# Studies on the geometry of Mori dream spaces
# (森夢空間の幾何学に関する研究)

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# Preface

This thesis consists of four chapters. All of them are about the results of the author on the geometry of Mori dream spaces. The first chapter deals with foundational matters, and the other three chapters deal with more specific topics.

The second chapter is devoted to the study of projective varieties which admits a surjective morphism from a Mori dream space. We show that such a variety is also a Mori dream space, and see that its geometry is strongly controlled by that of the source of the morphism.

In the third chapter we consider the global Okounkov body of Mori dream spaces. Global Okounkov body is a cone which encodes some asymptotic information of line bundles on a variety. The question, roughly, is if the cone is rational polyhedral or not for Mori dream spaces. We verify it for surfaces, and reduce the problem to more naive one on the geometry of Mori dream spaces.

In the final chapter, we prove that a projective surface of globally $F$-regular type is of Fano type. Along the proof the anti-canonical MMP, whose existence is a specific phenomena of Mori dream spaces, plays a central role.

The first and the second chapters originates from [Ok1]. The third chapter is from [Ok2], and the last chapter is from [Ok3].

## Conventions

Unless otherwise stated, every variety is assumed to be projective over a field $k$, normal and geometrically connected. For the notations and terminologies of Mori dream spaces and Cox rings we follow [HK], and for those of (V)GIT we follow [DH] and [GIT].

# Acknowledgements

The author would like to express his deepest gratitude to his advisor Professor Yujiro Kawamata, under whose supervision the author has studied algebraic geometry for five years. The author appreciate his continuous warm encouragement, many suggestions, both technical ones and those on the attitude toward mathematics. Throughout the PhD course, he also learned many things from Professors Frédéric Campana, Osamu Fujino, Noboru Hara, Yongnam Lee, Yasunari Nagai, Yoshinori Namikawa, Keiji Oguiso, Hiromichi Takagi, Shunsuke Takagi, Yukinobu Toda, and Kei-ichi Watanabe. He is also indebted to them.

Besides he would like to thank his colleagues, Doctors Yoshinori Gongyo, Tomoyuki Hisamoto, Atsushi Ito, Atsushi Kanazawa, Kotaro Kawatani, Shouhei Ma, Makoto Miura, Shin-ichi Matsumura, Akiyoshi Sannai, Taro Sano, Kiwamu Watanabe, Kazunori Yasutake for uncountably many stimulating discussions and suggestions. Thanks also to the members of graduate students' room 457, especially to Yoshihiko Matsumoto, for comfortable and cheerful atmosphere.

Concerning the research of Chapter 2, the author would like to thank Doctor Tadakazu Sawada for informing him of the paper [RS] and of examples of quotients by rational vector fields, and Professor Young-Hoon Kiem for his nice question.

Concerning the research of Chapter 3, the author would like to thank Professor Hajime Kaji for giving him the opportunity to give a talk on the research at the Miyako-no-Seihoku Algebraic Geometry Symposium. Chapter 3 is based on the article which the author submitted to the proceedings of the symposium. He would also like to thank Doctor Dave Anderson for kindly answering his questions.

Concerning the research of Chapter 4, the author is indebted to Professor Shunsuke Takagi for informing him of [HWY, Lemma 2.1] and for many useful discussions. He would also like to thank Professor Osamu Fujino for answering his question concerning Remark 4.2.5.

During the research of [Ok1], which was the author's master thesis, a suggestion from Professor Shunsuke Takagi was crucial. He would like to thank him about it. He would also like to thank Professor Yongnam Lee for kindly answering his questions.

He is indebted to Doctor Kotaro Kawatani concerning the research of [Ok2].

He learned many things through the discussion with the co-authors of [GOST], namely Gongyo, Sannai, and Professor Takagi, during the preparation. He is also indebted to them.

Throughout the doctor course, the author was supported by the Grant-in-Aid for Scientific Research (KAKENHI No. 22-849) and the Grant-in-Aid for JSPS fellows.

Last but not least, he would like to thank his family for their invaluable support, kindly allowing him to become a mathematician, and trusting him.

January 10th, 2012

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# 1

# Basics on Mori dream spaces

## 1.1 Summary

The definition of Mori dream spaces and Cox rings were given in the paper [HK]. In the same paper, basic facts on Mori dream spaces were established. The purpose of this chapter is to give a review of their results, together with some refinements.

First we recall the definition of Mori dream spaces and the properties of line bundles on them, with an emphasis on the Zariski decompositions. Moreover we introduce a certain fan structure on the effective cone of a Mori dream space, which encodes the information of the Zariski decompositions (Definition-Proposition 1.2.8). To be precise, we say that two line bundles on a Mori dream space are strongly Mori equivalent if the negative parts of their Zariski decompositions have the same support and the positive parts define the same rational map (see Definition 1.2.11). With this notion, we prove that two line bundles on a Mori dream space are strongly Mori equivalent if and only if they are contained in the relative interior of the same cone of the fan (see §1.2.3). These are the contents of Section 1.2.

In Section 1.3 we recall the basics on the VGIT for actions of algebraic tori on affine varieties. This will be applied to the VGIT of Cox rings of Mori dream spaces, on which the dual tori of the Picard groups naturally act. In such a situation, a line bundle on the Mori dream space corresponds to a character of the tori and vice versa (modulo multiplication, in general). With these preparations we show that the information of the Zariski decomposition of a line bundle is equivalent to the information of the semi-stable locus defined by the character of the line bundle. In particular we prove in §1.3.2 that two line bundles are strongly Mori equivalent if and only if the corresponding characters have the same semi-stable loci.

In the paper [HK], they introduced a fan structure on the movable cone of a Mori dream space. Our fan structure on the effective cone extends that one. In [HK] they only considered the relative interiors of the maximal dimensional cones, but the basic ideas needed for the proof of our refined results implicitly appears in the paper. For toric varieties, our fan structure was classically known as the Gelfand-Kapranov-Zelevinsky decomposition introduced by Oda and Park [OP]. After the author finished the first version of the draft, he found that Professor Jürgen Hausen introduced in [H] the notion of GIT fan, starting from the VGIT of Cox rings. This seems to coincide with the fan structure which we introduce in this paper, which is defined by starting from the geometry of line bundles.

## 1.2 Definition and basic properties of Mori dream space

### 1.2.1 Reminder

In this section, we briefly recall the definition and some of the basic properties of Mori dream space which we need in this paper. For details, see [HK]. We follow the terminologies of the paper.

**Definition 1.2.1.** Let $X$ be a normal projective variety. A small $\mathbb{Q}$-factorial modification of $X$ is a small (i.e. isomorphic in codimension one) birational map $f : X \dashrightarrow Y$ to another normal $\mathbb{Q}$-factorial projective variety $Y$.

**Definition 1.2.2.** A normal projective variety $X$ is called a Mori dream space provided that the following conditions hold:

1. $X$ is $\mathbb{Q}$-factorial, $\mathrm{Pic}\,(X)$ is finitely generated and $\mathrm{Pic}\,(X)_{\mathbb{Q}} \simeq \mathrm{N}^1\,(X)_{\mathbb{Q}}$ holds.
2. $\mathrm{Nef}\,(X)$ is the affine hull of finitely many semi-ample line bundles.
3. There is a finite collection of small $\mathbb{Q}$-factorial modifications $f_i : X \dashrightarrow X_i$ such that each $X_i$ satisfies (1)(2) and $\mathrm{Mov}\,(X)$ is the union of the $f_i^*(\mathrm{Nef}\,(X_i))$.

**Remark 1.2.3.** The author is not sure if the assumption that the morphism $\mathrm{Pic}\,(X)_{\mathbb{Q}} \to \mathrm{N}^1\,(X)_{\mathbb{Q}}$ is isomorphic follows from the finite generation of $\mathrm{Pic}\,(X)$. We note here that it is the case at least when the base field $k$ is algebraically closed with uncountably many elements. In fact, finite generation of $\mathrm{Pic}\,(X)$ implies that of $\mathrm{Pic}^0\,(X)$. If $k$ is uncountable, we see that

$\mathrm{Pic}^0_{X/k}$, the connected component of the identity of the Picard scheme of $X$, should be zero dimensional. Hence $\mathrm{Pic}^0(X)$ is a point. By [FGIKNV, Corollary 9.6.17], the finiteness of $\mathrm{Pic}^{tors}(X)$, the subgroup of torsion elements of $\mathrm{Pic}(X)$, follows from this. By [FGIKNV, Theorem 9.6.3] and [FGIKNV, Exercise 9.6.11], a numerically trivial line bundle should be torsion in this case. Thus we see that $\mathrm{Pic}(X)_{\mathbb{Q}} \to \mathrm{N}^1(X)_{\mathbb{Q}}$ is isomorphic.

When $k = \mathbb{C}$, the assumption $h^1(\mathcal{O}_X) = 0$ is equivalent to saying that $\mathrm{Pic}(X)$ is finitely generated. In general, we only have the inequality $\dim \mathrm{Pic}(X) \leq h^1(\mathcal{O}_X)$ (see [FGIKNV, Corollary 9.5.13 and Remark 9.5.15].)

Let $X$ be a normal projective variety satisfying the condition (1) of Definition 1.2.2. We start with recalling the definition of Mori chambers (see [HK, Definition 1.3 and 1.4]):

**Definition 1.2.4.** Let $D_1$ and $D_2$ be two $\mathbb{Q}$-Cartier divisors on $X$ with finitely generated section rings. Then we say $D_1$ and $D_2$ are Mori equivalent if the rational maps

$$\varphi_{D_i} : X \dashrightarrow \mathrm{Proj}\,(R_X(\mathcal{O}_X(D_i)))\ (i = 1, 2)$$

are isomorphic: i.e. if there is an isomorphism between their target spaces which makes the obvious triangular diagram commutative.

Note that the rational map $\varphi_{D_i}$ above is the same as the Iitaka fibration of $D_i$ (in the sense of [L1, Theorem 2.1.33]).

**Definition 1.2.5.** A Mori chamber of $X$ is the closure of a Mori equivalence class in $\mathrm{Pic}(X)_{\mathbb{R}}$ with non-empty interior.

[HK, Proposition 1.11 (2)] gives the natural decomposition of the effective cone of a Mori dream space into Mori chambers:

**Proposition 1.2.6.** *There are finitely many contracting birational maps $g_i : X \dashrightarrow Y_i$, with $Y_i$ a Mori dream space, such that*

$$\mathrm{Eff}\,(X) = \bigcup_i g_i^* \mathrm{SA}\,(Y_i) * \mathrm{ex}\,(g_i)$$

*gives a decomposition of the effective cone into closed rational polyhedral subcones with disjoint interiors. Each $g_i^* \mathrm{SA}\,(Y_i) * \mathrm{ex}\,(g_i)$ is a Mori chamber of $X$.*

Above $\mathrm{ex}\,(g_i)$ denotes the cone spanned by the exceptional prime divisors of $g_i$, and $g_i^* \mathrm{SA}\,(Y_i) * \mathrm{ex}\,(g_i)$ denotes the join of the cones $g_i^* \mathrm{SA}\,(Y_i)$ and

$\operatorname{ex}(g_i)$. We use the notation $*$ to indicate that any element of the cone $g_i^* \operatorname{SA}(Y_i) * \operatorname{ex}(g_i)$ is written uniquely as the sum of the elements of the cones $g_i^* \operatorname{SA}(Y_i)$ and $\operatorname{ex}(g_i)$.

Here we point out some properties of the cone $\operatorname{ex}(g_i)$.

**Lemma 1.2.7.** *For any integral divisor $E \in \operatorname{ex}(g_i)$, $h^0(X, \mathcal{O}_X(E)) = 1$. Moreover $\operatorname{ex}(g_i)$ is simplicial and its extremal rays are those cones spanned by an exceptional prime divisor of $g_i$. In particular $N_1, N_2 \in \operatorname{ex}(g_i)$ has the same support if and only if they are contained in the relative interior of the same face of $\operatorname{ex}(g_i)$.*

*Proof.* For any $g_i$ exceptional effective divisor $E$, the natural map $\mathcal{O}_{Y_i} \to g_{i*}\mathcal{O}_X(E)$ is isomorphic. The first claim follows from this. The second and the third claims follow from the first one. □

### 1.2.2 The fan of Mori dream space

Next we introduce a fan structure on the effective cone of a Mori dream space:

**Definition-Proposition 1.2.8.** Let $X$ be a Mori dream space. The set of faces of Mori chambers of $X$ forms a fan whose support coincides with the effective cone of $X$. We denote it by $\operatorname{Fan}(X)$.

**Remark 1.2.9.** The fan structure on $\operatorname{Eff}(X)$ defined above is the extension of that on $\operatorname{Mov}(X)$ introduced in [HK, Proposition 1.11(3)].

*Proof.* All we have to show is that the intersection of two cones of $\operatorname{Fan}(X)$ is a face of each cone. Let $\sigma_1, \sigma_2 \in \operatorname{Fan}(X)$ be two cones. We show that $\sigma_1 \cap \sigma_2$ is a face of $\sigma_2$.

By the definition of a face, there exists classes of curves $\ell_i \in N_1(X)_\mathbb{R} \cong N^1(X)_\mathbb{R}^\vee$ $(i = 1, 2)$ such that

$$\mathcal{C}_i \subseteq \ell_i^{\geq 0} = \{D \in N^1(X)_\mathbb{R} | D.\ell_i \geq 0\}$$

and

$$\sigma_i = \mathcal{C}_i \cap \ell_i^\perp$$

holds for $i = 1, 2$.

Consider the following sequence of inclusions

$$\sigma_1 \cap \sigma_2 = (\mathcal{C}_1 \cap \mathcal{C}_2 \cap \ell_2^\perp) \cap \ell_1^\perp \subset \mathcal{C}_1 \cap \mathcal{C}_2 \cap \ell_2^\perp \subset \mathcal{C}_1 \cap \mathcal{C}_2 \subset \mathcal{C}_2.$$

From this we see that it is enough to show that $\mathcal{C}_1 \cap \mathcal{C}_2$ is a face of $\mathcal{C}_2$, since we know that a face of a face again is a face (see [Fu2, page 10(4)]).

Let $g_i : X \dashrightarrow Y_i$ ($i = 1, 2$) be the contracting birational map corresponding to $\mathcal{C}_i$. We know that $\mathcal{C}_i = \mathcal{P}_i * \mathcal{N}_i$, where $\mathcal{P}_i = g_i^* \operatorname{SA}(Y_i)$ and $\mathcal{N}_i = \operatorname{ex}(g_i)$.

We divide the proof into the following claims:

**Claim.** *1.* $\mathcal{C}_1 \cap \mathcal{C}_2 = (\mathcal{P}_1 \cap \mathcal{P}_2) * (\mathcal{N}_1 \cap \mathcal{N}_2)$.

*2.* $\mathcal{P}_1 \cap \mathcal{P}_2$ *is a face of* $\mathcal{P}_2$.

*3.* $\mathcal{N}_1 \cap \mathcal{N}_2$ *is a face of* $\mathcal{N}_2$.

*4.* $\mathcal{C}_1 \cap \mathcal{C}_2$ *is a face of* $\mathcal{C}_2$.

*Proof.* (4) follows from (1)(2)(3). (1) follows from the uniqueness of the Zariski decomposition of line bundles on a Mori dream space (see Proposition 1.2.10). (2) is stated in [HK, Proposition 1.11(3)]. We check (3). Let $A = \sum a_i E_i$ ($a_i \geq 0$) and $B = \sum b_i E_i$ ($b_i \geq 0$) be two elements of $\mathcal{N}_2$ such that $A + B \in \mathcal{N}_1$. By Lemma 1.2.7 $h^0(X, \mathcal{O}(E)) = 1$ holds for any $E \in \mathcal{N}_1$, which means that $\operatorname{Supp}(A + B) \subset \operatorname{Ex}(g_1)$. Hence we see $\operatorname{Supp}(A), \operatorname{Supp}(B) \subset \operatorname{Ex}(g_1)$, concluding the proof. □

□

### 1.2.3 Zariski decompositions and the fan

Next we give an explicit description for the Zariski decompositions (in the sense of Cutkosky-Kawamata-Moriwaki) of line bundles on a Mori dream space, which turns out to characterize Mori dream spaces[1]:

**Proposition 1.2.10.** *Let $X$ be a Mori dream space. Consider the decomposition of* $\operatorname{Eff}(X)$ *into the Mori chambers as in Proposition 1.2.6:*

$$\operatorname{Eff}(X) = \bigcup_{\text{finite}} \mathcal{C}.$$

*Then for each chamber $\mathcal{C}$ there exists a small $\mathbb{Q}$-factorial modification $f_i : X \dashrightarrow X_i$ of $X$ and two $\mathbb{Q}$-linear maps*

$$P, N : \mathcal{C} \to \operatorname{Eff}(X)$$

*such that for any $\mathbb{Z}$-divisor $D \in \mathcal{C}$, $D \sim_{\mathbb{Q}} P(D) + N(D)$ gives the Zariski decomposition of $D$ as a divisor on $X_i$; i.e.*

[1] the author would like to thank Professor Y.-H. Kiem for asking him if it could be the case.

