ satisfies Condition 4.6, (ii) the singularities of B_u of each type form an etale cover of U, and (iii) the number of components of B_u is constant. Then the 2-elementary $K3$ surfaces associated to the DPN pairs (Y, B_u)

have constant main invariant (r, a, δ), and one obtains a period map $p\colon U \to \mathcal{M}_{r,a,\delta}$. Since p is Aut(Y)-invariant, it descends to a rational map

$$\mathcal{P} : U/\mathrm{Aut}(Y) \dashrightarrow \mathcal{M}_{r,a,\delta}.$$

Here $U/\mathrm{Aut}(Y)$ stands for a rational quotient (see §2). In this section we try to explain how to calculate the degree of $\mathcal{P}$. Such calculations have been done in some cases (e.g., [26], [18], [3]), and the method to be explained is more or less a systemization of them. Roughly speaking, deg($\mathcal{P}$) expresses for a general $(X, \iota) \in \mathcal{M}_{r,a,\delta}$ how many contractions $X/\iota \to Y$ exist by which the branch of $X \to X/\iota$ is mapped to a member of U.

Below we exhibit a recipe of calculation supplemented by few typical examples. The recipe will be applied in the rest of the article to about fifty main invariants (r, a, δ). Since the materials are diverse, it seems hard to formulate those calculations into some general proposition. Instead, we shall give sufficient instruction for each (r, a, δ) and then leave the detail to the reader by referring to the recipe below, which we believe is not difficult to master.

We use a certain cover of $\mathcal{M}_{r,a,\delta}$. Let L_- be the 2-elementary lattice of signature $(2, 20-r)$ used in the definition of $\mathcal{M}_{r,a,\delta}$, and $\widetilde{\mathrm{O}}(L_-)$ be the group of isometries of L_- which act trivially on the discriminant group D_{L_-}. Let $\widetilde{\mathcal{M}}_{r,a,\delta}$ be the arithmetic quotient $\widetilde{\mathrm{O}}(L_-)\backslash\Omega_{L_-}$, which is irreducible for $\widetilde{\mathrm{O}}(L_-)$ has an element exchanging the two components of Ω_{L_-}. The natural projection

$$\pi : \widetilde{\mathcal{M}}_{r,a,\delta} \dashrightarrow \mathcal{M}_{r,a,\delta}$$

is a Galois covering. The Galois group is the orthogonal group $\mathrm{O}(D_{L_-})$ of the discriminant form, for the homomorphism $\mathrm{O}(L_-) \to \mathrm{O}(D_{L_-})$ is surjective ([29]) and $-1 \in \mathrm{O}(L_-)$ acts trivially on D_{L_-}. In particular, we have

$$\deg(\pi) = |\mathrm{O}(D_{L_-})|.$$

Since D_{L_-} is 2-elementary, one may calculate $|\mathrm{O}(D_{L_-})|$ by using [26] Corollary 2.4 and Lemma 2.5. We shall use standard notation for orthogonal/symplectic groups in characteristic 2: $\mathrm{O}^+(2n, 2)$, $\mathrm{O}^-(2n, 2)$, and $\mathrm{Sp}(2n, 2)$ (see [26], [13]).

The cover $\widetilde{\mathcal{M}}_{r,a,\delta}$ is birationally a moduli of 2-elementary $K3$ surfaces with lattice-marking. We fix a primitive embedding $L_- \subset \Lambda_{K3}$ where $\Lambda_{K3} = U^3 \oplus E_8^2$, an even hyperbolic 2-elementary lattice L_+ of main invariant (r, a, δ), and an isometry $L_+ \simeq (L_-)^\perp \cap \Lambda_{K3}$. Suppose that one is given a 2-elementary $K3$ surface $(X, \iota) \in \mathcal{M}_{r,a,\delta}$ and a lattice isometry $j\colon L_+ \to L_+(X, \iota)$. By Nikulin's theory [29], j can be extended to an isometry $\Phi\colon \Lambda_{K3} \to H^2(X, \mathbb{Z})$. Then $\Phi^{-1}(H^{2,0}(X))$ belongs to Ω_{L_-}, and we define the period of $((X, \iota), j)$ by $[\Phi^{-1}(H^{2,0}(X))] \in \widetilde{\mathcal{M}}_{r,a,\delta}$. Since the restriction $\Phi|_{L_+}$ is fixed, this definition does not depend on the choice of Φ. Two such objects $((X, \iota), j)$ and $((X', \iota'), j')$ have the same period in $\widetilde{\mathcal{M}}_{r,a,\delta}$ if and only if there exists a Hodge isometry $\Psi\colon H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ with $\Psi \circ j = j'$. The open set of $\widetilde{\mathcal{M}}_{r,a,\delta}$ over $\mathcal{M}_{r,a,\delta}$ parametrizes such periods of lattice-marked 2-elementary $K3$ surfaces.

In order to calculate $\deg(\mathcal{P})$, we construct a generically injective lift of $\mathcal{P}$,

$$\widetilde{\mathcal{P}} : \widetilde{U}/G \dashrightarrow \widetilde{\mathcal{M}}_{r,a,\delta},$$

where $\widetilde{U}$ is a certain cover of U and G is the identity component of $\mathrm{Aut}(Y)$. Then we compare the degree of the projection $\widetilde{U}/G \to U/\mathrm{Aut}(Y)$ with $|\mathrm{O}(D_{L_-})|$. Here is a more precise procedure.

(1) We define a cover $\widetilde{U} \to U$ in which $\widetilde{U}$ parametrizes pairs (B_u, μ) where $B_u \in U$ and μ is a "reasonable" labeling of the singularities, the branches at the D_{2n}-singularities, and the components of B_u.

(2) Let $(X, \iota) = p(B_u)$ be the 2-elementary $K3$ surface associated to (Y, B_u). By Lemma 4.7, the labeling μ and a natural basis of NS_Y induce a lattice marking j of $L_+(X, \iota)$. Actually, Lemma 4.7 implies an appropriate definition of the reference lattice L_+, and then j should be obtained naturally. Considering the period of $((X, \iota), j)$ as defined above, we obtain a lift $\tilde{p}\colon \widetilde{U} \to \widetilde{\mathcal{M}}_{r,a,\delta}$ of p. By Borel's extension theorem ([6]), $\tilde{p}$ is a morphism of varieties.

(3) One observes that the group G acts on $\widetilde{U}$ and that $\tilde{p}$ is G-invariant. Hence $\tilde{p}$ induces a rational map $\widetilde{\mathcal{P}} : \widetilde{U}/G \dashrightarrow \widetilde{\mathcal{M}}_{r,a,\delta}$, which is a lift of $\mathcal{P}$.

(4) We prove that $\widetilde{\mathcal{P}}$ is generically injective. For that it suffices to show that the $\tilde{p}$-fibers are G-orbits. If $\tilde{p}(B_u, \mu) = \tilde{p}(B_{u'}, \mu')$, we have a Hodge isometry $\Phi\colon H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ with $\Phi \circ j' = j$ for the associated $((X, \iota), j)$ and $((X', \iota'), j')$. By Lemma 4.8 Φ turns out to preserve the ample cones. Then we obtain an isomorphism $\varphi\colon X \to X'$ with $\varphi^* = \Phi$ by the Torelli theorem. Let $f\colon X \to Y$, $f'\colon X' \to Y$ be the right covering maps and $L \in \mathrm{Pic}(Y)$ be the bundle as in Lemma 4.5. Since $\varphi^*(f'^*L) = \Phi(f'^*L) = f^*L$, by Lemma 4.5 we obtain an automorphism ψ of Y with $\psi \circ f = f' \circ \varphi$. This will imply that $\psi(B_u, \mu) = (B_{u'}, \mu')$. Since ψ acts trivially on NS_Y, we have $\psi \in G$.

(5) Now assume that $\dim(U/\mathrm{Aut}(Y)) = 20 - r$. Since $\widetilde{\mathcal{M}}_{r,a,\delta}$ is irreducible, then $\widetilde{\mathcal{P}}$ is birational. Therefore $\deg(\mathcal{P})$ is equal to $|\mathrm{O}(D_{L_-})|$ divided by the degree of $\widetilde{U}/G \to U/\mathrm{Aut}(Y)$. The latter may be calculated geometrically.

In the above recipe the construction of $\widetilde{U}$ and $\widetilde{\mathcal{P}}$ is left rather ambiguous. It could be formulated generally using monodromies for the universal curve over U. But in order to give an effective account, we find it better to describe it by typical examples. (When $Y = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, the idea can also be found in [20].)

Example 4.9. We consider curves on $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Let $U \subset |\mathcal{O}_Q(3,3)| \times |\mathcal{O}_Q(1,1)|$ be the open set of pairs (C, H) such that C and H are smooth and transverse to each other. The space U parametrizes the nodal $-2K_Q$-curves $C + H$, and we obtain a period map $\mathcal{P}\colon U/\mathrm{Aut}(Q) \dashrightarrow \mathcal{M}_{8,6,1}$. In this case, the cover $\widetilde{U}$ should be

$$\widetilde{U} = \{(C, H, p_1, \cdots, p_6) \in U \times Q^6,\ \{p_i\}_{i=1}^6 = C \cap H\},$$

which parametrizes the curves $C + H$ endowed with a labeling of its six nodes $C \cap H$. The projection $\widetilde{U} \to U$ is an $\mathfrak{S}_6$-covering.

We shall prepare the lattice L_+. Let $\{u, v\}$ and $\{e_1, \cdots, e_6\}$ be natural basis of the lattices $U(2)$ and A_1^6 respectively. We define the vectors $f_1, f_2 \in (U(2) \oplus A_1^6)^\vee$ by $2f_1 = 3(u+v) - \sum_{i=1}^{6} e_i$ and $2f_2 = u + v - \sum_{i=1}^{6} e_i$. Then the overlattice $L_+ = \langle U(2) \oplus A_1^6, f_1, f_2 \rangle$ is even and 2-elementary of main invariant $(8, 6, 1)$.

For a $(C, H, \cdots, p_6) \in \widetilde{U}$, we let $(X, \iota) = \mathcal{P}(C, H)$ and $f : X \to Q$ be the right covering map. The fixed curve X^ι is decomposed as $X^\iota = F_1 + F_2$ such that $f(F_1) = C$ and $f(F_2) = H$. Then we define an embedding $j : L_+ \to L_+(X, \iota)$ of lattices by $u \mapsto [f^*\mathcal{O}_Q(1, 0)]$, $v \mapsto [f^*\mathcal{O}_Q(0, 1)]$, $e_i \mapsto [f^{-1}(p_i)]$, and $f_i \mapsto [F_i]$. By Lemma 4.7 we have $j(L_+) = L_+(X, \iota)$. This gives a lattice-marked 2-elementary $K3$ surface $((X, \iota), j)$, and we obtain a period map $\tilde{p} : \widetilde{U} \to \widetilde{\mathcal{M}}_{8,6,1}$.

The morphism $\tilde{p}$ is *not* $\mathrm{Aut}(Q)$-invariant for $\mathrm{Aut}(Q)$ may exchange $\mathcal{O}_Q(1, 0)$ and $\mathcal{O}_Q(0, 1)$. Rather $\tilde{p}$ is invariant under the identity component $G = (\mathrm{PGL}_2)^2$ of $\mathrm{Aut}(Q)$. We prove that the $\tilde{p}$-fibers are G-orbits. If two points $(C, H, \cdots, p_6)$ and $(C', H', \cdots, p_6')$ of $\widetilde{U}$ have the same $\tilde{p}$-period, we have a Hodge isometry $\Phi : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ with $\Phi \circ j' = j$ for the associated $((X, \iota), j)$ and $((X', \iota'), j')$. In particular, Φ maps $[(f')^*\mathcal{O}_Q(a, b)], [(f')^{-1}(p_i)], [F_j']$ respectively to $[f^*\mathcal{O}_Q(a, b)], [f^{-1}(p_i)], [F_j]$. By Lemma 4.8 Φ preserves the ample cones. Hence by the Torelli theorem we obtain an isomorphism $\varphi : X \to X'$ with $\varphi^* = \Phi$. Then we have an automorphism ψ of Q with $\psi \circ f = f' \circ \varphi$ by Lemma 4.5. Considering the branch loci of f and f', we have $\psi(C + H) = C' + H'$. Also we have $\psi(p_i) = p_i'$ because $\varphi(f^{-1}(p_i)) = (f')^{-1}(p_i')$. Since ψ leaves the two rulings of Q invariant, we have $\psi \in G$. This concludes that the $\tilde{p}$-fibers are G-orbits. In view of the equality $\dim(\widetilde{U}/G) = 12$, $\tilde{p}$ induces a birational lift $\widetilde{\mathcal{P}} : \widetilde{U}/G \dashrightarrow \widetilde{\mathcal{M}}_{8,6,1}$ of $\mathcal{P}$.

Since $L_+ \simeq U \oplus A_1^6$, we have $|\mathrm{O}(D_{L_+})| = 2 \cdot |\mathrm{Sp}(4, 2)| = 2 \cdot 6!$ by [26]. On the other hand, the projection $\widetilde{U}/G \to U/\mathrm{Aut}(Q)$ has degree $[G : \mathrm{Aut}(Q)] \cdot |\mathfrak{S}_6| = 2 \cdot 6!$ because $\mathrm{Aut}(Q)$ acts on U almost freely. Therefore $\mathcal{P}$ is birational.

Example 4.10. We consider curves on $\mathbb{F}_1$. Keeping the notation of §3, we let $U \subset |L_{3,2}| \times |L_{0,1}|$ be the locus of pairs (C, F) such that C is smooth, transverse to F and Σ respectively, and passes the point $p = F \cap \Sigma$. The locus U parametrizes the $-2K_{\mathbb{F}_1}$-curves $C + F + \Sigma$, whose singularities are the D_4-point p and the three nodes $C \cap (F + \Sigma) \backslash p$. Hence we obtain a period map $\mathcal{P} : U/\mathrm{Aut}(\mathbb{F}_1) \dashrightarrow \mathcal{M}_{9,3,1}$. In this case, $\widetilde{U}$ should be the double cover

$$\widetilde{U} = \{ (C, F, p_1, p_2) \in U \times (\mathbb{F}_1)^2, \ \{p_1, p_2\} = C \cap F \backslash \Sigma \}.$$

In $\widetilde{U}$ only the two nodes $C \cap F \backslash p$ are labelled, and the rest two singularities are left unmarked. However, the node $C \cap \Sigma \backslash p$ is distinguished from $C \cap F \backslash p$ by the irreducible decomposition of $C + F + \Sigma$, and the D_4-point p is evidently distinguished from the nodes. Thus $\widetilde{U}$ actually parametrizes the curves $C + F + \Sigma$ endowed with a complete and reasonable labeling of the singularities. Also the components of $C + F + \Sigma$ are distinguished by their classes in $NS_{\mathbb{F}_1}$, and this distinguishes the three branches of $C + F + \Sigma$ at p.

Let us prepare the lattice L_+. Let $\{h, e\}$ and $\{e_1, e_2, e_3\}$ be natural basis of the lattices $\langle 2 \rangle \oplus A_1$ and A_1^3 respectively. We denote the root basis of the D_4-lattice by

$\{f_4, e_5, e_6, e_7\}$ where $(f_4, e_i) = 1$ and $(e_i, e_j) = -2\delta_{ij}$. We put $e_4 = 2f_4 + \sum_{i=5}^{7} e_i$. Then we define the vectors $f_i \in (\langle 2\rangle \oplus A_1^4 \oplus D_4)^\vee$ by $2f_1 = 3h + 2(h-e) - \sum_{i=1}^{5} e_i$, $2f_2 = h - e - (e_1 + e_2 + e_4 + e_6)$, and $2f_3 = e - (e_3 + e_4 + e_7)$. The overlattice $L_+ = \langle \langle 2\rangle \oplus A_1^4 \oplus D_4, \{f_i\}_{i=1}^3 \rangle$ is even and 2-elementary of main invariant $(9, 3, 1)$.

For a $(C, F, p_1, p_2) \in \widetilde{U}$, let $(X, \iota) = \mathcal{P}(C, F)$ and $f\colon X \to \mathbb{F}_1$ be the right covering map. We denote $p = F \cap \Sigma$ as above. On X we define line bundles and (-2)-curves by $H = f^*L_{1,0}$, $E = f^*L_{1,-1}$, $E_i = f^{-1}(p_i)$ for $i \leq 2$, and $E_3 = f^{-1}(C \cap \Sigma \backslash p)$. By §4.2, the four (-2)-curves on X contracted by f to p form a Dynkin graph of type D_4. We denote by E_5, E_6, E_7 those over the infinitely near points of p given by C, F, Σ respectively. The remaining one, denoted by F_4, is a component of X^ι. We have an embedding $i\colon \langle 2\rangle \oplus A_1^4 \oplus D_4 \to L_+(X, \iota)$ by $h \mapsto [H]$, $e \to [E]$, $e_i \mapsto [E_i]$, and $f_4 \mapsto [F_4]$. The fixed curve X^ι is decomposed as $X^\iota = \sum_{j=1}^{4} F_j$ such that $f(F_1) = C$, $f(F_2) = F$, and $f(F_3) = \Sigma$. Then i extends to an isometry $j\colon L_+ \to L_+(X, \iota)$ by sending $f_j \mapsto [F_j]$. Thus we associate a lattice-marked 2-elementary $K3$ surface $((X, \iota), j)$, which defines a period map $\tilde{p}\colon \widetilde{U} \to \widetilde{\mathcal{M}}_{9,3,1}$.

Since $\mathrm{Aut}(\mathbb{F}_1)$ acts trivially on $NS_{\mathbb{F}_1}$, the morphism $\tilde{p}$ is $\mathrm{Aut}(\mathbb{F}_1)$-invariant. We show that the $\tilde{p}$-fibers are $\mathrm{Aut}(\mathbb{F}_1)$-orbits. Indeed, if $\tilde{p}(C, F, p_1, p_2) = \tilde{p}(C', F', p_1', p_2')$, we will obtain an isomorphism $\varphi\colon X \to X'$ with $\varphi^* \circ j' = j$ for the associated $((X, \iota), j)$ and $((X', \iota'), j')$, as in the previous example. Let $f\colon X \to \mathbb{F}_1$ and $f'\colon X' \to \mathbb{F}_1$ be the respective right covering maps. We fix a contraction $\pi\colon \mathbb{F}_1 \to \mathbb{P}^2$ of Σ. Since $\varphi^*(f')^*L_{1,0} \simeq f^*L_{1,0}$, by Lemma 4.5 we obtain an automorphism ψ of $\mathbb{P}^2$ such that $\psi \circ \pi \circ f = \pi \circ f' \circ \varphi$. The point is that ψ fixes $\pi(\Sigma)$, which is the unique D_6-singularity of the branch curves of both $\pi \circ f$ and $\pi \circ f'$. Therefore ψ lifts to an automorphism $\bar{\psi}$ of $\mathbb{F}_1$ with $\bar{\psi} \circ f = f' \circ \varphi$. The rest of the proof is similar to the previous example. Since $\dim(U/\mathrm{Aut}(\mathbb{F}_1)) = 11$, $\tilde{p}$ descends to a birational lift $\widetilde{\mathcal{P}}\colon \widetilde{U}/\mathrm{Aut}(\mathbb{F}_1) \dashrightarrow \widetilde{\mathcal{M}}_{9,3,1}$ of $\mathcal{P}$.

The projection $\widetilde{U}/\mathrm{Aut}(\mathbb{F}_1) \to U/\mathrm{Aut}(\mathbb{F}_1)$ is a double covering. On the other hand, since $L_+ \simeq U(2) \oplus E_7$, we have $\mathrm{O}(D_{L_+}) \simeq \mathfrak{S}_2$. Therefore $\mathcal{P}$ is birational.

Example 4.11. Let $U \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(2)|$ be the locus of pairs (C, Q) such that Q is a union of two distinct lines and C is a smooth quartic transverse to Q. The variety U parametrizes the sextics $C + Q$ having nine nodes, $C \cap Q$ and $\mathrm{Sing}(Q)$. Hence we obtain a period map $\mathcal{P}\colon U/\mathrm{PGL}_3 \dashrightarrow \mathcal{M}_{10,6,1}$. That the parity $\delta = 1$ follows from Lemma 4.4 (1). In this case, we define the cover $\widetilde{U}$ as the locus in $U \times (\mathbb{P}^2)^8$ of those $(C, Q, p_1, \cdots, p_8)$ such that $\{p_i\}_{i=1}^8 = C \cap Q$ and that $\{p_i\}_{i=1}^4$ belong to the same component of Q. Such a labeling $(p_1, \cdots, p_8)$ of the eight nodes takes into account the components of Q passing them. The remaining one node, $\mathrm{Sing}(Q)$, is naturally distinguished from those eight. Also the components of Q are labelled as $Q = L_1 + L_2$ where L_1 is the one passing $\{p_i\}_{i=1}^4$. In this way the nodes and components of $C + Q$ are labelled compatibly. The projection $\widetilde{U} \to U$ is an $\mathfrak{S}_2 \ltimes (\mathfrak{S}_4)^2$-covering.

We prepare the lattice L_+ as follows. Let h and $\{e_i\}_{i=1}^9$ be natural basis of the lattices $\langle 2\rangle$ and A_1^9 respectively. We define the vectors $f_1, f_2, f_3 \in (\langle 2\rangle \oplus A_1^9)^\vee$ by

$2f_1 = h - e_9 - \sum_{i=1}^{4} e_i$, $2f_2 = h - e_9 - \sum_{i=5}^{8} e_i$, and $2f_3 = 4h - \sum_{i=1}^{8} e_i$. Then the even overlattice $L_+ = \langle \langle 2 \rangle \oplus A_1^9,\ \{f_i\}_{i=1}^3 \rangle$ is 2-elementary of main invariant $(10, 6, 1)$.

For a $(C, Q, \cdots, p_8) \in \widetilde{U}$, let $(X, \iota) = \mathcal{P}(C, Q)$ and $f\colon X \to \mathbb{P}^2$ be the right covering map. The labeling induces an isometry $j\colon L_+ \to L_+(X, \iota)$ by $h \mapsto [f^*\mathcal{O}_{\mathbb{P}^2}(1)]$, $e_i \mapsto [f^{-1}(p_i)]$ for $i \le 8$, $e_9 \mapsto [f^{-1}(\mathrm{Sing}(Q))]$, and $f_j \mapsto [F_j]$ where F_j are the components of X^ι with $f(F_1) = L_1$, $f(F_2) = L_2$, and $f(F_3) = C$. Thus we obtain a period map $\tilde{p}\colon \widetilde{U} \to \widetilde{\mathcal{M}}_{10,6,1}$. One may recover the morphism f from $j(h)$ by Lemma 4.5, the points p_i from $j(e_i)$, and the sextic $C + Q$ from f as the branch locus. As before, these imply that $\tilde{p}$ descends to a generically injective lift $\widetilde{\mathcal{P}}\colon \widetilde{U}/\mathrm{PGL}_3 \dashrightarrow \widetilde{\mathcal{M}}_{10,6,1}$ of $\mathcal{P}$. Since $\dim(\widetilde{U}/\mathrm{PGL}_3) = 10$, $\widetilde{\mathcal{P}}$ is actually birational.

Since $L_+ \simeq U \oplus D_4 \oplus A_1^4$, we have $|\mathrm{O}(D_{L_+})| = 2^4 \cdot |\mathrm{O}^+(4, 2)| = 2^4 \cdot 72$ by [26]. On the other hand, the projection $\widetilde{U}/\mathrm{PGL}_3 \to U/\mathrm{PGL}_3$ has degree $2 \cdot (4!)^2$. This concludes that $\mathcal{P}$ is birational.

Example 4.12. We consider curves on $\mathbb{F}_2$. We keep the notation of §3. Let $U \subset |L_{3,1}|$ be the locus of smooth curves C such that the π-fiber F passing the point $p = C \cap \Sigma$ is tangent to C at p with multiplicity 2. The $-2K_{\mathbb{F}_2}$-curve $C + F + \Sigma$ has the D_6-singularity p, the node $q = C \cap F \backslash p$, and no other singularity. Taking the right resolution of $C + F + \Sigma$, we obtain a period map $\mathcal{P}\colon U/\mathrm{Aut}(\mathbb{F}_2) \dashrightarrow \mathcal{M}_{9,1,1}$. In this case, the cover $\widetilde{U}$ should be U itself. Indeed, the two singularities of $C + F + \Sigma$ are distinguished by their type, and the components of $C + F + \Sigma$ are distinguished by their classes in $NS_{\mathbb{F}_2}$. The latter also distinguishes the branches of $C + F + \Sigma$ at p. Everything is a priori labelled, and we need no additional marking.

We prepare the lattice L_+. Let $\{u, v\}$ and e_0 be natural basis of the lattices $U(2)$ and A_1 respectively, and $\{e_1, e_2, f_3, e_4, f_5, e_6\}$ be the root basis of the D_6-lattice whose numbering corresponds to the one given for the graph in p.11. We put $e_3 = 2f_3 + e_1 + e_2 + e_4$ and $e_5 = 2f_5 + e_3 + e_4 + e_6$. Then we define the vectors $f_1, f_2, f_4 \in (U(2) \oplus A_1 \oplus D_6)^\vee$ by $2f_1 = 3(u+v) + v - e_0 - e_1 - e_3 - e_5$, $2f_2 = v - e_0 - e_2 - e_3 - e_5$, and $2f_4 = u - v - e_5 - e_6$. The even overlattice $L_+ = \langle U(2) \oplus A_1 \oplus D_6,\ f_1, f_2, f_4 \rangle$ is 2-elementary of main invariant $(9, 1, 1)$.

For a curve $C \in U$, let $(X, \iota) = \mathcal{P}(C)$ and $f\colon X \to \mathbb{F}_2$ be the right covering map. Let Λ_p and Λ_q be the sublattices of $L_+(X, \iota)$ generated by the (-2)-curves contracted by f to p and q respectively. We have a canonical isometry $A_1 \to \Lambda_q$ mapping e_0 to the class of the (-2)-curve. For the D_6-point p, we assign e_1 and e_2 respectively to the branches of C and F. This uniquely determines an isometry $D_6 \to \Lambda_p$ mapping the root basis to the classes of the (-2)-curves. Also we define an isometry $U(2) \to f^*NS_{\mathbb{F}_2}$ by $a(u+v) + bv \mapsto [f^*L_{a,b}]$. These define an embedding $i\colon U(2) \oplus A_1 \oplus D_6 \hookrightarrow L_+(X, \iota)$. The fixed curve X^ι is decomposed as $X^\iota = \sum_{i=1}^{5} F_i$ such that $[F_3] = j(f_3), [F_5] = j(f_5)$, and $f(F_1) = C, f(F_2) = F, f(F_4) = \Sigma$. The assignment $f_i \mapsto [F_i]$ extends i to an isometry $j\colon L_+ \to L_+(X, \iota)$. Considering the period of $((X, \iota), j)$, we obtain a lift $\widetilde{\mathcal{P}}\colon U/\mathrm{Aut}(\mathbb{F}_2) \dashrightarrow \widetilde{\mathcal{M}}_{9,1,1}$ of $\mathcal{P}$. In this construction, one may recover the morphism f from the class $j(e+f)$ by Lemma 4.5 (desingularize the quadratic cone in $\mathbb{P}^3$). As before, this implies that $\widetilde{\mathcal{P}}$ is birational. Since $D_{L_+} \simeq \mathbb{Z}/2\mathbb{Z}$, we have actually $\widetilde{\mathcal{M}}_{9,1,1} = \mathcal{M}_{9,1,1}$. Therefore $\mathcal{P}$ is birational.

