p-進数体上のモチフィックホモロジーと類体論

Motivic Homology and Class Field Theory over p-adic Fields

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Abstract

The main goal of this thesis is to give a description of the abelian étale fundamental group of a smooth (not necessarily proper) variety U over a p-adic field k in case U has a smooth compactification that has a good reduction over k. The group SK_1 in the proper case is replaced with the motivic homology. We first construct a map between motivic homology and étale cohomolgy with compact supports where in certain degrees the latter one can be idetified with abelian étale fundamental group using Poincare duality. Following Yamazaki we construct a reciprocity map and calculate the kernel and cokernel using known results on the vanishing of Kato homology.

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Contents

Introduction

Let k be a finite extension of the p-adic numbers $\mathbb{Q}_p$. Local class field theory gives us the reciprocity map

$$k^* \to G_k$$

where G_k denotes the absolute Galois group of k. Now let X be a variety over k and let $X_{(a)}$ denote the set of points $x \in X$ such that $\dim\overline{\{x\}} = a$. For $x \in X_{(0)}$ we have a map $\pi_1^{ab}(x) \to \pi_1^{ab}(X)$ since the abelian étale fundamental group $\pi_1^{ab}(-)$ is covariantly functorial. Noting $G_{k(x)} \cong \pi_1^{ab}(x)$ we get a map

$$\rho'_X : \bigoplus_{x \in X_{(0)}} k(x)^* \to \pi_1^{ab}(X).$$

Calculating the kernel and the cokernel of this map is one of the main problems of the higher dimensional class field theory. In the case X is proper Bloch [1] and Saito [21] showed that this map factors through

$$SK_1(X) := coker[\bigoplus_{x \in X_{(1)}} K_2(k(x)) \xrightarrow{\partial} \bigoplus_{x \in X_{(0)}} k(x)^*]$$

where ∂ arises from the localization theory for Quillen's K-theory cf. [32].

In the case X is not proper over k, this does not hold. Therefore we need to find a larger quotient of $\bigoplus_{x \in X_{(0)}} k(x)^*$ such that the map ρ'_X factors through. In [35] Yamazaki defined such a quotient $C_1(X)$, which is recalled in Section 3, and defined a map

$$\rho_X : C_1(X) \to \pi_1^{ab}(X)$$

that is induced by ρ'_X. Let X be a smooth projective geometrically irreducible surface over k such that $\bar{X} := X \times_k \bar{k}$ is rational where $\bar{k}$ denotes an algebraic closure of k. Then for a dense open subset U of X Yamazaki showed an isomorphism [35, Thm. 6.5]

$$\rho_U/n : C_1(U)/n \to \pi_1^{ab}(U)/n$$

for all $n \in \mathbb{Z}_{>0}$. The goal of this paper to generalize this result to a scheme U (possibly non-proper) of any dimension that has a smooth compactification with good reduction over k. Main result (cf. Thm. 2.3.3) of this paper is the following theorem which implies the bijectivity of ρ_X/n in certain conditions.

Main Theorem. *Let U be a smooth variety of dimension d over a finite extension k of $\mathbb{Q}_p$. Assume there exists X such that U is an open variety of X that is projective, smooth and has good reduction over k. Then for all $n > 0$, we have a natural isomorphism*

$$c_{U,n}^{-1,-1} : H_{-1}^{M}(U, \mathbb{Z}/n(-1)) \xrightarrow{\cong} H_{\acute{e}t,c}^{2d+1}(U, \mathbb{Z}/n(d+1)).$$

The left hand side denotes the motivic homology group which is defined as certain homomorphism group in Voevodsky's triangulated category of motives [29] and it is shown to be isomorphic to $C_1(U)/n$ [35]. The right hand side is étale cohomology with compact supports and the Poincare duality identifies this group with $\pi_1^{ab}(U)/n$. The map $c_{U,n}^{i,j}$ is defined in Subsection 2.2 by using Ivorra's ℓ-adic realisation [8].

Main tool we use for the proof of the theorem is certain vanishing results on the Kato homology which is recalled in Subsection 1.3. In Subsection 2.1, it is shown to control the kernel and the cokernel of the maps $c_{U,n}^{i,-1}$ in case U is projective for all $i \in \mathbb{Z}$ by using a method due to Jannsen and Saito [12]. The proof of the main theorem is then follows by an induction argument given in Subsection 2.3. Throughout the paper a scheme X is always assumed to be smooth.

1 Preliminaries

In this section we first review some facts about motivic (co)homology and Kato Homology. We state some results which are used in the proof of The Main Theorem. Throughout the section we assume k to be a perfect field.

1.1 Higher Chow Groups

Higher Chow groups are algebraic analogue of the singular homology groups in topology defined by Bloch [2]. We briefly recall the definition and state some known facts.

Assume X is a quasi-projective scheme over k. Define

$$\Delta^q = Spec(k[t_0, \ldots, t_n]/(\sum t_i - 1)) \cong \mathbb{A}_k^q \qquad q \geq 0.$$

Let $z^i(X, q)$ denote the free abelian group generated by subvarities of $X \times_k \Delta^q$ of codimension i which meets all faces $X \times_k \Delta^t$ for $m \leq q$ properly. We have Bloch's cycle complex:

$$z^i(X,*) : \ldots \to z^i(X,2) \xrightarrow{\partial} z^i(X,1) \xrightarrow{\partial} z^i(X,0)$$

where $\partial = \sum(-1)^r \partial_r$ with

$$\partial_r : z^i(X,q) \to z^i(X,q-1)$$

the restriction map to the r^{th} codimension one face for $0 \leq r \leq q$.
We define the higher Chow Group as

$$CH^i(X,q) := H_q(z^i(X,*)).$$

We also define the finite coefficient version as

$$CH^i(X,q;\mathbb{Z}/n) := H_q(z^i(X,*)\otimes \mathbb{Z}/n).$$

If $i > q + dim(X)$ then we cannot have a subvariety of $X \times_k \Delta^q$ which has codimension i, so we have

$$CH^i(X,q) = 0 \text{ for } \ i > q + dim(X). \tag{1.1}$$

One has the coefficient sequence for finite coefficients

$$0 \to CH^i(X,q)/n \to CH^i(X,q;\mathbb{Z}/n\mathbb{Z}) \to CH^i(X,q-1)[n] \to 0.$$

Let $Y \subset X$ be closed and have pure codimension c. Then we have a localization sequence [2]

$$\ldots \to CH^i(X{-}Y,q{+}1) \to CH^{i-c}(Y,q) \to CH^i(X,q) \to CH^i(X{-}Y,q) \to CH^{i-c}(Y,q{-}1) \to \ldots$$

We also have funtoriality for higher Chow groups. Namely, higher Chow groups are covariant for proper morphisms and contravariant for flat morphisms.

1.2 Motivic Homology and Cohomology

We recall some facts about triangulated category of motives and define motivic (co)homology as certain homomorphism groups in this category.

Voevodsky [29] constructed $DM_{gm}(k)$ triangulated tensor category of motives over k which is equipped with a functor

$$\begin{aligned} M_{gm} : Sm(k) &\to DM_{gm}(k) \\ X &\mapsto M(X) \end{aligned}$$

where $Sm(k)$ denotes the category of smooth schemes over k and $M(X)$ is called the motive of X.

The category $DM_{gm}(k)$ has a tensor structure and a distinguished invertible object $\mathbb{Z}(1)$ called the Tate object [29, cf. Sec. 2]. For any object M in $DM_{gm}(k)$ we write $M(n) := M \otimes \mathbb{Z}(1)^{\otimes n}$.

