<u>博士論文題目</u>

Lie foliations transversely modeled on nilpotent Lie algebras

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1 Introduction

Throughout this paper, we suppose that all manifolds are connected, smooth and oriented and all foliations are smooth and transversely oriented.

Lie foliations were first defined by E. Fedida (cf. [4]) and have been studied by several authors. The structure theorem for Riemannian foliations by P. Molino (cf. [10]) motivates the study of Lie foliations. It says that the lifted foliation $\widehat{\mathcal{F}}$ of a Riemannian foliation $\mathcal{F}$ to its transverse orthogonal frame bundle is transversely parallelizable and the foliation $\widehat{\mathcal{F}}|_{\overline{L}}$ is a Lie foliation, where $\overline{L}$ is the closure of a leaf of $\widehat{\mathcal{F}}$.

To each Lie foliation $\mathcal{F}$, there are associated two Lie algebras, the model Lie algebra $\mathfrak{g}$ and the structure Lie algebra $\mathfrak{h}$. The structure Lie algebra $\mathfrak{h}$ is a subalgebra of the model Lie algebra $\mathfrak{g}$. The structure Lie algebra $\mathfrak{h}$ is uniquely determined by $\mathcal{F}$, while the model Lie algebra $\mathfrak{g}$ may not be uniquely determined.

We have a natural question to determine the pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ which can be realized as a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of a closed manifold M with structure Lie algebra $\mathfrak{h}$.

M. Llabrés studied a special case of this problem, that is, to determine the pair of $(\mathfrak{g}, m)$ which is realized as a Lie $\mathfrak{g}$-flow of a closed manifold M with the structure Lie algebra $\mathbb{R}^m$, where flows mean one-dimensional foliations.

E. Gallego, B. Herrera, M. Llabrés and A. Reventós completely solved this problem in the case where the dimension of the Lie algebras $\mathfrak{g}$ is three (cf. [5], [6]).

In this paper, we study the realizing problems of $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}, m)$ in the case where $\mathfrak{g}$ is nilpotent Lie algebras of general dimensions. The main theorems are the following.

Theorem 4.1 *Let $\mathfrak{g}$ be a nilpotent Lie algebra which has a rational structure. Then $(\mathfrak{g}, m)$ is realizable if and only if $m \leq \dim \mathfrak{c}(\mathfrak{g})$, where $\mathfrak{c}(\mathfrak{g})$ is the center of $\mathfrak{g}$.*

Theorem 5.3 *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{h})$ is realizable if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and the quotient Lie algebra $\mathfrak{h} \backslash \mathfrak{g}$ has a rational structure.*

2 Preliminaries

2.1 Basic definitions

Let $\mathcal{F}$ be a codimension q foliation of an n-dimensional closed manifold M. Let $\mathfrak{X}(\mathcal{F})$ be the set of vector fields which are tangent to the leaves of $\mathcal{F}$ and

$$L(M, \mathcal{F}) = \{\, X \in \mathfrak{X}(M) \mid [X, \mathfrak{X}(\mathcal{F})] \subset \mathfrak{X}(\mathcal{F}) \,\}$$

be the Lie algebra of projectable vector fields. Since $\mathfrak{X}(\mathcal{F})$ is an ideal of $L(M, \mathcal{F})$, the quotient $l(M, \mathcal{F}) = L(M, \mathcal{F})/\mathfrak{X}(\mathcal{F})$ is a Lie algebra. We call it the Lie algebra of transverse vector fields.

A family $\{\bar{X}_1, \ldots, \bar{X}_q\}$ of transverse vector fields which is linearly independent everywhere is called a transverse parallelism of $\mathcal{F}$. If there exists a transverse parallelism of $\mathcal{F}$, then the foliation $\mathcal{F}$ is called transversely parallelizable.

For a transverse parallelism $\{\bar{X}_1, \ldots, \bar{X}_q\}$, the $\Omega_b^0(M, \mathcal{F})$-submodule spanned by $\{\bar{X}_1, \ldots \bar{X}_q\}$ is a Lie subalgebra of $l(M, \mathcal{F})$, where

$$\Omega_b^0(M, \mathcal{F}) = \{\, f \in C^\infty(M) \mid {}^\forall X \in \mathfrak{X}(\mathcal{F}), X(f) = 0 \,\}$$

is the set of basic functions on $(M, \mathcal{F})$. Note that, in general, the $\mathbb{R}$ vector subspace spanned by $\{\bar{X}_1, \ldots, \bar{X}_q\}$ may not be a Lie subalgebra, that is, it may not be closed under the Lie bracket.

Definition 2.1 Let $\mathfrak{g}$ be a q-dimensional Lie algebra. A codimension q foliation $\mathcal{F}$ of M is a Lie $\mathfrak{g}$-foliation if there exists a transverse parallelism $\{\bar{X}_1, \ldots \bar{X}_q\}$ of $\mathcal{F}$ such that the $\mathbb{R}$ vector subspace spanned by $\{\bar{X}_1, \ldots \bar{X}_q\}$ is a Lie subalgebra of $l(M, \mathcal{F})$ and is isomorphic to $\mathfrak{g}$.

We call such transverse parallelisms transverse Lie $\mathfrak{g}$-parallelisms. If $\mathcal{F}$ is a one dimensional Lie $\mathfrak{g}$-foliation, we call it Lie $\mathfrak{g}$-flow.

Let $\mathcal{F}$ be a transversely parallelizable foliation of a closed manifold M and suppose that $\mathcal{F}$ is minimal, that is, each leaf of $\mathcal{F}$ is dense in M. Then there exists the Lie algebra $\mathfrak{g}$ which is uniquely determined by $\mathcal{F}$ such that $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation. In fact, for a transverse parallelism $\{\overline{X}_1, \ldots, \overline{X}_q\}$ of $\mathcal{F}$, the Lie algebra of transverse vector fields $l(M, \mathcal{F})$ coincides with the $\Omega_b^0(M, \mathcal{F})$-submodule of $l(M, \mathcal{F})$ spanned by $\{\overline{X}_1, \ldots, \overline{X}_q\}$. Since $\mathcal{F}$ is minimal, any basic function on $(M, \mathcal{F})$ is a constant function. Thus $l(M, \mathcal{F})$ coincides with the $\mathbb{R}$ vector subspace $\langle \overline{X}_1, \ldots, \overline{X}_q \rangle_{\mathbb{R}}$ spanned by $\{\overline{X}_1, \ldots, \overline{X}_q\}$ which is a finite dimensional Lie algebra over $\mathbb{R}$. Therefore $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation, where $\mathfrak{g}$ is the set of transverse vector fields $l(M, \mathcal{F})$.

For a transversely parallelizable foliation which is not necessarily minimal, we have the following structure theorem.

Theorem 2.2 ([12], Theorem 1) *Let $\mathcal{F}$ be a codimension q transversely parallelizable foliation of a closed manifold M. Then*

(i) *there exists a locally trivial fibration $\pi \colon M \to W$ such that each fiber is the closure of a leaf of $\mathcal{F}$.*

(ii) *There exists the Lie algebra $\mathfrak{h}$ which is uniquely determined by $\mathcal{F}$ such that, for each fiber F of the fibration π, the induced foliation $\mathcal{F}|_F$ is a Lie $\mathfrak{h}$-foliation.*

Moreover if $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation, then $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$.

The fibration $\pi \colon M \to W$ is called the basic fibration, W is called the basic manifold, the dimension of W is called the basic dimension, and the Lie algebra $\mathfrak{h}$ is called the structure Lie algebra of $(M, \mathcal{F})$.

Let F be a fiber of the basic fibration $\pi \colon M \to W$. Then $\mathcal{F}|_F$ is a minimal transversely parallelizable foliation. Hence the Lie algebra of transverse vector fields $l(F, \mathcal{F}|_F)$ is a $(q - \dim W)$-dimensional Lie algebra over $\mathbb{R}$. The structure Lie algebra $\mathfrak{h}$ is given by the Lie algebra of transverse vector fields $l(F, \mathcal{F}|_F)$.

Let $\mathcal{F}$ be a minimal Lie $\mathfrak{g}$-flow on a closed manifold M. Then the Lie algebra $\mathfrak{g}$ is abelian (cf. [1], Theorem 1). Therefore, the structure Lie algebra of a one-dimensional transversely parallelizable foliation is abelian and thus it is isomorphic to $\mathbb{R}^m$ for some m.

To end this subsection, we define the notions of realizability of a pair $(\mathfrak{g}, \mathfrak{h})$ of a Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{h}$ and of a pair $(\mathfrak{g}, m)$ of a Lie algebra $\mathfrak{g}$ and an integer m ($\leq \dim \mathfrak{g}$).

Definition 2.3 For a q-dimensional Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{h}$, a pair $(\mathfrak{g}, \mathfrak{h})$ is realizable if there exists a closed manifold M and a Lie $\mathfrak{g}$-foliation of M such that the structure Lie algebra is $\mathfrak{h}$.

Definition 2.4 For a q-dimensional Lie algebra $\mathfrak{g}$ and an integer m with $0 \leq m \leq q$, a pair $(\mathfrak{g}, m)$ is realizable if there exists a closed manifold M and a Lie $\mathfrak{g}$-flow on M such that the dimension of the structure Lie algebra is equal to m.

2.2   The Darboux cover

In this section, we construct a covering map, which is called the Darboux covering, for a given Lie $\mathfrak{g}$-foliation and introduce a structure theorem proved by E. Fedida (cf. [4] and [13]). The contents of this section are referred to [4], [12] and [13].

Let M be a closed manifold and $\mathfrak{g}$ be a q-dimensional Lie algebra. Let $\Omega^1(M; \mathfrak{g})$ be the set of $\mathfrak{g}$-valued differential 1-forms on M. A $\mathfrak{g}$-valued differential 1-form $\omega \in \Omega^1(M; \mathfrak{g})$ is a Maurer-Cartan form if ω satisfies the equation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0,$$

where $[\omega, \omega]$ is the $\mathfrak{g}$-valued differential 2-form on M defined by

$$[\omega, \omega](X, Y) = [\omega(X), \omega(Y)].$$

A $\mathfrak{g}$-valued 1-form ω is non-singular if $\omega_x \colon T_x M \to \mathfrak{g}$ is surjective at any point $x \in M$.

Let $\mathcal{F}$ be a codimension q foliation of M. If there exists a non-singular Maurer-Cartan form $\omega \in \Omega^1(M; \mathfrak{g})$ on M such that $T\mathcal{F} = \mathrm{Ker}(\omega)$, then $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation. In fact, we choose a basis $\{e_1, \dots, e_q\}$ of $\mathfrak{g}$ and define the Lie $\mathfrak{g}$-parallelism $\{\overline{X}_1, \dots, \overline{X}_q\}$ of $\mathcal{F}$ by

$$\overline{X}_i(x) = \overline{\omega}_x^{-1}(e_i),$$

where

$$\overline{\omega}_x \colon N_x \mathcal{F} \to \mathfrak{g}$$

is the linear isomorphism induced by $\omega_x \colon T_x M \to \mathfrak{g}$ at the point $x \in M$.

Conversely, suppose that $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation with a transverse Lie $\mathfrak{g}$-parallelism $\{\overline{X}_1, \dots, \overline{X}_q\}$. By definition, the $\mathbb{R}$ vector subspace

$\langle \overline{X}_1, \ldots, \overline{X}_q \rangle_{\mathbb{R}}$ of $l(M, \mathcal{F})$ is isomorphic to $\mathfrak{g}$. Define the linear isomorphism

$$\sigma_x \colon \mathfrak{g} = \langle \overline{X}_1, \ldots, \overline{X}_q \rangle_{\mathbb{R}} \to N_x \mathcal{F}$$

by

$$\sigma_x \left(\sum_{i=1}^{q} t_i \overline{X}_i \right) = \sum_{i=1}^{q} t_i \overline{X}_i(x)$$

and define $\omega \in \Omega^1(M, \mathfrak{g})$ by

$$\omega_x = \sigma_x^{-1} \circ \pi \colon T_x M \xrightarrow{\pi} N_x \mathcal{F} \xrightarrow{\sigma_x^{-1}} \mathfrak{g},$$

where $N_x \mathcal{F} = T_x M / T_x \mathcal{F}$ and $\pi \colon T_x M \to N_x \mathcal{F}$ is the natural projection. Then ω is a non-singular Maurer-Cartan form on M such that $T\mathcal{F} = \mathrm{Ker}(\omega)$ (cf.[13], Lemma 4.3).

