

博士論文題目

Discrete branching laws of Zuckerman's derived
functor modules

(Zuckerman 導来関手加群の離散的分岐則)

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DISCRETE BRANCHING LAWS OF ZUCKERMAN'S DERIVED FUNCTOR MODULES

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1. INTRODUCTION

Our object of study is branching laws of Zuckerman's derived functor modules $A_q(\lambda)$ with respect to symmetric pairs of real reductive Lie groups.

Let G_0 be a real reductive Lie group with Lie algebra \mathfrak{g}_0 . Fix a Cartan involution θ of G_0 so that the fixed set $K_0 := (G_0)^\theta$ is a maximal compact subgroup of G_0 . Write K for the complexification of K_0 , $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ for the Cartan decomposition with respect to θ , and $\mathfrak{g} := \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ for the complexification. Similar notation will be used for other Lie algebras. The cohomologically induced module $A_q(\lambda)$ is a (\mathfrak{g}, K) -module defined for a θ -stable parabolic subalgebra \mathfrak{q} of \mathfrak{g} and a character λ . The (\mathfrak{g}, K) -module $A_q(\lambda)$ is unitarizable under a certain condition on the parameter λ and therefore plays a large part in the study of the unitary dual of real reductive Lie groups.

The definition of $A_q(\lambda)$ involves the derived functor of the Zuckerman functor (or the Bernstein functor). This construction can be regarded as a special case of the cohomological induction of (\mathfrak{g}, K) -modules. Let (\mathfrak{g}, K) be a pair (Definition 2.1) and let $\mathcal{C}(\mathfrak{g}, K)$ be the category of (\mathfrak{g}, K) -modules. Suppose that (\mathfrak{h}, L) is a subpair of (\mathfrak{g}, K) and that K and L are reductive. Following the book of Knapp and Vogan [KV], we define the functors $P_{\mathfrak{h}, L}^{\mathfrak{g}, K}$ and $I_{\mathfrak{h}, L}^{\mathfrak{g}, K} : \mathcal{C}(\mathfrak{h}, L) \rightarrow \mathcal{C}(\mathfrak{g}, K)$ as $V \mapsto R(\mathfrak{g}, K) \otimes_{R(\mathfrak{h}, L)} V$ and $V \mapsto \text{Hom}_{R(\mathfrak{h}, L)}(R(\mathfrak{g}, K), V)$, respectively. See Section 2 for the definition of the Hecke algebra $R(\mathfrak{g}, K)$. Let V be a (\mathfrak{h}, L) -module. We define the cohomologically induced module as the (\mathfrak{g}, K) -module $(P_{\mathfrak{h}, L}^{\mathfrak{g}, K})_j(V)$ for $j \in \mathbb{N}$, where $(P_{\mathfrak{h}, L}^{\mathfrak{g}, K})_j$ is the j -th left derived functor of $P_{\mathfrak{h}, L}^{\mathfrak{g}, K}$. Similarly, we define $(I_{\mathfrak{h}, L}^{\mathfrak{g}, K})^j(V)$, where $(I_{\mathfrak{h}, L}^{\mathfrak{g}, K})^j$ is the j -th right derived functor of $I_{\mathfrak{h}, L}^{\mathfrak{g}, K}$. If \mathfrak{h} is a θ -stable parabolic subalgebra, which we denote by \mathfrak{q} , then these functors produce (\mathfrak{g}, K) -modules called $A_q(\lambda)$. To be more precise, write $\bar{\mathfrak{q}}$ for the complex conjugate of \mathfrak{q} and put $\mathfrak{l} := \mathfrak{q} \cap \bar{\mathfrak{q}}$. Then we have the Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ for the nilradical \mathfrak{u} of \mathfrak{q} . Suppose that G_0 has a complexification G and write \bar{Q} and L for the connected subgroups of G with Lie algebras $\bar{\mathfrak{q}}$ and \mathfrak{l} , respectively. For a one-dimensional $(\bar{\mathfrak{q}}, L \cap K)$ -module \mathbb{C}_λ , Zuckerman's derived functor module is defined by

$$A_q(\lambda) := (P_{\bar{\mathfrak{q}}, L \cap K}^{\mathfrak{g}, K})_s \left(\mathbb{C}_\lambda \otimes \bigwedge^{\text{top}} (\mathfrak{g}/\bar{\mathfrak{q}}) \right),$$

where $s = \dim(\mathfrak{u} \cap \mathfrak{k})$.

One of the fundamental problems in the representation theory is to decompose a given representation into irreducible constituents. In particular, branching problems ask how an irreducible representation decomposes when restricted to a subgroup. To begin with, we consider the restriction of (\mathfrak{g}, K) -modules to K , or equivalently, to the compact group K_0 . In this case, any irreducible (\mathfrak{g}, K) -module decomposes as a direct sum of irreducible representations of K and each K -type occurs with finite multiplicity. For $A_q(\lambda)$, an explicit branching law of the restriction $A_q(\lambda)|_K$ for weakly fair λ is known as generalized Blattner's formula (see Fact 2.13, [Bie, §II.7], [KV, §V.5]).

On the other hand, the restriction to a non-compact subgroup is more complicated. Let σ be an involution of G_0 that commutes with θ and let G'_0 be the identity component of $(G_0)^\sigma$. The pair (G_0, G'_0) is called a symmetric pair. Write \mathfrak{g}' for the complexified Lie algebra of G'_0 and write K' for the complexification of the maximal compact group $K'_0 := (G'_0)^\theta$ of G'_0 . If G'_0 is non-compact, the restriction $A_q(\lambda)|_{(\mathfrak{g}', K')}$ does not decompose into irreducible (\mathfrak{g}', K') -modules in general. In fact, $A_q(\lambda)|_{(\mathfrak{g}', K')}$ does not have any irreducible submodule in many cases.

Nevertheless, there are classes of (\mathfrak{g}, K) -modules which decompose into irreducible (\mathfrak{g}', K') -modules and explicit branching formulas were obtained for some particular representations ([DV10], [GW00], [Kob93], [Kob94], [Kob96], [Kob07], [KØ03], [Lok00], [ØS08], [Sek11], [Spe11]). In his series of papers [Kob93], [Kob94], [Kob98a], [Kob98b], Kobayashi introduced the notion of discretely decomposable (\mathfrak{g}', K') -modules and gave criteria for the discretely decomposable restrictions (see Fact 3.8). By virtue of this result, we can single out $A_q(\lambda)$ that decompose into irreducible (\mathfrak{g}', K') -modules. See [KO12] for a classification of the discretely decomposable restrictions $A_q(\lambda)|_{(\mathfrak{g}', K')}$. Recent developments on these subjects are discussed in [Kob11].

Our aim is to find a branching law of $A_q(\lambda)|_{(\mathfrak{g}', K')}$ when it is discretely decomposable. The main results of this thesis are Theorem 5.1 and explicit branching formulas in Sections 8 and 9. Theorem 5.1 gives a decomposition of the restriction $A_q(\lambda)|_{(\mathfrak{g}', K')}$ corresponding to the K' -orbit decomposition of a flag variety of K . Let $K/(\overline{Q} \cap K) = \bigsqcup_{j=1}^n Y_j$ be the K' -orbit decomposition and choose representatives $k_j \in K_0$ such that $Y_j = K'k_j(\overline{Q} \cap K)$. Put

$$\begin{aligned} \mathfrak{q}_j &:= \text{Ad}(k_j)\mathfrak{q}, & Q_j &:= k_j Q k_j^{-1}, \\ s_j &:= \dim K/(\overline{Q} \cap K) - \dim Y_j, & u_j &:= \dim(\overline{Q}_j \cap K') - \dim C'_j, \end{aligned}$$

where C'_j is a maximal reductive subgroup of $\overline{Q}_j \cap K'$. Taking conjugation by k_j , we regard the one-dimensional $(\overline{\mathfrak{q}}, L \cap K)$ -module $\mathbb{C}_\lambda \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\overline{\mathfrak{q}})$ as a $(\overline{\mathfrak{q}}_j \cap \mathfrak{g}', C'_j)$ -module by restriction. We now state Theorem 5.1:

Theorem. *Let (G_0, G'_0) be a symmetric pair of connected real reductive Lie groups and \mathfrak{q} a θ -stable parabolic subalgebra of \mathfrak{g} . Suppose that $A_{\mathfrak{q}}(\lambda)$ is non-zero and discretely decomposable as a (\mathfrak{g}', K') -module with λ in the weakly fair range. Then we have an equation of virtual characters of (\mathfrak{g}', K') -modules*

(1.1)

$$\begin{aligned} & [A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}] \\ &= \sum_{j=1}^n \sum_{d \in \mathbb{Z}_{\geq 0}} (-1)^{d+s_j+u_j} \left[(P_{\bar{\mathfrak{q}}_j \cap \mathfrak{g}', C'_j}^{\mathfrak{g}', K'})_d \left(\mathbb{C}_{\lambda} \otimes \bigwedge^{\text{top}} (\mathfrak{g}/\bar{\mathfrak{q}}) \otimes S(\mathfrak{g}/(\bar{\mathfrak{q}}_j + \mathfrak{g}')) \right) \right]. \end{aligned}$$

Loosely speaking, it describes the restriction of cohomologically induced modules in terms of the cohomological induction of the restriction:

$$\begin{array}{ccc} (\bar{\mathfrak{q}}, L \cap K) & \xrightarrow{\text{cohomological induction}} & (\mathfrak{g}, K) \\ \text{restriction} \downarrow & & \downarrow \text{restriction} \\ (\bar{\mathfrak{q}}_j \cap \mathfrak{g}', C'_j) & \xrightarrow{\text{cohomological induction}} & (\mathfrak{g}', K') \end{array}$$

Using this, we derive branching formulas of $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$ in Sections 8 and 9. We will see in each case that the restriction $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$ is a direct sum of derived functor (\mathfrak{g}', K') -modules so branching formulas take form of

$$A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')} \simeq \bigoplus_{\mathfrak{q}'} \bigoplus_{\lambda'} m(\mathfrak{q}', \lambda') A_{\mathfrak{q}'}(\lambda'), \quad m(\mathfrak{q}', \lambda') \in \mathbb{N}.$$

In [KO12], the classes of discretely decomposable restrictions $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$ were divided into two types. One is what we call discrete series type. This means that there exists a θ -stable Borel subalgebra \mathfrak{b} contained in \mathfrak{q} such that $A_{\mathfrak{b}}(\lambda)|_{(\mathfrak{g}', K')}$ is also discretely decomposable. We call the other classes isolated type. We treat isolated type in Section 8 and discrete series type in Section 9. In most cases of isolated type, the parabolic subgroup \bar{Q} is maximal and the K' -orbit decomposition of $K/(\bar{Q} \cap K)$ is rather simple. We can thus obtain explicit branching formulas from Theorem 5.1. For discrete series type, we use another expression (Theorems 9.1 and 9.2) in addition to Theorem 5.1. If the Levi subgroup $L_0 = N_{G_0}(\mathfrak{q})$ is of Hermitian type, we write a (\mathfrak{g}, K) -module $A_{\mathfrak{q}}(\lambda)$ as an alternating sum of $A_{\mathfrak{b}}(\mu)$ for a Borel subalgebra \mathfrak{b} using a BGG type resolution of one-dimensional \mathfrak{l}_0 -module. Then we describe the restriction $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$ as an alternating sum of cohomologically induced modules in Theorem 9.1. If \mathfrak{q} is “close” enough to the Borel subalgebra \mathfrak{b} , then it turns out that $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$ decomposes into a direct sum of (limit of) discrete series representations for G'_0 (Theorem 9.2).

For the proof of Theorem 5.1, we realize (\mathfrak{g}, K) -modules $A_{\mathfrak{q}}(\lambda)$ as the global sections of sheaves on complex partial flag varieties in Theorem 4.12, using \mathcal{D} -modules. The localization theory by Beilinson–Bernstein [BB81]

provides a realization of (\mathfrak{g}, K) -modules as K -equivariant twisted \mathcal{D} -modules on the full flag variety of \mathfrak{g} . A relation between cohomologically induced modules and twisted \mathcal{D} -modules on the complete flag variety was established by Hecht–Milićić–Schmid–Wolf [HMSW]. We now recall their theorem. Let G_0 be a connected real reductive Lie group and let (\mathfrak{g}, K) be the pair defined in the above way. Suppose that $\mathfrak{h} = \mathfrak{b}$ is a Borel subalgebra of \mathfrak{g} and L is a maximal reductive subgroup of the normalizer $N_K(\mathfrak{b})$. Let X be the full flag variety of \mathfrak{g} , Y the K -orbit through $\mathfrak{b} \in X$, and $i : Y \rightarrow X$ the inclusion map. Suppose that V is a (\mathfrak{b}, L) -module and \mathfrak{b} acts as scalars given by $\lambda \in \mathfrak{b}^* := \text{Hom}_{\mathbb{C}}(\mathfrak{b}, \mathbb{C})$. Write \mathcal{V}_Y for the corresponding locally free \mathcal{O}_Y -module on Y and view it as a twisted \mathcal{D} -module. Let $\mathcal{D}_{X,\lambda}$ be the ring of twisted differential operators on X corresponding to λ and define the $\mathcal{D}_{X,\lambda}$ -module direct image $i_+ \mathcal{V}_Y$. Then the following is called the duality theorem:

Theorem 1.1 ([HMSW]). *There is an isomorphism of (\mathfrak{g}, K) -modules*

$$H^s(X, i_+ \mathcal{V}_Y)^* \simeq (I_{\mathfrak{b}, L}^{\mathfrak{g}, K})^{u-s} \left(V^* \otimes \bigwedge^{\text{top}} (\mathfrak{g}/\mathfrak{b})^* \right)$$

for $s \in \mathbb{N}$ and $u = \dim K/L - \dim Y$.

See [Bie], [Cha93], [Kit12], [MP98], [Sch91] for further developments on this subject. Milićić–Pandžić [MP98] gave a more conceptual proof of Theorem 1.1 by using equivariant derived categories. In [Cha93] and [Kit12], Theorem 1.1 was extended to the case of partial flag varieties. In this thesis we will realize geometrically the cohomologically induced modules in the following setting. Let $i : K \rightarrow G$ be a homomorphism between complex linear algebraic groups. Suppose that K is reductive and the kernel of i is finite so that the pair (\mathfrak{g}, K) is defined. Let H be a closed subgroup of G . Put $M := i^{-1}(H)$ and take a Levi decomposition $M = L \ltimes U$. We write $i : Y = K/M \rightarrow G/H = X$ for the natural immersion. Let V be a (\mathfrak{h}, M) -module. We see V as a (\mathfrak{h}, L) -module by restriction and define the cohomologically induced module $(P_{\mathfrak{b}, L}^{\mathfrak{g}, K})_j(V)$. In this generality, we can no longer realize it as a (twisted) \mathcal{D} -module on $X = G/H$. Instead we use the tensor product of an $i^{-1} \mathcal{D}_X$ -module and an $i^{-1} \mathcal{O}_X$ -module associated with V which is equipped with a (\mathfrak{g}, K) -action (see Definition 4.5). We will prove that (Theorem 4.12)

Theorem. *Suppose that \mathcal{V} is an $i^{-1} \tilde{\mathfrak{g}}_X$ -module associated with V (see Definition 4.5). Then we have an isomorphism of (\mathfrak{g}, K) -modules*

$$H^s(Y, i^{-1} i_+ \mathcal{L} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}) \simeq (P_{\mathfrak{b}, L}^{\mathfrak{g}, K})_{u-s} \left(V \otimes \bigwedge^{\text{top}} (\mathfrak{g}/\mathfrak{h}) \right)$$

for $s \in \mathbb{N}$ and $u = \dim U$.

Here \mathcal{L} is the invertible sheaf on Y defined just before Theorem 4.12 and the direct image $i_+\mathcal{L}$ in the categories of \mathcal{D} -modules is defined as

$$i_*((\mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^\vee.$$

Hence its inverse image $i^{-1}i_+\mathcal{L}$ as a sheaf of abelian groups is given by

$$(\mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee.$$

For a (\mathfrak{h}, L) -module V , an $i^{-1}\mathcal{O}_X$ -module associated with V is constructed in Proposition 4.14. Therefore, Theorem 4.12 and Proposition 4.14 yield a geometric realization of cohomologically induced modules in the setting above.

This thesis is organized as follows. In Section 2 we recall the definition of cohomological induction and $A_q(\lambda)$ following [KV] and prove some basic properties on them. In Section 3 we define the notion of discrete decomposability and admissibility of (\mathfrak{g}, K) -modules. We recall Kobayashi's criterion for the discrete decomposability of the restriction $A_q(\lambda)|_{(\mathfrak{g}', K')}$ and prove that the criterion is also equivalent to that a certain subspace of \mathfrak{g}' is a θ -stable parabolic subalgebra (see Theorem 3.7). Section 4 is devoted to a geometric realization of cohomologically induced modules. By using this realization, we prove Theorem 5.1 in Section 5. In Section 6, we give certain isomorphisms and exact sequences among $A_q(\lambda)$ for $G_0 = U(m, n)$ and $G_0 = Sp(m, n)$ (Theorems 6.3 and 6.5). These will be applied in the subsequent sections to rewrite the right hand side of (1.1). We refer to [KO12], the classification of discretely decomposable $A_q(\lambda)$ in Section 7. We derive branching laws of $A_q(\lambda)|_{(\mathfrak{g}', K')}$ in Sections 8 and 9.

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2. COHOMOLOGICAL INDUCTION

In this section, we fix notation concerning cohomological induction and $A_q(\lambda)$ modules, following [KV].

Let K_0 be a compact Lie group. The complexification K of K_0 has a structure of reductive linear algebraic group. Since any locally finite action

of K_0 is uniquely extended to an algebraic action of K , the locally finite K_0 -modules are identified with the algebraic K -modules.

Define the Hecke algebra $R(K_0)$ as the space of K_0 -finite distributions on K_0 . For $S \in R(K_0)$, the pairing with a smooth function $f \in C(K_0)$ on K_0 is written as

$$\int_{K_0} f(k) dS(k).$$

The product of $S, T \in R(K_0)$ is given by convolution

$$S * T : f \mapsto \int_{K_0 \times K_0} f(kk') dS(k) dT(k').$$

The associative algebra $R(K_0)$ does not have the identity, but has an approximate identity (see [KV, Chapter I]). The locally finite K_0 -modules are identified with the approximately unital left $R(K_0)$ -modules. The action map $R(K_0) \times V \rightarrow V$ is given by

$$(S, v) \mapsto \int_{K_0} kv dS(k)$$

for a locally finite K_0 -module V . Here, kv is regarded as a smooth function on K_0 that takes values in V . If dk_0 denotes the Haar measure of K_0 , then $R(K_0)$ is identified with the K -finite smooth functions $C(K_0)_{K_0}$ by $f dk_0 \mapsto f$ and hence with the regular functions $\mathcal{O}(K)$ on K . As a \mathbb{C} -algebra, we have a canonical isomorphism

$$R(K_0) \simeq \bigoplus_{\tau \in \widehat{K}} \text{End}_{\mathbb{C}}(V_{\tau}),$$

where \widehat{K} is the set of equivalence classes of irreducible K -modules, and V_{τ} is a representation space of $\tau \in \widehat{K}$. Hence $R(K_0)$ depends only on the complexification K , so in what follows, we also denote $R(K_0)$ by $R(K)$.

The Hecke algebra $R(K)$ is generalized to $R(\mathfrak{g}, K)$ for the following pairs (\mathfrak{g}, K) .

Definition 2.1. Let \mathfrak{g} be a finite-dimensional complex Lie algebra and let K be a complex reductive linear algebraic group with Lie algebra \mathfrak{k} . Suppose that \mathfrak{k} is a Lie subalgebra of \mathfrak{g} and that an algebraic group homomorphism $\phi : K \rightarrow \text{Aut}(\mathfrak{g})$ is given. We say that (\mathfrak{g}, K) is a *pair* if the following two assumptions hold.

- The restriction $\phi(k)|_{\mathfrak{k}}$ is equal to the adjoint action $\text{Ad}(k)$ for $k \in K$.
- The differential of ϕ is equal to the adjoint action $\text{ad}_{\mathfrak{g}}(\mathfrak{k})$.

Remark 2.2. Let G be a complex algebraic group and K a reductive linear algebraic subgroup. Then the Lie algebra \mathfrak{g} of G and K form a pair with respect to the adjoint action $\phi(k) := \text{Ad}(k)$ for $k \in K$.

Definition 2.3. For a pair (\mathfrak{g}, K) , let V be a complex vector space with a Lie algebra action of \mathfrak{g} and an algebraic action of K . We say that V is a (\mathfrak{g}, K) -module if

- the differential of the action of K coincides with the restriction of the action of \mathfrak{g} to \mathfrak{k} ; and
- $(\phi(k)\xi)v = k(\xi(k^{-1}(v)))$ for $k \in K$, $\xi \in \mathfrak{g}$, and $v \in V$.

We write $\mathcal{C}(\mathfrak{g}, K)$ for the category of (\mathfrak{g}, K) -modules.

Let (\mathfrak{g}, K) be a pair in the sense of Definition 2.1. We extend the representation $\phi : K \rightarrow \text{Aut}(\mathfrak{g})$ to a representation on the universal enveloping algebra $\phi : K \rightarrow \text{Aut}(U(\mathfrak{g}))$. Define the Hecke algebra $R(\mathfrak{g}, K)$ as

$$R(\mathfrak{g}, K) := R(K) \otimes_{U(\mathfrak{k})} U(\mathfrak{g}).$$

The product is given by

$$(S \otimes \xi) \cdot (T \otimes \eta) = \sum_i (S * (\langle \xi^i, \phi(\cdot)^{-1} \xi \rangle T) \otimes \xi_i \eta)$$

for $S, T \in R(K)$ and $\xi, \eta \in U(\mathfrak{g})$. Here ξ_i is a basis of the linear span of $\phi(K)\xi$ and ξ^i is its dual basis. As in the group case, the (\mathfrak{g}, K) -modules are identified with the approximately unital left $R(\mathfrak{g}, K)$ -modules. The action map $R(\mathfrak{g}, K) \times V \rightarrow V$ is given by

$$(S \otimes \xi, v) \mapsto \int_{K_0} k(\xi v) dS(k)$$

for a (\mathfrak{g}, K) -module V .

Let (\mathfrak{g}, K) and (\mathfrak{h}, L) be pairs in the sense of Definition 2.1. Let $i : (\mathfrak{h}, L) \rightarrow (\mathfrak{g}, K)$ be a map between pairs, namely, a Lie algebra homomorphism $i_{\text{alg}} : \mathfrak{h} \rightarrow \mathfrak{g}$ and an algebraic group homomorphism $i_{\text{gp}} : L \rightarrow K$ satisfy the following two assumptions.

- The restriction of i_{alg} to the Lie algebra \mathfrak{l} of L is equal to the differential of i_{gp} .
- $\phi_K(l) \circ i_{\text{alg}} = i_{\text{alg}} \circ \phi_L(l)$ for $l \in L$, where ϕ_K denotes ϕ for (\mathfrak{g}, K) in Definition 2.1 and ϕ_L denotes ϕ for (\mathfrak{h}, L) .

We define the functors $P_{\mathfrak{h}, L}^{\mathfrak{g}, K}, I_{\mathfrak{h}, L}^{\mathfrak{g}, K} : \mathcal{C}(\mathfrak{h}, L) \rightarrow \mathcal{C}(\mathfrak{g}, K)$ by

$$\begin{aligned} P_{\mathfrak{h}, L}^{\mathfrak{g}, K} : V &\mapsto R(\mathfrak{g}, K) \otimes_{R(\mathfrak{h}, L)} V, \\ I_{\mathfrak{h}, L}^{\mathfrak{g}, K} : V &\mapsto (\text{Hom}_{R(\mathfrak{h}, L)}(R(\mathfrak{g}, K), V))_K, \end{aligned}$$

where $(\cdot)_K$ is the subspace of K -finite vectors. Then $P_{\mathfrak{h}, L}^{\mathfrak{g}, K}$ is right exact and $I_{\mathfrak{h}, L}^{\mathfrak{g}, K}$ is left exact. Write $(P_{\mathfrak{h}, L}^{\mathfrak{g}, K})_j$ for the j -th left derived functor of $P_{\mathfrak{h}, L}^{\mathfrak{g}, K}$ and write $(I_{\mathfrak{h}, L}^{\mathfrak{g}, K})^j$ for the j -th right derived functor of $I_{\mathfrak{h}, L}^{\mathfrak{g}, K}$. We can see that $I_{\mathfrak{h}, L}^{\mathfrak{g}, K}$ is the right adjoint functor of the forgetful functor $\text{For}_{\mathfrak{g}, K}^{\mathfrak{h}, L}(V) := R(\mathfrak{g}, K) \otimes_{R(\mathfrak{g}, K)} V \simeq V$ and $P_{\mathfrak{h}, L}^{\mathfrak{g}, K}$ is the left adjoint functor of $\text{For}_{\mathfrak{g}, K}^{\mathfrak{h}, L}(V) := \text{Hom}_{R(\mathfrak{g}, K)}(R(\mathfrak{g}, K), V)_L$.

If i_{alg} and i_{gp} are injective, we say (\mathfrak{h}, L) is a subpair of (\mathfrak{g}, K) and identify (\mathfrak{h}, L) with its image.

The following properties are straight-forward.

Proposition 2.4. *Let $i : (\mathfrak{h}, L) \rightarrow (\mathfrak{g}, K)$ be a map between pairs and V a (\mathfrak{h}, L) -module.*

- (i) *For $k \in K$ we have another map of pairs $\text{Ad}(k) \circ i : (\mathfrak{h}, L) \rightarrow (\mathfrak{g}, K)$. If we define the functors ${}^k P_{\mathfrak{h}, L}^{\mathfrak{g}, K}$ and ${}^k I_{\mathfrak{h}, L}^{\mathfrak{g}, K}$ with respect to $\text{Ad}(k) \circ i$, they are canonically isomorphic to $P_{\mathfrak{h}, L}^{\mathfrak{g}, K}$ and $I_{\mathfrak{h}, L}^{\mathfrak{g}, K}$, respectively.*
- (ii) *Let $i' : (\mathfrak{g}, K) \rightarrow (\tilde{\mathfrak{g}}, \tilde{K})$ be another map of pairs. Then*

$$P_{\mathfrak{g}, K}^{\tilde{\mathfrak{g}}, \tilde{K}} \circ P_{\mathfrak{h}, L}^{\mathfrak{g}, K} \simeq P_{\mathfrak{h}, L}^{\tilde{\mathfrak{g}}, \tilde{K}}, \quad I_{\mathfrak{g}, K}^{\tilde{\mathfrak{g}}, \tilde{K}} \circ I_{\mathfrak{h}, L}^{\mathfrak{g}, K} \simeq I_{\mathfrak{h}, L}^{\tilde{\mathfrak{g}}, \tilde{K}}.$$

Moreover, we have convergences of spectral sequences

$$(P_{\mathfrak{g}, K}^{\tilde{\mathfrak{g}}, \tilde{K}})^{j'} \circ (P_{\mathfrak{h}, L}^{\mathfrak{g}, K})^j \Rightarrow (P_{\mathfrak{h}, L}^{\tilde{\mathfrak{g}}, \tilde{K}})^{j+j'}, \quad (I_{\mathfrak{g}, K}^{\tilde{\mathfrak{g}}, \tilde{K}})^{j'} \circ (I_{\mathfrak{h}, L}^{\mathfrak{g}, K})^j \Rightarrow (I_{\mathfrak{h}, L}^{\tilde{\mathfrak{g}}, \tilde{K}})^{j+j'}.$$

We also observe the following property concerning annihilator of (\mathfrak{g}, K) -modules.

Proposition 2.5. *Let (\mathfrak{h}, L) be a subpair of (\mathfrak{g}, K) and V a (\mathfrak{h}, L) -module. Suppose that \mathfrak{u} is a K -stable ideal of \mathfrak{g} and $\mathfrak{u} \subset \mathfrak{h}$. If V is annihilated by \mathfrak{u} , then $(P_{\mathfrak{h}, L}^{\mathfrak{g}, K})_j(V)$ is also annihilated by \mathfrak{u} .*

Proof. We simply write $P(V)$ for $(P_{\mathfrak{h}, L}^{\mathfrak{g}, K})_j(V)$. For a \mathfrak{u} -module W , write $a_W : \mathfrak{u} \otimes W \rightarrow W$ for the action map. We can see \mathfrak{u} as a (\mathfrak{g}, K) -module, or a (\mathfrak{h}, L) -module. Then there is Mackey isomorphism [KV, Theorem 2.103]

$$\Phi : P(\mathfrak{u} \otimes V) \xrightarrow{\sim} \mathfrak{u} \otimes P(V)$$

and the diagram

$$\begin{array}{ccc} P(\mathfrak{u} \otimes V) & \xrightarrow{\Phi} & \mathfrak{u} \otimes P(V) \\ P(a_V) \downarrow & \swarrow a_{P(V)} & \\ P(V) & & \end{array}$$

commutes by [KV, Theorem Proposition 3.77]. Hence $a_V = 0$ implies $a_{P(V)} = 0$. \square

In the context of unitary representations of real reductive Lie groups, we are especially interested in the (\mathfrak{g}, K) -modules cohomologically induced from one-dimensional representations of a certain type of parabolic subalgebras, which are called $A_q(\lambda)$ modules.

We say G_0 is a real linear reductive Lie group if G_0 is a closed subgroup of $GL(n, \mathbb{R})$ and stable under transpose. We say G_0 is a real reductive group if G_0 is a finite covering group of a real linear reductive Lie group.

Let G_0 be a connected real reductive Lie group with Lie algebra \mathfrak{g}_0 . Fix a Cartan involution θ so the θ -fixed point set $K_0 = G_0^\theta$ is a maximal compact subgroup of G_0 . Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the corresponding Cartan decomposition. We let θ also denote the induced involution on \mathfrak{g}_0 and its complex linear extension to \mathfrak{g} .

Let \mathfrak{q} be a parabolic subalgebra of \mathfrak{g} that is stable under θ . The normalizer $N_{G_0}(\mathfrak{q})$ of \mathfrak{q} in G_0 is denoted by L_0 . The complexified Lie algebra \mathfrak{l} of L_0 is

a Levi part of \mathfrak{q} . Let $\bar{x} \mapsto \bar{x}$ denote the complex conjugate with respect to the real form \mathfrak{g}_0 . Then we have $\mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{l}$ and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ for the nilradical \mathfrak{u} of \mathfrak{q} .

To describe θ -stable parabolic subalgebras of \mathfrak{g} , it is convenient to use the following convention:

Definition 2.6. Let $a \in \mathfrak{k}$ be a vector such that $\text{ad}(a)$ on \mathfrak{g} is semisimple with real eigenvalues. We say that a parabolic subalgebra \mathfrak{q} of \mathfrak{g} is given by a and write $\mathfrak{q} = \mathfrak{q}(a)$ if \mathfrak{q} is the sum of non-negative eigenspaces of $\text{ad}(a)$.

Then \mathfrak{q} is a θ -stable parabolic subalgebra and we get the Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, where \mathfrak{l} and \mathfrak{u} are the sums of zero and positive eigenspaces of $\text{ad}(a)$, respectively. We write $L(a)$, $\mathfrak{l}(a)$, and $\mathfrak{u}(a)$ for the corresponding Levi subgroup, Levi subalgebra, and the nilradical. Note that any θ -stable parabolic subalgebras are obtained in this way.

Let K_L be the complexification of $L_0 \cap K_0$. Since K_L is connected, one-dimensional (\mathfrak{l}, K_L) -modules are determined by the action of the center $\mathfrak{z}(\mathfrak{l})$ of \mathfrak{l} . Let \mathbb{C}_λ denote the one-dimensional (\mathfrak{l}, K_L) -module corresponding to $\lambda \in \mathfrak{z}(\mathfrak{l})^* := \text{Hom}_{\mathbb{C}}(\mathfrak{z}(\mathfrak{l}), \mathbb{C})$. With our normalization, the trivial representation corresponds to \mathbb{C}_0 . The top exterior product $\bigwedge^{\text{top}}(\mathfrak{g}/\bar{\mathfrak{q}})$ regarded as an (\mathfrak{l}, K_L) -module by the adjoint action corresponds to $\mathbb{C}_{2\rho(\mathfrak{u})}$ for $2\rho(\mathfrak{u}) := \text{Trace ad}_{\mathfrak{u}}(\cdot)$.

Definition 2.7. Let \mathbb{C}_λ be a one-dimensional (\mathfrak{l}, K_L) -module. We say λ is *unitary* if λ takes pure imaginary values on the center $\mathfrak{z}(\mathfrak{l}_0)$ of \mathfrak{l}_0 , or equivalently, if \mathbb{C}_λ is the underlying (\mathfrak{l}, K_L) -module of a unitary character of L_0 .

Let \mathbb{C}_λ be a one-dimensional (\mathfrak{l}, K_L) -module. We see $\mathbb{C}_{\lambda+2\rho(\mathfrak{u})} \simeq \mathbb{C}_\lambda \otimes \mathbb{C}_{2\rho(\mathfrak{u})}$ as a $(\bar{\mathfrak{q}}, K_L)$ -module (resp. a (\mathfrak{q}, K_L) -module) by letting $\bar{\mathfrak{u}}$ (resp. \mathfrak{u}) acts as zero. Then, for inclusion maps of pairs $(\bar{\mathfrak{q}}, K_L) \rightarrow (\mathfrak{g}, K)$ and $(\mathfrak{q}, K_L) \rightarrow (\mathfrak{g}, K)$, define the cohomologically induced modules $(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(\mathbb{C}_{\lambda+2\rho(\mathfrak{u})})$ and $(I_{\mathfrak{q}, K_L}^{\mathfrak{g}, K})^j(\mathbb{C}_{\lambda+2\rho(\mathfrak{u})})$.

The functor $P_{\mathfrak{g}, K_L}^{\mathfrak{g}, K}$ is called the Bernstein functor and denoted by $\Pi_{K_L}^K$. Since $P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K} \simeq \Pi_{K_L}^K \circ P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K_L}$ and $P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K_L}$ is exact, it follows that $(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j \simeq (\Pi_{K_L}^K)_j \circ P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K_L}$ for the j -th left derived functor $(\Pi_{K_L}^K)_j$ of $\Pi_{K_L}^K$. Therefore, we have

$$(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(\mathbb{C}_{\lambda+2\rho(\mathfrak{u})}) \simeq (\Pi_{K_L}^K)_j(U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} \mathbb{C}_{\lambda+2\rho(\mathfrak{u})}).$$

Similarly, $\Gamma_{K_L}^K := I_{\mathfrak{g}, K_L}^{\mathfrak{g}, K}$ is called the Zuckerman functor and we have

$$(I_{\mathfrak{q}, K_L}^{\mathfrak{g}, K})^j(\mathbb{C}_{\lambda+2\rho(\mathfrak{u})}) \simeq (\Gamma_{K_L}^K)^j(\text{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), \mathbb{C}_{\lambda+2\rho(\mathfrak{u})})_{K_L})$$

for the j -th right derived functor $(\Gamma_{K_L}^K)^j$ of $\Gamma_{K_L}^K$. Put $s = \dim(\mathfrak{u} \cap \mathfrak{k})$. We define

$$A_{\mathfrak{q}}(\lambda) := (P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_s(\mathbb{C}_{\lambda+2\rho(\mathfrak{u})}) \simeq (\Pi_{K_L}^K)_s(U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} \mathbb{C}_{\lambda+2\rho(\mathfrak{u})}).$$

We also define

$$\mathcal{L}_{\bar{q},j}^{\mathfrak{g}}(V) := (P_{\bar{q},K_L}^{\mathfrak{g},K})_j(V \otimes \mathbb{C}_{2\rho(\mathfrak{u})})$$

for an (\mathfrak{l}, K_L) -module V in this situation so that

$$A_{\mathfrak{q}}(\lambda) = \mathcal{L}_{\bar{q},s}^{\mathfrak{g}}(\mathbb{C}_{\lambda}).$$

We now discuss the positivity of the parameter λ . Let \mathfrak{h}_0 be a fundamental Cartan subalgebra of \mathfrak{l}_0 so that $\mathfrak{h}_0 \cap \mathfrak{k}_0$ is a Cartan subalgebra of $\mathfrak{l}_0 \cap \mathfrak{k}_0$. Choose a θ -stable positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ of the root system $\Delta(\mathfrak{g}, \mathfrak{h})$ such that $\Delta^+(\mathfrak{g}, \mathfrak{h}) \subset \Delta(\mathfrak{q}, \mathfrak{h})$ and put

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_{\alpha}.$$

We fix a non-degenerate invariant symmetric form $\langle \cdot, \cdot \rangle$ that is positive definite on the real span of the roots. In the following definition, we extend characters of $\mathfrak{z}(\mathfrak{l})$ to \mathfrak{h} by zero on $[\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{h}$.

Definition 2.8. For $\lambda \in \mathfrak{h}^*$ we say λ is in the *good range* (resp. *weakly good range*) if

$$\operatorname{Re} \langle \lambda + \rho(\mathfrak{u}), \alpha \rangle > 0 \text{ (resp. } \geq 0) \text{ for } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h})$$

and in the *fair range* (resp. *weakly fair range*) if

$$\operatorname{Re} \langle \lambda + \rho(\mathfrak{u}), \alpha|_{\mathfrak{z}(\mathfrak{l})} \rangle > 0 \text{ (resp. } \geq 0) \text{ for } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

Suppose that V is an (\mathfrak{l}, K_L) -module and has an infinitesimal character λ . We say V is *good* (resp. *weakly good*) if λ is in the good range (resp. weakly good range).

Definition 2.9. Let V be a (\mathfrak{g}, K) -module. We say V is *unitarizable* if V admits a Hermitian inner product with respect to which \mathfrak{g}_0 acts by skew-Hermitian operators on V .

The cohomologically induced modules and $A_{\mathfrak{q}}(\lambda)$ have the following properties.

Fact 2.10 ([KV]). *Let V be an (\mathfrak{l}, K_L) -module, which we regard as a $(\bar{\mathfrak{q}}, K_L)$ -module by letting $\bar{\mathfrak{u}}$ act as zero.*

- (i) *If V is of finite length, $\mathcal{L}_{\bar{\mathfrak{q}},j}^{\mathfrak{g}}(V)$ is of finite length as a (\mathfrak{g}, K) -module for any j .*
- (ii) *If V has infinitesimal character λ , then $\mathcal{L}_{\bar{\mathfrak{q}},j}^{\mathfrak{g}}(V)$ has infinitesimal character $\lambda + \rho(\mathfrak{u})$.*
- (iii) *If V is weakly good, then $\mathcal{L}_{\bar{\mathfrak{q}},j}^{\mathfrak{g}}(V) = 0$ for $j \neq s$.*
- (iv) *If V is irreducible and good (resp. weakly good), then $\mathcal{L}_{\bar{\mathfrak{q}},s}^{\mathfrak{g}}(V)$ is irreducible (resp. irreducible or zero).*
- (v) *If V is unitarizable and weakly good, then $\mathcal{L}_{\bar{\mathfrak{q}},s}^{\mathfrak{g}}(V)$ is unitarizable.*

Fact 2.11 ([KV]). *Let \mathbb{C}_{λ} be a one-dimensional (\mathfrak{l}, K_L) -module.*

- (i) *$A_{\mathfrak{q}}(\lambda)$ is of finite length as a (\mathfrak{g}, K) -module and has infinitesimal character $\lambda + \rho(\mathfrak{n})$.*

- (ii) If λ is in the weakly fair range, then $\mathcal{L}_{\bar{q},j}^{\mathfrak{g}}(\mathbb{C}_\lambda) = 0$ for $j \neq s$.
- (iii) If \mathbb{C}_λ is good (resp. weakly good), $A_{\bar{q}}(\lambda)$ is irreducible (resp. irreducible or zero).
- (iv) If λ is unitary and in the weakly fair range, then $A_{\bar{q}}(\lambda)$ is unitarizable.

In the next proposition we suppose that G_0 is compact so $\mathfrak{g} = \mathfrak{k}$. Then it turns out that $A_{\bar{q}}(\lambda)$ is an irreducible finite-dimensional representation of G_0 or zero. To be more precise, let $F^K(\mu)$ denote the irreducible K -module with highest weight μ for a dominant integral weight μ .

Proposition 2.12 ([KV, Corollary 4.160]). *Suppose that G_0 is compact so $\mathfrak{g} = \mathfrak{k}$ and let W be the Weyl group. Then*

$$\mathcal{L}_{\bar{q},j}^{\mathfrak{k}}(\mathbb{C}_\lambda) \simeq F^K(\mu)$$

if there exists $w \in W$ such that $w(\lambda + \rho(\mathfrak{n})) = \mu + \rho(\mathfrak{n})$ and $l(w) = s - j$, and

$$\mathcal{L}_{\bar{q},j}^{\mathfrak{k}}(\mathbb{C}_\lambda) = 0$$

if otherwise.

The proposition can be viewed as an algebraic analog of the Bott–Borel–Weil theorem.

For non-compact G_0 , the K -type of $A_{\bar{q}}(\lambda)$, namely, the branching of restriction of $A_{\bar{q}}(\lambda)$ to K is known as generalized Blattner’s formula. Put $\mathfrak{t} := \mathfrak{h} \cap \mathfrak{k}$, which is a Cartan subalgebra of \mathfrak{k} . We set the positive system as $\Delta^+(\mathfrak{k}, \mathfrak{t}) = \Delta(\mathfrak{n} \cap \mathfrak{k}, \mathfrak{t})$.

Fact 2.13 ([KV, Theorem 5.64]). *Let $F^K(\mu)$ be the irreducible K -module with highest weight $\mu \in \mathfrak{t}^*$. Then it follows that*

$$\begin{aligned} & \sum_{j=0}^s (-1)^j \dim \operatorname{Hom}_K((P_{\bar{q},K_L}^{\mathfrak{g},K})_j(\mathbb{C}_\lambda), F^K(\mu)) \\ &= \sum_{j=0}^s (-1)^j \sum_{p=0}^{\infty} \dim \operatorname{Hom}_{K_L}(S^p(\mathfrak{u} \cap \mathfrak{p}) \otimes \mathbb{C}_\lambda, H^j(\bar{\mathfrak{u}} \cap \mathfrak{k}, F^K(\mu))). \end{aligned}$$

In particular, if λ is in the weakly fair range,

$$\begin{aligned} & \dim \operatorname{Hom}_K(A_{\bar{q}}(\lambda), F^K(\mu)) \\ &= \sum_{j=0}^s (-1)^{s-j} \sum_{p=0}^{\infty} \dim \operatorname{Hom}_{K_L}(S^p(\mathfrak{u} \cap \mathfrak{p}) \otimes \mathbb{C}_{\lambda+2\rho(\mathfrak{u})}, H^j(\bar{\mathfrak{u}} \cap \mathfrak{k}, F^K(\mu))). \end{aligned}$$

We can rewrite this formula as follows. Consider the positive system of $\Delta(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t})$ induced from $\Delta^+(\mathfrak{k}, \mathfrak{t})$. For a dominant integral weight $\mu \in \mathfrak{t}^*$ for K_L , let $m(\mu)$ be the multiplicity of $F^{K_L}(\mu)$ in $S(\mathfrak{u} \cap \mathfrak{p}) \otimes \mathbb{C}_{\lambda+2\rho(\mathfrak{u} \cap \mathfrak{p})}$. Then for weakly fair λ

$$\dim \operatorname{Hom}_K(A_{\bar{q}}(\lambda), F^K(\mu)) = \sum_w (-1)^{l(w)} m(w(\mu + \rho(\mathfrak{n} \cap \mathfrak{k})) - \rho(\mathfrak{n} \cap \mathfrak{k})),$$

where w runs over the element of the Weyl group of K such that $w(\mu + \rho(\mathfrak{n} \cap \mathfrak{k})) - \rho(\mathfrak{n} \cap \mathfrak{k})$ is dominant for the positive system of $\Delta(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t})$.

3. DISCRETELY DECOMPOSABLE (\mathfrak{g}, K) -MODULES

Let G_0 be a connected real reductive Lie group with Lie algebra \mathfrak{g}_0 . Let θ be a Cartan involution of G_0 and write $K_0 := G_0^\theta$ for the corresponding maximal compact subgroups of G_0 . The Cartan decomposition is written as $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$. We denote by $\mathfrak{g}, \mathfrak{k}$, etc. the complexifications of $\mathfrak{g}_0, \mathfrak{k}_0$, etc. Let θ also denote the induced actions on \mathfrak{g}_0 and their complex linear extensions to \mathfrak{g} .

Definition 3.1. Let V be a (\mathfrak{g}, K) -module. We say that V is *discretely decomposable* if V admits a filtration $\{F_p V\}_{p \in \mathbb{N}}$ such that $V = \bigcup_{p \in \mathbb{N}} F_p V$ and $F_p V$ is of finite length as a (\mathfrak{g}, K) -module for any $p \in \mathbb{N}$.

If V is unitarizable and discretely decomposable, then V is an algebraic direct sum of irreducible (\mathfrak{g}, K) -modules (see [Kob98b, Lemma 1.3]).

Let $\widehat{G}_0^{\text{adm}}$ be the set of isomorphism classes of irreducible (\mathfrak{g}, K) -modules. If V is a (\mathfrak{g}, K) -module of finite-length and $\pi \in \widehat{G}_0^{\text{adm}}$, we write $[V](\pi)$ for the multiplicities of π appearing in the composition series of V . More generally, for a discretely decomposable (\mathfrak{g}, K) -module V , let $[V] : \widehat{G}_0^{\text{adm}} \rightarrow \mathbb{N} \cup \{\infty\}$ be the function given as $[V](\pi) := \sup\{[F_p V](\pi) : p \in \mathbb{N}\}$, where $F_p V$ is a filtration as in Definition 3.1. It is easy to see that $[V]$ does not depend on the choice of filtration $F_p V$. For two functions $c_1, c_2 : \widehat{G}_0^{\text{adm}} \rightarrow \mathbb{N} \cup \{\infty\}$, we write $c_1 \leq c_2$ if $c_1(\pi) \leq c_2(\pi)$ holds for any $\pi \in \widehat{G}_0^{\text{adm}}$.

Definition 3.2. Let V be a discretely decomposable (\mathfrak{g}, K) -module. We say V is (\mathfrak{g}, K) -*admissible* if $[V](\pi) < \infty$ for any $\pi \in \widehat{G}_0^{\text{adm}}$.

Lemma 3.3. Let $\{V_n\}_{n \in \mathbb{N}}$ be an inductive system of (\mathfrak{g}, K) -admissible modules. Then $\liminf_n [V_n](\pi) \geq \lim_{\rightarrow n} [V_n](\pi)$ holds for any $\pi \in \widehat{G}_0^{\text{adm}}$.

Proof. Suppose that $\lim_{\rightarrow n} [V_n](\pi) \geq m$ for $m \in \mathbb{N}$. This means that there exists a submodule W of $\lim_{\rightarrow n} V_n$, which is of finite length and $[W](\pi) \geq m$. Then we can find an integer n_0 such that the image of V_{n_0} in V contains W , which implies $[V_n](\pi) \geq [W](\pi) \geq m$ for $n \geq n_0$. \square

Definition 3.4. Let (\mathfrak{h}, L) be a pair (\mathfrak{h} is not necessarily reductive). Suppose that a (\mathfrak{h}, L) -module V has a filtration of submodules $\{F_p V\}_{p \in \mathbb{N}}$ such that $V = \bigcup_{p \in \mathbb{N}} F_p V$ and $F_p V / F_{p-1} V$ is an irreducible (\mathfrak{h}, L) -module for any $p \in \mathbb{N}$. We write $s(V) := \text{gr}_F V = \bigoplus_{p \in \mathbb{N}} F_p V / F_{p-1} V$ for the associated graded module.

The (\mathfrak{h}, L) -module $s(V)$ does not depend on the filtration (if at least one such filtration exists).

In the following two lemmas, we see admissibility of cohomologically induced modules under certain assumptions.

Lemma 3.5. *Let \mathfrak{q} be a θ -stable parabolic subalgebra of \mathfrak{g} and define $\mathfrak{l}, \mathfrak{u}, K_L$ as in Section 2. Suppose that V is a $(\bar{\mathfrak{q}}, K_L)$ -module on which $\bar{\mathfrak{u}}$ acts locally nilpotently. If $\text{For}_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{l}, K_L}(V)$ is (\mathfrak{l}, K_L) -admissible, then $(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(V)$ is (\mathfrak{g}, K) -admissible for any $j \in \mathbb{N}$.*

Proof. Since the $\bar{\mathfrak{u}}$ -action on V is locally nilpotent and $\text{For}_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{l}, K_L}(V)$ is (\mathfrak{l}, K_L) -admissible, there exists a filtration $\{F_p V\}_{p \in \mathbb{N}}$ of V such that the successive quotients $F_p V / F_{p-1} V$ are irreducible $(\bar{\mathfrak{q}}, K_L)$ -modules. Then $\bar{\mathfrak{u}}$ acts as zero on the associated graded module $s(V) = \text{gr}_F V$. Fact 2.10 (i) implies that $(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(F_p V / F_{p-1} V)$ and $(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(s(V))$ are discretely decomposable as (\mathfrak{g}, K) -module. Using the exact sequence

$$(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(F_{p-1} V) \rightarrow (P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(F_p V) \rightarrow (P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(F_p V / F_{p-1} V),$$

we conclude that $(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(F_p V)$ is discretely decomposable and that

$$[(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(F_p V)] \leq \sum_{i=0}^p [(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(F_i V / F_{i-1} V)] \leq [(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(s(V))].$$

Therefore, the isomorphism $\varinjlim_p (P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(F_p V) \simeq (P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(V)$ and Lemma 3.3 give $[(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(V)] \leq [(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(s(V))]$. It follows from Fact 2.10 (i) and (ii) that $(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(s(V))$ is (\mathfrak{g}, K) -admissible and hence $(P_{\bar{\mathfrak{q}}, K_L}^{\mathfrak{g}, K})_j(V)$ is also (\mathfrak{g}, K) -admissible. \square

Lemma 3.6. *Let K be a reductive group and M an algebraic subgroup with Levi decomposition $M = L \ltimes U$. Let V be an (\mathfrak{m}, L) -module and $a \in \mathfrak{l}$ a semisimple element such that every eigenspaces of a in V is finite-dimensional. Then $(P_{\mathfrak{m}, L}^{\mathfrak{k}, K})_j(V)$ is K -admissible for any $j \in \mathbb{N}$.*

Proof. It is enough to show that $\dim \text{Hom}_K((P_{\mathfrak{m}, L}^{\mathfrak{k}, K})_j(V), F^K(\mu)) < \infty$ for any μ , where $F^K(\mu)$ is an irreducible K -module with highest weight μ . By [KV, Proposition 5.113],

$$\begin{aligned} \text{Hom}_K((P_{\mathfrak{m}, L}^{\mathfrak{k}, K})_j(V), F^K(\mu)) &\simeq \text{Ext}_{\mathfrak{m}, L}^j(V, \text{For}_{\mathfrak{k}, K}^{\mathfrak{m}, L}(F^K(\mu))) \\ &\simeq H_j(\mathfrak{m}, L; V \otimes \text{For}_{\mathfrak{k}, K}^{\mathfrak{m}, L}(F^K(\mu))^*) \\ &\simeq H_j(\mathfrak{u}; V \otimes \text{For}_{\mathfrak{k}, K}^{\mathfrak{m}, L}(F^K(\mu))^*)^L. \end{aligned}$$

Using standard resolution, we see that $H_j(\mathfrak{u}; V \otimes \text{For}_{\mathfrak{k}, K}^{\mathfrak{m}, L}(F^K(\mu))^*)$ is a subquotient of $\bigwedge^j \mathfrak{u} \otimes V \otimes \text{For}_{\mathfrak{k}, K}^{\mathfrak{m}, L}(F^K(\mu))^*$. By our assumption, the 0-eigenspace of a in $\bigwedge^j \mathfrak{u} \otimes V \otimes \text{For}_{\mathfrak{k}, K}^{\mathfrak{m}, L}(F^K(\mu))^*$ is finite-dimensional. We therefore conclude that $H_j(\mathfrak{u}; V \otimes \text{For}_{\mathfrak{k}, K}^{\mathfrak{m}, L}(F^K(\mu))^*)^L$ is finite-dimensional. \square

Let σ be an involution of G_0 and G'_0 the identity component of the fixed point set G_0^σ . We replace the Cartan involution θ by its G_0 -conjugation if necessary so that θ and σ commute. Then the restriction of θ gives a Cartan involution of G'_0 . We write $K'_0 := (G'_0)^\theta$ and $\mathfrak{g}'_0 = \mathfrak{k}'_0 + \mathfrak{p}'_0$ for the maximal compact subgroup and the Cartan decomposition with respect to θ .

We write $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}'}$ for the nilpotent cones of \mathfrak{g} and \mathfrak{g}' , respectively. Let $\text{pr}_{\mathfrak{g} \rightarrow \mathfrak{g}'}$ denote the projection from \mathfrak{g} onto \mathfrak{g}' along $\mathfrak{g}^{-\sigma}$.

Theorem 3.7. *Let (G_0, G'_0) be a symmetric pair of connected real reductive Lie groups defined by an involution σ . Let \mathfrak{q} be a θ -stable parabolic subalgebra of \mathfrak{g} . Then the following three conditions are equivalent.*

- (i) $A_{\mathfrak{q}}(\lambda)$ is non-zero and discretely decomposable as a (\mathfrak{g}', K') -module for some λ in the weakly fair range.
- (ii) $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable as a (\mathfrak{g}', K') -module for any λ in the weakly fair range.
- (iii) For any element $k \in K$, the subspace

$$\mathfrak{q}' := N_{\mathfrak{p}'}(\text{Ad}(k)\mathfrak{q} \cap \mathfrak{p}') + (\text{Ad}(k)\mathfrak{q} \cap \mathfrak{p}')$$

is a θ -stable parabolic subalgebra of \mathfrak{g}' , where $N_{\mathfrak{p}'}(\text{Ad}(k)\mathfrak{q} \cap \mathfrak{p}')$ is the normalizer of $\text{Ad}(k)\mathfrak{q} \cap \mathfrak{p}'$ in \mathfrak{k}' .

The proof is based on the following criterion for the discrete decomposability ([Kob98b, § 4], see also [KO12, Theorem 2.8]).

In the following fact, we take a σ -stable Cartan subalgebra \mathfrak{t}_0 of \mathfrak{k}_0 such that $\mathfrak{t}_0^{-\sigma}$ is a maximal abelian subalgebra of $\mathfrak{k}^{-\sigma}$. We choose a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ that is compatible with some positive system of the restricted root system $\Sigma(\mathfrak{k}, \sqrt{-1}\mathfrak{t}_0^{-\sigma})$.

Fact 3.8. *In the setting above, suppose that \mathfrak{q} is given by a $\Delta^+(\mathfrak{k}, \mathfrak{t})$ -dominant vector $a \in \sqrt{-1}\mathfrak{t}_0$. Then the following conditions are equivalent.*

- (i) $A_{\mathfrak{q}}(\lambda)$ is non-zero and discretely decomposable as a (\mathfrak{g}', K') -module for some λ in the weakly fair range.
- (ii) $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable as a (\mathfrak{g}', K') -module for any λ in the weakly fair range.
- (iii) $\text{pr}_{\mathfrak{g} \rightarrow \mathfrak{g}'}(\mathfrak{u} \cap \mathfrak{p}) \subset \mathcal{N}_{\mathfrak{g}'}$ for the nilradical \mathfrak{u} of \mathfrak{q} .
- (iv) $\text{pr}_{\mathfrak{g} \rightarrow \mathfrak{g}'}(\text{Ad}(K)\mathfrak{u} \cap \mathfrak{p}) \subset \mathcal{N}_{\mathfrak{g}'}$ for the nilradical \mathfrak{u} of \mathfrak{q} .
- (v) $\sigma\alpha(a) \geq 0$ whenever $\alpha \in \Delta(\mathfrak{p}, \mathfrak{t})$ satisfies $\alpha(a) > 0$.

We use the following lemma for the proof of Theorem 3.7.

Lemma 3.9. *Let V be a finite-dimensional vector space with a non-degenerate symmetric bilinear form. For subspaces $V_1 \subset V_2 \subset V$, we denote by $V_1^{\perp V_2}$ the set of all vectors in V_2 that are orthogonal to V_1 .*

Suppose that X is a subspace of V such that $V = X \oplus X^{\perp V}$. Let p be the projection onto X along $X^{\perp V}$. Then for any subspace $W \subset V$, it follows that

$$(W \cap X)^{\perp X} = p(W^{\perp V}).$$

Proof. We have

$$(W \cap X)^{\perp X} = (W \cap X)^{\perp V} \cap X = (W^{\perp V} + X^{\perp V}) \cap X = p(W^{\perp V}),$$

so the assertion is verified. \square

By Fact 3.8, it is enough to prove that:

Lemma 3.10. *Let (G_0, G'_0) be a symmetric pair and let \mathfrak{q} be a θ -stable parabolic subalgebra of \mathfrak{g} . Then the following two conditions are equivalent.*

(i) *The subspace*

$$\mathfrak{q}' := N_{\mathfrak{g}'}(\mathfrak{q} \cap \mathfrak{p}') + (\mathfrak{q} \cap \mathfrak{p}')$$

is a θ -stable parabolic subalgebra of \mathfrak{g}' .

(ii) $\text{pr}_{\mathfrak{g} \rightarrow \mathfrak{g}'}(\mathfrak{u} \cap \mathfrak{p}) \subset \mathcal{N}_{\mathfrak{g}'}$ *for the nilradical \mathfrak{u} of \mathfrak{q} .*

Proof. First of all, \mathfrak{q}' defined above is a subalgebra of \mathfrak{g} because $[\mathfrak{q} \cap \mathfrak{p}', \mathfrak{q} \cap \mathfrak{p}'] \subset \mathfrak{q} \cap \mathfrak{p}' \subset N_{\mathfrak{g}'}(\mathfrak{q} \cap \mathfrak{p}')$.

Choose a non-degenerate invariant symmetric form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} such that the subspaces \mathfrak{t}' , $\mathfrak{t}^{-\sigma}$, \mathfrak{p}' , and $\mathfrak{p}^{-\sigma}$ are mutually orthogonal. We use the letter $^\perp$ for orthogonal spaces with respect to $\langle \cdot, \cdot \rangle$ as in Lemma 3.9.

Assume that (i) holds. The subspaces $\mathfrak{u} = \mathfrak{q}^\perp$ and $\mathfrak{u}' = \mathfrak{q}'^\perp$ are the nilradicals of \mathfrak{q} and \mathfrak{q}' , respectively. Because \mathfrak{q} and \mathfrak{q}' are θ -stable, we have $(\mathfrak{q} \cap \mathfrak{p})^\perp = \mathfrak{u} \cap \mathfrak{p}$ and $(\mathfrak{q}' \cap \mathfrak{p}')^\perp = \mathfrak{u}' \cap \mathfrak{p}'$. In view of Lemma 3.9 and $\mathfrak{q} \cap \mathfrak{p}' = \mathfrak{q}' \cap \mathfrak{p}'$, we get

$$\text{pr}_{\mathfrak{g} \rightarrow \mathfrak{g}'}(\mathfrak{u} \cap \mathfrak{p}) = \text{pr}_{\mathfrak{g} \rightarrow \mathfrak{g}'}((\mathfrak{q} \cap \mathfrak{p})^\perp) = (\mathfrak{q} \cap \mathfrak{p}')^\perp = (\mathfrak{q}' \cap \mathfrak{p}')^\perp = \mathfrak{u}' \cap \mathfrak{p}'.$$

The right side is contained in $\mathcal{N}_{\mathfrak{g}'}$. This shows (ii).

Assume that (ii) holds. As we have seen above,

$$\text{pr}_{\mathfrak{g} \rightarrow \mathfrak{g}'}(\mathfrak{u} \cap \mathfrak{p}) = \text{pr}_{\mathfrak{g} \rightarrow \mathfrak{g}'}((\mathfrak{q} \cap \mathfrak{p})^\perp) = (\mathfrak{q} \cap \mathfrak{p}')^\perp.$$

Since the vector space $(\mathfrak{q} \cap \mathfrak{p}')^\perp$ is contained in the nilpotent cone of \mathfrak{g}' , the bilinear form $\langle \cdot, \cdot \rangle$ is zero on $(\mathfrak{q} \cap \mathfrak{p}')^\perp$ and hence $(\mathfrak{q} \cap \mathfrak{p}')^\perp \subset \mathfrak{q} \cap \mathfrak{p}'$. Then it follows that $N_{\mathfrak{g}'}(\mathfrak{q} \cap \mathfrak{p}') = [(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')^\perp]^\perp$. Indeed, for $x \in \mathfrak{t}'$,

$$\begin{aligned} x \in [(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')^\perp]^\perp &\Leftrightarrow \langle x, [(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')^\perp] \rangle = \{0\} \\ &\Leftrightarrow \langle [x, (\mathfrak{q} \cap \mathfrak{p}')], (\mathfrak{q} \cap \mathfrak{p}')^\perp \rangle = \{0\} \\ &\Leftrightarrow [x, (\mathfrak{q} \cap \mathfrak{p}')] \in \mathfrak{q} \cap \mathfrak{p}' \\ &\Leftrightarrow x \in N_{\mathfrak{g}'}(\mathfrak{q} \cap \mathfrak{p}'). \end{aligned}$$

Put $\mathfrak{q}' := N_{\mathfrak{g}'}(\mathfrak{q} \cap \mathfrak{p}') + (\mathfrak{q} \cap \mathfrak{p}')$. Then

$$\mathfrak{q}'^\perp = N_{\mathfrak{g}'}(\mathfrak{q} \cap \mathfrak{p}')^\perp + (\mathfrak{q} \cap \mathfrak{p}')^\perp = [(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')^\perp] + (\mathfrak{q} \cap \mathfrak{p}')^\perp.$$

Since $[(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')^\perp] \subset [(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')] \subset \mathfrak{q} \cap \mathfrak{p}' \subset N_{\mathfrak{g}'}(\mathfrak{q} \cap \mathfrak{p}')$, we see that $\mathfrak{q}'^\perp \subset \mathfrak{q}'$. We therefore have $\langle x, y \rangle = 0$ for $x, y \in \mathfrak{q}'^\perp$. Moreover, \mathfrak{q}'^\perp is a subalgebra of \mathfrak{g}' because

$$\langle [\mathfrak{q}'^\perp, \mathfrak{q}'^\perp], \mathfrak{q}' \rangle = \langle \mathfrak{q}'^\perp, [\mathfrak{q}'^\perp, \mathfrak{q}'] \rangle \subset \langle \mathfrak{q}'^\perp, \mathfrak{q}' \rangle = \{0\}.$$

As a consequence, \mathfrak{q}'^\perp is a solvable Lie algebra and hence contained in some Borel subalgebra \mathfrak{b}' of \mathfrak{g}' . Write \mathfrak{n}' for the nilradical of \mathfrak{b}' so $\mathfrak{n}' = \mathfrak{b}'^\perp$. Let $M := N_{K'}(\mathfrak{q} \cap \mathfrak{p}')$ be the normalizer of $\mathfrak{q} \cap \mathfrak{p}'$, which is an algebraic subgroup of K' . Then M has a Levi decomposition with reductive part M_R

and unipotent part M_U (see [Hoc, §VIII.4] for the Levi decomposition). If we denote by \mathfrak{m}_R and \mathfrak{m}_U the Lie algebras of M_R and M_U , respectively, then the bilinear form $\langle \cdot, \cdot \rangle$ is non-degenerate on \mathfrak{m}_R and zero on \mathfrak{m}_U . We then conclude that the nilradical of $N_{\mathfrak{p}'}(\mathfrak{q} \cap \mathfrak{p}')$ equals the radical of $N_{\mathfrak{p}'}(\mathfrak{q} \cap \mathfrak{p}')$ with respect to the bilinear form. As a result, $[(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'}] = N_{\mathfrak{p}'}(\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'}$ is the nilradical of $N_{\mathfrak{p}'}(\mathfrak{q} \cap \mathfrak{p}')$ and hence $[(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'}] \subset \mathfrak{n}'$. Since $(\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'} \subset \mathcal{N}_{\mathfrak{g}'} \cap \mathfrak{b}' = \mathfrak{n}'$, it follows that $\mathfrak{q}'^{\perp \mathfrak{g}'} = [(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'}] + (\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'} \subset \mathfrak{n}'$. Hence we see that $\mathfrak{q}' \supset \mathfrak{n}'^{\perp \mathfrak{g}'} = \mathfrak{b}'$ and \mathfrak{q}' is a parabolic subalgebra of \mathfrak{g}' , showing (i). \square

Retain the notation and the assumption of Theorem 3.7 and suppose that the equivalent conditions in Theorem 3.7 are satisfied. Let \mathcal{Q} be the set of all θ -stable parabolic subalgebras \mathfrak{q}'_i of \mathfrak{g}' such that $\mathfrak{q}'_i \cap \mathfrak{p}' = \mathfrak{q} \cap \mathfrak{p}'$. Then the parabolic subalgebra $\mathfrak{q}' = N_{\mathfrak{p}'}(\mathfrak{q} \cap \mathfrak{p}') + (\mathfrak{q} \cap \mathfrak{p}')$ given in Theorem 3.7 is a unique maximal element of \mathcal{Q} .

On the other hand, a minimal element \mathfrak{q}'' of \mathcal{Q} is constructed as follows. For the parabolic subalgebra \mathfrak{q}' defined above, put $\mathfrak{l}' = \mathfrak{q}' \cap \overline{\mathfrak{q}'}$, which is a Levi component of \mathfrak{q}' . The θ -stable reductive subalgebra \mathfrak{l}' decomposes as

$$\mathfrak{l}' = \bigoplus_{i \in I} \mathfrak{l}'_i \oplus \mathfrak{z}(\mathfrak{l}'),$$

where \mathfrak{l}'_i are simple Lie algebras and $\mathfrak{z}(\mathfrak{l}')$ is the center of \mathfrak{l}' . Put $I_c := \{i \in I : \mathfrak{l}'_i \subset \mathfrak{k}'\}$ and define

$$(3.1) \quad \mathfrak{l}'_c := \bigoplus_{i \in I_c} \mathfrak{l}'_i \oplus (\mathfrak{z}(\mathfrak{l}') \cap \mathfrak{k}'), \quad \mathfrak{l}'_n := \bigoplus_{i \notin I_c} \mathfrak{l}'_i \oplus (\mathfrak{z}(\mathfrak{l}') \cap \mathfrak{p}').$$

Then we have

$$\mathfrak{l}' = \mathfrak{l}'_c \oplus \mathfrak{l}'_n, \quad \mathfrak{l}'_n = [(\mathfrak{l}' \cap \mathfrak{p}'), (\mathfrak{l}' \cap \mathfrak{p}')] + \mathfrak{l}' \cap \mathfrak{p}', \quad \mathfrak{l}'_c \subset \mathfrak{k}'.$$

Take a Borel subalgebra $\mathfrak{b}(\mathfrak{l}'_c)$ of \mathfrak{l}'_c and define

$$(3.2) \quad \mathfrak{q}'' := \mathfrak{b}(\mathfrak{l}'_c) \oplus \mathfrak{l}'_n \oplus \mathfrak{u}'.$$

We claim that \mathfrak{q}'' is a minimal element of \mathcal{Q} and every minimal element is obtained in this way. Indeed, since any element \mathfrak{q}'_i of \mathcal{Q} is contained in \mathfrak{q}' , the parabolic subalgebra \mathfrak{q}'_i decomposes as $(\mathfrak{q}'_i \cap \mathfrak{l}') \oplus \mathfrak{u}'$. The condition $\mathfrak{q}'_i \cap \mathfrak{p}' = \mathfrak{q} \cap \mathfrak{p}'$ implies that $\mathfrak{q}'_i \supset \mathfrak{l}' \cap \mathfrak{p}'$ and hence $\mathfrak{q}'_i \supset \mathfrak{l}'_n$. As a consequence, the set \mathcal{Q} consists of the Lie algebras $\mathfrak{q}(\mathfrak{l}'_c) \oplus \mathfrak{l}'_n \oplus \mathfrak{u}'$ for parabolic subalgebras $\mathfrak{q}(\mathfrak{l}'_c)$ of \mathfrak{l}'_c . Our claim follows from this. In particular, a minimal element of \mathcal{Q} is unique up to inner automorphisms of \mathfrak{l}'_c .

There is another easy way to get an element of \mathcal{Q} .

Proposition 3.11. *Suppose that \mathfrak{t}_0 is a σ -stable Cartan subalgebra of \mathfrak{k}_0 and \mathfrak{q} is given by a vector $a \in \sqrt{-1}\mathfrak{t}_0$. Then the parabolic subalgebra $\mathfrak{q}'(a + \sigma(a))$ of \mathfrak{g}' given by $a + \sigma(a)$ belongs to \mathcal{Q} .*

Proof. The condition Fact 3.8 (iii) implies Fact 3.8 (v) although we do not impose the assumption just before Fact 3.8. In fact, the argument in

[Kob98b, § 4] to prove that implication is still valid without the assumption. Therefore, $\alpha \in \Delta(\mathfrak{p}, \mathfrak{t})$ satisfies $\alpha(a + \sigma(a)) \geq 0$ if and only if $\alpha(a) \geq 0$ and $\alpha(\sigma(a)) \geq 0$. We have

$$\mathfrak{q} \cap \mathfrak{p}' = \sum_{\alpha(a) \geq 0, \alpha(\sigma(a)) \geq 0} (\mathfrak{p}_\alpha + \mathfrak{p}_{\sigma\alpha}) \cap \mathfrak{p}'$$

and hence $\mathfrak{q} \cap \mathfrak{p}' = \mathfrak{q}'(a + \sigma(a)) \cap \mathfrak{p}'$. \square

Remark 3.12. In the proposition above, the parabolic subalgebra $\mathfrak{q}'(a + \sigma(a))$ depends on the choice of a .