The motivic cohomology (respectively homology) is given as the following homomorphism groups in $DM_{gm}(k)$

$$H^i_M(X,\mathbb{Z}(j)) = Hom_{DM_{gm}(k)}(M(X),\mathbb{Z}(j)[i])$$

and

$$H_i^M(X,\mathbb{Z}(j)) = Hom_{DM_{gm}(k)}(\mathbb{Z}(j)[i],M(X))$$

Note that $\mathbb{Z}(j)$ does not have a geometrical meaning if $j<0$ and $Hom_{DM_{gm}(k)}(M(X),\mathbb{Z}(j)[i])$ (resp. $Hom_{DM_{gm}(k)}(\mathbb{Z}(j)[i],M(X)))$ is a formal notation for $Hom_{DM_{gm}(k)}(M(X)(-j),\mathbb{Z}[i])$ (resp. $H_i^M(X,\mathbb{Z}(j)) = Hom_{DM^{eff}_{gm}(k)}(\mathbb{Z}[i],M(X)(-j)))$.

Analogously motivic (co)homology groups with finite coefficients are defined as homomorphism groups in $DM_{gm}(k,\mathbb{Z}/n)$ as [20, cf. Lec. 14]

$$H^i_M(X,\mathbb{Z}/n(j)) = Hom_{DM_{gm}(k,\mathbb{Z}/n)}(M(X,\mathbb{Z}/n),\mathbb{Z}/n(j)[i])$$

and

$$H_i^M(X,\mathbb{Z}/n(j)) = Hom_{DM_{gm}(k,\mathbb{Z}/n)}(\mathbb{Z}/n(j)[i],M(X,\mathbb{Z}/n))$$

where $M(X,\mathbb{Z}/n) = M(X)\otimes^{\mathbb{L}}\mathbb{Z}/n$ and $\mathbb{Z}/n(1) = \mathbb{Z}(1)\otimes\mathbb{Z}/n$. For a smooth variety X, motivic cohomology and higher Chow groups agree [30] with the following indices:

$$H^i_M(X,\mathbb{Z}(j)) \cong CH^j(X,2j-i). \tag{1.2}$$

Moreover, if X is proper smooth variety of pure dimension d, we have

$$H^i_M(X,\mathbb{Z}(j)) \cong H^M_{2d-i}(X,\mathbb{Z}(d-j)). \tag{1.3}$$

If we take X to be $Spec(k)$ then motivic cohomology is also related with the Milnor K-group of k by

$$H^i_M(X,\mathbb{Z}(i)) \cong K_i^M(k). \tag{1.4}$$

Let Z be a smooth closed subscheme of X everywhere of codimension c. There is a distinguished triangle [29, Sec. 2.2]

$$M(X-Z) \to M(X) \to M(Z)(c)[2c] \to M(X-Z)[1].$$

This gives rise to the following long exact sequence

$$\begin{aligned} \dots H_{i+1}^M(X,\mathbb{Z}/n(j)) &\to H_{i+1-2c}^M(Z,\mathbb{Z}/n(j-c)) \to H_i^M(X-Z,\mathbb{Z}/n(j)) \\ &\to H_i^M(X,\mathbb{Z}/n(j)) \to H_{i-2c}^M(Z,\mathbb{Z}/n(j-c)) \to \dots \end{aligned} \tag{1.5}$$

Now, we give a proof of the following known facts.

Lemma 1.2.1. *Let X be a smooth variety over k of pure dimension d.*
(a) If $j < 0$ or $i > j + d$ or $i > 2j$ then $H_M^i(X, \mathbb{Z}(j)) = 0$.
(b) If k has characteristic 0 and $i < j$ then $H_i^M(X, \mathbb{Z}(j)) = 0$.

Proof. (a) Consider the isomorphism (1.2)

$$H_M^i(X,\mathbb{Z}(j)) \cong CH^j(X, 2j-i).$$

The right hand side vanishes if $j < 0$ or $i > 2j$ by definition of higher Chow groups and it vanishes for $i > j + d$ by (1.1).
(b) Let $\bar{X}$ be a smooth compactification of X. Write $Z = \bar{X} - X$ There is a stratification

$$\bar{X} \supset Z_m \supset Z_{m-1} \supset \dots \supset Z_0.$$

where $Z_m = Z$, $Z_{i-1} = (Z_i)_{sing}$ for $0 \geq i \geq m$ and Z_0 is non-singular. We do induction on m and dimension of X.

The case $m = -1$ meaning X is projective, follows from part (a) and the isomorphism between motivic cohomology and homology (1.3). Also if the dimension of X is 0, X is again proper and the result follows similarly.

Let X be of dimension d with a stratification of length m. Write $X' := \bar{X} - Z_{m-1}$, $Z' := Z - Z_{m-1}$ and $c := codim_{X'} Z'$. Then Z' is smooth and we have the exact sequence (1.5) for $i < j$

$$H_{i+1-2c}^M(Z',\mathbb{Z}/n(j-c)) \to H_i^M(X,\mathbb{Z}/n(j)) \to H_i^M(X',\mathbb{Z}/n(j)).$$

The left group vanishes by the induction assumption on dimension and the right group vanishes on the induction assumption on m. Which implies $H_i^M(X, \mathbb{Z}/n(j)) = 0$ for $i < j$ as desired. □

We end this section by stating the Beilinson-Lichtenbaum conjecture which is shown to hold by the works of Suslin-Voevodsky [28] and Geisser-Levine [6] assuming Bloch-Kato conjecture. Thanks to the recent result of Rost and Voevodsky [26,30,33] proving the Bloch-Kato conjecture, it now holds unconditionally.

Theorem 1.2.2. *(Beilinson-Lichtenbaum Conjecture) [6, 28] Let X be a smooth variety over k, and let $\pi : X_{ét} \to X_{Zar}$ be the natural map of sites. Let $j \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{>0}$.*

(1) There is a canonical isomorphism

$$\pi^*\mathbb{Z}(j)_X \otimes^{\mathbb{L}} \mathbb{Z}/n \xrightarrow{\cong} \mathbb{Z}/n(j)_{ét} \tag{1.6}$$

where $\mathbb{Z}/n(j)_{ét}$ is the étale motivic complex. In particular we get

$$H^i_M(X, \mathbb{Z}/n(j)) \to H^i_{ét}(X, \mathbb{Z}/n(j)) \quad \textit{for any non-negative integer } i. \tag{1.7}$$

(2) The above map (1.6) induces an isomorphism

$$\mathbb{Z}(j)_X \otimes^{\mathbb{L}} \mathbb{Z}/n \to \tau_{\leq j} R\pi_* \mathbb{Z}/n(j)_{ét}.$$

In particular, the map (1.7) is an isomorphism if $i \leq j$. For $i = j+1$ it is an injection.

Remark 1.2.3. For n invertible in k we have a quasi-isomorphism $\mathbb{Z}/n(a)_{ét} \cong \mu_n^{\otimes a}$ [20, Cor. 6.4]. For $n = p^r$ where p is the characteristics of the field k, we have a quasi-isomorphism $\mathbb{Z}/n(a)_{ét} \cong W_r\Omega^a_{X,log}[-a]$ [5, Sec. 1.3] where $W_r\Omega^a_{X,log}$ denotes the logarithmic part of the Hodge-Witt sheaf [7].

1.3 Kato Homology

Let k be a global field and $P(k)$ denotes the set of finite places of k. There is an exact sequence

$$0 \to Br(k) \to \bigoplus_{v \in P(k)} Br(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0.$$

The injectivity of the first map is called the Hasse principle for central simple algebras over k. In [15] Kato formulated conjectures to generalize this principle to function fields and varieties over k.

