Two-dimensional stochastic Navier-Stokes equations derived from a certain variational problem
(ある変分問題から導かれる二次元確率ナビエ・ストークス方程式)

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Chapter 1

Introduction

It is known that the Euler equation, which describes the motion of a fluid without viscosity, can be derived from a certain variational problem. In fact, Arnold [2] studies the variational problem for a certain action functional defined on a set of integral curves taking values in volume preserving diffeomorphisms and shows that the time derivative of its critical point satisfies the Euler equation. Later, in [13], the case where the integral curve is perturbed by some random force is studied. They consider the random action functional similarly to the deterministic case and show that the random velocity field $u(t,x) = (u^1(t,x), \cdots, u^n(t,x))$, which is formally derived from its random critical point, satisfies the following stochastic Navier-Stokes equation:

$$\frac{\partial u}{\partial t} - \mu \triangle u + (u \cdot \nabla)u + \sqrt{2\mu}\nabla u \cdot \dot{B}(t) + \nabla p = 0, \quad t > 0, x \in \mathbb{R}^n, \tag{1.1}$$

$$\text{div} u = 0, \quad t > 0, x \in \mathbb{R}^n, \tag{1.2}$$

where $p = p(t,x)$ denotes the pressure term, $\mu > 0$ is a constant and $\dot{B}(t) = \frac{d}{dt}(B^1(t), \cdots, B^n(t))$ is the distributional derivative of the n-dimensional Brownian motion $B(t) = (B^1(t), \cdots, B^n(t))$. Furthermore, it is also shown that if a weak solution u of (1.1)-(1.2) exists, then its expectation satisfies the Reynolds equation. However, the existence of weak solutions of (1.1)-(1.2) is not discussed in [13]. In addition, (1.1) does not satisfy the coercivity condition (see Condition 2.2.1), which is usually required to prove the existence of weak solutions. Indeed, the existence of weak solutions of this type of equation (1.1)-(1.2) is not known so far. There are many known results obtained by several authors about weak solutions of the various types of stochastic Navier-Stokes equations satisfying the coercivity condition. In this thesis, we study (1.1)-(1.2) on the following two regions:

1. two-dimensional torus $\mathbb{T}^2$,

2. two-dimensional Euclidean space $\mathbb{R}^2$.

We consider the initial value problem on each region described above, formulate the notion of weak solutions and prove the existence of them. As pointed above, the coercivity condition plays an important role in showing the uniform a priori estimate which leads to tightness, which is precisely stated in Chapter 2 and 3. However, if we consider (1.1)-(1.2) on $\mathbb{T}^2$ or $\mathbb{R}^2$, although the coercivity condition is missing, the tightness can be proven. Hence, the existence of the weak solution of (1.1)-(1.2) is shown on these two regions. In particular, on $\mathbb{R}^2$, it is more difficult to obtain tightness than the case of bounded domains. The uniqueness of the weak solutions is not discussed in this paper.

This thesis is organized as follows: In Chapter 2, the existence of weak solutions on $\mathbb{T}^2$ is discussed and in Chapter 3, the case of $\mathbb{R}^2$ is studied.

Chapter 2

Construction of weak solutions of a certain stochastic Navier-Stokes equation

2.1 Introduction

We study the initial value problem of the following type of stochastic Navier-Stokes equation for the velocity field $u = u(t,x) = (u^1(t,x), u^2(t,x))$, $t \geq 0$, $x = (x_1, x_2)$ and the pressure term $p = p(t,x)$ on a two-dimensional torus $\mathbb{T}^2$:

$$\frac{\partial u^i}{\partial t} + \sum_{j=1}^{2} \left(u^j \frac{\partial u^i}{\partial x_j} + \sqrt{2\mu} \frac{\partial u^i}{\partial x_j} \dot{B}^j(t) \right) - \mu \Delta u^i + \frac{\partial p}{\partial x_i} = 0, \quad t > 0, x \in \mathbb{T}^2, i = 1,2, \tag{2.1}$$

with the incompressibility condition:

$$\operatorname{div} u \equiv \sum_{j=1}^{2} \frac{\partial u^j}{\partial x_j} = 0, \quad t > 0, x \in \mathbb{T}^2, \tag{2.2}$$

under the initial condition:

$$u(0,x) = u_0(x), \quad x \in \mathbb{T}^2, \tag{2.3}$$

where $\mu > 0$ is a constant and $\dot{B}(t) = \frac{d}{dt}B(t)$ is a formal derivative of the two dimensional Brownian motion $B(t) = (B^1(t), B^2(t))$. We solve the equation (2.1) - (2.3) in the class of u's satisfying $\int_{\mathbb{T}^2} u dx = 0$. We assume that u_0 is a $\mathbf{V}$-valued deterministic function where $\mathbf{V} = \mathbf{W}^{1,2}(\mathbb{T}^2; \mathbb{R}^2) \cap \mathbf{H}$, $\mathbf{W}^{1,2}(\mathbb{T}^2; \mathbb{R}^2)$ denotes the usual $\mathbb{R}^2$-valued Sobolev space, see Section 2.2, and $\mathbf{H}$ is the family of $\mathbb{R}^2$-valued square integrable functions on $\mathbb{T}^2$ which are of divergence free and have mean zero, that is,

$$\mathbf{H} = \left\{ u \in \mathbf{L}^2(\mathbb{T}^2; \mathbb{R}^2) \mid \operatorname{div} u = 0, \int_{\mathbb{T}^2} u dx = 0 \right\},$$

where $\operatorname{div} u$ is defined in a distribution sense.

The equation (2.1) - (2.3) appears in a certain variational problem, see Appendix. The solution of the equation (2.1) - (2.3) will be defined in a weak sense (see Definition 2.2.1). The aim of this paper is to show the existence of the weak solution of (2.1) - (2.3) under a suitable assumption on the initial condition, which will be described in Section 2.2. Our main result will be formulated in Theorem 2.2.1.

Several authors have discussed the existence of solutions of stochastic Navier-Stokes equations which fulfill the coercivity condition ([12],[15]). Note that (2.1) does not satisfy the coercivity condition (see Condition 2.2.1), therefore we can not directly apply their results to our equation. In this paper we use the following method. First we construct a solution of the equation with the diffusion coefficient μ replaced by $\frac{2+\delta}{2}\mu$ for each $\delta > 0$ by the Galerkin's method. This is possible since the modified equation satisfies the coercivity condition for each $\delta > 0$. In the second step, we take the limit $\delta \to 0$ to construct a weak solution of our equation by showing a uniform estimate which implies the tightness of the distributions $\mathcal{L}(u_n^\delta)$ of the solutions $(u_n^\delta)_{n,\delta}$ of the approximating finite dimensional equation in $L^2(0,T;\mathbf{H})$ for $T > 0$. Similar approach can be found in a construction of weak solutions of two-dimensional stochastic Euler equations ([4],[5],[6],[7]). The cases of a bounded domain with Dirichlet boundary condition, an unbounded domain and the periodic boundary condition are discussed in [4],[6] and [7], respectively. However, the case with the stochastic term containing ∇u as in our equation is not studied in these papers.

Our method does not apply for higher dimensional case. Indeed, by applying Itô's formula for $|u_n^\delta(t)|_{\mathbf{H}}^2$, it is easy to see the following uniform estimate :

$$\sup_{n,\delta}\left\{\mathbf{E}\left\{|u_n^\delta(t)|_{\mathbf{H}}^2\right\} + \delta\mu\int_0^t \mathbf{E}\left\{||u_n^\delta(s)||_{\mathbf{V}}^2\right\} ds\right\} < \infty.$$

This estimate does not imply that $(u_n^\delta(t))_{\delta>0}$ has a strongly convergent subsequence in $\mathbf{H}$. However, on the two-dimensional torus, we can show such statement relying on the identity:

$$\sum_{j=1}^{2}\int_{\mathbb{T}^2}\frac{\partial(u\cdot\nabla u)}{\partial x_j}\cdot\frac{\partial u}{\partial x_j}\,dx = 0, \quad \text{for } u \in \mathbf{C}_\sigma^\infty, \tag{2.4}$$

where $\mathbf{C}_\sigma^\infty$ is a family of infinitely differentiable $\mathbb{R}^2$-valued functions which are of divergence free and have mean zero, that is,

$$\mathbf{C}_\sigma^\infty = \left\{u \in \mathbf{C}^\infty(\mathbb{T}^2;\mathbb{R}^2) \mid \operatorname{div} u = 0, \int_{\mathbb{T}^2} u dx = 0\right\}.$$

The contents of this paper are as follows: In Section 2.2, we introduce several notations and describe our main result. In Section 2.3, we give the proof of it. In Appendix, we explain the background and the reason why we consider the equation (2.1) - (2.3).

2.2 Notations and formulation of the result

In this section, we precisely formulate our problem. We denote the inner product of $\mathbf{H}$ by $\langle \cdot, \cdot \rangle$, that is,

$$\langle u, v \rangle = \sum_{j=1}^{2} \int_{\mathbb{T}^2} u^j(x) v^j(x) dx, \quad u, v \in \mathbf{H},$$

and the norm of $\mathbf{H}$ by $|\cdot|_{\mathbf{H}}$. We also denote the inner product of $\mathbf{V}$ by $\langle\langle \cdot, \cdot \rangle\rangle$, that is,

$$\langle\langle u, v \rangle\rangle = \sum_{j=1}^{2} \langle \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \rangle, \quad u, v \in \mathbf{V},$$

and the norm of $\mathbf{V}$ by $||\cdot||_{\mathbf{V}}$. Recall that $\mathbf{V} = \mathbf{W}^{1,2}(\mathbb{T}^2; \mathbb{R}^2) \cap \mathbf{H}$ and

$$\mathbf{W}^{1,2}(\mathbb{T}^2; \mathbb{R}^2) = \left\{ u \in \mathbf{L}^2(\mathbb{T}^2; \mathbb{R}^2) \mid \frac{\partial u}{\partial x_j} \in \mathbf{L}^2(\mathbb{T}^2; \mathbb{R}^2), \quad j = 1, 2 \right\},$$

where $\frac{\partial u}{\partial x_j}, j = 1, 2$ are defined in a distribution sense. For $\alpha \in (0, 1)$, we denote by $\mathbf{W}^{\alpha,2}(0, T; \mathbf{H})$ the fractional Sobolev space which is the family of $u \in L^2(0, T; \mathbf{H})$ such that

$$\int_0^T \int_0^T \frac{|u(t) - u(s)|_{\mathbf{H}}^2}{|t - s|^{1+2\alpha}} dt ds < \infty,$$

holds. We also denote the norm of $\mathbf{W}^{\alpha,2}(0, T; \mathbf{H})$ by $||\cdot||_{\mathbf{W}^{\alpha,2}(0,T;\mathbf{H})}^2$, that is,

$$||u||_{\mathbf{W}^{\alpha,2}(0,T;\mathbf{H})}^2 = \int_0^T |u(t)|_{\mathbf{H}}^2 dt + \int_0^T \int_0^T \frac{|u(t) - u(s)|_{\mathbf{H}}^2}{|t - s|^{1+2\alpha}} dt ds.$$

The weak form of our equation is formulated as follows:

$$\begin{aligned} &\sum_{i=1}^{2} \left(\int_{\mathbb{T}^2} u^i(t, x) \phi^i(x) dx - \int_{\mathbb{T}^2} u_0^i(x) \phi^i(x) dx \right) \qquad (2.5) \\ &= \sum_{i,j=1}^{2} \int_0^t \int_{\mathbb{T}^2} u^i(s, x) u^j(s, x) \frac{\partial \phi^i(x)}{\partial x_j} ds dx \\ &+ \sqrt{2\mu} \sum_{i,j=1}^{2} \int_0^t \left(\int_{\mathbb{T}^2} u^i(s, x) \frac{\partial \phi^i(x)}{\partial x_j} dx \right) dB_s^j + \mu \sum_{i=1}^{2} \int_0^t \left(\int_{\mathbb{T}^2} u^i(s, x) \Delta \phi^i(x) dx \right) ds, \end{aligned}$$

for all $\phi \in \mathbf{C}_\sigma^\infty$ and $t \geq 0$. Note that the term containing $\frac{\partial p}{\partial x_i}$, $i = 1, 2$ appearing in (2.1) vanishes because

$$\sum_{i=1}^{2} \int_{\mathbb{T}^2} \frac{\partial p}{\partial x_i}(t,x)\phi^i(x)dx = -\int_{\mathbb{T}^2} p(t,x) \text{ div } \phi(x)dx = 0,$$

holds.

Now we consider the following abstract evolution equation corresponding to (2.5):

$$\begin{cases} du(t) + \{Au(t) + B(u(t),u(t))\}\, dt + Gu(t)\, dB_t = 0, & t > 0, \\ u(0) = u_0. \end{cases} \tag{2.6}$$

The operators A, B and G are defined as follows. Let A be the linear operator with domain $D(A) = \mathbf{W}^{2,2}(\mathbb{T}^2; \mathbb{R}^2) \cap \mathbf{V}$ such that

$$A : D(A) \to \mathbf{H}, \quad Au = -\mu \mathbb{P}\Delta u,$$

where $\mathbf{W}^{2,2}(\mathbb{T}^2; \mathbb{R}^2)$ is the Sobolev space consisting of all $u \in \mathbf{L}^2(\mathbb{T}^2; \mathbb{R}^2)$ such that $\frac{\partial u}{\partial x_j} \in \mathbf{W}^{1,2}(\mathbb{T}^2; \mathbb{R}^2)$ for $j = 1, 2$ and $\mathbb{P}$ is a projection from $\mathbf{L}^2(\mathbb{T}^2; \mathbb{R}^2)$ onto $\mathbf{H}$. Note that A is a nonnegative self adjoint linear operator of $\mathbf{H}$. We denote by $(\lambda_j)_{j=1,2,\ldots}$ its eigenvalues and by $(e_j)_{j=1,2,\ldots}$ the corresponding eigenfunctions. Note that $e_j \in \mathbf{C}_\sigma^\infty$ for all j and we can assume that the eigenvalues satisfy $0 < \lambda_1 \leq \lambda_2 \leq \cdots$. We define the bilinear operator B such that

$$B : \mathbf{V} \times \mathbf{V} \to \mathbf{V}', \quad B(v,w) = \mathbb{P}(v \cdot \nabla)w,$$

where $\mathbf{V}'$ is the dual space of $\mathbf{V}$. The linear operator G is given by

$$G : \mathbf{V} \to \mathbf{L}_{\mathrm{H.S}}(\mathbb{R}^2; \mathbf{H}), \quad Gv = \sqrt{2\mu}\mathbb{P}\nabla v,$$

where $\mathbf{L}_{\mathrm{H.S}}(\mathbb{R}^2; \mathbf{H})$ denotes the family of Hilbert-Schmidt operators from $\mathbb{R}^2$ to $\mathbf{H}$. Note that the adjoint operator $(Gv)^*$ of Gv belongs to $\mathbf{L}_{\mathrm{H.S}}(\mathbf{H}; \mathbb{R}^2)$ and

$$(Gv)^*\,\phi = -\sqrt{2\mu}\left(\langle \frac{\partial \phi}{\partial x_1}, v\rangle, \langle \frac{\partial \phi}{\partial x_2}, v\rangle\right) \quad \text{for } \phi \in \mathbf{C}_\sigma^\infty.$$

In this setting, we give the definition of the weak solution of our equation (2.6).

Definition 2.2.1. *We say that $\{u(t), B(t)\}_{t\geq 0}$ is a weak solution of the stochastic Navier-Stokes equation (2.6) with the initial value u_0 if*

1. *$\{u(t)\}_{t\geq 0}$ is an adapted process defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.*

2. *$u \in L^2(0,T; \mathbf{V}) \cap L^\infty(0,T; \mathbf{H})$, P-a.s. for $T > 0$.*

3. $\{B(t), \{\mathcal{F}_t\}\}_{t\geq 0}$ *is a two-dimensional Brownian motion on* $(\Omega, \mathcal{F}, P)$.

4. *For every* $T > 0$ *and* $\phi \in \mathbf{C}_\sigma^\infty$, *P-a.s.,*

$$\langle u(t), \phi\rangle - \langle u_0, \phi\rangle =$$
$$- \int_0^t \langle A^*\phi, u(s)\rangle ds + \int_0^t \langle B(u(s), \phi), u(s)\rangle ds - \int_0^t (Gu(s))^* \phi \, dB(s),$$

holds for a.e.-$t \in [0, T]$.

Let us recall that we say the equation (2.6) satisfies the coercivity condition if the following Condition 2.2.1 holds:

Condition 2.2.1. *([12])* $G : \mathbf{V} \to \mathbf{L}_{H.S}(\mathbb{R}^2, \mathbf{H})$ *is continuous and*

$$2\langle Av, v\rangle - |Gv|^2_{\mathbf{L}_{H.S}(\mathbb{R}^2,\mathbf{H})} \geq \delta\mu||v||^2_{\mathbf{V}} - \lambda_0|v|^2_{\mathbf{H}} - \rho,$$

for all $v \in \mathbf{V}$ *and for some* $\delta \in (0, 2]$, $\lambda_0 \geq 0$ *and* $\rho \geq 0$.