We note here some observations on Lie algebras for later use.

Lemma 3.13. *Retain the notation and the assumption above. Then*

$$\mathfrak{q} \cap \mathfrak{g}' = (\mathfrak{q} \cap \mathfrak{l}'_c) \oplus \mathfrak{l}'_n \oplus \mathfrak{u}'.$$

Proof. From $\mathfrak{q} \cap \mathfrak{k}' \subset N_{\mathfrak{k}'}(\mathfrak{q} \cap \mathfrak{p}')$ and $\mathfrak{q} \cap \mathfrak{p}' = \mathfrak{q}' \cap \mathfrak{p}'$, we have $\mathfrak{q} \cap \mathfrak{g}' \subset \mathfrak{q}'$. From the proof of Theorem 3.7, we have

$$\begin{aligned} \mathfrak{u}' = \mathfrak{q}'^{\perp \mathfrak{g}'} &= [(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'}] + (\mathfrak{q} \cap \mathfrak{p}')^{\perp \mathfrak{p}'} \\ &\subset [(\mathfrak{q} \cap \mathfrak{p}'), (\mathfrak{q} \cap \mathfrak{p}')] + (\mathfrak{q} \cap \mathfrak{p}') \subset \mathfrak{q} \cap \mathfrak{g}'. \end{aligned}$$

Moreover, $\mathfrak{l}'_n = [(\mathfrak{l}' \cap \mathfrak{p}'), (\mathfrak{l}' \cap \mathfrak{p}')] + (\mathfrak{l}' \cap \mathfrak{p}')$ and $\mathfrak{l}' \cap \mathfrak{p}' \subset \mathfrak{q}' \cap \mathfrak{p}' = \mathfrak{q} \cap \mathfrak{p}'$ imply that $\mathfrak{l}'_n \subset \mathfrak{q} \cap \mathfrak{g}'$. Hence $\mathfrak{q} \cap \mathfrak{g}'$ decomposes as $\mathfrak{q} \cap \mathfrak{g}' = (\mathfrak{q} \cap \mathfrak{l}'_c) \oplus \mathfrak{l}'_n \oplus \mathfrak{u}'$. \square

Lemma 3.14. *Retain the notation and the assumption above. We assume moreover that $\mathfrak{q} \cap \mathfrak{k}'$ is a parabolic subalgebra of \mathfrak{k}' . Then $\mathfrak{q} \cap \mathfrak{g}'$ is a parabolic subalgebra of \mathfrak{g}' .*

Proof. By Lemma 3.13, $\mathfrak{q} \cap \mathfrak{g}' = (\mathfrak{q} \cap \mathfrak{l}'_c) \oplus \mathfrak{l}'_n \oplus \mathfrak{u}'$. Our assumption implies that $\mathfrak{q} \cap \mathfrak{l}'_c$ is a parabolic subalgebra of \mathfrak{l}'_c . Therefore, $\mathfrak{q} \cap \mathfrak{g}'$ is a parabolic subalgebra of \mathfrak{g}' . \square

4. LOCALIZATION OF COHOMOLOGICAL INDUCTION

In this section, we give a geometric realization of cohomologically induced modules in a general setting, which will be applied in the next section for the study of branching laws.

Let G be a complex linear algebraic group acting on a variety (or more generally a scheme) X . Let $a : G \times X \rightarrow X$ be the action map and $p_2 : G \times X \rightarrow X$ the second projection. Write \mathcal{O}_X for the structure sheaf of X and a^*, p_2^* for the inverse image functors as \mathcal{O} -modules. We say that an \mathcal{O}_X -module \mathcal{M} is G -equivariant if there is an isomorphism $a^*\mathcal{M} \simeq p_2^*\mathcal{M}$ satisfying the cocycle condition. For a G -equivariant \mathcal{O}_X -module \mathcal{M} , the G -action on \mathcal{M} differentiates to a \mathfrak{g} -action on \mathcal{M} .

If G acts on a smooth variety X , then the infinitesimal action is defined as a Lie algebra homomorphism from the Lie algebra \mathfrak{g} of G to the space of vector fields $\mathcal{T}(X)$ on X . Denote the image of $\xi \in \mathfrak{g}$ by $\xi_X \in \mathcal{T}(X)$. Then ξ_X gives a first order differential operator on the structure sheaf \mathcal{O}_X . Let

$\tilde{\mathfrak{g}}_X := \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$. This module becomes a Lie algebroid in a natural way (see [BB93, §1.2]): the Lie bracket is defined by

$$[f \otimes \xi, g \otimes \eta] = fg \otimes [\xi, \eta] + f\xi_X(g) \otimes \eta - g\eta_X(f) \otimes \xi$$

for $f, g \in \mathcal{O}_X$ and $\xi, \eta \in \mathfrak{g}$. Here $f \in \mathcal{O}_X$ means that f is a local section of \mathcal{O}_X . Similar notation will be used for other sheaves. Write $U(\tilde{\mathfrak{g}}_X) (\simeq \mathcal{O}_X \otimes U(\mathfrak{g}))$ for the universal enveloping algebra of $\tilde{\mathfrak{g}}_X$. Then a $U(\tilde{\mathfrak{g}}_X)$ -module is identified with an \mathcal{O}_X -module \mathcal{M} with a \mathfrak{g} -action satisfying $\xi(fm) = \xi_X(f)m + f(\xi m)$ for $\xi \in \mathfrak{g}$, $f \in \mathcal{O}_X$, and $m \in \mathcal{M}$.

Let \mathcal{T}_X be the sheaf of vector fields on X and let $p : \tilde{\mathfrak{g}}_X (= \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}) \rightarrow \mathcal{T}_X$ be the map given by $f \otimes \xi \mapsto f\xi_X$. Then the kernel $\mathcal{H} := \ker p$ is isomorphic to the G -equivariant locally free \mathcal{O}_X -module with typical fiber \mathfrak{h} . Let \mathcal{D}_X be the ring of differential operators on X . The map p extends to $p : U(\tilde{\mathfrak{g}}_X) \rightarrow \mathcal{D}_X$ and descends to an isomorphism of algebras

$$(4.1) \quad U(\tilde{\mathfrak{g}}_X)/U(\tilde{\mathfrak{g}}_X)\mathcal{H} \xrightarrow{\sim} \mathcal{D}_X.$$

Suppose that $X = G$ and the action of G on X is the product from left:

$$G \rightarrow \text{Aut}(G), \quad g \mapsto (g' \mapsto gg'),$$

where $\text{Aut}(G)$ is the automorphism group of G as a complex variety. In this case we write the vector field ξ_X as ξ_G^L , which is a right invariant vector field on G . Similarly, if the action of G on $X = G$ is the product from right:

$$G \rightarrow \text{Aut}(G), \quad g \mapsto (g' \mapsto g'g^{-1}),$$

we write the vector field ξ_X as ξ_G^R , which is a left invariant vector field on G . Let ξ_1, \dots, ξ_n be a basis of \mathfrak{g} and write $\xi^1, \dots, \xi^n \in \mathfrak{g}^*$ for the dual basis. Define regular functions α_i^j, β_i^j on G for $1 \leq i, j \leq n$ by

$$(4.2) \quad \alpha_j^i(g) := \langle \xi^i, \text{Ad}(g^{-1})\xi_j \rangle, \quad \beta_i^j(g) := \langle \xi^j, \text{Ad}(g)\xi_i \rangle.$$

Then it follows that

$$(\xi_j)_G^L = - \sum_{i=1}^n \alpha_j^i \cdot (\xi_i)_G^R, \quad (\xi_i)_G^R = - \sum_{j=1}^n \beta_i^j \cdot (\xi_j)_G^L, \quad \sum_{j=1}^n \alpha_j^i \beta_k^j = \delta_k^i.$$

We see $(\xi_j)_G^L$ as a differential operator on G . Then the function $(\xi_j)_G^L(\beta_i^j)$ on G is written as

$$(\xi_j)_G^L(\beta_i^j) = - \langle \xi^j, [\xi_j, \text{Ad}(\cdot)\xi_i] \rangle.$$

Hence

$$(4.3) \quad \sum_{j=1}^n (\xi_j)_G^L(\beta_i^j) = - \sum_{j=1}^n \langle \xi^j, [\xi_j, \text{Ad}(\cdot)\xi_i] \rangle = \text{Trace ad}(\text{Ad}(\cdot)\xi_i) = \text{Trace ad}(\xi_i).$$

Let H be a complex algebraic subgroup of G . The quotient $X := G/H$ is defined as a smooth algebraic variety (see [Bor, §II.6]). Denote by $\pi : G \rightarrow X$ the quotient map. Let V be a complex vector space with an algebraic action

of H . We define the G -equivariant quasi-coherent sheaf \mathcal{V}_X as the subsheaf of $\pi_* \mathcal{O}_G \otimes V$ given by

$$(4.4) \quad \mathcal{V}_X(U) := \{f \in \mathcal{O}(\pi^{-1}(U)) \otimes V : f(gh) = h^{-1}f(g)\}$$

for an open set $U \subset X$. Here, we identify sections of $\mathcal{O}(\pi^{-1}(U)) \otimes V$ with regular V -valued functions on $\pi^{-1}(U)$. Analogous identification will be used for other varieties. The \mathcal{O}_X -module \mathcal{V}_X corresponds to the G -equivariant vector bundle with typical fiber V . The category of G -equivariant quasi-coherent \mathcal{O}_X -modules is equivalent to the category of algebraic H -modules, and \mathcal{V}_X is the \mathcal{O}_X -module which corresponds to V via this equivalence. It also corresponds to the associated bundle $G \times_H V \rightarrow G/H$. The local sections of \mathcal{V}_X can be identified with the V -valued regular functions f on open subsets of G satisfying $f(gh) = h^{-1} \cdot f(g)$ for $h \in H$. We often use this identification in the following. Note that \mathcal{V}_X is locally free if V is finite-dimensional. Indeed, let v_1, \dots, v_n be a basis of V and take local sections $\tilde{v}_1, \dots, \tilde{v}_n$ such that $\tilde{v}_i(e) = v_i$ for the identity element $e \in G$. Then the map $\mathcal{O}_X^{\oplus n} \rightarrow \mathcal{V}_X$ given by $(f_i)_i \mapsto \sum_{i=1}^n f_i \tilde{v}_i$ is defined near the base point $eH \in G/H$ and is an isomorphism on some open neighborhood of eH . The G -equivariant structure on \mathcal{O}_G by the left translation induces a G -equivariant structure on \mathcal{V}_X . By differentiating it, the infinitesimal action of $\xi \in \mathfrak{g}$ is given by $f \mapsto \xi_G^L f$.

We write $\text{Ind}_H^G(V)$ for the space of global sections $\Gamma(X, \mathcal{V}_X)$ regarded as an algebraic G -module. Then by the Frobenius reciprocity,

$$\text{Hom}_G(W, \text{Ind}_H^G(V)) \xrightarrow{\sim} \text{Hom}_H(W, V)$$

for any algebraic G -module W .

Lemma 4.1. *If G and H are reductive, then*

$$R(G) \otimes_{R(H)} V \simeq \text{Ind}_H^G(V)$$

as G -modules.

Proof. We give the H -action on $\mathcal{O}(G) \otimes_{\mathbb{C}} V$ by $h(f \otimes v) \mapsto f(\cdot h) \otimes hv$. The H -module $\mathcal{O}(G) \otimes_{\mathbb{C}} V$ decomposes as a direct sum of irreducible factors because H is reductive. From the definition of \mathcal{V}_X , the space of global sections $\text{Ind}_H^G(V)$ is equal to the set of H -invariant elements $(\mathcal{O}(G) \otimes_{\mathbb{C}} V)^H$. With the identification $\mathcal{O}(G) \simeq R(G)$, we see that the canonical surjective map $R(G) \otimes_{\mathbb{C}} V \rightarrow R(G) \otimes_{R(H)} V$ is the projection onto the H -invariants. Hence we have

$$R(G) \otimes_{R(H)} V \simeq (\mathcal{O}(G) \otimes_{\mathbb{C}} V)^H \simeq \text{Ind}_H^G(V)$$

as G -modules. □

Suppose that H' is another algebraic subgroup of G such that $H \subset H'$. Let $X' := G/H'$ and $S := H'/H$ be the quotient varieties and $\varpi : X \rightarrow X'$ the canonical map. Write \mathcal{V}_S for the \mathcal{O}_S -module associated with V . Let

$W := \text{Ind}_H^{H'}(V)$ and let $\mathcal{W}_{X'}$ be the $\mathcal{O}_{X'}$ -module associated with the H' -module W .

The following lemma is immediate from the definition, which indicates ‘induction by stages’ in our setting.

Lemma 4.2. *In the setting above, there is a canonical G -equivariant isomorphism $\varpi_* \mathcal{V}_X \rightarrow \mathcal{W}_{X'}$.*

In the rest of this section we will work in the following setting.

Setting 4.3. Let $i : K \rightarrow G$ be a homomorphism of complex linear algebraic groups with finite kernel. Let H be a closed algebraic subgroup of G . Write $M := i^{-1}(H)$, which is an algebraic subgroup of K , and write $X := G/H$ and $Y := K/M$ for the quotient varieties. The map $i : K \rightarrow G$ induces an injective morphism between the quotient varieties $i : Y \rightarrow X$ and an injective homomorphism between Lie algebras $di : \mathfrak{k} \rightarrow \mathfrak{g}$. We identify \mathfrak{k} with its image $di(\mathfrak{k})$ and regard \mathfrak{k} as a subalgebra of \mathfrak{g} .

In particular, (\mathfrak{g}, K) and (\mathfrak{h}, M) become pairs in the sense of Definition 2.1, where \mathfrak{h} is the Lie algebra of H .

Let $e \in K$ be the identity element and let $o := eM \in Y$ be the base point of Y . Write

$$\begin{aligned} \mathcal{I}_Y &:= \{f \in \mathcal{O}_X : f(y) = 0 \text{ for } y \in Y\}, \\ \mathcal{I}_o &:= \{f \in \mathcal{O}_X : f(o) = 0\}, \end{aligned}$$

so \mathcal{I}_Y is the defining ideal of the closure \overline{Y} of Y . It follows that $i^{-1}\mathcal{O}_X/\mathcal{I}_Y \simeq \mathcal{O}_Y$. Here i^{-1} denotes the inverse image functor for the sheaves of abelian groups. For an $i^{-1}\mathcal{O}_X$ -module \mathcal{M} , the support of the sheaf $\mathcal{M}/(i^{-1}\mathcal{I}_o)\mathcal{M}$ is contained in $\{o\}$ so it is regarded as a vector space.

Let \mathcal{I}_Y^p the p -th power of \mathcal{I}_Y and let Y_p be the scheme $(Y, i^{-1}\mathcal{O}_X/(\mathcal{I}_Y)^p)$ for $p \geq 1$. If locally we have $X = \text{Spec } A$, $Y = \text{Spec } I$, and Y is closed in X , then $Y_p = \text{Spec}(A/I^p)$. The scheme Y_1 is identified with the algebraic variety Y . If \mathcal{M} is an $i^{-1}\mathcal{O}_X$ -module, then the sheaf $\mathcal{M}/(i^{-1}\mathcal{I}_Y)^p\mathcal{M}$ can be viewed as an \mathcal{O}_{Y_p} -module.

We can easily see that $i^{-1}(\mathcal{I}_Y^p/\mathcal{I}_Y^{p+1})$ is isomorphic to the K -equivariant \mathcal{O}_Y -module associated with the dual of the p -th symmetric tensor product $S^p(\mathfrak{g}/(\mathfrak{h} + \mathfrak{k}))^*$ with the coadjoint action of M . Let $\mathcal{T}_{X/Y}$ be the sheaf of vector fields in X tangent to Y , namely

$$\mathcal{T}_{X/Y} := \{\xi \in \mathcal{T}_X : \xi(\mathcal{I}_Y) \subset \mathcal{I}_Y\}.$$

Then $\xi \in \mathcal{T}_X$ operates on \mathcal{O}_X and induces an \mathcal{O}_Y -homomorphism

$$\xi : i^{-1}(\mathcal{I}_Y/\mathcal{I}_Y^2) \rightarrow i^{-1}(\mathcal{O}_X/\mathcal{I}_Y) \simeq \mathcal{O}_Y.$$

This gives an isomorphism of locally free \mathcal{O}_Y -modules

$$i^{-1}(\mathcal{T}_X/\mathcal{T}_{X/Y}) \simeq \text{Hom}_{\mathcal{O}_Y}(i^{-1}(\mathcal{I}_Y/\mathcal{I}_Y^2), \mathcal{O}_Y),$$

which correspond to the normal bundle of Y in X .

The ring of differential operators \mathcal{D}_X has the filtration given by

$$F_p \mathcal{D}_X := \{D \in \mathcal{D}_X : D(\mathcal{I}_Y^{p+1}) \subset \mathcal{I}_Y\},$$

which is called the filtration by normal degree with respect to i . A section of $F_p \mathcal{D}_X$ is locally written as $\sum \eta_1 \cdots \eta_r \xi_1 \cdots \xi_q$, where $q \leq p$, $\xi_1, \dots, \xi_q \in \mathcal{T}_X$, and $\eta_1, \dots, \eta_r \in \mathcal{T}_{X/Y}$. Let $G_p \mathcal{D}_X (\subset \mathcal{D}_X)$ be the sheaf of differential operators on X with rank equal or less than p . For $D \in G_p \mathcal{D}_X$, the differential operator $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ induces an \mathcal{O}_Y -homomorphism

$$i^{-1}(\mathcal{I}_Y^p / \mathcal{I}_Y^{p+1}) \rightarrow i^{-1}(\mathcal{O}_X / \mathcal{I}_Y) \simeq \mathcal{O}_Y,$$

which we denote by $\gamma(D)$. Write

$$i^{-1}(\mathcal{I}_Y^p / \mathcal{I}_Y^{p+1})^\vee := \text{Hom}_{\mathcal{O}_Y}(i^{-1}(\mathcal{I}_Y^p / \mathcal{I}_Y^{p+1}), \mathcal{O}_Y)$$

for the dual of $i^{-1}(\mathcal{I}_Y^p / \mathcal{I}_Y^{p+1})$. The map $D \mapsto \gamma(D)$ gives an isomorphism of \mathcal{O}_Y -modules

$$(4.5) \quad i^{-1}G_p \mathcal{D}_X / i^{-1}(G_p \mathcal{D}_X \cap F_{p-1} \mathcal{D}_X) \simeq i^{-1}(\mathcal{I}_Y^p / \mathcal{I}_Y^{p+1})^\vee.$$

They are also isomorphic to the p -th symmetric tensor of the locally free \mathcal{O}_Y -module $i^{-1}(\mathcal{I}_Y / \mathcal{I}_Y^2)^\vee$.

Let \mathcal{M} be a left \mathcal{D}_Y -module. The Lie algebra \mathfrak{k} acts on \mathcal{M} by η_Y for $\eta \in \mathfrak{k}$. Write Ω_X and Ω_Y for the canonical sheaves of X and Y , respectively. The push-forward by i is defined by

$$i_+ \mathcal{M} := i_*((\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} i^* \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^\vee.$$

Here, we write i_* for the push-forward of \mathcal{O} -modules or \mathbb{C} -modules and i_+ for the push-forward of \mathcal{D} -modules. i^* denotes the pull-back of \mathcal{O} -modules. It follows from the definition that

$$i^{-1}i_+ \mathcal{M} \simeq (\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X) \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee.$$

By using the filtration by normal degree, we define the $(i^{-1}\mathcal{O}_X)$ -module

$$(4.6) \quad F_p i^{-1}i_+ \mathcal{M} := (\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}F_p \mathcal{D}_X) \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee$$

for $p \geq 0$. This is well-defined because $\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}F_p \mathcal{D}_X$ is stable under the left \mathcal{D}_Y -action. We see that $i^{-1}F_p \mathcal{D}_X$ is a flat $(i^{-1}\mathcal{O}_X)$ -module, $\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}F_p \mathcal{D}_X$ is a left flat \mathcal{D}_Y -module, and $i^{-1}\Omega_X^\vee$ is a flat $(i^{-1}\mathcal{O}_X)$ -module. Hence the $(i^{-1}\mathcal{O}_X)$ -modules $F_p i^{-1}i_+ \mathcal{M}$ form a filtration of $i^{-1}i_+ \mathcal{M}$.

Consider the restriction of the \mathfrak{g} -action on $i_+ \mathcal{M}$ to \mathfrak{k} . For $\eta \in \mathfrak{k}$, the vector field η_X is tangent to Y . Hence the \mathfrak{k} -action stabilizes each $F_p i^{-1}i_+ \mathcal{M}$ and it induces an action on the quotient $F_p i^{-1}i_+ \mathcal{M} / F_{p-1} i^{-1}i_+ \mathcal{M}$. Moreover, $F_p \mathcal{D}_X \cdot \mathcal{I}_Y \subset F_{p-1} \mathcal{D}_X$ implies that $i^{-1}\mathcal{I}_Y \cdot F_p i^{-1}i_+ \mathcal{M} \subset F_{p-1} i^{-1}i_+ \mathcal{M}$. Therefore $F_p i^{-1}i_+ \mathcal{M} / F_{p-1} i^{-1}i_+ \mathcal{M}$ carries an \mathcal{O}_Y -module structure. Write $\Omega_{X/Y} := \Omega_Y^\vee \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X$ for the relative canonical sheaf. The K -equivariant structures on the \mathcal{O}_Y -modules $\Omega_{X/Y}^\vee$ and $i^{-1}(\mathcal{I}^p / \mathcal{I}^{p+1})$ give \mathfrak{k} -actions on them.

Lemma 4.4. *There is an isomorphism of \mathcal{O}_Y -modules*

$$F_p i^{-1} i_+ \mathcal{M} / F_{p-1} i^{-1} i_+ \mathcal{M} \simeq \mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_{X/Y}^\vee \otimes_{\mathcal{O}_Y} i^{-1} (\mathcal{I}_Y^p / \mathcal{I}_Y^{p+1})^\vee$$

that commutes with the actions of \mathfrak{k} . Here, the \mathfrak{k} -action on the right side is given by the tensor product of the action on each factors defined above.

Proof. The inverse image $i^* \mathcal{D}_X := \mathcal{O}_Y \otimes_{i^{-1} \mathcal{O}_X} i^{-1} \mathcal{D}_X$ of \mathcal{D}_X in the category of \mathcal{O} -modules has a left \mathcal{D}_Y -module structure. The action map

$$\mathcal{D}_Y \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes_{i^{-1} \mathcal{O}_X} i^{-1} \mathcal{D}_X) \rightarrow \mathcal{O}_Y \otimes_{i^{-1} \mathcal{O}_X} i^{-1} \mathcal{D}_X$$

induces a morphism of left \mathcal{D}_Y -modules

$$(4.7) \quad \begin{aligned} & \mathcal{D}_Y \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes_{i^{-1} \mathcal{O}_X} i^{-1} (G_p \mathcal{D}_X / (G_p \mathcal{D}_X \cap F_{p-1} \mathcal{D}_X))) \\ & \rightarrow \mathcal{O}_Y \otimes_{i^{-1} \mathcal{O}_X} i^{-1} (F_p \mathcal{D}_X / F_{p-1} \mathcal{D}_X). \end{aligned}$$

We give the inverse map of (4.7). Any section of $F_p \mathcal{D}_X / F_{p-1} \mathcal{D}_X$ is represented by a sum of section of the form $\eta_1 \cdots \eta_r \xi_1 \cdots \xi_p$ for $\xi_1, \dots, \xi_p \in \mathcal{T}_X$ and $\eta_1, \dots, \eta_r \in \mathcal{T}_{X/Y}$. The inverse map

$$\begin{aligned} & \mathcal{O}_Y \otimes_{i^{-1} \mathcal{O}_X} i^{-1} (F_p \mathcal{D}_X / F_{p-1} \mathcal{D}_X) \\ & \rightarrow \mathcal{D}_Y \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes_{i^{-1} \mathcal{O}_X} i^{-1} (G_p \mathcal{D}_X / (G_p \mathcal{D}_X \cap F_{p-1} \mathcal{D}_X))) \end{aligned}$$

is given by

$$f \otimes \eta_1 \cdots \eta_r \xi_1 \cdots \xi_p \mapsto f(\eta_1)|_Y \cdots (\eta_r)|_Y \otimes (1 \otimes \xi_1 \cdots \xi_p).$$

Hence (4.7) is an isomorphism.

By using (4.5) and (4.7), we obtain isomorphisms of \mathcal{O}_Y -modules:

$$(4.8) \quad \begin{aligned} & F_p i^{-1} i_+ \mathcal{M} / F_{p-1} i^{-1} i_+ \mathcal{M} \\ & \simeq (\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} (\mathcal{O}_Y \otimes_{i^{-1} \mathcal{O}_X} i^{-1} (F_p \mathcal{D}_X / F_{p-1} \mathcal{D}_X)) \otimes_{i^{-1} \mathcal{O}_X} i^{-1} \Omega_X^\vee \\ & \simeq (\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} (\mathcal{D}_Y \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes_{i^{-1} \mathcal{O}_X} i^{-1} (G_p \mathcal{D}_X / (G_p \mathcal{D}_X \cap F_{p-1} \mathcal{D}_X)))) \\ & \quad \otimes_{i^{-1} \mathcal{O}_X} i^{-1} \Omega_X^\vee \\ & \simeq (\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{O}_Y} i^{-1} (G_p \mathcal{D}_X / (G_p \mathcal{D}_X \cap F_{p-1} \mathcal{D}_X)) \otimes_{i^{-1} \mathcal{O}_X} i^{-1} \Omega_X^\vee \\ & \simeq \mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_{X/Y}^\vee \otimes_{\mathcal{O}_Y} i^{-1} (\mathcal{I}_Y^p / \mathcal{I}_Y^{p+1})^\vee. \end{aligned}$$

We now show that this map commutes with the \mathfrak{k} -actions. Take a section

$$(m \otimes \omega) \otimes (1 \otimes D) \otimes \omega' \in (\mathcal{M} \otimes_{\mathcal{O}} \Omega_Y) \otimes_{\mathcal{D}} (\mathcal{O}_Y \otimes_{i^{-1} \mathcal{O}} i^{-1} F_p \mathcal{D}_X) \otimes_{i^{-1} \mathcal{O}} i^{-1} \Omega_X^\vee$$

for $m \in \mathcal{M}$, $\omega \in \Omega_Y$, $D \in G_p \mathcal{D}_X$, and $\omega' \in \Omega_X^\vee$. Since any section of $F_p i^{-1} i_+ \mathcal{M} / F_{p-1} i^{-1} i_+ \mathcal{M}$ is represented by a sum of sections of this form, it is enough to see the commutativity for this section. Under the isomorphisms (4.8), the section $(m \otimes \omega) \otimes (1 \otimes D) \otimes \omega'$ corresponds to $m \otimes (\omega \otimes \omega') \otimes \gamma(D) \in$

$\mathcal{M} \otimes_{\mathcal{O}} \Omega_{X/Y}^{\vee} \otimes_{\mathcal{O}} i^{-1}(\mathcal{I}^p/\mathcal{I}^{p+1})^{\vee}$. For $\eta \in \mathfrak{k}$, the \mathfrak{k} -action on $i^{-1}i_+\mathcal{M}$ is given by

$$\begin{aligned} & (m \otimes \omega) \otimes (1 \otimes D) \otimes \omega' \\ \mapsto & (m \otimes \omega) \otimes (1 \otimes D(-\eta_X)) \otimes \omega' + (m \otimes \omega) \otimes (1 \otimes D) \otimes \eta_X \omega' \\ = & (m \otimes \omega) \otimes (1 \otimes (-\eta_X)D) \otimes \omega' + (m \otimes \omega) \otimes (1 \otimes [\eta_X, D]) \otimes \omega' \\ & + (m \otimes \omega) \otimes (1 \otimes D) \otimes \eta_X \omega'. \end{aligned}$$

Since $\eta_X|_Y = \eta_Y$, it follows that

$$\begin{aligned} & (m \otimes \omega) \otimes (1 \otimes (-\eta_X)D) \otimes \omega' = (m \otimes \omega)(-\eta_Y) \otimes (1 \otimes D) \otimes \omega' \\ & = (\eta_Y m \otimes \omega) \otimes (1 \otimes D) \otimes \omega' + (m \otimes \omega(-\eta_Y)) \otimes (1 \otimes D) \otimes \omega'. \end{aligned}$$

As a result, the action of η is given by

$$\begin{aligned} & \eta \cdot ((m \otimes \omega) \otimes (1 \otimes D) \otimes \omega') \\ = & (\eta_Y m \otimes \omega) \otimes (1 \otimes D) \otimes \omega' + (m \otimes \omega(-\eta_Y)) \otimes (1 \otimes D) \otimes \omega' \\ & + (m \otimes \omega) \otimes (1 \otimes [\eta_X, D]) \otimes \omega' + (m \otimes \omega) \otimes (1 \otimes D) \otimes \eta_X \omega'. \end{aligned}$$

Since $[\eta_X, D] \in G_p \mathcal{D}_X$, the section $\eta \cdot ((m \otimes \omega) \otimes (1 \otimes D) \otimes \omega')$ corresponds to

$$\begin{aligned} & \eta_Y m \otimes (\omega \otimes \omega') \otimes \gamma(D) + m \otimes \eta_Y(\omega \otimes \omega') \otimes \gamma(D) \\ & + m \otimes (\omega \otimes \omega') \otimes \gamma([\eta_X, D]). \end{aligned}$$

Thus, the commutativity follows from $\gamma([\eta_X, D]) = \eta \cdot \gamma(D)$. \square

The inverse image $i^{-1}U(\tilde{\mathfrak{g}}_X)$ of $U(\tilde{\mathfrak{g}}_X)$ is a sheaf of algebras on Y and an $i^{-1}\mathcal{O}_X$ -bimodule. We will call $i^{-1}U(\tilde{\mathfrak{g}}_X)$ -modules simply $i^{-1}\tilde{\mathfrak{g}}_X$ -modules. The K -action on $i^{-1}\tilde{\mathfrak{g}}_X$ is given by $f \otimes \xi \mapsto (k \cdot f) \otimes \text{Ad}(i(k))(\xi)$ for $f \in i^{-1}\mathcal{O}_X$, $\xi \in \mathfrak{g}$, $k \in K$. Suppose that \mathcal{M} is an $i^{-1}\tilde{\mathfrak{g}}_X$ -module and let $i^{-1}\tilde{\mathfrak{g}}_X \otimes \mathcal{M} \rightarrow \mathcal{M}$ be the action map. Then the inclusion $\mathfrak{g} \cdot (\mathcal{I}_Y)^p \subset (\mathcal{I}_Y)^{p-1}$ induces a map $i^{-1}\tilde{\mathfrak{g}}_X \otimes \mathcal{M}/(i^{-1}\mathcal{I}_Y)^p \mathcal{M} \rightarrow \mathcal{M}/(i^{-1}\mathcal{I}_Y)^{p-1} \mathcal{M}$. The K -actions on X and Y induce a K -action on Y_p . Since Y is K -stable in X , we have $\mathfrak{k} \cdot (\mathcal{I}_Y)^p \subset (\mathcal{I}_Y)^p$. Therefore, we can define a \mathfrak{k} -action on $\mathcal{M}/(i^{-1}\mathcal{I}_Y)^p \mathcal{M}$. Similarly, we have $\mathfrak{h} \cdot \mathcal{I}_o \subset \mathcal{I}_o$ and we can equip $\mathcal{M}/(i^{-1}\mathcal{I}_o) \mathcal{M}$ with a \mathfrak{h} -module structure.

Definition 4.5. Let V be a (\mathfrak{h}, M) -module. We say an $i^{-1}\tilde{\mathfrak{g}}_X$ -module \mathcal{V} is *associated with* V if $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p \mathcal{V}$ is a K -equivariant quasi-coherent \mathcal{O}_{Y_p} -module for all $p \geq 1$ and the following five assumptions hold.

(1) The canonical map

$$\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p \mathcal{V} \rightarrow \mathcal{V}/(i^{-1}\mathcal{I}_Y)^{p-1} \mathcal{V}$$

commutes with K -actions for $p \geq 2$.

(2) $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p \mathcal{V}$ is a flat \mathcal{O}_{Y_p} -module for $p \geq 1$.

(3) The action map $i^{-1}\tilde{\mathfrak{g}}_X \otimes \mathcal{V}/(i^{-1}\mathcal{I}_Y)^p \mathcal{V} \rightarrow \mathcal{V}/(i^{-1}\mathcal{I}_Y)^{p-1} \mathcal{V}$ commutes with K -actions for $p \geq 2$. Here K acts on $i^{-1}\tilde{\mathfrak{g}}_X \otimes \mathcal{V}/(i^{-1}\mathcal{I}_Y)^p \mathcal{V}$ by diagonal.

- (4) The \mathfrak{k} -action on $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ induced from the \mathfrak{g} -action on \mathcal{V} coincides with the differential of the K -action on $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ for $p \geq 1$.
- (5) There is an isomorphism $\iota : \mathcal{V}/(i^{-1}\mathcal{I}_o)\mathcal{V} \xrightarrow{\sim} V$ which commutes with \mathfrak{h} -actions and M -actions.

Remark 4.6. The \mathfrak{g} -action and the K -action on \mathcal{V} induce a \mathfrak{h} -action and an M -action on $\mathcal{V}/(i^{-1}\mathcal{I}_o)\mathcal{V}$. The conditions (3) and (4) imply that $\mathcal{V}/(i^{-1}\mathcal{I}_o)\mathcal{V}$ becomes a (\mathfrak{h}, M) -module.

Example 4.7. Suppose that V is an H -module and define the G -equivariant quasi-coherent \mathcal{O}_X -module \mathcal{V}_X as (4.4). The G -action on \mathcal{V}_X induces a \mathfrak{g} -action and a K -action on \mathcal{V}_X . Then by regarding V as a (\mathfrak{h}, M) -module, $i^{-1}\mathcal{V}_X$ is associated with V .

We will construct an $i^{-1}\tilde{\mathfrak{g}}_X$ -module associated with an arbitrary (\mathfrak{h}, M) -module in the end of this section.

Example 4.8. Let \mathcal{V} and \mathcal{W} be $i^{-1}\tilde{\mathfrak{g}}_X$ -modules associated with (\mathfrak{h}, M) -modules V and W , respectively. Then the tensor product $\mathcal{V} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{W}$ is associated with the (\mathfrak{h}, M) -module $V \otimes W$.

We can define the pull-back of $i^{-1}\tilde{\mathfrak{g}}_X$ -modules associated with V in the following way. Let K', G', H' be another triple of algebraic groups satisfying the assumptions in Setting 4.3. In particular, the map $i' : K' \rightarrow G'$ induces a morphism of the quotient varieties $i' : K'/M' \rightarrow G'/H'$, where $M' := (i')^{-1}(H')$. Suppose that $\varphi_K : K' \rightarrow K$ and $\varphi : G' \rightarrow G$ are homomorphisms such that the diagram

$$\begin{array}{ccc} K' & \xrightarrow{i'} & G' \\ \varphi_K \downarrow & & \downarrow \varphi \\ K & \xrightarrow{i} & G \end{array}$$

commutes and that $\varphi(H') \subset H$. Then $\varphi_K(M') \subset M$. The maps φ, φ_K induce morphisms $\varphi : X' := G'/H' \rightarrow X$, $\varphi_K : Y' := K'/M' \rightarrow Y$ and $\varphi_p : Y'_p := (Y', (i')^{-1}\mathcal{O}_{X'}/(\mathcal{I}_{Y'})^p) \rightarrow Y_p$. We get the commutative diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ \varphi_K \downarrow & & \downarrow \varphi \\ Y & \xrightarrow{i} & X \end{array}$$

Suppose that \mathcal{V} is an $i^{-1}\tilde{\mathfrak{g}}_X$ -module associated with a (\mathfrak{h}, M) -module V . Let $\mathcal{V}' := (i')^{-1}\mathcal{O}_{X'} \otimes_{(\varphi \circ i')^{-1}\mathcal{O}_X} \varphi_K^{-1}\mathcal{V}$. We define a \mathfrak{g}' -action on \mathcal{V}' by $\xi(f \otimes v) = \xi_{X'}(f) \otimes v + f \otimes \varphi(\xi)v$ for $\xi \in \mathfrak{g}'$, $f \in (i')^{-1}\mathcal{O}_{X'}$, and $v \in \varphi_K^{-1}\mathcal{V}$ so that \mathcal{V}' becomes an $(i')^{-1}\tilde{\mathfrak{g}}_{X'}$ -module. Since

$$\mathcal{V}'/((i')^{-1}\mathcal{I}_{Y'})^p\mathcal{V}' \simeq (i')^{-1}\mathcal{O}_{X'}/(\mathcal{I}_{Y'})^p \otimes_{(\varphi \circ i')^{-1}\mathcal{O}_X} \varphi_K^{-1}\mathcal{V} \simeq \varphi_p^*(\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}),$$

the sheaf $\mathcal{V}'/((i')^{-1}\mathcal{I}_{Y'})^p\mathcal{V}'$ is a K' -equivariant quasi-coherent $\mathcal{O}_{Y'}$ -module. We can easily show the following proposition by checking the five assumptions in Definition 4.5.

Proposition 4.9. *Let V be a (\mathfrak{h}, M) -module and \mathcal{V} an $i^{-1}\tilde{\mathfrak{g}}_X$ -module associated with V . Then the $(i')^{-1}\tilde{\mathfrak{g}}'_{X'}$ -module $(i')^{-1}\mathcal{O}_{X'} \otimes_{(\varphi \circ i')^{-1}\mathcal{O}_X} \varphi_K^{-1}\mathcal{V}$ is associated with the (\mathfrak{h}', M') -module $\text{For}_{\mathfrak{h}, M}^{\mathfrak{h}', M'}(V)$.*

In the next lemma, we assume that K and M are complex reductive linear algebraic groups. In particular, $Y := K/M$ is an affine variety by Matsushima's criterion. We endow $\mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X$ with a left $i^{-1}\tilde{\mathfrak{g}}_X$ -module structure by $\xi(f \otimes D) = f \otimes D(-\xi_X)$ for $\xi \in \mathfrak{g}$.

Let V be a (\mathfrak{h}, M) -module and \mathcal{V} an $i^{-1}\mathcal{O}_X$ -module associated with V . Define the (\mathfrak{g}, K) -module $R(\mathfrak{g}, K) \otimes_{R(\mathfrak{h}, M)} V$ as in Section 2.

The lemma relates these two modules.

Lemma 4.10. *Under the assumptions above, there is an isomorphism of (\mathfrak{g}, K) -modules*

$$\Gamma(Y, \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \simeq R(\mathfrak{g}, K) \otimes_{R(\mathfrak{h}, M)} V.$$

(See the remark below for the definition of the (\mathfrak{g}, K) -action on the left side.)

Remark 4.11. Since $\mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X$ and \mathcal{V} have $i^{-1}\tilde{\mathfrak{g}}_X$ -module structures, the tensor product $\mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$ becomes an $i^{-1}\tilde{\mathfrak{g}}_X$ -module. This gives a \mathfrak{g} -action on the space of global sections $\Gamma(Y, \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})$. In order to define a K -action, we use the filtration $F_p\mathcal{D}_X$ defined above. By definition, $\mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*F_p\mathcal{D}_X$ is annihilated by $(i^{-1}\mathcal{I}_Y)^{p+1}$ and hence is regarded as a quasi-coherent $\mathcal{O}_{Y_{p+1}}$ -module. We therefore have

$$(4.9) \quad \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*F_p\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \simeq \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*F_p\mathcal{D}_X \otimes_{\mathcal{O}_{Y_q}} \mathcal{V}/(i^{-1}\mathcal{I}_Y)^q\mathcal{V}$$

for $p < q$. Since $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^q\mathcal{V}$ is a flat \mathcal{O}_{Y_q} -module by Definition 4.5 (2), the map

$$\mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*F_{p-1}\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \rightarrow \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*F_p\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$$

is injective. We let K act on the right side of (4.9) by diagonal. Then it gives a K -action on $\Gamma(Y, \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*F_p\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})$. Using the isomorphisms

$$\begin{aligned} \Gamma(Y, \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) &\simeq \Gamma(Y, (\varinjlim_p \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*F_p\mathcal{D}_X) \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \\ &\simeq \Gamma(Y, \varinjlim_p (\mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*F_p\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})) \\ &\simeq \varinjlim_p \Gamma(Y, \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*F_p\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}), \end{aligned}$$

we define a K -action on $\Gamma(Y, \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})$. With these actions, $\Gamma(Y, \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})$ becomes a (\mathfrak{g}, K) -module because of Definition 4.5 (3) and (4).

Proof. Using the right $i^{-1}\mathcal{D}_X$ -module structure of $i^*\mathcal{D}_X$, we define a \mathfrak{g} -action ρ on the sheaf $i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$ by

$$\rho(\xi)(D \otimes v) := D(-\xi_X) \otimes v + D \otimes \xi v$$

for $\xi \in \mathfrak{g}$, $D \in i^*\mathcal{D}_X$, and $v \in \mathcal{V}$. Moreover, the sheaf $i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$ is K -equivariant. We denote this K -action and also its infinitesimal \mathfrak{k} -action by ν . By Definition 4.5 (4), the \mathfrak{k} -action ν is given by

$$\nu(\eta)(D \otimes v) = \eta_Y D \otimes v - D \eta_X \otimes v + D \otimes \eta v$$

for $\eta \in \mathfrak{k}$. Here $\eta_Y D$ and $D \eta_X$ are defined by the $(\mathcal{D}_Y, i^{-1}\mathcal{D}_X)$ -bimodule structure on $i^*\mathcal{D}_X$. Then $\Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})$ is a weak Harish-Chandra module in the sense of [BL95], namely,

$$(4.10) \quad \nu(k)\rho(\xi)\nu(k^{-1}) = \rho(\text{Ad}(k)\xi)$$

for $k \in K$ and $\xi \in \mathfrak{g}$. Put $\omega(\eta) := \nu(\eta) - \rho(\eta)$ for $\eta \in \mathfrak{k}$. Then $\omega(\eta)$ is given by

$$\omega(\eta)(D \otimes v) = \eta_Y D \otimes v.$$

Since Y is an affine variety, $\Gamma(Y, \mathcal{D}_Y)$ is generated by $U(\mathfrak{k})$ and $\mathcal{O}(Y)$ as an algebra. Therefore,

$$\begin{aligned} & \Gamma(Y, \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \\ & \simeq \mathcal{O}(Y) \otimes_{\Gamma(Y, \mathcal{D}_Y)} \Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \\ & \simeq \Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) / \omega(\mathfrak{k})\Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}). \end{aligned}$$

Let $e \in K$ be the identity element. Write $o := eM \in Y$ for the base point and $i_{o,Y} : \{o\} \rightarrow Y$ for the inclusion map. Let \mathcal{I}_o be the maximal ideal of \mathcal{O}_Y corresponding to o . The geometric fiber of $i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$ at o is given by

$$\begin{aligned} W &:= (i_{o,Y})^*(i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \\ &\simeq \Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) / \mathcal{I}_o(Y)\Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}). \end{aligned}$$

The actions ρ and ν on $i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$ induce a \mathfrak{g} -action ρ_o and an M -action ν_o on W . With these actions, W becomes a (\mathfrak{g}, M) -module. To show this, it is enough to see that ρ_o and ν_o agree on \mathfrak{m} . This follows from

$$\omega(\eta)\Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \subset \mathcal{I}_o(Y)\Gamma(Y, i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})$$

for $\eta \in \mathfrak{m}$.

We claim that $W \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V$ as a (\mathfrak{g}, M) -module. Put $i_{o,X} := i \circ i_{o,Y}$. Then

$$\begin{aligned} W &\simeq (i_{o,X})^*\mathcal{D}_X \otimes_{(i_{o,X})^{-1}\mathcal{O}_X} (i_{o,X})^{-1}\mathcal{V} \\ &\simeq (i_{o,X})^{-1}((i_{o,X})_*\mathcal{O}_{\{o\}} \otimes_{\mathcal{O}_X} \Omega_X) \otimes_{(i_{o,X})^{-1}\mathcal{O}_X} (i_{o,X})^{-1}\mathcal{V}. \end{aligned}$$

Let $\{F'_p \mathcal{D}_X\}$ be the filtration by normal degree with respect to $i_{o,X}$. Define the filtration

$$F_p W := (i_{o,X})^* F'_p \mathcal{D}_X \otimes_{(i_{o,X})^{-1}\mathcal{O}_X} (i_{o,X})^{-1}\mathcal{V}$$

of W . Then $F_p W$ is (\mathfrak{h}, M) -stable and there is an isomorphism of (\mathfrak{h}, M) -modules

$$F_p W / F_{p-1} W \simeq (i_{o,X})^{-1} (\mathcal{I}_o^p / \mathcal{I}_o^{p+1})^\vee \otimes V$$

by Lemma 4.4. The isomorphism $F_0 W \simeq V$ induces a (\mathfrak{g}, M) -homomorphism $\varphi : U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V \rightarrow W$. Let $U_p(\mathfrak{g})$ be the standard filtration of $U(\mathfrak{g})$. Then $(U_p(\mathfrak{g})U(\mathfrak{h})) \otimes_{U(\mathfrak{h})} V$ is a filtration of the (\mathfrak{h}, M) -module $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V$ and there is an isomorphism of (\mathfrak{h}, M) -modules:

$$(U_p(\mathfrak{g})U(\mathfrak{h})) \otimes_{U(\mathfrak{h})} V / (U_{p-1}(\mathfrak{g})U(\mathfrak{h})) \otimes_{U(\mathfrak{h})} V \simeq S^p(\mathfrak{g}/\mathfrak{h}) \otimes V.$$

In view of the proof of Lemma 4.4, we see that the map on the successive quotient

$$\varphi_p : (U_p(\mathfrak{g})U(\mathfrak{h})) \otimes_{U(\mathfrak{h})} V / (U_{p-1}(\mathfrak{g})U(\mathfrak{h})) \otimes_{U(\mathfrak{h})} V \rightarrow F_p W / F_{p-1} W$$

induced by φ is an isomorphism. Hence φ is an isomorphism.

As a K -equivariant \mathcal{O}_Y -module, $i^* \mathcal{D}_Y \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}$ is isomorphic to the \mathcal{O}_Y -module \mathcal{W}_Y associated with the M -module W . Hence we can see global sections $\Gamma(Y, i^* \mathcal{D}_Y \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V})$ as W -valued regular functions on K . Let f be a W -valued regular function on K such that $f(kh) = \nu_o(h^{-1})f(k)$ for $k \in K$ and $h \in M$. The \mathfrak{g} -action ρ at e is given by $(\rho(\xi)f)(e) = \rho_o(\xi)(f(e))$. Hence (4.10) implies that

$$(\rho(\xi)f)(k) = (\nu(k)\rho(\text{Ad}(k^{-1})\xi)\nu(k^{-1})f)(k) = \rho_o(\text{Ad}(k^{-1})\xi)(f(k)).$$

Let ξ_1, \dots, ξ_n be a basis of \mathfrak{g} and write $\xi^1, \dots, \xi^n \in \mathfrak{g}^*$ for its dual basis. Under the isomorphism $\Gamma(Y, \mathcal{W}_Y) \simeq R(K) \otimes_{R(M)} W$ given in Lemma 4.1, the \mathfrak{g} -action ρ on $R(K) \otimes_{R(M)} W$ is given by

$$(4.11) \quad \rho(\xi)(S \otimes w) = \sum_{i=1}^n \langle \xi^i, \text{Ad}(\cdot)^{-1} \xi \rangle S \otimes \rho_o(\xi_i)w$$

for $S \in R(K)$ and $w \in W$. If we define ρ on $R(K) \otimes_{\mathbb{C}} W$ by this equation, then ρ commutes with the canonical surjective map

$$p : R(K) \otimes_{\mathbb{C}} W \rightarrow R(K) \otimes_{R(M)} W.$$

The K -action ν is given by the left translation of $R(K)$:

$$\nu(k)(S \otimes w) = (kS) \otimes w.$$

Hence ν also lifts to the action on $R(K) \otimes_{\mathbb{C}} W$ and commutes with p . Let η_1, \dots, η_m be a basis of \mathfrak{k} and write $\eta^1, \dots, \eta^m \in \mathfrak{k}^*$ for its dual basis. Define the regular functions α_j^i and β_i^j on K with respect to η_i as in (4.2). Then the \mathfrak{k} -action ω is given by

$$\begin{aligned} \omega(\eta_j)(S \otimes w) &= \nu(\eta_j)(S \otimes w) - \rho(\eta_j)(S \otimes w) \\ &= ((\eta_j)_K^L S) \otimes w - \sum_{i=1}^m \alpha_j^i S \otimes \rho_o(\eta_i)w. \end{aligned}$$

Here, we identify $R(K)$ with $\mathcal{O}(K)$, and give actions of differential operators on K .

We have

$$\begin{aligned} & \Gamma(Y, \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^* \mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \\ & \simeq \Gamma(Y, i^* \mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) / \omega(\mathfrak{k}) \Gamma(Y, i^* \mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \\ & \simeq (R(K) \otimes_{R(M)} W) / \omega(\mathfrak{k}) (R(K) \otimes_{R(M)} W). \end{aligned}$$

We note that the \mathfrak{k} -actions ρ and ν agree on the quotient $(R(K) \otimes_{R(M)} W) / \omega(\mathfrak{k}) (R(K) \otimes_{R(M)} W)$ and hence it becomes a (\mathfrak{g}, K) -module.

The equation $\sum_{j=1}^m \alpha_j^i \beta_k^j = \delta_k^i$ implies that $\omega(\mathfrak{k}) (R(K) \otimes_{\mathbb{C}} W)$ is generated by the elements of the form $\sum_{j=1}^m \omega(\eta_j) (\beta_k^j S \otimes w)$ for $S \in R(K)$ and $w \in W$. We observe from (4.3) that $\sum_{j=1}^m (\eta_j)_K^L (\beta_k^j) = 0$ because $\text{Trace ad}(\cdot) = 0$ for the reductive Lie algebra \mathfrak{k} . Therefore,

$$(\eta_k)_K^R = - \sum_{j=1}^m \beta_k^j (\eta_j)_K^L = - \sum_{j=1}^m (\eta_j)_K^L \beta_k^j$$

as differential operators on K . Then

$$\begin{aligned} \sum_{j=1}^m \omega(\eta_j) (\beta_k^j S \otimes w) &= \sum_{j=1}^m (\eta_j)_K^L \beta_k^j S \otimes w + \sum_{i,j=1}^m (\alpha_j^i \beta_k^j S \otimes \rho_o(\eta_i) w) \\ &= -(\eta_k)_K^R S \otimes w + S \otimes \rho_o(\eta_k) w. \end{aligned}$$

Consequently, the kernel of the map

$$R(K) \otimes_{\mathbb{C}} W \rightarrow (R(K) \otimes_{R(M)} W) / \omega(\mathfrak{k}) (R(K) \otimes_{R(M)} W)$$

is generated by $\text{Ker } p$ and $-(\eta_k)_K^R S \otimes w + S \otimes \rho_o(\eta_k) w$. Hence

$$\begin{aligned} & (R(K) \otimes_{R(M)} W) / \omega(\mathfrak{k}) (R(K) \otimes_{R(M)} W) \\ & \simeq R(K) \otimes_{R(\mathfrak{k}, M)} W \\ & \simeq R(\mathfrak{g}, K) \otimes_{R(\mathfrak{g}, M)} W. \end{aligned}$$

From (4.11), we see that the isomorphism

$$(R(K) \otimes_{R(M)} W) / \omega(\mathfrak{k}) (R(K) \otimes_{R(M)} W) \simeq R(\mathfrak{g}, K) \otimes_{R(\mathfrak{g}, M)} W$$

commutes with the (\mathfrak{g}, K) -actions. Therefore,

$$\begin{aligned} \Gamma(Y, \mathcal{O}_Y \otimes_{\mathcal{D}_Y} i^* \mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) &\simeq R(\mathfrak{g}, K) \otimes_{R(\mathfrak{g}, M)} W \\ &\simeq R(\mathfrak{g}, K) \otimes_{R(\mathfrak{h}, M)} V \end{aligned}$$

and the lemma is proved. \square

In the rest of this section we assume that K is reductive but M is not necessarily reductive. Let $M = L \ltimes U$ be a Levi decomposition of M , where L is a maximal reductive subgroup of M and U is the unipotent radical of M . The corresponding decomposition of the Lie algebra is $\mathfrak{m} = \mathfrak{l} \oplus \mathfrak{u}$.

Let V be a (\mathfrak{h}, M) -module. We can see V as a (\mathfrak{h}, L) -module by restriction and then define the cohomologically induced module $(P_{\mathfrak{h}, L}^{\mathfrak{g}, K})_j(V)$ as in Section 2.

In order to state the main theorem, we need a shift of modules by a character (or an invertible sheaf) that we will define in the following. Write $\bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{l})$ for the top exterior product of $\mathfrak{k}/\mathfrak{l}$ and view it as a one-dimensional L -module by the adjoint action. Since K and L are reductive, the identity component of L acts trivially on $\bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{l})$. We extend the L -action on $\bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{l})$ to an M -action by letting U act trivially. Define \mathcal{L} as the K -equivariant locally free \mathcal{O}_Y -module on $Y := K/M$ whose typical fiber is isomorphic to the M -module $\bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{l})$. The K -action on \mathcal{L} differentiates to a \mathfrak{k} -action. Then \mathcal{L} becomes a $U(\mathfrak{k}_Y)$ -module and the kernel of the map $\tilde{\mathfrak{k}}_Y \rightarrow \mathcal{T}_Y$ acts by zero because the identity component of M acts trivially on $\bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{l})$. Therefore, \mathcal{L} has a structure of left \mathcal{D}_Y -module via the isomorphism (4.1) for Y .

Recall that the direct image $i_+\mathcal{L}$ of \mathcal{L} by i in the category of left \mathcal{D} -modules is defined as

$$i_+\mathcal{L} := i_*((\mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^\vee,$$

where i_* is the direct image functor for sheaves of abelian groups, Ω_Y is the canonical sheaf of Y , and Ω_X^\vee is the dual of the canonical sheaf of X . Via the map $p : U(\tilde{\mathfrak{g}}_X) \rightarrow \mathcal{D}_X$, we can see $i_+\mathcal{L}$ as a $\tilde{\mathfrak{g}}_X$ -module. The inverse image $i^{-1}i_+\mathcal{L}$ as a sheaf of abelian groups is

$$i^{-1}i_+\mathcal{L} = (\mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} i^*\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee,$$

which has an $i^{-1}\tilde{\mathfrak{g}}_X$ -module structure. We note that the functor $i^{-1}i_+$ is exact.

Let $\{F_p\mathcal{D}_X\}$ be the filtration by normal degree with respect to i . It induces a filtration of $i^{-1}i_+\mathcal{L}$:

$$F_p i^{-1}i_+\mathcal{L} := (\mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_Y} i^*F_p\mathcal{D}_X \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\Omega_X^\vee.$$

Then we can define a (\mathfrak{g}, K) -action on the cohomology space $H^s(Y, i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V})$ in a way similar to Remark 4.11.

The following is the main theorem in this section.

Theorem 4.12. *In Setting 4.3, we assume that K is reductive. Let $M = L \ltimes U$ be a Levi decomposition. Suppose that V is a (\mathfrak{h}, M) -module and that \mathcal{V} is an $i^{-1}\tilde{\mathfrak{g}}_X$ -module associated with V (Definition 4.5). Then we have an isomorphism of (\mathfrak{g}, K) -modules*

$$H^s(Y, i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) \simeq (P_{\mathfrak{h}, L}^{\mathfrak{g}, K})_{u-s} \left(V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}) \right)$$

for $s \in \mathbb{N}$ and $u = \dim U$.

Proof. Let $\tilde{X} := G/L$ and $\tilde{Y} := K/L$ be the quotient varieties. We have the commutative diagram:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{i}} & \tilde{X} \\ \pi_K \downarrow & & \downarrow \pi \\ Y & \xrightarrow{i} & X \end{array}$$

where the maps are defined canonically.

Let $i_+ : \mathbf{D}^b(\mathcal{D}_Y) \rightarrow \mathbf{D}^b(\mathcal{D}_X)$ denote the direct image functor between the bounded derived categories of left \mathcal{D} -modules. Similarly for \tilde{i}_+ , π_+ , and $(\pi_K)_+$. We have $\pi_+ \circ \tilde{i}_+ \simeq i_+ \circ (\pi_K)_+$. Since π_K is a smooth morphism and the fiber is isomorphic to the affine space \mathbb{C}^u , it follows that $(\pi_K)_+ \Omega_{\tilde{Y}}^\vee \simeq \mathcal{L}[u]$ (see [HMSW]). Here $\mathcal{L}[u] \in \mathbf{D}^b(\mathcal{D}_Y)$ is the complex $(\cdots \rightarrow 0 \rightarrow \mathcal{L} \rightarrow 0 \rightarrow \cdots)$, concentrated in degree $-u$. Therefore, $i_+(\pi_K)_+ \Omega_{\tilde{Y}}^\vee \simeq i_+ \mathcal{L}[u]$ in $\mathbf{D}^b(\mathcal{D}_X)$.

Since L is reductive, the varieties \tilde{X} and \tilde{Y} are affine by Matsushima's criterion. Hence the functor \tilde{i}_+ is exact for quasi-coherent \mathcal{D} -modules and π_* is exact for quasi-coherent \mathcal{O} -modules.

Denote by $\mathcal{T}_{\tilde{X}/X}$ the sheaf of local vector fields on \tilde{X} tangent to the fiber of π , and denote by $\Omega_{\tilde{X}/X}$ the top exterior product of its dual $\mathcal{T}_{\tilde{X}/X}^\vee$. We note that there is a natural isomorphism $\Omega_{\tilde{X}/X} \simeq \Omega_{\tilde{X}} \otimes_{\mathcal{O}_{\tilde{X}}} \pi^* \Omega_X^\vee$. Recall that for $\mathcal{M} \in \mathbf{D}^b(\mathcal{D}_{\tilde{X}})$ the direct image $\pi_* \mathcal{M}$ is defined as

$$\pi_* \mathcal{M} = \pi_* ((\mathcal{M} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}}) \otimes_{\mathcal{D}_{\tilde{X}}}^L \pi^* \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^\vee.$$

The left $\mathcal{D}_{\tilde{X}}$ -module $\pi^* \mathcal{D}_X$ has the resolution (see [HMSW, Appendix A.3.3]):

$$(4.12) \quad \mathcal{D}_{\tilde{X}} \otimes_{\mathcal{O}_{\tilde{X}}} \bigwedge^\bullet \mathcal{T}_{\tilde{X}/X} \rightarrow \pi^* \mathcal{D}_X,$$

where the boundary map ∂ on $\mathcal{D}_{\tilde{X}} \otimes_{\mathcal{O}_{\tilde{X}}} \bigwedge^\bullet \mathcal{T}_{\tilde{X}/X}$ is given as

$$\begin{aligned} & D \otimes \tilde{\xi}_1 \wedge \cdots \wedge \tilde{\xi}_d \\ & \mapsto \sum_{i=1}^d (-1)^{i+1} D \tilde{\xi}_i \otimes \tilde{\xi}_1 \wedge \cdots \wedge \widehat{\tilde{\xi}_i} \wedge \cdots \wedge \tilde{\xi}_d \\ & + \sum_{1 \leq i < j \leq d} (-1)^{i+j} D \otimes [\tilde{\xi}_i, \tilde{\xi}_j] \wedge \tilde{\xi}_1 \wedge \cdots \wedge \widehat{\tilde{\xi}_i} \wedge \cdots \wedge \widehat{\tilde{\xi}_j} \wedge \cdots \wedge \tilde{\xi}_d. \end{aligned}$$

The right $\pi^{-1} \mathcal{D}_{\tilde{X}}$ -module structure is not canonically defined on the complex, but the \mathfrak{g} -action can be described as

$$\xi(D \otimes \tilde{\xi}_1 \wedge \cdots \wedge \tilde{\xi}_d) = -D \xi_{\tilde{X}} \otimes \tilde{\xi}_1 \wedge \cdots \wedge \tilde{\xi}_d + D \otimes \xi(\tilde{\xi}_1 \wedge \cdots \wedge \tilde{\xi}_d)$$

for $\xi \in \mathfrak{g}$. Here we use the \mathfrak{g} -action on $\bigwedge \mathcal{T}_{\tilde{X}/X}$ induced from the G -equivariant structure.

By using the resolution (4.12), the direct image $\pi_+ \tilde{i}_+ \Omega_Y^\vee$ is given as the complex

$$\pi_* \left(\tilde{i}_+ \Omega_Y^\vee \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}} \otimes_{\mathcal{O}_{\tilde{X}}} \bigwedge^\bullet \mathcal{T}_{\tilde{X}/X} \right) \otimes_{\mathcal{O}_X} \Omega_X^\vee.$$

As a result, we have

$$i_+ \mathcal{L}[u] \simeq \pi_* \left(\tilde{i}_+ \Omega_Y^\vee \otimes_{\mathcal{O}_{\tilde{X}}} \bigwedge^\bullet \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right)$$

and hence

(4.13)

$$\begin{aligned} i^{-1} i_+ \mathcal{L}[u] &\simeq i^{-1} \pi_* \left(\tilde{i}_+ \Omega_Y^\vee \otimes_{\mathcal{O}_{\tilde{X}}} \bigwedge^\bullet \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \\ &\simeq i^{-1} \pi_* \tilde{i}_* \left(\tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \bigwedge^\bullet \mathcal{T}_{\tilde{X}/X} \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \Omega_{\tilde{X}/X} \right) \\ &\simeq (\pi_K)_* \left(\tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \left(\bigwedge^\bullet \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \right). \end{aligned}$$

There is a natural morphism of complexes of $i^{-1} \mathcal{O}_X$ -modules

(4.14)

$$\begin{aligned} \psi : (\pi_K)_* \left(\tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \left(\bigwedge^\bullet \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \right) \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V} \\ \rightarrow (\pi_K)_* \left(\tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \left(\bigwedge^\bullet \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \otimes_{\pi_K^{-1} i^{-1} \mathcal{O}_X} \pi_K^{-1} \mathcal{V} \right). \end{aligned}$$

We claim that ψ is an isomorphism. Indeed, if $F_p \tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee$ denotes the filtration of $\tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee$ defined in a way similar to $F_p i^{-1} i_+ \mathcal{L}$, then we get a map

$$\begin{aligned} \psi_p : (\pi_K)_* \left(F_p \tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \left(\bigwedge^\bullet \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \right) \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V} \\ \rightarrow (\pi_K)_* \left(F_p \tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \left(\bigwedge^\bullet \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \otimes_{\pi_K^{-1} i^{-1} \mathcal{O}_X} \pi_K^{-1} \mathcal{V} \right). \end{aligned}$$

It is enough to show that ψ_p is an isomorphism for all $p \geq 0$ because $\varinjlim_p F_p \tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee \simeq \tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee$. Since the ideal $\pi_K^{-1} (i^{-1} \mathcal{I}_Y)^{p+1}$ of $\pi_K^{-1} i^{-1} \mathcal{O}_X$ annihilates $F_p \tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee$, we have

$$\begin{aligned} (\pi_K)_* \left(F_p \tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \left(\bigwedge^\bullet \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \right) \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V} \\ \simeq (\pi_K)_* \left(F_p \tilde{i}^{-1} \tilde{i}_+ \Omega_Y^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \left(\bigwedge^\bullet \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \right) \otimes_{\mathcal{O}_{Y_{p+1}}} (\mathcal{V} / (i^{-1} \mathcal{I}_Y)^{p+1} \mathcal{V}). \end{aligned}$$

By Definition 4.5 (2), $\mathcal{V} / (i^{-1} \mathcal{I}_Y)^{p+1} \mathcal{V}$ is a flat $\mathcal{O}_{Y_{p+1}}$ -module. Hence the projection formula shows that ψ_p is an isomorphism and the claim is now verified.

The successive quotient of the filtration

$$F_p \mathcal{M} := F_p \tilde{i}^{-1} \tilde{i}_+ \Omega_{\tilde{Y}}^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \left(\bigwedge^d \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \otimes_{\pi_K^{-1} i^{-1} \mathcal{O}_X} \pi_K^{-1} \mathcal{V}$$

is

$$(F_p \tilde{i}^{-1} \tilde{i}_+ \Omega_{\tilde{Y}}^\vee / F_{p-1} \tilde{i}^{-1} \tilde{i}_+ \Omega_{\tilde{Y}}^\vee) \otimes_{\mathcal{O}_{\tilde{Y}}} \tilde{i}^* \left(\bigwedge^d \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \otimes_{\mathcal{O}_{\tilde{Y}}} \pi_K^* (\mathcal{V} / (i^{-1} \mathcal{I}_Y) \mathcal{V}),$$

which is a quasi-coherent $\mathcal{O}_{\tilde{Y}}$ -module. Since \tilde{Y} is affine, it follows that $H^s(\tilde{Y}, F_p \mathcal{M} / F_{p-1} \mathcal{M}) = 0$ for $s > 0$. Hence $H^s(\tilde{Y}, F_p \mathcal{M}) = 0$ and

$$H^s \left(\tilde{Y}, \tilde{i}^{-1} \tilde{i}_+ \Omega_{\tilde{Y}}^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \left(\bigwedge^d \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \otimes_{\pi_K^{-1} i^{-1} \mathcal{O}_X} \pi_K^{-1} \mathcal{V} \right) = 0$$

for $s > 0$. By (4.13) and (4.14), we conclude that

$$\begin{aligned} & H^s(Y, i^{-1} i_+ \mathcal{L} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}) \\ & \simeq H^{s-u} \Gamma \left(\tilde{Y}, \tilde{i}^{-1} \tilde{i}_+ \Omega_{\tilde{Y}}^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \left(\bigwedge^d \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \otimes_{\pi_K^{-1} i^{-1} \mathcal{O}_X} \pi_K^{-1} \mathcal{V} \right). \end{aligned}$$

Since $\tilde{i}^{-1} \tilde{i}_+ \Omega_{\tilde{Y}}^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \Omega_{\tilde{X}} \simeq \mathcal{O}_{\tilde{Y}} \otimes_{\mathcal{D}_{\tilde{Y}}} \tilde{i}^* \mathcal{D}_{\tilde{X}}$, we have

$$\begin{aligned} & \tilde{i}^{-1} \tilde{i}_+ \Omega_{\tilde{Y}}^\vee \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \left(\bigwedge^d \mathcal{T}_{\tilde{X}/X} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_{\tilde{X}/X} \right) \otimes_{\pi_K^{-1} i^{-1} \mathcal{O}_X} \pi_K^{-1} \mathcal{V} \\ & \simeq \mathcal{O}_{\tilde{Y}} \otimes_{\mathcal{D}_{\tilde{Y}}} \tilde{i}^* \mathcal{D}_{\tilde{X}} \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \tilde{i}^{-1} \bigwedge^d \mathcal{T}_{\tilde{X}/X} \otimes_{\pi_K^{-1} i^{-1} \mathcal{O}_X} \pi_K^{-1} (\mathcal{V} \otimes_{i^{-1} \mathcal{O}_X} i^{-1} \Omega_X^\vee). \end{aligned}$$

If we put

$$\mathcal{V}^{-d} := \tilde{i}^{-1} \bigwedge^d \mathcal{T}_{\tilde{X}/X} \otimes_{\pi_K^{-1} i^{-1} \mathcal{O}_X} \pi_K^{-1} (\mathcal{V} \otimes_{i^{-1} \mathcal{O}_X} i^{-1} \Omega_X^\vee),$$

then we obtain

$$(4.15) \quad H^s(Y, i^{-1} i_+ \mathcal{L} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}) \simeq H^{s-u} \Gamma(\tilde{Y}, \mathcal{O}_{\tilde{Y}} \otimes_{\mathcal{D}_{\tilde{Y}}} \tilde{i}^* \mathcal{D}_{\tilde{X}} \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \mathcal{V}^\bullet).$$

The boundary map

$$\partial : \mathcal{O}_{\tilde{Y}} \otimes_{\mathcal{D}_{\tilde{Y}}} \tilde{i}^* \mathcal{D}_{\tilde{X}} \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \mathcal{V}^{-d} \rightarrow \mathcal{O}_{\tilde{Y}} \otimes_{\mathcal{D}_{\tilde{Y}}} \tilde{i}^* \mathcal{D}_{\tilde{X}} \otimes_{\tilde{i}^{-1} \mathcal{O}_{\tilde{X}}} \mathcal{V}^{-d+1}$$

is given by

$$\begin{aligned} & f \otimes D \otimes \tilde{\xi}_1 \wedge \cdots \wedge \tilde{\xi}_d \otimes v \\ & \mapsto \sum_{i=1}^d (-1)^{i+1} f \otimes D \tilde{\xi}_i \otimes \tilde{\xi}_1 \wedge \cdots \wedge \widehat{\tilde{\xi}_i} \wedge \cdots \wedge \tilde{\xi}_d \otimes v \\ & + \sum_{1 \leq i < j \leq d} (-1)^{i+j} f \otimes D \otimes [\tilde{\xi}_i, \tilde{\xi}_j] \wedge \tilde{\xi}_1 \wedge \cdots \wedge \widehat{\tilde{\xi}_i} \wedge \cdots \wedge \widehat{\tilde{\xi}_j} \wedge \cdots \wedge \tilde{\xi}_d \otimes v, \end{aligned}$$

where $f \in \mathcal{O}_{\tilde{Y}}$, $D \in \tilde{i}^* \mathcal{D}_{\tilde{X}}$, $\tilde{\xi}_1, \dots, \tilde{\xi}_d \in \tilde{i}^{-1} \mathcal{T}_{\tilde{X}/X}$, and $v \in \pi_K^{-1} (\mathcal{V} \otimes_{i^{-1} \mathcal{O}_X} i^{-1} \Omega_X^\vee)$.