Let k be any field, X be an excellent scheme over k and $n \geq 1$, i be integers. Let $X_{(a)}$ denote the the set $\{x \in X | dim\overline{\{x\}} = a\}$. Assume for any prime divisors p of n, any $x \in X_{(0)}$ such that characteristic of the field $k(x)$ is $p > 0$, we have $[k(x) : k(x)^p] \leq p^i$. Kato defined complexes [15]

$$KC^{(i)}(X, \mathbb{Z}/n) : \ldots \xrightarrow{\partial} \bigoplus_{x \in X_{(a)}} H^{a+i+1}(k(x), \mathbb{Z}/n(a+i)) \xrightarrow{\partial} \ldots \xrightarrow{\partial} \bigoplus_{x \in X_{(1)}} H^{2+i}(k(x), \mathbb{Z}/n(1+i)) \xrightarrow{\partial} \bigoplus_{x \in X_{(0)}} H^{i+1}(k(x), \mathbb{Z}/n(i)) \tag{1.8}$$

where $\mathbb{Z}/n(a) = \mu_n^{\otimes a}$ if $ch(k(x), n) = 1$ and $\mathbb{Z}/n(a) = W_m\Omega^a_{X,log}[-a]$ if $n = p^m$. Here $W_m\Omega^a_{X,log}$ denotes the logarithmic part of the Hodge-Witt sheaf [7].

Now for a scheme X separated of finite type over a perfect field k (which is the main case of interest in this paper) we consider the étale homology defined by

$$H_a(X, \mathbb{Z}/n(e)) := H^{-a}(X_{ét}, Rf^!\mathbb{Z}/n(-e)).$$

Here $f : X \to Spec(k)$ is the structure map and $Rf^!$ is the functor defined in [24, XVIII, 3.1.4]. If $n = p^m$ then we only consider the case $e = 0$ since $\mathbb{Z}/n(-e) = W_m\Omega^{-e}_{k,log}[e]$ vanishes if $e < 0$ and is not defined if $e > 0$. Therefore we consider the two cases either $char(k) = 0$ and e arbitrary or $char(k) = p > 0$ and $e = 0$. Then it is shown in [14] that the complex $KC^{(i)}(X, \mathbb{Z}/n)$ agrees up to well-defined signs, to the complex $E^1_{*,-i-1}$ arising from the niveau spectral sequence constructed by Bloch-Ogus [3], associated to the étale homology $H_a(X, \mathbb{Z}/n(-i))$. The maps ∂ in the above complex 1.8 in that case are just the d_1 differentials of the spectral sequence. We return to this point in section 2.

Definition 1.3.1. Kato homology of X with coefficients in $\mathbb{Z}/n$ is defined as

$$KH_a^{(i)}(X, \mathbb{Z}/n) = H_a(KC_*^{(i)}(X, \mathbb{Z}/n)).$$

To see the analogy let X be a projective smooth connected curve over $\mathbb{F}_p$ with function field k or $X = Spec(\mathcal{O}_k)$ where $\mathcal{O}_k$ is the integer ring of a number field k. Since $H^2(k, \mathbb{Z}/n(1)) \cong Br(k)[n]$ the complex $C^0(X, \mathbb{Z}/n)$ is then reduced to

$$Br(k)[n] \to \bigoplus_{x \in X_{(0)}} H^1(k(x), \mathbb{Z}/n).$$

In case $char(k) > 0$ or k is totally imaginary the classical Hasse Principle for the global field k is equivalent to $KH_1^{(0)}(X, \mathbb{Z}/n) = 0$. Kato stated the following conjecture

Conjecture 1.3.2. *[15] Let X be a connected proper smooth variety over a finite field $\mathbb{F}_p$. Then $KH_a^{(0)}(X, \mathbb{Z}/n) = 0$ for $a \geq 1$ and $KH_0^{(0)}(X, \mathbb{Z}/n) = \mathbb{Z}/n$.*

Kerz and Saito [17] proved the conjecture in case $(n, p) = 1$. Also Jannsen and Saito [12] showed that the conjecture holds for a projective smooth variety of dimension less than or equal to 4 for all n.

Let A be an henselian discrete valuation ring with finite residue field k and quotient field K. Let Y be a scheme flat of finite type over $S := Spec(A)$. Write $Y_\eta := Y \times_{Spec(A)} K$ and $Y_s := Y \times_{Spec(A)} k$ for generic and special fiber respectively. Assume the following diagram commutes and the squares are cartesian

$$\begin{array}{ccccc} Y_\eta & \longrightarrow & Y & \longleftarrow & Y_s \\ \downarrow f_\eta & & \downarrow f & & \downarrow f_s \\ \eta & \longrightarrow & S & \longleftarrow & s \end{array}$$

Kato defined canonical residue maps

$$\partial_Y^a : KH_a^{(1)}(Y_\eta, \mathbb{Z}/n) \to KH_a^{(0)}(Y_s, \mathbb{Z}/n)$$

Note that if $Y = Spec(A)$ this map reduces to

$$H^2(K, \mathbb{Z}/n(1)) \to H^1(k, \mathbb{Z}/n)$$

which is known to be an isomorphism. Kato generalized this to the following conjecture and proved it for Y of dimension 2 [15, Prop. 5.2].

Conjecture 1.3.3. *[15] Let Y be regular proper flat over A. Then ∂_Y^a is an isomorphism for all $a \geq 0$.*

This allows one to calculate Kato homology of the generic fiber from the one of the special fiber. It is an isomorphism for $a = 0$ or 1 by Jannsen-Saito [12, Thm 1.5]. Also Kerz-Saito [17, Thm. 1.8] proved that the conjecture holds when one restricts to n prime to the residue field characteristics. The following theorem is a direct consequence of these results combined with the results related to vanishing of the Kato homology of the special fiber.

Theorem 1.3.4. *Let X be a proper smooth variety over a finite extension K of $\mathbb{Q}_p$ with good reduction over K.*

(a) For all n prime to the residue field characteristic we have

$$KH_a^{(1)}(X, \mathbb{Z}/n) = 0 \text{ for } a \neq 0$$

and

$$KH_0^{(1)}(X, \mathbb{Z}/n) = \mathbb{Z}/n.$$

(b) For all $n > 0$, we have

$$KH_1^{(1)}(X, \mathbb{Z}/n) = 0.$$

Furthermore if X is projective with dimension less than or equal to 2, we also have

$$KH_2^{(1)}(X, \mathbb{Z}/n) = 0.$$

Proof. Part (a) follows from Kerz-Saito [17, Thm. 1.8]. The first statement of part (b) follows from [11, Thm.1.5] which shows the Conjecture 1.3.3 for the degree 1. Then the result follows using [12, Thm. 0.4] which shows the vanishing of the Kato homology of the special fiber.

We now show the second statement. By [11, Cor. 6.9] $KH_2^{(1)}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for a projective smooth variety X. Here $KH_i^{(1)}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is defined taking the inductive limit $\varinjlim_\nu KH_i^{(1)}(X, \mathbb{Z}/\ell^\nu)$ by for a fixed prime ℓ. Using Bloch-Kato conjecture we get an exact sequence [11, Lemma 7.3]

$$0 \to KH_{i+1}^{(1)}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)/\ell^\nu \to KH_i^{(1)}(X, \mathbb{Z}/\ell^\nu) \to KH_i^{(1)}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^\nu] \to 0.$$

This gives the desired result since for a surface $H_3^K(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ vanishes by definition. $\square$

2 | Main Theorem

In this section we give a proof of main theorem. First we will obtain the necessary tools which will be used in the proof. An important result is the comparison of higher Chow groups and étale cohomology which is essential for the proof of the main theorem. The main idea is to compare the niveau spectral sequence related to both theories. After establishing this comparison we will prove our main theorem by an induction argument on the dimension.

