Quasi-morphisms on the group of area-preserving diffeomorphisms of the 2-disk

2次元円板の面積保存微分同相群上の擬準同型

石田 智彦

QUASI-MORPHISMS ON THE GROUP OF AREA-PRESERVING DIFFEOMORPHISMS OF THE 2-DISK

TOMOHIKO ISHIDA

ABSTRACT. Recently Gambaudo and Ghys proved that there exist infinitely many quasi-morphisms on the group $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ of area-preserving diffeomorphisms of the 2-disk D^2. For the proof, they constructed a homomorphism from the space of quasi-morphisms on the braid group to the space of quasi-morphisms on $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$. In this paper, we study this homomorphism and prove its injectivity. We give several applications of our result to stable commutator length of some elements of $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ and conjugation-invariant norms on $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$.

CONTENTS

1. Introduction 1
2. Preliminaries 3
2.1. Hamiltonian diffeomorphisms 3
2.2. Braid groups 4
3. Gambaudo and Ghys' construction and proof of the main theorem 6
4. Kernel of the homomorphism Γ_n 12
5. Applications to stable commutator length 15
5.1. The case of the 2-disk 16
5.2. The case of the 2-sphere 19
6. Comparison with other quasi-morphisms 21
6.1. Ruelle's quasi-morphism 21
6.2. Calabi quasi-morphisms 22
6.3. Linearly independence 23
7. Conjugation-invariant norms 24
7.1. Conjugation-generated norms 24
7.2. Autonomous norm, fragmentation norm, Hofer norm and L^p norm 25
8. Extension to the group which does not fix the boundary 26
References 27

1. INTRODUCTION

For a group G, a function $\phi\colon G \to \mathbb{R}$ is called a *quasi-morphism* if the real valued function on $G \times G$ defined by

$$(g, h) \mapsto \phi(gh) - \phi(g) - \phi(h)$$

2000 *Mathematics Subject Classification.* Primary 37C15, Secondary 37E30.
Key words and phrases. area-preserving diffeomorphisms, symplectomorphisms, quasi-morphisms, pseudo-characters.

is bounded. The real number

$$D(\phi) = \sup_{g,h\in G} |\phi(gh) - \phi(g) - \phi(h)|$$

is called the *defect* of ϕ. We denote the $\mathbb{R}$-vector space of quasi-morphisms on the group G by $\hat{Q}(G)$. By definition, bounded functions on groups are quasi-morphisms. Hence we denote the set of bounded functions on the group G by $C_b^1(G;\mathbb{R})$ and consider the quotient space $\hat{Q}(G)/C_b^1(G;\mathbb{R})$. A quasi-morphism $\phi\colon G \to \mathbb{R}$ is said to be *homogeneous* if the equation

$$\phi(g^p) = p\,\phi(g)$$

holds for any $g \in G$ and $p \in \mathbb{Z}$. We denote by $Q(G)$ the subspace of $\hat{Q}(G)$ consisting of homogeneous quasi-morphisms. For any quasi-morphism ϕ, a homogeneous quasi-morphism $\tilde{\phi}$ is defined by setting

$$\tilde{\phi}(g) = \lim_{p\to\infty} \frac{1}{p}\phi(g^p).$$

The limit always exists for each element g of G. The new function $\tilde{\phi}$ is in fact a quasi-morphism equal to the original quasi-morphism ϕ as an element of $\hat{Q}(G)/C_b^1(G;\mathbb{R})$. Thus we can identify the quotient space $\hat{Q}(G)/C_b^1(G;\mathbb{R})$ with $Q(G)$. Homogeneous quasi-morphisms are invariant under conjugations. Therefore we are interested in $Q(G)$ rather than $\hat{Q}(G)$.

Let $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ be the group of area-preserving C^∞-diffeomorphisms of the 2-disk D^2, which are the identity on a neighborhood of the boundary. On the vector space $Q(\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2))$, the following theorem is known.

Theorem 1.1 (Entov-Polterovich [16], Gambaudo-Ghys [19]). *The vector space* $Q(\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2))$ *is infinite dimensional.*

To prove Theorem 1.1, Entov and Polterovich explicitly constructed uncountably many quasi-morphisms on $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$, which are linearly independent. We will briefly introduce their construction in Section 6. After that Gambaudo and Ghys constructed countably many quasi-morphisms on $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ by a different idea, which is to consider the suspension of area-preserving diffeomorphisms of the disk and average the value of the signature of the braids appearing in the suspension. By generalizing their strategy Brandenbursky [7] defined the homomorphism

$$\Gamma_n\colon Q(P_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)),$$

which we review in Section 3. Here, $P_n(D^2)$ denotes the pure braid group on n-strands.

Let $B_n(D^2)$ be the braid group on n-strands. The natural inclusion $i\colon P_n(D^2) \to B_n(D^2)$ induces the homomorphism $Q(i)\colon Q(B_n(D^2)) \to Q(P_n(D^2))$. In this paper, we study the homomorphism Γ_n and prove the following theorem.

Theorem 1.2. *The composition*

$$\Gamma_n \circ Q(i)\colon Q(B_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2))$$

is injective.

Since it is known that $Q(B_n(D^2))$ is an infinite dimensional space [4], Theorem 1.2 gives alternative proof of Theorem 1.1.

Acknowledgments. This paper is written under the supervision of Professor Takashi Tsuboi. The author wishes to thank him for many helpful advices and careful reading of the manuscript. The author also thanks to Professors Étienne Ghys and Shigeyuki Morita for their warmly encouragement and to Professor Shigenori Matsumoto for valuable suggestions. The author is very grateful to Morimichi Kawasaki for his interest in this work and for useful discussions. Works in Section 6 could not be done without him. The author is supported by JSPS Research Fellowships for Young Scientists (23·1352).

2. Preliminaries

In this section, we recall basic notions which will be needed.

2.1. **Hamiltonian diffeomorphisms.** In this subsection, we recall the definition of Hamiltonian diffeomorphisms and the Calabi homomorphism on the group of Hamiltonian diffeomorphisms.

A *symplectic manifold* is a pair (M, Ω) of a $2n$-dimensional C^∞-manifold M and non-degenerate closed 2-form $\Omega \in A^2(M;\mathbb{R})$ on M. Here, the 2-form Ω is called the *symplectic form*. A diffeomorphism h of (M, Ω) is a *symplectomorphism* if h preserves the 2-form Ω. We denote the group of symplectomorphisms of (M, Ω) by $\mathrm{Symp}^\infty(M, \Omega)$.

Suppose that a compactly supported C^∞-function $H\colon M \times [0,1] \to \mathbb{R}$ is given. We denote by H^t the function on M defined by

$$H^t(x) = H(x,t).$$

The *Hamiltonian vector field* $\{X^t\}_{t\in[0,1]}$ generated by H is the vector field defined by

$$dH^t(Y) = \Omega(X^t, Y),$$

for any vector field Y. Since the 2-form Ω is non-degenerate, $\{X^t\}_{t\in[0,1]}$ is well-defined. A symplectomorphism $h \in \mathrm{Symp}^\infty(M, \Omega)$ is a *Hamiltonian diffeomorphism* generated by $H\colon M \times [0,1] \to \mathbb{R}$ if h can be represented as a time one map of the flow of the time-dependent Hamiltonian vector field generated by H. We also say that $H\colon M \times [0,1] \to \mathbb{R}$ is a *Hamiltonian function* of h. We denote by $\mathrm{Ham}_C^\infty(M)$ the subgroup of $\mathrm{Symp}^\infty(M, \Omega)$ consisting of compactly supported Hamiltonian diffeomorphisms. Clearly the subgroup $\mathrm{Ham}_C^\infty(M)$ is contained in the identity component $\mathrm{Symp}_C(M, \Omega)_0$ of the group of compactly supported symplectomorphisms of (M, Ω).

For a manifold M, if its first cohomology group $H_C^1(M;\mathbb{R})$ with compact support is trivial, then the group $\mathrm{Ham}_C^\infty(M)$ coincides with $\mathrm{Symp}_C(M, \Omega)_0$. That is for any compactly supported symplectomorphism h, if there exist paths from the identity map to h, then we can choose a path generated by a time-dependent Hamiltonian vector field. In fact, if $\{h_t\}_{t\in[0,1]}$ is a path in $\mathrm{Symp}^\infty(M, \Omega)$ such that h_0 is the identity map and $h_1 = h$, then the vector field X^t defined by

$$X^t\,\big|_{h_t(x)} = \frac{dh_t}{dt}(x)$$

satisfies

$$\mathcal{L}_{X^t}\Omega = 0,$$

where $\mathcal{L}_{X^t}$ means the Lie derivative by X^t. Since $d\Omega = 0$, by the Cartan's formula we have

$$d(\iota_{X^t}\Omega) = 0,$$

where ι_{X^t} means the interior product with X^t. Since the symplectic manifold (M, Ω) satisfies $H^1_C(M;\mathbb{R}) = 0$, the 1-form $\iota_{X^t}\Omega$ is exact and thus there exist a compactly supported C^∞-function H^t such that $\iota_{X^t}\Omega = dH^t$.

A symplectic manifold (M, Ω) is *exact* if the symplectic form Ω is exact. In particular, if the manifold M has a trivial second cohomology, then (M, Ω) is exact for any symplectic form Ω. When (M, Ω) is exact and not closed, there exists a group homomorphism

$$\mathrm{Cal}\colon \mathrm{Ham}^\infty_C(M) \to \mathbb{R}$$

defined by

$$\mathrm{Cal}(h) = \int_{M\times[0,1]} H\Omega^n dt,$$

which was introduced by Calabi [12] and called the *Calabi homomorphism*. This homomorphism $\mathrm{Cal}\colon \mathrm{Ham}^\infty_C(M) \to \mathbb{R}$ is well-defined. That is, the value $\mathrm{Cal}(h)$ is independent of the choice of the Hamiltonian function H^t. Banyaga showed the following theorem.

Theorem 2.1 (Banyaga [2]).

(i) *If the symplectic manifold (M, Ω) is closed, then the group* $\mathrm{Ham}^\infty_C(M)$ *is simple.*

(ii) *If the symplectic manifold (M, Ω) is exact and not closed, then the kernel of the Calabi homomorphism* $\mathrm{Cal}\colon \mathrm{Ham}^\infty_C(M) \to \mathbb{R}$ *is simple.*

In this paper, we mainly consider two groups. One is the group $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ of area-preserving diffeomorphisms of D^2, which are the identity on a neighborhood of the boundary. The other one is the identity component $\mathrm{Diff}^\infty_\Omega(S^2)_0$ of the group of area-preserving diffeomorphisms of S^2. Since both D^2 and S^2 are 2-dimensional, these two groups are equal to the groups of symplectomorphisms of D^2 and S^2, respectively. Furthermore, since both D^2 and S^2 have trivial first cohomology groups with compact support, any area-preserving diffeomorphism on D^2 or S^2 is Hamiltonian.

2.2. **Braid groups.** In this subsection, we recall the definition and properties of braid groups following [6].

Let M be a manifold of dimension greater than 1. The n-fold *configuration space* $X_n(M)$ of M is the set of ordered distinct n points. That is,

$$X_n(M) = \{(x_1, \dots, x_n); x_i \in M \text{ for } i = 1, \dots, n \text{ and } x_i \neq x_j \text{ if } i \neq j\}.$$

Clearly the configuration space $X_n(M)$ can be considered as a submanifold of the product manifold M^n. Let $\mathfrak{S}_n$ be the symmetric group of n symbols. The group $\mathfrak{S}_n$ acts on $X_n(M)$ by the permutation. The *braid group* $B_n(M)$ and the *pure braid group* $P_n(M)$ of M on n-strands are the fundamental groups of the quotient space $X_n(M)/\mathfrak{S}_n$ and of $X_n(M)$, respectively. Note that the pure braid group $P_n(M)$ is a subgroup of the braid group $B_n(M)$. An element of $B_n(M)$ is called a *braid* and braid is *pure* if it is in $P_n(M)$.

Choose a base point $x^0 = (x^0_1, \dots, x^0_n)$ of the configuration space $X_n(M)$. For a braid $\beta \in B_n(M)$, choose a loop $l\colon [0,1] \to X_n(M)/\mathfrak{S}_n$ which represents β such that $l(0) = l(1) = [x^0]$. Then we uniquely have the lift $\tilde{l}\colon [0,1] \to X_n(M)$ of l such that $\tilde{l}(0) = x^0$. This lift $\tilde{l}(t) = (\tilde{l}_1(t), \dots, \tilde{l}_n(t))$ gives the n arcs $\tilde{l}_1, \dots, \tilde{l}_n$ in $M \times [0,1]$. We call each arc $\tilde{l}_i$ the i-th *strand* of the braid β and their union *geometric braid*. There exists $\bar{\beta} \in \mathfrak{S}_n$ such that

$$\bar{\beta}^{-1}\tilde{l}(1) = x^0.$$

This correspondence between $\beta \in B_n(M)$ and $\bar{\beta} \in \mathfrak{S}_n$ defines the canonical projection $B_n(M) \to \mathfrak{S}_n$ whose kernel is $P_n(M)$.

