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博士論文

Uniform Representability of the Brauer Group of Diagonal Cubic Surfaces

対角的3次曲面のBrauer群の統一的な表示可能性について

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Abstract

The Brauer group of a scheme is an effective tool for studying its arithmetic and geometric properties. For such purposes, we need to know not only the structure of the Brauer group as an abelian group, but also explicit generators such as those represented by norm residue symbols.

Yu. I. Manin first studied such problems for diagonal cubic surfaces. He determined the structure of the Brauer group of some diagonal cubic surfaces and found its symbolic generators.

In this dissertation, we generalize his result. We introduce the notion of uniformity for generators and prove the following two results: first, diagonal cubic surfaces of a particular form have such uniform generators represented by a norm residue symbol; secondly, in general, diagonal cubic surfaces have no uniform generator. The latter result states that there is a limit to extend the Manin's result stated above.

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Chapter 1

Introduction

A central object of this dissertation is the Brauer group of fields and schemes. It is named after Richard Brauer [Bra29], a twentieth-century German and American mathematician.

We first review the history of the Brauer group. Let k be a (commutative) field. Algebras over k are began to study in the nineteenth century. On October 16, 1843, W. R. Hamilton [Ham44] discovered the first and the most famous example of skew fields, namely the field of quaternions, which is now denoted by $\mathbb{H}$. Historically, however, O. Rodrigues [Rod40] had already reached the notion of quaternion in his study about the transformation of coordinates of spaces in 1840. For details, see [Alt89]. The skew field $\mathbb{H}$ is an $\mathbb{R}$-algebra defined to be a 4-dimensional $\mathbb{R}$-vector space $\mathbb{R} \oplus \mathbb{R}\, i \oplus \mathbb{R}\, j \oplus \mathbb{R}\, k$ with the following multiplication rules:

$$i^2 = j^2 = k^2 = ijk = -1.$$

This algebra is an example of a central simple algebra over $\mathbb{R}$, that is, an $\mathbb{R}$-algebra

- which is finite-dimensional as an $\mathbb{R}$-vector space;
- which has no nontrivial two-sided ideal;
- whose center is equal to $\mathbb{R}$.

Here a natural problem occurs:

> How many central simple algebras are over a fixed field k (up to k-isomorphisms)?

Related to this problem, J. H. M. Wedderburn [Wed08] proved the following striking theorem:

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> Let k be a field. For any central simple algebra A over k, there exists a unique positive integer n and a unique central skew field D up to k-isomorphisms such that A is isomorphic to the matrix algebra $M_n(D)$ over D with rank n as k-algebras.

In a succession of works in the 1920's due to many mathematicians including, for example, E. Artin, R. Brauer and E. Noether, they reached the notion of similarity of central simple algebras over k and introduced a group structure into the set of their isomorphism classes. This is what is now called the Brauer group of k and denoted by $\mathrm{Br}(k)$. The Brauer group of fields appears in various contexts. For example, the following theorem, called the Albert-Brauer-Hasse-Noether theorem plays an important role in class field theory:

> Let k be a global field and Ω is the set of all places of k. We have the following exact sequence:
>
> $$0 \to \mathrm{Br}(k) \to \bigoplus_{v \in \Omega} \mathrm{Br}(k_v) \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0,$$
>
> where k_v is the v-adic completion of k and inv_v is the invariant map of k_v.

A generalization of the Brauer group of fields was proposed by G. Azumaya [Azu51]. He generalized central simple algebras over fields to those over local rings and defined the Brauer group $\mathrm{Br}(R)$ of a local ring R. In honor of his work, such algebras are now called Azumaya algebras.

The notion of the Brauer group of schemes were also considered. In a series of papers [Gro68a], [Gro68b], [Gro68c], A. Grothendieck introduced two types of Brauer groups of a scheme X, that is, the Brauer group $\mathrm{Br}_{\mathrm{Az}}(X)$ defined by using Azumaya algebras on X and the cohomological Brauer group $\mathrm{Br}(X)$ of X. These groups relate each other. For example, we always have an injection from $\mathrm{Br}_{\mathrm{Az}}(X)$ to $\mathrm{Br}(X)$.

In the following of this chapter, Brauer groups always mean cohomological ones. For Brauer groups defined by using Azumaya algebras, see Section 3.2. See also [Gro68a], [Gro68b], [Gro68c], [Mil80] (Chapter IV).

Let X be a variety over a field k. The (cohomological) Brauer group $\mathrm{Br}(X)$ is defined to be the second étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ of X with coefficient $\mathbb{G}_m$. It is an important problem to understand $\mathrm{Br}(X)$ since this group plays an important role in studying the arithmetic and the geometry of X. For example, this group appears as the so-called Brauer-Manin obstruction [Man71], a tool for studying the Hasse principle for rational points and zero-cycles on varieties over a number field. In [CTKS87], using this obstruction, J.-L. Colliot-Thélène, D. Kanevsky, and J.-J. Sansuc studied the Hasse principle for rational points on diagonal cubic surfaces. For relations between Brauer groups and zero-cycles on varieties over p-adic field, see, for example, [Lic69], [CT95b], [SS]. Besides, $\mathrm{Br}(X)$ also appears in studies of the rationality problem of X. For example, M. Artin and D. Mumford [AM72] constructed some examples of non-rational unirational varieties by showing the nontriviality of unramified cohomology groups, which are kinds of generalizations of Brauer groups. For other results and applications, see, for example, [CTS87], [CTO89], [Sko01].

Assume X is a proper, smooth, geometrically integral and geometrically rational variety over k. In order to apply Brauer groups to such studies as stated above, we have to answer the following two natural questions on $\mathrm{Br}(X)$:

(1) How do we determine the structure of $\mathrm{Br}(X)$?

(2) How do we represent elements in $\mathrm{Br}(X)$?

First we look at the question (1). Let $\pi\colon X \to \operatorname{Spec} k$ be the structure morphism. By the geometrical rationality of X, the group $\mathrm{Br}(X)/\mathrm{Br}(k) := \mathrm{Br}(X)/\pi^*\mathrm{Br}(k)$ injects into the Galois cohomology group $H^1(k, \mathrm{Pic}(\overline{X}))$, where $\overline{X}$ is the base change of X to a separable closure of k. Moreover, if we assume that $X(k) \neq \emptyset$ or that the cohomological dimension $\mathrm{cd}(k)$ of k is less than or equal to 2, these groups are isomorphic. Thus, if we know the Galois action on $\mathrm{Pic}(\overline{X})$, it is possible to compute $H^1(k, \mathrm{Pic}(\overline{X}))$ and hence we have an answer to the question (1). On the other hand, if $H^1(k, \mathrm{Pic}(\overline{X})) \neq 0$, $X(k) = \emptyset$ and $\mathrm{cd}(k)$ is greater than 2, it is a difficult problem to determine the structure of $\mathrm{Br}(X)/\mathrm{Br}(k)$.

Next, we look at the question (2). One way to represent elements of $\mathrm{Br}(X)$ is to use norm residue symbols. The Brauer group of X can be considered as a subgroup of $\mathrm{Br}(k(X))$, where $k(X)$ is the function field of X. On the other hand, for any field K and a positive integer n prime to $\mathrm{ch}(K)$, we have the norm residue map

$$\{\cdot,\cdot\}_n\colon K_2^M(K) \to H^2(K, \mu_n^{\otimes 2}),$$

where $K_2^M(K)$ is the second Milnor K-group of K. Hence, if k contains a primitive n-th root of unity, we have the composite map

$$K_2^M(k(X)) \xrightarrow{\{\cdot,\cdot\}_n} H^2(k(X), \mu_n^{\otimes 2}) \cong H^2(k(X), \mu_n) \hookrightarrow \mathrm{Br}(k(X)),$$

and can represent n-torsion elements in $\mathrm{Br}(X)$ in terms of norm residue symbols. In a particular case, for example, when $\mathrm{ch}(k) = 0$ and k contains all roots of unity, by a theorem of A. S. Merkurjev and A. A. Suslin ([MS82], Theorem 11.5), such representation exists for all elements of $\mathrm{Br}(X)$. Such representation is useful to calculate some objects relating to Brauer groups such as the Brauer-Manin obstruction, so it is an important question to find such one.

However, there are some problems. First, it is difficult to decide which element in $\mathrm{Br}(k(X))$ belongs to $\mathrm{Br}(X)$. Moreover, even if we know that an element $e \in \mathrm{Br}(X)$ can be represented by norm residue symbols, it does not necessarily mean that we can represent e as a symbol $\{f, g\}_n$, not a sum of symbols.

In [Man86], under the assumption k contains a primitive cubic root ζ of unity, Yu. I. Manin gave a complete answer to the above questions for smooth projective cubic surfaces V over k defined by a homogeneous equation $x^3 + y^3 + z^3 + dt^3 = 0$ with $d \in k^* \setminus (k^*)^3$. See also [CTS87]. In this case, we have the following:

Theorem 1.0.1 (Manin). (1) $\mathrm{Br}(V)/\mathrm{Br}(k) \cong (\mathbb{Z}/3\mathbb{Z})^2$;

(2) *the elements*

$$e_1 = \left\{ d, \frac{x+\zeta y}{x+y} \right\}_3, \quad e_2 = \left\{ d, \frac{x+z}{x+y} \right\}_3 \in \mathrm{Br}(k(V))$$

are contained in $\mathrm{Br}(V)$;

(3) *the images of* e_1 *and* e_2 *in* $\mathrm{Br}(V)/\mathrm{Br}(k)$ *are generators of this group.*

Using these symbolic representations, S. Saito and K. Sato [SS] recently computed the degree-zero part of the Chow group of zero-cycles on such cubic surfaces over p-adic fields explicitly, even in the case $p = 3$.

In this dissertation, we study the problem in a more general setting where the equation of V is of the forms $x^3 + y^3 + cz^3 + dt^3 = 0$ and $x^3 + by^3 + cz^3 + dt^3 = 0$.

In the sequel of this chapter, k is always assumed to be of characteristic 0 and contain a fixed primitive cubic root ζ of unity. First, we prove the following theorem, which gives an answer to the questions (1) and (2) for the case $x^3 + y^3 + cz^3 + dt^3 = 0$.

Theorem 1.0.2 (Theorem 5.1.1). *Let k be as above and V be the cubic surface over k defined by an equation $x^3 + y^3 + cz^3 + dt^3 = 0$, where c and $d \in k^*$. Assume that c, d, cd and d/c are not contained in $(k^*)^3$. Then*

(1) *the group* $\mathrm{Br}(V)/\mathrm{Br}(k)$ *is isomorphic to* $\mathbb{Z}/3\mathbb{Z}$;

(2) *the element*

$$e_1 = \left\{ \frac{d}{c}, \frac{x+\zeta y}{x+y} \right\}_3 \in \mathrm{Br}(k(V))$$

is contained in $\mathrm{Br}(V)$;

(3) *the image of* e_1 *in* $\mathrm{Br}(V)/\mathrm{Br}(k)$ *is a generator of this group.*

The claim (1) is essentially due to [CTKS87]. Recently Colliot-Thélène and O. Wittenberg found a symbolic generator of $\mathrm{Br}(X)/\mathrm{Br}(k)$ for an affine surface $X : x^3 + y^3 + 2z^3 = at^3$ when k does not contain a primitive cubic root of unity ([CTW12], Proposition 2.1). In this case, our symbolic generator was also appeared in the proof of that proposition.

We note that in the result of Manin and Theorem 1.0.2, we can take generators *uniformly*. More precisely, for example in Theorem 1.0.2, let c and d be indeterminates, $F = k(c,d)$, V the cubic surface $x^3 + y^3 + cz^3 + dt^3 = 0$ over F, and

$$e(c,d) = \left\{ \frac{d}{c}, \frac{x+\zeta y}{x+y} \right\}_3$$

an element in $\mathrm{Br}(V)$. Let $P = (c_0, d_0)$ be a point in $k^* \times k^*$ with c_0, d_0, $c_0 d_0$ and d_0/c_0 not contained in $(k^*)^3$, and V_P the surface defined by $x^3 + y^3 + c_0 z^3 + d_0 t^3 = 0$. If we want a symbolic generator of $\mathrm{Br}(V_P)/\mathrm{Br}(k)$, we can get it by *specializing* $e(c,d)$ at P.

We denote this element by $\mathrm{sp}(e(c,d);P)$. A precise definition of the specialization will be given in Subsection 3.2.5. In general, it is not necessary that the Brauer group of a given variety has such uniform generators. However, it is desirable that we take uniform generators if they exist, since we can make a calculation *simultaneously* for varieties defined by the same form of equations by using them.

Concerning the problem whether symbolic generators can be chosen uniformly or not, we prove the following non-existence result. Let $F = k(b,c,d)$, where b, c, d are indeterminates over k, and let V be the projective cubic surface over F defined by the equation $x^3 + by^3 + cz^3 + dt^3 = 0$. For $P = (b_0, c_0, d_0) \in k^* \times k^* \times k^*$, let V_P be the projective cubic surface over k defined by the equation $x^3 + b_0y^3 + c_0z^3 + d_0t^3 = 0$. For $e \in \mathrm{Br}(V)$, we will define its specialization at P, $\mathrm{sp}(e;P) \in \mathrm{Br}(V_P)$. Put

$$\mathcal{P}_k = \{P \in (\mathbb{G}_{m,k})^3(k) \mid \mathrm{Br}(V_P)/\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}.\}.$$

Note ([CTKS87]) that $\mathrm{Br}(V_P)/\mathrm{Br}(k)$ is isomorphic to either of $0, \mathbb{Z}/3\mathbb{Z}$ and $(\mathbb{Z}/3\mathbb{Z})^2$ and that Manin dealt with the last case, as we stated before.

We can state our main result:

Theorem 1.0.3 (Corollary 6.2.3). *Let k be a field of characteristic 0 and containing a primitive cubic root of unity, $F = k(b,c,d)$ and V be the projective cubic surface over F defined by the equation*

$$x^3 + by^3 + cz^3 + dt^3 = 0.$$

Assume $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$. Then there is no element $e \in \mathrm{Br}(V)$ satisfying the following condition:

> *there exists a dense open subset $W \subset (\mathbb{G}_{m,k})^3$ such that $\mathrm{sp}(e;\cdot)$ is defined on $W(k) \cap \mathcal{P}_k$ and for all $P \in W(k) \cap \mathcal{P}_k$, $\mathrm{sp}(e;P)$ is a generator of $\mathrm{Br}(V_P)/\mathrm{Br}(k)$.*

We note that the assumption $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$ implies the Zariski density of $\mathcal{P}_k \subset (\mathbb{G}_{m,k})^3$ (see Proposition 6.2.1), which is essentially necessary to prove Theorem 1.0.3. We easily see that this assumption holds for various fields, for example, all finitely generated fields over $\mathbb{Q}(\zeta)$ and $\mathbb{Q}_p(\zeta)$ for any prime number p, and hence this is a mild assumption.

Theorem 1.0.3 is an consequence of the following

Theorem 1.0.4 (Theorem 6.1.3). *Let k, F and V as above. Then*

$$\mathrm{Br}(V)/\mathrm{Br}(F) = 0.$$

Note that this vanishingness does not follow directly from the argument stated before since $H^1(F, \mathrm{Pic}(\overline{V})) \neq 0$, $V(F) = \emptyset$ and $\mathrm{cd}(F) \geq 3$. As far as we know, this would be the first example of computation of Brauer groups for such varieties.

This dissertation is written in the following fashion. It consists of six chapters including this chapter.

In Chapter 2, we fix the notation and recall some classical results on cohomology of groups. In particular, we review residue maps, which is intensively used in the proof of Theorem 1.0.3.

In Chapter 3, we review the definition and basic properties of Brauer groups. First, we deal with the Brauer group of fields — its definition and its expression by using symbols. Secondly, we consider the Brauer group of schemes. We give three descriptions of them. Here we also define specialization of Brauer groups.

In Chapter 4, we focus on the diagonal cubic surfaces, especially their Picard groups and Galois action on them. Some results of this chapter is found in [CTKS87] or in [Man86], but for convenience of the reader, we include them here. At the end of this chapter, we also recall an explicit description of a differential appearing in a spectral sequence, which is used in the proof of Theorem 1.0.3.

In Chapter 5, we obtain a uniform symbolic generator of the Brauer group of surfaces of the form $x^3 + y^3 + cz^3 + dt^3 = 0$.

In Chapter 6, we prove that there is no uniform generator for the case $x^3 + by^3 + cz^3 + dt^3 = 0$ by showing the non-vanishingness of the image of a particular cocycle in a certain cohomology group. In the last section of this chapter, we also discuss the condition $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$ appearing in Theorem 1.0.3.

Chapter 2

Preliminaries

2.1 Notation

For a group A and $f \in \mathrm{End}(A)$, we denote by ${}_f A$ the kernel of f. For a prime number p, we denote its p-primary subgroup by $A\{p\}$.

The term "ring" always means an associative commutative ring with unit. For a ring R, we denote its unit group by R^*. For a non-commutative ring R, we denote by R^{op} its opposite ring. This is the same as R as sets and its addition $+_{R^{\mathrm{op}}}$ is also the same as the addition in R, but its multiplication $\cdot_{R^{\mathrm{op}}}$ is defined to be:

$$a \cdot_{R^{\mathrm{op}}} b := b \cdot_R a.$$

The term "field" always mean a commutative field. If we want to say about not only commutative but non-commutative fields, we use the term "skew fields". For a field k, we we denote a separable closure of k by $\bar{k}$. We fix such a field $\bar{k}$ and each algebraic separable extension of k is always considered as a subfield of this $\bar{k}$. If k is a discrete valuation field, the field k^{ur} denotes the maximal unramified extension of k. We denote the characteristic and the cohomological dimension of k by $\mathrm{ch}(k)$ and $\mathrm{cd}(k)$. We define the condition $(*)$ as follows:

$$k \text{ contains a primitive cubic root of unity and } \mathrm{ch}(k) = 0. \qquad (*)$$

When we assume that $(*)$ holds for k, we always fix one of primitive cubic roots of unity and denote it by ζ.

For a profinite group G and a (left) G-module A, $H^q(G, A)$ denotes the q-th cohomology group of G with coefficients in A. When L is a Galois extension of k and G is the Galois group $\mathrm{Gal}(L/k)$, we denote its q-th cohomology group by $H^q(L/k, A)$. In particular, when $G = G_k := \mathrm{Gal}(\bar{k}/k)$, we denote this group by simply $H^q(k, A)$. A^G denotes the G-invariant part of A and we denote it by $A^{L/k}$ when $G = \mathrm{Gal}(L/k)$. $C^q(G, A)$ (resp. $B^q(G, A)$ and $Z^q(G, A)$) denotes the group of inhomogeneous q-cochains (resp. q-coboundaries and q-cocycles). If ϕ is an element in $Z^q(G, A)$, $[\phi]$ denotes the class of ϕ in $H^q(G, A)$. Similarly, $\hat{H}^q(G, A)$ denotes the q-th Tate cohomology group of G with coefficients in A. Other notations are defined palallelly as in the case of $H^q(G, A)$.

All schemes are assumed to be separated. For a scheme X, we denote $X^{(i)}$ be the set of all points in X with codimension i. The field $\kappa(x)$ denotes the residue field of $x \in X$. The groups $\mathrm{Div}(X)$ and $\mathrm{Pic}(X)$ denotes the group of Cartier divisors on X and the group of isomorphism classes of invertible sheaves on X. For a regular scheme X, we identify these groups with the group of Weil divisors and the group of linearly equivalent classes of Weil divisors respectively. For a Zariski sheaf $\mathcal{F}$ on X, we denote its stalk at $x \in X$ by $\mathcal{F}_x$. For an étale sheaf $\mathcal{F}$, we denote its (étale) stalk at $x \in X$ by $\mathcal{F}_{\overline{x}}$. Fix a positive integer n, we denote by $\mu_{n,X}$ the étale sheaf of n-th root of unity on X. Put

$$\mu_{n,X}^{\otimes j} := \begin{cases} \mu_{n,X} \otimes \cdots \otimes \mu_{n,X} & j > 0 \\ \mathbb{Z}/n\mathbb{Z} & j = 0 \\ \mathrm{Hom}(\mu_{n,X}^{\otimes -j}, \mathbb{Z}/n\mathbb{Z}) & j < 0. \end{cases}$$

The sheaf $\mathbb{G}_{m,X}$ denotes the étale sheaf of multiplicative units on X. We often omit X for simplicity. $H^q(X, \cdot)$ always denotes the q-th étale cohomology group of X, that is, the q-th right derived functor of the global section functor $\Gamma(X, \cdot)$. If R is a ring and $X = \operatorname{Spec} R$ is an affine scheme, we denote simply by $H^q(R, \cdot)$ the group $H^q(\operatorname{Spec} R, \cdot)$. In particular, if k is a field and $X = \operatorname{Spec} k = \{x\}$, for an abelian sheaf $\mathcal{F}$ on X, we have the following natural isomorphism:

$$H^q(k, \mathcal{F}) \cong H^q(k, \mathcal{F}_{\overline{x}}),$$

where the left-hand side is an étale cohomology group and the right hand side is a Galois cohomology group. In this dissertation, we always identify these groups by this isomorphism.

For a field k, a variety over k means a separated scheme of finite type over k. For a variety X over k and a given field extension L/k, we denote $X \times_k L$ by X_L. In the case where $L = \overline{k}$, we denote $X_{\overline{k}}$ by $\overline{X}$. If X is integral, we denote by $k(X)$ its function field or the residue field of its generic point.

Finally, All spectral sequences are assumed to be the first quadrant spectral sequences.

2.2 Cohomology of groups

2.2.1 Cohomology of finite cyclic groups

At first, we focus on finite cyclic groups. Let G be a finite cyclic group of order n and fix a generator σ of G. Let A be a G-module. Then we have the following theorem:

Theorem 2.2.1. *Let $[\chi]$ be an generator of the Tate cohomology group $\hat{H}^2(G, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. Then we have the following isomorphisms*

$$\cdot \cup [\chi] \colon \hat{H}^i(G, A) \to \hat{H}^{i+2}(G, A)$$

for all $i \in \mathbb{Z}$. Moreover these isomorphisms are compatible with connecting homomorphisms and change of modules.

For a proof, see [NSW00], Proposition 1.7.1. We recall an explicit description of these isomorphisms in the case $i = -1$ and 0. The following exact sequence of trivial G-modules

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

induces the long exact sequence

$$H^1(G,\mathbb{Q}) \to H^1(G,\mathbb{Q}/\mathbb{Z}) \to H^2(G,\mathbb{Z}) \to H^2(G,\mathbb{Q}).$$

Since the cohomology groups $H^q(G,\mathbb{Q}) = 0$ for $q > 0$ by the unique divisibility of $\mathbb{Q}$, the above sequence yields the isomorphism

$$H^1(G,\mathbb{Q}/\mathbb{Z}) \cong H^2(G,\mathbb{Z}).$$

We take the image of

$$\chi\colon G \to \mathbb{Q}/\mathbb{Z}; \quad \sigma^i \to \frac{i}{n}$$

under $H^1(G,\mathbb{Q}/\mathbb{Z}) \cong H^2(G,\mathbb{Z}) \cong \hat{H}^2(G,\mathbb{Z})$ as a generating cocycle $\hat{\chi}$ of $\hat{H}^2(G,\mathbb{Z})$. First, we look at the case $i = -1$.

Proposition 2.2.2. (1) *The map*

$$\hat{Z}^{-1}(G,A) \to A; \quad \phi \mapsto \phi(1)$$

induces a functorial isomorphism $\hat{H}^{-1}(G,A) \cong {}_{N_G}A/I_GA$. Here, N_G is the norm map of G and $I_G\colon A \to A$ maps x to $(\sigma - 1)x$.