2.1 Comparison Theorem

Following a method due to Jannsen-Saito [12] we prove the following comparison theorem between higher Chow groups and étale cohomology. The idea is to compare the niveau spectral sequence for higher Chow groups and étale cohomology using Beilinson-Lichtenbaum conjecture 1.2.2. We see that in the case X is over $\mathbb{Q}_p$, Kato homology $KH_*^{(1)}(X,\mathbb{Z}/n)$ comes into play and it is shown to control the kernel and cokernel of the map $CH^{d+1}(X,i;\mathbb{Z}/n) \to H^{2d+1}_{\acute{e}t}(X,\mathbb{Z}/n(d+1))$ which is important for our main application to class field theory.

Before we state and prove the comparison theorem we make a small detour to étale homology of a variety over a field of characteristic 0 and construct the spectral sequence we use in the proof of the theorem. We follow similar steps as in [10, Section 4]. Let d denote the dimension of X. We fix an integer e and we consider the étale homology theory given by [11, cf. Sec. 2]

$$H_a(X,\mathbb{Z}/n(e)) := H^{-a}(X_{\acute{e}t}, Rf^!\mathbb{Z}/n(-e))$$

where $f : X \to Spec(k)$ is the structure map. Here $\mathbb{Z}/n(i)$ is the usual i fold Tate twist of the constant sheaf $\mathbb{Z}/n$. If f is smooth, then Poincare duality [24, XVIII, 3.2.5] gives us an isomorphism of sheaves

$$Rf^!\mathbb{Z}/n(-e) \cong \mathbb{Z}/n(d-e)[2d].$$

Taking cohomology one gets the so called purity isomorphism

$$H_a(X, \mathbb{Z}/n(e)) \cong H^{2d-a}(X, \mathbb{Z}/n(d-e)).$$

There is a niveau spectral sequence associated to this homology theory [3] given by

$$E^1_{p,q}(X, \mathbb{Z}/n(e)) = \bigoplus_{x \in X_{(p)}} H_{p+q}(x, \mathbb{Z}/n(e)) \Rightarrow H_{p+q}(X, \mathbb{Z}/n(e)).$$

By definition $H_a(x, \mathbb{Z}/n(e)) = \varinjlim_{U \subseteq \overline{\{x\}}} H_a(U, \mathbb{Z}/n(e))$. The limit runs through all the dense open subsets U of $\overline{\{x\}}$. Since k is perfect $\overline{\{x\}}$ is generically smooth and we can run the limit on smooth U. Combining this with the above purity isomorphism we get

$$\begin{aligned} H_a(x, \mathbb{Z}/n(e)) := \varinjlim_{U \subseteq \overline{\{x\}}} H_a(U, \mathbb{Z}/n(e)) &\cong \varinjlim_{U \subseteq \overline{\{x\}}} H^{2p-a}(U, \mathbb{Z}/n(p-e)) \\ &\cong H^{2p-a}(k(x), \mathbb{Z}/n(p-e)). \end{aligned}$$

The last isomorphism follows since $\varinjlim_{U \subseteq \overline{\{x\}}} U = Spec(k(x))$ and étale cohomology commutes with the limit.

Using this we can rewrite the spectral sequence as

$$E^1_{p,q}(X, \mathbb{Z}/n(e)) = \bigoplus_{x \in X_{(p)}} H^{p-q}(x, \mathbb{Z}/n(p-e)) \Rightarrow H^{2d-p-q}(X, \mathbb{Z}/n(d-e)).$$

In [14], it is shown that $E^1_{*,-i-1}(X, \mathbb{Z}/n(-i))$ is quasi-isomorphic to the complex $KC^{(i)}(X, \mathbb{Z}/n)$.

Now we prove the following lemma which allows us to compare two first quadrant spectral sequences which only differ on the horizontal axis.

Lemma 2.1.1. *Let $E = \bigoplus E^1_{a,b}$ and $\bar{E} = \bigoplus \bar{E}^1_{a,b}$ be two first quadrant spectral sequences that strongly converge to H_* and $\bar{H}_*$ respectively. Assume $E^1_{a,0} = 0$ for all a and we have a map $\rho : E \to \bar{E}$ of spectral sequences such that*

$$\rho^1_{a,b} : E^1_{a,b} \to \bar{E}^1_{a,b}$$

is an isomorphism for $b \geq 1$. Then we have a long exact sequence

$$\ldots \to \bar{H}_{n+1} \to \bar{E}^2_{n+1,0} \to H_n \to \bar{H}_n \to \bar{E}^2_n \to H_{n-1} \to \ldots$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} E^r_{a+r,b-r+1} & \longrightarrow & E^r_{a,b} & \longrightarrow & E^r_{a-r,b+r-1} \\ \downarrow \rho^r_{a+r,b-r+1} & & \downarrow \rho^r_{a,b} & & \downarrow \rho^r_{a-r,b+r-1} \\ \bar{E}^r_{a+r,b-r+1} & \longrightarrow & \bar{E}^r_{a,b} & \longrightarrow & \bar{E}^r_{a-r,b+r-1} \end{array} \qquad (*)$$

Note that injectivity of $\rho^r_{a,b}$ and surjectivity of $\rho^r_{a+r,b-r+1}$ implies injectivity of $\rho^{r+1}_{a,b}$. Also surjectivity of $\rho^r_{a,b}$ and injectivity of $\rho^r_{a-r,b+r-1}$ implies surjectivity of $\rho^{r+1}_{a,b}$. Using this and applying induction on r we see that $\rho^r_{a,b}$ is surjective if $b \geq 1$ and injective if $b \geq r-1$. This implies $\rho^\infty_{a,b} : E^\infty_{a,b} \to \bar{E}^\infty_{a,b}$ is surjective if $b \geq 1$.

Now we look at the kernel of $\rho^\infty_{a,b}$ for $b \geq 1$. Consider the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & E^{b+1}_{a,b} & \longrightarrow & E^{b+1}_{a-b-1,2b} \\ \downarrow & & \downarrow \rho^{b+1}_{a,b} & & \downarrow \rho^{b+1}_{a-b-1,2b} \\ \bar{E}^{b+1}_{a+b+1,0} & \longrightarrow & \bar{E}^{b+1}_{a,b} & \longrightarrow & \bar{E}^{b+1}_{a-b-1,2b} \end{array}$$

From the above calculation we see that both $\rho^{b+1}_{a,b}$ and $\rho^{b+1}_{a-b-1,2b}$ are isomorphisms. This implies the kernels of the right horizontal maps are isomorphic. Since $E^{b+2}_{a,b} = Ker(d^{b+1}_{a,b})$ and $\bar{E}^{b+2}_{a,b} = Ker(\bar{d}^{b+1}_{a,b})/Im(\bar{d}^{b+1}_{a+b+1,0})$ we have the following exact sequence

$$\bar{E}^{b+1}_{a+b+1,0} \to E^{b+2}_{a,b} \to \bar{E}^{b+2}_{a,b} \to 0$$

and since the first map is the same as $\bar{d}^{b+1}_{a+b+1,0}$ the kernel of it is $\bar{E}^{b+2}_{a+b+1,0}$. This gives the following exact sequence

$$0 \to \bar{E}^{b+2}_{a+b+1,0} \to \bar{E}^{b+1}_{a+b+1,0} \to E^{b+2}_{a,b} \to \bar{E}^{b+2}_{a,b} \to 0$$

If we write the above diagram $(*)$ for the r-th sheet where $r \geq b+2$, the left hand side vanishes for both spectral sequences. This implies for ∞ terms:

$$0 \to \bar{E}^{b+2}_{a+b+1,0} \to \bar{E}^{b+1}_{a+b+1,0} \to E^\infty_{a,b} \to \bar{E}^\infty_{a,b} \to 0$$

Finally if we look at the case $a = 0$ we get

$$0 \to \bar{E}^{\infty}_{b+1,0} \to \bar{E}^{b+1}_{b+1,0} \to E^{\infty}_{0,b} \to \bar{E}^{\infty}_{0,b} \to 0$$

since $\bar{E}^{\infty}_{a+b+1,0} = \bar{E}^{a+b+2}_{a+b+1,0}$.