However, our equation (2.6) does not satisfy Condition 2.2.1, because

$$2\langle Au(t), u(t)\rangle - |Gu(t)|^2_{\mathbf{L}_{\mathbf{H.S}}(\mathbb{R}^2,\mathbf{H})} = 0. \tag{2.7}$$

Namely, in our case, $\delta = 0$. Now we are ready to describe our main result.

Theorem 2.2.1 (Existence of the weak solution). *There exists a weak solution* $\{u(t), B(t)\}_{t\geq 0}$ *of the stochastic Navier-Stokes equation (2.6) with the initial value* $u_0 \in \mathbf{V}$.

2.3 Proof of the main result

In this section, we give the proof of Theorem 2.2.1. We divide it into four steps. Before proving Theorem 2.2.1 we prepare the following lemma.

lemma 2.3.1. *If* $u \in \mathbf{C}_\sigma^\infty$, *then* $\langle\langle u, u\cdot\nabla u\rangle\rangle = 0$ *holds.*

Proof. For any $\mathbb{R}^2$-valued function u on $\mathbb{R}^2$ which is of divergence free, we can choose a C^2-function ϕ on $\mathbb{R}^2$ such that $u = \nabla^\perp\phi$ holds, where $\nabla^\perp\phi = (-\partial_2\phi, \partial_1\phi)$ and $\partial_k\phi$ denotes $\dfrac{\partial\phi}{\partial x_k}$, $k = 1, 2$, (see [1]). Then

$$\begin{aligned}
&\langle\langle u, u\cdot\nabla u\rangle\rangle \qquad (2.8)\\
=&\langle\partial_1 u, \partial_1(u\cdot\nabla u)\rangle + \langle\partial_2 u, \partial_2(u\cdot\nabla u)\rangle\\
=&\langle\partial_1 u, \partial_1 u\cdot\nabla u\rangle + \langle\partial_1 u, u\cdot\nabla(\partial_1 u)\rangle + \langle\partial_2 u, \partial_2 u\cdot\nabla u\rangle + \langle\partial_2 u, u\cdot\nabla(\partial_2 u)\rangle,
\end{aligned}$$

holds. The second and the forth terms in the last line of (2.8) are equal to 0 due to the incompressibility condition. Therefore, it suffices to show $\langle \partial_1 u, \partial_1 u \cdot \nabla u \rangle + \langle \partial_2 u, \partial_2 u \cdot \nabla u \rangle = 0$ to prove this lemma. Indeed, this is obtained by the following calculation:

$$\begin{aligned}&\langle \partial_1 u, \partial_1 u \cdot \nabla u \rangle + \langle \partial_2 u, \partial_2 u \cdot \nabla u \rangle \\ = &\sum_{j,k,l=1}^{2} \int_{\mathbb{T}^2} \partial_l \left(\nabla^{\perp}\phi(x)\right)^k \partial_l \left(\nabla^{\perp}\phi(x)\right)^j \partial_j \left(\nabla^{\perp}\phi(x)\right)^k dx = 0.\end{aligned}$$

Thus, the proof is complete. □

proof of Theorem 2.2.1.

Step 1: Approximation by finite dimensional S.D.E.s

By normalization, we can assume that $(e_j)_{j=1,2,\dots}$ is a complete orthonormal system of $\mathbf{H}$ and that each e_j $(j \geq 1)$ belongs to $\mathbf{C}_\sigma^\infty$. We denote by $\mathbf{H}_n$ the linear subspace of $\mathbf{H}$ spanned by $\{e_1, \cdots, e_n\}$ and by Π_n the orthogonal projection from $\mathbf{H}$ onto $\mathbf{H}_n$. We set

$$A_\delta = -\frac{2+\delta}{2}\mu \mathbb{P}\Delta, \quad \delta > 0.$$

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space on which a two-dimensional $\mathcal{F}_t$-Brownian motion $\{B_t\}_{t\geq 0}$ is defined. Then, we consider the following finite dimensional stochastic differential equations on $\mathbf{H}_n$:

$$\begin{cases} du_n^\delta(t) + \left\{A_\delta u_n^\delta(t) + \Pi_n B(u_n^\delta(t), u_n^\delta(t))\right\} dt + \Pi_n G u_n^\delta(t)\, dB_t = 0, \qquad t > 0, \\ u_n^\delta(0) = \Pi_n u_0, \end{cases} \tag{2.9}$$

namely,

$$u_n^\delta(t) - \Pi_n u_0 + \int_0^t \left\{A_\delta u_n^\delta(s) + \Pi_n B(u_n^\delta(s), u_n^\delta(s))\right\} ds + \int_0^t \Pi_n G u_n^\delta(s)\, dB_s = 0. \tag{2.10}$$

We can expand $u_n^\delta(t) \in \mathbf{H}_n$ as $u_n^\delta(t) = \sum_{j=1}^n u_j^{\delta,n}(t) e_j$, where $u_j^{\delta,n}(t) = \langle u_n^\delta(t), e_j \rangle$. By taking the scalar product with e_j, $1 \leq j \leq n$, we find

$$u_j^{\delta,n}(t) = \langle \Pi_n u_0, e_j \rangle + \int_0^t F_j^{\delta,n}(u_n^\delta(s)) ds + \int_0^t \sigma_j^n(u_n^\delta(s))\, dB_s, \quad 1 \leq j \leq n, \tag{2.11}$$

where $F_j^{\delta,n}(u_n) = -\langle A_\delta u_n + \Pi_n B(u_n, u_n), e_j \rangle$ and $\sigma_j^n(u_n) = -(\Pi_n G u_n)^* e_j$ for $u_n \in \mathbf{H}_n$.

Let us assume that u_n^δ and v_n^δ satisfy the equation (2.11) for each $\delta > 0$ and $n \geq 1$. Then, by using the following inequality (see [18]):

$$|\langle (u \cdot \nabla)v, w \rangle| \leq C ||u||_{\mathbf{V}} ||v||_{\mathbf{V}} ||w||_{\mathbf{V}}, \quad \text{for } u, v, w \in \mathbf{C}_\sigma^\infty,$$

we have

$$|\sigma_j^n(u_n) - \sigma_j^n(v_n)|_{\mathbb{R}^2} \le C_1||u_n - v_n||_{\mathbf{V}},$$
$$|\langle A_\delta u_n - A_\delta v_n, e_j\rangle| \le C_1||u_n - v_n||_{\mathbf{V}},$$
$$|\langle \Pi_n B(u_n, u_n) - \Pi_n B(v_n, v_n), e_j\rangle| \le C_1||u_n - v_n||_{\mathbf{V}},$$

for every $u_n, v_n \in \mathbf{H}_n$, with some $C_1 = C_1(\delta, n) > 0$, where $|\cdot|_{\mathbb{R}^2}$ represents the Euclidean norm of $\mathbb{R}^2$. Hence, we have

$$|F_j^{\delta,n}(u_n) - F_j^{\delta,n}(v_n)| \le 2C_1||u_n - v_n||_{\mathbf{V}}, \quad u_n, v_n \in \mathbf{H}_n.$$

Therefore, for any $\delta > 0$, $n \ge 1$ and $T > 0$, we see that there exists a unique solution u_n^δ of (2.10) and u_n^δ belongs to $C(0,T;\mathbf{H}_n)$, P-a.s.

Step 2: A priori estimate

Note that $\{\sqrt{\frac{\mu}{\lambda_j}}e_j\}_{j=1,2,\ldots}$ is an orthonormal system of $\mathbf{V}$ since $\langle\langle \psi, e_j\rangle\rangle = \mu^{-1}\lambda_j\langle \psi, e_j\rangle$ holds for any $j = 1, 2, \cdots$ and $\psi \in \mathbf{C}_\sigma^\infty$. By applying Itô's formula for $\langle u_n^\delta(t), e_j\rangle^2$, $j = 1, \cdots, n$, we have

$$\begin{aligned}\langle u_n^\delta(t), e_j\rangle^2 &= \langle u_n^\delta(0), e_j\rangle^2 + (2+\delta)\mu\int_0^t \langle u_n^\delta(s), e_j\rangle\langle \Delta u_n^\delta(s), e_j\rangle ds \\ &- 2\sqrt{2\mu}\sum_{k=1}^2\int_0^t \langle u_n^\delta(s), e_j\rangle\langle \partial_k u_n^\delta(s), e_j\rangle dB_s^k - 2\int_0^t \langle u_n^\delta(s), e_j\rangle\langle u_n^\delta(s)\cdot\nabla u_n^\delta(s), e_j\rangle ds \\ &+ 2\mu\sum_{k=1}^2\int_0^t \langle \partial_k u_n^\delta(s), e_j\rangle^2 ds, \quad t \in [0,T],\ P\text{-a.s.}\end{aligned} \tag{2.12}$$

for $\delta > 0$, $n \ge 1$ and $T > 0$, where $\partial_k u$ denotes $\frac{\partial u}{\partial x_k}$. Note that the projection Π_n defined in Step 1 does not appear in (2.12).

By multiplying (2.12) by $\mu^{-2}\lambda_j^2$, we have

$$\langle\langle u_n^\delta(t), e_j\rangle\rangle^2 = \langle\langle u_n^\delta(0), e_j\rangle\rangle^2 + (I) + (II) + (III) + (IV),$$

where

$$(I) = (2+\delta)\mu \int_0^t \langle\langle u_n^\delta(s), e_j\rangle\rangle\langle\langle \Delta u_n^\delta(s), e_j\rangle\rangle ds,$$

$$(II) = -2\sqrt{2\mu}\sum_{k=1}^{2}\int_0^t \langle\langle u_n^\delta(s), e_j\rangle\rangle\langle\langle \partial_k u_n^\delta(s), e_j\rangle\rangle dB_s^k,$$

$$(III) = -2\int_0^t \langle\langle u_n^\delta(s), e_j\rangle\rangle\langle\langle u_n^\delta(s)\cdot\nabla u_n^\delta(s), e_j\rangle\rangle ds,$$

$$(IV) = 2\mu\sum_{k=1}^{2}\int_0^t \langle\langle \partial_k u_n^\delta(s), e_j\rangle\rangle^2 ds.$$

Thus,

$$\begin{aligned}\sum_{j=1}^{n}\frac{\mu}{\lambda_j}(I) =&(2+\delta)\mu\sum_{j=1}^{n}\int_0^t \langle\langle u_n^\delta(s), \sqrt{\frac{\mu}{\lambda_j}}e_j\rangle\rangle\langle\langle \Delta u_n^\delta(s), \sqrt{\frac{\mu}{\lambda_j}}e_j\rangle\rangle ds \\ =&(2+\delta)\mu\int_0^t \langle\langle u_n^\delta(s), \Delta u_n^\delta(s)\rangle\rangle ds \\ =&-(2+\delta)\mu\sum_{k=1}^{2}\int_0^t \left|\left|\partial_k u_n^\delta(s)\right|\right|_V^2 ds,\end{aligned} \tag{2.13}$$

holds for any $\delta > 0$ and $n \geq 1$, where we have used Parseval's formula at the second line and then the integration by parts formula at the last line. In addition, we have

$$\sum_{j=1}^{n}\frac{\mu}{\lambda_j}(IV) \leq 2\mu\sum_{k=1}^{2}\int_0^t \left|\left|\partial_k u_n^\delta(s)\right|\right|_{\mathbf{V}}^2 ds, \tag{2.14}$$

for any $\delta > 0$ and $n \geq 1$. By (2.13) and (2.14), we obtain

$$\sum_{j=1}^{n}\left(\frac{\mu}{\lambda_j}(I) + \frac{\mu}{\lambda_j}(IV)\right) \leq -\delta\mu\sum_{k=1}^{2}\int_0^t \left|\left|\partial_k u_n^\delta(s)\right|\right|_{\mathbf{V}}^2 ds. \tag{2.15}$$

In addition,

$$\sum_{j=1}^{n}\frac{\mu}{\lambda_j}(III) = -2\int_0^t \langle\langle u_n^\delta(s), u_n^\delta(s)\cdot\nabla u_n^\delta(s)\rangle\rangle ds, \tag{2.16}$$

holds for any $\delta > 0$ and $n \geq 1$. Thus, we have

$$\begin{aligned}&||u_n^\delta(t)||_{\mathbf{V}}^2 - ||u_n^\delta(0)||_{\mathbf{V}}^2 \\ &\leq -\delta\mu\sum_{k=1}^{2}\int_0^t \left|\left|\partial_k u_n^\delta(s)\right|\right|_{\mathbf{V}}^2 ds - 2\int_0^t \langle\langle u_n^\delta(s), u_n^\delta(s)\cdot\nabla u_n^\delta(s)\rangle\rangle ds + \sum_{j=1}^{n}\frac{\mu}{\lambda_j}(II),\end{aligned} \tag{2.17}$$

for any $t \in [0,T]$, P-a.s. for $T > 0$. Lemma 2.3.1 shows that the second term of the right hand side of (2.17) equals to 0. Then, by taking the expectation of (2.17), we have

$$\mathbf{E}^P \left\{ ||u_n^\delta(t)||_{\mathbf{V}}^2 \right\} \leq \mathbf{E}^P \left\{ ||u_n^\delta(0)||_{\mathbf{V}}^2 \right\} - \delta\mu \sum_{k=1}^{2} \int_0^t \mathbf{E}^P \left\{ \left|\left|\partial_k u_n^\delta(s)\right|\right|_{\mathbf{V}}^2 \right\} ds \leq ||u_0||_{\mathbf{V}}^2, \tag{2.18}$$

for $t \in [0,T]$.

Since we assume that u_0 is a $\mathbf{V}$-valued function,

$$\sup_{n\geq 1, \delta > 0} \mathbf{E}^P \left\{ \int_0^T ||u_n^\delta(t)||_{\mathbf{V}}^2 dt \right\} < \infty, \tag{2.19}$$

holds for each $T > 0$. Similarly, by applying Itô's formula for $|u_n^\delta(t)|_{\mathbf{H}}^2$ and then taking the expectation, we have

$$\mathbf{E}^P \left\{ \sup_{s\in[0,T]} |u_n^\delta(s)|_{\mathbf{H}}^2 \right\} \leq |u_0|_{\mathbf{H}}^2 + \mathbf{E}^P \left\{ \sup_{s\in[0,T]} \left| \int_0^s 2\sqrt{2\mu} \left(\nabla u_n^\delta(s')\right)^* u_n^\delta(s')\, dB_{s'} \right| \right\}. \tag{2.20}$$

Then, the right hand side of (2.20) is bounded from above by

$$\begin{aligned} & |u_0|_{\mathbf{H}}^2 + C\mathbf{E}^P \left\{ \left(\int_0^T 8\mu ||u_n^\delta(s)||_{\mathbf{V}}^2 |u_n^\delta(s)|_{\mathbf{H}}^2\, ds \right)^{\frac{1}{2}} \right\} \\ & \leq |u_0|_{\mathbf{H}}^2 + C\mathbf{E}^P \left\{ \left(\sup_{s\in[0,T]} |u_n^\delta(s)|_{\mathbf{H}}^2 \right)^{\frac{1}{2}} \left(8\mu \int_0^T ||u_n^\delta(s)||_{\mathbf{V}}^2\, ds \right)^{\frac{1}{2}} \right\} \\ & \leq |u_0|_{\mathbf{H}}^2 + C\mathbf{E}^P \left\{ \frac{1}{2C} \left(\sup_{s\in[0,T]} |u_n^\delta(s)|_{\mathbf{H}}^2 \right) + \frac{C}{2} \left(8\mu \int_0^T ||u_n^\delta(s)||_{\mathbf{V}}^2\, ds \right) \right\}, \end{aligned} \tag{2.21}$$

for some $C > 0$, where we have used Burkholder-Davis-Gundy inequality at the first line and Young's inequality at the third line in (2.21). Thus,

$$\sup_{n\geq 1, \delta > 0} \mathbf{E}^P \left\{ \sup_{s\in[0,T]} |u_n^\delta(s)|_{\mathbf{H}}^2 ds \right\} < \infty, \tag{2.22}$$

holds for each $T > 0$. However, if $p > 2$, due to the condition (2.7) we only have the following uniform estimate with respect to $n \geq 1$:

$$\sup_{n\geq 1} \mathbf{E}^P \left\{ \sup_{s\in[0,T]} |u_n^\delta(s)|_{\mathbf{H}}^p ds \right\} < \infty, \quad p > 2. \tag{2.23}$$

In fact, by applying Itô's formula for $|u_n^\delta(t)|^p$,

$$d|u_n^\delta(t)|_{\mathbf{H}}^p \leq p|u_n^\delta(t)|_{\mathbf{H}}^{p-2} \langle u_n^\delta(t), du_n^\delta(t) \rangle + \frac{1}{2} p(p-1) |u_n^\delta(t)|_{\mathbf{H}}^{p-2} |\Pi_n G u_n^\delta(t)|_{\mathrm{L}(\mathbb{R}^2;\mathbf{H})}^2.$$

In short,

$$|u_n^\delta(t)|_{\mathbf{H}}^p \leq |\Pi_n u_0|_{\mathbf{H}}^p + \left(-\frac{2+\delta}{2}\mu p + \mu p(p-1)\right) \int_0^t |u_n^\delta(s)|_{\mathbf{H}}^{p-2} ||u_n^\delta(s)||_{\mathbf{V}}^2 ds$$
$$+ \text{martingale}.$$

Then, $\mathbf{E}^P \left\{|u_n^\delta(s)|_{\mathbf{H}}^p\right\} < \infty$ holds if

$$-\frac{2+\delta}{2}\mu p + \mu p(p-1) \leq 0, \quad \text{that is, } p \in [0, 2+\frac{\delta}{2}].$$

This means that if $p > 2$, the uniform estimate of (2.23) with respect to $\delta > 0$ cannot be expected.