Let us compute the cohomological induction $(P_{\mathfrak{h}, L}^{\mathfrak{g}, K})_s(V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}))$ by using the standard resolution ([KV, §II.7]). The standard resolution is a

projective resolution of the (\mathfrak{h}, L) -module $V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h})$ given by the complex

$$U(\mathfrak{h}) \otimes_{U(\mathfrak{l})} \left(\bigwedge^{\bullet}(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}) \right),$$

where the boundary map

$$\partial' : U(\mathfrak{h}) \otimes_{U(\mathfrak{l})} \left(\bigwedge^d(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}) \right) \rightarrow U(\mathfrak{h}) \otimes_{U(\mathfrak{l})} \left(\bigwedge^{d-1}(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}) \right)$$

is

$$\begin{aligned} & D \otimes \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_d \otimes v \\ \mapsto & \sum_{i=1}^d (-1)^{i+1} (D \xi_i \otimes \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \bar{\xi}_d \otimes v - D \otimes \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \bar{\xi}_d \otimes \xi_i v) \\ & + \sum_{1 \leq i < j \leq d} (-1)^{i+j} D \otimes [\bar{\xi_i}, \bar{\xi_j}] \wedge \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \bar{\xi}_d \otimes v \end{aligned}$$

for $D \in U(\mathfrak{h})$, $\xi_1, \dots, \xi_d \in \mathfrak{h}$, and $v \in V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h})$. Therefore,

(4.16)

$$\begin{aligned} (P_{\mathfrak{h}, L}^{\mathfrak{g}, K})_{u-s} \left(V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}) \right) & \simeq H^{s-u} P_{\mathfrak{h}, L}^{\mathfrak{g}, K} \left(U(\mathfrak{h}) \otimes_{U(\mathfrak{l})} \left(\bigwedge^{\bullet}(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}) \right) \right) \\ & \simeq H^{s-u} R(\mathfrak{g}, K) \otimes_{R(L)} \left(\bigwedge^{\bullet}(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}) \right), \end{aligned}$$

where the boundary map

$$\partial' : R(\mathfrak{g}, K) \otimes_{R(L)} \left(\bigwedge^d(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}) \right) \rightarrow R(\mathfrak{g}, K) \otimes_{R(L)} \left(\bigwedge^{d-1}(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}) \right)$$

is given by

$$\begin{aligned} & D \otimes \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_d \otimes v \\ \mapsto & \sum_{i=1}^d (-1)^{i+1} (D \xi_i \otimes \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \bar{\xi}_d \otimes v - D \otimes \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \bar{\xi}_d \otimes \xi_i v) \\ & + \sum_{1 \leq i < j \leq d} (-1)^{i+j} D \otimes [\bar{\xi_i}, \bar{\xi_j}] \wedge \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \bar{\xi}_d \otimes v \end{aligned}$$

for $D \in R(\mathfrak{g}, K)$, $\xi_1, \dots, \xi_d \in \mathfrak{h}$, and $v \in V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h})$.

Put

$$V^{-d} := \bigwedge^d(\mathfrak{h}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h})$$

for simplicity. We identify the fiber of $\mathcal{T}_{\tilde{X}/X}$ with $\mathfrak{h}/\mathfrak{l}$ in the following way:

if a vector field $\tilde{\xi} \in \mathcal{T}_{\tilde{X}/X}$ equals $-\xi_{\tilde{X}}$ at the base point $eL \in \tilde{X}$ for $\xi \in \mathfrak{h}$, then ξ takes the value $\bar{\xi} \in \mathfrak{h}/\mathfrak{l}$ at $e \in G$. Similarly, the fiber of $\Omega_{\tilde{X}/X}^V$ is

identified with $\bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h})$. Then \mathcal{V}^{-d} is associated with V^{-d} by Example 4.7 and Example 4.8. By Lemma 4.10, we have

$$\Gamma(\tilde{Y}, \mathcal{O}_{\tilde{Y}} \otimes_{\mathcal{D}_{\tilde{Y}}} i^* \mathcal{D}_{\tilde{X}} \otimes_{i^{-1}\mathcal{O}_{\tilde{X}}} \mathcal{V}^{-d}) \simeq R(\mathfrak{g}, K) \otimes_{R(L)} V^{-d}.$$

From (4.15) and (4.16) it is enough to show that the isomorphisms φ given in the proof of Lemma 4.10 for $V = V^{-d}$, $0 \leq d \leq \dim(\mathfrak{h}/\mathfrak{l})$ commute with the boundary maps. This is reduced to the commutativity of the following diagram:

$$(4.17) \quad \begin{array}{ccc} U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^{-d} & \xrightarrow{\partial'} & U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^{-d+1} \\ \varphi^d \downarrow & & \downarrow \varphi^{d-1} \\ i_o^*(i^* \mathcal{D}_{\tilde{X}} \otimes_{i^{-1}\mathcal{O}_{\tilde{X}}} \mathcal{V}^{-d}) & \xrightarrow{\partial} & i_o^*(i^* \mathcal{D}_{\tilde{X}} \otimes_{i^{-1}\mathcal{O}_{\tilde{X}}} \mathcal{V}^{-d+1}) \end{array}$$

Here, φ^d is the map φ given in the proof of Lemma 4.10 for $V' = V^{-d}$.

Let us prove that the diagram (4.17) commutes. A section $f \in i^* \mathcal{D}_{\tilde{X}} \otimes_{i^{-1}\mathcal{O}_{\tilde{X}}} \mathcal{V}^{-d}$ defines a section of $i_o^*(i^* \mathcal{D}_{\tilde{X}} \otimes_{i^{-1}\mathcal{O}_{\tilde{X}}} \mathcal{V}^{-d})$ and hence defines an element of $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^{-d}$ via the isomorphism φ^d . We write $i_o^* f \in U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V^{-d}$ for this element. Put $Z := H/L$ and write $i_Z : Z \rightarrow \tilde{X}$ for the inclusion map. Then $i_Z(Z) = \pi^{-1}(\{o\})$ and there is a canonical isomorphism $i_Z^* \mathcal{T}_{\tilde{X}/X} \simeq \mathcal{T}_Z$. For $\xi_1, \dots, \xi_d \in \mathfrak{h}$ and $v \in V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h})$, put

$$m := \bar{\xi}_1 \wedge \dots \wedge \bar{\xi}_d \otimes v \in V^{-d}.$$

We will choose sections $\tilde{\xi}_i \in \tilde{i}^{-1} \mathcal{T}_{\tilde{X}/X}$ and $\tilde{v} \in \pi_K^{-1}(\mathcal{V} \otimes_{i^{-1}\mathcal{O}_X} i^{-1} \Omega_X^\vee)$ on a neighborhood of the base point $o \in \tilde{Y}$ in the following way. Take $\tilde{\xi}_i \in \mathcal{T}_{\tilde{X}/X}$ such that $\tilde{\xi}_i|_Z \in i_Z^* \mathcal{T}_{\tilde{X}/X}$ corresponds to $-(\xi_i)_Z$. Then it gives a section of $\tilde{i}^{-1} \mathcal{T}_{\tilde{X}/X}$, which we denote by the same letter $\tilde{\xi}_i$. We take a section $\tilde{v} \in \pi_K^{-1}(\mathcal{V} \otimes_{i^{-1}\mathcal{O}_X} i^{-1} \Omega_X^\vee)$ on a neighborhood of o such that $i_o^* \tilde{v}$ corresponds to v . Define a section $\tilde{m} \in \mathcal{V}^{-d}$ in a neighborhood of o as

$$\tilde{m} := \tilde{\xi}_1 \wedge \dots \wedge \tilde{\xi}_d \otimes \tilde{v} \in \mathcal{V}^{-d}.$$

Then the element $\varphi^d(1 \otimes m)$ is represented by the section

$$1 \otimes \tilde{m} \in i^* \mathcal{D}_{\tilde{X}} \otimes_{i^{-1}\mathcal{O}_{\tilde{X}}} \mathcal{V}^{-d},$$

in other words, $i_o^*(1 \otimes \tilde{m}) = 1 \otimes m$.

We have

$$\begin{aligned} & \partial(1 \otimes \tilde{m}) \\ &= \sum_{i=1}^d (-1)^{i+1} (\tilde{\xi}_i \otimes \tilde{\xi}_1 \wedge \dots \wedge \widehat{\tilde{\xi}_i} \wedge \dots \wedge \tilde{\xi}_d \otimes \tilde{v}) \\ &+ \sum_{1 \leq i < j \leq d} (-1)^{i+j} (1 \otimes [\tilde{\xi}_i, \tilde{\xi}_j] \wedge \tilde{\xi}_1 \wedge \dots \wedge \widehat{\tilde{\xi}_i} \wedge \dots \wedge \widehat{\tilde{\xi}_j} \wedge \dots \wedge \tilde{\xi}_d \otimes \tilde{v}) \end{aligned}$$

and

$$\begin{aligned} & \partial'(1 \otimes m) \\ &= \sum_{i=1}^d (-1)^{i+1} (\xi_i \otimes \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \bar{\xi}_d \otimes v - 1 \otimes \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \bar{\xi}_d \otimes \xi_i v) \\ &+ \sum_{1 \leq i < j \leq d} (-1)^{i+j} (1 \otimes [\bar{\xi}_i, \bar{\xi}_j] \wedge \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \bar{\xi}_d \otimes v). \end{aligned}$$

Since $\tilde{\xi}_i|_Z$ corresponds to $-(\xi_i)_Z$, the vector fields $\tilde{\xi}_i$ and $(\xi_i)_{\tilde{X}}$ have the relation $\tilde{\xi}_i = -(\xi_i)_{\tilde{X}}$ at o . Recall that the \mathfrak{g} -action on $\mathcal{T}_{\tilde{X}/X}$ is defined as the differential of the G -equivariant structure on it. Hence our choice implies that $\xi_i \cdot \tilde{\xi}_j|_Z = -([\xi_i, \xi_j])_Z$. As a result,

$$\begin{aligned} & i_o^*(\tilde{\xi}_i \otimes \tilde{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \tilde{\xi}_d \otimes \tilde{v}) \\ &= i_o^*(\rho(\xi_i)(1 \otimes \tilde{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \tilde{\xi}_d \otimes \tilde{v})) - i_o^*(1 \otimes \tilde{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \tilde{\xi}_d \otimes \xi_i \tilde{v}) \\ &\quad - \sum_{1 \leq i < j \leq d} i_o^*(1 \otimes \tilde{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \tilde{\xi}_{j-1} \wedge (\xi_i \cdot \tilde{\xi}_j) \wedge \tilde{\xi}_{j+1} \wedge \cdots \wedge \tilde{\xi}_d \otimes \tilde{v}) \\ &\quad - \sum_{1 \leq j < i \leq d} i_o^*(1 \otimes \tilde{\xi}_1 \wedge \cdots \wedge \tilde{\xi}_{j-1} \wedge (\xi_i \cdot \tilde{\xi}_j) \wedge \tilde{\xi}_{j+1} \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \tilde{\xi}_d \otimes \tilde{v}) \\ &= \xi_i \otimes \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \bar{\xi}_d \otimes v - 1 \otimes \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \bar{\xi}_d \otimes \xi_i v \\ &\quad + \sum_{1 \leq i < j \leq d} (-1)^{j+1} (1 \otimes [\bar{\xi}_i, \bar{\xi}_j] \wedge \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \bar{\xi}_d \otimes v) \\ &\quad + \sum_{1 \leq j < i \leq d} (-1)^j (1 \otimes [\bar{\xi}_i, \bar{\xi}_j] \wedge \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \bar{\xi}_d \otimes v). \end{aligned}$$

Moreover, $[\tilde{\xi}_i, \tilde{\xi}_j]|_Z$ corresponds to $[-(\xi_i)_Z, -(\xi_j)_Z] = ([\xi_i, \xi_j])_Z$. Hence

$$\begin{aligned} & i_o^*(1 \otimes [\tilde{\xi}_i, \tilde{\xi}_j] \wedge \tilde{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \tilde{\xi}_d \otimes \tilde{v}) \\ &= -1 \otimes [\bar{\xi}_i, \bar{\xi}_j] \wedge \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \bar{\xi}_d \otimes v. \end{aligned}$$

We thus conclude that

$$\begin{aligned} & (\varphi^{d-1})^{-1} \circ \partial \circ \varphi^d(1 \otimes m) \\ &= i_o^*(\partial(1 \otimes \tilde{m})) \\ &= i_o^* \left(\sum_{i=1}^d (-1)^{i+1} (\tilde{\xi}_i \otimes \tilde{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \tilde{\xi}_d \otimes \tilde{v}) \right. \\ &\quad \left. + \sum_{1 \leq i < j \leq d} (-1)^{i+j} (1 \otimes [\tilde{\xi}_i, \tilde{\xi}_j] \wedge \tilde{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \tilde{\xi}_d \otimes \tilde{v}) \right) \\ &= \sum_{i=1}^d (-1)^{i+1} \left(\xi_i \otimes \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \bar{\xi}_d \otimes v - 1 \otimes \bar{\xi}_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \bar{\xi}_d \otimes \xi_i v \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{1 \leq i < j \leq d} (-1)^{j+1} (1 \otimes \overline{[\xi_i, \xi_j]} \wedge \overline{\xi_1} \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \overline{\xi_d} \otimes v) \\
& + \sum_{1 \leq j < i \leq d} (-1)^j (1 \otimes \overline{[\xi_i, \xi_j]} \wedge \overline{\xi_1} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \overline{\xi_d} \otimes v) \Big) \\
& + \sum_{1 \leq i < j \leq d} (-1)^{i+j+1} (1 \otimes \overline{[\xi_i, \xi_j]} \wedge \overline{\xi_1} \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \overline{\xi_d} \otimes v) \\
& = \sum_{i=1}^d (-1)^{i+1} (\xi_i \otimes \overline{\xi_1} \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \overline{\xi_d} \otimes v - 1 \otimes \overline{\xi_1} \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \overline{\xi_d} \otimes \xi_i v) \\
& + \sum_{1 \leq i < j \leq d} (-1)^{i+j} (1 \otimes \overline{[\xi_i, \xi_j]} \wedge \overline{\xi_1} \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \overline{\xi_d} \otimes v) \\
& = \partial'(1 \otimes m).
\end{aligned}$$

Since ∂ , ∂' and φ^d commute with \mathfrak{g} -actions,

$$\partial(\varphi^d(D \otimes m)) = D\partial(\varphi^d(1 \otimes m)) = D\varphi^{d-1}(\partial'(1 \otimes m)) = \varphi^{d-1}(\partial'(D \otimes m))$$

for $D \in U(\mathfrak{g})$. Consequently, the diagram (4.17) commutes and the proof of the theorem is complete. \square

Corollary 4.13. *In Setting 4.3, suppose that $i : Y \rightarrow X$ is an open immersion, or equivalently, $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$. Then we have an isomorphism of K -modules*

$$\mathrm{For}_{\mathfrak{g}, K}^{\mathfrak{k}, K}(P_{\mathfrak{h}, L}^{\mathfrak{g}, K})_j(V) \simeq (P_{\mathfrak{m}, L}^{\mathfrak{k}, K})_j(V)$$

for any (\mathfrak{h}, M) -module V and $j \in \mathbb{N}$.

We now construct an $i^{-1}\widetilde{\mathfrak{g}}_X$ -module \mathcal{V} associated with a (\mathfrak{h}, M) -module V . Let \mathcal{V}_Y be the K -equivariant quasi-coherent \mathcal{O}_Y -module with typical fiber the M -module V . Let $p : \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g} \rightarrow \mathcal{T}_X$ be the map given by $f \otimes \xi \mapsto f\xi_X$ and put $\mathcal{H} := \ker p$. The \mathcal{O}_X -module \mathcal{H} is G -equivariant with typical fiber \mathfrak{h} . Hence a section $\xi \in \mathcal{H}$ is identified with a \mathfrak{h} -valued regular function on a subset of G satisfying $\xi(gh) = \mathrm{Ad}(h^{-1})(\xi(g))$ for $h \in H$. Let $\xi, \xi' \in \mathcal{H}$. By regarding $\widetilde{\mathfrak{g}}_X = \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$ as a submodule of $U(\widetilde{\mathfrak{g}}_X) = \mathcal{O}_X \otimes_{\mathbb{C}} U(\mathfrak{g})$, we have $[\xi, \xi'] := \xi\xi' - \xi'\xi \in \mathcal{H}$ and $[\xi, \xi'](g) = [\xi(g), \xi'(g)]$ with the identification above. If we write $\xi = \sum_i f_i \otimes \xi_i$ for $f_i \in \mathcal{O}_X$ and $\xi_i \in \mathfrak{g}$, then $\xi(g) = \sum_i f_i(g) \mathrm{Ad}(g^{-1})(\xi_i)$.

Let \mathcal{A} be the subalgebra of $i^{-1}U(\widetilde{\mathfrak{g}}_X) = i^{-1}\mathcal{O}_X \otimes U(\mathfrak{g})$ generated by $i^{-1}\mathcal{H}$, $1 \otimes \mathfrak{k}$, and $i^{-1}\mathcal{O}_X \otimes 1$. We view $i^{-1}U(\widetilde{\mathfrak{g}}_X)$ as an $i^{-1}\mathcal{O}_X$ -module and consider the inverse image $\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X) (\simeq \mathcal{O}_Y \otimes U(\mathfrak{g}))$ of $U(\widetilde{\mathfrak{g}}_X)$. Let $\overline{\mathcal{A}}$ be the image of the map $\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} \mathcal{A} \rightarrow \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)$ so that $\overline{\mathcal{A}} \simeq \mathcal{A}/(\mathcal{A} \cap (i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g})))$. Since $\mathcal{A} \cdot (i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g})) \subset i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g})$ in the algebra $i^{-1}U(\widetilde{\mathfrak{g}}_X)$, the algebra structure of \mathcal{A} induces that of $\overline{\mathcal{A}}$, and $\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)$ becomes a left $\overline{\mathcal{A}}$ -module.

We give a left $\overline{\mathcal{A}}$ -module structure on \mathcal{V}_Y in the following way. We view a local section of \mathcal{V}_Y as a V -valued regular function on a subset of K and

define a $(1 \otimes i^{-1}\mathcal{H})$ -action and an $(\mathcal{O}_Y \otimes 1)$ -action by

$$\begin{aligned} ((1 \otimes \xi)v)(k) &= \xi(i(k))v(k), \\ (f \otimes 1)v &= fv \end{aligned}$$

for $\xi \in i^{-1}\mathcal{H}$, $v \in \mathcal{V}_Y$, $f \in \mathcal{O}_Y$, and $k \in K$; define a $(1 \otimes \mathfrak{k})$ -action on \mathcal{V}_Y by differentiating the K -action on \mathcal{V}_Y . These actions are compatible in the following sense: if $f_i \in i^{-1}\mathcal{O}_X$, $\eta_i \in \mathfrak{k}$ and $\xi \in i^{-1}\mathcal{H}$ satisfy

$$\sum_i (f_i \otimes \eta_i) - \xi \in i^{-1}\mathcal{I}_Y \otimes \mathfrak{g},$$

then we have

$$(4.18) \quad \sum_i (f_i|_Y \otimes 1)((1 \otimes \eta_i)v) = (1 \otimes \xi)v$$

for $v \in \mathcal{V}_Y$. In the proposition below, we will see that these actions give a well-defined $\overline{\mathcal{A}}$ -module structure.

Let $\mathcal{V} := \text{Hom}_{\overline{\mathcal{A}}}(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X), \mathcal{V}_Y)$, namely, \mathcal{V} consists of the sections $v \in \text{Hom}_{\mathbb{C}}(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X), \mathcal{V}_Y)$ satisfying

$$\begin{aligned} v((1 \otimes \xi)(f \otimes D)) &= (1 \otimes \xi)(v(f \otimes D)), \\ v((1 \otimes \eta)(f \otimes D)) &= (1 \otimes \eta)(v(f \otimes D)), \text{ and} \\ v(f'f \otimes D) &= (f' \otimes 1)(v(f \otimes D)) \end{aligned}$$

for $f, f' \in \mathcal{O}_Y$, $D \in U(\mathfrak{g})$, $\eta \in \mathfrak{k}$, and $\xi \in i^{-1}\mathcal{H}$. We endow \mathcal{V} with an $i^{-1}\widetilde{\mathfrak{g}}_X$ -module structure by giving $(f \otimes D) \cdot v$ as

$$((f \otimes D) \cdot v)(f' \otimes D') = v(f' \otimes (1 \otimes D')(f \otimes D))$$

for $v \in \mathcal{V}$, $f \in i^{-1}\mathcal{O}_X$, $f' \in \mathcal{O}_Y$, and $D, D' \in U(\mathfrak{g})$.

Proposition 4.14. *Let V be a (\mathfrak{h}, M) -module. Then the left $\overline{\mathcal{A}}$ -action on \mathcal{V}_Y given above is well-defined, and the $i^{-1}\widetilde{\mathfrak{g}}_X$ -module*

$$\mathcal{V} := \text{Hom}_{\overline{\mathcal{A}}}(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} U(\widetilde{\mathfrak{g}}_X), \mathcal{V}_Y)$$

is associated with V in the sense of Definition 4.5.

Proof. Let $k_0 \in K$ and $y_0 := k_0M \in Y$. We fix a trivialization near y_0 in the following way. Take sections $\xi_1, \dots, \xi_n \in i^{-1}\mathcal{H}$ on a neighborhood U of y_0 in Y such that the map

$$(i^{-1}\mathcal{O}_X)^{\oplus n}|_U \rightarrow (i^{-1}\mathcal{H})|_U, \quad (f_1, \dots, f_n) \mapsto \sum_{i=1}^n f_i \xi_i$$

is an isomorphism. Take elements $\eta_1, \dots, \eta_m \in \mathfrak{k}$ such that they form a basis of the quotient space $\mathfrak{k}/\text{Ad}(k_0)(\mathfrak{m})$ and take $\zeta_1, \dots, \zeta_l \in \mathfrak{g}$ such that

$\eta_1, \dots, \eta_m, \zeta_1, \dots, \zeta_l$ form a basis of the quotient space $\mathfrak{g}/\text{Ad}(i(k_0))\mathfrak{h}$. Replacing U if necessary, we get an isomorphism

$$(4.19) \quad (i^{-1}\mathcal{O}_X)^{\oplus n+m+l}|_U \rightarrow (i^{-1}\mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g})|_U,$$

$$(f_1, \dots, f_n, g_1, \dots, g_m, h_1, \dots, h_l) \mapsto \sum_{i=1}^n f_i \xi_i + \sum_{i=1}^m (g_i \otimes \eta_i) + \sum_{i=1}^l (h_i \otimes \zeta_i).$$

For integers $s, t \geq 0$, let

$$I_{s,t} := \{i = (i(1), \dots, i(s)) : 1 \leq i(1) \leq \dots \leq i(s) \leq t\}, \quad I_t := \prod_{s=0}^{\infty} I_{s,t}.$$

If $s = 0$, the set $I_{0,t}$ consists of one element $()$. For $i = (i(1), \dots, i(s)) \in I_{s,l}$, we put $\zeta_i := 1 \otimes \zeta_{i(1)} \cdots \zeta_{i(s)} \in i^{-1}\mathcal{O}_X \otimes U(\mathfrak{g})$. If $s = 0$ and $i = ()$ then put $\zeta_i := 1 \otimes 1$. In the same way, for $i' = (i'(1), \dots, i'(s)) \in I_{s,n}$ and $i'' = (i''(1), \dots, i''(s)) \in I_{s,m}$, put $\xi_{i'} := \xi_{i'(1)} \cdots \xi_{i'(s)}$ and $\eta_{i''} := 1 \otimes \eta_{i''(1)} \cdots \eta_{i''(s)}$. From the isomorphism (4.19) and the Poincaré–Birkhoff–Witt theorem, we see that a section of $i^{-1}U(\tilde{\mathfrak{g}}_X)|_U$ is uniquely written as

$$\sum_{i \in I_l, i' \in I_n, i'' \in I_m} f_{i,i',i''} \xi_{i'} \eta_{i''} \zeta_i,$$

where $f_{i,i',i''} \in i^{-1}\mathcal{O}_X$, and $f_{i,i',i''} = 0$ except for finitely many (i, i', i'') . Hence a section of $(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\tilde{\mathfrak{g}}_X))|_U$ is uniquely written as a finite sum $\sum_{i,i',i''} f_{i,i',i''} \xi_{i'} \eta_{i''} \zeta_i$ for $f_{i,i',i''} \in \mathcal{O}_Y$.

Lemma 4.15. *The subsheaf $\bar{\mathcal{A}}|_U$ of $\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\tilde{\mathfrak{g}}_X)$ consists of the sections written as a finite sum*

$$\sum_{i' \in I_n, i'' \in I_m} f_{i',i''} \otimes \xi_{i'} \eta_{i''}$$

for $f_{i',i''} \in \mathcal{O}_Y$.

Proof. It is enough to prove that for any section $a \in \bar{\mathcal{A}}|_U$ there exist functions $f_{i',i''} \in i^{-1}\mathcal{O}_X$ such that

$$(4.20) \quad a - \sum_{i',i''} f_{i',i''} \xi_{i'} \eta_{i''} \in i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g}).$$

For this we observe relations in the algebra $i^{-1}U(\tilde{\mathfrak{g}}_X)$. By our choice of ξ_1, \dots, ξ_n and η_1, \dots, η_m , we can find $f_i, g_i \in i^{-1}\mathcal{O}_X$ for each $\eta \in \mathfrak{k}$ such that

$$(1 \otimes \eta) - \left(\sum_{i=1}^n f_i \xi_i + \sum_{i=1}^m g_i \otimes \eta_i \right) \in i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g}).$$

We also have

$$[\xi_i, f \otimes 1] = 0, \quad [1 \otimes \eta, 1 \otimes \eta'] = 1 \otimes [\eta, \eta'], \quad [1 \otimes \eta, f \otimes 1] = (\eta_X(f)) \otimes 1$$

for $f \in i^{-1}\mathcal{O}_X$, $\eta, \eta' \in \mathfrak{k}$. Further $[\xi_i, \xi_j], [1 \otimes \eta_i, \xi_j] \in i^{-1}\mathcal{H}$ and hence there exist $f_{i,j,k}, g_{i,j,k} \in i^{-1}\mathcal{O}_X$ such that

$$[\xi_i, \xi_j] = \sum_{k=1}^n f_{i,j,k} \xi_k, \quad [1 \otimes \eta_i, \xi_j] = \sum_{k=1}^n g_{i,j,k} \xi_k.$$

Since \mathcal{A} is generated by $i^{-1}\mathcal{H}$, $1 \otimes \mathfrak{k}$ and $i^{-1}\mathcal{O}_X \otimes 1$, we can prove (4.20) by using these relations iteratively and using $\mathcal{A}(i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g})) \subset i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g})$. \square

From the lemma above and its proof, we see that the algebra $\overline{\mathcal{A}}$ is generated by $\mathcal{O}_Y \otimes 1$, $1 \otimes \xi_1, \dots, 1 \otimes \xi_n$, and $1 \otimes \mathfrak{k}$ with the relations:

$$\begin{aligned} 1 \otimes \eta &= \sum_{i=1}^n f_i \otimes \xi_i + \sum_{i=1}^m g_i \otimes \eta_i, \\ [1 \otimes \xi_i, f \otimes 1] &= 0, \quad [1 \otimes \eta, 1 \otimes \eta'] = 1 \otimes [\eta, \eta'], \quad [1 \otimes \eta, f \otimes 1] = (\eta_Y(f)) \otimes 1, \\ [1 \otimes \xi_i, 1 \otimes \xi_j] &= \sum_{k=1}^n f_{i,j,k} \otimes \xi_k, \quad [1 \otimes \eta_i, 1 \otimes \xi_j] = \sum_{k=1}^n g_{i,j,k} \otimes \xi_k, \end{aligned}$$

where $f_i, g_i, f_{i,j,k}, g_{i,j,k}$ are the restrictions to Y of the corresponding functions in the proof of Lemma 4.15 and $f \in \mathcal{O}_Y$, $\eta, \eta' \in \mathfrak{k}$. We can check that these relations are compatible with the action on \mathcal{V}_Y (see (4.18)) and hence the $\overline{\mathcal{A}}$ -action on \mathcal{V}_Y is well-defined.

By Lemma 4.15, $(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\mathfrak{g}_X))|_U$ is a free $\overline{\mathcal{A}}|_U$ -algebra with basis $1 \otimes \zeta_i$. Therefore, the map

$$\phi : \mathcal{V}|_U \rightarrow \prod_{i \in I_l} \mathcal{V}_Y|_U$$

given by $\phi(v) = (v(1 \otimes \zeta_i))_i$ is bijective.

Our choice of ζ_1, \dots, ζ_l implies that they form a basis of the normal tangent space of U in X . Since ϕ is bijective, we see that

$$\phi((i^{-1}\mathcal{I}_Y)^p \mathcal{V}|_U) = \prod_{s=p}^{\infty} \prod_{i \in I_{s,l}} \mathcal{V}_Y|_U,$$

and hence

$$(\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p \mathcal{V})|_U \simeq \prod_{s=0}^{p-1} \prod_{i \in I_{s,l}} \mathcal{V}_Y|_U.$$

If we endow the right side of the last isomorphism with \mathcal{O}_{Y_p} -module structure via the isomorphism, it is written as follows. Let $f \in i^{-1}\mathcal{O}_X$ and $v = (v_i)_i$. For a subset $A \subset \{1, \dots, s\}$ with $A = \{a(1), \dots, a(t)\}$, $a(1) < \dots < a(t)$ and for $\mathbf{i} = (i(1), \dots, i(s)) \in I_{s,l}$, let $\{b(1), \dots, b(s-t)\} = \{1, \dots, s\} \setminus A$ with $b(1) < \dots < b(s-t)$ and put $\mathbf{i}' := (i(b(1)), \dots, i(b(s-t))) \in I_{s-t,l}$. Then the \mathbf{i} -term of $f \cdot v$ is given as

$$(4.21) \quad (f \cdot v)_{\mathbf{i}} = \sum_{A \subset \{1, \dots, s\}} ((\zeta_{i(a(1))})_X \cdots (\zeta_{i(a(t))})_X f)|_U \cdot v_{\mathbf{i}'}.$$

On the right side here, we use the \mathcal{O}_Y -action on \mathcal{V}_Y . This $i^{-1}\mathcal{O}_X$ -action on $\prod_{s=0}^{p-1} \prod_{i \in I_{s,l}} \mathcal{V}_Y|_U$ induces an \mathcal{O}_{Y_p} -action.

We now show that $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ is a quasi-coherent and flat \mathcal{O}_{Y_p} -module. Suppose first that $\mathcal{V}_Y|_U$ is a free \mathcal{O}_U -module on U so there exist sections $v_j \in \Gamma(U, \mathcal{V}_Y)$, $j \in J$ such that the map $\mathcal{O}_U^{\oplus J} \rightarrow \mathcal{V}_Y|_U$, $(f_j)_{j \in J} \mapsto \sum_{j \in J} f_j v_j$ is bijective. We define the map

$$\psi : (\mathcal{O}_{Y_p}|_U)^{\oplus J} \rightarrow \prod_{s=0}^{p-1} \prod_{i \in I_{s,l}} \mathcal{V}_Y|_U$$

by giving the i -term of $\psi(f)$ for $i = (i(1), \dots, i(s)) \in I_{s,l}$ and $f = (f_j)_{j \in J}$ as

$$\psi(f)_i = \sum_{j \in J} ((\zeta_{i(1)})_X \cdots (\zeta_{i(s)})_X f_j)|_U \cdot v_j.$$

Then ψ is an isomorphism of $\mathcal{O}_{Y_p}|_U$ -modules and hence $(\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V})|_U$ is a free $\mathcal{O}_{Y_p}|_U$ -module.

For general case, we write V as a union of finite-dimensional M -submodules: $V = \bigcup_{\alpha} V^{\alpha}$. Then the K -equivariant quasi-coherent \mathcal{O}_Y -module \mathcal{V}_Y^{α} with fiber V^{α} is locally free. If we define the \mathcal{O}_{Y_p} -module structure on $\prod_{s=0}^{p-1} \prod_{i \in I_{s,l}} \mathcal{V}_Y^{\alpha}|_U$ as in (4.21), then the preceding argument proves that it is a locally free $\mathcal{O}_{Y_p}|_U$ -module. Since \mathcal{V}_Y is the union of \mathcal{V}_Y^{α} , we see that $(\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V})|_U$ is isomorphic to the union of $\prod_{s=0}^{p-1} \prod_{i \in I_{s,l}} \mathcal{V}_Y^{\alpha}|_U$ as an $\mathcal{O}_{Y_p}|_U$ -module. Hence $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ is a quasi-coherent and flat \mathcal{O}_{Y_p} -module.

We define a K -action on \mathcal{V} by

$$(k \cdot v)(f \otimes D) = k \cdot (v((k^{-1} \cdot f) \otimes \text{Ad}(i(k)^{-1})D))$$

for $k \in K$, $v \in \mathcal{V}$, $f \in \mathcal{O}_Y$, and $D \in U(\mathfrak{g})$. This action descends to a K -action on $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$ and makes it a K -equivariant \mathcal{O}_{Y_p} -module. From this definition, it immediately follows that the maps $\mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V} \rightarrow \mathcal{V}/(i^{-1}\mathcal{I}_Y)^{p-1}\mathcal{V}$ and $i^{-1}\tilde{\mathfrak{g}}_X \otimes \mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V} \rightarrow \mathcal{V}/(i^{-1}\mathcal{I}_Y)^{p-1}\mathcal{V}$ commute with K -actions for all $p > 0$.

We have checked conditions (1), (2) and (3) of Definition 4.5. We can verify the condition (4) by computing the \mathfrak{k} -action as

$$\begin{aligned} (\eta \cdot v)(f \otimes D) &= v(f \otimes D\eta) \\ &= -v(f \otimes [\eta, D]) + v((1 \otimes \eta)(f \otimes D)) - v((\eta_Y(f)) \otimes D) \\ &= -v(f \otimes [\eta, D]) + (1 \otimes \eta)(v(f \otimes D)) - v((\eta_Y(f)) \otimes D) \end{aligned}$$

for $\eta \in \mathfrak{k}$, $v \in \mathcal{V}$, $f \in \mathcal{O}_Y$, and $D \in U(\mathfrak{g})$.

For the condition (5), we get an isomorphism of vector spaces $\iota : \mathcal{V}/(i^{-1}\mathcal{I}_o)\mathcal{V} \simeq V$ by taking fiber of the isomorphism $\phi : \mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V} \simeq \mathcal{V}_Y$ at o . The map ι is written as $\iota(v) = (v(1 \otimes 1))(e)$ for $v \in \mathcal{V}$. For $\xi \in \mathfrak{h}$, there exists a section $\xi' \in i^{-1}\mathcal{H}$ near the base point o such that $1 \otimes \xi - \xi' \in i^{-1}\mathcal{I}_o \otimes \mathfrak{g}$, or equivalently, $\xi'(e) = \xi$. Then

$$\iota(\xi v) = ((\xi v)(1 \otimes 1))(e) = (v(1 \otimes \xi))(e) = (v(\xi'))(e) = \xi(v(1 \otimes 1)(e)) = \xi \iota(v).$$

Moreover, we have

$$\iota(mv) = ((mv)(1 \otimes 1))(e) = (m(v(1 \otimes 1)))(e) = m(v(1 \otimes 1)(e)) = m\iota(v)$$

for $m \in M$ and hence ι commutes with (\mathfrak{h}, M) -actions. \square

Remark 4.16. The $i^{-1}\tilde{\mathfrak{g}}_X$ -module \mathcal{V} constructed above in this section has the following universal property. If \mathcal{V}' is another $i^{-1}\tilde{\mathfrak{g}}_X$ -module associated with V , then there exists a canonical map $\mathcal{V}' \rightarrow \mathcal{V}$ such that the induced map

$$V \simeq \mathcal{V}'/(i^{-1}\mathcal{I}_o)\mathcal{V}' \rightarrow \mathcal{V}/(i^{-1}\mathcal{I}_o)\mathcal{V} \simeq V$$

is the identity map. Moreover, it also induces an isomorphism

$$\mathcal{V}'/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}' \rightarrow \mathcal{V}/(i^{-1}\mathcal{I}_Y)^p\mathcal{V}$$

for any $p \in \mathbb{N}$. Therefore, the tensor product $i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}'$ is unique up to canonical isomorphism.

Let V be a (\mathfrak{h}, M) -module and \mathcal{V} an $i^{-1}\tilde{\mathfrak{g}}_X$ -module associated with V . Since $\mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V}$ is a K -equivariant quasi-coherent \mathcal{O}_Y -module with typical fiber V , there is a canonical isomorphism $\mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V} \simeq \mathcal{V}_Y$. We view $\mathcal{H} := \ker(p : \mathcal{O}_X \otimes \mathfrak{g} \rightarrow \mathcal{T}_X)$ as a subsheaf of $U(\tilde{\mathfrak{g}}_X)$. Since $\mathcal{H}(\mathcal{I}_Y \otimes U(\mathfrak{g})) \subset \mathcal{I}_Y \otimes U(\mathfrak{g})$, the $i^{-1}\mathcal{H}$ -action on \mathcal{V} induces one on $\mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V}$. By regarding local sections of these equivariant modules as vector-valued regular functions, this action is written as

$$(4.22) \quad (\xi v)(k) = \xi(i(k))v(k)$$

for $\xi \in i^{-1}\mathcal{H}$, $v \in \mathcal{V}$ and $k \in K$. Indeed, since the action map $i^{-1}\mathcal{H} \otimes \mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V} \rightarrow \mathcal{V}/(i^{-1}\mathcal{I}_Y)\mathcal{V}$ commutes with K -actions by Definition 4.5 (3), it is enough to prove (4.22) for $k = e$. This follows from $\mathcal{H}(\mathcal{I}_o \otimes U(\mathfrak{g})) \subset \mathcal{I}_o \otimes U(\mathfrak{g})$ and Definition 4.5 (5).

The \mathcal{O}_Y -modules \mathcal{L} , \mathcal{V}_Y , Ω_Y , and $i^*\Omega_X^\vee$ are K -equivariant with typical fiber $\bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{l})$, V , $\bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{m})^*$, and $\bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h})$, respectively. Hence the tensor product $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^*\Omega_X^\vee$ is also K -equivariant and has typical fiber $\bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{l}) \otimes V \otimes \bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{m})^* \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h})$. We give a right $i^{-1}\mathcal{H}$ -module structure, a right \mathfrak{k} -module structure, and a right \mathcal{O}_Y -module structure on the sheaf $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^*\Omega_X^\vee$ by

$$\begin{aligned} ((f \otimes v \otimes \omega \otimes \omega')\xi)(k) &= -f(k) \otimes (\xi(i(k))v(k)) \otimes \omega(k) \otimes \omega'(k) \\ &\quad - f(k) \otimes v(k) \otimes \omega(k) \otimes \text{ad}(\xi(i(k)))\omega'(k), \\ (f \otimes v \otimes \omega \otimes \omega')\eta &= -(\eta f) \otimes v \otimes \omega \otimes \omega' - f \otimes (\eta v) \otimes \omega \otimes \omega' \\ &\quad - f \otimes v \otimes (\eta\omega) \otimes \omega' - f \otimes v \otimes \omega \otimes (\eta\omega'), \\ (f \otimes v \otimes \omega \otimes \omega')f' &= f'f \otimes v \otimes \omega \otimes \omega' \end{aligned}$$

for $f \in \mathcal{L}$, $\xi \in i^{-1}\mathcal{H}$, $\eta \in \mathfrak{k}$, $v \in \mathcal{V}_Y$, $\omega \in \Omega_Y$, $\omega' \in i^*\Omega_X^\vee$, $f' \in \mathcal{O}_Y$, and $k \in K$. These actions are compatible: if $f_i \in i^{-1}\mathcal{O}_X$, $\eta_i \in \mathfrak{k}$ and $\xi \in i^{-1}\mathcal{H}$

satisfy

$$\sum_i (f_i \otimes \eta_i) - \xi \in i^{-1}\mathcal{I}_Y \otimes U(\mathfrak{g}),$$

then we have

$$\sum_i ((f \otimes v \otimes \omega \otimes \omega') f_i|_Y) \eta_i = (f \otimes v \otimes \omega \otimes \omega') \xi.$$

Therefore, we can prove by the same argument as above that these actions define a right $\overline{\mathcal{A}}$ -module structure on $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^* \Omega_X^\vee$.

By using this right $\overline{\mathcal{A}}$ -module structure, we consider the sheaf

$$(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^* \Omega_X^\vee) \otimes_{\overline{\mathcal{A}}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)),$$

which has a right $i^{-1}\widetilde{\mathfrak{g}}_X$ -module structure. We view it as a left $i^{-1}\widetilde{\mathfrak{g}}_X$ -module via the anti-isomorphism

$$S : U(\widetilde{\mathfrak{g}}_X) \rightarrow U(\widetilde{\mathfrak{g}}_X), \quad f \otimes 1 \mapsto f \otimes 1, \quad 1 \otimes \xi \mapsto -1 \otimes \xi$$

for $f \in \mathcal{O}_X, \xi \in \mathfrak{g}$.

Proposition 4.17. *Let \mathcal{L} be as in Section 4. Let \mathcal{V} be an $i^{-1}\widetilde{\mathfrak{g}}_X$ -module associated with a (\mathfrak{h}, M) -module V . Then there exists a K -equivariant isomorphism of $i^{-1}\widetilde{\mathfrak{g}}_X$ -modules*

$$i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \simeq (\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^* \Omega_X^\vee) \otimes_{\overline{\mathcal{A}}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)).$$

Proof. Let $F_p i^{-1}i_+\mathcal{L}$ be the filtration of $i^{-1}i_+\mathcal{L}$ as in Section 4. Then $F_0 i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$ is regarded as a subsheaf of $i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}$ (see Remark 4.11). We have

$$\begin{aligned} F_0 i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} &\simeq F_0 i^{-1}i_+\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V} / (i^{-1}\mathcal{I}_Y) \mathcal{V} \\ &\simeq \mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^* \Omega_X^\vee \otimes_{\mathcal{O}_Y} \mathcal{V} / (i^{-1}\mathcal{I}_Y) \mathcal{V}. \end{aligned}$$

Therefore, we get an isomorphism of K -equivariant \mathcal{O}_Y -modules

$$(4.23) \quad \begin{aligned} \psi_0 : \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^* \Omega_X^\vee &\xrightarrow{\sim} F_0 i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}, \\ f \otimes v \otimes \omega \otimes \omega' &\mapsto (f \otimes \omega \otimes \omega') \otimes v. \end{aligned}$$

Here $v \in \mathcal{V}_Y$ and we choose a section of \mathcal{V} that is sent to $v \in \mathcal{V}_Y \simeq \mathcal{V} / (i^{-1}\mathcal{I}_Y) \mathcal{V}$ by the quotient map, which we denote by the same letter $v \in \mathcal{V}$. Write $\mathcal{V}'_Y := \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^* \Omega_X^\vee$ for simplicity. The isomorphism (4.23) extends to the homomorphism of $i^{-1}\widetilde{\mathfrak{g}}_X$ -modules

$$\begin{aligned} \psi : \mathcal{V}'_Y \otimes_{\mathbb{C}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)) &\rightarrow i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}, \\ v \otimes (1 \otimes (f \otimes D)) &\mapsto S(f \otimes D) \cdot \psi_0(v). \end{aligned}$$

We can check that the map ψ descends to

$$\overline{\psi} : \mathcal{V}'_Y \otimes_{\overline{\mathcal{A}}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)) \rightarrow i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}.$$

Let

$$\pi : \mathcal{V}'_Y \otimes_{\mathbb{C}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X)) \rightarrow \mathcal{V}'_Y \otimes_{\overline{\mathcal{A}}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\widetilde{\mathfrak{g}}_X))$$

be the quotient map and put

$$\mathcal{V}_p := \pi \left(\mathcal{V}'_Y \otimes_{\mathbb{C}} (\mathcal{O}_Y \otimes_{\mathbb{C}} U_p(\mathfrak{g})) \right),$$

where $\{U_p(\mathfrak{g})\}_{p \in \mathbb{N}}$ is the standard filtration of $U(\mathfrak{g})$. We have

$$\bar{\psi}(\mathcal{V}_p) = \psi(\mathcal{V}'_Y \otimes_{\mathbb{C}} (\mathcal{O}_Y \otimes_{\mathbb{C}} U_p(\mathfrak{g}))) \subset F_p i^{-1} i_+ \mathcal{L} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}.$$

Let us take an open set $U \subset Y$ and elements $\zeta_1, \dots, \zeta_l \in \mathfrak{g}$ as in the proof of Proposition 4.14 and use the same notation. Then by an argument similar to the proof of Proposition 4.14, we obtain a bijective map of sheaves

$$\prod_{s=0}^p \prod_{i \in I_{s,l}} \mathcal{V}'_Y|_U \simeq \mathcal{V}_p|_U,$$

$$(v_i)_i \mapsto \sum_i \pi(v_i \otimes (1 \otimes \zeta_i))$$

and hence we have

$$\prod_{i \in I_{p,l}} \mathcal{V}'_Y|_U \simeq \mathcal{V}_p / \mathcal{V}_{p-1}|_U.$$

We also see that

$$(F_p i^{-1} i_+ \mathcal{L} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}) / (F_{p-1} i^{-1} i_+ \mathcal{L} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}) \simeq (F_p i^{-1} i_+ \mathcal{L} / F_{p-1} i^{-1} i_+ \mathcal{L}) \otimes_{\mathcal{O}_Y} \mathcal{V}_Y$$

and

$$F_p i^{-1} i_+ \mathcal{L} / F_{p-1} i^{-1} i_+ \mathcal{L} \simeq \mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^* \Omega_X^\vee \otimes_{\mathcal{O}_Y} i^{-1} ((\mathcal{I}_Y)^p / (\mathcal{I}_Y)^{p+1})$$

by Lemma 4.4. Since ζ_i for $i \in I_{p,l}$ give a trivialization of $i^{-1} ((\mathcal{I}_Y)^p / (\mathcal{I}_Y)^{p+1})$, we conclude that the map

$$\mathcal{V}_p / \mathcal{V}_{p-1} \rightarrow (F_p i^{-1} i_+ \mathcal{L} \otimes \mathcal{V}) / (F_{p-1} i^{-1} i_+ \mathcal{L} \otimes \mathcal{V})$$

induced by $\bar{\psi}$ on the successive quotient is an isomorphism. Therefore the map $\bar{\psi}$ is also an isomorphism. We can also see that $\bar{\psi}$ commutes with K -action. Hence the proposition follows. \square

Let $\lambda \in \mathfrak{h}^*$ such that $\text{Ad}^*(h)\lambda = \lambda$ for $h \in H$. For a section $\xi \in \mathcal{H}$, we define a function $f_{\xi,\lambda} \in \mathcal{O}_X$ as

$$f_{\xi,\lambda}(gH) = \lambda(\xi(g)).$$

Let \mathcal{I}_λ be the two-sided ideal of the sheaf $U(\tilde{\mathfrak{g}}_X) = \mathcal{O}_X \otimes U(\mathfrak{g})$ generated by $\xi - (f_{\xi,\lambda} \otimes 1)$ for all $\xi \in \mathcal{H}$. We define the ring of twisted differential operators as

$$\mathcal{D}_{X,\lambda} := U(\tilde{\mathfrak{g}}_X) / \mathcal{I}_\lambda.$$

Let $\mu := \lambda|_{\mathfrak{m}}$ and define $\mathcal{D}_{Y,\mu}$ similarly. Then we can define the direct image of a left $\mathcal{D}_{Y,\mu}$ -module \mathcal{M} by

$$i_+ \mathcal{M} := i_* ((\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y) \otimes_{\mathcal{D}_{Y,-\mu}} i^* \mathcal{D}_{X,-\lambda}) \otimes_{\mathcal{O}_X} \Omega_X^\vee.$$

Suppose that V is a (\mathfrak{h}, M) -module and \mathfrak{h} acts on V by $\lambda \in \mathfrak{h}^*$. The K -equivariant \mathcal{O}_Y -module $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y$ has a natural structure of left $\mathcal{D}_{Y, \mu}$ -module. Therefore, we can define the direct image $i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y)$ as a left $\mathcal{D}_{X, \lambda}$ -module.

Proposition 4.18. *Suppose that V is a (\mathfrak{h}, M) -module and \mathfrak{h} acts on V by $\lambda \in \mathfrak{h}^*$ such that $\text{Ad}^*(h)\lambda = \lambda$ for $h \in H$. Let \mathcal{V} be an $i^{-1}\tilde{\mathfrak{g}}_X$ -module associated with V . Then we have a K -equivariant isomorphism of $i^{-1}\tilde{\mathfrak{g}}_X$ -modules*

$$i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \simeq i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y).$$

Proof. We define a filtration $F_p i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y)$ of $i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y)$ in the same way as $F_p i^{-1}i_+\mathcal{L}$. Then

$$F_0 i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y) \simeq \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Y} i^* \Omega_X^\vee.$$

By using the same argument as in Proposition 4.17, we define a map of $i^{-1}\tilde{\mathfrak{g}}_X$ -modules

$$\mathcal{V}'_Y \otimes_{\mathbb{C}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\tilde{\mathfrak{g}}_X)) \rightarrow i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y)$$

and we see that it induces an isomorphism

$$\mathcal{V}'_Y \otimes_{\mathbb{A}} (\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}U(\tilde{\mathfrak{g}}_X)) \simeq i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y).$$

Hence

$$i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \simeq i^{-1}i_+(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{V}_Y)$$

by Proposition 4.17. \square

Recall that \mathcal{L} is the K -equivariant invertible sheaf on $Y = K/M$ with typical fiber $\bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{l})$. We view a one-dimensional vector space $\bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{l})^*$ as a (\mathfrak{h}, M) -module in the following way: \mathfrak{h} acts as zero; the Levi component L of M acts as the coadjoint action $\bigwedge \text{Ad}^*$; the unipotent radical U of M acts trivially. Let \mathcal{L}' be an $i^{-1}\tilde{\mathfrak{g}}_X$ -module associated with $\bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{l})^*$. Then $\mathcal{L}'/(i^{-1}\mathcal{I}_Y)\mathcal{L}'$ is isomorphic to the dual of \mathcal{L} . Therefore, by Proposition 4.18 we have

$$i^{-1}i_+\mathcal{L} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{L}' \simeq i^{-1}i_+\mathcal{V}_Y.$$

Example 4.8 shows that the $i^{-1}\tilde{\mathfrak{g}}_X$ -module $\mathcal{V} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{L}'$ is associated with $V \otimes \bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{l})^*$.

Theorem 4.19. *In Setting 4.3, we assume that K is reductive. Suppose that V is a (\mathfrak{h}, M) -module and \mathfrak{h} acts on V by $\lambda \in \mathfrak{h}^*$ such that $\text{Ad}^*(h)\lambda = \lambda$ for $h \in H$. Let $M = L \ltimes U$ be a Levi decomposition. Then*

$$\begin{aligned} H^s(Y, i^{-1}i_+\mathcal{V}_Y) &\simeq (P_{\mathfrak{h}, L}^{\mathfrak{g}, K})_{u-s} \left(V \otimes \bigwedge^{\text{top}}(\mathfrak{k}/\mathfrak{l})^* \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}) \right) \\ &\simeq (I_{\mathfrak{g}, L}^{\mathfrak{g}, K})^{y+s} P_{\mathfrak{h}, L}^{\mathfrak{g}, L} \left(V \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}) \right) \end{aligned}$$

for $s \in \mathbb{N}$, $u = \dim U$, and $y = \dim Y$.

Proof. The first isomorphism follows from Theorem 4.12 and the argument above. Since the functor $P_{\mathfrak{h},L}^{\mathfrak{g},L}$ is exact, $(P_{\mathfrak{h},L}^{\mathfrak{g},K})_{u-s} \simeq (P_{\mathfrak{g},L}^{\mathfrak{g},K})_{u-s} \circ P_{\mathfrak{h},L}^{\mathfrak{g},L}$. Hence the duality ([KV, Theorem 3.5])

$$(P_{\mathfrak{g},L}^{\mathfrak{g},K})_{\dim(K/L)-s} \left(\cdot \otimes \bigwedge^{\text{top}} (\mathfrak{k}/\mathfrak{l})^* \right) \simeq (I_{\mathfrak{g},L}^{\mathfrak{g},K})^s(\cdot)$$

and $\dim K/L = \dim U + \dim Y$ give the second isomorphism. \square

By Theorem 4.19 we obtain the convergence of spectral sequence

$$(4.24) \quad \begin{aligned} H^s(X, R^t i_+ \mathcal{V}_Y) &\Rightarrow (P_{\mathfrak{h},L}^{\mathfrak{g},K})_{u-s-t} \left(V \otimes \bigwedge^{\text{top}} (\mathfrak{k}/\mathfrak{l})^* \otimes \bigwedge^{\text{top}} (\mathfrak{g}/\mathfrak{h}) \right) \\ &\simeq (I_{\mathfrak{g},L}^{\mathfrak{g},K})^{y+s+t} P_{\mathfrak{h},L}^{\mathfrak{g},L} \left(V \otimes \bigwedge^{\text{top}} (\mathfrak{g}/\mathfrak{h}) \right). \end{aligned}$$

Here $R^t i_+$ is the higher direct image functor for a twisted left \mathcal{D} -module.

We now see that this spectral sequence implies results of [HMSW] and [Kit12].

Example 4.20. Let G_0 be a connected real semisimple Lie group with a maximal compact subgroup K_0 and the complexified Lie algebra \mathfrak{g} . Let K be the complexification of K_0 and G the inner automorphism group of \mathfrak{g} . There is a canonical homomorphism $i : K \rightarrow G$, which has finite kernel. Suppose that H is a Borel subgroup of G . Let us apply Setting 4.3. Then $X = G/H$ is the full flag variety of \mathfrak{g} . Since L is abelian and K is connected, L acts trivially on $\bigwedge^{\text{top}} (\mathfrak{k}/\mathfrak{l})^*$. Moreover in this case it is known that Y is affinely embedded in X . Therefore, $R^t i_+ \simeq 0$ for $t > 0$ and the spectral sequence (4.24) collapses. We thus get the duality theorem [HMSW].

Example 4.21. Let G_0 be a connected real semisimple Lie group with a maximal compact subgroup K_0 . We define K , G , and $i : K \rightarrow G$ as in the previous example. Suppose that H is a parabolic subgroup of G and apply Setting 4.3. Then $X = G/H$ is a partial flag variety of \mathfrak{g} . In this case Y is not necessarily affinely embedded in X . Let \tilde{X} be the full flag variety of \mathfrak{g} and let $p : \tilde{X} \rightarrow X$ be the natural surjective map. Then we have an isomorphism $H^s(\tilde{X}, p^* \mathcal{M}) \simeq H^s(X, \mathcal{M})$ for any \mathcal{O}_X -module \mathcal{M} . Hence (4.24) becomes

$$H^s(\tilde{X}, p^* R^t i_+ \mathcal{V}_Y) \Rightarrow (I_{\mathfrak{g},L}^{\mathfrak{g},K})^{y+s+t} P_{\mathfrak{h},L}^{\mathfrak{g},L} \left(V \otimes \bigwedge^{\text{top}} (\mathfrak{g}/\mathfrak{h}) \right),$$

which is [Kit12, Theorem 5.4 (12)].

Let V be any (\mathfrak{h}, M) -module and \mathcal{V} an $i^{-1} \tilde{\mathfrak{g}}_X$ -module associated with V . Since $i^{-1} i_+ \mathcal{L} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{L}' \simeq i^{-1} i_+ \mathcal{O}_Y$, we have

$$i^{-1} i_+ \mathcal{L} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{L}' \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V} \simeq i^{-1} i_+ \mathcal{O}_Y \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}.$$

We can thus rewrite Theorem 4.12 as

Theorem 4.22. *In Setting 4.3, we assume that K is reductive. Let $M = L \ltimes U$ be a Levi decomposition. Suppose that V is a (\mathfrak{h}, M) -module and that \mathcal{V} is an $i^{-1}\widetilde{\mathfrak{g}}_X$ -module associated with V (Definition 4.5). Then*

$$\begin{aligned} H^s(Y, i^{-1}i_+ \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}) &\simeq (P_{\mathfrak{h}, L}^{\mathfrak{g}, K})_{u-s} \left(V \otimes \bigwedge^{\text{top}} (\mathfrak{k}/\mathfrak{l})^* \otimes \bigwedge^{\text{top}} (\mathfrak{g}/\mathfrak{h}) \right) \\ &\simeq (I_{\mathfrak{g}, L}^{\mathfrak{g}, K})^{y+s} P_{\mathfrak{h}, L}^{\mathfrak{g}, L} \left(V \otimes \bigwedge^{\text{top}} (\mathfrak{g}/\mathfrak{h}) \right) \end{aligned}$$

for $s \in \mathbb{N}$, $u = \dim U$, $y = \dim Y$.

5. DECOMPOSITION OF $A_{\mathfrak{q}}(\lambda)$

In this section we decompose a restriction $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$ in terms of the orbit decomposition of a flag variety, which will be the starting point of the more detailed study of branching laws in the subsequent sections.

We retain the setting of Section 3. We assume that G_0 is linear for simplicity so we can take a connected complex reductive algebraic group G with Lie algebra \mathfrak{g} which contains G_0 as a subgroup. Let K be the connected subgroup of G with Lie algebra \mathfrak{k} . Let \mathfrak{q} be a θ -stable parabolic subalgebra of \mathfrak{g} . Write $\bar{\mathfrak{q}}$ for its complex conjugate and put $\mathfrak{l} := \mathfrak{q} \cap \bar{\mathfrak{q}}$. Then we have the Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ for the nilradical \mathfrak{u} of \mathfrak{q} . Write \bar{Q} and L for the connected subgroups of G with Lie algebras $\bar{\mathfrak{q}}$ and \mathfrak{l} , respectively. For a one-dimensional $(\bar{\mathfrak{q}}, L \cap K)$ -module \mathbb{C}_{λ} , Zuckerman's derived functor module is defined by

$$A_{\mathfrak{q}}(\lambda) := (P_{\bar{\mathfrak{q}}, L \cap K}^{\mathfrak{g}, K})_s(\mathbb{C}_{\lambda+2\rho(\mathfrak{u})}),$$

where $s = \dim(\mathfrak{u} \cap \mathfrak{k})$.

Let σ be an involution of G_0 and let G'_0 be the identity component of the fixed subgroup of σ . Write G' and K' for the connected subgroups of G with Lie algebras \mathfrak{g}' and \mathfrak{k}' . In the following theorem we give a decomposition of the restriction $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$ corresponding to the K' -orbit decomposition of a flag variety of K . The quotient varieties G/\bar{Q} and $K/(\bar{Q} \cap K)$ are partial flag varieties of G and K , respectively. Then $K/(\bar{Q} \cap K) \simeq K_0/(L_0 \cap K_0)$ has only finitely many K' -orbits. Let $K/(\bar{Q} \cap K) = \bigsqcup_{j=1}^n Y_j$ be the orbit decomposition and choose representatives $k_j \in K_0$ such that $Y_j = K'k_j(\bar{Q} \cap K)$. Put

$$\begin{aligned} \mathfrak{q}_j &:= \text{Ad}(k_j)\mathfrak{q}, \quad Q_j := k_j Q k_j^{-1}, \\ s_j &:= \dim K/(\bar{Q} \cap K) - \dim Y_j, \quad u_j := \dim(\bar{Q}_j \cap K') - \dim C'_j, \end{aligned}$$

where C'_j is a maximal reductive subgroup of $\bar{Q}_j \cap K'$. Taking conjugation by k_j , we regard the one-dimensional $(\bar{\mathfrak{q}}, L \cap K)$ -module $\mathbb{C}_{\lambda+2\rho(\mathfrak{u})}$ as a $(\bar{\mathfrak{q}}_j \cap \mathfrak{g}', C'_j)$ -module, which we denote also by $\mathbb{C}_{\lambda+2\rho(\mathfrak{u})}$.

Theorem 5.1. *Let (G_0, G'_0) be a symmetric pair of connected real reductive Lie groups and \mathfrak{q} a θ -stable parabolic subalgebra of \mathfrak{g} . Suppose that $A_{\mathfrak{q}}(\lambda)$ is non-zero and discretely decomposable as a (\mathfrak{g}', K') -module with λ in the*

weakly fair range. Let $K/(\overline{Q} \cap K) = \bigsqcup_{j=1}^n Y_j$ be the K' -orbit decomposition and define \bar{q}_j, C'_j, s_j as above. Then $(P_{\bar{q}_j \cap \mathfrak{g}', C'_j}^{\mathfrak{g}', K'})_d(\mathbb{C}_{\lambda+2\rho(u)} \otimes S(\mathfrak{g}/(\bar{q}_j + \mathfrak{g}')))$ is (\mathfrak{g}', K') -admissible for any j, d and we have

$$(5.1) \quad [A_q(\lambda)|_{(\mathfrak{g}', K')}] = \sum_{j=1}^n \sum_{d \in \mathbb{Z}_{\geq 0}} (-1)^{d+s_j+u_j} \left[(P_{\bar{q}_j \cap \mathfrak{g}', C'_j}^{\mathfrak{g}', K'})_d(\mathbb{C}_{\lambda+2\rho(u)} \otimes S(\mathfrak{g}/(\bar{q}_j + \mathfrak{g}')) \right].$$

Proof. We first prove that $(P_{\bar{q}_j \cap \mathfrak{g}', C'_j}^{\mathfrak{g}', K'})_d(\mathbb{C}_{\lambda+2\rho(u)} \otimes S(\mathfrak{g}/(\bar{q}_j + \mathfrak{g}')))$ is (\mathfrak{g}', K') -admissible. Put $W := \mathbb{C}_{\lambda+2\rho(u)} \otimes S(\mathfrak{g}/(\bar{q}_j + \mathfrak{g}'))$. It is known that there exists a σ -stable Cartan subalgebra \mathfrak{t} of \mathfrak{k} which is contained in \bar{q}_j . We can find $a \in \mathfrak{t}$ such that \bar{q}_j is given by $-a$. Then $a + \sigma(a) \in \bar{q}_j \cap \mathfrak{g}'$ and all the eigenvalues of $\text{ad}(a + \sigma(a))$ in $\mathfrak{g}/(\bar{q}_j + \mathfrak{g}')$ are positive. Indeed, $[a, \sigma(a)] = 0$ and the eigenvalues of $\text{ad}(a)$ (resp. $\text{ad}(\sigma(a))$) in \mathfrak{g}/\bar{q}_j (resp. $\mathfrak{g}/\sigma(\bar{q}_j)$) are positive. So our claim follows from the inclusion $\bar{q}_j + \sigma(\bar{q}_j) \subset \bar{q}_j + \mathfrak{g}'$. Define $\mathfrak{q}'_j := N_{\mathfrak{g}'}(\mathfrak{q}_j \cap \mathfrak{p}') + (\mathfrak{q}_j \cap \mathfrak{p}')$ as in Theorem 3.7. Put $\mathfrak{l}'_j := \mathfrak{q}'_j \cap \bar{\mathfrak{q}}'_j$ and write L'_j for the connected subgroup of G' with Lie algebra \mathfrak{l}'_j . By replacing C'_j and $a + \sigma(a)$ with their $(\overline{Q}_j \cap K')$ -conjugates, we may assume that $C'_j \subset L'_j$ and $a + \sigma(a) \in \mathfrak{c}'_j$. Let $s(W)$ be the semisimplification of W as in Definition 3.4. By Proposition 2.4, it suffices to prove that $(P_{\bar{q}'_j, L'_j \cap K'}^{\mathfrak{g}', K'})_d(P_{\bar{q}_j \cap \mathfrak{g}', C'_j}^{\bar{\mathfrak{q}}'_j, L'_j \cap K'})_{d'}(s(W))$ is (\mathfrak{g}', K') -admissible for all $d, d' \in \mathbb{N}$. Lemma 3.13 implies that $\bar{\mathfrak{q}}'_j = \bar{q}_j \cap \mathfrak{g}' + \mathfrak{l}'_j \cap \mathfrak{k}'$. Then by Corollary 4.13,

$$\text{For } \bar{\mathfrak{q}}'_j, L'_j \cap K' \quad (P_{\bar{q}_j \cap \mathfrak{g}', C'_j}^{\bar{\mathfrak{q}}'_j, L'_j \cap K'})_{d'}(s(W)) \simeq (P_{\bar{q}_j \cap \mathfrak{l}'_j \cap \mathfrak{k}', C'_j}^{\mathfrak{l}'_j \cap \mathfrak{k}', L'_j \cap K'})_{d'}(s(W)).$$

Since all the eigenspaces of $a + \sigma(a)$ in $s(W)$ are finite-dimensional, the assumption of Lemma 3.6 is satisfied. As a consequence, $(P_{\bar{q}_j \cap \mathfrak{l}'_j \cap \mathfrak{k}', C'_j}^{\mathfrak{l}'_j \cap \mathfrak{k}', L'_j \cap K'})_{d'}(s(W))$ is $(L'_j \cap K')$ -admissible. In particular, $\text{For } \bar{\mathfrak{q}}'_j, L'_j \cap K' \quad (P_{\bar{q}_j \cap \mathfrak{g}', C'_j}^{\bar{\mathfrak{q}}'_j, L'_j \cap K'})_{d'}(s(W))$ is $(\mathfrak{l}'_j, L'_j \cap K')$ -admissible. Since \bar{u}'_j annihilates $s(W)$, Proposition 2.5 implies that it also annihilates $(P_{\bar{q}_j \cap \mathfrak{g}', C'_j}^{\bar{\mathfrak{q}}'_j, L'_j \cap K'})_{d'}(s(W))$. Using Lemma 3.5, we conclude that $(P_{\bar{q}_j, L'_j \cap K'}^{\mathfrak{g}', K'})_d(P_{\bar{q}_j \cap \mathfrak{g}', C'_j}^{\bar{\mathfrak{q}}'_j, L'_j \cap K'})_{d'}(s(W))$ is (\mathfrak{g}', K') -admissible.

For a smooth algebraic variety X , let $\mathcal{M}_{\text{qc}}(\mathcal{D}_X)$ be the category of quasi-coherent left \mathcal{D}_X -modules. It has enough injectives and any injective object is a flabby sheaf (see [HTT] and [Har]). Hence cohomology groups of quasi-coherent \mathcal{D}_X -modules are calculated by injective resolutions in $\mathcal{M}_{\text{qc}}(\mathcal{D}_X)$. Let $\mathbf{D}_{\text{qc}}^b(\mathcal{D}_X)$ be the bounded derived category of $\mathcal{M}_{\text{qc}}(\mathcal{D}_X)$.

Write $X := G/\overline{Q}$ and $Y := K/(\overline{Q} \cap K)$ for the partial flag varieties of G and K , respectively. Let $i : Y \rightarrow X$ be the natural inclusion map. Then the situation here agrees with Setting 4.3 by putting $H = \overline{Q}$. Take an $i^{-1}\mathcal{O}_X$ -module \mathcal{V}_λ associated with the one-dimensional $(\bar{q}, L \cap K)$ -module \mathbb{C}_λ . For example we can take \mathcal{V}_λ as in Proposition 4.14. If we define \mathcal{L} as we did before Theorem 4.12, we have $\mathcal{L} \simeq \mathcal{O}_Y$ because $L \cap K$ is connected. We also

have $\bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{h}) \simeq \mathbb{C}_{2\rho(u)}$ as a $(\bar{q}, L \cap K)$ -module. Then Theorem 4.12 gives a realization of $A_q(\lambda)$ on Y :

$$(5.2) \quad \Gamma(Y, i^{-1}i_+ \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}_\lambda) \simeq A_q(\lambda).$$

In addition we have

$$H^d(Y, i^{-1}i_+ \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}_\lambda) = 0$$

for $d \neq 0$ because of Fact 2.11 (ii).

We now decompose the structure sheaf \mathcal{O}_Y using the K' -orbit decomposition $Y = \bigsqcup_{j=1}^n Y_j$. We may assume that the index j is chosen to satisfy: if the closure of Y_j contains $Y_{j'}$, then $j \leq j'$. In particular, Y_1 is the unique open dense orbit in Y and hence $s_1 = 0$. Put $Y_{\leq p} := \bigcup_{j=1}^p Y_j$. Then $Y_{\leq p}$ is an open K' -stable subvariety of Y . We write

$$i_p : Y_p \rightarrow Y, \quad i_{\leq p} : Y_{\leq p} \rightarrow Y, \quad i_{p,\leq p} : Y_p \rightarrow Y_{\leq p}, \quad j_p : Y_{\leq p-1} \rightarrow Y_{\leq p}$$

for the natural inclusion maps. For each $2 \leq p \leq n$, there are inclusion maps

$$Y_p \xrightarrow{i_{p,\leq p}} Y_{\leq p} \xleftarrow{j_p} Y_{\leq p-1}.$$

The map $i_{p,\leq p}$ is closed and j_p is open. Therefore, we have a distinguished triangle in $\mathbf{D}_{\text{qc}}^b(\mathcal{D}_{Y_{\leq p}})$:

$$(i_{p,\leq p})_+ (i_{p,\leq p})^\dagger \mathcal{O}_{Y_{\leq p}} \rightarrow \mathcal{O}_{Y_{\leq p}} \rightarrow (j_p)_+ (j_p)^{-1} \mathcal{O}_{Y_{\leq p}} \rightarrow (i_{p,\leq p})_+ (i_{p,\leq p})^\dagger \mathcal{O}_{Y_{\leq p}}[1],$$

where $(i_{p,\leq p})^\dagger := (i_{p,\leq p})^*[s_{p-1} - s_p]$ and $(i_{p,\leq p})_+$ is the direct image functor between the derived categories. Since $(i_{p,\leq p})^* \mathcal{O}_{Y_{\leq p}} \simeq \mathcal{O}_{Y_p}$ and $j_p^{-1} \mathcal{O}_{Y_{\leq p}} \simeq \mathcal{O}_{Y_{\leq p-1}}$, it becomes

$$(i_{p,\leq p})_+ \mathcal{O}_{Y_p}[s_{p-1} - s_p] \rightarrow \mathcal{O}_{Y_{\leq p}} \rightarrow (j_p)_+ \mathcal{O}_{Y_{\leq p-1}} \rightarrow (i_{p,\leq p})_+ \mathcal{O}_{Y_p}[s_{p-1} - s_p + 1].$$

By applying the functor $(i_p)_+ : \mathbf{D}_{\text{qc}}^b(\mathcal{D}_{Y_{\leq p}}) \rightarrow \mathbf{D}_{\text{qc}}^b(\mathcal{D}_Y)$, we get a distinguished triangle in $\mathbf{D}_{\text{qc}}^b(\mathcal{D}_Y)$:

$$(5.3) \quad (i_p)_+ \mathcal{O}_{Y_p}[s_{p-1} - s_p] \rightarrow (i_{\leq p})_+ \mathcal{O}_{Y_{\leq p}} \rightarrow (i_{\leq p-1})_+ \mathcal{O}_{Y_{\leq p-1}} \rightarrow (i_p)_+ \mathcal{O}_{Y_p}[s_{p-1} - s_p + 1].$$

We decompose $\Gamma(Y, i^{-1}i_+ \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}_\lambda)$ using (5.3). In order to do this we need the following lemma.