The classical Artin's braid group is the braid group $B_n(D^2)$ and its presentation is given by the following theorem.

Theorem 2.2 (Artin [1]).

$$B_n(D^2) = \left\langle \sigma_1, \dots, \sigma_{n-1};\ \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for each } i = 1, \dots, n-2 \end{array} \right\rangle.$$

Here, σ_i corresponds to the braid that i-th and $(i+1)$-st strands cross just once like Figure 1. The pure braid group $P_n(D^2)$ of D^2 is generated by pure braid $A_{i,j}$'s

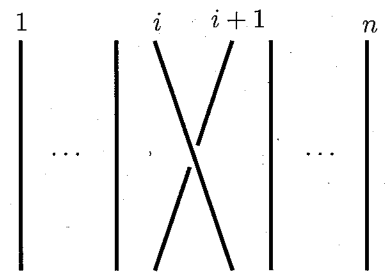

FIGURE 1. standard generator σ_i of $B_n(D^2)$

for $1 \leq i < j \leq n$ defined by

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2}\dots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\dots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}.$$

The pure braids $A_{i,j}$ twists only the i-th and the j-th strands (see Figure 2).

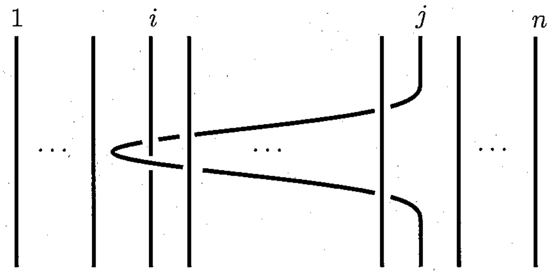

FIGURE 2. pure braid $A_{i,j}$

A presentation of the braid group $B_n(S^2)$ of S^2 is given by the following theorem.

Theorem 2.3 (Fadell-Van Buskirk [17]).

$$B_n(S^2) = \left\langle \delta_1, \dots, \delta_{n-1};\ \begin{array}{l} \delta_i\delta_j = \delta_j\delta_i \ \textit{if}\ |i-j| \geq 2 \\ \delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1}\ \textit{for each}\ i = 1, \dots, n-2 \\ \delta_1 \dots \delta_{n-2}\delta_{n-1}^2\delta_{n-2} \dots \delta_1 = 1 \end{array} \right\rangle .$$

Note that there exists a projection $B_n(D^2) \to B_n(S^2)$ which sends each σ_i to δ_i.

3. Gambaudo and Ghys' construction and proof of the main theorem

In this section, we review Gambaudo and Ghys' construction [19] of quasi-morphisms on the group $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ in a generalized form and prove Theorem 1.2.

For any $g \in \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ and for almost all $x = (x_1, \dots, x_n) \in X_n(D^2)$, we define the pure braid $\gamma(g; x)$ as follows. First we set the loop $l(g; x) \colon [0,1] \to X_n(D^2)$ by

$$l(g;x)(t) = \begin{cases} \{(1-3t)x_i^0 + 3tx_i\} & (0 \leq t \leq \frac{1}{3}) \\ \{g_{3t-1}(x_i)\} & (\frac{1}{3} \leq t \leq \frac{2}{3}) \\ \{(3-3t)g(x_i) + (3t-2)x_i^0\} & (\frac{2}{3} \leq t \leq 1) \end{cases},$$

where $\{g_t\}_{t\in[0,1]}$ is a Hamiltonian isotopy such that g_0 is the identity and $g_1 = g$. Of course for some $x \in X_n(D^2)$ the equation

$$(1-s)x_i^0 + sx_i = (1-s)x_j^0 + sx_j \quad \text{or} \quad sg(x_i) + (1-s)x_i^0 = sg(x_j) + (1-s)x_j^0$$

may hold and thus the loop $l(g;x)$ may not be defined. However, for almost every x the loop $l(g;x)$ is well-defined. For any $g \in \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ and almost all $x = (x_1, \dots, x_n) \in X_n(D^2)$ for which the loop $l(g;x)$ is well-defined, we define the braid $\gamma(g;x)$ to be the braid represented by the loop $l(g;x)$.

Furthermore, the braid $\gamma(g;x)$ is independent of the choice of the flow $\{g_t\}$. This is because of the fact the group $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ is contractible, which is easily proved from the contractibility of the diffeomorphism group $\mathrm{Diff}^\infty(D^2, \partial D^2)$ of D^2 [28] and the homotopy equivalence between $\mathrm{Diff}^\infty(D^2, \partial D^2)$ and $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ [23]. For a quasi-morphism ϕ on the pure braid group $P_n(D^2)$ on n-strands, we define the function $\hat{\Gamma}_n(\phi) \colon \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2) \to \mathbb{R}$ by

$$\hat{\Gamma}_n(\phi)(g) = \int_{x \in X_n(D^2)} \phi(\gamma(g;x))dx,$$

where the form dx means the volume form on $X_n(D^2)$ induced from the volume form Ω^n on the n-fold product space of D^2. Then the following lemma ensures that the homomorphism $\hat{\Gamma}_n \colon \hat{Q}(P_n(D^2)) \to \hat{Q}(\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2))$ is well-defined.

Lemma 3.1 (Brandenbursky [7]). *The function $\phi(\gamma(g; \cdot))$ is integrable on $X_n(D^2)$ for any $\phi \in \hat{Q}(P_n(D^2))$ and $g \in \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$.*

Proof. For a path $l \colon [0,1] \to X_n(D^2)$, and for each pair (i,j) such that $1 \leq i < j \leq n$, we define the map $\lambda_{i,j} \colon [0,1] \to S^1$ by

$$\lambda_{i,j}(t) = \frac{p_j(l(t)) - p_i(l(t))}{\|p_j(l(t)) - p_i(l(t))\|},$$

where the map $p_i\colon X_2(D^2) \to D^2$ is the natural i-th projection for $i = 1, \dots, n$. For $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we set $v_\theta \in S^1$ by $v_\theta = (\cos\theta, \sin\theta)$ and define $\chi_{i,j}(\theta)$ by

$$\chi_{i,j}(\theta) = \#\{t \in [0,1]; \lambda_{i,j}(t) = v_\theta\}.$$

for each pair (i,j) such that $1 \leq i < j \leq n$. Then $\chi_{i,j}(\theta)$ is well-defined for almost every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. By the change of variables, we have

$$\int_0^1 \left\| \frac{d}{dt}\lambda_{i,j}(t) \right\| dt = \int_{\mathbb{R}/2\pi\mathbb{Z}} \chi_{i,j}(\theta) d\theta. \tag{3.1}$$

We set the number $\Lambda_{i,j}(l)$ by

$$\Lambda_{i,j}(l) = \frac{1}{2\pi}\int_0^1 \left\| \frac{d}{dt}\lambda_{i,j}(t) \right\| dt,$$

which is the value of the integrals in Equation (3.1) divided by 2π.

For each braid $\gamma \in B_n(D^2)$, let us denote by $\mathrm{length}_{B_n}(\gamma)$ the word length of γ with respect to the standard generators $\{\sigma_i\}$ of $B_n(D^2)$. Suppose that $l\colon [0,1] \to X_n(D^2)$ is a loop and let $\gamma \in P_n(D^2)$ be the pure braid represented by l. For every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ such that $\chi_{i,j}(\theta)$ and $\chi_{i,j}(\theta+\pi)$ are well-defined for all pairs (i,j), the number

$$\sum_{i<j}(\chi_{i,j}(\theta) + \chi_{i,j}(\theta+\pi))$$

is equal to the number of double-points in the projected image of the geometric braid represented by the loop $l\colon [0,1] \to X_n(D^2)$ to the plane in $D^2 \times [0,1]$ perpendicular to the vector v_θ. The projected image of the geometric braid represented by the loop $l\colon [0,1] \to X_n(D^2)$ can be regarded as a braid diagram and the number of double points is the word length of the presentation given by the braid diagram. Since the braid represented by the braid diagram is conjugate to γ by an element of $P_n(D^2)$, we have the inequality

$$\min_{\alpha \in P_n(D^2)} \mathrm{length}_{B_n}(\alpha\gamma\alpha^{-1}) \leq \sum_{i<j}(\chi_{i,j}(\theta) + \chi_{i,j}(\theta+\pi))$$

for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ such that $\chi_{i,j}(\theta)$ and $\chi_{i,j}(\theta+\pi)$ are well-defined for all pairs (i,j). Hence we have

$$\min_{\alpha \in P_n(D^2)} \mathrm{length}_{B_n}(\alpha\gamma\alpha^{-1}) \leq \frac{1}{\pi}\sum_{i<j}\int_{\mathbb{R}/2\pi\mathbb{Z}} \chi_{i,j}(\theta)d\theta = 2\sum_{i<j}\Lambda_{i,j}(l). \tag{3.2}$$

Now we choose the path $l\colon [0,1] \to X_n(D^2)$ to be the loop $l(g;x)$, which represents the pure braid $\gamma(g;x)$. For a path $\{g_t\}_{t\in[0,1]}$ in $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ such that g_0 is the identity and $g_1 = g$, we define the map $L\colon X_2(D^2) \to \mathbb{R}$ by

$$L(x_1, x_2) = \frac{1}{2\pi}\int_0^1 \left\| \frac{d}{dt}\left(\frac{g_t(x_2) - g_t(x_1)}{\|g_t(x_2) - g_t(x_1)\|} \right) \right\| dt.$$

Let $l_1, l_2\colon [0,1] \to X_n(D^2)$ be the paths defined by

$$l_1(t) = l\left(\frac{t}{3}\right) = \{(1-t)x_i^0 + tx_i\} \quad \text{and} \quad l_2(t) = l\left(\frac{2+t}{3}\right) = \{(1-t)g(x_i) + tx_i^0\}.$$

Then the equality

$$\Lambda_{i,j}(l(g;x)) = \Lambda_{i,j}(l_1) + L(x_i, x_j) + \Lambda_{i,j}(l_2)$$

holds. Since

$$p_j(l_1(t)) - p_i(l_1(t)) = (1-t)(x_j^0 - x_i^0) + t(x_j - x_i),$$

the j-th strand of the braid $\gamma(g;x)$ turns at most half around its i-th strand in positive or negative direction from $t=0$ to $t=\frac{1}{3}$ and thus we have $\Lambda_{i,j}(l_1) \leq \frac{1}{2}$. Similarly $\Lambda_{i,j}(l_2) \leq \frac{1}{2}$ and thus we have

$$\Lambda_{i,j}(l(g;x)) \leq L(x_i, x_j) + 1. \tag{3.3}$$

By Inequalities (3.2) and (3.3), we have

$$\min_{\alpha \in P_n(D^2)} \mathrm{length}_{B_n}(\alpha\gamma(g;x)\alpha^{-1}) \leq 2\sum_{i<j}(L(x_i,x_j)+1). \tag{3.4}$$

On the other hand, for any pure braid $\gamma \in P_n(D^2)$,

$$|\phi(\gamma)| \leq (D(\phi)+M)\mathrm{length}_{P_n}(\gamma), \tag{3.5}$$

where length_{P_n} means the word length with respect to the generators $\{A_{i,j}\}$ of P_n and $M = \max|\phi(A_{i,j})|$. For any pure braids α and $\gamma \in P_n(D^2)$,

$$|\phi(1) - \phi(\alpha) - \phi(\alpha^{-1})| \leq D(\phi),$$

where 1 is the trivial braid, and

$$|\phi(\alpha\gamma\alpha^{-1}) - \phi(\alpha) - \phi(\gamma) - \phi(\alpha^{-1})| \leq 2D(\phi).$$

Hence we have the inequality

$$|\phi(\gamma)| \leq |\phi(\alpha\gamma\alpha^{-1})| + |\phi(1)| + 3D(\phi). \tag{3.6}$$

It is known (see for example [14]) that there exist two constants K_1 and K_2 such that

$$\mathrm{length}_{P_n}(\gamma) \leq K_1\mathrm{length}_{B_n}(\gamma) + K_2. \tag{3.7}$$

By Inequalities (3.5), (3.6) and (3.7), the inequality

$$\begin{aligned}|\phi(\gamma)| &\leq |\phi(\alpha\gamma\alpha^{-1})| + |\phi(1)| + 3D(\phi)\\ &\leq (D(\phi)+M)\mathrm{length}_{P_n}(\alpha\gamma\alpha^{-1}) + |\phi(1)| + 3D(\phi)\\ &\leq (D(\phi)+M)(K_1\mathrm{length}_{B_n}(\alpha\gamma\alpha^{-1}) + K_2) + |\phi(1)| + 3D(\phi)\end{aligned}$$

holds for any pure braids α and $\gamma \in P_n(D^2)$. By Inequality (3.4), we have

$$\begin{aligned}|\phi(\gamma(g;x))| \leq\ & 2K_1(D(\phi)+M)\sum_{i<j} L(x_i,x_j)\\ &+ (D(\phi)+M)(n(n-1)+K_2) + |\phi(1)| + 3D(\phi)\end{aligned} \tag{3.8}$$

and hence in order to show the integrability of the function $\phi(\gamma(g;\cdot))$, it is sufficient to prove that the function $L\colon X_2(D^2) \to \mathbb{R}$ is integrable over $X_2(D^2)$.