Finally, by proceeding similarly to [12], it is easy to see that we have

$$\sup_{n\geq 1, \delta\in(0,1]} \mathbf{E}^P \left\{||u_n^\delta||_{\mathbf{W}^{\alpha,2}(0,T;\mathbf{V}')}\right\} < \infty, \quad \alpha \in (0, \frac{1}{2}), \tag{2.24}$$

for each $T > 0$. Indeed, let us set

$$\begin{aligned} u_n^\delta(t) =& \Pi_n u_0 - \int_0^t A_\delta u_n^\delta(s) ds - \int_0^t \Pi_n B(u_n^\delta(s), u_n^\delta(s)) ds - \int_0^t \Pi_n G u_n^\delta(s)\, dB_s \\ =& J_0^n + J_1^{n,\delta}(t) + J_2^{n,\delta}(t) + J_3^{n,\delta}(t). \end{aligned}$$

Then, we have

$$\sup_{n\geq 1} |J_0^n|_{\mathbf{H}} < \infty, \quad \sup_{n\geq 1, \delta\in(0,1]} \mathbf{E}^{\mathbf{P}} \left\{||J_1^{n,\delta}||_{\mathbf{W}^{1,2}(0,T;\mathbf{V}')}\right\} < \infty, \tag{2.25}$$

$$\sup_{n\geq 1, \delta>0} \mathbf{E}^{\mathbf{P}} \left\{||J_2^{n,\delta}||_{\mathbf{W}^{1,2}(0,T;\mathbf{V}')}\right\} < \infty, \tag{2.26}$$

$$\sup_{n\geq 1, \delta>0} \mathbf{E}^{\mathbf{P}} \left\{||J_3^{n,\delta}||_{\mathbf{W}^{\alpha,2}(0,T;\mathbf{H})}\right\} < \infty, \quad \alpha \in (0, \frac{1}{2}). \tag{2.27}$$

Thus, (2.24) follows. By (2.19) and (2.24), $\{\mathcal{L}(u_n^\delta)\}_{n\geq 1, \delta\in(0,1]}$ is tight in $L^2(0,T;\mathbf{H})$, where $\mathcal{L}(u_n^\delta)$ is the law of u_n^δ (see [12], Theorem 2.1).

On the other hand, we can choose some $\alpha \in (0, \frac{1}{2})$ and $p > \frac{1}{\alpha}$ such that for each $T > 0$ and $\delta > 0$,

$$\sup_{n\geq 1} \mathbf{E}^P \left\{||J_3^{n,\delta}||_{\mathbf{W}^{\alpha,p}(0,T;\mathbf{V}')}\right\} < \infty, \tag{2.28}$$

holds. Indeed, let us set

$$\begin{aligned} &\mathbf{E}^{\mathbf{P}} \left\{||J_3^{n,\delta}||_{\mathbf{W}^{\alpha,p}(0,T;\mathbf{V}')}^p\right\} \\ &= \mathbf{E}^{\mathbf{P}} \left\{\int_0^T |J_3^{n,\delta}(t)|_{\mathbf{V}'}^p dt\right\} + \mathbf{E}^{\mathbf{P}} \left\{\int_0^T \int_0^T \frac{|J_3^{n,\delta}(t) - J_3^{n,\delta}(s)|_{\mathbf{V}'}^p}{|t-s|^{1+p\alpha}} dt ds\right\} \\ &\equiv (I) + (II). \end{aligned}$$

Then, by (2.23) we have

$$(I) \leq C \int_0^T \mathbf{E}^{\mathbf{P}} \left\{ \left(\int_0^t |u_n^\delta(s)|_{\mathbf{H}}^2 ds \right)^{\frac{p}{2}} \right\} dt \leq C' \sup_{n \geq 1} \mathbf{E}^{\mathbf{P}} \left\{ \sup_{s \in [0,T]} |u_n^\delta(s)|_{\mathbf{H}}^p \right\} < \infty,$$

for some C and C' depending on $p \geq 1$, $T > 0$ and $\delta > 0$. Similarly, we have

$$\begin{aligned}(II) &\leq C \int_0^T \int_0^T \frac{1}{|t-s|^{1+p\alpha}} \mathbf{E}^{\mathbf{P}} \left\{ \left(\int_s^t |u_n^\delta(s')|_{\mathbf{H}}^2 ds' \right)^{\frac{p}{2}} \right\} dsdt \\ &\leq C \left(\int_0^T \int_0^T |t-s|^{\frac{p}{2}-1-p\alpha} dsdt \right) \sup_{n \geq 1} \mathbf{E}^{\mathbf{P}} \left\{ \sup_{s \in [0,T]} |u_n^\delta(s)|_{\mathbf{H}}^p \right\},\end{aligned}$$

for some C depending on $p \geq 1$, $T > 0$ and $\delta > 0$. Thus, (2.28) follows if we choose suitable $\alpha \in (0, \frac{1}{2})$ and $p > \frac{1}{\alpha}$. By (2.25), (2.26) and (2.28), $\{\mathcal{L}(u_n^\delta)\}_{n \geq 1}$ is tight in $C(0,T;D(A)')$ for each $\delta > 0$, where $D(A)'$ is the dual space of $D(A)$ (see [12], Theorem 2.2). As a result, for each $\delta > 0$, we can construct the weak solution $\{u^\delta(t), B^\delta(t)\}_{t \geq 0}$ defined on a certain probability space $(\Omega^\delta, \mathcal{F}^\delta, \mathbf{P}^\delta, (\mathcal{F}_t^\delta)_{t \geq 0})$ of the following abstract stochastic differential equation:

$$\begin{cases} du^\delta(t) + \{A_\delta u^\delta(t) + B(u^\delta(t), u^\delta(t))\} \, dt + Gu^\delta(t) \, dB^\delta(t) = 0, & t > 0, \\ u^\delta(0) = u_0, \end{cases} \tag{2.29}$$

(see [12], Definition 3.1 and Theorem 3.2).

Step 3: Compactness argument

Note that $u^\delta \in \Omega_T \equiv L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}) \cap C(0,T;D(A)')$, $\mathbf{P}^\delta$-a.s. for $\delta > 0$. Then, by (2.19), (2.22) and Fatou's lemma, it is easy to see that

$$\sup_{\delta > 0} \mathbf{E}^{\mathbf{P}^\delta} \left\{ \sup_{t \in [0,T]} |u^\delta(t)|_{\mathbf{H}}^2 \right\} < \infty, \tag{2.30}$$

and

$$\sup_{\delta > 0} \mathbf{E}^{\mathbf{P}^\delta} \{ \int_0^T ||u^\delta(t)||_{\mathbf{V}}^2 dt \} < \infty, \tag{2.31}$$

hold. Let us set

$$\begin{aligned}u^\delta(t) =& \Pi_n u_0 - \int_0^t A_\delta u^\delta(s) ds - \int_0^t B(u^\delta(s), u^\delta(s)) ds - \int_0^t Gu^\delta(s) \, dB^\delta(s) \\ \equiv& J_0 + J_1^\delta(t) + J_2^\delta(t) + J_3^\delta(t),\end{aligned}$$

in $\mathbf{V}'$. Then, by Fatou's lemma we also have

$$\begin{aligned}&|J_0|_{\mathbf{H}} < \infty, \quad \sup_{\delta \in (0,1]} \mathbf{E}^{\mathbf{P}^\delta} \{ ||J_1^\delta||_{\mathbf{W}^{1,2}(0,T;\mathbf{V}')} \} < \infty, \\ &\sup_{\delta > 0} \mathbf{E}^{\mathbf{P}^\delta} \{ ||J_2^\delta||_{\mathbf{W}^{1,2}(0,T;\mathbf{V}')} \} < \infty, \quad \sup_{\delta > 0} \mathbf{E}^{\mathbf{P}^\delta} \{ ||J_3^\delta||_{\mathbf{W}^{\alpha,2}(0,T;\mathbf{H})} \} < \infty.\end{aligned} \tag{2.32}$$

Thus, by proceeding similarly to [12], we see

$$\sup_{\delta\in(0,1]} \mathbf{E}^{\mathbf{P}^\delta}\{||u^\delta||_{\mathbf{W}^{\alpha,2}(0,T;\mathbf{V}')}\} < \infty, \tag{2.33}$$

holds. Let us choose any sequence $\{\delta(n)\}_{n\geq1}$ such that $\delta(n)$ converges to 0 as $n\to\infty$. Two estimates (2.31) and (2.33) show that the family of $Q^\delta = \mathcal{L}(u^\delta, B^\delta)$ is tight in $L^2(0,T;\mathbf{H})\times C(0,T;\mathbb{R}^2)$, where $\mathcal{L}(u^\delta, B^\delta)$ is the joint law of u^δ and B^δ. Thus, we can find a subsequence $\{\delta(n_k)\}_{k\geq1}$ and a probability measure Q on $L^2(0,T;\mathbf{H})\times C(0,T;\mathbb{R}^2)$ such that $Q^{\delta(n_k)}$ converges to Q weakly in $L^2(0,T;\mathbf{H})\times C(0,T;\mathbb{R}^2)$ as $k\to\infty$. By Skorohod's embedding theorem, we see that there exist a filtered probability space $(\Omega', \mathcal{F}', \{\mathcal{F}'_t\}_{t\geq0}, \mathbb{P}')$ and random variables $\tilde{\xi}_k$, $\tilde{B}_k$, $\tilde{\xi}$, $\tilde{B}$ defined on Ω' in such a way that $\mathcal{L}(\tilde{\xi}_k, \tilde{B}_k)$ and $\mathcal{L}(\tilde{\xi}, \tilde{B})$ are equal to $Q^{\delta(n_k)}$ and Q, respectively, and $(\tilde{\xi}_k, \tilde{B}_k)$ converges to $(\tilde{\xi}, \tilde{B})$ in $L^2(0,T;\mathbf{H})\times C(0,T;\mathbb{R}^2)$, $\mathbb{P}'$-a.s. (see [8]). For each $k\geq 1$ and $\phi\in\mathbf{C}_\sigma^\infty$, we set

$$\begin{aligned}\widetilde{M}_k^\phi(t) =&\langle\tilde{\xi}_k(t),\phi\rangle - \langle u_0,\phi\rangle \\ &- \frac{2+\delta(n_k)}{2}\mu\int_0^t\langle\tilde{\xi}_k(s),\Delta\phi\rangle ds - \int_0^t\langle(\tilde{\xi}_k(s)\cdot\nabla)\phi,\tilde{\xi}_k(s)\rangle ds, \quad 0\leq t\leq T.\end{aligned} \tag{2.34}$$

Then, since $u^{\delta(n_k)}$ is a solution of (2.29) driven by a Brownian motion $B^{\delta(n_k)}$ and the law of $(u^{\delta(n_k)}, B^{\delta(n_k)})$ is equal to that of $(\tilde{\xi}_k, \tilde{B}_k)$ for each $k\geq1$, we see $\widetilde{M}_k^\phi$ is a continuous $\mathcal{F}'_t$-martingale whose quadratic variation is given by

$$\begin{aligned}\left\langle\left\langle\widetilde{M}_k^\phi, \widetilde{M}_k^\phi\right\rangle\right\rangle(t) &= \int_0^t(-G\tilde{\xi}_k(s))^*\phi\,((-G\tilde{\xi}_k(s))^*\phi)^* ds \\ &= \int_0^t 2\mu\left\{\langle\partial_1\phi,\tilde{\xi}_k(s)\rangle^2 + \langle\partial_2\phi,\tilde{\xi}_k(s)\rangle^2\right\} ds.\end{aligned} \tag{2.35}$$

Moreover, we have

$$\sup_{k\geq1}\mathbf{E}^{\mathbb{P}'}\left\{\sup_{t\in[0,T]}\left|\widetilde{M}_k^\phi(t)\right|^2\right\} < \infty, \tag{2.36}$$

for every $T>0$. Indeed, (2.36) is easily obtained by (2.31). In particular, $\left\{\widetilde{M}_k^\phi\right\}_{k\geq1}$ is a square integrable martingale. It is easy to check that

$$\begin{aligned}
&\mathbf{E}^{\mathbb{P}'}\left\{\sup_{t\in[0,T]}\left|\widetilde{M}^{\phi}_{k_1}(t)-\widetilde{M}^{\phi}_{k_2}(t)\right|^2\right\}\\
&\le 4\mathbf{E}^{\mathbb{P}'}\left\{\left|\widetilde{M}^{\phi}_{k_1}(T)-\widetilde{M}^{\phi}_{k_2}(T)\right|^2\right\}\\
&=4\mathbf{E}^{\mathbb{P}'}\left\{\int_0^T\left|(-G\tilde{\xi}_{k_1}(s))^*\phi-(-G\tilde{\xi}_{k_2}(s))^*\phi\right|^2_{\mathbb{R}^2}ds\right\}\\
&\le 8\mu\left(|\partial_1\phi|^2_{\mathbf{H}}+|\partial_2\phi|^2_{\mathbf{H}}\right)\mathbf{E}^{\mathbb{P}'}\left\{\int_0^T|\tilde{\xi}_{k_1}(s)-\tilde{\xi}_{k_2}(s)|^2_{\mathbf{H}}ds\right\}\\
&\longrightarrow 0,\quad \text{as } k_1,k_2\to\infty,
\end{aligned}$$

holds for each $T>0$ and $\phi\in\mathbf{C}^\infty_\sigma$, where we have used Doob's inequality at the first inequality and Lebesgue's theorem at the last. Therefore, for $T>0$ and $\phi\in\mathbf{C}^\infty_\sigma$, we can choose a subsequence $\{\widetilde{M}^{\phi}_{k'}\}$ converging $\mathbb{P}'$-a.s. uniformly on $t\in[0,T]$. We denote its limit by $\widetilde{M}^{\phi}$. Obviously,

$$\mathbf{E}^{\mathbb{P}'}\left\{\sup_{t\in[0,T]}\left|\widetilde{M}^{\phi}_{k'}(t)-\widetilde{M}^{\phi}(t)\right|^2\right\}\to 0, \tag{2.37}$$

as $k'\to\infty$. Furthermore, since $\tilde{\xi}_k\to\tilde{\xi}$ in $L^2(0,T;\mathbf{H})$, $\mathbb{P}'$-a.s. we can choose a subsequence $\{k''\}$ of $\{k'\}$ such that we have $\mathbb{P}'$-a.s.,

$$\lim_{k''\to\infty}\langle\tilde{\xi}_{k''}(t),\phi\rangle=\langle\tilde{\xi}(t),\phi\rangle, \tag{2.38}$$

for a.e.-$t\in[0,T]$ and any $\phi\in\mathbf{C}^\infty_\sigma$. By (2.31) and the fact that the family of $\{\tilde{\xi}_k\}$ is bounded in $L^2(0,T;\mathbf{H})$, we see

$$\begin{aligned}
\lim_{k\to\infty}\mathbf{E}^{\mathbb{P}'}\left\{\left\langle\left\langle\widetilde{M}^{\phi}_k,\widetilde{M}^{\phi}_k\right\rangle\right\rangle(t)\right\}&=\mathbf{E}^{\mathbb{P}'}\left\{\int_0^t(-G\tilde{\xi}(s))^*\phi\left((-G\tilde{\xi}(s))^*\phi\right)^*ds\right\}\\
&=\mathbf{E}^{\mathbb{P}'}\left\{\int_0^t2\mu\left\{\langle\partial_1\phi,\tilde{\xi}(s)\rangle^2+\langle\partial_2\phi,\tilde{\xi}(s)\rangle^2\right\}ds\right\},
\end{aligned}$$

holds for any $t\in[0,T]$ and $\phi\in\mathbf{C}^\infty_\sigma$. Consequently, it is easy to see that $\widetilde{M}^{\phi}$ is a square integrable continuous $\mathcal{F}'_t$-martingale whose quadratic variation is given by

$$\begin{aligned}
\left\langle\left\langle\widetilde{M}^{\phi},\widetilde{M}^{\phi}\right\rangle\right\rangle(t)&=\int_0^t(-G\tilde{\xi}(s))^*\phi\left((-G\tilde{\xi}(s))^*\phi\right)^*ds\\
&=\int_0^t2\mu\left\{\langle\partial_1\phi,\tilde{\xi}(s)\rangle^2+\langle\partial_2\phi,\tilde{\xi}(s)\rangle^2\right\}ds,\qquad t\in[0,T],\ \mathbb{P}'\text{-a.s.},
\end{aligned} \tag{2.39}$$

for any $T>0$ and $\phi\in\mathbf{C}^\infty_\sigma$. Thus, by applying the representation theorem for the martingale $\widetilde{M}^{\phi}$, there exist another filtered probability space $(\Omega'',\mathcal{F}'',\mathbb{P}'',\{\mathcal{F}''_t\}_{t\ge0})$ and a two-dimensional

$\mathcal{F}'_t \times \mathcal{F}''_t$-Brownian motion $\widetilde{\widetilde{B}} = (\widetilde{\widetilde{B^1}}, \widetilde{\widetilde{B^2}})$ defined on $(\Omega' \times \Omega'', \mathcal{F}' \times \mathcal{F}'', \mathbb{P}' \times \mathbb{P}'')$ such that for any $\phi \in \mathbf{C}^\infty_\sigma$, we have that $\mathbb{P}' \times \mathbb{P}''$-a.s.,

$$\begin{aligned}\widetilde{M}^\phi(t,\omega',\omega'') &= \int_0^t (-G\tilde{\xi}(s,\omega',\omega''))^* \phi \, d\widetilde{\widetilde{B}}(s,\omega',\omega'') \\ &= \sum_{j=1}^2 \int_0^t \sqrt{2\mu} \langle \partial_j \phi, \tilde{\xi}(s,\omega',\omega'')\rangle d\widetilde{\widetilde{B^j}}(s,\omega',\omega''), \quad t \in [0,T],\end{aligned} \tag{2.40}$$

where $\widetilde{M}^\phi(t,\omega',\omega'') = \widetilde{M}^\phi(t,\omega')$ and $\tilde{\xi}(t,\omega',\omega'') = \tilde{\xi}(t,\omega')$, (see [11]).