Lemma 5.2. *The functor from $\mathbf{D}_{\text{qc}}^b(\mathcal{D}_Y)$ to the category of left $U(\mathfrak{g})$ -modules defined by*

$$\mathcal{M} \mapsto H^d(Y, i^{-1}i_+ \mathcal{M} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}_\lambda), \quad d \in \mathbb{N}$$

is a cohomological functor.

Proof. Since the functor $\mathcal{M} \mapsto i^{-1}i_+ \mathcal{M} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}_\lambda$ is exact, it is enough to show that $H^d(Y, i^{-1}i_+ \mathcal{M} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{V}_\lambda) = 0$ if \mathcal{M} is injective and $d > 0$. Let $\mathcal{M} \in \mathcal{M}_{\text{qc}}(\mathcal{D}_Y)$ be an injective object. Then \mathcal{M} is a flabby sheaf. Let

$F_p i^{-1} i_+ \mathcal{M}$ be the filtration of \mathcal{M} defined in (4.6). By Lemma 4.4, it follows that

$$\begin{aligned} & (F_p i^{-1} i_+ \mathcal{M} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}_\lambda) / (F_{p-1} i^{-1} i_+ \mathcal{M} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}_\lambda) \\ & \simeq (F_p i^{-1} i_+ \mathcal{M} / F_{p-1} i^{-1} i_+ \mathcal{M}) \otimes_{\mathcal{O}_Y} (\mathcal{V}_\lambda / (i^{-1} \mathcal{I}_Y) \mathcal{V}_\lambda) \\ & \simeq \mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_{X/Y}^\vee \otimes_{\mathcal{O}_Y} i^{-1} (\mathcal{I}_Y^p / \mathcal{I}_Y^{p+1})^\vee \otimes_{\mathcal{O}_Y} (\mathcal{V}_\lambda / (i^{-1} \mathcal{I}_Y) \mathcal{V}_\lambda). \end{aligned}$$

Since $\Omega_{X/Y}^\vee \otimes_{\mathcal{O}_Y} i^{-1} (\mathcal{I}_Y^p / \mathcal{I}_Y^{p+1})^\vee \otimes_{\mathcal{O}_Y} (\mathcal{V}_\lambda / (i^{-1} \mathcal{I}_Y) \mathcal{V}_\lambda)$ is locally free and the flabbiness is a local property, we conclude that $(F_p i^{-1} i_+ \mathcal{M} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}_\lambda) / (F_{p-1} i^{-1} i_+ \mathcal{M} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}_\lambda)$ is also a flabby sheaf. Therefore,

$$H^d(Y, (F_p i^{-1} i_+ \mathcal{M} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}_\lambda) / (F_{p-1} i^{-1} i_+ \mathcal{M} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}_\lambda)) = 0$$

for all $p \in \mathbb{N}$ and $d > 0$, which proves $H^d(Y, i^{-1} i_+ \mathcal{M} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}_\lambda) = 0$ for $d > 0$. \square

We put

$$\begin{aligned} M_p^d &:= H^d(Y, i^{-1} i_+(i_p)_+ \mathcal{O}_{Y_p} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}_\lambda), \text{ and} \\ M_{\leq p}^d &:= H^d(Y, i^{-1} i_+(i_{\leq p})_+ \mathcal{O}_{Y_{\leq p}} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}_\lambda). \end{aligned}$$

Lemma 5.2 and (5.3) yield the following long exact sequence of (\mathfrak{g}', K') -modules.

$$\begin{aligned} (5.4) \quad & \rightarrow M_p^{d+s_{p-1}-s_p} \rightarrow M_{\leq p}^d \rightarrow M_{\leq p-1}^d \\ & \rightarrow M_p^{d+s_{p-1}-s_p+1} \rightarrow M_{\leq p}^{d+1} \rightarrow M_{\leq p-1}^{d+1} \\ & \rightarrow \dots \end{aligned}$$

We now describe the (\mathfrak{g}', K') -module M_p^d in terms of cohomological induction. Let $R(i_p)_*$ be the right derived functor of $(i_p)_*$. The isomorphisms

$$\begin{aligned} & i^{-1} i_+(i_p)_+ \mathcal{O}_{Y_p} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}_\lambda \\ & \simeq R(i_p)_* (i_p^{-1} i^{-1} i_+(i_p)_+ \mathcal{O}_{Y_p}) \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}_\lambda \\ & \simeq R(i_p)_* (i_p^{-1} i^{-1} i_+(i_p)_+ \mathcal{O}_{Y_p} \otimes_{i_p^{-1} i^{-1} \mathcal{O}_X} i_p^{-1} \mathcal{V}_\lambda) \end{aligned}$$

give

$$H^d(Y, i^{-1} i_+(i_p)_+ \mathcal{O}_{Y_p} \otimes_{i^{-1} \mathcal{O}_X} \mathcal{V}_\lambda) \simeq H^d(Y_p, i_p^{-1} i^{-1} i_+(i_p)_+ \mathcal{O}_{Y_p} \otimes_{i_p^{-1} i^{-1} \mathcal{O}_X} i_p^{-1} \mathcal{V}_\lambda).$$

Let X_p be the G' -orbit in X that contains Y_p . Write $i_{Y,X} : Y_p \rightarrow X_p$ and $j : X_p \rightarrow X$ for the natural inclusion maps. Taking $k_p \overline{Q} \in G/\overline{Q}$ as base point, we get isomorphisms:

$$\begin{array}{ccccc} Y_p & \xrightarrow{i_{Y,X}} & X_p & \xrightarrow{j} & X \\ \downarrow \mathbb{R} & & \downarrow \mathbb{R} & & \downarrow \mathbb{R} \\ K'/(k_p \overline{Q} k_p^{-1} \cap K') & \longrightarrow & G'/(k_p \overline{Q} k_p^{-1} \cap G') & \longrightarrow & G/(k_p \overline{Q} k_p^{-1}) \end{array}$$

The filtration $F_m j^{-1} \mathcal{D}_X$ by normal degree with respect to j induces the filtration $i_{Y,X}^{-1} F_m j^{-1} j_+ (i_{Y,X})_+ \mathcal{O}_{Y_p} \otimes_{i_p^{-1} i^{-1} \mathcal{O}_X} i_p^{-1} \mathcal{V}_\lambda$. By Lemma 4.4, there is an isomorphism

$$\begin{aligned} & F_m j^{-1} j_+ (i_{Y,X})_+ \mathcal{O}_{Y_p} / F_{m-1} j^{-1} j_+ (i_{Y,X})_+ \mathcal{O}_{Y_p} \\ & \simeq (i_{Y,X})_+ \mathcal{O}_{Y_p} \otimes_{\mathcal{O}_{X_p}} j^{-1} (\mathcal{I}_{X_p}^m / \mathcal{I}_{X_p}^{m+1})^\vee \otimes_{\mathcal{O}_{X_p}} \Omega_{X/X_p}^\vee, \end{aligned}$$

which respects \mathfrak{g}' -actions. Hence the successive quotient of the filtration $i_{Y,X}^{-1} F_m j^{-1} j_+ (i_{Y,X})_+ \mathcal{O}_{Y_p} \otimes_{i_p^{-1} i^{-1} \mathcal{O}_X} i_p^{-1} \mathcal{V}_\lambda$ is isomorphic to

$$i_{Y,X}^{-1} (i_{Y,X})_+ \mathcal{O}_{Y_p} \otimes_{i_{Y,X}^{-1} \mathcal{O}_{X_p}} i_{Y,X}^{-1} (j^{-1} (\mathcal{I}_{X_p}^m / \mathcal{I}_{X_p}^{m+1})^\vee \otimes_{\mathcal{O}_{X_p}} \Omega_{X/X_p}^\vee) \otimes_{i_{Y,X}^{-1} \mathcal{O}_{X_p}} i_p^{-1} \mathcal{V}_\lambda.$$

Since $i_{Y,X}^{-1} (j^{-1} (\mathcal{I}_{X_p}^m / \mathcal{I}_{X_p}^{m+1})^\vee \otimes_{\mathcal{O}_{X_p}} \Omega_{X/X_p}^\vee) \otimes_{i_{Y,X}^{-1} \mathcal{O}_{X_p}} i_p^{-1} \mathcal{V}_\lambda$ is an $(i \circ i_p)^{-1} \mathcal{O}_X$ -module associated with the $(\bar{\mathfrak{q}}_p, C'_p)$ -module $S^m(\mathfrak{g}/(\bar{\mathfrak{q}}_p + \mathfrak{g}')) \otimes \bigwedge^{\text{top}}(\mathfrak{g}/(\bar{\mathfrak{q}}_p + \mathfrak{g}')) \otimes \mathbb{C}_\lambda$, Theorem 4.12 implies

$$\begin{aligned} & H^d(Y_p, i_{Y,X}^{-1} (i_{Y,X})_+ \mathcal{O}_{Y_p} \otimes_{i_{Y,X}^{-1} \mathcal{O}_{X_p}} i_{Y,X}^{-1} (j^{-1} (\mathcal{I}_{X_p}^m / \mathcal{I}_{X_p}^{m+1})^\vee \otimes_{\mathcal{O}_{X_p}} \Omega_{X/X_p}^\vee) \otimes_{i_{Y,X}^{-1} \mathcal{O}_{X_p}} i_p^{-1} \mathcal{V}_\lambda) \\ & \simeq (P_{\bar{\mathfrak{q}}_p \cap \mathfrak{g}', C'_p}^{\mathfrak{g}', K'})_{u_p-d} (S^m(\mathfrak{g}/(\bar{\mathfrak{q}}_p + \mathfrak{g}')) \otimes \bigwedge^{\text{top}}(\mathfrak{g}/(\bar{\mathfrak{q}}_p + \mathfrak{g}')) \otimes \mathbb{C}_\lambda \otimes \bigwedge^{\text{top}}(\mathfrak{g}'/(\bar{\mathfrak{q}}_p \cap \mathfrak{g}')))) \\ & \simeq (P_{\bar{\mathfrak{q}}_p \cap \mathfrak{g}', C'_p}^{\mathfrak{g}', K'})_{u_p-d} (\mathbb{C}_\lambda \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\bar{\mathfrak{q}}_p) \otimes S^m(\mathfrak{g}/(\bar{\mathfrak{q}}_p + \mathfrak{g}')))) \\ & \simeq (P_{\bar{\mathfrak{q}}_p \cap \mathfrak{g}', C'_p}^{\mathfrak{g}', K'})_{u_p-d} (\mathbb{C}_{\lambda+2\rho(u)} \otimes S^m(\mathfrak{g}/(\bar{\mathfrak{q}}_p + \mathfrak{g}')))). \end{aligned}$$

By using long exact sequences associated with the filtration $i_{Y,X}^{-1} F_m j^{-1} j_+ (i_{Y,X})_+ \mathcal{O}_{Y_p}$, we get

$$[M_p^d] \leq [(P_{\bar{\mathfrak{q}}_p \cap \mathfrak{g}', C'_p}^{\mathfrak{g}', K'})_{u_p-d} (\mathbb{C}_{\lambda+2\rho(u)} \otimes S(\mathfrak{g}/(\bar{\mathfrak{q}}_p + \mathfrak{g}')))]$$

and hence M_p^d is (\mathfrak{g}', K') -admissible. We also conclude that

$$\sum_d (-1)^d [M_p^d] = \sum_d (-1)^{d+u_p} [(P_{\bar{\mathfrak{q}}_p \cap \mathfrak{g}', C'_p}^{\mathfrak{g}', K'})_d (\mathbb{C}_{\lambda+2\rho(u)} \otimes S(\mathfrak{g}/(\bar{\mathfrak{q}}_p + \mathfrak{g}')))].$$

Since $M_{\leq n}^s \simeq A_q(\lambda)|_{(\mathfrak{g}', K')}$ and $M_{\leq n}^d = 0$ for $d \neq s$, the long exact sequence (5.4) gives

$$\begin{aligned} & [A_q(\lambda)|_{(\mathfrak{g}', K')}] \\ & = \sum_{j=1}^n \sum_{d \in \mathbb{Z}_{\geq 0}} (-1)^{d+s_j+u_j} [(P_{\bar{\mathfrak{q}}_j \cap \mathfrak{g}', C'_j}^{\mathfrak{g}', K'})_d (\mathbb{C}_{\lambda+2\rho(u)} \otimes S(\mathfrak{g}/(\bar{\mathfrak{q}}_j + \mathfrak{g}')))], \end{aligned}$$

so the theorem is proved. \square

Remark 5.3. We can see from our proof that the theorem remains true if we replace $W := \mathbb{C}_{\lambda+2\rho(u)} \otimes S(\mathfrak{g}/(\bar{\mathfrak{q}}_j + \mathfrak{g}'))$ by $s(W)$ (see Definition 3.4).

Remark 5.4. Even if the group G_0 is not linear, we can find a reductive algebraic group G with Lie algebra \mathfrak{g} and the map $i : K \rightarrow G$ with finite kernel. Then we can realize $A_q(\lambda)$ on the flag variety as in the last section and prove Theorem 5.1 by the same argument.

From the proof of Theorem 5.1, we can prove the following inequality.

Theorem 5.5. *Under the notation and the assumptions in Theorem 5.1, suppose that Y_1 is the open K' -orbit in $K/(\overline{Q} \cap K)$. Then we have*

$$(5.5) \quad [A_q(\lambda)|_{(\mathfrak{g}', K')}] \leq \left[(P_{\overline{q}_1 \cap \mathfrak{g}', C'_1}^{\mathfrak{g}', K'})_{u_1} (\mathbb{C}_{\lambda+2\rho(u)} \otimes S(\mathfrak{g}/(\overline{q}_1 + \mathfrak{g}')) \right].$$

In [Osh11], we observed that the right side of (5.5) can be written as a direct sum of derived functor modules $A_{q''}(\lambda')$, where q'' is a minimal element of \mathcal{Q} defined in (3.2).

Our proof of Theorem 5.1 consists of the first part which we proved the admissibility of cohomologically induced modules and the second part which we proved the equation (5.1). Therefore, if we assume the admissibility, we get an equation like (5.1) in a more general setting.

Theorem 5.6. *Using the notation in Theorem 5.1, let V be an $(\mathfrak{l}, L \cap K)$ -module. We assume $(P_{\overline{q}_j \cap \mathfrak{g}', C'_j}^{\mathfrak{g}', K'})_d(V|_{(\overline{q}_j \cap \mathfrak{g}', C'_j)} \otimes S(\mathfrak{g}/(\overline{q}_j + \mathfrak{g}')))$ is (\mathfrak{g}', K') -admissible for any j and d . Then $(P_{\overline{q}, L \cap K}^{\mathfrak{g}, K})_d(V)|_{(\mathfrak{g}', K')}$ is also (\mathfrak{g}', K') -admissible and*

$$\begin{aligned} & \sum_{d \in \mathbb{Z}_{\geq 0}} (-1)^{d+s} [(P_{\overline{q}, L \cap K}^{\mathfrak{g}, K})_d(V)|_{(\mathfrak{g}', K')}] \\ &= \sum_{j=1}^n \sum_{d \in \mathbb{Z}_{\geq 0}} (-1)^{d+s_j+u_j} \left[(P_{\overline{q}_j \cap \mathfrak{g}', C'_j}^{\mathfrak{g}', K'})_d(V|_{(\overline{q}_j \cap \mathfrak{g}', C'_j)} \otimes S(\mathfrak{g}/(\overline{q}_j + \mathfrak{g}')) \right]. \end{aligned}$$

If $n = 1$, namely K' acts transitively on $K/(\overline{Q} \cap K)$, then $\overline{Q} \cap K'$ is a parabolic subgroup of K' . Then $s_1 = 0$, $u_1 = s$, and Lemma 3.14 shows that $\overline{Q} \cap G'$ is a parabolic subgroup of G' .

Corollary 5.7. *In the setting of Theorem 5.1, we assume that K' acts transitively on $K/(\overline{Q} \cap K)$. Let V be an $(\mathfrak{l}, L \cap K)$ -module such that $\mathcal{L}_{\overline{q} \cap \mathfrak{g}', d}^{\mathfrak{g}'}(V|_{\overline{q} \cap \mathfrak{g}'} \otimes \wedge^{\text{top}}(\mathfrak{g}/(\overline{q} + \mathfrak{g}')) \otimes S(\mathfrak{g}/(\overline{q} + \mathfrak{g}')))$ is (\mathfrak{g}', K') -admissible for any d . Then*

$$\begin{aligned} & \sum_{d \in \mathbb{Z}_{\geq 0}} (-1)^d [\mathcal{L}_{\overline{q}, d}^{\mathfrak{g}}(V)|_{(\mathfrak{g}', K')}] \\ &= \sum_{d \in \mathbb{Z}_{\geq 0}} (-1)^d \left[\mathcal{L}_{\overline{q} \cap \mathfrak{g}', d}^{\mathfrak{g}'} \left(V|_{\overline{q} \cap \mathfrak{g}'} \otimes \wedge^{\text{top}}(\mathfrak{g}/(\overline{q} + \mathfrak{g}')) \otimes S(\mathfrak{g}/(\overline{q} + \mathfrak{g}')) \right) \right]. \end{aligned}$$

Proof. We put $\mathfrak{q}' := \mathfrak{q} \cap \mathfrak{g}'$, which is a θ -stable parabolic subalgebra of \mathfrak{g}' . Write u' for the nilradical of \mathfrak{q}' . The corollary follows from $\mathbb{C}_{2\rho(u)-2\rho(u')} \simeq \wedge^{\text{top}}(\mathfrak{g}/(\overline{q} + \mathfrak{g}'))$. \square

Corollary 5.8. *In the setting of Theorem 5.1, we assume that K' acts transitively on $K/(\overline{Q} \cap K)$. Suppose that $A_q(\lambda)$ is non-zero and discretely decomposable as a (\mathfrak{g}', K') -module with λ in the weakly fair range. Then*

$$[A_q(\lambda)|_{(\mathfrak{g}', K')}] = \sum_{d \in \mathbb{Z}_{\geq 0}} (-1)^{d+s} \left[\mathcal{L}_{\overline{q} \cap \mathfrak{g}', d}^{\mathfrak{g}'} \left(\mathbb{C}_{\lambda} \otimes \wedge^{\text{top}}(\mathfrak{g}/(\overline{q} + \mathfrak{g}')) \otimes S(\mathfrak{g}/(\overline{q} + \mathfrak{g}')) \right) \right].$$

6. SMALL REPRESENTATIONS OF $U(m, n)$

In subsequent sections, we derive branching formulas of $A_q(\lambda)|_{(\mathfrak{g}', K')}$ based on Theorem 5.1. For that process, we need certain isomorphisms between cohomologically induced modules, which we show in this section.

Let m and n be positive integers and let

$$G_0 := U(m, n) = \{g \in GL(m+n, \mathbb{C}) : {}^t \bar{g} I_{m,n} g = I_{m,n}\}, \quad \text{and}$$

$$K_0 := U(m) \times U(n) = \left\{ \begin{pmatrix} A & O \\ O & B \end{pmatrix} : A \in U(m), B \in U(n) \right\},$$

where $I_{m,n}$ is the diagonal matrix $\text{diag}(\underbrace{1, \dots, 1}_m, \underbrace{-1, \dots, -1}_n)$. Their complexifications are $G = GL(m+n, \mathbb{C})$ and $K = GL(m, \mathbb{C}) \times GL(n, \mathbb{C})$ with Lie algebras $\mathfrak{g} = M(m+n, \mathbb{C})$ and $\mathfrak{k} = M(m, \mathbb{C}) \oplus M(n, \mathbb{C})$. We write $e_{i,j} (\in \mathfrak{g})$ for the $(m+n)$ by $(m+n)$ matrix with 1 in the (i, j) -entry and 0 elsewhere. Choose a Cartan subalgebra \mathfrak{t} of \mathfrak{k} (or of \mathfrak{g}) as diagonal matrices. Write $\epsilon_i (1 \leq i \leq m+n)$ for the element of \mathfrak{t}^* which sends $t = \text{diag}(t_1, \dots, t_{m+n}) \in \mathfrak{t}$ to $t_i \in \mathbb{C}$. The roots of \mathfrak{t} in \mathfrak{k} and \mathfrak{g} are given as

$$\Delta(\mathfrak{k}, \mathfrak{t}) = \{\pm(\epsilon_i - \epsilon_j)\}_{1 \leq i < j \leq m} \cup \{\pm(\epsilon_{m+i} - \epsilon_{m+j})\}_{1 \leq i < j \leq n} \text{ and}$$

$$\Delta(\mathfrak{g}, \mathfrak{t}) = \{\pm(\epsilon_i - \epsilon_j)\}_{1 \leq i < j \leq m+n}.$$

Choose their positive systems as

$$\Delta^+(\mathfrak{k}, \mathfrak{t}) = \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq m} \cup \{\epsilon_{m+i} - \epsilon_{m+j}\}_{1 \leq i < j \leq n} \text{ and}$$

$$\Delta^+(\mathfrak{g}, \mathfrak{t}) = \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq m+n}.$$

Let $V_1 = \mathbb{C}^{m+n}$ with standard coordinate z_1, \dots, z_{m+n} , on which G acts naturally. In other words, V_1 is a representation space of $F^G(\epsilon_1)$, the irreducible representation of G with highest weight ϵ_1 . The infinitesimal action of \mathfrak{g} on V_1 is given as

$$\mathfrak{g} \ni e_{i,j} \mapsto (e_{i,j})_{V_1} = -z_j \frac{\partial}{\partial z_i}.$$

Let $W_1 := \{v \in V_1 : z_i(v) = 0 \text{ for } 1 \leq i \leq m\} = \mathbb{C}^n$ and $i_1 : W_1 \rightarrow V_1$ the natural inclusion map. Consider $(i_1)_+ \mathcal{O}_{W_1}$, the direct image by i_1 of the structure sheaf as a \mathcal{D} -module. Since W_1 is K -stable, $\Gamma(V_1, (i_1)_+ \mathcal{O}_{W_1})$ is equipped with a (\mathfrak{g}, K) -module structure. We note that $H^d(V_1, (i_1)_+ \mathcal{O}_{W_1}) = 0$ for $d > 0$ because i is closed immersion and V_1 is affine. It is easy to see that V_1 decomposes into two G -orbits $V_1^o = \{0\}$ and $V_1' = V_1 \setminus \{0\}$. Similarly, W_1 decomposes into two K -orbits $W_1^o = \{0\}$ and $W_1' = W_1 \setminus \{0\}$. Write $i_1^o : W_1^o \rightarrow V_1$ and $i_1' : W_1' \rightarrow V_1$ for natural inclusions. As in the proof of Theorem 5.1, we get a long exact sequence

$$(6.1) \quad \begin{aligned} & \rightarrow H^{d-n}(V_1, (i_1^o)_+ \mathcal{O}_{W_1^o}) \rightarrow H^d(V_1, (i_1)_+ \mathcal{O}_{W_1}) \rightarrow H^d(V_1', (i_1')_+ \mathcal{O}_{W_1'}) \\ & \rightarrow H^{d-n+1}(V_1, (i_1^o)_+ \mathcal{O}_{W_1^o}) \rightarrow \dots \end{aligned}$$

Choose $v = {}^t(0, \dots, 0, 1) \in V_1$ as a base point. Then the stabilizer of v in G is

$$S = \left\{ \begin{pmatrix} A & 0 \\ b & 1 \end{pmatrix} : A \in GL(m+n-1, \mathbb{C}), b \in \mathbb{C}^{m+n-1} \right\}$$

Hence $V'_1 \simeq G/S$ and $W'_1 \simeq K/(S \cap K)$. The group

$$K_S = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix} : A \in GL(m, \mathbb{C}), B \in GL(n-1, \mathbb{C}) \right\}$$

is a maximal reductive subgroup of $S \cap K$. By Theorem 4.12,

$$H^d(V'_1, (i'_1)_+ \mathcal{O}_{W'_1}) \simeq (P_{\mathfrak{s}, K_S}^{\mathfrak{g}, K})_{n-d-1} \left(\bigwedge^{\text{top}} (\mathfrak{g}/\mathfrak{s}) \right).$$

We define the parabolic subgroup

$$Q_1 := \left\{ \begin{pmatrix} A & * \\ 0 & b \end{pmatrix} : A \in GL(m+n-1, \mathbb{C}), b \in \mathbb{C}^* \right\}$$

so that \mathfrak{q}_1 is given by $-e_{m+n, m+n}$ and then

$$\begin{aligned} \bar{\mathfrak{q}}_1 &= \left\{ \begin{pmatrix} A & 0 \\ * & b \end{pmatrix} : A \in M(m+n-1, \mathbb{C}), b \in \mathbb{C}^* \right\}, \\ L_1 &= \left\{ \begin{pmatrix} A & 0 \\ 0 & b \end{pmatrix} : A \in GL(m+n-1, \mathbb{C}), b \in \mathbb{C}^* \right\}, \\ (L_1)_0 &:= N_{U(m, n)}(\mathfrak{q}_1) \simeq U(m, n-1) \times U(1). \end{aligned}$$

Since $\bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{s}) \simeq \bigwedge^{\text{top}}(\mathfrak{g}/\bar{\mathfrak{q}}_1)$ as a $(\bar{\mathfrak{q}}_1, L_1 \cap K)$ -module, Mackey isomorphism gives

$$(P_{\mathfrak{s}, K_S}^{\bar{\mathfrak{q}}_1, L_1 \cap K})_d \left(\bigwedge^{\text{top}}(\mathfrak{g}/\mathfrak{s}) \right) \simeq (P_{\mathfrak{s}, K_S}^{\bar{\mathfrak{q}}_1, L_1 \cap K})_d(\mathbb{C}) \otimes \bigwedge^{\text{top}}(\mathfrak{g}/\bar{\mathfrak{q}}_1),$$

where \mathbb{C} is the trivial (\mathfrak{s}, K_S) -module. In light of Proposition 2.5 and $\bar{Q}_1/S \simeq (L_1 \cap K)/K_S \simeq \mathbb{C}^*$, we get

$$(P_{\mathfrak{s}, K_S}^{\bar{\mathfrak{q}}_1, L_1 \cap K})_d(\mathbb{C}) \simeq \begin{cases} \bigoplus_{\lambda \in \mathbb{Z}} \mathbb{C}_{\lambda \epsilon_{m+n}} & \text{if } d = 0, \\ 0 & \text{if } d > 0. \end{cases}$$

Consequently,

$$H^d(V'_1, (i'_1)_+ \mathcal{O}_{W'_1}) \simeq \bigoplus_{\lambda \in \mathbb{Z}} (P_{\bar{\mathfrak{q}}_1, L_1 \cap K}^{\mathfrak{g}, K})_{n-d-1}(\mathbb{C}_{\lambda \epsilon_{m+n} + 2\rho(u_1)}) \simeq \bigoplus_{\lambda \in \mathbb{Z}} \mathcal{L}_{\bar{\mathfrak{q}}_1, n-d-1}^{\mathfrak{g}}(\mathbb{C}_{\lambda \epsilon_{m+n}}).$$

We can also see that

$$H^d(V_1, (i_1^o)_+ \mathcal{O}_{W_1^o}) \simeq \begin{cases} \bigwedge^{\text{top}} V_1 \otimes \bigoplus_{\lambda \geq 0} S^\lambda(V_1) \simeq \bigoplus_{\lambda \geq 0} F^G(\lambda \epsilon_1 + \epsilon) & \text{if } d = 0, \\ 0 & \text{if } d > 0, \end{cases}$$

where

$$\epsilon = \epsilon_1 + \dots + \epsilon_{m+n}.$$

We decompose the long exact sequence (6.1) into eigenspaces of the center of \mathfrak{g} . Let $I_{m+n} = \text{diag}(\underbrace{1, \dots, 1}_{m+n}) \in \mathfrak{g}$. For $\lambda \in \mathbb{Z}$, write $\Gamma(V_1, (i_1)_+ \mathcal{O}_{W_1})_\lambda$ for the λ -eigenspace of I_{m+n} in $\Gamma(V_1, (i_1)_+ \mathcal{O}_{W_1})$. Then we obtain:

Lemma 6.1. *We have*

$$\mathcal{L}_{\bar{q}_1, d}^{\mathfrak{g}}(\mathbb{C}_{\lambda \epsilon_{m+n}}) = 0$$

if $d \neq 0, n-1$. For $n > 1$ we have isomorphisms

$$\begin{aligned} A_{q_1}(\lambda \epsilon_{m+n}) &= \mathcal{L}_{\bar{q}_1, n-1}^{\mathfrak{g}}(\mathbb{C}_{\lambda \epsilon_{m+n}}) \simeq \Gamma(V_1, (i_1)_+ \mathcal{O}_{W_1})_\lambda, \\ \mathcal{L}_{\bar{q}_1, 0}^{\mathfrak{g}}(\mathbb{C}_{\lambda \epsilon_{m+n}}) &\simeq \begin{cases} F^G((\lambda - m - n)\epsilon_1 + \epsilon) & \text{if } \lambda \geq m+n, \\ 0 & \text{if } \lambda < m+n. \end{cases} \end{aligned}$$

For $n = 1$ we have a short exact sequence

$$0 \rightarrow \Gamma(V_1, (i_1)_+ \mathcal{O}_{W_1})_\lambda \rightarrow A_{q_1}(\lambda \epsilon_{m+n}) \rightarrow F^G((\lambda - m - n)\epsilon_1 + \epsilon) \rightarrow 0$$

if $\lambda \geq m+n$ and an isomorphism

$$A_{q_1}(\lambda \epsilon_{m+n}) \simeq \Gamma(V_1, (i_1)_+ \mathcal{O}_{W_1})_\lambda$$

if $\lambda < m+n$.

Let us describe $\Gamma(V_1, (i_1)_+ \mathcal{O}_{W_1})$ using coordinates on V_1 . Put

$$x_i := z_i \quad (1 \leq i \leq m) \quad \text{and} \quad y_i := z_{m+i} \quad (1 \leq i \leq n).$$

Via identifications

$$\begin{aligned} \mathcal{O}_{W_1} &\simeq \Omega_{W_1}^\vee, \quad 1 \mapsto (dy_1 \wedge \dots \wedge dy_n)^{-1}, \\ \mathcal{O}_{V_1} &\simeq \Omega_{V_1}, \quad 1 \mapsto dz_1 \wedge \dots \wedge dz_{m+n}, \end{aligned}$$

we have

$$(i_1)_+ \mathcal{O}_{W_1} \simeq (i_1)_*(\mathcal{O}_{W_1} \otimes_{\mathcal{D}_{W_1}} i_1^* \mathcal{D}_{V_1}).$$

There is an element $1 = 1 \otimes 1 \in \Gamma(V_1, (i_1)_*(\mathcal{O}_{W_1} \otimes_{\mathcal{D}_{W_1}} i_1^* \mathcal{D}_{V_1}))$, which generates $\Gamma(V_1, (i_1)_*(\mathcal{O}_{W_1} \otimes_{\mathcal{D}_{W_1}} i_1^* \mathcal{D}_{V_1}))$ as a $\mathcal{D}(V_1)$ -module and its annihilator is the left ideal of $\mathcal{D}(V_1)$ generated by x_1, \dots, x_m and $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$. Hence if we write I for this ideal

$$\Gamma(V_1, (i_1)_+ \mathcal{O}_{W_1}) \simeq \mathcal{D}(V_1)/I \simeq \mathbb{C}\left[y_1, \dots, y_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right].$$

Define $V_2 := V_1^*$, namely, let V_2 be the representation space of $F^G(-\epsilon_{m+n})$. Let $z_1^*, \dots, z_{m+n}^* \in V_2$ be the dual basis of $z_1, \dots, z_{m+n} \in V_1$. The infinitesimal action of \mathfrak{g} on V_2 is given as

$$\mathfrak{g} \ni e_{i,j} \mapsto (e_{i,j})_{V_2} = z_i^* \frac{\partial}{\partial z_j^*} + \delta_{i,j} \left(= \frac{\partial}{\partial z_j^*} z_i^* \right).$$

Let $W_2 := \{v \in V_2 : z_{m+i}^*(v) = 0 \text{ for } 1 \leq i \leq n\} = \mathbb{C}^m$ and $i_2 : W_2 \rightarrow V_2$ the natural inclusion map. We define the parabolic subgroup

$$Q_2 := \left\{ \begin{pmatrix} a & * \\ 0 & B \end{pmatrix} : a \in \mathbb{C}^*, B \in GL(m+n-1, \mathbb{C}) \right\}$$

so that \mathfrak{q}_2 is given by $e_{1,1}$ and then

$$\begin{aligned}\bar{\mathfrak{q}}_2 &= \left\{ \begin{pmatrix} a & 0 \\ * & B \end{pmatrix} : a \in \mathbb{C}, B \in M(m+n-1, \mathbb{C}) \right\}, \\ L_2 &= \left\{ \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} : a \in \mathbb{C}^*, B \in GL(m+n-1, \mathbb{C}) \right\}, \\ (L_2)_0 &:= N_{U(m,n)}(\mathfrak{q}_2) \simeq U(1) \times U(m-1, n).\end{aligned}$$

Then in the same way as before, we can see the following lemma.

Lemma 6.2. *We have*

$$\mathcal{L}_{\bar{\mathfrak{q}}_2, d}^{\mathfrak{g}}(\mathbb{C}_{\lambda \epsilon_1}) = 0$$

if $d \neq 0, m-1$. For $n > 1$ we have isomorphisms

$$\begin{aligned}A_{\mathfrak{q}_2}(\lambda \epsilon_1) &= \mathcal{L}_{\bar{\mathfrak{q}}_2, m-1}^{\mathfrak{g}}(\mathbb{C}_{\lambda \epsilon_1}) \simeq \Gamma(V_2, (i_2)_+ \mathcal{O}_{W_2})_{\lambda}, \\ \mathcal{L}_{\bar{\mathfrak{q}}_2, 0}^{\mathfrak{g}}(\mathbb{C}_{\lambda \epsilon_1}) &\simeq \begin{cases} F^G((\lambda + m + n)\epsilon_{m+n} - \epsilon) & \text{if } \lambda \leq -m - n, \\ 0 & \text{if } \lambda > -m - n. \end{cases}\end{aligned}$$

For $n = 1$ we have a short exact sequence

$$0 \rightarrow \Gamma(V_2, (i_2)_+ \mathcal{O}_{W_2})_{\lambda} \rightarrow A_{\mathfrak{q}_2}(\lambda \epsilon_1) \rightarrow F^G((\lambda + m + n)\epsilon_{m+n} - \epsilon) \rightarrow 0$$

if $\lambda \leq -m - n$ and an isomorphism

$$A_{\mathfrak{q}_2}(\lambda \epsilon_1) \simeq \Gamma(V_2, (i_2)_+ \mathcal{O}_{W_2})_{\lambda}$$

if $\lambda > -m - n$.

Define the algebra isomorphism $\Phi : \mathcal{D}(V_1) \rightarrow \mathcal{D}(V_2)$ as

$$z_i \mapsto \frac{\partial}{\partial z_i^*}, \quad \frac{\partial}{\partial z_i} \mapsto -z_i^* \quad \text{for } 1 \leq i \leq m+n.$$

Putting

$$x_i^* := z_i^* \quad (1 \leq i \leq m) \quad \text{and} \quad y_i^* := z_{m+i}^* \quad (1 \leq i \leq n),$$

we see that $\Phi(I)$ is the left ideal generated by $\frac{\partial}{\partial x_1^*}, \dots, \frac{\partial}{\partial x_m^*}$ and y_1^*, \dots, y_n^* . Therefore, we get

$$\Gamma(V_2, (i_2)_+ \mathcal{O}_{W_2}) \simeq \mathcal{D}(V_2)/\Phi(I).$$

Let \det be the one-dimensional representation of G given by $G \ni g \mapsto \det(g) \in \mathbb{C}^*$. Its differential is given by $\mathfrak{g} \ni \xi \mapsto \text{Trace}(\xi) \in \mathbb{C}$. In the diagram

$$\begin{array}{ccccc} \mathfrak{g} & \longrightarrow & \mathcal{D}(V_1) & \longrightarrow & \mathcal{D}(V_1)/I \\ & \searrow & \downarrow \Phi & & \downarrow \Phi \\ & & \mathcal{D}(V_2) & \longrightarrow & \mathcal{D}(V_2)/\Phi(I) \end{array}$$

the right square is commutative and the left triangle is commutative up to Trace, namely, $\Phi(\xi_{V_1}) = \xi_{V_2} + \text{Trace}(\xi)$ for $\xi \in \mathfrak{g}$. As a result, Φ induces an isomorphism of (\mathfrak{g}, K) -modules

$$\Gamma(V_1, (i_1)_+ \mathcal{O}_{W_1}) \simeq \Gamma(V_2, (i_2)_+ \mathcal{O}_{W_2}) \otimes \det$$

and hence

$$\Gamma(V_1, (i_1)_+ \mathcal{O}_{W_1})_\lambda \simeq \Gamma(V_2, (i_2)_+ \mathcal{O}_{W_2})_{\lambda-m-n} \otimes \det$$

for $\lambda \in \mathbb{Z}$. We conclude from Lemmas 6.1 and 6.2 that:

Theorem 6.3. *In the setting above, we have*

$$\mathcal{L}_{\mathfrak{q}_1, d}^{\mathfrak{g}}(\mathbb{C}_{\lambda \epsilon_{m+n}}) \simeq \begin{cases} A_{\mathfrak{q}_1}(\lambda \epsilon_{m+n}) & \text{if } d = n - 1, \\ F^G((\lambda - m - n)\epsilon_1 + \epsilon) & \text{if } n > 1, d = 0, \text{ and } \lambda \geq m + n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathcal{L}_{\mathfrak{q}_2, d}^{\mathfrak{g}}(\mathbb{C}_{\lambda \epsilon_1}) \simeq \begin{cases} A_{\mathfrak{q}_2}(\lambda \epsilon_1) & \text{if } d = m - 1, \\ F^G((\lambda + m + n)\epsilon_{m+n} - \epsilon) & \text{if } m > 1, d = 0, \text{ and } \lambda \leq -m - n, \\ 0 & \text{otherwise.} \end{cases}$$

For $\lambda \geq \frac{m+n}{2}$, we have an injective map

$$\varphi : A_{\mathfrak{q}_2}((\lambda - m - n)\epsilon_1 + \epsilon) \rightarrow A_{\mathfrak{q}_1}(\lambda \epsilon_{m+n})$$

and

$$\text{Coker } \varphi \simeq \begin{cases} F^G((\lambda - m - n)\epsilon_1 + \epsilon) & \text{if } n = 1 \text{ and } \lambda \geq m + n, \\ 0 & \text{otherwise.} \end{cases}$$

For $\lambda \leq \frac{m+n}{2}$, we have an injective map

$$\varphi : A_{\mathfrak{q}_1}(\lambda \epsilon_{m+n}) \rightarrow A_{\mathfrak{q}_2}((\lambda - m - n)\epsilon_1 + \epsilon)$$

and

$$\text{Coker } \varphi \simeq \begin{cases} F^G(\lambda \epsilon_{m+n}) & \text{if } m = 1 \text{ and } \lambda \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 6.4. For the parabolic subalgebra \mathfrak{q}_1 , the parameter $\lambda \epsilon_{m+n}$ is in the weakly fair range if and only if $\lambda \leq \frac{m+n}{2}$. For \mathfrak{q}_2 , the parameter $\lambda \epsilon_1$ (or $\lambda \epsilon_1 + \epsilon$) is in the weakly fair range if and only if $\lambda \geq -\frac{m+n}{2}$.

Suppose that m and n are even and put $m' = \frac{m}{2}$ and $n' = \frac{n}{2}$. In what follows, we consider the restriction of the representations of $U(m, n)$ given above to the symmetric subgroup $Sp(m', n')$. We set

$$\begin{aligned} G'_0 &:= Sp(m', n') = \{g \in U(m, n) : {}^t g J_{m, n} g = J_{m, n}\}, \\ K'_0 &:= G'_0 \cap K_0 = Sp(m') \times Sp(n'), \end{aligned}$$

where $J_{m,n} = \begin{pmatrix} O & I_{m'} & & \\ -I_{m'} & O & & \\ & & O & I_{n'} \\ & & -I_{n'} & O \end{pmatrix}$. Their complexifications are $G' =$

$Sp(m' + n', \mathbb{C})$ and $K' = Sp(m', \mathbb{C}) \times Sp(n', \mathbb{C})$. Choose a Cartan subalgebra \mathfrak{t}' of \mathfrak{k}' (or of \mathfrak{g}') as diagonal matrices in \mathfrak{k}' , namely,

$$\mathfrak{t}' := \{t = \text{diag}(t_1, \dots, t_{m'}, -t_1, \dots, -t_{m'}, t_{m'+1}, \dots, t_{m'+n'}, -t_{m'+1}, \dots, -t_{m'+n'})\}$$

Write ϵ'_i ($1 \leq i \leq m' + n'$) for the element of $(\mathfrak{t}')^*$ which sends above t to $t'_i \in \mathbb{C}$. The roots of \mathfrak{t}' in \mathfrak{k}' and \mathfrak{g}' are given as

$$\Delta(\mathfrak{k}', \mathfrak{t}') = \{\pm \epsilon'_i \pm \epsilon'_j\}_{1 \leq i < j \leq m'} \cup \{\pm \epsilon'_{m'+i} \pm \epsilon'_{m'+j}\}_{1 \leq i < j \leq n'} \cup \{\pm 2\epsilon'_i\}_{1 \leq i \leq m'+n'}$$

and

$$\Delta(\mathfrak{g}', \mathfrak{t}') = \{\pm \epsilon'_i \pm \epsilon'_j\}_{1 \leq i < j \leq m'+n'} \cup \{\pm 2\epsilon'_i\}_{1 \leq i \leq m'+n'}.$$

We see that $G/\overline{Q}_1 \simeq G'/(\overline{Q}_1 \cap G')$ and $K/(\overline{Q}_1 \cap K) \simeq K'/(\overline{Q}_1 \cap K')$. The intersection $\mathfrak{q}'_1 := \mathfrak{q}_1 \cap \mathfrak{g}'$ is the parabolic subalgebra of \mathfrak{g}' given by $e_{m+n', m+n'} - e_{m+n, m+n}$. Therefore, Corollary 5.8 gives (we also see this in Section 8)

$$A_{\mathfrak{q}_1}(\lambda \epsilon_{m+n})|_{(\mathfrak{g}', K')} \simeq A_{\mathfrak{q}'_1}(-\lambda \epsilon'_{m'+n'}).$$

Similarly, $\mathfrak{q}'_2 := \mathfrak{q}_1 \cap \mathfrak{g}'$ is the parabolic subalgebra of \mathfrak{g}' given by $e_{1,1} - e_{m'+1, m'+1}$. Then it follows that

$$A_{\mathfrak{q}_2}(\lambda \epsilon_1 + \epsilon)|_{(\mathfrak{g}', K')} \simeq A_{\mathfrak{q}'_2}(\lambda \epsilon'_1).$$

We can see that

$$(L'_1)_0 := N_{Sp(m', n')}(\mathfrak{q}'_1) \simeq Sp(m', n' - 1) \times U(1),$$

$$(L'_2)_0 := N_{Sp(m', n')}(\mathfrak{q}'_2) \simeq U(1) \times Sp(m' - 1, n').$$

We now conclude from Theorem 6.3 that

Theorem 6.5. *In the setting above, we have*

$$\mathcal{L}_{\mathfrak{q}'_1, d}^{\mathfrak{g}'}(\mathbb{C}_{\lambda \epsilon'_{m'+n'}}) \simeq \begin{cases} A_{\mathfrak{q}'_1}(\lambda \epsilon'_{m'+n'}) & \text{if } d = n - 1, \\ F^{G'}(-(\lambda + m + n)\epsilon'_1) & \text{if } d = 0, \text{ and } \lambda \leq -m - n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathcal{L}_{\mathfrak{q}'_2, d}^{\mathfrak{g}'}(\mathbb{C}_{\lambda \epsilon'_1}) \simeq \begin{cases} A_{\mathfrak{q}'_2}(\lambda \epsilon'_1) & \text{if } d = m - 1, \\ F^{G'}(-(\lambda + m + n)\epsilon'_1) & \text{if } d = 0, \text{ and } \lambda \leq -m - n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, there is an isomorphism of (\mathfrak{g}', K') -modules.

$$A_{\mathfrak{q}'_1}(-\lambda \epsilon'_{m'+n'}) \simeq A_{\mathfrak{q}'_2}((\lambda - m - n)\epsilon'_1).$$

Remark 6.6. For the parabolic subalgebra \mathfrak{q}'_1 , the parameter $\lambda\epsilon'_{m'+n'}$ is in the weakly fair range if and only if $\lambda \geq -m' - n'$. For \mathfrak{q}'_2 , the parameter $\lambda\epsilon'_1$ is in the weakly fair range if and only if $\lambda \geq -m' - n'$.

7. CLASSIFICATION

We retain the setting of Section 3. In particular, $(\mathfrak{g}_0, \mathfrak{g}'_0)$ is a symmetric pair. We say that a triple $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q})$ satisfies the discrete decomposability condition if one of (hence any of) the conditions in Fact 3.8 holds. In [KO12], we got a classification of all the triples satisfying discrete decomposability condition by checking Fact 3.8 (v). We recall the classification and present a list of such triples in this section.

In order to do this, we prepare some terminology and setups.

Definition 7.1. We say the pair $(\mathfrak{g}_0, \mathfrak{g}'_0)$ is an *irreducible symmetric pair* if one of the following holds.

- (1) \mathfrak{g}_0 is simple.
- (2) \mathfrak{g}_0 is simple and $\mathfrak{g}_0 \simeq \mathfrak{g}_0 \oplus \mathfrak{g}_0$; σ acts by switching the factors.

Proposition 7.2. Let \mathfrak{q}_1 and \mathfrak{q}_2 be θ -stable parabolic subalgebras of \mathfrak{g} such that $\mathfrak{q}_1 \subset \mathfrak{q}_2$. If $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q}_1)$ satisfies the discrete decomposability condition, then so does $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q}_2)$.

Definition 7.3. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be a real non-compact simple Lie algebra. We say \mathfrak{g}_0 is of *Hermitian type* and the symmetric pair $(\mathfrak{g}_0, \mathfrak{k}_0)$ is a *Hermitian symmetric pair* if the center \mathfrak{z}_K of \mathfrak{k}_0 is one-dimensional.

If \mathfrak{g}_0 is of Hermitian type, then \mathfrak{p} decomposes into the direct sum of two irreducible submodules $\mathfrak{p} = \mathfrak{p}_+ + \mathfrak{p}_-$ as a K -module. The Riemannian symmetric space G_0/K_0 becomes a Hermitian symmetric space by choosing \mathfrak{p}_- as a holomorphic tangent space at the base point.

Definition 7.4. Suppose that \mathfrak{g}_0 is a simple Lie algebra of Hermitian type. A θ -stable parabolic subalgebra \mathfrak{q} of \mathfrak{g} is said to be *holomorphic* (resp. *anti-holomorphic*) if $\mathfrak{q} \supset \mathfrak{p}_+$ (resp. $\mathfrak{q} \supset \mathfrak{p}_-$).

We fix a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ with respect to a Cartan subalgebra \mathfrak{t} of \mathfrak{k} and present the set of weights $\Delta(\mathfrak{p}, \mathfrak{t})$ for each simple Lie algebra \mathfrak{g} . We will set vectors $\epsilon_i \in \mathfrak{t}^*$ and $e_i \in \mathfrak{t}$. If \mathfrak{g} is not equal to $\mathfrak{su}(m, n)$, $\mathfrak{sl}(2n, \mathbb{C})$, $\mathfrak{e}_{6(2)}$, or $\mathfrak{e}_{7(-25)}$, then $\{\epsilon_i\}$ is a basis of \mathfrak{t}^* and $\{e_i\}$ is a dual basis of $\{\epsilon_i\}$. We also write down the conditions for $a \in \sqrt{-1}\mathfrak{t}_0$ to be $\Delta^+(\mathfrak{k}, \mathfrak{t})$ -dominant in terms of the coordinates a_i , which are used in Tables 1, 3, and 4.

Setting 7.5. Let $\mathfrak{g}_0 = \mathfrak{su}(m, n)$. Choose $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ such that

$$\begin{aligned} \Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq m} \cup \{\epsilon_{m+i} - \epsilon_{m+j}\}_{1 \leq i < j \leq n}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm(\epsilon_i - \epsilon_{m+j})\}_{1 \leq i \leq m, 1 \leq j \leq n}. \end{aligned}$$

Define $e_1, \dots, e_{m+n} \in \mathfrak{t}$ such that $(\epsilon_i - \epsilon_j)(e_k) = \delta_{ik} - \delta_{jk}$ and then $e_1 + \dots + e_{m+n} = 0$. The dominant condition on $a = a_1 e_1 + \dots + a_{m+n} e_{m+n} \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq a_2 \geq \dots \geq a_m$ and $a_{m+1} \geq a_{m+2} \geq \dots \geq a_{m+n}$.

Setting 7.6. Let $\mathfrak{g}_0 = \mathfrak{so}(2m, 2n)$. Choose $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ such that

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq m} \cup \{\epsilon_{m+i} \pm \epsilon_{m+j}\}_{1 \leq i < j \leq n}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm \epsilon_i \pm \epsilon_{m+j}\}_{1 \leq i \leq m, 1 \leq j \leq n}.\end{aligned}$$

Denote by $e_1, \dots, e_{m+n} \in \mathfrak{t}$ the dual basis of $\epsilon_1, \dots, \epsilon_{m+n}$. The dominant condition on $a = a_1 e_1 + \dots + a_{m+n} e_{m+n} \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq \dots \geq a_{m-1} \geq |a_m|$ and $a_{m+1} \geq \dots \geq a_{m+n-1} \geq |a_{m+n}|$.

Setting 7.7. Let $\mathfrak{g}_0 = \mathfrak{so}(2m, 2n+1)$. Choose $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ such that

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq m} \cup \{\epsilon_{m+i} \pm \epsilon_{m+j}\}_{1 \leq i < j \leq n} \cup \{\epsilon_{m+i}\}_{1 \leq i \leq n}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm \epsilon_i \pm \epsilon_{m+j}\}_{1 \leq i \leq m, 1 \leq j \leq n} \cup \{\pm \epsilon_i\}_{1 \leq i \leq m}.\end{aligned}$$

Denote by $e_1, \dots, e_{m+n} \in \mathfrak{t}$ the dual basis of $\epsilon_1, \dots, \epsilon_{m+n}$. The dominant condition on $a = a_1 e_1 + \dots + a_{m+n} e_{m+n} \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq \dots \geq a_{m-1} \geq |a_m|$ and $a_{m+1} \geq \dots \geq a_{m+n} \geq 0$.

Setting 7.8. Let $\mathfrak{g}_0 = \mathfrak{so}(2m+1, 2n)$. Choose $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ such that

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq m} \cup \{\epsilon_{m+i} \pm \epsilon_{m+j}\}_{1 \leq i < j \leq n} \cup \{\epsilon_i\}_{1 \leq i \leq m}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm \epsilon_i \pm \epsilon_{m+j}\}_{1 \leq i \leq m, 1 \leq j \leq n} \cup \{\pm \epsilon_{m+i}\}_{1 \leq i \leq n}.\end{aligned}$$

Denote by $e_1, \dots, e_{m+n} \in \mathfrak{t}$ the dual basis of $\epsilon_1, \dots, \epsilon_{m+n}$. The dominant condition on $a = a_1 e_1 + \dots + a_{m+n} e_{m+n} \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq \dots \geq a_m \geq 0$ and $a_{m+1} \geq \dots \geq a_{m+n-1} \geq |a_{m+n}|$.

Setting 7.9. Let $\mathfrak{g}_0 = \mathfrak{so}(2m+1, 2n+1)$. Choose $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ such that

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq m} \cup \{\epsilon_{m+i} \pm \epsilon_{m+j}\}_{1 \leq i < j \leq n} \cup \{\epsilon_i\}_{1 \leq i \leq m} \cup \{\epsilon_{m+i}\}_{1 \leq i \leq n}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm \epsilon_i \pm \epsilon_{m+j}\}_{1 \leq i \leq m, 1 \leq j \leq n} \cup \{\pm \epsilon_i\}_{1 \leq i \leq m} \cup \{\pm \epsilon_{m+i}\}_{1 \leq i \leq n} \cup \{0\}.\end{aligned}$$

Denote by $e_1, \dots, e_{m+n} \in \mathfrak{t}$ the dual basis of $\epsilon_1, \dots, \epsilon_{m+n}$. The dominant condition on $a = a_1 e_1 + \dots + a_{m+n} e_{m+n} \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq \dots \geq a_m \geq 0$ and $a_{m+1} \geq \dots \geq a_{m+n} \geq 0$.

Setting 7.10. Let $\mathfrak{g}_0 = \mathfrak{sp}(m, n)$. Choose $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ such that

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq m} \cup \{\epsilon_{m+i} \pm \epsilon_{m+j}\}_{1 \leq i < j \leq n} \cup \{2\epsilon_i\}_{1 \leq i \leq m} \cup \{2\epsilon_{m+i}\}_{1 \leq i \leq n}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm \epsilon_i \pm \epsilon_{m+j}\}_{1 \leq i \leq m, 1 \leq j \leq n}.\end{aligned}$$

Denote by $e_1, \dots, e_{m+n} \in \mathfrak{t}$ the dual basis of $\epsilon_1, \dots, \epsilon_{m+n}$. The dominant condition on $a = a_1 e_1 + \dots + a_{m+n} e_{m+n} \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq \dots \geq a_m \geq 0$ and $a_{m+1} \geq \dots \geq a_{m+n} \geq 0$.

Setting 7.11. Let $\mathfrak{g}_0 = \mathfrak{so}^*(2n)$. Choose $\epsilon_1, \dots, \epsilon_n \in \mathfrak{t}^*$ such that

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq n}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm(\epsilon_i + \epsilon_j)\}_{1 \leq i < j \leq n}.\end{aligned}$$

Denote by $e_1, \dots, e_n \in \mathfrak{t}$ the dual basis of $\epsilon_1, \dots, \epsilon_n$. The dominant condition on $a = a_1 e_1 + \dots + a_n e_n \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq \dots \geq a_n$.

Setting 7.12. Let $\mathfrak{g}_0 = \mathfrak{sp}(n, \mathbb{R})$. Choose $\epsilon_1, \dots, \epsilon_n \in \mathfrak{t}^*$ such that

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq n}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm 2\epsilon_i\}_{1 \leq i \leq n} \cup \{\pm(\epsilon_i + \epsilon_j)\}_{1 \leq i < j \leq n}.\end{aligned}$$

Denote by $e_1, \dots, e_n \in \mathfrak{t}$ the dual basis of $\epsilon_1, \dots, \epsilon_n$. The dominant condition on $a = a_1 e_1 + \dots + a_n e_n \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq \dots \geq a_n$.

Setting 7.13. Let $\mathfrak{g}_0 = \mathfrak{sl}(2n, \mathbb{C})$. Choose $\epsilon_1, \dots, \epsilon_{2n} \in \mathfrak{t}^*$ such that

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq 2n}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm(\epsilon_i - \epsilon_j)\}_{1 \leq i < j \leq 2n} \cup \{0\}.\end{aligned}$$

Define $e_1, \dots, e_{2n} \in \mathfrak{t}$ such that $(\epsilon_i - \epsilon_j)(e_k) = \delta_{ik} - \delta_{jk}$ and then $e_1 + \dots + e_{2n} = 0$. The dominant condition on $a = a_1 e_1 + \dots + a_{2n} e_{2n} \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq \dots \geq a_{2n}$.

Setting 7.14. Let $\mathfrak{g}_0 = \mathfrak{so}(2n, \mathbb{C})$. Choose $\epsilon_1, \dots, \epsilon_n \in \mathfrak{t}^*$ such that

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq n}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm \epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq n} \cup \{0\}.\end{aligned}$$

Denote by $e_1, \dots, e_n \in \mathfrak{t}$ the dual basis of $\epsilon_1, \dots, \epsilon_n$. The dominant condition on $a = a_1 e_1 + \dots + a_n e_n \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq \dots \geq a_{n-1} \geq |a_n|$.

For real exceptional Lie algebras, we follow the notation of [Hel, Chapter X].

Setting 7.15. Let $\mathfrak{g}_0 = \mathfrak{f}_{4(4)} (\equiv \mathfrak{f}_4^1)$ so that $\mathfrak{k} = \mathfrak{sp}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. Choose $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathfrak{t}^*$ such that

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq 3} \cup \{2\epsilon_i\}_{1 \leq i \leq 3} \cup \{2\epsilon_4\}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4\} \cup \{\pm \epsilon_i \pm \epsilon_4\}_{1 \leq i \leq 3}.\end{aligned}$$

Denote by $e_1, e_2, e_3, e_4 \in \mathfrak{t}$ the dual basis of $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$. The dominant condition on $a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq a_2 \geq a_3 \geq 0$ and $a_4 \geq 0$.

Setting 7.16. Let $\mathfrak{g}_0 = \mathfrak{f}_{4(-20)} (\equiv \mathfrak{f}_4^2)$ so that $\mathfrak{k} = \mathfrak{so}(9, \mathbb{C})$. Choose $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathfrak{t}^*$ such that

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq 4} \cup \{\epsilon_i\}_{1 \leq i \leq 4}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \left\{ \frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \right\}.\end{aligned}$$

Denote by $e_1, e_2, e_3, e_4 \in \mathfrak{t}$ the dual basis of $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$. The dominant condition on $a = a_1 e_1 + \dots + a_4 e_4 \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0$.

Setting 7.17. Let $\mathfrak{g}_0 = \mathfrak{e}_{6(2)} (\equiv \mathfrak{e}_6^2)$ so that $\mathfrak{k} = \mathfrak{sl}(6, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. Choose $\epsilon_1, \dots, \epsilon_7 \in \mathfrak{t}^*$ such that

$$\Delta^+(\mathfrak{k}, \mathfrak{t}) = \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq 6} \cup \{2\epsilon_7\},$$

$$\Delta(\mathfrak{p}, \mathfrak{t}) = \left\{ \frac{1}{2} \left(\sum_{i=1}^6 (-1)^{k(i)} \epsilon_i \right) \pm \epsilon_7 : k(i) \in \{0, 1\}, k(1) + \dots + k(6) = 3 \right\}.$$

Define $e_1, \dots, e_7 \in \mathfrak{t}$ such that $(\epsilon_i - \epsilon_j)(e_k) = \delta_{ik} - \delta_{jk}$, $\epsilon_7(e_7) = 1$, and $(\epsilon_i - \epsilon_j)(e_7) = \epsilon_7(e_k) = 0$ for $1 \leq i, j, k \leq 6$. Then $e_1 + \dots + e_6 = 0$. The dominant condition on $a = a_1 e_1 + \dots + a_7 e_7 \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq \dots \geq a_6$ and $a_7 \geq 0$.

Setting 7.18. Let $\mathfrak{g}_0 = \mathfrak{e}_{6(-14)} (\equiv \mathfrak{e}_6^3)$ so that $\mathfrak{k} = \mathfrak{so}(10, \mathbb{C}) \oplus \mathbb{C}$. Choose $\epsilon_1, \dots, \epsilon_6 \in \mathfrak{t}^*$ such that

$$\Delta^+(\mathfrak{k}, \mathfrak{t}) = \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq 5},$$

$$\Delta(\mathfrak{p}, \mathfrak{t}) = \left\{ \frac{1}{2} \left(\sum_{i=1}^6 (-1)^{k(i)} \epsilon_i \right) : k(1) + \dots + k(6) \text{ odd} \right\}.$$

Denote by $e_1, \dots, e_6 \in \mathfrak{t}$ the dual basis of $\epsilon_1, \dots, \epsilon_6$. The dominant condition on $a = a_1 e_1 + \dots + a_6 e_6 \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq \dots \geq a_4 \geq |a_5|$.

Setting 7.19. Let $\mathfrak{g}_0 = \mathfrak{e}_{7(-5)} (\equiv \mathfrak{e}_7^2)$ so that $\mathfrak{k} = \mathfrak{so}(12, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. Choose $\epsilon_1, \dots, \epsilon_7 \in \mathfrak{t}^*$ such that

$$\Delta^+(\mathfrak{k}, \mathfrak{t}) = \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq 6} \cup \{2\epsilon_7\},$$

$$\Delta(\mathfrak{p}, \mathfrak{t}) = \left\{ \frac{1}{2} \left(\sum_{i=1}^6 (-1)^{k(i)} \epsilon_i \right) \pm \epsilon_7 : k(1) + \dots + k(6) \text{ odd} \right\}.$$

Denote by $e_1, \dots, e_7 \in \mathfrak{t}$ the dual basis of $\epsilon_1, \dots, \epsilon_7$. The dominant condition on $a = a_1 e_1 + \dots + a_7 e_7 \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_1 \geq \dots \geq a_5 \geq |a_6|$ and $a_7 \geq 0$.

Setting 7.20. Let $\mathfrak{g}_0 = \mathfrak{e}_{7(-25)} (\equiv \mathfrak{e}_7^3)$ so that $\mathfrak{k} = \mathfrak{e}_6^{\mathbb{C}} \oplus \mathbb{C}$. Choose $\epsilon_1, \dots, \epsilon_8 \in \mathfrak{t}^*$ such that

$$\Delta^+(\mathfrak{k}, \mathfrak{t}) = \{\epsilon_i \pm \epsilon_j\}_{1 \leq j < i \leq 5}$$

$$\cup \left\{ \frac{1}{2} \left(\epsilon_8 - \epsilon_7 - \epsilon_6 + \sum_{i=1}^5 (-1)^{k(i)} \epsilon_i \right) : k(1) + \dots + k(5) \text{ even} \right\},$$

$$\Delta(\mathfrak{p}, \mathfrak{t}) = \{\pm \epsilon_6 \pm \epsilon_i\}_{1 \leq i \leq 5} \cup \{\pm(\epsilon_8 - \epsilon_7)\}$$

$$\cup \left\{ \pm \frac{1}{2} \left(\epsilon_8 - \epsilon_7 + \epsilon_6 + \sum_{i=1}^5 (-1)^{k(i)} \epsilon_i \right) : k(1) + \dots + k(5) \text{ odd} \right\}.$$

Define $e_1, \dots, e_8 \in \mathfrak{t}$ such that $\epsilon_i(e_j) = \delta_{ij}$ for $1 \leq i \leq 6$, $1 \leq j \leq 8$ and that $(\epsilon_8 - \epsilon_7)(e_i) = \delta_{i8} - \delta_{i7}$ for $1 \leq i \leq 8$. Then $e_8 + e_7 = 0$. The dominant condition on $a = a_1 e_1 + \dots + a_8 e_8 \in \sqrt{-1}\mathfrak{t}_0$ amounts to that $a_5 \geq \dots \geq a_2 \geq |a_1|$ and $a_8 - a_7 - a_6 - a_5 - a_4 - a_3 - a_2 + a_1 \geq 0$.

Setting 7.21. Let $\mathfrak{g}_0 = \mathfrak{e}_{8(-24)} (\equiv \mathfrak{e}_8^2)$ so that $\mathfrak{k} = \mathfrak{e}_7^{\mathbb{C}} \oplus \mathfrak{sl}(2, \mathbb{C})$. Choose $\epsilon_1, \dots, \epsilon_8 \in \mathfrak{t}^*$ such that

$$\begin{aligned} \Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i \pm \epsilon_j\}_{1 \leq j < i \leq 6} \cup \{\epsilon_8 \pm \epsilon_7\} \\ &\cup \left\{ \frac{1}{2} \left(\epsilon_8 - \epsilon_7 + \sum_{i=1}^6 (-1)^{k(i)} \epsilon_i \right) : k(1) + \dots + k(6) \text{ odd} \right\}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm \epsilon_7 \pm \epsilon_i\}_{1 \leq i \leq 6} \cup \{\pm \epsilon_8 \pm \epsilon_i\}_{1 \leq i \leq 6} \\ &\cup \left\{ \pm \frac{1}{2} \left(\epsilon_8 + \epsilon_7 + \sum_{i=1}^6 (-1)^{k(i)} \epsilon_i \right) : k(1) + \dots + k(6) \text{ even} \right\}. \end{aligned}$$

Denote by $e_1, \dots, e_8 \in \mathfrak{t}$ the dual basis of $\epsilon_1, \dots, \epsilon_8$. The dominant condition on $a = a_1 e_1 + \dots + a_8 e_8 \in \sqrt{-1} \mathfrak{t}_0$ amounts to that $a_6 \geq \dots \geq a_2 \geq |a_1|$ and $a_8 - a_7 - a_6 - a_5 - a_4 - a_3 - a_2 + a_1 \geq 0$.

Theorem 7.22 ([KO12, Theorem 4.1]). *Let $(\mathfrak{g}_0, \mathfrak{g}'_0)$ be an irreducible symmetric pair defined by an involution σ such that σ commutes with θ and let \mathfrak{q} be a θ -stable parabolic subalgebra of \mathfrak{g} , not equal to \mathfrak{g} . Suppose that λ is in the weakly fair range and that $A_{\mathfrak{q}}(\lambda)$ is non-zero. Then $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable as a (\mathfrak{g}', K') -module if and only if one of the following conditions on the triple $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q})$ holds.*

- (1) \mathfrak{g}_0 is compact.
- (2) $\sigma = \theta$.
- (3) $\mathfrak{g}_0 = \mathfrak{g}_0 \oplus \mathfrak{g}_0$ and $\mathfrak{q} = \mathfrak{q}_1 \oplus \mathfrak{q}_2$. Further, \mathfrak{g}_0 is of Hermitian type and both of the parabolic subalgebras \mathfrak{q}_1 and \mathfrak{q}'_2 of \mathfrak{g} are holomorphic, or they are anti-holomorphic (see Table 1 for holomorphic and anti-holomorphic parabolic subalgebras).
- (4) The symmetric pair $(\mathfrak{g}_0, \mathfrak{g}'_0)$ is of holomorphic type (see Table 2 for the classification) and the parabolic subalgebra \mathfrak{q} is either holomorphic or anti-holomorphic.
- (5) The triple $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q})$ is isomorphic to one of those listed in Table 3 or in Table 4, where the parabolic subalgebra \mathfrak{q} is given by the conditions on a .

In Tables 1, 3, and 4, we have assumed that the defining element a of \mathfrak{q} is dominant with respect to $\Delta^+(\mathfrak{k}, \mathfrak{t})$ (see Settings 7.5 to 7.21 for concrete conditions on the coordinates of a) and list only additional conditions for the discrete decomposability.

Remark 7.23. The triples $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q})$ in Table 3 have the following property: there exists a θ -stable Borel subalgebra \mathfrak{b} contained in \mathfrak{q} such that $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{b})$ also satisfies the discrete decomposability condition. This is also the case for (1), (2), (3), and (4) in Theorem 7.22. Then Proposition 7.2 implies that every θ -stable parabolic subalgebra containing \mathfrak{b} satisfies the discrete decomposability condition. We call such triples $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q})$ *discrete series type*. The triples in Table 3 together with (1), (2), (3), and (4) in Theorem 7.22 give all triples of discrete series type.

Remark 7.24. The remaining case is (5) in Theorem 7.22 for Table 4. We call triples $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q})$ in Table 4 *isolated type*. For generic m, n and k , discrete series type and isolated type are exclusive. However, for particular m, n or k there may be overlaps.