(2) *For each $x \in {}_{N_G}A$, we define a 1-cocycle $\Phi_x \in Z^1(G,A)$ to be*

$$\Phi_x\colon \sigma^i \mapsto \sum_{j=0}^{n-1} a(i,-j)\cdot \sigma^j x,$$

where for all i and $j \in \mathbb{Z}$,

$$a(i,j) = \left\lfloor \frac{i+j}{n} \right\rfloor - \left\lfloor \frac{i}{n} \right\rfloor - \left\lfloor \frac{j}{n} \right\rfloor.$$

Then this correspondence induces the composite of the isomorphisms

$${}_{N_G}A/I_GA \overset{(1)}{\cong} \hat{H}^{-1}(G,A) \xrightarrow{\cdot\cup[\hat{\chi}]} H^1(G,A).$$

Proof. (1) This follows easily from the definition of $\hat{H}^{-1}(G,A)$. See [NSW00], Chapter 1, §2.

(2) This follows from a straightforward calculation. For explicit formulas of the cup product of inhomogeneous cochains, see [NSW00], Proposition 1.4.8.

□

We now consider the case $i = 0$.

Proposition 2.2.3. (1) *The map*

$$\hat{Z}^0(G,A) \to A; \quad \phi \mapsto \phi(1)$$

induces a functorial isomorphism $\hat{H}^0(G,A) \cong A^G/N_GA$.

(2) *For* $x \in A^G$, *we define a 2-cocycle* $\Phi_x \in Z^2(G,A)$ *to be*

$$\Phi_x \colon (\sigma^i, \sigma^j) \mapsto a(i,j)x.$$

Then this correspondence induces the composite of the isomorphisms

$$A^G/N_GA \overset{(1)}{\cong} \hat{H}^0(G,A) \xrightarrow{\cdot \cup [\hat{\chi}]} H^2(G,A).$$

Proof. For a proof, see the same reference cited in Proposition 2.2.2. □

Finally, we note the following:

Proposition 2.2.4. *For a exact sequence of G-modules*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

we have the following surjection

$$\hat{H}^{-1}(G,C) \to \mathrm{Ker}(\hat{H}^0(G,A) \to \hat{H}^0(G,B)); \quad [\phi] \mapsto \left[\sigma \mapsto \phi\left(\sum_{\tau \in G} \tau\right)\right].$$

Moreover, if we identify the left-hand side group as ${}_{N_G}C/I_GC$, *and the right-hand side group as* $(A \cap N_GB)/N_GA$, *the above homomorphism maps* $c \in {}_{N_G}C$ *to* $f^{-1}(N_G(b)) \in A^G$, *where* $b \in B$ *is an element satisfying the relation* $g(b) = c$.

Proof. This is a part of the "exact hexagon of Tate cohomology". See [NSW00], Proposition 1.7.2. □

2.2.2 A triviality lemma

In the sequel of this dissertation, we sometimes use the following lemma:

Lemma 2.2.5. *Let G be a finite group and* $E = \{e_i\}_{i \in I}$ *a G-set. Put*

$$A = \bigoplus_{i \in I} \mathbb{Z}\, e_i$$

and define the G-action on A by $g \cdot (\sum_i a_i e_i) := \sum_i a_i (g \cdot e_i)$. *Then we have*

$$H^1(G,A) = 0.$$

Proof. If we consider the action of G on E, the set E may split into some G-orbits: $E = \coprod_{\lambda\in\Lambda} E_\lambda$. If we define $A_\lambda = \bigoplus_{e\in E_\lambda} \mathbb{Z}e$, then each A_λ is also a G-module and we have the decomposition $A = \bigoplus_{\lambda\in\Lambda} A_\lambda$ as G-modules. Hence we have $H^1(G,A) = \bigoplus_{\lambda\in\Lambda} H^1(G,A_\lambda)$, so we may assume the action of G on E is transitive.

Put $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote the differentials $C^0(G,A) \to C^1(G,A)$ and $C^0(G,A_{\mathbb{Q}}) \to C^1(G,A_{\mathbb{Q}})$ by the same symbol d. For any 1-cocycle $\phi\colon G \to A$, there exists $\psi \in A_{\mathbb{Q}}$ such that $d\psi = \phi$. Indeed, if we take

$$\psi = -\frac{1}{\#G}\sum_{\sigma\in G}\phi(\sigma),$$

we can show that this map satisfies the relation $d\psi = \phi$ by using the cocycle condition on ϕ.

At this point, ψ is an element in $A_{\mathbb{Q}}$. Now we show that we can get a cochain $\widetilde{\psi} \in A$ satisfying $d\widetilde{\psi} = \phi$ by shifting ψ properly. Put $\psi = \sum_{i\in I} a_i e_i$, where $a_i \in \mathbb{Q}$. For all $\sigma \in G$ and for all $i \in I$, we define $\sigma(i) \in I$ as the unique element satisfying $\sigma e_{\sigma(i)} = e_i$. Since the element

$$\phi(\sigma) = (d\psi)(\sigma) = \sigma\psi - \psi = \sum_{i\in I}(a_{\sigma(i)} - a_i)e_i$$

is in A for all $\sigma \in G$, we have $a_{\sigma(i)} - a_i \in \mathbb{Z}$ for all $i \in I$. By the transitivity of the action of G on E, this means that the class of a_i in $\mathbb{Q}/\mathbb{Z}$ is independent of $i \in I$. Let $a \in \mathbb{Q}$ be a representative of this class. We put

$$\widetilde{\psi} = \sum_{i\in I}(a_i - a)e_i.$$

Then $\widetilde{\psi}$ is in A, and noting that $\sum_{i\in I} e_i \in A^G$, we have $\phi = d\widetilde{\psi} \in B^1(G,A)$. This completes the proof of Lemma 2.2.5. □

2.2.3 A residual exact sequence

We introduce a tool intensively used in Section 6.1. First, we recall the Hochschild-Serre spectral sequence:

Lemma 2.2.6. *Let G be a profinite group, H a closed normal subgroup of G and A be a G-module. Then we have the following spectral sequence:*

$$E_2^{p,q} = H^p(G/H, H^q(H,A)) \Rightarrow H^{p+q}(G,A).$$

Proof. See [NSW00], Theorem 2.4.1. □

As an application of this spectral sequence, we have the following:

Proposition 2.2.7. *Let k be a complete discrete valuation field, κ its residue field, p the characteristic of κ, and $I \subset G_K$ its inertia. Then for any torsion G_κ-module C with $C\{p\} = 0$, we have the following exact sequence*

$$0 \to H^i(\kappa, C) \to H^i(k,C) \xrightarrow{r} H^{i-1}(\kappa, \mathrm{Hom}(I,C)) \to 0.$$

Here the second map is induced by the canonical map $G_k \to G_\kappa$, and r is defined as follows:

For a normalized cocycle $\phi \in Z^i(k, C)$ satisfying

$$\begin{aligned} &\text{for all } i \geq 2, g_i \equiv g_i' \mod I \\ &\Rightarrow \phi(g_1, g_2, \ldots, g_n) = \phi(g_1, g_2', \ldots, g_n'), \end{aligned} \tag{2.1}$$

define $r\phi \in Z^{i-1}(\kappa, \mathrm{Hom}(I, C))$ as:

$$\text{for all } h \in I, \quad (r\phi)(\overline{g_1}, \ldots, \overline{g_{n-1}})(h) = \phi(h, g_1, \ldots, g_{n-1}),$$

where g_i are lifts of $\overline{g_i}$ to G_k.

Proof. We note that we can take a representative cocycle of any class in $H^i(k, C)$ which is normalized and satisfies the above condition. For details and a proof, see [SH53] and [GMS03], III, Theorem 6.1. □

2.3 Étale cohomology

We recall some classical results on étale cohomology, which frequently appears in the following argument.

2.3.1 Results on Limits

Lemma 2.3.1. *Let X be a quasi-compact scheme, I a pseudo filtered small category and $(\mathcal{F}_i)_{i \in I}$ a direct system of abelian sheaves on X indexed by I. Then*

$$\varinjlim_i H^q(X, \mathcal{F}_i) \cong H^q(X, \varinjlim_i \mathcal{F}_i)$$

for all $q \geq 0$. In particular, for a set I and a collection of abelian sheaves $(\mathcal{F}_i)_{i \in I}$ indexed by I, we have

$$\bigoplus_i H^q(X, \mathcal{F}_i) \cong H^q(X, \bigoplus_i \mathcal{F}_i)$$

for all $q \geq 0$.

Proof. See [Mil80], III, Remark 3.6. □

Lemma 2.3.2. *Let I be a filtered preordered set and $(X_i)_{i \in I}$ be a projective system of quasi-compact schemes whose transitive maps are affine, $X = \varprojlim_i X_i$. Let $\mathcal{F}_{i_0}$ be an abelian sheaf on X_{i_0} for an element $i_0 \in I$. For $i \geq i_0$, $\mathcal{F}_i$ denotes the inverse images of $\mathcal{F}_{i_0}$ on X_i. $\mathcal{F}$ denotes the inverse image of $\mathcal{F}_{i_0}$ on X. Then*

$$\varinjlim_i H^q(X_i, \mathcal{F}_i) \cong H^q(X, \mathcal{F})$$

for all $q \geq 0$.

Proof. See [AGV72], VII, Corollaire 5.8. □

2.3.2 Spectral sequences

Lemma 2.3.3 (Leray spectral sequence). *Let $f\colon X \longrightarrow Y$ be a morphism of schemes and $\mathcal{F}$ an abelian sheaf on X. Then we have the following spectral sequence:*

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Proof. See [Mil80], III, Theorem 1.18. □

Lemma 2.3.4 (Hochschild-Serre spectral sequence). *Let X be a variety over a field k, $\mathcal{F}$ be a sheaf on X and L/k a Galois extension of fields. Then we have the following spectral sequence:*

$$E_2^{p,q} = H^p(L/k, H^q(X_L, \mathcal{F})) \Rightarrow H^q(X, \mathcal{F}).$$

Proof. See [Mil80], III, Theorem 2.20. □

Chapter 3

Brauer groups

3.1 Brauer group of fields

We recall some basic facts about the Brauer group of fields.

3.1.1 Definition

Definition 3.1.1 (Central simple algebra). Let k be a (commutative) field. A central simple algebra A over k is defined to be a k-algebra satisfying the following:

(1) As a k-vector space, A is finitely dimensional.

(2) A is simple, that is, A has no nontrivial two-sided ideal.

(3) The center $C(A)$ of A is equal to k.

We have the following result due to Wedderburn [Wed08]:

Proposition 3.1.2. *Let k be a field. For each central simple algebra A, there exist a unique (up to k-isomorphisms) skew field D and a unique integer $n > 0$ such that $A \cong M_n(D)$.*

Now we define the notion of Morita equivalence, which is named after a Japanese mathematician Kiiti Morita [Mor58]:

Definition 3.1.3. Let k be a field, A and B central simple algebras over k. Take skew fields D and E and integers m and n such that $A \cong M_m(D)$ and $B \cong M_m(E)$. Then A and B are (Morita) equivalent if D is k-isomorphic to E as skew fields.

For a given central simple algebra A over k, we denote its equivalent class by $[A]$.

Let k be a field. A and B are central simple algebras over k. Then the tensor product $A \otimes_k B$ is again a central simple algebra over k. Moreover, the equivalent class of $A \otimes_k B$ is only independent of that of A and B. Therefore we can define an operation "$+$" on the set $\mathrm{Br}(k)$ of all equivalent classes of central simple algebras over k.

Lemma 3.1.4. *Let k be a field.*

(1) *The operation $+$ is commutative;*

(2) *the equivalent class $[k]$ of k is a unit with respect to $+$;*

(3) *for each central simple algebra A over k, the equivalent class of the opposite ring A^{op} of A is an inverse element with respect to $+$.*

By the above lemma, the set $\mathrm{Br}(k)$ has an abelian group structure.

Definition 3.1.5. Let k be a field. The Brauer group of k is defined to be the abelian group $(\mathrm{Br}(k), +)$.

By the definition of Morita equivalence, the group $\mathrm{Br}(k)$ can be seen as the classification space of finite dimensional central skew fields over k. For a field extension L/k and a simple central algebra A over k, the correspondence $A \mapsto A \otimes_k L$ yields a group homomorphism $\mathrm{res}_{L/k}\colon \mathrm{Br}(k) \to \mathrm{Br}(L)$. In other words, we obtain a functor:

$$\mathrm{Br}\colon (\text{Category of fields}) \to (\mathrm{Ab}).$$

One of the most important observations for Brauer groups is the following cohomological interpretation:

Proposition 3.1.6. *Let k be a field. We have the following natural isomorphism:*

$$\mathrm{Br}(k) \cong H^2(k, \overline{k}^*).$$

Proof. See [NSW00], Theorem 6.3.4. □

In the following of this dissertation, we always identify the Brauer group $\mathrm{Br}(k)$ with $H^2(k, \overline{k}^*)$.

3.1.2 Symbolic elements

Some elements of Brauer groups are represented by "symbols". In many contexts, these symbols are more useful for computation than central simple algebras or Galois 2-cocycles.

Definition 3.1.7. Let k be a field. For $b \in k^*$ and a cyclic character $\chi \in \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z})$. By the following isomorphism (see Subsection 2.2.1)

$$\mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \cong H^1(k, \mathbb{Q}/\mathbb{Z}) \overset{\delta}{\cong} H^2(k, \mathbb{Z})$$

and the following cup product

$$H^0(k, \overline{k}^*) \otimes H^2(k, \mathbb{Z}) \to H^2(k, \overline{k}^*),$$

we obtain the element $b \cup \delta(\chi) \in H^2(k, \overline{k}^*) = \mathrm{Br}(k)$. We denote it by (χ, b). An element of the form is often called a symbol.

We can define another type of symbolic elements of Brauer groups. We first recall the definition of Milnor K-groups of a ring.

Definition 3.1.8. Let R be a ring and q be a positive integer. The q-th Milnor K-group $K_q^M(R)$ of R is defined to be the quotient of

$$R^* \otimes_{\mathbb{Z}} R^* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R^* \quad (q \text{ times})$$

by its subgroup generated by elements of the forms:

(1) $a_1 \otimes \cdots \otimes a \otimes -a \otimes \cdots \otimes a_q, \quad a, a_1, \ldots, a_q \in R^*$;

(2) $a_1 \otimes \cdots \otimes a \otimes 1-a \otimes \cdots \otimes a_q, \quad a, 1-a, a_1, \ldots, a_q \in R^*$.

We denote the coset of $a_1 \otimes \cdots \otimes a_q$ by $\{a_1, \ldots, a_q\}$.

Note that the definition above yields the skew-symmetricity as follows:

$$\begin{aligned}\{a,b\} &= \{a,b\} + \{a,-a\} + \{b,-b\} + \{b,a\} - \{b,a\} \\ &= \{ab,-ab\} - \{b,a\} \\ &= -\{b,a\}.\end{aligned}$$

We also recall Kummer theory. For a proof, see [NSW00], Theorem 6.2.1.

Lemma 3.1.9 (Kummer sequence). *Let n be a positive integer and k a field with characteristic prime to n. Then the following sequence*

$$1 \to \mu_n \to \overline{k}^* \xrightarrow{n} \overline{k}^* \to 1$$

is exact. Here $\mu_n := \{x \in \overline{k}^* \mid x^n = 1\}$ and the map $n \colon \overline{k}^* \to \overline{k}^*$ maps x to x^n.

Lemma 3.1.10 (Hilbert's Theorem 90). *Let L/k be a Galois extension. Then we have*

$$H^1(L/k, L^*) = 0.$$

As a corollary of these lemmas, we obtain

Corollary 3.1.11 (Kummer theory). *Let n be a positive integer and k a field with characteristic prime to n. Then we have*

$$k^*/(k^*)^n \cong H^1(k, \mu_n).$$

Now we define norm residue symbols.

Definition 3.1.12. Let n, q be positive integers and k be a field with characteristic prime to n. Let a and $b \in k^*$. Using the connecting homomorphism

$$h = h_n^1 \colon k^* = H^0(k, \overline{k}^*) \to H^1(k, \mu_n)$$

induced by the Kummer sequence and the following cup product

$$H^1(k, \mu_n) \otimes \cdots \otimes H^1(k, \mu_n) \to H^q(k, \mu_n^{\otimes q}),$$

we obtain

$$h_n^q : k^* \otimes \cdots k^* \longrightarrow H^q(k, \mu_n^{\otimes q});$$
$$a_1 \otimes \cdots \otimes a_q \mapsto h(a_1) \cup \cdots \cup h(a_q).$$

We can see that this map factors through the q-th Milnor K-group $K_q^M(k)$. We denote the element $h_n^q(\{a_1, \ldots, a_q\})$ by $\{a_1, \ldots a_q\}_n$ and call it a (norm residue) symbol.

Remark 3.1.13. The homomorphisms h_n^q induces the following isomorphism:

$$K_q^M(k)/n \cong H^q(k, \mu_n^{\otimes q})$$

for any field k with characteristic prime to n, what is now known as the Voevodsky-Rost Theorem.

In the case k contains the group μ_n, we can obtain elements of Brauer group from Milnor K-groups.

Definition 3.1.14. Let n be a positive integer. Let k be a field containing a primitive n-th root ζ_n of unity. Let a and $b \in k^*$. Using the following isomorphism of (trivial) Galois modules

$$\mu_n^{\otimes 2} \cong \mu_n; \quad \zeta_n^i \otimes \zeta_n^j \mapsto \zeta_n^{ij},$$

we obtain an element in $H^2(k, \mu_n)$ corresponding to $\{a, b\}_n \in H^2(k, \mu_n^{\otimes 2})$. We also denote this element by $\{a, b\}_n$. By abuse of notation, its image in $H^2(k, \overline{k}^*) = \mathrm{Br}(k)$ under the natural inclusion $H^2(k, \mu_n) \hookrightarrow H^2(k, \overline{k}^*)$ is also denoted by the same symbol.

Next we see the relation between two symbols defined above. Let k be a field containing a primitive n-th root of unity. For $a \in k^*$, let $\chi_{n,a}$ denote the image of a under the following homomorphism:

$$\begin{aligned} \chi_n : k^* = H^0(k, \overline{k}^*) &\longrightarrow H^1(k, \mu_n) \\ &\cong H^1(k, \mathbb{Z}/n\mathbb{Z}) \quad (\mu_n \cong \mathbb{Z}/n\mathbb{Z}; \quad \zeta^i \mapsto i) \\ &\hookrightarrow H^1(k, \mathbb{Q}/\mathbb{Z}). \quad (\mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}; \quad 1 \mapsto 1/n) \end{aligned}$$

Proposition 3.1.15. *Let k be a field containing a primitive n-th root ζ_n of unity. Let a and $b \in k^*$. Then we have the following:*

$$\{a, b\}_n = (\chi_{n,a}, b).$$

Proof. See [Ser68], XIV, Proposition 5. □

3.1.3 Residue maps

Let R be a discrete valuation ring, with fractional field k and perfect residue field κ. Then there is a residue map:

$$\mathrm{res}_R : \mathrm{Br}(k) \longrightarrow H^1(\kappa, \mathbb{Q}/\mathbb{Z}).$$

In the following, we see the definition of this map.

Definition 3.1.16. Let R, k and κ be as above. Assume that R is complete. The map res_R is defined to be:

$$H^2(k,\overline{k}^*) \cong H^2(k^{\mathrm{ur}}/k, k^{\mathrm{ur}*}) \xrightarrow{v} H^2(k^{\mathrm{ur}}/k, \mathbb{Z}) \cong H^2(\kappa, \mathbb{Z}) \cong H^1(\kappa, \mathbb{Q}/\mathbb{Z}).$$

Here the first isomorphism is a consequence of the following Hochschild-Serre spectral sequence (see Lemma 2.2.6)

$$H^p(k^{\mathrm{ur}}/k, H^q(k^{\mathrm{ur}}, \overline{k}^*)) \Rightarrow H^{p+q}(k, \overline{k}^*)$$

and a theorem due to Lang ([Lan52] Theorem 12), which states that the maximal unramified extension of a complete discrete valuation field with perfect residue field is a C_1-field. The second map v is induced by the normalized valuation on k^{ur}, the third one is defined by using an isomorphism $\mathrm{Gal}(k^{\mathrm{ur}}/k) \cong G_\kappa$, and the last one is the same one as in Subsection 2.2.1.

For general discrete valuation rings, we define res_R as follows.

Definition 3.1.17. Let R, k and κ be as above. Let $\hat{R}$ be a completion of R with respect to its valuation, $\hat{k}$ be the fractional field of $\hat{R}$ and $\hat{\kappa}$ be its residue field. Note that we have a natural isomorphism $\kappa \cong \hat{\kappa}$. Then we define res_R to be:

$$H^2(k,\overline{k}^*) \to H^2(\hat{k},\overline{\hat{k}}^*) \xrightarrow{\mathrm{res}_{\hat{R}}} H^1(\hat{\kappa}, \mathbb{Q}/\mathbb{Z}) \cong H^1(\kappa, \mathbb{Q}/\mathbb{Z}).$$

A virtue of symbolic representation of elements of Brauer groups is that we can compute their images under the above residue map explicitly:

Proposition 3.1.18. *Let R, k and κ be as above. Let v be the normalized valuation on k and n be a positive integer prime to $\mathrm{ch}(\kappa)$. Assume that k contains a primitive n-th root ζ_n of unity. Using an isomorphism*

$$\mathbb{Z}/n\mathbb{Z} \cong \mu_n; \quad 1 \mapsto \zeta_n,$$

we identify $H^1(\kappa, \mathbb{Z}/n\mathbb{Z})$ with $H^1(\kappa, \mu_n) \cong \kappa^*/(\kappa^*)^n$. For $a, b \in k^*$, we have:

$$\mathrm{res}_R(\{a,b\}_n) = (-1)^{v(a)v(b)} \overline{\left(\frac{a^{v(b)}}{b^{v(a)}}\right)} \in \kappa^*/(\kappa^*)^n,$$

where $\overline{r} \in \kappa$ is the image of $r \in R$.

Proof. We may assume that R is complete. Let π be a uniformizer of R. By the formula

$$\{\pi,\pi\} = \{-\pi,\pi\} + \{-1,\pi\} = \{-1,\pi\} \in K_2^M(k),$$

and the skew-symmetricity of symbols, that is, $\{a,b\} = -\{b,a\}$ for all $a, b \in k^*$, it suffice to prove

$$\mathrm{res}_R(\{a,b\}_n) = \overline{a}^{v(b)},$$

for all $a \in R^*$ and $b \in k^*$.

We can take

$$\phi\colon G_k^2 \to \overline{k}^*; \quad (\sigma,\tau) \mapsto (\sqrt[n]{b})^{i(\tau)-i(\sigma\tau)+i(\sigma)}$$

as a 2-cocycle corresponding to $\{a,b\}_n = (\chi_{n,a}, b)$. Here for each $\sigma \in G_k$, we fix an integer $i(\sigma)$ as follows:

$$\frac{\sigma\sqrt[n]{a}}{\sqrt[n]{a}} = \zeta_n^{i(\sigma)}.$$

This cocycle is considered as a cocycle in $Z^2(k^{\mathrm{ur}}/k, (k^{\mathrm{ur}})^*)$ since $i(\sigma)$ is determined only by the class modulo $G_{k^{\mathrm{ur}}}$ because $\sqrt[n]{a} \in k^{\mathrm{ur}}$, and since $\phi(\sigma,\tau)$ is in $k^* \subset (k^{\mathrm{ur}})^*$ for all $\sigma, \tau \in G_k$.

The cocycle in $Z^2(k^{\mathrm{ur}}/k, \mathbb{Z})$ corresponding to ϕ is the following:

$$v \circ \phi\colon G(k^{\mathrm{ur}}/k)^2 \to \mathbb{Z}; \quad (\sigma,\tau) \mapsto \left(\frac{i(\tau)}{n} - \frac{i(\sigma\tau)}{n} + \frac{i(\sigma)}{n}\right) v(b).$$

Hence we can choose a cocycle $\in H^1(\kappa, \mathbb{Z}/n\mathbb{Z})$ corresponding to $v \circ \phi$ as follows:

$$G_\kappa \to \mathbb{Z}/n\mathbb{Z}; \quad \sigma \mapsto i(\sigma)\cdot v(b) \mod n\mathbb{Z}.$$

Considering the following correspondences

$$\mathbb{Z}/n\mathbb{Z} \cong \mu_n; \quad i(\sigma)\cdot v(b) \leftrightarrow \left(\frac{\sigma\sqrt[n]{\overline{a}}}{\sqrt[n]{\overline{a}}}\right)^{v(b)}$$

and

$$\kappa^*/(\kappa^*)^n \cong H^1(\kappa, \mu_n); \quad a \leftrightarrow \left[\sigma \mapsto \left(\frac{\sigma\sqrt[n]{\overline{a}}}{\sqrt[n]{\overline{a}}}\right)\right],$$

we find that the corresponding element is $\overline{a}^{v(b)}$, which completes the proof of this proposition. □

3.2 Brauer group of schemes

3.2.1 Azumaya algebras and cohomological Brauer groups

As a generalization of the Brauer group of fields, Azumaya [Azu51] considered the notion of "central simple algebras over a local ring", what is now called Azumaya algebras. His approach was generalized to schemes by Grothendieck [Gro68a], [Gro68b], [Gro68c]. He constructed Brauer groups of schemes in two ways. One is the way using Azumaya algebras over a scheme, and the other is the way using a cohomological method. We recall these definitions and some basic properties.