Step 4: Construction of a weak solution

We will check the convergence of the right hand side of (2.34). As we mentioned in Step 3, it is clear that $\mathbb{P}' \times \mathbb{P}''$-a.s.,

$$\lim_{k'' \to \infty} \langle \tilde{\xi}_{k''}(t), \phi \rangle = \langle \tilde{\xi}(t), \phi \rangle, \tag{2.41}$$

hold for a.e.-$t \in [0,T]$ and $\phi \in \mathbf{C}^\infty_\sigma$.

Similarly, we see $\mathbb{P}' \times \mathbb{P}''$-a.s.,

$$\lim_{k'' \to \infty} \frac{2 + \delta(n_{k''})}{2} \mu \int_0^t \langle \tilde{\xi}_{k''}(s), \Delta\phi \rangle ds = \mu \int_0^t \langle \tilde{\xi}(s), \Delta\phi \rangle ds, \tag{2.42}$$

holds for any $t \in [0,T]$ and $\phi \in \mathbf{C}^\infty_\sigma$. Indeed, this is easily obtained by the fact that the family of $\{\tilde{\xi}_k\}$ is a bounded set in $L^2(0,T;\mathbf{H})$, $\mathbb{P}' \times \mathbb{P}''$-a.s. Thus, (2.42) holds.

Next we are concerned with the non-linear term. It is enough to check

$$\int_0^t \int_{\mathbb{T}^2} \left\{ (\tilde{\xi}_{k''}(s))^i (\tilde{\xi}_{k''}(s))^j - (\tilde{\xi}(s))^i (\tilde{\xi}(s))^j \right\} \frac{\partial \phi_i}{\partial x_j} dx ds \to 0, \quad \text{as } k'' \to \infty, \tag{2.43}$$

for each $i,j = 1,2$. Indeed, (2.43) is easily obtained by the triangular inequality while using the fact that the family of $\{\tilde{\xi}_k\}$ is a bounded set in $L^2(0,T;\mathbf{H})$ and ϕ belongs to $\mathbf{C}^\infty_\sigma$. Thus, we have $\mathbb{P}' \times \mathbb{P}''$-a.s.,

$$\lim_{k'' \to \infty} \int_0^t \langle (\tilde{\xi}_{k''}(s) \cdot \nabla)\phi, \tilde{\xi}_{k''}(s) \rangle ds = \int_0^t \langle (\tilde{\xi}(s) \cdot \nabla)\phi, \tilde{\xi}(s) \rangle ds,$$

for any $t \in [0,T]$ and $\phi \in \mathbf{C}^\infty_\sigma$.

As a result, for any $\phi \in \mathbf{C}^\infty_\sigma$, we find that $\mathbb{P}' \times \mathbb{P}''$-a.s., $\widetilde{M}^\phi_{k''}(t)$ converges to

$$\widetilde{M}^\phi(t) = \langle \tilde{\xi}(t), \phi \rangle - \langle u_0, \phi \rangle - \mu \int_0^t \langle \tilde{\xi}(s), \Delta\phi \rangle ds - \int_0^t \langle (\tilde{\xi}(s) \cdot \nabla)\phi, \tilde{\xi}(s) \rangle ds, \tag{2.44}$$

for a.e.-$t \in [0,T]$.

From (2.40) and (2.44), it follows that $\{\tilde{\xi}(t), \tilde{\tilde{B}}(t)\}_{t\geq 0}$ on $(\Omega'\times\Omega'', \mathcal{F}'\times\mathcal{F}'', \{\mathcal{F}'_t\times\mathcal{F}''_t\}_{t\geq 0}, \mathbb{P}'\times\mathbb{P}'')$ satisfies the properties 3 and 4 of Definition 2.2.1. The properties 1 and 2 of Definition 2.2.1 are checked as follows. It is clear that $\tilde{\xi}(t)$ is an $\mathcal{F}'_t \times \mathcal{F}''_t$-adapted process. Finally, we will show $\tilde{\xi} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$, $\mathbb{P}' \times \mathbb{P}''$-a.s. Indeed, it follows by (2.31), (2.30) and Fatou's lemma. Thus, $\{\tilde{\xi}(t), \tilde{\tilde{B}}(t)\}_{t\geq 0}$ on $(\Omega' \times \Omega'', \mathcal{F}' \times \mathcal{F}'', \{\mathcal{F}'_t \times \mathcal{F}''_t\}_{t\geq 0}, \mathbb{P}' \times \mathbb{P}'')$ is a weak solution of (2.6) with the initial value u_0. This concludes the proof of Theorem 2.2.1.

□

Appendix. Background of our problem

In this appendix, we will explain our motivation. We denote by $\mathrm{Diff}(\mathbb{R}^n)$ the family of volume preserving diffeomorphisms of $\mathbb{R}^n$. Let $\Phi(t) = (\Phi_1(t), \cdots, \Phi_n(t))$, $t \in [0,1]$ be an integral curve which takes values in $\mathrm{Diff}(\mathbb{R}^n)$ and $\Psi^0, \Psi^1 \in \mathrm{Diff}(\mathbb{R}^n)$ be given. If we consider the action functional J defined by

$$J(\Phi) = \int_0^1 \int_{\mathbb{R}^n} \sum_{j=1}^n |\frac{\partial \Phi_j(t,x)}{\partial t}|^2 dx\, dt, \tag{2.45}$$

under the condition $\Phi(0) = \Psi^0$ and $\Phi(1) = \Psi^1$, it is known that the time derivative

$$u(t,x) = (u^1(t,x), \cdots, u^n(t,x)) = (\frac{\partial \widetilde{\Phi}_1}{\partial t}(t, \widetilde{\Phi}^{-1}(t,x)), \cdots, \frac{\partial \widetilde{\Phi}_n}{\partial t}(t, \widetilde{\Phi}^{-1}(t,x)),$$

of a stationary point $\widetilde{\Phi}(t,x) = (\widetilde{\Phi}_1(t,x), \cdots, \widetilde{\Phi}_n(t,x))$ of J satisfies the Euler equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \operatorname{div} u = 0, & t > 0,\ x \in \mathbb{R}^n, \end{cases} \tag{2.46}$$

where $p = p(t,x)$ is the pressure term, see [13].

In [13], the case where the integral curve appearing above is affected by some random force is studied, that is, for an n-dimensional Brownian motion $B = (B^1, \cdots, B^n)$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\Psi^0, \Psi^1 \in \mathrm{Diff}(\mathbb{R}^n)$, a.s., the following random action functional J_B is introduced:

$$J_B(\Phi) = \int_{\mathbb{R}^n} \int_0^1 \sum_{j=1}^n |\frac{\partial \Phi_j(t,x)}{\partial t} + \sqrt{2\mu}\frac{dB_t^j}{dt}|^2 dxdt, \tag{2.47}$$

where $\Phi(0,\omega) = \Psi^0(\omega)$ and $\Phi(1,\omega) = \Psi^1(\omega)$. By proceeding similarly to the deterministic case,

$$\begin{aligned} u(t,x,\omega) &= \left(u^1(t,x,\omega), \cdots, u^n(t,x,\omega)\right) \\ &= \left(\frac{\partial \bar{\Phi}_1^B}{\partial t}(t, \tilde{\Phi}^{-1}(t,x),\omega), \cdots, \frac{\partial \bar{\Phi}_n^B}{\partial t}(t, \tilde{\Phi}^{-1}(t,x),\omega)\right), \quad t > 0,\ x \in \mathbb{R}^n,\ \omega \in \Omega, \end{aligned}$$

would satisfy

$$\begin{cases} \dfrac{\partial(u^i+\sqrt{2\mu}\dot{B}^i)}{\partial t} + \displaystyle\sum_{j=1}^n \left(u^j \frac{\partial u^i}{\partial x_j} + \sqrt{2\mu}\frac{\partial u^i}{\partial x_j} \circ \dot{B}^j \right) + \frac{\partial p}{\partial x_i} = 0, & t>0,\ x \in \mathbb{R}^n, \\ & i=1,\cdots,n, \\ \operatorname{div} u = 0, & t>0,\ x\in\mathbb{R}^n, \end{cases} \tag{2.48}$$

where $\bar{\Phi}^B$ is the random stationary point of J_B:

$$\begin{aligned} &\bar{\Phi}^B(t,x,\omega) \\ &= (\bar{\Phi}_1^B(t,x,\omega)+\sqrt{2\mu}\dot{B}^1(t,\omega),\cdots,\bar{\Phi}_n^B(t,x,\omega)+\sqrt{2\mu}\dot{B}^n(t,\omega)), \quad t>0,\ x\in\mathbb{R}^n,\ \omega\in\Omega. \end{aligned}$$

Note that the notation $\circ$ appearing in the stochastic term means the Stratonovich sense. It is seen in [13] that if there exists a weak solution $u(t,x,\omega)$ of the equation (2.48) with the initial value $u_0 \in \mathbf{W}^{1,2}(\mathbb{R}^n;\mathbb{R}^n)$ satisfying div $u_0 = 0$, its expectation $\overline{u}(t,x) = \int_\Omega u(t,x,\omega)P(d\omega)$ satisfies the following Reynolds equation:

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t} - \mu\Delta\overline{u} + (\overline{u}\cdot\nabla)\overline{u} + \nabla p = -\overline{(u-\overline{u})\cdot\nabla)(u-\overline{u})}, & t>0,\ x\in\mathbb{R}^n, \\ \operatorname{div}\bar{u} = 0, & t>0,\ x\in\mathbb{R}^n, \end{cases}$$

However, the existence of the weak solution of (2.48) is not shown in [13]. Note that a Stratonovich integral can be rewritten into an Itô integral by using the following formula (see [13]):

$$\int_0^t \frac{\partial u^i}{\partial x_j}(s)\circ dB^j(s) = \int_0^t \frac{\partial u^i}{\partial x_j}(s)dB^j(s) + \frac{1}{2}\langle\langle M_{\frac{\partial u^i}{\partial x_j}}, B^j\rangle\rangle(t),$$

where $M_{\frac{\partial u^i}{\partial x_j}}$ denotes the martingale part determined uniquely by the decomposition of the process $\frac{\partial u^i}{\partial x_j}$ and $\langle\langle M_{\frac{\partial u^i}{\partial x_j}}, B^j\rangle\rangle$ the quadratic variation of $M_{\frac{\partial u^i}{\partial x_j}}$ and B^j. Thus, we arrive at the following stochastic Navier-Stokes equation:

$$\begin{cases} \dfrac{\partial u^i}{\partial t} + \sqrt{2\mu}\ddot{B}_t^i + \displaystyle\sum_{j=1}^n \left(u^j\frac{\partial u^i}{\partial x_j} + \sqrt{2\mu}\frac{\partial u^i}{\partial x_j}\dot{B}_t^j \right) - \mu\Delta u^i + \frac{\partial p}{\partial x_i} = 0, & t>0,\ x\in\mathbb{R}^n, \\ & i=1,\cdots,n, \\ \operatorname{div} u = 0, & t>0,\ x\in\mathbb{R}^n. \end{cases} \tag{2.49}$$

In this paper, we discuss the equation (2.49) on a two-dimensional torus $\mathbb{T}^2$, in which we disregard the term $\ddot{B}^i(t)$, $i=1,2$. This is reasonable because $\sum_{i=1}^2 \int_{\mathbb{T}^2} \ddot{B}^i(t)\phi^i(x)dx$ formally vanishes by the property of $\int_{\mathbb{T}^2}\phi(x)dx = 0$, $\phi\in\mathbf{C}_\sigma^\infty$.

Chapter 3

Weak solutions of non coercive stochastic Navier-Stokes equations in $\mathbb{R}^2$

3.1 Introduction

In this paper we study the following type of the stochastic Navier-Stokes equation with respect to $u = (u^1(t,x), u^2(t,x)), t > 0, x \in \mathbb{R}^2$:

$$\frac{\partial u}{\partial t} - \mu\triangle u + (u\cdot\nabla)u + \sqrt{2\mu}\nabla u\cdot\dot{B}(t) + \nabla p = 0, \quad t > 0, x \in \mathbb{R}^2, \tag{3.1}$$

$$\mathrm{div} u = 0, \quad t > 0, x \in \mathbb{R}^2, \tag{3.2}$$

$$u(0) = u_0, \quad x \in \mathbb{R}^2, \tag{3.3}$$

where $p = p(t,x)$ denotes the pressure term, $\mu > 0$ is a constant and $\dot{B}(t) = \frac{d}{dt}(B^1(t), B^2(t))$ the distributional derivative of the two-dimensional Brownian motion $B(t) = (B^1(t), B^2(t))$. Furthermore, u_0 is a deterministic $\mathbf{V}(\mathbb{R}^2)$-valued function on $\mathbb{R}^2$ with compact support. Here $\mathbf{V}(\mathbb{R}^2)$ is the set of functions defined as follows (see Section 3.2):

$$\mathbf{V}(\mathbb{R}^2) = \mathbf{W}^{1,2}(\mathbb{R}^2;\mathbb{R}^2) \cap \mathbf{H}(\mathbb{R}^2),$$

where

$$\mathbf{H}(\mathbb{R}^2) = \{u \in \mathbf{L}^2(\mathbb{R}^2;\mathbb{R}^2) \,|\, \mathrm{div}\ u = 0\}.$$

Note that equation (3.1)-(3.3) are formally derived as the Euler-Lagrange equation satisfied by a critical point of a random energy functional defined on the space of volume preserving diffeomorphisms in $\mathbb{R}^2$ perturbed by Brownian motion (see [13], [20]). In [13], the velocity defined as the time derivative of the associated stationary point satisfies the stochastic Navier-Stokes equation (3.1)-(3.3) and as a result, it is shown that the expectation of the solution of (3.1)-(3.3) satisfies the Reynolds equation. In [20], the existence of the weak solution of (3.1)-(3.3) in the case of the two-dimensional torus is studied.

On the other hand, [9] considers an energy functional different from that of [20], [13], and shows that the deterministic Navier-Stokes equation is related to its stationary point. In this paper, we try to study the equation (3.1)-(3.3) not in the case of a two-dimensional torus but on the whole space $\mathbb{R}^2$.

In comparing with the case of stochastic Navier-Stokes equations on a bounded domain, the case of unbounded domains requires more efforts because of the lack of compactness.

In addition, our equation does not satisfy the coercivity condition which usually gives the tightness. Let us explain briefly our strategy taken in this paper to construct the solution of the equation (3.1)-(3.3). First, we consider a family of modified equations with $2l$-period ($l \in \mathbb{N}$) in each variable whose viscosity coefficient is slightly larger than $\mu > 0$, that is, $\frac{2+\delta}{2}\mu$, $\delta > 0$, so that the approximating equations satisfy the coercivity condition. We construct its weak solution $u^{l,\delta}$ by using a standard Galerkin approximation. For suitable cutoff functions $\chi_R \uparrow 1_{\mathbb{R}^2}$, it can be shown that the family $u^{R,l,\delta} = \chi_R u^{l,\delta}$ is uniformly bounded in the space $L^2(\Omega, L^2(0,T; \mathbf{V}(\mathbb{R}^2))$ with respect to R,l and δ. Finally, we take a limit of $u^{R,l,\delta}$ as $\delta \to 0$ and $R \to \infty$ simultaneously and show that its limit satisfies the equation (3.1)-(3.3) in a weak sense.