Remark 7.25. Let $\mathfrak{g}_0 = \mathfrak{so}(2m, 2n)$. For $m = 2$ and $n \neq 2$, we write in Table 3 as

$$\begin{aligned}\mathfrak{g}'_0 &= \mathfrak{u}(2, n)_1 & \text{if } \sigma(e_1) = -e_2, \\ \mathfrak{g}'_0 &= \mathfrak{u}(2, n)_2 & \text{if } \sigma(e_1) = e_2.\end{aligned}$$

For $m = n = 2$, we write in Table 3 as

$$\begin{aligned}\mathfrak{g}'_0 &= \mathfrak{u}(2, 2)_{11} & \text{if } \sigma(e_1) = -e_2 \text{ and } \sigma(e_3) = -e_4, \\ \mathfrak{g}'_0 &= \mathfrak{u}(2, 2)_{12} & \text{if } \sigma(e_1) = -e_2 \text{ and } \sigma(e_3) = e_4, \\ \mathfrak{g}'_0 &= \mathfrak{u}(2, 2)_{21} & \text{if } \sigma(e_1) = e_2 \text{ and } \sigma(e_3) = -e_4, \\ \mathfrak{g}'_0 &= \mathfrak{u}(2, 2)_{22} & \text{if } \sigma(e_1) = e_2 \text{ and } \sigma(e_3) = e_4.\end{aligned}$$

\mathfrak{g}_0	$a = a_1 e_1 + a_2 e_2 + \cdots$	See Settings 7.5 to 7.21.
	holomorphic	anti-holomorphic
$\mathfrak{su}(m, n)$	$a_m \geq a_{m+1}$	$a_{m+n} \geq a_1$
$\mathfrak{so}(2, 2n)$	$a_1 \geq a_2$	$-a_1 \geq a_2$
$\mathfrak{so}(2, 2n+1)$	$a_1 \geq a_2$	$-a_1 \geq a_2$
$\mathfrak{so}^*(2n)$	$a_{n-1} + a_n \geq 0$	$a_1 + a_2 \leq 0$
$\mathfrak{sp}(n, \mathbb{R})$	$a_n \geq 0$	$a_1 \leq 0$
$\mathfrak{e}_{6(-14)}$	$a_6 \geq a_1 + a_2 + a_3 + a_4 + a_5$	$-a_6 \geq a_1 + a_2 + a_3 + a_4 - a_5$
$\mathfrak{e}_{7(-25)}$	$a_6 \geq a_5$	$a_8 \leq a_7$

TABLE 1. holomorphic parabolic subalgebras

\mathfrak{g}_0	\mathfrak{g}'_0
$\mathfrak{su}(m, n) \ m \neq n$	$\mathfrak{su}(k, l) \oplus \mathfrak{su}(m-k, n-l) \oplus \mathfrak{u}(1)$
$\mathfrak{su}(n, n)$	$\mathfrak{su}(k, l) \oplus \mathfrak{su}(n-k, n-l) \oplus \mathfrak{u}(1)$
	$\mathfrak{so}^*(2n)$
	$\mathfrak{sp}(n, \mathbb{R})$
$\mathfrak{so}(2, 2n)$	$\mathfrak{so}(2, k) \oplus \mathfrak{so}(2n-k)$
	$\mathfrak{u}(1, n)$
$\mathfrak{so}(2, 2n+1)$	$\mathfrak{so}(2, k) \oplus \mathfrak{so}(2n-k+1)$
$\mathfrak{so}^*(2n)$	$\mathfrak{u}(m, n-m)$
	$\mathfrak{so}^*(2m) \oplus \mathfrak{so}^*(2n-2m)$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{u}(m, n-m)$
	$\mathfrak{sp}(m, \mathbb{R}) \oplus \mathfrak{sp}(n-m, \mathbb{R})$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10) \oplus \mathfrak{so}(2)$
	$\mathfrak{so}(2, 8) \oplus \mathfrak{so}(2)$
	$\mathfrak{su}(4, 2) \oplus \mathfrak{su}(2)$
	$\mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$
	$\mathfrak{su}(5, 1) \oplus \mathfrak{sl}(2, \mathbb{R})$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-78)} \oplus \mathfrak{so}(2)$
	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$
	$\mathfrak{so}(2, 10) \oplus \mathfrak{sl}(2, \mathbb{R})$
	$\mathfrak{su}(6, 2)$
	$\mathfrak{so}^*(12) \oplus \mathfrak{su}(2)$

TABLE 2. symmetric pairs of holomorphic type

\mathfrak{g}_0	\mathfrak{g}'_0	$a = a_1 e_1 + a_2 e_2 + \dots$ See Settings 7.5 to 7.21.
$\mathfrak{su}(m, n)$	$\mathfrak{su}(m, k) \oplus \mathfrak{su}(n - k) \oplus \mathfrak{u}(1)$	$a_{m+n} \geq a_1,$ $a_l \geq a_{m+1}$ and $a_{m+n} \geq a_{l+1} (1 \leq l \leq m-1),$ or $a_m \geq a_{m+1}$
$\mathfrak{su}(2, 2n)$ $n \neq 1$	$\mathfrak{sp}(1, n)$	$a_1 \geq a_3$ and $a_{2n+2} \geq a_2$
$\mathfrak{su}(2, 2)$	$\mathfrak{sp}(1, 1)$	$a_1 \geq a_3 \geq a_4 \geq a_2$ or $a_3 \geq a_1 \geq a_2 \geq a_4$
$\mathfrak{so}(2m, 2n)$	$\mathfrak{so}(2m, k) \oplus \mathfrak{so}(2n - k)$	$ a_m \geq a_{m+1} $
$\mathfrak{so}(2m, 2n + 1)$	$\mathfrak{so}(2m, k) \oplus \mathfrak{so}(2n - k + 1)$	$ a_m \geq a_{m+1}$
$\mathfrak{so}(4, 2n)$ $n \neq 2$	$\mathfrak{u}(2, n)_1$ $\mathfrak{u}(2, n)_2$	$-a_2 \geq a_3 $ $a_2 \geq a_3 $
$\mathfrak{so}(4, 4)$	$\mathfrak{u}(2, 2)_{11}$ $\mathfrak{u}(2, 2)_{12}$ $\mathfrak{u}(2, 2)_{21}$ $\mathfrak{u}(2, 2)_{22}$	$-a_2 \geq a_3$ or $-a_4 \geq a_1$ $-a_2 \geq a_3$ or $a_4 \geq a_1$ $a_2 \geq a_3$ or $-a_4 \geq a_1$ $a_2 \geq a_3$ or $a_4 \geq a_1$
$\mathfrak{sp}(m, n)$	$\mathfrak{sp}(m, k) \oplus \mathfrak{sp}(n - k)$	$a_m \geq a_{m+1}$
$\mathfrak{f}_4(4)$	$\mathfrak{sp}(2, 1) \oplus \mathfrak{su}(2)$ $\mathfrak{so}(5, 4)$	$a_1 + a_2 + a_3 \leq a_4$ $a_1 + a_2 + a_3 \leq a_4$
$\mathfrak{e}_{6(2)}$	$\mathfrak{so}(6, 4) \oplus \mathfrak{so}(2)$ $\mathfrak{su}(4, 2) \oplus \mathfrak{su}(2)$ $\mathfrak{sp}(3, 1)$ $\mathfrak{f}_4(4)$	$a_1 + a_2 + a_3 - a_4 - a_5 - a_6 \leq 2a_7$ $a_1 + a_2 + a_3 - a_4 - a_5 - a_6 \leq 2a_7$ $a_1 + a_2 + a_3 - a_4 - a_5 - a_6 \leq 2a_7$ $a_1 + a_2 + a_3 - a_4 - a_5 - a_6 \leq 2a_7$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(8, 4) \oplus \mathfrak{su}(2)$ $\mathfrak{su}(6, 2)$ $\mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2)$	$a_1 + a_2 + a_3 + a_4 + a_5 - a_6 \leq 2a_7$ $a_1 + a_2 + a_3 + a_4 + a_5 - a_6 \leq 2a_7$ $a_1 + a_2 + a_3 + a_4 + a_5 - a_6 \leq 2a_7$
$\mathfrak{e}_{8(-24)}$	$\mathfrak{so}(12, 4)$ $\mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2)$	$a_7 \geq a_6$ $a_7 \geq a_6$

TABLE 3. $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q})$ of discrete series type

\mathfrak{g}_0	\mathfrak{g}'_0	$a = a_1 e_1 + a_2 e_2 + \cdots$ See Settings 7.5 to 7.21.
$\mathfrak{su}(2m, 2n)$	$\mathfrak{sp}(m, n)$	$(a_1, \dots, a_{2m}; a_{2m+1}, \dots, a_{2m+2n})$ $= (s, 0, \dots, 0; t, 0, \dots, 0), (0, \dots, 0, -s; 0, \dots, 0, -t),$ $(s, 0, \dots, 0, -t; 0, \dots, 0) \text{ or } (0, \dots, 0; s, 0, \dots, 0, -t)$ $\text{mod } \mathbb{I}_{2m+2n} \quad (s, t \geq 0)$
$\mathfrak{so}(2m+1, 2n)$	$\mathfrak{so}(2m+1, k) \oplus \mathfrak{so}(2n-k)$	$a_{m+1} = \cdots = a_{m+n} = 0$
$\mathfrak{so}(2m+1, 2n+1)$	$\mathfrak{so}(2m+1, k) \oplus \mathfrak{so}(2n-k+1)$	$a_{m+1} = \cdots = a_{m+n} = 0$
$\mathfrak{so}(2m, 2n)$	$\mathfrak{u}(m, n)$	$(a_1, \dots, a_m; a_{m+1}, \dots, a_{m+n})$ $= (s, 0, \dots, 0; 0, \dots, 0)$ $\text{or } (0, \dots, 0; s, 0, \dots, 0)$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2n-2) \oplus \mathfrak{so}(2)$	$(a_1, \dots, a_n) = \underbrace{(s, \dots, s, -s, \dots, -s)}_k$ $(1 \leq \exists k \leq n-1)$
	$\mathfrak{u}(n-1, 1)$	$(a_1, \dots, a_n) = \underbrace{(s, \dots, s, -s, \dots, -s)}_k$ $(1 \leq \exists k \leq n-1)$
$\mathfrak{sp}(m, n)$	$\mathfrak{sp}(k, l) \oplus \mathfrak{sp}(m-k, n-l)$ $k, l, m-k, n-l \geq 1$	$(a_1, \dots, a_m; a_{m+1}, \dots, a_{m+n})$ $= (s, 0, \dots, 0; 0, \dots, 0)$ $\text{or } (0, \dots, 0; s, 0, \dots, 0)$
	$\mathfrak{sp}(m, k) \oplus \mathfrak{sp}(n-k)$	$(a_1, \dots, a_m; a_{m+1}, \dots, a_{m+n})$ $= (0, \dots, 0; s, 0, \dots, 0),$ $a_{l-1} \geq a_{m+1} \text{ and } a_l = a_{m+2} = 0 \quad (2 \leq \exists l \leq m)$
$\mathfrak{sl}(2n, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{C})$	$(a_1, \dots, a_{2n}) = (s, 0, \dots, 0) \text{ or } (0, \dots, 0, s)$ $\text{mod } \mathbb{I}_{2n}$
	$\mathfrak{su}^*(2n)$	$(a_1, \dots, a_{2n}) = (s, 0, \dots, 0) \text{ or } (0, \dots, 0, s)$ $\text{mod } \mathbb{I}_{2n}$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2n-1, \mathbb{C})$	$(a_1, \dots, a_n) = (s, \dots, s)$
	$\mathfrak{so}(2n-1, 1)$	$(a_1, \dots, a_n) = (s, \dots, s)$
$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(8, 1)$	$(a_1, a_2, a_3, a_4) = (s, s, s, s) \text{ or } (s, s, 0, 0)$
$\mathfrak{e}_{6(2)}$	$\mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$	$(a_1, \dots, a_7) = (s, s, s, s, t, t, 0),$ $\text{or } (s, s, t, t, t, t, 0)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(2, 8) \oplus \mathfrak{so}(2)$	$(a_1, \dots, a_6) = (s, s, s, s, s, s),$ $\text{or } (s, s, s, s, -s, -s)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{f}_{4(-20)}$	$(a_1, \dots, a_6) = (s, s, 0, 0, 0, 0),$ $\text{or } (a_1, \dots, a_6) = (s, s, s, s, t, t)$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$	$(a_1, \dots, a_7) = (s, s, s, s, s, s, 0)$ $s, t \in \mathbb{R}, \quad \mathbb{I}_n = \underbrace{(1, \dots, 1)}_n$

TABLE 4. $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q})$ of isolated type

8. BRANCHING LAWS FOR ISOLATED TYPE

In what follows, we give branching formulas of $A_q(\lambda)|_{(\mathfrak{g}', K')}$ for symmetric pairs (G_0, G'_0) in a case-by-case way using the classification (Theorem 7.22 and Tables 1 to 4). It is easy to see that the branching law of $A_q(\lambda)|_{(\mathfrak{g}', K')}$ is reduced to the case where $(\mathfrak{g}_0, \mathfrak{g}'_0)$ is an irreducible symmetric pair (Definition 7.1). Suppose that $(\mathfrak{g}_0, \mathfrak{g}'_0)$ is an irreducible symmetric pair. According to Theorem 7.22 and following remarks, the triples $(\mathfrak{g}_0, \mathfrak{g}'_0, q)$ with the discrete decomposability condition are divided into two classes: discrete series type and isolated type. We treat the latter case in this section and the former case in the next section.

We derive branching formulas of $A_q(\lambda)|_{(\mathfrak{g}', K')}$ from Theorem 5.1. We will see that the right side of equation (5.1) equals (on the level of virtual (\mathfrak{g}', K') -modules) a sum of derived functor modules for G'_0 . This will be done according to the following procedure in each case.

- (1) Find the K' -orbit decomposition $K/(\overline{Q} \cap K) = Y_1 \sqcup \cdots \sqcup Y_n$.
- (2) For each orbit Y_i find the corresponding subgroups \overline{Q}_j and \overline{Q}'_j . Here \overline{Q}'_j is the parabolic subgroup of G' with Lie algebra $\mathfrak{q}'_j = N_{\mathfrak{p}'}(\mathfrak{q}_j \cap \mathfrak{p}') + (\mathfrak{q}_j \cap \mathfrak{p}')$. See Section 3.
- (3) Compute the $(\mathfrak{q}'_j, L'_j \cap K')$ -module $(P_{\mathfrak{q}_j \cap \mathfrak{g}', C'_j}^{\mathfrak{q}'_j, L'_j \cap K'})_d(s(W))$, where

$$W := \mathbb{C}_{\lambda+2\rho(u)} \otimes S(\mathfrak{g}/(\mathfrak{q}_j + \mathfrak{g}')).$$

See Remark 5.3.

- (4) Show that the right side of (5.1) can be written as a sum of derived functor modules.

For each K' -orbit Y_j we replace \mathfrak{t} by $\text{Ad}(k_j)\mathfrak{t}$ so that \mathfrak{q}_j is given by $-a \in \mathfrak{t}$. We therefore choose n different Cartan subalgebras of \mathfrak{k} and we write down the σ -actions on these Cartan subalgebras.

We describe step (3) here in more detail. We may temporarily assume that L'_j has a decomposition $L'_{j,c} \times L'_{j,n}$ which corresponds to the decomposition of the Lie algebra $\mathfrak{l}'_j = \mathfrak{l}'_{j,c} \oplus \mathfrak{l}'_{j,n}$ as (3.1).

Lemma 8.1. *Under the assumption above, suppose that S is a closed reductive subgroup of $L'_{j,c}$ and that $L'_{j,c} \subset K'$. Let V_c be an S -module and V_n an $(\mathfrak{l}'_{j,n}, L'_{j,n} \cap K')$ -module. Then*

$$(P_{\mathfrak{s}+\mathfrak{l}'_{j,n}, S \times (L'_{j,n} \cap K')}^{\mathfrak{l}'_j, L'_j \cap K'})_d(V_c \boxtimes V_n) \simeq \begin{cases} \text{Ind}_S^{L'_{j,c}}(V_c) \boxtimes V_n & \text{if } d = 0, \\ 0 & \text{if } d \neq 0. \end{cases}$$

The lemma will be applied to $S \times L'_{j,n} = \overline{Q} \cap G'$.

We write $\mathfrak{q}'(a)$ for the parabolic subalgebra given by $a \in \mathfrak{p}'$. Also write $L'(a)$, $\mathfrak{l}'(a)$, and $u'(a)$ for the corresponding Levi subgroup, Levi subalgebra, and the nilradical.

8.1. $\mathfrak{su}(2m, 2n) \downarrow \mathfrak{sp}(m, n)$.

Let m and n be positive integers. Set $\mathfrak{g}_0 := \mathfrak{u}(2m, 2n)$ and $\mathfrak{g}'_0 := \mathfrak{sp}(m, n)$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose a standard basis $\epsilon_1, \dots, \epsilon_{2m+2n} \in \mathfrak{t}^*$ so that

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{t}) &= \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq 2m} \cup \{\epsilon_{2m+i} - \epsilon_{2m+j}\}_{1 \leq i < j \leq 2n}, \\ \Delta(\mathfrak{p}, \mathfrak{t}) &= \{\pm(\epsilon_i - \epsilon_{2m+j})\}_{1 \leq i \leq 2m, 1 \leq j \leq 2n}\end{aligned}$$

and let $e_1, \dots, e_{2m+2n} \in \mathfrak{t}$ be the dual basis of $\epsilon_1, \dots, \epsilon_{2m+2n}$. Also we choose a Cartan subalgebra \mathfrak{t}' of \mathfrak{k}' and $\epsilon'_1, \dots, \epsilon'_{m+n} \in (\mathfrak{t}')^*$ such that

$$\begin{aligned}\Delta(\mathfrak{k}', \mathfrak{t}') &= \{\pm\epsilon'_i \pm \epsilon'_j\}_{1 \leq i < j \leq m} \cup \{\pm\epsilon'_{m+i} \pm \epsilon'_{m+j}\}_{1 \leq i < j \leq n} \cup \{\pm 2\epsilon'_i\}_{1 \leq i \leq m} \cup \{\pm 2\epsilon'_{m+i}\}_{1 \leq i \leq n}, \\ \Delta(\mathfrak{p}', \mathfrak{t}') &= \{\pm\epsilon'_i \pm \epsilon'_{m+j}\}_{1 \leq i \leq m, 1 \leq j \leq n}.\end{aligned}$$

Denote by $e'_1, \dots, e'_{m+n} \in \mathfrak{t}'$ the dual basis of $\epsilon'_1, \dots, \epsilon'_{m+n}$. Suppose that \mathfrak{q} is given by $a = a_1 e_1 + \dots + a_{2m+2n} e_{2m+2n}$. For every \mathfrak{q} in Table 4 the representation $A_{\mathfrak{q}}(\lambda)$ is defined for $U(2m, 2n)$ up to central character so we let $G_0 := U(2m, 2n)$, $G'_0 := Sp(m, n)$ and then $G = GL(2m+2n, \mathbb{C})$, $K = GL(2m, \mathbb{C}) \times GL(2n, \mathbb{C})$, $G' = Sp(m+n, \mathbb{C})$, $K' = Sp(m, \mathbb{C}) \times Sp(n, \mathbb{C})$.

Let $a = e_1$ and $\mathfrak{q} := \mathfrak{q}(a)$. Then $\mathfrak{l}_0(a) = \mathfrak{u}(1) \oplus \mathfrak{u}(2m-1, 2n)$. Let V be the dual representation of the natural representation of $GL(2m, \mathbb{C})$, namely, $V = F^{GL(2m, \mathbb{C})}(-\epsilon_{2m})$. Then $K/(\overline{Q} \cap K) \simeq \mathbb{P}(V) = \{V_1 \subset V : \dim V_1 = 1\}$. Since $Sp(m, \mathbb{C})$ acts transitively on $\mathbb{P}(V)$, there is only one K' -orbit in $K/(\overline{Q} \cap K)$, which implies that $\overline{\mathfrak{q}} \cap \mathfrak{k}'$ is a parabolic subalgebra of \mathfrak{k}' . Hence by Lemma 3.14, $\overline{\mathfrak{q}} \cap \mathfrak{g}'$ is a parabolic subalgebra of \mathfrak{g}' . Similarly let W be the dual representation of the natural representation of $GL(2n, \mathbb{C})$. Then $G/\overline{Q} \simeq \mathbb{P}(V \oplus W)$ and $G'/(\overline{Q} \cap G') \simeq G/\overline{Q}$. We give a σ -action on \mathfrak{t} and a relation with \mathfrak{t}' as

$$\begin{aligned}(8.1) \quad & \sigma e_i = -e_{2m-i+1} \quad (1 \leq i \leq 2m), \quad \sigma e_{2m+i} = -e_{2m+2n-i+1} \quad (1 \leq i \leq 2n), \\ & e_i - e_{2m-i+1} = e'_i \quad (1 \leq i \leq m), \quad \text{and } e_{2m+i} - e_{2m+2n-i+1} = e'_{m+i} \quad (1 \leq i \leq n) \\ & \text{so that } \epsilon_1|_{\mathfrak{t}'} = \epsilon'_1. \text{ Therefore, we see that } \overline{\mathfrak{q}}' := \overline{\mathfrak{q}} \cap \mathfrak{g}' \text{ is the parabolic} \\ & \text{subalgebra of } \mathfrak{g}' \text{ given by } -\epsilon'_1. \text{ We have } 2\rho(\mathfrak{u}) = (2m+2n)\epsilon_1 - \epsilon \text{ and} \\ & 2\rho(\mathfrak{u}') = (2m+2n)\epsilon'_1, \text{ where } \epsilon := \epsilon_1 + \dots + \epsilon_{2m+2n}. \text{ Hence } \lambda\epsilon_1 \text{ is weakly} \\ & \text{fair if and only if } \lambda \geq -m-n \text{ and that } \lambda\epsilon'_1 \text{ is weakly fair if and only if} \\ & \lambda \geq -m-n. \text{ Since } \mathfrak{g} = \overline{\mathfrak{q}} + \mathfrak{g}', \text{ Corollary 5.8 yields}\end{aligned}$$

$$A_{\mathfrak{q}(e_1)}(\lambda\epsilon_1)|_{(\mathfrak{g}', K')} \simeq A_{\mathfrak{q}'(e'_1)}(\lambda\epsilon'_1)$$

for $\lambda \in \mathbb{Z}$, $\lambda \geq -m-n$. We note that $\mathfrak{l}'_0(e'_1) = \mathfrak{u}(1) \oplus \mathfrak{sp}(m-1, n)$.

Let $a = e_1 + e_{2m+1}$ and $\mathfrak{q} := \mathfrak{q}(a)$ so that $\mathfrak{l}_0(a) = \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2m-1, 2n-1)$. Then $K/(\overline{Q} \cap K) \simeq \mathbb{P}(V) \times \mathbb{P}(W)$ in the previous notation. We see that K' acts transitively on $K/(\overline{Q} \cap K)$. The σ -action on \mathfrak{t} is given as (8.1). Then $\overline{\mathfrak{q}}' := \overline{\mathfrak{q}} \cap \mathfrak{g}'$ is given by $-(\epsilon'_1 + \epsilon'_{m+1})$. The quotient $\mathfrak{g}/(\overline{\mathfrak{q}} + \mathfrak{g}')$ is the one-dimensional \mathfrak{l}' -module corresponding to $\epsilon'_1 + \epsilon'_{m+1}$. We have $2\rho(\mathfrak{u}) = (2m+2n)(\epsilon_1 + \epsilon_{2m+1}) - 2\epsilon$ and $2\rho(\mathfrak{u}') = (2m+2n-1)(\epsilon'_1 + \epsilon'_{m+1})$. The parameter $\lambda(\epsilon_1 + \epsilon_{2m+1})$ is weakly fair for $\mathfrak{q}(e_1 + e_{2m+1})$ if and only if $\lambda \geq -m-n$ and

$\lambda(\epsilon'_1 + \epsilon'_{m+1})$ is weakly fair for $q'(e'_1 + e'_{m+1})$ if and only if $\lambda \geq -m - n + \frac{1}{2}$. By Corollary 5.8, we obtain

$$A_{q(e_1+e_{2m+1})}(\lambda(\epsilon_1 + \epsilon_{m+1}))|_{(g', K')} \simeq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} A_{q'(e'_1+e'_{m+1})}((\lambda + k + 1)(\epsilon'_1 + \epsilon'_{m+1}))$$

for $\lambda \in \mathbb{Z}$, $\lambda \geq -m - n$. We note that $l'_0(e'_1 + e'_{m+1}) = u(1, 1) \oplus \mathfrak{sp}(m-1, n-1)$.

Let $a = 2e_1 + e_{2m+1}$ and $q := q(a)$ so that $l_0(a) = u(1)^2 \oplus u(2m-1, 2n-1)$. Then $K/(\overline{Q} \cap K) \simeq \mathbb{P}(V) \times \mathbb{P}(W)$, on which K' acts transitively. The σ -action on \mathfrak{t} is given as (8.1). Then $\bar{q}' := \bar{q} \cap g'$ is given by $-(2e'_1 + e'_{m+1})$. The quotient $\mathfrak{g}/(\bar{q} + g')$ is the one-dimensional l' -module corresponding to $\epsilon'_1 + \epsilon'_{m+1}$. We have $2\rho(u) = (2m+2n+1)\epsilon_1 + (2m+2n-1)\epsilon_{2m+1} - 2\epsilon$ and $2\rho(u') = (2m+2n)\epsilon'_1 + (2m+2n-2)\epsilon'_{m+1}$. The parameter $\lambda_1\epsilon_1 + \lambda_2\epsilon_{2m+1}$ is weakly fair for $q(2e_1 + e_{2m+1})$ if and only if $\lambda_1 \geq \lambda_2 - 1 \geq -m - n - \frac{1}{2}$ and $\lambda_1\epsilon'_1 + \lambda_2\epsilon'_{m+1}$ is weakly fair for $q'(2e'_1 + e'_{m+1})$ if and only if $\lambda_1 \geq \lambda_2 - 1 \geq -m - n$. By Corollary 5.8, we have

$$\begin{aligned} & A_{q(2e_1+e_{2m+1})}(\lambda_1\epsilon_1 + \lambda_2\epsilon_{2m+1})|_{(g', K')} \\ & \simeq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} A_{q'(2e'_1+e'_{m+1})}((\lambda_1 + k + 1)\epsilon'_1 + (\lambda_2 + k + 1)\epsilon'_{m+1}) \end{aligned}$$

for $\lambda_1 \geq \lambda_2 - 1 \geq -m - n$. We note that $l'_0(2e'_1 + e'_{m+1}) = u(1)^2 \oplus \mathfrak{sp}(m-1, n-1)$.

Let $a = e_1 - e_{2m}$, $m > 1$ and $q := q(a)$ so that $l_0(a) = u(1)^2 \oplus u(2m-2, 2n)$. Consider the restriction $A_q(\lambda_1\epsilon_1 + \lambda_2\epsilon_{2m})|_{(g', K')}$. The generalized flag variety $K/(\overline{Q} \cap K)$ is identified with

$$\{V_1 \subset V_{2m-1} \subset V : \dim V_1 = 1, \dim V_{2m-1} = 2m-1\}.$$

Write $v(\alpha) \in V \oplus W$ for an α -weight vector. Then \overline{Q} is the stabilizer group of $\mathbb{C}v(-\epsilon_1)$ and $\mathbb{C}v(-\epsilon_1) + \cdots + \mathbb{C}v(-\epsilon_{2m-1}) + \mathbb{C}v(-\epsilon_{2m+1}) + \cdots + \mathbb{C}v(-\epsilon_{2m+2n})$ in G . Each K' -orbit corresponds to a symplectic structure $\langle \cdot, \cdot \rangle$ on V . Therefore, $K/(\overline{Q} \cap K)$ decomposes into two K' -orbits Y_1 and Y_2 according to

$$Y_1 \leftrightarrow \langle V_1, V_{2m-1} \rangle \neq 0, \quad Y_2 \leftrightarrow \langle V_1, V_{2m-1} \rangle = 0.$$

If $\langle V_1, V_{2m-1} \rangle \neq 0$, the space V_{2m-1}^\perp orthogonal to V_{2m-1} does not equal V_1 . Then $V_1 + V_{2m-1}^\perp$ is a two-dimensional isotropic subspace of V . For a Cartan subalgebra \mathfrak{t} corresponding to the orbit Y_1 , we give a σ -action and a relation with \mathfrak{t}' as

$$\sigma e_{2i-1} = -e_{2i} \quad (1 \leq i \leq m+n), \quad e_1 - e_2 = e'_1, \quad \text{and} \quad e_{2m-1} - e_{2m} = e'_2.$$

so that $\epsilon_1|_{\mathfrak{t}'} = \epsilon'_1$ and $\epsilon_{2m}|_{\mathfrak{t}'} = -\epsilon'_2$. Let \bar{q}'_1 be the parabolic subalgebra of g' given by $-(e'_1 + e'_2)$ so that

$$G'/\overline{Q}'_1 \simeq \{V_2 \subset V : \dim V_2 = 2, V_2 \text{ is isotropic}\}.$$

The corresponding Levi subgroup is $L'_1 \simeq GL(2, \mathbb{C}) \times Sp(m+n-2, \mathbb{C})$ and then $L'_{1,c} \simeq GL(2, \mathbb{C})$, $L'_{1,n} \simeq Sp(m+n-2, \mathbb{C})$. Then \overline{Q}'_1 is the stabilizer

subgroup of the two-dimensional isotropic subspace $\mathbb{C}v(-\epsilon'_1) + \mathbb{C}v(-\epsilon'_2)$ in G' , and $\bar{Q}_1 \cap G'$ is identified with the stabilizer subgroup of the two one-dimensional subspaces $\mathbb{C}v(-\epsilon'_1)$ and $\mathbb{C}v(-\epsilon'_2)$ in G' . Therefore, $\bar{U}'_1 \subset \bar{Q}_1 \cap G' \subset \bar{Q}'_1$ and $\bar{Q}_1 \cap L'_1 \simeq GL(1, \mathbb{C}) \times GL(1, \mathbb{C}) \times Sp(m+n-2, \mathbb{C})$. We see that $\mathfrak{g} = \bar{\mathfrak{q}}_1 + \mathfrak{g}'$ and $2\rho(\mathfrak{u}) = (2m+2n-1)(\epsilon_1 - \epsilon_{2m})$. The parameter $\lambda_1\epsilon_1 + \lambda_2\epsilon_{2m}$ is weakly fair for \mathfrak{q} if and only if $\lambda_1, -\lambda_2 \geq -m-n+\frac{1}{2}$. Hence the term in the right side of (5.1) for $j=1$ is

$$\sum_d (-1)^{d+1} [(P_{\bar{\mathfrak{q}}_1 \cap \mathfrak{g}', C'_1}^{\mathfrak{g}', K'})_d (\mathbb{C}_{(\lambda_1+2m+2n-1)\epsilon'_1 + (-\lambda_2+2m+2n-1)\epsilon'_2})].$$

If we write $\bar{\mathfrak{b}}$ for the parabolic subalgebra of $\mathfrak{l}'_{1,c}$ given by $-\epsilon'_1$, then

$$\begin{aligned} & \text{Ind}_{\bar{Q}_1 \cap L'_{1,c}}^{L'_{1,c}} (\mathbb{C}_{(\lambda_1+2m+2n-1)\epsilon'_1 + (-\lambda_2+2m+2n-1)\epsilon'_2}) \\ & \simeq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} F^{GL(2, \mathbb{C})}((\mu_1 + 2m + 2n + k - 1)\epsilon'_1 + (\mu_2 + 2m + 2n - k - 1)\epsilon'_2) \\ & \simeq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} (P_{\bar{\mathfrak{b}}, \bar{Q}_1 \cap L'_{1,c}}^{\mathfrak{l}'_{1,c}, L'_{1,c}})_1 (\mathbb{C}_{(\mu_1+2m+2n+k)\epsilon'_1 + (\mu_2+2m+2n-k-2)\epsilon'_2}), \end{aligned}$$

where

$$(\mu_1, \mu_2) := \begin{cases} (\lambda_1, -\lambda_2) & \text{if } \lambda_1 \geq -\lambda_2, \\ (-\lambda_2, \lambda_1) & \text{if } \lambda_1 < -\lambda_2. \end{cases}$$

Then Lemma 8.1 and Proposition 2.4 imply that

$$\begin{aligned} & \sum_d (-1)^{d+1} [(P_{\bar{\mathfrak{q}}_1 \cap \mathfrak{g}', C'_1}^{\mathfrak{g}', K'})_d (\mathbb{C}_{(\lambda_1+2m+2n-1)\epsilon'_1 + (\lambda_2+2m+2n-1)\epsilon'_2})] \\ & = \sum_d (-1)^d \sum_{k \in \mathbb{Z}_{\geq 0}} [(P_{\mathfrak{q}'(-2e'_1 - e'_2), L'(2e'_1 + e'_2) \cap K'}^{\mathfrak{g}', K'})_d (\mathbb{C}_{(\mu_1+2m+2n+k)\epsilon'_1 + (\mu_2+2m+2n-k-2)\epsilon'_2})] \\ & = \sum_d (-1)^d \sum_{k \in \mathbb{Z}_{\geq 0}} [\mathcal{L}_{\mathfrak{q}'(-2e'_1 - e'_2), d}^{\mathfrak{g}'} (\mathbb{C}_{(\mu_1+k)\epsilon'_1 + (\mu_2-k)\epsilon'_2})]. \end{aligned}$$

The orbit Y_2 is closed and $\bar{\mathfrak{q}}_2 \cap \mathfrak{g}'$ is the parabolic subalgebra of \mathfrak{g}' given by $-\epsilon'_1$. For a Cartan subalgebra \mathfrak{t} corresponding to the orbit Y_2 , we give a σ -action and a relation with \mathfrak{t}' as (8.1). We see that $\mathfrak{g}/(\bar{\mathfrak{q}}_2 + \mathfrak{g}')$ is $(2m+2n-2)$ -dimensional and isomorphic to $F^{L'(e'_1)}(\epsilon'_1 + \epsilon'_2)$ as an $L'(e'_1)$ -module. Hence the term in the right side of (5.1) for $j=2$ is

$$\sum_d (-1)^{d+1} \sum_{k \in \mathbb{Z}_{\geq 0}} [\mathcal{L}_{\mathfrak{q}'(-e'_1), d}^{\mathfrak{g}'} (F^{L'(e'_1)}((\lambda_1 - \lambda_2 + 2m + 2n + k - 2)\epsilon'_1 + k\epsilon'_2))].$$

By Theorem 6.5,

$$\begin{aligned}
& \mathcal{L}_{q'(-2e'_1-e'_2) \cap l'(e'_1), d}^{l'(e'_1)}(\mathbb{C}_{\nu_1 e'_1 + \nu_2 e'_2}) \\
& \simeq \begin{cases} F^{L'}(e'_1)(\nu_1 e'_1 + (-\nu_2 - 2m - 2n + 2)e'_2) & \text{if } d = 0 \text{ and } \nu_2 \leq -2m - 2n + 2, \\ 0 & \text{if } d = 0 \text{ and } \nu_2 > -2m - 2n + 2, \\ 0 & \text{if } d \neq 2m - 3, 0, \end{cases} \\
& \mathcal{L}_{q'(-2e'_1-e'_{m+1}) \cap l'(e'_1), d}^{l'(e'_1)}(\mathbb{C}_{\nu_1 e'_1 + \nu_2 e'_{m+1}}) \\
& \simeq \begin{cases} F^{L'}(e'_1)(\nu_1 e'_1 + (-\nu_2 - 2m - 2n + 2)e'_2) & \text{if } d = 0 \text{ and } \nu_2 \leq -2m - 2n + 2, \\ 0 & \text{if } d = 0 \text{ and } \nu_2 > -2m - 2n + 2, \\ 0 & \text{if } d \neq 2n - 1, 0, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
& \mathcal{L}_{q'(-2e'_1-e'_2) \cap l'(e'_1), 2m-3}^{l'(e'_1)}(\mathbb{C}_{\nu_1 e'_1 + \nu_2 e'_2}) \\
& \simeq \mathcal{L}_{q'(-2e'_1-e'_{m+1}) \cap l'(e'_1), 2n-1}^{l'(e'_1)}(\mathbb{C}_{\nu_1 e'_1 + (-\nu_2 - 2m - 2n + 2)e'_{m+1}}).
\end{aligned}$$

We see that $2\rho(u'(2e'_1 + e'_2)) = (2m + 2n)e'_1 + (2m + 2n - 2)e'_2$ and $2\rho(u'(2e'_1 + e'_{m+1})) = (2m + 2n)e'_1 + (2m + 2n - 2)e'_{m+1}$. Hence $\nu_1 e'_1 + \nu_2 e'_2$ is weakly fair for $q'(2e'_1 + e'_2)$ if and only if $\nu_1 \geq \nu_2 - 1 \geq -m - n$ and $\nu_1 e'_1 + \nu_{m+1} e'_{m+1}$ is weakly fair for $q'(2e'_1 + e'_{m+1})$ if and only if $\nu_1 \geq \nu_{m+1} - 1 \geq -m - n$. As a result,

$$\sum_d (-1)^d [\mathcal{L}_{q'(-2e'_1-e'_2), d}^{g'}(\mathbb{C}_{(\mu_1+k)e'_1 + (\mu_2-k)e'_2})] = [A_{q'(2e'_1+e'_2)}((\mu_1+k)e'_1 + (\mu_2-k)e'_2)]$$

for $0 \leq k \leq \mu_2 + m + n - 1$,

$$\begin{aligned}
& \sum_d (-1)^d [\mathcal{L}_{q'(-2e'_1-e'_2), d}^{g'}(\mathbb{C}_{(\mu_1+k)e'_1 + (\mu_2-k)e'_2})] \\
& = [A_{q'(2e'_1+e'_{m+1})}((\mu_1+k)e'_1 + (-\mu_2 - 2m - 2n + k + 2)e'_{m+1})]
\end{aligned}$$

for $\mu_2 + m + n - 1 < k < \mu_2 + 2m + 2n - 2$, and

$$\begin{aligned}
& \sum_d (-1)^d [\mathcal{L}_{q'(-2e'_1-e'_2), d}^{g'}(\mathbb{C}_{(\mu_1+k)e'_1 + (\mu_2-k)e'_2})] \\
& - \sum_d (-1)^d [\mathcal{L}_{q'(-e'_1), d}^{g'}(F^{L'}(e'_1)((\mu_1+k)e'_1 + (-\mu_2 - 2m - 2n + k + 2)e'_2))] \\
& = [A_{q'(2e'_1+e'_{m+1})}((\mu_1+k)e'_1 + (-\mu_2 - 2m - 2n + k + 2)e'_{m+1})]
\end{aligned}$$

for $\mu_2 + 2m + 2n - 2 \leq k$.

Therefore, Theorem 5.1 yields

$$\begin{aligned} & A_{\mathfrak{q}(e_1 - e_{2m})}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_{2m})|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{k=0}^{\mu_2 + m + n - 1} A_{\mathfrak{q}'(2e'_1 + e'_2)}((\mu_1 + k)\epsilon'_1 + (\mu_2 - k)\epsilon'_2) \\ & \quad \oplus \bigoplus_{\mu_2 + m + n - 1 < k} A_{\mathfrak{q}'(2e'_1 + e'_{m+1})}((\mu_1 + k)\epsilon'_1 + (-\mu_2 - 2m - 2n + k + 2)\epsilon'_{m+1}) \end{aligned}$$

for $\lambda_1, -\lambda_2 \geq -m - n + 1$. We note that $\mathfrak{l}'_0(2e'_1 + e'_2) = \mathfrak{u}(1)^2 \oplus \mathfrak{sp}(m - 2, n)$ and $\mathfrak{l}'_0(2e'_1 + e'_{m+1}) = \mathfrak{u}(1)^2 \oplus \mathfrak{sp}(m - 1, n - 1)$.

Let $m = 1$, $a = e_1 - e_2$ and $\mathfrak{q} := \mathfrak{q}(a)$ so that $\mathfrak{l}_0(a) = \mathfrak{u}(1)^2 \oplus \mathfrak{u}(2n)$. Consider the restriction $A_{\mathfrak{q}}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2)|_{(\mathfrak{g}', K')}$. Then K' acts transitively on $K/(\overline{Q} \cap K) \simeq \mathbb{P}^1$. The σ -action on \mathfrak{t} is given as (8.1). Then $\bar{\mathfrak{q}}' := \bar{\mathfrak{q}} \cap \mathfrak{g}'$ is given by $-e'_1$. The quotient $\mathfrak{g}/(\bar{\mathfrak{q}} + \mathfrak{g}')$ is $2n$ -dimension and isomorphic to $F^{L'(e'_1)}(\epsilon'_1 + \epsilon'_2)$ as an $L'(e'_1)$ -module. We see that $2\rho(\mathfrak{u}) = (2n + 1)(\epsilon_1 - \epsilon_2)$ and hence the parameter $\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2$ is weakly fair for $\mathfrak{q}(e_1 - e_2)$ if and only if $\lambda_1, -\lambda_2 \geq -n - \frac{1}{2}$. Then Corollary 5.8 yields

$$A_{\mathfrak{q}(e_1 - e_2)}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2)|_{(\mathfrak{g}', K')} \simeq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} A_{\mathfrak{q}'(2e'_1 + e'_2)}((\lambda_1 - \lambda_2 + 2n + k)\epsilon'_1 + k\epsilon'_2)$$

for $\lambda_1, -\lambda_2 \geq -n$. We note $\mathfrak{l}'_0(2e'_1 + e'_2) = \mathfrak{u}(1)^2 \oplus \mathfrak{sp}(n)$.

8.2. $\mathfrak{so}(2m + 1, 2n) \downarrow \mathfrak{so}(2m + 1, k) \oplus \mathfrak{so}(2n - k)$.

Let m, n and k be positive integers. Set $\mathfrak{g}_0 := \mathfrak{so}(2m + 1, 2n)$ and $\mathfrak{g}'_0 := \mathfrak{so}(2m + 1, k) \oplus \mathfrak{so}(2n - k)$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose $e_1, \dots, e_{m+n} \in \mathfrak{t}$ and $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ as in Setting 7.8. We may assume that $e_1, \dots, e_{m+n} \in \mathfrak{g}'$ if k is even and $e_1, \dots, e_{m+n-1} \in \mathfrak{g}'$ if k is odd. They form a basis of a Cartan subalgebra \mathfrak{t}' of \mathfrak{k}' . Put $k' = \lfloor \frac{k}{2} \rfloor$ and $l = n - \lfloor \frac{k}{2} \rfloor$. We restrict ϵ_i to \mathfrak{t}' and use the same notation. Then

$$\begin{aligned} \Delta(\mathfrak{k}', \mathfrak{t}') &= \{\pm \epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq m} \cup \{\pm \epsilon_{m+i} \pm \epsilon_{m+j}\}_{1 \leq i < j \leq k'} \\ &\quad \cup \{\pm \epsilon_{m+k'+i} \pm \epsilon_{m+k'+j}\}_{1 \leq i < j \leq l} \cup \{\pm \epsilon_i\}_{1 \leq i \leq m} \cup \{\pm \epsilon_{m+i}\}_{1 \leq i \leq k'} \cup \{\pm \epsilon_{m+k'+i}\}_{1 \leq i \leq l}, \\ \Delta(\mathfrak{p}', \mathfrak{t}') &= \{\pm \epsilon_i \pm \epsilon_{m+j}\}_{1 \leq i \leq m, 1 \leq j \leq k'} \cup \{\pm \epsilon_{m+i}\}_{1 \leq i \leq k'} \cup \{\pm \epsilon_i\}_{1 \leq i \leq m} \cup \{0\}, \end{aligned}$$

where the terms in the brackets appear if k is odd. For every \mathfrak{q} in Table 4 the representation $A_{\mathfrak{q}}(\lambda)$ is defined for $SO(2m + 1, 2n)_0$, the identity component of $SO(2m + 1, 2n)$, so we let $G_0 := SO(2m + 1, 2n)_0$, $G'_0 := SO(2m + 1, k)_0 \times SO(2n - k)$ and then $G = SO(2m + 2n + 1, \mathbb{C})$, $K = SO(2m + 1, \mathbb{C}) \times SO(2n, \mathbb{C})$, $G' = SO(2m + k + 1, \mathbb{C}) \times SO(2n - k, \mathbb{C})$, $K' = SO(2m + 1, \mathbb{C}) \times SO(k, \mathbb{C}) \times SO(2n - k, \mathbb{C})$. Suppose that \mathfrak{q} is given by $a = a_1 e_1 + \dots + a_m e_m$ with $a_1 \geq \dots \geq a_m \geq 0$. Then K' acts transitively on $K/(\overline{Q} \cap K)$ and $\bar{\mathfrak{q}} \cap \mathfrak{g}'$ is given by $-a \in \mathfrak{t}'$. Put $a_0 = \min\{a_i : a_i > 0\}$, $p = \min\{i : a_i = a_0\} - 1$, and $q = \max\{i : a_i = a_0\} - p$. Then \mathfrak{l}_0 is a direct sum of $\mathfrak{so}(2m - 2p - 2q + 1, 2n)$ and a compact Lie algebra. Consider the restriction $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$, where $\lambda = \lambda_1 \epsilon_1 + \dots + \lambda_{p+q} \epsilon_{p+q}$. It is easy to show that $A_{\mathfrak{q}}(\lambda) = 0$ if there exist $1 \leq i < j \leq p + q$ such that $\lambda_i < \lambda_j$ (see Lemma 9.6), so we assume that

$\lambda_1 \geq \dots \geq \lambda_{p+q}$. Under this assumption, the weakly fair condition amounts to $\lambda_{p+1} \geq -m-n+p+\frac{q}{2}$. Put $b := e_1 + \dots + e_{p+q}$ and then $\mathfrak{q}'(b) = \mathfrak{q}(b) \cap \mathfrak{g}'$. The corresponding Levi subgroups are $L(b) = GL(p+q, \mathbb{C}) \times SO(2m+2n-2p-2q+1, \mathbb{C})$ and $L'(b) = GL(p+q, \mathbb{C}) \times SO(2m+k-2p-2q+1, \mathbb{C}) \times SO(2n-k, \mathbb{C})$. If we put $s(b) := \dim(\mathfrak{u}'(b) \cap \mathfrak{k})$, we have

$$A_{\mathfrak{q}}(\lambda) \simeq \mathcal{L}_{\mathfrak{q}(-b), s(b)}^{\mathfrak{g}}(F^{L(b)}(\lambda)).$$

The quotient $\mathfrak{g}/(\mathfrak{q}(-b) + \mathfrak{g}')$ is isomorphic to $F^{L'(b)}(\epsilon_1 + \epsilon_{m+k'+1})$. Hence Corollary 5.7 gives

$$\begin{aligned} & [\mathcal{L}_{\mathfrak{q}(-b), s(b)}^{\mathfrak{g}}(F^{L(b)}(\lambda))|_{(\mathfrak{g}', K')}] \\ &= [\mathcal{L}_{\mathfrak{q}'(-b), s(b)}^{\mathfrak{g}'}(F^{L(b)}(\lambda)|_{L'(b)} \otimes S(F^{L'(b)}(\epsilon_1 + \epsilon_{m+k'+1})) \otimes \mathbb{C}_{(2n-k)(\epsilon'_1 + \dots + \epsilon'_{p+q})})]. \end{aligned}$$

For $\mu = \sum_{i=1}^{m+k'+l} \mu_i \epsilon_i$, let $m(\mu)$ be the multiplicity of $F^{L'(b)}(\mu)$ in the irreducible decomposition of $F^{L(b)}(\lambda)|_{L'(b)} \otimes S(F^{L'(b)}(\epsilon_1 + \epsilon_{m+k'+1})) \otimes \mathbb{C}_{(2n-k)(\epsilon_1 + \dots + \epsilon_{p+q})}$, namely,

$$\begin{aligned} (8.2) \quad & F^{L(b)}(\lambda)|_{L'(b)} \otimes S(F^{L'(b)}(\epsilon_1 + \epsilon_{m+k'+1})) \otimes \mathbb{C}_{(2n-k)(\epsilon_1 + \dots + \epsilon_{p+q})} \\ & \simeq \bigoplus_{\mu} m(\mu) F^{L'(b)}(\mu). \end{aligned}$$

Put $r = \min\{q-1, 2n-k\}$ and

$$a' = a'_1 e_1 + \dots + a'_{p+q} e_{p+q} + a'_{m+k'+1} e_{m+k'+1} + \dots + a'_{m+k'+l} e_{m+k'+l}$$

with $a'_1 > \dots > a'_{p+r+1} = \dots = a'_{p+q} > 0$ and $a'_{m+k'+1} > \dots > a'_{m+k'+l} > 0$. Then

$$\begin{aligned} L'(a') &\simeq GL(1, \mathbb{C})^{p+r+l} \times GL(q-r, \mathbb{C}) \times SO(2m+k-2p-2q+1, \mathbb{C}), \\ \mathfrak{l}'_0(a') &\simeq \mathfrak{u}(1)^{p+r+l} \oplus \mathfrak{u}(q-r) \oplus \mathfrak{so}(2m-2p-2q+1, k), \end{aligned}$$

and it turns out that $m(\mu) \neq 0$ only if $\mu_{p+r+1} = \dots = \mu_{p+q} > \mu_{p+q+1} = \dots = \mu_{m+k'} = 0$. Consequently, we obtain

$$A_{\mathfrak{q}(a)}(\lambda)|_{(\mathfrak{g}', K')} \simeq \bigoplus_{\mu} m(\mu) A_{\mathfrak{q}'(a')}(\mu),$$

where $m(\mu)$ is given by (8.2).

8.3. $\mathfrak{so}(2m+1, 2n+1) \downarrow \mathfrak{so}(2m+1, k) \oplus \mathfrak{so}(2n-k+1)$.

This is similar to the previous case. Let m, n and k be positive integers. Set $\mathfrak{g}_0 := \mathfrak{so}(2m+1, 2n+1)$ and $\mathfrak{g}'_0 := \mathfrak{so}(2m+1, k) \oplus \mathfrak{so}(2n-k+1)$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose $e_1, \dots, e_{m+n} \in \mathfrak{t}$ and $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ as in Setting 7.9. Denote by $e_1, \dots, e_{m+n} \in \mathfrak{t}$ the dual basis of $\epsilon_1, \dots, \epsilon_{m+n}$. We may assume that $e_1, \dots, e_{m+n} \in \mathfrak{g}'$. They form a basis of a Cartan subalgebra \mathfrak{t}' of \mathfrak{k}' . Put $k' = \lfloor \frac{k}{2} \rfloor$ and $l = n - \lceil \frac{k-1}{2} \rceil$. Then

$$\begin{aligned} \Delta(\mathfrak{k}', \mathfrak{t}') &= \{\pm \epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq m} \cup \{\pm \epsilon_{m+i} \pm \epsilon_{m+j}\}_{1 \leq i < j \leq k'} \\ &\quad \cup \{\pm \epsilon_{m+k'+i} \pm \epsilon_{m+k'+j}\}_{1 \leq i < j \leq l} \cup \{\pm \epsilon_i\}_{1 \leq i \leq m} \cup \{\pm \epsilon_{m+i}\}_{1 \leq i \leq k'}, \\ \Delta(\mathfrak{p}', \mathfrak{t}') &= \{\pm \epsilon_i \pm \epsilon_{m+j}\}_{1 \leq i \leq m, 1 \leq j \leq k'} \cup \{\pm \epsilon_i\}_{1 \leq i \leq m} \cup \{\pm \epsilon_{m+i}\}_{1 \leq i \leq k'} \cup \{0\} \end{aligned}$$

if k is odd and

$$\begin{aligned} \Delta(\mathfrak{k}', \mathfrak{t}') &= \{\pm \epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq m} \cup \{\pm \epsilon_{m+i} \pm \epsilon_{m+j}\}_{1 \leq i < j \leq k'} \\ &\quad \cup \{\pm \epsilon_{m+k'+i} \pm \epsilon_{m+k'+j}\}_{1 \leq i < j \leq l} \cup \{\pm \epsilon_i\}_{1 \leq i \leq m} \cup \{\pm \epsilon_{m+k'+i}\}_{1 \leq i \leq l}, \\ \Delta(\mathfrak{p}', \mathfrak{t}') &= \{\pm \epsilon_i \pm \epsilon_{m+j}\}_{1 \leq i \leq m, 1 \leq j \leq k'} \cup \{\pm \epsilon_{m+i}\}_{1 \leq i \leq k'} \end{aligned}$$

if k is even. For every \mathfrak{q} in Table 4 the representation $A_{\mathfrak{q}}(\lambda)$ is defined for $SO(2m+1, 2n+1)_0$ so we let $G_0 := SO(2m+1, 2n+1)_0$, $G'_0 := SO(2m+1, k)_0 \times SO(2n-k+1)$ and then $G = SO(2m+2n+2, \mathbb{C})$, $K = SO(2m+1, \mathbb{C}) \times SO(2n+1, \mathbb{C})$, $G' = SO(2m+k+1, \mathbb{C}) \times SO(2n-k+1, \mathbb{C})$, $K' = SO(2m+1, \mathbb{C}) \times SO(k, \mathbb{C}) \times SO(2n-k+1, \mathbb{C})$. Suppose that \mathfrak{q} is given by $a = a_1 e_1 + \cdots + a_m e_m$ with $a_1 \geq \cdots \geq a_m$. Put $a_0 = \min\{a_i : a_i > 0\}$, $p = \min\{i : a_i = a_0\} - 1$, and $q = \max\{i : a_i = a_0\} - p$. Then \mathfrak{l}_0 is a direct sum of $\mathfrak{so}(2m-2p-2q+1, 2n+1)$ and a compact Lie algebra. Consider the restriction $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$, where $\lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_{p+q} \epsilon_{p+q}$. Put $b := e_1 + \cdots + e_{p+q}$ and then $\mathfrak{q}'(b) = \mathfrak{q}(b) \cap \mathfrak{g}'$. Then $L(b) = GL(p+q, \mathbb{C}) \times SO(2m+2n-2p-2q+2, \mathbb{C})$ and $L'(b) = GL(p+q, \mathbb{C}) \times SO(2m+k-2p-2q+1, \mathbb{C}) \times SO(2n-k+1, \mathbb{C})$. For $\mu = \sum_{i=1}^{m+k'+l} \mu_i \epsilon_i$, we give $m(\mu)$ by the irreducible decomposition

$$\begin{aligned} (8.3) \quad F^{L(b)}(\lambda)|_{L'(b)} \otimes S(F^{L'(b)}(\epsilon_1 + \epsilon_{m+k'+1})) \otimes \mathbb{C}_{(2n-k+1)(\epsilon_1 + \cdots + \epsilon_{p+q})} \\ \simeq \bigoplus_{\mu} m(\mu) F^{L'(b)}(\mu). \end{aligned}$$

Put $r = \min\{q-1, 2n-k+1\}$ and

$$a' = a'_1 e_1 + \cdots + a'_{p+q} e_{p+q} + a'_{m+k'+1} e_{m+k'+1} + \cdots + a'_{m+k'+l} e_{m+k'+l}$$

with $a'_1 > \cdots > a'_{p+r+1} = \cdots = a'_{p+q} > 0$ and $a'_{m+k'+1} > \cdots > a'_{m+k'+l} > 0$. Then

$$\begin{aligned} L'(a') &\simeq GL(1, \mathbb{C})^{p+r+l} \times GL(q-r, \mathbb{C}) \times SO(2m+k-2p-2q+1, \mathbb{C}), \\ \mathfrak{l}'_0(a') &\simeq \mathfrak{u}(1)^{p+r+l} \oplus \mathfrak{u}(q-r) \oplus \mathfrak{so}(2m-2p-2q+1, k), \end{aligned}$$

and it turns out that $m(\mu) \neq 0$ only if $\mu_{p+r+1} = \cdots = \mu_{p+q} > \mu_{p+q+1} = \cdots = \mu_{m+k'} = 0$. Consequently, we obtain

$$A_{\mathfrak{q}(a)}(\lambda)|_{(\mathfrak{g}', K')} \simeq \bigoplus_{\mu} m(\mu) A_{\mathfrak{q}'(a')}(\mu),$$

where $m(\mu)$ is given by (8.3).

8.4. $\mathfrak{so}(2m, 2n) \downarrow \mathfrak{u}(m, n)$.

Let m and n be positive integers. Set $\mathfrak{g}_0 := \mathfrak{so}(2m, 2n)$ and $\mathfrak{g}'_0 := \mathfrak{u}(m, n)$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose $e_1, \dots, e_{m+n} \in \mathfrak{t}$ and $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ as in Setting 7.6. Also we choose a Cartan subalgebra \mathfrak{t}' of \mathfrak{k}' and $\epsilon'_1, \dots, \epsilon'_{m+n} \in (\mathfrak{t}')^*$ such that

$$\begin{aligned} \Delta(\mathfrak{k}', \mathfrak{t}') &= \{\pm(\epsilon'_i - \epsilon'_j)\}_{1 \leq i < j \leq m} \cup \{\pm(\epsilon'_{m+i} - \epsilon'_{m+j})\}_{1 \leq i < j \leq n}, \\ \Delta(\mathfrak{p}', \mathfrak{t}') &= \{\pm(\epsilon'_i - \epsilon'_{m+j})\}_{1 \leq i \leq m, 1 \leq j \leq n} \end{aligned}$$

and $(\epsilon_1, \dots, \epsilon_{m+n})$ is K -conjugate to $(\epsilon'_1, \dots, \epsilon'_{m+n})$. Denote by $e'_1, \dots, e'_{m+n} \in \mathfrak{t}'$ the dual basis of $\epsilon'_1, \dots, \epsilon'_{m+n}$. Suppose that \mathfrak{q} is given by $a = e_1$ so that $\mathfrak{l}_0 = \mathfrak{u}(1) \oplus \mathfrak{so}(2m-2, 2n)$. Then the representation $A_{\mathfrak{q}}(\lambda)$ is defined for $SO(2m, 2n)_0$ so we let $G_0 := SO(2m, 2n)_0$, $G'_0 := U(m, n)$, and then $G = SO(2m+2n, \mathbb{C})$, $K = SO(2m, \mathbb{C}) \times SO(2n, \mathbb{C})$, $G' = GL(m+n, \mathbb{C})$, $K' = GL(m, \mathbb{C}) \times GL(n, \mathbb{C})$. Consider the restriction $A_{\mathfrak{q}}(\lambda\epsilon_1)|_{(\mathfrak{g}', K')}$.

Suppose that $m > 1$. Let V be the natural representation of $SO(2m, \mathbb{C})$, namely, $V = F^{SO(2m, \mathbb{C})}(\epsilon_1)$. Similarly, let $W = F^{SO(2n, \mathbb{C})}(\epsilon_{m+1})$. Then V has a natural symmetric bilinear form and

$$K/(\overline{Q} \cap K) \simeq \{V_1 \subset V : \dim V_1 = 1, V_1 \text{ is isotropic}\}.$$

Write $v(\alpha) \in V \oplus W$ for an α -weight vector. Then \overline{Q} is the stabilizer group of $\mathbb{C}v(-\epsilon_1)$ in G . The subgroup $GL(m, \mathbb{C}) \subset SO(2m, \mathbb{C})$ is characterized as an m -dimensional isotropic subspace of V . Hence each K' -orbit corresponds to an m -dimensional isotropic subspace V_m of V . Therefore, $K/(\overline{Q} \cap K)$ decomposes into three K' -orbits Y_1, Y_2 , and Y_3 according to

$$Y_1 \leftrightarrow V_1 \not\subset V_m, V_m^\perp, \quad Y_2 \leftrightarrow V_1 \subset V_m, \quad Y_3 \leftrightarrow V_1 \subset V_m^\perp.$$

For a Cartan subalgebra \mathfrak{t} corresponding to the orbit Y_1 , we give a σ -action and a relation with \mathfrak{t}' as

$$\begin{aligned} \sigma e_i &= -e_{m-i+1}, \quad e_i - e_{m-i+1} = e'_i - e'_{m-i+1} \quad (1 \leq i \leq \lfloor \frac{m}{2} \rfloor), \text{ and} \\ \sigma e_{\frac{m}{2}} &= e_{\frac{m}{2}} \quad (\text{if } m \text{ is even}). \end{aligned}$$

Let $\overline{\mathfrak{q}}'_1$ be the parabolic subalgebra of \mathfrak{g}' given by $-(e'_1 - e'_m)$. The corresponding Levi subgroup is $L'_1 \simeq GL(1, \mathbb{C}) \times GL(m+n-2, \mathbb{C}) \times GL(1, \mathbb{C})$. and then $L'_{1,c} \simeq GL(1, \mathbb{C})^2$, $L'_{1,n} \simeq GL(m+n-2, \mathbb{C})$. We see that \overline{Q}'_1 is the stabilizer subgroup of the two one-dimensional subspaces $\mathbb{C}v(-\epsilon'_1)$ and $\mathbb{C}v(\epsilon'_m)$ in G' and $\overline{Q}_1 \cap G'$ is identified with the stabilizer subgroup of $\mathbb{C}(v(-\epsilon'_1) + v(\epsilon'_m))$ in G' . Therefore, $\overline{U}'_1 \subset \overline{Q}_1 \cap G' \subset \overline{Q}'_1$ and $\overline{Q}_1 \cap L'_1 \simeq GL(1, \mathbb{C}) \times GL(m+n-2, \mathbb{C})$. Here, the Lie algebra of $GL(1, \mathbb{C})$ -component of $\overline{Q}_1 \cap L'_1$ is spanned by the vector $e'_1 - e'_m$. We see that $\mathfrak{g} = \overline{\mathfrak{q}}_1 + \mathfrak{g}'$ and $2\rho(\mathfrak{u}) = (2m+2n-2)\epsilon_1$. The parameter $\lambda\epsilon_1$ is weakly fair for \mathfrak{q} if and only if $\lambda \geq -m-n+1$. Hence the term in the right side of (5.1) for $j=1$ is

$$\sum_d (-1)^{d+1} [(P_{\overline{\mathfrak{q}}_1 \cap \mathfrak{g}', C'_1}^{\mathfrak{g}', K'})_d (\mathbb{C}_{\lambda+2m+2n-2})].$$

Here, $\mathbb{C}_{\lambda+2m+2n-2}$ denotes the one-dimensional representation of $\overline{Q}_1 \cap L'$ such that $e'_1 - e'_m$ acts by $\lambda + 2m + 2n - 2$ and the $GL(m+n-2, \mathbb{C})$ -component acts trivially. Then Lemma 8.1 implies that

$$\begin{aligned} & \sum_d (-1)^{d+1} [(P_{\overline{\mathfrak{q}}_1 \cap \mathfrak{g}', C'_1}^{\mathfrak{g}', K'})_d (\mathbb{C}_{\lambda+2m+2n-2})] \\ &= \sum_d (-1)^{d+1} \sum_{k \in \mathbb{Z}} [(P_{\mathfrak{q}'(-e'_1+e'_m), L'(e'_1-e'_m) \cap K'}^{\mathfrak{g}', K'})_d (\mathbb{C}_{(\lambda+2m+2n+k-2)\epsilon'_1+k\epsilon'_m})] \end{aligned}$$

$$= \sum_d (-1)^{d+1} \sum_{k \in \mathbb{Z}} [\mathcal{L}_{\mathfrak{q}'(-e'_1+e'_m),d}^{\mathfrak{g}'}(\mathbb{C}_{(\lambda+m+n+k-1)\epsilon'_1+(m+n+k-1)\epsilon'_m})].$$

The orbit Y_2 is closed and $\bar{\mathfrak{q}}_2 \cap \mathfrak{g}'$ is the parabolic subalgebra of \mathfrak{g}' given by $-e'_1$. We see that $\mathfrak{g}/(\bar{\mathfrak{q}}_2 + \mathfrak{g}')$ is $(m+n-1)$ -dimensional and isomorphic to $F^{L'(e'_1)}(\epsilon'_1 + \epsilon'_2)$ as an $L'(e'_1)$ -module. Hence the term in the right side of (5.1) for $j = 2$ is

$$\sum_d (-1)^d \sum_{k \in \mathbb{Z}_{\geq 0}} [\mathcal{L}_{\mathfrak{q}'(-e'_1),d}^{\mathfrak{g}'}(F^{L'(e'_1)}((\lambda+m+n+k-2)\epsilon'_1 + k\epsilon'_2 + \epsilon'))],$$

where $\epsilon' := \epsilon'_1 + \dots + \epsilon'_{m+n}$. Similarly, the term in the right side of (5.1) for $j = 3$ is

$$\sum_d (-1)^d \sum_{k \in \mathbb{Z}_{\geq 0}} [\mathcal{L}_{\mathfrak{q}'(e'_m),d}^{\mathfrak{g}'}(F^{L'(-e'_m)}(-(\lambda+m+n+k-2)\epsilon'_m - k\epsilon'_{m+n} - \epsilon'))].$$

By Theorem 6.3,

$$\begin{aligned} & A_{\mathfrak{q}(e_1)}(\lambda\epsilon_1)|_{(\mathfrak{g}',K')} \\ & \simeq \bigoplus_{\substack{k < -\lambda - \frac{m+n-1}{2} \\ k \in \mathbb{Z}}} A_{\mathfrak{q}'(-2e'_m - e'_{m+n})}((k+1)\epsilon'_m + (\lambda+m+n+k-1)\epsilon'_{m+n} - \epsilon') \\ & \quad \oplus \bigoplus_{\substack{-\lambda - \frac{m+n-1}{2} \leq k \leq \frac{m+n-1}{2} \\ k \in \mathbb{Z}}} A_{\mathfrak{q}'(e'_1 - e'_m)}((\lambda+k)\epsilon'_1 + k\epsilon'_m) \\ & \quad \oplus \bigoplus_{\substack{\frac{m+n-1}{2} < k \\ k \in \mathbb{Z}}} A_{\mathfrak{q}'(2e'_1 + e'_{m+1})}((\lambda+k-1)\epsilon'_1 + (-m-n+k+1)\epsilon'_{m+1} + \epsilon'). \end{aligned}$$

We note that

$$\begin{aligned} \mathfrak{l}'_0(-2e'_m - e'_{m+n}) & \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(m-1, n-1), \\ \mathfrak{l}'_0(e'_1 - e'_m) & \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(m-2, n), \\ \mathfrak{l}'_0(2e'_1 + e'_{m+1}) & \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(m-1, n-1). \end{aligned}$$

If $m = 1$, then K' acts transitively on $K/(\bar{\mathfrak{Q}} \cap K)$. We set $e'_1 = e_1$ and then $\bar{\mathfrak{q}}' := \bar{\mathfrak{q}} \cap \mathfrak{g}'$ is given by $-e'_1$. We can see that $\mathfrak{g}/(\bar{\mathfrak{q}} + \mathfrak{g}') \simeq F^{L(e'_1)}(\epsilon'_1 + \epsilon'_2)$. Therefore,

$$A_{\mathfrak{q}(e_1)}(\lambda\epsilon_1)|_{(\mathfrak{g}',K')} \simeq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} A_{\mathfrak{q}'(2e'_1 + e'_2)}((\lambda+n+k-1)\epsilon'_1 + k\epsilon'_2 + \epsilon').$$

We note $\mathfrak{l}'_0(2e'_1 + e'_2) \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(n-1)$.

8.5. $\mathfrak{so}^*(2n) \downarrow \mathfrak{so}^*(2n-2) \oplus \mathfrak{so}(2)$.