Therefore we have the following proposition.

Proposition 2.5 ([13], Lemma 4.3) *Let $\mathcal{F}$ be a codimension q foliation of a closed manifold M and $\mathfrak{g}$ be a q-dimensional Lie algebra. Then $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation if and only if there exists a non-singular Maurer-Cartan form $\omega \in \Omega^1(M; \mathfrak{g})$ such that $T\mathcal{F} = \mathrm{Ker}(\omega)$.*

Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-foliation of a closed manifold M and $\omega \in \Omega^1(M; \mathfrak{g})$ be a non-singular Maurer-Cartan form with $\mathrm{Ker}(\omega) = T\mathcal{F}$. Let G be the simply connected Lie group with the Lie algebra $\mathfrak{g}$. For any projectable vector field $X \in L(M, \mathcal{F})$, the map $\omega(X) \colon M \to \mathfrak{g}$ is basic, that is, $Z(\omega(X)) = 0$ for any $Z \in \mathfrak{X}(\mathcal{F})$. The map $\omega(X)$ is a constant map if the corresponding transverse vector field $\overline{X}$ belong to $\mathfrak{g}$. Consider the subalgebra of L_ω of $L(M, \mathcal{F})$ defined by

$$L_\omega = \{ X \in L(M, \mathcal{F}) \mid \omega(X) \text{ is constant} \}$$

and the trivial left-principal G-bundle

$$pr_1 \colon M \times G \to M.$$

Then, for any $X \in L_\omega$, we can define the left-invariant vector field $\widetilde{X}$ on $M \times G$ by

$$\widetilde{X} = X + \omega(X),$$

where we identify the set of left-invariant vector fields on G with $\mathfrak{g}$. The correspondence $X \mapsto \widetilde{X}$ is $\mathbb{R}$-linear and we have

$$[\widetilde{X}, \widetilde{Y}] = \widetilde{[X, Y]} \text{ for any } X, Y \in L_\omega.$$

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For each $(x, g) \in M \times G$, we define the subspace $H_{(x,g)}$ of $T_{(x,g)}(M \times G)$ by

$$H_{(x,g)} = \{ \widetilde{X}(x, g) \mid X \in L_\omega \}.$$

Then H defines a connection on the principal G-bundle $pr_1 \colon M \times G \to M$. Since $[\widetilde{X}, \widetilde{Y}] = \widetilde{[X, Y]}$ for any $X, Y \in L_\omega$, the connection H is a flat connection. Hence the connection H defines the foliation $\mathcal{G}$ of $M \times G$. Fix a leaf M' of $\mathcal{G}$. Then the projections $pr_1 \colon M \times G \to M$ and $pr_2 \colon M \times G \to G$ induce maps $p_1 \colon M' \to M$ and $p_2 \colon M' \to G$.

Theorem 2.6 ([4]) *The map $p_1 \colon M' \to M$ is a covering map and the map $p_2 \colon M' \to G$ is a locally trivial fibration. Moreover, the foliation $\mathcal{F}'$ of M' defined by the pull-back of $\mathcal{F}$ by p_1 is the simple foliation defined by p_2, that is, the leaf space $M'/\mathcal{F}'$ is a Hausdorff space and the foliation $\mathcal{F}'$ is given by the connected components of the fibers of p_2.*

The pair $(M', \mathcal{F}')$ is called the Darboux covering of the Lie $\mathfrak{g}$-foliation $(M, \mathcal{F})$.

Let x_0 be an arbitrary point in M and fix a point $x_0' \in p_1^{-1}(x_0)$. Then the set

$$\Gamma = \{ g \in G \mid g(x_0') \in M' \}$$

is a subgroup of G and depends neither on the choice of $x_0 \in M$ nor on the choice of $x_0' \in p_1^{-1}(x_0)$ (cf. [13], Lemma 4.5). We call the subgroup Γ the holonomy group of the Darboux covering $(M', \mathcal{F}')$.

Fix a base point x_0 in M and let M' be the leaf of $\mathcal{G}$ which contains the point $x_0' = (x_0, e)$, where e is the identity element of G. Then the action of $\pi_1(M, x_0)$ on M' by the covering transformation defines the homomorphism

$$h \colon \pi_1(M, x_0) \to G$$

by

$$h(\gamma) = p_2(\gamma \cdot x_0'),$$

that is, $h(\gamma)$ is the element of G satisfying $\gamma \cdot \widetilde{x}_0 = (x_0, h(\gamma))$. This homomorphism $h \colon \pi_1(M, x_0) \to G$ is called the holonomy homomorphism of the Lie $\mathfrak{g}$-foliation $\mathcal{F}$. The image of h coincides with the holonomy group Γ.

For an arbitrary element $x' = (x, g) \in M'$ and $\gamma \in \pi_1(M, x_0)$, since

$$p_1(\gamma \cdot x') = p_1(x') = p_1(x, h(\gamma)g)$$

and $\gamma \cdot x_0' = (x_0, h(\gamma)e)$, we have $\gamma \cdot (x, g) = (x, h(\gamma)g)$. Thus $p_2 \colon M' \to G$ is h-equivariant, that is, the map $p_2 \colon M' \to G$ satisfies

$$p_2(\gamma \cdot x') = h(\gamma)p_2(x')$$

for any $\gamma \in \pi_1(M, x_0)$ and any $x' \in M'$.

Let K be the closure of the holonomy group Γ in G. Then the space $K \backslash G$ is a manifold and the projection $f \colon G \to K \backslash G$ is a locally trivial fibration. Since $p_2 \colon M' \to G$ is a locally trivial fibration, the map

$$f \circ p_2 \colon M' \to K \backslash G$$

is also a locally trivial fibration. Since p_2 is h-equivariant, the locally trivial fibration $f \circ p_2$ induces the locally trivial fibration

$$\psi \colon M \to K \backslash G.$$

Proposition 2.7 ([4]) *The closure of the leaves of $\mathcal{F}$ are the fibers of the fibration $\psi \colon M \to K \backslash G$.*

Proof. Let L be a leaf of $\mathcal{F}$ and L' be a leaf of $\mathcal{F}'$ such that $p_1(L') = L$. Then $p_1^{-1}(L) = \pi_1(M, x_0) \cdot L'$. By Theorem 2.6, p_2 maps the leaf L' of $\mathcal{F}'$ to a point $g \in G$. Since p_2 is h-equivariant and the holonomy group Γ coincides with the image of h, we have

$$p_2(p_1^{-1}(L)) = \Gamma \cdot p_2(L') = \Gamma \cdot g.$$

We consider the subset $p_2(p_1^{-1}(\overline{L}))$ of G, where $\overline{L}$ is the closure of the leaf L in M. Since $\overline{L}$ is a closed subset and saturated by $\mathcal{F}$, the inverse image $p_1^{-1}(\overline{L})$ is a closed subset and saturated by $\mathcal{F}'$. Thus $p_2(p_1^{-1}(\overline{L}))$ is a closed subset of G which contains $\Gamma \cdot g$. Since $p_2(p_1^{-1}(\overline{L}))$ is closed, $p_2(p_1^{-1}(\overline{L}))$ contains $K \cdot g$. Hence $p_2(p_1^{-1}(\overline{L}))$ is a union of right cosets of K. Therefore $p_1^{-1}(\overline{L})$ is a closed subset and saturated by the the fibers of the fibration $f \circ p_2 \colon M' \to K \backslash G$.

Set $\bar{g} = f(g) \in K \backslash G$ and consider $\psi^{-1}(\bar{g})$. Since $p_1^{-1}(\overline{L})$ is saturated by the fibers of the fibration $f \circ p_2$, the closure $\overline{L}$ is saturated by the fibers of the fibration $\psi \colon M \to K \backslash G$. Hence $\psi^{-1}(\bar{g}) = p_1((f \circ p_2)^{-1}(\bar{g}))$ is contained in $\overline{L}$. On the other hand, the leaf L is contained in the fiber $\psi^{-1}(\bar{g})$. Since $\psi^{-1}(\bar{g})$ is a closed subset of M, we have $\overline{L} \subset \psi^{-1}(\bar{g})$. Therefore $\overline{L}$ coincides with the fiber $\psi^{-1}(\bar{g})$. $\qquad\square$

By Proposition 2.7, the basic manifold for a Lie $\mathfrak{g}$-foliation is of the form $K\backslash G$, where K is the closure of the holonomy group. Moreover we can describe the structure Lie algebra $\mathfrak{h}$ of a Lie $\mathfrak{g}$-foliation by the holonomy group Γ.

Proposition 2.8 ([12], Remark 2) *The structure Lie algebra $\mathfrak{h}$ is isomorphic to the Lie algebra of K which is the closure of the holonomy group Γ.*

Proof. Let l be the codimension of the fibers of $\psi\colon M \to K\backslash G$ and let F be the fiber of ψ at $f(e) \in K\backslash G$. Then F is an $(n-l)$-dimensional closed submanifold of M and $\mathcal{F}|_F$ is a codimension $(q-l)$ foliation of F, where q is the codimension of $\mathcal{F}$.

Let $\mathfrak{k}$ be the Lie algebra of K, which is a subalgebra of $\mathfrak{g}$. Fix a basis $\{e_1, \ldots, e_q\}$ of the Lie algebra $\mathfrak{g}$ such that $\{e_1, \ldots, e_{q-l}\}$ is a basis of $\mathfrak{k}$. Let $X_1, \ldots, X_q$ be projectable vector fields of $\mathcal{F}$ such that the corresponding transverse vector fields $\overline{X}_1, \ldots, \overline{X}_q$ satisfy

$$\overline{\omega}_x(\overline{X}_i(x)) = e_i \quad (i = 1, \ldots, q)$$

for any $x \in M$, where $\overline{\omega}_x\colon N_x\mathcal{F} \to \mathfrak{g}$ is the isomorphism induced by $\omega_x\colon T_xM \to \mathfrak{g}$. Fix a point $x \in F$ and consider the map

$$f_* \circ \omega_x\colon T_xM \to \mathfrak{g} \to \mathfrak{k}\backslash\mathfrak{g}.$$

Then the tangent space T_xF coincides with the subspace $\mathrm{Ker}(f_* \circ \omega_x)$. Since

$$\omega_x(X_i) = \overline{\omega}_x(\overline{X}_i) = e_i$$

and $\mathfrak{k} = \langle e_1, \ldots, e_{q-l}\rangle_{\mathbb{R}}$, projectable vector fields $X_1, \ldots, X_{q-l}$ are tangent to the subspace $\mathrm{Ker}(f_* \circ \omega_x)$ at each $x \in F$. Thus $\overline{X}_1, \ldots, \overline{X}_{q-l}$ are transverse vector fields of $(F, \mathcal{F}|_F)$. Therefore $\{\overline{X}_1, \ldots, \overline{X}_{q-l}\}$ is a transverse parallelism of $\mathcal{F}|_F$. The structure Lie algebra $\mathfrak{h}$ is given by the Lie algebra of transverse vector fields $l(F, \mathcal{F}|_F)$, which coincides with $\langle \overline{X}_1, \ldots, \overline{X}_{q-l}\rangle_{\mathbb{R}}$. By definition, $\langle \overline{X}_1, \ldots, \overline{X}_{q-l}\rangle_{\mathbb{R}}$ is isomorphic to $\mathfrak{k}$. Therefore the structure Lie algebra $\mathfrak{h}$ is isomorphic to $\mathfrak{k}$. $\square$

We take the universal covering

$$p\colon \widetilde{M} \to M.$$

Then $M' = \mathrm{Ker}(h)\backslash\widetilde{M}$. Let

$$p'\colon \widetilde{M} \to M' = \mathrm{Ker}(h)\backslash\widetilde{M}$$

be the covering map and

$$D = p_2 \circ p' \colon \widetilde{M} \to G$$

be the locally trivial fibration. Then the map D is h-equivariant and the foliation $\widetilde{\mathcal{F}}$ of $\widetilde{M}$ defined by the pull-back of $\mathcal{F}$ by p coincides with the simple foliation defined by D. Therefore we have the following theorem.