The integrability of the function $L\colon X_2(D^2) \to \mathbb{R}$ is proved by Gambaudo and Lagrange [20]. In fact,

$$\left\|\frac{d}{dt}\left(\frac{g_t(x_2)-g_t(x_1)}{\|g_t(x_2)-g_t(x_1)\|}\right)\right\| = \frac{|(\frac{d}{dt}(g_t(x_2)-g_t(x_1))) \wedge (g_t(x_2)-g_t(x_1))|}{\|g_t(x_2)-g_t(x_1)\|^2},$$

where the symbol $\wedge$ means the wedge product;

$$(a,b)\wedge(c,d) = ad - bc.$$

Hence by the Cauchy-Schwarz inequality,

$$\left\| \frac{d}{dt}\left(\frac{g_t(x_2)-g_t(x_1)}{\|g_t(x_2)-g_t(x_1)\|}\right)\right\| \leq \frac{2\sup\|\frac{d}{dt}g_t(x_0)\|}{\|g_t(x_2)-g_t(x_1)\|},$$

where the supremum is taken over $(t, x_0) \in [0,1] \times D^2$. Hence we have

$$\begin{aligned} L(x_1,x_2) &= \frac{1}{2\pi}\int_0^1 \left\| \frac{d}{dt}\left(\frac{g_t(x_2)-g_t(x_1)}{\|g_t(x_2)-g_t(x_1)\|}\right)\right\| dt \\ &\leq C\int_0^1 \frac{1}{\|g_t(x_2)-g_t(x_1)\|}dt, \end{aligned}$$

where

$$C = \frac{\sup\|\frac{d}{dt}g_t(x_0)\|}{\pi}.$$

Therefore,

$$\int_{x\in X_2(D^2)} L(x_1,x_2)dx \leq C\int_{x\in X_2(D^2)}\int_0^1 \frac{dtdx}{\|g_t(x_2)-g_t(x_1)\|},$$

Since each diffeomorphism g_t is area-preserving, we have

$$\int_{x\in X_2(D^2)}\int_0^1 \frac{dtdx}{\|g_t(x_2)-g_t(x_1)\|} = \int_{x\in X_2(D^2)} \frac{dx}{\|x_2-x_1\|}.$$

Since the integral of the right hand side converges, the integrability of the function $L\colon X_2(D^2) \to \mathbb{R}$ follows. □

The function $\hat{\Gamma}_n(\phi)\colon \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2) \to \mathbb{R}$ is also a quasi-morphism. In fact, the equation

$$\gamma(gh;x) = \gamma(h;x)\gamma(g;h_*x)$$

holds, where the map $h_*\colon X_n(D^2) \to X_n(D^2)$ is the diagonal action of the area-preserving diffeomorphism h on $X_n(D^2)$ and thus we have

$$|\phi(\gamma(gh;x)) - \phi(\gamma(h;x)) - \phi(\gamma(g;h_*x))| \leq D(\phi).$$

Since h is area-preserving,

$$\begin{aligned} &\left|\hat{\Gamma}_n(\phi)(gh) - \hat{\Gamma}_n(\phi)(g) - \hat{\Gamma}_n(\phi)(h)\right| \\ &= \left|\int_{x\in X_n(D^2)} (\phi(\gamma(gh;x)) - \phi(\gamma(h;x)) - \phi(\gamma(g;x)))dx\right| \\ &\leq \int_{x\in X_n(D^2)} |\phi(\gamma(gh;x)) - \phi(\gamma(h;x)) - \phi(\gamma(g;h_*x))|dx \\ &\leq D(\phi)\mathrm{area}(D^2). \end{aligned} \tag{3.9}$$

The map $\hat{\Gamma}_n\colon \hat{Q}(P_n(D^2)) \to \hat{Q}(\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2))$ is clearly $\mathbb{R}$-lenear. Since quasi-morphisms obtained in this way are not homogeneous, we define $\Gamma_n\colon Q(P_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2))$ by

$$\Gamma_n(\phi)(g) = \lim_{p\to\infty}\frac{1}{p}\hat{\Gamma}_n(\phi)(g^p).$$

Remark 3.2. By Inequality (3.9), it is also ensured that the homomorphism $\Gamma_n\colon Q(P_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$ maps the classical linking number homomorphism $\mathrm{lk}_n\colon B_n(D^2) \to \mathbb{Z}$ on the braid group to a homomorphism on $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$. In fact, the image $\Gamma_n(\mathrm{lk}_n)$ of $\mathrm{lk}\colon B_n(D^2) \to \mathbb{Z}$ coincides with a constant multiple of the classical Calabi homomorphism on $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ [18] and in this sense quasi-morphisms obtained in this way can be considered as generalizations of the Calabi homomorphism. By the proof of Lemma 3.1, it is observed that quasi-morphisms obtained by the homomorphism $\hat{\Gamma}_n\colon \hat{Q}(P_n(D^2)) \to \hat{Q}(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$ can be defined on the group of area-preserving C^1-diffeomorphisms of D^2, as well as the Calabi homomorphism.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let us suppose that a homogeneous quasi-morphism $\phi \in Q(B_n(D^2))$ is non-trivial. Then there exists a braid $\beta \in B_n(D^2)$ such that $\phi(\beta) \neq 0$. We may assume that β is pure. It is sufficient to prove that the homogeneous quasi-morphism $\Gamma_n(\phi) \in \hat{Q}(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$ is also non-trivial. That is, there exists an area-preserving diffeomorphism $g \in \mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ such that

$$\Gamma_n(\phi)(g) \neq 0.$$

Since the braid β is pure, it can be written as a composition of $A_{i,j}$'s and their inverses. We take n disjoint subsets U_i's of D^2. Furthermore, for a pair of (i,j), we take subsets $V_{i,j}$ and $W_{i,j}$ of D^2 such that $U_i \cup U_j \subset W_{i,j} \subset V_{i,j}$, $U_k \cap V_{i,j} = \emptyset$ if $k \neq i,j$ and $V_{i,j}, W_{i,j}$ are diffeomorphic to D^2 as Figure 3. Let $\{h_t\}_{t\in[0,1]}$ be a path

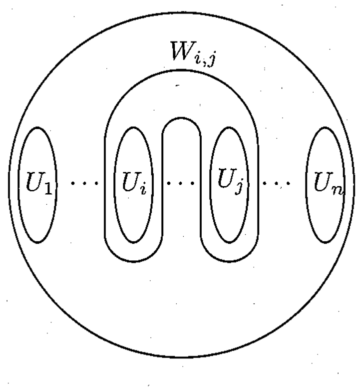

FIGURE 3. domains U_i's and $W_{i,j}$

in $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ such that the support of h_t is contained in the interior of $V_{i,j}$ and rotates $W_{i,j}$ once. Then for $h = h_1$ and $(x_1,\dots,x_n) \in X_n(D^2)$ such that $x_i \in U_i$, the braid $\gamma(h;x)$ is conjugate to the pure braid $A_{i,j}$ for $(x_1,\dots,x_n) \in X_n(D^2)$ such that $x_i \in U_i$. Taking paths $\{h_t\}$'s constructed above for the all $A_{i,j}$'s which present β and composing them, we have a path $\{g_t\}_{t\in[0,1]}$ in $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ with $g_0 = \mathrm{id}$ which twists U_i's in the form of the pure braid β. If we set $g = g_1$, then g is the identity on U_i's and $\gamma(g;(x_1,\dots,x_n)) = \beta$ for $(x_1,\dots,x_n) \in X_n(D^2)$ such that

$x_i \in U_i$. Then by setting $U = U_1 \cup \cdots \cup U_n$, we have

$$\Gamma_n(\phi)(g) = \lim_{p\to\infty} \frac{1}{p}\left(\int_{x\in X_n(U)} \phi(\gamma(g^p;x))dx + \int_{x\in X_n(D^2)\setminus X_n(U)} \phi(\gamma(g^p;x))dx\right)$$
$$= \int_{x\in X_n(U)} \phi(\gamma(g;x))dx + \lim_{p\to\infty}\frac{1}{p}\int_{x\in X_n(D^2)\setminus X_n(U)} \phi(\gamma(g^p;x))dx.$$

If we denote the first term of the equation by Y and set $a_i = \mathrm{area}(U_i)$ and $[n] = \{1,\ldots,n\}$, then Y is written as

$$\int_{x\in X_n(U)} \phi(\gamma(g;x))dx = \sum_{F\colon [n]\to[n]} \left(\prod_{i=1}^{n} a_{F(i)}\right) x_F,$$

where $x_F = \phi(\gamma_F)$ and $\gamma_F = \gamma(g;x)$ for x in the case that each x_i is in $U_{F(i)}$. The real numbers x_F's have the following properties.

(i) For two maps F and $G\colon [n]\to[n]$, if $\#F^{-1}(i) = \#G^{-1}(i)$ for each $1\le i\le n$ then $x_F = x_G$.

(ii) If a map $F\colon [n]\to[n]$ is bijective, then x_F is non-zero.

The property (i) follows from the invariance of ϕ under conjugation and the property (ii) follows because $\phi(\beta)$ is non-zero. Therefore, the coefficient of $a_1\ldots a_n$ in Y is non-zero. Since the polynomial Y is not identically 0, we can choose a_i's such that Y is non-zero.

Note that if we replace a_i's by bigger ones fixing the ratio of any two of them the term Y stays non-zero. On the other hand, the values $\phi(\gamma(g;x))$ is bounded. In fact, for each word $A_{i,j}$ we set the path $\{h_t\}$ in $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ such that an open set $W_{i,j}$ rotates once, for $\{h_t\}$ above and any $(x_1,x_2)\in X_2(D^2)$,

$$L(x_1,x_2) = \frac{1}{2\pi}\int_0^1 \left\|\frac{d}{dt}\left(\frac{h_t(x_2)-h_t(x_1)}{\|h_t(x_2)-h_t(x_1)\|}\right)\right\| dt$$

is bounded by 2. Hence for an area-preserving diffeomorphism $g\in \mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ we constructed, the inequality

$$L(x_1,x_2) = \frac{1}{2\pi}\int_0^1 \left\|\frac{d}{dt}\left(\frac{g_t(x_2)-g_t(x_1)}{\|g_t(x_2)-g_t(x_1)\|}\right)\right\| dt \le 2\ \mathrm{length}_{P_n}(\beta).$$

holds. By Inequality (3.8) we have

$$|\phi(\gamma(g^p;x))| \le 2n(n-1)K_1(D(\phi)+M)p\ \mathrm{length}_{P_n}(\beta)$$
$$+ (D(\phi)+M)(n(n-1)+K_2) + |\phi(1)| + 3D(\phi).$$

Therefore,

$$\lim_{p\to\infty}\frac{1}{p}\left|\int_{x\in X_n(D^2)\setminus X_n(U)} \phi(\gamma(g^p;x))dx\right|$$
$$\le \lim_{p\to\infty}\frac{1}{p}\mathrm{area}(X_n(D^2)\setminus X_n(U))$$
$$\{2n(n-1)K_1(D(\phi)+M)p\ \mathrm{length}_{P_n}(\beta)$$
$$+ (D(\phi)+M)(n(n-1)+K_2) + |\phi(1)| + 3D(\phi)\}$$
$$= 2\mathrm{area}(X_n(D^2)\setminus X_n(U))n(n-1)K_1(D(\phi)+M)\ \mathrm{length}_{P_n}(\beta)$$
$$\to 0 \quad (\text{as} \quad a_1+\cdots+a_n \to \mathrm{area}(D^2)).$$

This completes the proof. □

As we noted in Remark 3.2, The homomorphism $\hat{\Gamma}_n$ maps any homomorphism on $P_n(D^2)$ to a homomorphism on $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ up to a bounded function. Hence the homomorphism

$$Q(P_n(D^2))/H^1(P_n(D^2);\mathbb{R}) \to Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))/H^1(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2);\mathbb{R})$$

is also induced. By an argument similar to the proof of Theorem 1.2, the following proposition holds.