Let F_pH_q, $F_p\bar{H}_q$ denote the corresponding filtrations of H_q and $\bar{H}_q$ respectively. Convergence gives us the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{p-1}H_q & \longrightarrow & F_pH_q & \longrightarrow & E^{\infty}_{p,q-p} & \longrightarrow & 0 \\
 & & \downarrow \rho^q_{p-1} & & \downarrow \rho^q_p & & \downarrow \rho^{\infty}_{p,q-p} & & \\
0 & \longrightarrow & F_{p-1}\bar{H}_q & \longrightarrow & F_p\bar{H}_q & \longrightarrow & \bar{E}^{\infty}_{p,q-p} & \longrightarrow & 0
\end{array}$$

For $p = 0$, the left hand groups vanish. Therefore $ker(\rho^q_0) \cong Ker(\rho^{\infty}_{0,q})$ which is isomorphic to $ker_0 := \bar{E}^{q+1}_{q+1,0}/\ \bar{E}^{\infty}_{q+1,0}$. Also since $\rho^{\infty}_{0,q}$ is surjective, we have $F_0H_q/(\bar{E}^{q+1}_{q+1,0}/\ \bar{E}^{\infty}_{q+1,0}) \cong F_0\bar{H}_q$. Combining this with the above commutative diagram, we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_0H_q/ker_0 & \longrightarrow & F_1H_q/ker_0 & \longrightarrow & E^{\infty}_{1,q-1} & \longrightarrow & 0 \\
 & & \downarrow \rho^q_0 & & \downarrow \rho^q_1 & & \downarrow \rho^{\infty}_{1,q-1} & & \\
0 & \longrightarrow & F_0\bar{H}_q & \longrightarrow & F_1\bar{H}_q & \longrightarrow & \bar{E}^{\infty}_{1,q-1} & \longrightarrow & 0
\end{array}$$

The upper row is still exact. Also since ker_0 injects into $ker(\rho^q_1)$, we still have commutativity. The left hand map is an isomorphism which implies as before $ker(\rho^q_1) \cong Ker(\rho^{\infty}_{1,q-1})$ which is isomorphic to $ker_1 := \bar{E}^q_{q+1,0}/\ \bar{E}^{q+1}_{q,0}$. Noting $ker_1 \cong (\bar{E}^q_{q+1,0}/\bar{E}^{\infty}_{q,0})/(\bar{E}^{q+1}_{q+1,0}/\bar{E}^{\infty}_{q,0})$, we see that

$$(F_1H_q/ker_0)/ker_1 \cong F_1H_q/(\bar{E}^q_{q+1,0}/\bar{E}^{\infty}_{q+1,0}) \cong F_1\bar{H}_q.$$

Applying the same method gives $ker(\rho^q_{q-1} : F_{q-1}H_q \to F_{q-1}\bar{H}_q) \cong \bar{E}^2_{q+1,0}/\ \bar{E}^{\infty}_{q+1,0}$. Finally we look at the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{q-1}H_q & \longrightarrow & H_q & \longrightarrow & 0 & & \\
 & & \downarrow \rho^q_{q-1} & & \downarrow \rho^q_X & & \downarrow & & \\
0 & \longrightarrow & F_{q-1}\bar{H}_q & \longrightarrow & \bar{H}_q & \longrightarrow & \bar{E}^{\infty}_{q,0} & \longrightarrow & 0
\end{array}$$

By snake lemma we find $ker(\rho_X^q) = ker(\rho_{q-1}^q) \cong \bar{E}^2_{q+1,0}/\ \bar{E}^\infty_{q+1,0}$ and $coker(\rho_X^q) \cong \bar{E}^\infty_{q,0}$. This gives us exact sequences, for all $q \geq 0$

$$0 \to \ \bar{E}^\infty_{q+1,0} \to \ \bar{E}^2_{q+1,0} \to H_q \to \bar{H}_q \to \ \bar{E}^\infty_{q,0} \to 0$$

Since $E^\infty_{q,0}$ injects into $E^2_{q,0}$ we can combine these short exact sequences to get the desired long exact sequence. □

Theorem 2.1.2. *Let X be a smooth variety of pure dimension d over a field k of characteristic 0.*

(a) The map

$$\rho_X^{i,j} : CH^j(X, i; \mathbb{Z}/n) \to H^{2j-i}_{\acute{e}t}(X, \mathbb{Z}/n(j))$$

is an isomorphism for $j \geq d + cd(k)$, where $cd(k)$ denotes the cohomological dimension of k.

(b) Let $j = d + c$ where $c = cd(k) - 1$. Then we have a long exact sequence

$$\begin{aligned} \dots \ \to \ & KH^{(c)}_{q-c+2}(X, \mathbb{Z}/n) \to CH^{d+c}(X, q; \mathbb{Z}/n) \to H^{2d-q+2c}_{\acute{e}t}(X, \mathbb{Z}/n(d+c)) \\ \to \ & KH^{(c)}_{q-c+1}(X, \mathbb{Z}/n) \to CH^{d+c}(X, q-1; \mathbb{Z}/n) \to H^{2d-q+2c+1}_{\acute{e}t}(X, \mathbb{Z}/n(d+c)) \to \dots \end{aligned}$$

In particular if k is a finite extension of $\mathbb{Q}_p$ we have an exact sequence

$$\dots \to KH^{(1)}_{q+1}(X, \mathbb{Z}/n) \to CH^{d+1}(X, q; \mathbb{Z}/n) \to H^{2d-q+2}_{\acute{e}t}(X, \mathbb{Z}/n(d+1)) \to KH^{(1)}_q(X, \mathbb{Z}/n) \to \dots$$

Proof. We follow essentially the same steps as Jannsen-Saito [11, Pf. of Lemma 6.2] with a shift because of the change in cohomological dimension.

Using the localization sequence for higher Chow groups [2, 19] we get a niveau spectral sequence

$${}^{CH}E^1_{a,b}(j-d) = \bigoplus_{x \in X_{(a)}} CH^{a+j-d}(x, a+b; \mathbb{Z}/n) \Rightarrow CH^j(X, a+b; \mathbb{Z}/n).$$

Also for étale cohomology using purity we get

$${}^{\acute{e}t}E^1_{a,b}(j-d) = \bigoplus_{x \in X_{(a)}} H^{a-b+2j-2d}_{\acute{e}t}(x, \mathbb{Z}/n(a+j-d)) \Rightarrow H^{2j-a-b}_{\acute{e}t}(X, \mathbb{Z}/n(j)).$$

We note that this is the same as the above spectral sequence $E^1_{p,q}(X, \mathbb{Z}/n(-c))$ with a shift given by $b = q + 2c$. We have cycle class maps

$$\rho_x^{a+j-d,a+b} : CH^{a+j-d}(x, a+b; \mathbb{Z}/n) \to H_{ét}^{a-b+2j-2d}(x, \mathbb{Z}/n(a+j-d)).$$

These maps give us a map of spectral sequences. This follows from the construction of niveau spectral sequence and the fact that cycle class maps are functorial with respect to the localization sequence.