- $P(D) \in \mathrm{SA}\,(X_i)$.
- $N(D) \geq 0$.
- *The natural map*

$$H^0(X, \mathcal{O}_X(mP(D))) \to H^0(X, \mathcal{O}_X(mD)), \tag{1.1}$$

*which is defined by the multiplication of a non-zero global section of the line bundle $\mathcal{O}_X(mN(D))$ is isomorphic for every sufficiently divisible positive integer $m$.*

*Zariski decomposition of $D$ is unique up to $\mathbb{Q}$-linear equivalence.*

*Conversely a normal projective variety satisfying Definition 1.2.2 (1) and having a decomposition of its effective cone into finitely many chambers $\mathcal{C}$ on which Zariski decompositions are $\mathbb{Q}$-linear actually is a Mori dream space.*

*Proof.* Let $\mathcal{C}$ be a Mori chamber. Then we have a birational contraction $g_i : X \dashrightarrow Y_i$ to another Mori dream space $Y_i$. We can replace $X$ with one of its SQMs so that $g_i$ becomes a morphism by (3) of Definition 1.2.2. Now we define the maps $P, N$ as follows:

- $P(D) = g_i^* g_{i*} D$.
- $N(D) = D - P(D)$.

By Lemma 1.2.7 $h^0(X, \mathcal{O}_X(mN(D))) = 1$ holds for all sufficiently divisible positive integer $m$. Thus the map (1.1) is uniquely defined up to constant. When $m$ is sufficiently divisible so that $mP(D)$ is a $\mathbb{Z}$-divisor, it is easy to see that this map has the required properties.

The uniqueness of the Zariski decomposition follows from the fact that the positive parts are movable.

The last statement can be shown by checking the finite generation of a Cox ring via exactly the same argument as in Lemma 2.2.2.

□

Now we define a stronger version of the Mori equivalence relation, which is closely related to the fan of Mori dream spaces defined above:

**Definition 1.2.11.** Let $X$ be a Mori dream space. Two line bundles $L, M$ are said to be strongly Mori equivalent if they are Mori equivalent and

$$\mathrm{Supp}\,(N(L)) = \mathrm{Supp}\,(N(M))$$

holds, where $N(L)$ (resp. $N(M)$) is the negative part of the Zariski decomposition of $L$ (resp. $M$).

The notion of strong Mori equivalence can be re-formulated in the following way.

**Lemma 1.2.12.** *Let $L$, $M$ be two line bundles on a Mori dream space $X$. $L$ and $M$ are strongly Mori equivalent if and only if they are Mori equivalent and $\mathbb{B}(L) = \mathbb{B}(M)$ holds, where $\mathbb{B}$ denotes the stable base locus of the line bundle.*

*Proof.* 'if' part is trivial, so we prove the 'only if' part.

Since $L$ and $M$ are Mori equivalent, the positive parts $P(L)$ and $P(M)$, which are movable, are the pull-backs of some ample divisors under the same contracting rational map $\varphi : X \dashrightarrow Y$. Therefore the stable base loci of $P(L)$ and $P(M)$ are the same as the locus of indeterminacy of the rational map $\varphi$. Now note that the stable base locus of $L$ is the union of the support of $N(L)$ and the stable base locus of $P(L)$. The same thing holds for $M$, concluding the proof. □

Now we state the relationship between the notion of strong Mori equivalence and the fan structure of Mori dream spaces.

**Proposition 1.2.13.** *For a Mori dream space $X$, a strong Mori equivalence class coincides with the relative interior of a cone of* $\mathrm{Fan}(X)$ *and vice versa.*

*Proof.* Let $\mathcal{C}$ be a Mori chamber and let $\mathcal{C} = \mathcal{P} * \mathcal{N}$ be the Zariski decomposition of the chamber. By an elementary fact on convex cones, the join of a face of $\mathcal{P}$ with a face of $\mathcal{N}$ is a face of $\mathcal{C}$, and any face of $\mathcal{C}$ is of this form. Moreover if $C$ is a face of $\mathcal{C}$ and $C = P * N$ is the decomposition,

$$C^{relint} = P^{relint} * N^{relint}$$

holds.

Recall also that the relative interior of a face of $P$ is a strong Mori equivalence class. This follows from the fact that two semi-ample line bundles are Mori equivalent if and only if the set of curves contracted by the morphisms coincide.

The same thing also holds for $N$ by Lemma 1.2.7.

Now Proposition 1.2.13 follows immediately from these facts.

□

## 1.3 Mori dream space and GIT revisited

In this section, we re-establish the relation between the variation of GIT quotients (VGIT for short) of Cox rings and the geometry of Mori dream spaces.

To be precise, we give an explicit description for VGIT of actions of tori on affine varieties with torsion divisor class groups. After that we establish the correspondence between the fan structure defined in Definition-Proposition 1.2.8 , and the GIT chambers and cells. This refines a result of [HK], in which they thought of the correspondence between the GIT chambers and the Mori chambers. The results of this section will be used later in the proof of Theorem 2.1.2 via GIT.

**Remark 1.3.1.** In [HK] they quoted some results on VGIT from the paper [DH], in which every variety acted on by a reductive group is assumed to be proper. Since we have to deal with affine varieties defined by Cox rings, we need another version: namely, VGIT for affine varieties. One big difference from the proper case is the fact that for a 1-parameter subgroup $\lambda$ of the group and a point $x$ on the variety, the limit point $\lim_{t\to 0}\lambda(t)x$ does not exist in general. As a consequence, the wall defined by the point $x$ may not be a convex set, contrary to the proper case.

### 1.3.1 VGIT of torus actions on affine varieties

The purpose of this subsection is to give an explicit description for the VGIT of actions of algebraic tori on affine varieties. First we fix some notations.

Let $G$ be a reductive group acting on a normal affine variety $V$. Assume for simplicity that only finitely many elements of $G$ acts on $V$ trivially.

We denote by $\chi(G)$ the space of characters of $G$, and by $\chi_\bullet(G)$ the space of 1-parameter subgroups of $G$. Note that there is a natural pairing

$$\langle \chi, \lambda \rangle = n,$$

where $\chi \in \chi(G)$, $\lambda \in \chi_\bullet(G)$ and $(\chi \circ \lambda)(t) = t^n$.

In the rest of this paper, for a character $\chi \in \chi(G)$ we denote by $U_\chi := V^{ss}(L_\chi)$ the semi-stable locus of $V$ with respect to the linearization $L_\chi$. Here $L_\chi$ is the linearized line bundle on $V$ whose underlying line bundle is trivial and the action of $G$ is given by the formula

$$(g \cdot f)(x) = \chi(g) f(g^{-1}x)$$

for $g \in G$, $f \in \mathcal{O}_V$ and $x \in V$.

We denote by $q_\chi : U_\chi \to Q_\chi = U_\chi//G$ the quotient map.

Now we recall the numerical criterion of stability for affine varieties [K, Proposition 2.5]:

**Proposition 1.3.2.** *Let $G$, $V$ as above. Let $\chi \in \chi(G)$ be a character of $G$. Then*

1. *$x \in V$ is $L_\chi$ semi-stable $\iff$ $\langle \chi, \lambda \rangle \geq 0$ holds for any 1-parameter subgroup $\lambda \in \chi_\bullet(G) \setminus \{0\}$ such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists.*

2. *$x \in V$ is $L_\chi$ stable $\iff$ $\langle \chi, \lambda \rangle > 0$ holds for any 1-parameter subgroup $\lambda \in \chi_\bullet(G) \setminus \{0\}$ such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists.*

We rephrase this criterion in the case when $G$ is an algebraic torus $T$. Let $V \subset \mathbb{A}$ be a $T$-equivariant embedding of $V$ into an affine space. It is well-known that $\mathbb{A}$ admits a weight decomposition $\mathbb{A} = \bigoplus_{\chi \in \chi(T)} \mathbb{A}_\chi$, where $\mathbb{A}_\chi = \{x \in \mathbb{A} | g \cdot x = \chi(g)x \ \forall\ g \in T\}$.

Take a point $x \in V \subset \mathbb{A}$. According to the decomposition above, there exists a unique decomposition $x = \sum x_\chi$. Now since $V$ is closed in $\mathbb{A}$, we see

$$\begin{aligned} & \lim_{t\to 0} \lambda(t) \cdot x \ \text{ exists in V} \\ \iff & \lim_{t\to 0} \lambda(t) \cdot x \ \text{ exists in } \mathbb{A} \\ \iff & \langle \chi, \lambda \rangle \geq 0 \ \text{ holds for all } \chi \text{ such that } x_\chi \neq 0. \end{aligned}$$

Let $\mathrm{st}\,(x) = \{\chi \in \chi(T) | x_\chi \neq 0\} \subset \chi(T)$ be the state set of the point $x \in V$. Note that there are only finitely many possibility for the set $\mathrm{st}\,(x)$, since it is a subset of the finite set $\mathrm{st}\,(\mathbb{A}) = \{\chi \in \chi(T) | \mathbb{A}_\chi \neq 0\}$.

Denote by $\mathcal{D}_x \subset \chi(T)_\mathbb{R}$ the cone spanned by $\mathrm{st}\,(x)$. Then

**Proposition 1.3.3.** *For a character $\chi \in \chi(T)$,*

1. *$x \in V$ is semi-stable with respect to $L_\chi$ if and only if $\chi \in \mathcal{D}_x$ holds.*

2. *$x \in V$ is stable with respect to $L_\chi$ if and only if $\chi \in \mathcal{D}_x^\circ$, where $\mathcal{D}_x^\circ$ denotes the interior of the set $\mathcal{D}_x$ (possibly empty).*

*Proof.* This is almost tautological. From Proposition 1.3.2 and the argument above, $x \in V$ is semi-stable with respect to $\chi$ if and only if $\langle \chi, \lambda \rangle \geq 0$ holds for any 1-parameter subgroup $\lambda$ which is semi-positive definite on the cone $\mathcal{D}_x$: i.e. the set of such characters $\chi$ is the double dual cone of the cone $\mathcal{D}_x$. Since $\mathcal{D}_x$ is rational polyhedral, the double dual coincides with itself by [Fu2, (1) on page 9].

Stable case can be checked similarly. □

Set $C = C^T(V) = \bigcup_{x\in V} \mathcal{D}_x$. We define the following notions according to [DH].

**Definition 1.3.4.** A wall defined by $x \in V$ is the set $\partial\mathcal{D}_x$. A GIT chamber is a connected component of the set $C \setminus \bigcup_{x\in V} \partial\mathcal{D}_x$.

Two characters $\chi, \chi'$ are said to be wall equivalent if $V^{sss}(\chi) = V^{sss}(\chi')$ holds. A connected component of a wall equivalence class, which is not a chamber, is called a (GIT) cell.

**Definition 1.3.5.** Two characters $\chi, \chi'$ are said to be GIT equivalent if $V^{ss}(\chi) = V^{ss}(\chi')$ holds.

Via similar arguments as in [DH, Theorem 3.3.2] and [DH, Lemma 3.3.10], we can check the following

**Lemma 1.3.6.** *1. A GIT chamber is a GIT equivalence class.*

*2. For any GIT chamber $\mathcal{C}$,*

$$\mathcal{C} = \bigcap_{x \in V^s(\mathcal{C})} \mathcal{D}_x^\circ$$

*holds.*

*3. A cell is contained in a GIT equivalence class.*

### 1.3.2 Strong Mori equivalence = GIT equivalence

Let $X$ be a Mori dream space, and fix a Cox ring $R = R_X(\Gamma)$, where $\Gamma$ is a finitely generated group of Weil divisors as usual.

The purpose of this subsection is to show that the following three kinds of sets are the same:

- the relative interior of a cone of Fan $(X)$.
- a strong Mori equivalence class.
- a GIT equivalence class.

In the paper [HK], they proved this fact only for the cones of maximal dimension. We need the refined version as above for the two proofs of Theorem 2.1.2.

We first recall some basic facts about the relationship between Mori dream space and VGIT (see [HK] for detail).

**Reminder 1.3.7.** Set $V_X = \mathrm{Spec}\,(R_X(\Gamma))$. Recall that the torus $T_X := Hom_{gp}(\Gamma, k^*)$ acts naturally on $V_X$ as follows: for any element $g \in T$, a divisor $D \in \Gamma$ and $f \in H^0(X, \mathcal{O}_X(D))$, set

$$g \cdot f = g(D)f.$$

As stated in [HK, Theorem 2.3], we have a natural isomorphism

$$\psi : \chi(T_X)_{\mathbb{R}} \to \Gamma_{\mathbb{R}} \cong \mathrm{Pic}\,(X)_{\mathbb{R}}$$

in such a way that $D \in \Gamma$ corresponds to the character $ev_D : g \mapsto g(D)$.

For any $D \in \Gamma$, it is easy to see that $Q_{\mathrm{ev}_D} = \operatorname{Proj} R_X(\mathcal{O}_X(D))$. In particular, if we take an ample $A \in \Gamma$, $q_{\mathrm{ev}_A} = \mathrm{id}_X$ holds.

By the universal property of categorical quotients, we obtain the rational map

$$X = Q_{\mathrm{ev}_A} \dashrightarrow Q_{\mathrm{ev}_D},$$

and it is easy to see that this coincides with the rational map $\varphi_{\mathcal{O}_X(D)} : X \dashrightarrow \operatorname{Proj}(R_X(\mathcal{O}_X(D)))$.

Summing up, we obtain the following commutative diagram.

$$\begin{array}{ccccc}
V^{ss}(\mathrm{ev}_A) & \xleftarrow{\supset} & V^{ss}(\mathrm{ev}_A) \bigcap V^{ss}(\mathrm{ev}_D) & \xrightarrow{\subset} & V^{ss}(\mathrm{ev}_D) \\
\Big\downarrow {\scriptstyle /T \,|\, q_{\mathrm{ev}_A}} & & \Big\downarrow {\scriptstyle /T} & & \Big\downarrow {\scriptstyle //T \,|\, q_{\mathrm{ev}_D}} \\
V^{ss}(\mathrm{ev}_A)/T & \xleftarrow{\supset} & V^{ss}(\mathrm{ev}_A) \bigcap V^{ss}(\mathrm{ev}_D)/T & \longrightarrow & V^{ss}(\mathrm{ev}_D)//T \\
\Big\downarrow \cong & & & & \Big\downarrow \cong \\
X & & \overset{\varphi_D}{\dashrightarrow} & & \operatorname{Proj} R_X(\mathcal{O}_X(D))
\end{array}$$

In the diagram, $\subset$ denotes an open immersion and $/T$ (resp. $//T$) denotes a geometric (resp. categorical) quotient by the torus $T$.

Now we can state our main observation:

**Proposition 1.3.8.** *Two line bundles $L, M$ on $X$ are strongly Mori equivalent if and only if $U_{\mathrm{ev}_L} = U_{\mathrm{ev}_M}$ i.e. $\mathrm{ev}_L$ and $\mathrm{ev}_M$ are GIT equivalent. It is also equivalent to saying that the two line bundles are contained in the relative interior of the same cone of* Fan$(X)$.

*Proof.* The last line follows from Proposition 1.2.13. For the first line, the arguments in [HK, Proof of Theorem 2.3] literally works: in the proof they only proved that the relative interiors of the Mori chambers are identified (via $\psi$) with the GIT chambers, but the arguments can be applied more generally to arbitrary strong Mori equivalence classes.

We only sketch the proof (see [HK, Proof of Theorem 2.3] for detail).