We first look at the definition of Brauer groups defined by using Azumaya algebras.

Definition 3.2.1. Let X be a scheme. An $\mathcal{O}_X$-algebra $\mathcal{A}$ is called an Azumaya algebra $\mathcal{A}$ over X if it is locally free $\mathcal{O}_X$-module and $\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is a central simple algebra over $\kappa(x)$ for all $x \in X$.

For a scheme X and an $\mathcal{O}_X$-module $\mathcal{F}$, we denote by $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})$ the following $\mathcal{O}_X$-algebra defined by

$$U \rightsquigarrow \mathrm{End}_{\mathcal{O}_U}(\mathcal{F}|_U).$$

For a scheme X and a positive integer n, we denote by $M_n(\mathcal{O}_X)$ the sheaf of all matrices defined by

$$U \rightsquigarrow M_n(\mathcal{O}_X(U)).$$

Proposition 3.2.2. *Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra which is finite type as an $\mathcal{O}_X$-module. Then the following are equivalent:*

(1) *$\mathcal{A}$ is an Azumaya algebra.*

(2) *$\mathcal{A}$ is a locally free $\mathcal{O}_X$-module and the canonical homomorphism*

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}); \quad a \otimes b \longmapsto (c \longmapsto acb)$$

is isomorphic, where $\mathcal{A}^{\mathrm{op}}$ is the opposite sheaf of $\mathcal{O}_X$-algebra $\mathcal{A}$, whose section on $U \subset X$ is defined to be

$$\mathcal{A}^{\mathrm{op}}(U) := (\mathcal{A}(U))^{\mathrm{op}}.$$

(3) *For all $x \in X$, there exist a positive integer n, a open neighborhood U of x and a finite étale surjective morphism $\pi : U' \longrightarrow U$ such that $\pi^*(\mathcal{A}|_U) \cong M_n(\mathcal{O}_{U'})$ as $\mathcal{O}_{U'}$-algebras, where $\pi^*(\mathcal{A}|_U)$ is the pull-back of $\mathcal{A}|_U$ as $\mathcal{O}_U$-module.*

Proof. See [Gro68a] and [Mil80], IV, Proposition 2.1. □

Definition 3.2.3. Let X be a scheme. Two Azumaya algebras $\mathcal{A}$ and $\mathcal{A}'$ are equivalent if there exist locally free $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{E}'$ of finite rank over $\mathcal{O}_X$, such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}').$$

This is in fact an equivalence relation. Likewise in the case of fields, we can define an operation $+$ on the set $\mathrm{Br}_{\mathrm{Az}}(X)$ of all equivalent classes of Azumaya algebras on X to be:

$$[\mathcal{A}] + [\mathcal{A}'] = [\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}'].$$

We also have the following lemma:

Lemma 3.2.4. *Let X be a scheme.*

(1) *The operation $+$ is commutative.*

(2) *The equivalent class $[\mathcal{O}_X]$ of $\mathcal{O}_X$ is a unit with respect to $+$.*

(3) *For each Azumaya algebra $\mathcal{A}$ on X, the equivalent class of the opposite algebra $\mathcal{A}^{\mathrm{op}}$ of $\mathcal{A}$ is an inverse element with respect to $+$.*

Therefore we get the Brauer group of a scheme:

Definition 3.2.5. Let X be a scheme. The Brauer group of X is defined to be the abelian group $(\mathrm{Br}_{\mathrm{Az}}(X), +)$.

For a morphism $f\colon X \to Y$ of schemes and an Azumaya algebra $\mathcal{A}$ on Y, $f^*\mathcal{A}$ is also an Azumaya algebra on X. Moreover, its class $[f^*\mathcal{A}]$ is determined only by the class $[\mathcal{A}]$. These facts imply

$$\mathrm{Br}_{\mathrm{Az}}\colon (\text{Schemes}) \to (\text{Ab})$$

is a contravariant functor.

On the other hand, we can also consider the cohomological Brauer group:

Definition 3.2.6. Let X be a scheme. the cohomological Brauer group $\mathrm{Br}(X)$ of X is defined to be:

$$\mathrm{Br}(X) := H^2(X, \mathbb{G}_m).$$

Moreover, for a morphism $f\colon X \to Y$ of schemes, we have a natural homomorphism

$$\mathrm{Br}(f)\colon H^2(Y, \mathbb{G}_m) \to H^2(X, f^*\mathbb{G}_m) \to H^2(X, \mathbb{G}_m).$$

This yields the following contravariant functor

$$\mathrm{Br}\colon (\text{Schemes}) \to (\text{Ab}).$$

For $f\colon X \to Y$, we often denote $\mathrm{Br}(f)$ simply by f^*.

At the end of this subsection, we state a result on relations between $\mathrm{Br}_{\mathrm{Az}}(X)$ and $\mathrm{Br}(X)$.

Proposition 3.2.7. *There exists a natural injective homomorphism* $\delta\colon \mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(X)$.

Proof. See [Mil80], Chapter IV and [Gir71]. □

In the following of this dissertation, Brauer group of a scheme X always means the cohomological one.

3.2.2 Unramified Brauer groups

In this subsection, we recall a description of the Brauer group of a scheme as an unramified Brauer group. The main reference of this subsection is [CT95a]. Recall that for a scheme X, we denote by $X^{(i)}$ the set of all points in X with codimension i. We say an integer n is invertible on X if n is prime to $\mathrm{ch}(\kappa(x))$ for all $x \in X$.

We prove the following theorem:

Theorem 3.2.8. *Let X be a noetherian regular integral scheme, η be the generic point of X. Then for each prime ℓ which is invertible on X, we have the following exact sequence:*

$$0 \to \mathrm{Br}(X)\{\ell\} \to \mathrm{Br}(\kappa(\eta))\{\ell\} \xrightarrow{\oplus \partial_x} \bigoplus_{x \in X^{(1)}} H^1(\kappa(x), \mathbb{Q}/\mathbb{Z})\{\ell\}.$$

Here, the map ∂_x is the residue map at x defined below.

We first see the injectivity. This holds without restriction to ℓ-primary part.

Proposition 3.2.9. *Let X and η be as above. we have the canonical inclusion*

$$\mathrm{Br}(X) \to \mathrm{Br}(\kappa(\eta)).$$

Proof. We have the following exact sequence:

$$0 \to \mathbb{G}_m \to j_* \mathbb{G}_{m,\eta} \to \mathrm{Div}_X \to 0, \tag{3.1}$$

where $j : \eta = \operatorname{Spec} \kappa(\eta) \hookrightarrow X$ and $\mathrm{Div}_X = \mathrm{Coker}(\mathbb{G}_m \to j_* \mathbb{G}_{m,\eta})$ is the sheaf of Cartier divisor on X. By the regularity of X, we have

$$\mathrm{Div}_X = \bigoplus_{x \in X^{(1)}} i_{x*} \mathbb{Z},$$

where $i_x : x = \operatorname{Spec} \kappa(x) \hookrightarrow X$. Thus we get the exact sequence

$$\cdots \to \bigoplus_{x \in X^{(1)}} H^1(X, i_{x*} \mathbb{Z}) \to H^2(X, \mathbb{G}_m) \to H^2(X, j_* \mathbb{G}_{m,\eta}) \to \cdots$$

because of the commutativity of direct sums and étale cohomology. Therefore the claim is a consequence of the following two lemmas:

Lemma 3.2.10. *We have*

$$H^2(X, j_* \mathbb{G}_{m,\eta}) \hookrightarrow H^2(\kappa(\eta), \mathbb{G}_m).$$

Proof. We have the following exact sequence

$$H^0(X, R^1 j_* \mathbb{G}_m) \to H^2(X, j_* \mathbb{G}_m) \to H^2(\kappa(\eta), \mathbb{G}_m)$$

by the Leray spectral sequence

$$H^p(X, R^q j_* \mathbb{G}_{m,\eta}) \Rightarrow H^{p+q}(\kappa(\eta), \mathbb{G}_m).$$

Hence it suffices to show $R^1 j_* \mathbb{G}_m = 0$. For $x \in X$, we have $(R^1 j_* \mathbb{G}_m)_{\overline{x}} = H^1(K_{\overline{x}}, \mathbb{G}_m)$, where $K_{\overline{x}}$ is the fractional field of $\mathcal{O}_{X,\overline{x}}$. This group is trivial by Hilbert's Theorem 90 and hence we have $R^1 j_* \mathbb{G}_m = 0$. □

Lemma 3.2.11. $H^1(X, i_{x*} \mathbb{Z}) = 0$.

Proof. The same argument in Lemma 3.2.10 and the fact $H^1(k, \mathbb{Z}) = 0$ for any field k imply

$$R^1 i_{x*} \mathbb{Z} = 0.$$

This and the Leray spectral sequence

$$H^p(X, R^q i_{x*} \mathbb{Z}) \Rightarrow H^{p+q}(\kappa(x), \mathbb{Z}).$$

yield

$$H^1(X, i_{x*} \mathbb{Z}) \cong H^1(\kappa(x), \mathbb{Z}) = 0.$$

□

This completes the proof of Proposition 3.2.9. □

Secondly, we consider the exactness at $\mathrm{Br}(\kappa(\eta))\{\ell\}$. We begin with the following lemma:

Lemma 3.2.12 (Mayer-Vietoris exact sequence). *Let X be a scheme, $\{U_1, U_2\}$ be a Zariski open covering of X and $\mathcal{F}$ be a étale sheaf on X. Put $U_{12} = U_1 \cap U_2$. Then we have the following long exact sequence:*

$$\to H^q(X,\mathcal{F}) \xrightarrow{\alpha^q} H^q(U_1,\mathcal{F}) \oplus H^q(U_2,\mathcal{F}) \xrightarrow{\beta^q} H^q(U_{12},\mathcal{F}) \xrightarrow{\delta^q} H^{q+1}(X,\mathcal{F}) \to,$$

where α^q maps s to $(s|_{U_1}, -s|_{U_2})$ and β^q maps (s,t) to $s|_{U_{12}} + t|_{U_{12}}$ and δ^q is defined below.

Proof. We have the following spectral sequence ([Mil80], III, Proposition 2.7.):

$$E_2^{p,q} = \check{H}^p(\{U_i\}/X, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(X,\mathcal{F}),$$

where $\check{H}^p(\{U_i\}/X, \cdot)$ is the p-th Čech cohomology with respect to the covering $\{U_i\}/X$ and $\mathcal{H}^q(\mathcal{F})$ is the presheaf on X defined to be:

$$V \rightsquigarrow H^q(V,\mathcal{F}).$$

By explicit computation, we know:

$$E_2^{p,q} = \begin{cases} \mathrm{Ker}\,\beta^q & p = 0, \\ H^q(U_{12},\mathcal{F})/\mathrm{Im}\beta^q & p = 1, \\ 0 & p \geq 2. \end{cases}$$

Since $E_2^{p,q} = 0$ for $p \geq 2$, we have the short exact sequence

$$0 \to E_2^{1,q} \to E^{q+1} \to E_2^{0,q+1} \to 0.$$

Put $\delta^q \colon H^q(U_{12},\mathcal{F}) \to E_2^{1,q} \to E^{q+1}$. The claim follows immediately these results. □

Now we recall the following purity theorem due to O. Gabber.

Theorem 3.2.13 (Absolute purity theorem, ([Fuj02], Theorem 2.1.1)). *Let $i\colon Y \hookrightarrow X$ be a closed immersion of noetherian regular schemes of pure codimension c. Let n be an integer which is invertible on X. Then*

$$R^q i^!(\mu_n^{\otimes j}) \cong \begin{cases} 0 & q \neq 2c \\ \mu_n^{\otimes j-c} & q = 2c. \end{cases}$$

Theorem 3.2.14 (Semi-purity theorem ([Fuj02], Section 8)). *Let X be a noetherian regular scheme and n be an integer which is invertible on X. Then for (not necessarily regular) closed subscheme Y of pure codimension $\geq c$ and for all $q < 2c$,*

$$R^q i^!(\mu_n^{\otimes j}) = 0.$$

Lemma 3.2.15. *For a discrete valuation ring R with quotient field k and residue field κ and for an integer n coprime to $\operatorname{ch}\kappa$, we have the following exact sequence:*

$$H^2(R,\mu_n^{\otimes j}) \to H^2(k,\mu_n^{\otimes j}) \xrightarrow{\partial_R} H^1(\kappa,\mu_n^{\otimes j-1}).$$

Proof. Put $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec}\kappa$. Then $c = 1$ and the complement $U = \operatorname{Spec} k$. We have the following spectral sequence ([Mil80], VI, Section 5.):

$$E_2^{p,q} = H^p(Y, R^q i^!(\mu_n^{\otimes j})) \Rightarrow H_Y^{p+q}(X,\mu_n^{\otimes j}). \tag{3.2}$$

Applying Theorem 3.2.13 to this spectral sequence, we get

$$H^p(Y,\mu_n^{\otimes j-1}) \cong H_Y^{p+2}(X,\mu_n^{\otimes j}).$$

On the other hand, we have the localization sequence ([Mil80], III Proposition 1.25.):

$$\to H_Y^i(X,\mu_n^{\otimes j}) \to H^i(X,\mu_n^{\otimes j}) \to H^i(U,\mu_n^{\otimes j}) \to H_Y^{i+1}(X,\mu_n^{\otimes j}) \to . \tag{3.3}$$

Combining these results, we have the following exact sequence, which is called "Gysin sequence":

$$\to H^{i-2}(Y,\mu_n^{\otimes j-1}) \to H^i(X,\mu_n^{\otimes j}) \to H^i(U,\mu_n^{\otimes j}) \to H^{i-1}(Y,\mu_n^{\otimes j-1}) \to .$$

If we put $i = 2$, then the claim is obtained. □

Proposition 3.2.16. *For a noetherian regular integral scheme X and an integer n which is invertible on X, we have the following exact sequence:*

$$H^2(X,\mu_n^{\otimes j}) \to H^2(\kappa(\eta),\mu_n^{\otimes j}) \xrightarrow{\oplus\partial_x} \bigoplus_{x\in X^{(1)}} H^1(\kappa(x),\mu_n^{\otimes j-1}),$$

where $\partial_x = \partial_{\mathcal{O}_{X,x}}$ in Lemma 3.2.15.

Proof. The above sequence is a complex since for each $x \in X^{(1)}$, we have the factorization

$$H^2(X,\mu_n^{\otimes j}) \to H^2(\mathcal{O}_{X,x},\mu_n^{\otimes j}) \to H^2(\kappa(\eta),\mu_n^{\otimes j}) \to H^1(\kappa(x),\mu_n^{\otimes j-1})$$

and the composite of the middle and right maps is the zero map by Lemma 3.2.15.

Take $\alpha \in \operatorname{Ker}(H^2(\kappa(\eta),\mu_n^{\otimes j}) \to \bigoplus_{x\in X^{(1)}} H^1(\kappa(x),\mu_n^{\otimes j-1})$. By the commutativity of étale cohomology and filtered direct limit (Lemma 2.3.2), there exists an open $U \subset X$ and $\alpha_U \in H^2(U,\mu_n^{\otimes j})$ such that $\alpha_U \mapsto \alpha$. Put $Y := X \setminus U$ and $Y_1 \subset Y$ to be the union of irreducible components of Y with codimension 1 in X. Write Y_1 as $Y_1 = \bigcup_{i=1}^N D_i$, where D_i is a prime divisor. Since Y is noetherian, this union is indeed finite. Let x_i be the generic point of D_i. Since $\partial_{x_1}(\alpha) = 0$, there exists a $\alpha_{x_1} \in H^2(\mathcal{O}_{X,x_1},\mu_n^{\otimes j})$ such that $\alpha_{x_1} \mapsto \alpha$ by Lemma 3.2.15. Moreover, we can take a open neighborhood V_1 of x_1 and $\alpha_{V_1} \in H^2(V_1, \mu_n^{\otimes j})$ such that $\alpha_{V_1} \mapsto \alpha_{x_1}$. we would like to show α_U and α_{V_1} can be patched. Since $\varprojlim_{x_1 \in V}(U \cap V) = \operatorname{Spec} \kappa(\eta)$, we can assume that $\alpha_U|_{U\cap V_1} = \alpha_{V_1}|_{U \cap V_1}$ by replacing V_1 with a smaller neighborhood of x_1. Hence we know there exists $\alpha_{U\cup V_1}$ which is a extension of α_U and α_{V_1} by Proposition 3.2.12.

Applying the same argument to $V_2, \ldots, V_N$, we obtain an element

$$\alpha_{U\cup V_1 \cup \cdots \cup V_N} \in H^2(U \cup V_1 \cup \cdots \cup V_N, \mu_n^{\otimes j})$$

whose image in $H^2(\kappa(\eta), \mu_n^{\otimes j})$ is α. Replace $U \cup V_1 \cup \cdots \cup V_N$ with U. Then the new $Y = X \setminus U$ has codimension at least 2 in X. Now we make a similar argument in Lemma 3.2.15. By the semi-purity theorem (Theorem 3.2.14) and the spectral sequence (3.2), we have

$$H_Y^p(X, \mu^{\otimes j}) = 0$$

for all $p \leq 3$. Therefore the localization sequence (3.3) yields the isomorphism

$$H^2(X, \mu_n^{\otimes j}) \cong H^2(X \setminus Y, \mu_n^{\otimes j}),$$

which means that we can get an element $\alpha_X \in H^2(X, \mu_n^{\otimes j})$ such that $\alpha_X \mapsto \alpha$. This complete the proof of the proposition. □

Proof of Theorem 3.2.8. In Proposition 3.2.16, put $n = \ell^m$ and $j = 1$. Then we have:

$$H^2(X, \mu_{\ell^m}) \to {}_{\ell^m}\mathrm{Br}(\kappa(\eta)) \xrightarrow{\oplus \partial_x} \bigoplus_{x \in X^{(1)}} H^1(\kappa(x), \mathbb{Z}/\ell^m\mathbb{Z}).$$

On the other hand, by the Kummer sequence and Proposition 3.2.9, the left map above factors as follows:

$$H^2(X, \mu_{\ell^m}) \twoheadrightarrow {}_{\ell^m}\mathrm{Br}(X) \hookrightarrow {}_{\ell^m}\mathrm{Br}(\kappa(\eta)).$$

Combining these sequence, we obtain the exact sequence:

$$0 \to {}_{\ell^m}\mathrm{Br}(X) \to {}_{\ell^m}\mathrm{Br}(\kappa(\eta)) \xrightarrow{\oplus \partial_x} \bigoplus_{x \in X^{(1)}} H^1(k(x), \mathbb{Z}/\ell^m\mathbb{Z}).$$

Taking the direct limit of this sequence with respect to m, we complete the proof of Theorem 3.2.8. □

3.2.3 A fundamental exact sequence

We first recall the following:

Lemma 3.2.17 (Hilbert's Theorem 90). *Let X be a scheme. Then we have canonical isomorphisms:*

$$\mathrm{Pic}(X) \cong H^1_{\mathrm{Zar}}(X, \mathcal{O}_X^*) \cong H^1(X, \mathbb{G}_m),$$

where H^*_{Zar} is Zariski cohomology and $\mathcal{O}_X^*$ is the Zariski sheaf of multiplicative units on X.

Proof. See [Mil80], III, Proposition 4.9. □

By this isomorphism, we always identify $H^1(X, \mathbb{G}_m)$ with $\mathrm{Pic}(X)$.

Let X be a variety over a field k. By the Hochschild-Serre spectral sequence

$$H^p(\bar{k}/k, H^q(\overline{X}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m),$$

we have the following exact sequence

$$0 \to \mathrm{Br}_1(X)/\,\mathrm{Br}(k) \to H^1(k, \mathrm{Pic}(\overline{X})) \xrightarrow{d^{1,1}} H^3(k, \bar{k}^*), \tag{3.4}$$

where

$$\mathrm{Br}_1(X) := \mathrm{Ker}(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})),$$
$$\mathrm{Br}_1(X)/\,\mathrm{Br}(k) := \mathrm{Br}_1(X)/\pi^*\,\mathrm{Br}(k).$$

Hence we know that $\mathrm{Br}_1(X)/\,\mathrm{Br}(k)$ has an inclusion into $H^1(k, \mathrm{Pic}(\overline{X}))$. It is not clear whether this inclusion is an isomorphism or not. However, here are the following sufficient conditions:

Lemma 3.2.18. *Let X be a variety over a field k. If $\mathrm{cd}(k) \leq 2$ or $X(k) \neq \emptyset$, then*

$$\mathrm{Br}_1(X)/\,\mathrm{Br}(k) \cong H^1(k, \mathrm{Pic}(\overline{X})).$$

Proof. First if $\mathrm{cd}(k) \leq 2$, we have $H^3(k, \bar{k}^*) = 0$ and the claim immediately follows from the above exact sequence.

Next we assume $X(k) \neq \emptyset$. We can extend the above sequence to the following complex

$$0 \to \mathrm{Br}_1(X)/\,\mathrm{Br}(k) \to H^1(k, \mathrm{Pic}(\overline{X})) \to H^3(k, \bar{k}^*) \to H^3(X, \mathbb{G}_m).$$

By the assumption $X(k) \neq \emptyset$, the map

$$H^3(k, \bar{k}^*) \to H^3(X, \mathbb{G}_m)$$

is injective, which implies the surjectivity of $\mathrm{Br}_1(X)/\,\mathrm{Br}(k) \to H^1(k, \mathrm{Pic}(\overline{X}))$. This completes the proof of this lemma. □

3.2.4 Another description of Brauer groups

We use the following result in Chapter 5. Let X be a variety over a field k. We describe the following Brauer group

$$\mathrm{Br}(X_L/X) := \mathrm{Ker}(\mathrm{Br}(X) \to \mathrm{Br}(X_L)),$$

where L/k is a Galois extension. The claim is:

Proposition 3.2.19. *Let X be a smooth, geometrically integral variety over a field k and let L be a Galois extension of k. Put $G := \mathrm{Gal}(L/k)$. Then we have the following exact sequence*

$$0 \to \mathrm{Br}(X_L/X) \to H^2(G, L(X)^*) \to H^2(G, \mathrm{Div}(X_L)),$$

where the third map is naturally induced by

$$\mathrm{div}\colon L(X)^* \to \mathrm{Div}(X_L); \quad f \mapsto \mathrm{div}(f).$$

Proof. Let $j\colon \eta = \mathrm{Spec}\, k(X) \to X$ be the generic point of X. By taking cohomology of the exact sequence (3.1), we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Br}(X) & \longrightarrow & H^2(X, j_* \mathbb{G}_m) & \longrightarrow & H^2(X, \mathrm{Div}_X) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Br}(X_L) & \longrightarrow & H^2(X_L, j_* \mathbb{G}_m) & \longrightarrow & H^2(X_L, \mathrm{Div}_X). \end{array}$$

Taking the kernel of each column, we obtain the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Br}(X_L/X) \\ &\to \mathrm{Ker}(H^2(X, j_* \mathbb{G}_m) \to H^2(X_L, j_* \mathbb{G}_m)) \\ &\to \mathrm{Ker}(H^2(X, \mathrm{Div}_X) \to H^2(X_L, \mathrm{Div}_X)). \end{aligned}$$

Hence it suffices to prove:

Lemma 3.2.20. *we have the following commutative diagram:*

$$\begin{array}{ccc} H^2(G, L(X)^*) & \xrightarrow{\cong} & \mathrm{Ker}(H^2(X, j_* \mathbb{G}_m) \to H^2(X_L, j_* \mathbb{G}_m)) \\ \downarrow & & \downarrow \\ H^2(G, \mathrm{Div}(X_L)) & \xrightarrow{\cong} & \mathrm{Ker}(H^2(X, \mathrm{Div}_X) \to H^2(X_L, \mathrm{Div}_X)). \end{array}$$

Proof. Applying the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q(X_L, \cdot)) \Rightarrow E^{p+q} = H^{p+q}(X, \cdot)$$

to sheaves $\mathcal{F}_1 \to \mathcal{F}_2$ on X, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} E_2^{0,1}(\mathcal{F}_1) & \longrightarrow & E_2^{2,0}(\mathcal{F}_1) & \longrightarrow & \mathrm{Ker}(E^2 \to E_2^{0,2})(\mathcal{F}_1) & \longrightarrow & E_2^{1,1}(\mathcal{F}_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E_2^{0,1}(\mathcal{F}_2) & \longrightarrow & E_2^{2,0}(\mathcal{F}_2) & \longrightarrow & \mathrm{Ker}(E^2 \to E_2^{0,2})(\mathcal{F}_2) & \longrightarrow & E_2^{1,1}(\mathcal{F}_2). \end{array}$$

Put $\mathcal{F}_1 := j_* \mathbb{G}_{m,\eta}$ and $\mathcal{F}_2 := \mathrm{Div}_X$. Noting that $H^1(L(X), \mathbb{G}_m) = 0$ by Hilbert's Theorem 90, and that $H^1(X_L, \mathrm{Div}_X) = 0$ by Lemma 3.2.11, we obtain the desired diagram. □

This completes the proof of Proposition 3.2.19. □

3.2.5 Specialization of Brauer groups

In Chapter 5, we will see that the Brauer group of surfaces of the form $x^3 + y^3 + cz^3 + dt^3 = 0$ has a *uniform* symbolic generator, that is, if we put

$$e(c,d) = \left\{ \frac{d}{c}, \frac{x+\zeta y}{x+y} \right\}_3,$$

where c and d are considered as *indeterminates* and if we want a symbolic generator of $\mathrm{Br}(V_P)/\mathrm{Br}(k)$, where V_P is the surface of the form $x^3 + y^3 + c_0 z^3 + d_0 t^3 = 0$ with c_0 and $d_0 \in k^*$, we can get it by specializing $e(c,d)$ at $(c,d) = (c_0, d_0)$.