5. The case $k = 0$

In this section we prove that the spaces $\mathcal{M}_{r,r,1}$ with $2 \leq r \leq 9$ are rational. The quotient $Y = X/\iota$ of a general $(X, \iota) \in \mathcal{M}_{r,r,1}$ is a del Pezzo surface of degree $10 - r$. Let $\mathcal{M}_{DP}(d)$ be the moduli of del Pezzo surfaces of degree d (we exclude $\mathbb{P}^1 \times \mathbb{P}^1$). By the correspondence between 2-elementary $K3$ surfaces and right DPN pairs, we have a fibration $\mathcal{M}_{r,r,1} \dashrightarrow \mathcal{M}_{DP}(10 - r)$ whose fiber over Y is birational to $|-2K_Y|/\mathrm{Aut}(Y)$. When $r \leq 5$, Y has no moduli so that $\mathcal{M}_{r,r,1} \sim |-2K_Y|/\mathrm{Aut}(Y)$. The rationality of $\mathcal{M}_{r,r,1}$ is reduced to the $\mathrm{Aut}(Y)$-representation $H^0(-2K_Y)$. By contrast, when $r \geq 6$, $\mathcal{M}_{DP}(10 - r)$ has positive dimension and $\mathrm{Aut}(Y)$ is a small finite group. We then analyze the fibration $\mathcal{M}_{r,r,1} \dashrightarrow \mathcal{M}_{DP}(10-r)$ or the fixed curve map $\mathcal{M}_{r,r,1} \to \mathcal{M}_{11-r}$ to deduce the rationality.

5.1. $\mathcal{M}_{2,2,1}$ and one-nodal sextics.

Let Y be the blow-up of $\mathbb{P}^2$ at one point $p \in \mathbb{P}^2$. By associating to a smooth curve $B \in |-2K_Y|$ the double cover of Y branched over B, we have a birational map

$$\mathcal{P} : |-2K_Y|/\mathrm{Aut}(Y) \dashrightarrow \mathcal{M}_{2,2,1}.$$

Indeed, $\mathcal{P}$ is generically injective by the correspondence between 2-elementary $K3$ surfaces and right DPN pairs, and is dominant because $|-2K_Y|/\mathrm{Aut}(Y)$ has dimension 18. One may identify $|-2K_Y|$ with the linear system of plane sextics singular at p, and $\mathrm{Aut}(Y)$ with the stabilizer of p in PGL_3.

Proposition 5.1. *The quotient $|-2K_Y|/\mathrm{Aut}(Y)$ is rational. Therefore $\mathcal{M}_{2,2,1}$ is rational.*

Proof. Let $\Sigma \subset Y$ be the (-1)-curve. Let $\varphi\colon |-2K_Y| \dashrightarrow |\mathcal{O}_\Sigma(2)|$ be the $\mathrm{Aut}(Y)$-equivariant map defined by $B \to B|_\Sigma$. The group $\mathrm{Aut}(Y)$ acts on $|\mathcal{O}_\Sigma(2)|$ almost transitively. For two disinct points $p_1, p_2 \in \Sigma$, the fiber $\varphi^{-1}(p_1 + p_2)$ is an open set of the linear system $\mathbb{P}V$ of $-2K_Y$-curves passing p_1 and p_2. If $G \subset \mathrm{Aut}(Y)$ is the stabilizer of $p_1 + p_2$, by the slice method 2.3 we have $|-2K_Y|/\mathrm{Aut}(Y) \sim \mathbb{P}V/G$.

Let $F_i \subset Y$ be the strict transform of the line passing p with tangent p_i. Set $W = |\mathcal{O}_{F_1}(3)| \times |\mathcal{O}_{F_2}(3)|$ and consider the G-equivariant map

$$\psi : \mathbb{P}V \dashrightarrow W, \quad B \mapsto (B|_{F_1} - p_1, B|_{F_2} - p_2). \tag{5.1}$$

The fiber $\psi^{-1}(D_1, D_2)$ over a general $(D_1, D_2) \in W$ is an open set of a linear subspace of $\mathbb{P}V$. Since $-2K_Y$ is $\mathrm{Aut}(Y)$-linearized, G acts on V so that ψ is G-birational to the projectivization of a G-linearized vector bundle over an open set of W. The group G acts on W almost freely. Indeed, for a general $(D_1, D_2) \in W$ the four points $p_1 + D_1$ on $F_1 \simeq \mathbb{P}^1$ are not projectively equivalent to the four points $p_2 + D_2$ on F_2, and any nontrivial $g \in \mathrm{PGL}_2$ with $g(p_1 + D_1) = p_1 + D_1$ does not fix p_1. Clearly an automorphism of Y acting trivially on $F_1 + F_2$ must be trivial. Thus we may apply the no-name method to see that $\mathbb{P}V/G \sim \mathbb{P}^{16} \times (W/G)$. Since $\dim(W/G) = 2$, W/G is rational. □

Note that the natural moduli map $f\colon |-2K_Y|/\mathrm{Aut}(Y) \dashrightarrow \mathcal{M}_9$ is generically injective, for the normalization of a one-nodal plane sextic has only one g^2_6, the restriction of $|\mathcal{O}_{\mathbb{P}^2}(1)|$ (see [2] Appendix A.20). The composition $f \circ \mathcal{P}^{-1}$ is nothing but the fixed curve map $\mathcal{M}_{2,2,1} \to \mathcal{M}_9$. Therefore

Corollary 5.2. *The fixed curve map $\mathcal{M}_{2,2,1} \to \mathcal{M}_9$ is generically injective with a generic image being the locus of non-hyperelliptic, non-trigonal, non-bielliptic curves possessing g^2_6.*

5.2. $\mathcal{M}_{3,3,1}$ **and two-nodal sextics.** Let Y be the blow-up of $\mathbb{P}^2$ at two distinct points p_1, p_2. As in §5.1, we have a natural birational map $|-2K_Y|/\mathrm{Aut}(Y) \dashrightarrow \mathcal{M}_{3,3,1}$ by the double cover construction. One may identify $|-2K_Y|$ with the linear system of plane sextics singular at p_1, p_2, and $\mathrm{Aut}(Y)$ with the stabilizer of $p_1 + p_2$ in PGL_3. In this form, Casnati and del Centina [7] proved that $|-2K_Y|/\mathrm{Aut}(Y)$ is rational. Their proof is based on a direct calculation of an invariant field. Here we shall present another simple proof.

Proposition 5.3 ([7])**.** *The quotient $|-2K_Y|/\mathrm{Aut}(Y)$ is rational. Therefore $\mathcal{M}_{3,3,1}$ is rational.*

Proof. Let $E_i \subset Y$ be the (-1)-curve over p_i and let $W = |\mathcal{O}_{E_1}(2)| \times |\mathcal{O}_{E_2}(2)|$. We consider the map $\varphi\colon |-2K_Y| \dashrightarrow W$, $B \mapsto (B|_{E_1}, B|_{E_2})$, which is $\mathrm{Aut}(Y)$-equivariant. The identifications $E_i = \mathbb{P}(T_{p_i}\mathbb{P}^2)$ show that $\mathrm{Aut}(Y)$ acts on W almost transitively. If $\mathbf{q} = (q_{11} + q_{12}, q_{21} + q_{22})$ is a general point of W, let $L_{ij} \subset \mathbb{P}^2$ be the line passing p_i with tangent q_{ij}. Then the stabilizer $G \subset \mathrm{Aut}(Y)$ of $\mathbf{q}$ is identified with the group of $g \in \mathrm{PGL}_3$ which preserve $\sum_{i,j} L_{ij}$ and $p_1 + p_2$. In particular, $G \simeq \mathfrak{S}_2 \ltimes (\mathfrak{S}_2)^2$. The fiber $\varphi^{-1}(\mathbf{q})$ is an open set of the linear system $\mathbb{P}V$ of sextics passing $\{q_{ij}\}_{i,j}$. By the slice method we have $|-2K_Y|/\mathrm{Aut}(Y) \sim \mathbb{P}V/G$.

The net $\mathbb{P}V_0 = \overline{p_1p_2} + \sum_{i,j} L_{ij} + |\mathcal{O}_{\mathbb{P}^2}(1)|$ is a G-invariant linear subspace of $\mathbb{P}V$. Since G is finite, we can decompose the G-representation V as $V = V_0 \oplus V_0^{\perp}$. The projection $\mathbb{P}V \dashrightarrow \mathbb{P}V_0$ from $V_0^{\perp}$ is a G-linearized vector bundle. Since G acts on $\mathbb{P}V_0$ almost freely, by the no-name method we have $\mathbb{P}V/G \sim \mathbb{C}^{15} \times (\mathbb{P}V_0/G)$. The quotient $\mathbb{P}V_0/G$ is clearly rational. □

As in Corollary 5.2, we have the following.

Corollary 5.4. *The fixed curve map $\mathcal{M}_{3,3,1} \to \mathcal{M}_8$ is generically injective with a generic image being the locus of non-hyperelliptic, non-trigonal, non-bielliptic curves possessing g^2_6.*

5.3. $\mathcal{M}_{4,4,1}$ **and three-nodal sextics.** Let p_1, p_2, p_3 be three linearly independent points in $\mathbb{P}^2$. The blow-up Y of $\mathbb{P}^2$ at $p_1 + p_2 + p_3$ is a sextic del Pezzo surface. As before, we have a birational equivalence $|-2K_Y|/\mathrm{Aut}(Y) \sim \mathcal{M}_{4,4,1}$. One may identify $|-2K_Y|$ with the linear system of plane sextics singular at $p_1 + p_2 + p_3$, and $\mathrm{Aut}(Y)$ with the group $\mathfrak{S}_2 \ltimes G$ where G is the stabilizer of $p_1 + p_2 + p_3$ in PGL_3 and $\mathfrak{S}_2$ is generated by the standard Cremona transformation based at $p_1 + p_2 + p_3$. In this form, $|-2K_Y|/\mathrm{Aut}(Y)$ is proved to be rational by Casnati and del Centina [7].

Proposition 5.5 ([7])**.** *The space $\mathcal{M}_{4,4,1}$ is rational.*

Remark 5.6. The proof in [7] is by a direct calculation of an invariant field. Actually, it is also possible to give a geometric proof as in the previous sections, but not so short.

As noted in [7], the natural map $|-2K_Y|/\mathrm{Aut}(Y) \dashrightarrow \mathcal{M}_7$ is generically injective. This is a consequence of the classical fact that a generic three-nodal plane sextic has exactly two g^2_6, $|\mathcal{O}_{\mathbb{P}^2}(1)|$ and its transformation by the Cremona map. Thus

Corollary 5.7. *The fixed curve map for $\mathcal{M}_{4,4,1}$ is generically injective with a generic image being the locus of non-hyperelliptic, non-trigonal, non-bielliptic curves possessing g^2_6.*

5.4. $\mathcal{M}_{5,5,1}$ **and a quintic del Pezzo surface.** Let Y be a quintic del Pezzo surface. As in the previous sections, we have a birational map $|-2K_Y|/\mathrm{Aut}(Y) \dashrightarrow \mathcal{M}_{5,5,1}$ by the double cover construction (see also [3]). Shepherd-Barron [34] proved that $|-2K_Y|/\mathrm{Aut}(Y)$ is rational. Also it is classically known that the natural map $|-2K_Y|/\mathrm{Aut}(Y) \dashrightarrow \mathcal{M}_6$ is birational (cf. [34]). Therefore

Proposition 5.8 ([34], [3])**.** *The space $\mathcal{M}_{5,5,1}$ is rational. The fixed curve map $\mathcal{M}_{5,5,1} \to \mathcal{M}_6$ is birational.*

5.5. $\mathcal{M}_{6,6,1}$ **and genus five curves.** Let $\mathbb{P}V = |\mathcal{O}_{\mathbb{P}^4}(2)|$ and $\mathbb{G}(1, \mathbb{P}V)$ be the Grassmannian of pencils of quadrics in $\mathbb{P}^4$. Let $\mathcal{E} \to \mathbb{G}(1, \mathbb{P}V)$ be the universal quotient bundle. The fiber $\mathcal{E}_l$ over a pencil $l = \mathbb{P}W$ is the linear space $H^0(\mathcal{O}_{\mathbb{P}^4}(2))/W$. A general pencil l defines a smooth $(2,2)$ complete intersection Y_l in $\mathbb{P}^4$, which is an anticanonical model of a quartic del Pezzo surface (cf. [11]). One has the identification $\mathcal{E}_l = H^0(\mathcal{O}_{Y_l}(2)) = H^0(-2K_{Y_l})$ by a general property of complete intersections. Hence a general point of the bundle $\mathbb{P}\mathcal{E}$ corresponds to a pair (Y_l, B) of a quartic del Pezzo surface Y_l and a $-2K_{Y_l}$-curve B. This defines a period map $\mathbb{P}\mathcal{E} \dashrightarrow \mathcal{M}_{6,6,1}$, which descends to a rational map $\mathcal{P}\colon \mathbb{P}\mathcal{E}/\mathrm{PGL}_5 \dashrightarrow \mathcal{M}_{6,6,1}$. Since $Y_l \subset \mathbb{P}^4$ is an anticanonical model, we see that $\mathcal{P}$ is generically injective. The equality $\dim(\mathbb{P}\mathcal{E}/\mathrm{PGL}_5) = 14$ shows that $\mathcal{P}$ is birational.

The $-2K_{Y_l}$-curve on Y_l defined by a general point of $\mathbb{P}\mathcal{E}_l$ is a $(2,2,2)$ complete intersection in $\mathbb{P}^4$, which is a canonical genus five curve. We study $\mathbb{P}\mathcal{E}/\mathrm{PGL}_5$ from this viewpoint.

Let $\mathbb{G}(2, \mathbb{P}V)$ be the Grassmannian of nets of quadrics in $\mathbb{P}^4$, and $\mathcal{F} \to \mathbb{G}(2, \mathbb{P}V)$ be the universal sub bundle. The fiber $\mathcal{F}_P$ over a net $P = \mathbb{P}U$ is the linear subspace U of V. The bundle $\mathbb{P}\mathcal{E}$ parametrizes pairs (W, L) of a 2-plane $W \subset V$ and a line $L \subset V/W$, while the bundle $\mathbb{P}\mathcal{F}^\vee$ parametrizes pairs (U, H) of a 3-plane $U \subset V$ and a 2-plane $H \subset U$. These two objects canonically correspond by $(W, L) \mapsto (\langle W, L\rangle, W)$ and $(U, H) \mapsto (H, U/H)$. Thus we have a canonical PGL_5-isomorphism $\mathbb{P}\mathcal{E} \simeq \mathbb{P}\mathcal{F}^\vee$.

Let $P = \mathbb{P}U$ be a general point of $\mathbb{G}(2, \mathbb{P}V)$, and $B \subset \mathbb{P}^4$ be the $g = 5$ curve defined by P. One may identify $V = H^0(\mathcal{O}_{\mathbb{P}^4}(2))$ with $S^2H^0(K_B)$, and $\mathcal{F}_P = U$ with the kernel of the natural linear map $S^2H^0(K_B) \to H^0(2K_B)$, which is surjective by M. Noether. Let $\pi\colon \mathcal{X}_5 \to \mathcal{M}_5$ be the universal curve (over the open locus of curves with no automorphism), K_π be the relative canonical bundle for π, and $\mathcal{G}$ be the kernel of the bundle map $S^2\pi_* K_\pi \to \pi_* K_\pi^2$. Since $\mathbb{G}(2, \mathbb{P}V)/\mathrm{PGL}_5$ is naturally birational to $\mathcal{M}_5$, the above identification gives rise to the birational equivalence

$$(5.2) \qquad \mathbb{P}\mathcal{F}^\vee/\mathrm{PGL}_5 \sim \mathbb{P}\mathcal{G}^\vee \sim \mathcal{M}_5 \times \mathbb{P}^2.$$

The moduli space $\mathcal{M}_5$ is rational by Katsylo [16]. Therefore

Proposition 5.9. *The quotient $\mathbb{P}\mathcal{F}^\vee/\mathrm{PGL}_5$ is rational. Hence $\mathcal{M}_{6,6,1}$ is rational.*

The following assertion is also obtained.

Corollary 5.10. *The fixed curve map $\mathcal{M}_{6,6,1} \to \mathcal{M}_5$ is dominant, with the fiber over a general $B \in \mathcal{M}_5$ being birationally identified with $\mathbb{P}U^\vee$ where U is the kernel of the natural map $S^2H^0(K_B) \to H^0(2K_B)$.*

The first bundle structure $\mathbb{P}\mathcal{E} \to \mathbb{G}(1, \mathbb{P}V)$ corresponds to the quotient surface map $\mathcal{M}_{6,6,1} \dashrightarrow \mathcal{M}_{DP}(4)$, $(X, \iota) \mapsto X/\iota$, while the second one $\mathbb{P}\mathcal{F}^\vee \to \mathbb{G}(2, \mathbb{P}V)$ corresponds to the fixed curve map. The latter is easier to handle with thanks to the absence of automorphism of general $g = 5$ curves.

5.6. $\mathcal{M}_{7,7,1}$ **and cubic surfaces.** If $Y \subset \mathbb{P}^3$ is a cubic surface, the restriction map $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(-2K_Y)$ is isomorphic. Hence a general point of $U = |\mathcal{O}_{\mathbb{P}^3}(3)| \times |\mathcal{O}_{\mathbb{P}^3}(2)|$ corresponds to a pair (Y, B) of a smooth cubic surface Y and a smooth $-2K_Y$-curve B, which gives a 2-elementary $K3$ surface of type $(7, 7, 1)$. This induces a period map $\mathcal{P}\colon U/\mathrm{PGL}_4 \dashrightarrow \mathcal{M}_{7,7,1}$. Since cubic surfaces are anticanonically embedded, $\mathcal{P}$ is generically injective. By the equality $\dim(U/\mathrm{PGL}_4) = 13$, we see that $\mathcal{P}$ is birational.

Proposition 5.11. *The quotient U/PGL_4 is rational. Therefore $\mathcal{M}_{7,7,1}$ is rational.*

Proof. For a general $(Y, Q) \in U$ the intersection $B = Y \cap Q$ is a canonical $g = 4$ curve. This induces a rational map $f\colon U/\mathrm{PGL}_4 \dashrightarrow \mathcal{M}_4$, which is identified with the fixed curve map $\mathcal{M}_{7,7,1} \to \mathcal{M}_4$. The f-fiber over a general $B \in \mathcal{M}_4$ is the linear system of cubics containing B, for the quadric containing B is unique. Therefore, if $\pi\colon \mathcal{X}_4 \to \mathcal{M}_4$ is the universal curve (over an open locus) and $\mathcal{E}$ is the kernel of the bundle map $S^3\pi_* K_\pi \to \pi_* K_\pi^3$ where K_π is the relative canonical bundle, then we have the birational equivalence

$$(5.3) \qquad U/\mathrm{PGL}_4 \sim \mathbb{P}\mathcal{E} \sim \mathcal{M}_4 \times \mathbb{P}^4.$$

The rationality of $\mathcal{M}_4$ is proved by Shepherd-Barron [32]. □

Corollary 5.12. *The fixed curve map $\mathcal{M}_{7,7,1} \to \mathcal{M}_4$ is dominant, with the fiber over a general B being birationally identified with the projectivization of the kernel of the natural map $S^3H^0(K_B) \to H^0(3K_B)$.*

One may also deduce the rationality of $\mathcal{M}_{7,7,1}$ by applying the no-name lemma to the projection $U \to |\mathcal{O}_{\mathbb{P}^3}(3)|$ and resorting to the rationality of $\mathcal{M}_{DP}(3)$ (cf. [12]).

5.7. $\mathcal{M}_{8,8,1}$ **and quadric del Pezzo surfaces.** As before, one may use the fibration $\mathcal{M}_{8,8,1} \dashrightarrow \mathcal{M}_{DP}(2)$, $(X, \iota) \mapsto X/\iota$, to reduce the rationality of $\mathcal{M}_{8,8,1}$ to that of $\mathcal{M}_{DP}(2)$ due to Katsylo [17]. However, in order to describe the fixed curve map, we here adopt a more roundabout approach.

Recall that if Y is a quadric del Pezzo surface, its anticanonical map $\phi\colon Y \to \mathbb{P}^2$ is a double covering branched over a smooth quartic Γ. The correspondence $Y \mapsto \Gamma$ induces a birational map $\mathcal{M}_{DP}(2) \dashrightarrow \mathcal{M}_3$. The covering transformation i of ϕ is the *Geiser involution* of Y. Its fixed curve Y^i belongs to $|-2K_Y|$. Let $V_- \subset H^0(-2K_Y)$ be the line given by Y^i and let $V_+ = \phi^* H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \subset H^0(-2K_Y)$. We have the i-decomposition $H^0(-2K_Y) = V_+ \oplus V_-$ where $i^*|_{V_\pm} = \pm 1$.

Lemma 5.13. *Every smooth curve $B \in |-2K_Y|$ has genus 3, and the map $\phi|_B\colon B \to \mathbb{P}^2$ is a canonical map of B. In particular, B is hyperelliptic if and only if $B \in \mathbb{P}V_+$.*

Proof. The first sentence follows from the adjunction formula $-K_Y|_B \simeq K_B$ and the vanishings $h^0(K_Y) = h^1(K_Y) = 0$. By the ι-decomposition of $H^0(-2K_Y)$ we have $i(B) = B$ if and only if $B \in \mathbb{P}V_\pm$. Hence $\phi|_B$ is generically injective (actually an embedding) unless $B \in \mathbb{P}V_+$, which proves the last assertion. □

Let $M_\Gamma \subset |\mathcal{O}_{\mathbb{P}^2}(4)|$ be the cone over the locus of double conics $2Q$, $Q \in |\mathcal{O}_{\mathbb{P}^2}(2)|$, with vertex $\Gamma \in |\mathcal{O}_{\mathbb{P}^2}(4)|$. It is the closure of the locus of smooth quartics tangent to Γ at eight points lying on a conic.

Lemma 5.14. *The morphism $\phi_*\colon |-2K_Y| \to |\mathcal{O}_{\mathbb{P}^2}(4)|$ induces an isomorphism between $|-2K_Y|/i$ and M_Γ which maps the pencils $\langle \phi^* Q, Y^i \rangle$, $Q \in |\mathcal{O}_{\mathbb{P}^2}(2)|$, to the pencils $\langle 2Q, \Gamma \rangle$.*

Proof. If $B \in \langle \phi^* Q, Y^i \rangle$, then $B|_{Y^i} = \phi^* Q|_{Y^i} = \phi^{-1}(Q \cap \Gamma)$ so that $\phi_* B$ is tangent to Γ at $Q \cap \Gamma$. Hence $\phi_*(|-2K_Y|) \subset M_\Gamma$, and the equality $\dim M_\Gamma = \dim |-2K_Y|$ proves the assertion. □

Now let $U \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(4)|$ be the locus of pairs (Γ, Γ') such that Γ and Γ' are smooth and tangent to each other at eight points lying on a conic. For a $(\Gamma, \Gamma') \in U$ we take the double cover $\phi\colon Y \to \mathbb{P}^2$ branched over Γ. By Lemma 5.14 we have $\phi^* \Gamma' = B + i(B)$ for a smooth $B \in |-2K_Y|$ where i is the Geiser involution of Y. Taking the 2-elementary $K3$ surface corresponding to $(Y, B) \simeq (Y, i(B))$, we obtain a well-defined morphism $U \to \mathcal{M}_{8,8,1}$, which descends to a rational map $\mathcal{P}\colon U/\mathrm{PGL}_3 \dashrightarrow \mathcal{M}_{8,8,1}$.

Proposition 5.15. *The map $\mathcal{P}$ is birational.*

Proof. By the birational equivalence $\mathcal{M}_{DP}(2) \sim \mathcal{M}_3$ and Lemma 5.14, the first projection $U \to |\mathcal{O}_{\mathbb{P}^2}(4)|$, $(\Gamma, \Gamma') \mapsto \Gamma$, induces a fibration $U/\mathrm{PGL}_3 \to \mathcal{M}_{DP}(2)$ whose fiber over a general $Y \in \mathcal{M}_{DP}(2)$ is an open set of $|-2K_Y|/i$. Since $\mathrm{Aut}(Y) = \langle i \rangle$ for a general Y, this shows that $\mathcal{P}$ is generically injective. The equality $\dim(U/\mathrm{PGL}_3) = 12$ concludes the proof. □

Proposition 5.16. *The quotient U/PGL_3 is rational. Hence $\mathcal{M}_{8,8,1}$ is rational.*

Proof. We have a PGL_3-equivariant dominant morphism

$$\varphi : U \to |\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(2)|, \qquad (\Gamma, \Gamma') \mapsto (\Gamma, Q),$$

where Q is the unique conic with $2Q$ contained in the pencil $\langle \Gamma, \Gamma' \rangle$. The φ-fiber over a general (Γ, Q) is identified with an open set of the pencil $\langle 2Q, \Gamma \rangle$ in $|\mathcal{O}_{\mathbb{P}^2}(4)|$. Then we may use the no-name lemma 2.5 to see that

$$U/\mathrm{PGL}_3 \sim \mathbb{P}^1 \times (|\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(2)|)/\mathrm{PGL}_3.$$

The quotient $(|\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(2)|)/\mathrm{PGL}_3$ is rational by the slice method for the projection $|\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(2)| \to |\mathcal{O}_{\mathbb{P}^2}(2)|$ and Katsylo's theorem 2.2. □

By $\mathcal{P}$ we identify $\mathcal{M}_{8,8,1}$ birationally with U/PGL_3. The first projection $U \to |\mathcal{O}_{\mathbb{P}^2}(4)|$, $(\Gamma, \Gamma') \mapsto \Gamma$, induces the quotient surface map

$$Q : \mathcal{M}_{8,8,1} \dashrightarrow \mathcal{M}_{DP}(2) \sim \mathcal{M}_3, \qquad (X, \iota) \mapsto X/\iota.$$

On the other hand, the second projection $U \to |\mathcal{O}_{\mathbb{P}^2}(4)|$, $(\Gamma, \Gamma') \mapsto \Gamma'$, induces the fixed curve map $F \colon \mathcal{M}_{8,8,1} \to \mathcal{M}_3$. Let J be the rational involution of $\mathcal{M}_{8,8,1}$ induced by the involution $(\Gamma, \Gamma') \mapsto (\Gamma', \Gamma)$ of U. Now we know that

Proposition 5.17. *The two fibrations $F, Q \colon \mathcal{M}_{8,8,1} \dashrightarrow \mathcal{M}_3$ are exchanged by the involution J of $\mathcal{M}_{8,8,1}$. In particular, the generic fiber $F^{-1}(\Gamma')$ of F is birationally identified with the cone $M_{\Gamma'}$ in $\mathbb{P}(S^4H^0(K_{\Gamma'}))$.*

The fixed locus of J contains the locus $\mathcal{B} \subset \mathcal{M}_{8,8,1}$ of pairs $(\Gamma, \Gamma) \in U$, $\Gamma \in |\mathcal{O}_{\mathbb{P}^2}(4)|$. Generically, $\mathcal{B}$ may be characterized either as the locus of (1) the vertices of F-fibers $M_{\Gamma'}$, $\Gamma' \in \mathcal{M}_3$, of (2) the vertices of Q-fibers $|-2K_Y|/i$, $Y \in \mathcal{M}_{DP}(2)$, and of (3) quadruple covers of $\mathbb{P}^2$ branched over smooth quartics. Via the last description, $\mathcal{B}$ admits the structure of a ball quotient by a result of Kondō [19].

