Proper actions and designs on homogeneous spaces
(等質空間上の固有な作用とデザイン)

Takayuki Okuda
(奥田隆幸)

Organization of this thesis

This thesis is consisted of six chapters as follows. In Chapter 1, 2, 3 and 4, we study proper actions on some pseudo-Riemannian non-compact homogeneous spaces. In Chapter 5, our interesting is in real minimal nilpotent orbits in certain simple Lie algebras. The theme in Chapter 6 is to construct designs on compact homogeneous spaces.

We give an abstract for each chapter below:

Chapter 1: For a semisimple symmetric pair (G, H), we show that G/H admits a non virtually abelian discontinuous group if and only if G/H admits a proper $SL(2, \mathbb{R})$-action. Furthermore, we also classify (G, H) satisfying the equivalent conditions above.

Chapter 2: For a semisimple Lie algebra $\mathfrak{g}$, we show a kind of prodigality of $\mathfrak{sl}_2$-triples in $\mathfrak{g}$. The main result of Chapter 2 is used in Chapter 1.

Chapter 3: The claim of the main theorem in Chapter 1 does not holds for non-symmetric (G, H) in general. We give an example of it.

Chapter 4: Let (G, H_1) and (G, H_2) be symmetric pairs with simple G. We classify (G, H_1, H_2) with proper diagonal G-action on $G/H_1 \times G/H_2$.

Chapter 5: Let $\mathfrak{g}$ be a simple Lie algebra isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{f}_{4(-20)}$ or $\mathfrak{e}_{6(-26)}$ $(k \geq 2,\ n \geq 5,\ p \geq q \geq 1)$. Then it is known that the complex minimal nilpotent orbit in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$ does not meets $\mathfrak{g}$. We show that there uniquely exists a real minimal nilpotent orbit in $\mathfrak{g}$ and determine the complexification of it.

Chapter 6: We show a new construction of spherical designs on S^3 from a given design on S^2. We also generalize it to a construction of designs on a compact Lie group G.

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Contents

Part I

Proper actions on pseudo-Riemannian homogeneous spaces

Chapter 1

Classification of semisimple symmetric spaces with proper $SL(2,\mathbb{R})$-actions

We give a complete classification of irreducible symmetric spaces for which there exist proper $SL(2,\mathbb{R})$-actions as isometries, using the criterion for proper actions by T. Kobayashi [Math. Ann. '89] and combinatorial techniques of nilpotent orbits. In particular, we classify irreducible symmetric spaces that admit surface groups as discontinuous groups, combining this with Benoist's theorem [Ann. Math. '96].

1.1 Introduction

The aim of this chapter is to classify semisimple symmetric spaces G/H that admit isometric proper actions of non-compact simple Lie group $SL(2,\mathbb{R})$, and also those of surface groups $\pi_1(\Sigma_g)$. Here, isometries are considered with respect to the natural pseudo-Riemannian structure on G/H.

We motivate our work in one of the fundamental problems on locally symmetric spaces, stated below:

Problem 1.1.1 (See [20]). *Fix a simply connected symmetric space $\widetilde{M}$ as a model space. What discrete groups can arise as the fundamental groups of complete affine manifolds M which are locally isomorphic to the space $\widetilde{M}$?*

By a theorem of É. Cartan, such M is represented as the double coset space $\Gamma\backslash G/H$. Here $\widetilde{M} = G/H$ is a simply connected symmetric space and

$\Gamma \simeq \pi_1(M)$ a discrete subgroup of G acting properly discontinuously and freely on $\widetilde{M}$.

Conversely, for a given symmetric pair (G, H) and an abstract group Γ with discrete topology, if there exists a group homomorphism $\rho : \Gamma \to G$ for which Γ acts on G/H properly discontinuously and freely via ρ, then the double coset space $\rho(\Gamma)\backslash G/H$ becomes a C^∞-manifold such that the natural quotient map

$$G/H \to \rho(\Gamma)\backslash G/H$$

is a C^∞-covering. The double coset manifold $\rho(\Gamma)\backslash G/H$ is called a Clifford–Klein form of G/H, which is endowed with a locally symmetric structure through the covering. We say that G/H *admits* Γ *as a discontinuous group* if there exists such ρ.

Then Problem 1.1.1 may be reformalized as:

Problem 1.1.2. *Fix a symmetric pair* (G, H). *What discrete groups does* G/H *admit as discontinuous groups?*

For a compact subgroup H of G, the action of any discrete subgroup of G on G/H is automatically properly discontinuous. Thus our interest is in non-compact H, for which not all discrete subgroups Γ of G act properly discontinuously on G/H. Problem 1.1.2 is non-trivial, even when $\widetilde{M} = \mathbb{R}^n$ regarded as an affine symmetric space, i.e. $(G, H) = (GL(n, \mathbb{R}) \ltimes \mathbb{R}^n, GL(n, \mathbb{R}))$. In this case, the long-standing conjecture (Auslander's conjecture) states that such discrete group Γ will be virtually polycyclic if the Clifford–Klein form M is compact (see [1, 3, 11, 43]). On the other hand, as was shown by E. Calabi and L. Markus [7] in 1962, no infinite discrete subgroup of $SO_0(n+1, 1)$ acts properly discontinuously on the de Sitter space $SO_0(n+1, 1)/SO_0(n, 1)$. More generally, if G/H does not admit any infinite discontinuous group, we say that a Calabi–Markus phenomenon occurs for G/H.

For the rest of this chapter, we consider the case that G is a linear semisimple Lie group. In this setting, a systematic study of Problem 1.1.2 for the general homogeneous space G/H was initiated in the late 1980s by T. Kobayashi [15, 16, 17]. One of the fundamental results of Kobayashi in [15] is a criterion for proper actions, including a criterion for the Calabi–Markus phenomenon on homogeneous spaces G/H. More precisely, he showed that the following four conditions on G/H are equivalent: the space G/H admits an infinite discontinuous group; the space G/H admits a proper $\mathbb{R}$-action; the space G/H admits the abelian group $\mathbb{Z}$ as a discontinuous group; and $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} > \operatorname{rank}_{\mathbb{R}} \mathfrak{h}$.

Furthermore, Y. Benoist [5] obtained a criterion for the existence of infinite non-virtually abelian discontinuous groups for G/H.

Obviously, such discontinuous groups exist if there exists a Lie group homomorphism $\Phi : SL(2,\mathbb{R}) \to G$ such that $SL(2,\mathbb{R})$ acts on G/H properly via Φ. We prove that the converse statement also holds when G/H is a semisimple symmetric space. More strongly, our first main theorem gives a characterization of symmetric spaces G/H that admit proper $SL(2,\mathbb{R})$-actions:

Theorem 1.1.3 (see Theorem 1.2.2). *Suppose that G is a connected linear semisimple Lie group. Then the following five conditions on a symmetric pair (G,H) are equivalent:*

(*i*) *There exists a Lie group homomorphism $\Phi : SL(2,\mathbb{R}) \to G$ such that $SL(2,\mathbb{R})$ acts on G/H properly via Φ.*

(*ii*) *For some $g \geq 2$, the symmetric space G/H admits the surface group $\pi_1(\Sigma_g)$ as a discontinuous group, where Σ_g is a closed Riemann surface of genus g.*

(*iii*) *G/H admits an infinite discontinuous group Γ which is not virtually abelian (i.e. Γ has no abelian subgroup of finite index).*

(*iv*) *There exists a complex nilpotent orbit $\mathcal{O}^{G_\mathbb{C}}_{nilp}$ in $\mathfrak{g}_\mathbb{C}$ such that $\mathcal{O}^{G_\mathbb{C}}_{nilp} \cap \mathfrak{g} \neq \emptyset$ and $\mathcal{O}^{G_\mathbb{C}}_{nilp} \cap \mathfrak{g}^c = \emptyset$, where $\mathfrak{g}^c$ is the c-dual of the symmetric pair $(\mathfrak{g},\mathfrak{h})$ (see* (1.2.1) *for definition).*

(*v*) *There exists a complex antipodal hyperbolic orbit $\mathcal{O}^{G_\mathbb{C}}_{hyp}$ in $\mathfrak{g}_\mathbb{C}$ (see Definition* 1.2.3) *such that $\mathcal{O}^{G_\mathbb{C}}_{hyp} \cap \mathfrak{g} \neq \emptyset$ and $\mathcal{O}^{G_\mathbb{C}}_{hyp} \cap \mathfrak{g}^c = \emptyset$.*

The implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is straightforward and easy. The non-trivial part of Theorem 1.1.3 is the implication (iii) $\Rightarrow$ (i).