To make this notion of uniformity precise, we define specialization as follows. Let k be a field, $\mathcal{O}_F$ a polynomial ring over k with r variables, F its fractional field and $f_1, \ldots, f_m$ homogeneous polynomials in $\mathcal{O}_F[x_0, \ldots, x_n]$. Let $\mathcal{X}$ be the projective scheme over $\mathcal{O}_F$ defined as:

$$\mathcal{X} = \mathrm{Proj}\,(\mathcal{O}_F[x_0, \ldots, x_n]/(f_1, \ldots, f_m)) \xrightarrow{\pi} \mathrm{Spec}\,\mathcal{O}_F\,.$$

Let $\pi_F\colon X := \mathcal{X}_F \longrightarrow \mathrm{Spec}\,F$ be the base change of π to $\mathrm{Spec}\,F$. Assume that X is smooth over F. Let $e \in \mathrm{Br}(X)$ be an arbitrary element. If $(S_i)_{i \in I}$ is the projective system of the non-empty affine open subschemes in $\mathbb{A}_k^r = \mathrm{Spec}\,\mathcal{O}_F$, we have

$$\varprojlim_i (\mathcal{X} \times_{\mathbb{A}_k^r} S_i) \cong X,$$

and there exists a non-empty affine open subscheme S and $\widetilde{e} \in \mathrm{Br}(\mathcal{X} \times_{\mathbb{A}_k^r} S)$ satisfying that $\mathcal{X} \times_{\mathbb{A}_k^r} S$ is smooth over S and that

$$\mathrm{res}^S_{\mathrm{Spec}\,F}(\widetilde{e}) = e,$$

where $\mathrm{res}^S_{\mathrm{Spec}\,F}\colon \mathrm{Br}(\mathcal{X} \times_{\mathbb{A}_k^r} S) \longrightarrow \mathrm{Br}(X)$. This follows from [Gro67] (Proposition 17.7.8) and Lemma 2.3.2. For a given $P \in S(k)$, we have the following diagram:

$$\begin{array}{ccccc} X_P & \xrightarrow{P} & \mathcal{X} \times_{\mathbb{A}_k^r} S & \longleftarrow & X \\ \downarrow{\scriptstyle \pi_0} & \square & \downarrow{\scriptstyle \pi_S} & \square & \downarrow{\scriptstyle \pi_F} \\ \mathrm{Spec}\,k & \xrightarrow{P} & S & \longleftarrow & \mathrm{Spec}\,F \end{array}$$

We define the *specialization of e at P* as

$$\mathrm{sp}(e;P) := P^*\widetilde{e} \in \mathrm{Br}(X_P).$$

We now see that the above definition is independent of S and $\widetilde{e}$. We define a triple $(S, \widetilde{e}, P)$ as the following data:

- S: a non-empty affine open in $\mathbb{A}_k^r$ such that $\mathcal{X} \times_{\mathbb{A}_k^r} S$ is smooth over S,
- $\widetilde{e}$: a lift of e in $\mathrm{Br}(\mathcal{X} \times_{\mathbb{A}_k^r} S)$,

- P: a k-valued point of S.

Given another triple $(S', \widetilde{e}', P)$ with the same point P. Since $\mathbb{A}_k^r$ is separated, $S'' := S \cap S'$ is also affine and $P \in S''(k)$. Therefore it suffices to consider the case $S' \subset S$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} & & \mathrm{Br}(\mathcal{X} \times_{\mathbb{A}_k^r} S') & & \\ & \overset{\mathrm{res}_{S'}^{S}}{\nearrow} & & \overset{\mathrm{res}_{\mathrm{Spec}\,F}^{S'}}{\searrow} & \\ \mathrm{Br}(\mathcal{X} \times_{\mathbb{A}_k^r} S) & & \xrightarrow{\quad \mathrm{res}_{\mathrm{Spec}\,F}^{S} \quad} & & \mathrm{Br}(X). \end{array}$$

Since $\mathcal{X} \times_{\mathbb{A}_k^r} S$, $\mathcal{X} \times_{\mathbb{A}_k^r} S'$ and X are regular, the restriction maps are all injective by Proposition 3.2.9 and hence

$$\mathrm{res}_{S'}^{S} \widetilde{e} = \widetilde{e}' \in \mathrm{Br}(\mathcal{X} \times_{\mathbb{A}_k^r} S'),$$

which implies the well-definedness of the specialization.

Chapter 4

Diagonal cubic surfaces

4.1 Generalities of cubic surfaces

In this section, let k be an algebraically closed field. We review some facts about cubic surfaces. We mainly refer to [Har77], Chapter V and [Man86], Chapter IV.

For a smooth projective surface V, we have the intersection product:

$$\mathrm{Div}(V) \times \mathrm{Div}(V) \to \mathbb{Z}; \quad (C, D) \mapsto C.D$$

We call $C.D$ the intersection number of C and D. For details, see [Har77], Section V.1.

We denote by K_V a canonical divisor of V. We look at basic properties of smooth cubic surfaces:

Proposition 4.1.1 ([Man86], Theorem 24.1). *Let V be a smooth projective cubic surface over k in $\mathbb{P}^3$. Then:*

(1) *V is rational, that is, birational to $\mathbb{P}^2$.*

(2) *The anticanonical divisor $-K_V$ is ample. Moreover, we have $\mathcal{O}_V(-K_V) \cong \mathcal{O}_V(1)$, under a natural embedding $V \hookrightarrow \mathbb{P}^3$.*

Definition 4.1.2. (1) A del Pezzo surface V over k is a smooth projective surface with ample anticanonical divisor.

(2) the degree d of a del Pezzo surface V is the self-intersection number $K_V.K_V$

Remark 4.1.3. (1) In particular, smooth cubic surfaces are del Pezzo surfaces of degree 3.

(2) In general, varieties with ample anticanonical divisor are called *Fano varieties*.

The fundamental result in this section is the following:

Theorem 4.1.4 ([Man86], Theorem 24.4). *Let V be a del Pezzo surface of degree 3. Then V is isomorphic to a blow-up of $\mathbb{P}^2$ at six points $P_1, \ldots, P_6$ such that:*

(1) *There is no line in* $\mathbb{P}^2$ *passing through three of* $P_1, \ldots, P_6$*;*

(2) *There is no conic in* $\mathbb{P}^2$ *passing through all* $P_1, \ldots, P_6$.

Conversely, any surface obtained by blowing-up such six points are a del Pezzo surface of degree 3.

We fix our notation. Let V be a smooth cubic surface. Let $P_1, \ldots, P_6$ be six points in $\mathbb{P}^2$ such that the blow-up of $\mathbb{P}^2$ at these points is isomorphic to V, and $\pi\colon V \longrightarrow \mathbb{P}^2$ be the blow-down. Let $E_1, \ldots, E_6$ be the exceptional curves corresponding to $P_1, \ldots, P_6$ respectively. We denote by $[E_i]$ the class of E_i in $\mathrm{Pic}(V)$, $i = 1, \ldots, 6$.
As consequences of Theorem 4.1.4, we have the following:

Proposition 4.1.5 ([Har77], Chapter V, Proposition 4.8). *Let* V *be the cubic surface as above and* l *the class of the inverse image of a line in* $\mathbb{P}^2$. *Then*:

(1) $\mathrm{Pic}(V) \cong \mathbb{Z}^7$*, generated by* $[E_1], \ldots, [E_6], l$.

(2) *The intersection matrix with respect to bases in* (1) *is*

$$\mathrm{diag}(-1, -1, -1, -1, -1, -1, 1),$$

where $\mathrm{diag}(a_1, \ldots, a_n)$ *denotes the diagonal matrix with entries* $a_1, \ldots, a_n$.

(3) *The hyperplane section is* $3l - \sum_{i=1}^{6}[E_i]$.

Proposition 4.1.6 ([Har77], Chapter V, Theorem 4.9). *Let* V *be the cubic surface as above. There are exactly 27 lines on* V. *They are*:

(i) *The exceptional curves* E_i *(six of these);*

(ii) *The strict transform* F_{ij} *of the line in* $\mathbb{P}^2$ *passing through* P_i *and* P_j *(fifteen of these);*

(iii) *The strict transform* G_i *of the conic in* $\mathbb{P}^2$ *passing through five* $P_j, j \neq i$ *(six of these).*

Proposition 4.1.7 ([Har77], Chapter V, Proposition 4.10.). *Let* V *be the cubic surfaces as above. Let* $E'_1, \ldots, E'_6$ *be any subset of six mutually skew lines in 27 lines on* V. *Then there exists a morphism* $\pi'\colon V \longrightarrow \mathbb{P}^2$ *such that*

- V *is isomorphic to the blow-up of* $\mathbb{P}^2$ *with six point* $P'_1, \ldots, P'_6$ *(no 3 colinear and not all 6 on a conic),*
- $E'_1, \ldots, E'_6$ *are the exceptional curves for* π'.

In the sequel of this Chapter, we are concerned with diagonal cubic surfaces, in particular, their Picard groups and their Galois structures. We fix our notation. Let k be a field satisfying $(*)$ (for definition, see Section 2.1). Let V be the smooth projective surface over k defined by a homogeneous equation

$$ax^3 + by^3 + cz^3 + dt^3 = 0,$$

where a, b, c and d are in k^*. Let $\pi\colon V \to \operatorname{Spec} k$ denote the structure morphism. Now we put

$$\lambda = \frac{b}{a}, \quad \mu = \frac{c}{a} \quad \text{and} \quad \nu = \frac{ad}{bc},$$

and then we can write as the equation of V

$$x^3 + \lambda y^3 + \mu z^3 + \lambda\mu\nu t^3 = 0.$$

4.2 Group structure of Brauer groups

We first prove the following:

Proposition 4.2.1. *We have* $\operatorname{Br}(\overline{V}) = 0$.

By the geometrical rationality of V and the birational invariance of the Brauer group of proper schemes of dimension two ([Gro68c], Corollaire 7.5), it suffices to show $\operatorname{Br}(\overline{\mathbb{P}^2}) = 0$. The triviality of the Brauer group of projective spaces is a well known result, but we cannot find its published proof as far as we know. So we include a proof of this fact. The proof given here is taught by T. Yamazaki [Yam12]. We now prove the $\mathbb{A}^1$-invariance of Brauer groups:

Proposition 4.2.2. *Let X be a regular integral variety over a field K with* $\operatorname{ch}(K) = 0$. *Then we have*

$$\pi^*\colon \operatorname{Br}(X) \cong \operatorname{Br}(X \times \mathbb{A}^1),$$

where $\pi\colon X \times \mathbb{A}^1 \to X$ is the first projection.

Proof. We first consider the case $X = \operatorname{Spec} K$.

Lemma 4.2.3. *Let $\pi\colon \mathbb{A}^1_K \to \operatorname{Spec} K$ be the structure morphism. Then we have:*

$$R^q\pi_* \mathbb{G}_m = \begin{cases} \mathbb{G}_m & q = 0, \\ 0 & q = 1, 2. \end{cases}$$

Proof. For a finite separable extension L/K, we have a functorial isomorphism

$$\pi_* \mathbb{G}_m(\operatorname{Spec} L) = \mathbb{G}_m(\mathbb{A}^1_L) = L^* = \mathbb{G}_m(\operatorname{Spec} L),$$

which proves the case $q = 0$.

For $q \geq 1$, we have

$$(R^q\pi_* \mathbb{G}_m)_{\overline{x}} = H^q(\mathbb{A}^1_{\overline{K}}, \mathbb{G}_m).$$

The case $q = 1$ follows from the fact $\operatorname{Pic}(\mathbb{A}^1_{\overline{K}}) = 0$. Moreover the case $q = 2$ follows from Proposition 3.2.9 and the fact that the Brauer group of a C_1-field is trivial. □

The claim $\mathrm{Br}(K) \cong \mathrm{Br}(\mathbb{A}^1_K)$ follows from this lemma and the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(K, R^q\pi_* \mathbb{G}_m) \Rightarrow H^{p+q}(\mathbb{A}^1_K, \mathbb{G}_m).$$

We next deal with general cases. Put

$$C(X) := \mathrm{Coker}(\pi\colon \mathrm{Br}(X) \longrightarrow \mathrm{Br}(X \times \mathbb{A}^1)).$$

By the existence of the zero-section $s_{0,X}\colon X \longrightarrow X \times \mathbb{A}^1$ of π, We have the direct decomposition with respect to $s_{0,X}$:

$$\mathrm{Br}(X \times \mathbb{A}^1) = \mathrm{Br}(X) \oplus C(X).$$

This decomposition is compatible with a natural inclusion $\mathrm{Spec}\, k(X) \times \mathbb{A}^1 \hookrightarrow X \times \mathbb{A}^1$ and therefore we have an injection $C(X) \hookrightarrow C(k(X))$. On the other hand, we have already proved $C(k(X)) = 0$. Hence $C(X) = 0$ and we obtain the desired isomorphism

$$\mathrm{Br}(X) \cong \mathrm{Br}(X \times \mathbb{A}^1).$$

□

Proof of Proposition 4.2.1. Applying the result of Proposition 4.2.2 to $\mathbb{A}^2_{\overline{k}} \longrightarrow \mathbb{A}^1_{\overline{k}} \longrightarrow \mathrm{Spec}\,\overline{k}$, we have $\mathrm{Br}(\mathbb{A}^2_{\overline{k}}) \cong \mathrm{Br}(\overline{k}) = 0$. Moreover, by Proposition 3.2.9, we have $\mathrm{Br}(\mathbb{P}^2_{\overline{k}}) \hookrightarrow \mathrm{Br}(\mathbb{A}^2_{\overline{k}})$. These implies the statement of Proposition 4.2.1. □

Hence we rewrite the exact sequence in Section 3.2.3 as follows:

$$0 \longrightarrow \mathrm{Br}(V)/\mathrm{Br}(k) \longrightarrow H^1(k, \mathrm{Pic}(\overline{V})) \stackrel{d^{1,1}}{\longrightarrow} H^3(k, \overline{k}^*) \quad \text{(exact)}. \tag{4.1}$$

This sequence plays a fundamental role in the sequel of this dissertation.

The structure of the group $H^1(k, \mathrm{Pic}(\overline{V}))$ is well-known:

Proposition 4.2.4 ([CTKS87], Proposition 1.).

$$H^1(k, \mathrm{Pic}(\overline{V})) \cong \begin{cases} 0 & \textit{if one of } \nu, \nu/\lambda, \nu/\mu \textit{ is a cube in } k^*, \\ (\mathbb{Z}/3\mathbb{Z})^2 & \textit{if exactly three of } \lambda, \mu, \lambda/\mu, \lambda\mu\nu, \lambda\nu, \mu\nu \\ & \textit{are cubes in } k^*, \\ \mathbb{Z}/3\mathbb{Z} & \textit{otherwise.} \end{cases}$$

We do not deal with a complete proof of this proposition, but some tools used in this proof are introduced in the sequel of this chapter.

At the end of the section, We define some extensions of k which are often used. Let α, α' and γ be solutions in $\overline{k}$ of equations $X^3 - \lambda = 0$, $X^3 - \mu = 0$ and $X^3 - \nu = 0$ respectively. Put $\beta = \alpha\gamma$ and $\beta' = \alpha'\gamma$. We define a field k' and k'' as $k(\alpha, \gamma)$ and $k'(\alpha')$.

4.3 Twenty seven lines and Picard groups

There are twenty seven lines on the surface $V_{k''}$:

$$\begin{aligned}
L(i)&: x+\zeta^i\alpha y = z+\zeta^i\beta t = 0,\\
L'(i)&: x+\zeta^i\alpha y = z+\zeta^{i+1}\beta t = 0,\\
L''(i)&: x+\zeta^i\alpha y = z+\zeta^{i+2}\beta t = 0,\\
M(i)&: x+\zeta^i\alpha' z = y+\zeta^{i+1}\beta' t = 0,\\
M'(i)&: x+\zeta^i\alpha' z = y+\zeta^{i+2}\beta' t = 0,\\
M''(i)&: x+\zeta^i\alpha' z = y+\zeta^{i}\beta' t = 0,\\
N(i)&: x+\zeta^i\alpha\beta' t = y+\zeta^{i+2}\alpha^{-1}\alpha' z = 0,\\
N'(i)&: x+\zeta^i\alpha\beta' t = y+\zeta^{i}\alpha^{-1}\alpha' z = 0,\\
N''(i)&: x+\zeta^i\alpha\beta' t = y+\zeta^{i+1}\alpha^{-1}\alpha' z = 0,
\end{aligned} \tag{4.2}$$

where i is either $0, 1$ or 2. Since six lines $L(0)$, $L(1)$, $L(2)$, $M(0)$, $M(1)$ and $M(2)$ are mutually skew, we can get a $\overline{k}$-morphism $\pi\colon \overline{V} \longrightarrow \overline{\mathbb{P}^2}$ by blowing down these six lines by Proposition 4.1.7. We define $l \in \mathrm{Pic}(\overline{V})$ as the inverse image of a line in $\overline{\mathbb{P}^2}$, which generates $\mathrm{Pic}(\overline{\mathbb{P}^2}) \cong \mathbb{Z}$. Then we can obtain generators of $\mathrm{Pic}(\overline{V}) \cong \mathbb{Z}^7$:

$$[L(0)],\quad [L(1)],\quad [L(2)],\quad [M(0)],\quad [M(1)],\quad [M(2)], \text{ and } l, \tag{4.3}$$

where $[D]$ denotes the class of $D \in \mathrm{Div}(\overline{V})$ in $\mathrm{Pic}(\overline{V})$. Let H be the hyperplane section of $\overline{V}$ defined by the equation $x = 0$,

$$[L] = [L(0)] + [L(1)] + [L(2)], \text{ and } [M] = [M(0)] + [M(1)] + [M(2)].$$

Note that we have the following relation by Proposition 4.1.5:

$$[H] = 3l - [L] - [M]. \tag{4.4}$$

We prepare the following lemma:

Lemma 4.3.1. *Let X be a variety over K and L/K be any Galois extension. Assume $X(K) \neq \emptyset$. Then we have*

$$\mathrm{Pic}(X_K) \cong \mathrm{Pic}(X_L)^{L/K}.$$

Proof. By the Hochschild-Serre spectral sequence

$$H^p(L/K, H^q(X_L, \mathbb{G}_m)) \Rightarrow H^{p+q}(X_K, \mathbb{G}_m),$$

we get the following exact sequence

$$0 \longrightarrow H^1(L/K, L^*) \longrightarrow \mathrm{Pic}(X_K) \longrightarrow \mathrm{Pic}(X_L)^{L/K} \longrightarrow \mathrm{Br}(K) \longrightarrow \mathrm{Br}(X).$$

Now we have $H^1(L/K, L^*) = 0$ by Hilbert's Theorem 90, and the map $\mathrm{Br}(K) \longrightarrow \mathrm{Br}(X)$ is injective by the assumption $X(K) \neq \emptyset$. The desired isomorphism is an easy consequence of these facts. □

Applying this lemma to $V_{k''}$ and $\bar{k}/k''$, we have

$$\mathrm{Pic}(V_{k''}) \cong \mathrm{Pic}(\overline{V})^{G_{k''}}.$$

The class $[L(i)]$ and $[M(i)]$ ($i = 0,1,2$) are $G_{k''}$-invariant. Moreover this fact, the torsion-freeness of $\mathrm{Pic}(\overline{V})$ and the relation (4.4) yield that l is also $G_{k''}$-invariant. Therefore we have $\mathrm{Pic}(V_{k''}) \cong \mathbb{Z}^7$. As its generators, we take the classes corresponding to $[L(i)]$, $[M(i)]$ and $l \in \mathrm{Pic}(\overline{V})$. By abuse of notation, we use the same symbols as in the case of $\mathrm{Pic}(\overline{V})$.

4.4 Galois cohomology of Picard groups

Here, we analyse the group $H^1(k, \mathrm{Pic}(\overline{V}))$.

4.4.1 Reduction to finite extension

First, we represent this group as a cohomology of finite groups. We have the inflation-restriction exact sequence:

$$0 \to H^1(k'/k, \mathrm{Pic}(\overline{V})^{\bar{k}/k'}) \to H^1(k, \mathrm{Pic}(\overline{V})) \to H^1(k', \mathrm{Pic}(\overline{V}))^{k'/k}.$$

Here we have the following

Proposition 4.4.1.

$$H^1(k'/k, (\mathrm{Pic}(V_{k'}))) \cong H^1(k, \mathrm{Pic}(\overline{V})).$$

Proof. To show the statement, it suffices to prove the following two claims:

(1) $H^1(k', \mathrm{Pic}(\overline{V})) = 0$

(2) $\mathrm{Pic}(V_{k'}) \cong \mathrm{Pic}(\overline{V})^{\bar{k}/k'}$

First we show the claim (1). We have another inflation-restriction exact sequence

$$0 \to H^1(k''/k', \mathrm{Pic}(\overline{V})^{\bar{k}/k''}) \to H^1(k', \mathrm{Pic}(\overline{V})) \to H^1(k'', \mathrm{Pic}(\overline{V}))^{k''/k'}.$$

As we have seen in Section 4.3, $\mathrm{Pic}(\overline{V})$ is a trivial $G_{k''}$-module. Hence we have

$$H^1(k'', \mathrm{Pic}(\overline{V})) = \mathrm{Hom}(G_{k''}, \mathbb{Z}^7) = 0.$$

Moreover, since the action of $\mathrm{Gal}(k''/k')$ on $\mathrm{Pic}(V_{k''}) \cong \mathbb{Z}^7$ maps to one basis to another,

$$H^1(k''/k', \mathrm{Pic}(V_{k''})) = 0$$

follows from Lemma 2.2.5. This completes the proof of (1).