5.8. **$\mathcal{M}_{9,9,1}$ and del Pezzo surfaces of degree** 1. Let Y be a del Pezzo surface of degree 1. The bi-anticanonical map $\phi_{-2K_Y} \colon Y \to \mathbb{P}^3$ is a degree 2 morphism onto a quadratic cone Q, which maps the base point of $|-K_Y|$ to the vertex p_0 of Q, and which is branched over a smooth curve $C \in |\mathcal{O}_Q(3)|$ and p_0. In view of this, we may define a map $|\mathcal{O}_Q(3)| \times |\mathcal{O}_Q(1)| \dashrightarrow \mathcal{M}_{9,9,1}$ as follows. For a general $(C, H) \in |\mathcal{O}_Q(3)| \times |\mathcal{O}_Q(1)|$ we take the double cover $Y \to Q$ branched over C and p_0, and let $B \in |-2K_Y|$ be the pullback of H. (Equivalently, we take the double cover of the desingularization $\mathbb{F}_2$ of Q branched over $C + \Sigma$, and then contract the (-1)-curve over Σ.) Then we associate the 2-elementary $K3$ surface corresponding to the right DPN pair (Y, B). This construction, being $\mathrm{Aut}(Q)$-invariant, defines a period map $\mathcal{P} \colon (|\mathcal{O}_Q(3)| \times |\mathcal{O}_Q(1)|)/\mathrm{Aut}(Q) \dashrightarrow \mathcal{M}_{9,9,1}$. Since the double cover $Y \to Q$ is a bi-anticanonical map of Y, $\mathcal{P}$ is generically injective. Then $\mathcal{P}$ is birational because $(|\mathcal{O}_Q(3)| \times |\mathcal{O}_Q(1)|)/\mathrm{Aut}(Q)$ has dimension 11.

Proposition 5.18. *The quotient $(|\mathcal{O}_Q(3)| \times |\mathcal{O}_Q(1)|)/\mathrm{Aut}(Q)$ is rational. Therefore $\mathcal{M}_{9,9,1}$ is rational.*

Proof. We may apply the no-name lemma 2.5 to the projection $|\mathcal{O}_Q(3)| \times |\mathcal{O}_Q(1)| \to |\mathcal{O}_Q(3)|$ to see that $(|\mathcal{O}_Q(3)| \times |\mathcal{O}_Q(1)|)/\mathrm{Aut}(Q) \sim \mathbb{P}^3 \times (|\mathcal{O}_Q(3)|/\mathrm{Aut}(Q))$. The quotient $|\mathcal{O}_Q(3)|/\mathrm{Aut}(Q)$ is birational to $\mathcal{M}_{DP}(1)$, which is rational by Dolgachev [12]. □

6. The case $g \geq 7$

In this section we prove that the spaces $\mathcal{M}_{r,a,\delta}$ with $g \geq 7$ and $k > 0$ are rational. In §6.1 we construct birational period maps using trigonal curves of fixed Maroni invariant. Then we prove the rationality for each space.

6.1. **Period maps and Maroni loci.** We first construct birational period maps for $\mathcal{M}_{r,r-2,\delta}$ with $2 \leq r \leq 5$ using curves on the Hirzebruch surface $\mathbb{F}_{6-r}$. We keep the notation of §3. Let $U_r \subset |L_{3,r-2}|$ be the open set of smooth curves which are transverse to the $(r-6)$-curve Σ. For each $C \in U_r$ the curve $C + \Sigma$ belongs to $|-2K_{\mathbb{F}_{6-r}}|$ and has the $r-2$ nodes $C \cap \Sigma$ as the singularities. Considering the 2-elementary $K3$ surfaces associated to the DPN pairs $(\mathbb{F}_{6-r}, C+\Sigma)$, we obtain a period map $\mathcal{P}_r \colon U_r/\mathrm{Aut}(\mathbb{F}_{6-r}) \dashrightarrow \mathcal{M}_{r,r-2,\delta}$.

Proposition 6.1. *The map $\mathcal{P}_r$ is birational.*

Proof. Of course one may follow the recipe in §4.3, but here a more direct proof is possible. Indeed, by Proposition 3.8, $U_r/\mathrm{Aut}(\mathbb{F}_{6-r})$ is naturally birational to the moduli $\mathcal{T}_{12-r,6-r}$ of trigonal curves of genus $12-r$ and scroll invariant $6-r$. Then $\mathcal{P}_r$ is generically injective because the fixed curve map for $\mathcal{M}_{r,r-2,\delta}$ gives the left inverse. By comparison of dimensions, $\mathcal{P}_r$ is birational. □

Corollary 6.2. *The fixed curve map for $\mathcal{M}_{r,r-2,\delta}$ with $2 \leq r \leq 5$ is generically injective, with a generic image the Maroni locus $\mathcal{T}_{12-r,6-r}$ of Maroni invariant* 2.

Next we construct a birational period map for $\mathcal{M}_{6,2,0}$ using curves on $\mathbb{F}_3$. Let $U \subset |L_{3,0}|\times|L_{0,1}|$ be the open set of pairs (C, F) such that C is smooth and transverse to F. For each $(C, F) \in U$ the curve $C + F + \Sigma$ belongs to $|-2K_{\mathbb{F}_3}|$ and has the four nodes $F \cap (C + \Sigma)$ as the singularities. By taking the right resolution of $C + F + \Sigma$, we obtain a period map $\mathcal{P} \colon U/\mathrm{Aut}(\mathbb{F}_3) \dashrightarrow \mathcal{M}_{6,2,0}$.

Proposition 6.3. *The map $\mathcal{P}$ is birational.*

Proof. We proceed as in Example 4.10. Consider the following $\mathfrak{S}_3$-cover of U:

$$\widetilde{U} = \{(C, F, p_1, p_2, p_3) \in U \times (\mathbb{F}_3)^3,\ \{p_1, p_2, p_3\} = C \cap F\}.$$

The variety $\widetilde{U}$ parametrizes the curves $C + F + \Sigma$ equipped with labelings of the three nodes $C \cap F$. The rest one node $F \cap \Sigma$ is distinguished from those three by the irreducible decomposition of $C + F + \Sigma$. Therefore we will obtain a generically injective lift $\widetilde{\mathcal{P}} \colon \widetilde{U}/\mathrm{Aut}(\mathbb{F}_3) \dashrightarrow \widetilde{\mathcal{M}}_{6,2,0}$ of $\mathcal{P}$. Since $\widetilde{U}/\mathrm{Aut}(\mathbb{F}_3)$ has dimension 14, $\widetilde{\mathcal{P}}$ is birational. The projection $\widetilde{U}/\mathrm{Aut}(\mathbb{F}_3) \dashrightarrow U/\mathrm{Aut}(\mathbb{F}_3)$ is an $\mathfrak{S}_3$-covering, while the projection $\widetilde{\mathcal{M}}_{6,2,0} \dashrightarrow \mathcal{M}_{6,2,0}$ is an $\mathrm{O}(D_{L_+})$-covering for the lattice $L_+ = U \oplus D_4$. It is easy to see that $\mathrm{O}(D_{L_+}) \simeq \mathfrak{S}_3$. Hence $\mathcal{P}$ is birational. □

The quotient $|L_{3,0}|/\mathrm{Aut}(\mathbb{F}_3)$ is birationally identified with the Maroni locus $\mathcal{T}_{7,3}$, and the fibers of the projection $(|L_{3,0}| \times |L_{0,1}|)/\mathrm{Aut}(\mathbb{F}_3) \dashrightarrow |L_{3,0}|/\mathrm{Aut}(\mathbb{F}_3)$ are the trigonal pencils. Therefore

Corollary 6.4. *The fixed curve map for $\mathcal{M}_{6,2,0}$ gives a dominant map $\mathcal{M}_{6,2,0} \dashrightarrow \mathcal{T}_{7,3}$ whose general fibers are identified with the trigonal pencils.*

Remark 6.5. The locus $\mathcal{T}_{7,3}$ is the complement of $\mathcal{T}_{7,1}$ in the moduli of trigonal curves. Corollary 6.4 may also be obtained from Corollary 6.2 for $r = 5$ by extending the fixed curve map for $\mathcal{M}_{5,3,1}$ to a component of the discriminant divisor.

Remark 6.6. The Maroni loci $\mathcal{T}_{g,n}$ in this section may be characterized in terms of special divisors: a trigonal curve C of genus $8 \le g \le 10$ (resp. $g = 7$) has Maroni invariant 2 (resp. 1) if and only if $W^1_{g-3}(C)$ (resp. $W^1_5(C)$) is irreducible. This follows from Maroni's description of $W^r_d(C)$ (see [22] Proposition 1).

6.2. **$\mathcal{M}_{2,0,0}$ and Jacobian $K3$ surfaces.** By a *Jacobian $K3$ surface* we mean a $K3$ surface endowed with an elliptic fibration and with its zero section.

Proposition 6.7. *The space $\mathcal{M}_{2,0,0}$ is birational to the moduli space of Jacobian $K3$ surfaces. Therefore $\mathcal{M}_{2,0,0}$ is rational.*

Proof. In Proposition 6.1 we saw that a general member (X, ι) of $\mathcal{M}_{2,0,0}$ is canonically a double cover $f : X \to \mathbb{F}_4$ branched over a smooth curve $C + \Sigma$, $C \in |L_{3,0}|$. The natural projection $\mathbb{F}_4 \to \mathbb{P}^1$ gives rise to an elliptic fibration $X \to \mathbb{P}^1$, and the (-2)-curve $E = f^{-1}(\Sigma)$ is its section. The involution ι is the inverse map of the fibration with respect to E. In this way a general member of $\mathcal{M}_{2,0,0}$ is equipped with the structure of a Jacobian $K3$ surface. Conversely, for a Jacobian $K3$ surface $(X \to \mathbb{P}^1, E)$ such that every singular fiber is irreducible, the inversion map ι of $X/\mathbb{P}^1$ with respect to E is a non-symplectic involution of X, and the quotient of $X \to \mathbb{P}^1$ by ι is the natural projection $\mathbb{F}_n \to \mathbb{P}^1$ of a Hirzebruch surface $\mathbb{F}_n$. As the image of E in $\mathbb{F}_n$ is a (-4)-curve, we have $n = 4$. Thus our first assertion is verified. It is known that the moduli of Jacobian $K3$ surfaces is birational to the quotient

$$(H^0(\mathcal{O}_{\mathbb{P}^1}(8)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(12)))/\mathbb{C}^\times \times \mathrm{SL}_2,$$

via the Weierstrass forms of elliptic fibrations (for example, see [24]). By Katsylo's theorem 2.2 this quotient is rational. □

6.3. **The rationality of $\mathcal{M}_{3,1,1}$.** By Proposition 6.1 we have a birational equivalence $\mathcal{M}_{3,1,1} \sim |L_{3,1}|/\mathrm{Aut}(\mathbb{F}_3)$ for the bundle $L_{3,1}$ on $\mathbb{F}_3$.

Proposition 6.8. *The quotient $|L_{3,1}|/\mathrm{Aut}(\mathbb{F}_3)$ is rational. Hence $\mathcal{M}_{3,1,1}$ is rational.*

Proof. We apply the slice method to the map $|L_{3,1}| \dashrightarrow \Sigma$, $C \mapsto C|_\Sigma$. Let $G \subset \mathrm{Aut}(\mathbb{F}_3)$ be the stabilizer of a point $p \in \Sigma$, and $\mathbb{P}V \subset |L_{3,1}|$ be the linear system of curves passing p. Then we have $|L_{3,1}|/\mathrm{Aut}(\mathbb{F}_3) \sim \mathbb{P}V/G$. The group G is connected and solvable by Proposition 3.3, and G acts linearly on V by Proposition 3.2. Hence $\mathbb{P}V/G$ is rational by Miyata's theorem 2.1. □

6.4. **The rationality of $\mathcal{M}_{4,2,1}$.** By Proposition 6.1 we have a birational equivalence $\mathcal{M}_{4,2,1} \sim |L_{3,2}|/\mathrm{Aut}(\mathbb{F}_2)$ for the bundle $L_{3,2}$ on $\mathbb{F}_2$.

Proposition 6.9. *The quotient $|L_{3,2}|/\mathrm{Aut}(\mathbb{F}_2)$ is rational. Hence $\mathcal{M}_{4,2,1}$ is rational.*

Proof. First we define an $\mathrm{Aut}(\mathbb{F}_2)$-equivariant map $\varphi_1 : |L_{3,2}| \dashrightarrow S^2\mathbb{F}_2$ to the symmetric product of $\mathbb{F}_2$ as follows. For a general $C \in |L_{3,2}|$, let $C|_\Sigma = p_1 + p_2$ and F_i be the π-fiber passing p_i. We have a unique involution ι_{F_i} of F_i which fixes p_i and exchanges the two points $C|_{F_i} - p_i$. Then we let $q_i \in F_i$ be the fixed point of ι_{F_i} other than p_i, and set $\varphi_1(C) = q_1 + q_2$. The φ_1-fiber over a general $q_1 + q_2 \in S^2\mathbb{F}_2$ is the *linear* space $\mathbb{P}V_1$ of curves C with $C|_\Sigma = p_1 + p_2$ and $\iota_i(C|_{F_i}) = C|_{F_i}$, where F_i is the π-fiber passing q_i, p_i is $F_i \cap \Sigma$, and ι_i is the involution of F_i fixing q_i and p_i. If $G_1 \subset \mathrm{Aut}(\mathbb{F}_2)$ is the stabilizer of $q_1 + q_2$, by the slice method we have

$$|L_{3,2}|/\mathrm{Aut}(\mathbb{F}_2) \sim \mathbb{P}V_1/G_1.$$

Next we consider the G_1-equivariant map

$$\varphi_2 : \mathbb{P}V_1 \dashrightarrow \mathbb{P}(T_{p_1}\mathbb{F}_2) \times \mathbb{P}(T_{p_2}\mathbb{F}_2), \quad C \mapsto (T_{p_1}C, T_{p_2}C).$$

The φ_2-fiber over a general (v_1, v_2) is the sub linear system $\mathbb{P}V_2 \subset \mathbb{P}V_1$ of curves passing v_i or singular at p_i. One checks that G_1 acts almost transitively on $\mathbb{P}(T_{p_1}\mathbb{F}_2) \times \mathbb{P}(T_{p_2}\mathbb{F}_2)$. If $G_2 \subset G_1$ is the stabilizer of (v_1, v_2), by the slice method we have

$$\mathbb{P}V_1/G_1 \sim \mathbb{P}V_2/G_2.$$

We analyze the G_2-representation V_2 by using the coordinate system $\{U_i\}_{i=1}^4$ of $\mathbb{F}_2$ introduced in §3.4. We may assume that q_1 (resp. q_2) is the origin $(x_3, y_3) = (0,0)$ in U_3 (resp. $(x_4, y_4) = (0,0)$ in U_4). Then p_i is the origin $(x_i, y_i) = (0,0)$ in U_i, $i = 1,2$. We may also assume that $v_i \in \mathbb{P}(T_{p_i}\mathbb{F}_2)$ is expressed as $v_i = \mathbb{C}(x_i + y_i)$ by the coordinate (x_i, y_i) of U_i around p_i. Let $g_{\alpha,s}$, h_β, and j be the automorphisms of $\mathbb{F}_2$ described in the equations (3.4), (3.5), and (3.6) respectively. We set $\rho = g_{-\sqrt{-1},0} \circ h_{\sqrt{-1}}$.

Lemma 6.10. *For p_i, q_i, v_i as above, the stabilizer G_2 is given by*

$$G_2 = \langle j \rangle \ltimes (\langle \rho \rangle \ltimes \{g_{1,\lambda XY}\}_{\lambda\in\mathbb{C}}) \simeq \mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/4\mathbb{Z} \ltimes \mathbb{C}).$$

Here j acts on $\mathbb{Z}/4\mathbb{Z} \ltimes \mathbb{C}$ by $(-1, 1)$, and ρ acts on $\mathbb{C}$ by -1.

Proof. First observe that j, ρ, and $g_{1,\lambda XY}$ are contained in G_2. Conversely, we set $G_2' = \{g \in G_2, g(q_i) = q_i\}$. The quotient G_2/G_2' is $\mathbb{Z}/2\mathbb{Z}$ generated by j. The G_2'-action on Σ is contained in $\{h_\beta|_\Sigma\}_{\beta\in\mathbb{C}^\times}$. Hence by the sequence (3.1) every element of G_2' is written as $g_{\alpha,s} \circ h_\beta$ for some $g_{\alpha,s} \in R$ and $\beta \in \mathbb{C}^\times$. Since $g_{\alpha,0}$ and h_β fix q_i, we have $g_{1,s}(q_i) = q_i$ so that $s = \lambda XY$ for some $\lambda \in \mathbb{C}$. This implies that $g_{\alpha,0} \circ h_\beta \in G_2'$. Then $g_{\alpha,0} \circ h_\beta$ acts on U_1 by $(x_1, y_1) \mapsto (\beta x_1, \alpha^{-1} y_1)$, and on U_2 by $(x_2, y_2) \mapsto (\beta^{-1} x_2, \alpha^{-1}\beta^2 y_2)$. Thus we must have $\alpha\beta = 1$ and $\beta^4 = 1$. □

Next we describe the linear system $\mathbb{P}V_2$ by the coordinates. By Proposition 3.9 the space $H^0(L_{3,2})$ is isomorphic to the vector space $\{\sum_{i=0}^3 f_i(x_1)y_1^i, \deg f_i \leq 2i+2\}$ by restriction to U_1. We shall express the polynomials f_i as $f_i(x) = \sum_{j=0}^{2i+2} a_{ij}x^j$.

Lemma 6.11. *The linear subspace $V_2 \subset H^0(L_{3,2})$ is defined by the equations*

$$a_{00} = a_{02} = a_{20} = a_{26} = 0, \quad a_{01} = a_{10} = a_{14}. \tag{6.1}$$

Proof. Let $C \in |L_{3,2}|$ be defined by $\sum_{i=0}^3 f_i(x_1)y_1^i = 0$. The condition that $p_1 \in C$ (resp. $p_2 \in C$) is expressed by $a_{00} = 0$ (resp. $a_{02} = 0$). The condition on the tangent $T_{p_1}C$ (resp. $T_{p_2}C$) is written as $a_{01} = a_{10}$ (resp. $a_{01} = a_{14}$). The restriction of C to the fiber $\{x_1 = 0\}$ (resp. $\{x_2 = 0\}$) is given by $\sum_{i=0}^3 a_{i0}y_1^i = 0$ (resp.

$\sum_{i=0}^{3} a_{i,2i+2}y_2^i = 0$). Since these polynomials should be anti-symmetric, we have $a_{20} = 0$ and $a_{26} = 0$. □

Using the coordinate U_3, we identify V_2 with the subspace of $\{\sum_{i=0}^{3} f_i(x_3)y_3^{3-i}\}$ defined by (6.1), and calculate the G_2-action on it. The coefficients $a_{01} = a_{10} = a_{14}, a_{11}, a_{12}, \cdots$ of the polynomials are basis of the dual space $V_2^\vee$. The subspace $\mathbb{C}\langle a_{01}, a_{12}\rangle \subset V_2^\vee$ is G_2-invariant. Indeed, we have

$$\begin{aligned} g_{1,\lambda XY} &: (a_{01}, a_{12}) \mapsto (a_{01}, a_{12} - 3\lambda a_{01}), \\ \rho &: (a_{01}, a_{12}) \mapsto (-a_{01}, a_{12}), \\ j &: (a_{01}, a_{12}) \mapsto (a_{01}, a_{12}). \end{aligned}$$

Therefore the annihilator of $\mathbb{C}\langle a_{01}, a_{12}\rangle$,

$$V_3 = \{F(x_3, y_3) \in V_2, a_{01} = a_{12} = 0\},$$

is a G_2-invariant subspace of V_2. We want to apply the slice method to the projection $\mathbb{P}V_2 \dashrightarrow \mathbb{P}(V_2/V_3)$ from V_3, which is G_2-equivariant. We have the coordinate $(a_{01}, a_{12})\colon V_2/V_3 \to \mathbb{C}^2$ for V_2/V_3. Then $a = a_{01}^{-1}a_{12}$ is an inhomogeneous coordinate of $\mathbb{P}(V_2/V_3)$. By the above calculation, G_2 acts on $\mathbb{P}(V_2/V_3)$ by

$$g_{1,\lambda XY}(a) = a - 3\lambda, \qquad \rho(a) = -a, \qquad j(a) = a.$$

This shows that G_2 acts on the affine line $\mathbb{P}(V_2/V_3)\backslash\{a_{01} = 0\}$ transitively with the stabilizer of the point $p_0 = \{a_{12} = 0\}$ being $G_3 = \langle j\rangle \ltimes \langle\rho\rangle$. The fiber of the projection $\mathbb{P}V_2 - \mathbb{P}V_3 \to \mathbb{P}(V_2/V_3)$ over p_0 is the hyperplane $\{a_{12} = 0\} \subset \mathbb{P}V_2$ minus $\mathbb{P}V_3$, which is naturally G_3-isomorphic to the G_3-representation $V_3' = \mathrm{Hom}(\mathbb{C}x_3y_3^3, V_3)$. Hence by the slice method we have

$$\mathbb{P}V_2/G_2 \sim V_3'/G_3.$$

Let $V_4 \subset V_3'$ be the subspace

$$V_4 = (\mathbb{C}x_3y_3^3)^\vee \otimes \mathbb{C}\langle x_3y_3^2, x_3^3y_3^2\rangle.$$

One checks that V_4 is G_3-invariant, and that with respect to the given basis G_3 acts on V_4 by $j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\rho = \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$. In particular, G_3 acts on V_4 effectively and so almost freely. Since G_3 is a finite group, we have a G_3-decomposition $V_3' = V_4 \oplus V_4^\perp$. Applying the no-name lemma 2.4 to the projection $V_3' \to V_4$ from $V_4^\perp$, we have

$$V_3'/G_3 \sim \mathbb{C}^{14} \times (V_4/G_3).$$

Since $\dim V_4 = 2$, the quotient V_4/G_3 is rational. This completes the proof of Proposition 6.9. □

6.5. **The rationality of $\mathcal{M}_{5,3,1}$.** By Corollary 6.2, $\mathcal{M}_{5,3,1}$ is birational to $\mathcal{T}_{7,1}$. In [21] we proved that the moduli of trigonal curves of genus 7 is rational. Therefore

Proposition 6.12. *The moduli space $\mathcal{M}_{5,3,1}$ is rational.*

6.6. **The rationality of $\mathcal{M}_{6,2,0}$.** Recall from Proposition 6.3 that we have a birational equivalence $\mathcal{M}_{6,2,0} \sim (|L_{3,0}| \times |L_{0,1}|)/\mathrm{Aut}(\mathbb{F}_3)$.

Proposition 6.13. *The quotient* $(|L_{3,0}| \times |L_{0,1}|)/\mathrm{Aut}(\mathbb{F}_3)$ *is rational. Therefore* $\mathcal{M}_{6,2,0}$ *is rational.*

Proof. Applying the slice method to the projection $|L_{3,0}| \times |L_{0,1}| \to |L_{0,1}|$, we have $(|L_{3,0}| \times |L_{0,1}|)/\mathrm{Aut}(\mathbb{F}_3) \sim |L_{3,0}|/G$ where $G \subset \mathrm{Aut}(\mathbb{F}_3)$ is the stabilizer of a point $p \in \Sigma$. As in the proof of Proposition 6.8, we see that $|L_{3,0}|/G$ is rational. □

7. The case $g = 6$

In this section we prove that the spaces $\mathcal{M}_{r,a,\delta}$ with $g = 6$ and $k > 0$ are rational. We will find curves of Clifford index 1, i.e., trigonal curves and plane quintics as the main fixed curves.

7.1. $\mathcal{M}_{6,4,0}$ **and plane quintics.** Let $U \subset |\mathcal{O}_{\mathbb{P}^2}(5)| \times |\mathcal{O}_{\mathbb{P}^2}(1)|$ be the open set of pairs (C, L) such that C is smooth and transverse to L. The 2-elementary $K3$ surface associated to the sextic $C + L$ has parity $\delta = 0$. Indeed, if $(Y, B_1 + B_2)$ is the corresponding right DPN pair, we have $B_1 - B_2 \in 4NS_Y$. Thus we obtain a period map $\mathcal{P}\colon U/\mathrm{PGL}_3 \to \mathcal{M}_{6,4,0}$.

Proposition 7.1. *The period map* $\mathcal{P}$ *is birational.*

Proof. We consider the following $\mathfrak{S}_5$-cover of U.

$$\widetilde{U} = \{(C, L, p_1, \cdots, p_5) \in U \times (\mathbb{P}^2)^5,\ C \cap L = \{p_1, \cdots, p_5\}\}.$$

By $\widetilde{U}$ the sextics $C + L$ are endowed with complete labelings of the nodes. Since $\dim(U/\mathrm{PGL}_3) = 14$, this induces a birational lift $\widetilde{U}/\mathrm{PGL}_3 \to \widetilde{\mathcal{M}}_{6,4,0}$ of $\mathcal{P}$ by the recipe in §4.3. The projection $\widetilde{U}/\mathrm{PGL}_3 \to U/\mathrm{PGL}_3$ has degree $|\mathfrak{S}_5|$ because PGL_3 acts almost freely on U. On the other hand, $\widetilde{\mathcal{M}}_{6,4,0}$ is an $\mathrm{O}(D_{L_+})$-cover of $\mathcal{M}_{6,4,0}$ for the lattice $L_+ = U(2) \oplus D_4$. By [26] we have $|\mathrm{O}(D_{L_+})| = |\mathrm{O}^-(4, 2)| = 5!$. This proves the proposition. □

Proposition 7.2. *The quotient* $(|\mathcal{O}_{\mathbb{P}^2}(5)| \times |\mathcal{O}_{\mathbb{P}^2}(1)|)/\mathrm{PGL}_3$ *is rational. Therefore* $\mathcal{M}_{6,4,0}$ *is rational.*

Proof. We may apply the no-name lemma 2.5 to the projection $|\mathcal{O}_{\mathbb{P}^2}(5)| \times |\mathcal{O}_{\mathbb{P}^2}(1)| \to |\mathcal{O}_{\mathbb{P}^2}(5)|$. Indeed, the tautological bundle on $|\mathcal{O}_{\mathbb{P}^2}(5)|$ is SL_3-linearized where the element $e^{\frac{2\pi i}{3}} I$ acts by the multiplication by $e^{\frac{2\pi i}{3}}$. The group PGL_3 acts almost freely on $|\mathcal{O}_{\mathbb{P}^2}(5)|$. Hence we have

$$(|\mathcal{O}_{\mathbb{P}^2}(5)| \times |\mathcal{O}_{\mathbb{P}^2}(1)|)/\mathrm{PGL}_3 \sim \mathbb{P}^2 \times (|\mathcal{O}_{\mathbb{P}^2}(5)|/\mathrm{PGL}_3).$$

The quotient $|\mathcal{O}_{\mathbb{P}^2}(5)|/\mathrm{PGL}_3$ is rational by Shepherd-Barron [33]. □

Corollary 7.3. *The fixed curve map for* $\mathcal{M}_{6,4,0}$ *is a dominant map onto the locus of plane quintics, i.e., non-hyperelliptic curves having* g^2_5*, and its general fibers are identified with the* g^2_5.