By theorem 1.2.2 the above map is an isomorphism if $a + j - d \geq a - b + 2j - 2d$ i.e. for $b \geq j - d$. If $b < j - d$ then ${}^{CH}E^1_{a,b} = 0$ by (1.1). Also for $b < 2(j-d) - cd(k)$, ${}^{ét}E^1_{a,b}(j-d) = 0$ since $cd(k(x)) = a + cd(k)$. So if $(j - d) \geq cd(k)$, the two spectral sequences are isomorphic and we get the desired isomorphism.

Now we show the second statement. Let $j = d + c$, where $c = cd(k) - 1$. In that case the only difference occurs on the line $b = c - 1$ where ${}^{CH}E^1_{a,b}(c) = 0$ but ${}^{ét}E^1_{a,b}(c)$ might not be 0. We can apply Lemma 2.1.1 after a shift of indexes to get

$$\begin{aligned} \cdots \to\ & {}^{ét}E^2_{q-c+2,c-1}(c) \to CH^{d+c}(X, q; \mathbb{Z}/n) \to H_{ét}^{2d-q+2c}(X, \mathbb{Z}/n(d+c)) \\ \to\ & {}^{ét}E^2_{q-c+1,c-1}(c) \to CH^{d+c}(X, q-1; \mathbb{Z}/n) \to H_{ét}^{2d-q+2c+1}(X, \mathbb{Z}/n(d+c)) \to \cdots \end{aligned}$$

This implies the desired sequence since ${}^{ét}E^2_{q,c-1}(c) = KH^{(c)}_q(X, \mathbb{Z}/n)$. Next statement follows directly from the fact that $cd(k) = 2$ for a finite extension k of $\mathbb{Q}_p$. □

2.2 The map $c^{i,j}_{X,n}$

In this section we give a construction of the following map

$$c^{i,j}_{X,n} : H_i^M(X, \mathbb{Z}/n(j)) \to H^{2d-i}_{ét,c}(X, \mathbb{Z}/n(d-j))$$

for a smooth and irreducible variety X of dimension d over k and for any i, j and positive n prime to characteristic of k.

Let $\pi : X \to Spec(k)$ be the structure map. Fix n such that $(n, ch(k)) = 1$. We have a canonical functor

$$\begin{aligned} Sm(k)^{op} &\to D^b_c(k, \mathbb{Z}/n) \\ X &\mapsto R\pi_*(\mathbb{Z}/n)_X \end{aligned}$$

where $D^b_c(k, \mathbb{Z}/n)$ denotes the derived category of complexes of étale sheaves of $\mathbb{Z}/n$ modules with bounded constructible cohomology sheaves and $(\mathbb{Z}/n)_X$ is the constant sheaf on X. Ivorra showed that [8, cf. Theorem 4.3] this extends to a canonical tensor triangulated functor

$$\begin{aligned} DM_{gm}(k,\mathbb{Z}/n)^{op} &\to D^b_c(k,\mathbb{Z}/n) \\ M(X) &\mapsto R\pi_*(\mathbb{Z}/n)_X \end{aligned}$$

This functor sends $\mathbb{Z}/n(j)[i]$ to $\mathbb{Z}/n(-j)[-i]$ [8, cf. Lemma 4.7]. This induces the following map

$$Hom_{DM_{gm}(k,\mathbb{Z}/n)}(\mathbb{Z}/n(j)[i], M(X)) \to Hom_{D^b_c(k,\mathbb{Z}/n)}(R\pi_*(\mathbb{Z}/n)_X, \mathbb{Z}/n(-j)[-i]).$$

Left hand side is isomorphic to motivic homology $H^M_i(X,\mathbb{Z}/n(j))$ by definition. By the following lemma we get the desired maps $c^{i,j}_{X,n}$.

Lemma 2.2.1. *There is a canonical isomorphism*

$$Hom_{D^b_c(k,\mathbb{Z}/n)}(R\pi_*(\mathbb{Z}/n)_X, \mathbb{Z}/n(-j)[-i]) \cong H^{2d-i}_{ét,c}(X,\mathbb{Z}/n(d-j))$$

Proof. $\mathbb{Z}/n$ is a dualizing complex on $Spec(k)_{ét}$ by [25, Expose I, Thm. 3.4.1], i.e. writing $D(F) = Hom_{D^b_c(k,\mathbb{Z}/n)}(F,\mathbb{Z}/n)$ for any $F \in D^b_c(k,\mathbb{Z}/n)$ we have a functorial isomorphism

$$Hom_{D^b_c(k,\mathbb{Z}/n)}(G \otimes^{\mathbb{L}} F, \mathbb{Z}/n) \cong Hom_{D^b_c(k,\mathbb{Z}/n)}(G, D(F)) \tag{2.1}$$

Noting $R\pi^!\mathbb{Z}/n \cong (\mathbb{Z}/n)_X(d)[2d]$ [24, cf. 3.2.5 XVIII] where $R\pi^!$ is the right adjoint of $R\pi_!$, we get an isomorphism by [25, Expose I, Prop. 1.12]

$$D(R\pi_!(\mathbb{Z}/n)_X) \cong R\pi_*(\mathbb{Z}/n)_X(d)[2d]$$

Since $D \circ D$ is the identity we get

$$D(R\pi_*(\mathbb{Z}/n)_X) \cong R\pi_!(\mathbb{Z}/n)_X(d)[2d] \tag{2.2}$$

This gives us the desired isomorphism since

$$\begin{aligned} Hom_{D^b_c(k,\mathbb{Z}/n)}(R\pi_*(\mathbb{Z}/n)_X, \mathbb{Z}/n(-j)[-i]) &\cong Hom_{D^b_c(k,\mathbb{Z}/n)}(\mathbb{Z}/n, D(R\pi_*(\mathbb{Z}/n)_X)(-j)[-i]) \\ &\cong Hom_{D^b_c(k,\mathbb{Z}/n)}(\mathbb{Z}/n, R\pi_!(\mathbb{Z}/n(d-j)[2d-i])) \\ &\cong H^{2d-i}_{ét,c}(X, \mathbb{Z}/n(d-j)). \end{aligned}$$

□

Here the first isomorphism follows from 2.1 and the second one follows from 2.2.

If X is projective then we have an isomorphism $H^M_i(X, \mathbb{Z}/n(j)) \cong CH^{d-j}(X, i-2j; \mathbb{Z}/n)$. In that case from [9, Prop. 3.5] $c^{i,j}_{X,n}$ agrees with the cycle class map [1,6]

$$CH^{d-j}(X, i-2j; \mathbb{Z}/n) \to H^{2d-i}_{ét}(X, \mathbb{Z}/n(d-j)).$$

Remark 2.2.2. This map is first constructed by Schmidt and Spiess in a different way in [27].

2.3 Proof of The Main Theorem

Using Thm. 2.1.2 we prove the following proposition which gives us isomorphism between motivic homology and étale cohomology with compact supports in certain degrees.

Proposition 2.3.1. *Let U be a smooth variety of pure dimension d over a field k of characteristic 0. Then we have isomorphisms*

$$c^{i,-a}_{X,n} : H^M_i(U, \mathbb{Z}/n(-a)) \to H^{2d-i}_{ét,c}(U, \mathbb{Z}/n(d+a)) \quad \textit{for any } a \geq cd(k).$$

is an isomorphism where $cd(k)$ denotes the cohomological dimension of the field k.