By using Theorem 1.1.3, we give a complete classification of semisimple symmetric spaces G/H that admit a proper $SL(2,\mathbb{R})$-action. As is clear for (iv) or (v) in Theorem 1.1.3, it is sufficient to work at the Lie algebra level. Recall that the classification of semisimple symmetric pairs $(\mathfrak{g},\mathfrak{h})$ was accomplished by M. Berger [6]. Our second main theorem is to single out which symmetric pairs among his list satisfy the equivalent conditions in Theorem 1.1.3:

Theorem 1.1.4. *Suppose G is a simple Lie group. Then, the two conditions below on a symmetric pair (G, H) are equivalent:*

(i) (G, H) *satisfies one of (therefore, all of) the equivalent conditions in Theorem* 1.1.3.

(ii) *The pair* $(\mathfrak{g}, \mathfrak{h})$ *belongs to Table* 1.3 *in Appendix 1.A.*

The existence problem for compact Clifford–Klein forms has been actively studied in the last two decades since Kobayashi's paper [15]. The properness criteria of Kobayashi and Benoist yield necessary conditions on (G, H) for the existence [5, 15]. See also [24, 28, 30, 33, 45] for some other methods for the existence problem of compact Clifford–Klein forms. The recent developments on this topic can be found in [21, 22, 27, 31].

We go back to semisimple symmetric pair (G, H). By Kobayashi's criterion [15, Corollary 4.4], the Calabi–Markus phenomenon occurs for G/H if and only if $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} = \operatorname{rank}_{\mathbb{R}} \mathfrak{h}$ holds. (see Fact 1.2.6 for more details). In particular, G/H does not admit compact Clifford–Klein forms in this case unless G/H itself is compact. In Section 1.2, we give the list, as Table 1.2, of symmetric pair $(\mathfrak{g}, \mathfrak{h})$ with simple $\mathfrak{g}$ which does not appear in Table 1.3 and $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} > \operatorname{rank}_{\mathbb{R}} \mathfrak{h}$, i.e. $(\mathfrak{g}, \mathfrak{h})$ does not satisfy the equivalent conditions in Theorem 1.1.3 with $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} > \operatorname{rank}_{\mathbb{R}} \mathfrak{h}$. Apply a theorem of Benoist [5, Corollary 1], we see G/H does not admit compact Clifford–Klein forms if $(\mathfrak{g}, \mathfrak{h})$ is in Table 1.2 (see Corollary 1.2.8). In this table, we find some "new" examples of semisimple symmetric spaces G/H that do not admit compact Clifford–Klein forms, for which we can not find in the existing literature as follows:

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{su}^*(4m+2)$	$\mathfrak{sp}(m+1, m)$
$\mathfrak{su}^*(4m)$	$\mathfrak{sp}(m, m)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{f}_{4(4)}$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{sp}(3, 1)$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{so}(4m+2, \mathbb{C})$	$\mathfrak{so}(2m+2, 2m)$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(2)}$

Table 1.1: Examples of G/H without compact Clifford–Klein forms

We remark that Table 1.1 is the list of symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ in Table 1.2 which are neither in Benoist's examples [5, Example 1] nor in Kobayashi's examples [17, Example 1.7, Table 4.4], [19, Table 5.18].

The proof of the non-trivial implication (iii) $\Rightarrow$ (i) in Theorem 1.1.3 is given by reducing it to an equivalent assertion on complex adjoint orbits, namely, (v) $\Rightarrow$ (iv). The last implication is proved by using the Dynkin–Kostant classification of $\mathfrak{sl}_2$-triples (equivalently, complex nilpotent orbits) in $\mathfrak{g}_{\mathbb{C}}$. We note that the proof does not need Berger's classification of semisimple symmetric pairs.

The reduction from (iii) $\Rightarrow$ (i) to (v) $\Rightarrow$ (iv) in Theorem 1.1.3 is given by proving (i) $\Leftrightarrow$ (iv) and (iii) $\Leftrightarrow$ (v) as follows. We show the equivalence (i) $\Leftrightarrow$ (iv) by combining Kobayashi's properness criterion [15] and a result of J. Sekiguchi for real nilpotent orbits in [38] with some observations on complexifications of real hyperbolic orbits. The equivalence (iii) $\Leftrightarrow$ (v) is obtained from Benoist's criterion [5].

As a refinement of the equivalence (i) $\Leftrightarrow$ (iv) in Theorem 1.1.3, we give a bijection between real nilpotent orbits $\mathcal{O}^G_{\text{nilp}}$ in $\mathfrak{g}$ such that the complexifications of $\mathcal{O}^G_{\text{nilp}}$ do not intersect the another real form $\mathfrak{g}^c$ and Lie group homomorphisms $\Phi : SL(2,\mathbb{R}) \to G$ for which the $SL(2,\mathbb{R})$-actions on G/H via Φ are proper, up to inner automorphisms of G (Theorem 1.10.1).

Concerning the proof of Theorem 1.1.4, for a given semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$, we give an algorithm to check whether or not the condition (v) in Theorem 1.1.3 holds, by using Satake diagrams of $\mathfrak{g}$ and $\mathfrak{g}^c$.

The chapter is organized as follows. In Section 1.2, we set up notation and state our main theorems. The next section contains a brief summary of Kobayashi's properness criterion [15] and Benoist's criterion [5] as preliminary results. We prove Theorem 1.1.3 in Section 1.4. The proof is based on some theorems, propositions and lemmas which are proved in Section 1.5 to Section 1.8 (see Section 1.4 for more details). Section 1.9 is about the algorithm for our classification. The last section establishes the relation between proper $SL(2,\mathbb{R})$-actions on G/H and real nilpotent orbits in $\mathfrak{g}$.

The main results of this chapter were announced in [34] with a sketch of the proofs.

1.2 Main results

Throughout this chapter, we shall work in the following:

Setting 1.2.1. *G is a connected linear semisimple Lie group, σ is an involutive automorphism on G, and H is an open subgroup of $G^\sigma := \{ g \in G \mid \sigma g = g \}$.*

This setting implies that G/H carries a pseudo-Riemannian structure g for which G acts as isometries and G/H becomes a symmetric space with respect to the Levi-Civita connection. We call (G, H) a semisimple symmetric pair. Note that g is positive definite, namely $(G/H, g)$ is Riemannian, if and only if H is compact.

Since G is a connected linear Lie group, we can take a connected complexification, denoted by $G_\mathbb{C}$, of G. We write $\mathfrak{g}_\mathbb{C}$, $\mathfrak{g}$ and $\mathfrak{h}$ for Lie algebras of $G_\mathbb{C}$, G and H, respectively. The differential action of σ on $\mathfrak{g}$ will be denoted by the same letter σ. Then $\mathfrak{h} = \{ X \in \mathfrak{g} \mid \sigma X = X \}$, and we also call $(\mathfrak{g}, \mathfrak{h})$ a semisimple symmetric pair. Let us denote by $\mathfrak{q} := \{ X \in \mathfrak{g} \mid \sigma X = -X \}$, and write the c-dual of $(\mathfrak{g}, \mathfrak{h})$ for

$$\mathfrak{g}^c := \mathfrak{h} + \sqrt{-1}\mathfrak{q}. \tag{1.2.1}$$

Then both $\mathfrak{g}$ and $\mathfrak{g}^c$ are real forms of $\mathfrak{g}_\mathbb{C}$. We note that the complex conjugation corresponding to $\mathfrak{g}^c$ on $\mathfrak{g}_\mathbb{C}$ is the anti $\mathbb{C}$-linear extension of σ on $\mathfrak{g}_\mathbb{C}$, and the semisimple symmetric pair $(\mathfrak{g}^c, \mathfrak{h})$ is the same as $(\mathfrak{g}, \mathfrak{h})^{ada}$ (which coincides with $(\mathfrak{g}, \mathfrak{h})^{dad}$; see [35, Section 1] for the notation).

For an abstract group Γ with discrete topology, we say that G/H *admits Γ as a discontinuous group* if there exists a group homomorphism $\rho : \Gamma \to G$ such that Γ acts properly discontinuously and freely on G/H via ρ (then ρ is injective and $\rho(\Gamma)$ is discrete in G, automatically). For such Γ-action on G/H, the double coset space $\Gamma\backslash G/H$, which is called a *Clifford–Klein form of G/H*, becomes a C^∞-manifold such that the quotient map

$$G/H \to \rho(\Gamma)\backslash G/H$$

is a C^∞-covering. In our context, the freeness of the action is less important than the properness of it (see [15, Section 5] for more details).

Here is the first main result:

Theorem 1.2.2. *In Setting 1.2.1, the following ten conditions on a semisimple symmetric pair (G, H) are equivalent:*

(*i*) *There exists a Lie group homeomorphism* $\Phi : SL(2,\mathbb{R}) \to G$ *such that* $SL(2,\mathbb{R})$ *acts properly on* G/H *via* Φ.

(*ii*) *For any* $g \geq 2$, *the symmetric space* G/H *admits the surface group* $\pi_1(\Sigma_g)$ *as a discontinuous group, where* Σ_g *is a closed Riemann surface of genus* g.