Fix a character $\chi$ which corresponds to an ample line bundle on $X$. For an arbitrary character $y \in C^T(V) \cap \chi(T)$, let $\psi(y) = P + N$ be the Zariski decomposition of the corresponding $\mathbb{Q}$-line bundle.

Then we can show that

$$U_\chi \setminus U_y = q_\chi^{-1}(\operatorname{Supp}(N))$$

holds in codimension one.

This follows from the following equality

$$H^0(X, \mathcal{O}_X(m\psi(y))) = H^0(U_\chi, L_y^{\otimes m})^T,$$

which is the same equation as (2.3.2) in [HK, Proof of Theorem 2.3], as pointed out there. From this we can immediately conclude that GIT equivalence implies the strong Mori equivalence. Conversely if we assume the strong Mori equivalence of $\psi(y)$ and $\psi(z)$ for two characters $y$ and $z$, then we see that $Q_y = Q_z$ and that $U_y$ and $U_z$ coincide in codimension one. The rest of the arguments is precisely the same as in [HK, Proof of Theorem 2.3]. □

**Corollary 1.3.9.** *Any GIT equivalence class is contained in a GIT cell. Combined with (3) of Lemma 1.3.6, this means that a GIT equivalence class and a cell are the same thing in this case.*

*Proof.* Take $\sigma \in \text{Fan}(X)$. If $\sigma^{relint}$ is not contained in a cell, the stable loci are not constant on it: i.e. there exists a point $x \in V_X$ such that $\mathcal{D}_x^\circ \cap \sigma^{relint} \neq \emptyset$ but $\sigma^{relint} \not\subset \mathcal{D}_x^\circ$. Since $\mathcal{D}_x$ and $\sigma$ are rational polyhedral cones, this means $\sigma^{relint} \not\subset \mathcal{D}_x$, contradicting the fact that $\sigma^{relint}$ is a GIT equivalence class. □

# 2

# Images of Mori dream spaces

## 2.1 Summary

The purpose of this chapter is to study the geometry of the images of morphisms from Mori dream spaces. We rely on the ring-theoretic viewpoint of the geometry of Mori dream spaces, and from this perspective what we do is the study of the behavior of multi-section rings under surjective morphisms.

The first result is the following

**Theorem 2.1.1.** *Let $X, Y$ be normal $\mathbb{Q}$-factorial projective varieties, and $f : X \to Y$ be a surjective morphism. If $X$ is a Mori dream space, then so is $Y$.*

We give an application of Theorem 2.1.1. When $\operatorname{char} k = 0$, it is known (e.g. see [FG2, Corollary 5.2]) that the image of a variety of Fano type again is a variety of Fano type. Recall that a variety of Fano type is a Mori dream space at least in characteristic zero ([BCHM, Corollary 1.3.2]).

Theorem 2.1.1 can be used in another proof of this result. In [GOST, Theorem 1.2], it was shown that a Mori dream space of globally $F$-regular type is a variety of Fano type and vice versa. Since the property of being of globally $F$-regular type is also preserved under surjective morphisms, by using Theorem 2.1.1, we can avoid the arguments in [FG1] which uses the theory of variation of Hodge structures (see [GOST] for detail).

We also expect that Theorem 2.1.1 will be useful to construct new examples of Mori dream spaces (see Example 2.7.2).

Here we explain the structure of the proof. Theorem 2.1.1 will be proven by checking that a Cox ring of $Y$ is of finite type over the base field (see Fact 2.4.3). We deduce it from the finite generation of a Cox ring of $X$. By taking the Stein factorization of $f$, the proof is divided into two parts: the case when $f$ is an algebraic fiber space (§2.2) and the case when $f$ is a finite morphism (§2.3). A finite morphism will be further decomposed into the separable part and the purely inseparable part, and treated independently (but with somewhat similar ideas). Combining them, Theorem 2.1.1 will be proven in §2.4.

In [B], it was shown that the projective GIT quotient of an invariant open subset of a Mori dream space by an action of a reductive group is a Mori dream space. Our proof of Theorem 2.1.1 for finite morphisms seems to have something in common with the arguments in [B].

Now let $f : X \to Y$ be a surjective morphism between Mori dream spaces. We denote the fan of $X$ (resp. $Y$) by $\mathrm{Fan}\,(X)$ (resp. $\mathrm{Fan}\,(Y)$). We establish the comparison theorem of the fan of $Y$ with that of $X$.

To see this, note that we can regard $\mathrm{Pic}\,(Y)_{\mathbb{R}}$ as a subspace of $\mathrm{Pic}\,(X)_{\mathbb{R}}$ via the natural injection

$$f^* : \mathrm{Pic}\,(Y)_{\mathbb{R}} \to \mathrm{Pic}\,(X)_{\mathbb{R}},$$

so that we can restrict $\mathrm{Fan}\,(X)$ to the fan on $\mathrm{Pic}\,(Y)_{\mathbb{R}}$ by intersecting the cones of $\mathrm{Fan}\,(X)$ with $\mathrm{Pic}\,(Y)_{\mathbb{R}}$. Then we have the following theorem.

**Theorem 2.1.2.** *With the same assumptions as in Theorem 2.1.1, the fan of $Y$ coincides with the restriction of the fan of $X$ to the subspace $\mathrm{Pic}\,(Y)_{\mathbb{R}} \subset \mathrm{Pic}\,(X)_{\mathbb{R}}$: i.e.*

$$\mathrm{Fan}\,(Y) = \mathrm{Fan}\,(X)|_{\mathrm{Pic}\,(Y)_{\mathbb{R}}}$$

*holds.*

See Example 2.7.1 for an illustration of Theorem 2.1.2.

We give two proofs to Theorem 2.1.2 corresponding to the two characterizations of the relative interiors of the cones explained above. These are treated respectively in §2.5 and §2.6. In both of the proofs, a result on the behavior of multi-section rings under finite morphisms (Theorem 2.3.1) is repeatedly used.

In the final section, we extend our results to $\mathbb{Q}$-factorial Mori dream spaces and Mori dream regions.

## 2.2 Finite generation of multi-section rings on Mori dream space

We first prepare the notation for multi-section rings.

**Definition 2.2.1.** Let $X$ be a normal variety with $H^0(X, \mathcal{O}_X) = k$. Let $\Gamma \subset \mathrm{Wdiv}\,(X)$ be a sub-semigroup of Weil divisors. The multi-section ring $R_X(\Gamma)$ associated to $\Gamma$ is the $\Gamma$-graded $k$-algebra defined by

$$R_X(\Gamma) = \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D)).$$

Similarly for a divisor $D$ on $X$, we define the section ring of $D$ by

$$R_X(\mathcal{O}_X(D)) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD)).$$

In this section we prove the finiteness of multi-section rings on a Mori dream space. This has first been proven in [B, Theorem 1.2] by using the finite generation theorem for invariant subrings. Our proof is based on the Zariski decompositions on Mori dream spaces, hence is more geometric.

**Lemma 2.2.2.** *Let $X$ be a Mori dream space. Let $\Gamma \subset \mathrm{Wdiv}\,(X)$ be a finitely generated group of Weil divisors. Then the multi-section ring $R_X(\Gamma)$ is of finite type over $k$. More generally, for any open subset $U \subset X$, $R_U(\Gamma|_U)$ is of finite type over $k$.*

*Proof.* We may assume that the natural map $\Gamma \to \mathrm{Div}\,(X)_{\mathbb{Q}}$ is injective. To see this we borrow some ideas from [B].

In general we can find a splitting $\Gamma = \Gamma_0 \oplus \Gamma_1$ such that $\Gamma_0$ coincides with the kernel of $\Gamma \to \mathrm{Div}\,(X)_{\mathbb{Q}}$. Then we see

$$R_X(\Gamma) \cong R_X(\Gamma_1)[\Gamma_0].$$

Note that $\Gamma_1$ maps injectively to $\mathrm{Pic}\,(X)_{\mathbb{Q}}$ and that $\Gamma_0$ is a finitely generated free abelian group. Thus we may assume $\Gamma = \Gamma_1$.

Let $\mathcal{C}$ be a Mori chamber. Set $\Gamma_{\mathcal{C}} = \Gamma \cap \mathcal{C}$. Note that it is a finitely generated semigroup. Let $g_i : X \dashrightarrow Y_i$ be the birational map corresponding to $\mathcal{C}$.

Recall from the proof of Proposition 1.2.10 that for all $D \in \Gamma_{\mathcal{C}}$

$$D = g_i^* g_{i*} D + (D - g_i^* g_{i*} D),$$

as an equality of $\mathbb{Q}$-divisors, gives the Zariski decomposition of $D$. Since $\Gamma$ is finitely generated and there are only finitely many Mori chambers, there exists a positive integer $m > 0$ such that for any Mori chamber $\mathcal{C}$ and $D \in (m\Gamma)_{\mathcal{C}}$, the positive and the negative parts of the above decomposition are both $\mathbb{Z}$-divisors.

We can replace $\Gamma$ with $m\Gamma$, since $R_X(m\Gamma) \subset R_X(\Gamma)$ is finite.

With these preparations, we can compute $R_X(\Gamma_{\mathcal{C}})$ as follows:

$$R_X(\Gamma_{\mathcal{C}}) \cong R_Y(P(\Gamma_{\mathcal{C}}))[N(\Gamma_{\mathcal{C}})].$$

We construct an isomorphism $\varphi$ from the LHS to the RHS. Choose $0 \neq s_D \in H^0(X, \mathcal{O}_X(D))$ for each $D \in N(\Gamma_{\mathcal{C}})$ such that $s_D \otimes s_{D'} = s_{D+D'}$ holds for all $D, D'$. Given $s \in H^0(X, \mathcal{O}_X(D))$, where $D \in \Gamma_{\mathcal{C}}$, set

$$\varphi(s) = s \otimes s_{N(D)}^{-1} \chi^{N(D)}.$$

Above $\chi^{N(D)}$ is the monomial corresponding to $N(D) \in N(\Gamma_{\mathcal{C}})$. Due to the property of the Zariski decompositions, $\varphi$ is an isomorphism.

Now since $P(\Gamma_{\mathcal{C}})$ is a finitely generated semigroup of semi-ample divisors, and $N(\Gamma_{\mathcal{C}})$ is a finitely generated semigroup, we see that $R_X(\Gamma_{\mathcal{C}})$ is of finite type over $k$ (see [HK, Lemma 2.8]).

Since there are only finitely many chambers, $R_X(\Gamma)$ itself is finitely generated over $k$ (by the union of finite sets of generators for each $R_X(\Gamma_{\mathcal{C}})$).

Finally, the conclusion for general open subsets $U$ follows from the case when $U = X$ (see the last two paragraphs of the proof of [B, Theorem 1.2]). Note that this is the only place we need the finite generation theorem for invariant subrings. □

## 2.3 Finite generation of multi-section rings under a finite morphism

In this section we prove that finite generation of multi-section rings is invariant under finite morphisms.

**Theorem 2.3.1.** *Let $f : X \to Y$ be a finite surjective morphism. Let $\Gamma \subset$ Wdiv $(Y)$ be a finitely generated semigroup of Weil divisors. Then the natural morphism of multi-section rings $R_Y(\Gamma) \subset R_X(f^*\Gamma)$ is finite if one of the rings is of finite type over $k$. Moreover, $R_Y(\Gamma)$ is of finite type over $k$ if and only if $R_X(f^*\Gamma)$ is.*

See Definition 2.2.1 for the definition of multi-section rings.

### 2.3.1 Preliminary for the proof of Theorem 2.3.1

In the proof of Theorem 2.3.1, we frequently use universal torsors. We prepare some notations here.

**Definition 2.3.2.** Let $\Gamma \subset \mathrm{Wdiv}(Y)$ be a finitely generated semigroup of Weil divisors. We set

$$\mathcal{S}_Y(\Gamma) = \bigoplus_{D \in \Gamma} \mathcal{O}_Y(D),$$

and call it the universal torsor associated to $\Gamma$.

**Remark 2.3.3.**

1. Note that
$$H^0(Y, \mathcal{S}_Y) = \bigoplus_{D \in \Gamma} H^0(Y, \mathcal{O}_Y(D)) = R_Y(\Gamma)$$
holds.

2. $R_Y(\Gamma)$ does not change if we replace $Y$ with the non-singular locus of $Y$. When $f : X \to Y$ is a finite morphism, we may assume that $X, Y$ are non-singular by removing suitable closed subsets of codimensions at least two. Hence we will freely assume that the varieties involved are non-singular.

3. Note that if we assume that $Y$ is non-singular, then $\mathcal{S}_Y$ is a flat $\mathcal{O}_Y$ module.

4. $R_X(f^*\Gamma) = H^0(Y, f_*\mathcal{S}_X(f^*\Gamma))$, and we can calculate
$$f_*\mathcal{S}_X(f^*\Gamma) = f_* \bigoplus_{D \in \Gamma} \mathcal{O}_X(f^*D) \cong \bigoplus_{D \in \Gamma} \mathcal{O}_Y(D) \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X = \mathcal{S}_Y(\Gamma) \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X.$$

Now we go back to the proof Theorem 2.3.1. 'if' part of the second claim follows from the first claim, so we prove the first claim of Theorem 2.3.1 and the 'only if' part of its second claim.

Let
$$X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y \tag{2.1}$$
be the decomposition of $f$ into the purely inseparable part $g$ and the separable part $h$ (i.e. $\tilde{Y}$ is the normalization of $Y$ in the separable closure of $k(Y)$ in $k(X)$.)

Therefore we may assume that $f$ is either purely inseparable or separable. We treat each case separately in the following two subsections, although the ideas are basically the same.

### 2.3.2 Purely inseparable case

Assume that $f$ is a purely inseparable morphism. We can divide the extension $k(Y) \subset k(X)$ into subextensions of degree $p$, so that we may assume that $\deg(f) = p$.

The key idea for this case is to describe $f$ as a "uniform geometric quotient" by an action of a rational vector field on $X$:

**Proposition 2.3.4.** *Let $f : X \to Y$ be a finite surjective morphism of degree $p$ between normal varieties. Then there exists a rational vector field on $X$ i.e. $\delta \in Der_{k(Y)}k(X)$ such that*

$$\mathcal{O}_Y = \mathcal{O}_X^{\delta} := \{f \in \mathcal{O}_X | \delta f = 0\}$$

*holds. Moreover this quotient is uniform; i.e. for a flat morphism $Z \to Y$, set $W = X \times_Y Z$ and $\delta_Z = \delta \otimes_{\mathcal{O}_Y} 1_{\mathcal{O}_Z}$. Then*

$$\mathcal{O}_Z = \mathcal{O}_W^{\delta_Z} \tag{2.2}$$

*holds.*

*Proof.* First half is well-known (see [RS, p. 1206]). We check the uniformity (2.2).

Consider the following sequence of $\mathcal{O}_Y$ modules, which is exact by definition:

$$0 \to \mathcal{O}_Y \to \mathcal{O}_X \xrightarrow{\delta} k(X) \tag{2.3}$$

(since $f$ is finite, we dropped $f_*$). Since $Z$ is flat over $Y$, by tensoring $\otimes_{\mathcal{O}_Y}\mathcal{O}_Z$ with (2.3) we obtain

$$0 \to \mathcal{O}_Z \to \mathcal{O}_W \xrightarrow{\delta_Z} k(X) \otimes_{\mathcal{O}_Y} \mathcal{O}_Z,$$

concluding the proof.

□

**Corollary 2.3.5.** *With the same notations as in Proposition 2.3.4, assume that $Z$ and $W$ are both normal varieties. Then*

$$H^0(W, \mathcal{O}_W)^p \subseteq H^0(Z, \mathcal{O}_Z) \subseteq H^0(W, \mathcal{O}_W)$$

*holds. In particular $H^0(W, \mathcal{O}_W)$ is of finite type over $k$ if and only if $H^0(Z, \mathcal{O}_Z)$ is, and in that case $H^0(Z, \mathcal{O}_Z) \subseteq H^0(W, \mathcal{O}_W)$ is finite.*

*Proof.* By Proposition 2.3.4, we see that $W \to Z$ is a quotient by the induced vector field $\delta_Z$ on $W$. Therefore

$$\mathcal{O}_Z = (\mathcal{O}_W)^{\delta_Z}$$

holds. Since a derivation kills the $p$-th powers of functions,

$$\mathcal{O}_W^p \subseteq (\mathcal{O}_W)^{\delta_Z} = \mathcal{O}_Z$$

holds. Taking $H^0$, we obtain the proof for the first line.