Next we prove the claim (2). Since $V_{k'}$ has a k'-rational point, for example, $(x : y : z : t) = (-\alpha : 1 : -\alpha\gamma : 1)$, this claim is a consequence of Lemma 4.3.1.

This completes the proof of Proposition 4.4.1. □

4.4.2 Representations of H^1 by using divisors

In this subsection, we describe the group $H^1(k'/k(\alpha), \mathrm{Pic}(V_{k'}))$ by using divisors on V. In Chapter 5, we consider cubic surfaces over k defined by $x^3 + y^3 + cz^3 + dt^3 = 0$. If one of cd and d/c is in $(k^*)^3$, we have $H^1(k, \mathrm{Pic}(\overline{V})) = 0$ and $\mathrm{Br}(V)/\mathrm{Br}(k) = 0$ by Theorem 4.2.4. Therefore we need not consider this case. If neither cd nor d/c is in $(k^*)^3$ and c is in $(k^*)^3$, such surfaces are isomorphic to surfaces defined by $x^3 + y^3 + z^3 + dt^3 = 0$ and Manin has already found the structure and the generators of their Brauer groups. Hence we may assume c, d, cd and $d/c \notin (k^*)^3$.

In Chapter 6, we consider surfaces $x^3 + by^3 + cz^3 + dt^3 = 0$ over $k(b, c, d)$, where b, c and d are indeterminates. Hence $k(\alpha, \gamma, \alpha')/k$ is an extension of degree 27.

Therefore in this subsection, we always assume one of the following conditions:

(1) $\lambda = 1$, and neither $\mu, \nu, \mu\nu$ nor ν/μ is cubic in k^*;

(2) $k(\alpha, \gamma, \alpha')$ is a field extension of k with degree 27.

Under the assumption (1), we have $k(\alpha) = k$ and hence

$$H^1(k'/k(\alpha), \mathrm{Pic}(V_{k'})) = H^1(k'/k, \mathrm{Pic}(V_{k'})),$$

which is isomorphic to $H^1(k, \mathrm{Pic}(\overline{V}))$ by Proposition 4.4.1. Under the assumption (2), these two groups $H^1(k'/k(\alpha), \mathrm{Pic}(V_{k'}))$ and $H^1(k'/k, \mathrm{Pic}(V_{k'}))$ are apparently different, but we will later see in Section 6.1 that these are isomorphic to each other.

Let s be the generator of $G = \mathrm{Gal}(k'/k(\alpha))$ such that $s\gamma = \zeta\gamma$ and w the generator of $\mathrm{Gal}(k''/k')$ such that $w\alpha' = \zeta\alpha'$. Note that G and $\mathrm{Gal}(k''/k')$ are isomorphic to $\mathbb{Z}/3\mathbb{Z}$ under one of the above assumptions and such elements s and w *do* exist.

By Lemma 4.3.1, we have $\mathrm{Pic}(V_{k'}) \cong \mathrm{Pic}(V_{k''})^{k''/k'}$; moreover using the explicit defining equations of divisors (4.2) and the equation (4.4) in Section 4.3, we see that

$$\mathrm{Pic}(V_{k'}) = \mathbb{Z}l \oplus \mathbb{Z}[L(0)] \oplus \mathbb{Z}[L(1)] \oplus \mathbb{Z}[L(2)] \oplus \mathbb{Z}[M].$$

Let $\mathcal{D}$ be the following free abelian group of rank 10:

$$\mathcal{D} = \mathbb{Z}H \oplus \bigoplus_{i=0}^{2} \mathbb{Z}L(i) \oplus \bigoplus_{i=0}^{2} \mathbb{Z}L'(i) \oplus \bigoplus_{i=0}^{2} \mathbb{Z}L''(i).$$

We see that $\mathcal{D}$ is a G-submodule of $\mathrm{Div}(V_{k'})$ by using (4.2). Let $\mathcal{D}_0$ be the G-submodule generated by the following five divisors:

$$\begin{aligned} D_1 &= \mathrm{div}(f_1) = (L(0) + L'(0) + L''(0)) - H, \\ D_2 &= \mathrm{div}(f_2) = (L(1) + L'(1) + L''(1)) - H, \\ D_3 &= \mathrm{div}(f_3) = (L(0) + L'(2) + L''(1)) - H, \\ D_4 &= \mathrm{div}(f_4) = (L(1) + L'(0) + L''(2)) - H, \\ D_5 &= \mathrm{div}(f_5) = (L(2) + L'(1) + L''(0)) - H, \end{aligned} \tag{4.5}$$

where

$$f_1 = \frac{x + \alpha y}{x},\ f_2 = \frac{x + \zeta\alpha y}{x},\ f_3 = \frac{z + \beta t}{x},\ f_4 = \frac{z + \zeta\beta t}{x},\ f_5 = \frac{z + \zeta^2\beta t}{x}.$$

Lemma 4.4.2. *Let $\mathcal{D}$ and $\mathcal{D}_0$ be as above. Then we have the following exact sequence of G-modules:*

$$0 \to \mathcal{D}_0 \to \mathcal{D} \to \mathrm{Pic}(V_{k'}) \to 0.$$

Proof. For the exactness at $\mathrm{Pic}(V_{k'})$, it suffices to show that we can write the classes l and $[M]$ as linear combinations of $[H]$, $[L(i)]$, $[L'(i)]$ and $[L''(i)]$. By Proposition 4.1.5, the intersection matrix with respect to the basis in (4.3) is:

$$\mathrm{diag}(-1,-1,-1,-1,-1,-1,1).$$

By using this matrix, (4.2) and (4.4), we can write $[L'(0)]$ and $[L''(0)]$ as follows:

$$[L'(0)] = 2l - [L(0)] - [L(1)] - [M], \quad [L''(0)] = l - [L(0)] - [L(2)],$$

which implies the surjectivity of $\mathcal{D} \to \mathrm{Pic}(V_{k''})$.

By definition, the exactness at $\mathcal{D}_0$ is trivial.

Finally we consider the exactness at $\mathcal{D}$. By the definition of Picard group, this is a complex. Moreover, comparing the rank of $\mathcal{D}_0$ with that of $\mathrm{Ker}(\mathcal{D} \to \mathrm{Pic}(V_{k'}))$, we know that the sequence is exact at $\mathcal{D}$. This completes the proof of this lemma. □

By this lemma, we have the following long exact sequence:

$$H^1(G,\mathcal{D}) \to H^1(G,\mathrm{Pic}(V_{k'})) \to H^2(G,\mathcal{D}_0) \to H^2(G,\mathcal{D}).$$

Applying Lemma 2.2.5 to the G-module $\mathcal{D}$, we have $H^1(G,\mathcal{D}) = 0$ and hence

$$\begin{aligned} H^1(G,\mathrm{Pic}(V_{k'})) &\cong H^1(G,\mathcal{D}\,/\,\mathcal{D}_0) \\ &\cong \mathrm{Ker}(H^2(G,\mathcal{D}_0) \to H^2(G,\mathcal{D})) \\ &\cong \frac{\mathcal{D}_0 \cap N_G\,\mathcal{D}}{N_G\,\mathcal{D}_0}, \end{aligned}$$

where the last isomorphism is a consequence of Proposition 2.2.3. Now we compute the last group.

Lemma 4.4.3. (1)

$$N_G\,\mathcal{D} = \mathbb{Z}\cdot 3H \oplus \mathbb{Z}\,N_G L(0) \oplus \mathbb{Z}\,N_G L(1) \oplus \mathbb{Z}\,N_G L(2).$$

(2)

$$\begin{aligned} \mathcal{D}_0 \cap N_G\,\mathcal{D} = &\,\mathbb{Z}\,N_G(L(1)-L(0)) \oplus \mathbb{Z}\,N_G(L(1)-L(2)) \\ &\oplus \mathbb{Z}\cdot 3(N_G L(1) - H). \end{aligned}$$

(3)

$$\frac{\mathcal{D}_0 \cap N_G\,\mathcal{D}}{N_G\,\mathcal{D}_0} \cong \mathbb{Z}\,/3\,\mathbb{Z},$$

and we can take the image of $N_G(L(1)-L(0))$ as a generator of this group.

Proof. The action of s on a divisor $D \in \mathrm{Pic}(V_{k'})$ is the following:

D	H	$L(i)$	$L'(i)$	$L''(i)$	D_1	D_2	D_3	D_4	D_5
sD	H	$L'(i)$	$L''(i)$	$L(i)$	D_1	D_2	D_4	D_5	D_3

(1) This follows immediately from the above table.
(2) Since

$$\begin{aligned} N_G(L(1)-L(0)) &= D_2 - D_1, \\ N_G(L(1)-L(2)) &= D_1 + 2D_2 - D_3 - D_4 - D_5, \\ 3(N_G(L(1)) - H) &= 3D_2 = N_G(D_2), \end{aligned}$$

we know that the right-hand side include in the left-hand side. We also see these three elements are $\mathbb{Z}$-basis of the right-hand side.

We assume that

$$D = \sum_{i=1}^{5} a_i D_i \in N_G \mathcal{D}.$$

Since D is expanded as follows:

$$\begin{aligned} D = &-\left(\sum_{i=1}^{5} a_i\right) H \\ &+ (a_1+a_3)L(0) + (a_1+a_4)L'(0) + (a_1+a_5)L''(0) \\ &+ (a_2+a_4)L(1) + (a_2+a_5)L'(1) + (a_2+a_3)L''(1) \\ &+ a_5 L(2) + a_3 L'(2) + a_4 L''(2), \end{aligned}$$

we have

$$\sum_{i=1}^{5} a_i = 3N \text{ for some } N \in \mathbb{Z}, \quad a_3 = a_4 = a_5$$

by (1). Then we can rewrite D as the following form:

$$\begin{aligned} D =& a_1 D_1 + (3(N-a_3) - a_1)D_2 + a_3(D_3 + D_4 + D_5) \\ =& -(a_1 + a_3) N_G(L(1) - L(0)) \\ & - a_3 N_G(L(1) - L(2)) + N \cdot 3(N_G L(1) - H), \end{aligned}$$

which says the left-hand side also include the right-hand side.
(3) The following equations

$$\begin{aligned} N_G D_1 &= -3(N_G(L(1) - L(0))) + 3(N_G L(1) - H), \\ N_G D_2 &= 3(N_G L(1) - H), \\ N_G D_3 &= N_G D_4 = N_G D_5 \\ &= -(N_G(L(1) - L(0))) - (N_G(L(1) - L(2))) + 3(N_G L(1) - H), \end{aligned}$$

yield

$$\frac{\mathcal{D}_0 \cap N_G \mathcal{D}}{N_G \mathcal{D}_0} \cong \mathbb{Z}/3\mathbb{Z} \cdot N_G(L(1) - L(0)).$$

□

Next we compute the 1-cocycle in $Z^1(G, \mathrm{Pic}(V_{k'}))$ corresponding to $N_G(L(1) - L(0))$ by using the the following commutative diagram:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(G,\mathcal{D}\,/\,\mathcal{D}_0) & \xleftarrow{\cong} & \hat{H}^{-1}(G,\mathcal{D}\,/\,\mathcal{D}_0) & \xleftarrow{\cong} & \dfrac{N_G\mathcal{D}\,/\,\mathcal{D}_0}{I_G(\mathcal{D}\,/\,\mathcal{D}_0)} & & \\
\downarrow & & \downarrow & & \downarrow & \searrow{\scriptstyle\cong} & \\
H^2(G,\mathcal{D}_0) & \xleftarrow{\cong} & \hat{H}^0(G,\mathcal{D}_0) & \xleftarrow{\cong} & \dfrac{\mathcal{D}_0^G}{N_G\mathcal{D}_0} & \xleftarrow{\supset} & \dfrac{\mathcal{D}_0\cap N_G\mathcal{D}}{N_G\mathcal{D}_0} \\
\downarrow & & \downarrow & & \downarrow & & \\
H^2(G,\mathcal{D}) & \xleftarrow{\cong} & \hat{H}^0(G,\mathcal{D}) & \xleftarrow{\cong} & \mathcal{D}^G\,/N_G\,\mathcal{D}, & &
\end{array}$$

where vertical lines are exact. By Proposition 2.2.4, the element in $\dfrac{N_G(\mathcal{D}\,/\,\mathcal{D}_0)}{I_G(\mathcal{D}\,/\,\mathcal{D}_0)}$ corresponding to $N_G(L(1) - L(0)) \in \dfrac{\mathcal{D}_0 \cap N_G\,\mathcal{D}}{N_G\,\mathcal{D}_0}$ is $[L(1) - L(0)]$. By Proposition 2.2.2, the element of $H^1(G, \mathrm{Pic}(V_{k'}))) \cong H^1(G, \mathcal{D}\,/\,\mathcal{D}_0)$ corresponding to the above element is the cocycle $\Phi_{[L(1)-L(0)]}$ explicitly written as:

$$\begin{aligned}
\Phi_{[L(1)-L(0)]}(1) &= 0, \\
\Phi_{[L(1)-L(0)]}(s) &= s[L(1) - L(0)] \\
&= [L'(1)] - [L'(0)] \\
&= [\mathrm{div}(f_5/f_1)] - [L(2)] + [L(0)] \\
&= [L(0)] - [L(2)], \\
\Phi_{[L(1)-L(0)]}(s^2) &= s[L(1) - L(0)] + s^2[L(1) - L(0)] \\
&= [L(0)] - [L(2)] + [L''(1)] - [L''(0)] \\
&= [L(0)] - [L(2)] + [\mathrm{div}(f_2/f_5)] - [L(1)] + [L(2)] \\
&= [L(0)] - [L(1)].
\end{aligned}$$

Therefore we get the following

Proposition 4.4.4. *we have an isomorphism*

$$H^1(k'/k(\alpha), \mathrm{Pic}(V_{k'})) \cong \mathbb{Z}\,/3\,\mathbb{Z},$$

and as a generator of the left hand side, we can take the class of the following cocycle:

$$1 \mapsto 0, \quad s \mapsto [L(0)] - [L(2)], \quad s^2 \mapsto [L(0)] - [L(1)].$$

4.5 An explicit description of $d^{1,1}$

In Section 6.1, we shall show the nontriviality of images of elements in $H^1(k, \mathrm{Pic}(\overline{V}))$ under the differential

$$d_{\overline{k}}^{1,1} : H^1(k, \mathrm{Pic}(\overline{V})) \to H^3(k, \overline{k}^*)$$

appearing in (4.1). In Subsection 4.4.1, we see that $H^1(k, \mathrm{Pic}(\overline{V}))$ is described by a cohomology of a finite group, and we have the following commutative diagram:

$$\begin{array}{ccc} H^1(k'/k, \mathrm{Pic}(V_{k'})) & \xrightarrow{d_{k'}^{1,1}} & H^3(k'/k, k'^*) \\ \cong \downarrow & & \downarrow i_{\overline{k}}^{k'} \\ H^1(k, \mathrm{Pic}(\overline{V})) & \xrightarrow{d_{\overline{k}}^{1,1}} & H^3(k, \overline{k}^*), \end{array}$$

which enables us to reduce the computation of $d_{\overline{k}}^{1,1}$ to that of the composite of $d_{k'}^{1,1}$ and the inflation $i_{\overline{k}}^{k'}$. We note that $i_{\overline{k}}^{k'}$ is not necessarily injective.

We now give one way of computing $d_{k'}^{1,1}$ explicitly. Let $\mathcal{D} \subset \mathrm{Div}(V_{k'})$ be a $\mathrm{Gal}(k'/k)$-submodule which generates $\mathrm{Pic}(V_{k'})$, and $\mathcal{D}_0$ the kernel of $\mathcal{D} \to \mathrm{Pic}(V_{k'})$. Then we have the following two exact sequence of $\mathrm{Gal}(k'/k)$-modules:

$$0 \to \mathcal{D}_0 \to \mathcal{D} \to \mathrm{Pic}(V_{k'}) \to 0,$$
$$0 \to k'^* \to \mathrm{div}^{-1}(\mathcal{D}_0) \to \mathcal{D}_0 \to 0.$$

Using these sequences, we obtain the following diagram:

$$\begin{array}{ccc} H^1(k'/k, \mathrm{Pic}(V_{k'})) & \xrightarrow{\partial} & H^2(k'/k, \mathcal{D}_0) \\ & \searrow^{d_{k'}^{1,1}} & \downarrow \delta \\ & & H^3(k'/k, k'^*), \end{array}$$

where ∂ and δ are connecting homomorphisms induced by the above exact sequences. For this diagram, we have:

Proposition 4.5.1. *The composite $\delta \circ \partial$ is equal to $d_{k'}^{1,1}$.*

Proof. See [KT08], Proposition 6.1, (i). □

Chapter 5

The case $x^3 + y^3 + cz^3 + dt^3 = 0$

5.1 A representability result

Let k be a field satisfying $(*)$. Let V be the cubic surface over k defined by a homogeneous equation $x^3 + y^3 + cz^3 + dt^3 = 0$, where c and $d \in k^*$. Moreover, we may assume the condition (1) in Subsection 4.4.2, that is, c, d, cd and d/c are not in $(k^*)^3$. Note that in this case $\lambda = \alpha^3 = 1$, $\nu = \gamma^3 = d/c$ and hence $k' = k(\alpha, \gamma) = k(\gamma)$ is a field extension of k with degree 3.

Our result in this section is:

Theorem 5.1.1. *Let k be a field satisfying $(*)$ and V the cubic surface over k defined by an equation $x^3 + y^3 + cz^3 + dt^3 = 0$, where c and $d \in k^*$. Assume that c, d, cd and d/c are not contained in $(k^*)^3$. Then we have the following:*

(1) *The group* $\mathrm{Br}(V)/\mathrm{Br}(k)$ *is isomorphic to* $\mathbb{Z}/3\mathbb{Z}$;

(2) *the element*
$$e_1 = \left\{ \frac{d}{c}, \frac{x+\zeta y}{x+y} \right\}_3 \in \mathrm{Br}(k(V))$$
is contained in $\mathrm{Br}(V)$;

(3) *the image of e_1 in* $\mathrm{Br}(V)/\mathrm{Br}(k)$ *is a generator of this group.*

Proof. Let $G = \mathrm{Gal}(k'/k)$ and $s \in G$ a generator such that $s\gamma = \zeta\gamma$. We start from the exact sequence (4.1):

$$0 \to \mathrm{Br}(V)/\mathrm{Br}(k) \to H^1(k, \mathrm{Pic}(\overline{V})) \to H^3(k, \overline{k}^*).$$

By Proposition 4.4.1 and Proposition 4.4.4, we have

$$H^1(k, \mathrm{Pic}(\overline{V})) \cong H^1(G, \mathrm{Pic}(V_{k'})) \cong \mathbb{Z}/3\mathbb{Z}$$

and a generating cocycle ϕ of $H^1(G, \mathrm{Pic}(V_{k'}))$ can be taken as:

$$\phi(1) = 0, \quad \phi(s) = [L(0)] - [L(2)], \quad \phi(s^2) = [L(0)] - [L(1)].$$

First we consider (1). This surface has a k-rational point $P=(1:-1:0:0)$. Therefore the claim of (1) follows from Lemma 3.2.18.

Next we consider (2). Let ϕ be as above. Computing the cocycle $\partial'\phi$, where

$$\partial' \colon H^1(G,\mathrm{Pic}(V_{k'})) \to H^2(G,k'(V)^*/k'^*)$$

is induced by the exact sequence of G-modules:

$$0 \to k'(V)^*/k'^* \to \mathrm{Div}(V_{k'}) \to \mathrm{Pic}(V_{k'}) \to 0,$$

we can show

$$\partial'\phi(s^i,s^j) = \left(\frac{f_2}{f_1}\right)^{a(i,j)} \in k'(V)^*/k'^*,$$

where

$$a(i,j) = \left[\frac{i+j}{3}\right] - \left[\frac{i}{3}\right] - \left[\frac{j}{3}\right]$$

is the map appearing in Proposition 2.2.2.

On the other hand, the symbol $\{v, f_2/f_1\}_3 \in \mathrm{Br}(k(V))$ is equal to

$$(\chi_{3,v}, f_2/f_1) \in H^2(k(V), \overline{k(V)}^*),$$

where $\chi_{3,v} \in \mathrm{Hom}(G_{k(V)}, \mathbb{Q}/\mathbb{Z})$ is the cyclic character of order 3 associated to v (see Subsection 3.1.2) and $(\cdot,\cdot)$ is the symbol introduced in Definition 3.1.7. Moreover, by the following commutative diagram

$$\begin{array}{ccc}
H^2(k(V),\mathbb{Z}) \otimes H^0(k(V),\overline{k(V)}^*) & \xrightarrow{\ \cup\ } & H^2(k(V),\overline{k(V)}^*) \\
\uparrow & & \uparrow \\
H^2(k'(V)/k(V),\mathbb{Z}) \otimes H^0(k'(V)/k(V),k'(V)^*) & \xrightarrow{\ \cup\ } & H^2(k'(V)/k(V),k'(V)^*) \\
\cong\uparrow & & \cong\uparrow \\
H^2(G,\mathbb{Z}) \otimes H^0(G,k'(V)^*) & \xrightarrow{\ \cup\ } & H^2(G,k'(V)^*),
\end{array}$$

$(\chi_{3,v}, f_2/f_1)$ can be considered as an element in $H^2(G,k'(V)^*)$, and we see that the corresponding cocycle is of the form

$$\left\{v, \frac{f_2}{f_1}\right\}_3 (s^i,s^j) = \left(\frac{f_2}{f_1}\right)^{a(i,j)} \in k'(V)^*.$$

Finally, we have the following commutative diagram with all rows and columns exact:

$$\begin{array}{ccccccc}
 & & \mathrm{Br}(k'/k) & = & \mathrm{Br}(k'/k) & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Br}(V_{k'}/V) & \longrightarrow & H^2(G,k'(V)^*) & \xrightarrow{\mathrm{div}} & H^2(G,\mathrm{Div}(V_{k'})) \\
 & & \vdots & & \downarrow & & \| \\
0 & \longrightarrow & H^1(G,\mathrm{Pic}(V_{k'})) & \xrightarrow{\partial'} & H^2(G,k'(V)^*/k'^*) & \xrightarrow{\mathrm{div}} & H^2(G,\mathrm{Div}(V_{k'})),
\end{array}$$

where the middle column is induced by the exact sequence

$$0 \to k'^* \to k'(V)^* \to k'(V)^*/k'^* \to 0,$$

the middle row is the result of Proposition 3.2.19, the bottom row is induced by the exact sequence

$$0 \to k'(V)^*/k'^* \to \mathrm{Div}(V_{k'}) \to \mathrm{Pic}(V_{k'}) \to 0,$$

and the triviality of its leftmost term follows from Lemma 2.2.5. Then the map

$$\mathrm{Br}(V'_k/V) \to H^1(G, \mathrm{Pic}(V_{k'}))$$

is naturally induced by

$$H^2(G, k'(V)^*) \to H^2(G, k'(V)^*/k'^*).$$

The fact that $\partial'\phi$ and $\{\nu, f_2/f_1\}_3$ coincide in $H^2(G, k'(V)^*/k'^*)$ yields

$$\mathrm{div}(\{\nu, f_1/f_2\}_3) = \mathrm{div} \circ \partial'\phi = 0.$$

This means

$$\{\nu, f_2/f_1\}_3 = \left\{\frac{d}{c}, \frac{x+\zeta y}{x+y}\right\}_3 \in \mathrm{Br}(V_{k'}/V) \subset \mathrm{Br}(V),$$

which completes the proof of (2).