(*iii*) *For some* $g \geq 2$, *the symmetric space* G/H *admits the surface group* $\pi_1(\Sigma_g)$ *as a discontinuous group.*

(*iv*) G/H *admits an infinite discontinuous group* Γ *which is not virtually abelian (i.e.* Γ *has no abelian subgroup of finite index).*

(*v*) G/H *admits a discontinuous group which is a free group generated by a unipotent element in* G.

(*vi*) *There exists a complex nilpotent adjoint orbit* $\mathcal{O}^{G_\mathbb{C}}_{nilp}$ *of* $G_\mathbb{C}$ *in* $\mathfrak{g}_\mathbb{C}$ *such that* $\mathcal{O}^{G_\mathbb{C}}_{nilp} \cap \mathfrak{g} \neq \emptyset$ *and* $\mathcal{O}^{G_\mathbb{C}}_{nilp} \cap \mathfrak{g}^c = \emptyset$.

(*vii*) *There exists a real antipodal hyperbolic adjoint orbit* $\mathcal{O}^G_{hyp}$ *of* G *in* $\mathfrak{g}$ *(defined below) such that* $\mathcal{O}^G_{hyp} \cap \mathfrak{h} = \emptyset$.

(*viii*) *There exists a complex antipodal hyperbolic adjoint orbit* $\mathcal{O}^{G_\mathbb{C}}_{hyp}$ *of* $G_\mathbb{C}$ *in* $\mathfrak{g}_\mathbb{C}$ *such that* $\mathcal{O}^{G_\mathbb{C}}_{hyp} \cap \mathfrak{g} \neq \emptyset$ *and* $\mathcal{O}^{G_\mathbb{C}}_{hyp} \cap \mathfrak{g}^c = \emptyset$.

(*ix*) *There exists an* $\mathfrak{sl}_2$*-triple* (A, X, Y) *in* $\mathfrak{g}$ *(i.e.* $A, X, Y \in \mathfrak{g}$ *with* $[A, X] = 2X$, $[A, Y] = -2Y$ *and* $[X, Y] = A$*) such that* $\mathcal{O}^G_A \cap \mathfrak{h} = \emptyset$, *where* $\mathcal{O}^G_A$ *is the real adjoint orbit through* A *of* G *in* $\mathfrak{g}$.

(*x*) *There exists an* $\mathfrak{sl}_2$*-triple* (A, X, Y) *in* $\mathfrak{g}_\mathbb{C}$ *such that* $\mathcal{O}^{G_\mathbb{C}}_A \cap \mathfrak{g} \neq \emptyset$ *and* $\mathcal{O}^{G_\mathbb{C}}_A \cap \mathfrak{g}^c = \emptyset$, *where* $\mathcal{O}^{G_\mathbb{C}}_A$ *is the complex adjoint orbit through* A *of* $G_\mathbb{C}$ *in* $\mathfrak{g}_\mathbb{C}$.

Theorem 1.1.3 is a part of this theorem.

The definitions of *hyperbolic* orbits and *antipodal* orbits are given here:

Definition 1.2.3. *Let* $\mathfrak{g}$ *be a complex or real semisimple Lie algebra. An element* X *of* $\mathfrak{g}$ *is said to be hyperbolic if the endomorphism* $\mathrm{ad}_\mathfrak{g}(X) \in \mathrm{End}(\mathfrak{g})$ *is diagonalizable with only real eigenvalues. We say that an adjoint orbit* $\mathcal{O}$ *in* $\mathfrak{g}$ *is hyperbolic if any (or some) element in* $\mathcal{O}$ *is hyperbolic. Moreover, an adjoint orbit* $\mathcal{O}$ *in* $\mathfrak{g}$ *is said to be antipodal if for any (or some) element* X *in* $\mathcal{O}$, *the element* $-X$ *is also in* $\mathcal{O}$.

A proof of Theorem 1.2.2 will be given in Section 1.4. Here is a short remark on it. In (i) $\Rightarrow$ (ix), the homomorphism Φ associates an $\mathfrak{sl}_2$-triples (A, X, Y) by the differential of Φ (see Section 1.4.1). The complex adjoint orbits in (viii) and (x) are obtained by the complexification of the real adjoint orbits in (vii) and (ix), respectively (see Section 1.4.3). In (x) $\Rightarrow$ (vi), the $\mathfrak{sl}_2$-triple (A, X, Y) in (x) associates a complex nilpotent orbit in (vi) by $\mathcal{O}_X^{G_\mathbb{C}} := \mathrm{Ad}(G_\mathbb{C}) \cdot X$ (see Section 1.4.4). The implication (i) $\Rightarrow$ (ii) is obvious if we take $\pi_1(\Sigma_g)$ inside $SL(2, \mathbb{R})$. The equivalence (iv) $\Leftrightarrow$ (vii) is a kind of paraphrase of Benoist's criterion [5, Theorem 1.1] on symmetric spaces (see Section 1.4.2). The key ingredient of Theorem 1.2.2 is the implication (iii) $\Rightarrow$ (i). We will reduce it to the implication (viii) $\Rightarrow$ (x). The condition (viii) will be used for a classification of (G, H) satisfying the equivalence conditions in Theorem 1.2.2 (see Section 1.9).

Remark 1.2.4. **(1)** *K. Teduka [40] gave a list of (G, H) satisfying the condition* (i) *in Theorem* 1.2.2 *in the special cases where $(\mathfrak{g}, \mathfrak{h})$ is a complex symmetric pair. He also studied proper $SL(2, \mathbb{R})$-actions on some non-symmetric spaces in [41].*

(2) *Y. Benoist [5, Theorem1.1] proved a criterion for the condition* (iv) *in a more general setting, than we treat here.*

(3) *The following condition on a semisimple symmetric pair (G, H) is weaker than the equivalent conditions in Theorem* 1.2.2:

- *There exists a real nilpotent adjoint orbit $\mathcal{O}^G_{nilp}$ of G in $\mathfrak{g}$ such that $\mathcal{O}^G_{nilp} \cap \mathfrak{h} = \emptyset$.*

For a discrete subgroup Γ of G, we say that a Clifford–Klein form $\Gamma \backslash G/H$ is *standard* if Γ is contained in closed reductive subgroup L of G (see Definition 1.3.1) acting properly on G/H (see [14]), and is *nonstandard* if not. See [13] for an example of a Zariski-dense discontinuous group Γ for G/H, which gives a nonstandard Clifford–Klein form. We obtain the following corollary to the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 1.2.2.

Corollary 1.2.5. *Let $g \geq 2$. Then, in Setting* 1.2.1, *the symmetric space G/H admits the surface group $\pi_1(\Sigma_g)$ as a discontinuous group if and only if there exists a discrete subgroup Γ of G such that $\Gamma \simeq \pi_1(\Sigma_g)$ and $\Gamma \backslash G/H$ is standard.*

Theorem 1.2.2 may be compared with the fact below for proper actions by the abelian group $\mathbb{R}$ consisting of hyperbolic elements:

Fact 1.2.6 (Criterion for the Calabi–Markus phenomenon). *In Setting* 1.2.1, *the following seven conditions on a semisimple symmetric pair* (G, H) *are equivalent*:

(*i*) *There exists a Lie group homomorphism* $\Phi : \mathbb{R} \to G$ *such that* $\mathbb{R}$ *acts properly on* G/H *via* Φ.

(*ii*) G/H *admits the abelian group* $\mathbb{Z}$ *as a discontinuous group.*

(*iii*) G/H *admits an infinite discontinuous group.*

(*iv*) G/H *admits a discontinuous group which is a free group generated by a hyperbolic element in* G.

(*v*) $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} > \operatorname{rank}_{\mathbb{R}} \mathfrak{h}$.

(*vi*) *There exists a real hyperbolic adjoint orbit* $\mathcal{O}^G_{hyp}$ *of* G *in* $\mathfrak{g}$ *such that* $\mathcal{O}^G_{hyp} \cap \mathfrak{h} = \emptyset$.

(*vii*) *There exists a complex hyperbolic adjoint orbit* $\mathcal{O}^{G_{\mathbb{C}}}_{hyp}$ *of* $G_{\mathbb{C}}$ *in* $\mathfrak{g}_{\mathbb{C}}$ *such that* $\mathcal{O}^{G_{\mathbb{C}}}_{hyp} \cap \mathfrak{g} \neq \emptyset$ *and* $\mathcal{O}^{G_{\mathbb{C}}}_{hyp} \cap \mathfrak{g}^c = \emptyset$.

The equivalence among (i), (ii), (iii), (iv) and (v) in Fact 1.2.6 was proved in a more general setting in T. Kobayashi [15, Corollary 4.4]. The real rank condition (v) serves as a criterion for the Calabi–Markus phenomenon (iii) in Fact 1.2.6 (cf. [7], [15]). We will give a proof of the equivalence among (v), (vi) and (vii) in Appendix 1.B.