Proof. Consider a smooth compactification X of U and write $Z := X - U$. There is a stratification

$$X \supset Z_m \supset Z_{m-1} \supset \ldots \supset Z_0.$$

where $Z_m = Z$, $Z_{i-1} = (Z_i)_{sing}$ for $0 \geq i \geq m$ and Z_0 is non-singular. We do induction on m and dimension d of U.

The cases $m=-1$ or $d=0$ implies U is projective so we have an isomorphism

$$H_i^M(U,\mathbb{Z}/n(-a)) \cong CH^{d+a}(U,2a+i;\mathbb{Z}/n) \xrightarrow{\simeq} H_{\acute{e}t,c}^{2d-i}(U,\mathbb{Z}/n(d+a))$$

where the first isomorphism follows from 1.2 and 1.3) and the second isomorphism follows from Thm. 2.1.2 since $a \geq cd(k)$.

To show it holds in general let U be of dimension d with a stratification of length m. Write $X' := X - Z_{m-1}$, $Z' := Z - Z_{m-1}$ and $c := codim_{X'}Z'$. Then consider the commutative diagram

$$\begin{array}{ccc}
H_{i+1}^M(X',\mathbb{Z}/n(-a)) & \xrightarrow{c_{X',n}^{i+1,-a}} & H_{\acute{e}t,c}^{2d-i-1}(X',\mathbb{Z}/n(d+a)) \\
\downarrow & & \downarrow \\
H_{i+1-2c}^M(Z',\mathbb{Z}/n(-a-c)) & \xrightarrow{c_{Z',n}^{i+1-2c,-a-c}} & H_{\acute{e}t,c}^{2d-i-1}(Z',\mathbb{Z}/n(d+a)) \\
\downarrow & & \downarrow \\
H_i^M(U,\mathbb{Z}/n(-a)) & \xrightarrow{c_{U',n}^{i,-a}} & H_{\acute{e}t,c}^{2d-i}(U,\mathbb{Z}/n(d+a)) \\
\downarrow & & \downarrow \\
H_i^M(X',\mathbb{Z}/n(-a)) & \xrightarrow{c_{X',n}^{i,-a}} & H_{\acute{e}t,c}^{2d-i}(X',\mathbb{Z}/n(d+a)) \\
\downarrow & & \downarrow \\
H_{i-2c}^M(Z',\mathbb{Z}/n(-a-c)) & \xrightarrow{c_{Z',n}^{i-2c,-a-c}} & H_{\acute{e}t,c}^{2d-i}(Z',\mathbb{Z}/n(d+a))
\end{array}$$

$c_{X',n}^{i+1,-a}$ and $c_{X',n}^{i,-a}$ are isomorphisms by the induction assumption on m, $c_{Z',n}^{i+1-2c,-a}$ and $c_{Z',n}^{i-2c,-a}$ are isomorphisms by the induction assumption on the dimension. Therefore by five lemma $c_{U',n}^{i,-a}$ is also an isomorphism. □

The following proposition gives the desired isomorphism in the compact case.

Proposition 2.3.2. *Let k be a finite extension of $\mathbb{Q}_p$. Let X be a proper smooth variety of dimension d with good reduction over k.*

(a) $c^{i,-1}_{X,n}$ is an isomorphism if $i \neq -2$ and injective if $i = -2$ for all n such that $(n,p) = 1$.

(b) If X is projective, then for all $n > 0$, $c^{i,-1}_{X,n}$ is an isomorphism if $i = -1$ and surjective if $i = 0$.

Proof. (a) This is a direct consequence of Thm. 2.1.2, the isomorphism (1.2) combined with Thm. 1.3.4(a).

(b) The surjectivity of the map $c^{-1,-1}_{X,n}$ follows from the vanishing of the Kato homology of X in degree 1, 1.3.4(b). Injectivity is shown in [13, Thm. 6] by using the vanishing of $KH^{(1)}_2(X, \mathbb{Z}/n)$ for a projective variety X of dimension less than or equal to 2 and doing induction on the dimension using Lefschetz pencils. Now we show the surjectivity of

$$c^{0,-1}_{X,n} : CH^{d+1}(X, 2; \mathbb{Z}/n) \to H^{2d}_{\acute{e}t}(X, \mathbb{Z}/n(d+1)).$$

Thm. 2.1.2(b) and Thm. 1.3.4(b) implies the surjectivity in case $dimX \leq 2$. We will do induction on the dimension of X. Assume $dimX > 2$. By [13, Cor. 0] there exist a projective smooth hyperplane section $Y \subset X$ which has good reduction. Now consider the following commutative diagram which arises from the corresponding localization sequences.

$$\begin{array}{ccc} CH^d(Y, 2; \mathbb{Z}/n) & \longrightarrow & H^{2d-2}_{\acute{e}t}(Y, \mathbb{Z}/n(d)) \\ \downarrow & & \downarrow \\ CH^{d+1}(X, 2; \mathbb{Z}/n) & \longrightarrow & H^{2d}_{\acute{e}t}(X, \mathbb{Z}/n(d+1)) \end{array}$$

Upper horizontal map is surjective by the induction assumption. Therefore in order to show the surjectivity of the lower horizontal map it is enough to show the surjectivity of the right vertical map. Consider the localization sequence

$$\ldots \to H^{2d-2}_{\acute{e}t}(Y, \mathbb{Z}/n(d)) \to H^{2d}_{\acute{e}t}(X, \mathbb{Z}/n(d+1)) \to H^{2d}_{\acute{e}t}(X - Y, \mathbb{Z}/n(d+1)) \to \ldots$$

We note that $X - Y$ is affine and $2d > d + 2$ since $d > 2$. This implies the vanishing of $H^{2d}_{\acute{e}t}(X - Y, \mathbb{Z}/n(d+1))$. Therefore the desired surjectivity follows completing the proof of the proposition. □

Now we are ready to prove our main theorem.

Theorem 2.3.3. *Let U be a smooth variety of dimension d over a finite extension k of $\mathbb{Q}_p$. Assume there exists X such that U is an open variety of X that is proper smooth and has good reduction over k. Then*

(a) For $(n,p)=1$ where p is the residue field characteristic of k, the map

$$c^{i,-1}_{U,n} : H^M_i(U, \mathbb{Z}/n(-1)) \to H^{2d-i}_{\text{ét},c}(U, \mathbb{Z}/n(d+1))$$

is an isomorphism for all $i \geq -1$.

(b) If also X is projective, the map

$$c^{i,-1}_{U,n} : H^M_{-1}(U, \mathbb{Z}/n(-1)) \to H^{2d-i}_{\text{ét},c}(U, \mathbb{Z}/n(d+1))$$

is an isomorphism if $i=-1$ and surjective if $i=0$.