Theorem 2.9 ([4]) *Let $\mathcal{F}$ be a codimension q Lie $\mathfrak{g}$-foliation of a closed manifold M and G be the simply connected Lie group with the Lie algebra $\mathfrak{g}$. Let $p \colon \widetilde{M} \to M$ be the universal covering of M. Fix a transverse Lie $\mathfrak{g}$-parallelism $\{\bar{X}_1, \ldots, \bar{X}_q\}$ of $(M, \mathcal{F})$. Then there exists a locally trivial fibration $D \colon \widetilde{M} \to G$ and a homomorphism $h \colon \pi_1(M) \to G$ such that*

(i) *$D(\alpha \cdot \widetilde{x}) = h(\alpha) \cdot D(\widetilde{x})$ for any $\alpha \in \pi_1(M)$ and any $\widetilde{x} \in \widetilde{M}$ and*

(ii) *the lifted foliation $\widetilde{\mathcal{F}} = \pi^* \mathcal{F}$ is given by the fibers of the fibration D.*

Moreover the pair (D, h) is unique modulo the equivalence relation generated by $(D, h) \sim (g \cdot D, g \cdot h \cdot g^{-1})$, where g is an element of G.

The fibration D is called the developing map, the homomorphism h is called the holonomy homomorphism and the image of h is called the holonomy group of the Lie $\mathfrak{g}$-foliation $\mathcal{F}$ with respect to the Lie $\mathfrak{g}$-parallelism $\{\bar{X}_1, \ldots, \bar{X}_q\}$.

Conversely, if there exist such D and h satisfying the condition (i) above, then the set of the fibers of D induces the Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of M such that the developing map is D and the holonomy homomorphism is h.

2.3 Basic cohomology

Let $\mathcal{F}$ be a codimension q foliation of a closed manifold M. We have defined the notion of basic functions in the section 2.1. Now we define the notion of basic k-forms. A differential k-form $\omega \in \Omega^k(M)$ on M is said to be basic if it satisfies

(i) $i_X \omega = 0$ and

(ii) $i_X d\omega = 0$

for every $X \in \mathfrak{X}(\mathcal{F})$, where i_X is the interior product by X. We denote the set of basic k-forms on M by $\Omega_b^k(M, \mathcal{F})$. For any basic k-form $\omega \in \Omega_b^k(M, \mathcal{F})$, the exterior derivative $d\omega$ is a basic $(k+1)$-form. Hence $\{\Omega_b^*(M, \mathcal{F}), d\}$ is a subcomplex of the de Rham complex $\{\Omega^*(M), d\}$. We denote the cohomology by $H_b^*(M, \mathcal{F})$ and call the basic cohomology of $(M, \mathcal{F})$. If k is greater than the codimension of $\mathcal{F}$, then $H_b^k(M, \mathcal{F}) = 0$.

Let $\mathcal{F}$ be a Riemannian foliation of M, that is, $\mathcal{F}$ is a foliation equipped with a $C^\infty(M)$-bilinear symmetric from $g_T \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ which satisfies

(i) $g_T(X, X)$ is non-negative for any $X \in \mathfrak{X}(M)$,

(ii) $\mathrm{Ker}(g_T) = T\mathcal{F}$ and

(iii) $L_X(g_T) = 0$ for any $X \in \mathfrak{X}(\mathcal{F})$,

where $\mathrm{Ker}(g_T) = \{\, X \in TM \mid {}^\forall Y \in TM, g_T(X, Y) = 0 \,\}$ and L_X is the Lie derivative. Such a g_T is called a transverse metric of $\mathcal{F}$. A. El Kacimi, V. Sergiescu and G. Hector [7] proved that if $\mathcal{F}$ is a codimension q Riemannian foliation of a closed manifold M, then

$$H_b^q(M, \mathcal{F}) = 0 \text{ or } \mathbb{R}.$$

They also proved that the cohomology $H_b^*(M, \mathcal{F})$ satisfies the Poincaré duality if and only if $H_b^q(M, \mathcal{F}) \neq 0$. The codimension q Riemannian foliation is said to be unimodular if $H_b^q(M, \mathcal{F}) \neq 0$.

Let $\mathcal{F}$ be a codimension q transversely parallelizable foliation, then a transverse parallelization $\{\overline{X}_1, \ldots, \overline{X}_q\}$ induces the metric g on the normal bundle $N\mathcal{F}$ by

$$g(\overline{X}_i, \overline{X}_j) = \delta_{ij}$$

and induces the transverse metric g_T of $\mathcal{F}$ by

$$g_T(X, Y)(x) = g(p(X(x)), p(Y(x))),$$

where $p \colon TM \to N\mathcal{F}$ is the natural projection. Hence transversely parallelizable foliations are Riemannian foliations. In particular, Lie $\mathfrak{g}$-foliations are Riemannian foliations.

If $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation with dense leaves, then the basic cohomology $H_b^*(M, \mathcal{F})$ is isomorphic to the cohomology $H^*(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. Hence, in this case, a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ is unimodular if and only if the Lie algebra $\mathfrak{g}$ is unimodular. M. Llabrés and A. Reventós [9] proved that

if a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ is unimodular then the Lie algebra $\mathfrak{g}$ is unimodular. It is not known whether the converse is true. However, if the Lie algebra $\mathfrak{g}$ is nilpotent, the converse is true.

Theorem 2.10 ([9], Corollary 3.3) *Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-foliation of a closed manifold M. If the Lie algebra $\mathfrak{g}$ is nilpotent, then the foliation $\mathcal{F}$ is unimodular.*

2.4 Commuting sheaves

In this subsection, we define the notion of commuting sheaves, which are introduced by P. Molino [11], and review a result by P. Molino and V. Sergiescu [14].

Let $\mathcal{F}$ be a codimension q transversely parallelizable foliation of a closed manifold M and U be an open subset of M. A transverse vector field $\overline{Z}_U \in l(U, \mathcal{F}|_U)$ is a local commuting transverse vector field if for any $\overline{X} \in l(M, \mathcal{F})$, the restriction $\overline{X}|_U$ of $\overline{X}$ to U commutes with $\overline{Z}_U$. We denote the set of local commuting transverse vector fields on U by $\mathrm{C}(U)$, which is a subalgebra of $l(U, \mathcal{F}|_U)$. If $U = M$, then $\mathrm{C}(M)$ coincides with the center $c(M, \mathcal{F})$ of $l(M, \mathcal{F})$.

For each $x \in M$, let $\mathrm{C}_x(M, \mathcal{F})$ be the set of germs at x of the local commuting transverse vector fields and let

$$\mathrm{C}(M, \mathcal{F}) = \bigcup_{x \in M} \mathrm{C}_x(M, \mathcal{F})$$

be the set of all these germs at different points of M. Then $\mathrm{C}(M, \mathcal{F})$ is a sheaf.

For each open subset U of M and each $\overline{Z}_U \in \mathrm{C}(U)$, we consider the subset

$$O(U, \overline{Z}_U) = \bigcup_{x \in U} \{ (\overline{Z}_U)_x \in \mathrm{C}_x(M, \mathcal{F}) \}$$

of $\mathrm{C}(M, \mathcal{F})$, where $(\overline{Z}_U)_x$ is the germ of $\overline{Z}_U$ at x. Then

$$\{ O(U, \overline{Z}_U) \mid U \subset M, \overline{Z}_U \in \mathrm{C}(U) \}$$

generates the sheaf topology of $\mathrm{C}(M, \mathcal{F})$. We call the sheaf $\mathrm{C}(M, \mathcal{F})$ equipped with the sheaf topology commuting sheaf of the foliation $\mathcal{F}$.

Let $\chi \colon \mathrm{C}(M, \mathcal{F}) \to M$ be the natural projection. Then this map χ is continuous and the Lie algebra $\mathrm{C}(U)$ can be identified with the set of continuous sections of the sheaf over U.

Proposition 2.11 ([13], Lemma 4.6) *Let U be a connected open subset of M, let x_0 be a point of U and let $\overline{Z}_U \in \mathrm{C}(U)$. If $\overline{Z}_U(x_0) = 0$, then $\overline{Z}_U = 0$.*

Proof. Fix a transverse parallelism $\{\overline{X}_1, \ldots, \overline{X}_q\}$ of $\mathcal{F}$. Consider the subset $X = \{\, x \in U \mid \overline{Z}_U(x) = 0 \,\}$ of U. Then X is closed and non-empty. We show that X is open. Let x be an element of X and consider an open neighborhood $U' \subset U$ of x such that U' is foliated by $\mathcal{F}$ and $\mathcal{F}|_{U'}$ is a simple foliation of U'. Let

$$\pi\colon U' \to \overline{U}' = U'/\mathcal{F}|_{U'}$$

be the projection. Then the transverse parallelism $\{\overline{X}_1, \ldots, \overline{X}_q\}$ of $(M, \mathcal{F})$ defines the parallelism $\{\pi_*(\overline{X}_1), \ldots, \pi_*(\overline{X}_q)\}$ of $\overline{U}'$. Thus, for any point $\pi(y) \in \overline{U}'$, there exists $t_1, \ldots, t_q \in \mathbb{R}$ such that

$$\phi_1^{t_1} \circ \cdots \circ \phi_q^{t_q}(\pi(x)) = \pi(y),$$

where $\phi_i^{t_i}$ is the local 1-parameter transformation group of $\overline{U}'$ generated by $\pi_*(\overline{X}_i)$. Since $\overline{Z}_U$ commutes with $\overline{X}_i$ for each i, the projected vector field $\pi_*(\overline{Z}_U)$ commutes with $\pi_*(\overline{X}_i)$ for each i. Since $\pi_*(\overline{Z}_U)(\pi(x)) = 0$ we have

$$
\begin{aligned}
\pi_*(\overline{Z}_U)(\pi(y)) &= \pi_*(\overline{Z}_U)(\phi_1^{t_1} \circ \cdots \circ \phi_q^{t_q}(\pi(x)) \\
&= (\phi_1^{t_1})_* \circ \cdots \circ (\phi_q^{t_q})_*(\overline{Z}_U(\pi(x))) \\
&= 0
\end{aligned}
$$

Since $\overline{Z}_U$ is a transverse vector field, this means $\overline{Z}_U(y) = 0$ for any $y \in U'$. $\qquad\square$

By Proposition 2.11, the set of local commuting transverse vector fields $\mathrm{C}(U)$ is a real Lie algebra of dimension less than or equal to q. Hence $\mathrm{C}_x(M, \mathcal{F})$ is also a real Lie algebra of dimension less than or equal to q. We call the dimension q' of $\mathrm{C}_x(M, \mathcal{F})$ the dimension of the fibers of the commuting sheaf. By Molino's Theorem (cf. [12], Theorem 2), the dimension of the fibers of the commuting sheaf coincides with the dimension of the structure Lie algebra of $(M, \mathcal{F})$.