So far, there are several known results about weak solutions of stochastic Navier-Stokes equations ([17], [3], [12], [15], [16], [19]). However, no results are known in the case where the equation does not satisfy the coercivity condition in a bounded or unbounded domain in $\mathbb{R}^n (n \geq 2)$. As a consequence, there is no easy way to obtain tightness of suitable Galerkin approximation in L^2-spaces.

[17] and [3] study the equation with a trace class Wiener process and a spatially homogeneous initial distribution and the existence of the spatially homogeneous weak solution in a weighted Sobolev space is proven. There are also several results about the case of the two-dimensional torus ([20], [14], [7]). Especially, [20] studies the case where the equation does not satisfy the coercivity condition in a two-dimensional torus, then shows that there exists a weak solution. On the other hand, [19] shows that there exists a spatially homogeneous weak solution of the equations in $\mathbb{R}^n (n \geq 2)$ with a spatially homogeneous H^1-valued initial distribution independent of the space-time white noise. [15] and [16] study the stochastic Navier-Stokes equations on $\mathbb{R}^n (n \geq 2)$ satisfying the coercivity condition. In this paper, we partly use the method which is studied in [19] and [3], that is, we construct the solution by taking the limit

of the sequence of periodic solutions. This paper is organized as follows: In Section 3.2, we introduce notations used in this paper and our main result. Section 3.3 and Section 3.4 contain the proof of our main results.

3.2 Notations and results

In this section we introduce for several notations appearing later. Set $\boldsymbol{T}_l = (-l,l)^2$, $l \in \mathbb{N}$. We denote by

$$\mathbf{C}^\infty_{per}(l) = \{u \in \mathbf{C}^\infty(\mathbb{R}^2;\mathbb{R}^2) \mid u \text{ is } 2l\text{-periodic in } (x_1,x_2) \in \mathbb{R}^2\},$$

the family of smooth vector fields u having period $2l$ in each variable $(x_1,x_2) \in \mathbb{R}^2$. We also denote by $\mathbf{C}^\infty_{per,\sigma}(l)$ the subspace of divergence free vector fields u satisfying and $\int_{\boldsymbol{T}_l} udx = 0$, that is,

$$\mathbf{C}^\infty_{per,\sigma}(l) = \{u \in \mathbf{C}^\infty_{per}(l) \mid \int_{\boldsymbol{T}_l} udx = 0,\ \operatorname{div} u = 0 \text{ in } \boldsymbol{T}_l\}.$$

We also denote the following function spaces:

$$\mathbf{C}^\infty_0 = \{u \in \mathbf{C}^\infty(\mathbb{R}^2;\mathbb{R}^2) \mid \operatorname{supp} u \text{ is compact}\},$$

$$\mathbf{C}^\infty_0(\Omega) = \{u \in \mathbf{C}^\infty_0 \mid \operatorname{supp} u \subset \Omega\},$$

$$\mathbf{C}^\infty_{0,\sigma} = \left\{u \in \mathbf{C}^\infty_0 \mid \int_{\mathbb{R}^2} udx = 0,\ \operatorname{div} u = 0 \text{ in } \mathbb{R}^2\right\},$$

We denote by $\mathbf{H}(l)$ the set of square integrable vector fields u on $\boldsymbol{T}_l$ which are of divergence zero and satisfy $\int_{\boldsymbol{T}_l} udx = 0$, that is,

$$\mathbf{H}(l) = \left\{u \in \mathbf{L}^2(\boldsymbol{T}_l;\mathbb{R}^2) \mid \int_{\boldsymbol{T}_l} udx = 0,\ \operatorname{div} u = 0 \text{ in } \boldsymbol{T}_l\right\}.$$

Let $\langle u,v\rangle_l = \sum_{i=1}^{2} \int_{\boldsymbol{T}_l} u^i(x)v^i(x)dx$ be its inner product and $|u|_l = \langle u,u\rangle_l^{\frac{1}{2}}$ its norm. In addition, we set

$$\mathbf{V}(l) = \mathbf{W}^{1,2}(\boldsymbol{T}_l;\mathbb{R}^2) \cap \mathbf{H}(l),$$

with its inner product $\langle\langle u,v\rangle\rangle_l = \sum_{j=1}^{2} \langle \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j}\rangle_l$ and associated norm $||u||_l = \langle\langle u,u\rangle\rangle_l^{\frac{1}{2}}$.

Let us set $\mathbb{Z}_0^2 = \mathbb{Z}^2 \backslash \{0\}$ and $\mathbb{T}_l^2 = \mathbb{R}^2 / 2l\mathbb{Z}^2$. Let

$$\mathbf{H}_{per}(l) = \left\{ u \in \mathbf{L}^2(\mathbb{T}_l^2; \mathbb{R}^2) \mid \int_{\mathbb{T}_l^2} u dx = 0, \ \operatorname{div} u = 0 \text{ in } \mathbb{R}^2 \right\},$$

be the Hilbert space with inner product $\langle u, v \rangle_{per} = \sum_{k \in \mathbb{Z}_0^2} \hat{u}(k)\hat{v}(k)$ and associated norm $|u|_{per} = (\sum_{k \in \mathbb{Z}_0^2} |\hat{u}(k)|^2)^{\frac{1}{2}}$, $u, v \in \mathbf{H}_{per}(l)$, where $\hat{u}(k)$ represents the $k = (k_1, k_2)$-th Fourier coefficient of the Fourier expansion of u. In addition, we set

$$\mathbf{V}_{per}(l) = \mathbf{W}^{1,2}(\mathbb{T}_l^2; \mathbb{R}^2) \cap \mathbf{H}_{per}(l),$$

with inner product $\langle\langle u, v \rangle\rangle_{per} = \sum_{k \in \mathbb{Z}_0^2} (\frac{\pi}{l}|k|)^2 \hat{u}(k)\hat{v}(k)$ and associated norm $||u||_{per} = (\sum_{k \in \mathbb{Z}^2} (\frac{\pi}{l}|k|)^2 |\hat{u}(k)|^2)^{\frac{1}{2}}$ for $u, v \in \mathbf{V}_{per}(l)$. Note that $|u|_{per} = |u|_l$ if $u \in \mathbf{H}_{per}(l)$ and $||u||_{per} = ||u||_l$ if $u \in \mathbf{V}_{per}(l)$. Similarly, let us set

$$\mathbf{H}(\mathbb{R}^2) = \left\{ u \in \mathbf{L}^2(\mathbb{R}^2; \mathbb{R}^2) \mid \operatorname{div} u = 0 \text{ in } \mathbb{R}^2 \right\},$$

with its inner product and the norm denoted by $\langle \cdot, \cdot \rangle$ and $| \cdot |$, respectively, and

$$\mathbf{V}(\mathbb{R}^2) = \mathbf{W}^{1,2}(\mathbb{R}^2; \mathbb{R}^2) \cap \mathbf{H}(\mathbb{R}^2),$$

with its inner product and the norm denoted by $\langle\langle \cdot, \cdot \rangle\rangle$ and $|| \cdot ||$, respectively. For an open set $\Omega \subset \mathbb{R}^2$, let us define

$$\mathbf{H}(\Omega) = \{ u \in \mathbf{H}(\mathbb{R}^2) \mid \text{ supp } u \subset \Omega \},$$
$$\mathbf{V}(\Omega) = \{ u \in \mathbf{V}(\mathbb{R}^2) \mid \text{ supp } u \subset \Omega \}.$$

In addition, define

$$H_\Omega = \{ u \mid |u|^2_{\mathbf{H}(\Omega)} := \int_\Omega |u(x)|^2 dx < \infty \},$$
$$V_\Omega = \{ u \mid ||u||^2_{\mathbf{V}(\Omega)} := \sum_{j=1}^2 \int_\Omega |\frac{\partial u(x)}{\partial x_j}|^2 dx < \infty \},$$
$$\mathbf{H}_\Omega = \{ u \in H_\Omega \mid \text{ div } u = 0 \},$$
$$\mathbf{V}_\Omega = \{ u \in V_\Omega \mid \text{ div } u = 0 \}.$$

We denote by H_{loc}, V_{loc} the set of vector-fields whose countable semi norms $||u||_{0,R}$, $||u||_{1,R}$ are finite for all $R \in \mathbb{N}$, respectively, that is,

$$H_{loc} = \{ u \in (\mathbf{C}_0^\infty)' \mid ||u||_{0,R} < \infty \text{ for all } R \in \mathbb{N} \},$$
$$V_{loc} = \{ u \in (\mathbf{C}_0^\infty)' \mid ||u||_{1,R} < \infty \text{ for all } R \in \mathbb{N} \},$$

where $||u||_{0,R}$, $||u||_{1,R}$ are defined as follows:

$$||u||_{0,R} = \int_{B_R} |u(x)|^2 dx,$$
$$||u||_{1,R} = \sum_{i=1}^{2} \int_{B_R} |\frac{\partial u(x)}{\partial x_i}|^2 dx,$$

where B_R is the open ball with radius $R \in \mathbb{N}$ centered at the origin. In addition, let us set

$$\mathbf{H}_{\text{loc}} = \{u \in H_{loc}| \text{ div } u(x) = 0, x \in \mathbb{R}^2\},$$
$$\mathbf{V}_{\text{loc}} = \{u \in V_{loc}| \text{ div } u(x) = 0, x \in \mathbb{R}^2\}.$$

and $\mathbf{V}_{\text{loc}}'$ the topological dual space of $\mathbf{V}_{\text{loc}}$. Note that the divergence appearing in each class is understood in the distributional sense. Let $Au = -\mu\mathbb{P}\Delta u$ be the Stokes operator with domain

$$D(A) = \mathbf{V}(\mathbb{R}^2) \cap \mathbf{W}^{2,2}(\mathbb{R}^2; \mathbb{R}^2),$$

where $\mathbb{P}$ represents the Leray projection. It is well known that A is a non negative self adjoint linear operator. Furthermore, let B be defined by

$$\langle B(u,v), w\rangle = \int_{\mathbb{R}^2} (u(x)\cdot\nabla)v(x)\cdot w(x)dx, \quad u,v,w \in \mathbf{C}_{0,\sigma}^{\infty},$$

and $G : \mathbf{V}_{\text{loc}} \to \mathrm{L}_{\mathrm{H.S}}(\mathbb{R}^2; \mathbf{H}_{\text{loc}})$ be defined by

$$Gu = \sqrt{2\mu}\nabla u,$$

where $\mathrm{L}_{\mathrm{H.S}}(\mathbb{R}^2; \mathbf{H})$ denotes the space of Hilbert-Schmidt operators from $\mathbb{R}^2$ to $\mathbf{H}$. By Sobolev's embedding theorem, we see that B can be uniquely extended to a $\mathbf{V}_{\text{loc}}'$-valued bilinear operator on $\mathbf{V}_{\text{loc}} \times \mathbf{V}_{\text{loc}}$. Indeed,

$$\begin{aligned} |\int_{B_R} u_i \frac{\partial u_j}{\partial x_i}\phi_j dx| &\le |u_i|_{L^4(B_R)} |\frac{\partial u_j}{\partial x_i}|_{L^2(B_R)} |\phi_j|_{L^4(B_R)} \\ &\le |u_i|_{H_0^1(B_R)} |u_j|_{H_0^1(B_R)} |\phi_j|_{H_0^1(B_R)}, \quad u,\phi \in C_0^{\infty}, \quad j = 1,2, \end{aligned}$$

holds. This implies B is a $\mathbf{V}_{\text{loc}}'$-valued bilinear operator on $\mathbf{V}_{\text{loc}} \times \mathbf{V}_{\text{loc}}$. In our equation, the noise is finite-dimensional and thus its covariance is trivially of finite trace, so the square root is Hilbert-Schmidt. The abstract stochastic evolution equation associated with (3.1)-(3.3) is defined as follows:

$$\begin{cases} du(t) + Au(t) + B(u(t), u(t)) + Gu(t)\cdot dB(t) = 0, \quad t > 0, \\ u(0) = u_0. \end{cases} \tag{3.4}$$

Definition 3.2.1. *We say $\{u(t), B(t)\}_{t\geq 0}$ is a weak solution of (3.4) if*

1. *$u(t)$ is an adapted process on a probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$,*

2. *$u \in L^2(0,T;\mathbf{V}_{loc}) \cap L^\infty(0,T;\mathbf{H}_{loc})$, a.s.*

3. *$\{B(t), \mathcal{F}_t\}_{t\geq 0}$ is a two-dimensional Brownian motion,*

4. *For every $\phi \in \mathbf{C}^\infty_{0,\sigma}$, P-a.s., the following equality*

$$\langle u(t), \phi\rangle - \langle u_0, \phi\rangle + \int_0^t \langle u(s), A\phi\rangle ds$$
$$= \int_0^t \langle B(u(s), \phi), u(s)\rangle ds - \int_0^t (Gu(s))^* \phi \cdot dB(s),$$

holds for a.e. $t \in [0,T]$.

Remark 3.2.1. *The term containing ∇p drops out in the weak form of the solution since $\int \nabla p \cdot \phi dx = -\int p \ div\phi dx = 0$ holds.*

Remark 3.2.2. *We can regard the second condition of the Definition 3.2.1 as*

$$u \in \bigcap_{R\in \mathbb{N}} L^2(0,T;\mathbf{V}_{B_R}) \cap L^\infty(0,T;\mathbf{H}_{B_R}), a.s.$$

Now we can formulate our main result in this paper.

Theorem 3.2.1. *Let $u_0 \in \mathbf{V}(\mathbb{R}^2)$ have compact support. Then, there exists a weak solution of (3.4).*

3.3 Proof of Theorem 3.2.1

We will separate the proof into four steps.

step 1

We denote by $A_{l,\delta}$ the Stokes operator with viscosity $\frac{2+\delta}{2}\mu$, that is,

$$A_{l,\delta}u = \frac{2+\delta}{2} A_l u,$$

where

$$A_l u = -\mu \mathbb{P} \triangle u,$$

with domain

$$D(A_l) = \mathbf{V}_{per}(l) \cap \mathbf{W}^{2,2}(\mathbb{T}_l^2; \mathbb{R}^2).$$

Note that A_l is a strictly positive definite self-adjoint operator and has a compact resolvent. Let $0 < \lambda_1^{(l)} \leq \lambda_2^{(l)} \leq \ldots$ be the eigenvalues of A_l and $e_1^{(l)}, e_2^{(l)}, \ldots$ the associated normalized eigenfunctions. Let us consider the following equation:

$$\begin{cases} du^{l,\delta}(t) + A_{l,\delta}u^{l,\delta}(t) + B(u^{l,\delta}(t), u^{l,\delta}(t)) + Gu^{l,\delta}(t) \cdot dB(t) = 0, \quad t > 0, \\ u^{l,\delta}(0) = u_0^{(l)}, \end{cases} \tag{3.5}$$

where $u_0^{(l)}$ is the Fourier expansion of u_0 in $\mathbf{H}_{per}(l)$, that is, $u_0^{(l)} = \sum_k \hat{u_0}(k)e_k^{(l)}$, where $\hat{u_0}(k)$ denotes the k-th Fourier coefficient. (3.5) satisfies the coercivity condition, hence there exists a weak solution $u^{l,\delta}$ for each $l \in \mathbb{N}$ and $\delta > 0$, that is, we can construct a weak solution $\{u^{l,\delta}(t), B(t)\}_{t\geq 0}$ on a probability space $(\Omega^{l,\delta}, \mathcal{F}^{l,\delta}, \mathbf{P}^{l,\delta}, \{\mathcal{F}_t^{l,\delta}\}_{t\geq 0})$ such that $\mathbf{P}^{l,\delta}$-a.s.,

$$u^{l,\delta} \in L^2(0,T; \mathbf{V}_{per}(l)) \cap L^\infty(0,T; \mathbf{H}_{per}(l)) \cap C([0,T]; D(A_{l,\delta})'), \tag{3.6}$$

holds (see Lemma 3.4.1). The proof is similar to [12]. Let $0 \leq \chi_R \leq 1$ be a $C_0^\infty(\mathbb{R})$-function which is equal to 1 in B_R, 0 outside B_{2R} and satisfies $|\chi_R'(x)| \leq c$ for some uniform constant $c > 0$. Let $u^{l,\delta,R} = \chi_R u^{l,\delta}$. Now let us obtain an a priori estimate of $\mathbf{E}^{\mathbf{P}^{l,\delta}}\{||u^{l,\delta,R}(t)||^2\}$. Let $\{K_m\}_{m\geq 1}$ be an increasing sequence of compact sets in $\mathbb{R}^2$ such that

$$K_1 \subset K_2 \subset \cdots \uparrow \mathbb{R}^2,$$

and assume that $K_m \subset K_{m+1}^i$, where K_{m+1}^i denotes the interior of K_{m+1}. For each K_m, choose two bounded open sets Ω_{K_m}, Ω'_{K_m} such that $K_m \subset \Omega_{K_m} \subset \bar{\Omega}_{K_m} \subset \Omega'_{K_m} \subset K_{m+1}^i$ holds. For each $R \in \mathbb{N}$ and compact set K, let us choose $l = l(R,K) \in \mathbb{N}$ such that $(-l,l)^2 \supset \Omega_K \cup B_{2R} \cup$ supp u_0 holds. Then,

$$\begin{aligned} ||u^{l,\delta,R}(t)||^2 =& ||\chi_R u^{l,\delta}(t)||^2 \\ =& |\chi_R u^{l,\delta}(t)|^2 + \sum_{j=1}^2 |\frac{\partial}{\partial x_j}\chi_R u^{l,\delta}(t)|^2 \\ \leq& C\left(|u^{l,\delta}(t)|_l^2 + ||u^{l,\delta}(t)||_l^2\right), \end{aligned} \tag{3.7}$$

holds for some constant $C > 0$, since χ_R and $\frac{\partial}{\partial x_j}\chi_R$ are bounded functions.