Proof. The proof follows similar steps as the proof of Prop. 2.3.1. Let the notation be the same as in the Prop. 2.3.1. Consider the following commutative diagram

$$\begin{array}{ccc}
H^M_{i+1}(X', \mathbb{Z}/n(-1)) & \xrightarrow{c^{i+1,-1}_{X',n}} & H^{2d-i-1}_{\text{ét},c}(X', \mathbb{Z}/n(d+1)) \\
\downarrow & & \downarrow \\
H^M_{i+1-2c}(Z', \mathbb{Z}/n(-1-c)) & \xrightarrow{c^{i+1-2c,-1-c}_{Z',n}} & H^{2d-i-1}_{\text{ét},c}(Z', \mathbb{Z}/n(d+1)) \\
\downarrow & & \downarrow \\
H^M_i(U, \mathbb{Z}/n(-1)) & \xrightarrow{c^{i,-1}_{U',n}} & H^{2d-i}_{\text{ét},c}(U, \mathbb{Z}/n(d+1)) \\
\downarrow & & \downarrow \\
H^M_i(X', \mathbb{Z}/n(-1)) & \xrightarrow{c^{i,-1}_{X',n}} & H^{2d-i}_{\text{ét},c}(X', \mathbb{Z}/n(d+1)) \\
\downarrow & & \downarrow \\
H^M_{i-2c}(Z', \mathbb{Z}/n(-1-c)) & \xrightarrow{c^{i-2c,-1-c}_{Z',n}} & H^{2d-i}_{\text{ét},c}(Z', \mathbb{Z}/n(d+1))
\end{array}$$

If $m=-1$, i.e. U is smooth proper and has good reduction over k, the result follows from Prop. 2.3.2. The maps $c^{i+1-2c,-1-c}_{Z',n}$ and $c^{i-2c,-1-c}_{Z',n}$ are isomorphisms by Prop. 2.3.1. In case $(n,p)=1$, $c^{i+1,-1}_{X',n}$ and $c^{i,-1}_{X',n}$ are isomorphisms by induction assumption on m which proves part (a) by five lemma. Similarly, if $i=-1$, then for all n, $c^{0,-1}_{X',n}$ is surjective and $c^{-1,-1}_{X',n}$ is an isomorphism by induction assumption on m which proves (b).

□

3 Class Field Theory

3.1 Wiesend's Tame Ideal Class Group

In this subsection we define Wiesend's tame ideal class group as defined in [35, Def. 1.1] based on an earlier work of Wiesend [33].

Let X be a variety over a field k. Take $y \in X_{(1)}$ and $C(y)$ be the closure of y in X. Denote by $\tilde{C}(y)$ the normalization of $C(y)$ and $\bar{C}(y)$ smooth completion of $\tilde{C}(y)$. Write $C_\infty(y) := \bar{C}(y) \backslash \tilde{C}(y)$. Define:

$$UK_r(y) := ker[K_r^M(k(y)) \to \bigoplus_{x \in C_\infty(y)} (K_{r-1}^M(k(x)) \oplus K_r^M(k(x)))].$$

where the x-component of the map is defined by $a \mapsto (\partial_x(a), \partial_x(\pi_x \cup a))$ where π_x is a uniformizer at x and $\partial_x(a)$ is the tame symbol at x. The kernel of the map, therefore the group $UK_r(y)$ does not depend on the choice of π_x. When $r = 1$, it is the group of rational functions on $\bar{C}(y)$ which takes value 1 at all points of $C_\infty(y)$.

Definition 3.1.1. [35, Def. 1.1] Wiesend's tame ideal class group in degree r is defined as:

$$C_r(X) := coker[\bigoplus_{y \in X_{(1)}} UK_{r+1}(y) \xrightarrow{\partial} \bigoplus_{x \in X_{(0)}} K_r^M(k(x))].$$

where ∂ is given by the composition of the natural injection $UK_{r+1}(y) \hookrightarrow K_{r+1}^M(k(y))$ followed by the boundary map of the Gersten complex of the Milnor K-sheaf.

Note that if X is proper $C_1(X) = SK_1(X)$ as desired. In [23, cf. Thm. 5.1] Schmidt showed that $C_0(X)$ is isomorphic to $h_0(X)$ where h denotes the Suslin's algebraic singular homology. Yamazaki [35, Thm. 1.3] extended this to higher degrees and proved the following theorem which allows us to use the tools from motivic homology and is central for this section.

Theorem 3.1.2. *[35, Thm. 1.3] Let X be a variety over a perfect field and let $r \geq 0$. Then we have a canonical isomorphism*

$$\phi_r : C_r(X) \xrightarrow{\cong} H^M_{-r}(X, \mathbb{Z}(-r)).$$

Here the map ϕ is defined as the composition

$$\phi_r : C_r(X) \to \bigoplus_{x \in X_{(0)}} K^M_r(k(x)) \cong \bigoplus_{x \in X_{(0)}} H^M_{-r}(x, \mathbb{Z}(-r)) \to H^M_{-r}(X, \mathbb{Z}(-r)).$$

The middle isomorphism follows from 1.3 and 1.4. The right hand map is given by the covariant functoriality of the motivic homology.

3.2 The reciprocity map

In this subsection we define the reciprocity map ρ_X. First we note the following, there is an exact sequence

$$0 \to H^M_i(X, \mathbb{Z}(j))/n \to H^M_i(X, \mathbb{Z}/n(j)) \to H^M_{i-1}(X, \mathbb{Z}/(j))[n] \to 0$$

Since $H^M_i(X, \mathbb{Z}(j))$ vanishes for $i < j$ (cf. 1.2.1), we have $H^M_i(X, \mathbb{Z}(i))/n \cong H^M_i(X, \mathbb{Z}/n(i))$. Using this and Thm. 3.1.2 we identify $C_r(X)/n$ with $H^M_{-r}(X, \mathbb{Z}/n(-r))$ in what follows.

The map $c^{-1,-1}_{X,n}$ now reads as

$$c^{-1,-1}_{X,n} : C_1(X)/n \to H^{2d+1}_{ét,c}(X, \mathbb{Z}/n(d+1))$$

The reciprocity map is defined as

$$\rho_X : C_1(X) \to \varprojlim_n C_1(X)/n \xrightarrow{(c^{-1,-1}_{X,n})_n} H^{2d+1}_{ét,c}(X, \hat{\mathbb{Z}}/n(d+1)) \cong \pi^{ab}_1(X)$$

where the right isomorphism is given by the Poincare duality. To see that this map is compatible with the map ρ'_X which is recalled in the introduction, it is enough to show the following diagram is commutative [35, Sec. 4.2].

$$\begin{array}{ccccc} & & \bigoplus_{x\in X_{(0)}} C_1(x) & \longrightarrow & C_1(X) \\ & \overset{\cong}{\nearrow} & \Big\downarrow \oplus\rho_x & & \Big\downarrow \rho_X \\ \bigoplus_{x\in X_{(0)}} k(x)^* & \xrightarrow{\oplus\rho'_x} & \bigoplus_{x\in X_{(0)}} \pi_1^{ab}(x) & \longrightarrow & \pi_1^{ab}(X) \end{array}$$

Here the composition of the two horizontal maps gives ρ'_X. The left triangle is commutative since the maps ρ_x and ρ'_x are the same. This follows since the reciprocity map and cycle class map agrees for a point x and the map $\phi_1 : C_1(x) \xrightarrow{\cong} H^M_{-1}(x, \mathbb{Z}(-1))$ is the identity. The commutativity of the right square follows since the maps $c^{i,j}_{X,n}$ are functorial with respect to scheme morphisms. Similarly we see that $ker(\rho_X/n) \cong ker(c^{-1,-1}_{X,n})$ and $coker(\rho_X/n) \cong coker(c^{-1,-1}_{X,n})$ [35, Thm. 4.2]. Using this we get the following theorem as a direct consequence of the Thm. 2.3.3(b).

Theorem 3.2.1. *Let U be smooth variety of dimension d over a finite extension k of $\mathbb{Q}_p$. Assume there exists a projective smooth variety X with good reduction over k such that U is an open variety of X. Then the map*

$$\rho_U/n : C_1(U)/n \to \pi_1^{ab}(U)/n.$$

is an isomorphism for all $n > 0$.

□

Remark 3.2.2. Without the good reduction assumption theorem does not hold. Sato [22] constructed a smooth projective surface X over a p-adic field where the above map is not injective.

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