Let $n \geq 2$. Set $\mathfrak{g}_0 := \mathfrak{so}^*(2n)$ and $\mathfrak{g}'_0 := \mathfrak{so}^*(2n-2) \oplus \mathfrak{so}(2)$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose $e_1, \dots, e_n \in \mathfrak{t}$ and $\epsilon_1, \dots, \epsilon_n \in \mathfrak{t}^*$ as in Setting 7.11. Also we choose a Cartan subalgebra \mathfrak{t}' of \mathfrak{k}' and $\epsilon'_1, \dots, \epsilon'_n \in (\mathfrak{t}')^*$ such that

$$\Delta(\mathfrak{k}', \mathfrak{t}') = \{\pm(\epsilon'_i - \epsilon'_j)\}_{1 \leq i < j \leq n-1}, \quad \Delta(\mathfrak{p}', \mathfrak{t}') = \{\pm(\epsilon'_i + \epsilon'_j)\}_{1 \leq i < j \leq n-1}$$

and $(\epsilon_1, \dots, \epsilon_n)$ is K -conjugate to $(\epsilon'_1, \dots, \epsilon'_n)$. Denote by $e'_1, \dots, e'_n \in \mathfrak{t}'$ the dual basis of $\epsilon'_1, \dots, \epsilon'_n$. Let $1 \leq k \leq n-1$ and put $a := e_1 + \dots + e_k - (e_{k+1} + \dots + e_n)$ and $\alpha := \epsilon_1 + \dots + \epsilon_k - (\epsilon_{k+1} + \dots + \epsilon_n)$. Suppose that \mathfrak{q} is given by a so that $\mathfrak{l}_0 = \mathfrak{u}(k, n-k)$. Then the representation $A_{\mathfrak{q}}(\lambda)$ is defined for the double covering group $\widetilde{SO^*(2n)}$ of $SO^*(2n)$ so we let $G_0 := \widetilde{SO^*(2n)}$ and $G'_0 := (\widetilde{SO^*(2n-2)} \times \widetilde{SO(2)})/\mathbb{Z}_2$. Consider the restriction $A_{\mathfrak{q}}(\lambda\alpha)|_{(\mathfrak{g}', K')}$. The parameter $\lambda\alpha$ is weakly fair for \mathfrak{q} if and only if $\lambda \geq \frac{-n+1}{2}$. Let V be the dual of the natural representation of $GL(n, \mathbb{C})$, namely, $V = F^{GL(n, \mathbb{C})}(-\epsilon_n)$. Then

$$K/(\overline{Q} \cap K) \simeq \{V_k \subset V : \dim V_k = k\}.$$

Each K' -orbit corresponds to a decomposition $V = V_{n-1} \oplus V_1$, where V_{n-1} is $(n-1)$ -dimensional and V_1 is one-dimensional. Therefore, $K/(\overline{Q} \cap K)$ decomposes into three K' -orbits Y_1 , Y_2 , and Y_3 according to

$$Y_1 \leftrightarrow V_1 \not\subset V_k \not\subset V_{n-1}, \quad Y_2 \leftrightarrow V_1 \subset V_k, \quad Y_3 \leftrightarrow V_k \subset V_{n-1}.$$

Put $b := (e'_1 + \dots + e'_{k-1}) - (e'_{k+1} + \dots + e'_{n-1})$ and $\beta := (\epsilon'_1 + \dots + \epsilon'_{k-1}) - (\epsilon'_{k+1} + \dots + \epsilon'_{n-1})$. Let $\overline{\mathfrak{q}}'_1$ be the parabolic subalgebra of \mathfrak{g}' given by $-b$. The corresponding Levi subgroup L'_1 is a covering group of $GL(n-2, \mathbb{C}) \times GL(1, \mathbb{C})^2$. Then $\overline{U}'_1 \subset \overline{Q}_1 \cap G' \subset \overline{Q}'_1$ and $\overline{Q}_1 \cap L'_1$ is a covering group of $GL(n-2, \mathbb{C}) \times GL(1, \mathbb{C})$. Here, the Lie algebra of $GL(1, \mathbb{C})$ -component of $\overline{Q}_1 \cap L'_1$ is spanned by the vector $e'_k + e'_n$. Hence the term in the right side of (5.1) for $j = 1$ is

$$\sum_d (-1)^{d+k(n-k)-1} [(P_{\overline{\mathfrak{q}}'_1 \cap \mathfrak{g}', C'_1}^{\mathfrak{g}', K'})_d(\mathbb{C}_{(\lambda+n-1)\alpha})].$$

Then Lemma 8.1 implies that

$$\begin{aligned} & \sum_d (-1)^{d+k(n-k)-1} [(P_{\overline{\mathfrak{q}}'_1 \cap \mathfrak{g}', C'_1}^{\mathfrak{g}', K'})_d(\mathbb{C}_{(\lambda+n-1)\alpha})] \\ &= \sum_d (-1)^{d+k(n-k)-1} \sum_{l \in \mathbb{Z}} [(P_{\mathfrak{q}'(-b), L'(b) \cap K'}^{\mathfrak{g}', K'})_d(\mathbb{C}_{(\lambda+n-1)\beta+l(\epsilon'_k - \epsilon'_n)})] \\ &= \sum_d (-1)^{d+k(n-k)-1} \sum_{l \in \mathbb{Z}} [\mathcal{L}_{\mathfrak{q}'(-b), d}^{\mathfrak{g}'}(\mathbb{C}_{\lambda\beta+l(\epsilon'_k - \epsilon'_n)})]. \end{aligned}$$

The orbit Y_2 is closed and $\overline{\mathfrak{q}}_2 \cap \mathfrak{g}'$ is the parabolic subalgebra of \mathfrak{g}' given by $-b + e'_k$. We see that $\mathfrak{g}/(\overline{\mathfrak{q}}_2 + \mathfrak{g}') \simeq F^{L'(b-e'_k)}(\epsilon'_1 + \epsilon'_n)$ as an $L'(b-e'_k)$ -module. Hence the term in the right side of (5.1) for $j = 2$ is

$$\sum_d (-1)^{d+k(n-k)} \sum_{l \in \mathbb{Z}_{\geq 0}} [\mathcal{L}_{\mathfrak{q}'(-b+e'_k), d}^{\mathfrak{g}'}(F^{L'(b-e'_k)}((\lambda+1)(\beta - \epsilon'_k) + l\epsilon'_1 + (\lambda+n+l-1)\epsilon'_n))].$$

Similarly, the term in the right side of (5.1) for $j = 3$ is

$$\sum_d (-1)^{d+k(n-k)} \sum_{l \in \mathbb{Z}_{\geq 0}} [\mathcal{L}_{\mathfrak{q}'(-b-e'_k), d}^{\mathfrak{g}'}(F^{L'(b+e'_k)}((\lambda+1)(\beta + \epsilon'_k) + l\epsilon'_1 - (\lambda+n+l-1)\epsilon'_n))].$$

By Theorem 6.3, if $2 \leq k \leq n-2$,

$$\begin{aligned}
& A_{\mathfrak{q}(a)}(\lambda\alpha)|_{(\mathfrak{g}', K')} \\
& \simeq \bigoplus_{\substack{l < -\lambda - \frac{n-1}{2} \\ l \in \mathbb{Z}}} A_{\mathfrak{q}'(e'_1+b-e'_k)}((- \lambda - n - l + 1)\epsilon'_1 + (\lambda + 1)(\beta - \epsilon'_k) - l\epsilon'_n) \\
& \quad \oplus \bigoplus_{\substack{-\lambda - \frac{n-1}{2} \leq l \leq \lambda + \frac{n-1}{2} \\ l \in \mathbb{Z}}} A_{\mathfrak{q}'(b)}(\lambda\beta + l\epsilon'_k - l\epsilon'_n) \\
& \quad \oplus \bigoplus_{\substack{\lambda + \frac{n-1}{2} < l \\ l \in \mathbb{Z}}} A_{\mathfrak{q}'(b+e'_k-e'_{n-1})}((\lambda + 1)(\beta + \epsilon'_k) + (\lambda + n - l - 1)\epsilon'_{n-1} - l\epsilon'_n).
\end{aligned}$$

If $k = n-1$,

$$\begin{aligned}
& A_{\mathfrak{q}(a)}(\lambda\alpha)|_{(\mathfrak{g}', K')} \\
& \simeq \bigoplus_{\substack{l < -\lambda - \frac{n-1}{2} \\ l \in \mathbb{Z}}} A_{\mathfrak{q}'(e'_1+b-e'_{n-1})}((- \lambda - n - l + 1)\epsilon'_1 + (\lambda + 1)(\beta - \epsilon'_{n-1}) - l\epsilon'_n) \\
& \quad \oplus \bigoplus_{\substack{-\lambda - \frac{n-1}{2} \leq l \leq \lambda \\ l \in \mathbb{Z}}} A_{\mathfrak{q}'(b)}(\lambda\beta + l\epsilon'_{n-1} - l\epsilon'_n).
\end{aligned}$$

We note that

$$\begin{aligned}
\mathfrak{l}'_0(e'_1 + b - e'_k) & \simeq \mathfrak{u}(1) \oplus \mathfrak{u}(k-2, n-k) \oplus \mathfrak{so}(2), \\
\mathfrak{l}'_0(b) & \simeq \mathfrak{u}(1) \oplus \mathfrak{u}(k-1, n-k-1) \oplus \mathfrak{so}(2), \\
\mathfrak{l}'_0(b + e'_k - e'_{n-1}) & \simeq \mathfrak{u}(1) \oplus \mathfrak{u}(k, n-k-2) \oplus \mathfrak{so}(2).
\end{aligned}$$

8.6. $\mathfrak{so}^*(2n) \downarrow \mathfrak{u}(n-1, 1)$.

Let $n \geq 2$. Set $\mathfrak{g}_0 := \mathfrak{so}^*(2n)$ and $\mathfrak{g}'_0 := \mathfrak{u}(n-1, 1)$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose $e_1, \dots, e_n \in \mathfrak{t}$ and $\epsilon_1, \dots, \epsilon_n \in \mathfrak{t}^*$ as in Setting 7.11. Also we choose a Cartan subalgebra \mathfrak{t}' of \mathfrak{k}' and $\epsilon'_1, \dots, \epsilon'_n \in (\mathfrak{t}')^*$ such that

$$\Delta(\mathfrak{k}', \mathfrak{t}') = \{\pm(\epsilon'_i - \epsilon'_j)\}_{1 \leq i < j \leq n-1}, \quad \Delta(\mathfrak{p}', \mathfrak{t}') = \{\pm(\epsilon'_i - \epsilon'_n)\}_{1 \leq i \leq n-1}$$

and $(\epsilon_1, \dots, \epsilon_n)$ is K -conjugate to $(\epsilon'_1, \dots, \epsilon'_{n-1}, -\epsilon'_n)$. Denote by $e'_1, \dots, e'_n \in \mathfrak{t}'$ the dual basis of $\epsilon'_1, \dots, \epsilon'_{n-1}$. Let $1 \leq k \leq n-1$ and put $a := e_1 + \dots + e_k - (e_{k+1} + \dots + e_n)$ and $\alpha := \epsilon_1 + \dots + \epsilon_k - (\epsilon_{k+1} + \dots + \epsilon_n)$. Suppose that \mathfrak{q} is given by a so that $\mathfrak{l}_0 = \mathfrak{u}(k, n-k)$. Consider the restriction $A_{\mathfrak{q}}(\lambda\alpha)|_{(\mathfrak{g}', K')}$. As in the previous case, $K/(\overline{Q} \cap K)$ decomposes into three K' -orbits Y_1, Y_2 , and Y_3 . Put $b := e'_1 + \dots + e'_{k-1}$, $c := e'_{k+1} + \dots + e'_{n-1}$, $\beta := \epsilon'_1 + \dots + \epsilon'_{k-1}$, and $\gamma := \epsilon'_{k+1} + \dots + \epsilon'_{n-1}$. Let $\overline{\mathfrak{q}}'_1$ be the parabolic subalgebra of \mathfrak{g}' given by $-(b-c)$. The corresponding Levi subgroup L'_1 is a covering group of $GL(k-1, \mathbb{C}) \times GL(n-k-1, \mathbb{C}) \times GL(2, \mathbb{C})$. Then $\overline{U}'_1 \subset \overline{Q}'_1 \cap G' \subset \overline{Q}'_1$ and $\overline{Q}'_1 \cap L'_1$ is a covering group of $GL(k-1, \mathbb{C}) \times GL(n-k-1, \mathbb{C}) \times SL(2, \mathbb{C})$.

We see that $\mathfrak{g}/(\bar{q}_1 + \mathfrak{g}') \simeq F^{GL(k-1, \mathbb{C})}(\epsilon'_1 + \epsilon'_2) \boxtimes F^{GL(n-k-1, \mathbb{C})}(-\epsilon'_{n-2} - \epsilon'_{n-1})$. Hence the term in the right side of (5.1) for $j = 1$ is

$$\sum_d (-1)^{d+k(n-k)-1} [(P_{\bar{q}_1 \cap \mathfrak{g}', C'_1}^{\mathfrak{g}', K'})_d(\mathbb{C}_{(\lambda+n-1)\alpha} \otimes S(\mathfrak{g}/(\bar{q}_1 + \mathfrak{g}')))].$$

Then Lemma 8.1 implies that

$$\begin{aligned} & \sum_d (-1)^{d+k(n-k)-1} [(P_{\bar{q}_1 \cap \mathfrak{g}', C'_1}^{\mathfrak{g}', K'})_d(\mathbb{C}_{(\lambda+n-1)\alpha} \otimes S(\mathfrak{g}/(\bar{q}_1 + \mathfrak{g}')))] \\ = & \sum_d (-1)^{d+k(n-k)-1} \sum_{l \in \mathbb{Z}} [\mathcal{L}_{\mathfrak{q}'(-b+c), d}^{\mathfrak{g}'}(\mathbb{C}_{(\lambda+k-2)\beta - (\lambda+n-k-2)\gamma + l(\epsilon'_k + \epsilon'_n)} \otimes S(\mathfrak{g}/(\bar{q}_1 + \mathfrak{g}')))]. \end{aligned}$$

Suppose that $k = 2p + 1$ and $n - k = 2q + 1$ are odd. Then

$$\begin{aligned} & A_{\mathfrak{q}(a)}(\lambda\alpha)|_{(\mathfrak{g}', K')} \\ \simeq & \bigoplus A_{\mathfrak{q}'(a')}((\lambda + k - 2)\beta + r_1(\epsilon'_1 + \epsilon'_2) + \cdots + r_p(\epsilon'_{k-2} + \epsilon'_{k-1}) + l(\epsilon'_k + \epsilon'_n) \\ & - (\lambda + n - k - 2)\gamma - s_1(\epsilon'_{k+1} + \epsilon'_{k+2}) - \cdots - s_q(\epsilon'_{n-2} + \epsilon'_{n-1})), \end{aligned}$$

where

$$a' := \sum_{i=1}^p (p - i + 1)(e'_{2i-1} + e'_{2i}) - \sum_{i=1}^q i(e'_{k+2i-1} + e'_{k+2i})$$

and the sum is taken over integers satisfying

$$r_1 \geq \cdots \geq r_p \geq 0, \quad 0 \leq s_1 \leq \cdots \leq s_q, \quad -(\lambda + n - k - 1) \leq l \leq \lambda + k - 1.$$

We note that $\mathfrak{l}'_0(a') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^{p+q}$.

Suppose that $k = 2p + 2$ is even and $n - k = 2q + 1$ is odd. Then

$$\begin{aligned} & A_{\mathfrak{q}(a)}(\lambda\alpha)|_{(\mathfrak{g}', K')} \\ \simeq & \bigoplus A_{\mathfrak{q}'(a')}((\lambda + k - 2)\beta + r_1(\epsilon'_1 + \epsilon'_2) + \cdots + r_p(\epsilon'_{k-3} + \epsilon'_{k-2}) + l(\epsilon'_k + \epsilon'_n) \\ & - (\lambda + n - k - 2)\gamma - s_1(\epsilon'_{k+1} + \epsilon'_{k+2}) - \cdots - s_q(\epsilon'_{n-2} + \epsilon'_{n-1})) \\ \oplus & \bigoplus A_{\mathfrak{q}'(a' + e'_k)}((\lambda + k - 2)(\beta - \epsilon'_{k-1}) + r_1(\epsilon'_1 + \epsilon'_2) + \cdots + r_p(\epsilon'_{k-3} + \epsilon'_{k-2}) \\ & + (l - 1)(\epsilon'_{k-1} + \epsilon'_k) - (\lambda + n - k - 2)\gamma - s_1(\epsilon'_{k+1} + \epsilon'_{k+2}) - \cdots - s_q(\epsilon'_{n-2} + \epsilon'_{n-1}) + (\lambda + k)\epsilon'_n), \end{aligned}$$

where

$$a' := \sum_{i=1}^p (p - i + 2)(e'_{2i-1} + e'_{2i}) + e'_{k-1} - \sum_{i=1}^q i(e'_{k+2i-1} + e'_{k+2i}),$$

and the sum is taken over integers satisfying

$$r_1 \geq \cdots \geq r_p \geq 0, \quad 0 \leq s_1 \leq \cdots \leq s_q, \quad -(\lambda + n - k - 1) \leq l \leq \lambda + k - 1$$

for the first term and

$$r_1 \geq \cdots \geq r_p \geq 0, \quad 0 \leq s_1 \leq \cdots \leq s_q, \quad \lambda + k \leq l \leq \lambda + k + r_p - 1$$

for the second term. We note that

$$\mathfrak{l}'_0(a') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^{p+q} \oplus \mathfrak{u}(1), \quad \mathfrak{l}'_0(a' + e'_k) \simeq \mathfrak{u}(2)^{p+q+1} \oplus \mathfrak{u}(1).$$

Suppose that $k = 2p + 2$ and $n - k = 2q + 2$ are even. Then

$$\begin{aligned}
& A_{q(a)}(\lambda\alpha)|_{(\mathfrak{g}', K')} \\
& \simeq \bigoplus A_{q'(a')}((\lambda + k - 2)\beta + r_1(\epsilon'_1 + \epsilon'_2) + \cdots + r_p(\epsilon'_{k-3} + \epsilon'_{k-2}) + l(\epsilon'_k + \epsilon'_n) \\
& \quad - (\lambda + n - k - 2)\gamma - s_1(\epsilon'_{k+2} + \epsilon'_{k+3}) - \cdots - s_q(\epsilon'_{n-2} + \epsilon'_{n-1})) \\
& \oplus \bigoplus A_{q'(a' + e'_k)}((\lambda + k - 2)(\beta - \epsilon'_{k-1}) + r_1(\epsilon'_1 + \epsilon'_2) + \cdots + r_p(\epsilon'_{k-3} + \epsilon'_{k-2}) \\
& \quad + (l - 1)(\epsilon'_{k-1} + \epsilon'_k) - (\lambda + n - k - 2)\gamma - s_1(\epsilon'_{k+2} + \epsilon'_{k+3}) - \cdots - s_q(\epsilon'_{n-2} + \epsilon'_{n-1}) + (\lambda + k)\epsilon'_n) \\
& \oplus \bigoplus A_{q'(a' - e'_k)}((\lambda + k - 2)\beta + r_1(\epsilon'_1 + \epsilon'_2) + \cdots + r_p(\epsilon'_{k-3} + \epsilon'_{k-2}) + (l + 1)(\epsilon'_k + \epsilon'_{k+1}) \\
& \quad - (\lambda + n - k - 2)(\gamma - \epsilon'_{k+1}) - s_1(\epsilon'_{k+2} + \epsilon'_{k+3}) - \cdots - s_q(\epsilon'_{n-2} + \epsilon'_{n-1}) - (\lambda + n - k)\epsilon'_n),
\end{aligned}$$

where

$$a' := \sum_{i=1}^p (p - i + 2)(e'_{2i-1} + e'_{2i}) + e'_{k-1} - e'_{k+1} - \sum_{i=1}^q (i + 1)(e'_{k+2i} + e'_{k+2i+1}),$$

and the sum is taken over integers satisfying

$$r_1 \geq \cdots \geq r_p \geq 0, \quad 0 \leq s_1 \leq \cdots \leq s_q, \quad -(\lambda + n - k - 1) \leq l \leq \lambda + k - 1$$

for the first term,

$$r_1 \geq \cdots \geq r_p \geq 0, \quad 0 \leq s_1 \leq \cdots \leq s_q, \quad \lambda + k \leq l \leq \lambda + k + r_p - 1$$

for the second term and

$$r_1 \geq \cdots \geq r_p \geq 0, \quad 0 \leq s_1 \leq \cdots \leq s_q, \quad -(\lambda + n - k + s_1 - 1) \leq l \leq -(\lambda + n - k - 1)$$

for the third term. We note that

$$\begin{aligned}
\mathfrak{l}'_0(a') & \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^{p+q} \oplus \mathfrak{u}(1)^2, \quad \mathfrak{l}'_0(a' + e'_k) \simeq \mathfrak{u}(2)^{p+q+1} \oplus \mathfrak{u}(1)^2, \\
\mathfrak{l}'_0(a' - e'_k) & \simeq \mathfrak{u}(2)^{p+q+1} \oplus \mathfrak{u}(1)^2.
\end{aligned}$$

These formulas can be verified by calculating K' -types of the both sides.

8.7. $\mathfrak{sp}(m, n) \downarrow \mathfrak{sp}(k, l) \oplus \mathfrak{sp}(m - k, n - l)$.

Let k, l, m , and n be integers such that $k, l, m - k$, and $n - l$ are positive. Set $\mathfrak{g}_0 := \mathfrak{sp}(m, n)$ and $\mathfrak{g}'_0 := \mathfrak{sp}(k, l) \oplus \mathfrak{sp}(m - k, n - l)$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose $e_1, \dots, e_{m+n} \in \mathfrak{t}$ and $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ as in Setting 7.10. We can choose \mathfrak{t} such that $\mathfrak{t} \subset \mathfrak{k}'$ and

$$\begin{aligned}
\Delta(\mathfrak{k}', \mathfrak{t}) &= \{\pm \epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq k} \cup \{\pm \epsilon_{k+i} \pm \epsilon_{k+j}\}_{1 \leq i < j \leq m-k} \cup \{\pm \epsilon_{m+i} \pm \epsilon_{m+j}\}_{1 \leq i < j \leq l} \\
&\quad \cup \{\pm \epsilon_{m+l+i} \pm \epsilon_{m+l+j}\}_{1 \leq i < j \leq n-l} \cup \{\pm 2\epsilon_i\}_{1 \leq i \leq k} \\
&\quad \cup \{\pm 2\epsilon_{k+i}\}_{1 \leq i \leq m-k} \cup \{\pm 2\epsilon_{m+i}\}_{1 \leq i \leq l} \cup \{\pm 2\epsilon_{m+l+i}\}_{1 \leq i \leq n-l}, \\
\Delta(\mathfrak{p}', \mathfrak{t}) &= \{\pm \epsilon_i \pm \epsilon_{m+j}\}_{1 \leq i \leq k, 1 \leq j \leq l} \cup \{\pm \epsilon_{k+i} \pm \epsilon_{m+l+j}\}_{1 \leq i \leq m-k, 1 \leq j \leq n-l}.
\end{aligned}$$

Then

$$\begin{aligned} A_{q(e_1)}(\lambda\epsilon_1)|_{(g',K')} &\simeq \bigoplus_{-k-l \leq r \leq \lambda+m+n-k-l} A_{q'(e_1+e_{k+1})}(r\epsilon_1 + (\lambda-r)\epsilon_{k+1}) \\ &\oplus \bigoplus_{\lambda+m+n-k-l < r} A_{q'(e_1+e_{m+l+1})}(r\epsilon_1 + (r-\lambda-2(m+n-k-l))\epsilon_{m+l+1}) \\ &\oplus \bigoplus_{r < -k-l} A_{q'(e_{m+1}+e_{k+1})}((-r-2(k+l))\epsilon_{m+1} + (\lambda-r)\epsilon_{k+1}). \end{aligned}$$

We note that

$$\begin{aligned} \mathfrak{l}_0(e_1) &\simeq \mathfrak{u}(1) \oplus \mathfrak{sp}(m-1, n), \\ \mathfrak{l}'_0(e_1 + e_{k+1}) &\simeq \mathfrak{u}(1)^2 \oplus \mathfrak{sp}(k-1, l) \oplus \mathfrak{sp}(m-k-1, n-l), \\ \mathfrak{l}'_0(e_1 + e_{m+l+1}) &\simeq \mathfrak{u}(1)^2 \oplus \mathfrak{sp}(k-1, l) \oplus \mathfrak{sp}(m-k, n-l-1), \\ \mathfrak{l}'_0(e_{m+1} + e_{k+1}) &\simeq \mathfrak{u}(1)^2 \oplus \mathfrak{sp}(k, l-1) \oplus \mathfrak{sp}(m-k-1, n-l). \end{aligned}$$

If $l = n$, then

$$\begin{aligned} A_{q(e_1)}(\lambda\epsilon_1)|_{(g',K')} &\simeq \bigoplus_{-k-n \leq r \leq \lambda} A_{q'(e_1+e_{k+1})}(r\epsilon_1 + (\lambda-r)\epsilon_{k+1}) \\ &\oplus \bigoplus_{r < -k-n} A_{q'(e_{m+1}+e_{k+1})}((-r-2(k+n))\epsilon_{m+1} + (\lambda-r)\epsilon_{k+1}). \end{aligned}$$

We note

$$\begin{aligned} \mathfrak{l}'_0(e_1 + e_{k+1}) &\simeq \mathfrak{u}(1)^2 \oplus \mathfrak{sp}(k-1, n) \oplus \mathfrak{sp}(m-k-1), \\ \mathfrak{l}'_0(e_{m+1} + e_{k+1}) &\simeq \mathfrak{u}(1)^2 \oplus \mathfrak{sp}(k, n-1) \oplus \mathfrak{sp}(m-k-1). \end{aligned}$$

We treat the remaining case, namely the case where $k = m$ and there exists $1 \leq p \leq m-1$ such that $a_p \geq a_{m+1} > 0$ and $a_{p+1} = a_{m+2} = 0$, in the next section.

8.8. $\mathfrak{sl}(2n, \mathbb{C}) \downarrow \mathfrak{sp}(n, \mathbb{C})$.

Let n be a positive integer. Set $\mathfrak{g}_0 := \mathfrak{gl}(2n, \mathbb{C})$ and $\mathfrak{g}'_0 := \mathfrak{sp}(n, \mathbb{C})$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose a standard basis $\epsilon_1, \dots, \epsilon_{2n} \in \mathfrak{t}^*$ so that

$$\Delta^+(\mathfrak{k}, \mathfrak{t}) = \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq 2n}, \quad \Delta(\mathfrak{p}, \mathfrak{t}) = \{\pm(\epsilon_i - \epsilon_j)\}_{1 \leq i < j \leq 2n} \cup \{0\}$$

and let $e_1, \dots, e_{2n} \in \mathfrak{t}$ be the dual basis of $\epsilon_1, \dots, \epsilon_{2n}$. We take a Cartan subalgebra \mathfrak{h} of \mathfrak{g} such that $\mathfrak{t} \subset \mathfrak{h}$. Choose a standard basis $\zeta_1, \dots, \zeta_{2n} \in (\mathfrak{h} \cap \mathfrak{p})^*$ so that

$$\Delta(\mathfrak{g}, \mathfrak{h}) = \{\pm((\epsilon_i + \zeta_i) - (\epsilon_j + \zeta_j))\}_{1 \leq i < j \leq 2n} \cup \{\pm((\epsilon_i - \zeta_i) - (\epsilon_j - \zeta_j))\}_{1 \leq i < j \leq 2n}$$

and let $f_1, \dots, f_{2n} \in \mathfrak{h} \cap \mathfrak{p}$ be the dual basis of $\zeta_1, \dots, \zeta_{2n}$. We let σ act on \mathfrak{h} by $\sigma e_{2i-1} = -e_{2i}$, $\sigma f_{2i-1} = -f_{2i}$ and put $e'_i := e_{2i-1} - e_{2i}$, $f'_i := f_{2i-1} - f_{2i}$. Then e'_i and f'_i form a basis of \mathfrak{h}^σ and $\mathfrak{h}' := \mathfrak{h}^\sigma$ is a Cartan subalgebra of \mathfrak{g}' . Let $\epsilon'_1, \dots, \epsilon'_n, \zeta'_1, \dots, \zeta'_n$ be the dual basis of $e'_1, \dots, e'_n, f'_1, \dots, f'_n$. Suppose that \mathfrak{q} is given by e_1 . Then G' acts transitively on G/\overline{Q} . Therefore

$$A_{q(e_1)}(\lambda\epsilon_1 + \mu\zeta_1)|_{(g',K')} \simeq A_{q'(e'_1)}(\lambda\epsilon'_1 + \mu\zeta'_1)$$

for $\lambda \in \mathbb{Z}$ and $\mu \in \sqrt{-1}\mathbb{R}$. We note that $\mathfrak{l}_0(e_1) \simeq \mathbb{C} \oplus \mathfrak{gl}(2n-1, \mathbb{C})$ and $\mathfrak{l}'_0(e'_1) \simeq \mathbb{C} \oplus \mathfrak{sp}(n-1, \mathbb{C})$.

8.9. $\mathfrak{sl}(2n, \mathbb{C}) \downarrow \mathfrak{su}^*(2n)$.

Set $\mathfrak{g}_0 := \mathfrak{gl}(2n, \mathbb{C})$ and $\mathfrak{g}'_0 := \mathfrak{su}^*(2n) \oplus \mathbb{R}$. We use the same notation as in the previous case for $\mathfrak{t}, \mathfrak{h}, \epsilon_1, \dots, \epsilon_{2n}, \zeta_1, \dots, \zeta_{2n} \in \mathfrak{h}^*$, and $e_1, \dots, e_{2n}, f_1, \dots, f_{2n} \in \mathfrak{h}$. We let σ act on \mathfrak{h} by $\sigma e_{2i-1} = -e_{2i}, \sigma f_{2i-1} = f_{2i}$ and put $e'_i := e_{2i-1} - e_{2i}, f'_i = f_{2i-1} + f_{2i}$. Then e'_i and f'_i form a basis of \mathfrak{h}^σ and $\mathfrak{h}' := \mathfrak{h}^\sigma$ is a Cartan subalgebra of \mathfrak{g}' . Let $\epsilon'_1, \dots, \epsilon'_n, \zeta'_1, \dots, \zeta'_n$ be the dual basis of $e'_1, \dots, e'_n, f'_1, \dots, f'_n$. Suppose that \mathfrak{q} is given by e_1 . Then K' acts transitively on $K/(\overline{Q} \cap K)$, $\overline{\mathfrak{q}} \cap \mathfrak{g}'$ is given by e'_1 , and $\mathfrak{g}/(\overline{\mathfrak{q}} + \mathfrak{g}')$ is isomorphic to $\mathbb{C}_{\epsilon'_1}$. We have

$$A_{\mathfrak{q}(e_1)}(\lambda \epsilon_1 + \mu \zeta_1)|_{(\mathfrak{g}', K')} \simeq \bigoplus_{m \in \mathbb{Z}_{\geq 0}} A_{\mathfrak{q}'(e'_1)}((\lambda + m + 1)\epsilon'_1 + \mu \zeta'_1)$$

for $\lambda \in \mathbb{Z}$ and $\mu \in \sqrt{-1}\mathbb{R}$. We note that $\mathfrak{l}_0(e_1) \simeq \mathbb{C} \oplus \mathfrak{gl}(2n-1, \mathbb{C})$ and $\mathfrak{l}'_0(e'_1) \simeq \mathbb{C} \oplus \mathfrak{su}^*(2n-2) \oplus \mathbb{R}$.

8.10. $\mathfrak{f}_{4(-20)} \downarrow \mathfrak{so}(8, 1)$.

Set $\mathfrak{g}_0 := \mathfrak{f}_{4(-20)}$ and $\mathfrak{g}'_0 := \mathfrak{so}(8, 1)$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose $e_1, \dots, e_4 \in \mathfrak{t}$ and $\epsilon_1, \dots, \epsilon_4 \in \mathfrak{t}^*$ as in Setting 7.16. We can choose \mathfrak{t} such that $\mathfrak{t} \subset \mathfrak{t}'$ and

$$\begin{aligned} \Delta(\mathfrak{t}', \mathfrak{t}) &= \{\pm \epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq 4}, \\ \Delta(\mathfrak{p}', \mathfrak{t}) &= \left\{ \frac{1}{2} \left(\sum_{i=1}^4 (-1)^{k(i)} \epsilon_i \right) : \sum_{i=1}^4 k(i) \text{ is even} \right\}. \end{aligned}$$

Put

$$\begin{aligned} e'_1 &:= \frac{1}{2}(e_1 + e_2 + e_3 + e_4), & e'_2 &:= \frac{1}{2}(e_1 + e_2 - e_3 - e_4), \\ e'_3 &:= \frac{1}{2}(e_1 - e_2 + e_3 - e_4), & e'_4 &:= \frac{1}{2}(-e_1 + e_2 + e_3 - e_4) \end{aligned}$$

and let $\epsilon'_1, \dots, \epsilon'_4$ be the dual basis of e'_1, \dots, e'_4 so that they agree with Setting 7.7.

Let \mathfrak{q} be given by $e_1 + e_2$ so that $\mathfrak{l}_0 = \mathfrak{sp}(2, 1) \oplus \mathfrak{u}(1)$. Then $2\rho(\mathfrak{u}) = 8(\epsilon_1 + \epsilon_2)$ and hence the parameter $\lambda(\epsilon_1 + \epsilon_2)$ is weakly fair for $\mathfrak{q}(e_1 + e_2)$ if and only if $\lambda \geq -4$. We can see that $A_{\mathfrak{q}(e_1 + e_2)}(\lambda(\epsilon_1 + \epsilon_2)) = 0$ for $\lambda < -2$. We have

$$\begin{aligned} & A_{\mathfrak{q}(e_1 + e_2)}(\lambda(\epsilon_1 + \epsilon_2))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{0 \leq n \leq \lambda + 2} A_{\mathfrak{q}'(2e_1 + e_2)}(\lambda \epsilon_1 + n \epsilon_2) \\ & \quad \oplus \bigoplus_{n \in \mathbb{Z}_{>0}} A_{\mathfrak{q}'(2e_1 + 2e_2 + e_3 - e_4)} \left(\left(\lambda + \frac{n}{2} + 1 \right) (\epsilon_1 + \epsilon_2) + \left(\frac{n}{2} - 1 \right) (\epsilon_3 - \epsilon_4) \right) \\ & \simeq \bigoplus_{0 \leq n \leq \lambda + 2} A_{\mathfrak{q}'(2e'_1 + 2e'_2 + e'_3 - e'_4)} \left(\frac{\lambda + n}{2} (\epsilon'_1 + \epsilon'_2) + \frac{\lambda - n}{2} (\epsilon'_3 - \epsilon'_4) \right) \end{aligned}$$

$$\oplus \bigoplus_{n \in \mathbb{Z}_{>0}} A_{q'(2e'_1+2e'_2+e'_3+e'_4)} \left(\left(\lambda + \frac{n}{2} + 1 \right) (\epsilon'_1 + \epsilon'_2) + \left(\frac{n}{2} - 1 \right) (\epsilon'_3 + \epsilon'_4) \right).$$

We note that

$$\begin{aligned} \mathfrak{l}'_0(2e'_1 + 2e'_2 + e'_3 - e'_4) &\simeq \mathfrak{u}(2)^2 (\simeq \mathfrak{u}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{sp}(1)), \\ \mathfrak{l}'_0(2e'_1 + 2e'_2 + e'_3 + e'_4) &\simeq \mathfrak{u}(2)^2 (\simeq \mathfrak{u}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{sp}(1)). \end{aligned}$$

Let \mathfrak{q} be given by $e_1 + e_2 + e_3 + e_4$ so that $\mathfrak{l}_0 = \mathfrak{so}(6, 1) \oplus \mathfrak{u}(1)$. Then $2\rho(\mathfrak{u}) = \frac{11}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)$ and hence the parameter $\lambda(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)$, $\lambda \in \frac{\mathbb{Z}}{2}$ is weakly fair for $\mathfrak{q}(e_1 + e_2 + e_3 + e_4)$ if and only if $\lambda \geq -\frac{11}{4}$. We can see that $A_{\mathfrak{q}(e_1+e_2+e_3+e_4)}(\lambda(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)) = 0$ for $\lambda < -\frac{3}{2}$. We have

$$\begin{aligned} &A_{\mathfrak{q}(e_1+e_2+e_3+e_4)}(\lambda(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4))|_{(\mathfrak{g}', K')} \\ &\simeq \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{0 \leq n \leq 2\lambda+3} A_{q'(e_1+e_2+e_3)} \left(\left(\lambda + \frac{m+1}{2} \right) (\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4) + n\epsilon_4 \right) \\ &\simeq \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{0 \leq n \leq 2\lambda+3} A_{q'(2e'_1+e'_2+e'_3+e'_4)} \left(\left(\lambda + \frac{m+n+1}{2} \right) \epsilon'_1 + \left(\lambda + \frac{m-n+1}{2} \right) (\epsilon'_2 + \epsilon'_3 + \epsilon'_4) \right). \end{aligned}$$

We note $\mathfrak{l}'_0(2e'_1 + e'_2 + e'_3 + e'_4) \simeq \mathfrak{u}(3) \oplus \mathfrak{u}(1)$.

8.11. $\mathfrak{e}_{6(2)} \downarrow \mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$.

Set $\mathfrak{g}_0 := \mathfrak{e}_{6(2)}$ and $\mathfrak{g}'_0 := \mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose $e_1, \dots, e_7 \in \mathfrak{t}$ and $\epsilon_1, \dots, \epsilon_7 \in \mathfrak{t}^*$ as in Setting 7.17. We can choose \mathfrak{t} such that $\mathfrak{t} \subset \mathfrak{k}'$ and

$$\begin{aligned} \Delta(\mathfrak{k}', \mathfrak{t}) &= \{\pm(\epsilon_i - \epsilon_j)\}_{1 \leq i < j \leq 5}, \\ \Delta(\mathfrak{p}', \mathfrak{t}) &= \left\{ \frac{1}{2} \left(\sum_{i=1}^5 (-1)^{k(i)} \epsilon_i + \epsilon_6 \right) + \epsilon_7 : k(i) \in \{0, 1\}, \sum_{i=1}^5 k(i) = 3 \right\} \\ &\cup \left\{ \frac{1}{2} \left(\sum_{i=1}^5 (-1)^{k(i)} \epsilon_i - \epsilon_6 \right) - \epsilon_7 : k(i) \in \{0, 1\}, \sum_{i=1}^5 k(i) = 2 \right\}. \end{aligned}$$

We put

$$e'_i := e_i + \frac{1}{2}e_6 + \frac{1}{4}e_7 \quad (1 \leq i \leq 5), \quad e'_6 := e_6 - \frac{1}{2}e_7.$$

Then e'_1, \dots, e'_6 form a basis of \mathfrak{t} . Denote by $\epsilon'_1, \dots, \epsilon'_6 \in \mathfrak{t}^*$ the dual basis of e'_1, \dots, e'_6 so that $\epsilon'_1, \dots, \epsilon'_5$ satisfy Setting 7.11 for \mathfrak{k}' and \mathfrak{p}' . We note that

$$\begin{aligned} \epsilon_i - \epsilon_j &= \epsilon'_i - \epsilon'_j \quad (1 \leq i, j \leq 5), \\ \epsilon_1 - \epsilon_6 &= \frac{1}{2}(\epsilon'_1 - \epsilon'_2 - \epsilon'_3 - \epsilon'_4 - \epsilon'_5) - \epsilon'_6, \\ \epsilon_7 &= \frac{1}{4}(\epsilon'_1 + \epsilon'_2 + \epsilon'_3 + \epsilon'_4 + \epsilon'_5) - \frac{1}{2}\epsilon'_6. \end{aligned}$$

Let \mathfrak{q} be given by $e_1 + e_2$ so that $\mathfrak{l}_0 = \mathfrak{so}(6, 4) \oplus \mathfrak{u}(1)$. Then $2\rho(\mathfrak{u}) = 4(2\epsilon_1 + 2\epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6)$ and hence the parameter $\lambda(2\epsilon_1 + 2\epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6)$

with $\lambda \in \frac{\mathbb{Z}}{3}$ is weakly fair for $\mathfrak{q}(e_1 + e_2)$ if and only if $\lambda \geq -2$. We have

$$\begin{aligned} & A_{\mathfrak{q}(e_1+e_2)}(\lambda(2\epsilon_1 + 2\epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{0 \leq n \leq 3\lambda+5} A_{\mathfrak{q}'(2e'_1+e'_2-e'_3-e'_4-e'_5)}\left(\left(\frac{3\lambda+m+n-1}{2}\right)\epsilon'_1\right. \\ & \quad \left.+ \left(\frac{3\lambda+m-n+3}{2}\right)(\epsilon'_2 - \epsilon'_3 - \epsilon'_4 - \epsilon'_5) + (-\lambda+m+n-1)\epsilon'_6\right) \\ & \oplus \bigoplus_{m, n \in \mathbb{Z}_{\geq 0}} A_{\mathfrak{q}'(e'_1+e'_2-e'_4-e'_5)}\left(\left(\frac{3\lambda+m+n+2}{2}\right)(\epsilon'_1 + \epsilon'_2 - \epsilon'_4 - \epsilon'_5)\right. \\ & \quad \left.+ \left(\frac{-3\lambda-m+n-6}{2}\right)\epsilon'_3 + (-\lambda+m-n-2)\epsilon'_6\right). \end{aligned}$$

We note that

$$\begin{aligned} \mathfrak{l}'_0(2e'_1 + e'_2 - e'_3 - e'_4 - e'_5) & \simeq \mathfrak{u}(1) \oplus \mathfrak{u}(1, 3) \oplus \mathfrak{so}(2), \\ \mathfrak{l}'_0(e'_1 + e'_2 - e'_4 - e'_5) & \simeq \mathfrak{u}(1) \oplus \mathfrak{u}(2, 2) \oplus \mathfrak{so}(2). \end{aligned}$$

8.12. $\mathfrak{e}_{6(-14)} \downarrow \mathfrak{so}(2, 8) \oplus \mathfrak{so}(2)$.

Set $\mathfrak{g}_0 := \mathfrak{e}_{6(-14)}$ and $\mathfrak{g}'_0 := \mathfrak{so}(2, 8) \oplus \mathfrak{so}(2)$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose $e_1, \dots, e_7 \in \mathfrak{t}$ and $\epsilon_1, \dots, \epsilon_7 \in \mathfrak{t}^*$ as in Setting 7.18. We can choose \mathfrak{t} such that $\mathfrak{t} \subset \mathfrak{k}'$ and

$$\Delta(\mathfrak{k}', \mathfrak{t}) = \{\pm \epsilon_i \pm \epsilon_j\}_{2 \leq i < j \leq 5},$$

$$\Delta(\mathfrak{p}', \mathfrak{t}) = \left\{ \frac{1}{2} \left(\sum_{i=1}^6 (-1)^{k(i)} \epsilon_i \right) : k(i) \in \{0, 1\}, k(1) + \dots + k(6) \text{ odd}, k(1) = k(6) \right\}.$$

Put

$$\begin{aligned} e'_1 &:= \frac{1}{2}(e_1 + 3e_6), & e'_2 &:= \frac{1}{2}(e_2 + e_3 + e_4 - e_5), & e'_3 &:= \frac{1}{2}(e_2 + e_3 - e_4 + e_5), \\ e'_4 &:= \frac{1}{2}(e_2 - e_3 + e_4 + e_5), & e'_5 &:= \frac{1}{2}(-e_2 + e_3 + e_4 + e_5), & e'_6 &:= e_1 - e_6. \end{aligned}$$

Then e'_1, \dots, e'_6 form a basis of \mathfrak{t} . Denote by $\epsilon'_1, \dots, \epsilon'_6 \in \mathfrak{t}^*$ the dual basis of e'_1, \dots, e'_6 so that $\epsilon'_1, \dots, \epsilon'_5$ satisfy Setting 7.6 for \mathfrak{k}' and \mathfrak{p}' .

Let \mathfrak{q} be given by $e_1 + \dots + e_6$ so that $\mathfrak{l}_0 = \mathfrak{so}^*(10) \oplus \mathfrak{u}(1)$. Then $2\rho(\mathfrak{u}) = 6(\epsilon_1 + \dots + \epsilon_5) + 2\epsilon_6$ and hence the parameter $\lambda(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \frac{1}{3}\epsilon_6)$ with $\lambda \in \frac{\mathbb{Z}}{2}$ is weakly fair for $\mathfrak{q}(e_1 + \dots + e_6)$ if and only if $\lambda \geq -3$. We have

$$\begin{aligned} & A_{\mathfrak{q}(e_1+\dots+e_6)}\left(\lambda\left(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \frac{1}{3}\epsilon_6\right)\right)|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{0 \leq n \leq 2\lambda+5} A_{\mathfrak{q}'(e'_1+2e'_2+e'_3+e'_4+e'_5)}\left(\left(\lambda + \frac{m+n-1}{2}\right)\epsilon'_2\right. \\ & \quad \left.+ \left(\lambda + \frac{m-n+3}{2}\right)(\epsilon'_1 + \epsilon'_3 + \epsilon'_4 + \epsilon'_5) + \left(\frac{2}{3}\lambda - m - n + 1\right)\epsilon'_6\right) \\ & \oplus \bigoplus_{m, n \in \mathbb{Z}_{\geq 0}} A_{\mathfrak{q}'(e'_2+e'_3+e'_4+e'_5)}\left(\left(\lambda + \frac{m+n}{2} + 1\right)(\epsilon'_2 + \epsilon'_3 + \epsilon'_4 + \epsilon'_5)\right) \end{aligned}$$

$$+ \left(\lambda + \frac{m-n}{2} + 3 \right) \epsilon'_1 + \left(\frac{2}{3} \lambda - m + n + 2 \right) \epsilon'_6.$$

We note that

$$\begin{aligned} \ell'_0(e'_1 + 2e'_2 + e'_3 + e'_4 + e'_5) &\simeq \mathfrak{u}(1) \oplus \mathfrak{u}(1, 3) \oplus \mathfrak{so}(2), \\ \ell'_0(e'_2 + e'_3 + e'_4 + e'_5) &\simeq \mathfrak{u}(1) \oplus \mathfrak{u}(4) \oplus \mathfrak{so}(2). \end{aligned}$$

8.13. $\mathfrak{e}_{6(-14)} \downarrow \mathfrak{f}_{4(-20)}$.

Set $\mathfrak{g}_0 := \mathfrak{e}_{6(-14)}$ and $\mathfrak{g}'_0 := \mathfrak{f}_{4(-20)}$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose $e_1, \dots, e_7 \in \mathfrak{t}$ and $\epsilon_1, \dots, \epsilon_7 \in \mathfrak{t}^*$ as in Setting 7.18. We let σ act on \mathfrak{t} by $\sigma e_i = e_i$ for $i = 1, 2, 3, 4$ and $\sigma e_i = -e_i$ for $i = 5, 6$. Then e_1, \dots, e_4 form a basis of \mathfrak{t}^σ and $\mathfrak{t}' := \mathfrak{t}^\sigma$ is a Cartan subalgebra of \mathfrak{g}' . Denote by $\epsilon_1, \dots, \epsilon_4 \in (\mathfrak{t}')^*$ the dual basis of e_1, \dots, e_4 so that they satisfy Setting 7.16 for \mathfrak{t}' and \mathfrak{p}' .

Let \mathfrak{q} be given by $e_1 + e_2$ so that $\mathfrak{l}_0 = \mathfrak{su}(2, 4) \oplus \mathfrak{u}(1)$. Then $2\rho(\mathfrak{u}) = 11(\epsilon_1 + \epsilon_2)$ and hence the parameter $\lambda(\epsilon_1 + \epsilon_2)$ with $\lambda \in \mathbb{Z}$ is weakly fair for $\mathfrak{q}(e_1 + e_2)$ if and only if $\lambda \geq -\frac{11}{2}$. We have

$$\begin{aligned} &A_{\mathfrak{q}(e_1+e_2)}(\lambda(\epsilon_1 + \epsilon_2))|_{(\mathfrak{g}', K')} \\ &\simeq \bigoplus_{0 \leq n \leq \lambda+3} A_{\mathfrak{q}'(2e_1+e_2)}(\lambda\epsilon_1 + n\epsilon_2) \\ &\oplus \bigoplus_{0, \lambda+4 \leq n} A_{\mathfrak{q}'(2e_1+2e_2+e_3+e_4)}\left(\left(\frac{\lambda+n}{2}\right)(\epsilon_1 + \epsilon_2) + \left(\frac{-\lambda+n}{2} - 3\right)(\epsilon_3 + \epsilon_4)\right). \end{aligned}$$

We note that

$$\ell'_0(2e_1 + e_2) \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{sp}(2), \quad \ell'_0(2e_1 + 2e_2 + e_3 + e_4) \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{sp}(1, 1).$$

Let \mathfrak{q} be given by $e_1 + e_2 + e_3 + e_4$ so that $\mathfrak{l}_0 = \mathfrak{so}(6, 2) \oplus \mathfrak{u}(1)^2$. Then $2\rho(\mathfrak{u}) = 8(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)$ and hence the parameter $\lambda_1(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) + \lambda_2(\epsilon_5 + \epsilon_6)$ with $\lambda_1 \in \frac{\mathbb{Z}}{2}$, $\lambda_2 \in \lambda_1 + \mathbb{Z}$ is weakly fair for $\mathfrak{q}(e_1 + e_2 + e_3 + e_4)$ if and only if $\lambda_1 + 4 \geq |\lambda_2|$. We can see that $A_{\mathfrak{q}(e_1+e_2+e_3+e_4)}(\lambda_1(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) + \lambda_2(\epsilon_5 + \epsilon_6)) = 0$ if $\lambda_1 + 4 = |\lambda_2|$. We have

$$\begin{aligned} &A_{\mathfrak{q}(e_1+e_2+e_3+e_4)}(\lambda_1(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) + \lambda_2(\epsilon_5 + \epsilon_6))|_{(\mathfrak{g}', K')} \\ &\simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{k=0}^{2\lambda_1+6} A_{\mathfrak{q}'(2e_1+2e_2+2e_3+e_4)} m(n, k) \left(\left(\lambda_1 + \frac{n}{2} + 1 \right) (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) - (k-2)\epsilon_4 \right), \end{aligned}$$

where

$$m(n, k) = \max\{0, \min\{\lambda_1 - |\lambda_2| + 4, k + 1, -k + 2\lambda_1 + 7, n + 1, n - k + \lambda_1 - |\lambda_2| + 4, n - 2k + 2\lambda_1 + 7\}\}.$$

We note that $\ell'_0(2e_1 + 2e_2 + 2e_3 + e_4) \simeq \mathfrak{u}(3) \oplus \mathfrak{u}(1)$.

Let \mathfrak{q} be given by $e_1 + e_2 + e_3 + e_4 + e_5 + e_6$ so that $\mathfrak{l}_0 = \mathfrak{so}^*(10) \oplus \mathfrak{u}(1)$. Then $2\rho(\mathfrak{u}) = 6(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) + 2\epsilon_6$ and hence the parameter

$\lambda(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \frac{1}{3}\epsilon_6)$ with $\lambda \in \frac{\mathbb{Z}}{2}$ is weakly fair for $\mathfrak{q}(e_1 + e_2 + e_3 + e_4)$ if and only if $\lambda \geq -3$. We have

$$\begin{aligned} & A_{\mathfrak{q}(e_1+e_2+e_3+e_4+e_5+e_6)} \left(\lambda \left(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \frac{1}{3}\epsilon_6 \right) \right) \Big|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{\mathfrak{q}'(e_1+e_2+e_3+e_4)} \left(\left(\lambda + \frac{n+1}{2} \right) (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) \right). \end{aligned}$$

We note that $\mathfrak{l}'_0(e_1 + e_2 + e_3 + e_4) \simeq \mathfrak{so}(6, 1) \oplus \mathfrak{u}(1)$.

8.14. $\mathfrak{e}_{7(-5)} \downarrow \mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$.

Set $\mathfrak{g}_0 := \mathfrak{e}_{7(-5)}$ and $\mathfrak{g}'_0 := \mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$. For a Cartan subalgebra \mathfrak{t} of \mathfrak{k} we choose $e_1, \dots, e_7 \in \mathfrak{t}$ and $\epsilon_1, \dots, \epsilon_7 \in \mathfrak{t}^*$ as in Setting 7.19. We can choose \mathfrak{t} such that $\mathfrak{t} \subset \mathfrak{k}'$ and

$$\begin{aligned} \Delta(\mathfrak{k}', \mathfrak{t}) &= \{\pm \epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq 5}, \\ \Delta(\mathfrak{p}', \mathfrak{t}) &= \left\{ \frac{1}{2} \left(\sum_{i=1}^6 (-1)^{k(i)} \epsilon_i \right) + (-1)^{k(7)} e_7 \right. \\ &\quad \left. : k(i) \in \{0, 1\}, k(1) + \dots + k(6) \text{ odd}, k(6) = k(7) \right\}. \end{aligned}$$

Put

$$e'_i := e_i \ (1 \leq i \leq 5), \quad e'_6 := \frac{1}{3}(e_6 + e_7), \quad e'_7 := e_6 - \frac{1}{2}e_7.$$

Denote by $\epsilon'_1, \dots, \epsilon'_7 \in \mathfrak{t}^*$ the dual basis of e'_1, \dots, e'_7 so that $\epsilon'_1, \dots, \epsilon'_6$ satisfy Setting 7.18 for \mathfrak{k}' and \mathfrak{p}' .

Let \mathfrak{q} be given by $e_1 + \dots + e_6$ so that $\mathfrak{l}_0 = \mathfrak{e}_{6(2)} \oplus \mathfrak{u}(1)$. Then $2\rho(\mathfrak{u}) = 9(\epsilon_1 + \dots + \epsilon_6)$ and hence the parameter $\lambda(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6)$ with $\lambda \in \frac{\mathbb{Z}}{2}$ is weakly fair for $\mathfrak{q}(e_1 + \dots + e_6)$ if and only if $\lambda \geq -\frac{9}{2}$. We can see that $A_{\mathfrak{q}(e_1+\dots+e_6)}(\lambda(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6)) = 0$ if $\lambda < -4$. We have

$$\begin{aligned} & A_{\mathfrak{q}(e_1+\dots+e_6)}(\lambda(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6)) \Big|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{k, l \in \mathbb{Z}_{\geq 0} \atop -\lambda-4 \leq n \leq \lambda+4 \atop \lambda-n \in \mathbb{Z}} \bigoplus A_{\mathfrak{q}'(e'_1+e'_2+e'_3+e'_4)} \left(\left(\lambda + \frac{k+l}{2} + 1 \right) (\epsilon'_1 + \epsilon'_2 + \epsilon'_3 + \epsilon'_4) \right. \\ & \quad \left. + \left(n + \frac{k-l}{2} \right) \left(\epsilon'_5 + \frac{1}{3}\epsilon'_6 \right) + (n-k+l)\epsilon'_7 \right). \end{aligned}$$

We note that $\mathfrak{l}'_0(e'_1 + e'_2 + e'_3 + e'_4) \simeq \mathfrak{so}(6, 2) \oplus \mathfrak{u}(1)^2 \oplus \mathfrak{so}(2)$.

9. BRANCHING LAWS FOR DISCRETE SERIES TYPE

Let $(\mathfrak{g}_0, \mathfrak{g}'_0)$ be an irreducible symmetric pair. In this section we suppose that $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q})$ satisfies discrete decomposability condition and that $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q})$ is of discrete series type. Then there exists a Borel subalgebra \mathfrak{b} of \mathfrak{g} such that $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{b})$ also satisfies the discrete decomposability condition.

If \mathfrak{g}_0 is compact, $A_{\mathfrak{q}}(\lambda)$ is an irreducible finite-dimensional representation of G_0 . Then the branching law is obtained by Kostant's branching formula ([Kna, Theorem 9.20]). If $\sigma = \theta$, the branching law is known as generalized

Blattner's formula (Fact 2.13). So we consider the case where \mathfrak{g}_0 is non-compact. Take a Cartan subalgebra \mathfrak{t} of \mathfrak{k} (and hence of \mathfrak{g}) such that $\mathfrak{t} \subset \mathfrak{b}$. We write \mathfrak{n} for the nilradical of \mathfrak{b} . We choose $\Delta^+(\mathfrak{g}, \mathfrak{t}) := \Delta(\mathfrak{b}, \mathfrak{t})$ as a positive system and let Π be the corresponding simple roots in $\Delta^+(\mathfrak{g}, \mathfrak{t})$. We define a grading on \mathfrak{g} by

$$\begin{aligned} \mathfrak{t} &\subset \mathfrak{g}(0), \\ \mathfrak{g}_{\pm\alpha} &\subset \mathfrak{g}(0) \text{ if } \alpha \in \Pi \text{ and } \mathfrak{g}_{\alpha} \subset \mathfrak{k}, \\ \mathfrak{g}_{\pm\alpha} &\subset \mathfrak{g}(\pm 1) \text{ if } \alpha \in \Pi \text{ and } \mathfrak{g}_{\alpha} \subset \mathfrak{p} \end{aligned}$$

and makes \mathfrak{g} a graded Lie algebra. By our classification (Theorem 7.22 and Table 3), it turns out that $\mathfrak{g}(n) = 0$ if $|n| > 2$. It follows that

$$\begin{aligned} \mathfrak{k} &= \mathfrak{g}(-2) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(2), & \mathfrak{p} &= \mathfrak{g}(-1) \oplus \mathfrak{g}(1), \\ \mathfrak{b} &\supset \mathfrak{g}(1) \oplus \mathfrak{g}(2), & \bar{\mathfrak{b}} &\supset \mathfrak{g}(-2) \oplus \mathfrak{g}(-1). \end{aligned}$$

We note that $\mathfrak{g}(\pm 2) = 0$ if and only if \mathfrak{b} is holomorphic. We write $\mathfrak{g}_{\leq 0} := \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0)$. Similar notation will be used for Lie subalgebras of \mathfrak{g} with respect to induced grading.

We assume $\mathfrak{l} \subset \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$. Then $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{l}(0)$ is a Levi subalgebra of \mathfrak{l} and \mathfrak{l}_0 is of Hermitian type. Let W_L and $W_{L \cap K}$ be the Weyl group of $\Delta(\mathfrak{l}, \mathfrak{t})$ and $\Delta(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t})$, respectively. Let $W_L^{L \cap K} (\subset W_L)$ be the set of minimal representatives of $W_{L \cap K} \backslash W_L$. Then it follows that $w\Delta(\mathfrak{l} \cap \mathfrak{n}, \mathfrak{t}) \supset \Delta(\mathfrak{l}(0) \cap \mathfrak{n}, \mathfrak{t})$ for $w \in W_L^{L \cap K}$.

There exists the following exact sequence of $(\mathfrak{l}, L \cap K)$ -modules (analog of the BGG resolution, see [Hum, §9.16])

$$0 \rightarrow M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_k \rightarrow \cdots \rightarrow M_{\dim \mathfrak{l}(1)} \rightarrow \mathbb{C} \rightarrow 0,$$

where

$$M_k := \bigoplus_{\substack{w \in W_L^{L \cap K} \\ l(w)=k}} U(\mathfrak{l}) \otimes_{U(\mathfrak{l}_{\leq 0})} F^{L \cap K}(w\rho(\mathfrak{l} \cap \mathfrak{n}) - \rho(\mathfrak{l} \cap \mathfrak{n}) + 2\rho(\mathfrak{l}(1))).$$

Put $s(L) := \dim(\mathfrak{l}(0) \cap \mathfrak{n})$. Since

$$\begin{aligned} &\mathcal{L}_{\mathfrak{l} \cap \bar{\mathfrak{b}}, d}^{\mathfrak{l}}(\mathbb{C}_{w\rho(\mathfrak{l} \cap \mathfrak{n}) - \rho(\mathfrak{l} \cap \mathfrak{n})}) \\ &\simeq \begin{cases} U(\mathfrak{l}) \otimes_{U(\mathfrak{l}_{\leq 0})} F^{L \cap K}(w\rho(\mathfrak{l} \cap \mathfrak{n}) - \rho(\mathfrak{l} \cap \mathfrak{n}) + 2\rho(\mathfrak{l}(1))) & \text{if } d = s(L), \\ 0 & \text{if } d \neq s(L), \end{cases} \end{aligned}$$

we get

$$\begin{aligned} [\mathbb{C}] &= \sum_{w \in W_L^{L \cap K}} (-1)^{l(w) + \dim \mathfrak{l}(1)} [\mathcal{L}_{\mathfrak{l} \cap \bar{\mathfrak{b}}, s(L)}^{\mathfrak{l}}(\mathbb{C}_{w\rho(\mathfrak{l} \cap \mathfrak{n}) - \rho(\mathfrak{l} \cap \mathfrak{n})})] \\ &= \sum_{w \in W_L^{L \cap K}} (-1)^{l(w) + \dim \mathfrak{l}(1)} \sum_d (-1)^{d + s(L)} [\mathcal{L}_{\mathfrak{l} \cap \bar{\mathfrak{b}}, d}^{\mathfrak{l}}(\mathbb{C}_{w\rho(\mathfrak{l} \cap \mathfrak{n}) - \rho(\mathfrak{l} \cap \mathfrak{n})})]. \end{aligned}$$

Suppose that $\lambda \in \mathfrak{t}^*$ is weakly fair for \mathfrak{q} . Then

$$\begin{aligned}
& [A_{\mathfrak{q}}(\lambda)] \\
&= \sum_d (-1)^{d+\dim(\mathfrak{u} \cap \mathfrak{k})} [\mathcal{L}_{\bar{\mathfrak{q}},d}^{\mathfrak{g}}(\mathbb{C}_{\lambda})] \\
&= \sum_{w \in W_L^{L \cap K}} (-1)^{l(w)+\dim \mathfrak{l}(1)} \sum_d \sum_{d'} (-1)^{d+d'+\dim(\mathfrak{u} \cap \mathfrak{k})+s(L)} [\mathcal{L}_{\bar{\mathfrak{q}},d'}^{\mathfrak{g}}(\mathcal{L}_{\mathfrak{l} \cap \bar{\mathfrak{b}},d}^{\mathfrak{l}}(\mathbb{C}_{\lambda+w\rho(\mathfrak{l} \cap \mathfrak{n})-\rho(\mathfrak{l} \cap \mathfrak{n})}))] \\
&= \sum_{w \in W_L^{L \cap K}} (-1)^{l(w)+\dim \mathfrak{l}(1)} \sum_d (-1)^{d+\dim(\mathfrak{n} \cap \mathfrak{k})} [\mathcal{L}_{\bar{\mathfrak{b}},d}^{\mathfrak{g}}(\mathbb{C}_{\lambda+w\rho(\mathfrak{l} \cap \mathfrak{n})-\rho(\mathfrak{l} \cap \mathfrak{n})})] \\
&= \sum_{w \in W_L^{L \cap K}} (-1)^{l(w)+\dim \mathfrak{l}(1)} \sum_{d,d'} (-1)^{d+d'+\dim(\mathfrak{n} \cap \mathfrak{k})} [\mathcal{L}_{\mathfrak{g}_{\leq 0},d}^{\mathfrak{g}}(\mathcal{L}_{\mathfrak{g}(0) \cap \bar{\mathfrak{b}},d'}^{\mathfrak{g}(0)}(\mathbb{C}_{\lambda+w\rho(\mathfrak{l} \cap \mathfrak{n})-\rho(\mathfrak{l} \cap \mathfrak{n})}))].
\end{aligned}$$

Let $G(0)$ be the connected subgroup of K with Lie algebra $\mathfrak{g}(0)$. We have the following description

$$\sum_{w \in W_L^{L \cap K}} (-1)^{l(w)+\dim \mathfrak{l}(1)+d+\dim(\mathfrak{n} \cap \mathfrak{g}(0))} [\mathcal{L}_{\mathfrak{g}(0) \cap \bar{\mathfrak{b}},d}^{\mathfrak{g}(0)}(\mathbb{C}_{\lambda+w\rho(\mathfrak{l} \cap \mathfrak{n})-\rho(\mathfrak{l} \cap \mathfrak{n})})] = \sum_{\mu} m(\mu) [F^{G(0)}(\mu)],$$

where $m(\mu) \in \mathbb{Z}$ and the sum is taken over integral dominant weight $\mu \in \mathfrak{t}^*$. Notice that $m(\mu) = 0$ but finitely many $\mu \in \mathfrak{t}^*$.

We may assume that \mathfrak{t} is σ -stable, $\mathfrak{t}' := \mathfrak{t}^{\sigma}$ is a Cartan subalgebra of \mathfrak{g}' , and $\mathfrak{b}' := \mathfrak{b} \cap \mathfrak{g}'$ is a Borel subalgebra of \mathfrak{g}' . Write \mathfrak{n}' for the nilradical of \mathfrak{b}' . Choose $\Delta^+(\mathfrak{g}', \mathfrak{t}') = \Delta(\mathfrak{b}', \mathfrak{t}')$ as a positive system and let Π' be the corresponding simple roots. We can observe from the classification that

$$\mathfrak{g}' = \bigoplus_{i=-2}^2 \mathfrak{g}'(i), \text{ where } \mathfrak{g}'(i) := \mathfrak{g}' \cap \mathfrak{g}(i), \quad \mathfrak{g}'(\pm 2) = \mathfrak{g}(\pm 2)$$

and \mathfrak{g}' is generated by $\mathfrak{g}'(0) \oplus \mathfrak{g}'(\pm 1)$ if \mathfrak{g}'_0 is non-compact. By Corollary 5.7, we obtain

$$\begin{aligned}
& \sum_d (-1)^d [\mathcal{L}_{\mathfrak{g}_{\leq 0},d}^{\mathfrak{g}}(F^{G(0)}(\mu))|_{(\mathfrak{g}',K')}] \\
&= \sum_d (-1)^d [\mathcal{L}_{\mathfrak{g}'_{\leq 0},d}^{\mathfrak{g}'}(F^{G(0)}(\mu)|_{G'(0)} \otimes \bigwedge^{\text{top}} (\mathfrak{g}'/(\mathfrak{g}_{\leq 0} + \mathfrak{g}')) \otimes S(\mathfrak{g}'/(\mathfrak{g}_{\leq 0} + \mathfrak{g}')))] \\
&= \sum_d (-1)^d [\mathcal{L}_{\mathfrak{g}'_{\leq 0},d}^{\mathfrak{g}'}(F^{G(0)}(\mu)|_{G'(0)} \otimes \bigwedge^{\text{top}} (\mathfrak{g}^{-\sigma}(1)) \otimes S(\mathfrak{g}^{-\sigma}(1)))].
\end{aligned}$$

Let

$$\sum_{\mu} m(\mu) [F^{G(0)}(\mu)|_{G'(0)} \otimes \bigwedge^{\text{top}} (\mathfrak{g}^{-\sigma}(1)) \otimes S(\mathfrak{g}^{-\sigma}(1))] = \sum_{\nu} n(\nu) [F^{G'(0)}(\nu)]$$

be a decomposition into irreducible $G'(0)$ -modules. Here $n(\nu) \in \mathbb{Z}$ and the sum is taken over integral dominant weight $\nu \in (\mathfrak{t}')^*$. Then we conclude the following.

Theorem 9.1. *Suppose that $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q})$ satisfies discrete decomposability condition and is of discrete series type. We use the notation above and assume that $\mathfrak{l} \subset \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$. Then*

$$[A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}] = \sum_d (-1)^{d+\dim \mathfrak{g}'(2)} \sum_{\nu} n(\nu) [\mathcal{L}_{\mathfrak{g}'_{\leq 0}, d}^{\mathfrak{g}'}(F^{G'(0)}(\nu))]$$

for weakly fair λ .

If \mathbb{C}_{ν} is good, then $\mathcal{L}_{\mathfrak{g}'_{\leq 0}, \dim \mathfrak{g}'(2)}^{\mathfrak{g}'}(F^{G'(0)}(\nu)) \simeq A_{\mathfrak{b}'}(\nu)$ and $\mathcal{L}_{\mathfrak{g}'_{\leq 0}, d}^{\mathfrak{g}'}(F^{G'(0)}(\nu)) = 0$ for $d \neq \dim \mathfrak{g}'(2)$. We therefore have

Theorem 9.2. *In the above assumptions and notation, we assume moreover that λ is unitary and that $\nu + \rho(\mathfrak{n}')$ is dominant whenever $n(\nu) \neq 0$. Then*

$$A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')} \simeq \bigoplus_{\nu} n(\nu) A_{\mathfrak{b}'}(\nu).$$

In particular, $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$ decomposes into (limit of) discrete series representations.

We now consider when the assumption of Theorem 9.2 holds, namely $\langle \nu + \rho(\mathfrak{n}'), \alpha \rangle \geq 0$ if $\alpha \in \Pi'$ and if $n(\nu) \neq 0$. We assume that the bilinear form on $(\mathfrak{t}')^*$ is induced from that on \mathfrak{t}^* . Since $\langle \nu, \alpha \rangle \geq 0$ always holds for $\alpha \in \Pi' \cap \Delta^+(\mathfrak{g}'(0), \mathfrak{t}')$ and since $\mathfrak{g}'(2) = [\mathfrak{g}'(1), \mathfrak{g}'(1)]$, it suffices to see whether $\langle \nu + \rho(\mathfrak{n}'), \beta \rangle \geq 0$ for $\beta \in \Pi' \cap \Delta(\mathfrak{g}'(1), \mathfrak{t}')$. If $\beta \in \Pi' \cap \Delta(\mathfrak{g}'(1), \mathfrak{t}')$, we can find $\beta_1 \in \Delta(\mathfrak{g}(1), \mathfrak{t})$ such that $\beta_1|_{\mathfrak{t}'} = \beta$. For $\gamma \in \mathfrak{t}^*$, Define

$$c(\lambda; \gamma) := \min\{\langle vw(\lambda + \rho(\mathfrak{n})), \gamma \rangle : v \in W_{G'(0)}, w \in W_L^{L \cap K}\}.$$

Lemma 9.3. *In the setting above, let $\gamma_1, \gamma_2 \in \mathfrak{t}^*$ be $\mathfrak{g}(0)$ -antidominant weights such that γ_1 and β_1 are in the same $W_{G(0)}$ -orbit and γ_2 and $\sigma\beta_1$ are in the same $W_{G(0)}$ -orbit. Then*

$$\langle \nu, \beta \rangle \geq \langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \beta \rangle + \frac{1}{2}(c(\lambda; \gamma_1) + c(\lambda; \gamma_2) - \langle \rho(\mathfrak{n}), \gamma_1 + \gamma_2 \rangle)$$

if $n(\nu) \neq 0$.

Proof. Suppose that $n(\nu) \neq 0$. Then there exists a dominant integral weight $\mu \in \mathfrak{t}^*$ such that $m(\mu) \neq 0$ and that $F^{G'(0)}(\nu)$ occurs in the irreducible decomposition of $F^{G(0)}(\mu)|_{G'(0)} \otimes \bigwedge^{\text{top}}(\mathfrak{g}^{-\sigma}(1)) \otimes S(\mathfrak{g}^{-\sigma}(1))$. Hence ν can be written as $\nu = \nu_1 + \nu_2 + 2\rho(\mathfrak{g}^{-\sigma}(1))$, where $F^{G'(0)}(\nu_1)$ occurs in $F^{G(0)}(\mu)|_{G'(0)}$ and ν_2 is a sum of weights in $S(\mathfrak{g}^{-\sigma}(1))$. Since $[\mathfrak{g}'(1), \mathfrak{g}^{-\sigma}(1)] = \mathfrak{g}^{-\sigma}(2) = 0$, we have $\langle \alpha, \beta \rangle \geq 0$ for $\alpha \in \Delta(\mathfrak{g}^{-\sigma}(1), \mathfrak{t}')$, which implies $\langle \nu_2, \beta \rangle \geq 0$. Therefore, the lemma is reduced to the inequality

$$\langle \nu_1, \beta \rangle \geq \frac{1}{2}(c(\lambda; \gamma_1) + c(\lambda; \gamma_2) - \langle \rho(\mathfrak{n}), \gamma_1 + \gamma_2 \rangle).$$

By $m(\mu) \neq 0$, there exist $w \in W_L^{L \cap K}$ and $v \in W_{G(0)}$ such that

$$\begin{aligned} \mu + \rho(\mathfrak{g}(0) \cap \mathfrak{n}) &= v(\lambda + w\rho(\mathfrak{l} \cap \mathfrak{n}) - \rho(\mathfrak{l} \cap \mathfrak{n}) + \rho(\mathfrak{g}(0) \cap \mathfrak{n})) \\ &= v(\lambda + w\rho(\mathfrak{n}) - \rho(\mathfrak{n}) + \rho(\mathfrak{g}(0) \cap \mathfrak{n})) \\ &= v(\lambda + w\rho(\mathfrak{n}) - \rho(\mathfrak{g}_{>0})) = vw(\lambda + \rho(\mathfrak{n})) - \rho(\mathfrak{g}_{>0}). \end{aligned}$$

Hence $\mu = vw(\lambda + \rho(\mathfrak{n})) - \rho(\mathfrak{n})$. From this, we see that if $\kappa \in \mathfrak{t}^*$ is a weight in $F^{G(0)}(\mu)$, then

$$\langle \kappa, \beta_1 \rangle \geq \langle \mu, \gamma_1 \rangle \geq c(\lambda; \gamma_1) - \langle \rho(\mathfrak{n}), \gamma_1 \rangle.$$

Similarly $\langle \kappa, \sigma\beta_1 \rangle \geq c(\lambda; \gamma_2) - \langle \rho(\mathfrak{n}), \gamma_2 \rangle$. Since $F^{G'(0)}(\nu_1)$ occurs in $F^{G(0)}(\mu)|_{G'(0)}$ there exists a weight $\kappa \in \mathfrak{t}^*$ such that $\kappa|_{\mathfrak{t}'} = \nu_1$. Therefore,

$$\langle \nu_1, \beta \rangle = \frac{1}{2} \langle \kappa, \beta_1 + \sigma\beta_1 \rangle \geq \frac{1}{2} (c(\lambda; \gamma_1) + c(\lambda; \gamma_2) - \langle \rho(\mathfrak{n}), \gamma_1 + \gamma_2 \rangle),$$

which gives the lemma. \square

Remark 9.4. If $\gamma_1 = \gamma_2$, or equivalently β_1 and $\sigma\beta_1$ lie in the same $W_{G(0)}$ -orbit, then the lemma becomes

$$(9.1) \quad \langle \nu, \beta \rangle \geq \langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \beta \rangle + c(\lambda; \gamma_1) - \langle \rho(\mathfrak{n}), \gamma_1 \rangle.$$

We give a lower bound of $c(\lambda; \gamma)$, which is useful for our purpose.