Let x_0 be an arbitrary point of M. Then there exists a connected open neighborhood U of x_0 such that $\dim \mathrm{C}(U) = q'$. Let $\overline{Z}_{U1}, \ldots, \overline{Z}_{Uq'} \in \mathrm{C}(U)$ be local commuting transverse vector fields which are linearly independent at each point of U. Then every element $\overline{Z}_U \in \mathrm{C}(U)$ can be

written as
$$\overline{Z}_U = a_1 \overline{Z}_{U1} + \cdots a_{q'} \overline{Z}_{Uq'},$$
where $a_i \in \mathbb{R}$. Therefore we can define the bijection $\phi \colon \chi^{-1}(U) \to U \times \mathbb{R}^{q'}$ by
$$\phi((\overline{Z}_U)_x) = (x, a_1, \ldots, a_{q'}),$$
where $\overline{Z}_U = \sum_{i=1}^{q'} a_i \overline{Z}_{Ui}$. This bijection is a homeomorphism if $U \times \mathbb{R}^{q'}$ is equipped with the product topology of that on U and the discrete topology on $\mathbb{R}^{q'}$. This homeomorphism $\phi \colon \chi^{-1}(U) \to U \times \mathbb{R}^{q'}$ gives a local trivialization of the sheaf $C(M, \mathcal{F})$. Thus the commuting sheaf is a locally trivial sheaf.

We say that the commuting sheaf is trivial if there exists a global trivialization $\phi \colon C(M, \mathcal{F}) \to M \times \mathbb{R}^{q'}$, in other words, there exist $\overline{Z}_1, \ldots, \overline{Z}_{q'} \in C(M) = c(M, \mathcal{F})$ which are linearly independent on M.

The end of this subsection, we introduce the following theorem which is proved by P. Molino and V. Sergiescu (cf. [14]).

Theorem 2.12 ([14], Theorem A) *Let $\mathcal{F}$ be a Riemannian flow on an n-dimensional closed manifold M. The following properties are equivalent:*

(i) *$\mathcal{F}$ is isometric.*

(ii) *$\mathcal{F}$ is unimodular.*

(iii) *The commuting sheaf of $(M, \mathcal{F})$ is trivial.*

Here a flow $\mathcal{F}$ is isometric if there exist a Riemannian metric g on M and a Killing vector field $X \in \mathfrak{X}(M)$ of g which has no singular points such that $\mathcal{F}$ is the orbits of X.

3 Nilpotent Lie algebras

3.1 Basic properties of Lie groups and their Lie algebras

In this section, we describe several basic properties of Lie groups and their Lie algebras.

Let $\mathfrak{g}$ be a q-dimensional Lie algebra over $\mathbb{R}$ and G be the q-dimensional simply connected Lie group with the Lie algebra $\mathfrak{g}$. Then we can define the exponential map
$$\exp \colon \mathfrak{g} \to G.$$

There exists a neighborhood V of $0 \in \mathfrak{g}$ such that $\exp|_V \colon V \to \exp(V)$ is a diffeomorphism. Thus the inverse is well-defined on a neighborhood U of $e \in G$. We denote it by $\log \colon U \to \mathfrak{g}$.

For any $X, Y \in \mathfrak{g}$, we define $X * Y \in \mathfrak{g}$ by

$$X * Y = \log(\exp X \cdot \exp Y).$$

This is well-defined near $X = Y = 0$. If $\mathfrak{g}$ is abelian, then $X * Y = X + Y$. In general, we have the following formula:

$$X * Y = \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{\substack{p_i + q_i > 0 \\ 1 \le i \le n}} \frac{(\sum_{i=1}^{n}(p_i + q_i))^{-1}}{p_1! q_1! \cdots p_n! q_n!}$$

$$\times (\mathrm{ad}X)^{p_1} (\mathrm{ad}Y)^{q_1} \cdots (\mathrm{ad}X)^{p_n} (\mathrm{ad}Y)^{q_n - 1} Y,$$

where $\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$ which is defined by

$$\mathrm{ad}(X)Y = [X, Y].$$

This formula is called the Campbell-Baker-Hausdorff formula (cf. [3], Chapter 1).

For any $g \in G$, let $\alpha_g \colon G \to G$ be the automorphism defined by

$$\alpha_g(x) = gxg^{-1}$$

and let $\mathrm{Ad}(g) \colon \mathfrak{g} \to \mathfrak{g}$ be the differential of α_g at $e \in G$. Then

$$\mathrm{Ad} \colon G \to \mathrm{Aut}(\mathfrak{g})$$

is a representation of G, which is called the adjoint representation of G.

Let H be a Lie group and $\mathfrak{h}$ be its Lie algebra. For any smooth homomorphism $f \colon G \to H$, the differential $df_e \colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras and satisfies

$$f(\exp X) = \exp df_e(X)$$

for any $X \in \mathfrak{g}$. Thus we have

$$\alpha_g(\exp X) = \exp \mathrm{Ad}(g)X.$$

Since $d\mathrm{Ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ coincides with the adjoint representation $\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, we have

$$\mathrm{Ad}(\exp X) = \exp \mathrm{ad}(X).$$

Therefore we have

$$\mathrm{Ad}(\exp X)Y = \exp \mathrm{ad}(X)Y = \sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{ad}X)^k Y.$$

3.2 Nilpotent Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. The descending central series of $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}^{(1)} = \mathfrak{g} \quad \text{and}$$
$$\mathfrak{g}^{(k)} = [\mathfrak{g}, \mathfrak{g}^{(k-1)}].$$

A Lie algebra $\mathfrak{g}$ is nilpotent if there exists an integer n such that $\mathfrak{g}^{(n)} = \{0\}$ and a connected Lie group G is nilpotent if the Lie algebra of G is nilpotent. If $\mathfrak{g}$ is a nilpotent Lie algebra and G is the simply connected nilpotent Lie group with the Lie algebra $\mathfrak{g}$, then the exponential map

$$\exp \colon \mathfrak{g} \to G$$

is a diffeomorphism. Thus the inverse map $\log \colon G \to \mathfrak{g}$ can be defined on the whole of G.

For a nilpotent Lie algebra $\mathfrak{g}$, there exists a special basis which is introduced by A. I. Mal'cev (cf. [3], Theorem 1.1.13 and [10], Section 2).

Theorem 3.1 ([10]) *Let $\mathfrak{g}$ be a q-dimensional nilpotent Lie algebra over $\mathbb{K}$ and let $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_k$ be ideals with $\dim \mathfrak{g}_j = l_j$, where $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. Then there exists a basis $\{X_1, \ldots, X_q\}$ of $\mathfrak{g}$ such that*

(i) *for each l, $\mathfrak{h}_l = \langle X_1, \ldots, X_l \rangle_{\mathbb{K}}$ is an ideal of $\mathfrak{g}$ and*

(ii) *for each $j \in \{1, \ldots, k\}$, $\mathfrak{h}_{l_j} = \mathfrak{g}_j$,*

where $\langle X_1, \ldots, X_l \rangle_{\mathbb{K}}$ is the $\mathbb{K}$ vector subspace of $\mathfrak{g}$ spanned by $\{X_1, \ldots, X_l\}$.

To prove Theorem 3.1, we prove the following lemma.

Lemma 3.2 ([3], Theorem 1.1.13) *Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbb{K}$, where $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. Let $\mathfrak{i}$ be a non-zero ideal of $\mathfrak{g}$. Then $\mathfrak{i} \cap \mathfrak{c}(\mathfrak{g}) \neq \{0\}$.*

Proof. We define a descending series of ideals $\mathfrak{i}^{(k)}$ of $\mathfrak{g}$ by

$$\mathfrak{i}^{(1)} = \mathfrak{i} \quad \text{and}$$
$$\mathfrak{i}^{(k)} = [\mathfrak{g}, \mathfrak{i}^{(k-1)}].$$

Then we have $\mathfrak{i}^{(k)} \subset \mathfrak{g}^{(k)}$. Since the Lie algebra $\mathfrak{g}$ is nilpotent, there exists the smallest integer $r \geq 1$ such that $\mathfrak{i}^{(r)} \neq \{0\}$ and $\mathfrak{i}^{(r+1)} = \{0\}$. Then we have $\{0\} \neq \mathfrak{i}^{(r)} \subset \mathfrak{i} \cap \mathfrak{c}(\mathfrak{g})$. $\qquad\qquad \square$

Proof of Theorem 3.1. We prove by induction on the dimension of $\mathfrak{g}$. If $\dim \mathfrak{g} = 1$, then the assertion is trivial. Suppose that it is true for any nilpotent Lie algebra of dimension less than q.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension q and $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_k$ be ideals with $\dim \mathfrak{g}_j = l_j$. By Lemma 3.2, there exists a non-trivial element $X_1 \in \mathfrak{g}_1 \cap \mathfrak{c}(\mathfrak{g})$. Let

$$\pi \colon \mathfrak{g} \to \mathfrak{g}/\langle X_1 \rangle_{\mathbb{K}}$$

be the natural projection. Since π is a surjective homomorphism, $\pi(\mathfrak{g}_i)$ is an ideal of $\mathfrak{g}/\langle X_1 \rangle_{\mathbb{K}}$. Since the dimension of $\mathfrak{g}/\langle X_1 \rangle_{\mathbb{K}}$ is less than q, by the hypothesis of induction, there exists a basis $\{\pi(X_2), \ldots, \pi(X_q)\}$ of $\mathfrak{g}/\langle X_1 \rangle$ which satisfies the conditions (i) and (ii). We consider the basis $\{X_1, X_2, \ldots, X_q\}$ of $\mathfrak{g}$. Then this basis satisfies the conditions (i) and (ii). In fact, since the subspace $\langle \pi(X_2), \ldots, \pi(X_l) \rangle_{\mathbb{K}}$ is an ideal of $\mathfrak{g}/\langle X_1 \rangle_{\mathbb{K}}$ and $X_1 \in \mathfrak{c}(\mathfrak{g})$, the subspace $\langle X_1, \ldots, X_l \rangle_{\mathbb{K}}$ is an ideal of $\mathfrak{g}$. Moreover since $\pi(\mathfrak{g}_j) = \langle \pi(X_2), \ldots, \pi(X_{l_j}) \rangle_{\mathbb{K}}$ and $X_1 \in \mathfrak{g}_1$, we have $\mathfrak{g}_j = \langle X_1, \ldots, X_{l_j} \rangle_{\mathbb{K}}$. $\square$

We call a basis satisfying (i) and (ii) a strong Mal'cev basis of $\mathfrak{g}$ through $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. We call a strong Mal'cev basis of $\mathfrak{g}$ through $\{0\}$ simply a strong Mal'cev basis of $\mathfrak{g}$.

Let G be the simply connected nilpotent Lie group with the Lie algebra $\mathfrak{g}$. Then G has a special coordinate $\phi \colon \mathbb{R}^q \to G$ which is defined by a strong Mal'cev basis.