On the other hand, since $u^{l,\delta}$ is a solution of (3.5), we obtain the following uniform estimate:

$$\mathbf{E}^{\mathbf{P}^{l,\delta}}\{||u^{l,\delta}(t)||_l^2\} \leq ||u_0^{(l)}||_l^2, \tag{3.8}$$

(see Lemma 3.4.2). By Parseval's formula,

$$||u_0^{(l)}||_l^2 \leq ||u_0||_l^2. \tag{3.9}$$

As a result, for any $R \in \mathbb{N}$ and compact set $K \subset \mathbb{R}^2$, we have

$$\mathbf{E}^{\mathbf{P}^{l(R,K),\delta}}\left\{||u^{l(R,K),\delta,R}(t)||^2\right\} \leq ||u_0||^2,$$

which means that

$$\sup_{R\in\mathbb{N},\delta>0,K\subset\mathbb{R}^2 \text{ compact}} \mathbf{E}^{\mathbf{P}^{l(R,K),\delta}}\left\{||u^{l(R,K),\delta,R}(t)||^2\right\} < \infty. \tag{3.10}$$

On the other hand, for any $\phi \in C_0^\infty(\Omega_K) \subseteq C_0^\infty((-l,l)^2)$ satisfying $||\phi||_{\mathbf{V}(\Omega_K)} \leq 1$, hence $||\phi||_{\mathbf{V}((-l,l)^2)} \leq 1$, we have that

$$\begin{aligned}
&\left|{}_{\mathbf{V}(\Omega_K)'}\langle u^{l,\delta,R}(t), \phi\rangle_{\mathbf{V}(\Omega_K)}\right| \\
=&\left|\int_{\Omega_K} u^{l,\delta,R}(t,x)\cdot\phi(x)dx\right| \leq ||u^{l,\delta,R}(t)||_{\mathbf{V}((-l,l)^2)'},
\end{aligned}$$

and thus,

$$||u^{l,\delta,R}(t)||_{\mathbf{V}(\Omega_K)'} \leq ||u^{l,\delta,R}(t)||_{\mathbf{V}((-l,l)^2)'}. \tag{3.11}$$

In addition, since $(-l,l)^2$ contains B_{2R}, it is easy to see that

$$||u^{l,\delta,R}(t)||_{\mathbf{V}((-l,l)^2)'} \leq C||u^{l,\delta}(t)||_{\mathbf{V}((-l,l)^2)'}, \tag{3.12}$$

holds for some constant $C > 0$ using the properties of the cut-off function χ_R.

On the other hand, we have the following estimate (see [12] or Lemma 3.4.1):

$$\mathbf{E}^{\mathbf{P}^{l,\delta}}\left\{||u^{l,\delta}||_{W^{\alpha,2}(0,T;\mathbf{V}((-l,l)^2)')}\right\} < \infty, \quad \alpha \in (0,\frac{1}{2}), \tag{3.13}$$

where

$$||v||^2_{W^{\alpha,2}(0,T;E)} = \int_0^T ||v(t)||_E^2 dt + \int_0^T\int_0^T \frac{||v(t)-v(s)||_E^2}{|t-s|^{1+2\alpha}}dtds,$$

and E is a Banach space. Thus, if we take $\alpha \in (0,\frac{1}{2})$, we see

$$\sup_{R\in\mathbb{N},\delta\in(0,1],K\subset\mathbb{R}^2 \text{ compact}} \mathbf{E}^{\mathbf{P}^{l(R,K),\delta}}\left\{||u^{l(R,K),\delta,R}||^2_{W^{\alpha,2}(0,T;\mathbf{V}(\Omega_K)')}\right\} < \infty. \tag{3.14}$$

step 2

Let $0 \leq \psi_{K_1} \leq 1$ be a $C_0^\infty(\mathbb{R})$-function which is equal to 1 on Ω_{K_1} and whose support is contained in Ω'_{K_1}. Set $\delta = \frac{1}{R}$. Then, the sequence of $(\psi_{K_1} u^{l(R),\frac{1}{R},R})_{R\in\mathbb{N}}$ is bounded in both $L^2(0,T;\mathbf{V}(\Omega_{K_1}))$ and $W^{\alpha,2}(0,T;\mathbf{V}(\Omega_{K_1})')$, $\alpha \in (0,\frac{1}{2})$, a.s. By the following inclusion:

$$\mathbf{V}(\Omega_{K_1}) \subset\subset \mathbf{H}(\Omega_{K_1}) \subset \mathbf{V}(\Omega_{K_1})',$$

$\{\mathcal{L}(\psi_{K_1} u^{l(R),\frac{1}{R},R})\}_{R\in\mathbb{N}}$ is tight in $L^2(0,T;\mathbf{H}(\Omega_{K_1}))$, hence in $L^2(0,T;\mathbf{H}(\mathbb{R}^2))$, where $\mathcal{L}(u)$ denotes the distribution of u. Thus, there exists a subsequence $\{R_n^{(1)}\}_{n\in\mathbb{N}}$ such that

$$(-l(R_n^{(1)}), l(R_n^{(1)}))^2 \supset \Omega_{K_2} \cup B_{2R} \cup \text{ supp } u_0,$$

and $\{\mathcal{L}(\psi_{K_1} u^{l(R_n^{(1)}),\frac{1}{R_n^{(1)}},R_n^{(1)}})\}_{n\in\mathbb{N}}$ is convergent in $L^2(0,T;\mathbf{H}(\Omega_{K_1}))$, hence, in $L^2(0,T;\mathbf{H}(\mathbb{R}^2))$.

Next, let $0 \leq \psi_{K_2} \leq 1$ be a $C_0^\infty(\mathbb{R})$-function which is equal to 1 on Ω_{K_2} and whose support is contained in Ω'_{K_2}. Then, the family $(\psi_{K_2} u^{l(R_n^{(1)}),\frac{1}{R_n^{(1)}},R_n^{(1)}})_{n\in\mathbb{N}}$ is bounded in both $L^2(0,T;\mathbf{V}(\Omega_{K_2}))$ and $W^{\alpha,2}(0,T;\mathbf{V}(\Omega_{K_2})')$, a.s. From the following inclusion,

$$\mathbf{V}(\Omega_{K_2}) \subset\subset \mathbf{H}(\Omega_{K_2}) \subset \mathbf{V}(\Omega_{K_2})',$$

the family $\{\mathcal{L}(\psi_{K_2} u^{l(R_n^{(1)}),\frac{1}{R_n^{(1)}},R_n^{(1)}})\}_{n\in\mathbb{N}}$ is tight in $L^2(0,T;\mathbf{H}(\Omega_{K_1}))$, hence in $L^2(0,T;\mathbf{H}(\mathbb{R}^2))$. Thus, there exists a subsequence $\{R_n^{(2)}\}_{n\in\mathbb{N}}$ of $\{R_n^{(1)}\}_{n\in\mathbb{N}}$ such that

$$(-l(R_n^{(2)}), l(R_n^{(2)}))^2 \supset \Omega_{K_3} \cup B_{2R} \cup \text{ supp } u_0,$$

and $\{\mathcal{L}(\psi_{K_2} u^{l(R_n^{(2)}),\frac{1}{R_n^{(2)}},R_n^{(2)}})\}_{n\in\mathbb{N}}$ is convergent in $L^2(0,T;\mathbf{H}(\Omega_{K_2}))$, hence, in $L^2(0,T;\mathbf{H}(\mathbb{R}^2))$. Note that

$$\mathcal{L}(\psi_{K_2} u^{l(R_n^{(1)}),\frac{1}{R_n^{(1)}},R_n^{(1)}}|_{K_1}) = \mathcal{L}(\psi_{K_1} u^{l(R_n^{(1)}),\frac{1}{R_n^{(1)}},R_n^{(1)}}|_{K_1}), \quad n \in \mathbb{N},$$

where $u|_K$ denotes the restriction of u to a set K. Similarly, we can extract a suitable subsequence $\{R_n^{(m)}\}_{n\in\mathbb{N}}$ of $\{R_n^{(m-1)}\}_{n\in\mathbb{N}}$ such that

$$(-l(R_n^{(m)}), l(R_n^{(m)}))^2 \supset \Omega_{K_{m+1}} \cup B_{2R} \cup \text{ supp } u_0,$$

and that $\{\mathcal{L}(\psi_{K_m} u^{l(R_n^{(m)}),\frac{1}{R_n^{(m)}},R_n^{(m)}})\}_{n\in\mathbb{N}}$ is convergent in $L^2(0,T;\mathbf{H}(\Omega_{K_m}))$, hence, in $L^2(0,T;\mathbf{H}(\mathbb{R}^2))$.

By utilization of the diagonal method, we see that for each $m \in \mathbb{N}$,

$$\{\mathcal{L}(\psi_{K_n} u^{l(R_n^{(n)}),\frac{1}{R_n^{(n)}},R_n^{(n)}}|_{K_m})\}_{n\in\mathbb{N}}$$

is a convergent sequence in $L^2(0,T;\mathbf{H}(\Omega_{K_m}))$, hence in $L^2(0,T;\mathbf{H}(\mathbb{R}^2))$. We denote by Q its limit. Since $L^2(0,T;\mathbf{H}(\mathbb{R}^2))\times C([0,T];\mathbb{R}^2)$ is a complete separable metric space, by Skorohod's embedding theorem, there exist another probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ on which $L^2(0,T;\mathbf{H}(\mathbb{R}^2))$-valued random variables X_n, X and the two-dimensional Brownian motion W are defined in a such way that the distribution of X_n and X are equal to $\mathcal{L}(\psi_{K_n} u^{l(R_n^{(n)}),\frac{1}{R_n^{(n)}},R_n^{(n)}})$ and Q, respectively. In addition, $\tilde{P}$-a.s.,

$$X_n|_{K_m} \to X|_{K_m},$$

in $L^2(0,T;\mathbf{H}(\mathbb{R}^2))$ for every $m\in\mathbb{N}$ and therefore $X_n\to X$, $\tilde{P}$-a.s., in particular.

step 3

Let us choose $\phi\in\mathbf{C}^\infty_{0,\sigma}$. Then, there exists $n_0\in\mathbb{N}$ such that $\operatorname{supp}\phi\subset B_{n_0}\subset K_n\subset(-l(R_n^{(n)}),l(R_n^{(n)}))^2$ for $n\geq n_0$. Clearly

$$\psi_{K_n}\chi_n\phi=\phi, \tag{3.15}$$

hence

$$\int_{(-l(R_n^{(n)}),l(R_n^{(n)}))^2}\psi_{K_n}\chi_n\phi(x)\,dx=0,$$

and

$$\operatorname{div}(\psi_{K_n}\chi_n\phi)=0.$$

Now we set

$$\begin{aligned}M_n^\phi(t)=&\langle u^{l(R_n^{(n)}),\frac{1}{R_n^{(n)}}}(t),\phi\rangle_{l(R_n^{(n)})}-\langle u_0^{l(R_n^{(n)})},\phi\rangle_{l(R_n^{(n)})}\\&-\int_0^t\frac{2+R_n^{(n)}{}^{-1}}{2}\mu\langle u^{l(R_n^{(n)}),\frac{1}{R_n^{(n)}}}(s),\Delta\phi\rangle_{l(R_n^{(n)})}ds\\&-\int_0^t\langle(u^{l(R_n^{(n)}),\frac{1}{R_n^{(n)}}}(s)\cdot\nabla)\phi,u^{l(R_n^{(n)}),\frac{1}{R_n^{(n)}}}(s)\rangle_{l(R_n^{(n)})}ds.\end{aligned}\tag{3.16}$$

Since $u^{l(R_n^{(n)}),\frac{1}{R_n^{(n)}}}$ is a weak solution of (3.5), we see

$$M_n^\phi(t)=\sqrt{2\mu}\sum_{j=1}^2\int_0^t\langle u^{l(R_n^{(n)}),\frac{1}{R_n^{(n)}}}(s),\frac{\partial\phi}{\partial x_j}\rangle_{l(R_n^{(n)})}dB^j(s).\tag{3.17}$$

If we choose ϕ satisfying (3.15), since supp ϕ is contained in B_{n_0}, $u^{l(R_n^{(n)}),\frac{1}{R_n^{(n)}}}$ appearing in each integral of both (3.16) and (3.17) can be replaced $\psi_{K_n}\chi_n u^{l(R_n^{(n)}),\frac{1}{R_n^{(n)}}}$, that is, $\psi_{K_n} u^{l(R_n^{(n)}),\frac{1}{R_n^{(n)}},R_n^{(n)}}$. Let us recall again that $\mathcal{L}(\psi_{K_n} u^{l(R_n^{(n)}),\frac{1}{R_n^{(n)}},R_n^{(n)}})$ and $\mathcal{L}(X_n)$ are the same. Now we set

$$\begin{aligned}\tilde{M}_n^{\phi}(t) = & \langle X_n(t), \phi\rangle_{l(R_n^{(n)})} - \langle u_0^{l(R_n^{(n)})}, \phi\rangle_{l(R_n^{(n)})} \\ & - \int_0^t \frac{2+R_n^{(n)-1}}{2}\mu\langle X_n(s), \triangle\phi\rangle_{l(R_n^{(n)})} ds \\ & - \int_0^t \langle (X_n(s)\cdot\nabla)\phi, X_n(s)\rangle_{l(R_n^{(n)})} ds.\end{aligned} \tag{3.18}$$

Then, we have

$$\tilde{M}_n^{\phi}(t) = \sqrt{2\mu}\sum_{j=1}^{2}\int_0^t \langle X_n(s), \frac{\partial\phi}{\partial x_j}\rangle_{l(R_n^{(n)})} dW^j(s).$$

Let us define the following filtrations:

$$\begin{aligned}& \mathcal{F}_n^0(t) = \sigma(X_n(s), s\le t),\ \ \mathcal{F}_n(t) = \bigcap_{\epsilon>0}\mathcal{F}_n^0(t+\epsilon), \\ & \mathcal{N}_n = \left\{A\subset\tilde{\Omega} \mid A\subset B\in\mathcal{F}_n^0(\infty) \text{ such that } \tilde{P}(B)=0\right\}, \\ & \tilde{\mathcal{F}}_n(t) = \sigma(\mathcal{F}_n(t)\cup\mathcal{N}_n),\ \ \tilde{\mathcal{F}}^0(t) = \sigma(\bigcup_{n\ge1}\tilde{\mathcal{F}}_n(t)), \\ & \mathcal{N} = \left\{A\subset\tilde{\Omega} \mid A\subset B\in\tilde{\mathcal{F}}^0(\infty) \text{ such that } \tilde{P}(B)=0\right\}, \\ & \tilde{\mathcal{F}}(t) = \sigma(\tilde{\mathcal{F}}^0(t)\cup\mathcal{N}).\end{aligned}$$