Proposition 3.3. *The map*

$$Q(B_n(D^2))/H^1(B_n(D^2);\mathbb{R}) \to Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))/H^1(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2);\mathbb{R})$$

induced by the composition $\Gamma_n \circ Q(i)\colon Q(B_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$ *is injective.*

The homomorphism $\Gamma_n\colon Q(P_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$ can be defined also for the 2-sphere S^2 instead of D^2 as Gambaudo and Ghys mentioned in their paper. Let $\mathrm{Diff}^\infty_\Omega(S^2)_0$ be the identity component of the group of area-preserving diffeomorphisms of S^2. Then we can choose a pure braid $\gamma(g;x) \in P_n(S^2)$ for any $g \in \mathrm{Diff}^\infty_\Omega(S^2)_0$ and almost every $x \in X_n(S^2)$ as in the case of the 2-disk. Since the group $\mathrm{Diff}^\infty_\Omega(S^2)_0$ is homotopy equivalent to $SO(3)$ [23][28] and its fundamental group has order 2, for any element g of $\mathrm{Diff}^\infty_\Omega(S^2)_0$ there exist two homotopy classes of paths connecting the identity and g in $\mathrm{Diff}^\infty_\Omega(S^2)_0$. However, for any homogeneous quasi-morphism ϕ on $P_n(S^2)$, the value $\phi(\gamma(g;x))$ is independent of the choice of the path. In fact, from a path which represents the generator of $\pi_1(\mathrm{Diff}^\infty_\Omega(S^2)_0)$ the pure braid $\xi_n = (\delta_1 \dots \delta_{n-1})^n$ is obtained. The pure braid ξ_n has order 2 and is in the center of $P_n(S^2)$. Hence the homomorphism $\Gamma_n\colon Q(P_n(S^2)) \to Q(\mathrm{Diff}^\infty_\Omega(S^2)_0)$ is defined. Since the braid group $B_n(S^2)$ of the 2-sphere on n-strands can be considered as a quotient group of the braid group $B_n(D^2)$ (Theorems 2.2, 2.3), by an argument similar to the proof of Theorem 1.2, we obtain the following theorem.

Theorem 3.4. *The composition*

$$\Gamma_n \circ Q(i)\colon Q(B_n(S^2)) \to Q(\mathrm{Diff}^\infty_\Omega(S^2)_0)$$

is injective.

The homomorphism $Q(i)$ in the statement of Theorem 3.4 is the one induced from the inclusion $i\colon P_n(S^2) \to B_n(S^2)$.

4. Kernel of the homomorphism Γ_n

The homomorphism $\Gamma_n\colon Q(P_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$ itself is not injective although Theorem 1.2 holds. In this section we study the kernel of the homomorphism Γ_n.

Let G be a group and H its finite index subgroup. We denote by $\overline{\beta}$ the image of an element $\beta \in G$ by the natural projection $G \to G/H$. For each left coset $\sigma \in G/H$ of G modulo H, we fix an element $\gamma_\sigma \in G$ such that $\overline{\gamma_\sigma} = \sigma$ and for any $\phi \in \hat{Q}(H)$ define the function $\hat{\mathcal{T}}(\phi)\colon G \to \mathbb{R}$ by

$$\hat{\mathcal{T}}(\phi)(\beta) = \frac{1}{(G:H)} \sum_{\sigma \in G/H} \phi(\gamma_{\overline{\beta\gamma_\sigma}}{}^{-1}\beta\gamma_\sigma).$$

Since $\gamma_{\overline{\beta\gamma_\sigma}}{}^{-1}\beta\gamma_\sigma$ is in H, the function $\hat{\mathcal{T}}(\phi)$ is well-defined on G.

Lemma 4.1. *For any quasi-morphism ϕ on H, the function $\hat{\mathcal{T}}(\phi)\colon G \to \mathbb{R}$ is also a quasi-morphism.*

Proof. Since the equality

$$\gamma_{\overline{\beta_1\beta_2\gamma_\sigma}}{}^{-1}\beta_1\beta_2\gamma_\sigma = (\gamma_{\overline{\beta_1\beta_2\gamma_\sigma}}{}^{-1}\beta_1\gamma_{\overline{\beta_2\gamma_\sigma}})(\gamma_{\overline{\beta_2\gamma_\sigma}}{}^{-1}\beta_2\gamma_\sigma)$$

holds, we have the inequality

$$\begin{aligned}
&|\hat{\mathcal{T}}(\phi)(\beta_1\beta_2) - \hat{\mathcal{T}}(\phi)(\beta_1) - \hat{\mathcal{T}}(\phi)(\beta_2)| \\
&= \frac{1}{(G:H)}\left| \sum_{\sigma\in G/H} \Big\{\phi((\gamma_{\overline{\beta_1\beta_2\gamma_\sigma}}{}^{-1}\beta_1\gamma_{\overline{\beta_2\gamma_\sigma}})(\gamma_{\overline{\beta_2\gamma_\sigma}}{}^{-1}\beta_2\gamma_\sigma)) \right. \\
&\qquad\qquad \left. - \phi(\gamma_{\overline{\beta_1\gamma_\sigma}}{}^{-1}\beta_1\gamma_\sigma) - \phi(\gamma_{\overline{\beta_2\gamma_\sigma}}{}^{-1}\beta_2\gamma_\sigma)\Big\} \right| \\
&= \frac{1}{(G:H)}\left| \sum_{\sigma\in G/H} \Big\{\phi((\gamma_{\overline{\beta_1\beta_2\gamma_\sigma}}{}^{-1}\beta_1\gamma_{\overline{\beta_2\gamma_\sigma}})(\gamma_{\overline{\beta_2\gamma_\sigma}}{}^{-1}\beta_2\gamma_\sigma)) \right. \\
&\qquad\qquad \left. - \phi(\gamma_{\overline{\beta_1\beta_2\gamma_\sigma}}{}^{-1}\beta_1\gamma_{\overline{\beta_2\gamma_\sigma}}) - \phi(\gamma_{\overline{\beta_2\gamma_\sigma}}{}^{-1}\beta_2\gamma_\sigma)\Big\} \right| \\
&\leq D(\phi).
\end{aligned}$$

Hence the function $\hat{\mathcal{T}}(\phi)\colon G \to \mathbb{R}$ is also a quasi-morphism. □

The map $\hat{\mathcal{T}}$ is a homomorphism between vector spaces from $\hat{Q}(H)$ to $\hat{Q}(G)$. We define $\mathcal{T}\colon Q(H) \to Q(G)$ by

$$\mathcal{T}(\phi)(\beta) = \lim_{p\to\infty} \frac{1}{p}\hat{\mathcal{T}}(\phi)(\beta^p).$$

Furthermore, the following proposition holds.

Proposition 4.2. *The homomorphism $\mathcal{T}\colon Q(H) \to Q(G)$ is independent of the choice of γ_σ's.*

Proof. Suppose that ϕ is a homogeneous quasi-morphism on H. If an element β is in H, then $\overline{\gamma_\sigma\beta} = \sigma$ for each $\sigma \in G/H$. For any $\beta \in G$ there exists an integer k such that β^k is in H and we have

$$\begin{aligned}
\mathcal{T}(\phi)(\beta) &= \lim_{p'\to\infty} \frac{1}{kp'}\hat{\mathcal{T}}(\phi)(\beta^{kp'}) \\
&= \lim_{p'\to\infty} \frac{1}{(G:H)kp'} \sum_{\sigma\in G/H} \phi(\gamma_\sigma^{-1}\beta^k\gamma_\sigma)^{p'} \\
&= \frac{1}{(G:H)k} \sum_{\sigma\in G/H} \phi(\gamma_\sigma^{-1}\beta^k\gamma_\sigma). \qquad (4.1)
\end{aligned}$$

Since ϕ is invariant under conjugations in H, the value $\phi(\gamma_\sigma^{-1}\beta^k\gamma_\sigma)$ depends only on σ. □

Let $Q(i)\colon Q(G) \to Q(H)$ be the homomorphism induced by the inclusion $i\colon H \to G$. As a corollary to the Equality (4.1), we have the following.

Corollary 4.3. *The composition*

$$\mathcal{T} \circ Q(i) \colon Q(G) \to Q(G)$$

is the identity on $Q(G)$. *Furthermore, we have the decomposition*

$$Q(H) = \mathrm{Ker}(\mathcal{T}) \oplus \mathrm{Im}(Q(i))$$

as vector spaces.

Remark 4.4. Of course, the homomorphism $\hat{\mathcal{T}}(\phi) \colon G \to \mathbb{R}$ can be defined using the right coset $H \backslash G$ instead of G/H by

$$\hat{\mathcal{T}}(\phi)(\beta) = \frac{1}{(G:H)} \sum_{\sigma \in G/H} \phi(\gamma_\sigma \beta \gamma_{\overline{\gamma_\sigma \beta}}^{-1}).$$

By an argument similar to the proof of Lemma 4.1 and Proposition 4.2, it is verified that this alternative definition is also well-defined and induces the same homomorphism $\mathcal{T} \colon Q(H) \to Q(G)$.

Remark 4.5. The homomorphism $\mathcal{T} \colon Q(H) \to Q(G)$ is just a straightforward generalization of transfer map, and it is also introduced in [22] and [29].

Since the pure braid groups $P_n(D^2)$ and $P_n(S^2)$ are finite index subgroups of the braid groups $B_n(D^2)$ and $B_n(S^2)$, respectively, the homomorphisms

$$\mathcal{T} \colon Q(P_n(D^2)) \to Q(B_n(D^2)) \quad \text{and} \quad \mathcal{T} \colon Q(P_n(S^2)) \to Q(B_n(S^2))$$

can be defined and Corollary 4.3 is true for $G = B_n(D^2), H = P_n(D^2)$ and $G = B_n(S^2), H = P_n(S^2)$, respectively.

The following proposition is the main result of this section.

Proposition 4.6. *The composition*

$$\Gamma_n \circ Q(i) \circ \mathcal{T} \colon Q(P_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2))$$

coincides with Γ_n. *In particular,* $\mathrm{Ker}(\Gamma_n) = \mathrm{Ker}(\mathcal{T})$ *and* $\mathrm{Im}(\Gamma_n) = \mathrm{Im}(\Gamma_n \circ Q(i))$.

Proof. By the Equality (4.1), for any homogeneous quasi-morphism $\phi \in Q(P_n(D^2))$ and any area-preserving diffeomorphism $g \in \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$,

$$\hat{\Gamma}_n \circ Q(i) \circ \mathcal{T}(\phi)(g) = \int_{x \in X_n(D^2)} \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \phi(\gamma_\sigma \gamma(g;x) \gamma_\sigma^{-1}) dx. \tag{4.2}$$

For any $\sigma \in \mathfrak{S}_n$ and almost all $x \in D^2$, we set the path $l \colon [0,1] \to X_n(D^2)$ by

$$l(t) = \begin{cases} \{(1-2t)x_i^0 + 2tx_i\} & (0 \le t \le \frac{1}{2}) \\ \{(2-2t)x_i + (2t-1)x_{\sigma(i)}^0\} & (\frac{1}{2} \le t \le 1) \end{cases}.$$

Considering the path l as a loop in the quotient space $X_n(D^2)/\mathfrak{S}_n$, we define the braid $\beta(\sigma;x)$ to be the braid represented by the loop l. Then by definition,

$$\beta(\sigma;x)\gamma(g;\sigma^{-1}(x))\beta(\sigma;g_*^{-1}x)^{-1} = \gamma(g;x),$$

where, the symmetric group $\mathfrak{S}_n$ acts on $X_n(D^2)$ by the permutation

$$\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Since the homomorphism $\mathcal{T}\colon Q(P_n(D^2)) \to Q(B_n(D^2))$ is defined independently to the choice of braids γ_σ's, we may choose γ_σ to be $\beta(\sigma; x)$. Hence we have

$$\begin{aligned}\gamma_\sigma\gamma(g;\sigma^{-1}(x))\gamma_\sigma^{-1} &= \beta(\sigma;x)\gamma(g;\sigma^{-1}(x))\beta(\sigma;x)^{-1}\\ &= \gamma(g;x)\beta(\sigma;g_*(x))\beta(\sigma;x)^{-1}. \qquad (4.3)\end{aligned}$$

As in the proof of Lemma 3.1, the function $\beta(\sigma;\cdot)$ is bounded on D^2 and by Equalities (4.2) and (4.3) we have

$$\begin{aligned}\Gamma_n \circ Q(i) \circ \mathcal{T}(\phi)(g) &= \lim_{p\to\infty} \frac{1}{p}\hat{\Gamma}_n \circ Q(i) \circ \mathcal{T}(\phi)(g^p)\\ &= \lim_{p\to\infty} \frac{1}{p} \int_{x\in X_n(D^2)} \phi(\gamma(g^p;x))dx\\ &= \Gamma_n(\phi)(g).\end{aligned}$$

Hence $\Gamma_n \circ Q(i) \circ \mathcal{T} = \Gamma_n$.