Next note that if $R$ is an algebra of finite type over $k$, then $R^p$ also is of finite type over $k$ and $R$ is a finitely generated module over $R^p$. 'only if' part of the second line follows from this. To see 'if' part, note that the extension $k(W) \subset k(Z)$ is finite and $Z$ is an integral extension of $W$. □

*Proof of Theorem 2.3.1 when $f$ is purely inseparable.*

Suppose that $f$ is purely inseparable. As mentioned before, we may assume that $\deg(f) = p$.

Note that we have a natural inclusion $\mathcal{S}_Y(\Gamma) \subset f_*\mathcal{S}_X(f^*\Gamma)$ of quasi-coherent sheaves of $\mathcal{O}_Y$ algebras, which in turn is the product of the natural map $\mathcal{O}_Y \subset f_*\mathcal{O}_X$ with $\mathrm{id}_{\mathcal{S}_Y(\Gamma)}$ (Remark 2.3.3 (4)).

By Remark 2.3.3 (2)(3), we may assume that $X, Y$ are non-singular and hence $\mathcal{S}_Y(\Gamma)$ is a torus bundle over $\mathcal{O}_Y$.

Therefore we can apply Corollary 2.3.5 for $f : X \to Y$ and the morphism $Z := \mathcal{S}pec_{\mathcal{O}_Y}\mathcal{S}_Y(\Gamma) \to Y$. Since $H^0(Z, \mathcal{O}_Z) = R_Y(\Gamma)$ and $H^0(W, \mathcal{O}_W) = R_X(f^*\Gamma)$ (see Remark 2.3.3 (1) and (4)), we obtain the desired conclusions. □

### 2.3.3 Separable case

Assume that $f$ is separable. This case is relatively easier; by passing to a Galois closure, we can describe $Y$ as a uniform geometric quotient by the Galois group, so that we can apply the finite generation theorem for invariant subrings.

*Proof of Theorem 2.3.1 when $f$ is separable.* Suppose that $f$ is separable. Let $k(W)$ be the Galois closure of $k(Y) \subset k(X)$, and let $W$ be the normalization of $X$ in $k(W)$. If we denote by $G$ the Galois group of $W/Y$, we see that $G$ acts on $W$ and $Y \cong W/G$. By removing suitable closed subsets, we assume that $X, Y, W$ are all non-singular.

Since $W/X$ also is Galois, it is the uniform geometric quotient of the action of $Gal(W/X)$ on $W$. Since $\mathcal{S}_X(\Gamma)$ is locally free on $X$, it is a flat $\mathcal{O}_X$

algebra, and hence we have

$$\mathcal{S}_X(\Gamma) = \mathcal{S}_W(\Gamma)^{Gal(W/X)}.$$

In particular we see that $R_X(\Gamma) = R_W(\Gamma)^{Gal(W/X)}$. Therefore $R_W(\Gamma)$ is an integral extension of $R_X(\Gamma)$.

Suppose that $R_X(\Gamma)$ is of finite type over $k$. By the finiteness theorem for integral closures, we see $R_W(\Gamma)$ also is of finite type over $k$.

Similarly, we can show that $R_Y(\Gamma) = R_W(\Gamma)^G$. Since $G$ is a finite group, we obtain the finiteness of $R_Y(\Gamma)$.

Finally, the finiteness of $R_Y(\Gamma) \subset R_X(f^*\Gamma)$ follows from these descriptions, concluding the proof. □

## 2.4 Proof of Theorem 2.1.1

We prove Theorem 2.1.1 by using the fact that the finite generation of the Cox ring characterizes Mori dream spaces.

First of all, we recall the definition of Cox rings.

**Definition 2.4.1.** Let $X$ be a normal projective variety with finitely generated $\mathrm{Cl}\,(X)$. Let $\Gamma \subset \mathrm{Wdiv}\,(X)$ be a finitely generated subgroup such that $\Gamma_{\mathbb{Q}}$ is naturally isomorphic to $\mathrm{Cl}\,(X)_{\mathbb{Q}}$. A Cox ring of $X$ is the multi-section ring

$$R_X(\Gamma) = \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D))$$

for such $\Gamma$.

**Remark 2.4.2.** Our definition of Cox rings depends on the choice of $\Gamma$, but the basic properties such as finite generation does not depend on the choice.

There is a canonical way to define Cox rings (see [ADHL, Section 4.2]), and the finite generation of the Cox ring in our sense is equivalent to the finite generation of their Cox ring. See [GOST, Remark 2.17] for detail.

The following is the characterization of Mori dream spaces via the finite generation of their Cox rings.

**Fact 2.4.3.** *[HK, Proposition 2.9] says that a normal projective variety satisfying Definition 1.2.2 (1) is a Mori dream space if and only if a Cox ring of the variety is of finite type over $k$.*

Now we go back to the proof of Theorem 2.1.1. First of all, we check the condition (1) of Definition 1.2.2 for $Y$;

**Lemma 2.4.4.** *Under the same assumption as Theorem 2.1.1,* $\mathrm{Pic}\,(Y)$ *is finitely generated and* $\mathrm{Pic}\,(Y)_{\mathbb{Q}} \simeq \mathrm{N}^1\,(Y)_{\mathbb{Q}}$ *holds.*

*Proof.* Let

$$X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y$$

be the Stein factorization of $f$. Let $L \in \mathrm{Pic}\,(Y)$ be a numerically trivial line bundle. Then $f^*L$ is also numerically trivial by the projection formula of intersection theory. By the assumption, there exists a positive integer $m$ such that $f^*L^{\otimes m} \cong \mathcal{O}_X$. By taking $g_*$ and using the projection formula, we see that $h^*L^{\otimes m} \cong \mathcal{O}_{\tilde{Y}}$. Now suppose $L \cong \mathcal{O}(D)$ for a Cartier divisor $D$. Since $h$ is a finite morphism, we have $h_*h^*D \sim \deg\,(h)D$. Hence $m\deg\,(h)D \sim h_*h^*mD \sim 0$. Thus we checked the second claim.

In order to show that the finite generation of $\mathrm{Pic}\,(Y)$, consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^{tors}\,(Y) & \longrightarrow & \mathrm{Pic}\,(Y) & \longrightarrow & \mathrm{Pic}^{free}\,(Y) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \\ 0 & \longrightarrow & \mathrm{Pic}^{tors}\,(X) & \longrightarrow & \mathrm{Pic}\,(X) & \longrightarrow & \mathrm{Pic}^{free}\,(X) & \longrightarrow & 0 \end{array}$$

In the diagram above, $\mathrm{Pic}^{tors}\,(X)$ is the torsion part of $\mathrm{Pic}\,(X)$ and $\mathrm{Pic}^{free}\,(X)$ is the quotient of $\mathrm{Pic}\,(X)$ by $\mathrm{Pic}^{tors}\,(X)$.

Using the similar arguments as above, we can show that the group homomorphism $f^* : \mathrm{Pic}^{free}\,(Y) \to \mathrm{Pic}^{free}\,(X)$ is injective. Hence we see that $\mathrm{Pic}^{free}\,(Y)$ is finitely generated.

Finally we see the finiteness of the torsion part. As we checked, there exists a non-zero constant $m$ such that for any numerically trivial line bundle $L$ on $Y$, $L^{\otimes m} \cong \mathcal{O}_Y$ holds. This means that $\mathrm{Pic}^0\,(Y)$ is contained in the subgroup of $m$-division points of $\mathrm{Pic}^0\,(Y)$, which is a finite set since $\mathrm{Pic}^0_{Y/k}$ with its reduced structure is an abelian variety (see [FGIKNV, Remark 9.5.25]). Together with [FGIKNV, Corollary 9.6.17], the finiteness of $\mathrm{Pic}^{tors}\,(Y)$ follows. □

**Remark 2.4.5.** Using similar arguments we can directly check the following lemma, which are worth noting.

For a surjective morphism $f : X \to Y$ between normal projective varieties, $f^* : \mathrm{Pic}\,(Y)_{\mathbb{R}} \to \mathrm{Pic}\,(X)_{\mathbb{R}}$ is injective. We regard $\mathrm{Pic}\,(Y)_{\mathbb{R}}$ as a subspace of $\mathrm{Pic}\,(X)_{\mathbb{R}}$ via the mapping $f^*$. Then

**Lemma 2.4.6.** *With the same assumptions as in Theorem 2.1.1, the following equalities hold:*

*1.* $\mathrm{Nef}(Y) = \mathrm{Nef}(X) \cap \mathrm{Pic}(Y)_{\mathbb{R}} = \mathrm{SA}(X) \cap \mathrm{Pic}(Y)_{\mathbb{R}} = \mathrm{SA}(Y)$.

*2.* $\mathrm{Eff}(Y) = \mathrm{Eff}(X) \cap \mathrm{Pic}(Y)_{\mathbb{R}}$.

Now we go back to the proof of Theorem 2.1.1.

*Proof of Theorem 2.1.1.* By Lemma 2.4.4 and Fact 2.4.3, it is enough to show the finiteness of a Cox ring of $Y$.

Set $\Gamma \subset \mathrm{Div}(Y)$ be a finitely generated subgroup of rank $\rho(Y)$ whose image in $\mathrm{Pic}(Y)$ is of finite index. By Lemma 2.2.2, we know that $R_X(f^*\Gamma)$ is of finite type over $k$. On the other hand

$$R_X(f^*\Gamma) \cong R_{\tilde{Y}}(h^*\Gamma)$$

holds, since $g$ is an algebraic fiber space. Since $h$ is finite, by Theorem 2.3.1 we see that $R_Y(\Gamma)$ is of finite type over $k$. □

## 2.5 Comparison of the fans -without GIT-

In this section we prove Theorem 2.1.2 via direct arguments. The problem is reduced to the following

**Theorem 2.5.1.** *Let $f : X \to Y$ be a surjective morphism between Mori dream spaces. Then two line bundles $L$ and $M$ on $Y$ are strongly Mori equivalent if and only if $f^*L$ and $f^*M$ are strongly Mori equivalent.*

See Definition 1.2.11 for the notion of strong Mori equivalence.

We first check that Theorem 2.1.2 actually follows from this.

*Proof of Theorem 2.1.2.* Take any $\sigma \in \mathrm{Fan}(Y)$. By Proposition 1.2.13 and Theorem 2.5.1, there exists a cone $\Sigma \in \mathrm{Fan}(X)$ such that

$$\sigma^{relint} = \Sigma^{relint} \cap \mathrm{Pic}(Y)_{\mathbb{R}}$$

holds. Since RHS is not empty, we can check

$$\sigma = \overline{(\Sigma^{relint} \cap \mathrm{Pic}(Y)_{\mathbb{R}})} = \Sigma \cap \mathrm{Pic}(Y)_{\mathbb{R}}.$$

Conversely, let $\Sigma \in \mathrm{Fan}(X)$ be a cone which intersects with $\mathrm{Pic}(Y)_{\mathbb{R}}$. Let $\Sigma'$ be the largest face of $\Sigma$ such that

$$\Sigma \cap \mathrm{Pic}(Y)_{\mathbb{R}} = \Sigma' \cap \mathrm{Pic}(Y)_{\mathbb{R}}$$

holds. Note that $\Sigma'^{relint} \cap \mathrm{Pic}(Y)_{\mathbb{R}} \neq \emptyset$ holds.

Again by Proposition 1.2.13 and Theorem 2.5.1, there exists a cone $\sigma \in$ Fan $(Y)$ such that

$$\Sigma'^{relint} \cap \operatorname{Pic}(Y)_{\mathbb{R}} = \sigma^{relint}.$$

Taking the closures, we obtain

$$\Sigma \cap \operatorname{Pic}(Y)_{\mathbb{R}} = \Sigma' \cap \operatorname{Pic}(Y)_{\mathbb{R}} = \sigma.$$

□

*Proof of Theorem 2.5.1.* Note first that the stable base loci of line bundles are compatible with pull-backs via surjective morphisms between projective varieties. Therefore, in view of Lemma 1.2.12, it is enough to show the following claim:

**Claim.** *$L, M$ are Mori equivalent if and only if $f^*L$ and $f^*M$ are Mori equivalent.*

Let

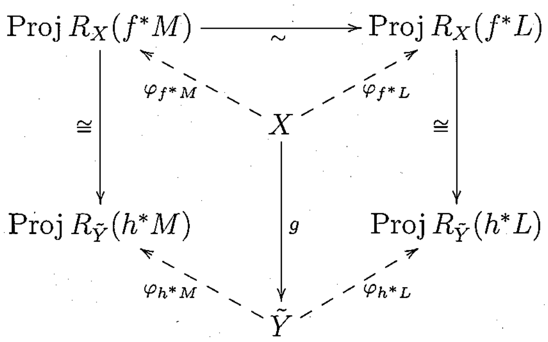
$$f^*L, f^*M \in \operatorname{Eff}(X) \cap f^* \operatorname{Pic}(Y)_{\mathbb{R}}$$

be Mori equivalent line bundles. We prove that $L, M$ are Mori equivalent.

First of all, take the Stein factorization of $f$:

$$X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y.$$

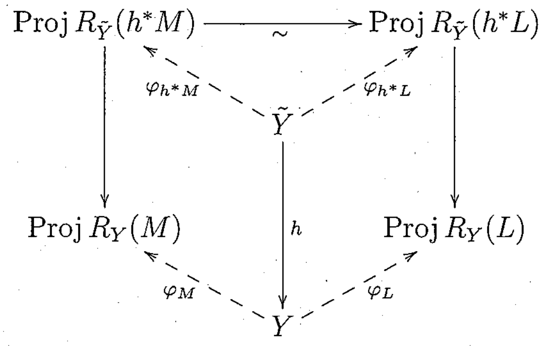
Consider the following diagram.




In the diagram above, the top horizontal arrow is an isomorphism which makes the upper triangle commutative, whose existence is guaranteed since $f^*L$ and $f^*M$ are Mori equivalent. Note that the two side vertical morphisms are isomorphisms, since $g$ is an algebraic fiber space.

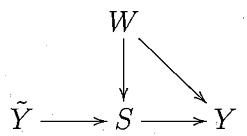
Now it is easy to see that the isomorphism from Proj $R_{\tilde{Y}}(h^*M)$ to Proj $R_{\tilde{Y}}(h^*L)$ which is obtained by composing the three isomorphisms in the diagram is compatible with the rational maps $\varphi_{h^*M}$ and $\varphi_{h^*L}$.

Therefore we obtain the following diagram.




In the diagram above, the top horizontal arrow is the isomorphism obtained as above, hence the top triangle is commutative. We can show that the two side vertical morphisms are finite, since $h$ is. In fact, by Theorem 2.3.1 $R_{\tilde{Y}}(h^*L)$ is finite over $R_Y(L)$ (take as $\Gamma$ the free abelian group generated by $L$. Then $R_Y(L) = R_Y(\Gamma)$ holds). The finiteness of the morphism $\mathrm{Proj}\, R_{\tilde{Y}}(h^*L) \to \mathrm{Proj}\, R_Y(L)$ follows from this.

Finally, there exists an isomorphism from $\mathrm{Proj}\, R_Y(M)$ to $\mathrm{Proj}\, R_Y(L)$ which are compatible with any other maps. In order to prove this, recall the following decomposition of the morphism $h$ from §2.3:



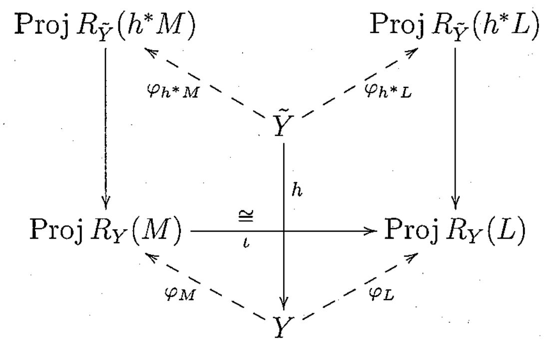

.

In the diagram above, $S$ is the separable closure of $\tilde{Y}/Y$ and $W$ is the Galois closure of $S/Y$.