Then, for $n_0\le m<n$ and any K_N containing supp ϕ,

$$\begin{aligned}& \mathbf{E}^{\tilde{P}}\left\{\sup_{t\in[0,T]}\left|\tilde{M}_m^{\phi}(t)-\tilde{M}_n^{\phi}(t)\right|^2\right\} \\ = & \mathbf{E}^{\tilde{P}}\left\{\sup_{t\in[0,T]}\left|\sqrt{2\mu}\int_0^t\sum_{j=1}^{2}\left\{\langle X_n(s),\frac{\partial\phi}{\partial x_j}\rangle_{l(R_n^{(n)})} - \langle X_m(s),\frac{\partial\phi}{\partial x_j}\rangle_{l(R_m^{(m)})}\right\} dW^j(s)\right|^2\right\} \\ = & \mathbf{E}^{\tilde{P}}\left\{\sup_{t\in[0,T]}\left|\sqrt{2\mu}\int_0^t\sum_{j=1}^{2}\left\{\int_{K_N}(X_n(s)-X_m(s))\cdot\frac{\partial\phi}{\partial x_j}dx\right\} dW^j(s)\right|^2\right\} \\ \le & 8\mu\sum_{j=1}^{2}\mathbf{E}^{\tilde{P}}\left\{\int_0^T\left|\int_{K_N}(X_n(s)-X_m(s))\cdot\frac{\partial\phi}{\partial x_j}dx\right|^2 ds\right\} \\ \to & 0, \quad \text{as } m,n\to\infty,\end{aligned}$$

where the last inequality comes from the fact that the sequence $\{X_n\}$ is compact in $L^2(0,T;\mathbf{H}_{K_N})$. This shows that $\{\tilde{M}_n^{\phi}\}_{n\in\mathrm{N}}$ is a Cauchy sequence in $L^2(\tilde{\Omega};C([0,T];\mathbb{R}^2))$. Hence, there exists a subsequence $\{\tilde{M}^{\phi}_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\tilde{M}^{\phi}_{n_k}=\tilde{M}^{\phi}$, uniformly in [0,T], $\tilde{P}$-a.s., for $T>0$. $\{\tilde{M}^{\phi}_{n}\}_{n\in\mathbb{N}}$ is a square martingale whose quadratic variation $\langle\langle\tilde{M}^{\phi}_n,\tilde{M}^{\phi}_n\rangle\rangle(t)$ is given by

$$\begin{aligned}\langle\langle\tilde{M}^{\phi}_n,\tilde{M}^{\phi}_n\rangle\rangle(t)&=2\mu\sum_{j=1}^{2}\int_0^t\langle X_n(s),\frac{\partial\phi}{\partial x_j}\rangle^2_{l(R_n^{(n)})}ds\\&=2\mu\sum_{j=1}^{2}\int_0^t\langle X_n(s),\frac{\partial\phi}{\partial x_j}\rangle^2 ds,\end{aligned}$$

since supp ϕ is contained in B_{n_0}. In addition, $\tilde{M}^{\phi}_n(t)$ is a continuous $\tilde{\mathcal{F}}(t)$-martingale. For any compact set K_N, we see $X_{n_k}|_{K_N}\to X|_{K_N}$ in $L^2(0,T;\mathbf{H}(\mathbb{R}^2))$, a.s. as $k\to\infty$. Hence, it follows that $\tilde{M}^{\phi}$ is also a square integrable continuous martingale whose quadratic variation $\langle\langle\tilde{M}^{\phi},\tilde{M}^{\phi}\rangle\rangle$ is given by

$$\langle\langle\tilde{M}^{\phi},\tilde{M}^{\phi}\rangle\rangle(t)=2\mu\sum_{j=1}^{2}\int_0^t\langle X(s),\frac{\partial\phi}{\partial x_j}\rangle^2 ds,$$

Thus, by the representation theorem of continuous martingales (see [11]), there exist another probability space with a filtration $(\tilde{\tilde{\Omega}},\tilde{\tilde{\mathcal{F}}},\tilde{\tilde{P}},(\tilde{\tilde{\mathcal{F}}}_t)_{t\geq 0})$ and two-dimensional $\tilde{\mathcal{F}}_t\times\tilde{\tilde{\mathcal{F}}}_t$-Brownian motion $\tilde{\tilde{B}}=(\tilde{\tilde{B}}^1,\tilde{\tilde{B}}^2)$ defined on $(\tilde{\Omega}\times\tilde{\tilde{\Omega}},\tilde{\mathcal{F}}\times\tilde{\tilde{\mathcal{F}}},\tilde{P}\times\tilde{\tilde{P}})$ such that $\tilde{P}\times\tilde{\tilde{P}}$-a.s.,

$$\tilde{M}^{\phi}(t,\tilde{\omega},\tilde{\tilde{\omega}})=\sqrt{2\mu}\sum_{j=1}^{2}\int_0^t\langle X(s,\tilde{\omega},\tilde{\tilde{\omega}}),\frac{\partial\phi}{\partial x_j}\rangle d\tilde{\tilde{B}}^j(s,\tilde{\omega},\tilde{\tilde{\omega}}),$$

holds for $t\in[0,T]$, where $\tilde{M}^{\phi}(t,\tilde{\omega},\tilde{\tilde{\omega}})=\tilde{M}^{\phi}(t,\tilde{\omega})$ and $X(s,\tilde{\omega},\tilde{\tilde{\omega}})=X(s,\tilde{\omega})$.

step 4

Note that the (R.H.S) of (3.18) can be rewritten to

$$\begin{aligned}&\langle X_n(t),\phi\rangle-\langle u_0^{l(R_n^{(n)})},\phi\rangle\\&-\int_0^t\frac{2+R_n^{(n)-1}}{2}\mu\langle X_n(s),\triangle\phi\rangle ds-\int_0^t\langle(X_n(s)\cdot\nabla)\phi,X_n(s)\rangle ds,\end{aligned}\tag{3.19}$$

for $n\geq n_0$ because of supp $\phi\subset B_{n_0}$. Now let us set (3.19) = (I) + (II) + (III) + (IV). Since $\tilde{P}$-a.s., thus $\tilde{P}\times\tilde{\tilde{P}}$-a.s., $X_n\to X$, in $L^2(0,T;\mathbf{H}_{\mathrm{loc}})$ holds by using (3.10), which means that (III) $\to-\int_0^t\mu\langle X(s),\triangle\phi\rangle ds$ and (IV) $\to-\int_0^t\langle(X(s)\cdot\nabla)\phi,X(s)\rangle ds$, $\tilde{P}\times\tilde{\tilde{P}}$-a.s., as $n\to\infty$. As for (I) and (II), by taking a subsequence (n'), we see $\tilde{P}$-a.s., thus $\tilde{P}\times\tilde{\tilde{P}}$-a.s., $X_{n'}\to X$, in $\mathbf{H}_{\mathrm{loc}}$ for a.e.-$t\in(0,T)$. Thus, (I) $\to\langle X(t),\phi\rangle$ for a.e.-$t\in(0,T)$ as $n'\to\infty$. Finally, since u_0 has a compact support, we see $u_0^{l(R_{n'}^{(n')})}\to u_0$, in $\mathbf{H}(\mathbb{R}^2)$. Thus, we have (II) $\to\langle u_0,\phi\rangle$. Thus, the 4th condition of Definition 3.2.1 is shown. The 3rd is shown already. The 1st is clear, that is, $X(t)$ is $\tilde{\mathcal{F}}\times\tilde{\tilde{\mathcal{F}}}$-adapted. The 2nd follows easily by using Fatou's lemma. Thus, the proof is complete.

3.4 Existence of weak solutions of (3.5)

In this section, we will give proofs about the a priori estimate (3.10) and Lemma 3.4.1 appearing in Theorem 3.2.1. Although the following lemma is similar to [12], we give the proof here for the reader's convenience.

lemma 3.4.1. *There exists a weak solution* $\{u^{l,\delta}, B(t)\}$ *of (3.5) on a probability space* $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geq 0})$. *Furthermore, we have* $\mathbf{P}$*-a.s.,*

$$u^{l,\delta} \in L^2(0,T;\mathbf{V}_{per}(l)) \cap L^\infty(0,T;\mathbf{H}_{per}(l)) \cap C([0,T];D(A_{l,\delta})'). \tag{3.20}$$

Proof. We will take several steps to prove this lemma.

step 1.

Let Π_n be the orthogonal projection onto the linear subspace spanned by $\{e_j^{(l)}\}_{|j|\leq n}$ Let us set $u_n^{l,\delta} = \Pi_n u^{l,\delta}$. Note that $u_n^{l,\delta}$ can be rewritten as a Fourier expansion with respect to $\{e_k^{(l)}\}_{k\in\mathbb{Z}_0^2}$, where $\mathbb{Z}_0^2 = \mathbb{Z}^2\backslash\{0\}$, that is, $u_n^{l,\delta} = \sum_{|k|\leq n} u_n^{l,\delta,k}(s)e_k^{(l)}$, where $u_n^{l,\delta,k}$ stands for the Fourier coefficient: $u_n^{l,\delta,k} = \langle u_n^{l,\delta}(s), e_k^{(l)}\rangle_l$. Let us set $u_0^{(l),j} = \langle \Pi_n u_0^{(l)}, e_j^{(l)}\rangle_l$, $u_n^{l,\delta,j}(t) = \langle u_n^{l,\delta}(t), e_j^{(l)}\rangle_l$. Now let us consider the following finite dimensional simultaneous stochastic integral equations:

$$\begin{aligned} u_n^{l,\delta,j}(t) = u_0^{(l),j} &+ \int_0^t F_j(u_n^{l,\delta,1}(s),\cdots,u_n^{l,\delta,n}(s))ds \\ &+ \int_0^t \sigma_j(u_n^{l,\delta,1}(s),\cdots,u_n^{l,\delta,n}(s))dB(s), \quad j=1,\cdots,n, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} F_j(u^1,\cdots,u^n) &= \langle -A_{l,\delta}e_j^{(l)} + \Pi_n B(\sum_{|k|\leq n} u^k e_k^{(l)}, e_j^{(l)}), \sum_{|k|\leq n} u^k e_k^{(l)}\rangle_l \\ \sigma_j(u^1,\cdots,u^n) &= -(\Pi_n G \sum_{|k|\leq n} u^k e_k^{(l)})^* e_j^{(l)}, \end{aligned}$$

that is,

$$u(t) = u_0 + \int_0^t F(u(s))ds + \int_0^t \sigma(u(s))dB(s),$$

where $u(t) = (u_n^{l,\delta,1}(t),\cdots,u_n^{l,\delta,n}(t))$, $u_0 = (u_0^{(l),1},\cdots,u_0^{(l),n})$, $F(u) = (F_1(u),\cdots,F_n(u))'$ and $\sigma(u) = (\sigma_1(u),\cdots,\sigma_n(u))'$. Let us set

$$T_R = \begin{cases} \inf\{t; |u(t)| \geq R\}, & \{\} \text{ is not empty,} \\ \infty, & \text{otherwise.} \end{cases}$$

Then,

$$\mathbf{E}\{|u(t\wedge T_R)|^2\} \leq |u_0|^2 + C\int_0^t \mathbf{E}\{|u(s\wedge T_R)|^2\}ds,$$

holds for some C independent of R. Thus, we have $\mathbf{E}\{|u(t)|^2\} < \infty$. Furthermore,

$$|F(u) - F(v)| \leq C_R|u - v|, \quad \text{for every } |u|, |v| \leq R,$$
$$|\sigma(u) - \sigma(v)| \leq C|u - v|,$$

holds for some constant $C, C_R > 0$. Therefore, (3.21) has a unique strong solution for each $\delta > 0$ and $n, l \in \mathbb{N}$.

step 2.

By applying Itô's formula to $|u_n^{l,\delta}(t)|_l^2$,

$$\sup_{n,l}\mathbf{E}\left\{\sup_{t\in[0,T]}|u_n^{l,\delta}(t)|_l^2 + \delta\mu\int_0^T ||u_n^{l,\delta}||_l^2\right\} < \infty. \tag{3.22}$$

In addition,

$$\sup_{n,l}\mathbf{E}\left\{\sup_{t\in[0,T]}|u_n^{l,\delta}(t)|_l^p\right\} < \infty, \quad \text{for } p \in (2, \frac{2+\delta}{2}]. \tag{3.23}$$

Indeed, by Itô's formula applied to $|u_n^{l,\delta}(t)|_l^p$, $p > 2$, it is easy to see that

$$\begin{aligned}
&|u_n^{l,\delta}(t)|_l^p \leq |\Pi_n u_0^{(l)}|_l^p \\
&+ \mu p\left(-\frac{2+\delta}{2} + p - 1\right)\int_0^t |u_n^{l,\delta}(s)|_l^{p-2}||u_n^{l,\delta}(s)||_l^2 ds \\
&+ p\int_0^t |u_n^{l,\delta}(s)|_l^{p-2}(\Pi_n G u_n^{l,\delta}(s))^* u_n^{l,\delta}(s)dB(s),
\end{aligned}$$

Then, $\sup_n \mathbf{E}\left\{|u_n^{l,\delta}(t)|_l^p\right\} < \infty$ holds if $\mathbf{E}\left\{|u_0^{(l)}|_l^p\right\} < \infty$ and

$$-\frac{2+\delta}{2} + p - 1 \leq 0, \quad \text{that is, } p \in [0, 2 + \frac{\delta}{2}].$$

Clearly, this condition ensures that

$$\sup_n \mathbf{E}\int_0^t |u_n^{l,\delta}(s)|_l^{p-2}||u_n^{l,\delta}(s)||_l^2 ds < \infty,$$

holds. Note that the following trivial inequality holds:

$$|Gu|^2_{\mathbf{L}_{\mathrm{H.S}}(\mathbb{R}^2;\mathbf{H}_{per})} \leq 2\mu||u||_l^2 + \lambda|u|_l^2, \quad u \in \mathbf{V}_{per}(l), \lambda > 0.$$

Then the stochastic term can be estimated as follows:

$$
\begin{aligned}
&\mathbf{E}\left\{\sup_{s\in[0,t]}\left|\int_0^s p|u_n^{l,\delta}(s')|_l^{p-2}(\Pi_n G u_n^{l,\delta}(s'))^* u_n^{l,\delta}(s')dB(s')\right|\right\}\\
\leq &C\mathbf{E}\left\{\left(\int_0^t p^2|u_n^{l,\delta}(s')|_l^{2p-4}|(\Pi_n G u_n^{l,\delta}(s'))^*|^2_{\mathbf{L_{H.S}}}|u_n^{l,\delta}(s')|_l^2 ds'\right)^{\frac{1}{2}}\right\}\\
\leq &C\mathbf{E}\left\{\left(\int_0^t p^2|u_n^{l,\delta}(s')|_l^{2p-2}\left(2\mu||u_n^{l,\delta}(s')||_l^2+\lambda|u_n^{l,\delta}(s')|_l^2\right)ds'\right)^{\frac{1}{2}}\right\}\\
\leq &C\mathbf{E}\{(\int_0^t(\sup_{s\in[0,t]}|u_n^{l,\delta}(s)|_l^p)(2\mu p^2||u_n^{l,\delta}(s')||_l^2|u_n^{l,\delta}(s')|_l^{p-2}\\
&+\lambda p^2(\sup_{\sigma\in[0,s']}|u_n^{l,\delta}(\sigma)|_l^p)ds')^{\frac{1}{2}}\}\\
\leq &\frac{1}{2}\mathbf{E}\left\{\sup_{s\in[0,t]}|u_n^{l,\delta}(s)|_l^p\right\}\\
&+C^2\mu p^2\mathbf{E}\left\{\int_0^t||u_n^{l,\delta}(s')||_l^2|u_n^{l,\delta}(s')|_l^{p-2}ds'\right\}\\
&+\frac{C^2}{2}\lambda p^2\int_0^t\mathbf{E}\left\{\sup_{\sigma\in[0,s']}|u_n^{l,\delta}(\sigma)|_l^p\right\}ds',
\end{aligned}
$$

where the first inequality comes from Burkholder's inequality. As a result, by Gronwall's lemma, (3.23) follows for $p\in(2,2+\frac{\delta}{2}]$.

In addition, the following estimate holds:

$$
\sup_{n\geq1,\delta\in(0,1]}\mathbf{E}^P\left\{||u_n^{l,\delta}||_{\mathbf{W}^{\alpha,2}(0,T;\mathbf{V}_{per}(l)')}\right\}<\infty,\quad \alpha\in(0,\frac{1}{2}),\tag{3.24}
$$

for $l\in\mathbb{N}$ and $T>0$. Indeed, let us set

$$
\begin{aligned}
u_n^{l,\delta}(t)=&\Pi_n u_0^{(l)}-\int_0^t A_{l,\delta}u_n^{l,\delta}(s)ds\\
&-\int_0^t\Pi_n B(u_n^{l,\delta}(s),u_n^{l,\delta}(s))ds-\int_0^t\Pi_n G u_n^{l,\delta}(s)\,dB_s\\
=&J_0^{l,n}+J_1^{n,l,\delta}(t)+J_2^{n,l,\delta}(t)+J_3^{n,l,\delta}(t).
\end{aligned}
$$

Then,

$$
\sup_{n\geq1}|J_0^{l,n}|_{\mathbf{H}}<\infty,\quad \sup_{n\geq1,\delta\in(0,1]}\mathbf{E}^{\mathbf{P}}\left\{||J_1^{n,l,\delta}||_{\mathbf{W}^{1,2}(0,T;\mathbf{V}_{per}(l)')}\right\}<\infty,\tag{3.25}
$$

$$
\sup_{n\geq1,\delta>0}\mathbf{E}^{\mathbf{P}}\left\{||J_2^{n,l,\delta}||_{\mathbf{W}^{1,2}(0,T;\mathbf{V}_{per}(l)')}\right\}<\infty,\tag{3.26}
$$

$$
\sup_{n\geq1,\delta>0}\mathbf{E}^{\mathbf{P}}\left\{||J_3^{n,l,\delta}||_{\mathbf{W}^{\alpha,2}(0,T;\mathbf{H}_{per}(l))}\right\}<\infty,\quad \alpha\in(0,\frac{1}{2}),\tag{3.27}
$$

hold (see [20]). These estimates (3.25), (3.26) and (3.27) imply (3.24). Thus, from (3.22) and (3.24), we see that $\{\mathcal{L}(u_n^{l,\delta})\}_n$ is tight in $L^2(0,T;\mathbf{H}_{per}(l))$.