The second main result is a classification of semisimple symmetric pairs (G, H) satisfying one of (therefore, all of) the equivalent conditions in Theorem 1.2.2.

If a semisimple symmetric pair (G, H) is irreducible, but G is not simple, then G/H admits a proper $SL(2,\mathbb{R})$-action, since the symmetric space G/H can be regarded as a complex simple Lie group. Therefore, the crucial case is on symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ with simple Lie algebra $\mathfrak{g}$.

To describe our classification, we denote by

$$S := \{ (\mathfrak{g}, \mathfrak{h}) \mid (\mathfrak{g}, \mathfrak{h}) \text{ is a semisimple symmetric pair with a simple Lie algebra } \mathfrak{g} \}$$

The set S was classified by M. Berger [6] up to isomorphisms. We also put

$$A := \{ (\mathfrak{g}, \mathfrak{h}) \in S \mid (\mathfrak{g}, \mathfrak{h}) \text{ satisfies one of the conditions in Theorem 1.2.2} \},$$
$$B := \{ (\mathfrak{g}, \mathfrak{h}) \in S \mid \operatorname{rank}_{\mathbb{R}} \mathfrak{g} > \operatorname{rank}_{\mathbb{R}} \mathfrak{h} \} \setminus A,$$
$$C := \{ (\mathfrak{g}, \mathfrak{h}) \in S \mid \operatorname{rank}_{\mathbb{R}} \mathfrak{g} = \operatorname{rank}_{\mathbb{R}} \mathfrak{h} \}.$$

Then $A \cap C = \emptyset$ by Fact 1.2.6, and we have

$$S = A \sqcup B \sqcup C.$$

One can easily determine the set C in S. Thus, to describe the classification of A, we only need to give the classification of B.

Here is our classification of the set B, namely, a complete list of $(\mathfrak{g}, \mathfrak{h})$ satisfying the following:

$$\begin{gathered}\mathfrak{g} \text{ is simple, } (\mathfrak{g}, \mathfrak{h}) \text{ is a symmetric pair with } \operatorname{rank}_{\mathbb{R}} \mathfrak{g} > \operatorname{rank}_{\mathbb{R}} \mathfrak{h} \\ \text{but does not satisfies the equivalent conditions in Theorem 1.2.2.}\end{gathered} \tag{1.2.2}$$

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sl}(2k, \mathbb{R})$	$\mathfrak{sp}(k, \mathbb{R})$
$\mathfrak{sl}(2k, \mathbb{R})$	$\mathfrak{so}(k, k)$
$\mathfrak{sl}(2k-1, \mathbb{R})$	$\mathfrak{so}(k, k-1)$
$\mathfrak{su}^*(4m+2)$	$\mathfrak{sp}(m+1, m)$
$\mathfrak{su}^*(4m)$	$\mathfrak{sp}(m, m)$
$\mathfrak{su}^*(2k)$	$\mathfrak{so}^*(2k)$
$\mathfrak{so}(2k-1, 2k-1)$	$\mathfrak{so}(i+1, i) \oplus \mathfrak{so}(j, j+1)$ $(i + j = 2k - 2)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{f}_{4(4)}$
$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(4, \mathbb{R})$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{sp}(3, 1)$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{sl}(n, \mathbb{C})$	$\mathfrak{so}(n, \mathbb{C})$
$\mathfrak{sl}(2k, \mathbb{C})$	$\mathfrak{sp}(k, \mathbb{C})$
$\mathfrak{sl}(2k, \mathbb{C})$	$\mathfrak{su}(k, k)$
$\mathfrak{so}(4m+2, \mathbb{C})$	$\mathfrak{so}(i, \mathbb{C}) \oplus \mathfrak{so}(j, \mathbb{C})$ $(i + j = 4m + 2,\ i, j \text{ are odd})$

$\mathfrak{so}(4m+2,\mathbb{C})$	$\mathfrak{so}(2m+2,2m)$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{sp}(4,\mathbb{C})$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{f}_{4,\mathbb{C}}$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(2)}$

Table 1.2: Classification of $(\mathfrak{g},\mathfrak{h})$ satisfying (1.2.2)

Here, $k \geq 2$, $m \geq 1$ and $n \geq 2$.

Theorem 1.1.4, which gives a classification of the set A, is obtained by Table 1.2.

Concerning our classification, we will give an algorithm to check whether or not a given symmetric pair $(\mathfrak{g},\mathfrak{h})$ satisfies the condition (viii) in Theorem 1.2.2. More precisely, we will determine the set of complex antipodal hyperbolic orbits in a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ (see Section 1.6.2) and introduce an algorithm to check whether or not a given such orbit meets a real form $\mathfrak{g}$ [resp. $\mathfrak{g}^c$] (see Section 1.7). Table 1.2 is obtained by using this algorithm (see Section 1.9).

Remark 1.2.7. **(1)** *Using* [5, *Theorem*1.1], *Benoist gave a number of examples of symmetric pairs* (G,H) *which do not satisfy the condition* (iv) *in Theorem* 1.2.2 *with* $\operatorname{rank}_{\mathbb{R}}\mathfrak{g} > \operatorname{rank}_{\mathbb{R}}\mathfrak{h}$ (*see* [5, *Example* 1]). *Table* 1.2 *gives its complete list.*

(2) *We take this opportunity to correct* [34, *Table* 2.6], *where the pair* $(\mathfrak{sl}(2k-1,\mathbb{R}),\mathfrak{so}(k,k-1))$ *was missing.*

We discuss an application of the main result (Theorem 1.2.2) to the existence problem of compact Clifford–Klein forms. As we explained in Introduction, a Clifford–Klein form of G/H is the double coset space $\Gamma\backslash G/H$ when Γ is a discrete subgroup of G acting on G/H properly discontinuously and freely. Recall that we say that a homogeneous space G/H *admits compact Clifford–Klein forms*, if there exists such Γ where $\Gamma\backslash G/H$ is compact. See also [5, 15, 16, 17, 19, 23, 24, 28, 30, 33, 39, 45] for preceding results for the existence problem for compact Clifford–Klein forms. Among them, there are three methods that can be applied to semisimple symmetric spaces to show the non-existence of compact Clifford–Klein forms:

- Using the Hirzebruch–Kobayashi–Ono proportionality principle [15, Proposition 4.10], [23].
- Using a comparison theorem of cohomological dimension [17, Theorem 1.5]. (A generalization of the criterion in [15] of the Calabi–Markus phenomenon.)
- Using a criterion for the non-existence of properly discontinuous actions of non-virtually abelian groups [5, Corollary 1].

As an immediate corollary of the third method and the description of the set B by Table 1.2, one concludes:

Corollary 1.2.8. *The simple symmetric space G/H does not admit compact Clifford–Klein forms if $(\mathfrak{g}, \mathfrak{h})$ is in Table 1.2.*

1.3 Preliminary results for proper actions

In this section, we recall results of T. Kobayashi [15] and Y. Benoist [5] in a form that we shall need. Our proofs of the equivalences (i) $\Leftrightarrow$ (x) and (iv) $\Leftrightarrow$ (viii) in Theorem 1.2.2 will be based on these results (see Section 1.4.1 and Section 1.4.2).

1.3.1 Kobayashi's properness criterion

Let G be a connected linear semisimple Lie group and write $\mathfrak{g}$ for the Lie algebra of G. First, we fix a terminology as follows:

Definition 1.3.1. *We say that a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is reductive in $\mathfrak{g}$ if there exists a Cartan involution θ of $\mathfrak{g}$ such that $\mathfrak{h}$ is θ-stable. Furthermore, we say that a closed subgroup H of G is reductive in G if H has only finitely many connected components and the Lie algebra $\mathfrak{h}$ of H is reductive in $\mathfrak{g}$.*

For simplicity, we call $\mathfrak{h}$ [resp. H] a reductive subalgebra of $\mathfrak{g}$ [resp. a reductive subgroup of G] if $\mathfrak{h}$ is reductive in $\mathfrak{g}$ [resp. H is reductive in G]. We call such (G, H) a reductive pair. Note that a reductive subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a reductive Lie algebra.