Then obviously $\mathrm{Ker}(\mathcal{T}) \subseteq \mathrm{Ker}(\Gamma_n)$ and $\mathrm{Im}(\Gamma_n) = \mathrm{Im}(\Gamma_n \circ Q(i))$ hold. If $\phi \in \mathrm{Ker}(\Gamma_n)$ then

$$\Gamma_n \circ Q(i) \circ \mathcal{T}(\phi) = \Gamma_n(\phi) = 0$$

and hence $\mathcal{T}(\phi) = 0$ by Theorem 1.2. Thus we have $\mathrm{Ker}(\Gamma_n) \subseteq \mathrm{Ker}(\mathcal{T})$. □

Proposition 4.6 also holds for the groups $B_n(S^2)$ and $P_n(S^2)$.

5. Applications to stable commutator length

In this section, we compute the lower bound of stable commutator lengths of some elements of $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ and $\mathrm{Diff}^\infty_\Omega(S^2)_0$ applying quasi-morphism on these groups.

For a perfect group G, the *commutator length* $\mathrm{cl}(g)$ of $g \in G$ is defined by the minimal number of commutators in G whose product is equal to g. Here, for the identity element id, we define $\mathrm{cl}(\mathrm{id}) = 0$. For $g \in G$, the *stable commutator length* $\mathrm{scl}(g)$ is defined by

$$\mathrm{scl}(g) = \lim_{p\to\infty} \frac{1}{p}\mathrm{cl}(g^p).$$

A perfect group G is *uniformly perfect* if there exist a natural number N such that any $g \in G$ can be written as a product of at most N commutators. When the group G is uniformly perfect, the stable commutator length on G is obviously trivial. Commutator lengths and stable commutator lengths are also invariant under conjugations and related with quasi-morphisms. The following theorem is called the Bavard's duality theorem.

Theorem 5.1 (Bavard [3]). *Suppose that G be a perfect group. For any $g \in G$, the equation*

$$\mathrm{scl}(g) = \sup_{\phi\in Q(G)} \frac{|\phi(g)|}{2D(\phi)}$$

holds.

In the light of the Bavard's duality theorem, if a perfect group G admits non-trivial quasi-morphism then G has elements which have non-trivial stable commutator lengths. In particular, G is not uniformly perfect. Therefore, the commutator subgroup of $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ is not uniformly perfect and nor is the group $\mathrm{Diff}^\infty_\Omega(S^2)_0$. In contrast, the group of ordinary C^∞-diffeomorphisms of D^2, which are the identity near the boundary and the identity component of the group of ordinary C^∞-diffeomorphisms of S^2 are uniformly perfect [11] and thus they admit no non-trivial quasi-morphisms. Hence we have to consider the groups of area-preserving diffeomorphisms to construct the non-trivial homomorphisms $\Gamma_n\colon Q(P_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2))$ or $\Gamma_n\colon Q(P_n(S^2)) \to Q(\mathrm{Diff}^\infty_\Omega(S^2)_0)$.

5.1. **The case of the 2-disk.** We identify D^2 with the subset
$$\{(x,y) \in \mathbb{R}^2; x^2+y^2 \le 1\}$$
of $\mathbb{R}^2$. Let $\omega\colon [0,1] \to \mathbb{R}$ be a C^∞-function which is equal to 0 on a neighborhood of 1 and constant on a neighborhood of 0. We define the area-preserving diffeomorphism $F_\omega \in \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ of D^2 by the map which rotate each $x \in D^2$ around 0 by the angle $\omega(|x|)$. Now we evaluate the value of $\Gamma_n(\phi)$ at F_ω for a homogeneous quasi-morphism $\phi \in Q(P_n(D^2))$. Let $\eta_{i,n}$ be the pure braid defined by $\eta_{i,n} = A_{1,i}A_{2,i}\dots A_{i-1,i}$ for $2 \le i \le n$ (see Figure 4) and $a(r)$ the area of the disk of radius r in D^2 centered at the origin. The following lemma is the same as that proved in [19].

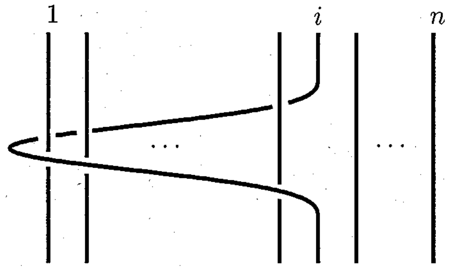

FIGURE 4. pure braid $\eta_{i,n}$

Lemma 5.2. *For any homogeneous quasi-morphism $\phi \in Q(B_n(D^2))$,*
$$\Gamma_n(\phi)(F_\omega) = \sum_{i=2}^{n} i\binom{n}{i}\phi(\eta_{i,n})\int_0^1 \omega(r)a(r)^{i-1}(1-a(r))^{n-i}da(r).$$

Proof. Since $\phi \in Q(B_n(D^2))$,
$$\begin{aligned}\Gamma_n(\phi)(F_\omega) &= \lim_{p\to\infty}\frac{1}{p}\int_{x\in X_n(D^2)}\phi(\gamma(F^p_\omega;x))dx\\ &= n!\lim_{p\to\infty}\frac{1}{p}\int_{|x_1|<\dots<|x_n|}\phi(\gamma(F^p_\omega;x))dx.\end{aligned}$$
Since the quasi-morphism ϕ is homogeneous, the value $\phi(\gamma(F^p_\omega;x))$ does not depend on the choice of the base point $x^0 \in X_n(D^2)$. Hence we may assume that $x^0_i = x_i$. We choose a Hamiltonian path g_t which connects the identity map with F_ω such that
$$g_t = F_{t\omega}.$$
Then we have a loop $l\colon [0,1] \to X_n(D^2)$ which represents the pure braid $\gamma(F_\omega;x)$. A path $l|_{[0,\frac{1}{3}]}$ represents the trivial braid and if t is in $[\frac{2}{3},1]$ then the i-th strand of the pure braid $\gamma(F_\omega;x)$ connects $F_\omega(x_i)$ to x^0_i by a straight line. When t is in

$[\frac{1}{3}, \frac{2}{3}]$, for each $i \geq 2$, the i-th strand of the pure braid $\gamma(F_\omega; x)$ rotates around its first, ..., $(i-1)$-st strands by the angle $\omega(|x_i|)$. Since $\eta_{i,n}$'s commute each other, we have

$$\gamma(F_\omega; x) = \eta_{2,n}^{[\omega(|x_2|)]} \cdots \eta_{n,n}^{[\omega(|x_n|)]} \gamma_\omega.$$

Here, γ_ω is the braid represented by the loop $l\colon [0,1] \to X_n(D^2)$ defined by

$$l(t) = \begin{cases} \{F_{2t(\omega - [\omega])}\} & (0 \leq t \leq \frac{1}{2}) \\ \{(2-2t)F_\omega(x_i) + (2t-1)x_i^0\} & (\frac{1}{2} \leq t \leq 1) \end{cases}.$$

By an argument similar to the proof of Lemma 3.1, the number $\Lambda_{i,j}(l)$ is bounded and hence $\phi(\gamma_\omega)$ is also bounded independently on $x \in D^2$ and $\omega\colon [0,1] \to \mathbb{R}$. Therefore,

$$\gamma(F_\omega^p; x) = \eta_{2,n}^{[p\omega(|x_2|)]} \cdots \eta_{n,n}^{[p\omega(|x_n|)]} \gamma_{p\omega}$$

and since $\phi(\gamma_{p\omega})$ is bounded,

$$\lim_{p\to\infty} \frac{1}{p} \phi(\gamma(F_\omega^p; x)) = \sum_{i=2}^{n} \phi(\eta_{i,n}) \omega(|x_i|).$$

Hence

$$\begin{aligned} \Gamma_n(\phi)(F_\omega) &= n! \lim_{p\to\infty} \frac{1}{p} \int_{|x_1|<\cdots<|x_n|} \phi(\gamma(F_\omega^p; x)) dx \\ &= n! \sum_{i=2}^{n} \phi(\eta_{i,n}) \int_{|x_1|<\cdots<|x_n|} \omega(|x_i|) dx. \end{aligned}$$

Since

$$\int_{|x_1|<\cdots<|x_n|} \omega(|x_i|) dx = \frac{1}{(i-1)!(n-i)!} \int_0^1 \omega(r) a(r)^{i-1} (1-a(r))^{n-i} da(r),$$

we have the required equality. □

Let $\eta\colon P_{n+1}(D^2) \to P_n(D^2)$ be the projection forgetting the $(n+1)$-st strand. Then the homomorphism $Q(\eta)\colon Q(P_n(D^2)) \to Q(P_{n+1}(D^2))$ is induced and it is easily checked that the composition

$$\Gamma_{n+1} \circ Q(\eta)\colon Q(P_n(D^2)) \to Q(\mathrm{Diff}_\Omega^\infty(D^2, \partial D^2))$$

coincides with $(n+1)\Gamma_n$. Hence $\mathrm{Im}(\Gamma_n) \subseteq \mathrm{Im}(\Gamma_{n+1})$. Lemma 5.2 implies that $\mathrm{Im}(\Gamma_n)$ is a proper subspace of $\mathrm{Im}(\Gamma_{n+1})$.

As we noted in Remark 3.2, any homomorphism on $P_n(D^2)$ is mapped to a multiple of the Calabi homomorphism on $\mathrm{Diff}_\Omega^\infty(D^2, \partial D^2)$ by the homomorphism $\Gamma_n\colon Q(P_n(D^2)) \to Q(\mathrm{Diff}_\Omega^\infty(D^2, \partial D^2))$. Hence we have the following proposition applying Lemma 5.2 to the case $n = 2$.

Proposition 5.3. *An area-preserving diffeomorphism $F_\omega \in \mathrm{Diff}_\Omega^\infty(D^2, \partial D^2)$ is in the commutator subgroup of* $\mathrm{Diff}_\Omega^\infty(D^2, \partial D^2)$ *if and only if*

$$\int_0^1 \omega(r) a(r) da(r) = 0.$$

In particular, in the case

$$\Omega = \frac{1}{\pi} dx \wedge dy,$$

area-preserving diffeomorphism $F_\omega \in \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ is in the commutator subgroup of $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ if and only if

$$\int_0^1 r^3 \omega(r) dr = 0.$$

Furthermore, applying Lemma 5.2 to the case $n = 3$, we have

$$\Gamma_3(\phi)(F_\omega) = 6\phi(\eta_{2,3}) \int \omega(r) a(r)(1 - a(r)) da(r) + 3\phi(\eta_{3,3}) \int \omega(r) a(r)^2 da(r).$$

By Theorem 5.1, Proposition 5.3 and Inequality (3.9), we have

$$\mathrm{scl}(F_\omega) \geq \frac{|6\phi(\eta_{2,3}) - 3\phi(\eta_{3,3})|}{2D(\phi)\mathrm{area}(D^2)} \left| \int_0^1 \omega(r) a(r)^2 da(r) \right|$$

for any area-preserving diffeomorphism F_ω, which is in the commutator subgroup of $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ and any homogeneous quasi-morphism $\phi \in Q(B_3(D^2))$ except homomorphisms. For example, the signature is a classical invariant of knots and links which is defined as the signature of the Seifert pairing (for a precise definition see [25]) and it gives rise to a quasi-morphism on braid groups by considering the closure of braids [19]. If we denote the homogenization of the signature quasi-morphism on $B_n(D^2)$ by Sign_n, then following [19]

$$\mathrm{Sign}_n(\eta_{i,n}) = \begin{cases} -i & (\text{if } i \text{ is even}) \\ 1 - i & (\text{if } i \text{ is odd}) \end{cases}.$$

On the other hand, the following lemma holds.

Lemma 5.4 ([13]). *For any group G and any quasi-morphism $\phi \in \hat{Q}(G)$,*

$$D(\tilde{\phi}) \leq 2D(\phi),$$

where $\tilde{\phi}$ is the homogenization of ϕ.

Since the defect of the signature quasi-morphism on $B_n(D^2)$ is bounded by $n-1$ [19], by Lemma 5.4 we have

$$D(\mathrm{Sign}_n) \leq 2(n-1).$$

Hence we have the following proposition.

Proposition 5.5. *Suppose that an area-preserving diffeomorphism F_ω is in the commutator subgroup of $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$. Then*

$$\mathrm{scl}(F_\omega) \geq \frac{3}{4\mathrm{area}(D^2)} \left| \int_0^1 \omega(r) a(r)^2 da(r) \right|.$$

In particular, stable commutator length on $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ is unbounded.