Proposition 3.3 ([3], Proposition 1.2.7) *Let $\mathfrak{g}$ be a q-dimensional nilpotent Lie algebra and let $\{X_1, \ldots, X_q\}$ be a strong Mal'cev basis of $\mathfrak{g}$. Define the map $\phi \colon \mathbb{R}^q \to G$ by*

$$\phi(t_1, \ldots, t_q) = \exp t_1 X_1 \cdots \exp t_q X_q.$$

Then

(i) *there exist polynomials $P_1(t), \ldots, P_q(t)$ in $t_1, \ldots, t_q$ such that*

$$\phi(t_1, \ldots, t_q) = \exp \left(\sum_{j=1}^{q} P_j(t) X_j \right).$$

(ii) *For each j, there exists a polynomial $Q_j(t)$ in $t_{j+1}, \ldots, t_q$ such that $P_j(t) = t_j + Q_j(t)$.*

(iii) *The map $\log \circ \phi \colon \mathbb{R}^q \to \mathfrak{g}$ is a polynomial diffeomorphism.*

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(iv) *Let $\mathfrak{g}_k$ be the ideal of $\mathfrak{g}$ generated by $\{X_1, \ldots, X_k\}$ over $\mathbb{R}$ and $G_k = \exp \mathfrak{g}_k$. Then*

$$G_k = \exp(\mathbb{R}X_1) \cdots \exp(\mathbb{R}X_k).$$

Proof. We prove (i) and (ii) by induction on the dimension q. If $q = 1$, then the assertion is trivial. We assume that this holds for any nilpotent Lie algebra of dimension less than q. Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension q. The ideal $\mathfrak{g}_1 = \mathbb{R}X_1$ is central in $\mathfrak{g}$. Let

$$\pi \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_1$$

be the natural projection. Then $\{\pi(X_2), \ldots, \pi(X_q)\}$ is a strong Mal'cev basis of $\mathfrak{g}/\mathfrak{g}_1$. By the hypothesis of induction, there exist polynomials $P_j(t)$ in $t_2, \ldots, t_q$ and $Q_j(t)$ in $t_{j+1}, \ldots, t_q$ such that

$$t_2\pi(X_2) * \cdots * t_q\pi(X_q) = \sum_{j=2}^{q} P_j(t)\pi(X_j)$$

and $P_j(t) = t_j + Q_j(t)$. Hence we have

$$t_2X_2 * \cdots * t_qX_q = \widetilde{P}(t_2, \ldots, t_q)X_1 + \sum_{j=2}^{q} P_j(t)X_j.$$

Since X_1 is central, $X_1 * Y = X_1 + Y$ for any $Y \in \mathfrak{g}$. Therefore we have

$$t_1X_1 * \cdots * t_nX_n = \sum_{j=1}^{q} P_j(t)X_j,$$

where $P_1(t) = t_1 + \widetilde{P_1}(t_2, \ldots, t_q)$. Hence (i) and (ii) hold.

By (ii), the map

$$\log \circ \phi(t_1, \ldots, t_n) = \sum_{j=1}^{q} P_j(t)X_j$$

is a polynomial diffeomorphism.

We prove that $G_k = \exp \mathbb{R}X_1 \cdots \exp \mathbb{R}X_k$. Since $\{X_1, \ldots, X_k\}$ is a strong Mal'cev basis of $\mathfrak{g}_k$, by (ii), the map $\phi_k \colon \mathbb{R}^k \to G_k$ defined by

$$\phi_k(t_1, \ldots, t_k) = \exp t_1X_1 \cdots \exp t_kX_k$$

is a diffeomorphism. Therefore we have

$$G_k = \phi_k(\mathbb{R}^k) = \exp \mathbb{R}X_1 \cdots \exp \mathbb{R}X_k.$$

$\square$

The map $\phi \colon \mathbb{R}^q \to G$ is called a strong Mal'cev coordinate.

3.3 Lattice of nilpotent Lie groups

Let G be a q-dimensional Lie group and Γ be a subgroup of G. We say that the subgroup Γ is uniform if the quotient space $\Gamma\backslash G$ is compact and the subgroup Γ is uniform lattice if Γ is uniform and discrete in G.

In general, it is difficult to know whether a given Lie group admits a uniform lattice. But for a simply connected nilpotent Lie group, Mal'cev proved the following theorem.

Theorem 3.4 ([10], Theorem 7) *Let G be a simply connected nilpotent Lie group and $\mathfrak{g}$ be the Lie algebra of G. Then G admits a lattice if and only if $\mathfrak{g}$ admits a rational structure, that is, $\mathfrak{g}$ has a basis with respect to which the structure constants are rational.*

On the other hand, Mal'cev also studied which group is realized as a uniform lattice of a simply connected nilpotent Lie group.

Theorem 3.5 ([10], Theorem 6) *A group Γ is isomorphic to a uniform lattice in a simply connected nilpotent Lie group if and only if*

(i) *Γ is finitely generated,*

(ii) *Γ is nilpotent and*

(iii) *Γ has no torsion.*

Let G_i be simply connected Lie groups and Γ_i be uniform lattices of G_i $(i = 1, 2)$. In general, it is not always true that any isomorphism from Γ_1 to Γ_2 extends to an isomorphism from G_1 to G_2. But if G_1 and G_2 are nilpotent, it is true.

Theorem 3.6 ([15], Theorem 2.11) *Let G_1 and G_2 be simply connected nilpotent Lie groups and Γ be a uniform lattice in G_1. Then any homomorphism $f \colon \Gamma \to G_2$ extends uniquely to a continuous homomorphism $\widetilde{f} \colon G_1 \to G_2$. In particular, for any lattices Γ_i in G_i $(i = 1, 2)$, any isomorphism $f \colon \Gamma_1 \to \Gamma_2$ extends to an isomorphism $\widetilde{f} \colon G_1 \to G_2$.*

Let G be a simply connected nilpotent Lie group with the Lie algebra $\mathfrak{g}$. Let $\rho \colon G \to \mathrm{GL}(n; \mathbb{R})$ be a representation of G. The representation ρ is faithful if ρ is injective and the representation ρ is unipotent if $\rho(g) \in \mathrm{GL}(n; \mathbb{R})$ is unipotent for any $g \in G$. For any simply connected nilpotent Lie group has a faithful unipotent representation (cf. [3], Theorem 1.1.11).

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Theorem 3.7 (Birkoff Embedding Theorem) *Let G be a simply connected nilpotent Lie group. Then there exists a faithful unipotent representation $\rho\colon G \to \mathrm{GL}(n;\mathbb{R})$ for some n.*

Let H be a subgroup of a simply connected nilpotent Lie group G. The subgroup H is Zariski dense in G if there exists a faithful unipotent representation $\rho\colon G \to \mathrm{GL}(n;\mathbb{R})$ such that $\rho(H)$ and $\rho(G)$ have the same Zariski closure in $\mathrm{GL}(n;\mathbb{C})$. We have the following theorem (cf. [15])

Theorem 3.8 ([15], Theorem 2.3) *Let H be a subgroup of a simply connected nilpotent Lie group G. Then the following conditions are equivalent.*

(i) *H is uniform in G.*

(ii) *H is Zariski dense in G.*

(iii) *For any faithful unipotent representation $\rho\colon G \to \mathrm{GL}(n;\mathbb{R})$, $\rho(H)$ and $\rho(G)$ have the same Zariski closure in $\mathrm{GL}(n;\mathbb{C})$.*

4   Nilpotent Lie flows

In this section, we prove the following theorem.

Theorem 4.1 *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension q which has a rational structure. Then $(\mathfrak{g}, m)$ is realizable if and only if $m \leq \dim \mathfrak{c}(\mathfrak{g})$, where $\mathfrak{c}(\mathfrak{g})$ is the center of $\mathfrak{g}$.*

First we prove the following lemma which is the key to prove the sufficiency.

Lemma 4.2 *Let $\mathfrak{g}$ be a q-dimensional nilpotent Lie algebra and $\mathcal{F}$ be a codimension q Lie $\mathfrak{g}$-foliation of a closed manifold M. Fix a transverse Lie $\mathfrak{g}$-parallelism $\{\overline{X}_1, \ldots, \overline{X}_q\}$ of $(M, \mathcal{F})$. Let $\overline{X} = \sum_{i=1}^{q} f_i \overline{X}_i$ be a transverse vector field, where each f_i is a basic function on $(M, \mathcal{F})$. If $\overline{X}$ is in the center $c(M, \mathcal{F})$ of $l(M, \mathcal{F})$ then each f_i is constant.*

To prove this we use Engel's Theorem (cf. [3], Theorem 1.1.9).

Theorem 4.3 (Engel's Theorem) *Let V be a q-dimensional vector space and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Assume that each element*

$X \in \mathfrak{g}$ *is a nilpotent endomorphism of* V. *Then there exists a basis* $\{x_1, \ldots, x_q\}$ *of* V *such that* $X \cdot V_k \subset V_{k+1}$ *for any* $X \in \mathfrak{g}$ *and any* $k = 1, \ldots, q$, *where* V_k *is the subspace of* V *spanned by* $\{x_k, \ldots, x_q\}$ *and* $V_{q+1} = \{0\}$.

Proof of Lemma 4.2. Since $\mathfrak{g}$ is a nilpotent Lie algebra, $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is a Lie subalgebra consisting of nilpotent endomorphisms. Therefore we can find a transverse Lie $\mathfrak{g}$-parallelism $\{\overline{Y}_1, \ldots, \overline{Y}_q\}$ of $\mathcal{F}$ satisfying the condition in Theorem 4.3. Since there exists a matrix $C = (c_{ij}) \in \mathrm{GL}(q; \mathbb{R})$ such that

$$\overline{Y}_j = \sum_{i=1}^{n} c_{ij} \overline{X}_i,$$

we may assume that the transverse Lie $\mathfrak{g}$-parallelism $\{\overline{X}_1, \ldots, \overline{X}_q\}$ satisfies the condition in Lemma 4.3.

Let $a_{ij}^k \in \mathbb{R}$ be the structure constants of $\mathfrak{g}$ with respect to the basis $\{\overline{X}_1, \ldots, \overline{X}_q\}$. Since $\mathrm{ad}(\overline{X}_i) \cdot \overline{X}_j \in V_{j+1}$, we have

$$[\overline{X}_i, \overline{X}_j] = \sum_{k=j+1}^{q} a_{ij}^k \overline{X}_k.$$

Since $\overline{X} = \sum_{j=1}^{q} f_j \overline{X}_j$ is in the center of $l(M, \mathcal{F})$, we have $[\overline{X}_i, \overline{X}] = 0$ which is written as follows

$$\sum_{j=1}^{q} \overline{X}_i(f_j) \overline{X}_j - \sum_{j=1}^{q} \sum_{k=j+1}^{q} f_j a_{ij}^k \overline{X}_k = 0$$

for each $i = 1, \ldots, q$. By comparing the coefficients of $\overline{X}_k$, we have

$$\overline{X}_i(f_1) = 0$$

$$\overline{X}_i(f_k) = \sum_{j=1}^{k-1} f_j a_{ij}^k$$

for each i and each $k \geq 2$.

We prove f_k is constant by induction on the index k. First, since $\overline{X}_i(f_1) = 0$ for $i = 1, \ldots, q$ and f_1 is a basic function, f_1 is constant.

Next, we assume that f_j is constant for $j \leq k$. In the equality

$$\overline{X}_i(f_{k+1}) = \sum_{j=1}^{k} f_j a_{ij}^{k+1},$$

the right hand side of the equation is constant by the induction hypothesis. Since f_{k+1} is a function on a closed manifold M, $\bar{X}_i(f_{k+1}) = 0$ for each i. Therefore f_{k+1} is constant. □

Proof of the "only if" part of Theorem 4.1. Assume that $(\mathfrak{g}, m)$ is realizable. Let $\mathcal{F}$ be a Lie $\mathfrak{g}$-flow on a $(q+1)$-dimensional closed manifold M with the m-dimensional structure Lie algebra. Fix a transverse Lie $\mathfrak{g}$-parallelism $\{\bar{X}_1, \ldots, \bar{X}_q\}$ of $(M, \mathcal{F})$. Since $\mathfrak{g}$ is nilpotent, by Theorem 2.10, the flow $\mathcal{F}$ is unimodular. Then, by Theorem 2.12, the commuting sheaf $C(M, \mathcal{F})$ is trivial. Hence there exist transverse vector fields $\bar{Z}_1, \ldots, \bar{Z}_m$ which are in the center of $l(M, \mathcal{F})$ and linearly independent at each $x \in M$. By Lemma 4.3, the coefficients of $\bar{Z}_i$ with respect to the transverse Lie $\mathfrak{g}$-parallelism $\{\bar{X}_1, \ldots, \bar{X}_q\}$ are constant. Therefore each $\bar{Z}_i$ is in the center of $\mathfrak{g}$. Since $\bar{Z}_1, \ldots, \bar{Z}_m$ are linearly independent, we have $m \leq \dim \mathfrak{c}(\mathfrak{g})$.