We give two examples relating to Theorem 1.2.2:

Example 1.3.2. *In Setting* 1.2.1*, the subgroup H is reductive in G since there exists a Cartan involution θ on $\mathfrak{g}$, which is commutative with σ (cf. [6]).*

Example 1.3.3. *Let $\mathfrak{l}$ be a semisimple subalgebra of $\mathfrak{g}$. Then any Cartan involution on $\mathfrak{l}$ can be extended to a Cartan involution on $\mathfrak{g}$ (cf. G. D. Mostow [32]) and the analytic subgroup L corresponding to $\mathfrak{l}$ is closed in G (cf. K. Yosida [44]). Therefore, $\mathfrak{l}$ [resp. L] is reductive in $\mathfrak{g}$ [resp. G].*

In the rest of this subsection, we follow the setting below:

Setting 1.3.4. *G is a connected linear semisimple Lie group, H and L are reductive subgroups of G.*

We denote by $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ the Lie algebras of G, H and L, respectively. Take a Cartan involution θ of $\mathfrak{g}$ which preserves $\mathfrak{h}$. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{h} = \mathfrak{k}(\mathfrak{h}) + \mathfrak{p}(\mathfrak{h})$ for the Cartan decomposition of $\mathfrak{g}$, $\mathfrak{h}$ corresponding to θ, $\theta|_{\mathfrak{h}}$, respectively. We fix a maximal abelian subspace $\mathfrak{a}_{\mathfrak{h}}$ of $\mathfrak{p}(\mathfrak{h})$ (i.e. $\mathfrak{a}_{\mathfrak{h}}$ is a maximally split abelian subspace of $\mathfrak{h}$), and extend it to a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ (i.e. $\mathfrak{a}$ is a maximally split abelian subspace of $\mathfrak{g}$). We write K for the maximal compact subgroup of G with its Lie algebra $\mathfrak{k}$, and denote the Weyl group acting on $\mathfrak{a}$ by $W(\mathfrak{g}, \mathfrak{a}) := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. Since $\mathfrak{l}$ is also reductive in $\mathfrak{g}$, we can take a Cartan involution θ' of $\mathfrak{g}$ preserving $\mathfrak{l}$. We write $\mathfrak{l} = \mathfrak{k}'(\mathfrak{l}) + \mathfrak{p}'(\mathfrak{l})$ for the Cartan decomposition of $\mathfrak{l}$ corresponding to $\theta'|_{\mathfrak{l}}$, and fix a maximal abelian subspace $\mathfrak{a}'_{\mathfrak{l}}$ of $\mathfrak{p}'(\mathfrak{l})$. Then there exists $g \in G$ such that $\mathrm{Ad}(g) \cdot \mathfrak{a}'_{\mathfrak{l}}$ is contained in $\mathfrak{a}$, and we put $\mathfrak{a}_{\mathfrak{l}} := \mathrm{Ad}(g) \cdot \mathfrak{a}'_{\mathfrak{l}}$. The subset $W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}_{\mathfrak{l}}$ of $\mathfrak{a}$ does not depend on a choice of such $g \in G$.

The following fact holds:

Fact 1.3.5 (T. Kobayashi [15, Theorem 4.1])**.** *In Setting* 1.3.4*, L acts on G/H properly if and only if*

$$\mathfrak{a}_{\mathfrak{h}} \cap W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}_{\mathfrak{l}} = \{0\}.$$

The proof of Fact 1.2.6 is reduced to Fact 1.3.5 (see [15]). However, to prove the equivalences between (v), (vi) and (vii) in Fact 1.2.6 we need an additional argument which will be described in Appendix 1.B.

1.3.2 Benoist's criterion

Let (G, H) be a reductive pair (see Definition 1.3.1). In this subsection, we use the notation $\mathfrak{g}$, $\mathfrak{h}$, θ, $\mathfrak{a}_{\mathfrak{h}}$, $\mathfrak{a}$ and $W(\mathfrak{g}, \mathfrak{a})$ as in the previous subsection.

Let us denote the restricted root system of $(\mathfrak{g}, \mathfrak{a})$ by $\Sigma(\mathfrak{g}, \mathfrak{a})$. We fix a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ of $\Sigma(\mathfrak{g}, \mathfrak{a})$, and put

$$\mathfrak{a}_+ := \{ A \in \mathfrak{a} \mid \xi(X) \geq 0 \text{ for any } \xi \in \Sigma^+(\mathfrak{g}, \mathfrak{a}) \}.$$

Then $\mathfrak{a}_+$ is a fundamental domain for the action of the Weyl group $W(\mathfrak{g}, \mathfrak{a})$. We write w_0 for the longest element in $W(\mathfrak{g}, \mathfrak{a})$ with respect to the positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. Then, by the action of w_0, every element in $\mathfrak{a}_+$ moves to $-\mathfrak{a}_+ := \{-A \mid A \in \mathfrak{a}_+\}$. In particular,

$$-w_0 : \mathfrak{a} \to \mathfrak{a}, \quad A \mapsto -(w_0 \cdot A)$$

is an involutive automorphism on $\mathfrak{a}$ preserving $\mathfrak{a}_+$. We put

$$\mathfrak{b} := \{ A \in \mathfrak{a} \mid -w_0 \cdot A = A \}, \quad \mathfrak{b}_+ := \mathfrak{b} \cap \mathfrak{a}_+.$$

Then the next fact holds:

Fact 1.3.6 (Y. Benoist [5, Theorem in Section 1.1]). *The following conditions on a reductive pair (G, H) are equivalent:*

(i) G/H admits an infinite discontinuous group which is not virtually abelian.

(ii) $\mathfrak{b}_+ \not\subset w \cdot \mathfrak{a}_{\mathfrak{h}}$ for any $w \in W(\mathfrak{g}, \mathfrak{a})$.

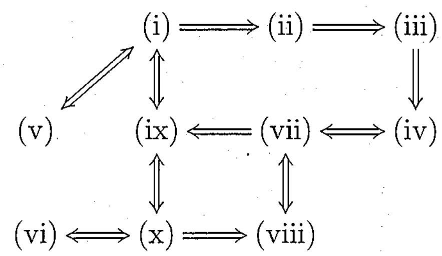
(iii) $\mathfrak{b}_+ \not\subset W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}_{\mathfrak{h}}$.

Remark 1.3.7. *Benoist showed* (i) $\Leftrightarrow$ (ii) *in Fact* 1.3.6. *The equivalence* (ii) $\Leftrightarrow$ (iii) *follows from the fact below (since $\mathfrak{b}_+$ is a convex set of $\mathfrak{a}$ and $w \cdot \mathfrak{a}_{\mathfrak{h}}$ is a linear subspace of $\mathfrak{a}$ for any $w \in W(\mathfrak{g}, \mathfrak{a})$).*

Fact 1.3.8. *Let U_1, U_2, $\ldots$, U_n be subspaces of a finite dimensional real vector space V and Ω a convex set of V. Then Ω is contained in $\bigcup_{i=1}^n U_i$ if and only if Ω is contained in U_k for some $k \in \{1, \ldots, n\}$.*

1.4 Proof of Theorem 1.2.2

We give a proof of Theorem 1.2.2 by proving the implications in the figure below:



In this section, to show the implications, we use some theorems, propositions and lemmas, which will be proved later in this chapter.

Notation: Throughout this chapter, for a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and its real form $\mathfrak{g}$, we denote a complex [resp. real] nilpotent, hyperbolic, antipodal hyperbolic adjoint orbit in $\mathfrak{g}_{\mathbb{C}}$ [resp. $\mathfrak{g}$] simply by a complex [resp. real] nilpotent, hyperbolic, antipodal hyperbolic orbit in $\mathfrak{g}_{\mathbb{C}}$ [resp. $\mathfrak{g}$].

1.4.1 Proof of (i) $\Leftrightarrow$ (ix) in Theorem 1.2.2

Our proof of the equivalence (i) $\Leftrightarrow$ (ix) in Theorem 1.2.2 starts with the next theorem, which will be proved in Section 1.5:

Theorem 1.4.1 (Corollary to Fact 1.3.5). *In Setting* 1.3.4, *the following conditions on* (G, H, L) *are equivalent*:

(*i*) *L acts on G/H properly,*

(*ii*) *There do not exist real hyperbolic orbits in $\mathfrak{g}$* (*see Definition* 1.2.3) *meeting both $\mathfrak{l}$ and $\mathfrak{h}$ other than the zero-orbit,*

where $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ are Lie algebras of G, H and L, respectively.

By using Theorem 1.4.1, we will prove the next proposition in Section 1.5:

Proposition 1.4.2. *Let* (G, H) *be a reductive pair* (*see Definition* 1.3.1). *Then there exists a bijection between the following two sets*:

- *The set of Lie group homomorphisms $\Phi : SL(2,\mathbb{R}) \to G$ such that $SL(2,\mathbb{R})$ acts on G/H properly via Φ,*
- *The set of $\mathfrak{sl}_2$-triples (A, X, Y) in $\mathfrak{g}$ such that the real adjoint orbit through A does not meet $\mathfrak{h}$.*

In Setting 1.2.1, the subgroup H of G is reductive in G (see Example 1.3.2). Hence, we obtain the equivalence (i) $\Leftrightarrow$ (ix) in Theorem 1.2.2.