Note that a smooth plane quintic has only one g^2_5.

7.2. $\mathcal{M}_{6,4,1}$ **and trigonal curves.** Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and $U \subset |O_Q(3,4)| \times |O_Q(1,0)|$ be the open set of pairs (C, F) such that C is smooth and transverse to F. The group $G = \mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts on U. For a $(C, F) \in U$ the 2-elementary $K3$ surface (X, ι) associated to the bidegree $(4, 4)$ curve $C + F$ has invariant $(r, a) = (6, 4)$. In order to calculate δ, we blow-up a point p in $C \cap F$ and then contract the two ruling fibers passing p to obtain an irreducible sextic with a node or a cusp and with a D_4-singularity. By Lemma 4.7 this shows that $L_+(X, \iota) \simeq \langle 2\rangle \oplus A_1 \oplus D_4$, and so (X, ι) has $\delta = 1$. Thus we obtain a period map $\mathcal{P}\colon U/G \to \mathcal{M}_{6,4,1}$.

Proposition 7.4. *The period map $\mathcal{P}$ is birational.*

Proof. Let $\widetilde{U} \subset U \times Q^4$ be the locus of $(C, F, p_1, \cdots, p_4)$ such that $C \cap F = \{p_1, \cdots, p_4\}$. As in Example 4.9, we see that $\mathcal{P}$ lifts to a birational map $\widetilde{U}/G \to \widetilde{\mathcal{M}}_{6,4,1}$. Note that $\dim(U/G) = 14$. The projection $\widetilde{U}/G \to U/G$ is an $\mathfrak{S}_4$-covering, while $\widetilde{\mathcal{M}}_{6,4,1}$ is an $\mathrm{O}(D_{L_+})$-cover of $\mathcal{M}_{6,4,1}$ for the lattice $L_+ = U \oplus A_1^4$. It is easy to calculate that $\mathrm{O}(D_{L_+}) \simeq \mathfrak{S}_4$. □

Proposition 7.5. *The quotient $(|O_Q(3,4)| \times |O_Q(1,0)|)/G$ is rational. Therefore $\mathcal{M}_{6,4,1}$ is rational.*

Proof. We may apply the no-name lemma 2.5 to the projection $|O_Q(3,4)| \times |O_Q(1,0)| \to |O_Q(3,4)|$. Indeed, the group $\tilde{G} = \mathrm{SL}_2 \times \mathrm{SL}_2$ acts on $H^0(O_Q(1,0))$ and on the natural hyperplane bundle $O_{\mathbb{P}V}(1)$ over $\mathbb{P}V = |O_Q(3,4)|$. The kernel of $\tilde{G} \to G$ is generated by $\iota_1 = (1, -1)$ and $\iota_2 = (-1, 1)$. Then ι_1 (resp. ι_2) acts by 1 (resp. -1) on both $H^0(O_Q(1,0))$ and $O_{\mathbb{P}V}(1)$. The G-action on $|O_Q(3,4)|$ is almost free, with the quotient $|O_Q(3,4)|/G$ birational to the moduli $\mathcal{T}_6$ of trigonal curves of genus 6 (see Proposition 3.8). Therefore

$$(|O_Q(3,4)| \times |O_Q(1,0)|)/G \sim \mathbb{P}^1 \times \mathcal{T}_6.$$

The space $\mathcal{T}_6$ is rational by Shepherd-Barron [32]. □

Corollary 7.6. *The fixed curve map for $\mathcal{M}_{6,4,1}$ gives a dominant map $\mathcal{M}_{6,4,1} \dashrightarrow \mathcal{T}_6$, whose fiber over a general C is identified with the pencil $|K_C - 2T|$ where T is the trigonal bundle.*

Remark 7.7. One might also deduce Corollaries 7.3 and 7.6 from the birational map $\mathcal{M}_{5,5,1} \dashrightarrow \mathcal{M}_6$ in §5.4, by extending it to the discriminant divisor (cf. [3]).

7.3. **The rationality of $\mathcal{M}_{5+k,5-k,\delta}$ with $k > 1$.** We construct 2-elementary $K3$ surfaces using curves on $\mathbb{F}_2$. We keep the notation of §3. For $2 \leq k \leq 5$ let $U_k \subset |L_{3,1}| \times |L_{0,1}|$ be the locus of pairs (C, F) such that (i) C is smooth, (ii) F intersects with C at $C \cap \Sigma$ with multiplicity $k - 2$, and (iii) $|C \cap F \backslash \Sigma| = 5 - k$. For $k = 2$, these conditions mean that F does not pass $C \cap \Sigma$ and is transverse to C. In particular, U_2 is open in $|L_{3,1}| \times |L_{0,1}|$. For $k \geq 3$, U_k is regarded as a sublocus of $|L_{3,1}|$ because F is uniquely determined by the point $C \cap \Sigma$.

Lemma 7.8. *The variety U_k has the expected dimension* $22 - k$.

Proof. Let $f(x_1, y_1) = 0$ be a defining equation of a smooth $C \in |L_{3,1}|$ as in Proposition 3.9. We normalize $F = \{x_1 = 0\}$. Then we have $(C, F) \in U_k$ if and only if the cubic polynomial $f(0,y_1)$ of y_1 is factorized as $f(0,y_1) = y_1^{k-2}g(y_1)$ such that $g(0) \neq 0$ and g has no multiple root. This proves the assertion. □

For a pair $(C,F) \in U_k$, the curve $B = C+F+\Sigma$ belongs to $|-2K_{\mathbb{F}_2}|$ and is singular at $(C \cap F) \cup (C \cap \Sigma) \cup (F \cap \Sigma)$. For $k = 2$ those points are distinct nodes of B. For $k \geq 3$ the point $C \cap \Sigma = F \cap \Sigma$ is a D_{2k-2}-singularity of B, and the rest $5-k$ points $C \cap F \backslash \Sigma$ are nodes. Hence the 2-elementary $K3$ surface associated to B has invariant $(r,a) = (5+k, 5-k)$, and we obtain a period map $\mathcal{P}_k\colon U_k/\mathrm{Aut}(\mathbb{F}_2) \dashrightarrow \mathcal{M}_{5+k,5-k,\delta}$ where $\delta = 1$ for $k \leq 4$ and $\delta = 0$ for $k = 5$.

Proposition 7.9. *The map $\mathcal{P}_k$ is birational.*

Proof. First we treat the case $k = 2$ using the recipe in §4.3. Let $\widetilde{U_2} \subset U_2 \times (\mathbb{F}_2)^3$ be the locus of (C, F, p_1, p_2, p_3) such that $C \cap F = \{p_i\}_{i=1}^3$. The space $\widetilde{U_2}$ parametrizes the $-2K_{\mathbb{F}_2}$-curves $B = C + F + \Sigma$ endowed with labelings of the three nodes $C \cap F$. The rest two nodes, $F \cap \Sigma$ and $C \cap \Sigma$, are distinguished by the irreducible decomposition of B, and the components of B are identified by their classes in $NS_{\mathbb{F}_2}$. Thus we will obtain a birational lift $\widetilde{U_2}/\mathrm{Aut}(\mathbb{F}_2) \dashrightarrow \widetilde{\mathcal{M}_{7,3,1}}$ of $\mathcal{P}_2$. The projection $\widetilde{U_2}/\mathrm{Aut}(\mathbb{F}_2) \dashrightarrow U_2/\mathrm{Aut}(\mathbb{F}_2)$ is an $\mathfrak{S}_3$-covering, while $\widetilde{\mathcal{M}_{7,3,1}}$ is an $\mathrm{O}(D_{L_+})$-cover of $\mathcal{M}_{7,3,1}$ for the lattice $L_+ = U \oplus A_1 \oplus D_4$. Since $\mathrm{O}(D_{L_+}) \simeq \mathfrak{S}_3$, the map $\mathcal{P}_2$ is birational.

For $k \geq 3$ a similar argument is possible (see Example 4.12 for $k = 4$), but we may also proceed as in the proof of Proposition 6.1: since U_k may be regarded as a sublocus of $|L_{3,1}|$, by Proposition 3.8 the quotient $U_k/\mathrm{Aut}(\mathbb{F}_2)$ is naturally birational to a sublocus of the Maroni divisor $\mathcal{T}_{6,2}$. Considering the fixed curve map for $\mathcal{M}_{5+k,5-k,\delta}$, we see that $\mathcal{P}_k$ is generically injective. By Lemma 7.8, $\mathcal{P}_k$ is birational. □

Proposition 7.10. *The quotient $U_k/\mathrm{Aut}(\mathbb{F}_2)$ is rational. Therefore $\mathcal{M}_{5+k,5-k,\delta}$ with $2 \leq k \leq 5$ is rational.*

Proof. This is analogous to the proofs of Propositions 6.8 and 6.13: we apply the slice method to the projection $U_k \to |L_{0,1}|$, whose fibers are open sets of linear subspaces of $|L_{3,1}|$ by the proof of Lemma 7.8. Then we resort to Miyata's theorem. Note that although $L_{3,1}$ may not be $\mathrm{Aut}(\mathbb{F}_2)$-linearized, one may use the group $\mathrm{GL}_2 \ltimes R$ in §3 instead of $\mathrm{Aut}(\mathbb{F}_2)$. □

We shall study the fixed curve maps. For $k \geq 3$ we let $\mathcal{U}_k \subset \mathcal{T}_{6,2}$ be the closure of the image of natural morphism $U_k \to \mathcal{T}_{6,2}$. In particular, $\mathcal{U}_3$ coincides to $\mathcal{T}_{6,2}$. By Proposition 7.9 we have

Corollary 7.11. *The fixed curve map for $\mathcal{M}_{7,3,1}$ is a dominant map $\mathcal{M}_{7,3,1} \dashrightarrow \mathcal{T}_{6,2}$ whose general fibers are birationally identified with the g_3^1. The fixed curve map for $\mathcal{M}_{5+k,5-k,\delta}$ with $k \geq 3$ is generically injective with a generic image $\mathcal{U}_k$.*

In fact, it is more natural to identify the fibers of $\mathcal{M}_{7,3,1} \dashrightarrow \mathcal{T}_{6,2}$ with the non-free pencils $|K_C - 2T|$ where T is the trigonal bundle, whose free part is $|T|$ (cf. Corollary 7.6). The loci $\mathcal{U}_k$ may be described in terms of special divisors as follows.

Proposition 7.12. *Let C be a trigonal curve of genus 6 with the trigonal bundle T.*

(1) *The curve C has scroll invariant 2 if and only if $W^1_4(C)$ is irreducible.*

(2) *When C has scroll invariant 2, we have $C \in \mathcal{U}_4$ if and only if $2K_C - 5T$ is contained in* $\mathrm{Sing}(\mathrm{Sing}W^1_5(C))$.

(3) *The locus $\mathcal{U}_5$ is the intersection of $\mathcal{U}_4$ with the theta-null divisor.*

Proof. (1) This follows from [22] Proposition 1 (or [2] V. A-9). Specifically, $W^1_4(C)$ consists of the curve $T+W_1(C)$ and the point K_C-2T, and K_C-2T is not contained in $T+W_1(C)$ if and only if C has scroll invariant 0. When C is an $L_{3,1}$-curve on $\mathbb{F}_2$, we have $K_C - 2T \sim T + p_0$ for the point $p_0 = C \cap \Sigma$.

(2) The locus $\mathcal{U}_4 \subset \mathcal{T}_{6,2}$ consists of those C whose trigonal map ramifies at p_0. By [22] Proposition 1 we have $W^1_5(C) = W_+ \cup W_-$ where $W_+ = T + W_2(C)$ and $W_- = K_C - W_+$ is the residual of W_+. Let $V_+ = T+p_0+W_1(C)$ and $V_- = 2T-W_1(C)$ be the residual of V_+. Then we see that

$$\mathrm{Sing}W^1_5(C) = W_+ \cap W_- = V_+ \cup V_-.$$

If $T - p_0 \sim p_1 + p_2$, we have $V_+ \cap V_- = \{T + p_0 + p_1, T + p_0 + p_2\}$. Since $2K_C - 5T \sim T + 2p_0$, this proves the assertion.

(3) A curve C in $\mathcal{T}_{6,2}$ has an effective even theta characteristic if and only if the residual involution on $W^1_5(C)$ has a fixed point. By the structure of $W^1_5(C)$ described above, this is exactly when $V_+ \cap V_-$ is one point, i.e., $p_1 = p_2$. When $C \in \mathcal{U}_4$, this is equivalent to the condition $T \sim 3p_0$. □

8. The case $g = 5$

In this section we prove that the spaces $\mathcal{M}_{r,a,\delta}$ with $g = 5$ and $k > 0$ are rational. The case $(k, \delta) = (4, 0)$ was settled by Kondō [18] using trigonal curves with vanishing theta-null. It turns out that the main fixed curves for other (k, δ) are also trigonal.

8.1. **The rationality of $\mathcal{M}_{7,5,1}$.** We construct 2-elementary $K3$ surfaces using curves on $\mathbb{F}_1$. Let $U \subset |L_{3,2}| \times |L_{1,0}|$ be the open set of pairs (C, H) such that C and H are smooth and transverse to each other. The 2-elementary $K3$ surface associated to the $-2K_{\mathbb{F}_1}$-curve $C + H$ has invariant $(g, k) = (5, 1)$. Thus we obtain a period map $\mathcal{P}\colon U/\mathrm{Aut}(\mathbb{F}_1) \dashrightarrow \mathcal{M}_{7,5,1}$.

Proposition 8.1. *The period map $\mathcal{P}$ is birational.*

Proof. As before, we consider an $\mathfrak{S}_5$-cover $\widetilde{U}$ of U whose fiber over a $(C, H) \in U$ corresponds to labelings of the five nodes $C \cap H$. Noticing that the blow-down $\phi\colon \mathbb{F}_1 \to \mathbb{P}^2$ contracts the (-1)-curve to the unique node of $\phi(C)$, one may proceed as in Example 4.10 to obtain a birational lift $\widetilde{U}/\mathrm{Aut}(\mathbb{F}_1) \dashrightarrow \widetilde{\mathcal{M}}_{7,5,1}$ of $\mathcal{P}$. The projection $\widetilde{\mathcal{M}}_{7,5,1} \dashrightarrow \mathcal{M}_{7,5,1}$ is an $\mathrm{O}(D_{L_+})$-covering for the lattice $L_+ = U \oplus A_1^5$. By [26] we have $|\mathrm{O}(D_{L_+})| = |\mathrm{O}^-(4, 2)| = 5!$, so that $\mathcal{P}$ is birational. □

Proposition 8.2. *The quotient $U/\mathrm{Aut}(\mathbb{F}_1)$ is rational. Hence $\mathcal{M}_{7,5,1}$ is rational.*

Proof. By Proposition 3.8, $|L_{3,2}|/\mathrm{Aut}(\mathbb{F}_1)$ is canonically birational to the moduli $\mathcal{T}_5$ of trigonal curves of genus 5. Since $L_{1,0}$ is $\mathrm{Aut}(\mathbb{F}_1)$-linearized, we may apply the no-name lemma 2.5 to the projection $|L_{3,2}| \times |L_{1,0}| \to |L_{3,2}|$. Then we have $U/\mathrm{Aut}(\mathbb{F}_1) \sim \mathbb{P}^2 \times \mathcal{T}_5$. The space $\mathcal{T}_5$ is rational by [21]. □

For every smooth $C \in |L_{3,2}|$, the restriction of $|L_{1,0}|$ to C is identified with the linear system $|K_C - T|$ where T is the trigonal bundle. Therefore

Corollary 8.3. *The fixed curve map for $\mathcal{M}_{7,5,1}$ gives a dominant map $\mathcal{M}_{7,5,1} \dashrightarrow \mathcal{T}_5$ whose general fibers are birationally identified with the residuals of the g^1_3.*

8.2. **The rationality of $\mathcal{M}_{6+k,6-k,1}$ with $k > 1$.** We consider curves on $\mathbb{F}_1$. For $2 \leq k \leq 5$ let $U_k \subset |L_{3,2}| \times |L_{0,1}|$ be the locus of pairs (C, F) such that (i) C is smooth and transverse to the (-1)-curve Σ, (ii) F intersects with C at one of $C \cap \Sigma$ with multiplicity $k-2$, and (iii) $|C \cap F \backslash \Sigma| = 5 - k$. When $k = 2$, the conditions (ii) and (iii) simply mean that F is transverse to $C + \Sigma$. In particular, U_2 is open in $|L_{3,2}| \times |L_{0,1}|$. The projection $U_k \to |L_{3,2}|$ is dominant for $k = 2, 3$, and generically injective for $k = 4, 5$. As in the proof of Lemma 7.8, one checks that U_k has the expected dimension $20 - k$.

For a $(C, F) \in U_k$, the curve $B = C + F + \Sigma$ belongs to $|-2K_{\mathbb{F}_1}|$. When $k = 2$, B has only nodes (the intersections of the components) as the sigularities. When $k \geq 3$, B has the D_{2k-2}-point $F \cap \Sigma$, the $5-k$ nodes $C \cap F \backslash \Sigma$, the one node $C \cap \Sigma \backslash F$, and no other singularity. Thus the 2-elementary $K3$ surface associated to B has invariant $(r, a) = (6 + k, 6 - k)$. When $k = 4$, we have parity $\delta = 1$ by Lemma 4.4 (3). Hence we obtain a period map $\mathcal{P}_k\colon U_k/\mathrm{Aut}(\mathbb{F}_1) \dashrightarrow \mathcal{M}_{6+k,6-k,1}$.

Proposition 8.4. *The map $\mathcal{P}_k$ is birational.*

Proof. When $k = 4, 5$, the singularities of the curves $B = C + F + \Sigma$ are a priori distinguished by their type and by the irreducible decomposition of B. Also the three branches at the D_{2k-2}-point are distinguished by the decomposition of B. Therefore $\mathcal{P}_k$ lifts to a birational map $U_k/\mathrm{Aut}(\mathbb{F}_1) \dashrightarrow \widetilde{\mathcal{M}}_{6+k,6-k,1}$ by the recipe in §4.3. One checks that $\widetilde{\mathcal{M}}_{6+k,6-k,1} = \mathcal{M}_{6+k,6-k,1}$ in these cases.

The case $k = 3$ is treated in Example 4.10.

For $k = 2$, we label the three nodes $C \cap F$ and the two nodes $C \cap \Sigma$ independently. This is realized by an $\mathfrak{S}_3 \times \mathfrak{S}_2$-cover $\widetilde{U}_2$ of U_2. The rest node $F \cap \Sigma$ of B is distinguished from those five. Then we will obtain a birational lift $\widetilde{U}_2/\mathrm{Aut}(\mathbb{F}_1) \dashrightarrow \widetilde{\mathcal{M}}_{8,4,1}$ of $\mathcal{P}_2$. By [26] we have $|\mathrm{O}(D_{L_+})| = 2 \cdot |\mathrm{Sp}(2, 2)| = 2 \cdot 3!$ for the lattice $L_+ = U \oplus D_4 \oplus A_1^2$. Therefore $\mathcal{P}_2$ is birational. □

Proposition 8.5. *The quotient $U_k/\mathrm{Aut}(\mathbb{F}_1)$ is rational. Therefore $\mathcal{M}_{6+k,6-k,1}$ with $k > 1$ is rational.*

Proof. The same proof as for Proposition 7.10 works. □

For completeness, we briefly explain the birational period map for $\mathcal{M}_{10,2,0}$ constructed by Kondō [18]. Let $U \subset |L_{3,2}|$ be the locus of smooth curves C which are tangent to Σ. The quotient $U/\mathrm{Aut}(\mathbb{F}_1)$ is identified with the theta-null divisor $\mathcal{T}'_5$ in the trigonal locus. For a $C \in U$ let F be the π-fiber passing $C \cap \Sigma$. The 2-elementary

$K3$ surface associated to the $-2K_{\mathbb{F}_1}$-curve $C + F + \Sigma$ belongs to $\mathcal{M}_{10,2,0}$. Then the period map $U/\mathrm{Aut}(\mathbb{F}_1) \to \mathcal{M}_{10,2,0}$ is birational. Kondō's proof is essentially along the line of §4.3. It can also be deduced using the fixed curve map.

We describe the fixed curve maps as rational maps $F_k : \mathcal{M}_{6+k,6-k,1} \dashrightarrow \mathcal{T}_5$ to the trigonal locus. By Proposition 8.4 we have

Corollary 8.6. *The map F_k is dominant for $k = 2, 3$, and is generically injective for $k = 4, 5$. The general fibers of F_2 are birationally identified with the g^1_3, and the general fibers of F_3 are identified with the two points $\mathrm{Sing}W^1_4(C)$, $C \in \mathcal{T}_5$.*

Proof. The F_3-fiber over a general $C \in |L_{3,2}|$ is identified with the two points $\{p_1, p_2\} = \Sigma \cap C$. Note that $K_C \sim 2T + p_1 + p_2$ for the trigonal bundle T. By [22], $W^1_4(C)$ consists of two residual components, $W_+ = T + W_1(C)$ and $W_- = K_C - W_+$. Then $\mathrm{Sing}W^1_4(C)$ is the intersection $W_+ \cap W_- = \{T + p_1, T + p_2\}$. □

A generic image of F_4 (resp. F_5) is the locus where the trigonal map ramifies (resp. totally ramifies) at the base point of one of $\mathrm{Sing}W^1_4(C)$. We also note that the theta-null divisor $\mathcal{T}'_5$ is exactly the locus where $\mathrm{Sing}W^1_4(C)$ is one point. Thus the double covering $F_3 \colon \mathcal{M}_{9,3,1} \dashrightarrow \mathcal{T}_5$ is the quotient by the residuation and is ramified at $\mathcal{M}_{10,2,0}$ over $\mathcal{T}'_5$

9. The case $g = 4$

In this section we study the case $g = 4$, $k > 0$. Our constructions are related to canonical models of genus 4 curves, i.e., curves on quadratic surfaces cut out by cubics. Except for §9.4 we shall use the following notation: Q is the surface $\mathbb{P}^1 \times \mathbb{P}^1$, $L_{a,b}$ is the bundle $\mathcal{O}_Q(a, b)$, and $\mathrm{Aut}(Q)_0 = \mathrm{PGL}_2 \times \mathrm{PGL}_2$ is the identity component of $\mathrm{Aut}(Q)$.

9.1. **The rationality of $\mathcal{M}_{8,6,1}$.** Let $U \subset |L_{3,3}| \times |L_{1,1}|$ be the open set of pairs (C, H) such that C and H are smooth and transverse to each other. Considering the 2-elementary $K3$ surfaces associated to the $-2K_Q$-curves $C+H$, we obtain a period map $\mathcal{P} \colon U/\mathrm{Aut}(Q) \dashrightarrow \mathcal{M}_{8,6,1}$. In Example 4.9 we proved that $\mathcal{P}$ is birational.

Proposition 9.1. *The quotient $U/\mathrm{Aut}(Q)$ is rational. Hence $\mathcal{M}_{8,6,1}$ is rational.*

Proof. Recall that the quotient $|L_{3,3}|/\mathrm{Aut}(Q)$ is naturally birational to the moduli $\mathcal{M}_4$ of genus 4 curves. This is a consequence of the fact that a general canonical genus 4 curve is a complete intersection of a cubic and a unique smooth quadric. For a smooth $C \in |L_{3,3}|$ the linear system $|L_{1,1}|$ is identified with $|K_C|$ by restriction. Then let $\pi \colon \mathcal{X}_4 \to \mathcal{M}_4$ be the universal genus 4 curve (over an open locus), and $\mathcal{E}$ be the bundle $\pi_* K_{\mathcal{X}_4/\mathcal{M}_4}$ over $\mathcal{M}_4$. The above remark implies that $(|L_{3,3}| \times |L_{1,1}|)/\mathrm{Aut}(Q)$ is birational to $\mathbb{P}\mathcal{E}$. Since $\mathcal{M}_4$ is rational ([32]), so is $\mathbb{P}\mathcal{E}$. □

Corollary 9.2. *The fixed curve map $\mathcal{M}_{8,6,1} \to \mathcal{M}_4$ is dominant with the general fibers birationally identified with the canonical systems.*

9.2. **The rationality of $\mathcal{M}_{9,5,1}$.** For a point $p \in Q$ we denote by D_p the union of the two ruling fibers meeting at p. Let $U \subset |L_{3,3}| \times Q$ be the open set of pairs (C, p) such that C is smooth and transverse to D_p. Taking the right resolution of the $-2K_Q$-curves $C + D_p$, we obtain a period map $\mathcal{P}\colon U/\mathrm{Aut}(Q) \dashrightarrow \mathcal{M}_{9,5,1}$.