The function field of $\mathrm{Proj}\, R_Y(M)$, as a subfield of the function field of $\mathrm{Proj}\, R_{\tilde{Y}}(h^*M)$, can be obtained by repeatedly taking the subfields of elements killed by vector fields which corresponds to a chain of degree $p$ subextensions of $\tilde{Y} \to S$ (see Proposition 2.3.4), taking the algebraic closure in $k(W)$ (see [L1, Example 2.1.12]), and taking the fields of invariants by the Galois group $G(W/Y)$ (see the arguments in §2.3.3). Note that these vector fields and Galois extensions depend only on the function fields of $\mathrm{Proj}\, R_{\tilde{Y}}(h^*M)$, $\tilde{Y}$ and $Y$. Therefore $\mathrm{Proj}\, R_Y(M)$ and $\mathrm{Proj}\, R_Y(L)$ should be isomorphic compatibly with the isomorphism between $R_{\tilde{Y}}(h^*M)$ and $R_{\tilde{Y}}(h^*L)$.

Conversely, take two Mori equivalent line bundles $L, M$ on $Y$. We show that $f^*L$ and $f^*M$ are Mori equivalent. For this, we can trace back the arguments above.

Consider the following diagram:




In the diagram above, the bottom horizontal arrow $\iota$ is an isomorphism which makes the bottom triangle commutative. The existence of such an isomorphism is guaranteed by the Mori equivalence of $L$ and $M$.

Note that the function field $k(\operatorname{Proj} R_{\tilde{Y}}(h^*L))$ is the algebraic closure of $k(\operatorname{Proj} R_Y(L))$ in $k(X)$. The same thing holds for $M$, so we see that the two function fields $k(\operatorname{Proj} R_{\tilde{Y}}(h^*L))$ and $k(\operatorname{Proj} R_{\tilde{Y}}(h^*M))$ coincide as subfields of $k(X)$.

By Theorem 2.3.1, three vertical morphisms are finite. Hence we see that $\operatorname{Proj} R_{\tilde{Y}}(h^*L)$ (resp. $\operatorname{Proj} R_{\tilde{Y}}(h^*M)$) is the normalization of $\operatorname{Proj} R_Y(L)$ (resp. $\operatorname{Proj} R_Y(M)$) in the same subfield of $k(\tilde{Y})$.Therefore the isomorphism $\iota$ lifts to an isomorphism between $\operatorname{Proj} R_{\tilde{Y}}(h^*L)$ and $\operatorname{Proj} R_{\tilde{Y}}(h^*M)$ making everything commutative. Since $g : X \to \tilde{Y}$ is an algebraic fiber space, this isomorphism guarantees the equivalence of $f^*L$ and $f^*M$, concluding the proof.

□

## 2.6 Comparison of the fans -via GIT-

In this section we prove Theorem 2.1.2 via the GIT interpretation of the relative interiors of the cones (see Proposition 1.3.8).

To carry out the proof, we prepare some notations. Let $f : X \to Y$ be a surjective morphism between Mori dream spaces. Fix a subgroup $\Gamma \subset \operatorname{Div}(Y)$ of rank $\rho(Y)$ which maps injectively to $\operatorname{Pic}(Y)_{\mathbb{Q}}$. Fix also a subgroup $\Gamma_X \subset \operatorname{Div}(X)$ of rank $\rho(X)$ mapping injectively to $\operatorname{Pic}(X)_{\mathbb{Q}}$, and $f^*\Gamma \subset \Gamma_X$.

For such a pair $(\Gamma, \Gamma_X)$, the natural ring homomorphism $f^* : R_Y(\Gamma) \to R_X(\Gamma_X)$ induces the morphism of affine varieties

$$V_f : V_X = \operatorname{Spec} R_X(\Gamma_X) \to V_Y = \operatorname{Spec} R_Y(\Gamma).$$

Set $T_X = \mathrm{Hom}(\Gamma_X, k^*)$ and $T_Y = \mathrm{Hom}(\Gamma, k^*)$. Via $f^*\Gamma \subset \Gamma_X$, we obtain the surjective morphism of algebraic tori $T_f : T_X \to T_Y$. $T_X$ (resp. $T_Y$) acts on $V_X$ (resp. $V_Y$) via the grading, and $V_f$ is an equivariant morphism with respect to these actions of tori and $T_f$.

The following is the main ingredient of the proof of Theorem 2.1.2:

**Proposition 2.6.1.** *Let $f : X \to Y$ be a surjective morphism between Mori dream spaces. Then if we choose an appropriate pair $(\Gamma, \Gamma_X)$ as above, the following holds:*

*Let $V_f : V_X \to V_Y$ be the associated morphism, and $\mathrm{ev}_L \in \chi(T_Y)$ be the character corresponding to a line bundle $L$ on $Y$. Then*

$$V_f^{-1}(V_Y^{ss}(\mathrm{ev}_L)) = V_X^{ss}(\mathrm{ev}_{f^*L}) \tag{2.4}$$

*holds.*

**Remark 2.6.2.**

$$V_f^{-1}(V_Y^{sss}(\mathrm{ev}_L)) = V_X^{sss}(\mathrm{ev}_{f^*L})$$

is not correct in general.

The conclusion of Proposition 2.6.1 does not hold for an arbitrary equivariant morphism between affine varieties. For example, consider the morphism

$$\varphi : \mathbb{A}^n \to \mathbb{A}^1; (x_1, \ldots, x_n) \mapsto x_1,$$

action of $\mathbb{G}_m$ on both sides with weights one, and the character $\chi$ of weight one. Then

$$(\mathbb{A}^n)^{ss}(\chi) = \mathbb{A}^n \setminus \{0\} \supsetneq (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^{n-1} = \varphi^{-1}((\mathbb{A}^1)^{ss}(\chi)).$$

The following is the GIT counterpart of Theorem 2.5.1

**Corollary 2.6.3.** *With the same assumptions as above, let $L, M$ be line bundles on $Y$. Then $V_Y^{ss}(\mathrm{ev}_L) = V_Y^{ss}(\mathrm{ev}_M)$ holds if and only if $V_X^{ss}(\mathrm{ev}_{f^*L}) = V_X^{ss}(\mathrm{ev}_{f^*M})$ holds.*

*Proof.* This follows from Proposition 2.6.1 and the surjectivity of $V_f$. □

Now it is clear that Theorem 2.1.2 immediately follows from Corollary 2.6.3, in view of Proposition 1.3.8 (see also the proof of Theorem 2.1.2 in Section 2.5).

In the rest of this section, we prove Proposition 2.6.1. The following lemma is the key to the proof:

**Lemma 2.6.4.** *Let $G$ be a reductive group. Let $\pi : Z \to W$ be a finite morphism between affine varieties such that $G$ acts on $Z$ and $W$ equivariantly. Let $\mathcal{L}$ be a linearization on $W$. Then*

$$\pi^{-1}(W^{ss}(\mathcal{L})) = Z^{ss}(\pi^* \mathcal{L})$$

*holds.*

*Proof.* See the proof for [GIT, Theorem 1.19] (and [GIT, Appendix to Chapter 1, §C] for positive characteristic cases). □

*Proof for Proposition 2.6.1.* We need some preparation. Take the Stein factorization

$$X \xrightarrow{g} \tilde{Y} \xrightarrow{h} Y$$

of $f$. Fix a subgroup $\Gamma_{\tilde{Y}} \subset \operatorname{Div}(\tilde{Y})$ of rank $\rho(\tilde{Y})$ which maps injectively to $\operatorname{Pic}(\tilde{Y})_{\mathbb{Q}}$ and containing $h^*\Gamma$. Similarly take $\Gamma_X \subset \operatorname{Div}(X)$ of rank $\rho(X)$ which maps injectively to $\operatorname{Pic}(X)_{\mathbb{Q}}$ and containing $g^*\Gamma_{\tilde{Y}}$. Define groups $Q$ and $R$ by the following exact sequences:

$$0 \to \Gamma \xrightarrow{h^*} \Gamma_{\tilde{Y}} \to Q \to 0$$
$$0 \to \Gamma_{\tilde{Y}} \xrightarrow{g^*} \Gamma_X \to R \to 0.$$

For simplicity we replace $\Gamma_{\tilde{Y}}$ and $\Gamma_X$ if necessary so that the quotients $Q$ and $R$ are both torsion free. Taking the duals of these sequences, we obtain the following exact sequences of algebraic tori:

$$0 \to T_Q \to T_{\tilde{Y}} \to T_Y \to 0$$
$$0 \to T_R \to T_X \to T_{\tilde{Y}} \to 0.$$

In particular, we obtain the following sequence of surjective group homomorphisms

$$T_X \xrightarrow{T_g} T_{\tilde{Y}} \xrightarrow{T_h} T_Y. \tag{2.5}$$

Next consider the following diagram.

$$\begin{array}{ccccc}
V_X & & & & \\
\downarrow {\scriptstyle \pi_R\ /T_R} & & & & \\
J & \xrightarrow[\cong]{} & V_{\tilde{Y}} & & \\
& & \downarrow {\scriptstyle \pi_Q\ /T_Q} & & \\
& & I & \xrightarrow[\text{finite}]{F} & V_Y
\end{array}$$

In the diagram above $I = \mathrm{Spec}\,(R_{\tilde{Y}}(h^*\Gamma))$ and $J = \mathrm{Spec}\,(R_X(g^*\Gamma_{\tilde{Y}}))$, and $\pi_R$ and $\pi_Q$ are natural projections. It is easy to see that they are categorical quotients by $T_R$ and $T_Q$, respectively.

Note that $T_X$ acts on $V_X$, $T_{\tilde{Y}}$ on $J$ and $V_{\tilde{Y}}$, and $T_Y$ on $I$ and $V_Y$. Moreover these actions are compatible with respect to the group homomorphisms (2.5). Since these homomorphisms are surjective, semi-stability of points on $V_Y$ with respect to the action of $T_Y$ is equivalent to the semi-stability with respect to the action of $T_X$ (similar for the points on $I, V_{\tilde{Y}}$, and $J$ respectively).

Now we go back to the proof of Proposition 2.6.1.

By Lemma 2.6.4, we see that

$$F^{-1}(V_Y^{ss}(ev_L)) = I^{ss}(ev_{h^*L})$$

holds, since $I \to V_Y$ is finite (Theorem 2.3.1).

Next we prove the following

**Claim.**

$$\pi_Q^{-1}(I^{ss}(ev_{h^*L})) = V_{\tilde{Y}}^{ss}(ev_{h^*L}).$$

*Proof of the claim.* The inclusion $\subseteq$ is the direct consequence of the definition of semi-stability, since $\pi_Q$ is affine. Conversely, suppose that $x \in V_{\tilde{Y}}^{ss}(ev_{h^*L})$. Then there exists a non-zero section $s \in R_{\tilde{Y}}(\Gamma_{\tilde{Y}})$ which is semi-invariant with respect to the character $ev_{h^*L}$ and $s(x) \neq 0$ holds. Note that semi-invariance of $s$ with respect to the character $ev_{h^*L}$ and the action of $T_{\tilde{Y}}$ implies that $s$ is the global section of some positive multiple of $h^*L$, hence $s \in R_{\tilde{Y}}(h^*\Gamma)$. Thus we obtain the other inclusion. □

Since $J \to V_{\tilde{Y}}$ is isomorphic, there is nothing to argue.

Finally by arguing as in Claim, we can show that

$$\pi_R^{-1}(J^{ss}(ev_{f^*L})) = V_X^{ss}(ev_{f^*L})$$

holds. Summing up, we obtain the desired equality. □

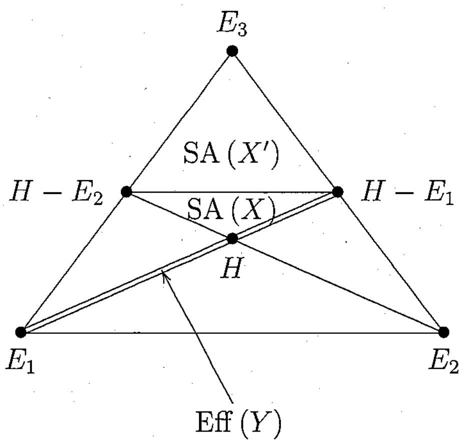
## 2.7 Examples

**Example 2.7.1.** We borrow from [AW, Example 5.5].

Let $X$ be the blow-up of $\mathbb{P}^3$ in two distinct points, say $p_1$ and $p_2$. $X$ is toric, hence is a Mori dream space. Let $E_1, E_2$ be the exceptional divisors corresponding to $p_1, p_2$ respectively, and let $\ell$ be the line passing through the points $p_1, p_2$. Let $E_3$ be the class of the strict transformation of a plane containing $\ell$. We can show that $X$ has a flopping contraction which contracts the strict transformation of the line $\ell$. Let $X'$ be the flop. Using the toric description, we see that this is an Atiyah flop.

The effective cone of $X$ is spanned by the divisors $E_i$, and the movable cone is the union of the semi-ample cones of $X$ and $X'$. SA$(X)$ is spanned by three divisors $H, H-E_1$, and $H-E_2$. SA$(X')$ is spanned by $H-E_1, H-E_2$, and $E_3$.

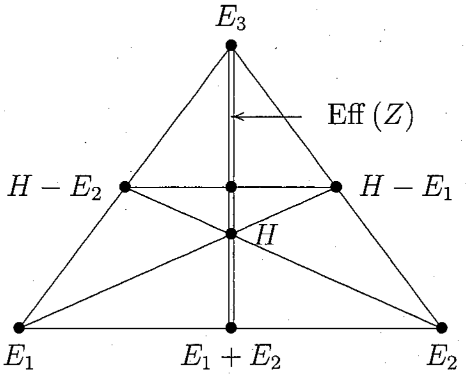
A slice of Eff$(X)$, together with its Mori chamber decomposition is described in the following figure:




Let $Y$ be the blow-up of $\mathbb{P}^3$ in $p_1$. Then the effective cone of $Y$, together with its decomposition into Mori chambers sits in Eff$(X)$ as indicated in the figure above (Eff$(Y)$ is denoted by the double line).

As indicated in the figure above, Eff$(Y)$ is mapped onto the cone spanned by $E_1$ and $H-E_1$. The cone spanned by $H$ and $H-E_1$ is the semi-ample cone of $Y$, and that spanned by $H$ and $E_1$ corresponds to the Mori chamber of $Y$ whose interior points correspond to the line bundles defining the birational contraction to $\mathbb{P}^3$.

Now take a coordinate on $\mathbb{P}^3$ such that $p_1 = (0:0:0:1)$ and $p_2 = (0:0:1:0)$. Consider the action of $\mathbb{Z}_2$ on $\mathbb{P}^3$ defined by $(x:y:z:w) \mapsto (x:y:w:z)$. This action lifts to $X$, and let $X \to Z$ be the quotient morphism. The effective cone of $Z$ together with its Mori chamber decomposition sits in that of $X$ as follows (Eff$(Z)$ is denoted by the double line):




As indicated in the diagram above, we can see that $Z$ has two Mori chambers other than the semi-ample cone (recall that $\mathrm{SA}(Z)$ coincides with the restriction of $\mathrm{SA}(X)$ to $\mathrm{Pic}(Z)_{\mathbb{R}}$. See Lemma 2.4.6(1)).

Let $Z'$ be the quotient of $X'$ by the involution induced from that on $X$. Again by Lemma 2.4.6(1), we can check that the Mori chamber of $Z$ obtained by restricting $\mathrm{SA}(X')$ is the semi-ample cone of $Z'$. The morphism defined by the ray separating $\mathrm{SA}(Z)$ and $\mathrm{SA}(Z')$ is the flipping contraction of $Z$ which contracts the image of $\ell$ under the quotient morphism $X \to Z$, and $Z'$ is the flip.

This example shows that a Mori chamber of the target space $Z$ is not necessarily a face of a Mori chamber of the source $X$.

**Example 2.7.2.** This example is well-known to experts, but we give a detailed explanation for the sake of completeness. The author learned this example from Doctor Tadakazu Sawada.

Suppose that $p > 0$. Let $\mathbb{A}^2_{x,y} \subset \mathbb{P}^2$ be an standard embedding of affine 2-plane with coordinate functions $x$ and $y$. Take $f = f(x,y) \in k[x,y]$. Consider the rational vector field defined by

$$\delta = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}.$$

According to [RS], we obtain the quotient of $\mathbb{P}^2$ by $\delta$. That is, we obtain a purely inseparable finite morphism $\pi : \mathbb{P}^2 \to Y$ of degree $p$ to a normal projective variety $Y$ such that $\mathcal{O}_Y = \mathcal{O}_{\mathbb{P}^2}^{\delta}$. It is easy to see that $k[x,y]^{\delta} = k[x^p, y^p, f(x,y)] \simeq k[X,Y,Z]/(Z^p - f(X,Y))$ holds.