1.4.2 Proof of (iv) $\Leftrightarrow$ (vii) in Theorem 1.2.2

We will prove the next theorem in Section 1.5:

Theorem 1.4.3 (Corollary to Fact 1.3.6). *The following conditions on a reductive pair (G, H) (see Definition 1.3.1) are equivalent:*

(*i*) *G/H admits an infinite discontinuous group that is not virtually abelian.*

(*ii*) *There exists a real antipodal hyperbolic orbit in $\mathfrak{g}$ that does not meet $\mathfrak{h}$.*

In Setting 1.2.1, the equivalence (iv) $\Leftrightarrow$ (vii) in Theorem 1.2.2 holds as a special case of Theorem 1.4.3.

1.4.3 Proofs of (x) $\Leftrightarrow$ (ix), (viii) $\Leftrightarrow$ (vii) and (x) $\Rightarrow$ (viii) in Theorem 1.2.2

Let $\mathfrak{g}_\mathbb{C}$ be a complex semisimple Lie algebra. We use the following convention for hyperbolic elements (see Definition 1.2.3):

$$\begin{aligned}
\mathcal{H} &:= \{\, A \in \mathfrak{g}_\mathbb{C} \mid A \text{ is a hyperbolic element in } \mathfrak{g}_\mathbb{C} \,\},\\
\mathcal{H}^a &:= \{\, A \in \mathcal{H} \mid \text{The complex adjoint orbit through } A \text{ is antipodal} \,\},\\
\mathcal{H}^n &:= \{\, A \in \mathfrak{g}_\mathbb{C} \mid \text{There exist } X, Y \in \mathfrak{g}_\mathbb{C}\\
&\qquad\qquad \text{such that } (A, X, Y) \text{ is an } \mathfrak{sl}_2\text{-triple} \,\}.
\end{aligned}$$

We also write $\mathcal{H}/G_\mathbb{C}$, $\mathcal{H}^a/G_\mathbb{C}$ for the sets of complex hyperbolic orbits and complex antipodal hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$, respectively. Let us denote by $\mathcal{H}^n/G_\mathbb{C}$ the set of complex adjoint orbits contained in $\mathcal{H}^n$.

The next lemma will be proved in Section 1.6.3:

Lemma 1.4.4. *For any $\mathfrak{sl}_2$-triple (A, X, Y) in $\mathfrak{g}_\mathbb{C}$, the element A of $\mathfrak{g}_\mathbb{C}$ is hyperbolic and the complex adjoint orbit through A in $\mathfrak{g}_\mathbb{C}$ is antipodal.*

By Lemma 1.4.4, we have

$$\mathcal{H}^n \subset \mathcal{H}^a \subset \mathcal{H}.$$

Hence, the implication (x) $\Rightarrow$ (viii) in Theorem 1.2.2 follows.

Further, for any subalgebra $\mathfrak{l}$ of $\mathfrak{g}_{\mathbb{C}}$, we also use the following convention:

$$\begin{aligned}
\mathcal{H}_{\mathfrak{l}} &:= \{\, A \in \mathcal{H} \mid \text{The complex adjoint orbit through } A \text{ meets } \mathfrak{l} \,\}, \\
\mathcal{H}^a_{\mathfrak{l}} &:= \mathcal{H}^a \cap \mathcal{H}_{\mathfrak{l}}, \\
\mathcal{H}^n_{\mathfrak{l}} &:= \mathcal{H}^n \cap \mathcal{H}_{\mathfrak{l}}.
\end{aligned}$$

Let us write $\mathcal{H}_{\mathfrak{l}}/G_{\mathbb{C}}$, $\mathcal{H}^a_{\mathfrak{l}}/G_{\mathbb{C}}$, $\mathcal{H}^n_{\mathfrak{l}}/G_{\mathbb{C}}$ for the sets of complex adjoint orbits contained in $\mathcal{H}$, $\mathcal{H}^a$, $\mathcal{H}^n$ meeting $\mathfrak{l}$, respectively.

Here, we fix a real form $\mathfrak{g}$, and set

$$\begin{aligned}
\mathcal{H}(\mathfrak{g}) &:= \{\, A \in \mathfrak{g} \mid A \text{ is a hyperbolic element in } \mathfrak{g} \,\}, \\
\mathcal{H}^a(\mathfrak{g}) &:= \{\, A \in \mathcal{H}(\mathfrak{g}) \mid \text{The real adjoint orbit through } A \text{ is antipodal} \,\}, \\
\mathcal{H}^n(\mathfrak{g}) &:= \{\, A \in \mathfrak{g} \mid \text{There exist } X, Y \in \mathfrak{g} \\
&\qquad\qquad \text{such that } (A, X, Y) \text{ is an } \mathfrak{sl}_2\text{-triple} \,\}.
\end{aligned}$$

We also write $\mathcal{H}(\mathfrak{g})/G$, $\mathcal{H}^a(\mathfrak{g})/G$, $\mathcal{H}^n(\mathfrak{g})/G$ for the sets of real adjoint orbits contained in $\mathcal{H}(\mathfrak{g})$, $\mathcal{H}^a(\mathfrak{g})$, $\mathcal{H}^n(\mathfrak{g})$, respectively.

Then the following proposition gives a one-to-one correspondence between real hyperbolic orbits and complex hyperbolic orbits with real points:

Proposition 1.4.5. *(i) The following map gives a one-to-one correspondence between $\mathcal{H}(\mathfrak{g})/G$ and $\mathcal{H}_{\mathfrak{g}}/G_{\mathbb{C}}$:*

$$\begin{aligned}
\mathcal{H}(\mathfrak{g})/G \to \mathcal{H}_{\mathfrak{g}}/G_{\mathbb{C}}, &\quad \mathcal{O}^G_{hyp} \mapsto \operatorname{Ad}(G_{\mathbb{C}}) \cdot \mathcal{O}^G_{hyp}, \\
\mathcal{H}_{\mathfrak{g}}/G_{\mathbb{C}} \to \mathcal{H}(\mathfrak{g})/G, &\quad \mathcal{O}^{G_{\mathbb{C}}}_{hyp} \mapsto \mathcal{O}^{G_{\mathbb{C}}}_{hyp} \cap \mathfrak{g}.
\end{aligned}$$

(ii) The bijection in (i) *gives the one-to-one correspondence below:*

$$\mathcal{H}^a(\mathfrak{g})/G \overset{1:1}{\longleftrightarrow} \mathcal{H}^a_{\mathfrak{g}}/G_{\mathbb{C}}.$$

(iii) The bijection in (i) *gives the one-to-one correspondence below:*

$$\mathcal{H}^n(\mathfrak{g})/G \overset{1:1}{\longleftrightarrow} \mathcal{H}^n_{\mathfrak{g}}/G_{\mathbb{C}}.$$

The proof of Proposition 1.4.5 will be given in Section 1.7.

In Setting 1.2.1, recall that both $\mathfrak{g}$ and $\mathfrak{g}^c$ are real forms of $\mathfrak{g}_\mathbb{C}$. In Section 1.8, we will prove the following proposition, which claims that a complex hyperbolic orbit meets $\mathfrak{h}$ if it meets both $\mathfrak{g}$ and $\mathfrak{g}^c$:

Proposition 1.4.6. *In Setting* 1.2.1, $\mathcal{H}_\mathfrak{g} \cap \mathcal{H}_{\mathfrak{g}^c} = \mathcal{H}_\mathfrak{h}$.

The equivalences (x) $\Leftrightarrow$ (ix) and (viii) $\Leftrightarrow$ (vii) in Theorem 1.2.2 follows from Proposition 1.4.5 and Proposition 1.4.6.

1.4.4 Proof of (vi) $\Leftrightarrow$ (x) in Theorem 1.2.2

The equivalence (vi) $\Leftrightarrow$ (x) in Theorem 1.2.2 can be obtained by the Jacobson–Morozov theorem and the lemma below (see Proposition 1.7.8 for a proof):

Lemma 1.4.7 (Corollary to J. Sekiguchi [38, Proposition 1.11]). *Let $\mathfrak{g}_\mathbb{C}$ be a complex semisimple Lie algebra and $\mathfrak{g}$ a real form of $\mathfrak{g}_\mathbb{C}$. Then the following conditions on an $\mathfrak{sl}_2$-triple (A, X, Y) in $\mathfrak{g}_\mathbb{C}$ are equivalent:*

(*i*) *The complex adjoint orbit through A in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{g}$.*

(*ii*) *The complex adjoint orbit through X in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{g}$.*

1.4.5 Proof of (vii) $\Rightarrow$ (ix) in Theorem 1.2.2

Let $\mathfrak{g}$ be a semisimple Lie algebra. In this subsection, we use $\mathcal{H}(\mathfrak{g})$, $\mathcal{H}^a(\mathfrak{g})$ and $\mathcal{H}^n(\mathfrak{g})$ as in Section 1.4.3.