Proposition 9.3. *The map $\mathcal{P}$ is birational.*

Proof. Let $\widetilde{U}$ be the space of those $(C, p, p_1, \cdots, p_6) \in U \times Q^6$ such that $C \cap D_p = \{p_i\}_{i=1}^6$ and that p_1, p_2, p_3 lie on the $(1,0)$-component of D_p. The space $\widetilde{U}$ parametrizes the curves $C + D_p$ endowed with labelings of the six nodes $C \cap D_p$ which take into account the decomposition of D_p. The rest node of $C+D_p$, the point p, is clearly distinguished from those six. Note that $\mathrm{Aut}(Q)$ does not act on $\widetilde{U}$, for the definition of $\widetilde{U}$ involves the distinction of the two rulings. Rather $\widetilde{U}$ is acted on by $\mathrm{Aut}(Q)_0$. As in Example 4.9, $\mathcal{P}$ lifts to a birational map $\widetilde{U}/\mathrm{Aut}(Q)_0 \dashrightarrow \widetilde{\mathcal{M}}_{9,5,1}$. The projection $\widetilde{U}/\mathrm{Aut}(Q)_0 \dashrightarrow U/\mathrm{Aut}(Q)$ has degree $2 \cdot |\mathfrak{S}_3 \times \mathfrak{S}_3|$, while $\widetilde{\mathcal{M}}_{9,5,1}$ is an $\mathrm{O}(D_{L_+})$-cover of $\mathcal{M}_{9,5,1}$ for the lattice $L_+ = U \oplus D_4 \oplus A_1^3$. By [26] we have $|\mathrm{O}(D_{L_+})| = |\mathrm{O}^+(4,2)| = 72$. □

Proposition 9.4. *The quotient $(|L_{3,3}| \times Q)/\mathrm{Aut}(Q)$ is rational. Therefore $\mathcal{M}_{9,5,1}$ is rational.*

Proof. Applying the slice method to the projection $|L_{3,3}| \times Q \to Q$, we have $(|L_{3,3}| \times Q)/\mathrm{Aut}(Q) \sim |L_{3,3}|/G$ where $G \subset \mathrm{Aut}(Q)$ is the stabilizer of a point $p \in Q$. Let F_1, F_2 be respectively the $(1,0)$- and the $(0,1)$-fiber passing p. We denote $V = |\mathcal{O}_{F_1}(3)| \times |\mathcal{O}_{F_2}(3)|$. We want to apply the no-name lemma to the G-equivariant map $\varphi\colon |L_{3,3}| \dashrightarrow V$, $C \mapsto (C|_{F_1}, C|_{F_2})$. If (D_1, D_2) is a general point of V, there is no isomorphism $F_1 \to F_2$ mapping (p, D_1) to (p, D_2). Since PGL_2 acts on $\mathbb{P}^1 \times |\mathcal{O}_{\mathbb{P}^1}(3)|$ almost freely, this deduces that G acts on V almost freely. A general fiber of φ is an open set of a linear subspace of $|L_{3,3}|$. Tensoring the natural $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ over V, we are able to use Proposition 2.5 to see that $|L_{3,3}|/G \sim \mathbb{P}^9 \times (V/G)$. The quotient V/G is rational because $\dim(V/G) = 2$. □

Corollary 9.5. *The fixed curve map $\mathcal{M}_{9,5,1} \to \mathcal{M}_4$ is dominant with the general fibers birationally identified with the products of the two trigonal pencils.*

9.3. **$\mathcal{M}_{10,4,1}$ and the universal genus 4 curve.** As in §9.2, for a point $p \in Q$ we denote by D_p the reducible bidegree $(1,1)$ curve singular at p. Let $U \subset |L_{3,3}| \times Q$ be the locus of pairs (C, p) such that C is smooth, passes p, and is transverse to each component of D_p. The space U is an open set of the universal bidegree $(3,3)$ curve. The bidegree $(4,4)$ curve $C + D_p$ has the D_4-point p, the four nodes $C \cap D_p \backslash p$, and no other singularity. The associated 2-elementary $K3$ surface has invariant $(g,k) = (4,3)$. It has parity $\delta = 1$ by Lemma 4.4 (2). Thus we obtain a period map $\mathcal{P}\colon U/\mathrm{Aut}(Q) \dashrightarrow \mathcal{M}_{10,4,1}$.

Proposition 9.6. *The map $\mathcal{P}$ is birational.*

Proof. As in §9.2, we consider the locus $\widetilde{U} \subset U \times Q^4$ of those $(C, p, p_1, \cdots, p_4)$ such that $C \cap D_p \backslash p = \{p_i\}_{i=1}^4$ and that p_1, p_2 lie on the $(1,0)$-component of D_p.

Since the three branches of $C + D_p$ at p are distinguished by the irreducible decomposition of $C + D_p$, then $\mathcal{P}$ lifts to a birational map $\widetilde{U}/\mathrm{Aut}(Q)_0 \dashrightarrow \widetilde{\mathcal{M}}_{10,4,1}$. The space $\widetilde{U}/\mathrm{Aut}(Q)_0$ is an $\mathfrak{S}_2 \ltimes (\mathfrak{S}_2)^2$-cover of $U/\mathrm{Aut}(Q)$, while $\widetilde{\mathcal{M}}_{10,4,1}$ is an $\mathrm{O}(D_{L_+})$-cover of $\mathcal{M}_{10,4,1}$ for the lattice $L_+ = U(2) \oplus E_7 \oplus A_1$. By [26] we have $|\mathrm{O}(D_{L_+})| = 2^2 \cdot |\mathrm{O}^+(2,2)| = 8$. □

Since $|L_{3,3}|/\mathrm{Aut}(Q)$ is naturally biational to $\mathcal{M}_4$, the quotient $U/\mathrm{Aut}(Q)$ is birational to the universal genus 4 curve, which is rational by Catanese [8]. Therefore

Proposition 9.7. *The moduli space $\mathcal{M}_{10,4,1}$ is rational.*

Corollary 9.8. *The fixed curve map $\mathcal{M}_{10,4,1} \to \mathcal{M}_4$ identifies $\mathcal{M}_{10,4,1}$ birationally with the universal genus 4 curve.*

9.4. $\mathcal{M}_{10,4,0}$ **and genus 4 curves with vanishing theta-null.** We construct 2-elementary $K3$ surfaces using curves on $\mathbb{F}_2$. We use the notation of §3. Let $U \subset |L_{3,0}| \times |L_{0,2}|$ be the open set of pairs (C, D) such that C and $D = F_1 + F_2$ are smooth and transverse to each other. The 2-elementary $K3$ surface (X, ι) associated to the nodal $-2K_{\mathbb{F}_2}$-curve $C + D + \Sigma$ has invariant $(g, k) = (4, 3)$. Since the strict transform of $C - D + \Sigma$ in $Y' = X/\iota$ belongs to $4NS_{Y'}$, (X, ι) has parity $\delta = 0$. Thus we obtain a period map $\mathcal{P}\colon U/\mathrm{Aut}(\mathbb{F}_2) \dashrightarrow \mathcal{M}_{10,4,0}$.

Proposition 9.9. *The map $\mathcal{P}$ is birational.*

Proof. Let $\widetilde{U} \subset U \times (\mathbb{F}_2)^6$ be the locus of those $(C, D, p_1, \cdots, p_6)$ such that $C \cap D = \{p_i\}_{i=1}^6$ and that p_1, p_2, p_3 are on the same component of D. By considering $\widetilde{U}$, the nodes of $C + D$ and the two components of D are labelled in a compatible way. As in Example 4.11, we see that $\mathcal{P}$ lifts to a birational map $\widetilde{U}/\mathrm{Aut}(\mathbb{F}_2) \dashrightarrow \widetilde{\mathcal{M}}_{10,4,0}$. The projection $\widetilde{U}/\mathrm{Aut}(\mathbb{F}_2) \to U/\mathrm{Aut}(\mathbb{F}_2)$ is an $\mathfrak{S}_2 \ltimes (\mathfrak{S}_3)^2$-covering, while $\widetilde{\mathcal{M}}_{10,4,0}$ is an $\mathrm{O}(D_{L_+})$-cover of $\mathcal{M}_{10,4,0}$ for the lattice $L_+ = U \oplus D_4^2$. By [26] we have $|\mathrm{O}(D_{L_+})| = |\mathrm{O}^+(4,2)| = 72$. This proves the proposition. □

The quotient $|L_{3,0}|/\mathrm{Aut}(\mathbb{F}_2)$ is naturally birational to the theta-null divisor $\mathcal{M}_4'$ in $\mathcal{M}_4$. Indeed, recall that the morphism $\phi_{L_{1,0}}\colon \mathbb{F}_2 \to \mathbb{P}^3$ associated to the bundle $L_{1,0}$ is the minimal desingularization of the quadratic cone $Q = \phi_{L_{1,0}}(\mathbb{F}_2)$, and that the restriction of $\phi_{L_{1,0}}$ to each smooth $C \in |L_{3,0}|$ is a canonical map of C. Then our claim follows from the fact that a non-hyperelliptic genus 4 curve has an effective even theta characteristic if and only if its canonical model lies on a singular quadric. In that case the half canonical pencil is given by the pencil of lines on Q, or equivalently, the pencil $|L_{0,1}|$.

Proposition 9.10. *The quotient $(|L_{3,0}| \times |L_{0,2}|)/\mathrm{Aut}(\mathbb{F}_2)$ is rational. Hence $\mathcal{M}_{10,4,0}$ is rational.*

Proof. Since both $L_{3,0}$ and $L_{0,2}$ are $\mathrm{Aut}(\mathbb{F}_2)$-linearized (Proposition 3.2) and since a general $C \in \mathcal{M}_4'$ has no automorphism, we may apply the no-name lemma 2.5 to the projection $|L_{3,0}| \times |L_{0,2}| \to |L_{3,0}|$ to see that

$$(|L_{3,0}| \times |L_{0,2}|)/\mathrm{Aut}(\mathbb{F}_2) \sim \mathbb{P}^2 \times (|L_{3,0}|/\mathrm{Aut}(\mathbb{F}_2)) \sim \mathbb{P}^2 \times \mathcal{M}_4'.$$

The space $\mathcal{M}_4'$ is rational by Dolgachev [12]. □

Corollary 9.11. *The fixed curve map for $\mathcal{M}_{10,4,0}$ is a dominant map onto $\mathcal{M}_4'$ whose general fibers are birationally identified with the symmetric products of the half-canonical pencils.*

9.5. **The rationality of $\mathcal{M}_{11,3,1}$ and $\mathcal{M}_{12,2,1}$.** Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$. For a point $p \in Q$ we denote by D_p the reducible bidegree $(1,1)$ curve singular at p. For $k = 4, 5$ let $U_k \subset |L_{3,3}| \times Q$ be the locus of pairs (C, p) such that (i) C is smooth with $p \in C$, (ii) the $(1,0)$-component of D_p is tangent to C at p with multiplicity $k-2$, and (iii) the $(0,1)$-component of D_p is transverse to C. The space U_k is acted on by $\mathrm{Aut}(Q)_0$ (but not by $\mathrm{Aut}(Q)$). The bidegree $(4,4)$ curve $C + D_p$ has the D_{2k-2}-singularity p, the $7-k$ nodes $C \cap D_p \backslash p$, and no other singularity. Taking the right resolution of $C + D_p$, we obtain a period map $\mathcal{P}_k \colon U_k/\mathrm{Aut}(Q)_0 \dashrightarrow \mathcal{M}_{7+k,7-k,1}$.

Proposition 9.12. *The map $\mathcal{P}_k$ is birational.*

Proof. We only have to distinguish the two intersection points other than p of C and the $(0,1)$-component of D_p. This defines a double cover $\widetilde{U}_k \to U_k$. The rest singularities of $C + D_p$ and the branches of $C + D_p$ at p are a priori labeled as before. Checking $\dim(U_k/\mathrm{Aut}(Q)_0) = 13 - k$, we see that $\mathcal{P}_k$ lifts to a birational map $\widetilde{U}_k/\mathrm{Aut}(Q)_0 \dashrightarrow \widetilde{\mathcal{M}}_{7+k,7-k,1}$. The variety $\widetilde{\mathcal{M}}_{7+k,7-k,1}$ is an $\mathrm{O}(D_{L_-})$-cover of $\mathcal{M}_{7+k,7-k,1}$ for the lattice $L_- = \langle 2 \rangle^2 \oplus E_8 \oplus A_1^{5-k}$. Then $\mathrm{O}(D_{L_-}) \simeq \mathfrak{S}_2$ for both $k = 4, 5$, so that $\mathcal{P}_k$ has degree 1. □

Proposition 9.13. *The quotient $U_k/\mathrm{Aut}(Q)_0$ is rational. Therefore $\mathcal{M}_{11,3,1}$ and $\mathcal{M}_{12,2,1}$ are rational.*

Proof. This is a consequence of the slice method for the projection $U_k \to Q$ and Miyata's theorem. The stabilizer of a point $p \in Q$ in $\mathrm{Aut}(Q)_0$ is isomorphic to $(\mathbb{C}^\times \ltimes \mathbb{C})^2$, which is connected and solvable. □

Corollary 9.14. *The fixed curve map $\mathcal{M}_{11,3,1} \to \mathcal{M}_4$ is finite and dominant, with the fiber over a general $C \in \mathcal{M}_4$ being the ramification points of the two trigonal maps $C \to \mathbb{P}^1$.*

For $k = 5$ the image of the natural map $U_5 \to \mathcal{M}_4$ consists of curves such that one of its trigonal maps has a total ramification point. Such a point is nothing but a Weierstrass point whose first non-gap is 3. Therefore

Corollary 9.15. *The fixed curve map for $\mathcal{M}_{12,2,1}$ is generically injective with a generic image being the locus of curves having a Weierstrass point whose first non-gap is* 3.

10. The case $g = 3$

In this section we study the case $g = 3$, $k > 0$. When $(k, \delta) \neq (2, 0)$, we use plane quartics to construct general members of $\mathcal{M}_{r,a,\delta}$. For $(k, \delta) = (2, 0)$ we find hyperelliptic curves as the main fixed curves.

10.1. **The rationality of $\mathcal{M}_{9,7,1}$.** Let $U \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(2)|$ be the open locus of pairs (C, Q) such that C and Q are smooth and transverse to each other. The 2-elementary $K3$ surfaces associated to the sextics $C+Q$ have invariant $(g, k) = (3, 1)$, and we obtain a period map $\mathcal{P}\colon U/\mathrm{PGL}_3 \dashrightarrow \mathcal{M}_{9,7,1}$.

Proposition 10.1. *The map $\mathcal{P}$ is birational.*

Proof. We consider an $\mathfrak{S}_8$-covering $\widetilde{U} \to U$ whose fiber over a $(C, Q) \in U$ corresponds to labelings of the eight nodes $C \cap Q$ of $C + Q$. As before, we see that $\mathcal{P}$ lifts to a birational map $\widetilde{U}/\mathrm{PGL}_3 \dashrightarrow \widetilde{\mathcal{M}}_{9,7,1}$. The projection $\widetilde{\mathcal{M}}_{9,7,1} \dashrightarrow \mathcal{M}_{9,7,1}$ is an $\mathrm{O}(D_{L_+})$-covering for the lattice $L_+ = U \oplus A_1^7$. By [26] we calculate $|\mathrm{O}(D_{L_+})| = |\mathrm{O}^+(6, 2)| = 8!$. □

Recall that we have a natural birational equivalence $|\mathcal{O}_{\mathbb{P}^2}(4)|/\mathrm{PGL}_3 \sim \mathcal{M}_3$, for a canonical model of a non-hyperelliptic genus 3 curve is a plane quartic.

Proposition 10.2. *The quotient $(|\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(2)|)/\mathrm{PGL}_3$ is rational. Therefore $\mathcal{M}_{9,7,1}$ is rational.*

Proof. Let $\pi\colon \mathcal{X}_3 \to \mathcal{M}_3$ be the universal curve (over an open set), K_π be the relative canonical bundle for π, and $\mathcal{E}$ be the bundle $\pi_* K_\pi^2$. As the restriction map $|\mathcal{O}_{\mathbb{P}^2}(2)| \to |2K_C|$ for a smooth quartic C is isomorphic, the quotient $(|\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(2)|)/\mathrm{PGL}_3$ is birational to $\mathbb{P}\mathcal{E}$. Since $\mathcal{M}_3$ is rational by Katsylo [17], so is $\mathbb{P}\mathcal{E}$. □

Corollary 10.3. *The fixed curve map $\mathcal{M}_{9,7,1} \to \mathcal{M}_3$ is dominant with general fibers being birationally identified with the bi-canonical systems.*

10.2. **The rationality of $\mathcal{M}_{10,6,1}$.** Let $U \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(2)|$ be the locus of pairs (C, Q) such that Q is the union of two distinct lines, and C is smooth and transverse to Q. In Example 4.11 we showed that the 2-elementary $K3$ surfaces associated to the sextics $C + Q$ have main invariant $(r, a, \delta) = (10, 6, 1)$, and that the induced period map $U/\mathrm{PGL}_3 \dashrightarrow \mathcal{M}_{10,6,1}$ is birational.

Proposition 10.4. *The quotient U/PGL_3 is rational. Hence $\mathcal{M}_{10,6,1}$ is rational.*

Proof. We keep the notation in the proof of Proposition 10.2. Let $\mathcal{F}$ be the bundle $\pi_* K_\pi$ over $\mathcal{M}_3$. By restriction, the space of singular plane conics is identified with the symmetric product of the canonical system of every smooth quartic. This implies that U/PGL_3 is birational to the symmetric product of $\mathbb{P}\mathcal{F}$ over $\mathcal{M}_3$. Since $\mathbb{P}\mathcal{F} \sim \mathcal{M}_3 \times \mathbb{P}^2$, we have $U/\mathrm{PGL}_3 \sim \mathcal{M}_3 \times S^2\mathbb{P}^2$. Then $S^2\mathbb{P}^2$ is birational to the quotient of $\mathbb{C}^2 \times \mathbb{C}^2$ by the permutation, and so is rational. Since $\mathcal{M}_3$ is rational ([17]), our assertion is proved. □

Corollary 10.5. *The fixed curve map $\mathcal{M}_{10,6,1} \to \mathcal{M}_3$ is dominant with general fibers being birationally identified with the symmetric products of the canonical systems.*

10.3. $\mathcal{M}_{10,6,0}$ **and hyperelliptic curves.** We construct 2-elementary $K3$ surfaces using curves on the Hirzebruch surface $\mathbb{F}_4$. We keep the notation of §3. Let $U \subset |L_{2,0}| \times |L_{1,0}|$ be the open set of pairs (C, H) such that C and H are smooth and transverse to each other. The 2-elementary $K3$ surface (X, ι) associated to the nodal $-2K_{\mathbb{F}_4}$-curve $C + H + \Sigma$ has invariant $(g, k) = (3, 2)$. Since the strict transform of $C - H - \Sigma$ in $Y' = X/\iota$ belongs to $4NS_{Y'}$, (X, ι) has parity $\delta = 0$. Thus we obtain a period map $\mathcal{P}\colon U/\mathrm{Aut}(\mathbb{F}_4) \dashrightarrow \mathcal{M}_{10,6,0}$.

Proposition 10.6. *The map $\mathcal{P}$ is birational.*

Proof. We consider an $\mathfrak{S}_8$-covering $\widetilde{U} \to U$ whose fiber over a $(C, H) \in U$ corresponds to labelings of the eight nodes $C \cap H$ of $C + H + \Sigma$. In the familiar way we will obtain a birational lift $\widetilde{U}/\mathrm{Aut}(\mathbb{F}_4) \dashrightarrow \widetilde{\mathcal{M}}_{10,6,0}$ of $\mathcal{P}$. The variety $\widetilde{\mathcal{M}}_{10,6,0}$ is an $\mathrm{O}(D_{L_+})$-cover of $\mathcal{M}_{10,6,0}$ for the lattice $L_+ = U(2) \oplus D_4^2$. By [26] we have $|\mathrm{O}(D_{L_+})| = |\mathrm{O}^+(6, 2)| = 8!$. □

Proposition 10.7. *The quotient* $(|L_{2,0}| \times |L_{1,0}|)/\mathrm{Aut}(\mathbb{F}_4)$ *is rational. Therefore* $\mathcal{M}_{10,6,0}$ *is rational.*

Proof. This is a consequence of the slice method for the projection $|L_{2,0}| \times |L_{1,0}| \to |L_{1,0}|$, Proposition 3.4, and Katsylo's theorem 2.2. □

Recall that the quotient $|L_{2,0}|/\mathrm{Aut}(\mathbb{F}_4)$ is birational to the moduli $\mathcal{H}_3$ of genus 3 hyperelliptic curves (Proposition 3.6), and that the stabilizer in $\mathrm{Aut}(\mathbb{F}_4)$ of a general $C \in |L_{2,0}|$ is generated by its hyperelliptic involution (Corollary 3.7). Since $\Sigma \cap C = \emptyset$, we have $L_{1,0}|_C \simeq L_{0,4}|_C \simeq 2K_C$. Then the restriction map $|L_{1,0}| \to |2K_C|$ is isomorphic because $h^0(L_{-1,0}) = h^1(L_{-1,0}) = 0$. These show that

Corollary 10.8. *The fixed curve map for $\mathcal{M}_{10,6,0}$ gives a dominant morphism $\mathcal{M}_{10,6,0} \to \mathcal{H}_3$ whose general fibers are birationally identified with the quotients of the bi-canonical systems by the hyperelliptic involutions.*

We note that the hyperelliptic locus $\mathcal{H}_3$ is the theta-null divisor of $\mathcal{M}_3$.

10.4. $\mathcal{M}_{11,5,1}$ **and the universal genus** 3 **curve.** Let $U \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(2)|$ be the locus of pairs (C, Q) such that C is smooth, Q is the union of distinct lines with $p = \mathrm{Sing}(Q)$ lying on C, and each component of Q is transverse to C. The point p is a D_4-singularity of the sextic $C + Q$, and the rest singularities of $C + Q$ are the six nodes $C \cap Q \backslash p$. The 2-elementary $K3$ surface associated to $C + Q$ has invariant $(g, k) = (3, 3)$. Thus we obtain a period map $\mathcal{P}\colon U/\mathrm{PGL}_3 \dashrightarrow \mathcal{M}_{11,5,1}$.

Proposition 10.9. *The map $\mathcal{P}$ is birational.*

Proof. Let $\widetilde{U} \subset U \times (\mathbb{P}^2)^6$ be the locus of those $(C, Q, p_1 \cdots, p_6)$ such that $C \cap Q \backslash \mathrm{Sing}(Q) = \{p_i\}_{i=1}^6$ and that p_1, p_2, p_3 lie on the same component of Q. For a point $(C, \cdots, p_6)$ of $\widetilde{U}$, the six nodes of $C + Q$ and the two components of Q are labelled in a compatible way. In particular, the three tangents at the D_4-point of $C + Q$ are also distinguished. Thus $\mathcal{P}$ lifts to a birational map $\widetilde{U}/\mathrm{PGL}_3 \dashrightarrow \widetilde{\mathcal{M}}_{11,5,1}$. The space $\widetilde{U}/\mathrm{PGL}_3$ is an $\mathfrak{S}_2 \ltimes (\mathfrak{S}_3)^2$-cover of U/PGL_3, while $\widetilde{\mathcal{M}}_{11,5,1}$ is an $\mathrm{O}(D_{L_+})$-cover of $\mathcal{M}_{11,5,1}$ for the lattice $L_+ = U \oplus D_4^2 \oplus A_1$. By [26] we calculate $|\mathrm{O}(D_{L_+})| = |\mathrm{O}^+(4, 2)| = 72$. □

Proposition 10.10. *The quotient U/PGL_3 is rational. Hence $\mathcal{M}_{11,5,1}$ is rational.*

Proof. Let $\pi\colon \mathcal{X}_3 \to \mathcal{M}_3$ be the universal genus 3 curve (over an open locus), and $\mathcal{E}$ be the subbundle of $\pi^*\pi_*K_\pi$ whose fiber over a (C,p) is $H^0(K_C - p)$. The natural map $U \to \mathcal{X}_3$, $(C,Q) \mapsto (C, \mathrm{Sing}(Q))$, shows that U/PGL_3 is birational to the symmetric product of $\mathbb{P}\mathcal{E}$ over $\mathcal{X}_3$. In particular, U/PGL_3 is birational to $\mathbb{P}^2 \times \mathcal{X}_3$. The space $\mathcal{X}_3$ is known to be rational: see [12] where this fact is attributed to Shepherd-Barron. It is a consequence of the slice method and Miyata's theorem (associate to a pointed quartic (C,p) the pointed tangent line (T_pC, p)). □

Corollary 10.11. *The fixed curve map for $\mathcal{M}_{11,5,1}$ lifts to a dominant map $\mathcal{M}_{11,5,1} \dashrightarrow \mathcal{X}_3$ whose fiber over a general (C,p) is birationally identified with the symmetric product of the pencil $|K_C - p|$.*

Remark 10.12. One can also prove Propositions 10.4 and 10.10 by considering the configuration $C \cap Q\backslash\mathrm{Sing}(Q)$ of points and using the no-name lemma. This avoids resorting to the rationality of $\mathcal{M}_3$.

10.5. **The rationality of $\mathcal{M}_{8+k,8-k,\delta}$ with $k \geq 4$.** For $4 \leq k \leq 6$ let $U_k \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(1)|^2$ be the locus of triplets (C, L_1, L_2) such that (i) C is smooth and passes $p = L_1 \cap L_2$, (ii) L_2 is transverse to C, and (iii) L_1 is tangent to C at p with multiplicity $k-2$ and transverse to C elsewhere. For $k = 5, 6$ the point p is an inflectional point of C of order $k-2$. The sextic $B = C + L_1 + L_2$ has the D_{2k-2}-point p, the $9-k$ nodes $(L_1 + L_2) \cap C\backslash p$, and no other singularity. Taking the right resolution of B, we obtain a period map $\mathcal{P}_k\colon U_k/\mathrm{PGL}_3 \dashrightarrow \mathcal{M}_{8+k,8-k,\delta}$. Here $\delta = 1$ for $k = 4, 5$, and $\delta = 0$ for $k = 6$.

Proposition 10.13. *The map $\mathcal{P}_k$ is birational.*

Proof. For $k = 4$ we label the three nodes $L_2 \cap C\backslash L_1$ and the two nodes $L_1 \cap C\backslash L_2$ independently. This is realized by an $\mathfrak{S}_3 \times \mathfrak{S}_2$-covering $\widetilde{U}_4 \to U_4$ as before. Note that L_1 and L_2 are distinguished by their intersection with C. Then we will obtain a birational lift $\widetilde{U}_4/\mathrm{PGL}_3 \dashrightarrow \widetilde{\mathcal{M}}_{12,4,1}$ of $\mathcal{P}_4$. The invariant lattice L_+ is isometric to $U(2) \oplus A_1^2 \oplus E_8$. By [26] we have $|\mathrm{O}(D_{L_+})| = 2 \cdot |\mathrm{Sp}(2,2)| = 12$. Therefore $\mathcal{P}_4$ is birational.

For $k = 5, 6$ we label only the three nodes $L_2 \cap C\backslash L_1$, which defines an $\mathfrak{S}_3$-covering $\widetilde{U}_k \to U_k$. The rest (at most one) node $L_1 \cap C\backslash L_2$ is obviously distinguished from those three. Therefore we will obtain a birational lift $\widetilde{U}_k/\mathrm{PGL}_3 \dashrightarrow \widetilde{\mathcal{M}}_{8+k,8-k,\delta}$ of $\mathcal{P}_k$. The anti-invariant lattice L_- is isometric to $U^2 \oplus D_4 \oplus A_1^{6-k}$. Then we have $\mathrm{O}(D_{L_-}) \simeq \mathfrak{S}_3$, which proves the proposition. □

Proposition 10.14. *The quotient U_k/PGL_3 is rational. Therefore $\mathcal{M}_{12,4,1}$, $\mathcal{M}_{13,3,1}$, and $\mathcal{M}_{14,2,0}$ are rational.*

Proof. Consider the projection $\pi\colon U_k \to |\mathcal{O}_{\mathbb{P}^2}(1)|^2, (C, L_1, L_2) \mapsto (L_1, L_2)$. The group PGL_3 acts almost transitively on $|\mathcal{O}_{\mathbb{P}^2}(1)|^2$ with the stabilizer of a general (L_1, L_2) being isomorphic to $(\mathbb{C}^\times \ltimes \mathbb{C})^2$. The π-fiber over (L_1, L_2) is an open set of the linear system of quartics which are tangent to L_1 at $L_1 \cap L_2$ with multiplicity $\geq k-2$. Thus our assertion follows from the slice method for π and Miyata's theorem. □

The fixed curve maps are described as follows. Recall that the ordinary inflectional points of a smooth quartic $C \subset \mathbb{P}^2$ are just the normal Weierstrass points of C, and the inflectional points of order 4 are just the Weierstrass points of weight 2. Then let $\mathcal{X}_3$ be the universal genus 3 curve, $\mathcal{U}_5 \subset \mathcal{X}_3$ the locus of normal Weierstrass points, and $\mathcal{U}_6 \subset \mathcal{X}_3$ the locus of Weierstrass points of weight 2.