In the case

$$\Omega = \frac{1}{\pi} dx \wedge dy,$$

the inequality of Proposition 5.5 is written as

$$\mathrm{scl}(F_\omega) \geq \frac{3}{2} \left| \int_0^1 r^5 \omega(r) dr \right|.$$

5.2. **The case of the 2-sphere.** We identify S^2 with $\mathbb{C} \cup \{\infty\}$. Let $\omega\colon [0,\infty] \to \mathbb{R}$ be a C^∞-function which is constant in a neighborhood of 0 and outside some compact set. We define $F_\omega \in \mathrm{Diff}^\infty_\Omega(S^2)_0$ by

$$F_\omega(z) = \begin{cases} \exp(2\sqrt{-1}\pi\omega(|z|))z & (\text{if } z \neq \infty) \\ \infty & (\text{if } z = \infty) \end{cases}.$$

Then F_ω is in $\mathrm{Diff}^\infty_\Omega(S^2)_0$. By an argument similar to the proof of Lemma 5.2, it is possible to evaluate the value of $\Gamma_n(\phi)$ at F_ω for a homogeneous quasi-morphism $\phi \in Q(P_n(S^2))$. Since $\mathrm{Diff}^\infty_\Omega(S^2)_0$ is perfect [2], for any ω, $\mathrm{scl}(F_\omega)$ can be defined and lower bounds like Proposition 5.5 can also be given.

We denote by $\tau_{i,n}$ the image of the pure braid $\eta_{i,n} \in B_n(D^2)$ by the projection $B_n(D^2) \to B_n(S^2)$. Then the equality

$$\Gamma_n(\phi)(F_\omega) = n! \sum_{i=2}^{n} \phi(\tau_{i,n}) \int_{|x_1|<\cdots<|x_n|} \omega(|x_i|)dx$$

holds. In the case of the 2-sphere S^2,

$$\int_{|x_1|<\cdots<|x_n|} \omega(|x_i|)dx = \frac{1}{(i-1)!(n-i)!} \int_0^\infty \omega(r)a(r)^{i-1}(1-a(r))^{n-i}da(r),$$

where $a(r)$ is the spherical area of the disk in $\mathbb{C}$ with radius r. Here, note that

$$(\sigma_{n-1}\ldots\sigma_1)\sigma_1\ldots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-1}\ldots\sigma_1(\sigma_{n-1}\ldots\sigma_1)^{-1} = \eta_{n,n}.$$

By Theorem 2.3, $\tau_{n,n}$ is the trivial braid. Therefore, we have the following lemma, which corresponds to Lemma 5.2 in the case of $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$.

Lemma 5.6. *For any homogeneous quasi-morphism $\phi \in Q(B_n(S^2))$,*

$$\Gamma_n(\phi)(F_\omega) = \sum_{i=2}^{n-1} i\binom{n}{i}\phi(\tau_{i,n}) \int_0^\infty \omega(r)a(r)^{i-1}(1-a(r))^{n-i}da(r).$$

In the case $n = 3$, $\phi(\tau_{2,3}) = 0$ for any $\phi \in Q(B_3(S^2))$. In fact, Theorem 2.3 implies $\delta_1^2 = \delta_2^{-2}$ and hence $\phi(\delta_1) = -\phi(\delta_2)$. On the other hand, $\phi(\delta_1) = \phi(\delta_2)$ because $(\delta_1\delta_2)\delta_1(\delta_1\delta_2)^{-1} = \delta_2$. Applying Lemma 5.6 to the case $n = 4$, we have

$$\begin{aligned}\Gamma_4(\phi)(F_\omega) &= 12\phi(\tau_{2,4}) \int_0^\infty \omega(r)a(r)(1-a(r))^2 da(r) \\ &\quad + 12\phi(\tau_{3,4}) \int_0^\infty \omega(r)a(r)^2(1-a(r))da(r).\end{aligned}$$

Here, by Theorem 2.3 we have $\tau_{3,4} = \delta_2\delta_1\delta_1\delta_2 = \delta_3^{-2}$. Since $\phi(\delta_1) = \phi(\delta_2) = \phi(\delta_3)$, we have $\phi(\tau_{3,4}) = -\phi(\tau_{2,4})$. Therefore,

$$\Gamma_4(\phi)(F_\omega) = 12\phi(\tau_{2,4}) \int_0^\infty \omega(r)a(r)(1-a(r))(1-2a(r))da(r)$$

and hence

$$\mathrm{scl}(F_\omega) \geq \frac{6|\phi(\tau_{2,4})|}{D(\phi)\mathrm{area}(S^2)} \left| \int_0^\infty \omega(r)a(r)(1-a(r))(1-2a(r))da(r) \right|$$

for any area-preserving diffeomorphism $F_\omega \in \mathrm{Diff}^\infty_\Omega(S^2)_0$ and any homogeneous quasi-morphism $\phi \in Q(B_4(S^2))$. In [19], Gambaudo and Ghys constructed quasi-morphisms on the pure braid group $P_n(S^2)$ of S^2 from quasi-morphisms on the pure braid group $P_{n-1}(D^2)$ of D^2 as follows. By the embedding $D^2 \cong \mathbb{C} \setminus \{\infty\} \to S^2$, we can consider D^2 as a subset of S^2. Moreover, by the embedding $X_{n-1}(D^2) \to X_n(S^2)$ defined by

$$(x_1^0, \dots, x_{n-1}^0) \mapsto (x_1^0, \dots, x_{n-1}^0, \infty)$$

we consider $X_{n-1}(D^2)$ as a subset of $X_n(S^2)$. The induced map $k_n \colon P_{n-1}(D^2) \to P_n(S^2)$ on fundamental groups is surjective and its kernel is generated by the square of $\zeta_{n-1} = (\sigma_1 \dots \sigma_{n-2})^{n-1}$. In fact,

$$\begin{aligned}
\xi_n &= (\delta_1 \dots \delta_{n-1})^n \\
&= (\delta_1 \dots \delta_{n-2})^{n-1}(\delta_{n-1} \dots \delta_2 \dots \delta_{n-1}) \\
&= (\delta_1 \dots \delta_{n-2})^{n-1} \\
&= k_n(\zeta_{n-1}).
\end{aligned}$$

We define the homomorphism $K_n \colon Q(P_{n-1}(D^2)) \to Q(P_n(S^2))$ by

$$K_n(\phi)(\gamma) = \phi(\tilde{\gamma}) - \frac{\phi(\zeta_{n-1})}{(n-1)(n-2)} \mathrm{lk}_{n-1}(\tilde{\gamma}),$$

where $\tilde{\gamma} \in P_{n-1}(D^2)$ means a braid which is in the inverse image of γ by the homomorphism $k_n \colon P_{n-1}(D^2) \to P_n(S^2)$ and $\mathrm{lk}_n \colon P_n(D^2) \to \mathbb{Z}$ the restriction of the unique homomorphism on $B_n(D^2)$ which maps each σ_i to 1. Since the pure braid ζ_{n-1} is in the center of $P_{n-1}(D^2)$ and $\mathrm{lk}_{n-1}(\zeta_{n-1}) = (n-1)(n-2)$, the homomorphism K_n defined above does not depend on the choice of $\tilde{\gamma}$. In fact,

$$\begin{aligned}
&\phi(\tilde{\gamma}\zeta_{n-1}^2) - \frac{\phi(\zeta_{n-1})}{(n-1)(n-2)} \mathrm{lk}_{n-1}(\tilde{\gamma}\zeta_{n-1}^2) \\
&= \phi(\tilde{\gamma}) + 2\phi(\zeta_{n-1}) - \frac{\phi(\zeta_{n-1})}{(n-1)(n-2)} (\mathrm{lk}_{n-1}(\tilde{\gamma}) + 2\mathrm{lk}_{n-1}(\zeta_{n-1})) \\
&= \phi(\tilde{\gamma}) - \frac{\phi(\zeta_{n-1})}{(n-1)(n-2)} \mathrm{lk}_{n-1}(\tilde{\gamma}).
\end{aligned}$$

Since

$$\begin{aligned}
&K_n(\phi)(\gamma_1\gamma_2) - K_n(\phi)(\gamma_1) - K_n(\phi)(\gamma_2) \\
&= \phi(\tilde{\gamma}_1\tilde{\gamma}_2) - \phi(\tilde{\gamma}_1) - \phi(\tilde{\gamma}_2) - \frac{\phi(\zeta_{n-1})}{(n-1)(n-2)} (\mathrm{lk}_{n-1}(\tilde{\gamma}_1\tilde{\gamma}_2) - \mathrm{lk}_{n-1}(\tilde{\gamma}_1) - \mathrm{lk}_{n-1}(\tilde{\gamma}_2)) \\
&= \phi(\tilde{\gamma}_1\tilde{\gamma}_2) - \phi(\tilde{\gamma}_1) - \phi(\tilde{\gamma}_2),
\end{aligned}$$

$K_n(\phi)$ is in fact a quasi-morphism and

$$D(K_n(\phi)) = D(\phi).$$

In the case that n is even the quasi-morphism $K_n(\mathrm{Sign}_{n-1})$ is invariant under conjugation not only by elements of $P_n(S^2)$ but also elements of $B_n(S^2)$ [19], unlike the case where n is odd. Hence we have

$$\begin{aligned}\mathcal{T}\circ K_4(\mathrm{Sign}_3) &= K_4(\mathrm{Sign}_3)(\tau_{2,4})\\ &= \mathrm{Sign}_3(\eta_{2,3}) - \frac{\mathrm{Sign}_3(\zeta_3)}{6}\mathrm{lk}_{n-1}(\eta_{2,3})\\ &= -2 - \frac{-4}{6}\cdot 2\\ &= -\frac{2}{3}.\end{aligned}$$

Since

$$D(\mathcal{T}\circ K_4(\mathrm{Sign}_3)) \leq D(K_4(\mathrm{Sign}_3)) = D(\mathrm{Sign}_3) \leq 4,$$

we have the following proposition.

Proposition 5.7. *For any area-preserving diffeomorphism* $F_\omega \in \mathrm{Diff}^\infty_\Omega(S^2)_0$,

$$\mathrm{scl}(F_\omega) \geq \frac{1}{\mathrm{area}(S^2)}\left|\int_0^\infty \omega(r)a(r)(1-a(r))(1-2a(r))da(r)\right|.$$

6. Comparison with other quasi-morphisms

In this section, we introduce quasi-morphisms on $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ other than ones obtained from the homomorphism $\Gamma_n\colon Q(P_n(D^2)) \to Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$ and verify that they are linearly independent. We continue to identify D^2 with the subset

$$\{x \in \mathbb{R}^2; |x|^2 \leq 1\}$$

of $\mathbb{R}^2$ and identify S^2 with the subset

$$\{(x_1,x_2,x_3) \in \mathbb{R}^3; x_1^2+x_2^2+x_3^2 = 1\}$$

of $\mathbb{R}^3$.

6.1. **Ruelle's quasi-morphism.** First we consider Ruelle's quasi-morphism, that was introduced in [26] and proved to be a quasi-morphism in [18]. For $g \in \mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$, choose a Hamiltonian isotopy $\{g_t\}_{t\in[0,1]}$ such that g_0 is the identity and $g_1 = g$. Then we can consider the differential $dg_t(x) \in SL(2,\mathbb{R})$ for each $x \in D^2$. We denote the first column of $dg_t(x)$ by $v_t(x)$ and define $u_t(x) \in S^1$ by

$$u_t(x) = \frac{v_t(x)}{\|v_t(x)\|}.$$

Considering a lift $\tilde{u}_t(x) \in \mathbb{R}$ of $u_t(x)$ and if we set $\mathrm{Ang}_g(x) = \tilde{u}_1(x) - \tilde{u}_0(x)$, then $\mathrm{Ang}_g(x)$ is determined independently of the choice of the lift. We define the function $r\colon \mathrm{Diff}^\infty_\Omega(D^2,\partial D^2) \to \mathbb{R}$ by

$$r(g) = \int_{x\in D^2} \mathrm{Ang}_g(x)dx.$$

The function $\mathrm{Ang}_g(x)\colon D^2 \to \mathbb{R}$ is integrable and the function $r\colon \mathrm{Diff}^\infty_\Omega(D^2,\partial D^2) \to \mathbb{R}$ is well-defined [26] . Since

$$|\mathrm{Ang}_{gh}(x) - \mathrm{Ang}_g(x) - \mathrm{Ang}_h(x)| < \frac{1}{2},$$

the function $r\colon \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2) \to \mathbb{R}$ is also a quasi-morphism. We define Ruelle's quasi-morphism $R\colon \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2) \to \mathbb{R}$ to be the homogenization of r. That is,

$$R(g) = \lim_{p\to\infty} \frac{1}{p} r(g^p).$$

The value of Ruelle's quasi-morphism at $F_\omega \in \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ is computed by Gambaudo and Ghys.