Set $f(x,y) = x^p y + x y^p$. By the Fedder's criterion for $F$-purity ([F, Proposition 1.7]), we can check that the singularity $(0 \in k[x,y]^{\delta}) \cong k[[X,Y,Z]]/(Z^p - X^pY - XY^p)$ is not $F$-pure. Therefore $Y$ is not globally $F$-regular, despite $\mathbb{P}^2$ is.

On the other hand we can show that $Y$ is a Mori dream space. Firstly Picard number of $Y$ is one since $\pi^* : \mathrm{Pic}\,(Y)_{\mathbb{R}} \to \mathrm{Pic}\,(\mathbb{P}^2)_{\mathbb{R}}$ is injective. We can also check that $Y$ is $\mathbb{Q}$-factorial:

**Claim.** *Let $f : X \to Y$ be a purely inseparable finite morphism between normal varieties. If $X$ is $\mathbb{Q}$-factorial, so is $Y$.*

*Proof.* Everything follows from the following lemma, which can be shown easily:

**Lemma 2.7.3.** *Under the same assumptions, let $(U, f_U)_U$ be a Cartier divisor on $X$, where $X = \bigcup U$ is an open covering of $X$ and $f_U \in k(X)$. Then the pushforward of the Weil divisor corresponding to $(U, f_U)_U$ corresponds to the Cartier divisor $(U, N(f_U))_U$, where $N = N_{X/Y} : k(X) \to k(Y)$ is the norm function.*

Let $D$ be a Weil divisor on $Y$. By assumption, there exists a positive integer $m$ such that $mf^*D = f^*mD$ is Cartier. By the subclaim above, we can show that $f_*f^*mD = m \deg(f) D$ and that $f_*f^*mD$ is Cartier. □

## 2.8 Amplifications

In this section, we extend our results

- to the case when varieties involved are not necessarily $\mathbb{Q}$-factorial.
- to Mori dream regions.

As an example of a Mori dream region, we treat the Shokurov polytopes in the final subsection.

### 2.8.1 Not necessarily $\mathbb{Q}$-factorial Mori dream space

In [AHL, §2], the notion of Mori dream space has been extended to not necessarily $\mathbb{Q}$-factorial normal projective varieties. In this subsection we call them not necessarily $\mathbb{Q}$-factorial Mori dream spaces, and show that our main results are also valid in that context.

**Definition 2.8.1.** Let $X$ be a normal projective variety. $X$ is said to be a not necessarily $\mathbb{Q}$-factorial Mori dream space if

1. $\mathrm{Cl}\,(X)$ is finitely generated, where $\mathrm{Cl}\,(X)$ denotes the Weil divisor class group of $X$.

2. A Cox ring of $X$ is of finite type over the base field.

Note that our definitions of not necessarily $\mathbb{Q}$-factorial Mori dream spaces and Cox rings coincide with those of $\mathbb{Q}$-factorial Mori dream spaces when the variety is $\mathbb{Q}$-factorial. Note also that the finite generation of a Cox ring is independent of the choice of $\Gamma$.

**Remark 2.8.2.** In [AHL, Theorem 2.3], they gave a characterization of not necessarily $\mathbb{Q}$-factorial Mori dream spaces via the properties of line bundles as in the original definition of Mori dream spaces. At the same time they proved the existence of a small $\mathbb{Q}$-factorization for such varieties. In the proof of [AHL, Lemma 2.4], which was used in the proof of [AHL, Theorem 2.3], they used the characteristic zero assumption so that they can apply the existence of resolutions of singularities. Actually we can avoid the use of resolutions as follows, so that their results work in arbitrary characteristics.

*Proof.* We give a proof of [AHL, Lemma 2.4] which works in arbitrary characteristics.

Take prime divisors $D_1, \dots, D_r$ on $X$ which generates $\mathrm{Cl}(X)$. Let $X' \to X$ be a birational proper morphism from a projective variety $X'$, whose existence is guaranteed by the Chow's lemma. Let $D'_i$ be the strict transforms $D_i$s. Now let $X'' \to X'$ be (the normalization of) the successive blow-ups of $X'$ along $D'_i$s. Let $D''_i$ be the total transform of $D'_i$. Note that $D''_i$ is a Cartier divisor by the construction of blow-up. Let $f : X'' \to X$ be the composition of the two morphisms, and note that $f_* D''_i = D_i$ holds.

If we regard $f$ as the birational morphism $\pi$ in the proof of [AHL, Lemma 2.4], the rest of the arguments works similarly. □

We go back to our results. First, Theorem 2.1.1 also holds in this case:

**Theorem 2.8.3.** *Let $X$ be a not necessarily $\mathbb{Q}$-factorial Mori dream space, and $X \to Y$ be a surjective morphism to another normal projective variety. Then $Y$ also is a not necessarily $\mathbb{Q}$-factorial Mori dream space.*

*Proof.* The proof is essentially the same as that for Theorem 2.1.1, so we only point out where should be modified in the original one. First of all, we can replace $X$ with its small $\mathbb{Q}$-factorization: i.e. there exists a small birational morphism $\tilde{X} \to X$ from a $\mathbb{Q}$-factorial normal projective variety $\tilde{X}$. This fact implicitly appears in [AHL, Theorem 2.3] and its proof.

We should check that $\dim_{\mathbb{Q}} \mathrm{Cl}(Y)_{\mathbb{Q}} < \infty$. Note that $\mathrm{Cl}(Y)$ does not change if we remove the singular locus of $Y$. If we remove the inverse image of this locus from $X$, the Weil divisor class group does not increase. The rest of the argument is the same.

In order to prove the finite generation of a Cox ring of $Y$, we take the Stein factorization of $X \to Y$ as before.

Nothing has to be changed for finite morphisms. For algebraic fiber spaces, we again remove the singular locus of $Y$ and its inverse image. Then we can apply Lemma 2.2.2, since any Weil divisor on a non-singular variety is Cartier so that it can be pulled-back. □

For a not necessarily $\mathbb{Q}$-factorial Mori dream space $X$, we can define the notion of Mori equivalence, Mori chambers and so on exactly in the same manner as before. On the other hand, we can take a small $\mathbb{Q}$-factorization $\tilde{X} \to X$ and the divisor class group and Mori equivalence do not change under this operation. Therefore we obtain

**Theorem 2.8.4.** *Let $X \to Y$ be a surjective morphism between not necessarily $\mathbb{Q}$-factorial Mori dream spaces. Then*

$$\mathrm{Fan}\,(Y) = \mathrm{Fan}\,(X)|_{\mathrm{Cl}\,(Y)_{\mathbb{R}}}$$

*holds.*

*Proof.* By taking suitable small $\mathbb{Q}$-factorizations of $X$ and $Y$, the morphism lifts to the one between ordinary Mori dream spaces. Thus we can reduce the problem to our original Theorem 2.1.2. □

### 2.8.2 Mori dream region

Let $X$ be a normal $\mathbb{Q}$-factorial projective variety.

There is a notion called Mori dream regions defined in [HK, Definition 2.12], which generalizes Mori dream spaces. In this subsection we check that Theorem 2.1.1 can be extended to Mori dream regions.

First we recall the definition of Mori dream regions from [HK, Definition 2.12]:

**Definition 2.8.5.** Let $X$ be a normal $\mathbb{Q}$-factorial projective variety. A cone $\mathcal{C} \subset \mathrm{Pic}\,(X)_{\mathbb{R}}$ spanned by finitely many line bundles is called a Mori dream region (Mori dream region) if the multi-section ring

$$R_X(\mathcal{C}) = \bigoplus_{D \in \mathcal{C} \cap \mathrm{Pic}\,(X)^{free}} H^0(X, \mathcal{O}_X(D))$$

is of finite type over the base field.

If the natural morphism $\mathrm{Pic}(X)_{\mathbb{Q}} \to \mathrm{N}^1(X)_{\mathbb{Q}}$ is isomorphic and $\mathcal{C} = \mathrm{Eff}(X)$, $\mathcal{C}$ is a Mori dream region if and only if $X$ is a Mori dream space.

As in the case of Mori dream space, Mori dream region can be characterized via the existence of a decomposition into finitely many rational polyhedral subcones such that on each of them the Zariski decomposition is $\mathbb{Q}$-linear:

**Proposition 2.8.6.** *Let $X$ be a normal $\mathbb{Q}$-factorial projective variety and $\mathcal{C} \subset \mathrm{Pic}(X)_{\mathbb{R}}$ be a cone spanned by finitely many line bundles. It is a Mori dream region if and only if the following conditions are satisfied:*

- *$\mathcal{C} \cap \mathrm{Eff}(X)$ is spanned by finitely many line bundles.*
- *The section ring of any line bundle of $\mathcal{C} \cap \mathrm{Eff}(X)$ is of finite type over the base field. In particular, $\mathbb{Q}$-effective line bundles admit the Zariski decompositions. I.e. for such a line bundle $L$ there exists a decomposition $L = P + N$ such that $P$ is movable and $N$ is effective, and for all sufficiently divisible positive integer $m$ all the global sections of $mL$ comes from that of $mP$ as in (1.1) of Proposition 1.2.10.*
- *There exists a decomposition of $\mathcal{C} \subset \mathrm{Pic}(X)_{\mathbb{R}}$ into finitely many rational polyhedral subcones such that on each of them the Zariski decomposition is $\mathbb{Q}$-linear.*

*Proof.* 'if' part is exactly the same as the proof of Lemma 2.2.2. For the 'only if' part, the second condition follows from [HK, 'If' part of Lemma 1.6]. For the first and the third conditions, see [CL, Theorem 3.5]. □

**Remark 2.8.7.** In [HK, Theorem 2.13], they claim that we can find a decomposition of $\mathcal{C}$ into chambers $\mathcal{C}_i$ so that for each of them we can find a contracting birational map $g_i : X \dashrightarrow Y_i$ such that

$$\mathcal{C}_i = \mathcal{C} \cap (g_i^* \mathrm{Nef}(Y_i) * \mathrm{ex}(g_i))$$

holds.

The author believe that it is not so easy to prove, since using that claim we can derive the existence of a minimal model from that of the canonical model. This is why he replaced [HK, Theorem 2.13] with Proposition 2.8.6.

Now we can show the following

**Corollary 2.8.8.** *Let $f : X \to Y$ be a surjective morphism between normal $\mathbb{Q}$-factorial projective varieties. Let $\mathcal{C} \subset \mathrm{Pic}(X)_{\mathbb{R}}$ be a finitely generated rational polyhedral cone which is a Mori dream region. Then $\mathcal{C}|_{\mathrm{Pic}(Y)_{\mathbb{R}}}$ also is a Mori dream region.*

The proof is the same as that for Theorem 2.1.1.

**Remark 2.8.9.** In general, we do not know if Theorem 2.1.2 holds for Mori dream regions. Here we give some observations to this problem.

Consider a rational polyhedral cone which is contained in the ample cone of a normal projective variety. It is clear that all the divisors in the cone are strongly Mori equivalent. Taking this cone as a Mori dream region, we see that we do not have the fan structure for Mori dream regions such that the relative interior of the cone of the fan is an equivalence class.

Moreover, we do not know if the GIT equivalence and the strong Mori equivalence coincide for arbitrary Mori dream regions or not. The reason is as follows.

By closely looking at the proof of Proposition 1.3.8, we see that the GIT equivalence implies the strong Mori-equivalence for arbitrary Mori dream regions, provided that they contain ample divisors. For the proof of the converse, it was essential that the unstable locus for ample divisors has codimension at least two. Even if we take an arbitrary Mori dream region which contains an ample divisor and consider the spec of the corresponding multisection ring, the unstable locus of ample divisors can have divisorial component: the Nef cone of the blow-up of $\mathbb{P}^2$ at a point gives such an example. The difference comes from the fact that [HK, Lemma 2.7] holds only for Cox rings.

Nevertheless, note that Corollary 2.6.3 and Theorem 2.5.1 holds for $\mathcal{C}$ and $\mathcal{C}|_{\mathrm{Pic}(Y)_{\mathbb{R}}}$, where $\mathcal{C}$ is an arbitrary Mori dream region on $X$.

# 3 Global Okounkov bodies of Mori dream spaces

## 3.1 Summary

Let $X$ be a projective variety of dimension $n$. A flag $Y_\bullet$ on $X$ is a sequence

$$Y_\bullet = Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_n = \{pt\}$$

of closed subvarieties such that each $Y_i$ is smooth at the point $Y_n$.

For a flag $Y_\bullet$ and a big line bundle $L$ on $X$, we can define the Okounkov body $\Delta_{Y_\bullet}(X, L)$, which is a compact convex set in $\mathbb{R}^n$ (see [LM, Definition 1.8]). It is known that the Euclidian volume of this body coincides with the volume of the line bundle $L$, up to the constant $n!$ ([LM, Theorem 2.3]). Therefore we can regard Okounkov body as a geometric refinement of the volume function for line bundles. Okounkov body depends on the choice of the flag $Y_\bullet$, but is determined by the numerical class of the big line bundle $L$ ([LM, Proposition 4.1(i)]).

We can also define the notion of global Okounkov body $\Delta_{Y_\bullet}(X)$, which is a closed convex cone in $\mathbb{R}^n \times N^1(X)_\mathbb{R}$ whose fiber over a big line bundle $L \in N^1(X)_\mathbb{R}$ coincides with the Okounkov body $\Delta_{Y_\bullet}(X, L)$ of $L$ ([LM, Theorem B]).

In [LM], Lazarsfeld and Mustaţă asked the following problem ([LM, Problem 7.1]):

**Problem 3.1.1.** Does a Mori dream space admit a flag with respect to which the global Okounkov body is rational polyhedral?

Problem 3.1.1 is known to be true for smooth projective toric varieties ([LM, Proposition 6.1 (ii)]); in that case, we choose the flag consisting of torus invariant strata. Since Mori dream space is a generalization of toric varieties, it is natural to ask if the same thing also holds for Mori dream spaces.

Another supporting evidence is as follows: it is known that for a Mori dream space $X$ the volume function $vol(\cdot) : \text{Eff}\,(X) \to \mathbb{R}$ is piecewise polynomial (this follows from Proposition 1.2.10 and the fact that the volume of a nef line bundle equals to its self-intersection number). Problem 3.1.1 can be regarded as a refinement of this fact.

The purpose of this chapter is to give a positive answer to Problem 3.1.1 and to propose an approach to the problem in higher dimensions.

We first establish a formula (see Lemma 3.3.1) which describes slices of an Okounkov body as the Okounkov body of certain line bundles on $Y_1$, the first piece of the flag $Y_\bullet$. This enables us to calculate Okounkov bodies inductively. As a first application of the formula, we obtain the following result:

**Lemma 3.1.2.** *Problem 3.1.1 is true for surfaces.*

To deal with higher dimensional cases, we define the notion of a good flag:

**Definition 3.1.3.** Let $X$ be a Mori dream space. A flag $Y_\bullet = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{pt\}$ is said to be good if the following conditions hold:

- $Y_i$ is the birational image of a Mori dream space, say $\tilde{Y}_i$, for $i = 1, \ldots, n-2$, such that there exists a sequence of closed immersions

$$\tilde{Y}_1 \supset \tilde{Y}_2 \supset \cdots \supset \tilde{Y}_{n-2}$$

  compatible with the projections to $Y_i$'s.

- $Y_i$ is not contained in the base loci of line bundles on $Y_{i-1}$ for all $i = 1, 2, \ldots, n-1$

- $Y_0$ is not contained in the images of the exceptional loci of small $\mathbb{Q}$-factorial modifications of $\tilde{Y}_i's$ or exceptional loci of birational morphisms $\tilde{Y}_i \to Y_i$ appearing in the first condition.

Note that the last two conditions are fulfilled if $Y_i$ is a general member of a base point free linear system.

**Remark 3.1.4.**

- Since only $\mathbb{P}^1$ is a Mori dream curve, we cannot expect in general that $Y_{n-1}$ is a Mori dream curve. But we do not need this because line bundles on a curve behaves quite nicely. This is the reason why we do not assume that $Y_{n-1}$ is a Mori dream space.