To prove the implication (vii) $\Rightarrow$ (ix), we use the next proposition and lemma:

Proposition 1.4.8. *We take*

$$\mathfrak{b} := \{ A \in \mathfrak{a} \mid -w_0 \cdot A = A \}, \quad \mathfrak{b}_+ := \mathfrak{b} \cap \mathfrak{a}_+$$

as in Section 1.3.2. *Then the following holds:*

(*i*) $\mathfrak{b} = \mathbb{R}\text{-span}(\mathfrak{a}_+ \cap \mathcal{H}^n(\mathfrak{g}))$.

(*ii*) $\mathcal{H}^a(\mathfrak{g}) = \mathrm{Ad}(G) \cdot \mathfrak{b}_+$.

Lemma 1.4.9. *Let* $(\mathfrak{g}, \mathfrak{h}, \sigma)$ *be a semisimple symmetric pair. We fix a Cartan involution* θ *on* $\mathfrak{g}$ *such that* $\theta\sigma = \sigma\theta$ *and denote by* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ *the Cartan decomposition of* $\mathfrak{g}$ *with respect to* θ*. Let us take* $\mathfrak{a}$ *and* $\mathfrak{a}_{\mathfrak{h}} = \mathfrak{a} \cap \mathfrak{h}$ *as in Section* 1.3.1*. We fix an ordering on* $\mathfrak{a}_{\mathfrak{h}}$ *and extend it to* $\mathfrak{a}$*, and put* $\mathfrak{a}_+$ *to the closed Weyl chamber of* $\mathfrak{a}$ *with respect to the ordering. Then*

$$\mathfrak{a}_+ \cap \mathcal{H}_{\mathfrak{h}}(\mathfrak{g}) \subset \mathfrak{a}_{\mathfrak{h}},$$

where $\mathcal{H}_{\mathfrak{h}}(\mathfrak{g})$ *is the set of hyperbolic elements in* $\mathfrak{g}$ *whose adjoint orbits meet* $\mathfrak{h}$.

Postponing the proof of Proposition 1.4.8 and Lemma 1.4.9 in later sections, we complete the proof of the implication (viii) $\Rightarrow$ (x) in Theorem 1.2.2.

Proof of (viii) $\Rightarrow$ (x) *in Theorem* 1.2.2. We shall prove that $\mathcal{H}^a(\mathfrak{g}) \subset \mathcal{H}_{\mathfrak{h}}(\mathfrak{g})$ under the assumption $\mathcal{H}^n(\mathfrak{g}) \subset \mathcal{H}_{\mathfrak{h}}(\mathfrak{g})$. By combining Proposition 1.4.8 (i), Lemma 1.4.9 with the assumption, we have

$$\mathfrak{b} \subset \mathfrak{a}_{\mathfrak{h}} (\subset \mathfrak{h}).$$

Therefore, by Proposition 1.4.8 (ii), we obtain that $\mathcal{H}^a(\mathfrak{g}) \subset \mathcal{H}_{\mathfrak{h}}(\mathfrak{g})$. □

We shall give a proof of Proposition 1.4.8 (i) in Section 1.7.5 by comparing Dynkin's classification of $\mathfrak{sl}_2$-triples in $\mathfrak{g}_{\mathbb{C}}$ [10] with the Satake diagram of the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$. The proof of Proposition 1.4.8 (ii) will be given in Section 1.5.1, and that of Lemma 1.4.9 in Section 1.8.

1.4.6 Proofs of (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (viii) and (i) $\Leftrightarrow$ (v) in Theorem 1.2.2

The implication (i) $\Rightarrow$ (ii) in Theorem 1.2.2 is deduced from the lifting theorem of surface groups (cf. [26]). The implication (iii) $\Rightarrow$ (viii) follows by the fact that the surface group of genus g is not virtually abelian for any $g \geq 2$.

The equivalence (i) $\Leftrightarrow$ (v) can be proved by the observation below: Let Γ_0 be the free group generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $SL(2, \mathbb{R})$; Then, for any free group Γ generated by a unipotent element in a linear semisimple Lie group G, there exists a Lie group homomorphism $\Phi : SL(2, \mathbb{R}) \to G$ such that $\Phi(\Gamma_0) = \Gamma$ (by the Jacobson–Morozov theorem); Furthermore, by [18, Lemma 3.2], for any closed subgroup H of G, the $SL(2, \mathbb{R})$-action on G/H via Φ is proper if and only if the Γ-action on G/H is properly discontinuous.

1.5 Real hyperbolic orbits and proper actions of reductive subgroups

In this section, we prove Theorem 1.4.1, Proposition 1.4.2, Theorem 1.4.3 and Proposition 1.4.8 (ii).

1.5.1 Kobayashi's properness criterion and Benoist's criterion rephrased by real hyperbolic orbits

In this subsection, Theorem 1.4.1 and Theorem 1.4.3 are proved as corollaries to Fact 1.3.5 and Fact 1.3.6, respectively. We also prove Proposition 1.4.8 (ii) in this subsection.

Let $\mathfrak{g}$ be a semisimple Lie algebra. The next fact for real hyperbolic orbits in $\mathfrak{g}$ (see Definition 1.2.3) is well known:

Fact 1.5.1. *Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}$ and a maximally split abelian subspace $\mathfrak{a}$ of $\mathfrak{g}$ (i.e. $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$). Then any real hyperbolic orbit $\mathcal{O}^G_{hyp}$ in $\mathfrak{g}$ meets $\mathfrak{a}$, and the intersection $\mathcal{O}^G_{hyp} \cap \mathfrak{a}$ is a single $W(\mathfrak{g}, \mathfrak{a})$-orbit, where $W(\mathfrak{g}, \mathfrak{a}) := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. In particular, we have a bijection*

$$\mathcal{H}(\mathfrak{g})/G \to \mathfrak{a}/W(\mathfrak{g}, \mathfrak{a}), \quad \mathcal{O}^G_{hyp} \mapsto \mathcal{O}^G_{hyp} \cap \mathfrak{a},$$

where $\mathcal{H}(\mathfrak{g})/G$ is the set of real hyperbolic orbits in $\mathfrak{g}$ and $\mathfrak{a}/W(\mathfrak{g}, \mathfrak{a})$ the set of $W(\mathfrak{g}, \mathfrak{a})$-orbits in $\mathfrak{a}$.

Let $\mathfrak{h}$ be a reductive subalgebra of $\mathfrak{g}$ (see Definition 1.3.1). Take a maximally split abelian subspace $\mathfrak{a}_\mathfrak{h}$ of $\mathfrak{h}$ and extend it to a maximally split abelian subspace $\mathfrak{a}$ of $\mathfrak{g}$ in a similar way as in Section 1.3.1. Then the following lemma holds:

Lemma 1.5.2. *A real hyperbolic orbit $\mathcal{O}^G_{hyp}$ in $\mathfrak{g}$ meets $\mathfrak{h}$ if and only if it meets $\mathfrak{a}_\mathfrak{h}$. In particular, we have a bijection*

$$\mathcal{H}_\mathfrak{h}(\mathfrak{g})/G \to \{\, \mathcal{O}^{W(\mathfrak{g},\mathfrak{a})} \in \mathfrak{a}/W(\mathfrak{g}, \mathfrak{a}) \mid \mathcal{O}^{W(\mathfrak{g},\mathfrak{a})} \cap \mathfrak{a}_\mathfrak{h} \neq \emptyset \,\}, \quad \mathcal{O}^G_{hyp} \mapsto \mathcal{O}^G_{hyp} \cap \mathfrak{a},$$

where $\mathcal{H}_\mathfrak{h}(\mathfrak{g})/G$ is the set of real hyperbolic orbits in $\mathfrak{g}$ meeting $\mathfrak{h}$.

Proof of Lemma 1.5.2. Suppose that $\mathcal{O}^G_{\text{hyp}}$ meets $\mathfrak{h}$, and we shall prove that $\mathcal{O}^G_{\text{hyp}}$ meets $\mathfrak{a}_\mathfrak{h}$. We write $\mathfrak{h} = \mathfrak{h}_{\text{ss}} + Z_\mathfrak{h}$ for the decomposition of $\mathfrak{h}$ by the semisimple part and the center of $\mathfrak{h}$. If $\mathfrak{h}$ is semisimple, i.e. if $\mathfrak{h} = \mathfrak{h}_{\text{ss}}$, then

$\mathcal{O}^G_{\mathrm{hyp}} \cap \mathfrak{h}$ contains some (in fact, a unique) hyperbolic adjoint orbit in $\mathfrak{h}$. Hence, our claim follows by Fact 1.5.1. Thus let us consider the cases where $Z_{\mathfrak{h}} \neq \{0\}$. The Cartan involution θ preserves $\mathfrak{h}_{\mathrm{ss}}$, $Z_{\mathfrak{h}}$, respectively. Hence, $\mathfrak{h}$ can be decomposed as

$$\mathfrak{h} = \mathfrak{h}_{\mathrm{ss}} \oplus (Z_{\mathfrak{h}} \cap \mathfrak{p}) \oplus (Z_{\mathfrak{h}} \cap \mathfrak{k}),$$

where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ corresponding to θ. Let X be an element in $\mathcal{O}^G_{\mathrm{hyp}} \cap \mathfrak{h}$ and we put

$$X = X_{\mathrm{ss}} + X_{\mathfrak{p}} + X_{\mathfrak{k}} \quad (X_{\mathrm{ss}} \in \mathfrak{h}_{\mathrm{ss}},\ X_{\mathfrak{p}} \in Z_{\mathfrak{h}} \cap \mathfrak{p}, X_{\mathfrak{k}} \in Z_{\mathfrak{h}} \cap \mathfrak{k}).$$

By Fact 1.5.1, we only need to show that X_{ss} is hyperbolic in $\mathfrak{h}_{\mathrm{ss}}$ and $X_{\mathfrak{k}} = 0$.