Lemma 6.1 (Gambaudo-Ghys [19]).

$$R(F_\omega) = 2\int_0^1 r\omega(r)dr.$$

6.2. **Calabi quasi-morphisms.** In this subsection, we introduce Calabi quasi-morphisms constructed in [16]. Here we assume the area forms Ω on the both of D^2 and S^2 are normalized and standard ones.

For a symplectic manifold (M, Ω) and any subset $U \subseteq M$, let $\mathrm{Ham}^\infty(U)$ be a subgroup of $\mathrm{Ham}^\infty_C(M)$ consisting of elements with support in U. A subset $U \subset M$ is *displaceable* if there exists $g \in \mathrm{Ham}^\infty_C(M)$ such that $\bar{U} \cap g(U) = \emptyset$. A quasi-morphism ϕ on $\mathrm{Ham}^\infty_C(M)$ is a *Calabi quasi-morphism* if for any displaceable subset $U \subset M$ diffeomorphic to the ball of the same dimension as M, the restriction of ϕ on $\mathrm{Ham}^\infty(U)$ coincides with the Calabi homomorphism on U.

In [16], Entov and Polterovich constructed a quasi-morphisms μ on $\mathrm{Diff}^\infty_\Omega(S^2)_0$ and uncountably many quasi-morphisms μ_ϵ on $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ by pulling back μ via embeddings $D^2 \to S^2$. Precisely, for $\epsilon \in (\frac{1}{2}, 1)$, considering the map $h_\epsilon\colon D^2 \to S^2$ which sends each circle $\{|x|^2 = c\}$ to the level set $\{x_3 = 1 - 2\epsilon c\}$, they defined

$$\mu_\epsilon = \frac{1}{\epsilon^2} h_\epsilon^* \mu.$$

Theorem 6.2 (Entov-Polterovich [16]). *The family* $\{\mu_\epsilon\}_{\epsilon\in(\frac{1}{2},1)}$ *of quasi-morphisms on* $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ *satisfies*

(i) *For any subset* $U \subset D^2$ *diffeomorphic to* D^2, *if* $\mathrm{area}(U) < \epsilon$ *then the restriction of* μ_ϵ *on* $\mathrm{Ham}^\infty(U)$ *coincides with the Calabi homomorphism of* U.

(ii) *For any finite subset* $I \subset (\frac{1}{2}, 1)$, *quasi-morphisms* $\{\mu_\epsilon\}_{\epsilon\in I}$ *are linearly independent.*

The values of these Calabi quasi-morphisms at $F_\omega \in \mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ are also calculated by Entov and Polterovich [16].

Lemma 6.3.

$$\mu_\epsilon(F_\omega) = \int_0^1 r^3\omega(r)dr + \frac{1}{\epsilon}\int_{1/2\epsilon}^1 \omega(r)dr.$$

Remark 6.4. Entov and Polterovich constructed a Calabi quasi-morphism not only on the sphere but also on $(S^2 \times S^2, \Omega \oplus \Omega)$ and $(\mathbb{C}P^n, \Omega_{\mathrm{FS}})$, where Ω_{FS} means the Fubini-Study form. Afterwards, Biran, Entov and Polterovich proved a theorem similar to Theorem 6.2 for the group $\mathrm{Symp}(D^{2n}, \partial D^{2n})_0$ of symplectomorphisms of any even-dimensional ball instead of $\mathrm{Diff}^\infty_\Omega(D^2, \partial D^2)$ [5].

6.3. **Linearly independence.** Comparing Lemmas 5.2, 6.1 and 6.3, we have the main result of this section.

Proposition 6.5. *Let $n \geq 3$. Suppose that V is a finite subset of $Q(B_n(D^2))$ consisting of linearly independent quasi-morphisms and $I \subset (\frac{1}{2}, 1)$ is a finite subset. Then, quasi-morphisms $\{\Gamma_n(\phi)\}_{\phi \in V}$, $\{\mu_\epsilon\}_{\epsilon \in I}$, R are linearly independent.*

Proof. Let $V = \{\phi_1, \dots, \phi_k\}$ and $I = \{\epsilon_1, \dots, \epsilon_l\}$. Suppose

$$\sum_{i=1}^{k} a_i \Gamma_n(\phi_i) + \sum_{j=1}^{l} b_j \mu_{\epsilon_j} + cR = 0, \tag{6.1}$$

where $a_1, \dots, a_k, b_1, \dots, b_l, c$ are real numbers. It is sufficient to prove that $a_1, \dots,$ a_k, b_1, $\dots$, b_l, c are equal to 0. If we set

$$\phi = \sum_{i=1}^{k} a_i \phi_i,$$

then by Lemma 5.2 we have

$$\begin{aligned}\sum_{i=1}^{k} a_i \Gamma_n(\phi_i)(F_\omega) &= \Gamma_n \phi(F_\omega) \\ &= \sum_{i=2}^{n} i \binom{n}{i} \phi(\eta_{i,n}) \int_0^1 \omega(r) r^{2i-1} (1-r^2)^{n-i} dr. \end{aligned} \tag{6.2}$$

If we set $b = b_1 + \cdots + b_l$, by Lemma 6.3 we have

$$\sum_{j=1}^{l} b_j \mu_{\epsilon_j}(F_\omega) = b \int_0^1 r^3 \omega(r) dr + \sum_{j=1}^{l} \frac{b_j}{\epsilon_j} \int_{1/2\epsilon_j}^1 \omega(r) dr. \tag{6.3}$$

Finally by Lemma 6.1 we have

$$cR(F_\omega) = 2c \int_0^1 r\omega(r) dr. \tag{6.4}$$

Substituting Equalities (6.2), (6.3) and (6.4) to Equality (6.1), we have

$$\begin{aligned}\sum_{i=2}^{n} i \binom{n}{i} \phi(\eta_{i,n}) \int_0^1 \omega(r) r^{2i-1} (1-r^2)^{n-i} dr + b \int_0^1 r^3 \omega(r) dr \\ + \sum_{j=1}^{l} \frac{b_j}{\epsilon_j} \int_{1/2\epsilon_j}^1 \omega(r) dr + 2c \int_0^1 r\omega(r) dr = 0. \end{aligned} \tag{6.5}$$

Since the Equality (6.5) holds for any C^∞-function ω which is equal to 0 on a neighborhood of 1 and constant on a neighborhood of 0, the coefficients of

$$\int_{1/2\epsilon_j}^1 \omega(r) dr \quad \text{and} \quad \int_0^1 r\omega(r) dr$$

must vanish. Hence $b_j = 0$ for each j and $c = 0$. Therefore, we have

$$\Gamma_n \phi(F_\omega) = 0.$$

By Theorem 1.2, we have $a_i = 0$ for each i. □

7. Conjugation-invariant norms

7.1. **Conjugation-generated norms.**

In this subsection, we define conjugation-generated norms on groups following [11] and summarize a relationship with quasi-morphisms.

Suppose that G is a simple group and $K \subseteq G$ is a symmetric subset. That is for any $g \in K$, its inverse g^{-1} is also in K. Since the group G is simple, any element g of G can be written as a product of conjugates of elements of K. We define for each $g \in G$ the number $q_K(g)$ by the minimal number of conjugates of elements of K whose product is equal to g. Here, for the identity element id, we define $q_K(\mathrm{id}) = 0$. The function $q_K \colon G \to \mathbb{N}$ is obviously invariant under conjugations. If we assume that G is non-abelian, then G is perfect. Moreover, if K is the set of commutators in G, then for each $g \in G$, the number $q_K(g)$ is the commutator length $\mathrm{cl}(g)$ of g. Hence the function $q_K \colon G \to \mathbb{N}$ is a generalization of the commutator length. For each symmetric subset $K \subseteq G$, the function $q_K \colon G \to \mathbb{N}$ defines a conjugation-invariant norm on G. Note that the commutator length is also a conjugation-invariant norm.

Certain quasi-morphisms give lower bounds of the norm q_K as follows. We define the vector subspace $Q(G,K) \subseteq Q(G)$ by

$$Q(G,K) = \{\phi \in Q(G); \phi(g) = 0 \text{ for any } g \in K\}.$$

Suppose that $g \in G$ is written as

$$g = f_1 \cdots f_n,$$

where $f_1, \ldots, f_n$ are conjugates of elements of K. Then for $\phi \in Q(G,K)$ the equation

$$|\phi(g) - \phi(f_1) - \cdots - \phi(f_n)| \leq (n-1)D(\phi)$$

holds. Since $\phi(f_i) = 0$ for any i, we have

$$1 + \frac{|\phi(g)|}{D(\phi)} \leq n.$$

Therefore, we have the following lemma.

Lemma 7.1. *For any $g \in G$ and $\phi \in Q(G,K)$,*

$$1 + \frac{|\phi(g)|}{D(\phi)} \leq q_K(g).$$

A simple group G is *uniformly simple* if there exists a natural number N such that if $q_{\{h,h^{-1}\}}(g) < N$ for any $g, h \in G \setminus \{\mathrm{id}\}$. By Lemma 7.1, if the vector space $Q(G,K)$ is non-trivial, then the function $q_K \colon G \to \mathbb{N}$ is unbounded. Therefore, if a symmetric set K is finite and G admits sufficiently many linearly independent quasi-morphisms which are not homomorphisms, then $q_K \colon G \to \mathbb{N}$ is a unbounded function. In particular, if G admits a non-trivial quasi-morphism then G is not uniformly simple. For example, $\mathrm{Ker}(\mathrm{Cal})$ and $\mathrm{Diff}^\infty_\Omega(S^2)_0$ are not uniformly simple and for any finite and symmetric subset $K \subseteq \mathrm{Ker}(\mathrm{Cal})$ or $K \subseteq \mathrm{Diff}^\infty_\Omega(S^2)_0$, the function $q_K \colon G \to \mathbb{N}$ is unbounded on $\mathrm{Ker}(\mathrm{Cal})$ or $\mathrm{Diff}^\infty_\Omega(S^2)_0$, respectively.

7.2. **Autonomous norm, fragmentation norm, Hofer norm and L^p norm.** For a symplectic manifold (M,Ω), a Hamiltonian diffeomorphism $h \in \mathrm{Ham}_C^\infty(M)$ is *autonomous* if the Hamiltonian function H^t of h can be chosen to be independent of t. If we denote by Aut the set of autonomous Hamiltonian diffeomorphisms, then it is a symmetric and in general infinite set and in the case that (M,Ω) is closed or exact we can define the *autonomous norm* q_{Aut} on $\mathrm{Ham}_C^\infty(M)$. For each $g \in \mathrm{Ham}_C^\infty(M)$, the number $q_{\mathrm{Aut}}(g)$ is the minimal number of autonomous diffeomorphisms whose product is equal to g.

For the autonomous norm on $\mathrm{Diff}_\Omega^\infty(D^2,\partial D^2)$ and $\mathrm{Diff}_\Omega^\infty(S^2)_0$, the following theorem is known.

Theorem 7.2 (Brandenbursky-Kędra [9], Gambaudo-Ghys [19]). *The autonomous norm q_{Aut} is unbounded on the groups $\mathrm{Diff}_\Omega^\infty(D^2,\partial D^2)$ and $\mathrm{Diff}_\Omega^\infty(S^2)_0$.*

For a manifold M which is given a volume-form and $\epsilon > 0$, we denote by $\mathrm{Frag}(\epsilon)$ the set of volume-preserving diffeomorphisms whose support is contained in the ball of area less than ϵ. The set $\mathrm{Frag}(\epsilon)$ is a symmetric and in general infinite set. In the case that the group of volume-preserving diffeomorphisms of M is simple, we can define the *fragmentation norm* $q_{\mathrm{Frag}(\epsilon)}$. By Theorem 6.2, if we assume the volume form Ω on D^2 is normalized, for any $\epsilon_0,\epsilon_1 \in (\frac{1}{2},1)$ such that $\epsilon_0 < \epsilon_1$, the quasi-morphism $\mu_{\epsilon_0} - \mu_{\epsilon_1}$ on $\mathrm{Diff}_\Omega^\infty(D^2,\partial D^2)$ vanishes on $\mathrm{Frag}(\epsilon) \subset \mathrm{Diff}_\Omega^\infty(D^2,\partial D^2)$. Hence we have the following theorem.

Theorem 7.3. *The fragmentation norm $q_{\mathrm{Frag}(\epsilon)}$ is unbounded on $\mathrm{Diff}_\Omega^\infty(D^2,\partial D^2)$ for any $\epsilon \in (0,1)$.*

On the other hand, the problem whether the fragmentation norm $q_{\mathrm{Frag}(\epsilon)}$ on $\mathrm{Diff}_\Omega^\infty(S^2)_0$ is bounded is still open.