- It seems that we should not assume that $Y_i$ itself is a Mori dream space. In fact, when we think of a rational homogeneous variety, which is a Mori dream space since it is Fano, a natural candidate for a good flag is the one consisting of Schubert subvarieties. Schubert varieties are not necessarily $\mathbb{Q}$-factorial, hence we have to pass to their Bott-Samelson resolutions.

With this notion, we can show

**Theorem 3.1.5.** *Let $X$ be a Mori dream space and $Y_\bullet$ is a good flag . Then $\Delta_{Y_\bullet}(X)$ is a rational polyhedral cone.*

The final part of the chapter is devoted to a discussion on how to construct such a flag.

Shin-Yao Jow proved ([S, Theorem 6]) that a sufficiently ample and very general divisor of a Mori dream space of dimension at least three again is a Mori dream space, provided that the ambient variety satisfies certain GIT condition. Therefore, the following naive expectation arises:

**Problem 3.1.6.** Let $X$ be a Mori dream space of dimension at least three (not necessarily satisfying the GIT condition above). Let $A$ be a sufficiently ample and very general divisor of $X$. Then $A$ also is a Mori dream space.

Note that in general a smooth ample divisor of a Mori dream space may not be a Mori dream space (see [Og] and [Ka] for such examples). So far Problem 3.1.6 has no approach.

Therefore we are forced to construct good flags on a case-by-case basis. We discuss two examples. The first one is a Mori dream 3-fold given in [KLM, Proposition 3.5]. In the paper an example of flags with respect to which the Okounkov body is not rational polyhedral, but we can find a good flag such that the global Okounkov body is rational polyhedral.

The second one is rational homogeneous varieties. There is a natural candidate for the flag $Y_\bullet$ so that the global Okounkov body is rational polyhedral, but there still remains some difficulty.

Here is a historical remark. The notion of Okounkov body first appeared in the works of Andrei Okounkov. His aim was to describe the multiplicities of irreducible representations appearing in a representation in terms of the volume of certain convex bodies so that he can prove that the log concavity holds for the multiplicities by using the Brunn-Minkowski inequality for convex bodies ([Ok] is an interesting survey).

Later Lazarsfeld and Mustaţă defined similar convex bodies (which they called the Okounkov body) for big line bundles and did a foundational work ([LM]). In this case, the volume of the convex body associated to a big line bundle coincides with the volume of the line bundle as mentioned above.

## 3.2 (Global) Okounkov body

In this section we recall the definition and first properties of (global) Okounkov bodies. Most of the subjects in this section was taken from [LM].

Consider any divisor $D$ on $X$. We begin by defining a function

$$\nu = \nu_{Y_\bullet} : H^0\big(X, \mathcal{O}_X(D)\big) \setminus \{0\} \longrightarrow \mathbf{Z}^n \quad , \quad s \mapsto \nu(s) = \big(\nu_1(s), \dots, \nu_n(s)\big).$$

Given

$$0 \neq s \in H^0\big(X, \mathcal{O}_X(D)\big),$$

set to begin with

$$\nu_1 = \nu_1(s) = \mathrm{ord}_{Y_1}(s).$$

After choosing a local equation for $Y_1$ in $X$, say $t_1$, $s$ determines a section

$$\tilde{s}_1 = s \otimes t_1^{-\nu_1} \in H^0\big(X, \mathcal{O}_X(D - \nu_1 Y_1)\big)$$

that does not vanish (identically) along $Y_1$, and so we get by restricting a non-zero section

$$s_1 \in H^0\big(Y_1, \mathcal{O}_{Y_1}(D - \nu_1 Y_1)\big).$$

Then take

$$\nu_2 = \nu_2(s) = \mathrm{ord}_{Y_2}(s_1).$$

In general, given integers $a_1, \dots, a_i \geq 0$ denote by $\mathcal{O}(D - a_1Y_1 - a_2Y_2 - \dots - a_iY_i)_{|Y_i}$ the line bundle

$$\mathcal{O}_X(D)_{|Y_i} \otimes \mathcal{O}_X(-a_1Y_1)_{|Y_i} \otimes \mathcal{O}_{Y_1}(-a_2Y_2)_{|Y_i} \otimes \dots \otimes \mathcal{O}_{Y_{i-1}}(-a_iY_i)_{|Y_i}$$

on $Y_i$. Suppose inductively that for $i \leq k$ one has constructed non-vanishing sections

$$s_i \in H^0\Big(Y_i, \mathcal{O}(D - \nu_1Y_1 - \nu_2Y_2 - \dots - \nu_iY_i)_{|Y_i}\Big),$$

with $\nu_{i+1}(s) = \mathrm{ord}_{Y_{i+1}}(s_i)$, so that in particular

$$\nu_{k+1}(s) = \mathrm{ord}_{Y_{k+1}}(s_k).$$

Dividing by the appropriate power of a local equation of $Y_{k+1}$ in $Y_k$ yields a section

$$\tilde{s}_{k+1} \in H^0\Big(Y_k, \mathcal{O}\big(D - \nu_1 Y_1 - \nu_2 Y_2 - \ldots - \nu_k Y_k\big)_{|Y_k} \otimes \mathcal{O}_{Y_k}(-\nu_{k+1} Y_{k+1})\Big)$$

not vanishing along $Y_{k+1}$. Then take

$$s_{k+1} \;=\; \tilde{s}_{k+1}|Y_{k+1} \;\in\; H^0\Big(Y_{k+1}, \mathcal{O}\big(D - \nu_1 Y_1 - \nu_2 Y_2 - \ldots - \nu_{k+1} Y_{k+1}\big)_{|Y_{k+1}}\Big)$$

to continue the process. Note that while the sections $\tilde{s}_i$ and $s_i$ will depend on the choice of a local equation of each $Y_i$ in $Y_{i-1}$, the values $\nu_i(s) \in \mathbf{N}$ do not.

**Definition 3.2.1. (Graded semigroup of a divisor).** The *graded semigroup* of $D$ is the sub-semigroup

$$\Gamma(D) \;=\; \Gamma_{Y_\bullet}(D) \;=\; \big\{(\nu_{Y_\bullet}(s), m)\,\big|\, 0 \neq s \in H^0\big(X, \mathcal{O}_X(mD)\big)\,,\, m \geq 0\big\}$$

of $\mathbf{N}^n \times \mathbf{N} = \mathbf{N}^{n+1}$. □

Writing $\Gamma = \Gamma(D)$, denote by

$$\Sigma(\Gamma) \;\subseteq\; \mathbf{R}^{n+1}$$

the intersection of all the closed convex cones containing $\Gamma$. The Okounkov body of $D$ is then the slice of this cone at the level one:

**Definition 3.2.2. (Okounkov body).** The *Okounkov body* of $D$ (with respect to the fixed flag $Y_\bullet$) is the compact convex set

$$\Delta_{Y_\bullet}(X, D) \;=\; \Sigma(\Gamma) \cap \big(\mathbf{R}^n \times \{1\}\big).$$

We view $\Delta(D)$ in the natural way as a closed convex subset of $\mathbf{R}^n$.

We can show that it is compact, and that it depends only on the numerical class of the divisor $D$ (see [LM] for detail).

**Remark 3.2.3.** $\Delta_{Y_\bullet}(X, L)$ does depend on the choice of the flag $Y_\bullet$. For example, even if $X$ is a toric Fano and $L$ is ample, we have to chose a suitable flag $Y_\bullet$ to make $\Delta_{Y_\bullet}(X, L)$ rational polyhedral (see [KLM, Example 3.4]).

There is a globalization of this notion (see [LM, §4.2]):

**Theorem-Definition 3.2.4.** *There exists a closed convex cone*

$$\Delta_{Y_\bullet}(X) \subseteq \mathbf{R}^n \times N^1(X)_{\mathbf{R}}$$

*characterized by the property that the fibre of $\Delta_{Y_\bullet}(X)$ over any big class $\xi \in N^1(X)_{\mathbf{Q}}$ is $\Delta_{Y_\bullet}(X, \xi)$, i.e.*

$$\mathrm{pr}_2^{-1}(\xi) \cap \Delta(X) = \Delta_{Y_\bullet}(X, \xi) \subseteq \mathbf{R}^n \times \{\xi\} = \mathbf{R}^n.$$

The cone $\Delta_{Y_\bullet}(X)$ is called the global Okounkov body of $X$ with respect to the flag $Y_\bullet$.

The following theorem says that the global Okounkov body is a refinement of the volume function:

**Theorem 3.2.5.** *If $D$ is any big divisor on $X$, then*

$$\mathrm{vol}_{\mathbf{R}^n}\big(\Delta(D)\big) = \frac{1}{n!} \cdot \mathrm{vol}_X(D).$$

The quantity on the right is the *volume* of $D$, defined as the limit

$$\mathrm{vol}_X(D) =_{\mathrm{def}} \lim_{m \to \infty} \frac{h^0\big(X, \mathcal{O}_X(mD)\big)}{m^n/n!}.$$

## 3.3 Inductive calculation of Okounkov bodies

### 3.3.1 Slices of Okounkov bodies

We recall some facts on the slices of Okounkov bodies.

**Lemma 3.3.1.** *Let $X$ be a normal projective variety and $L$ be a big line bundle on $X$. Let $Y_\bullet$ be a flag on $X$ such that $Y_1 \not\subset \mathbb{B}_+(L)$, where $\mathbb{B}_+(L)$ denotes the augmented base locus of the line bundle $L$ (see [L2, Definition 10.3.2]). Take some rational number $t \in \mathbb{Q}_{\geq 0}$ which satisfies the following properties:*

- *$L - tY_1$ is big*
- *$L - tY_1$ admits a Zariski decomposition*
- *$Y_1 \not\subset \mathbb{B}_+(P(L - tY_1))$,*

*where $P(L - tY_1)$ is the positive part of $L - tY_1$.*

*Then*

$$\Delta_{Y_\bullet}(L)_{\nu_1=t} := \Delta_{Y_\bullet}(L) \cap \{\nu_1 = t\} = \Delta_{Y_\bullet}(Y_1, P(L - tY_1)|_{Y_1}) \tag{3.1}$$

*holds up to the parallel transportation by the valuation vector of the restriction of the section of $N(L - tY_1)$.*

*Proof.* Since $Y_1 \not\subset \mathbb{B}_+(L)$ we have

$$\Delta_{Y_\bullet}(L)_{\nu_1=t} = \Delta_{Y_\bullet}(Y_1|X, L - tY_1),$$

where the right hand side denotes the restricted Okounkov body (see [LM, Theorem 4.24]). By a property of the Zariski decomposition, the right hand side of (3.1) is a parallel transportation of $\Delta_{Y_\bullet}(Y_1|X, P(L - tY_1))$ by the valuation vector mentioned in the statement of the lemma.

By [LM, Corollary 4.25 (i)], $(n-1)!$ times the volume of $\Delta_{Y_\bullet}(Y_1|X, P)$ equals to the restricted volume of $P$. By [ELMNP, Corollary 2.17], the restricted volume of $P$ equals to $(P^{n-1} \cdot Y_1)$ since $Y_1 \not\subseteq \mathbb{B}_+(P)$. On the other hand, since $P$ is nef, the volume of $P|_{Y_1}$, which in turn equals the $(n-1)!$ times the volume of the Okounkov body $\Delta_{Y_\bullet}(Y_1, P|_{Y_1})$, equals to $(P^{n-1} \cdot Y_1)$. Summing up, we see that $\Delta_{Y_\bullet}(Y_1|X, P) \subseteq \Delta_{Y_\bullet}(Y_1, P|_{Y_1})$ are closed convex bodies of the same volume. Hence they must coincide. □

### 3.3.2 A decomposition of the Effective cone

Fix a flag $Y_\bullet$ on a Mori dream space $X$ such that $Y_1$ avoids the base loci of effective line bundles on $X$. Consider the projection map $\pi$ defined by

$$\pi : \mathrm{Pic}\,(X)_{\mathbb{R}} \times \mathbb{R}^n \to \mathrm{Pic}\,(X)_{\mathbb{R}} \times \mathbb{R};\ (D, \nu_1, \nu_2, \ldots, \nu_n) \mapsto (D, \nu_1).$$

The purpose of this subsection is to prove the existence of a decomposition of $\pi(\Delta_{Y_\bullet}(X)) \subset \mathrm{Eff}\,(X) \times \mathbb{R}_{\geq 0}$ into finitely many rational polyhedral cones such that on each of the cones the function $\varphi : (D, t) \mapsto P(D - tY_1)$ is rationally linear.

Consider the linear mapping

$$T : \mathrm{Eff}\,(X) \times \mathbb{R}_{\geq 0} \to \mathrm{Pic}\,(X)_{\mathbb{R}}; (D, t) \mapsto D - tY_1.$$

By Lemma 3.5.2, for each Mori chamber $C \subset \mathrm{Pic}\,(X)_{\mathbb{R}}$, $\mathcal{C} = T^{-1}(C)$ is a rational polyhedral cone. Combined with Proposition 1.2.10, we see that $\varphi$ is rationally linear on $\mathcal{C}$. Therefore we have the decomposition $T^{-1}(\mathrm{Eff}\,(X)) = \bigcup \mathcal{C}$ into rational polyhedral subcones such that on each of the subcones $\varphi$ is rationally linear. Now we can show the following lemma.

**Lemma 3.3.2.** $\pi(\Delta_{Y_\bullet}(X)) = T^{-1}(\mathrm{Eff}\,(X))$ *holds.*

*Proof.* $T^{-1}(\mathrm{Eff}\,(X)) \subseteq \pi(\Delta_{Y_\bullet}(X))$ follows from Lemma 3.3.1. Conversely, choose a point $(D, t)$ from the interior of $\pi(\Delta_{Y_\bullet}(X))$.

Then we can apply [LM, Corollary A.3] to obtain

$$\pi^{-1}(D,t) \bigcap \Delta_{Y_\bullet}(X) = \Delta_{Y_\bullet}(Y_1|X, D - tY_1).$$

Moreover we know that the left hand side has an interior point. Hence $D - tY_1 \in \mathrm{Eff}\,(X)$ must hold. Since $T^{-1}(\mathrm{Eff}\,(X))$ is closed, this is enough to show the lemma. □

Thus we obtain the desired decomposition of $\pi(\Delta_{Y_\bullet}(X))$.

### 3.3.3 Proof of Lemma 3.1.2

*Proof of Lemma 3.1.2.* Let $Y_\bullet$ be a good flag. Consider the following projection

$$\pi : \mathrm{Pic}\,(X)_{\mathbb{R}} \times \mathbb{R}^2 \to \mathrm{Pic}\,(X)_{\mathbb{R}} \times \mathbb{R};\ (D, \nu_1, \nu_2) \mapsto (D, \nu_1).$$

Let $\mathcal{C} \subset \mathrm{Eff}\,(X) \times \mathbb{R}_{\geq 0}$ be a chamber as in §3.3.2. By Lemma 3.5.1, it is enough to show that $\pi^{-1}(\mathcal{C}) \bigcap \Delta_{Y_\bullet}(X)$ is a rational polyhedral cone. To prove it, it is enough to show that the function

$$(D,t) \mapsto \mathrm{Vol}\,(\Delta_{Y_\bullet}(D)_{\nu_1 = t}) \tag{3.2}$$

is rationally linear on $\mathcal{C}$, since

$$\Delta_{Y_\bullet}(D)_{\nu_1 = t} = [0, \mathrm{Vol}\,(\Delta_{Y_\bullet}(D)_{\nu_1 = t})]$$

holds.

But we know from Lemma 3.3.1 that the right hand side of (3.2) equals to

$\deg\,(\Delta_{Y_\bullet}(Y_1, P(D - tY_1)|_{Y_1})) = Y_1.(P(D - tY_1))$, which is clearly rationally linear in $(D, t)$. □

### 3.3.4 Proof of Theorem 3.1.5

*Proof of Theorem 3.1.5.* Consider the following projection

$$\pi : \mathrm{Pic}\,(X)_{\mathbb{R}} \times \mathbb{R}^n \to \mathrm{Pic}\,(X)_{\mathbb{R}} \times \mathbb{R};\ (D, \nu_1, \nu_2, \ldots, \nu_n) \mapsto (D, \nu_1).$$

Let $\mathcal{C} \subset \mathrm{Eff}\,(X) \times \mathbb{R}_{\geq 0}$ be a chamber as in §3.3.2. By Lemma 3.5.1, it is enough to show that $\pi^{-1}(\mathcal{C}) \bigcap \Delta_{Y_\bullet}(X)$ is a rational polyhedral convex cone.