Finally we consider (3). By the above argument, $[\phi]$ and $\{\nu, f_2/f_1\}_3$ coincide in $H^1(G, \mathrm{Pic}(V_{k'}))$. Hence we can take

$$\left\{\frac{d}{c}, \frac{x+\zeta y}{x+y}\right\}_3$$

as a generator of the group $\mathrm{Br}(V)/\mathrm{Br}(k)$. This completes the proof of Theorem 5.1.1. □

Remark 5.1.2. By using the explicit description of $d^{1,1}$ appearing in Section 4.5, we can also prove (1) by computing the image of ϕ in $H^3(k, \overline{k}^*)$ explicitly. Let $\mathcal{D}$ and $\mathcal{D}_0$ be as in Subsection 4.4.2. By using the diagram in Proposition 4.5.1, we compute the cocycle $\delta\partial\phi$ in $Z^3(k'/k, k'^*)$. If we take

$$0, \quad L(i) \in \mathcal{D}$$

as lifts of classes 0, $[L(i)]$, we have the following representation of $\partial\phi$:

$$\begin{aligned}
&\partial\phi(1,1) = 0, \ \partial\phi(1,s) = 0, && \partial\phi(1,s^2) = 0,\\
&\partial\phi(s,1) = 0, \ \partial\phi(s,s) = \mathrm{div}\frac{z+\zeta\gamma t}{x+\zeta^2 y}, && \partial\phi(s,s^2) = \mathrm{div}\frac{x+y}{z+\zeta^2\gamma t},\\
&\partial\phi(s^2,1) = 0, \partial\phi(s^2,s) = \mathrm{div}\frac{x+y}{z+\zeta\gamma t}, && \partial\phi(s^2,s^2) = \mathrm{div}\frac{z+\zeta^2\gamma t}{x+\zeta y}.
\end{aligned}$$

Again, if we take

$$1,\quad \frac{z+\zeta\gamma t}{x+\zeta^2 y} \in \operatorname{div}^{-1}(\mathcal{D}_0)$$

as lifts of divisors 0, $\operatorname{div}(z+\zeta\gamma t)/(x+\zeta^2 y)$ and so forth, we have:

$$\begin{aligned}
&\delta\partial\phi(1,s^{i_2},s^{i_3})=1, && \delta\partial\phi(s^{i_1},1,s^{i_3})=1, && \delta\partial\phi(s^{i_1},s^{i_2},1)=1,\\
&\delta\partial\phi(s,s,s)=1, && \delta\partial\phi(s,s,s^2)=-\mu, &&\\
&\delta\partial\phi(s,s^2,s)=1, && \delta\partial\phi(s,s^2,s^2)=-\mu^{-1}, &&\\
&\delta\partial\phi(s^2,s,s)=-\mu^{-1}, && \delta\partial\phi(s^2,s,s^2)=1, &&\\
&\delta\partial\phi(s^2,s^2,s)=-\mu, && \delta\partial\phi(s^2,s^2,s^2)=1, &&
\end{aligned}$$

where indices i_1, i_2 and i_3 take on any values in $\{0,1,2\}$. Now if we define the 2-cochain ψ in $C^2(k'/k,k'^*)$ as follows:

$$\psi(s^{i_1},s^{i_2})=\begin{cases}-\mu^{-1} & \text{if } (i_1,i_2)=(1,1),\\ 1 & \text{otherwise.}\end{cases}$$

We can show that $d\psi=\delta\partial\phi$, which means that $[\delta\partial\phi]$ is zero in $H^3(k'/k,k'^*)$, especially in $H^3(k,\overline{k}^*)$. Therefore we have $\operatorname{Br}(V)/\operatorname{Br}(k)\cong\mathbb{Z}/3\mathbb{Z}$ and complete the proof of (1). A similar calculation appears in the next section.

By using the specialization of Brauer groups, we can formulate Theorem 5.1.1 as follows:

Corollary 5.1.3. *Let k be a field satisfying $(*)$, $\mathcal{O}_F=k[c,d]$, F its fractional field and*

$$V=\operatorname{Proj}(F[x,y,z,t]/(x^3+y^3+cz^3+dt^3)).$$

Then

$$e_1=\left\{\frac{d}{c},\frac{x+\zeta y}{x+y}\right\}_3\in\operatorname{Br}(V)$$

is a uniform generator, that is, for all $P=(c_0,d_0)\in k^*\times k^*$ such that c_0, d_0, c_0d_0 and $d_0/c_0\notin(k^*)^3$, the specialization $\operatorname{sp}(e_1;P)$ is a generator of $\operatorname{Br}(V_P)/\operatorname{Br}(k)$.

Proof. We confirm that $\operatorname{sp}(e_1;P)$ is in fact the desired symbol, that is,

$$\left\{\frac{d_0}{c_0},\frac{x+\zeta y}{x+y}\right\}_3.$$

We define $S\subset\mathbb{A}^2_k$ and $\mathcal{V}$ to be:

$$\begin{aligned}
S&=\mathbb{G}_{m,k}\times\mathbb{G}_{m,k}=\operatorname{Spec}k[c^{\pm},d^{\pm}],\\
\mathcal{V}&=\operatorname{Proj}\mathcal{O}_F[x,y,z,t]/(x^3+y^3+cz^3+dt^3).
\end{aligned}$$

where the symbol $c^{\pm}$ is the abbreviation of c and c^{-1}. Put $\mathcal{V}\times S:=\mathcal{V}\times_{\mathbb{A}^2_k}S$.

Lemma 5.1.4. *$\mathcal{V} \times S$ is smooth over S.*

Proof. It suffices to show the flatness of $\mathcal{V} \to \mathbb{A}_k^2$ and the smoothness of the geometric fiber of each point $P \in S$. First, we consider the flatness. $\mathcal{V}_t$ is the affine open subscheme of $\mathcal{V}$ defined by $t \neq 0$. Its global section is:

$$\mathcal{O}_F[x,y,z]/(x^3+y^3+cz^3+d).$$

This is a free $\mathcal{O}_F$-module. In fact, we have

$$\begin{aligned}&\mathcal{O}_F[x,y,z]/(x^3+y^3+cz^3+d)\\&=\mathcal{O}_F[y,z][x]/(x^3+(y^3+cz^3+d))\\&\cong \mathcal{O}_F[y,z]\oplus\mathcal{O}_F[y,z]x\oplus\mathcal{O}_F[y,z]x^2.\end{aligned}$$

This is a free $\mathcal{O}_F[y,z]$-module, in particular, a free $\mathcal{O}_F$-module. Hence $\mathcal{V}_t$ is flat over $\mathbb{A}_k^2$. Since we can prove the flatness of other affine open subschemes defined by $x \neq 0, y \neq 0$ and $z \neq 0$ over $\mathbb{A}_k^2$ in a similar way, we complete the proof of the flatness.

Moreover, noting that c and d is units in $k[c^{\pm}, d^{\pm}]$ and using the Jacobian criterion for regularity, we can easily prove the smoothness of $\mathcal{V}_{\overline{\kappa(P)}} \to \operatorname{Spec}\overline{\kappa(P)}$ for each $P \in S$. □

Lemma 5.1.5. *the element e_1 can lift to $\widetilde{e_1} \in \operatorname{Br}(\mathcal{V} \times S)$.*

Proof. We have the following exact sequence (see Theorem 3.2.8):

$$0 \to \operatorname{Br}(\mathcal{V}\times S) \to \operatorname{Br}(F(V)) \xrightarrow{\oplus_x \partial_x} \bigoplus H^1(\kappa(x), \mathbb{Q}/\mathbb{Z}),$$

where the sum is taken over all points x of codimension one in $\mathcal{V} \times S$, and $\kappa(x)$ is the residue field of x.

By [CTSD94], Section 1.1, the kernel of this residue map ∂_x is equal to that of the residue map $\operatorname{res}_{\mathcal{O}_{X,x}}$ defined in Subsection 3.1.3. Therefore we can compute the image of

$$e_1 = \left\{\frac{d}{c}, \frac{x+\zeta y}{x+y}\right\}_3$$

in each $H^1(\kappa(x), \mathbb{Q}/\mathbb{Z})$ explicitly by using Proposition 3.1.18.

In order to prove the lemma, it suffices to consider the following two equation:

$$x+y=0, \quad x+\zeta y = 0$$

since the divisors $\{c=0\}$ and $\{d=0\}$ in $\mathbb{A}_k^2$ are already removed by the definition of S. For $i=0,1$, the closed subscheme in $\mathcal{V} \times S$ defined by $x+\zeta^i y$ is

$$C_i := \operatorname{Proj} k[c^{\pm}, d^{\pm}][x,y,z,t]/(x+\zeta^i y, cz^3+dt^3).$$

C_i is integral and we denote its generic point by η_i. We now prove the residue of e_1 at η_i is zero. We have

$$\kappa(\eta_i) = \mathrm{Frac}(k[c^{\pm}, d^{\pm}][x,y,z]/(x+\zeta^i y, cz^3+d)).$$

The local ring

$$\mathcal{O}_{\mathcal{V},\eta_i} \cong (k[c^{\pm}, d^{\pm}][x,y,z]/(x^3+y^3+cz^3+d))_{(x+\zeta^i y, cz^3+d)}$$

is a discrete valuation ring with uniformizer $x+\zeta^i y$. By Proposition 3.1.18, we have:

$$\begin{aligned}\partial_{\eta_i}\left(\left\{\frac{d}{c}, \frac{x+\zeta y}{x+y}\right\}_3\right) &= (-1)^{0\cdot\varepsilon_i}\overline{\left(\frac{\left(\frac{d}{c}\right)^{-\varepsilon_i}}{\left(\frac{x+\zeta y}{x+y}\right)^0}\right)} \\ &= \left(\frac{d}{c}\right)^{-\varepsilon_i} \\ &= (-z)^{-3\varepsilon_i} \\ &= 1 \in \kappa(\eta_i)^*/(\kappa(\eta_i)^*)^3,\end{aligned}$$

where

$$\varepsilon_i = \begin{cases} -1 & i=0, \\ 1 & i=1. \end{cases}$$

Hence the residue of e_1 along C_i is zero. This completes the proof of the lemma. □

Hence we can use the above S and $\tilde{e_1}$ to construct the specializations of e_1. Now we define the subscheme U of $\mathcal{V}\times S$ as follows:

$$U = \mathcal{V}\times S \setminus (D_+(x+y)\cup D_+(x+\zeta y)),$$

where $D_+(f)$ is the non-vanishing locus of a homogeneous polynomial f. Explicitly, if we put

$$R := \frac{k[c^{\pm}, d^{\pm}]\left[\frac{x}{x+y}, \frac{y}{x+y}, \frac{z}{x+y}, \frac{t}{x+y}, \frac{x+y}{x+\zeta y}\right]}{\left(\frac{x^3+y^3+cz^3+dt^3}{(x+y)^3}\right)},$$

then $U = \operatorname{Spec} R$. We have

$$\frac{d}{c}, \quad \frac{x+\zeta y}{x+y} \in \Gamma(U, \mathcal{O}_U)^*$$

and hence

$$\left\{\frac{d}{c}, \frac{x+\zeta y}{x+y}\right\}_3 \in \mathrm{Br}(U),$$

where

$$\begin{aligned}\{\cdot,\cdot\}_3 \colon \Gamma(U, \mathcal{O}_U)^* \otimes \Gamma(U, \mathcal{O}_U)^* &\\ \to H^1(U, \mu_3) \otimes H^1(U, \mu_3)&\\ \stackrel{\cup}{\to} H^2(U, \mu_3^{\otimes 2})&\\ \cong H^2(U, \mu_3)&\\ \to \mathrm{Br}(U)&\end{aligned}$$

is a norm residue symbol defined similarly as in the field case. Take any $P = (c_0, d_0) \in k^* \times k^*$ and put $R_P := R/(c - c_0, d - d_0)$. We have the canonical morphism $P \colon U_P := \operatorname{Spec} R_P \to U$ and the following commutative diagram:

$$\begin{array}{ccccc}
R_P^* \otimes R_P^* & \xleftarrow{P^* \otimes P^*} & R^* \otimes R^* & & \\
\downarrow {\scriptstyle \{\cdot,\cdot\}_3} & & \downarrow {\scriptstyle \{\cdot,\cdot\}_3} & & \\
\mathrm{Br}(U_P) & \xleftarrow{P^*} & \mathrm{Br}(U) & \longrightarrow & \mathrm{Br}(F(V)) \\
\uparrow {\scriptstyle \mathrm{res}^{V_P}_{U_P}} & & \uparrow {\scriptstyle \mathrm{res}^{\mathcal{V} \times S}_{U}} & & \uparrow {\scriptstyle \mathrm{res}^{V}_{F(V)}} \\
\mathrm{Br}(V_P) & \xleftarrow{P^*} & \mathrm{Br}(\mathcal{V} \times S) & \longrightarrow & \mathrm{Br}(V).
\end{array}$$

Therefore we get

$$\begin{aligned}\mathrm{res}^{V_P}_{U_P}(\mathrm{sp}(e_1; P)) &= \mathrm{res}^{V_P}_{U_P}(P^*(\tilde{e}_1)) \\ &= P^*(\mathrm{res}^{\mathcal{V} \times S}_{U}(\tilde{e}_1)) \\ &= P^*\left(\left\{\frac{d}{c}, \frac{x + \zeta y}{x + y}\right\}_3\right) \\ &= \left\{\frac{d_0}{c_0}, \frac{x + \zeta y}{x + y}\right\}_3\end{aligned}$$

and complete the proof of Corollary 5.1.3. □

Chapter 6

The case $x^3 + by^3 + cz^3 + dt^3 = 0$

6.1 A non-representability result

Before stating the main result of this dissertation, we introduce a condition on a field k, called $C(k)$. Let k be a field, $\mathcal{O}_F = k[\lambda, \mu, \nu]$ and

$$\mathcal{V} = \mathrm{Proj}(\mathcal{O}_F[x, y, z, t]/(x^3 + \lambda y^3 + \mu z^3 + \lambda\mu\nu t^3).$$

Put $(\mathbb{G}_m)^3 := \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m$. For a given $P \in (\mathbb{G}_{m,k})^3(k)$, $V_P = \mathcal{V} \times_{\mathcal{O}_F} \mathrm{Spec}\, k$ is the fiber of $\mathcal{V} / \mathcal{O}_F$ at P. For all $P \in (\mathbb{G}_{m,k})^3(k)$, V_P is smooth over k. We define the set $\mathcal{P}_k$ to be

$$\{P \in (\mathbb{G}_{m,k})^3(k) \mid \mathrm{Br}(V_P)/\, \mathrm{Br}(k) \cong \mathbb{Z}\,/3\mathbb{Z}\,.\}.$$

The condition $C(k)$ is defined to be the Zariski density of $\mathcal{P}_k$ in $(\mathbb{G}_{m,k})^3$. To prove the theorem below, we need to assume that $C(k)$ holds for k. However, we will show that the following conditions are equivalent:

Proposition 6.1.1 (Proposition 6.2.1). *Let k be a field satisfying $(*)$. Then the following conditions are equivalent:*

(1) $C(k)$*, that is,* $\mathcal{P}_k$ *is Zariski dense in* $(\mathbb{G}_{m,k})^3$*;*

(2) $\mathcal{P}_k$ *is non-empty;*

(3) $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$.

We easily see that the condition (3) holds for various fields, for example, all finitely generated fields over $\mathbb{Q}(\zeta)$ and $\mathbb{Q}_p(\zeta)$ for any prime number p, and hence this condition $C(k)$ is mild and reasonable. We prove Proposition 6.1.1 in Section 6.2.

The main result of this chapter is:

Theorem 6.1.2. *Let k be a field satisfying $(*)$ and $C(k)$, $F = k(\lambda, \mu, \nu)$ and V be the projective surface over F defined by the homogeneous equation $x^3 + \lambda y^3 + \mu z^3 + \lambda\mu\nu t^3 = 0$. Then there is no element $e \in \mathrm{Br}(V)$ satisfying the following property:*

there exists a dense open subset $W \subset (\mathbb{G}_{m,k})^3$ *such that* $\mathrm{sp}(e;\cdot)$ *is defined on* $W(k) \cap \mathcal{P}_k$ *and for all* $P \in W(k) \cap \mathcal{P}_k$, $\mathrm{sp}(e;P)$ *is a generator of* $\mathrm{Br}(V_P)/\mathrm{Br}(k)$.

This theorem is a consequence of the following theorem:

Theorem 6.1.3. *Let k be a field satisfying $(*)$, F and V be as in Theorem* 6.1.2. *Then*

$$\mathrm{Br}(V)/\mathrm{Br}(F) = 0.$$

Remark 6.1.4. Let V be a smooth cubic surface over k of the form $x^3 + by^3 + cz^3 + dt^3 = 0$. Assume $H^1(k, \mathrm{Pic}(\overline{V})) = \mathbb{Z}/3\mathbb{Z}$ and $V(k) = \emptyset$. We note some known results of the structure and the representability of $\mathrm{Br}(V)/\mathrm{Br}(k)$.

1. By a theorem of Merkurjev-Suslin [MS82], we always write a generator of $\mathrm{Br}(V)/\mathrm{Br}(k)$ as a sum of norm residue symbols.

2. If $\mathrm{cd}(k) \leq 2$, $\mathrm{Br}(V)/\mathrm{Br}(k)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ by Lemma 3.2.18. Its symbolic generator is not "uniform" by Theorem 6.1.2 and we do not know this can be written by *one* symbol $\{f,g\}_3$ for some f, $g \in k(V)^*$. However, here is a partial result.

   Proposition 6.1.5. *Let k and V as above. Assume that:*

   > *the restriction* $\mathrm{Br}(k) \to \mathrm{Br}(k(\gamma))$ *is surjective, where* $v = d/bc$ *and* $\gamma = \sqrt[3]{v}$.

   Then we have a generator of $\mathrm{Br}(V)/\mathrm{Br}(k)$ *of the form* $\{v,f\}_3$ *for some* $f \in k(V)^*$.

   Proof. We have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*\mathrm{Br}(k) & \longrightarrow & \mathrm{Br}(V) & \longrightarrow & \mathrm{Br}(V)/\mathrm{Br}(k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi^*\mathrm{Br}(k(\gamma)) & \longrightarrow & \mathrm{Br}(V_{k(\gamma)}) & \longrightarrow & \mathrm{Br}(V_{k(\gamma)})/\mathrm{Br}(k(\gamma)) & \longrightarrow & 0. \end{array}$$

   By Proposition 4.2.4, we have $\mathrm{Br}(V_{k(\gamma)})/\mathrm{Br}(k(\gamma)) = 0$. By usual diagram chasing and the surjectivity of the left vertical map, we can construct an element $\alpha \in \mathrm{Br}(V)$ whose image in $\mathrm{Br}(V)/\mathrm{Br}(k)$ is its generator and whose image in $\mathrm{Br}(V_{k(\gamma)})$ is zero. We consider α as an element in $\mathrm{Br}(k(V)(\gamma)/k(V))$. Then the claim is a consequence of the surjectivity of the following map:

$$\{v,\cdot\}_3 \colon k(V)^* \to \mathrm{Br}(k(V)(\gamma)/k(V)).$$

   □

   Some examples of k which satisfy the above condition for any V are:

   - a field k with $\mathrm{cd}(k) \leq 1$.
   - a p-adic field k.

In the former case, $\mathrm{Br}(k)$ and $\mathrm{Br}(k(\gamma))$ are zero and the condition holds trivially. In the latter case, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(k) & \longrightarrow & \mathrm{Br}(k(\gamma)) \\ \cong\downarrow & & \cong\downarrow \\ \mathbb{Q}/\mathbb{Z} & \xrightarrow{3} & \mathbb{Q}/\mathbb{Z} \end{array}$$

and the bottom map is surjective.

3. If $\mathrm{cd}(k) \geq 3$, it is difficult to determine whether $\mathrm{Br}(V)/\mathrm{Br}(k)$ is isomorphic 0 or $\mathbb{Z}/3\mathbb{Z}$. Our V/F has $\mathrm{cd}(F) \geq 3$ and $V(F) = \emptyset$ and $H^1(F, \mathrm{Pic}(\overline{V})) \neq 0$. As far as we know, Our result would be the first example of computation of the Brauer group of such varieties.

We first prove Theorem 6.1.3. The implication from Theorem 6.1.2 to Theorem 6.1.3 is given after the proof of Theorem 6.1.3.

Proof of Theorem 6.1.3. We recall some notations in Section 4.2. We define

$$\alpha = \sqrt[3]{\lambda}, \quad \gamma = \sqrt[3]{\nu}, \quad \alpha' = \sqrt[3]{\mu}, \quad \beta = \alpha\gamma.$$

Moreover we put

$$F' = F(\alpha, \gamma), \quad F'' = F'(\alpha') = F(\alpha, \gamma, \alpha').$$

We have the following exact sequence:

$$0 \longrightarrow \mathrm{Br}(V)/\mathrm{Br}(F) \longrightarrow H^1(F, \mathrm{Pic}(\overline{V})) \xrightarrow{d^{1,1}} H^3(F, \overline{F}^*).$$

Therefore, to prove the theorem, it suffices to show the image of all nonzero elements in $H^1(F, \mathrm{Pic}(\overline{V}))$ does not vanish in $H^3(F, \overline{F}^*)$.

Before proving this claim, we sketch the outline of its proof. In Step 1, we show

$$H^1(F, \mathrm{Pic}(\overline{V})) \cong H^1(F'/F, \mathrm{Pic}(V_{F'})) \cong \mathbb{Z}/3\mathbb{Z}$$

and find a generating cocycle $\phi \in H^1(F'/F, \mathrm{Pic}(V_{F'}))$. In Step 2, we compute the image of ϕ under the differential

$$d^{1,1}_{F'} \colon H^1(F'/F, \mathrm{Pic}(V_{F'})) \longrightarrow H^3(F'/F, (F')^*)$$

explicitly. Since the inflation $i^{F'}_{F} : H^3(F'/F, F'^*) \longrightarrow H^3(F, \overline{F}^*)$ does not necessarily injective, if we prove that $d^{1,1}_{F'}(\phi) \neq 0$, this is insufficient to prove the theorem. In Step 3, we consider $d^{1,1}_{F'}(\phi)$ as an element of $H^3(F''/F, \mu_3)$. In Step 4, by computing the residue of $d^{1,1}_{F'}(\phi)$ along a certain prime divisor D in $\mathbb{A}^3_k$ and replacing the base field k with the field adding all roots of unity, we reduce the proof to showing that a certain cocycle induced by ϕ is nontrivial in $H^2(k(D), \mu_3)$, where $k(D)$ is the function field of D. Finally, in Step 5, we again compute the residue of a cocycle

given in Step 4 along a certain prime divisor D' in $\mathbb{A}_k^2$ and check this is nonzero. These steps complete the proof of the theorem.
Step 1. Let $t \in \mathrm{Gal}(F'/F(\gamma))$ be the element satisfying $t\alpha = \zeta\alpha$ and $s \in \mathrm{Gal}(F'/F(\alpha))$ the element satisfying $s\gamma = \zeta\gamma$. These generate the group $\mathrm{Gal}(F'/F)$. By Proposition 4.2.4 and Proposition 4.4.1, we have

$$H^1(F'/F, \mathrm{Pic}(V_{F'})) \cong H^1(F, \mathrm{Pic}(\overline{V})) \cong \mathbb{Z}/3\mathbb{Z}.$$

We define a cochain $\phi \in C^1(F'/F, \mathrm{Pic}(V_{F'}))$ as follows:

$$\begin{aligned} \phi((st)^i) &= 0, \\ \phi(s(st)^i) &= [L(0)] - [L(2)], \\ \phi(s^2(st)^i) &= [L(0)] - [L(1)], \end{aligned}$$

where $i = 0, 1, 2$. We show ϕ is a cocycle and generates $H^1(F'/F, \mathrm{Pic}(V_{F'}))$. First, the class $[L(1)] - [L(0)]$ is st-invariant since

$$\begin{aligned} st([L(0)] - [L(1)]) &= [L'(1)] - [L'(2)] \\ &= [D_2] - [D_3] + [L(0)] - [L(1)] \\ &= [L(0)] - [L(1)]. \end{aligned}$$