Corollary 10.15. *The fixed curve map for $\mathcal{M}_{12,4,1}$ (resp. $\mathcal{M}_{8+k,8-k,\delta}$ with $k = 5, 6$) lifts to a dominant map $\mathcal{M}_{12,4,1} \dashrightarrow \mathcal{X}_3$ (resp. $\mathcal{M}_{8+k,8-k,\delta} \dashrightarrow \mathcal{U}_k$) whose general fibers are birationally identified with the pencils $|K_C - p|$.*

11. The case $g = 2$

In this section we treat the case $g = 2$, $k > 0$. In view of the unirationality result [20], we may assume $k < 9$. The case $(k, \delta) = (1, 0)$ is reduced to the rationality of $\mathcal{M}_{10,4,0}$ via the structure as an arithmetic quotient. The case $(k, \delta) = (1, 1)$ is settled by analyzing the quotient rational surfaces. The cases $2 \leq k \leq 4$ and $(k, \delta) = (5, 0)$ are studied using genus 2 curves on $\mathbb{F}_3$, and the case $k \geq 5$ with $\delta = 1$ is studied using cuspidal plane quartics.

11.1. **The rationality of $\mathcal{M}_{10,8,0}$.** Here we may take the same approach as the one for $\mathcal{M}_{10,10,0}$ by Kondō [18]. Recall that $\mathcal{M}_{10,8,0}$ is an open set of the modular variety $\mathcal{F}(\mathrm{O}(L_-))$ where L_- is the 2-elementary lattice $U^2 \oplus E_8(2)$. Since we have canonical isomorphisms $\mathrm{O}(L_-) \simeq \mathrm{O}(L_-^\vee) \simeq \mathrm{O}(L_-^\vee(2))$, the variety $\mathcal{F}(\mathrm{O}(L_-))$ is isomorphic to $\mathcal{F}(\mathrm{O}(L_-^\vee(2)))$. The lattice $L_-^\vee(2)$ is isometric to $U(2)^2 \oplus E_8$, so that $\mathcal{F}(\mathrm{O}(L_-^\vee(2)))$ is birational to $\mathcal{M}_{10,4,0}$. In §9.4 we proved that $\mathcal{M}_{10,4,0}$ is rational, and hence $\mathcal{M}_{10,8,0}$ is rational.

11.2. **The rationality of $\mathcal{M}_{10,8,1}$.** We construct the corresponding right DPN pairs starting from the Hirzebruch surface $\mathbb{F}_2$. For a smooth curve $C \in |L_{2,2}|$ transverse to Σ, let $f \colon Y \to \mathbb{F}_2$ be the double cover branched along C. Since we have a conic fibration $Y \to \mathbb{F}_2 \to \mathbb{P}^1$, the surface Y is rational. The curve $f^*\Sigma$ is a (-4)-curve on Y.

Lemma 11.1. *We have $|-2K_Y| = f^*\Sigma + f^*|L_{1,0}|$.*

Proof. By the ramification formula we have $-2K_Y \simeq f^*L_{2,-2}$. Since $|L_{2,-2}| = \Sigma + |L_{1,0}|$ and $\dim|L_{1,0}| = 3$, it suffices to show that $\dim|-2K_Y| = 3$. We take a smooth curve $H \in |L_{1,0}|$ transverse to C. The inverse image $D = f^*H$ is a smooth genus 2 curve disjoint from $f^*\Sigma$. Since $\mathcal{O}_D(f^*\Sigma) \simeq \mathcal{O}_D$, we have $-K_Y|_D \simeq K_D$ by the adjunction formula. Then the exact sequence

$$0 \to \mathcal{O}_Y \to -2K_Y \to 2K_D \oplus \mathcal{O}_{f^*\Sigma}(-4) \to 0$$

shows that $\dim|-2K_Y| = h^0(2K_D) = 3$. □

Thus the resolution of the bi-anticanonical map $Y \dashrightarrow \mathbb{P}^3$ of Y is given by the composition of $f \colon Y \to \mathbb{F}_2$ and the morphism $\phi \colon \mathbb{F}_2 \to \mathbb{P}^3$ associated to $L_{1,0}$. The image $\phi(\mathbb{F}_2)$ is a quadratic cone with vertex $\phi(\Sigma)$. In this way the quotient morphism f is recovered from the bi-anticanonical map of Y. It follows that

Lemma 11.2. *For two smooth curves $C, C' \in |L_{2,2}|$ transverse to Σ, the associated double covers Y, Y' are isomorphic if and only if C and C' are* $\mathrm{Aut}(\mathbb{F}_2)$*-equivalent.*

Now we let $U \subset |L_{2,2}| \times |L_{1,0}|$ be the open locus of pairs (C, H) such that C and H are smooth and that C is transverse to $H + \Sigma$. To a $(C, H) \in U$ we associate the right DPN pair (Y, B) where $f\colon Y \to \mathbb{F}_2$ is the double cover branched along C and $B = f^*(H + \Sigma)$. Since (Y, B) has invariant $(g, k) = (2, 1)$, we obtain a period map $U \to \mathcal{M}_{10,8,1}$. By Lemmas 11.1 and 11.2 the induced map $U/\mathrm{Aut}(\mathbb{F}_2) \dashrightarrow \mathcal{M}_{10,8,1}$ is generically injective. In view of the equality $\dim(U/\mathrm{Aut}(\mathbb{F}_2)) = 10$, we have a birational equivalence $U/\mathrm{Aut}(\mathbb{F}_2) \sim \mathcal{M}_{10,8,1}$.

Proposition 11.3. *The quotient $U/\mathrm{Aut}(\mathbb{F}_2)$ is rational. Hence $\mathcal{M}_{10,8,1}$ is rational.*

Proof. This follows from the slice method for the projection $|L_{2,2}| \times |L_{1,0}| \to |L_{1,0}|$, Proposition 3.4, and Katsylo's theorem 2.2. □

Remark 11.4. The above (Y, B) are (generically) right resolution of the plane sextics $C + Q$ where C is a one-nodal quartic and Q is a smooth conic transverse to C. Although the period map for this sextic model has degree > 1, we can analyze its fibers to derive the rationality of $\mathcal{M}_{10,8,1}$. This alternative approach may also be of some interest. We here give an outline for future reference.

Let $\widetilde{U} \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \times (\mathbb{P}^2)^8$ be the locus of $(C, p_1, \cdots, p_8)$ such that C is one-nodal and $\{p_i\}_{i=1}^8 = C \cap Q$ for a smooth conic Q. Using the labeling $\{p_i\}_{i=1}^8$, we obtain a birational map $\mathcal{U} = \widetilde{U}/\mathrm{PGL}_3 \dashrightarrow \widetilde{\mathcal{M}}_{10,8,1}$ as before. The projection $\widetilde{\mathcal{M}}_{10,8,1} \dashrightarrow \mathcal{M}_{10,8,1}$ is an $\mathfrak{S}_8 \ltimes (\mathbb{Z}/2)^6$-covering, which exceeds the the obvious $\mathfrak{S}_8$-symmetry of $\mathcal{U}$. In order to find the rest $(\mathbb{Z}/2)^6$-symmetry, let H be the group of even cardinality subsets of $\{1, \cdots, 8\}$ with the symmetric difference operation. For $\{i, j\} \in H$ and $(C, p_1, \cdots, p_8) \in \mathcal{U}$, consider the quadratic transformation $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ based at p_i, p_j, and $p_0 = \mathrm{Sing}(C)$. We set $C^+ = \varphi(C), p_i^+ = \varphi(\overline{p_0 p_i}), p_j^+ = \varphi(\overline{p_0 p_j})$, and $p_k^+ = \varphi(p_k)$ for $k \neq i, j$. Then we have $(C^+, p_1^+, \cdots, p_8^+) \in \mathcal{U}$, and this defines an action of H on $\mathcal{U}$. The element $\{1, \cdots, 8\} \in H$ acts on $\mathcal{U}$ trivially (it gives the covering transformation of the above $Y \to \mathbb{F}_2$). Now the period map $\mathcal{U} \dashrightarrow \mathcal{M}_{10,8,1}$ is $\mathfrak{S}_8 \ltimes H$-invariant, so that $\mathcal{M}_{10,8,1}$ is birational to $\mathcal{U}/(\mathfrak{S}_8 \ltimes H)$ by a degree comparison. We can prove that $\mathcal{U}/(\mathfrak{S}_8 \ltimes H)$ is rational.

11.3. **$\mathcal{M}_{11,7,1}$ and genus 2 curves on $\mathbb{F}_3$.** We construct 2-elementary $K3$ surfaces using curves on $\mathbb{F}_3$. Let $U \subset |L_{2,0}| \times |L_{1,1}|$ be the open set of pairs (C, D) such that C and D are smooth and transverse to each other. Then the curves $C + D + \Sigma$ belong to $|-2K_{\mathbb{F}_3}|$. Taking the right resolution of $C + D + \Sigma$, we obtain a period map $\mathcal{P}\colon U/\mathrm{Aut}(\mathbb{F}_3) \dashrightarrow \mathcal{M}_{11,7,1}$.

Proposition 11.5. *The map $\mathcal{P}$ is birational.*

Proof. We consider an $\mathfrak{S}_8$-covering $\widetilde{U} \to U$ whose fiber over a $(C, D) \in U$ corresponds to labelings of the eight nodes $C \cap D$. The rest node of $C + D + \Sigma$ is the one point $D \cap \Sigma$, which is obviously distinguished from those eight. Thus we obtain a birational lift $\widetilde{U}/\mathrm{Aut}(\mathbb{F}_3) \dashrightarrow \widetilde{\mathcal{M}}_{11,7,1}$ of $\mathcal{P}$. The projection $\widetilde{\mathcal{M}}_{11,7,1} \dashrightarrow \mathcal{M}_{11,7,1}$ is an $\mathrm{O}(D_{L_+})$-covering for the lattice $L_+ = U \oplus D_4 \oplus A_1^5$. By [26] we have $|\mathrm{O}(D_{L_+})| = |\mathrm{O}^+(6, 2)| = 8!$, which implies that $\mathcal{P}$ is birational. □

Proposition 11.6. *The quotient* $(|L_{2,0}| \times |L_{1,1}|)/\mathrm{Aut}(\mathbb{F}_3)$ *is rational. Therefore* $\mathcal{M}_{11,7,1}$ *is rational.*

Proof. We apply the slice method to the projection $|L_{2,0}| \times |L_{1,1}| \to |L_{1,1}|$, and then use Proposition 3.5 and Miyata's theorem. □

For every smooth $C \in |L_{2,0}|$ the linear system $|L_{1,1}|$ is identified with $|4K_C|$ by restriction. Indeed, we have $4K_C \simeq L_{0,4}|_C$ by the adjunction formula, and $L_{0,4}|_C \simeq L_{1,1}|_C$ because $C \cap \Sigma = \emptyset$. Then the vanishings $h^0(L_{-1,1}) = h^1(L_{-1,1}) = 0$ prove our claim. In view of Proposition 3.6 and Corollary 3.7, we have

Corollary 11.7. *The fixed curve map* $\mathcal{M}_{11,7,1} \to \mathcal{M}_2$ *is dominant with a general fiber being birationally identified with the quotient of* $|4K_C|$ *by the hyperelliptic involution.*

11.4. $\mathcal{M}_{12,6,1}$, $\mathcal{M}_{13,5,1}$, $\mathcal{M}_{14,4,0}$ **and genus** 2 **curves on** $\mathbb{F}_3$**.** We construct 2-elementary $K3$ surfaces using curves on $\mathbb{F}_3$. For $3 \leq k \leq 5$ let $U_k \subset |L_{2,0}| \times |L_{1,0}| \times |L_{0,1}|$ be the locus of triplets (C, H, F) such that (i) C and H are smooth and transverse to each other, (ii) F intersects with C at $p = F \cap H$ with multiplicity $k-3$ in case $k = 4, 5$, and is transverse to $C+H$ in case $k = 3$. It is easy to calculate $\dim U_k = 19-k$. For a $(C, H, F) \in U_k$ the curve $B = C+H+F+\Sigma$ belongs to $|-2K_{\mathbb{F}_3}|$. When $k = 3$, B has only nodes as the singularities. When $k = 4, 5$, B has the D_{2k-4}-singularity p, the nodes $C \cap H \backslash p$, $F \cap C \backslash p$, $F \cap \Sigma$, and no other sigularity. The 2-elementary $K3$ surface (X, ι) associated to B has invariant $(r, a) = (9 + k, 9 - k)$. When $k = 5$, (X, ι) has parity $\delta = 0$. Indeed, let (Y', B') be the corresponding right DPN pair. The curve B' has two components over p, say E_3 and E_5, whose numbering corresponds to the one for the vertices of the D_6-graph in p.11. Then the sum of $-E_3 + E_5$ and the strict transform of $C + F - H - \Sigma$ belongs to $4NS_{Y'}$, which proves $\delta = 0$. Thus we obtain a period map $\mathcal{P}_k \colon U_k/\mathrm{Aut}(\mathbb{F}_3) \dashrightarrow \mathcal{M}_{9+k,9-k,\delta}$ where $\delta = 1$ if $k = 3, 4$ and $\delta = 0$ if $k = 5$.

Proposition 11.8. *The map* $\mathcal{P}_k$ *is birational.*

Proof. For $k = 3$ we label the six nodes $C \cap H$ and the two nodes $C \cap F$ independently, which is realized by an $\mathfrak{S}_6 \times \mathfrak{S}_2$-covering $\widetilde{U_3} \to U_3$. The rest two nodes, $F \cap H$ and $F \cap \Sigma$, are distinguished by the irreducible decomposition of B. This defines a birational lift $\widetilde{U_3}/\mathrm{Aut}(\mathbb{F}_3) \dashrightarrow \widetilde{\mathcal{M}_{12,6,1}}$ of $\mathcal{P}_3$. The variety $\widetilde{\mathcal{M}_{12,6,1}}$ is an $\mathrm{O}(D_{L_+})$-cover of $\mathcal{M}_{12,6,1}$ for the lattice $L_+ = U \oplus A_1^2 \oplus D_4^2$. By [26] we have $|\mathrm{O}(D_{L_+})| = 2 \cdot |\mathrm{Sp}(4, 2)| = 2 \cdot 6!$. Therefore $\mathcal{P}_3$ has degree 1.

For $k = 4, 5$ we consider labelings the five nodes $C \cap H \backslash F$, which defines an $\mathfrak{S}_5$-cover $\widetilde{U_k}$ of U_k. The rest singularities of B and the branches at the D_{2k-4}-point are a priori distinguished. Thus we obtain a birational lift $\widetilde{U_k}/\mathrm{Aut}(\mathbb{F}_3) \dashrightarrow \widetilde{\mathcal{M}_{9+k,9-k,\delta}}$ of $\mathcal{P}_k$. The variety $\widetilde{\mathcal{M}_{9+k,9-k,\delta}}$ is an $\mathrm{O}(D_{L_-})$-cover of $\mathcal{M}_{9+k,9-k,\delta}$ for the lattice $L_- = U \oplus U(2) \oplus D_4 \oplus A_1^{5-k}$. We have $|\mathrm{O}(D_{L_-})| = |\mathrm{O}^-(4, 2)| = 5!$ for both $k = 4, 5$. Hence $\mathcal{P}_k$ is birational. □

Proposition 11.9. *The quotient* $U_k/\mathrm{Aut}(\mathbb{F}_3)$ *is rational. Therefore* $\mathcal{M}_{12,6,1}$, $\mathcal{M}_{13,5,1}$, *and* $\mathcal{M}_{14,4,0}$ *are rational.*

Proof. We apply the slice method to the projection $\pi_k \colon U_k \to |L_{1,0}| \times |L_{0,1}|$. By Propositions 3.4 and 3.3 the group $\mathrm{Aut}(\mathbb{F}_3)$ acts on $|L_{1,0}| \times |L_{0,1}|$ almost transitively with the stabilizer G of a general (H, F) being connected and solvable. The fiber $\pi_k^{-1}(H, F)$ is an open set of a linear subspace $\mathbb{P}V_k$ of $|L_{2,0}|$. Then $\mathbb{P}V_k/G$ is rational by Miyata's theorem. □

As in the paragraph just before Corollary 11.7, we see that for every smooth $C \in |L_{2,0}|$ the linear system $|L_{1,0}|$ is identified with $|3K_C|$ by restriction. When $k = 3$, we obtain a pointed genus 2 curve (C, p) by considering either of the two points $F \cap C$. When $k \geq 4$, the point $p = H \cap F$ determines F, and H is a general member of $|3K_C - p|$. For $k = 5$, p is a Weierstrass point of C. These infer the following.

Corollary 11.10. *Let $\mathcal{X}_2$ be the moduli of pointed genus 2 curves (C, p), and $\mathcal{W} \subset \mathcal{X}_2$ be the divisor of Weierstrass points. The fixed curve maps for $\mathcal{M}_{12,6,1}$, $\mathcal{M}_{13,5,1}$, and $\mathcal{M}_{14,4,0}$ lift to rational maps $\widetilde{F}_k : \mathcal{M}_{9+k,9-k,\delta} \dashrightarrow \mathcal{X}_2$. Then*

(1) $\widetilde{F}_3$ is dominant with a general fiber birationally identified with the quotient of $|3K_C|$ by the hyperelliptic involution.

(2) $\widetilde{F}_4$ is dominant with a general fiber birationally identified with $|3K_C - p|$.

(3) $\widetilde{F}_5$ is a dominant map onto $\mathcal{W}$ whose general fiber is birationally identified with the quotient of $|3K_C - p|$ by the hyperelliptic involution.

11.5. $\mathcal{M}_{9+k,9-k,1}$ **with** $k \geq 5$ **and cuspidal plane quartics.** Let $U \subset |\mathcal{O}_{\mathbb{P}^2}(4)|$ be the locus of plane quartics with an ordinary cusp and with no other singularity. For $5 \leq k \leq 8$ we denote by $U_k \subset U \times |\mathcal{O}_{\mathbb{P}^2}(1)|$ the locus of pairs (C, L) such that if $M \subset \mathbb{P}^2$ is the tangent line of C at the cusp, then L intersects with C at $C \cap M \backslash \mathrm{Sing}(C)$ with multiplicity $k - 5$, and is transverse to C elsewhere. The space U_k is of dimension $19 - k$. This is obvious for $5 \leq k \leq 7$. For $k = 8$, if we take the homogeneous coordinate $[X, Y, Z]$ of $\mathbb{P}^2$ and normalize $p = [0, 0, 1]$, $M = \{Y = 0\}$, and $L = \{Z = 0\}$, then quartics C having cusp at p with $(C.M)_p = 3$ and $(C.L)_{L \cap M} = 3$ are defined by the equations

$$a_{02}Y^2Z^2 + \sum_{i+j=3} a_{ij}X^iY^jZ + a_{13}XY^3 + a_{04}Y^4 = 0, \tag{11.1}$$

in which the coefficients a_* may be taken general. This shows that U_8 is of the expected dimension.

For a $(C, L) \in U_k$ the sextic $B = C + L + M$ has an E_7-singularity at $p = \mathrm{Sing}(C)$, $9 - k$ nodes at $C \cap L \backslash M$, a D_{2k-8}-singularity at $L \cap M$ (resp. two nodes at $L \cap M$ and $C \cap M \backslash p$) in case $k \geq 6$ (resp. $k = 5$), and no other singularity. Hence the 2-elementary $K3$ surface (X, ι) associated to B has invariant $(r, a) = (9 + k, 9 - k)$. When $k = 5$, (X, ι) has parity $\delta = 1$ by Lemma 4.4 (1). Thus we obtain a period map $\mathcal{P}_k \colon U_k \dashrightarrow \mathcal{M}_{9+k,9-k,1}$.

Proposition 11.11. *The map $\mathcal{P}_k$ is birational.*

Proof. We consider an $\mathfrak{S}_{9-k}$-covering $\widetilde{U}_k \to U_k$ whose fiber over a $(C, L) \in U_k$ corresponds to labelings the $9 - k$ nodes $C \cap L \backslash M$. As described above, the rest singular points of B are a priori distinguished by the type of singularity and by the irreducible decomposition of B. Also note that the two lines L and M are distinguished by their intersection with C. Therefore we will obtain a birational lift $\widetilde{U}_k/\mathrm{PGL}_3 \dashrightarrow \widetilde{\mathcal{M}}_{9+k,9-k,1}$ of $\mathcal{P}_k$. The anti-invariant lattice L_- is isometric to $U^2 \oplus A_1^{9-k}$. It is easy to check that $\mathrm{O}(D_{L_-}) \simeq \mathfrak{S}_{9-k}$. Since PGL_3 acts on U almost freely, this shows that $\mathcal{P}_k$ has degree 1. □

Proposition 11.12. *The quotient U_k/PGL_3 is rational. Therefore $\mathcal{M}_{9+k,9-k,1}$ with $5 \leq k \leq 8$ is rational.*

Proof. Let $V \subset \mathbb{P}^2 \times |\mathcal{O}_{\mathbb{P}^2}(1)|^2$ be the locus of triplets (p, M, L) such that $p \in M$. We have the PGL_3-equivariant map $\pi_k \colon U_k \to V$, $(C, L) \mapsto (\mathrm{Sing}(C), M, L)$, where M is the tangent line of C at the cusp. The group PGL_3 acts on V almost transitively with the stabilizer of a general point isomorphic to $\mathbb{C}^\times \times (\mathbb{C}^\times \ltimes \mathbb{C})$. Since a general π_k-fiber is an open set of a sub linear system of $|\mathcal{O}_{\mathbb{P}^2}(4)|$, the assertion follows from the slice method for π_k and Miyata's theorem. □

Let $\mathcal{X}_2$ be the moduli of pointed genus 2 curves (C, p). Recall (cf. [12]) that we have a birational map $\mathcal{X}_2 \dashrightarrow U/\mathrm{PGL}_3$ by associating to a pointed curve (C, p) the image $\phi(C) \subset \mathbb{P}^2$ by the linear system $|K_C + 2p|$. When p is not a Weierstrass point, $\phi(C)$ is a quartic with a cusp $\phi(p)$, and the projection from $\phi(p)$ gives the hyperelliptic map of C. Thus we see the following.

Corollary 11.13. *The fixed curve map for $\mathcal{M}_{9+k,9-k,1}$ with $5 \leq k \leq 8$ lifts to a rational map $\widetilde{F}_k : \mathcal{M}_{9+k,9-k,1} \dashrightarrow \mathcal{X}_2$. Then*

(1) *$\widetilde{F}_5$ is dominant with general fibers birationally identified with $|K_C + 2p|$.*

(2) *$\widetilde{F}_6$ is dominant with general fibers birationally identified with $|3p|$.*

(3) *$\widetilde{F}_7$ is birational.*

(4) *$\widetilde{F}_8$ is generically injective with a generic image being the divisor of those (C, p) with $5p - 2K_C$ effective minus the divisor of Weierstrass points.*

12. The case $g = 1$ (I)

In this section we study the case $g = 1$, $1 \leq k \leq 4$, $\delta = 1$. It seems (at least to the author) difficult to use the previous methods for this case. We obtain general members of $\mathcal{M}_{r,a,\delta}$ from plane sextics of the form $C_1 + C_2$ where C_1 is a nodal cubic and C_2 is a smooth cubic intersecting with C_1 at the node with suitable multiplicity. But the period map for this sextic model has degree > 1. In order to pass to a "canonical" construction, we find a Weyl group symmetry in this sextic model, which reflects the fact that the Galois group of $\widetilde{\mathcal{M}}_{r,a,\delta} \dashrightarrow \mathcal{M}_{r,a,\delta}$ is the Weyl group or its central quotient. This leads to regard the cubics C_i as anticanonical curves on del Pezzo surfaces of degree k.

12.1. **Weyl group actions on universal families.** We begin with constructing a universal family of marked del Pezzo surfaces with an equivariant action by the Weyl group. For $5 \leq d \leq 8$ let $U^d \subset (\mathbb{P}^2)^d$ be the open set of ordered d points in general position in the sense of [11]. There exists a geometric quotient $\mathcal{U}^d = U^d/\mathrm{PGL}_3$ of U^d by PGL_3 (see [13]). Since any $\mathbf{p} = (p_1, \cdots, p_d) \in U^d$ has trivial stabilizer, the quotient map $U^d \to \mathcal{U}^d$ is a principal PGL_3-bundle by Luna's etale slice theorem. Hence by [27] Proposition 7.1 the projection $\mathbb{P}^2 \times U^d \to U^d$ descends to a smooth morphism $\mathcal{X}^d \to \mathcal{U}^d$ where $\mathcal{X}^d$ is a geometric quotient of $\mathbb{P}^2 \times U^d$ by PGL_3. Here PGL_3 acts on $\mathbb{P}^2$ in the natural way. For $1 \leq i \leq d$ we have the section s_i of $\mathcal{X}^d \to \mathcal{U}^d$ induced by the i-th projection $U^d \to \mathbb{P}^2$, $(p_1, \cdots, p_d) \mapsto p_i$. Blowing-up the d sections s_i, we obtain a family $f\colon \mathcal{Y}^d \to \mathcal{U}^d$ of del Pezzo surfaces of degree $9-d$. The exceptional divisor over s_i, denoted by $\mathcal{E}_i$, is a family of (-1)-curves.

Let Λ_d be the lattice $\langle 1\rangle \oplus \langle -1\rangle^d$ with a natural orthogonal basis $h, e_1, \cdots, e_d$. For each $\mathbf{p} \in \mathcal{U}^d$, the Picard lattice of the f-fiber $\mathcal{Y}^d_{\mathbf{p}}$ over $\mathbf{p}$ is isometric to Λ_d by associating h to the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ and e_i to the class of the (-1)-curve $(\mathcal{E}_i)_{\mathbf{p}}$. This gives a trivialization $\varphi\colon \mathcal{U}^d \times \Lambda_d \to R^2 f_*\mathbb{Z}$ of the local system. Thus we have a universal family

$$(12.1) \qquad (f : \mathcal{Y}^d \to \mathcal{U}^d,\ \varphi)$$

of marked del Pezzo surfaces.