□

Next we prove "if" part of Theorem 4.1, that is, we construct an example of Lie $\mathfrak{g}$-flow with the m-dimensional structure Lie algebra. To construct examples, we use a strong Mal'cev basis for a nilpotent Lie algebra and the uniform lattice which is determined by the strong Mal'cev basis.

Lemma 4.4 ([3], Corollary 5.1.10) *Let $\mathfrak{g}$ be a q-dimensional nilpotent Lie algebra which has a rational structure. Let $\{X_1, \ldots, X_q\}$ be a strong Mal'cev basis with rational structure constants. Then there exists an integer λ such that the subset*

$$\Gamma = \exp \mathbb{Z}\lambda X_1 \cdots \exp \mathbb{Z}\lambda X_q$$

of G is a uniform lattice of G.

Proof. For each $g \in G$, let $\alpha_g \colon G \to G$ be the inner automorphism defined by $\alpha_g(x) = gxg^{-1}$. Then we have

$$\exp(\mathrm{Ad}(g)X) = \alpha_g(\exp X).$$

and

$$\mathrm{Ad}(\exp X)Y = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{ad}X)^k Y = (\exp \mathrm{ad}X)Y$$

for any $g \in G$ and $X, Y \in \mathfrak{g}$. Hence, for any $i < j$, we have

$$\exp tX_j \cdot \exp uX_i \cdot \exp(-tX_j) = \alpha_{\exp tX_j}(\exp uX_i)$$

$$= \exp(\mathrm{Ad}(\exp tX_j)uX_i)$$

$$= \exp\left(\sum_{k=0}^{q} \frac{1}{k!}(\mathrm{ad}\, tX_j)^k uX_i\right)$$

$$= \exp\left(uX_i + \sum_{k=1}^{q} u\frac{t^k}{k!}(\mathrm{ad}\, X_j)^k X_i\right)$$

$$= \exp(P_{i,j,1}(u,t)X_1)\cdots\exp(P_{i,j,i-1}(u,t)X_{i-1}) \cdot \exp uX_i,$$

where $P_{i,j,k}(u,t)$ are polynomials in t and u having rational coefficients. Let λ be a common multiple of all the denominators in these coefficients for all the $P_{i,j,k}$.

Let $\{Y_1,\ldots,Y_q\} = \{\lambda X_1,\ldots,\lambda X_q\}$ be the new strong Mal'cev basis of $\mathfrak{g}$. Then, for any $i < j$, we have

$$\exp tY_j \cdot \exp uY_i \cdot \exp(-tY_j)$$

$$= \exp R_{i,j,1}(t,u)Y_1\cdots\exp R_{i,j,i-1}(t,u)Y_{i-1} \cdot \exp uY_1,$$

where $R_{i,j,k}(u,t)$ are polynomials with integer coefficients.

Let

$$\Gamma_i = \exp \mathbb{Z}Y_1 \cdots \exp \mathbb{Z}Y_i$$

be the subset of G. We prove inductively that the subset Γ_i is a subgroup of G. It is obvious that Γ_1 is a subgroup of G. Assume that Γ_k is a subgroup of G for any $k < m$. We prove that Γ_m is a subgroup of G. By the hypothesis of induction, it is sufficient to show that

$$\exp n_m Y_m \cdot \exp n_i Y_i \in \Gamma_m$$

for any $i < m$ and $n_m, n_i \in \mathbb{Z}$. But this holds since

$$\exp n_m Y_m \cdot \exp n_i Y_i \cdot \exp(-n_m Y_m) \in \Gamma_i$$

for $i < m$ and $n_m, n_i \in \mathbb{Z}$.

Hence Γ_i is a subgroup of G and thus $\Gamma = \Gamma_q$ is a subgroup of G. Since the strong Mal'cev coordinate $\phi \colon \mathbb{R}^q \to G$ is a diffeomorphism, the subgroup Γ is a discrete subgroup of G. Since $G = \phi([0,1]^q)\Gamma$, the subgroup Γ is uniform. $\qquad\square$

Proof of the "if" part of Theorem 4.1. By Lemma 4.4, there exists a strong Mal'cev basis $\{X_1, \ldots, X_q\}$ of $\mathfrak{g}$ through the center $\mathfrak{c}(\mathfrak{g})$ with integer structure constants such that

$$\Delta = \exp \mathbb{Z}X_1 \cdots \exp \mathbb{Z}X_q$$

is a uniform lattice of G, where G is the simply connected nilpotent Lie group with the Lie algebra $\mathfrak{g}$. Let $\phi \colon \mathbb{R}^q \to G$ be the strong Mal'cev coordinate.

Fix an element $X = \sum\limits_{i=1}^{l} b_i X_i \in \mathfrak{c}(\mathfrak{g})$ satisfying the following conditions

$$b_1, \ldots, b_m, 1 \text{ are linearly independent over } \mathbb{Q} \text{ and}$$
$$b_{m+1}, \ldots, b_l \in \mathbb{Q}.$$

Consider the uniform lattice $\Delta \times \mathbb{Z}$ of the $(q+1)$-dimensional Lie group $G \times \mathbb{R}$ and the quotient manifold $M = (\Delta \times \mathbb{Z})\backslash(G \times \mathbb{R})$. We define the submersion $D \colon G \times \mathbb{R} \to G$ by

$$D(g, t) = g \cdot \exp(tX)$$

and define the homomorphism $h \colon \Delta \times \mathbb{Z} \to G$ by

$$h = D|_{\Delta \times \mathbb{Z}}.$$

Then the pair (D, h) satisfies the equivariance condition

$$D((\delta, n) \cdot (g, t)) = h(\delta, n) \cdot D(g, t)$$

for any $(\delta, n) \in \Delta \times \mathbb{Z}$ and $(g, t) \in G \times \mathbb{R}$. Thus the pair (D, h) defines a Lie $\mathfrak{g}$-flow $\mathcal{F}$ on M. The holonomy group is

$$\Gamma = \{\, \delta \cdot \exp(nX) \mid \delta \in \Delta, n \in \mathbb{Z} \,\}.$$

By the definition of Δ, since X commutes with any $Y \in \mathfrak{g}$, we have

$$\Gamma = \{\, \prod_{i=1}^{l} \exp(n_i + b_i n)X_i \cdot \prod_{i=l+1}^{q} \exp n_i X_i \mid n_i, n \in \mathbb{Z} \,\}.$$

The inverse image of Γ by ϕ is

$$\phi^{-1}(\Gamma) = \{\, (n_1 + b_1 n, \ldots, n_l + b_l n, n_{l+1}, \ldots, n_q) \mid n_i, n \in \mathbb{Z} \,\}$$
$$= (\mathbb{Z}^l + \mathbb{Z} \begin{pmatrix} b_1 \\ \vdots \\ b_l \end{pmatrix}) \oplus \mathbb{Z}^{q-l}.$$

By the choice of b_i, we have

$$\dim \overline{\Gamma} = \dim \overline{\phi^{-1}(\Gamma)} = m.$$

Thus the dimension of the structure Lie algebra of $\mathcal{F}$ is equal to m. Therefore $(M, \mathcal{F})$ is a Lie $\mathfrak{g}$-flow on M with an m-dimensional structure Lie algebra. $\qquad \square$

For general nilpotent Lie algebras, this result does not hold. First, there are constructed nilpotent Lie algebras which have no rational structures (cf. [2]).

Example 4.5 ([2], Lemma) Let $c_{ij}^k, 1 \leq i, j \leq m, 1, \leq k \leq n$ be real numbers such that $c_{ij}^k = -c_{ji}^k$. Assume that c_{ij}^k are algebraically independent over $\mathbb{Q}$. Let $\mathfrak{g}$ be the Lie algebra defined by a basis

$$\{X_1, \ldots, X_m, Y_1, \ldots, Y_n\}$$

with the products

$$[X_i, X_j] = \sum_{k=1}^n c_{ij}^k Y_k$$

for $i, j = 1, \ldots, m$ and all other products being zero. Then $\mathfrak{g}$ is nilpotent a Lie algebra and $[\mathfrak{g}, \mathfrak{g}] = \langle Y_1, \ldots, Y_n \rangle_{\mathbb{R}}$. This Lie algebra $\mathfrak{g}$ has no rational structure if $(n/2)(m^2 - m) > m^2 + n^2$.

Next we prove the Lie algebra $\mathfrak{g}$ constructed above cannot be realized as a Lie $\mathfrak{g}$-flow if m and n satisfy $(n/2)(m^2 - m) > (m+1)^2 + (n+1)^2$.

Proposition 4.6 *Suppose that $(n/2)(m^2-m) > (m+1)^2+(n+1)^2$, then the Lie algebra $\mathfrak{g}$ constructed above cannot be realized as a Lie $\mathfrak{g}$-flow.*

Proof. Suppose that there exists a Lie $\mathfrak{g}$-flow $\mathcal{F}$ of a closed manifold M. Let $D\colon \widetilde{M} \to G$ be the developing map, $h\colon \pi_1(M) \to G$ be the holonomy homomorphism of $\mathcal{F}$ and Γ be the holonomy group of $\mathcal{F}$. If $h\colon \pi_1(M) \to G$ is not injective, then each leaf of $\mathcal{F}$ is closed in M. Thus the closure $\overline{\Gamma}$ of Γ coincides with Γ. Hence the holonomy group Γ is a discrete subgroup of G. Moreover, since M is compact, Γ is uniform in G. Hence Γ is a uniform lattice of G. This contradicts Theorem 3.4.

Therefore $h\colon \pi_1(M) \to G$ is injective. Since Γ is a finitely generated, by Theorem 3.5, there exists a simply connected nilpotent Lie group G' and a uniform lattice Δ of G' which is isomorphic to Γ. Let $f\colon \Delta \to$

Γ be an isomorphism. By Theorem 3.6, f extends to an continuous homomorphism $\widetilde{f}\colon G' \to G$.

Consider the homomorphism

$$\widetilde{f}_*\colon \mathfrak{g}' \to \mathfrak{g}$$

induced by $\widetilde{f}$, where $\mathfrak{g}'$ is the Lie algebra of G'. We show that $\widetilde{f}_*$ is surjective. Since the group Γ is uniform in G, the subgroup Γ is Zariski dense in G by Theorem 3.8. Therefore the $\mathbb{R}$ vector subspace V of $\mathfrak{g}$ which is spanned by $\log \Gamma$ coincides with $\mathfrak{g}$. Similarly the $\mathbb{R}$ vector subspace V' of $\mathfrak{g}'$ which is spanned by $\log \Delta$ coincides with $\mathfrak{g}'$. Since $\widetilde{f}|_\Delta = f$ is an isomorphism,

$$\widetilde{f}_*|_{\log \Delta}\colon \log \Delta \to \log \Gamma$$

is bijective.

Let $Z = \sum a_i Z_i$ be an arbitrary element of $\mathfrak{g}$, where $a_i \in \mathbb{R}$ and $Z_i \in \log \Gamma$. Then the element $X = \sum a_i \widetilde{f}_*^{-1}(Z_i) \in \mathfrak{g}'$ satisfies that $\widetilde{f}_*(X) = Z$. Therefore the map $\widetilde{f}_*$ is surjective.