On the other hand, as for $J_3^{n,l,\delta}$, we can choose some $\alpha \in (0,1)$, $p \geq 1$ satisfying $\alpha p > 1$ such that

$$\sup_{n\geq 1} \mathbf{E}^{\mathbf{P}} \left\{ ||J_3^{n,l,\delta}||_{\mathbf{W}^{\alpha,p}(0,T;\mathbf{V}_{per}(l)')} \right\} < \infty, \tag{3.28}$$

holds for $l \in \mathbb{N}$, $\delta > 0$ and $T > 0$. Indeed, (3.28) is true if we choose $p = 2 + \frac{\delta}{2}$ and $\alpha \in (\frac{1}{p}, \frac{1}{2})$. Therefore, (3.25), (3.26) and (3.28) imply that $\{\mathcal{L}(u_n^{l,\delta})\}_n$ is also tight in $C([0,T]; D(A_l)')$. Now that we have the tightness, the rest of the proof is similar to [12] and we can obtain a weak solution $u^{l,\delta}$. The proof is complete.

□

lemma 3.4.2. *The following estimate holds:*

$$\mathbf{E}^{\mathbf{P}^{l,\delta}} \left\{ ||u^{l,\delta}(t)||_l^2 \right\} \leq ||u_0^{(l)}||_l^2. \tag{3.29}$$

Proof. Here we use the same notations as introduced in the proof of Lemma 3.4.1. By applying Itô's formula to $\langle u_n^{l,\delta}(t), e_j^{(l)}\rangle_l^2$,

$$\begin{aligned} &\langle u_n^{l,\delta}(t), e_j^{(l)}\rangle_l^2 - \langle u_n^{l,\delta}(t), e_j^{(l)}\rangle_l^2 \\ &= (2+\delta)\mu \int_0^t \langle u_n^{l,\delta}(s), e_j^{(l)}\rangle_l \langle \Delta u_n^{l,\delta}(s), e_j^{(l)}\rangle_l ds \\ &\quad - 2\int_0^t \langle \Pi_n(u_n^{l,\delta}(s)\cdot \nabla u_n^{l,\delta}(s)), e_j^{(l)}\rangle_l \langle u_n^{l,\delta}(s), e_j^{(l)}\rangle_l ds + \text{martingale} \\ &\quad + 2\mu \int_0^t \langle e_j^{(l)}, \frac{\partial u_n^{l,\delta}(s)}{\partial x_1}\rangle_l^2 + \langle e_j^{(l)}, \frac{\partial u_n^{l,\delta}(s)}{\partial x_2}\rangle_l^2 ds, \end{aligned} \tag{3.30}$$

where $u_n^{l,\delta}(s) \in \mathbf{C}_{per,\sigma}^{\infty}(l)$ and we use the integration by parts in the last term. Let us multiply (3.30) by $(\lambda_j^{(l)}\mu^{-1})^2$,

$$\begin{aligned} &\langle\langle u_n^{l,\delta}(t), e_j^{(l)}\rangle\rangle_l^2 - \langle\langle u_n^{l,\delta}(t), e_j^{(l)}\rangle\rangle_l^2 \\ &= (2+\delta)\mu \int_0^t \langle\langle u_n^{l,\delta}(s), e_j^{(l)}\rangle\rangle_l \langle\langle \Delta u_n^{l,\delta}(s), e_j^{(l)}\rangle\rangle_l ds \\ &\quad - 2\int_0^t \langle\langle \Pi_n(u_n^{l,\delta}(s)\cdot \nabla u_n^{l,\delta}(s)), e_j^{(l)}\rangle\rangle_l \langle\langle u_n^{l,\delta}(s), e_j^{(l)}\rangle\rangle_l ds + \text{martingale} \\ &\quad + 2\mu \int_0^t \langle\langle e_j^{(l)}, \frac{\partial u_n^{l,\delta}(s)}{\partial x_1}\rangle\rangle_l^2 + \langle\langle e_j^{(l)}, \frac{\partial u_n^{l,\delta}(s)}{\partial x_2}\rangle\rangle_l^2 ds, \end{aligned} \tag{3.31}$$

holds. Since $\{(\mu\lambda_j^{(l)^{-1}})^{\frac{1}{2}} e_j^{(l)}\}_{j\in\mathbb{Z}_0^2}$ is orthonormal system in $\mathbf{V}_{per}(l)$, multiply (3.31) by $\mu\lambda_j^{(l)^{-1}}$, then sum from $j = 1$ to $|n|$, we have

$$\begin{aligned} & ||u_n^{l,\delta}(t)||_l^2 - ||u_n^{l,\delta}(0)||_l^2 \\ \leq & (2+\delta)\mu \int_0^t \langle\langle u_n^{l,\delta}(s), \triangle u_n^{l,\delta}(s)\rangle\rangle_l ds \\ & - 2\int_0^t \langle\langle \Pi_n(u_n^{l,\delta}(s)\cdot\nabla u_n^{l,\delta}(s)), u_n^{l,\delta}(s)\rangle\rangle_l ds + \text{martingale} \\ & + 2\mu \int_0^t ||\frac{\partial u_n^{l,\delta}(s)}{\partial x_1}||_l^2 + ||\frac{\partial u_n^{l,\delta}(s)}{\partial x_2}||_l^2 ds, \end{aligned} \tag{3.32}$$

Note that the integrand of the second term of (R.H.S.) is equal to $\langle\langle (u_n^{l,\delta}(s)\cdot\nabla u_n^{l,\delta}(s)), u_n^{l,\delta}(s)\rangle\rangle_l$. As for the first term of (R.H.S.), we have

$$\langle\langle u_n^{l,\delta}(s), \triangle u_n^{l,\delta}(s)\rangle\rangle_l = -||\frac{\partial u_n^{l,\delta}(s)}{\partial x_1}||_l^2 - ||\frac{\partial u_n^{l,\delta}(s)}{\partial x_2}||_l^2,$$

by using the integration by parts formula. On the other hand, we see

$$\langle\langle \Pi_n(u_n^{l,\delta}(s)\cdot\nabla u_n^{l,\delta}(s)), u_n^{l,\delta}(s)\rangle\rangle_l = 0. \tag{3.33}$$

Indeed, in the case of two dimensional torus, there exists a stream function $\phi(s)$ satisfying $u_n^{l,\delta}(s) = \nabla^{\perp}\phi(s)$. (3.33) is shown by using such ϕ (see also [20] Lemma 3.1 or [10] Proposition 6.3). However, it does not hold in the case of higher dimension in general. As a result,

$$\mathbf{E}^{P^{l,\delta}}\left\{\left\|u_n^{l,\delta}(t)\right\|_l^2\right\} \leq ||u_0^{(l)}||_l^2.$$

Thus, the conclusion is obtained easily (see also [20] Section 3). The proof is complete. □

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論文の内容の要旨

1. 論文題目: Two-dimensional stochastic Navier-Stokes equations derived from a certain variational problem
(ある変分問題から導かれる二次元確率ナビエ・ストークス方程式)

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オイラー方程式は粘性係数が 0 である流体の運動を記述する偏微分方程式として知られている。この方程式は、以下のような変分原理から導くことも可能である。$\mathbb{R}^n$ 上の体積を保存する微分同相写像に値を取る関数 $\Phi(t), t \in [0,1]$ に関し、次のような汎関数 (action functional), J:

$$J(\Phi) = \int_0^1 \int_{\mathbb{R}^n} \sum_{j=1}^{n} |\frac{\partial \Phi_j(t,x)}{\partial t}|^2 dx\, dt,$$

を考える。この汎関数 J の条件 $\Phi(0) = \Psi^0$ かつ $\Phi(1) = \Psi^1$ の下での下限を与える Φ を求める変分問題を考える時、その停留点 $\tilde{\Phi}(t)$ に対して、速度場 (velocity field) $u(t)$ を $u(t) := \dfrac{\partial \tilde{\Phi}(t)}{\partial t}$ で定義すれば、u がオイラー方程式を満たすことは V.I. Arnold ([1]) の結果としてよく知られている。一方、$\Phi(t)$ に、ある確率空間 $(\Omega, \mathcal{F}, P)$ で定義された n 次元ブラウン運動 $B(t,\omega) = (B^1(t,\omega), \cdots, B^n(t,\omega))$ の効果 $\sqrt{2\mu}B(t,\omega)$, $\mu > 0$ を付加した時の対応するランダムな変分問題については Inoue, Funaki ([2]) で議論された。[2] では、そのランダムな臨界点 $\tilde{\Phi}(t,x,\omega)$ から与えられるランダムな速度場 $u(t,x,\omega)$ は、非粘性ではなく粘性を持ったあるランダムな方程式を満たすと考えられ、実際、形式的には次のような確率ナビエ・ストークス方程式を満足すると述べている。

$$\begin{cases} \dfrac{\partial u^i}{\partial t} + \displaystyle\sum_{j=1}^{n} \left(u^j \frac{\partial u^i}{\partial x_j} + \sqrt{2\mu} \frac{\partial u^i}{\partial x_j} \dot{B}_t^j \right) - \mu \Delta u^i + \dfrac{\partial p}{\partial x_i} = 0, & t > 0,\ x \in \mathbb{R}^n, \quad i = 1, \cdots, n, \\ \operatorname{div} u = 0, & t > 0,\ x \in \mathbb{R}^n. \end{cases} \tag{1}$$

初期値 u_0 がランダムでないベクトル場で、かつ、$\mathbf{W}^{1,2}(\mathbb{R}^n; \mathbb{R}^n)$-値であるような確率ナビエ・ストークス方程式 (1) の初期値問題に対して、弱解 $u(t,x,\omega)$ が存在すると仮定すれば、$u(t,x,\omega)$ の平均値 $\langle u \rangle(t,x) = \displaystyle\int_\Omega u(t,x,\omega) P(d\omega)$ が次のようなレイノルズ (Reynolds) 方程式 (2) を満たすことは、平均の線型性と確率積分のマルチンゲール性に注意すれば容易に確かめられる。

$$\begin{cases} \dfrac{\partial \langle u \rangle}{\partial t} - \mu \triangle \langle u \rangle + (\langle u \rangle \cdot \nabla) \langle u \rangle + \nabla p = -\langle ((u - \langle u \rangle) \cdot \nabla)(u - \langle u \rangle) \rangle, & t > 0, x \in \mathbb{R}^n \\ \operatorname{div} \langle u \rangle = 0, & t > 0, x \in \mathbb{R}^n \end{cases} \tag{2}$$

しかしながら、(1) の弱解の存在については [2] で議論されていない。しかも方程式 (1) の特徴は、確率ナビエ・ストークス方程式の弱解の構成時に通常必要とされる強圧的条件 (coercivity condition) を満たしていないため、このタイプの方程式の弱解の存在についてはこれまでのところ知られていない。強圧的条件を満たす場合の弱解の存在については Flandoli, Gatarek([3]) の結果などよく知られている。本論文では強圧的条件が満たされていない (1) の弱解の構成を試みる。強圧的条件を満たさない 3 次元以上の場合は弱解の構成は難しい。しかしながら、2 次元の周期境界条件、および、2 次元全空間の場合には弱解が存在することを示す。弱解の一意性に関しては本論文では扱わない。本論文の構成は以下の通りである。

1. 第1章は序文である。

2. 第2章は2次元周期境界条件の場合の (1) の弱解の存在を証明する。

3. 第3章は Wilhelm Stannat 氏との共同研究であり、2次元全空間 $\mathbb{R}^2$ の場合の (1) を考察し、その弱解が存在することを示す。

1　2次元トーラス上の確率ナビエ・ストークス方程式の弱解の構成

2次元トーラス $\mathbb{T}^2$ において (1) の弱解を構成する。Hilbert 空間 $\mathbf{H}$ を

$$\mathbf{H} = \{u \in L^2(\mathbb{T}^2; \mathbb{R}^2) | \int_{\mathbb{T}^2} u(x)dx = 0,\ \mathrm{div}\ u = 0\},$$

、Sobolev 空間 $\mathbf{V}$ を $\mathbf{V} = \mathbf{W}^{1,2}(\mathbb{T}^2; \mathbb{R}^2) \cap \mathbf{H}$ と定義する。解の定義は、弱解、すなわち、適当な関数空間の試験関数との積分で表された weak form として定義され、かつ、確率空間 $(\Omega, \mathcal{F}, P)$ とその上に定義された $u(t)$ およびブラウン運動 $B(t)$ を求めることとする。主定理を述べる。

定理 1.1. 初期値 u_0 が $\mathbf{V}$ に属していれば、*(1)* の弱解が存在する。

証明は次のような4段階で進められる。

- 第1段階. Galerkin 近似による有限次元確率微分方程式の解の存在、一意性の証明
- 第2段階. アプリオリ評価の実施
- 第3段階. 確率分布の tightness の証明
- 第4段階. 極限以降による解の構築

2次元周期境界条件と限らない場合の方程式 (1) は強圧的条件を満足しない。そこで、粘性係数 μ を $\frac{2+\delta}{2}\mu$, $\delta > 0$ とした同条件を満たす修正された方程式を考えれば、第2段階で伊藤の公式により、$\delta > 0$ に関する $L^2(\Omega, L^2(0, T; \mathbf{H}))$ の一様評価のみ得ることができる。従って $L^2(\Omega, L^2(0, T; \mathbf{H}))$ における強収束部分列の存在を期待することはできず、第4段階で移流項 (convection term) の収束を論じるには上述の一様評価では弱すぎる。しかしながら、2次元周期境界条件の場合は、$\delta > 0$ に関して $L^2(\Omega, L^2(0, T; \mathbf{V}))$ での一様評価を得ることができ、その結果、解を構成することが可能である。

2　$\mathbb{R}^2$ 上の強圧的でない確率ナビエ・ストークス方程式の弱解

本章では、(1) を全空間 $\mathbb{R}^2$ で考える。弱解の定義は2次元周期境界条件の場合とほぼ同様であるが、試験関数のクラスは、発散ゼロであり、かつ、$\mathbb{R}^2$ 上での積分が 0 であるコンパクトな台をもつ $\mathbf{C}^\infty$ 級ベクトル場全体と定義する。ここで積分が 0 の条件は、弱解を構成する手法で必要とされるものである。理由は、周期 $2l, (l \in \mathbb{N})$ の周期境界条件をもつ方程式の弱解を考え、周期解に関する評価から $\mathbb{R}^2$ 全空間での弱解を構成する手法を取っているためである。Hilbert 空間 $\mathbf{H}(\mathbb{R}^2)$ を、

$$\mathbf{H}(\mathbb{R}^2) = \left\{ u \in \mathbf{L}^2(\mathbb{R}^2; \mathbb{R}^2) |\ \mathrm{div}\ u = 0, \int_{\mathbb{R}} u dx = 0 \right\},$$

Sobolev 空間 $\mathbf{V}(\mathbb{R}^2)$ を $\mathbf{V}(\mathbb{R}^2) = \mathbf{W}^{1,2}(\mathbb{R}^2; \mathbb{R}^2) \cap \mathbf{H}(\mathbb{R}^2)$ と定義する。主定理を述べる。

定理 2.1. 初期値 u_0 がコンパクトな台をもち、かつ $\mathbf{V}(\mathbb{R}^2)$ に属していれば、*(1)* の弱解が存在する。

証明の手順は、定理 1.1 の方針に基本的に従うが、領域が $\mathbb{R}^2$ 全体でコンパクトでないため工夫を要する。周期 $2l$, $l \in \mathbb{N}$ の周期境界条件を持つ方程式の弱解を考える。さらに、半径 $R \in \mathbb{N}$ の球 $B_R = \{x \in \mathbb{R}^2 | |x| \leq R\}$ 上では 1、B_{2R} の外側では恒等的に 0 となる $[0, 1]$-値 $\mathbf{C}^\infty$-級 cutoff 関数 χ_R を上述の弱解に作用させる。2次元周期境界条件の場合の手法を利用することで、cutoff された弱解の $L^2(\Omega; L^2(0, T; \mathbf{V}(\mathbb{R}^2)))$ ノルムの2乗が、初期値 u_0 の $\mathbf{V}(\mathbb{R}^2)$ ノルムの2乗で上から評価できる。我々の目的とする弱解は、l および R の極限を取り構成できる。

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