Recall that the Weyl group W_d is the group of isometries of Λ_d which fix the vector $3h - \sum_{i=1}^d e_i$. Let $w \in W_d$. For each $\mathbf{p} \in \mathcal{U}^d$ the marked del Pezzo surface $(\mathcal{Y}^d_{\mathbf{p}}, \varphi_{\mathbf{p}})$ is transformed by w to $(\mathcal{Y}^d_{\mathbf{p}}, \varphi_{\mathbf{p}} \circ w^{-1})$, which is isomorphic to $(\mathcal{Y}^d_{\mathbf{q}}, \varphi_{\mathbf{q}})$ for a $\mathbf{q} \in \mathcal{U}^d$. Indeed, the classes $\varphi_{\mathbf{p}} \circ w^{-1}(e_i)$ are represented by disjoint (-1)-curves E_i' on $\mathcal{Y}^d_{\mathbf{p}}$ which define a blow-down $\mathcal{Y}^d_{\mathbf{p}} \to \mathbb{P}^2$. Then $\mathbf{q}$ is the blown-down points of $E_1', \cdots, E_d'$. We set $w(\mathbf{p}) = \mathbf{q}$. By construction, we have an isomorphism $\mu_w\colon \mathcal{Y}^d_{\mathbf{p}} \to \mathcal{Y}^d_{w(\mathbf{p})}$ with $(\mu_w)_* \circ \varphi_{\mathbf{p}} \circ w^{-1} = \varphi_{w(\mathbf{p})}$. The last equality characterizes μ_w uniquely because the cohomological representation $\mathrm{Aut}(\mathcal{Y}^d_{\mathbf{p}}) \to \mathrm{O}(\mathrm{Pic}(\mathcal{Y}^d_{\mathbf{p}}))$ is injective. This ensures that $(w'w)(\mathbf{p}) = w'(w(\mathbf{p}))$ and that $\mu_{w'} \circ \mu_w = \mu_{w'w}$ for $w, w' \in W_d$. In this way we obtain an equivariant action of W_d on the family $f\colon \mathcal{Y}^d \to \mathcal{U}^d$. The W_d-action on $\mathcal{U}^d$ is the Cremona representation. In the following we will refer to [13] for the basic properties of the Cremona representation for each d.

On $\mathcal{Y}^d$ we have two natural W_d-linearized vector bundles: the relative tangent bundle T_f and the relative anticanonical bundle K_f^{-1}. The direct image $f_*K_f^{-1}$ is a W_d-linearized bundle over $\mathcal{U}^d$. The fiber of the $\mathbb{P}^{9-d}$-bundle $\mathbb{P}(f_*K_f^{-1})$ over a point $\mathbf{p} = [(p_1, \cdots, p_d)]$ of $\mathcal{U}^d$ is identified with the linear system of plane cubics passing the d points $p_1, \cdots, p_d$ of $\mathbb{P}^2$.

12.2. $\mathcal{M}_{11,9,1}$ and del Pezzo surfaces of degree 1.

12.2.1. *A period map.* Let $f\colon \mathcal{Y}^8 \to \mathcal{U}^8$ be the family of marked del Pezzo surfaces of degree 1 constructed above. The Cremona representation of the Weyl group W_8 has kernel $\mathbb{Z}/2$ whose generator w_0 acts on the f-fibers by the Bertini involutions. The quotient $W_8/\langle w_0\rangle$ is isomorphic to $\mathrm{O}^+(8,2)$ and acts on $\mathcal{U}^8$ almost freely.

We let $\mathcal{V}^8 \subset \mathbb{P}(f_*K_f^{-1})$ be the locus of singular anticanonical curves. The locus $\mathcal{V}^8$ is invariant under the W_8-action on $\mathbb{P}(f_*K_f^{-1})$, and the projection $\mathcal{V}^8 \to \mathcal{U}^8$ is of degree 12. We denote by $\mathcal{E}$ the pullback of the bundle $f_*K_f^{-1}$ by $\mathcal{V}^8 \to \mathcal{U}^8$. An open set of the variety $\mathbb{P}\mathcal{E}$ parametrizes triplets $(\mathcal{Y}^8_{\mathbf{p}}, C_1, C_2)$ of a marked del Pezzo surface $\mathcal{Y}^8_{\mathbf{p}} = Y$, a nodal $-K_Y$-curve C_1, and a smooth $-K_Y$-curve C_2. Notice that C_1 is irreducible and transverse to C_2 because $(C_1.C_2) = 1$. Taking a right resolution of the DPN pair $(Y, C_1 + C_2)$, we obtain a 2-elementary $K3$ surface of invariant $(g,k) = (1,1)$. This defines a period map $\mathcal{P}\colon \mathbb{P}\mathcal{E} \dashrightarrow \mathcal{M}_{11,9,1}$.

Proposition 12.1. *The map $\mathcal{P}$ is W_8-invariant and descends to a birational map $\mathbb{P}\mathcal{E}/W_8 \dashrightarrow \mathcal{M}_{11,9,1}$.*

Proof. Two W_8-equivalent points of $\mathbb{P}\mathcal{E}$ give rise to isomorphic DPN pairs so that $\mathcal{P}$ is W_8-invariant. We shall show that $\mathcal{P}$ lifts to a birational map $\mathbb{P}\mathcal{E} \dashrightarrow \widetilde{\mathcal{M}}_{11,9,1}$. For a point $(\mathbf{p}, C_1, C_2)$ of $\mathbb{P}\mathcal{E}$, we have the marking $\varphi_{\mathbf{p}}$ of the Picard lattice of $\mathcal{Y}^8_{\mathbf{p}}$ induced by $\mathbf{p}$. The curve $C_1 + C_2$ has two nodes, namely the intersection point $C_1 \cap C_2$ and the node of C_1, which are clearly distinguished. This induces a marking of the invariant lattice of $(X,\iota) = \mathcal{P}(\mathbf{p}, C_1, C_2)$, which defines a lift $\widetilde{\mathcal{P}}\colon \mathbb{P}\mathcal{E} \dashrightarrow \widetilde{\mathcal{M}}_{11,9,1}$ of $\mathcal{P}$. In order to show that $\widetilde{\mathcal{P}}$ is birational, the key point is that the blow-down $\pi\colon \mathcal{Y}^8_{\mathbf{p}} \to \mathbb{P}^2$ defined by $\mathbf{p}$ translates the triplet $(\mathbf{p}, C_1, C_2)$ into the plane sextic $\pi(C_1 + C_2)$ endowed with a labeling of the nine nodes $\pi(C_1) \cap \pi(C_2)$. Indeed, $\pi(C_1) \cap \pi(C_2)$ consists of the eight ordered points $\mathbf{p}$ and the rest one point $\pi(C_1 \cap C_2)$. The first eight nodes recover the marked del Pezzo surface, and the ninth is determined as the base point of the associated cubic pencil. Note also that the unlabeled node $\mathrm{Sing}(\pi(C_1))$ of $\pi(C_1 + C_2)$ is clearly distinguished from those nine. In this way $\mathbb{P}\mathcal{E}$ is birationally identified with the PGL_3-quotient of the space of such nodal sextics with labelings. Now one may follow the recipe in §4.3 to see that $\widetilde{\mathcal{P}}$ is birational.

We compare the two projections $\mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{E}/W_8$ and $\widetilde{\mathcal{M}}_{11,9,1} \dashrightarrow \mathcal{M}_{11,9,1}$. The Bertini involution $w_0 \in W_8$ acts trivially on $\mathbb{P}\mathcal{E}$ because it acts trivially on the anticanonical pencils $|-K_Y|$. Hence $\mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{E}/W_8$ is an $\mathrm{O}^+(8,2)$-covering. On the other hand, $\widetilde{\mathcal{M}}_{11,9,1}$ is an $\mathrm{O}(D_{L_+})$-cover of $\mathcal{M}_{11,9,1}$ for the lattice $L_+ = U \oplus A_1^9$. Since $\mathrm{O}(D_{L_+}) \simeq \mathrm{O}^+(8,2)$, this finishes the proof. □

12.2.2. *The rationality.* The W_8-action on $\mathbb{P}\mathcal{E}$ gets rid of the markings of del Pezzo surfaces. This implies that the quotient $\mathbb{P}\mathcal{E}/W_8$ is birationally a moduli of triplets (Y, C_1, C_2) where Y is an (unmarked) del Pezzo surface of degree 1, and C_1 (resp. C_2) is a singular (resp. smooth) $-K_Y$-curve on Y. We consider the blow-up $\widehat{Y} \to Y$ at the base point $C_1 \cap C_2$ of $|-K_Y|$. The quotient of $\widehat{Y}$ by the Bertini involution is the Hirzebruch surface $\mathbb{F}_2$, and the quotient morphism $\phi\colon \widehat{Y} \to \mathbb{F}_2$ is branched over the (-2)-curve Σ and over a smooth $L_{3,0}$-curve Γ. Note that $\phi^{-1}(\Sigma)$ is the exceptional curve of $\widehat{Y} \to Y$. The pencil $|L_{0,1}|$ on $\mathbb{F}_2$ is pulled-back by ϕ to the pencil $|-K_{\widehat{Y}}| = |-K_Y|$. Then $\phi(C_1)$ is an $L_{0,1}$-fiber tangent to Γ, and $\phi(C_2)$ is an $L_{0,1}$-fiber transverse to Γ. If we let $U \subset |L_{3,0}| \times |L_{0,1}|^2$ be the locus of triplets (Γ, F_1, F_2) such that Γ is smooth and F_1 (resp. F_2) is tangent (resp. transverse) to Γ, we thus obtain a rational map

$$\mathbb{P}\mathcal{E}/W_8 \dashrightarrow U/\mathrm{Aut}(\mathbb{F}_2).$$

Since this construction may be reversed, the map is birational.

Proposition 12.2. *The quotient $U/\mathrm{Aut}(\mathbb{F}_2)$ is rational. Thus $\mathcal{M}_{11,9,1}$ is rational.*

Proof. Let V be the fiber product $\mathbb{F}_2 \times_{\mathbb{P}^1} \mathbb{F}_2$, where $\pi \colon \mathbb{F}_2 \to \mathbb{P}^1$ is the natural projection. We consider the $\mathrm{Aut}(\mathbb{F}_2)$-equivariant map

$$\varphi : U \to V \times |L_{0,1}|, \qquad (\Gamma, F_1, F_2) \mapsto ((p, q), F_2),$$

where p is the point of tangency of Γ and F_1, and $q = \Gamma|_{F_1} - 2p$. By §3, the group $\mathrm{Aut}(\mathbb{F}_2)$ acts on $V \times |L_{0,1}|$ almost transitively. If we normalize $F_1 = \pi^{-1}([0,1])$ and $F_2 = \pi^{-1}([1,0])$, the stabilizer G of a general point $((p,q), F_2)$ with $p, q \in F_1$ is described by the exact sequence $0 \to H \to G \to \mathbb{C}^\times \to 1$, where $H = \{s \in H^0(\mathcal{O}_{\mathbb{P}^1}(2)), s([0,1]) = 0\}$. In particular, G is connected and solvable. By the slice method we have $U/\mathrm{Aut}(\mathbb{F}_2) \sim \varphi^{-1}((p,q), F_2)/G$. The fiber $\varphi^{-1}((p,q), F_2)$ is an open set of the linear system $\mathbb{P}W \subset |L_{3,0}|$ of curves Γ with $\Gamma|_{F_1} = 2p + q$. By Miyata's theorem the quotient $\mathbb{P}W/G$ is rational. □

12.3. $\mathcal{M}_{12,8,1}$ and quadric del Pezzo surfaces.

12.3.1. *A period map.* Let us begin with few remarks. Let Y be a quadric del Pezzo surface and $\phi \colon Y \to \mathbb{P}^2$ be the anticanonical map, which is branched along a smooth quartic Γ. The pullback $\phi^* L$ of a line $L \subset \mathbb{P}^2$ is singular if and only if L is tangent to Γ. Therefore the reduced curve $\phi^{-1}(\Gamma)$, the fixed locus of the Geiser involution, is the locus of singular points of anticanonical curves on Y. For every $p \in \phi^{-1}(\Gamma)$ we have a unique singular $-K_Y$-curve C_p with $p \in \mathrm{Sing}(C_p)$, which is the pullback of the tangent line L_p of Γ at $\phi(p)$. The singular curve C_p is irreducible and nodal if and only if L_p is an ordinary tangent line. In this case, the Geiser involution exchanges the two tangents of C_p at p. On the other hand, C_p has only one tangent at p if and only if $\phi(p)$ is an inflectional point, i.e., a Weierstrass point of Γ.

Now let $f \colon \mathcal{Y}^7 \to \mathcal{U}^7$ be the family of marked quadric del Pezzo surfaces constructed in §12.1. The Cremona representation of the Weyl group W_7 has kernel $\mathbb{Z}/2$ whose generator w_0 acts on the f-fibers by the Geiser involutions. The quotient $W_7/\langle w_0 \rangle$ is isomorphic to $\mathrm{Sp}(6,2)$ and acts on $\mathcal{U}^7$ almost freely.

Let $C \subset \mathcal{Y}^7$ be the fixed locus of w_0. We define a double cover $\widetilde{C}$ of C as the locus in $(\mathbb{P}T_f)|_C$ of triplets $(\mathbf{p}, p, v)$ such that $v \in \mathbb{P}(T_p \mathcal{Y}^7_{\mathbf{p}})$ is a tangent at p of the anticanonical curve on $\mathcal{Y}^7_{\mathbf{p}}$ singular at p. The locus $\widetilde{C}$ is invariant under the W_7-action on $(\mathbb{P}T_f)|_C$. In particular, the Geiser involution w_0 acts on $\widetilde{C}$ by the covering transformation of $\widetilde{C} \to C$. The branch divisor of $\widetilde{C} \to C$ is the family of the Weierstrass points.

We pull back the bundle $f_* K_f^{-1}$ on $\mathcal{U}^7$ by the projection $\widetilde{C} \to \mathcal{U}^7$, and consider its subbundle $\mathcal{E}$ whose fiber over a $(\mathbf{p}, p, v) \in \widetilde{C}$ is the vector space of anticanonical forms on $\mathcal{Y}^7_{\mathbf{p}}$ vanishing at p. An open set of the variety $\mathbb{P}\mathcal{E}$ parametrizes quadruples $(\mathcal{Y}^7_{\mathbf{p}}, C_1, v, C_2)$ such that $\mathcal{Y}^7_{\mathbf{p}} = Y$ is a marked quadric del Pezzo surface, C_1 is an irreducible nodal $-K_Y$-curve, v is one of the tangents of C_1 at the node, and C_2 is a smooth $-K_Y$-curve passing the node of C_1. Then $C_1 + C_2$ is a $-2K_Y$-curve with the D_4-singularity $C_1 \cap C_2$ and with no other singularity. Its two tangents given by C_1 at the D_4-point are distinguished by v. The 2-elementary $K3$ surface associated to the DPN pair $(Y, C_1 + C_2)$ has invariant $(g, k) = (1, 2)$. Thus we obtain a period map $\mathcal{P}\colon \mathbb{P}\mathcal{E} \dashrightarrow \mathcal{M}_{12,8,1}$.

Proposition 12.3. *The map $\mathcal{P}$ is W_7-invariant and descends to a birational map $\mathbb{P}\mathcal{E}/W_7 \dashrightarrow \mathcal{M}_{12,8,1}$.*

Proof. The W_7-invariance of $\mathcal{P}$ is straightforward. The marking $\varphi_{\mathbf{p}}$ of $\mathrm{Pic}(\mathcal{Y}^7_{\mathbf{p}})$ and the labeling ν for the D_4-point $C_1 \cap C_2$ induce a marking of the invariant lattice of $(X, \iota) = \mathcal{P}(\mathbf{p}, C_1, \nu, C_2)$. This defines a lift $\widetilde{\mathcal{P}}\colon \mathbb{P}\mathcal{E} \dashrightarrow \widetilde{\mathcal{M}}_{12,8,1}$ of $\mathcal{P}$. As in §12.2, the blow-down $\pi\colon \mathcal{Y}^7_{\mathbf{p}} \to \mathbb{P}^2$ determined by $\mathbf{p}$ translates the quadruple $(\mathbf{p}, C_1, \nu, C_2)$ into the plane sextic $\pi(C_1 + C_2)$ endowed with a labeling of its seven nodes, $\mathbf{p}$, and of the two tangents at its D_4-point $\pi(C_1 \cap C_2)$ given by $\pi(C_1)$. Applying the recipe in §4.3 for such sextics with labelings, we see that $\widetilde{\mathcal{P}}$ is birational. The projection $\widetilde{\mathcal{M}}_{12,8,1} \dashrightarrow \mathcal{M}_{12,8,1}$ is an $\mathrm{O}(D_{L_-})$-covering for the lattice $L_- = U \oplus U(2) \oplus A_1^6$. By [26] we have $|\mathrm{O}(D_{L_-})| = 2 \cdot |\mathrm{Sp}(6, 2)| = |W_7|$. Since W_7 acts on $\mathbb{P}\mathcal{E}$ almost freely, this concludes that the induced map $\mathbb{P}\mathcal{E}/W_7 \dashrightarrow \mathcal{M}_{12,8,1}$ is birational. □

12.3.2. *The rationality.* We shall prove that $\mathbb{P}\mathcal{E}/W_7$ is rational. The Weyl group W_7 acts on the markings of del Pezzo surfaces, and the Geiser involution $w_0 \in W_7$ is the covering transformation of $\widetilde{C} \to C$. These facts infer that $\mathbb{P}\mathcal{E}/W_7$ is birationally a moduli space of triplets (Y, p, C) where Y is an (unmarked) quadric del Pezzo surface, $p \in Y$ is a fixed point of the Geiser involution, and C is a $-K_Y$-curve passing p. The anticanonical map $Y \to \mathbb{P}^2$ translates (Y, p, C) into the triplet (Γ, q, L) such that Γ is a smooth plane quartic, $q \in \Gamma$, and L is a line passing q. If $U \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \times \mathbb{P}^2 \times |\mathcal{O}_{\mathbb{P}^2}(1)|$ denotes the space of such triplets (Γ, q, L), we obtain a birational equivalence

$$\mathbb{P}\mathcal{E}/W_7 \sim U/\mathrm{PGL}_3.$$

In Proposition 10.14 (for $k = 4$) we proved that this quotient U/PGL_3 is rational. Therefore

Proposition 12.4. *The moduli space $\mathcal{M}_{12,8,1}$ is rational.*

The final step of the proof shows that we have a natural birational map $\mathcal{M}_{12,8,1} \dashrightarrow \mathcal{M}_{12,4,1}$.

12.4. $\mathcal{M}_{13,7,1}$ and cubic surfaces.

12.4.1. *A period map.* We begin with the remark that for every point p of a cubic del Pezzo surface Y, there uniquely exists a $-K_Y$-curve C_p singular at p. Indeed, if we embed Y in $\mathbb{P}^3$ naturally, C_p is the intersection of Y with the tangent plane of Y at p. When p is generic, C_p is irreducible and nodal. For later use, we also explain an alternative construction. The blow-up $\widehat{Y}$ of Y at a general $p \in Y$ is a quadric del Pezzo surface. Let $E \subset \widehat{Y}$ be the (-1)-curve over p, and $\phi\colon \widehat{Y} \to \mathbb{P}^2$ be the anticanonical map of $\widehat{Y}$ which is branched over a smooth quartic Γ. Then $\phi(E)$ is a bitangent of Γ, and $\phi^*\phi(E) = E + \iota(E)$ where ι is the Geiser involution of $\widehat{Y}$. Since $\phi^*\phi(E)$ is a $-K_{\widehat{Y}}$-curve, the image of $\iota(E)$ in Y gives the desired curve C_p. The fact that ϕ and ι are given by the projection $Y \subset \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from p connects these two constructions.

Let $f\colon \mathcal{Y}^6 \to \mathcal{U}^6$ be the family of marked cubic surfaces constructed in §12.1. The Weyl group W_6 acts on $\mathcal{U}^6$ almost freely because a generic cubic surface has no automorphism. We shall denote a point of the variety $\mathbb{P}T_f$ as $(\mathbf{p}, p, v)$ where $\mathbf{p} \in \mathcal{U}^6$, $p \in \mathcal{Y}^6_{\mathbf{p}}$, and $v \in \mathbb{P}(T_p\mathcal{Y}^6_{\mathbf{p}})$. Then we let $\mathcal{Z} \subset \mathbb{P}T_f$ be the locus of $(\mathbf{p}, p, v)$ such that v is one of the tangents of the anticanonical curve C_p singular at p. The locus $\mathcal{Z}$ is W_6-invariant and is a double cover of $\mathcal{Y}^6$. The branch divisor of $\mathcal{Z} \to \mathcal{Y}^6$ is the family of the Hessian quartics restricted to the marked cubic surfaces.

We pull-back the vector bundle $f_*K_f^{-1}$ on $\mathcal{U}^6$ by the projection $\mathcal{Z} \to \mathcal{U}^6$, and consider its subbundle $\mathcal{E}$ whose fiber over a $(\mathbf{p}, p, v) \in \mathcal{Z}$ is the space of anticanonical forms on $\mathcal{Y}^6_{\mathbf{p}}$ which vanish at p and whose first derivatives at p vanish at v. Then $\mathcal{E}$ is a W_6-linearized vector bundle over $\mathcal{Z}$. An open set of the variety $\mathbb{P}\mathcal{E}$ parametrizes triplets $(\mathcal{Y}^6_{\mathbf{p}}, p, C)$ where $\mathcal{Y}^6_{\mathbf{p}} = Y$ is a marked cubic surface, $p \in Y$ is such that the singular $-K_Y$-curve C_p is irreducible and nodal, and C is a smooth $-K_Y$-curve with $(C.C_p)_p = 3$. By v is chosen which branch of C_p at p is tangent to C. The curve $C_p + C$ has the D_6-singularity p as its only singularity. Thus, taking the right resolution of the DPN pairs $(Y, C_p + C)$, we obtain a period map $\mathcal{P}\colon \mathbb{P}\mathcal{E} \dashrightarrow \mathcal{M}_{13,7,1}$.

Proposition 12.5. *The map $\mathcal{P}$ is W_6-invariant and descends to a birational map $\mathbb{P}\mathcal{E}/W_6 \dashrightarrow \mathcal{M}_{13,7,1}$.*

Proof. This is analogous to Propositions 12.1 and 12.3: for a triplet $(\mathbf{p}, C_p, C)$ in $\mathbb{P}\mathcal{E}$, the blow-down $\pi\colon \mathcal{Y}^6_{\mathbf{p}} \to \mathbb{P}^2$ determined by $\mathbf{p}$ translates $(\mathbf{p}, C_p, C)$ into the plane sextic $\pi(C_p + C)$ endowed with the labeling $\mathbf{p}$ of its six nodes. The rest singularity of $\pi(C_p + C)$ is the D_6-point $\pi(p)$, at which the three branches of $\pi(C_p + C)$ are a priori distinguished by the irreducible decomposition of $\pi(C_p + C)$ and by the intersection multiplicity at $\pi(p)$. By the recipe in §4.3 we see that $\mathcal{P}$ lifts to a birational map $\mathbb{P}\mathcal{E} \dashrightarrow \widetilde{\mathcal{M}}_{13,7,1}$. The variety $\widetilde{\mathcal{M}}_{13,7,1}$ is an $\mathrm{O}(D_{L_-})$-cover of $\mathcal{M}_{13,7,1}$ for the lattice $L_- = U \oplus U(2) \oplus A_1^5$. Since $\mathrm{O}(D_{L_-}) \simeq \mathrm{O}^-(6,2) \simeq W_6$, our assertion is proved. □

12.4.2. *The rationality.* We shall prove that $\mathbb{P}\mathcal{E}/W_6$ is rational. First we apply the no-name method to the W_6-linearized bundle $\mathcal{E}$ over $\mathcal{Z}$ to see that

$$\mathbb{P}\mathcal{E}/W_6 \sim \mathbb{P}^1 \times (\mathcal{Z}/W_6).$$

The variety $\mathcal{Z}$ is a moduli of triplets $(\mathcal{Y}^6_{\mathbf{p}}, p, v)$ where $\mathcal{Y}^6_{\mathbf{p}} = Y$ is a marked cubic surface, $p \in Y$, and $v \in \mathbb{P}(T_pY)$ is one of the tangents of the singular $-K_Y$-curve C_p. The W_6-action takes off the markings of surfaces. Let $\widehat{Y} \to Y$ be the blow-up at p, and E be the (-1)-curve over p. When p is generic, $\widehat{Y}$ is a quadric del Pezzo surface. As explained in the beginning of §12.4.1, if we regard v as a point q of E, then q is contained in $E \cap \iota(E)$ where ι is the Geiser involution of $\widehat{Y}$. By applying the anticanonical map $\widehat{Y} \to \mathbb{P}^2$ of $\widehat{Y}$, the new triplet $(\widehat{Y}, E, q)$ is then translated into a triplet (Γ, L, P) of a smooth quartic Γ, a bitangent L of Γ, and a point P in $\Gamma \cap L$. Therefore, if $U \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \times \mathbb{P}^2$ denotes the space of pairs (Γ, P) of a smooth quartic

Γ and a tangent point P of a bitangent of Γ, we have a rational map

$$\mathcal{Z}/W_6 \dashrightarrow U/\mathrm{PGL}_3. \tag{12.2}$$

Conversely, for a point (Γ, P) of U, we take the double cover $\phi\colon \widehat{Y} \to \mathbb{P}^2$ branched along Γ. If L is the tangent line of Γ at P, we have $\phi^*L = E + \iota(E)$ for a (-1)-curve E and the Geiser involution ι of $\widehat{Y}$. Since the triplet $(\widehat{Y}, E, \phi^{-1}(P))$ is isomorphic to $(\widehat{Y}, \iota(E), \phi^{-1}(P))$ by ι, we may contract either E or $\iota(E)$ to obtain a well-defined inverse map of (12.2). Thus $\mathcal{Z}/W_6$ is birational to U/PGL_3.

Proposition 12.6. *The quotient U/PGL_3 is rational. Hence $\mathcal{M}_{13,7,1}$ is rational.*

Proof. For a point (Γ, P) of U, let Q be the another tangent point of the bitangent at P. This defines a PGL_3-equivariant map $\varphi\colon U \to \mathbb{P}^2 \times \mathbb{P}^2$, $(\Gamma, P) \mapsto (P, Q)$. The PGL_3-action on $\mathbb{P}^2 \times \mathbb{P}^2$ is almost transitive with the stabilizer G of a general (P, Q) isomorphic to $(\mathbb{C}^\times \ltimes \mathbb{C})^2$. By the slice method we have $U/\mathrm{PGL}_3 \sim \varphi^{-1}(P, Q)/G$. The fiber $\varphi^{-1}(P, Q)$ is an open set of the linear system $\mathbb{P}V$ of quartics Γ with $\Gamma|_{\overline{PQ}} = 2P + 2Q$. By Miyata's theorem $\mathbb{P}V/G$ is rational. □

12.5. $\mathcal{M}_{14,6,1}$ **and quartic del Pezzo surfaces.**

12.5.1. *A period map.* We first note the following.