Using this and (4.5), we can check this is a cocycle. Moreover, since the image of ϕ under the restriction

$$\mathbb{Z}/3\mathbb{Z} \cong H^1(F'/F, \mathrm{Pic}(V_{F'})) \to H^1(\langle s \rangle, \mathrm{Pic}(V_{F'})) \cong \mathbb{Z}/3\mathbb{Z}$$

is the generating cocycle appearing in Proposition 4.4.4, ϕ is also a non-zero element, in fact, a generator of $H^1(F'/F, \mathrm{Pic}(V_{F'}))$.
Step 2. In the following argument, by using Proposition 4.5.1, we compute the cocycle $\delta\partial\phi$ in $Z^3(F'/F, F'^*)$, where ϕ is the cocycle defined above and

$$\begin{aligned} \partial &: H^1(F'/F, \mathrm{Pic}(V_{F'})) \to H^2(F'/F, \mathcal{D}_0), \\ \delta &: H^2(F'/F, \mathcal{D}_0) \to H^3(F'/F, F'^*) \end{aligned}$$

are connecting homomorphisms appearing in Section 4.5. First we compute the cocycle $\partial\phi \in Z^2(F'/F, \mathcal{D}_0)$. Let $\mathcal{D}$ and $\mathcal{D}_0$ be as in Subsection 4.4.2. We take

$$0, \quad L(0) - L(2), \quad L(0) - L(1) \in \mathcal{D}$$

as lifts of 0, $[L(0)] - [L(2)]$ and $[L(0)] - [L(1)] \in \mathrm{Pic}(V_{F'})$ respectively. We have

$$\mathrm{div}\frac{x + \zeta^2\alpha y}{x} = \mathrm{div}\left(-\frac{\mu f_3 f_4 f_5}{f_1 f_2}\right) \in \mathcal{D}_0$$

and the following table of the action of $\mathrm{Gal}(F'/F)$ on two divisors $L(0) - L(2)$ and $L(0) - L(1)$:

D	$L(2) - L(0)$	$L(0) - L(1)$
sD	$L'(2) - L'(0)$	$L'(0) - L'(1)$
s^2D	$L''(2) - L''(0)$	$L''(0) - L''(1)$
tD	$L(0) - L(1)$	$L(1) - L(2)$
stD	$L'(0) - L'(1)$	$L'(1) - L'(2)$
s^2tD	$L''(0) - L''(1)$	$L''(1) - L''(2)$
t^2D	$L(1) - L(2)$	$L(2) - L(0)$
st^2D	$L'(1) - L'(2)$	$L'(2) - L'(0)$
s^2t^2D	$L''(1) - L''(2)$	$L''(2) - L''(0)$

From the construction of the map ∂, we get the following equations

$$\begin{array}{lll}
\partial\phi(1,1) = 0, & \partial\phi(1,s) = 0, & \partial\phi(1,s^2) = 0, \\
\partial\phi(s,1) = 0, & \partial\phi(s,s) = \operatorname{div}\dfrac{z+\zeta\beta t}{x+\zeta^2\alpha y}, & \partial\phi(s,s^2) = \operatorname{div}\dfrac{x+\alpha y}{z+\zeta^2\beta t}, \\
\partial\phi(s^2,1) = 0, & \partial\phi(s^2,s) = \operatorname{div}\dfrac{x+\alpha y}{z+\zeta\beta t}, & \partial\phi(s^2,s^2) = \operatorname{div}\dfrac{z+\zeta^2\beta t}{x+\zeta\alpha y}, \\
\partial\phi(t,1) = 0, & \partial\phi(t,s) = 0, & \partial\phi(t,s^2) = 0, \\
\partial\phi(st,1) = 0, & \partial\phi(st,s) = \operatorname{div}\dfrac{z+\zeta^2\beta t}{x+\alpha y}, & \partial\phi(st,s^2) = \operatorname{div}\dfrac{x+\zeta\alpha y}{z+\beta t}, \\
\partial\phi(s^2t,1) = 0, & \partial\phi(s^2t,s) = \operatorname{div}\dfrac{x+\zeta\alpha y}{z+\zeta^2\beta t}, & \partial\phi(s^2t,s^2) = \operatorname{div}\dfrac{z+\beta t}{x+\zeta^2\alpha y}, \\
\partial\phi(t^2,1) = 0, & \partial\phi(t^2,s) = 0, & \partial\phi(t^2,s^2) = 0, \\
\partial\phi(st^2,1) = 0, & \partial\phi(st^2,s) = \operatorname{div}\dfrac{z+\beta t}{x+\zeta\alpha y}, & \partial\phi(st^2,s^2) = \operatorname{div}\dfrac{x+\zeta^2\alpha y}{z+\zeta\beta t}, \\
\partial\phi(s^2t^2,1) = 0, & \partial\phi(s^2t^2,s) = \operatorname{div}\dfrac{x+\zeta^2\alpha y}{z+\beta t}, & \partial\phi(s^2t^2,s^2) = \operatorname{div}\dfrac{z+\zeta\beta t}{x+\alpha y},
\end{array}$$

$$\partial\phi(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}) = \partial\phi(s^{i_1}t^{j_1}, s^{i_2-j_2}),$$

where the indices i_1, i_2, j_1 and j_2 can take on any values in $\{0,1,2\}$.

Sending this cocycle under δ, we get $\delta\partial\phi$ in $Z^3(F'/F, F'^*)$. If we take

$$1, \quad \frac{x+\zeta^i\alpha y}{z+\zeta^j\beta t}, \quad \frac{z+\zeta^j\beta t}{x+\zeta^i\alpha y} \in \operatorname{div}^{-1}(\mathcal{D}_0)$$

as lifts of 0, $\operatorname{div}\dfrac{x+\zeta^i\alpha y}{z+\zeta^j\beta t}$ and $\operatorname{div}\dfrac{z+\zeta^j\beta t}{x+\zeta^i\alpha y} \in \mathcal{D}_0$ respectively, this cocycle is determined by the following equations:

$$\begin{aligned}
&\delta\partial\phi(t^{j_1}, s^{i_2}t^{j_2}, s^{i_3}t^{j_3}) = 1,\\
&\delta\partial\phi(s^{i_1}t^{j_1}, 1, s^{i_3}t^{j_3}) = 1,\\
&\delta\partial\phi(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}, 1) = 1,\\
&\delta\partial\phi(st^{j_1}, s, s) = 1, && \delta\partial\phi(st^{j_1}, s, s^2) = -\mu,\\
&\delta\partial\phi(st^{j_1}, s^2, s) = 1, && \delta\partial\phi(st^{j_1}, s^2, s^2) = -\mu^{-1},\\
&\delta\partial\phi(st^{j_1}, t, s) = 1, && \delta\partial\phi(st^{j_1}, t, s^2) = -\mu^{-1},\\
&\delta\partial\phi(st^{j_1}, st, s) = -\mu^{-1}, && \delta\partial\phi(st^{j_1}, st, s^2) = -\mu,\\
&\delta\partial\phi(st^{j_1}, s^2t, s) = -\mu, && \delta\partial\phi(st^{j_1}, s^2t, s^2) = 1,\\
&\delta\partial\phi(st^{j_1}, t^2, s) = -\mu, && \delta\partial\phi(st^{j_1}, t^2, s^2) = 1,\\
&\delta\partial\phi(st^{j_1}, st^2, s) = -\mu^{-1}, && \delta\partial\phi(st^{j_1}, st^2, s^2) = 1,\\
&\delta\partial\phi(st^{j_1}, s^2t^2, s) = 1, && \delta\partial\phi(st^{j_1}, s^2t^2, s^2) = 1,\\
&\delta\partial\phi(s^2t^{j_1}, s, s) = -\mu^{-1}, && \delta\partial\phi(s^2t^{j_1}, s, s^2) = 1,\\
&\delta\partial\phi(s^2t^{j_1}, s^2, s) = -\mu, && \delta\partial\phi(s^2t^{j_1}, s^2, s^2) = 1,\\
&\delta\partial\phi(s^2t^{j_1}, t, s) = 1, && \delta\partial\phi(s^2t^{j_1}, t, s^2) = -\mu,\\
&\delta\partial\phi(s^2t^{j_1}, st, s) = 1, && \delta\partial\phi(s^2t^{j_1}, st, s^2) = 1,\\
&\delta\partial\phi(s^2t^{j_1}, s^2t, s) = 1, && \delta\partial\phi(s^2t^{j_1}, s^2t, s^2) = -\mu^{-1},\\
&\delta\partial\phi(s^2t^{j_1}, t^2, s) = -\mu^{-1}, && \delta\partial\phi(s^2t^{j_1}, t^2, s^2) = 1,\\
&\delta\partial\phi(s^2t^{j_1}, st^2, s) = 1, && \delta\partial\phi(s^2t^{j_1}, st^2, s^2) = -\mu,\\
&\delta\partial\phi(s^2t^{j_1}, s^2t^2, s) = -\mu, && \delta\partial\phi(s^2t^{j_1}, s^2t^2, s^2) = -\mu^{-1},
\end{aligned}$$

$$\delta\partial\phi(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}, s^{i_3}t^{j_3}) = \delta\partial\phi(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}, s^{i_3-j_3}),$$

where the indices i_1, i_2, i_3, j_1, j_2 and j_3 can take on any values in $\{0,1,2\}$.
Step 3. Let $i_F^{F'}$ be the following inflation

$$i_F^{F'} : H^3(F'/F, F'^*) \to H^3(F, \overline{F}^*).$$

The class $i_F^{F'}\delta\partial[\phi]$ in $H^3(F, \overline{F}^*)$ is a 3-torsion element, hence by the Kummer sequence, comes from $H^3(F, \mu_3)$. Now we want to find a finite extension K over F such that $i_F^{F'}\delta\partial[\phi]$ comes from $H^3(K/F, \mu_3)$. In fact, we can take $K = F''$:

Proposition 6.1.6. *The class $i_F^{F'}\delta\partial[\phi] \in H^3(F, \overline{F}^*)$ comes from $H^3(F''/F, \mu_3)$.*

Proof. We have the following exact sequence of $\mathrm{Gal}(F''/F)$-modules

$$1 \to \mu_3 \to F''^* \xrightarrow{3} (F''^*)^3 \to 1,$$

and hence the following commutative diagram

$$\begin{array}{ccccc} & & H^3(F'/F, F'^*) & & \\ & & \Big\downarrow {\scriptstyle i^{F'}_{F''}} & & \\ H^3(F''/F, \mu_3) & \longrightarrow & H^3(F''/F, F''^*) & \xrightarrow{3} & H^3(F''/F, (F''^*)^3) \\ \Big\downarrow & & \Big\downarrow {\scriptstyle i^{F''}_{F}} & & \Big\downarrow \\ H^3(F, \mu_3) & \longrightarrow & H^3(F, \overline{F}^*) & \xrightarrow{3} & H^3(F, \overline{F}^*), \end{array}$$

where $i^{F''}_{F'}$ and $i^{F''}_{F}$ are inflations and each row is exact. The class $i^{F'}_{F}\delta\partial[\phi]$ is the image of $i^{F'}_{F''}\delta\partial[\phi] \in H^3(F''/F, F''^*)$ under $i^{F''}_{F}$. Therefore to prove the claim, it suffices to show $i^{F'}_{F''}\delta\partial[\phi]$ vanishes in $H^3(F''/F, (F''^*)^3)$. Let w be the generator of $\mathrm{Gal}(F''/F')$ defined as in Subsection 4.4.2. The image of $i^{F'}_{F''}\delta\partial[\phi]$ under $3\colon H^3(F''/F, F''^*) \to H^3(F''/F, (F''^*)^3)$ is the class of the following cocycle:

$$(s^{i_1}t^{j_1}w^{k_1}, s^{i_2}t^{j_2}w^{k_2}, s^{i_3}t^{j_3}w^{k_3}) \mapsto \delta\partial\phi(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}, s^{i_3}t^{j_3})^3,$$

and what we have to prove is that this cocycle is in $B^3(F''/F, (F''^*)^3)$. Define $\psi \in C^2(F''/F, (F''^*)^3)$ to be:

$$\begin{array}{ll} \psi(t^{j_1}w^{k_1}, s^{i_2}t^{j_2}w^{k_2}) = 1, & \psi(s^{i_1}t^{j_1}w^{k_1}, w^{k_2}) = 1, \\ \psi(st^{j_1}w^{k_1}, sw^{k_2}) = -\mu^{-1}, & \psi(st^{j_1}w^{k_1}, s^2w^{k_2}) = -\mu, \\ \psi(s^2t^{j_1}w^{k_1}, sw^{k_2}) = -\mu, & \psi(s^2t^{j_1}w^{k_1}, s^2w^{k_2}) = -\mu^{-1}, \end{array}$$

$$\psi(s^{i_1}t^{j_1}w^{k_1}, s^{i_2}t^{j_2}w^{k_2}) = \psi(s^{i_1}t^{j_1}w^{k_1}, s^{i_2-j_2}w^{k_2}),$$

where indices i_*, j_* and k_* can take on any values in $\{0,1,2\}$. Then we can easily see $d\psi = (i^{F'}_{F''}\delta\partial\phi)^3$ in $C^3(F''/F, (F''^*)^3)$ and hence the class of $(i^{F'}_{F''}\delta\partial\phi)^3$ vanishes in $H^3(F''/F, (F''^*)^3)$. This completes the proof of Proposition 6.1.6. □

By using this cochain ψ, we can construct the class in $H^3(F''/F, \mu_3)$ whose image in $H^3(F, \overline{F}^*)$ is $i^{F''}_{F}\delta\partial[\phi]$. We have the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^2(F''/F, \mu_3) & \xrightarrow{f^2} & C^2(F''/F, F''^*) & \xrightarrow{g^2} & C^2(F''/F, (F''^*)^3) & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle d^2} & & \Big\downarrow {\scriptstyle d^2} & & \Big\downarrow {\scriptstyle d^2} & & \\ 0 & \longrightarrow & C^3(F''/F, \mu_3) & \xrightarrow{f^3} & C^3(F''/F, F''^*) & \xrightarrow{g^3} & C^3(F''/F, (F''^*)^3) & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle d^3} & & \Big\downarrow {\scriptstyle d^3} & & \Big\downarrow {\scriptstyle d^3} & & \\ 0 & \longrightarrow & C^4(F''/F, \mu_3) & \xrightarrow{f^4} & C^4(F''/F, F''^*) & \xrightarrow{g^4} & C^4(F''/F, (F''^*)^3) & \longrightarrow & 0. \end{array}$$

If $\widetilde{\psi} \in C^2(F''/F, F''^*)$ is a lift of ψ, we see that

$$i^{F'}_{F''}\delta\partial\phi - d^2\widetilde{\psi} \in \mathrm{Ker}\, g^3 = \mathrm{Im} f^3,$$

that is, there exists a cochain $\Phi \in C^3(F''/F, \mu_3)$ such that

$$f^3\Phi = i^{F'}_{F''}\delta\partial\phi - d^2\widetilde{\psi}.$$

By abuse of notation, the symbol $-d^2\widetilde{\psi}$ means the multiplicative inverse of $d^2\widetilde{\psi}$. Moreover, by construction, Φ is a cocycle and $f^3[\Phi] = [i^{F'}_{F''}\delta\partial\phi - d^2\widetilde{\psi}] = i^{F'}_{F''}\delta\partial[\phi]$. Therefore the class $[\Phi]$ is what we need. As a lift $\widetilde{\psi}$ of ψ, we can take the following cochain:

$$\begin{array}{ll} \widetilde{\psi}(t^{j_1}w^{k_1}, s^{i_2}t^{j_2}w^{k_2}) = 1, & \widetilde{\psi}(s^{i_1}t^{j_1}w^{k_1}, w^{k_2}) = 1, \\ \widetilde{\psi}(st^{j_1}w^{k_1}, sw^{k_2}) = -\alpha'^{-1}, & \widetilde{\psi}(st^{j_1}w^{k_1}, s^2w^{k_2}) = -\alpha', \\ \widetilde{\psi}(s^2t^{j_1}w^{k_1}, sw^{k_2}) = -\alpha', & \widetilde{\psi}(s^2t^{j_1}w^{k_1}, s^2w^{k_2}) = -\alpha'^{-1}, \end{array}$$

$$\widetilde{\psi}(s^{i_1}t^{j_1}w^{k_1}, s^{i_2}t^{j_2}w^{k_2}) = \widetilde{\psi}(s^{i_1}t^{j_1}w^{k_1}, s^{i_2-j_2}w^{k_2}).$$

Hence we can write Φ explicitly as follows:

$$(s^{i_1}t^{j_1}w^{k_1}, s^{i_2}t^{j_2}w^{k_2}, s^{i_3}t^{j_3}w^{k_3}) \mapsto \frac{\widetilde{\psi}(s^{i_2}t^{j_2}w^{k_2}, s^{i_3}t^{j_3}w^{k_3})}{w^{k_1}\widetilde{\psi}(s^{i_2}t^{j_2}w^{k_2}, s^{i_3}t^{j_3}w^{k_3})} \in \mu_3.$$

Step 4. First, we prepare the following lemma:

Lemma 6.1.7. *Let K be a field of characteristic 0. For $i > 1$, we have:*

$$H^i(K, \mathbb{Q}/\mathbb{Z}(1)) \cong H^i(K, \overline{K}^*),$$

where $\mathbb{Q}/\mathbb{Z}(1) = \varinjlim_n \mu_n$ be the group of all roots of unity.

Proof. Consider the following exact sequence

$$1 \to \mathbb{Q}/\mathbb{Z}(1) \xrightarrow{\iota} \overline{K}^* \to \operatorname{Coker}\iota \to 1,$$

we have the long exact sequence

$$H^{i-1}(K, \operatorname{Coker}\iota) \to H^i(K, \mathbb{Q}/\mathbb{Z}(1)) \to H^i(K, \overline{K}^*) \to H^i(K, \operatorname{Coker}\iota).$$

Since $\operatorname{Coker}\iota$ is uniquely divisible, the group $H^i(K, \operatorname{Coker}\iota) = 0$ for all $i > 0$. Therefore we get the isomorphism which we have to show. □

By this lemma, for any prime divisor $D \subset \mathbb{A}^3_k = \operatorname{Spec} k[\lambda, \mu, \nu]$, we have the following commutative diagram:

$$\begin{array}{ccc} H^3(F''/F, \mu_3) & & \\ \downarrow{\scriptstyle i^{F''}_F} & & \\ H^3(F, \mu_3) & \xrightarrow{\operatorname{res}_D} & H^2(k(D), \mathbb{Z}/3\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^3(F, \mathbb{Q}/\mathbb{Z}(1)) & \xrightarrow{\operatorname{res}_D} & H^2(k(D), \mathbb{Q}/\mathbb{Z}) \\ \cong\downarrow & & \\ H^3(F, \overline{F}^*), & & \end{array}$$

where $F = k(\lambda, \mu, \nu)$ is considered as the function field of $\mathbb{A}^3_k$, $k(D)$ is the function field of D, and res_D are residue maps associated to D. The definition of res_D is described just below.

Recall that our goal is to prove the nontriviality of $i^{F'}_{F}\delta\partial[\phi] \in H^3(F, \overline{F}^*)$. To prove this, by the above diagram, it suffices to show:

$$\text{There exists } D \subset \mathbb{A}^3_k \text{ such that } \mathrm{res}_D(i^{F''}_{F}[\Phi]) \neq 0 \in H^2(k(D), \mathbb{Q} / \mathbb{Z}). \tag{6.1}$$

Now we look at the definition of

$$\mathrm{res}_D \colon H^3(F, \mu_3) \to H^2(k(D), \mathbb{Z} / 3\,\mathbb{Z}).$$

The lower res_D for $\mathbb{Q} / \mathbb{Z}$ is defined in the same way. In the sequel, D always denotes the divisor $\{\mu = 0\} \subset \mathbb{A}^3_k$. Let $\mathcal{O}_D$ be the completion of the local ring $k[\lambda, \mu, \nu]_{(\mu)}$ at its maximal ideal and F_D its fractional field. Note that μ is a uniformizer of $\mathcal{O}_D$ and the residue field of $\mathcal{O}_D$ is isomorphic to $k(D) = k(\lambda, \nu)$.

There is the canonical isomorphism

$$\iota \colon \mathrm{Hom}(G_{F^{\mathrm{ur}}_D}, \mu_3) = H^1(F^{\mathrm{ur}}_D, \mu_3) \cong F^{\mathrm{ur}*}_D / (F^{\mathrm{ur}*}_D)^3 \cong \mathbb{Z} / 3\,\mathbb{Z},$$

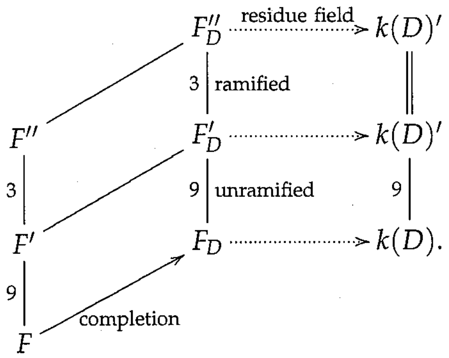
where the middle isomorphism is induced by Kummer sequence and the right one is given by normalized valuation on F^{ur}_D. Then res_D is given by

$$\begin{aligned} H^3(F, \mu_3) &\xrightarrow{c} H^3(F_D, \mu_3) \\ &\xrightarrow{r} H^2(k(D), \mathrm{Hom}(G_{F^{\mathrm{ur}}_D}, \mu_3)) \\ &\overset{H^2(\iota)}{\cong} H^2(k(D), \mathbb{Z} / 3\,\mathbb{Z}), \end{aligned}$$

where c is the restriction and r is the map defined in Proposition 2.2.7.

Now we describe the class $rc[i^{F''}_{F}\Phi] \in H^2(k(D), \mathrm{Hom}(G_{F^{\mathrm{ur}}_D}, \mu_3))$ explicitly. By the definition of r and the fact $i^{F''}_{F}\Phi$ originally comes from the cocycle Φ of $\mathrm{Gal}(F''/F)$, we would naturally expect that $rci^{F''}_{F}\Phi$ also comes from a cocycle of the Galois group of "the residue field of F'' along to D" over $k(D)$. In fact, we find that it is true.

Before stating the claim, we introduce some field extensions. Let $k(D)'$, F''_D, F'_D be the same notation as in Section 4.2. Moreover, by abuse of notation, we denote the elements in $\mathrm{Gal}(F''_D/F_D)$ corresponding to s, t and $w \in \mathrm{Gal}(F''/F)$ by the same symbols. To make our situation clear, we give the following diagram of field extensions:



We have the following:

Lemma 6.1.8. *If we define the cochain*

$$\overline{r\Phi} \in C^2(k(D)'/k(D), \mathrm{Hom}(\mathrm{Gal}(F''_D/F'_D), \mu_3))$$

as

$$\overline{r\Phi}(\bar{s}^{i_1}\bar{t}^{j_1}, \bar{s}^{i_2}\bar{t}^{j_2})(w^k) := \Phi(w^k, s^{i_1}t^{j_1}, s^{i_2}t^{j_2}),$$

where $\bar{s}$ and $\bar{t}$ are the images of s and t under the natural map

$$\mathrm{Gal}(F''_D/F_D) \to \mathrm{Gal}(k(D)'/k(D))$$

and the above Φ is naturally considered as an element in $C^3(F''_D/F_D, \mu_3)$.

Then $\overline{r\Phi}$ is a cocycle and its image under the map

$$i^{k(D)'}_{k(D)} : H^2(k(D)'/k(D), \mathrm{Hom}(\mathrm{Gal}(F''_D/F'_D), \mu_3)) \to H^2(k(D), \mathrm{Hom}(G_{F_D^{\mathrm{ur}}}, \mu_3))$$

is $rc[i^{F''}_F \Phi]$.

Proof. First we prove the cocycle condition of $\overline{r\Phi}$. We have

$$\begin{aligned}
&(d\overline{r\Phi})(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}, s^{i_3}t^{j_3})(w^k)\\
=&s^{i_1}t^{j_1}\Phi(w^k, s^{i_2}t^{j_2}, s^{i_3}t^{j_3}) - \Phi(w^k, s^{i_1+i_2}t^{j_1+j_2}, s^{i_3}t^{j_3})\\
&+ \Phi(w^k, s^{i_1}t^{j_1}, s^{i_2+i_3}t^{j_2+j_3}) - \Phi(w^k, s^{i_1}t^{j_1}, s^{i_2}t^{j_2})\\
=&(\widetilde{\psi}(s^{i_2}t^{j_2}, s^{i_3}t^{j_3}) - w^k\widetilde{\psi}(s^{i_2}t^{j_2}, s^{i_3}t^{j_3}))\\
&- (\widetilde{\psi}(s^{i_1+i_2}t^{j_1+j_2}, s^{i_3}t^{j_3}) - w^k\widetilde{\psi}(s^{i_1+i_2}t^{j_1+j_2}, s^{i_3}t^{j_3}))\\
&+ (\widetilde{\psi}(s^{i_1}t^{j_1}, s^{i_2+i_3}t^{j_2+j_3}) - w^k\widetilde{\psi}(s^{i_1}t^{j_1}, s^{i_2+i_3}t^{j_2+j_3}))\\
&- (\widetilde{\psi}(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}) - w^k\widetilde{\psi}(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}))\\
=&(d^2\widetilde{\psi})(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}, s^{i_3}t^{j_3}) - w^k(d^2\widetilde{\psi})(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}, s^{i_3}t^{j_3})\\
=&1.
\end{aligned}$$

The last equality follows from the fact that

$$(d^2\widetilde{\psi})(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}, s^{i_3}t^{j_3}) = (i^{F'}_{F''}\delta\partial\phi - f^3\Phi)(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}, s^{i_3}t^{j_3})$$

is in F^* and hence w-invariant.