Lemma 9.5. *Suppose that λ is weakly fair and $\gamma \in \Delta(\mathfrak{g}(1), \mathfrak{t})$. Then*

$$c(\lambda; \gamma) \geq \min\{\langle \rho(\mathfrak{l} \cap \mathfrak{n}), \gamma' \rangle : \gamma' \in \Delta(\mathfrak{q}, \mathfrak{t}), |\gamma| = |\gamma'|\}.$$

If moreover $\mathfrak{l} \subset \mathfrak{g}(0)$, then

$$c(\lambda; \gamma) \geq \min\{\langle \rho(\mathfrak{l} \cap \mathfrak{n}), \gamma' \rangle : \gamma' \in \Delta(\mathfrak{g}(1), \mathfrak{t}), |\gamma| = |\gamma'|\}.$$

Proof. We have

$$\begin{aligned} c(\lambda; \gamma) &= \min\{\langle vw(\lambda + \rho(\mathfrak{n})), \gamma \rangle : v \in W_{G'(0)}, w \in W_L^{L \cap K}\} \\ &= \min\{\langle \lambda + \rho(\mathfrak{n}), w^{-1}v^{-1}\gamma \rangle : v \in W_{G'(0)}, w \in W_L^{L \cap K}\}. \end{aligned}$$

Since $v^{-1}\gamma \in \Delta(\mathfrak{g}(1), \mathfrak{t}) \subset \Delta(\mathfrak{q}, \mathfrak{t})$, we see that $w^{-1}v^{-1}\gamma \in \Delta(\mathfrak{q}, \mathfrak{t})$. If we put $\gamma' := w^{-1}v^{-1}\gamma$, the first assertion follows from $\langle \gamma', \lambda + \rho(\mathfrak{n}) \rangle \geq \langle \gamma', \rho(\mathfrak{l} \cap \mathfrak{n}) \rangle$. If $\mathfrak{l} \subset \mathfrak{g}(0)$, then $\gamma' \in \Delta(\mathfrak{g}(1), \mathfrak{t})$, which implies the second assertion. \square

There is a one-to-one correspondence between the parabolic subalgebras containing \mathfrak{b} and the subsets of Π . We write $\mathfrak{q}(S)$ for the parabolic subalgebra corresponding to $S \subset \Pi$. Also write $\mathfrak{l}(S)$ and $\mathfrak{u}(S)$ for the corresponding Levi subalgebra and nilradical. A subset $S \subset \Pi$ is regarded as a graph via Dynkin diagram. Let S_n be the union of the connected components of S that intersect $\Pi \cap \Delta(\mathfrak{g}(1), \mathfrak{t})$. Then it follows that $\mathfrak{l}(S_n) + (\mathfrak{l}(S) \cap \mathfrak{k}) = \mathfrak{l}(S)$ and we get an isomorphism $A_{\mathfrak{q}(S)}(\lambda) \simeq A_{\mathfrak{q}(S_n)}(\lambda)$. However, λ may not be weakly fair for $\mathfrak{q}(S_n)$ even if it is for $\mathfrak{q}(S)$.

Lemma 9.6. *Let $\mathfrak{q} = \mathfrak{q}(S)$ with $S \subset \Pi$. If there exists $\alpha \in \Pi$ such that $\langle \lambda + \rho(\mathfrak{n}), \alpha \rangle \leq 0$ and α is not adjacent to S , then $A_{\mathfrak{q}}(\lambda) = 0$.*

Proof. Since α is not adjacent to S , we have $l(\{\alpha\} \cup S_n) = l(\{\alpha\}) + l(S_n)$ and

$$\mathcal{L}_{\bar{q}(S_n) \cap l(\{\alpha\} \cup S_n), 1}^{l(\{\alpha\} \cup S_n)}(\mathbb{C}_\lambda)|_{l(\{\alpha\})} \simeq \mathcal{L}_{\bar{q}(S_n) \cap l(\{\alpha\}), 1}^{l(\{\alpha\})}(\mathbb{C}_\lambda) = 0.$$

Hence $A_{q(S_n)}(\lambda) = 0$. \square

We thus assume that the parameter λ does not admit a root $\alpha \in \Delta(\mathfrak{g}(0), \mathfrak{t})$ as in the lemma above.

In the following calculations, we normalize the bilinear form on \mathfrak{t}^* . If \mathfrak{g} is simply laced, we suppose $\langle \alpha, \alpha \rangle = 2$ for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$. If \mathfrak{g} is not simply laced, we suppose $\langle \alpha, \alpha \rangle = 2$ for a long root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$ so that $\langle \alpha, \alpha \rangle = 1$ for a short root α (notice that type G_2 does not appear in our list).

9.1. $\mathfrak{f}_{4(4)} \downarrow \mathfrak{sp}(2, 1) \oplus \mathfrak{su}(2)$.

Let $\mathfrak{g}_0 := \mathfrak{f}_{4(4)}$ and $\mathfrak{g}'_0 := \mathfrak{sp}(1, 2) \oplus \mathfrak{su}(2)$. We write the Dynkin diagram of \mathfrak{g} and label the simple roots:

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \bullet & \circ & \circ & \circ \\ & \longrightarrow & \longrightarrow & \longrightarrow \end{array}$$

Here, the painted root α_1 corresponds to a root in $\mathfrak{g}(1)$. Other roots α_2, α_3 and α_4 are roots in $\mathfrak{g}(0)$.

In this case, we can take $\mathfrak{t} \subset \mathfrak{t}'$ and hence $\mathfrak{t}' = \mathfrak{t}$. We let $\sigma = 1$ on \mathfrak{g}_{α_i} if $i = 2, 3$ and $\sigma = -1$ on \mathfrak{g}_{α_i} if $i = 1, 4$. Then $\Pi' = \{\alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3 + 2\alpha_4\}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}) = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$. Put $\beta := \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. Then β is a short root and $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \beta \rangle = 5$. We see that $\gamma := \alpha_1 + \alpha_2 + \alpha_3$ is $\mathfrak{g}(0)$ -antidominant and lies in the same $W_{G(0)}$ -orbit with β . By a direct calculation, we can verify that

$$\{\langle \rho(l \cap \mathfrak{n}), \gamma' \rangle : \gamma' \in \Delta(\mathfrak{q}, \mathfrak{t}), \gamma' \text{ is short} \} \geq -\frac{5}{2}$$

if $\mathfrak{q} \neq \mathfrak{g}$. Hence Lemma 9.5 gives $c(\lambda; \gamma) \geq -\frac{5}{2}$. As a consequence of (9.1),

$$\langle \nu + \rho(\mathfrak{n}'), \beta \rangle = \langle \nu, \beta \rangle + \frac{1}{2} \geq 5 + c(\lambda; \gamma) - \langle \rho(\mathfrak{n}), \gamma \rangle + \frac{1}{2} \geq 5 - \frac{5}{2} - \frac{3}{2} + \frac{1}{2} \geq 0$$

if $n(\nu) \neq 0$ and the assumption in Theorem 9.2 is fulfilled.

9.2. $\mathfrak{f}_{4(4)} \downarrow \mathfrak{so}(5, 4)$.

Let $\mathfrak{g}_0 := \mathfrak{f}_{4(4)}$ and $\mathfrak{g}'_0 := \mathfrak{so}(4, 5)$. We label the simple roots as in the previous case. We let $\sigma = 1$ on \mathfrak{g}_{α_i} if $i = 1, 2, 3$ and $\sigma = -1$ on \mathfrak{g}_{α_4} . Then $\Pi' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4\}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}) = \{\alpha_1\}$. Put $\beta := \alpha_1$. Then β is a $\mathfrak{g}(0)$ -antidominant long root and $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \beta \rangle = 2$. The assumption of Theorem 9.2 is fulfilled if $c(\lambda; \beta) \geq -2$.

Suppose that $\mathfrak{q} = \mathfrak{q}(S)$ with $S \subset \Pi$. Put $S^c := \Pi \setminus S$. If $S^c \neq \{\alpha_1\}$ or $\{\alpha_4\}$, then it turns out that $c(\lambda; \beta) \geq -2$. To analyze the remaining cases, we choose a basis $\epsilon_i \in \mathfrak{t}^*$ such that

$$\alpha_1 = -\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4, \quad \alpha_2 = 2\epsilon_3, \quad \alpha_3 = \epsilon_2 - \epsilon_3, \quad \alpha_4 = \epsilon_1 - \epsilon_2,$$

and let $e_i \in \mathfrak{t}$ be the dual basis. Then they satisfy Setting 7.15. Let

$$e'_1 := \frac{1}{2}(e_1 + e_4), \quad e'_2 := \frac{1}{2}(-e_1 + e_4), \quad e'_3 := \frac{1}{2}(e_2 + e_3), \quad e'_4 := \frac{1}{2}(e_2 - e_3)$$

and $\epsilon'_i \in \mathfrak{t}^*$ the dual basis. Then they satisfy Setting 7.7 for \mathfrak{k}' and \mathfrak{p}' .

Suppose that $S^c = \{\alpha_1\}$. Then \mathfrak{q} is given by e_4 and $\mathfrak{l}_0 \simeq \mathfrak{sp}(3) \oplus \mathfrak{u}(1)$ is compact. We have

$$A_{\mathfrak{q}(e_4)}(\lambda \epsilon_4)|_{(\mathfrak{g}', K')} \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{\mathfrak{q}'(2e'_1 + 2e'_2 + e'_3 + e'_4)}\left(\frac{\lambda + n + 4}{2}(\epsilon'_1 + \epsilon'_2) + \frac{n}{2}(\epsilon'_3 + \epsilon'_4)\right).$$

We note $\mathfrak{l}'_0(2e'_1 + 2e'_2 + e'_3 + e'_4) \simeq \mathfrak{u}(2)^2$.

Suppose that $S^c = \{\alpha_4\}$. Then \mathfrak{q} is given by $e_1 + e_4$ so that $\mathfrak{l}_0 \simeq \mathfrak{so}(2, 5) \oplus \mathfrak{u}(1)$. By using Theorem 5.1, we conclude that

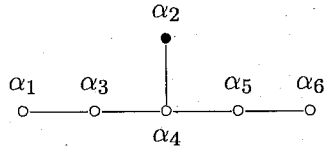
$$\begin{aligned} & A_{\mathfrak{q}(e_1 + e_4)}(\lambda(\epsilon_1 + \epsilon_4))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{0 \leq n \leq \lambda + 5} A_{\mathfrak{q}'(2e'_1 + e'_2 + e'_3 + e'_4)}\left(\left(\lambda + \frac{m - n}{2} + 2\right)\epsilon'_1 + \left(\frac{m + n}{2} - 1\right)(\epsilon'_2 + \epsilon'_3 + \epsilon'_4)\right) \\ & \oplus \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{0 \leq n \leq \lambda + 4} A_{\mathfrak{q}'(2e'_1 + 2e'_2 + 2e'_3 + e'_4)}\left(\left(\frac{\lambda + m + n}{2} + 1\right)(\epsilon'_1 + \epsilon'_2 + \epsilon'_3) + \left(\frac{\lambda + m - n}{2} + 2\right)\epsilon'_4\right). \end{aligned}$$

We note that

$$\mathfrak{l}'_0(2e'_1 + e'_2 + e'_3 + e'_4) \simeq \mathfrak{u}(1) \oplus \mathfrak{u}(1, 2), \quad \mathfrak{l}'_0(2e'_1 + 2e'_2 + 2e'_3 + e'_4) \simeq \mathfrak{u}(2, 1) \oplus \mathfrak{u}(1).$$

9.3. $\mathfrak{e}_{6(2)} \downarrow \mathfrak{so}(6, 4) \oplus \mathfrak{so}(2)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{6(2)}$ and $\mathfrak{g}'_0 := \mathfrak{so}(4, 6) \oplus \mathfrak{so}(2)$. We write the Dynkin diagram of \mathfrak{g} and label the simple roots:



Here, the painted root α_2 corresponds to a root in $\mathfrak{g}(1)$. Other roots are in $\mathfrak{g}(0)$.

In this case, we can take $\mathfrak{t} \subset \mathfrak{k}'$ and hence $\mathfrak{t}' = \mathfrak{t}$. We let $\sigma = 1$ on \mathfrak{g}_{α_i} if $i = 2, 3, 4, 5$ and $\sigma = -1$ on \mathfrak{g}_{α_i} if $i = 1, 6$. Then $\Pi' = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}) = \{\alpha_2\}$. The only $\mathfrak{g}(0)$ -antidominant root in $\mathfrak{g}(1)$ is α_2 and $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \alpha_2 \rangle = 4$. Hence the assumption of Theorem 9.2 is fulfilled if $c(\lambda; \alpha_2) \geq -4$.

Suppose that $\mathfrak{q} = \mathfrak{q}(S)$ for $S \subset \Pi$. We can verify $c(\lambda; \alpha_2) \geq -4$ unless $\Pi \setminus S = \{\alpha_1\}, \{\alpha_6\}, \{\alpha_2\}, \{\alpha_1, \alpha_6\}$. Choose $\epsilon_i \in \mathfrak{t}^*$ such that

$$\begin{aligned} \alpha_1 &= \epsilon_1 - \epsilon_2, & \alpha_2 &= \frac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6) + \epsilon_7, \\ \alpha_3 &= \epsilon_2 - \epsilon_3, & \alpha_4 &= \epsilon_3 - \epsilon_4, & \alpha_5 &= \epsilon_4 - \epsilon_5, & \alpha_6 &= \epsilon_5 - \epsilon_6 \end{aligned}$$

and $e_i \in \mathfrak{t}$ so that they satisfy Setting 7.17. Let

$$\begin{aligned} e'_1 &:= \frac{1}{2}(e_1 - e_6 + e_7), & e'_2 &:= \frac{1}{2}(-e_1 + e_6 + e_7), & e'_3 &:= \frac{1}{2}(e_2 + e_3 - e_4 - e_5), \\ e'_4 &:= \frac{1}{2}(e_2 - e_3 + e_4 - e_5), & e'_5 &:= \frac{1}{2}(e_2 - e_3 - e_4 + e_5), & e'_6 &:= e_1 + e_6 \end{aligned}$$

and $\epsilon'_i \in \mathfrak{t}^*$ the dual basis. Then $\epsilon'_1, \dots, \epsilon'_5$ satisfy Setting 7.6 for \mathfrak{k}' and \mathfrak{p}' .

Suppose that $S^c = \{\alpha_2\}$. Then \mathfrak{q} is given by e_7 and $\mathfrak{l}_0 \simeq \mathfrak{su}(6) \oplus \mathfrak{u}(1)$. We have

$$\begin{aligned} & A_{\mathfrak{q}(e_7)}(\lambda \epsilon_7)|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{m, n \in \mathbb{Z}_{\geq 0}} \bigoplus_{k=0}^{\min\{m, n\}} A_{\mathfrak{q}'(2e'_1 + 2e'_2 + e'_3 + e'_4)} \left(\frac{\lambda + m + n + 8}{2} (\epsilon'_1 + \epsilon'_2) \right. \\ & \quad \left. + \frac{m + n - 2k}{2} (\epsilon'_3 + \epsilon'_4) + \frac{m - n}{2} \epsilon'_5 + (m - n) \epsilon'_6 \right). \end{aligned}$$

We note $\mathfrak{l}'_0(2e'_1 + 2e'_2 + e'_3 + e'_4) \simeq \mathfrak{u}(2)^2 \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(2)$.

Suppose that $S^c = \{\alpha_1\}$. Then \mathfrak{q} is given by $2e_1 + e_7$ and $\mathfrak{l}_0 \simeq \mathfrak{so}^*(10) \oplus \mathfrak{u}(1)$. By using Theorem 5.1, we conclude that

$$\begin{aligned} & A_{\mathfrak{q}(2e_1 + e_7)} \left(\lambda (\epsilon_1 + \epsilon_7) - \frac{\lambda}{6} \epsilon \right) \Big|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{0 \leq n \leq \lambda + 5} A_{\mathfrak{q}'(2e'_1 + e'_2 + e'_3 + e'_4 + e'_5)} \left(\left(\lambda + \frac{m - n}{2} + 2 \right) \epsilon'_1 \right. \\ & \quad \left. + \left(\frac{m + n}{2} - 1 \right) (\epsilon'_2 + \epsilon'_3 + \epsilon'_4 + \epsilon'_5) + \left(\frac{2}{3} \lambda + m - n + 4 \right) \epsilon'_6 \right) \\ & \oplus \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{\mathfrak{q}'(e'_1 + e'_2 + e'_3 + e'_4)} \left(\left(\frac{\lambda + m + n}{2} + 1 \right) (\epsilon'_1 + \epsilon'_2 + \epsilon'_3 + \epsilon'_4) \right. \\ & \quad \left. + \left(\frac{\lambda + m - n}{2} + 3 \right) \epsilon'_5 + \left(-\frac{1}{3} \lambda + m - n - 2 \right) \epsilon'_6 \right), \end{aligned}$$

where $\epsilon = \epsilon_1 + \dots + \epsilon_6$. We note that

$$\begin{aligned} \mathfrak{l}'_0(2e'_1 + e'_2 + e'_3 + e'_4 + e'_5) &\simeq \mathfrak{u}(1) \oplus \mathfrak{u}(1, 3) \oplus \mathfrak{so}(2), \\ \mathfrak{l}'_0(e'_1 + e'_2 + e'_3 + e'_4) &\simeq \mathfrak{u}(2, 2) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(2). \end{aligned}$$

Suppose that $S^c = \{\alpha_1, \alpha_6\}$. Then \mathfrak{q} is given by $e_1 - e_6 + e_7$ and $\mathfrak{l}_0 \simeq \mathfrak{so}^*(8) \oplus \mathfrak{u}(1)^2$. It follows that $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$ is isomorphic to a direct sum of (\mathfrak{g}', K') -modules $A_{\mathfrak{q}'(3e'_1 + 2e'_2 + e'_3 + e'_4)}(\lambda')$, $A_{\mathfrak{q}'(3e'_1 + 2e'_2 + 2e'_3 + e'_4)}(\lambda')$, and $A_{\mathfrak{q}'(3e'_1 + 3e'_2 + 2e'_3 + e'_4)}(\lambda')$. We note that

$$\begin{aligned} \mathfrak{l}'_0(3e'_1 + 2e'_2 + e'_3 + e'_4) &\simeq \mathfrak{u}(1)^3 \oplus \mathfrak{u}(2) \oplus \mathfrak{so}(2), \\ \mathfrak{l}'_0(3e'_1 + 2e'_2 + 2e'_3 + e'_4) &\simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(1, 1) \oplus \mathfrak{so}(2), \\ \mathfrak{l}'_0(3e'_1 + 3e'_2 + 2e'_3 + e'_4) &\simeq \mathfrak{u}(1)^3 \oplus \mathfrak{u}(2) \oplus \mathfrak{so}(2). \end{aligned}$$

9.4. $\mathfrak{e}_{6(2)} \downarrow \mathfrak{su}(4, 2) \oplus \mathfrak{su}(2)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{6(2)}$ and $\mathfrak{g}'_0 := \mathfrak{su}(2, 4) \oplus \mathfrak{su}(2)$. We label the simple roots as in the previous case. We let $\sigma = 1$ on \mathfrak{g}_{α_i} if $i = 1, 2, 3, 4, 6$ and $\sigma = -1$ on \mathfrak{g}_{α_5} . Then $\Pi' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}) = \{\alpha_2, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\}$. We have $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \alpha_2 \rangle = \langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \rangle = 6$ and hence the assumption of Theorem 9.2 is fulfilled if $c(\lambda; \alpha_2) \geq -6$.

Suppose that $\mathfrak{q} = \mathfrak{q}(S)$ with $S \subset \Pi$. Put $S^c := \Pi \setminus S$. If $S^c \neq \{\alpha_1\}$ or $\{\alpha_6\}$, it turns out that $c(\lambda; \alpha_2) \geq -6$.

Choose $\epsilon_i \in \mathfrak{t}^*$ and e_i as in the previous case. Let

$$\begin{aligned} e'_1 &:= -\frac{1}{2}(e_5 + e_6) + \frac{1}{2}e_7, & e'_2 &:= e_1 + \frac{1}{2}(e_5 + e_6), \\ e'_3 &:= e_2 + \frac{1}{2}(e_5 + e_6), & e'_4 &:= e_3 + \frac{1}{2}(e_5 + e_6), \\ e'_5 &:= e_4 + \frac{1}{2}(e_5 + e_6), & e'_6 &:= -\frac{1}{2}(e_5 + e_6) - \frac{1}{2}e_7, & e'_7 &:= e_5 - e_6 \end{aligned}$$

and define $\epsilon'_1, \dots, \epsilon'_7 \in \mathfrak{t}^*$ such that $(\epsilon'_i - \epsilon'_j)(e'_k) = \delta_{ik} - \delta_{jk}$, $\epsilon'_7(e'_7) = 1$, and $(\epsilon'_i - \epsilon'_j)(e'_7) = \epsilon'_7(e'_i) = 0$ for $1 \leq i, j, k \leq 6$.

Suppose that $S^c = \{\alpha_1\}$. Then \mathfrak{q} is given by $2e_1 + e_7$ and $\mathfrak{l}_0 \simeq \mathfrak{so}^*(10) \oplus \mathfrak{u}(1)$. By using Theorem 5.1, we conclude that

$$\begin{aligned} & A_{\mathfrak{q}(2e_1+e_7)} \left(\lambda(\epsilon_1 + \epsilon_7) - \frac{\lambda}{6}\epsilon \right) \Big|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{0 \leq n \leq \lambda+5} A_{\mathfrak{q}'(2e'_1+2e'_2+e'_3-e'_6+e'_7)} \left((\lambda + m - n + 4)(\epsilon'_1 + \epsilon'_2) + m\epsilon'_3 \right. \\ & \quad \left. + (-n + 1)\epsilon'_6 + (m + n + 1)\epsilon'_7 - \frac{1}{6}(2\lambda + 3m - 3n + 9)\epsilon' \right) \\ & \oplus \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{n=1}^m A_{\mathfrak{q}'(2e'_1+e'_2+e'_3-e'_6+e'_7)} \left((m - 2)\epsilon'_1 + (m - n)(\epsilon'_2 + \epsilon'_3) \right. \\ & \quad \left. - (\lambda + n + 4)\epsilon'_6 + (\lambda + m + n + 6)\epsilon'_7 - \frac{1}{6}(-\lambda + 3m - 3n - 6)\epsilon' \right) \\ & \oplus \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{n=0}^m A_{\mathfrak{q}'(2e'_1+2e'_2+e'_3-e'_5-e'_6+e'_7)} \left((\lambda + m + n + 5)(\epsilon'_1 + \epsilon'_2) + m\epsilon'_3 \right. \\ & \quad \left. + n\epsilon'_4 + \epsilon'_5 + \epsilon'_6 + (m - n)\epsilon'_7 - \frac{1}{6}(2\lambda + 3m + 3n + 12)\epsilon' \right), \end{aligned}$$

where $\epsilon = \epsilon_1 + \dots + \epsilon_6$ and $\epsilon' = \epsilon'_1 + \dots + \epsilon'_6$. We note that

$$\begin{aligned} \mathfrak{l}'_0(2e'_1 + 2e'_2 + e'_3 - e'_6 + e'_7) &\simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(2) \oplus \mathfrak{u}(1), \\ \mathfrak{l}'_0(2e'_1 + e'_2 + e'_3 - e'_6 + e'_7) &\simeq \mathfrak{u}(1) \oplus \mathfrak{u}(2)^2 \oplus \mathfrak{u}(1), \\ \mathfrak{l}'_0(3e'_1 + 3e'_2 + 2e'_3 + e'_4) &\simeq \mathfrak{u}(1, 1)^2 \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1). \end{aligned}$$

9.5. $\mathfrak{e}_{6(2)} \downarrow \mathfrak{sp}(3, 1)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{6(2)}$ and $\mathfrak{g}'_0 := \mathfrak{sp}(1, 3)$. We label the simple roots as in the previous case. We let

$$\sigma\alpha_1 = \alpha_6, \quad \sigma\alpha_3 = \alpha_5, \quad \sigma = 1 \text{ on } \mathfrak{g}_{\alpha_4}, \quad \sigma = -1 \text{ on } \mathfrak{g}_{\alpha_2}.$$

Then $\mathfrak{t}' := \mathfrak{t}^\sigma$ is a Cartan subalgebra of \mathfrak{k}' and $\Pi' = \{\alpha_1|_{\mathfrak{t}'}, \alpha_3|_{\mathfrak{t}'}, \alpha_4|_{\mathfrak{t}'}, (\alpha_2 + \alpha_3 + \alpha_4)|_{\mathfrak{t}'}\}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}) = \{(\alpha_2 + \alpha_3 + \alpha_4)|_{\mathfrak{t}'}\}$. We have $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), (\alpha_2 + \alpha_3 + \alpha_4)|_{\mathfrak{t}'} \rangle = 7$. If $\mathfrak{q} \neq \mathfrak{g}$, then $c(\lambda; \alpha_2) \geq -7$ and the assumption of Theorem 9.2 is fulfilled.

9.6. $\mathfrak{e}_{6(2)} \downarrow \mathfrak{f}_{4(4)}$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{6(2)}$ and $\mathfrak{g}'_0 := \mathfrak{f}_{4(4)}$. We label the simple roots as in the previous case. We let

$$\sigma\alpha_1 = \alpha_6, \quad \sigma\alpha_3 = \alpha_5, \quad \sigma = 1 \text{ on } \mathfrak{g}_{\alpha_4}, \quad \sigma = 1 \text{ on } \mathfrak{g}_{\alpha_2}.$$

Then $\mathfrak{t}' := \mathfrak{t}^\sigma$ is a Cartan subalgebra of \mathfrak{k}' and $\Pi' = \{\alpha_1|_{\mathfrak{t}'}, \alpha_3|_{\mathfrak{t}'}, \alpha_4|_{\mathfrak{t}'}, \alpha_2|_{\mathfrak{t}'}\}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}) = \{\alpha_2|_{\mathfrak{t}'}\}$. We have $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \alpha_2|_{\mathfrak{t}'} \rangle = 3$ and hence the assumption of Theorem 9.2 is fulfilled if $c(\lambda; \alpha_2) \geq -3$.

Suppose that $\mathfrak{q} = \mathfrak{q}(S)$ with $S \subset \Pi$. Put $S^c := \Pi \setminus S$. The inequality $c(\lambda; \alpha_2) < -3$ may hold only if $|S^c| = 1$, $S^c = \{\alpha_1, \alpha_6\}, \{\alpha_1, \alpha_3\}, \{\alpha_5, \alpha_6\}$. Choose $\epsilon_i \in \mathfrak{t}^*$ and $e_i \in \mathfrak{t}$ as above. Let

$$e'_1 := e_1 - e_6, \quad e'_2 := e_2 - e_5, \quad e'_3 := e_3 - e_4, \quad e'_4 := e_7,$$

and $\epsilon'_i \in (\mathfrak{t}')^*$ the dual basis. Then $\epsilon'_1, \dots, \epsilon'_4$ satisfy Setting 7.15 for \mathfrak{k}' and \mathfrak{p}' .

Suppose that $S^c = \{\alpha_2\}$. Then \mathfrak{q} is given by e_7 and $\mathfrak{l}_0 \simeq \mathfrak{su}(6) \oplus \mathfrak{u}(1)$. We have

$$A_{\mathfrak{q}(e_7)}(\lambda\epsilon_7)|_{(\mathfrak{g}', K')} \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{\mathfrak{q}'(e'_1 + 2e'_4)}(n\epsilon'_1 + (\lambda + n + 6)\epsilon'_4).$$

We note $\mathfrak{l}'_0(e'_1 + 2e'_4) \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{sp}(2)$.

Suppose that $S^c = \{\alpha_1\}$. Then \mathfrak{q} is given by $2e_1 + e_7$ and $\mathfrak{l}_0 \simeq \mathfrak{so}^*(10) \oplus \mathfrak{u}(1)$. Since K' acts transitively on $K/(\overline{Q} \cap K)$ in this case, Corollary 5.8 yields

$$A_{\mathfrak{q}(2e_1 + e_7)}\left(\lambda(\epsilon_1 + \epsilon_7) - \frac{\lambda}{6}\epsilon\right)\Big|_{(\mathfrak{g}', K')} \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{\mathfrak{q}'(e'_1 + e'_4)}((\lambda + n + 1)(\epsilon'_1 + \epsilon'_4)).$$

where $\epsilon = \epsilon_1 + \dots + \epsilon_6$. We note $\mathfrak{l}'_0(e'_1 + e'_4) \simeq \mathfrak{u}(1) \oplus \mathfrak{so}(2, 5)$.

Suppose that $S^c = \{\alpha_1, \alpha_6\}$. Then \mathfrak{q} is given by $e_1 - e_6 + e_7$ and $\mathfrak{l}_0 \simeq \mathfrak{so}^*(8) \oplus \mathfrak{u}(1)^2$. We can see that $A_{\mathfrak{q}(e_1 - e_6 + e_7)}(\lambda_1\epsilon_1 + \lambda_2\epsilon_6 + (\lambda_1 - \lambda_2)\epsilon_7 - \frac{\lambda_1 + \lambda_2}{6}\epsilon) = 0$ if $\lambda_1 \leq -4$ or $-\lambda_2 \leq -4$. For $\lambda_1 \geq -\lambda_2 \geq -3$,

$$\begin{aligned} & A_{\mathfrak{q}(e_1 - e_6 + e_7)}\left(\lambda_1\epsilon_1 + \lambda_2\epsilon_6 + (\lambda_1 - \lambda_2)\epsilon_7 - \frac{\lambda_1 + \lambda_2}{6}\epsilon\right)\Big|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{k=0}^{\lambda_1 - \lambda_2 + 6} m(n, k) A_{\mathfrak{q}'(2e'_1 + e'_2 + 3e'_4)}((\lambda_1 - \lambda_2 + n - k + 4)\epsilon'_1 + (k - 2)\epsilon'_2 + (\lambda_1 - \lambda_2 + n + 2)\epsilon'_4), \end{aligned}$$

where

$$m(n, k) = \max\{0, \min\{-\lambda_2 + 4, k + 1, -k + \lambda_1 - \lambda_2 + 7, \\ n + 1, n - k - \lambda_2 + 4, n - 2k + \lambda_1 - \lambda_2 + 7\}\}.$$

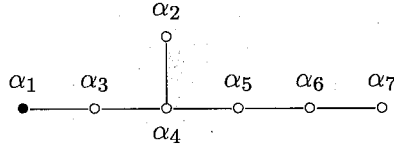
We note $\ell'_0(2e'_1 + e'_2 + 3e'_4) \simeq \mathfrak{u}(1) \oplus \mathfrak{u}(1, 2)$.

Suppose that $S^c = \{\alpha_1, \alpha_3\}$. Then \mathfrak{q} is given by $3e_1 + e_2 + 2e_7$ and $\mathfrak{l}_0 \simeq \mathfrak{u}(1, 4) \oplus \mathfrak{u}(1)$. It follows that $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$ is isomorphic to a direct sum (with multiplicity) of (\mathfrak{g}', K') -modules $A_{\mathfrak{q}'(2e'_1 + e'_2 + 4e'_4)}(\lambda')$ and $A_{\mathfrak{q}'(3e'_1 + 2e'_2 + e'_3 + 6e'_4)}(\lambda')$. We note that

$$\begin{aligned} \ell'_0(2e'_1 + e'_2 + 4e'_4) &\simeq \mathfrak{u}(2) \oplus \mathfrak{u}(1)^2, \\ \ell'_0(3e'_1 + 2e'_2 + e'_3 + 6e'_4) &\simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(1)^2. \end{aligned}$$

9.7. $\mathfrak{e}_{7(-5)} \downarrow \mathfrak{so}(8, 4) \oplus \mathfrak{su}(2)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{7(-5)}$ and $\mathfrak{g}'_0 := \mathfrak{so}(4, 8) \oplus \mathfrak{su}(2)$. We write the Dynkin diagram of \mathfrak{g} and label the simple roots:



Here, the painted root α_1 corresponds to a root in $\mathfrak{g}(1)$. Other roots are in $\mathfrak{g}(0)$.

In this case, we can take $\mathfrak{t} \subset \mathfrak{k}'$ and hence $\mathfrak{t}' = \mathfrak{t}$. We let $\sigma = 1$ on \mathfrak{g}_{α_i} if $i \neq 6$ and $\sigma = -1$ on \mathfrak{g}_{α_6} . Then $\Pi' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7\}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}) = \{\alpha_1\}$. The only $\mathfrak{g}(0)$ -antidominant root in $\mathfrak{g}(1)$ is α_1 and $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \alpha_1 \rangle = 8$. Hence the assumption of Theorem 9.2 is fulfilled if $c(\lambda; \alpha_1) \geq -8$.

Suppose that $\mathfrak{q} = \mathfrak{q}(S)$ for $S \subset \Pi$. Then the inequality $c(\lambda; \alpha_1) \geq -8$ fails only if $\Pi \setminus S = \{\alpha_7\}$. Choose $\epsilon_i \in \mathfrak{t}^*$ such that

$$\begin{aligned} \alpha_1 &= \frac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 + \epsilon_6) + \epsilon_7, & \alpha_2 &= \epsilon_5 + \epsilon_6, & \alpha_3 &= \epsilon_5 - \epsilon_6, \\ \alpha_4 &= \epsilon_4 - \epsilon_5, & \alpha_5 &= \epsilon_3 - \epsilon_4, & \alpha_6 &= \epsilon_2 - \epsilon_3, & \alpha_7 &= \epsilon_1 - \epsilon_2 \end{aligned}$$

and $e_i \in \mathfrak{t}$ so that they satisfy Setting 7.19. Let

$$\begin{aligned} e'_1 &:= \frac{1}{2}(e_1 + e_2 + e_7), & e'_2 &:= \frac{1}{2}(-e_1 - e_2 + e_7), & e'_3 &:= \frac{1}{2}(e_3 + e_4 + e_5 - e_6), \\ e'_4 &:= \frac{1}{2}(e_3 + e_4 - e_5 + e_6), & e'_5 &:= \frac{1}{2}(e_3 - e_4 + e_5 + e_6), \\ e'_6 &:= \frac{1}{2}(e_3 - e_4 - e_5 - e_6), & e'_7 &:= e_1 - e_2 \end{aligned}$$

and $\epsilon'_i \in \mathfrak{t}^*$ the dual basis. Then $\epsilon'_1, \dots, \epsilon'_6$ satisfy Setting 7.6.

Let $\Pi \setminus S = \{\alpha_7\}$. Then \mathfrak{q} is given by $2e_1 + e_7$ and $\mathfrak{l}_0 \simeq \mathfrak{e}_{6(-14)} \oplus \mathfrak{u}(1)$. We have

$$\begin{aligned}
& A_{\mathfrak{q}(2e_1+e_7)}(\lambda(\epsilon_1 + \epsilon_7))|_{(\mathfrak{g}', K')} \\
& \simeq \bigoplus_{m=0}^{\lambda+4} \bigoplus_{\substack{0 \leq n \leq \lambda-m+4 \\ m+n+\lambda \in 2\mathbb{Z}}} A_{\mathfrak{q}'(3e'_1+2e'_2+e'_3+e'_4+e'_5+e'_6+e'_7)} \left(\frac{\lambda+m}{2} \epsilon'_1 + \frac{\lambda-m}{2} \epsilon'_2 \right. \\
& \quad \left. + \frac{n}{2} (\epsilon'_3 + \epsilon'_4 + \epsilon'_5 + \epsilon'_6) + m\epsilon'_7 \right) \\
& \oplus \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{\substack{\lambda-m+6 \leq n \leq \lambda+m+6 \\ 0, -\lambda+m-10 \leq n \\ m+n+\lambda \in 2\mathbb{Z}}} A_{\mathfrak{q}'(2e'_1+e'_2+e'_3+e'_4+e'_5+e'_7)} \left(\frac{\lambda+m}{2} \epsilon'_1 \right. \\
& \quad \left. + \left(\frac{n}{2} - 1 \right) (\epsilon'_2 + \epsilon'_3 + \epsilon'_4 + \epsilon'_5) + \left(\frac{\lambda-m}{2} + 4 \right) \epsilon'_6 + m\epsilon'_7 \right) \\
& \oplus \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{\substack{0 \leq n \leq -\lambda+m-12 \\ m+n+\lambda \in 2\mathbb{Z}}} A_{\mathfrak{q}'(3e'_1+2e'_2+e'_3+e'_4+e'_5-e'_6+e'_7)} \left(\frac{\lambda+m}{2} \epsilon'_1 \right. \\
& \quad \left. + \left(\frac{-\lambda+m}{2} - 8 \right) \epsilon'_2 + \frac{n}{2} (\epsilon'_3 + \epsilon'_4 + \epsilon'_5 - \epsilon'_6) + m\epsilon'_7 \right) \\
& \oplus \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{\substack{0, \lambda+m+8 \leq n \\ m+n+\lambda \in 2\mathbb{Z}}} A_{\mathfrak{q}'(2e'_1+2e'_2+2e'_3+2e'_4+e'_5+e'_7)} \left(\left(\frac{n}{2} - 2 \right) (\epsilon'_1 + \epsilon'_2 + \epsilon'_3 + \epsilon'_4) \right. \\
& \quad \left. + \left(\frac{\lambda+m}{2} + 4 \right) \epsilon'_5 + \left(\frac{\lambda-m}{2} + 4 \right) \epsilon'_6 + m\epsilon'_7 \right).
\end{aligned}$$

We note that

$$\begin{aligned}
\mathfrak{l}'_0(3e'_1 + 2e'_2 + e'_3 + e'_4 + e'_5 + e'_6 + e'_7) & \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(4) \oplus \mathfrak{u}(1), \\
\mathfrak{l}'_0(2e'_1 + e'_2 + e'_3 + e'_4 + e'_5 + e'_7) & \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(1, 3) \oplus \mathfrak{u}(1), \\
\mathfrak{l}'_0(3e'_1 + 2e'_2 + e'_3 + e'_4 + e'_5 - e'_6 + e'_7) & \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(4) \oplus \mathfrak{u}(1), \\
\mathfrak{l}'_0(2e'_1 + 2e'_2 + 2e'_3 + 2e'_4 + e'_5 + e'_7) & \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(2, 2) \oplus \mathfrak{u}(1).
\end{aligned}$$

9.8. $\mathfrak{e}_{7(-5)} \downarrow \mathfrak{su}(6, 2)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{7(-5)}$ and $\mathfrak{g}'_0 := \mathfrak{su}(2, 6)$. We label the simple roots as in the previous case. We let $\sigma = 1$ on \mathfrak{g}_{α_i} if $i \neq 2$ and $\sigma = -1$ on \mathfrak{g}_{α_2} . Then $\Pi' = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}) = \{\alpha_1, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$. We can compute that $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \beta \rangle = 10$ for any $\beta \in \Delta(\mathfrak{g}^{\sigma}(1), \mathfrak{t})$. Hence the assumption of Theorem 9.2 is fulfilled if $c(\lambda; \alpha_1) \geq -10$.

Suppose that $\mathfrak{q} = \mathfrak{q}(S)$ for $S \subset \Pi$. Then the inequality $c(\lambda; \alpha_1) \geq -10$ fails only if $\Pi \setminus S = \{\alpha_7\}$. Choose $\epsilon_i \in \mathfrak{t}^*$ and $e_i \in \mathfrak{t}$ as above. Let

$$\begin{aligned}
e'_1 &:= \frac{1}{4}(-e_1 - e_2 - e_3 - e_4 - e_5 - e_6) + \frac{1}{2}e_7, \\
e'_2 &:= \frac{1}{4}(e_1 + e_2 + e_3 + e_4 + e_5 - 3e_6), \quad e'_3 := \frac{1}{4}(e_1 + e_2 + e_3 + e_4 - 3e_5 + e_6),
\end{aligned}$$

$$\begin{aligned}
e'_4 &:= \frac{1}{4}(e_1 + e_2 + e_3 - 3e_4 + e_5 + e_6), & e'_5 &:= \frac{1}{4}(e_1 + e_2 - 3e_3 + e_4 + e_5 + e_6), \\
e'_6 &:= \frac{1}{4}(e_1 - 3e_2 + e_3 + e_4 + e_5 + e_6), & e'_7 &:= \frac{1}{4}(-3e_1 + e_2 + e_3 + e_4 + e_5 + e_6), \\
e'_8 &:= \frac{1}{4}(-e_1 - e_2 - e_3 - e_4 - e_5 - e_6) - \frac{1}{2}e_7
\end{aligned}$$

and define $\epsilon'_i \in \mathfrak{t}^*$ such that $(\epsilon'_i - \epsilon'_j)(\epsilon'_k) = \delta_{ik} - \delta_{jk}$ for $1 \leq i, j, k \leq 8$.

Let $\Pi \setminus S = \{\alpha_7\}$. Then \mathfrak{q} is given by $2e_1 + e_7$ and $\mathfrak{l}_0 \simeq \mathfrak{e}_{6(-14)} \oplus \mathfrak{u}(1)$. Using Theorem 5.1, we conclude that

$$\begin{aligned}
& A_{\mathfrak{q}(2e_1+e_7)}(\lambda(\epsilon_1 + \epsilon_7))|_{(\mathfrak{g}', K')} \\
& \simeq \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \bigoplus_{\substack{0 \leq n \leq \frac{m}{2} \\ n \in \mathbb{Z}}} \bigoplus_{k=0}^{\lambda+8} A_{\mathfrak{q}'(2e'_1+2e'_2+e'_3+e'_4-e'_7-e'_8)}((\lambda + m - k + 7)(\epsilon'_1 + \epsilon'_2) \\
& + (m - n)(\epsilon'_3 + \epsilon'_4) + n(\epsilon'_5 + \epsilon'_6) + (-k + 1)(\epsilon'_7 + \epsilon'_8) - \frac{1}{4}(\lambda + 2m - 2k + 8)\epsilon'_2),
\end{aligned}$$

where $\epsilon' = \epsilon'_1 + \dots + \epsilon'_8$. We note that $\mathfrak{l}'_0(2e'_1 + 2e'_2 + e'_3 + e'_4 - e'_7 - e'_8) \simeq \mathfrak{u}(1, 1)^2 \oplus \mathfrak{su}(2)^2 \oplus \mathfrak{u}(1)$.

9.9. $\mathfrak{e}_{7(-5)} \downarrow \mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{7(-5)}$ and $\mathfrak{g}'_0 := \mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2)$. We label the simple roots as in the previous case. We let $\sigma = 1$ on \mathfrak{g}_{α_i} if $i \neq 1, 2$ and $\sigma = -1$ on \mathfrak{g}_{α_1} and \mathfrak{g}_{α_2} . Then $\Pi' = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}) = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$. We can compute that $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \beta \rangle = 6$ for any $\beta \in \Delta(\mathfrak{g}^\sigma(1), \mathfrak{t})$. Hence the assumption of Theorem 9.2 is fulfilled if $c(\lambda; \alpha_1) \geq -6$.

Suppose that $\mathfrak{q} = \mathfrak{q}(S)$ for $S \subset \Pi$. If $S^c \neq \{\alpha_1\}$, $\{\alpha_6\}$, $\{\alpha_7\}$, or $\{\alpha_6, \alpha_7\}$, then $c(\lambda; \alpha_1) \geq -6$ holds. Choose $\epsilon_i \in \mathfrak{t}^*$ and $e_i \in \mathfrak{t}$ as above. Let

$$\begin{aligned}
e'_1 &:= \frac{1}{6}(e_1 + e_2 + e_3 + e_4 + e_5 - 5e_6), & e'_2 &:= \frac{1}{6}(e_1 + e_2 + e_3 + e_4 - 5e_5 + e_6), \\
e'_3 &:= \frac{1}{6}(e_1 + e_2 + e_3 - 5e_4 + e_5 + e_6), & e'_4 &:= \frac{1}{6}(e_1 + e_2 - 5e_3 + e_4 + e_5 + e_6), \\
e'_5 &:= \frac{1}{6}(e_1 - 5e_2 + e_3 + e_4 + e_5 + e_6), & e'_6 &:= \frac{1}{6}(-5e_1 + e_2 + e_3 + e_4 + e_5 + e_6), \\
e'_7 &:= e_7, & e'_8 &:= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6)
\end{aligned}$$

and define $\epsilon'_i \in \mathfrak{t}^*$ such that $(\epsilon'_i - \epsilon'_j)(\epsilon'_k) = \delta_{ik} - \delta_{jk}$ for $1 \leq i, j, k \leq 6$ and that $\epsilon'_i(\epsilon'_j) = \delta_{ij}$ for $i = 7, 8$ and $1 \leq j \leq 8$.

Let $\Pi \setminus S = \{\alpha_1\}$ and then \mathfrak{q} is given by e_7 . Since $\mathfrak{l}_0 \simeq \mathfrak{so}(12) \oplus \mathfrak{u}(1)$ is compact, we get

$$\begin{aligned}
& A_{\mathfrak{q}(e_7)}(\lambda\epsilon_7)|_{(\mathfrak{g}', K')} \\
& \simeq \bigoplus_{m, n \in \mathbb{Z}_{\geq 0}} \bigoplus_{k=0}^{\min\{m, n\}} A_{\mathfrak{q}'(e'_1 - e'_6 + 2e'_7)}((m - k)\epsilon'_1 - (n - k)\epsilon'_6)
\end{aligned}$$

$$+ (\lambda + m + n + 12)\epsilon'_7 + (m - n)\epsilon'_8 - \frac{m - n}{6}\epsilon'_9,$$

where $\epsilon' = \epsilon'_1 + \dots + \epsilon'_6$. We note that $\mathfrak{l}'_0(\epsilon'_1 - \epsilon'_6 + 2\epsilon'_7) \simeq \mathfrak{u}(4) \oplus \mathfrak{u}(1)^2 \oplus \mathfrak{so}(2)$.

Let $\Pi \setminus S = \{\alpha_7\}$. Then \mathfrak{q} is given by $2e_1 + e_7$ and $\mathfrak{l}_0 \simeq \mathfrak{e}_{6(-14)} \oplus \mathfrak{u}(1)$. Using Theorem 5.1, we conclude that

$$\begin{aligned} & A_{\mathfrak{q}(2e_1+e_7)}(\lambda(\epsilon_1 + \epsilon_7))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{m=0}^{\lambda+8} \bigoplus_{n, k \in \mathbb{Z}_{\geq 0}} A_{\mathfrak{q}'(\epsilon'_1 - \epsilon'_6 + \epsilon'_7)} \left(\left(\lambda - m + \frac{n+k}{2} + 5 \right) \epsilon'_1 + \frac{n-k}{2} (\epsilon'_2 + \epsilon'_3 + \epsilon'_4 + \epsilon'_5) \right. \\ & \quad \left. + \left(-m - \frac{n+k}{2} + 3 \right) \epsilon'_6 + (\lambda + n + k + 2) \epsilon'_7 \right. \\ & \quad \left. + \left(\frac{\lambda}{2} - m - n + k + 4 \right) \epsilon'_8 - \frac{1}{6} (\lambda - 2m + 2n - 2k + 8) \epsilon'_9 \right), \end{aligned}$$

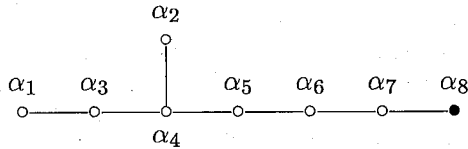
where $\epsilon' = \epsilon'_1 + \dots + \epsilon'_6$. We note that $\mathfrak{l}'_0(\epsilon'_1 - \epsilon'_6 + \epsilon'_7) \simeq \mathfrak{so}(6, 2) \oplus \mathfrak{u}(1)^2 \oplus \mathfrak{so}(2)$.

Suppose that $S^c = \{\alpha_6, \alpha_7\}$. Then \mathfrak{q} is given by $3e_1 + e_2 + 2e_7$ and $\mathfrak{l}_0 \simeq \mathfrak{so}(2, 8) \oplus \mathfrak{u}(1)^2$. It follows that $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$ is isomorphic to a direct sum of (\mathfrak{g}', K') -modules $A_{\mathfrak{q}'(2e'_1 + e'_2 - e'_5 - 2e'_6 + 4e'_7)}(\lambda')$, and $A_{\mathfrak{q}'(3e'_1 + 2e'_2 + e'_3 - e'_4 - 2e'_5 - 3e'_6 + 6e'_7)}(\lambda')$. We note that

$$\begin{aligned} \mathfrak{l}'_0(2e'_1 + e'_2 - e'_5 - 2e'_6 + 4e'_7) & \simeq \mathfrak{u}(2) \oplus \mathfrak{u}(1)^4 \oplus \mathfrak{so}(2), \\ \mathfrak{l}'_0(3e'_1 + 2e'_2 + e'_3 - e'_4 - 2e'_5 - 3e'_6 + 6e'_7) & \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(1)^4 \oplus \mathfrak{so}(2). \end{aligned}$$

9.10. $\mathfrak{e}_{8(-24)} \downarrow \mathfrak{so}(12, 4)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{8(-24)}$ and $\mathfrak{g}'_0 := \mathfrak{so}(4, 12)$. We write the Dynkin diagram of \mathfrak{g} and label the simple roots:



Here, the painted root α_8 corresponds to a root in $\mathfrak{g}(1)$. Other roots are in $\mathfrak{g}(0)$.

In this case, we can take $\mathfrak{t} \subset \mathfrak{k}'$ and hence $\mathfrak{t}' = \mathfrak{t}$. We let $\sigma = 1$ on \mathfrak{g}_{α_i} if $i \neq 1$ and $\sigma = -1$ on \mathfrak{g}_{α_1} . Then $\Pi' = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}) = \{\alpha_8\}$. The only $\mathfrak{g}(0)$ -antidominant root in $\mathfrak{g}(1)$ is α_8 and $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \alpha_8 \rangle = 16$. If $\mathfrak{q} \neq \mathfrak{g}$, then $c(\lambda; \alpha_8) \geq -16$ and the assumption of Theorem 9.2 is fulfilled.

9.11. $\mathfrak{e}_{8(-24)} \downarrow \mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{8(-24)}$ and $\mathfrak{g}'_0 := \mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2)$. We label the simple roots as in the previous case. We let $\sigma = 1$ on \mathfrak{g}_{α_i} if $i \neq 1, 8$ and $\sigma = -1$ on \mathfrak{g}_{α_1} and \mathfrak{g}_{α_8} . Then $\Pi' = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}) = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8\}$. We can compute that $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \beta \rangle = 12$

for any $\beta \in \Delta(\mathfrak{g}^\sigma(1), \mathfrak{t})$. Hence the assumption of Theorem 9.2 is fulfilled if $c(\lambda; \alpha_8) \geq -12$.

Suppose that $\mathfrak{q} = \mathfrak{q}(S)$ for $S \subset \Pi$. Then the inequality $c(\lambda; \alpha_8) \geq -12$ fails only if $\Pi \setminus S = \{\alpha_8\}$. Choose $\epsilon_i \in \mathfrak{t}^*$ such that

$$\begin{aligned} \alpha_1 &= \frac{1}{2}(\epsilon_8 - \epsilon_7 - \epsilon_6 - \epsilon_5 - \epsilon_4 - \epsilon_3 - \epsilon_2 + \epsilon_1), & \alpha_2 &= \epsilon_2 + \epsilon_1, & \alpha_3 &= \epsilon_2 - \epsilon_1, \\ \alpha_4 &= \epsilon_3 - \epsilon_2, & \alpha_5 &= \epsilon_4 - \epsilon_3, & \alpha_6 &= \epsilon_5 - \epsilon_4, & \alpha_7 &= \epsilon_6 - \epsilon_5, & \alpha_8 &= \epsilon_7 - \epsilon_6, \end{aligned}$$

and $e_i \in \mathfrak{t}$ so that they satisfy Setting 7.21. Let

$$\begin{aligned} e'_1 &:= e_6, & e'_2 &:= e_5, & e'_3 &:= e_4, & e'_4 &:= e_3, & e'_5 &:= e_2, & e'_6 &:= -e_1, \\ e'_7 &:= e_8 + e_7, & e'_8 &:= e_8 - e_7 \end{aligned}$$

and $\epsilon'_i \in \mathfrak{t}^*$ the dual basis. Then $\epsilon'_1, \dots, \epsilon'_7$ satisfy Setting 7.19.

Let $\Pi \setminus S = \{\alpha_8\}$ and then \mathfrak{q} is given by $e_8 + e_7$. Since $\mathfrak{l}_0 \simeq \mathfrak{e}_{7(-133)} \oplus \mathfrak{u}(1)$ is compact, Corollary 5.8 implies that $A_{\mathfrak{q}(e_8+e_7)}(\lambda(\epsilon_8+\epsilon_7))|_{(\mathfrak{g}', K')}$ decomposes into a direct sum of Zuckerman's modules of the form $A_{\mathfrak{q}'(2e'_1+e'_2+2e'_7+e'_8)}(m\epsilon'_1+n\epsilon'_2+k\epsilon'_7+l\epsilon'_8)$. We note $\mathfrak{l}'_0(2e'_1+e'_2+2e'_7+e'_8) \simeq \mathfrak{so}(8) \oplus \mathfrak{u}(1)^4$.

9.12. $\mathfrak{su}(2, 2n) \downarrow \mathfrak{sp}(1, n)$.

Let $\mathfrak{g}_0 := \mathfrak{u}(2, 2n)$ and $\mathfrak{g}'_0 := \mathfrak{sp}(1, n)$. We write the Dynkin diagram of \mathfrak{g} and label the simple roots:

$$\begin{array}{ccccccc} \alpha_1 & & \alpha_2 & & & & \alpha_{2n} & \alpha_{2n+1} \\ \bullet & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \text{---} & \bullet \end{array}$$

Here, the painted roots α_1 and α_{2n+1} correspond to roots in $\mathfrak{g}(1)$. Other roots are in $\mathfrak{g}(0)$.

We let $\sigma\alpha_i = \alpha_{2n-i+2}$ and $\sigma = 1$ on $\mathfrak{g}_{\alpha_{n+1}}$. Then $\mathfrak{t}' := \mathfrak{t}^\sigma$ is a Cartan subalgebra of \mathfrak{t}' and $\Pi' = \{\alpha_i|_{\mathfrak{t}'}\}_{1 \leq i \leq n+1}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}') = \{\alpha_1|_{\mathfrak{t}'}\}$. The $\mathfrak{g}(0)$ -antidominant roots in $\mathfrak{g}(1)$ are α_1 and α_{2n+1} . We have $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \alpha_1|_{\mathfrak{t}'} \rangle = n$. Hence the assumption of Theorem 9.2 is fulfilled if $c(\lambda; \alpha_1) + c(\lambda; \alpha_{2n+1}) \geq -(2n-1)$.

Suppose that $\mathfrak{q} = \mathfrak{q}(S)$ for $S \subset \Pi$. Then the inequality $c(\lambda; \alpha_1) + c(\lambda; \alpha_{2n+1}) \geq -(2n-1)$ fails only if $|\Pi \setminus S| = 1$. Take a standard basis $\epsilon_i \in \mathfrak{t}^*$ such that $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq 2n+1$ and let $e_i \in \mathfrak{t}$ be the dual basis. Let $\sigma\epsilon_i = \epsilon_{2n-i+2}$. Put $e'_i := e_i - e_{2n-i+3} \in \mathfrak{t}'$ for $1 \leq i \leq n+1$ and $\epsilon'_i \in (\mathfrak{t}')^*$ the dual basis. Then ϵ'_i satisfy Setting 7.10. We denote $\sum_{i=1}^{n+1} a_i \epsilon'_i$ by $(a_1; a_2, \dots, a_{n+1})$.

Let $\Pi \setminus S = \{\alpha_1\}$ and then \mathfrak{q} is given by e_1 . By Corollary 5.8,

$$A_{\mathfrak{q}(e_1)}(\lambda\epsilon_1)|_{(\mathfrak{g}', K')} \simeq A_{\mathfrak{q}'(e'_1)}(\lambda\epsilon'_1).$$

We note $\mathfrak{l}_0(e_1) \simeq \mathfrak{u}(1) \oplus \mathfrak{u}(2n, 1)$ and $\mathfrak{l}'_0(e'_1) \simeq \mathfrak{u}(1) \oplus \mathfrak{sp}(n)$.

Let $\Pi \setminus S = \{\alpha_m\}$ for $2 \leq m \leq n+1$. Then \mathfrak{q} is given by $e_1 + \cdots + e_m$ and $\mathfrak{l}_0 \simeq \mathfrak{u}(1, m-1) \oplus \mathfrak{u}(2n-m+1, 1)$. Put $k := \lfloor \frac{m-1}{2} \rfloor$. If m is odd,

$$\begin{aligned} & A_{\mathfrak{q}(e_1+\cdots+e_m)}(\lambda(\epsilon_1+\cdots+\epsilon_m))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus A_{\mathfrak{q}'(a')}((\lambda; r_1, r_1, r_2, r_2, \dots, r_k, r_k, 0, \dots, 0)) \\ & \oplus \bigoplus A_{\mathfrak{q}'(a'+e'_2)}((r_1-1; r_1-1, \lambda+2, r_2, r_2, \dots, r_k, r_k, 0, \dots, 0)), \end{aligned}$$

where $a' := (k+1)e'_1 + \sum_{i=1}^k (k-i+1)(e'_{2i} + e'_{2i+1})$ and the sum is taken over $\lambda+1 \geq r_1 \geq \cdots \geq r_k \geq 0$ for the first term and $r_1 \geq \lambda+2 \geq r_2 \geq \cdots \geq r_k \geq 0$ for the second term. We note

$$\begin{aligned} \mathfrak{l}'_0(a') & \simeq \mathfrak{u}(1) \oplus \mathfrak{u}(2)^k \oplus \mathfrak{sp}(n-2k), \\ \mathfrak{l}'_0(a' + e'_2) & \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(2)^{k-1} \oplus \mathfrak{sp}(n-2k), \end{aligned}$$

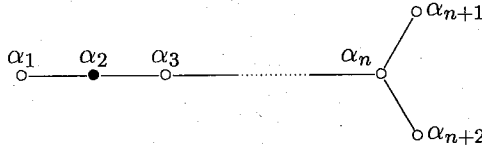
If m is even,

$$\begin{aligned} & A_{\mathfrak{q}(e_1+\cdots+e_m)}(\lambda(\epsilon_1+\cdots+\epsilon_m))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus A_{\mathfrak{q}'(a')}((\lambda+l+1; \lambda+l+1, r_1, r_1, \dots, r_k, r_k, 0, \dots, 0)), \end{aligned}$$

where $a' := \sum_{i=1}^{k+1} (k-i+2)(e'_{2i+1} + e'_{2i+2})$ and the sum is taken over $\lambda+2 \geq r_1 \geq \cdots \geq r_k \geq 0$ and $l \in \mathbb{Z}_{\geq 0}$. We note $\mathfrak{l}'_0(a') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^k \oplus \mathfrak{sp}(n-2k-1)$.

9.13. $\mathfrak{so}(4, 2n) \downarrow \mathfrak{u}(2, n)$.

Let $\mathfrak{g}_0 := \mathfrak{so}(4, 2n)$ and $\mathfrak{g}'_0 := \mathfrak{u}(2, n)$. We write the Dynkin diagram of \mathfrak{g} and label the simple roots:



Here, the painted root α_2 corresponds to a root in $\mathfrak{g}(1)$. Other roots are in $\mathfrak{g}(0)$.

In this case, we can take $\mathfrak{t} \subset \mathfrak{t}'$ and hence $\mathfrak{t}' = \mathfrak{t}$. We let $\sigma = 1$ on \mathfrak{g}_{α_i} if $3 \leq i \leq n+1$ and $\sigma = -1$ on \mathfrak{g}_{α_i} if $i = 1, 2, n+2$. Then $\Pi' = \{\alpha_3, \dots, \alpha_{n+1}, \alpha_1 + \alpha_2, \alpha_2 + \cdots + \alpha_n + \alpha_{n+2}\}$. Hence $\Pi' \cap \Delta(\mathfrak{g}(1), \mathfrak{t}) = \{\alpha_1 + \alpha_2, \alpha_2 + \cdots + \alpha_n + \alpha_{n+2}\}$. The only $\mathfrak{g}(0)$ -antidominant root in $\mathfrak{g}(1)$ is α_2 . We have $\langle 2\rho(\mathfrak{g}^{-\sigma}(1)), \beta \rangle = n$ for any $\beta \in \Delta(\mathfrak{g}'(1), \mathfrak{t}')$. Hence the assumption of Theorem 9.2 is fulfilled if $c(\lambda; \alpha_2) \geq -n$.

Suppose that $\mathfrak{q} = \mathfrak{q}(S)$ for $S \subset \Pi$. Then the inequality $c(\lambda; \alpha_2) \geq -n$ fails only if $\Pi \setminus S = \{\alpha_i\}$ for $i = 1, n+1$, or $n+2$. Take $\epsilon_i \in \mathfrak{t}^*$ such that

$$\alpha_1 = \epsilon_1 + \epsilon_2, \quad \alpha_2 = -\epsilon_2 - \epsilon_3, \quad \alpha_i = \epsilon_i - \epsilon_{i+1} \quad (3 \leq i \leq n+1), \quad \alpha_{n+2} = \epsilon_{n+1} + \epsilon_{n+2}$$

and $e_i \in \mathfrak{t}$ the dual basis so that they satisfy Setting 7.6. We denote $\sum_{i=1}^{n+2} a_i \epsilon_i$ by $(a_1; a_3, \dots, a_{n+2}; a_2)$.

Let $\Pi \setminus S = \{\alpha_1\}$. Then \mathfrak{q} is given by e_1 and $\mathfrak{l}_0 \simeq \mathfrak{u}(1) \oplus \mathfrak{so}(2, 2n)$. By Theorem 5.1,

$$\begin{aligned} & A_{\mathfrak{q}(e_1)}((\lambda; 0, \dots, 0; 0))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus_{\substack{-\frac{n+1}{2} \leq m \leq \lambda + \frac{n+1}{2} \\ m \in \mathbb{Z}}} A_{\mathfrak{q}'(e_1 - e_2)}((\lambda - m; 0, \dots, 0; -m)) \\ & \oplus \bigoplus_{\substack{\lambda + \frac{n+1}{2} < m \\ m \in \mathbb{Z}}} A_{\mathfrak{q}'(-2e_2 - e_{n+2})}((-1; -1, \dots, -1, \lambda - m + n; -m)) \\ & \oplus \bigoplus_{\substack{m < -\frac{n+1}{2} \\ m \in \mathbb{Z}}} A_{\mathfrak{q}'(2e_1 + e_3)}((\lambda - m; -m - n, 1, \dots, 1; 1)). \end{aligned}$$

We note that

$$\begin{aligned} \mathfrak{l}'_0(e_1 - e_2) & \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(n), \quad \mathfrak{l}'_0(-2e_2 - e_{n+2}) \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(1, n - 1), \\ \mathfrak{l}'_0(2e_1 + e_3) & \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(n - 1, 1). \end{aligned}$$

Let $\Pi \setminus S = \{\alpha_{n+2}\}$. Then \mathfrak{q} is given by $e_1 - e_2 + e_3 + e_4 + \dots + e_{n+2}$ and $\mathfrak{l}_0 \simeq \mathfrak{u}(2, n)$. Put $a := e_1 - e_2 + e_3 + e_4 + \dots + e_{n+2}$.

If $n = 2m$ is even, then

$$\begin{aligned} & A_{\mathfrak{q}(a)}((\lambda; \lambda, \dots, \lambda; -\lambda))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus A_{\mathfrak{q}'(a')}((\lambda; r_1, r_1, r_2, r_2, \dots, r_m, r_m; -\lambda)) \\ & \oplus \bigoplus A_{\mathfrak{q}'(a' + e_3)}((r_1 - 1; r_1 - 1, \lambda + 2, r_2, r_2, r_3, r_3, \dots, r_m, r_m; -\lambda)) \\ & \oplus \bigoplus A_{\mathfrak{q}'(a' - e_{n+2})}((\lambda; r_1, r_1, r_2, r_2, \dots, r_{m-1}, r_{m-1}, -\lambda - 2, r_m + 1; r_m + 1)) \\ & \oplus \bigoplus A_{\mathfrak{q}'(a' + e_3 - e_{n+2})}((r_1 - 1; r_1 - 1, \lambda + 2, r_2, r_2, \dots, r_{m-1}, r_{m-1}, -\lambda - 2, r_m + 1; r_m + 1)), \end{aligned}$$

where $a' := me_1 - e_2 + \sum_{i=1}^m (m - i)(e_{2i+1} + e_{2i+2})$ and the sum is taken over

$$\lambda + 1 \geq r_1 \geq \dots \geq r_m \geq -\lambda - 1$$

for the first term,

$$r_1 \geq \lambda + 2 \geq r_2 \geq \dots \geq r_m \geq -\lambda - 1$$

for the second term,

$$\lambda + 1 \geq r_1 \geq \dots \geq r_{m-1} \geq -\lambda - 2 \geq r_m$$

for the third term, and

$$r_1 \geq \lambda + 2 \geq r_2 \geq \dots \geq r_{m-1} \geq -\lambda - 2 \geq r_m$$

for the fourth term. We note that

$$\begin{aligned} \mathfrak{l}'_0(a') & \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(2)^m, \quad \mathfrak{l}'_0(a' + e_3) \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(1)^2 \oplus \mathfrak{u}(2)^{m-1}, \\ \mathfrak{l}'_0(a' - e_{n+2}) & \simeq \mathfrak{u}(1)^2 \oplus \mathfrak{u}(2)^{m-1} \oplus \mathfrak{u}(1, 1), \quad \mathfrak{l}'_0(a' + e_3) \simeq \mathfrak{u}(1, 1)^2 \oplus \mathfrak{u}(1)^2 \oplus \mathfrak{u}(2)^{m-2}. \end{aligned}$$

If $n = 2m + 1$ is odd, then

$$\begin{aligned} & A_{\mathfrak{q}(a)}((\lambda; \lambda, \dots, \lambda; -\lambda))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus A_{\mathfrak{q}'(a')}((\lambda + k + 1; \lambda + k + 1, r_1, r_1, \dots, r_m, r_m; -\lambda)) \\ & \oplus \bigoplus A_{\mathfrak{q}'(a' - e_{n+2})}((\lambda + k + 1; \lambda + k + 1, r_1, r_1, \dots, r_{m-1}, r_{m-1}, -\lambda - 2, r_m + 1; r_m + 1)), \end{aligned}$$

where $a' := m(e_1 + e_3) - e_2 + \sum_{i=1}^m (m - i)(e_{2i+2} + e_{2i+3})$, the sum is taken over

$$\lambda + 2 \geq r_1 \geq \dots \geq r_m \geq -\lambda - 1, \quad k \in \mathbb{Z}_{\geq 0}$$

for the first term and

$$\lambda + 2 \geq r_1 \geq \dots \geq r_{m-1} \geq -\lambda - 2 \geq r_m, \quad k \in \mathbb{Z}_{\geq 0}$$

for the second term. We note that

$$\mathfrak{l}'_0(a') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^m \oplus \mathfrak{u}(1), \quad \mathfrak{l}'_0(a' - e_{n+2}) \simeq \mathfrak{u}(1, 1)^2 \oplus \mathfrak{u}(2)^{m-1} \oplus \mathfrak{u}(1).$$

Let $\Pi \setminus S = \{\alpha_{n+1}\}$. Then \mathfrak{q} is given by $e_1 - e_2 + e_3 + e_4 + \dots + e_{n+1} - e_{n+2}$ and $\mathfrak{l}_0 \simeq \mathfrak{u}(2, n)$. Put $a := e_1 - e_2 + e_3 + e_4 + \dots + e_{n+1} - e_{n+2}$.