Lemma 12.7. *Let Y be a quartic del Pezzo surface and let (p, v) be a general point of $\mathbb{P}TY$, where $p \in Y$ and $v \in \mathbb{P}(T_pY)$. Then there uniquely exists an irreducible nodal $-K_Y$-curve $C_{p,v}$ whose node is p with one of the tangents being v.*

Proof. Let $Y' \to Y$ be the blow-up at p with the exceptional curve D', and $\widehat{Y} \to Y'$ be the blow-up at $v \in D'$ with the exceptional curve E. Then $\widehat{Y}$ is the blow-up of $\mathbb{P}^2$ at seven points in almost general position in the sense of [11], and the strict transform D of D' is the unique (-2)-curve on $\widehat{Y}$. Hence the anticanonical map $\phi\colon \widehat{Y} \to \mathbb{P}^2$ is the composition of the contraction $\widehat{Y} \to \widehat{Y}_0$ of D and a double covering $\widehat{Y}_0 \to \mathbb{P}^2$ branched along a quartic Γ with exactly one node. The curve $L = \phi(E)$ is a line passing the node and tangent to Γ elsewhere. We have $\phi^*L = E + \iota(E) + D$ where ι is the involution of $\widehat{Y}$ induced by the covering transformation of $\widehat{Y}_0 \to \mathbb{P}^2$. Then the image of $\iota(E)$ in Y is the desired curve $C_{p,v}$. The uniqueness of $C_{p,v}$ follows from intersection calculation. □

Pulling back the pencil of lines passing $L \cap \Gamma \backslash \mathrm{Sing}(\Gamma)$, we obtain the pencil of $-K_{\widehat{Y}}$-curves passing $E \cap \iota(E)$. Then its image in $|-K_Y|$ is the pencil $l_{p,v}$ of $-K_Y$-curves whose general member C is smooth and passes p and v with $(C.C_{p,v})_p = 4$. (There is another pencil of $-K_Y$-curves passing (p, v) with $(C.C_{p,v})_p = 4$: the one of those singular at p.)

Now let $f\colon \mathcal{Y}^5 \to \mathcal{U}^5$ be the family of marked quartic del Pezzo surfaces constructed in §12.1. The kernel of the Cremona representation of the Weyl group W_5 is isomorphic to $(\mathbb{Z}/2)^4$, which is the automorphism group of a general f-fiber. Let $\mathbb{P}T_f^0 \subset \mathbb{P}T_f$ be the open set of triplets $(\mathbf{p}, p, v)$ such that there exists an anticanonical curve $C_{p,v}$ on $\mathcal{Y}^5_{\mathbf{p}}$ as in Lemma 12.7.

Let $\mathcal{F}$ be the pullback of the bundle $f_*K_f^{-1}$ by the natural projection $\mathbb{P}T_f^0 \to \mathcal{U}^5$. We consider the subbundle $\mathcal{E}$ of $\mathcal{F}$ such that the fiber of $\mathbb{P}\mathcal{E}$ over a $(\mathbf{p}, p, v)$ is the pencil $l_{p,v}$ described above. By the uniqueness of $C_{p,v}$, $\mathcal{E}$ is invariant under the W_5-action on $\mathcal{F}$. An open set of $\mathbb{P}\mathcal{E}$ parametrizes triplets $(\mathcal{Y}^5_{\mathbf{p}}, C_{p,v}, C)$ such that $\mathcal{Y}^5_{\mathbf{p}} = Y$ is a marked quartic del Pezzo surface, $C_{p,v}$ is an irreducible nodal $-K_Y$-curve, and C is a smooth $-K_Y$-curve with $(C.C_{p,v})_p = 4$. The $-2K_Y$-curve $C_{p,v} + C$ has the D_8-point p as its only singularity. The 2-elementary $K3$ surface (X, ι) associated to the DPN pair $(Y, C_{p,v} + C)$ has invariant $(g, k) = (1, 4)$. We show that (X, ι) has parity $\delta = 1$. If $\pi\colon Y \to \mathbb{P}^2$ is the blow-down given by $\mathbf{p}$, the sextic $\pi(C_{p,v} + C)$ has the five nodes $\mathbf{p}$, the D_8-point $\pi(p)$, and no other singularity. Then we may apply Lemma 4.4 (3). Thus we obtain a period map $\mathcal{P}\colon \mathbb{P}\mathcal{E} \dashrightarrow \mathcal{M}_{14,6,1}$.

Proposition 12.8. *The period map $\mathcal{P}$ is W_5-invariant and descends to a birational map $\mathbb{P}\mathcal{E}/W_5 \dashrightarrow \mathcal{M}_{14,6,1}$.*

Proof. This is proved in the same way as Propositions 12.1, 12.3, and 12.5. The projection $\widetilde{\mathcal{M}}_{14,6,1} \dashrightarrow \mathcal{M}_{14,6,1}$ has degree $|\mathrm{O}(D_{L_-})|$ for the lattice $L_- = U \oplus U(2) \oplus A_1^4$, which by [26] is equal to $2^4 \cdot |\mathrm{O}^-(4, 2)| = 2^4 \cdot 5! = |W_5|$. □

12.5.2. *The rationality.* We shall prove that $\mathbb{P}\mathcal{E}/W_5$ is rational. By the no-name method for the W_5-linearized bundle $\mathcal{E}$ over $\mathbb{P}T_f^0$, we have

$$\mathbb{P}\mathcal{E}/W_5 \sim \mathbb{P}^1 \times (\mathbb{P}T_f/W_5).$$

The variety $\mathbb{P}T_f/W_5$ is the moduli of triplets (Y, p, v) where Y is an (unmarked) quartic del Pezzo surface, $p \in Y$, and $v \in \mathbb{P}(T_pY)$. As in the proof of Lemma 12.7, for a general (Y, p, v) we blow-up Y at p and v and then apply the anticanonical map to obtain a one-nodal quartic Γ and a line L passing the node of Γ and tangent to Γ elsewhere. If $U \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(1)|$ is the space of such pairs (Γ, L), we thus obtain a rational map

$$\mathbb{P}T_f/W_5 \dashrightarrow U/\mathrm{PGL}_3. \tag{12.3}$$

Conversely, given a $(\Gamma, L) \in U$, we take the double cover $\widehat{Y}_0 \to \mathbb{P}^2$ branched along Γ and then the minimal desingularization $\widehat{Y} \to \widehat{Y}_0$. The covering transformation of $\widehat{Y}_0 \to \mathbb{P}^2$ induces an involution ι of $\widehat{Y}$. The pullback of L to $\widehat{Y}$ is written as $E + \iota(E) + D$ where D is the (-2)-curve over the double point of $\widehat{Y}_0$, and E is a (-1)-curve with $(E.D) = (E.\iota(E)) = 1$. Notice that ι exchanges E and $\iota(E)$, and leaves D invariant. Then we contract E (or equivalently, $\iota(E)$) and D successively to obtain a point (Y, p, v) of $\mathbb{P}T_f/W_5$. This gives a well-defined inverse map of (12.3), and thus $\mathbb{P}T_f/W_5$ is birational to U/PGL_3.

Proposition 12.9. *The quotient U/PGL_3 is rational. Hence $\mathcal{M}_{14,6,1}$ is rational.*

Proof. We have the PGL_3-equivariant morphism

$$\varphi : U \to \mathbb{P}^2 \times \mathbb{P}^2, \qquad (\Gamma, L) \mapsto (\mathrm{Sing}(\Gamma), L \cap \Gamma \backslash \mathrm{Sing}(\Gamma)).$$

As in the proof of Proposition 12.6, we have $U/\mathrm{PGL}_3 \sim \varphi^{-1}(P, Q)/(\mathbb{C}^\times \ltimes \mathbb{C})^2$ for a general point $(P, Q) \in \mathbb{P}^2 \times \mathbb{P}^2$. Then $\varphi^{-1}(P, Q)$ is identified with an open set of a linear system of quartics, so that our assertion follows from Miyata's theorem. □

Remark 12.10. Del Pezzo surfaces in this section are not the quotient surfaces of 2-elementary $K3$ surfaces, but rather their "canonical" blow-down — compare with §5 where del Pezzo surfaces appear as the quotient surfaces. These two series are related by mirror symmetry for lattice-polarized $K3$ surfaces.

Remark 12.11. The Weyl group symmetry translated to the moduli of labelled sextics $C_1 + C_2$ is generated by the renumbering of labelings and by quadratic transformations based at ordinary intersection points of C_1 and C_2 (cf. Remark 11.4).

13. The case $g = 1$ (II)

In this section we treat the case $g = 1, k \geq 4, (k, \delta) \neq (4, 1)$. By the unirationality result [20] we may assume $k \leq 7$. For $k \geq 5$ we use plane cubics with a chosen inflectional point to construct birational period maps.

13.1. **The rationality of $\mathcal{M}_{14,6,0}$.** The space $\mathcal{M}_{14,6,0}$ is proven to be rational in the same way as for $\mathcal{M}_{10,8,0}$. The anti-invariant lattice L_- for $\mathcal{M}_{14,6,0}$ is isometric to $U(2)^2 \oplus D_4$. Then we have $L_-^{\vee}(2) \simeq U^2 \oplus D_4$, so that $\mathcal{M}_{14,6,0}$ is birational to $\mathcal{M}_{14,2,0}$. In §10.5 we proved that $\mathcal{M}_{14,2,0}$ is rational.

13.2. **The rationality of $\mathcal{M}_{15,5,1}$.** Let $U \subset |\mathcal{O}_{\mathbb{P}^2}(3)| \times |\mathcal{O}_{\mathbb{P}^2}(2)| \times \mathbb{P}^2$ be the locus of triplets (C, Q, p) such that (i) C is smooth, (ii) p is an inflectional point of C, and (iii) Q is smooth, passes p, and is transverse to C. If $L \subset \mathbb{P}^2$ is the tangent line of C at p, the sextic $B = C + Q + L$ has a D_8-singularity at p, five nodes at $C \cap Q \backslash p$, one node at $Q \cap L \backslash p$, and no other singularity. The 2-elementary $K3$ surface associated to the sextic B has invariant $(g, k) = (1, 5)$. Thus we obtain a period map $\mathcal{P}\colon U/\mathrm{PGL}_3 \to \mathcal{M}_{15,5,1}$.

Proposition 13.1. *The period map $\mathcal{P}$ is birational.*

Proof. We consider an $\mathfrak{S}_5$-cover $\widetilde{U}$ of U whose fiber over a $(C, Q, p) \in U$ corresponds to labelings of the five nodes $C \cap Q \backslash p$. As in the previous sections, $\mathcal{P}$ lifts to a birational map $\widetilde{U}/\mathrm{PGL}_3 \dashrightarrow \widetilde{\mathcal{M}}_{15,5,1}$. The variety $\widetilde{\mathcal{M}}_{15,5,1}$ is an $\mathrm{O}(D_{L_-})$-cover of $\mathcal{M}_{15,5,1}$ for the lattice $L_- = U \oplus U(2) \oplus A_1^3$. Since PGL_3 acts on U almost freely and since $|\mathrm{O}(D_{L_-})| = |\mathrm{O}^-(4, 2)| = 5!$ by [26], $\mathcal{P}$ has degree 1. □

Proposition 13.2. *The quotient U/PGL_3 is rational. Hence $\mathcal{M}_{15,5,1}$ is rational.*

Proof. Let $V \subset |\mathcal{O}_{\mathbb{P}^2}(2)| \times |\mathcal{O}_{\mathbb{P}^2}(1)| \times \mathbb{P}^2$ be the locus of triplets (Q, L, p) such that $p \in Q \cap L$. We have a PGL_3-equivariant map $\pi\colon U \to V$ defined by $(C, Q, p) \mapsto (Q, L, p)$, where L is the tangent line of C at p. A general π-fiber is an open set of a linear system $\mathbb{P}W$ of cubics. The group PGL_3 acts on V almost transitively with the stabilizer G of a general point (Q, L, p) isomorphic to $\mathbb{C}^{\times}$. Indeed, since Q is anti-canonically embedded, G is identified with the group of automorphisms of $Q \simeq \mathbb{P}^1$ fixing the two points $Q \cap L$. By the slice method we have $U/\mathrm{PGL}_3 \sim \mathbb{P}W/G$, and $\mathbb{P}W/G$ is clearly rational. □

13.3. **The rationality of $\mathcal{M}_{16,4,1}$ and $\mathcal{M}_{17,3,1}$.** For $k = 6, 7$, let $U_k \subset |\mathcal{O}_{\mathbb{P}^2}(3)| \times |\mathcal{O}_{\mathbb{P}^2}(1)|^3$ be the locus of quadruplets (C, L_1, L_2, L_3) such that (i) C is smooth, (ii) L_1, L_2, L_3 are linearly independent, (iii) $p = L_1 \cap L_2$ is an inflectional point of C with tangent line L_1, (iv) L_2, L_3 are respectively transverse to C, and (v) C passes (resp. does not pass) $L_2 \cap L_3$ for $k = 7$ (resp. $k = 6$). When $k = 6$, the singularities of the sextic $B = C + \sum_{i=1}^{3} L_i$ are

$$\mathrm{Sing}(B) = p + (L_2 \cap C \backslash p) + (L_3 \cap C) + (L_3 \cap L_1) + (L_3 \cap L_2),$$

where p is a D_8-singularity and the rest points are nodes. When $k = 7$, denoting $q = L_2 \cap L_3$, we have

$$\mathrm{Sing}(B) = p + q + (L_2 \cap C \backslash \{p, q\}) + (L_3 \cap C \backslash q) + (L_3 \cap L_1),$$

where p is a D_8-singularity, q is a D_4-singularity, and the rest points are nodes. The 2-elementary $K3$ surface associated to B has invariant $(r, a) = (10 + k, 10 - k)$, and we obtain a period map $\mathcal{P}_k \colon U_k/\mathrm{PGL}_3 \to \mathcal{M}_{10+k,10-k,1}$.

Proposition 13.3. *The map $\mathcal{P}_k$ is birational.*

Proof. For $k = 6$ we label the three nodes $L_3 \cap C$ and the two nodes $L_2 \cap C \backslash p$ independently, which is realized by an $\mathfrak{S}_3 \times \mathfrak{S}_2$-cover $\widetilde{U}_6$ of U_6. For $k = 7$ we distinguish the two nodes $L_3 \cap C \backslash q$, which defines a double cover $\widetilde{U}_7$ of U_7. Note that in both cases, the three lines L_i are distinguished by their intersection properties, and hence the rest nodes are a priori labelled. As before, we see that $\mathcal{P}_k$ lifts to a birational map $\widetilde{U}_k/\mathrm{PGL}_3 \dashrightarrow \widetilde{\mathcal{M}}_{10+k,10-k,1}$. Then $\widetilde{\mathcal{M}}_{10+k,10-k,1}$ is an $\mathrm{O}(D_{L_-})$-cover of $\mathcal{M}_{10+k,10-k,1}$ for the lattice $L_- = U \oplus U(2) \oplus A_1^{8-k}$. By [26] we have $|\mathrm{O}(D_{L_-})| = 12, 2$ for $k = 6, 7$ respectively, which concludes the proof. □

Proposition 13.4. *The quotient U_k/PGL_3 is rational. Therefore $\mathcal{M}_{16,4,1}$ and $\mathcal{M}_{17,3,1}$ are rational.*

Proof. This follows from the slice method for the projection $\pi \colon U_k \to |\mathcal{O}_{\mathbb{P}^2}(1)|^3$. General π-fibers are open subsets of linear systems of cubics, and PGL_3 acts on $|\mathcal{O}_{\mathbb{P}^2}(1)|^3$ almost transitively with a general stabilizer isomorphic to $(\mathbb{C}^\times)^2$. □

Remark 13.5. Degenerating the above sextic models, one sees that

(1) $\mathcal{M}_{18,2,1}$ is birational to the Kummer modular surface for $\mathrm{SL}_2(\mathbb{Z})$,
(2) $\mathcal{M}_{18,2,0}$ is birational to the pullback of the Kummer modular surface for $\Gamma_0(2)$ by the Fricke involution,
(3) $\mathcal{M}_{19,1,1}$ is birational to the modular curve for $\Gamma_0(3)$,

via their fixed curve maps. Note that $\mathcal{M}_{18,2,1}$ and $\mathcal{M}_{18,2,0}$ are Heegner divisors of $\mathcal{M}_{17,3,1}$, though they look like boundary divisors of toroidal compactification.

14. The case $g = 0$

In this section we study the case $g = 0$, $k > 4$. In view of the unirationality result [20], we treat only $\mathcal{M}_{16,6,1}$ and $\mathcal{M}_{17,5,1}$. We use cuspidal plane cubics to obtain birational period maps.

14.1. **The rationality of $\mathcal{M}_{16,6,1}$.** Let $V \subset |\mathcal{O}_{\mathbb{P}^2}(3)|$ be the variety of cuspidal plane cubics. Let $U \subset V \times |\mathcal{O}_{\mathbb{P}^2}(2)|$ be the open set of pairs (C, Q) such that Q is smooth and transverse to $C + L$, where L is the tangent line of C at the cusp. Then the sextic $B = C + Q + L$ has an E_7-singularity at the cusp of C, eight nodes at $C \cap Q$ and $L \cap Q$, and no other singularity. The associated 2-elementary $K3$ surface has invariant $(g, k) = (0, 5)$, and we obtain a period map $\mathcal{P}\colon U/\mathrm{PGL}_3 \dashrightarrow \mathcal{M}_{16,6,1}$.

Proposition 14.1. *The map $\mathcal{P}$ is birational.*

Proof. We label the six nodes $C \cap Q$ and the two nodes $L \cap Q$ independently. This defines an $\mathfrak{S}_6 \times \mathfrak{S}_2$-cover $\widetilde{U}$ of U. By the familiar method we see that $\mathcal{P}$ lifts to a birational map $\widetilde{U}/\mathrm{PGL}_3 \dashrightarrow \widetilde{\mathcal{M}}_{16,6,1}$. The variety $\widetilde{\mathcal{M}}_{16,6,1}$ is an $\mathrm{O}(D_{L_-})$-cover of $\mathcal{M}_{16,6,1}$ for the lattice $L_- = U(2)^2 \oplus A_1^2$. By [26] we calculate $|\mathrm{O}(D_{L_-})| = 2 \cdot |\mathrm{Sp}(4, 2)| = 2 \cdot 6!$, which implies that $\deg(\mathcal{P}) = 1$. □

Proposition 14.2. *The quotient U/PGL_3 is rational. Hence $\mathcal{M}_{16,6,1}$ is rational.*

Proof. The slice method for the projection $U \to V$ shows that $U/\mathrm{PGL}_3 \sim |\mathcal{O}_{\mathbb{P}^2}(2)|/G$, where G is the stabilizer of a $C \in V$. Since $G \simeq \mathbb{C}^\times$, the quotient $|\mathcal{O}_{\mathbb{P}^2}(2)|/G$ is clearly rational. □

Remark 14.3. General members of $\mathcal{M}_{16,6,1}$ can also be obtained from six general lines on $\mathbb{P}^2$. Matsumoto-Sasaki-Yoshida [23] studied this sextic model, and showed that the period map is the quotient map by the association involution. Therefore $\mathcal{M}_{16,6,1}$ is birational to the moduli of double-sixers (cf. [13]), which is rational by Coble [10] (see [4] for a proof by Dolgachev). This is an alternative approach for the rationality of $\mathcal{M}_{16,6,1}$. Conversely, this section may offer another proof of the result of Coble.

14.2. **The rationality of $\mathcal{M}_{17,5,1}$.** Let V be the space of cuspidal plane cubics and $U \subset V \times |\mathcal{O}_{\mathbb{P}^2}(2)|$ be the locus of pairs (C, Q) such that Q is the union of distinct lines and transverse to $C+L$, where L is the tangent line of C at the cusp. The sextic $C + Q + L$ has an E_7-singularity at the cusp of C, nine nodes at $C \cap Q$, $L \cap Q$, and $\mathrm{Sing}(Q)$, and no other singularity. Hence the associated 2-elementary $K3$ surface has invariant $(g, k) = (0, 6)$, and we obtain a period map $\mathcal{P}\colon U/\mathrm{PGL}_3 \to \mathcal{M}_{17,5,1}$.

Proposition 14.4. *The map $\mathcal{P}$ is birational.*

Proof. Let $\widetilde{U} \subset U \times (\mathbb{P}^2)^6$ be the locus of $(C, Q, p_1, \cdots, p_6)$ such that $C \cap Q = \{p_i\}_{i=1}^6$ and that p_1, p_2, p_3 belong to the same component of Q. By $\widetilde{U}$, the six nodes $C \cap Q$ and the two components of Q are labeled compatibly. As before, $\mathcal{P}$ lifts to a birational map $\widetilde{U}/\mathrm{PGL}_3 \to \widetilde{\mathcal{M}}_{17,5,1}$. Since PGL_3 acts on U almost freely, $\widetilde{U}/\mathrm{PGL}_3$ is an $\mathfrak{S}_2 \ltimes (\mathfrak{S}_3)^2$-cover of U/PGL_3. On the other hand, $\widetilde{\mathcal{M}}_{17,5,1}$ is an $\mathrm{O}(D_{L_-})$-cover of $\mathcal{M}_{17,5,1}$ for the lattice $L_- = U(2)^2 \oplus A_1$. By [26] we calculate $|\mathrm{O}(D_{L_-})| = |\mathrm{O}^+(4, 2)| = 2 \cdot (3!)^2$. □

Proposition 14.5. *The quotient U/PGL_3 is rational. Hence $\mathcal{M}_{17,5,1}$ is rational.*

Proof. Consider the PGL_3-equivariant map $\pi\colon U \to V \times \mathbb{P}^2$, $(C, Q) \mapsto (C, \mathrm{Sing}(Q))$. A general fiber $\pi^{-1}(C, p)$ is an open set of the net of conics singular at p. It is straightforward to see that PGL_3 acts on $V \times \mathbb{P}^2$ almost freely. Then we may apply the no-name lemma 2.5 to see that

$$U/PGL_3 \sim \mathbb{P}^2 \times ((V \times \mathbb{P}^2)/PGL_3).$$

The quotient $(V \times \mathbb{P}^2)/PGL_3$ is of dimension 1 and so is rational. □

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References

[1] Alexeev, V.; Nikulin, V. V. *Del Pezzo and K3 surfaces.* MSJ Memoirs **15**, 2006.

[2] Arbarello, E.; Cornalba, M.; Griffiths, P. A.; Harris, J. *Geometry of algebraic curves, I.* Springer, 1985.

[3] Artebani, M; Kondō, S. *The moduli of curves of genus six and K3 surfaces.* Trans. Amer. Math. Soc. **363** (2011), 1445-1462.

[4] Bauer, I.; Verra, A. *The rationality of the moduli spaces of genus-4 curves endowed with an order-3 subgroup of their Jacobian.* Michigan Math. J. **59** (2010) 483–504.

[5] Bogomolov, F. A.; Katsylo, P. I. *Rationality of some quotient varieties.* Mat. Sb. (N.S.) **126(168)** (1985), 584–589.

[6] Borel, A. *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem.* J. Diff. Geom. **6** (1972), 543–560.

[7] Casnati, G.; del Centina, A. *On certain spaces associated to tetragonal curves of genus 7 and 8.* Commutative algebra and algebraic geometry (Ferrara), 35–45, Lect. Notes in Pure and Appl. Math **206**, Dekker, 1999.

[8] Catanese, F. *On the rationality of certain moduli spaces related to curves of genus 4.* Algebraic geometry (Ann Arbor, Mich., 1981), 30–50, Lecture Notes in Math., **1008**, Springer, 1983.

[9] Chernousov, V.; Gille, P.; Reichstein, Z. *Resolving G-torsors by abelian base extensions.* J. Algebra **296** (2006), 561–581.

[10] Coble, A. *Point sets and allied Cremona groups I.* Trans. Amer. Math. Soc. **16** (1915) 155–198.

[11] Demazure, M. *Surfaces de del Pezzo, II, III, IV, V.* Sèminaire sur les Singularités des Surfaces, 21–69, Lecture Notes in Math., **777**, Springer, 1980.

[12] Dolgachev, I. V. *Rationality of fields of invariants.* Algebraic geometry, Bowdoin, 1985, 3–16, Proc. Symp. Pure Math., **46**, Part 2, Amer. Math. Soc., Providence, 1987.

[13] Dolgachev, I.; Ortland, D. *Point sets in projective spaces and theta functions.* Astérisque No. 165 (1988).

[14] Katsylo, P. I. *Rationality of the moduli spaces of hyperelliptic curves.* Izv. Akad. Nauk SSSR. **48** (1984), 705–710.

[15] Katsylo, P. I. *Rationality of fields of invariants of reducible representations of* SL_2. Mosc. Univ. Math. Bull. **39** (1984) 80–83.

[16] Katsylo, P. I. *Rationality of the variety of moduli of curves of genus 5.* Mat. Sb. 182 (1991), 457–464

[17] Katsylo, P. *Rationality of the moduli variety of curves of genus 3.* Comment. Math. Helv. **71** (1996), 507–524.

[18] Kondō, S. *The rationality of the moduli space of Enriques surfaces.* Compositio Math. **91** (1994), 159–173.

[19] Kondō, S. *A complex hyperbolic structure for the moduli space of curves of genus three.* J. Reine Angew. Math. **525** (2000), 219–232.

[20] Ma, S. *The unirationality of the moduli spaces of 2-elementary K3 surfaces (with an Appendix by Ken-Ichi Yoshikawa).* arXiv:1011.1963, to appear in Proc. London. Math. Soc.

[21] Ma, S. *The rationality of the moduli spaces of trigonal curves of odd genus.* arXiv:1012.0983.

[22] Martens, G.; Schreyer, F.-O. *Line bundles and syzygies of trigonal bundles.* Abh. Math. Sem. Univ. Hamburg **86** (1986), 169–189.

[23] Matsumoto, K.; Sasaki, T.; Yoshida, M. *The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type* (3, 6). Internat. J. Math. **3** (1992), no. 1.

[24] Miranda, R. *The moduli of Weierstrass fibrations over* $\mathbb{P}^1$. Math. Ann. **255** (1981), 379–394.

[25] Miyata, T. *Invariants of certain groups. I.* Nagoya Math. J. **41** (1971), 69–73.

[26] Morrison, D. R.; Saitō, M.-H. *Cremona transformations and degrees of period maps for K3 surfaces with ordinary double points.* Algebraic geometry, Sendai, 1985, 477–513, Adv. Stud. Pure Math. **10**, 1987.

[27] Mumford, D.; Fogarty, J.; Kirwan, F. *Geometric invariant theory.* Third edition. Springer, 1994.

[28] Nakayama, N. *Classification of log del Pezzo surfaces of index two.* J. Math. Sci. Univ. Tokyo **14** (2007), no. 3, 293–498.

[29] Nikulin, V.V. *Integral symmetric bilinear forms and some of their applications.* Math. USSR Izv. **14** (1980), 103–167.

[30] Nikulin, V.V. *Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections.* J. Soviet Math. **22** (1983), 1401–1476.

[31] Popov, V.L.; Vinberg, E.B. *Invariant theory.* in: Algebraic Geometry, IV, in: Encyclopaedia Math. Sci., vol. **55**, Springer, 1994, 123–284.

[32] Shepherd-Barron, N. I. *The rationality of certain spaces associated to trigonal curves.* Algebraic geometry, Bowdoin, 1985, 165–171, Proc. Symp. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[33] Shepherd-Barron, N. I. *The rationality of some moduli spaces of plane curves.* Compositio Math. **67** (1988), 51–88.

[34] Shepherd-Barron, N. I. *Invariant theory for* S_5 *and the rationality of* M_6. Compositio Math. **70** (1989), 13–25.

[35] Yoshikawa, K.-I. *K3 surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space.* Invent. Math. **156** (2004), 53–117.

[36] Yoshikawa, K.-I. *K3 surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space II.* arXiv:1007.2830, to appear in J. Reine. Angew. Math.

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