Since G is a simply connected nilpotent Lie group, G is contractible. Since $D\colon \widetilde{M} \to G$ is a locally trivial fibration and G is contractible, $\widetilde{M}$ is contractible. Hence M is aspherical. Since $\pi_1(M)$ is isomorphic to Γ and Γ is isomorphic to Δ, the manifold M is homotopy equivalent to $\Delta \backslash \widetilde{G}$. Therefore $\dim \widetilde{G} = \dim M = \dim G + 1$. Thus we have

$$\dim \mathrm{Ker}(f_*) = \dim \widetilde{G} - \dim G = 1.$$

By Lemma 3.2, $\mathrm{Ker}(f_*)$ is in the center of $\mathfrak{g}'$. Therefore

$$0 \to \mathrm{Ker}(f_*) \to \mathfrak{g}' \xrightarrow{f_*} \mathfrak{g} \to 0$$

is a central extension. Let T is a non-trivial element of $\mathrm{Ker}(f_*) \subset \mathfrak{g}'$ and let $s\colon \mathfrak{g} \to \mathfrak{g}'$ be a linear section of f_*. Define

$$\psi\colon \mathfrak{g} \wedge \mathfrak{g} \to \mathbb{R} \simeq \langle T \rangle_\mathbb{R} \simeq \mathrm{Ker} f_*$$

by

$$\psi(X, Y)T = s([X, Y]) - [s(X), s(Y)].$$

We consider the new Lie algebra $\widetilde{\mathfrak{g}}$ by $\mathfrak{g} \oplus \langle T \rangle_R$ with the product

$$[(Z_1, a_1 T), (Z_2, a_2 T)] = ([Z_1, Z_2], \psi(Z_1, Z_2)T).$$

Then $\widetilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}'$ via $\Psi \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}'$ which is defined by

$$\Psi(Z, aT) = s(Z) + aT.$$

If $\psi = 0$, that is, $\widetilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{g} \oplus \langle T \rangle_{\mathbb{R}}$ as a Lie algebra, then we have

$$[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] = \langle Y_1, \dots, Y_n \rangle_{\mathbb{R}}.$$

If $\psi \neq 0$, then we have $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] = \langle Y_1, \dots, Y_n, T \rangle_{\mathbb{R}}$

First we assume that $[\widetilde{\mathfrak{g}}, \mathfrak{g}] = \langle Y_1, \dots, Y_n, T \rangle_{\mathbb{R}}$. Let $\widetilde{c}_{ij}^k$ be the structure constants of $\widetilde{\mathfrak{g}}$ with respect to the basis

$$\{X_1, \dots, X_m, Y_1, \dots, Y_n, T\}.$$

Since $[X_i, X_j] = \sum\limits_{k=1}^{m} c_{ij}^k Y_k + \psi(X_i, X_j) T$, we have $\widetilde{c}_{ij}^k = c_{ij}^k$ for $i, j = 1, \dots, m$ and $k = 1, \dots, n$. Since G' has a uniform lattice, by Theorem 3.4, the Lie algebra $\mathfrak{g}' \simeq \widetilde{\mathfrak{g}}$ has a rational structure.

Let $\{Z_1, \dots, Z_{n+m+1}\}$ be a basis of $\widetilde{\mathfrak{g}}$ with rational structure constants $\{d_{ij}^k\}$. We may assume that $\pi(Z_1), \dots, \pi(Z_m)$ are linearly independent, where

$$\pi \colon \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}} / [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$$

is the natural projection. Let $C = \langle Z_1, \dots, Z_m \rangle_{\mathbb{R}}$ be the vector subspace of $\widetilde{\mathfrak{g}}$ which is a complement of $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$. Since $\widetilde{\mathfrak{g}} = C \oplus [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$, for each $j = 1, \dots, n+1$, there exist $V_j \in C$ and $T_j \in [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ such that $Z_{m+j} = V_j + T_j$. Then

$$\mathcal{B} = \{Z_1, \dots, Z_m, T_1, \dots, T_{n+1}\}$$

is a new basis of $\widetilde{\mathfrak{g}}$ and $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] = \langle T_1, \dots, T_{n+1} \rangle_{\mathbb{R}}$. Since

$$[Z_i, Z_j] = \sum_{k=1}^{m} d_{ij}^k Z_k + \sum_{k=1}^{n+1} d_{ij}^{m+k}(V_k + T_k)$$

$$= \sum_{k=1}^{m} d_{ij}^k Z_k + \sum_{k=1}^{n+1} d_{ij}^{m+k} V_k + \sum_{k=1}^{n+1} d_{ij}^{m+k} T_k$$

and $[Z_i, Z_j] \in \langle T_1, \dots, T_{n+1} \rangle_{\mathbb{R}}$, we have

$$[Z_i, Z_j] = \sum_{k=1}^{n+1} d_{ij}^{m+k} T_k.$$

On the other hand, since $C' = \langle X_1, \dots, X_m \rangle_{\mathbb{R}}$ is a complement of $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$, there exists $S_i \in C'$ such that

$$S_i - Z_i = \sum_{k=1}^{m+1} \lambda_{ik} T_k \in [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$$

for $i = 1, \ldots, m$. Consider the new basis

$$\mathcal{B}' = \{S_1, \ldots, S_m, T_1, \ldots, T_{n+1}\}$$

of $\widetilde{\mathfrak{g}}$. Since $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] \subset \mathfrak{c}(\widetilde{\mathfrak{g}})$, we have

$$
\begin{aligned}
[S_i, S_j] &= [Z_i + \sum_{k=1}^{n+1} \lambda_{ik} T_k, Z_j + \sum_{k=1}^{n+1} \lambda_{jk} T_k] \\
&= [Z_i, Z_j] \\
&= \sum_{k=1}^{n+1} d_{ij}^{k+m} T_k
\end{aligned}
$$

Thus the structure constants of $\widetilde{\mathfrak{g}}$ for the basis $\mathcal{B}$ are the same as for the basis $\mathcal{B}'$.

Since $\{S_1, \ldots, S_m\}$ is a basis of C', there exists a non-singular matrix $A = (a_{ij}) \in \mathrm{GL}(m; \mathbb{R})$ such that $S_j = \sum_{i=1}^{m} a_{ij} X_i$. Similarly, there exists a non-singular matrix $B = (b_{ij}) \in \mathrm{GL}(n+1; \mathbb{R})$ such that $T_j = \sum_{i=1}^{n+1} b_{ij} Y_i$, where $Y_{n+1} = T$. Since $[S_i, S_j] = \sum_{k=1}^{n+1} d_{ij}^k T_k$, by comparing the coefficients of Y_k, we have

$$\sum_{p=1}^{m} \sum_{q=1}^{m} a_{pi} a_{qj} c_{pq}^k = \sum_{r=1}^{n+1} d_{ij}^{m+r} b_{kr}$$

for $i, j = 1, \ldots, m$ and $k = 1, \ldots n+1$, where $c_{ij}^{n+1} = \psi(X_i, X_j)$. Let $(\bar{a}_{ij})$ be the inverse of A and $(\bar{b}_{ij})$ be the inverse of B. Then, for each $i, j = 1, \ldots, m$ and $k = 1, \ldots, n+1$, we have

$$c_{ij}^k = \sum_{p=1}^{m} \sum_{q=1}^{m} \sum_{r=1}^{n+1} \bar{a}_{pi} \bar{a}_{qj} d_{pq}^{m+r} b_{kr}.$$

Therefore c_{ij}^k is in the field $\mathbb{Q}(a_{ij}, b_{ij})$. The transcendental degree of $\mathbb{Q}(a_{ij}, b_{ij})$ is at most $m^2 + (n+1)^2$. This contradicts the assumption that c_{ij}^k are algebraically independent over $\mathbb{Q}$ and

$$(n/2)(m^2 - m) > (m+1)^2 + (n+1)^2.$$

In the case where $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] = \langle Y_1, \ldots, Y_n \rangle_{\mathbb{R}}$, by the same argument, we can prove that there exist $(a_{ij}') \in \mathrm{GL}(m+1; \mathbb{R})$ and $(b_{ij}') \in \mathrm{GL}(n; \mathbb{R})$ such that $c_{ij}^k \in \mathbb{Q}(a_{ij}', b_{ij}')$. The transcendental degree of $\mathbb{Q}(a_{ij}', b_{ij}')$ is at most $(m+1)^2 + n^2$. This is a contradiction. $\qquad\square$

If $n = 4$ and $m \geq 8$, then the inequality

$$(n/2)(m^2 - m) > (m + 1)^2 + (n + 1)^2$$

holds. Therefore, for any $q \geq 12$. there exists a q-dimensional nilpotent Lie algebra $\mathfrak{g}$ such that $(\mathfrak{g}, m)$ is not realizable for any m.

On the other hand, there exists a nilpotent Lie algebra $\mathfrak{g}$ which has no rational structures such that $(\mathfrak{g}, m)$ is realizable for some m. We prove this in the next section.

5   Nilpotent Lie foliations

In this section we consider which pair $(\mathfrak{g}, \mathfrak{h})$ can be realized as a Lie $\mathfrak{g}$-foliation of a closed manifold with the structure Lie algebra $\mathfrak{h}$.

Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathcal{F}$ be a Lie $\mathfrak{g}$-foliation of a closed manifold M. Let Γ be the holonomy group of $\mathcal{F}$. By Theorem 2.8, the structure Lie algebra $\mathfrak{h}$ is isomorphic to the Lie algebra of the closure $\overline{\Gamma}$ of Γ. Since Γ is uniform in G, the closure $\overline{\Gamma}$ of Γ is also uniform in G.

For any closed uniform subgroup of a nilpotent Lie group has the following property.

Lemma 5.1 ([15], Corollary 3) *Let H be a closed uniform subgroup of a nilpotent Lie group G. Then the identity component H_e of H is a normal subgroup of G.*

Let Γ be the holonomy group of a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of a closed manifold M. Then Γ is finitely generated and uniform subgroup of the simply connected Lie group G. For a simply connected nilpotent Lie group, the converse is true.

Lemma 5.2 *Let G be a simply connected nilpotent Lie group and $\mathfrak{g}$ be its Lie algebra. Let Γ be a finitely generated uniform subgroup of G. Then there exists a closed manifold M and a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of M such that the holonomy group is Γ.*

Proof. Since G is a simply connected nilpotent Lie group, the subgroup Γ of G is nilpotent and torsion-free. Thus, by Theorem 3.5, there exists a simply connected nilpotent Lie group G' and a uniform lattice Δ of G' such that Δ is isomorphic to Γ. Let $f \colon \Delta \to \Gamma$ be an isomorphism. By Theorem 3.6, f extends to an continuous homomorphism $\widetilde{f} \colon G' \to G$.

Consider the homomorphism

$$\widetilde{f}_* \colon \mathfrak{g}' \to \mathfrak{g}$$

induced by $\widetilde{f}$, where $\mathfrak{g}'$ is the Lie algebra of G'. By the proof of Proposition 4.6, the map $\widetilde{f}_*$ is surjective. Hence, the map $\widetilde{f} \colon \widetilde{G} \to G$ is a submersion. The submersion

$$\widetilde{f} \colon G' \to G$$

and the homomorphism

$$\widetilde{f}|_\Delta = f \colon \Delta \to \Gamma$$

define a Lie $\mathfrak{g}$-foliation of a closed manifold $M = \Delta \backslash G'$ with the holonomy group Γ. $\qquad\square$

Theorem 5.3 *Let $\mathfrak{g}$ be a q-dimensional nilpotent Lie algebra and $\mathfrak{h}$ be an l-dimensional subalgebra of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{h})$ is realizable if and only if $\mathfrak{h}$ is an ideal and $\mathfrak{h} \backslash \mathfrak{g}$ has a rational structure.*

Proof. Suppose that there exists a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of a closed manifold M with the structure Lie algebra $\mathfrak{h}$. Let Γ be the holonomy group of $\mathcal{F}$. Then $\mathfrak{h}$ is the Lie algebra of $\overline{\Gamma}_e$, where $\overline{\Gamma}_e$ is the identity component of the closure of Γ. Since $\overline{\Gamma}$ is uniform and closed, by Lemma 5.1, $\overline{\Gamma}_e$ is a normal subgroup of G. Therefore $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. The homogeneous space $\overline{\Gamma} \backslash G$ is compact and diffeomorphic to $(\overline{\Gamma}_e \backslash \overline{\Gamma}) \backslash (\overline{\Gamma}_e \backslash G)$. Hence the discrete subgroup $\overline{\Gamma}_e \backslash \overline{\Gamma}$ of $\overline{\Gamma}_e \backslash G$ is a uniform lattice. Therefore, by Theorem 3.4, $\mathfrak{g}/\mathfrak{h}$ has a rational structure.