Since Φ is normalized and satisfies that the values

$$\Phi(s^{i_1}t^{j_1}w^{k_1}, s^{i_2}t^{j_2}w^{k_2}, s^{i_3}t^{j_3}w^{k_3})$$

are independent of k_2 and k_3, we know that $ci^{F''}_{F}\Phi$ is normalized and satisfies the condition (2.1). Hence we can apply Proposition 2.2.7 to this cocycle and we have

$$rci^{F''}_{F}(\overline{g_1}, \overline{g_2})(h) = \Phi(w^k, s^{i_1}t^{j_1}w^{k_1}, s^{i_2}t^{j_2}w^{k_2}),$$

where $w^k, s^{i_1}t^{j_1}w^{k_1}$ and $s^{i_2}t^{j_2}w^{k_2}$ is the restriction of $h \in G_{F_D^{\mathrm{ur}}}$ and $g_1, g_2 \in G_{F_D}$ to F'' respectively. Therefore we obtain

$$i^{k(D)'}_{k(D)}[\overline{r\Phi}] = rc[i^{F''}_{F}\Phi].$$

□

Applying the following isomorphisms of $\mathrm{Gal}(k(D)'/k(D))$-modules

$$\mathrm{Hom}(\mathrm{Gal}(F''_D/F'_D), \mu_3) \cong \mathbb{Z}/3\mathbb{Z}; \quad (w \mapsto \zeta^k) \mapsto k \mod 3$$

and of $G_{k(D)}$-modules

$$\begin{array}{ccc}
\mathrm{Hom}(G_{F_D^{\mathrm{ur}}}, \mu_3) & \xrightarrow{\cong} & \mathbb{Z}/3\mathbb{Z} \\
\downarrow & & \downarrow \\
\mathrm{Hom}(G_{F_D^{\mathrm{ur}}}, \mathbb{Q}/\mathbb{Z}(1)) & \xrightarrow{\cong} & \mathbb{Q}/\mathbb{Z},
\end{array}$$

to the following diagram

$$\begin{array}{ccc}
H^2(k(D)'/k(D), \mathrm{Hom}(\mathrm{Gal}(F''_D/F'_D), \mu_3)) & \longrightarrow & H^2(k(D), \mathrm{Hom}(G_{F_D^{\mathrm{ur}}}, \mu_3)) \\
 & & \downarrow \\
 & & H^2(k(D), \mathrm{Hom}(G_{F_D^{\mathrm{ur}}}, \mathbb{Q}/\mathbb{Z}(1))),
\end{array}$$

we obtain:

$$\begin{array}{ccc}
H^2(k(D)'/k(D), \mathbb{Z}/3\mathbb{Z}) & \longrightarrow & H^2(k(D), \mathbb{Z}/3\mathbb{Z}) \\
 & & \downarrow \\
 & & H^2(k(D), \mathbb{Q}/\mathbb{Z}).
\end{array}$$

For a field K of characteristic 0, we denote $\widetilde{K}$ by $\bigcup_{n>0} K(\zeta_n)$, where ζ_n is a primitive n-th root of unity. Noting that $k(D)' = k(D)(\alpha, \gamma)$ and that α and γ are transcendental over k, we have $k(D)' \cap \widetilde{k(D)} = k(D)$ and therefore

$$\begin{aligned} \mathrm{Gal}(\widetilde{k(D)'}/\widetilde{k(D)}) &\xrightarrow{\cong} \mathrm{Gal}(k(D)'/k(D)' \cap \widetilde{k(D)}) \\ &= \mathrm{Gal}(k(D)'/k(D)). \end{aligned}$$

We fix an isomorphism $\mathbb{Q}/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}(1)$ as trivial $\widetilde{k(D)}$-modules. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^2(k(D)'/k(D), \mathbb{Z}/3\mathbb{Z}) & \longrightarrow & H^2(k(D), \mathbb{Q}/\mathbb{Z}) \\ \cong\downarrow & & \downarrow \\ H^2(\widetilde{k(D)'}/\widetilde{k(D)}, \mathbb{Z}/3\mathbb{Z}) & \longrightarrow & H^2(\widetilde{k(D)}, \mathbb{Q}/\mathbb{Z}) \\ \cong\downarrow & & \downarrow\cong \\ H^2(\widetilde{k(D)'}/\widetilde{k(D)}, \mu_3) & \longrightarrow & H^2(\widetilde{k(D)}, \mathbb{Q}/\mathbb{Z}(1)) \\ \downarrow & & \downarrow\cong \\ H^2(\widetilde{k(D)}, \mu_3) & \longrightarrow & H^2(\widetilde{k(D)}, \overline{k(D)}^*). \end{array}$$

Since the bottom map in the above diagram is injective by Hilbert's Theorem 90, in order to prove the claim (6.1), it suffices to show:

$$[\overline{r\Phi}] \in H^2(k(D)'/k(D), \mathbb{Z}/3\mathbb{Z}) \text{ does not vanish in } H^2(\widetilde{k(D)}, \mu_3).$$

Step 5. For simplicity, we put $E = \widetilde{k(D)} = \widetilde{k}(\lambda, \nu)$ and $E' = \widetilde{k(D)'} = E(\alpha, \gamma)$. we define the cocycle $\Psi \in Z^2(E'/E, \mu_3)$ as follows:

$$\begin{array}{ll} \Psi(t^{j_1}, s^{i_2}t^{j_2}) = 1 & \Psi(s^{i_1}t^{j_1}, (st)^{i_2}) = 1 \\ \Psi(st^{j_1}, s(st)^{i_2}) = \zeta & \Psi(st^{j_1}, s^2(st)^{i_2}) = \zeta^2 \\ \Psi(s^2t^{j_1}, s(st)^{i_2}) = \zeta^2 & \Psi(s^2t^{j_1}, s^2(st)^{i_2}) = \zeta. \end{array}$$

We can easily see that Ψ is the image of

$$\overline{r\Phi} \in H^2(k(D)'/k(D), \mathrm{Hom}(\mathrm{Gal}(F''_D/F'_D), \mu_3)) \cong H^2(k(D)'/k(D), \mathbb{Z}/3\mathbb{Z})$$

under the isomorphism $H^2(k(D)'/k(D), \mathbb{Z}/3\mathbb{Z}) \xrightarrow{\cong} H^2(E'/E, \mu_3)$ in the above diagram.

To prove Theorem 6.1.3, it suffices to show that the image of $[\Psi] \in H^2(E'/E, \mu_3)$ under

$$i_E^{E'} : H^2(E'/E, \mu_3) \to H^2(E, \mu_3)$$

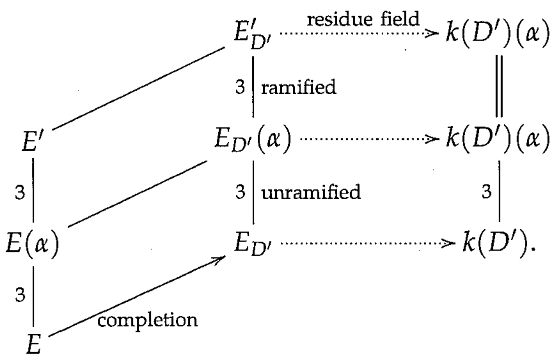
is nonzero. Now we consider the residue of $i_E^{E'}[\Psi]$ along to the divisor $D' = \{\nu = 0\} \subset \mathbb{A}_k^2$. The claim is:

Proposition 6.1.9. *The image of $i_E^{E'}[\Upsilon]$ under the residue map*

$$\mathrm{res}_{D'}\colon H^2(E, \mu_3) \to H^1(k(D'), \mathbb{Z}/3\mathbb{Z})$$

is nonzero.

Proof. We fix notations. Let $\mathcal{O}_{D'}$ be the completion of the local ring $k[\lambda, \nu]_{(\nu)}$ at its maximal ideal and $E_{D'}$ its fractional field. Note that ν is a uniformizer of $\mathcal{O}_{D'}$ and the residue field of $\mathcal{O}_{D'}$ is isomorphic to $k(D') = k(\lambda)$. Let $E'_{D'}$ be the same notation as in Section 4.2. By abuse of notation, we denote the elements in $\mathrm{Gal}(E'_{D'}/E_{D'})$ corresponding to s and $t \in \mathrm{Gal}(E'/E)$ as the same symbols. To make our situation clear, we give the following diagram of field extensions:



Now $\mathrm{res}_{D'}$ is given by

$$\begin{aligned} H^2(E, \mu_3) &\xrightarrow{c} H^2(E_{D'}, \mu_3) \\ &\xrightarrow{r} H^1(k(D'), \mathrm{Hom}(G_{E_{D'}^{\mathrm{ur}}}, \mu_3)) \\ &\xrightarrow{\cong} H^1(k(D'), \mathbb{Z}/3\mathbb{Z}). \end{aligned}$$

To describe $\mathrm{res}_{D'}(i_E^{E'}[\Upsilon])$ explicitly, the problem is that $ci_E^{E'}\Upsilon$ does not satisfy the condition (2.1) in Proposition 2.2.7. So we have to replace Υ with an appropriate cocycle satisfying (2.1).

Lemma 6.1.10. *Let $\Theta \in C^1(E'/E, \mu_3)$ be the following cochain:*

$$\Theta(s^i t^j) = \begin{cases} 1 & i = 0 \\ \zeta & i = 1, 2. \end{cases}$$

Put $\Upsilon' := \Upsilon - d\Theta \in Z^2(E'/E, \mu_3)$. Then $ci_E^{E'}\Upsilon' \in H^2(E_{D'}, \mu_3)$ is normalized and satisfies the condition (2.1) *in Proposition* 2.2.7.

Proof. By straightforward calculation, we see that

$$\Psi'(t^{j_1}, s^{i_2}t^{j_2}) = 1$$

$$\Psi'(st^{j_1}, s^{i_2}t^{j_2}) = \begin{cases} 1 & j_2 = 0 \\ \zeta^2 & j_2 = 1 \\ \zeta & j_2 = 2 \end{cases}$$

$$\Psi'(s^2t^{j_1}, s^{i_2}t^{j_2}) = \begin{cases} 1 & j_2 = 0 \\ \zeta & j_2 = 1 \\ \zeta^2 & j_2 = 2, \end{cases}$$

from which the claim easily follows. □

We also have a similar result to Lemma 6.1.8:

Lemma 6.1.11. *If we define the cochain*

$$\overline{r\Psi'} \in C^1(k(D')(\alpha)/k(D'), \mathrm{Hom}(\mathrm{Gal}(E'_{D'}/E_{D'}(\alpha)), \mu_3))$$

as

$$\overline{r\Psi'}(\bar{t}^j)(s^i) := \Psi'(s^i, t^j),$$

where $\bar{t}$ is the image of t under the natural map

$$\mathrm{Gal}(E'_{D'}/E_{D'}) \to \mathrm{Gal}(k(D')(\alpha)/k(D')),$$

then $\overline{r\Psi'}$ is a cocycle and its image under the map

$$i^{k(D')(\alpha)}_{k(D')} : H^1(k(D')(\alpha)/k(D'), \mathrm{Hom}(\mathrm{Gal}(E'_{D'}/E_{D'}(\alpha)), \mu_3)) \to H^1(k(D'), \mathrm{Hom}(G_{E^{\mathrm{ur}}_{D'}}, \mu_3))$$

is $rci^{E'}_{E}[\Psi]$.

Proof. The claim follows from similar calculations in Lemma 6.1.8. □

We now go back to the proof of Proposition 6.1.9. Applying the following commutative diagram of trivial $\mathrm{Gal}(k(D')(\alpha)/k(D'))$-modules and trivial $G_{k(D')}$-modules

$$\begin{array}{ccc} \mathrm{Hom}(\mathrm{Gal}(E'_{D'}/E_{D'}(\alpha)), \mu_3) & \xrightarrow{\cong} & \mathrm{Hom}(G_{E^{\mathrm{ur}}_{D'}}, \mu_3) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{Z}/3\mathbb{Z} & = & \mathbb{Z}/3\mathbb{Z} \end{array}$$

to the inflation $i^{k(D')(\alpha)}_{k(D')}$, we obtain

$$\mathrm{Hom}(\mathrm{Gal}(k(D')(\alpha)/k(D')), \mathbb{Z}/3\mathbb{Z}) \to \mathrm{Hom}(G_{k(D')}, \mathbb{Z}/3\mathbb{Z}),$$

which is injective. Moreover, we have

$$\overline{r\Psi'} \neq 0 \in \mathrm{Hom}(k(D')(\alpha)/k(D'), \mathrm{Hom}(\mathrm{Gal}(E'_{D'}/E_{D'}(\alpha)), \mu_3))$$

by the definition in Lemma 6.1.11. Therefore $\mathrm{res}_{D'}(i^{E'}_{E}[\Psi]) \neq 0$, which completes the proof of Proposition 6.1.9. □

Theorem 6.1.3 is a consequence of Proposition 6.1.9. □

Proof of Theorem 6.1.2. Suppose we have an element $e \in \mathrm{Br}(V)$ satisfying the property stated in Theorem 6.1.2:

> there exists a dense open subset $W \subset (\mathbb{G}_{m,k})^3$ such that $\mathrm{sp}(e;\cdot)$ is defined on $W(k) \cap \mathcal{P}_k$ and for all $P \in W(k) \cap \mathcal{P}_k$, $\mathrm{sp}(e;P)$ is a generator of $\mathrm{Br}(V_P)/\,\mathrm{Br}(k)$.

By Theorem 6.1.3, we have

$$\mathrm{Br}(V)/\,\mathrm{Br}(F) = 0$$

and hence there exists an element $e' \in \mathrm{Br}(F)$ such that $\pi_F^* e' = e$. Since we have the isomorphism

$$\varinjlim_i \mathrm{Br}(S_i) = \mathrm{Br}(F),$$

where (S_i) is the projective system of all non-empty open affine subschemes in $\mathbb{A}_k^3$, there exists a non-empty affine open subscheme S and $\widetilde{e'} \in \mathrm{Br}(S)$ such that $\widetilde{e'}$ is a lift of e' and $\mathcal{V} \times_{\mathbb{A}_k^3} S$ is smooth over S. Since S and W is not empty, $S \cap W$ is also a non-empty Zariski open set in $(\mathbb{G}_{m,k})^3$. Moreover, $\mathcal{P}_k$ is a Zariski dense set in $(\mathbb{G}_{m,k})^3$ by the condition $C(k)$. These facts imply that there exists a point $P \in (S \cap W)(k) \cap \mathcal{P}_k$. For this point P, we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Br}(V_P) & \xleftarrow{P^*} & \mathrm{Br}(\mathcal{V} \times_{\mathbb{A}_k^3} S) & \hookrightarrow & \mathrm{Br}(V) \\
\uparrow{\scriptstyle \pi_P^*} & & \uparrow{\scriptstyle \pi_S^*} & & \uparrow{\scriptstyle \pi_F^*} \\
\mathrm{Br}(k) & \xleftarrow{P^*} & \mathrm{Br}(S) & \hookrightarrow & \mathrm{Br}(F)
\end{array}$$

and hence we can take $\pi_S^* \widetilde{e'}$ as a lift of e. Then we get

$$\mathrm{sp}(e;P) = P^*(\pi_S^* \widetilde{e'}) = \pi_P^* P^* \widetilde{e'} \in \pi_P^* \,\mathrm{Br}(k).$$

This means that $\mathrm{sp}(e;P)$ is zero in the group $\mathrm{Br}(V_P)/\,\mathrm{Br}(k)$, which contradicts the assumption $\mathrm{sp}(e;P)$ is a generator of $\mathrm{Br}(V_P)/\,\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}$. Therefore we see that there is no such element e, and complete the proof of Theorem 6.1.2. □

6.2 The condition $C(k)$

At the end of this chapter, we now concentrate on the condition $C(k)$. It is an important problem to clarify when $C(k)$ holds. Recall

$$\{P \in (\mathbb{G}_{m,k})^3(k) \mid \mathrm{Br}(V_P)/\,\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}.\}.$$

Here we give some equivalent conditions:

Proposition 6.2.1 (Proposition 6.1.1). *Let k be a field satisfying $(*)$. Then the following are equivalent:*

(1) *$C(k)$, that is, $\mathcal{P}_k$ is Zariski dense in $(\mathbb{G}_{m,k})^3$;*

(2) *$\mathcal{P}_k$ is non-empty;*

(3) $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$.

To prove this proposition, we have to prepare the following lemma.

Lemma 6.2.2. *Let S, S' and S'' be infinite subsets of k^*. Then $S\times S'\times S''$ is Zariski dense in $(\mathbb{G}_{m,k})^3$.*

Proof. Put $\mathcal{P}=S\times S'\times S''$. Suppose $\mathcal{P}$ would not be a Zariski dense set in $(\mathbb{G}_{m,k})^3$. Then there exists a non-empty open $U\subset(\mathbb{G}_{m,k})^3$ such that $\mathcal{P}\cap U=\emptyset$. Since this condition holds for any smaller open subscheme than U, we may take as U an affine open subscheme of the form:

$$U=\operatorname{Spec} k[\lambda^{\pm},\mu^{\pm},\nu^{\pm}]_f,\quad 0\neq f\in k[\lambda,\mu,\nu],$$

where the symbol $\lambda^{\pm}$ is the abbreviation of λ and λ^{-1} and so forth. Write f as

$$f=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{m,n}(\lambda)\mu^m\nu^n,$$

where $a_{m,n}(\lambda)\in k[\lambda]$, and $a_{m,n}(\lambda)\equiv 0$ for sufficiently large m and n. Then there exists $\lambda_0\in S$ such that $f(\lambda_0,\mu,\nu)\neq 0\in k[\mu,\nu]$. Otherwise, we would have $f(\lambda_0,\mu,\nu)\equiv 0$ in $k[\mu,\nu]$ for all $\lambda_0\in S$, which is equivalent to

$$a_{m,n}(\lambda_0)=0 \text{ for all } m,n\in\mathbb{N} \text{ and for all } \lambda_0\in S.$$

However, since S is infinite, this means that $a_{m,n}(\lambda)\equiv 0$ in $k[\lambda]$ for all m and $n\in\mathbb{N}$. Therefore $f\equiv 0$, which contradicts the definition of f.

Now we write $f(\lambda_0,\mu,\nu)=\sum_{n=0}^{\infty}b_m(\mu)\nu^n$, where $b_m(\mu)=0$ for sufficiently large m. By the same argument, there exist $\mu_0\in S'$ such that $f(\lambda_0,\mu_0,\nu)\neq 0$ in $k[\nu]$. Moreover, by the same one again, we can choose $\nu_0\in S''$ such that $f(\lambda_0,\mu_0,\nu_0)\neq 0$. If we put $P:=(\lambda_0,\mu_0,\nu_0)$, then we find that $P\in\mathcal{P}\cap U$, which contradicts the assumption $\mathcal{P}\cap U\neq\emptyset$. This completes the proof of this lemma. □

Proof of Proposition 6.2.1. (1) ⇒ (2). This is a trivial implication.
(2) ⇒ (3). We prove the contrapositive statement.

If we assume $\dim_{\mathbb{F}_3} k^*/(k^*)^3=1$, we can take $v\in k^*\setminus(k^*)^3$. Then the equation of diagonal cubic surfaces is essentially equal to one of the following:

$$\begin{aligned}&x^3+y^3+z^3+t^3=0, && x^3+y^3+z^3+vt^3=0,\\ &x^3+y^3+z^3+v^2t^3=0, && x^3+y^3+vz^3+vt^3=0,\\ &x^3+y^3+vz^3+v^2t^3=0,\end{aligned}$$

all of which have a k-rational point. Moreover, we can easily see by Proposition 4.2.4 that $H^1(k,\operatorname{Pic}(\overline{V_P}))\cong 0$ or $(\mathbb{Z}/3\mathbb{Z})^2$ for all $P\in(\mathbb{G}_{m,k})^3$, and therefore

$$\forall P\in(\mathbb{G}_{m,k})^3,\ \operatorname{Br}(V_P)/\operatorname{Br}(k)\cong 0 \text{ or } (\mathbb{Z}/3\mathbb{Z})^2$$

by Lemma 3.2.18.

If we assume $\dim_{\mathbb{F}_3} k^*/(k^*)^3 = 0$, the equation of diagonal cubic surfaces is essentially equal to:

$$x^3 + y^3 + z^3 + t^3 = 0.$$

Hence the same result holds by the same reason. Hence we have $\mathcal{P}_k = \emptyset$.
(3) $\Rightarrow$ (1). We first construct a subset $\mathcal{P}$ of $\mathbb{A}_k^3$ satisfying the following three conditions:

(i) $\mathcal{P}$ is Zariski dense in $\mathbb{A}_k^3$;

(ii) $P \in \mathcal{P} \Rightarrow V_P(k) \neq \emptyset$;

(iii) $P \in \mathcal{P} \Rightarrow H^1(k, \mathrm{Pic}(\overline{V_P})) \cong \mathbb{Z}/3\mathbb{Z}$.

Since $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$, we can take two linearly independent elements v_1 and v_2. Now we define $\mathcal{P}$ as

$$\mathcal{P} = S \times S' \times S'', \quad S = (k^*)^3, \quad S' = v_1(k^*)^3, \quad S'' = v_2(k^*)^3.$$

We show that $\mathcal{P}$ satisfies the above three conditions. First, by Lemma 6.2.2 and the assumption that $(k^*)^3$ is infinite, the condition (i) holds. Secondly, for $P = (\lambda_0, \mu_0, \nu_0) \in \mathcal{P}$, we can take $\lambda_0' \in k^*$ such that $(\lambda_0')^3 = \lambda_0$, and have a rational point $(\lambda_0' : -1 : 0 : 0) \in V_P(k)$. Hence the condition (ii) holds. Finally, by the choice of v_1 and $v_2 \in k^*$ and Proposition 4.2.4, we can see that the condition (iii) holds.

The conditions (ii), (iii) and Lemma 3.2.18 imply $\mathrm{Br}(V_P)/\mathrm{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}$ for all $P \in \mathcal{P}$, that is $\mathcal{P} \subset \mathcal{P}_k$. Hence the condition (i) yields the condition (3).

This completes the proof of Proposition 6.2.1. □

Using this proposition, we obtain:

Corollary 6.2.3. *Let k be a field satisfying $(*)$ and $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$. Let $F = k(\lambda, \mu, \nu)$ and V the projective surface over F defined by the homogeneous equation $x^3 + \lambda y^3 + \mu z^3 + \lambda\mu\nu t^3 = 0$. Then there is no element $e \in \mathrm{Br}(V)$ satisfying the following property:*

there exists a dense open subset $W \subset (\mathbb{G}_{m,k})^3$ such that $\mathrm{sp}(e; \cdot)$ is defined on $W(k) \cap \mathcal{P}_k$ and for all $P \in W(k) \cap \mathcal{P}_k$, $\mathrm{sp}(e; P)$ is a generator of $\mathrm{Br}(V_P)/\mathrm{Br}(k)$.

At the end of this subsection, we give some examples of k satisfying the assumption of Corollary 6.2.3:

(1) a field finitely generated over $\mathbb{Q}(\zeta)$ or $\mathbb{Q}_p(\zeta)$.

(2) a function field of an integral variety over $\mathbb{C}$ or $\mathbb{R}$ of dimension ≥ 1.

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