Consider the following linear mapping:

$$\varphi : \mathrm{Pic}\,(X)_{\mathbb{R}} \times \mathbb{R}^n \to \mathrm{Pic}\,(Y_1)_{\mathbb{R}} \times \mathbb{R}^{n-1}; (D, t, \nu') \mapsto (P(D-tY_1)|_{Y_1}, \nu' - \nu'(N(D - tY_1))),$$

where $\nu' = (\nu_2, \nu_3, \dots, \nu_n)$. Let $\varphi_{\mathcal{C}}$ be the restriction of $\varphi$ to $\pi^{-1}(\mathcal{C})$. Recall that $\varphi_{\mathcal{C}}$ is rationally linear. Now since $\Delta_{Y_\bullet}(X, D)_{\nu_1 = t} = \Delta_{Y_\bullet}(Y_1, P(D - tY_1)|_{Y_1}) + \nu'(N(D - tY_1))$, for each $(D, t, \nu') \in \pi^{-1}(\mathcal{C})$

$$(D, t, \nu') \in \Delta_{Y_\bullet}(X) \iff \varphi_{\mathcal{C}}(D, t, \nu') \in \Delta_{Y_\bullet}(Y_1).$$

Therefore $\pi^{-1}(\mathcal{C}) \bigcap \Delta_{Y_\bullet}(X) = \varphi_{\mathcal{C}}^{-1}(\Delta_{Y_\bullet}(Y_1))$. Since we assumed that $Y_\bullet$ is a good flag, $Y_1$ is the image of a birational morphism from a Mori dream space $\tilde{Y}_1$ and we can identify $\Delta_{Y_\bullet}(Y_1)$ with a subcone of the global Okounkov body of $\tilde{Y}_1$ (with respect to the flag obtained by $\tilde{Y}_i$'s). Hence it is a rational polyhedral cone. By Lemma 3.5.2, it follows that $\varphi_{\mathcal{C}}^{-1}(\Delta_{Y_\bullet}(Y_1))$ also is a rational polyhedral cone. □

## 3.4 Discussions about good flags

In this section, we discuss the existence problem of good flags.

As mentioned before, Problem 3.1.6 is true if $X$ satisfies the small unstable locus condition (see [S]).

If we take $(\mathbb{P}^1)^4$ as $X$, then it does not satisfy the condition. Moreover a general hypersurface of multi-degree $(2, 2, 2, 2)$ of $X$ is not a Mori dream space ([Og]). This example shows that we can not expect that an arbitrary ample divisor which is general in its linear system is a Mori dream space. So far we know nothing about Problem 3.1.6. In the rest of this subsection, we consider two examples of Mori dream spaces which (almost) admit good flags.

The first example is a Mori dream space given in [KLM, Proposition 3.5]. We use the same notations as in the paper.

Let $H$ be a sufficiently general member. We see that $H \to \pi(H)$ is a blow-up of $\pi(H) \cong \mathbb{P}^2$ in $\pi(H) \bigcap (C_1 \bigcup C_2) = \{8 \text{ points}\}$. By [TVV, Example 1.1(a)], $H$ turns out to be a Mori dream space. Set $Y_1 = H$, and choose $Y_2, Y_3$ sufficiently generally. This gives a good flag for $X$, and $\Delta_{Y_\bullet}(X)$ is a rational polyhedral cone by Theorem 3.1.5.

The second one is a rational homogeneous variety. In this case, the flag $Y_\bullet$ consisting of Schubert varieties seems to be the most natural one. We cannot expect that Schubert varieties are $\mathbb{Q}$-factorial, but still we can take their Bott-Samelson resolutions. The author heard from Dave Anderson that it has been proven that Bott-Samelson varieties are log Fano, hence are Mori dream spaces.

The problem is that we cannot expect that $Y_i$ is smooth at $Y_n$. In this sense $Y_\bullet$ is an "almost" good flag. If we use the chain of Bott-Samelson resolutions in the definition of the valuation, then the "global Okounkov body" defined by the valuation is rational polyhedral, though the valuation is not defined by a good flag in the usual sense.

## 3.5 Some combinatorial lemmas

In this section, we recall some elementary combinatorial facts which we need.

**Lemma 3.5.1.** *Let $\Delta \subset \mathbb{R}^{p+q}$ be a closed cone. Let $\pi : \mathbb{R}^{p+q} \to \mathbb{R}^p$ be the natural projection, and assume that $\pi(\Delta)$ is a rational polyhedral cone. Suppose furthermore that $\pi(\Delta)$ is decomposed into finitely many rational polyhedral cones, and for each cone $C$ $\pi^{-1}(C) \bigcap \Delta$ is rational polyhedral. Then $\Delta$ itself is a rational polyhedral cone.*

**Lemma 3.5.2.** *Let $T : \mathbb{R}^p \to \mathbb{R}^q$ be a linear mapping defined over $\mathbb{Q}$, and let $\Delta \subset \mathbb{R}^q$ be a rational polyhedral cone. Then so is $T^{-1}(\Delta)$. In particular, if we restrict $T$ to another rational polyhedral cone $\Delta' \subset \mathbb{R}^p$, the same conclusion holds: i.e. $(T|_{\Delta'})^{-1}(\Delta) = \Delta' \bigcap T^{-1}(\Delta)$ is a rational polyhedral cone.*

# 4

# Surfaces of globally $F$-regular type are of Fano type

## 4.1 Summary

It was proven ([SS, Theorem 1.2]) that a variety of Fano type defined over a field of characteristic zero is of globally $F$-regular type. In the same paper, they asked if the converse is true or not [SS, Question 7.1].

Since a variety of Fano type in characteristic zero is a Mori dream space [BCHM, Corollary 1.3.2], the converse to [SS, Theorem 1.2], if true, implies that a variety of globally $F$-regular type is a Mori dream space. If we assume in advance that the variety is a Mori dream space, we can prove the converse ([GOST, Theorem 1.2]).

The purpose of this chapter is to prove the converse for surfaces, without assuming that the variety is a Mori dream space.

**Theorem 4.1.1.** *Let $X$ be a projective surface defined over a field of characteristic zero. Suppose that $X$ is of globally $F$-regular type. Then $X$ is of Fano type.*

The strategy of our proof is an extension of that for the proof of [SS, Theorem 1.2]. We find a $(-K_X)$-MMP such that after finitely many divisorial contractions we obtain a model on which the anti-canonical divisor is semi-ample and big, and the singularity is at worst log terminal.

In the proof of [SS, Theorem 1.2] we took the existence of a $(-K_X)$-MMP for granted, since $X$ was assumed to be a Mori dream space. In the proof of Theorem 4.1.1, we first show that a globally $F$-regular surface is a Mori dream space (Proposition 4.2.4). Thus we get an anti-canonical MMP for $X_\mu$, where $X_\mu$ is a reduction of $X$ to a positive characteristic. The point is that we can lift it to characteristic zero. Since each step of the MMP is defined by a globally generated line bundles, it is enough to lift the line bundles and the global sections. These follow from a vanishing theorem for nef line bundles on globally $F$-regular varieties and some deformation theory. The rest of the argument is the same as the proof of [GOST, Theorem 1.2].

## 4.2 Preparations

**Definition 4.2.1.** Let $X$ be a normal projective variety, and $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$. The pair $(X, \Delta)$ is called a log Fano pair if it is a klt pair and $-(K_X + \Delta)$ is ample.

A normal projective variety $X$ is said to be of Fano type if there exists an effective $\mathbb{Q}$-divisor $\Delta$ on $X$ such that $(X, \Delta)$ is a log Fano pair.

**Lemma 4.2.2.** *Let $A$ be a complete discrete valuation ring, and $f : X_A \to \operatorname{Spec} A$ be a projective morphism whose geometric fibers are integral normal schemes. Set $\operatorname{Spec} A = \{\xi, \mu\}$, where $\xi$ (resp. $\mu$) is the generic (resp. the closed) point of $\operatorname{Spec} A$. We denote by $X_\xi$ ($X_\mu$) the generic (closed) fiber of $f$. Suppose that $H^2(X_\mu, \mathcal{O}_{X_\mu}) = 0$ holds. Then any line bundle $L_\mu$ on $X_\mu$ extends to a line bundle $L_A$ on $X_A$ so that $L_A|_{X_\mu} \simeq L_\mu$.*

*Proof.* Let $m$ be the maximal ideal of $A$. Set $A_n = A/m^{n+1}$ and $X_n = X_A \otimes_A A_n$. Consider the formal scheme $\hat{X}$ obtained by taking the completion of $X_A$ along $X_\mu$. If we could extend $L_\mu$ to a coherent sheaf $\hat{L_A}$ on $\hat{X}$, we can find a coherent sheaf $L_A$ on $X_A$ which restricts to $\hat{L_A}$ by [FGIKNV, Theorem 8.4.2]. Since every fiber of $L_A$ at a closed point of $X_A$ is one dimensional (recall that $L_A$ is an extension of the line bundle $L_\mu$), we see that $L_A$ is an invertible sheaf.

In order to construct $\hat{L_A}$, as is well known, it is enough to show that we can lift $L_n$ to $L_{n+1}$ for any $n \geq 0$. Set $L_0 = L_\mu$.

By [FGIKNV, Theorem 8.5.3 (b)], the obstruction to extending $L_n$ to a coherent sheaf on $X_{n+1}$ lies in $Ext^2_{X_n}(L_n, L_n) \simeq H^2(X_n, \mathcal{O}_{X_n})$, which is vanishing by the assumption (note that the normal bundle of $X_n$ in $X_{n+1}$ is trivial). Therefore we get an extension $L_{n+1}$ of $L_n$. $\square$

The following was proven in [SS, Theorem 1.2].

**Theorem 4.2.3.** *Let $X$ be a normal $\mathbb{Q}$-factorial projective variety defined over an $F$-finite field which is globally $F$-regular. Then $X$ is of Fano type. In particular, $-K_X$ is big.*

**Proposition 4.2.4.** *Let $X$ be a projective surface defined over an $F$-finite field, which is of globally $F$-regular type. Then $X$ is a Mori dream space.*

*Proof.* By [HWY, Lemma 2.1], the minimal resolution of $X$ is also globally $F$-regular type. By Theorem 2.1.1, it is enough to show that the minimal resolution is a Mori dream space. Hence we may assume that $X$ itself is smooth.

By [Sm, Corollary 4.3], it follows that $H^1(X, \mathcal{O}_X) = 0$. Moreover by Theorem 4.2.3, we see that $2K_{X_\mu}$ has no global section. By the characterization of rational surfaces ([Z]), we see that $X_\mu$ is rational. Since $-K_{X_\mu}$ is big by Theorem 4.2.3, we conclude that $X_\mu$ is a Mori dream space by [TVV, Theorem 1]. □

**Remark 4.2.5.** By Theorem 4.2.3, Proposition 4.2.4 would follow from the cone and contraction theorems for klt surface pairs in positive characteritcics, whose reference was unclear to the author. Recently Professor Fujino informed me of a preprint by Hiromu Tanaka in preparation, in which the author establishes such results.

The following is a key for the proof.

**Proposition 4.2.6.** *Let $A$ and $f : X_A \to \operatorname{Spec} A$ be as in Lemma 4.2.2. Assume that $X_\mu$ is globally $F$-regular. Suppose that there exists an algebraic fiber space $g_\mu : X_\mu \to Y_\mu$ to a normal projective variety $Y_\mu$. Then $g_\mu$ extends to an algebraic fiber space $g_A : X_A \to Y_A$ over $A$.*

*Proof.* Let $L_\mu$ be a line bundle on $X_\mu$ whose complete linear system defines the morphism $g_\mu$. Note in particular that $L_\mu$ is nef. Since $X_\mu$ is globally $F$-regular, we see $H^2(X_\mu, \mathcal{O}_{X_\mu}) = 0$ and $H^1(X_\mu, L_\mu) = 0$ by [Sm, Corollary 4.3]. By Lemma 4.2.2, $L_\mu$ lifts to a line bundle $L_A$ on $X_A$. By [Har, Chapter III, Corollary 12.9], we see that $H^1(X_A, L_A) = 0$ holds. Applying [Har, Chapter III, Theorem 12.11 (b)] for $i = 1$, and then [Har, Chapter III, Theorem 12.11 (a)] for $i = 0$, $H^0(X_A, L_A)$ is a free $A$ module and $H^0(X_A, L_A) \otimes_A k(\mu) \simeq H^0(X_\mu, L_\mu)$. Therefore we see that $L_A$ is globally generated over $\operatorname{Spec} A$ and its complete linear system defines an algebraic fiber space $g_A : X_A \to Y_A$, which restricts to $g_\mu$. □

In the final step, we need the following easy

**Proposition 4.2.7.** *Let $X$ be a normal projective variety over a field $k$, and $k \subset K$ be an extension of fields. If the base change $X_K = X \otimes_k K$ is a Mori dream space, so is $X$.*

*Proof.* Let $\Gamma \subset \operatorname{Div} X$ be a finitely generated group of Cartier divisors on $X$ which defines a Cox ring of $X$. Then

$$R_X(\Gamma) \otimes_k K \cong R_{X_K}(\Gamma_K)$$

holds, where $\Gamma_K$ is the pull-back of $\Gamma$ to $X_K$.

By Lemma 2.2.2, we see that $R_{X_K}(\Gamma_K) \cong R_X(\Gamma) \otimes_k K$ is of finite type over $K$. Since the finite generation descends under the base field extension, we see that $R_X(\Gamma)$ is of finite type over $k$, concluding the proof. $\square$

## 4.3 Proof of Theorem 4.1.1

*Proof of Theorem 4.1.1.* Let $\pi : X_A \to \operatorname{Spec} A$ be a model of $X$, and assume that $X_\mu$ is globally $F$-regular for a closed point $\mu \in \operatorname{Spec} A$. By Proposition 4.2.4, $X_\mu$ is a Mori dream space.

Since $X_\mu$ is a Mori dream space, we can run a $(-K_{X_\mu})$-MMP, say $X_\mu = X_{\mu,0} \to X_{\mu,1} \to \cdots \to X_{\mu,\ell}$ so that $-K_{X_{\mu,\ell}}$ is nef and big. Since each morphism is an algebraic fiber space, we see that all $X_{\mu,i}$ are globally $F$-regular type and Mori dream space ([GOST, Lemma 2.12] and Theorem 2.1.1). Since $X_{\mu,\ell}$ is a Mori dream space, $-K_{X_{\mu,\ell}}$ is semi-ample.

Let $A_\mu$ be the localization of $A$ at $\mu$, and let $\hat{A}$ be the completion of $A_\mu$ by its maximal ideal. We take the base change of $X_A \to \operatorname{Spec} A$ by $A \subset \hat{A}$. As in Lemma 4.2.2, we set $\xi$ (resp. $\mu$) the generic (closed) point of $\operatorname{Spec} \hat{A}$. Here we note that the generic fiber $X_\xi = X_A \otimes_A Q(\hat{A})$ is still of globally $F$-regular type.

By Proposition 4.2.6, the MMP described above lifts over $\operatorname{Spec} \hat{A}$. Taking its restriction to characteristic 0, we obtain a $(-K_{X_\xi})$-MMP $X_\xi = X_{\xi,0} \to X_{\xi,1} \to \cdots \to X_{\xi,\ell}$ such that $-K_{X_{\xi,\ell}}$ is semi-ample and big.

Using the similar arguments as the proof of [GOST, Theorem 1.2], we see that $X_\xi$ is a variety of Fano type. Therefore by the cone theorem (see [KM]), we see that $X_\xi$ is a Mori dream surface. By Proposition 4.2.7, $X$ itself is a Mori dream surface. Hence by [GOST, Theorem 1.2], we see that it is of Fano type. $\square$

## 4.4 What about higher dimensions?

We point out the remaining obstructions to extending our approach to higher dimensional cases.

1. We do not know if a globally $F$-regular variety is a Mori dream space. We suspect it is true, but is verified so far only for surfaces as seen above.

2. In dimensions greater than 2, we may have flips in anti-canonical MMPs. We can lift the flipping contraction to characteristic zero as proven

above, but we do not know if we can lift the flipped morphism.

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