Suppose that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{h} \backslash \mathfrak{g}$ has a rational structure. Let $p \colon \mathfrak{g} \to \mathfrak{h} \backslash \mathfrak{g}$ be the natural projection and $\{p(Z_1), \ldots, p(Z_{q-l})\}$ be a strong Mal'cev basis of $\mathfrak{h} \backslash \mathfrak{g}$ with rational structure constants. By Theorem 4.4, we may assume that

$$\exp \mathbb{Z} p(Z_1) \cdots \exp \mathbb{Z} p(Z_{q-l})$$

is a uniform lattice of $H \backslash G$. Let $\{X_1, \ldots, X_q\}$ be a strong Mal'cev basis of $\mathfrak{g}$ through $\mathfrak{h}$. Consider the new basis

$$\mathcal{B} = \{X_1, \ldots, X_l, Z_1, \ldots, Z_{q-l}\}$$

of $\mathfrak{g}$. We show this basis $\mathcal{B}$ is also strong Mal'cev basis of $\mathfrak{g}$ through $\mathfrak{h}$. Since $\{X_1, \ldots, X_q\}$ is a strong Mal'cev basis of $\mathfrak{g}$, for each $i \leq l$, the $\mathbb{R}$

vector subspace $\langle X_1, \ldots, X_i \rangle_{\mathbb{R}}$ of $\mathfrak{g}$ which is spanned by $\{X_1, \ldots, X_i\}$ is an ideal of $\mathfrak{g}$. Since $p(Z_1), \ldots, p(Z_{q-l})$ is a strong Mal'cev basis of $\mathfrak{h} \backslash \mathfrak{g}$, we have

$$[p(Z_i), p(Z_j)] \in \langle p(Z_1), \ldots, p(Z_i) \rangle_{\mathbb{R}}$$

for any $j > i$. Hence we have

$$[Z_i, Z_j] \in \mathfrak{h} \oplus \langle Z_1, \ldots, Z_i \rangle_{\mathbb{R}}$$
$$= \langle X_1, \ldots, X_l, Z_1, \ldots, Z_i \rangle_{\mathbb{R}}.$$

Therefore $\mathcal{B}$ is a strong Mal'cev basis of $\mathfrak{g}$ through $\mathfrak{h}$.

Fix an irrational number a and let Γ be the subgroup of G generated by

$$\{\exp X_1, \exp a X_1, \ldots, \exp X_l, \exp a X_l, \exp Z_1, \ldots, \exp Z_{q-l}\}.$$

Then Γ is finitely generated and uniform. We show that the identity component $\overline{\Gamma}_e$ of the closure of Γ coincides with $H = \exp \mathfrak{h}$. Let

$$\phi \colon \mathbb{R}^q = \mathbb{R}^l \times \mathbb{R}^{q-l} \to \mathfrak{g}$$

be the strong Mal'cev coordinate with respect to the basis $\mathcal{B}$, which is defined by

$$\phi(s_1, \ldots, s_l, t_1 \ldots, t_{q-l})$$
$$= \exp s_1 X_1 \cdots \exp s_l X_l \cdot \exp t_1 Z_1 \cdots \exp t_{q-l} Z_{q-l}.$$

Consider the subset

$$V = \{ \sum_{i=1}^{l} (n_i + m_i a) X_i + \sum_{i=1}^{q-l} k_i Z_i \mid n_i, m_i, k_i \in \mathbb{Z} \}$$
$$= (\mathbb{Z} + a\mathbb{Z}) X_1 \oplus \cdots \oplus (\mathbb{Z} + a\mathbb{Z}) X_l \oplus \mathbb{Z} Z_1 \oplus \cdots \oplus \mathbb{Z} Z_{q-l}$$

Then we have $\phi(V) \subset \Gamma$. Since the closure of V is

$$\overline{V} = \bigoplus_{i=1}^{l} \mathbb{R} X_i \oplus \bigoplus_{i=1}^{q-l} \mathbb{Z} Z_i$$
$$= \mathfrak{h} \oplus \bigoplus_{i=1}^{q-l} \mathbb{Z} Z_i$$

and ϕ is a diffeomorphism, we have

$$\phi(\mathfrak{h} \oplus \{0\}) = \phi(\overline{V}_0) = \overline{\phi(V)}_e \subset \overline{\Gamma}_e.$$

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On the other hand, since $\mathcal{B}$ is a strong Mal'cev basis through $\mathfrak{h}$, we have

$$H = \phi(\mathfrak{h} \oplus \{0\}).$$

Hence H is contained in $\overline{\Gamma}_e$.

We show that $\overline{\Gamma}_e$ is contained in H. Let $\widetilde{p} \colon G \to H \backslash G$ be the natural projection and Γ' be the subset of G generated by

$$\{H, \exp Z_1, \ldots, \exp Z_{q-l}\}.$$

Then Γ is contained in Γ'. Since H is a normal subgroup of G, we have

$$\begin{aligned}\widetilde{p}(\Gamma') &= \langle \widetilde{p}(\exp Z_1), \ldots, \widetilde{p}(\exp Z_{q-l}) \rangle \\ &= \langle \exp p(Z_1), \ldots, \exp p(Z_{q-l}) \rangle.\end{aligned}$$

By definition of $\{p(Z_1), \ldots, p(Z_{q-l})\}$, the subset

$$\exp \mathbb{Z} p(Z_1) \cdots \exp \mathbb{Z} p(Z_{q-l}).$$

is a uniform lattice of $H \backslash G$ and thus we have

$$\langle \exp p(Z_1), \ldots, \exp p(Z_{q-l}) \rangle = \exp \mathbb{Z} p(Z_1) \cdots \exp \mathbb{Z} p(Z_{q-l}).$$

Hence $\widetilde{p}(\Gamma')$ is discrete in $H \backslash G$. Therefore $\widetilde{p}^{-1}(\widetilde{p}(\Gamma'))$ is a closed subgroup of G. On the other hand, since H is a normal subgroup of G, we have

$$\widetilde{p}^{-1}(\widetilde{p}(\Gamma')) = H \cdot \Gamma' = \Gamma'.$$

Thus Γ' is a closed subgroup of G. Therefore we have

$$\overline{\Gamma}_e \subset \overline{\Gamma'}_e = \Gamma'_e = H.$$

Since Γ is a finitely generated and uniform subgroup of G, by Lemma 5.2, there exists a Lie $\mathfrak{g}$-foliation of a closed manifold M with the holonomy group Γ. The structure Lie algebra coincides with the Lie algebra of $\overline{\Gamma}_e = H$. Therefore the structure Lie algebra is $\mathfrak{h}$. $\qquad\square$

Let $\mathcal{F}$ be a Lie foliation, then $\mathcal{F}$ is minimal if and only if the holonomy group is dense in G. Thus we have the following corollary.

Corollary 5.4 *For any nilpotent Lie algebra $\mathfrak{g}$, there exists a minimal Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of a closed manifold M.*

Proof. Since $\mathfrak{g}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{g} = \{0\}$ has the rational structure, by Theorem 5.3, $(\mathfrak{g}, \mathfrak{g})$ is realizable. Therefore there exists a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of a closed manifold M with the structure Lie algebra $\mathfrak{g}$. Let Γ be the holonomy group of $\mathcal{F}$. Then the Lie algebra of $\overline{\Gamma}_e$ coincides with $\mathfrak{g}$. Therefore we have $\overline{\Gamma}_e = G$. This means that the Lie $\mathfrak{g}$-foliation $\mathcal{F}$ is minimal. $\qquad\square$

Corollary 5.5 *There exists a nilpotent Lie algebra $\mathfrak{g}$ which has no rational structures such that $\mathfrak{g}$ can be realized as a Lie $\mathfrak{g}$-flow, that is, $(\mathfrak{g}, m)$ is realizable for some m.*

Proof. Fix an q'-dimensional nilpotent Lie algebra $\mathfrak{g}'$ which has no rational structures. Let G' be the simply connected nilpotent Lie group with the Lie algebra $\mathfrak{g}'$. By the proof of Lemma 5.2, there exists a simply connected nilpotent Lie group G'' with dimension q'' which admits a uniform lattice and there exists a submersion homomorphism $F\colon G'' \to G'$. If $q'' = q'$, then $F\colon G'' \to G'$ is isomorphism. This contradicts the assumption that $\mathfrak{g}'$ has no rational structures. Hence $q'' > q'$.

Consider the induced homomorphism

$$F_*\colon \mathfrak{g}'' \to \mathfrak{g}'.$$

Since F is submersion, F_* is surjective. Let $l > 0$ be the dimension of $\mathrm{Ker}(F_*)$. Since G'' has a uniform lattice, by Theorem 3.4, the Lie algebra $\mathfrak{g}''$ of G'' admits a rational structure. Let $\{X_1, \ldots, X_{q''}\}$ be a strong Mal'cev basis of $\mathfrak{g}''$ through $\mathrm{Ker}(F_*)$ and define ideals $\mathfrak{h}_k$ ($0 \leq k \leq l$) of $\mathfrak{g}''$ by

$$\mathfrak{h}_0 = \{0\} \text{ and}$$
$$\mathfrak{h}_k = \mathbb{R}\text{-span}\{X_1, \ldots, X_k\}.$$

Let

$$p_k\colon \mathfrak{g}''/\mathfrak{h}_k \to \mathfrak{g}''/\mathfrak{h}_{k+1}$$

be the natural projection. Then we have the sequence of nilpotent Lie algebras

$$\mathfrak{g}'' = \mathfrak{g}''/\mathfrak{h}_0 \xrightarrow{p_0} \mathfrak{g}''/\mathfrak{h}_1 \xrightarrow{p_1} \ldots \xrightarrow{p_{l-1}} \mathfrak{g}''/\mathfrak{h}_l \xrightarrow{F_*} \mathfrak{g}'.$$

Since $\mathfrak{h}_l = \mathrm{Ker}(F_*)$, the nilpotent Lie algebra $\mathfrak{g}''/\mathfrak{h}_l$ is isomorphic to $\mathfrak{g}'$ via F_*. Since $\mathfrak{g}''$ has a rational structure and $\mathfrak{g}'$ has no rational structures,

there exists $k < l$ such that $\mathfrak{g}''/\mathfrak{h}_k$ has a rational structure and $\mathfrak{g}''/\mathfrak{h}_{k+1}$ has no rational structures.

Let G_k'' and G_{k+1}'' be simply connected nilpotent Lie groups with Lie algebras $\mathfrak{g}_k''$ and $\mathfrak{g}_{k+1}''$, respectively. Since

$$\dim \mathfrak{g}''/\mathfrak{h}_k - \dim \mathfrak{g}''/\mathfrak{h}_{k+1} = 1,$$

we have

$$\dim G_k'' - \dim G_{k+1}'' = 1.$$

Since $\mathfrak{g}''/\mathfrak{h}_k$ has a rational structure, by Theorem 3.4, there exists a lattice Δ in G_k''. Since $p_k \colon \mathfrak{g}''/\mathfrak{h}_k \to \mathfrak{g}''/\mathfrak{h}_{k+1}$ is surjective, the map

$$D = \exp \circ p_k \circ \log \colon G_k'' \to G_{k+1}''$$

is a submersion homomorphism. Then the submersion D and the homomorphism

$$h = D|_\Delta \colon \Delta \to G_{k+1}''$$

define a Lie $\mathfrak{g}''/\mathfrak{h}_{k+1}$-flow on $M = \Delta\backslash G_k''$.

$\square$

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