For a compactly supported Hamiltonian function H^t on a symplectic manifold (M,Ω), we set

$$\mathrm{Osc}(H^t) = \max_{x\in M} H^t(x) - \min_{x\in M} H^t(x).$$

The *Hofer norm* $\rho(h)$ of $h \in \mathrm{Ham}_C^\infty(M)$, which was introduced in [21], is defined by

$$\rho(h) = \inf_{H^t} \int_0^1 \mathrm{Osc}(H^t)dt.$$

Here, the infimum is taken over all Hamiltonian function which generates h.

In the case that (M,Ω) is exact and not closed, the group $\mathrm{Ham}_C^\infty(M)$ admits the Calabi homomorphism and it is seen that

$$\rho(h) \geq \mathrm{Cal}(h).$$

Hence the Hofer norm is unbounded. Furthermore, the following theorem is also known.

Theorem 7.4 (Polterovich [24]). *The Hofer norm ρ is unbounded on the group $\mathrm{Diff}_\Omega^\infty(S^2)_0$.*

In the case that the manifold M is compact and endowed with a Riemannian structure, we can consider the L^p-norm $\mathcal{L}_p\colon \mathrm{Symp}^\infty(M,\Omega)_0 \to \mathbb{R}$ on the identity component $\mathrm{Symp}^\infty(M,\Omega)_0$ of the group of symplectomorphisms of M. For a path

$\{g_t\}_{t\in[0,1]}$ in $\mathrm{Symp}^\infty(M,\Omega)_0$ we set

$$\mathcal{L}_p(\{g_t\}) = \int_0^1 \left(\int_{x\in M} \left| \frac{g_t(x)}{dt} \right|^p \Omega^n \right)^{1/p} dt.$$

For a symplectomorphism $g \in \mathrm{Symp}^\infty(M,\Omega)_0$, we define

$$\mathcal{L}_p(g) = \inf_{\{g_t\}} \mathcal{L}_p(\{g_t\}),$$

where the infimum is taken over all path $\{g_t\}$ such that g_0 is the identity map and $g_1 = g$.

For the L^p-norm on $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ and $\mathrm{Diff}^\infty_\Omega(S^2)_0$, the following theorem is known.

Theorem 7.5 (Eliashberg-Ratiu [15]). *The L^p-norm $\mathcal{L}_p$ is unbounded on the groups $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ and $\mathrm{Diff}^\infty_\Omega(S^2)_0$.*

Remark 7.6. The L^p-norm can be defined for groups of volume-preserving diffeomorphisms of any Riemannian manifold. Eliashberg and Ratiu proved Theorem 7.5 for surfaces and for manifolds with positive first Betti number in [15]. Furthermore, Brandenbursky and Kędra proved Theorem 7.5 in more general case [8][10]. On the other hand, the L^p-norm for the group of volume-preserving diffeomorphisms is bounded if $p \geq 2$ and the manifold is simply connected and has a dimension 3 or greater [27].

8. Extension to the group which does not fix the boundary

Let $\mathrm{Diff}^\infty_\Omega(D^2)$ be the group of area-preserving diffeomorphisms, which are rotations on a neighborhood of the boundary. Obviously the group $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ is a normal subgroup of $\mathrm{Diff}^\infty_\Omega(D^2)$. In this section, we consider extending quasi-morphisms on $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ to quasi-morphisms on $\mathrm{Diff}^\infty_\Omega(D^2)$.

Let $p\colon \mathrm{Diff}^\infty_\Omega(D^2) \to SO(2)$ be the projection of diffeomorphisms on ∂D^2. Then its kernel is $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$. Choose a section $s\colon SO(2) \to \mathrm{Diff}^\infty_\Omega(D^2)$ which is a homomorphism. For example, for the rotation R_θ by the angle θ in $SO(2)$, if we define $s(R_\theta)$ by the rotation of the disk D^2 by angle θ then $s\colon SO(2) \to \mathrm{Diff}^\infty_\Omega(D^2)$ is a section. For each $g \in \mathrm{Diff}^\infty_\Omega(D^2)$, a diffeomorphism $g \circ (s \circ p(g))^{-1}$ is in $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$. For each $\phi \in \hat{Q}(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$, we define a function $\tilde{\phi}\colon \mathrm{Diff}^\infty_\Omega(D^2) \to \mathbb{R}$ by

$$\tilde{\phi}(g) = \phi(g \circ (s \circ p(g))^{-1}).$$

Proposition 8.1. *For any quasi-morphism $\phi \in Q(P_n(D^2))$, the function*

$$\widetilde{\Gamma_n(\phi)}\colon \mathrm{Diff}^\infty_\Omega(D^2) \to \mathbb{R}$$

is a quasi-morphism on $\mathrm{Diff}^\infty_\Omega(D^2)$.

For the proof of Proposition 8.1, we show the following lemma.

Lemma 8.2. *For any quasi-morphism ϕ on $P_n(D^2)$, the quasi-morphism $\Gamma_n(\phi) \in Q(\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2))$ is invariant under conjugations by elements of $\mathrm{Diff}^\infty_\Omega(D^2)$.*

Proof. By Proposition 4.6, we may assume $\phi \in Q(B_n(D^2))$. For any area-preserving diffeomorphisms $g \in \mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$ and $f \in \mathrm{Diff}^\infty_\Omega(D^2)$ and for almost every $x \in X_n(D^2)$, there exists a braid $\beta_x \in B_n(D^2)$ such that

$$\gamma(fgf^{-1};x) = \beta_x \gamma(g; f_*^{-1}x)\beta_x^{-1}.$$

Therefore,

$$\begin{aligned}\Gamma_n(\phi)(fgf^{-1}) &= \lim_{p\to\infty}\frac{1}{p}\int_{x\in X_n(D^2)}\phi(\gamma(fg^pf^{-1};x))dx \\ &= \lim_{p\to\infty}\frac{1}{p}\int_{x\in X_n(D^2)}\phi(\beta_x\gamma(g^p;f_*^{-1}(x))\beta_x^{-1})dx \\ &= \lim_{p\to\infty}\frac{1}{p}\int_{x\in X_n(D^2)}\phi(\gamma(g^p;x))dx \\ &= \Gamma_n(\phi)(g).\end{aligned}$$

□

Proof of Proposition 8.1. Let us denote $(s\circ p)(g)\in \mathrm{Diff}^\infty_\Omega(D^2)$ by g' for each $g\in \mathrm{Diff}^\infty_\Omega(D^2)$. Then for any $g,h\in \mathrm{Diff}^\infty_\Omega(D^2)$,

$$\widetilde{\Gamma_n(\phi)}(g\circ h) = \Gamma_n(\phi)(g\circ h\circ((g\circ h)')^{-1}).$$

Since the section $s\colon SO(2)\to \mathrm{Diff}^\infty_\Omega(D^2)$ is a homomorphism,

$$\begin{aligned}g\circ h\circ((g\circ h)')^{-1} &= g\circ h\circ (h')^{-1}\circ (g')^{-1} \\ &= (g\circ (g')^{-1})\circ (g'\circ (h\circ (h')^{-1})\circ (g')^{-1}).\end{aligned}$$

Therefore,

$$\begin{aligned}&|\widetilde{\Gamma_n(\phi)}(g\circ h) - \widetilde{\Gamma_n(\phi)}(g) - \widetilde{\Gamma_n(\phi)}(h)| \\ &= |\Gamma_n(\phi)((g\circ (g')^{-1})\circ (g'\circ (h\circ (h')^{-1})\circ (g')^{-1})) \\ &\qquad - \Gamma_n(\phi)(g\circ (g')^{-1}) - \Gamma_n(\phi)(h\circ (h')^{-1})| \\ &= |\Gamma_n(\phi)((g\circ (g')^{-1})\circ (g'\circ (h\circ (h')^{-1})\circ (g')^{-1})) \\ &\qquad - \Gamma_n(\phi)(g\circ (g')^{-1}) - \Gamma_n(\phi)(g'\circ (h\circ (h')^{-1})\circ (g')^{-1}))| \\ &\le D(\Gamma_n(\phi)).\end{aligned}$$

This completes the proof. □

Remark 8.3. Instead of $\mathrm{Diff}^\infty_\Omega(D^2)$ and $\mathrm{Diff}^\infty_\Omega(D^2,\partial D^2)$, for any group G and its normal subgroup H, if there exists a section $G/H\to G$ which is a homomorphism and the statement similar to Lemma 8.2 holds, then the statement similar to Proposition 8.1 holds.

References

[1] E. Artin, *Theorie der Zöpfe*, Abh. Math. Sem. Univ. Hamburg **4** (1925), 47–72.

[2] A. Banyaga, *Sur la structure du groupe des diffeomorphismes qui préservent une forme symplectique*, Comment. Math. Helv. **53** (1978), no. 2, 174–227.

[3] C. Bavard, *Longueur stable des commutateurs*, Enseign. Math. **37** (1991), no. 1-2, 109–150.

[4] M. Bestvina and K. Fujiwara, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. **6** (2002), 69–89.

[5] P. Biran, M. Entov, and L. Polterovich, *Calabi quasimorphisms for the symplectic ball*, Commun. Contemp. Math. **6** (2004), no. 5, 793–802.

[6] J. Birman, *Braids, links, and mapping class groups*, Annals of Mathematics Studies, no. 82, Princeton University Press, Princeton, N.J., 1974.

[7] M. Brandenbursky, *On quasi-morphisms from knot and braid invariants*, J. Knot Theory Ramifications **20** (2011), no. 10, 1397–1417.

[8] ———, *Quasi-morphisms and L^p-metrics on groups of volume-preserving diffeomorphisms*, J. Topol. Anal. **4** (2012), no. 2, 255–270.

[9] M. Brandenbursky and J. Kędra, *On the autonomous metric on the group of area-preserving diffeomorphisms of the* 2*-disc*, arXiv:1207.0624, to appear in Algebr. Geom. Topol.
[10] ———, *Quasi-isometric embeddings into diffeomorphism groups*, arXiv:1108:5126, to appear in Groups, Geom. Dyn.
[11] D. Burago, S. Ivanov, and L. Polterovich, *Conjugation-invariant norms on groups of geometric origin*, Groups of diffeomorphisms, Adv. Stud. Pure Math., vol. 52, Math. Soc. Japan, Tokyo, 2008, pp. 221–250.
[12] E. Calabi, *On the group of automorphisms of a symplectic manifold*, Problems in analysis (Lectures at the Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, N.J., 1969), Princeton Univ. Press, Princeton, N.J., 1970, pp. 1–26.
[13] D. Calegeri, *scl*, MSJ Memoirs, vol. 20, Math. Soc. Japan, Tokyo, 2009.
[14] P. de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics., University of Chicago Press, Chicago, IL, 2000., 2000.
[15] Yakov. Eliashberg and T. Ratiu, *The diameter of the symplectomorphism group is infinite*, Invent. Math. **103** (1991), no. 2, 327–340.
[16] M. Entov and L. Polterovich, *Calabi quasimorphism and quantum homology.*, Int. Math. Res. Not. (2003), no. 30, 1635–1676.
[17] E. Fadell and J. Van Buskirk, *The braid groups of E^2 and S^2*, Duke. Math. J. **29** (1962), no. 2, 243–257.
[18] J-. M. Gambaudo and É. Ghys, *Enlacements asymptotiques*, Topology **36** (1997), no. 6, 1355–1379.
[19] ———, *Commutators and diffeomorphisms of surfaces*, Ergodic Theory Dynam. Systems **24** (2004), no. 5, 1591–1617.
[20] J-. M. Gambaudo and M. Lagrange, *Topological lower bounds on the distance between area preserving diffeomorphisms*, Bol. Soc. Brasil. Mat. (N.S.) **31** (2000), no. 1, 9–27.
[21] H. Hofer, *On the topological properties of symplectic maps.*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), no. 1-2, 25–38.
[22] A. V. Malyutin, *Operators in the spaces of pseudocharacters of braid groups*, Algebra i Analiz **21** (2009), no. 2, 136–165.
[23] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), no. 2, 286–294.
[24] L. Polterovich, *Hofer's diameter and lagrangian intersections*, Internat. Math. Res. Notices (1998), no. 4, 217–223.
[25] D. Rolfsen, *Knots and links*, Mathematics Lecture Series, vol. 7, Publish or Perish, Inc., Houston, TX, 1990.
[26] D. Ruelle, *Rotation numbers for diffeomorphisms and flows*, Ann. Inst. H. Poincaré Phys. Théor. **42** (1985), no. 1, 109–115.
[27] A. Shnirelman, *The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid*, Mat. Sb. (N.S.) **128(170)** (1985), no. 1, 82–109.
[28] S. Smale, *Diffeomorphisms of the* 2*-sphere*, Proc. Amer. Math. Soc. **10** (1959), no. 4, 621–626.
[29] A. Walker, *Surface maps into free groups*, Ph.D. thesis, California Institute of Technology, 2012.

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.
E-mail address: ishidat@ms.u-tokyo.ac.jp