If $n = 2m + 2$ is even, then

$$\begin{aligned} & A_{\mathfrak{q}(a)}((\lambda; \lambda, \dots, \lambda, -\lambda; -\lambda))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus A_{\mathfrak{q}'(a')}((\lambda + k + 1; \lambda + k + 1, r_1, r_1, \dots, r_m, r_m, -\lambda - l - 1; -\lambda - l - 1)), \end{aligned}$$

where $a' := m(e_1 + e_3) - (e_2 + e_{n+2}) + \sum_{i=1}^m (m - i)(e_{2i+2} + e_{2i+3})$ and the sum is taken over

$$\lambda + 2 \geq r_1 \geq \dots \geq r_m \geq -\lambda - 2, \quad k, l \in \mathbb{Z}_{\geq 0}.$$

We note $\mathfrak{l}'_0(a') \simeq \mathfrak{u}(1, 1)^2 \oplus \mathfrak{u}(2)^m$.

If $n = 2m + 1$ is odd, then

$$\begin{aligned} & A_{\mathfrak{q}(a)}((\lambda; \lambda, \dots, \lambda, -\lambda; -\lambda))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus A_{\mathfrak{q}'(a')}((\lambda; r_1, r_1, \dots, r_m, r_m, -\lambda - k - 1; -\lambda - k - 1)) \\ & \oplus \bigoplus A_{\mathfrak{q}'(a' + e_3)}((r_1 - 1; r_1 - 1, \lambda + 2, r_2, r_2, \dots, r_m, r_m, -\lambda - k - 1; -\lambda - k - 1)), \end{aligned}$$

where $a' := me_1 - (e_2 + e_{n+2}) + \sum_{i=1}^m (m - i)(e_{2i+1} + e_{2i+2})$, the sum is taken over

$$\lambda + 1 \geq r_1 \geq \dots \geq r_m \geq -\lambda - 2, \quad k \in \mathbb{Z}_{\geq 0}$$

for the first term and

$$r_1 \geq \lambda + 2 \geq r_2 \geq \dots \geq r_m \geq -\lambda - 2, \quad k \in \mathbb{Z}_{\geq 0}$$

for the second term. We note

$$\mathfrak{l}'_0(a') \simeq \mathfrak{u}(1) \oplus \mathfrak{u}(2)^m \oplus \mathfrak{u}(1, 1), \quad \mathfrak{l}'_0(a' + e_3) \simeq \mathfrak{u}(1, 1)^2 \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(2)^{m-1}.$$

9.14. $\mathfrak{u}(m, n) \downarrow \mathfrak{u}(m, k) \oplus \mathfrak{u}(n - k)$.

First, we consider the restriction from $\mathfrak{u}(m, n)$ to $\mathfrak{u}(m, n - 1)$. Let $\mathfrak{g}_0 := \mathfrak{u}(m, n)$ and $\mathfrak{g}'_0 := \mathfrak{u}(m, n - 1)$. This is not a symmetric pair but we use similar notation for the subalgebra \mathfrak{g}'_0 . We take a standard basis $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ and its dual basis $e_1, \dots, e_{m+n} \in \mathfrak{t}$. We suppose that $e_1, \dots, e_{m+n-1} \in \mathfrak{t}'$ so they form a basis of a Cartan subalgebra \mathfrak{t}' of \mathfrak{g}' .

We let $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and write Dynkin diagrams of \mathfrak{g} and \mathfrak{g}' :

$$\begin{array}{ccccccc} \alpha_1 & \alpha_2 & & \alpha_{m-1} & \alpha_m & \alpha_{m+1} & \alpha_{m+n-1} \\ \circ & \circ & \cdots & \circ & \bullet & \circ & \circ \\ \alpha_1 & \alpha_2 & & \alpha_{m-1} & \alpha_m & \alpha_{m+1} & \alpha_{m+n-2} \\ \circ & \circ & \cdots & \circ & \bullet & \circ & \circ \end{array}$$

Here, the painted root α_m corresponds to a root in $\mathfrak{g}(1)$. Other roots are in $\mathfrak{g}(0)$. We denote $\sum_{i=1}^{m+n} a_i \epsilon_i \in \mathfrak{t}^*$ by $(a_1, \dots, a_m; a_{m+1}, \dots, a_{m+n})$ and $\sum_{i=1}^{m+n-1} a_i \epsilon_i \in (\mathfrak{t}')^*$ by $(a_1, \dots, a_m; a_{m+1}, \dots, a_{m+n-1})$.

Let $S := \{\alpha_i : m - p + 1 \leq i \leq m + q - 1\}$. Then the parabolic subalgebra $\mathfrak{q}(S)$ is holomorphic and $\mathfrak{l}_0(S) \simeq \mathfrak{u}(p, q) \oplus \mathfrak{u}(1)^{m+n-p-q}$. We have

$$\begin{aligned} & A_{\mathfrak{q}(S)}((\lambda_1, \dots, \lambda_{m-p}, \underbrace{\mu, \dots, \mu}_p; \underbrace{\mu, \dots, \mu}_q, \lambda_{m+q+1}, \dots, \lambda_{m+n}))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus A_{\mathfrak{q}'(S'_1)}((r_1, \dots, r_{m-p}, \underbrace{\mu, \dots, \mu}_p; \underbrace{\mu, \dots, \mu}_{q-1}, s_1, \dots, s_{n-q})) \\ & \oplus \bigoplus A_{\mathfrak{q}'(S'_2)}((r_1, \dots, r_{m-p+1}, \underbrace{\mu+1, \dots, \mu+1}_{p-1}; \underbrace{\mu+1, \dots, \mu+1}_q, s_1, \dots, s_{n-q-1})), \end{aligned}$$

where $S'_1 := \{\alpha_i : m - p + 1 \leq i \leq m + q - 2\}$ and $S'_2 := \{\alpha_i : m - p + 2 \leq i \leq m + q - 1\}$, the sum is taken over

$$\begin{aligned} r_1 & \geq \lambda_1 + 1 \geq r_2 \geq \lambda_2 + 1 \geq \dots \geq r_{m-p} \geq \lambda_{m-p} + 1, \\ \mu + \frac{p+q}{2} & \geq s_1 \geq \lambda_{m+q+1} \geq s_2 \geq \lambda_{m+q+2} \geq \dots \geq s_{n-q} \geq \lambda_{m+n} \end{aligned}$$

for the first term and

$$\begin{aligned} r_1 & \geq \lambda_1 + 1 \geq \dots \geq r_{m-p} \geq \lambda_{m-p} + 1 \geq r_{m-p+1} > \mu - \frac{p+q}{2} + 1, \\ \lambda_{m+q+1} & \geq s_1 \geq \lambda_{m+q+2} \geq s_2 \geq \dots \geq \lambda_{m+n-1} \geq s_{n-q-1} \geq \lambda_{m+n} \end{aligned}$$

for the second term. We note $\mathfrak{l}'_0(S'_1) \simeq \mathfrak{u}(p, q-1) \oplus \mathfrak{u}(1)^{m+n-p-q}$ and $\mathfrak{l}'_0(S'_2) \simeq \mathfrak{u}(p-1, q) \oplus \mathfrak{u}(1)^{m+n-p-q}$.

Let l be an integer such that $0 \leq l \leq m$. We let simple roots for \mathfrak{g} be

$$\begin{aligned} \alpha_i & = \epsilon_i - \epsilon_{i+1} \quad (1 \leq i \leq l-1), \quad \alpha_l = \epsilon_l - \epsilon_{m+1}, \\ \alpha_{l+i} & = \epsilon_{m+i} - \epsilon_{m+i+1} \quad (1 \leq i \leq n-1), \quad \alpha_{l+n} = \epsilon_{m+n} - \epsilon_{l+1}, \\ \alpha_{l+n+i} & = \epsilon_{l+i} - \epsilon_{l+i+1} \quad (1 \leq i \leq m-l-1) \end{aligned}$$

and let simple roots for \mathfrak{g}' be

$$\begin{aligned}\beta_i &= \epsilon_i - \epsilon_{i+1} \quad (1 \leq i \leq l-1), \quad \beta_l = \epsilon_l - \epsilon_{m+1}, \\ \beta_{l+i} &= \epsilon_{m+i} - \epsilon_{m+i+1} \quad (1 \leq i \leq n-2), \quad \beta_{l+n-1} = \epsilon_{m+n-1} - \epsilon_{l+1}, \\ \beta_{l+n+i-1} &= \epsilon_{l+i} - \epsilon_{l+i+1} \quad (1 \leq i \leq m-l-1).\end{aligned}$$

Then the Dynkin diagrams of \mathfrak{g} and \mathfrak{g}' are:

$$\begin{array}{ccccccccccc} \alpha_1 & & \alpha_{l-1} & \alpha_l & \alpha_{l+1} & & \alpha_{l+n-1} & \alpha_{l+n} & \alpha_{l+n+1} & & \alpha_{m+n-1} \\ \circ & \cdots & \circ & \bullet & \circ & \cdots & \circ & \bullet & \circ & \cdots & \circ \\ \text{and} & & & & & & & & & & \\ \beta_1 & & \beta_{l-1} & \beta_l & \beta_{l+1} & & \beta_{l+n-2} & \beta_{l+n-1} & \beta_{l+n} & & \beta_{m+n-2} \\ \circ & \cdots & \circ & \bullet & \circ & \cdots & \circ & \bullet & \circ & \cdots & \circ \end{array}$$

Here, the painted roots α_l and α_{l+n} correspond to roots in $\mathfrak{g}(1)$. Other roots are in $\mathfrak{g}(0)$. We denote $\sum_{i=1}^{m+n} a_i \epsilon_i \in \mathfrak{t}^*$ by $(a_1, \dots, a_l; a_{m+1}, \dots, a_{m+n}; a_{l+1}, \dots, a_m)$ and $\sum_{i=1}^{m+n-1} a_i \epsilon_i \in (\mathfrak{t}')^*$ by $(a_1, \dots, a_l; a_{m+1}, \dots, a_{m+n-1}; a_{l+1}, \dots, a_m)$.

Let $S := \{\alpha_i : l-p+1 \leq i \leq l+q-1\} \cup \{\alpha_i : l+n-r+1 \leq i \leq l+n+s-1\}$ for non-negative integers p, q, r, s such that

$$p \leq l, \quad s \leq m-l, \quad p+q \geq 1, \quad r+s \geq 1, \quad \text{and } q+r \leq n.$$

Then $\mathfrak{l}_0(S) \simeq \mathfrak{u}(p, q) \oplus \mathfrak{u}(r, s) \oplus \mathfrak{u}(1)^{m+n-p-q-r-s}$. We have

(9.2)

$$\begin{aligned} & A_{\mathfrak{q}(S)}((\lambda_1, \dots, \lambda_{l-p}, \underbrace{\mu, \dots, \mu}_p; \underbrace{\mu, \dots, \mu}_q, \lambda_{m+q+1}, \dots, \lambda_{m+n-r}, \\ & \quad \underbrace{\nu, \dots, \nu}_r; \underbrace{\nu, \dots, \nu}_s, \lambda_{l+s+1}, \dots, \lambda_m))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus A_{\mathfrak{q}'(S'_1)}((t_1, \dots, t_{l-p}, \underbrace{\mu, \dots, \mu}_p; \underbrace{\mu, \dots, \mu}_{q-1}, u_1, \dots, u_{n-q-r+1}, \\ & \quad \underbrace{\nu, \dots, \nu}_{r-1}; \underbrace{\nu, \dots, \nu}_s, v_1, \dots, v_{m-l-s})) \\ & \oplus \bigoplus A_{\mathfrak{q}'(S'_2)}((t_1, \dots, t_{l-p+1}, \underbrace{\mu+1, \dots, \mu+1}_{p-1}; \underbrace{\mu+1, \dots, \mu+1}_q, u_1, \dots, u_{n-q-r}, \\ & \quad \underbrace{\nu, \dots, \nu}_{r-1}; \underbrace{\nu, \dots, \nu}_s, v_1, \dots, v_{m-l-s})) \\ & \oplus \bigoplus A_{\mathfrak{q}'(S'_3)}((t_1, \dots, t_{l-p}, \underbrace{\mu, \dots, \mu}_p; \underbrace{\mu, \dots, \mu}_{q-1}, u_1, \dots, u_{n-q-r}, \\ & \quad \underbrace{\nu-1, \dots, \nu-1}_r; \underbrace{\nu-1, \dots, \nu-1}_{s-1}, v_1, \dots, v_{m-l-s+1})) \\ & \oplus \bigoplus A_{\mathfrak{q}'(S'_4)}((t_1, \dots, t_{l-p+1}, \underbrace{\mu+1, \dots, \mu+1}_{p-1}; \underbrace{\mu+1, \dots, \mu+1}_q, u_1, \dots, u_{n-q-r-1}, \\ & \quad \underbrace{\nu-1, \dots, \nu-1}_r; \underbrace{\nu-1, \dots, \nu-1}_{s-1}, v_1, \dots, v_{m-l-s+1})), \end{aligned}$$

where

$$\begin{aligned} S'_1 &:= \{\beta_i : l-p+1 \leq i \leq l+q-2\} \cup \{\beta_i : l+n-r+1 \leq i \leq l+n+s-2\}, \\ S'_2 &:= \{\beta_i : l-p+2 \leq i \leq l+q-1\} \cup \{\beta_i : l+n-r+1 \leq i \leq l+n+s-2\}, \\ S'_3 &:= \{\beta_i : l-p+1 \leq i \leq l+q-2\} \cup \{\beta_i : l+n-r \leq i \leq l+n+s-3\}, \\ S'_4 &:= \{\beta_i : l-p+2 \leq i \leq l+q-1\} \cup \{\beta_i : l+n-r \leq i \leq l+n+s-3\}, \end{aligned}$$

and the sum is taken over

$$\begin{aligned} t_1 &\geq \lambda_1 + 1 \geq t_2 \geq \lambda_2 + 1 \geq \cdots \geq t_{l-p} \geq \lambda_{l-p} + 1, \\ \mu + \frac{p+q}{2} &\geq u_1 \geq \lambda_{m+q+1} \geq \cdots \geq u_{n-q-r} \geq \lambda_{m+n-r} \geq u_{n-q-r+1} \geq \nu - \frac{r+s}{2}, \\ \lambda_{l+s+1} - 1 &\geq v_1 \geq \lambda_{l+s+2} - 1 \geq v_2 \geq \cdots \geq \lambda_m - 1 \geq v_{m-l-s} \end{aligned}$$

for the first term,

$$\begin{aligned} t_1 &\geq \lambda_1 + 1 \geq \cdots \geq t_{l-p} \geq \lambda_{l-p} + 1 \geq t_{l-p+1} > \mu - \frac{p+q}{2} + 1, \\ \lambda_{m+q+1} &\geq u_1 \geq \lambda_{m+q+2} \geq u_2 \geq \cdots \geq \lambda_{m+n-r} \geq u_{n-q-r} \geq \nu - \frac{r+s}{2}, \\ \lambda_{l+s+1} - 1 &\geq v_1 \geq \lambda_{l+s+2} - 1 \geq v_2 \geq \cdots \geq \lambda_m - 1 \geq v_{m-l-s} \end{aligned}$$

for the second term,

$$\begin{aligned} t_1 &\geq \lambda_1 + 1 \geq t_2 \geq \lambda_2 + 1 \geq \cdots \geq t_{l-p} \geq \lambda_{l-p} + 1, \\ \mu + \frac{p+q}{2} &\geq u_1 \geq \lambda_{m+q+1} \geq u_2 \geq \lambda_{m+q+2} \geq \cdots \geq u_{n-q-r} \geq \lambda_{m+n-r}, \\ \nu - \frac{r+s}{2} - 1 &> v_1 \geq \lambda_{l+s+1} - 1 \geq v_2 \geq \cdots \geq \lambda_m - 1 \geq v_{m-l-s+1} \end{aligned}$$

for the third term, and

$$\begin{aligned} t_1 &\geq \lambda_1 + 1 \geq \cdots \geq t_{l-p} \geq \lambda_{l-p} + 1 \geq t_{l-p+1} > \mu - \frac{p+q}{2} + 1, \\ \lambda_{m+q+1} &\geq u_1 \geq \lambda_{m+q+2} \geq u_2 \geq \lambda_{m+q+3} \geq \cdots \geq u_{n-q-r-1} \geq \lambda_{m+n-r}, \\ \nu - \frac{r+s}{2} - 1 &> v_1 \geq \lambda_{l+s+1} - 1 \geq v_2 \geq \cdots \geq \lambda_m - 1 \geq v_{m-l-s+1} \end{aligned}$$

for the fourth term. We note that

$$\begin{aligned} \ell'_0(S'_1) &\simeq u(p, q-1) \oplus u(r-1, s) \oplus u(1)^{m+n-p-q-r-s}, \\ \ell'_0(S'_2) &\simeq u(p-1, q) \oplus u(r-1, s) \oplus u(1)^{m+n-p-q-r-s}, \\ \ell'_0(S'_3) &\simeq u(p, q-1) \oplus u(r, s-1) \oplus u(1)^{m+n-p-q-r-s}, \\ \ell'_0(S'_4) &\simeq u(p-1, q) \oplus u(r, s-1) \oplus u(1)^{m+n-p-q-r-s}. \end{aligned}$$

We remark that the second and the fourth terms do not appear if $p = 0$ and the first term does not appear if $q + r = n$. Similarly for the case q, r , or $s = 0$.

We now consider the general case. Let $\mathfrak{g}_0 := u(m, n)$ and $\mathfrak{g}'_0 := u(m, k) \oplus u(n-k)$ for positive integers m, n, k such that $n-k > 0$. We take a standard basis $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ and $e_1, \dots, e_{m+n} \in \mathfrak{t}$ as before. We may assume that

$\mathfrak{t} \subset \mathfrak{k}'$. Let simple roots α_i for \mathfrak{g} be as in the previous case and let simple roots for the $\mathfrak{u}(m, k)$ -component of \mathfrak{g}' be

$$\begin{aligned}\beta_i &= \epsilon_i - \epsilon_{i+1} \quad (1 \leq i \leq l-1), \quad \beta_l = \epsilon_l - \epsilon_{m+1}, \\ \beta_{l+i} &= \epsilon_{m+i} - \epsilon_{m+i+1} \quad (1 \leq i \leq k-1), \quad \beta_{l+k} = \epsilon_{m+k} - \epsilon_{l+1}, \\ \beta_{l+k+i} &= \epsilon_{l+i} - \epsilon_{l+i+1} \quad (1 \leq i \leq m-l-1).\end{aligned}$$

Then the Dynkin diagrams for \mathfrak{g} and $\mathfrak{u}(m, k)$ are:

$$\begin{array}{ccccccccccc} \alpha_1 & & \alpha_{l-1} & \alpha_l & \alpha_{l+1} & & \alpha_{l+n-1} & \alpha_{l+n} & \alpha_{l+n+1} & & \alpha_{m+n-1} \\ \circ & \cdots & \circ & \bullet & \circ & \cdots & \circ & \bullet & \circ & \cdots & \circ \end{array}$$

and

$$\begin{array}{ccccccccccc} \beta_1 & & \beta_{l-1} & \beta_l & \beta_{l+1} & & \beta_{l+k-1} & \beta_{l+k} & \beta_{l+k+1} & & \beta_{m+k-1} \\ \circ & \cdots & \circ & \bullet & \circ & \cdots & \circ & \bullet & \circ & \cdots & \circ \end{array}$$

We denote $\sum_{i=1}^{m+n} a_i \epsilon_i \in \mathfrak{t}^*$ by $(a_1, \dots, a_l; a_{m+1}, \dots, a_{m+n}; a_{l+1}, \dots, a_m)$. When we regard $\sum_{i=1}^{m+n} a_i \epsilon_i \in \mathfrak{t}^*$ as a weight for \mathfrak{g}' , we denote it by

$$(a_1, \dots, a_l; a_{m+1}, \dots, a_{m+k}; a_{l+1}, \dots, a_m; a_{m+k+1}, \dots, a_{m+n})'.$$

Let S be as in the previous case, namely, $S := \{\alpha_i : l-p+1 \leq i \leq l+q-1\} \cup \{\alpha_i : l+n-r+1 \leq i \leq l+n+s-1\}$ for non-negative integers p, q, r, s such that

$$p \leq l, \quad s \leq m-l, \quad \text{and} \quad q+r \leq n.$$

Then the formula (9.2) implies that the restriction of

$$A_{q(S)}((\lambda_1, \dots, \lambda_{l-p}, \underbrace{\mu, \dots, \mu}_p; \underbrace{\mu, \dots, \mu}_q, \lambda_{m+q+1}, \dots, \lambda_{m+n-r}, \underbrace{\nu, \dots, \nu}_r; \underbrace{\nu, \dots, \nu}_s, \lambda_{l+s+1}, \dots, \lambda_m))$$

decomposes into a direct sum (with multiplicity) of

$$A_{q'(S')}((t_1, \dots, t_{l-p'}, \underbrace{\mu+p-p', \dots, \mu+p-p'}_{p'}, \underbrace{\mu+p-p', \dots, \mu+p-p'}_{q'}, u_1, \dots, u_{k-q'-r'}, \underbrace{\nu-s+s', \dots, \nu-s+s'}_{r'}, \underbrace{\nu-s+s', \dots, \nu-s+s'}_{s'}, v_1, \dots, v_{m-l-s'}, w_1, \dots, w_{n-k})),$$

where p', q', r', s' are non-negative integers such that

$$\begin{aligned}p' &\leq p, \quad q' \leq q, \quad r' \leq r, \quad s' \leq s, \\ p' + q' &= p + q - (n - k) \text{ if } p + q > n - k, \quad p' = q' = 0 \text{ if } p + q \leq n - k, \\ r' + s' &= r + s - (n - k) \text{ if } r + s > n - k, \quad r' = s' = 0 \text{ if } r + s \leq n - k,\end{aligned}$$

$S' := \{\beta_i : l-p'+1 \leq i \leq l+q'-1\} \cup \{\beta_i : l+k-r'+1 \leq i \leq l+k+s'-1\}$, and the parameters are in the weakly fair range. We note

$\ell'_0(S') \simeq \mathbf{u}(p', q') \oplus \mathbf{u}(r', s') \oplus \mathbf{u}(1)^{m+n-p'-q'-r'-s'}$. Put

$$\begin{aligned} \lambda := & (\lambda_1, \dots, \lambda_{l-p}, \underbrace{\mu, \dots, \mu}_p; \underbrace{\mu, \dots, \mu}_q, \lambda_{m+q+1}, \dots, \lambda_{m+n-r}, \\ & \underbrace{\nu, \dots, \nu}_r; \underbrace{\nu, \dots, \nu}_s, \lambda_{l+s+1}, \dots, \lambda_m), \\ \lambda' := & (t_1, \dots, t_{l-p'}, \underbrace{\mu+p-p', \dots, \mu+p-p'}_{p'}, \underbrace{\mu+p-p', \dots, \mu+p-p'}_{q'}, u_1, \dots, u_{k-q'-r'}, \\ & \underbrace{\nu-s+s', \dots, \nu-s+s'}_{r'}, \underbrace{\nu-s+s', \dots, \nu-s+s'}_{s'}, v_1, \dots, v_{m-l-s'}, w_1, \dots, w_{n-k})' \end{aligned}$$

and write

$$m(k, l, m, n, p, q, r, s, p', q', r', s', S, S', \lambda, \lambda')$$

for the multiplicity of $A_{q'(S')}(\lambda')$ in $A_{q(S)}(\lambda)|_{(\mathfrak{g}', K')}$. We now reduce the computation of multiplicity to lower rank cases. Let p'', q'', r'', s'', c, d be non-negative integers such that

$$\begin{aligned} p'' &\leq p', \quad q'' \leq q', \quad r'' \leq r', \quad s'' \leq s', \\ p'' + q'' &= 2c, \quad r'' + s'' = 2d. \end{aligned}$$

We use similar notation for the restriction $\mathbf{u}(m-p''-s'', n-q''-r'') \downarrow \mathbf{u}(m-p''-s'', k-q''-r'') \oplus \mathbf{u}(n-k)$. Then if we put

$$\begin{aligned} \kappa := & (\lambda_1+2c, \dots, \lambda_{l-p}+2c, \underbrace{\mu+c, \dots, \mu+c}_{p-p''}; \underbrace{\mu+c, \dots, \mu+c}_{q-q''}, \lambda_{m+q+1}, \dots, \lambda_{m+n-r}, \\ & \underbrace{\nu-d, \dots, \nu-d}_{r-r''}; \underbrace{\nu-d, \dots, \nu-d}_{s-s''}, \lambda_{l+s+1}-2d, \dots, \lambda_m-2d), \\ \kappa' := & (t_1+2c, \dots, t_{l-p'}+2c, \underbrace{\mu+p-p'+c, \dots, \mu+p-p'+c}_{p'-p''}; \underbrace{\mu+p-p'+c, \dots, \mu+p-p'+c}_{q'-q''}, \\ & u_1, \dots, u_{k-q'-r'}, \underbrace{\nu-s+s'-d, \dots, \nu-s+s'-d}_{r'-r''}; \underbrace{\nu-s+s'-d, \dots, \nu-s+s'-d}_{s'-s''}, \\ & v_1-2d, \dots, v_{m-l-s'}-2d; w_1+c-d, \dots, w_{n-k}+c-d)', \end{aligned}$$

(9.2) implies that

$$\begin{aligned} & m(k, l, m, n, p, q, r, s, p', q', r', s', S, S', \lambda, \lambda') \\ &= m(k-q''-r'', l-p'', m-p''-s'', n-q''-r'', \\ & \quad p-p'', q-q'', r-r'', s-s'', p'-p'', q'-q'', r'-r'', s'-s'', T, T', \kappa, \kappa'), \end{aligned}$$

where

$$\begin{aligned} T &:= \{\alpha_i : l-p+1 \leq i \leq l-p''+q-q''-1\} \\ &\quad \cup \{\alpha_i : l+n-p''-q''-r+1 \leq i \leq l+n-p''-q''-r''-s''+s-1\}, \\ T' &:= \{\beta_i : l-p'+1 \leq i \leq l-p''+q'-q''-1\} \\ &\quad \cup \{\beta_i : l+k-p''-q''-r'+1 \leq i \leq l+k-p''-q''-r''-s''+s'-1\}. \end{aligned}$$

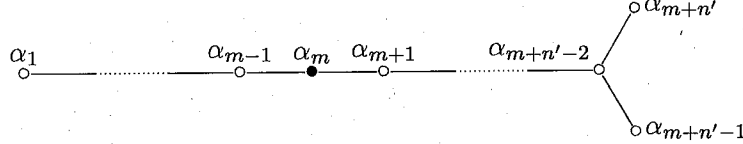
Therefore, the determination of the multiplicity $m(\cdot)$ is reduced to the case where $p' + q' \leq 1$ and $r' + s' \leq 1$. This is equivalent to $p + q - 1, r + s - 1 \leq n - k$, which implies the assumption of Theorem 9.2. Hence we can compute the multiplicity by using Theorem 9.2.

We remark that the (\mathfrak{g}', K') -modules $A_{q'(S')}(\lambda)$ can be isomorphic to each other among different S' . So we need to avoid overlap for the branching formula. This occurs when the parameter is on the boundary of weakly fair range. We can see this phenomenon from Theorem 6.3.

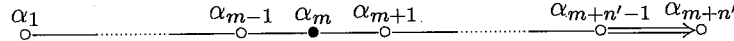
Let $S := \{\alpha_i : l - p + 1 \leq i \leq l + n + q - 1\}$ for non-negative integers p, q so that $\mathfrak{l}_0(S) \simeq \mathfrak{u}(p + q, n) \oplus \mathfrak{u}(1)^{m-p-q}$. Then Corollary 5.8 implies that $A_{q(S)}(\lambda)|_{(\mathfrak{g}', K')}$ decomposes into a direct sum (with multiplicity) of (\mathfrak{g}', K') -modules $A_{q'(S')}(\lambda')$, where $S' := \{\beta_i : l - p + 1 \leq i \leq l + k + q - 1\}$. Note $\mathfrak{l}'_0(S') \simeq \mathfrak{u}(p + q, k) \oplus \mathfrak{u}(1)^{m+n-p-q-k}$. The multiplicity is obtained in a way similar to the case $\mathfrak{so}(2m + 1, n) \downarrow \mathfrak{so}(2m + 1, k) \oplus \mathfrak{so}(n - k)$ treated in Section 8.

9.15. $\mathfrak{so}(2m, n) \downarrow \mathfrak{so}(2m, k) \oplus \mathfrak{so}(n - k)$.

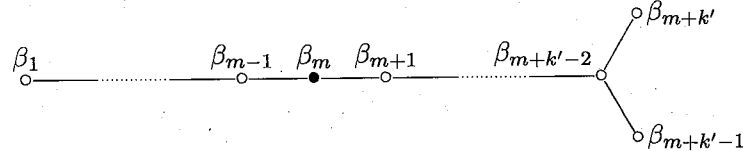
Let $\mathfrak{g}_0 := \mathfrak{so}(2m, n)$ and $\mathfrak{g}'_0 := \mathfrak{so}(2m, k) \oplus \mathfrak{so}(n - k)$. We assume $k > 2$ to simplify the notation. The case of $k \leq 2$ is similar. Put $n' := \lfloor \frac{n}{2} \rfloor$, $k' := \lfloor \frac{k}{2} \rfloor$, and $l := \lfloor \frac{n-k}{2} \rfloor$. We write the Dynkin diagram of \mathfrak{g} and label the simple roots:



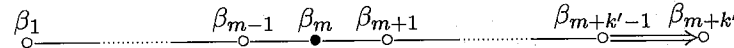
if n is even and



if n is odd. Similarly, the Dynkin diagram of the $\mathfrak{so}(2m, k)$ -component of \mathfrak{g}' is:



if k is even and



if k is odd.

Let p and q be non-negative integers such that $p \leq m$ and $q \leq n'$. Put $S := \{\alpha_i : m - p + 1 \leq i \leq m + q - 1\}$ and $\mathfrak{q} := \mathfrak{q}(S)$. The Levi subalgebra $\mathfrak{l}_0(S)$ is isomorphic to $\mathfrak{u}(p, q) \oplus \mathfrak{u}(1)^{m+n'-p-q}$. As in the case $\mathfrak{u}(m, n) \downarrow \mathfrak{u}(m, k) \oplus \mathfrak{u}(n - k)$, the restriction $A_{q(S)}(\lambda)|_{(\mathfrak{g}', K')}$ is isomorphic to a direct sum (with multiplicity) of (\mathfrak{g}', K') -modules $A_{q'(S')}(\lambda')$. The possible subsets $S' \subset \Pi'$ are given as follows:

$$S' = \emptyset, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1)^{m+k'+l}$$

if $p + q \leq n - k$;

$$S' = \{\beta_i : m - p' + 1 \leq i \leq m + q' - 1\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(p', q') \oplus \mathfrak{u}(1)^{m+k'+l-p-q+n-k}$$

for integers p' and q' such that

$$0 \leq p' \leq p, \quad 0 \leq q' \leq q, k', \quad \text{and} \quad p' + q' = p + q - (n - k);$$

if $p + q > n - k$; in addition if k is even and if $(p', q') = (p', k')$ satisfies the conditions above, then

$$S' = \{\beta_i : m - p' + 1 \leq i \leq m + k' - 2\} \cup \{\beta_{m+k'}\},$$

$$\mathfrak{l}'_0(S') \simeq \mathfrak{u}(p', q') \oplus \mathfrak{u}(1)^{m+k'+l-p-q+n-k}$$

is also possible. The determination of the multiplicity can be reduced to lower rank cases as well.

Let p be a non-negative integer such that $p \leq m$. Put $S := \{\alpha_i : m - p + 1 \leq i \leq m + n'\}$ and $\mathfrak{q} := \mathfrak{q}(S)$. The Levi subalgebra $\mathfrak{l}_0(S)$ is isomorphic to $\mathfrak{so}(2p, n) \oplus \mathfrak{u}(1)^{m-p}$. This case is similar to the case $\mathfrak{so}(2m+1, n) \downarrow \mathfrak{so}(2m+1, k) \oplus \mathfrak{so}(n-k)$ in Section 8. By applying Corollary 5.8, $A_{\mathfrak{q}(S)}(\lambda)$ decomposes into a direct sum (with multiplicity) of (\mathfrak{g}', K') -modules $A_{\mathfrak{q}'(S')}(\lambda')$, where $S' = \{\beta_i : m - p + 1 \leq i \leq m + k'\}$ and $\mathfrak{l}'_0(S') \simeq \mathfrak{so}(2p, k) \oplus \mathfrak{u}(1)^{m-p+l}$.

9.16. $\mathfrak{sp}(m, n) \downarrow \mathfrak{sp}(m, k) \oplus \mathfrak{sp}(n - k)$.

Let $\mathfrak{g}_0 := \mathfrak{sp}(m, n)$ and $\mathfrak{g}'_0 := \mathfrak{sp}(m, k) \oplus \mathfrak{sp}(n - k)$. We write the Dynkin diagram of \mathfrak{g} and label the simple roots:

$$\begin{array}{ccccccc} \alpha_1 & & \alpha_{m-1} & \alpha_m & \alpha_{m+1} & & \alpha_{m+n-1} & \alpha_{m+n} \\ \circ & \cdots & \circ & \bullet & \circ & \cdots & \circ & \circ \end{array}$$

Similarly, the Dynkin diagram of the $\mathfrak{sp}(m, k)$ -component of \mathfrak{g}' is:

$$\begin{array}{ccccccc} \beta_1 & & \beta_{m-1} & \beta_m & \beta_{m+1} & & \beta_{m+k-1} & \beta_{m+k} \\ \circ & \cdots & \circ & \bullet & \circ & \cdots & \circ & \circ \end{array}$$

Let p and q be non-negative integers such that $p \leq m$ and $q \leq n$. Put $S := \{\alpha_i : m - p + 1 \leq i \leq m + q - 1\}$ and $\mathfrak{q} := \mathfrak{q}(S)$. The Levi subalgebra $\mathfrak{l}_0(S)$ is isomorphic to $\mathfrak{u}(p, q) \oplus \mathfrak{u}(1)^{m+n-p-q}$. As in the case $\mathfrak{u}(m, n) \downarrow \mathfrak{u}(m, k) \oplus \mathfrak{u}(n - k)$, the restriction $A_{\mathfrak{q}(S)}(\lambda)|_{(\mathfrak{g}', K')}$ is isomorphic to a direct sum (with multiplicity) of (\mathfrak{g}', K') -modules $A_{\mathfrak{q}'(S')}(\lambda')$. The possible subsets $S' \subset \Pi'$ are given as follows:

$$S' = \emptyset, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1)^{m+n}$$

if $p + q \leq 2(n - k)$, and

$$S' = \{\beta_i : m - p' + 1 \leq i \leq m + q' - 1\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(p', q') \oplus \mathfrak{u}(1)^{m+n-p-q+2(n-k)}$$

for integers p' and q' such that

$$0 \leq p' \leq p, \quad 0 \leq q' \leq q, \quad p' + q' = p + q - 2(n - k)$$

if $p + q > 2(n - k)$. The determination of the multiplicity can be reduced to lower rank cases.

Let p be a non-negative integer such that $p \leq m$. Put $S := \{\alpha_i : m - p + 1 \leq i \leq m + n\}$ and $\mathfrak{q} := \mathfrak{q}(S)$. The Levi subalgebra $\mathfrak{l}_0(S)$ is isomorphic to

$\mathfrak{sp}(p, n) \oplus \mathfrak{u}(1)^{m-p}$. This case is similar to the case $\mathfrak{so}(2m+1, n) \downarrow \mathfrak{so}(2m+1, k) \oplus \mathfrak{so}(n-k)$ in Section 8. By applying Corollary 5.8, $A_{\mathfrak{q}(S)}(\lambda)$ decomposes into a direct sum (with multiplicity) of (\mathfrak{g}', K') -modules $A_{\mathfrak{q}'(S')}(\lambda')$, where $S' = \{\beta_i : m-p+1 \leq i \leq m+k\}$ and $\mathfrak{l}'_0(S') \simeq \mathfrak{sp}(p, k) \oplus \mathfrak{u}(1)^{m-p+n-k}$.

Let us consider the case where \mathfrak{q} is of isolated type, which was postponed in the last section. We take a basis $\epsilon_1, \dots, \epsilon_{m+n} \in \mathfrak{t}^*$ and its dual basis $e_1, \dots, e_{m+n} \in \mathfrak{t}$ as in Setting 7.10. We let $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq m+n-1$ and $\alpha_{m+n} = 2\epsilon_{m+n}$. Similarly, let $\beta_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq m+k-1$ and $\beta_{m+k} = 2\epsilon_{m+k}$. Then they agree with the previous notation. Suppose that \mathfrak{q} is given by $a = \sum_{i=1}^{m+n} a_i e_i$ such that

$$a_1 \geq \dots \geq a_p \geq a_{m+1} > a_{p+1} = \dots = a_m = a_{m+2} = \dots = a_{m+n} = 0$$

for some $1 \leq p \leq m-1$. We denote $\sum_{i=1}^{m+n} a_i \epsilon_i \in \mathfrak{t}^*$ by $(a_1, \dots, a_m; a_{m+1}, \dots, a_{m+n})$. When we regard $\sum_{i=1}^{m+n} a_i \epsilon_i \in \mathfrak{t}^*$ as a weight for \mathfrak{g}' , then we denote it by $(a_1, \dots, a_m; a_{m+1}, \dots, a_{m+k}; a_{m+k+1}, \dots, a_{m+n})$. Although the triple $(\mathfrak{g}_0, \mathfrak{g}'_0, \mathfrak{q})$ is not of discrete series type, we can prove analogs of Theorems 9.1 and 9.2 in this case. First, we assume that $a_p > a_{m+1}$ so that $\mathfrak{l}_0(a)$ is a direct sum of $\mathfrak{sp}(m-p, n-1)$ and a compact Lie algebra. The parameter λ for $\mathfrak{q} := \mathfrak{q}(a)$ can be written as $\lambda = (\lambda_1, \dots, \lambda_p, 0, \dots, 0; \lambda_{m+1}, 0, \dots, 0)$. We put $b := e_1 + \dots + e_{p+1}$. Then $L(b)$ is isomorphic to $GL(p+1, \mathbb{C}) \times Sp(m-p+n-1, \mathbb{C})$ up to covering. We put $s(b) := \dim(\mathfrak{u}(b) \cap \mathfrak{k})$. If $\lambda_1 \geq \dots \geq \lambda_p \geq -\lambda_{m+1} - 2(m-p+n)$, Theorem 6.5 gives

$$\begin{aligned} & A_{\mathfrak{q}(a)}((\lambda_1, \dots, \lambda_p, 0, \dots, 0; \lambda_{m+1}, 0, \dots, 0)) \\ & \simeq A_{\mathfrak{q}(a+a_{m+1}(e_{p+1}-e_{m+1}))}((\lambda_1, \dots, \lambda_p, -\lambda_{m+1}-2(m-p+n), 0, \dots, 0; 0, \dots, 0)) \\ & \simeq \mathcal{L}_{\mathfrak{q}(-b), s(b)}^{\mathfrak{g}}(F^{L(b)}(\lambda_1, \dots, \lambda_p, -\lambda_{m+1}-2(m-p+n), 0, \dots, 0; 0, \dots, 0)). \end{aligned}$$

Since K' acts transitively on $K/(\mathcal{Q}(-b) \cap K)$, the restriction

$$[A_{\mathfrak{q}(a)}((\lambda_1, \dots, \lambda_p, 0, \dots, 0; \lambda_{m+1}, 0, \dots, 0))|_{(\mathfrak{g}', K')}]$$

can be written as a sum of $[A_{\mathfrak{q}'(c')}(\lambda')]$, where $c' = \sum_{i=1}^{p+1} (p-i+2)e_i + \sum_{i=1}^{n-k} (n-k-i+1)e_{m+k+i}$ and hence $\mathfrak{l}'_0(c') \simeq \mathfrak{sp}(m-p-1, n) \oplus \mathfrak{u}(1)^{p+1}$. If there exists $0 \leq q \leq p-1$ such that $a_q > a_{q+1} = \dots = a_p = a_{m+1}$, then the Levi subalgebra $\mathfrak{l}_0(a)$ is a direct sum of $\mathfrak{u}(p-q, 1) \oplus \mathfrak{sp}(m-p, n-1)$ and a compact Lie algebra. By using BGG type resolution of one-dimensional representation of $\mathfrak{l}_0(a)$ for the $\mathfrak{u}(p-q, 1)$ -component, we can describe $[A_{\mathfrak{q}(a)}(\lambda)|_{(\mathfrak{g}', K')}]$ as an alternating sum of $[A_{\mathfrak{q}'(c')}(\lambda')]$, which is analogous to Theorem 9.1.

On the other hand, by considering the case $n-k=1$, it turns out that the restriction $A_{\mathfrak{q}(a)}(\lambda)|_{(\mathfrak{g}', K')}$ can be written as

$$\begin{aligned} & A_{\mathfrak{q}(a)}((\lambda_1, \dots, \lambda_q, \underbrace{\mu, \dots, \mu}_{p-q}, \underbrace{0, \dots, 0}_{m-p}, \underbrace{\mu, 0, \dots, 0}_{n-1}))|_{(\mathfrak{g}', K')} \\ & \simeq \bigoplus m_1(r, s, t) A_{\mathfrak{q}'(a')}((r_1, \dots, r_{q+2(n-k)-1}, \underbrace{\mu+2(n-k)-1, \dots, \mu+2(n-k)-1}_{p-q-2(n-k)+1})). \end{aligned}$$

$$\begin{aligned}
& \underbrace{0, \dots, 0}_{m-p}; s, \underbrace{0, \dots, 0}_{k-1}; t_1, \dots, t_{n-k}) \\
& \oplus \bigoplus m_2(r, s, t) A_{q'(a'+e_{p+1}-e_{m+1})}((r_1, \dots, r_{q+2(n-k)-1}, \\
& \quad \underbrace{\mu+2(n-k)-1, \dots, \mu+2(n-k)-1}_{p-q-2(n-k)+1}, s, \underbrace{0, \dots, 0}_{m-p-1}, \underbrace{0, \dots, 0}_k; t_1, \dots, t_{n-k})) \\
& \oplus \bigoplus m_3(r, s, t) A_{q'(a'+e_{q+2(n-k)}+e_{m+1})}((r_1, \dots, r_{q+2(n-k)-1}, s, \\
& \quad \underbrace{\mu+2(n-k), \dots, \mu+2(n-k)}_{p-q-2(n-k)}, \underbrace{0, \dots, 0}_{m-p}, \underbrace{\mu+2(n-k), 0, \dots, 0}_{k-1}; t_1, \dots, t_{n-k}))
\end{aligned}$$

if $p - q + 1 > 2(n - k)$ and

$$\begin{aligned}
& A_{q(a)}((\lambda_1, \dots, \lambda_q, \underbrace{\mu, \dots, \mu}_{p-q}, \underbrace{0, \dots, 0}_{m-p}, \underbrace{\mu, 0, \dots, 0}_{n-1}))|_{(\mathfrak{g}', K')} \\
& \simeq \bigoplus m_1(r, s, t) A_{q'(b')}((r_1, \dots, r_p, \underbrace{0, \dots, 0}_{m-p}, s, \underbrace{0, \dots, 0}_{k-1}; t_1, \dots, t_{n-k})) \\
& \oplus \bigoplus m_2(r, s, t) A_{q'(b'+e_{p+1}-e_{m+1})}((r_1, \dots, r_p, s, \underbrace{0, \dots, 0}_{m-p-1}, \underbrace{0, \dots, 0}_k; t_1, \dots, t_{n-k}))
\end{aligned}$$

if $p - q + 1 \leq 2(n - k)$. Here,

$$\begin{aligned}
a' &= \sum_{i=1}^{q+2(n-k)-1} (q+2(n-k)-i+3)e_i + \sum_{i=q+2(n-k)}^p 2e_i + e_{m+1} + \sum_{i=1}^{n-k} (n-k-i+1)e_{m+k+i}, \\
b' &= \sum_{i=1}^p (p-i+2)e_i + e_{m+1} + \sum_{i=1}^{n-k} (n-k-i+1)e_{m+k+i}
\end{aligned}$$

and hence

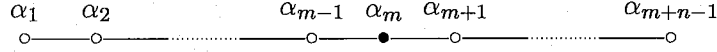
$$\begin{aligned}
\mathfrak{l}'_0(a') &\simeq \mathfrak{u}(p-q-2(n-k)+1) \oplus \mathfrak{sp}(m-p, k-1) \oplus \mathfrak{u}(1)^{q+3(n-k)}, \\
\mathfrak{l}'_0(a'+e_{p+1}-e_{m+1}) &\simeq \mathfrak{u}(p-q-2(n-k)+1) \oplus \mathfrak{sp}(m-p-1, k) \oplus \mathfrak{u}(1)^{q+3(n-k)}, \\
\mathfrak{l}'_0(a'+e_{q+2(n-k)}+e_{m+1}) &\simeq \mathfrak{u}(p-q-2(n-k), 1) \oplus \mathfrak{sp}(m-p, k-1) \oplus \mathfrak{u}(1)^{q+3(n-k)}, \\
\mathfrak{l}'_0(b') &\simeq \mathfrak{sp}(m-p, k-1) \oplus \mathfrak{u}(1)^{p+n-k+1}, \\
\mathfrak{l}'_0(b'+e_{p+1}-e_{m+1}) &\simeq \mathfrak{sp}(m-p-1, k) \oplus \mathfrak{u}(1)^{p+n-k+1}.
\end{aligned}$$

As in the case $\mathfrak{u}(m, n) \downarrow \mathfrak{u}(m, k) \oplus \mathfrak{u}(n-k)$, the determination of multiplicity $m_i(r, s, t)$ can be reduced to lower rank cases.

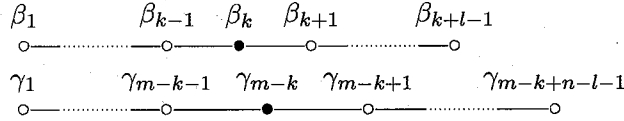
The remaining cases are Theorem 7.22 (3) and (4). We can write $A_{q(S)}(\lambda)|_{(\mathfrak{g}, K)}$ as a direct sum of derived functor modules $A_{q'(S')}(\lambda')$ in these cases as well. In what follows, we only write down $S' \subset \Pi'$ for each $S \subset \Pi$ such that $A_{q'(S')}(\lambda')$ can appear in the branching law of $A_{q(S)}(\lambda)|_{(\mathfrak{g}, K)}$. As in the previous cases, painted roots denote roots in $\mathfrak{g}(1)$.

9.17. $\mathfrak{u}(m, n) \downarrow \mathfrak{u}(k, l) \oplus \mathfrak{u}(m-k, n-l)$.

Let $\mathfrak{g}_0 := \mathfrak{u}(m, n)$ and $\mathfrak{g}'_0 := \mathfrak{u}(k, l) \oplus \mathfrak{u}(m-k, n-l)$. We label the simple roots of \mathfrak{g} as



and those of \mathfrak{g}' as



Suppose that $S = \{\alpha_i : m-p+1 \leq i \leq m+q-1\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{u}(p, q) \oplus \mathfrak{u}(1)^{m+n-p-q}$. Let $S' \subset \Pi'$ be a subset such that $A_{q'(S')}(\lambda')$ occurs in the branching law of $A_{q(S)}(\lambda)|_{(\mathfrak{g}', K')}$. Put $S'_1 := S' \cap \{\beta_i\}_{1 \leq i \leq k+l-1}$ and $S'_2 := S' \cap \{\gamma_i\}_{1 \leq i \leq m-k+n-l-1}$. Then

$$S'_1 = \{\beta_i : k-p'+1 \leq i \leq k+q'-1\}$$

for integers p' and q' such that

$$0 \leq p' \leq p, k, \quad 0 \leq q' \leq q, l, \quad \text{and } p' + q' = p + q - (m - k + n - l)$$

if $p + q > m - k + n - l$, and $S'_1 = \emptyset$ if $p + q \leq m - k + n - l$. Similarly,

$$S'_2 = \{\gamma_i : m - k - r' + 1 \leq i \leq m - k + s' - 1\}$$

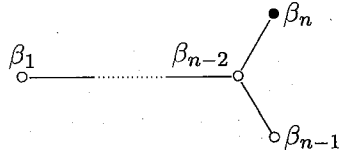
for integers r' and s' such that

$$0 \leq r' \leq p, m - k, \quad 0 \leq s' \leq q, n - l, \quad \text{and } r' + s' = p + q - (k + l)$$

if $p + q > k + l$, and $S'_2 = \emptyset$ if $p + q \leq k + l$.

9.18. $\mathfrak{u}(n, n) \downarrow \mathfrak{so}^*(2n)$.

Let $\mathfrak{g}_0 := \mathfrak{u}(n, n)$ and $\mathfrak{g}'_0 := \mathfrak{so}^*(2n)$. We label the simple roots of \mathfrak{g} as in the previous case and those of \mathfrak{g}' as



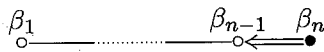
Suppose that $S = \{\alpha_i : n-p+1 \leq i \leq n+q-1\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{u}(p, q) \oplus \mathfrak{u}(1)^{2n-p-q}$. Then

$$S' = \{\beta_i : 2n - p - q + 1 \leq i \leq n\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{so}(2(p+q-n))^* \oplus \mathfrak{u}(1)^{2n-p-q}$$

if $p + q \geq n + 2$ and $S' = \emptyset$ if $p + q < n + 2$.

9.19. $\mathfrak{u}(n, n) \downarrow \mathfrak{sp}(n, \mathbb{R})$.

Let $\mathfrak{g}_0 := \mathfrak{u}(n, n)$ and $\mathfrak{g}'_0 := \mathfrak{sp}(n, \mathbb{R})$. We label the simple roots of \mathfrak{g} as in the previous case and those of \mathfrak{g}' as



Suppose that $S = \{\alpha_i : n - p + 1 \leq i \leq n + q - 1\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{u}(p, q) \oplus \mathfrak{u}(1)^{2n-p-q}$. Then

$$S' = \{\beta_i : 2n - p - q + 1 \leq i \leq n\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{sp}(p + q - n, \mathbb{R}) \oplus \mathfrak{u}(1)^{2n-p-q}$$

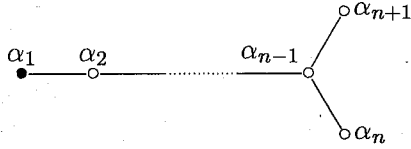
if $p + q \geq n + 1$ and $S' = \emptyset$ if $p + q < n + 1$.

9.20. $\mathfrak{so}(2, n) \downarrow \mathfrak{so}(2, k) \oplus \mathfrak{so}(n - k)$.

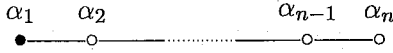
This is a special case of $\mathfrak{so}(2m, n) \downarrow \mathfrak{so}(2m, k) \oplus \mathfrak{so}(n - k)$ above.

9.21. $\mathfrak{so}(2, 2n) \downarrow \mathfrak{u}(1, n)$.

Let $\mathfrak{g}_0 := \mathfrak{so}(2, 2n)$ and $\mathfrak{g}'_0 := \mathfrak{u}(1, n)$. We label the simple roots of \mathfrak{g} as



We may assume that $\mathfrak{t} \subset \mathfrak{k}'$ and simple roots of \mathfrak{g}' are



Suppose that $S = \{\alpha_i : 1 \leq i \leq n\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{u}(1, n)$. Then

$$S' = \{\alpha_{2i} : 1 \leq i \leq m\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(2)^m \oplus \mathfrak{u}(1),$$

$$S' = \{\alpha_1\} \cup \{\alpha_{2i} : 2 \leq i \leq m\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^{m-1} \oplus \mathfrak{u}(1)$$

if $n = 2m$ is even, and

$$S' = \{\alpha_{2i-1} : 1 \leq i \leq m\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^{m-1}$$

if $n = 2m - 1$ is odd.

Suppose that $S = \{\alpha_i : 1 \leq i \leq n - 1\} \cup \{\alpha_{n+1}\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{u}(1, n)$.

Then

$$S' = \{\alpha_{2i-1} : 1 \leq i \leq m\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^{m-1} \oplus \mathfrak{u}(1)$$

if $n = 2m$ is even, and

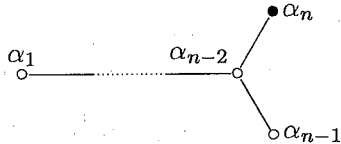
$$S' = \{\alpha_{2i} : 1 \leq i \leq m - 1\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(2)^{m-1} \oplus \mathfrak{u}(1)^2,$$

$$S' = \{\alpha_1\} \cup \{\alpha_{2i} : 2 \leq i \leq m - 1\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^{m-2} \oplus \mathfrak{u}(1)^2$$

if $n = 2m - 1$ is odd.

9.22. $\mathfrak{so}^*(2n) \downarrow \mathfrak{u}(m, n - m)$.

Let $\mathfrak{g}_0 := \mathfrak{so}^*(2n)$ and $\mathfrak{g}'_0 := \mathfrak{u}(m, n - m)$. We label the simple roots of \mathfrak{g} as



and those of \mathfrak{g}' as



Suppose that $S = \{\alpha_i : p+1 \leq i \leq n\}$ for $p \leq n-2$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{so}(2n-2p)^* \oplus \mathfrak{u}(1)^p$. Then

$$S' = \{\beta_i : m-p'+1 \leq i \leq m+q'-1\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(p', q') \oplus \mathfrak{u}(1)^{2p}$$

for integers p' and q' such that

$$0 \leq p' \leq m, \quad 0 \leq q' \leq n-m, \quad \text{and} \quad p' + q' = n-2p$$

if $n > 2p$, and $S' = \emptyset$ if $n \leq 2p$.

Suppose that $S = \{\alpha_i : 1 \leq i \leq n-2\} \cup \{\alpha_n\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{u}(n-1, 1)$.

Then

$$S' = \{\beta_{2i-1} : 1 \leq i \leq k\} \cup \{\beta_m\} \cup \{\beta_{m+2i+1} : 1 \leq i \leq l\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^{k+l} \oplus \mathfrak{u}(1)^2,$$

$$S' = \{\beta_{2i-1} : 1 \leq i \leq k\} \cup \{\beta_{m-1}\} \cup \{\beta_{m+2i+1} : 1 \leq i \leq l\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(2)^{k+l+1} \oplus \mathfrak{u}(1)^2,$$

$$S' = \{\beta_{2i-1} : 1 \leq i \leq k\} \cup \{\beta_{m+1}\} \cup \{\beta_{m+2i+1} : 1 \leq i \leq l\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(2)^{k+l+1} \oplus \mathfrak{u}(1)^2$$

if $m = 2k+2$ and $n-m = 2l+2$ are even;

$$S' = \{\beta_{2i-1} : 1 \leq i \leq k\} \cup \{\beta_m\} \cup \{\beta_{m+2i+1} : 1 \leq i \leq l\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^{k+l} \oplus \mathfrak{u}(1),$$

$$S' = \{\beta_{2i-1} : 1 \leq i \leq k\} \cup \{\beta_{m+1}\} \cup \{\beta_{m+2i+1} : 1 \leq i \leq l\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(2)^{k+l+1} \oplus \mathfrak{u}(1)$$

if $m = 2k+1$ is odd and $n-m = 2l+2$ is even;

$$S' = \{\beta_{2i-1} : 1 \leq i \leq k\} \cup \{\beta_m\} \cup \{\beta_{m+2i} : 1 \leq i \leq l\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^{k+l} \oplus \mathfrak{u}(1),$$

$$S' = \{\beta_{2i-1} : 1 \leq i \leq k\} \cup \{\beta_{m-1}\} \cup \{\beta_{m+2i} : 1 \leq i \leq l\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(2)^{k+l+1} \oplus \mathfrak{u}(1)$$

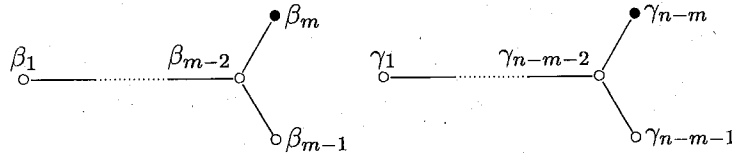
if $m = 2k+2$ is even and $n-m = 2l+1$ is odd;

$$S' = \{\beta_{2i-1} : 1 \leq i \leq k\} \cup \{\beta_m\} \cup \{\beta_{m+2i} : 1 \leq i \leq l\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^{k+l}$$

if $m = 2k+1$ and $n-m = 2l+1$ are odd.

9.23. $\mathfrak{so}(2n)^* \downarrow \mathfrak{so}(2m)^* \oplus \mathfrak{so}(2n-2m)^*$.

Let $\mathfrak{g}_0 := \mathfrak{so}(2n)^*$ and $\mathfrak{g}'_0 := \mathfrak{so}(2m)^* \oplus \mathfrak{so}(2n-2m)^*$. We label the simple roots of \mathfrak{g} as in the previous case and those of \mathfrak{g}' as



For $S' \subset \Pi'$, we put $S'_1 := S' \cap \{\beta_i\}_{1 \leq i \leq m}$ and $S'_2 := S' \cap \{\gamma_i\}_{1 \leq i \leq n-m}$.

Suppose that $S = \{\alpha_i : p+1 \leq i \leq n\}$ for $p \leq n-2$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{so}(2n-2p)^* \oplus \mathfrak{u}(1)^p$. Then $S'_1 = \{\beta_i : p+1 \leq i \leq m\}$ if $p < m-1$ and $S'_1 = \emptyset$ if $p \geq m-1$. Similarly for S'_2 .

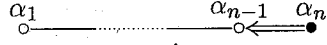
Suppose that $S = \{\alpha_i : p+1 \leq i \leq n-2\} \cup \{\alpha_n\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{u}(n-p-1, 1) \oplus \mathfrak{u}(1)^p$. Then

$$S'_1 = \{\beta_i : p+n-m \leq i \leq m-2\}, \quad \{\beta_i : p+n-m+1 \leq i \leq m-2\} \cup \{\beta_m\}$$

if $p \leq 2m - n - 2$ and $S'_1 = \emptyset$ if $p > 2m - n - 2$. Similarly for S'_2 .

9.24. $\mathfrak{sp}(n, \mathbb{R}) \downarrow \mathfrak{u}(m, n - m)$.

Let $\mathfrak{g}_0 := \mathfrak{sp}(n, \mathbb{R})$ and $\mathfrak{g}'_0 := \mathfrak{u}(m, n - m)$. We label the simple roots of \mathfrak{g} as



and those of \mathfrak{g}' as



Suppose that $S = \{\alpha_i : p+1 \leq i \leq n\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{sp}(n-p, \mathbb{R}) \oplus \mathfrak{u}(1)^p$. Then

$$S' = \{\beta_i : m - p' + 1 \leq i \leq m + q' - 1\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(p', q') \oplus \mathfrak{u}(1)^{2p}$$

for integers p' and q' such that

$$0 \leq p' \leq m, \quad 0 \leq q' \leq n - m, \quad \text{and} \quad p' + q' = n - 2p$$

if $n > 2p$, and $S' = \emptyset$ if $n \leq 2p$.

9.25. $\mathfrak{sp}(n, \mathbb{R}) \downarrow \mathfrak{sp}(m, \mathbb{R}) \oplus \mathfrak{sp}(n - m, \mathbb{R})$.

Let $\mathfrak{g}_0 := \mathfrak{sp}(n, \mathbb{R})$ and $\mathfrak{g}'_0 := \mathfrak{sp}(m, \mathbb{R}) \oplus \mathfrak{sp}(n - m, \mathbb{R})$. We label the simple roots of \mathfrak{g} as in the previous case and those of \mathfrak{g}' as

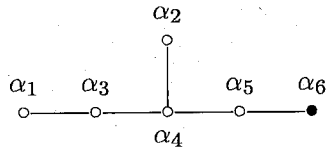


For $S' \subset \Pi'$, we put $S'_1 := S' \cap \{\beta_i\}_{1 \leq i \leq m}$ and $S'_2 := S' \cap \{\gamma_i\}_{1 \leq i \leq n-m}$.

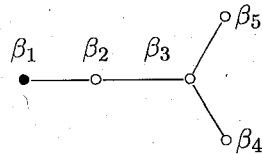
Suppose that $S = \{\alpha_i : p+1 \leq i \leq n\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{sp}(n-p, \mathbb{R}) \oplus \mathfrak{u}(1)^p$. Then $S'_1 = \{\beta_i : p+1 \leq i \leq m\}$ if $p \leq m-1$ and $S'_1 = \emptyset$ if $p > m-1$. Similarly for S'_2 .

9.26. $\mathfrak{e}_{6(-14)} \downarrow \mathfrak{so}(2, 8) \oplus \mathfrak{so}(2)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{6(-14)}$ and $\mathfrak{g}'_0 := \mathfrak{so}(2, 8) \oplus \mathfrak{so}(2)$. We label the simple roots of \mathfrak{g} as



and those of \mathfrak{g}' as



Suppose $S^c = \{\alpha_1\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{so}(2, 8) \oplus \mathfrak{u}(1)$. Then

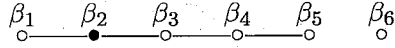
$$\begin{aligned} S' &= \{\beta_2, \beta_3, \beta_4\}, & \mathfrak{l}'_0(S') &\simeq \mathfrak{u}(4) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(2), \\ S' &= \{\beta_1, \beta_2, \beta_3\}, & \mathfrak{l}'_0(S') &\simeq \mathfrak{u}(1, 3) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(2), \\ S' &= \{\beta_2, \beta_3, \beta_5\}, & \mathfrak{l}'_0(S') &\simeq \mathfrak{u}(4) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(2). \end{aligned}$$

Suppose $S^c = \{\alpha_2\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{u}(1, 5)$. Then

$$S' = \{\beta_1, \beta_3\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(2).$$

9.27. $\mathfrak{e}_{6(-14)} \downarrow \mathfrak{su}(4, 2) \oplus \mathfrak{su}(2)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{6(-14)}$ and $\mathfrak{g}'_0 := \mathfrak{su}(4, 2) \oplus \mathfrak{su}(2)$. We label the simple roots of \mathfrak{g} as in the previous case and those of \mathfrak{g}' as

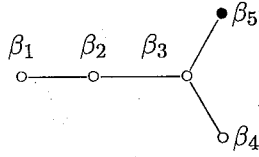


Suppose $S^c = \{\alpha_1\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{so}(2, 8) \oplus \mathfrak{u}(1)$. Then

$$\begin{aligned} S' &= \{\beta_3, \beta_5\}, & \mathfrak{l}'_0(S') &\simeq \mathfrak{u}(2)^2 \oplus \mathfrak{u}(1)^2, \\ S' &= \{\beta_2, \beta_5\}, & \mathfrak{l}'_0(S') &\simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2) \oplus \mathfrak{u}(1)^2, \\ S' &= \{\beta_1, \beta_5\}, & \mathfrak{l}'_0(S') &\simeq \mathfrak{u}(2)^2 \oplus \mathfrak{u}(1)^2. \end{aligned}$$

9.28. $\mathfrak{e}_{6(-14)} \downarrow \mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{6(-14)}$ and $\mathfrak{g}'_0 := \mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$. We label the simple roots of \mathfrak{g} as in the previous case and those of \mathfrak{g}' as



Suppose $S^c = \{\alpha_1\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{so}(2, 8) \oplus \mathfrak{u}(1)$. Then

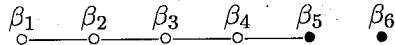
$$\begin{aligned} S' &= \{\beta_2, \beta_3, \beta_5\}, & \mathfrak{l}'_0(S') &\simeq \mathfrak{u}(3, 1) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(2), \\ S' &= \{\beta_1, \beta_2, \beta_3\}, & \mathfrak{l}'_0(S') &\simeq \mathfrak{u}(4) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(2). \end{aligned}$$

Suppose $S^c = \{\alpha_2\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{u}(1, 5)$. Then

$$\begin{aligned} S' &= \{\beta_1, \beta_3\}, & \mathfrak{l}'_0(S') &\simeq \mathfrak{u}(2)^2 \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(2), \\ S' &= \{\beta_1, \beta_5\}, & \mathfrak{l}'_0(S') &\simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(2). \end{aligned}$$

9.29. $\mathfrak{e}_{6(-14)} \downarrow \mathfrak{su}(5, 1) \oplus \mathfrak{sl}(2, \mathbb{R})$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{6(-14)}$ and $\mathfrak{g}'_0 := \mathfrak{su}(5, 1) \oplus \mathfrak{sl}(2, \mathbb{R})$. We label the simple roots of \mathfrak{g} as in the previous case and those of \mathfrak{g}' as

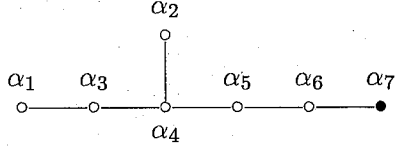


Suppose $S^c = \{\alpha_1\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{so}(2, 8) \oplus \mathfrak{u}(1)$. Then

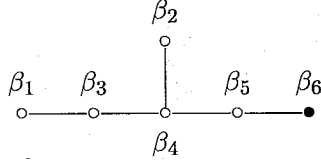
$$S' = \{\beta_3, \beta_5\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2) \oplus \mathfrak{u}(1)^2.$$

9.30. $\mathfrak{e}_{7(-25)} \downarrow \mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{7(-25)}$ and $\mathfrak{g}'_0 := \mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$. We label the simple roots of \mathfrak{g} as



and those of \mathfrak{g}' as



Suppose $S^c = \{\alpha_1\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{so}(2, 10) \oplus \mathfrak{u}(1)$. Then

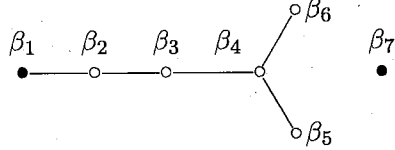
$$S' = \{\beta_3, \beta_4, \beta_5\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(4) \oplus \mathfrak{u}(1)^2 \oplus \mathfrak{so}(2),$$

$$S' = \{\beta_4, \beta_5, \beta_6\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(3, 1) \oplus \mathfrak{u}(1)^2 \oplus \mathfrak{so}(2),$$

$$S' = \{\beta_2, \beta_4, \beta_5\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(4) \oplus \mathfrak{u}(1)^2 \oplus \mathfrak{so}(2).$$

9.31. $\mathfrak{e}_{7(-25)} \downarrow \mathfrak{so}(2, 10) \oplus \mathfrak{sl}(2, \mathbb{R})$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{7(-25)}$ and $\mathfrak{g}'_0 := \mathfrak{so}(2, 10) \oplus \mathfrak{sl}(2, \mathbb{R})$. We label the simple roots of \mathfrak{g} as in the previous case. We let $\beta_1 = \alpha_7$, $\beta_2 = \alpha_6$, $\beta_3 = \alpha_5$, $\beta_4 = \alpha_4$, $\beta_5 = \alpha_2$, $\beta_6 = \alpha_3$, $\beta_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ and then the Dynkin diagram of \mathfrak{g}' is



Suppose $S^c = \{\alpha_1\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{so}(2, 10) \oplus \mathfrak{u}(1)$. Then

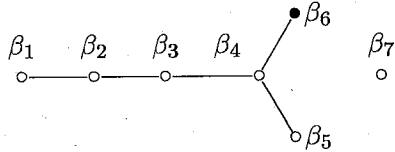
$$S' = \{\beta_1, \beta_3, \beta_6\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^2 \oplus \mathfrak{u}(1).$$

9.32. $\mathfrak{e}_{7(-25)} \downarrow \mathfrak{su}(6, 2)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{7(-25)}$ and $\mathfrak{g}'_0 := \mathfrak{su}(6, 2)$. In this case, for any holomorphic $\mathfrak{q} (\neq \mathfrak{g})$ and a weakly fair parameter λ , the restriction $A_{\mathfrak{q}}(\lambda)|_{(\mathfrak{g}', K')}$ decomposes into holomorphic discrete series representations.

9.33. $\mathfrak{e}_{7(-25)} \downarrow \mathfrak{so}^*(12) \oplus \mathfrak{su}(2)$.

Let $\mathfrak{g}_0 := \mathfrak{e}_{7(-25)}$ and $\mathfrak{g}'_0 := \mathfrak{so}^*(12) \oplus \mathfrak{su}(2)$. We label the simple roots of \mathfrak{g} as in the previous case and those of \mathfrak{g}' as



Suppose $S^c = \{\alpha_1\}$ so that $\mathfrak{l}_0(S) \simeq \mathfrak{so}(2, 10) \oplus \mathfrak{u}(1)$. Then

$$S' = \{\beta_1, \beta_3, \beta_6\}, \quad \mathfrak{l}'_0(S') \simeq \mathfrak{u}(1, 1) \oplus \mathfrak{u}(2)^2 \oplus \mathfrak{u}(1).$$

Let \mathfrak{q}_1 and \mathfrak{q}_2 be holomorphic parabolic subalgebras of \mathfrak{g} . For the branching law of tensor product of highest weight modules $A_{\mathfrak{q}_1}(\lambda_1) \otimes A_{\mathfrak{q}_2}(\lambda_2)$ we can use the isomorphism

$$A_{\mathfrak{q}_1}(\lambda_1) \otimes A_{\mathfrak{q}_2}(\lambda_2) = \mathcal{L}_{\bar{\mathfrak{q}}_1, s}^{\mathfrak{g}}(\mathbb{C}_{\lambda_1}) \otimes A_{\mathfrak{q}_2}(\lambda_2) \simeq \mathcal{L}_{\bar{\mathfrak{q}}_1, s}^{\mathfrak{g}}(\mathbb{C}_{\lambda_1} \otimes A_{\mathfrak{q}_2}(\lambda_2)|_{\mathfrak{l}_1}),$$

where $s = \dim(\mathfrak{u}_1 \cap \mathfrak{k})$. By taking a sequence $\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_n = \mathfrak{l}_1$ such that $(\mathfrak{g}_i, \mathfrak{g}_{i+1})$ are symmetric pairs, the restriction $A_{\mathfrak{q}_2}(\lambda_2)|_{\mathfrak{l}_1}$ is reduced to the previous cases.

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