First, we prove that X_{ss} is hyperbolic in $\mathfrak{h}_{\mathrm{ss}}$, i.e. $\mathrm{ad}_{\mathfrak{h}_{\mathrm{ss}}} X_{\mathrm{ss}} \in \mathrm{End}(\mathfrak{h}_{\mathrm{ss}})$ is diagonalizable with only real eigenvalues. Since $X_{\mathfrak{p}} + X_{\mathfrak{k}}$ is in $Z_{\mathfrak{h}}$, we have

$$\mathrm{ad}_{\mathfrak{h}_{\mathrm{ss}}} X_{\mathrm{ss}} = (\mathrm{ad}_{\mathfrak{g}} X)|_{\mathfrak{h}_{\mathrm{ss}}}.$$

Recall that X is hyperbolic in $\mathfrak{g}$, that is, $\mathrm{ad}_{\mathfrak{g}} X$ is diagonalizable with only real eigenvalues. Then $\mathrm{ad}_{\mathfrak{h}_{\mathrm{ss}}} X_{\mathrm{ss}}$ is also diagonalizable with only real eigenvalues.

Finally, we show that $X_{\mathfrak{k}} = 0$. By the argument above, there is no loss of generality in assuming that X_{ss} is in $\mathfrak{h}_{\mathrm{ss}} \cap \mathfrak{a}_{\mathfrak{h}}$. In particular, X_{ss} and $X_{\mathfrak{p}}$ are in $\mathfrak{p}$. Thus, by Fact 1.5.1, $X_{\mathrm{ss}} + X_{\mathfrak{p}}$ is hyperbolic in $\mathfrak{g}$. Then

$$X_{\mathfrak{k}} = X - (X_{\mathrm{ss}} + X_{\mathfrak{p}})$$

is also hyperbolic in $\mathfrak{g}$ since X and $X_{\mathrm{ss}} + X_{\mathfrak{p}}$ are commutative. It is known that for any element $Y_{\mathfrak{k}}$ in $\mathfrak{k}$, the element $\mathrm{ad}_{\mathfrak{g}} Y_{\mathfrak{k}}$ in $\mathrm{End}(\mathfrak{g})$ has only pure imaginary eigenvalues. Hence, we have $\mathrm{ad}_{\mathfrak{g}} X_{\mathfrak{k}} = 0$, and $X_{\mathfrak{k}}$ must be zero since $\mathfrak{g}$ is semisimple. □

We now prove Theorem 1.4.1 as a corollary to Fact 1.3.5.

Proof of Theorem 1.4.1. In Setting 1.3.4, by Fact 1.5.1 and Lemma 1.5.2, we have a bijection between the following two sets:

- The set of $W(\mathfrak{g}, \mathfrak{a})$-orbits in $\mathfrak{a}$ meeting both $\mathfrak{a}_{\mathfrak{h}}$ and $\mathfrak{a}_{\mathfrak{l}}$,
- The set of real hyperbolic orbits in $\mathfrak{g}$ meeting both $\mathfrak{h}$ and $\mathfrak{l}$.

Hence, our claim follows from Fact 1.3.5. □

To prove Theorem 1.4.3, we shall show the next lemma:

Lemma 1.5.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then a real hyperbolic orbit in $\mathfrak{g}$ is antipodal if and only if it meets $\mathfrak{b}_+$ (see Section 1.3.2 for the notation). In particular, we have a bijection*

$$\mathcal{H}^a(\mathfrak{g})/G \to \{\, \mathcal{O}^{W(\mathfrak{g},\mathfrak{a})} \in \mathfrak{a}/W(\mathfrak{g},\mathfrak{a}) \mid \mathcal{O}^{W(\mathfrak{g},\mathfrak{a})} \cap \mathfrak{b}_+ \neq \emptyset \},\ \mathcal{O}^G_{hyp} \mapsto \mathcal{O}^G_{hyp} \cap \mathfrak{a},$$

where $\mathcal{H}^a(\mathfrak{g})/G$ is the set of real antipodal hyperbolic orbits in $\mathfrak{g}$.

Proof of Lemma 1.5.3. By Fact 1.5.1, any real hyperbolic orbit $\mathcal{O}^G_{\mathrm{hyp}}$ in $\mathfrak{g}$ meets $\mathfrak{a}_+$ with a unique element A_0 in $\mathcal{O}^G_{\mathrm{hyp}} \cap \mathfrak{a}_+$. It remains to prove that $-A_0$ is in $\mathcal{O}^G_{\mathrm{hyp}}$ if and only if $-w_0 \cdot A_0 = A_0$. First, we suppose that $-A_0 \in \mathcal{O}^G_{\mathrm{hyp}}$. Then the element $-A_0$ of $-\mathfrak{a}_+$ is conjugate to A_0 under the action of $W(\mathfrak{g},\mathfrak{a})$ by Fact 1.5.1. Recall that both $\mathfrak{a}_+$ and $-\mathfrak{a}_+$ are fundamental domains of $\mathfrak{a}$ for the action of $W(\mathfrak{g},\mathfrak{a})$, and $w_0 \cdot \mathfrak{a}_+ = -\mathfrak{a}_+$. Hence, we obtain that $-w_0 \cdot A_0 = A_0$. Conversely, we assume that $-A_0 = w_0 \cdot A_0$. In particular, $-A_0$ is in $W(\mathfrak{g},\mathfrak{a}) \cdot A_0$. This implies that $-A_0$ is also in $\mathcal{O}^G_{\mathrm{hyp}}$. □

We are ready to prove Theorem 1.4.3.

Proof of Theorem 1.4.3. In the setting of Fact 1.3.6, by Fact 1.5.1, Lemma 1.5.2 and Lemma 1.5.3, we have a bijection between the following two sets:

- The set of $W(\mathfrak{g},\mathfrak{a})$-orbits in $\mathfrak{a}$ which meet $\mathfrak{b}_+$ but not $\mathfrak{a}_{\mathfrak{h}}$.
- The set of real antipodal hyperbolic orbits in $\mathfrak{g}$ that do not meet $\mathfrak{h}$.

Hence, our claim follows from Fact 1.3.6. □

Proposition 1.4.8 (ii) is also obtained by Lemma 1.5.3 as follows:

Proof of Proposition 1.4.8 (ii). The first claim of Lemma 1.5.3 means that an adjoint orbit $\mathcal{O}$ in $\mathfrak{g}$ is real antipodal hyperbolic if and only if $\mathcal{O}$ is in $\mathrm{Ad}(G) \cdot \mathfrak{b}_+$. Thus we have $\mathcal{H}^a(\mathfrak{g}) = \mathrm{Ad}(G) \cdot \mathfrak{b}_+$. □

1.5.2 Lie group homomorphisms from $SL(2,\mathbb{R})$

In this subsection, we prove Proposition 1.4.2 by using Theorem 1.4.1.

Let G be a connected linear semisimple Lie group and write $\mathfrak{g}$ for its Lie algebra. Then the next lemma holds:

Lemma 1.5.4. *Any Lie algebra homomorphism $\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$ can be uniquely lifted to $\Phi : SL(2,\mathbb{R}) \to G$ (i.e. Φ is the Lie group homomorphism with its differential ϕ). In particular, we have a bijection between the following two sets:*

- *The set of Lie group homomorphism from $SL(2,\mathbb{R})$ to G,*
- *The set of $\mathfrak{sl}_2$-triples in $\mathfrak{g}$.*

Proof of Lemma 1.5.4. The uniqueness follows from the connectedness of $SL(2,\mathbb{R})$. We shall lift ϕ. Let us denote by

$$\phi_{\mathbb{C}} : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}$$

the complexification of ϕ. Recall that G is linear. Then we can take a complexification $G_{\mathbb{C}}$ of G. Since $SL(2,\mathbb{C})$ is simply-connected, the Lie algebra homomorphism $\phi_{\mathbb{C}}$ can be lifted to

$$\Phi_{\mathbb{C}} : SL(2,\mathbb{C}) \to G_{\mathbb{C}}.$$

Then $\Phi_{\mathbb{C}}(SL(2,\mathbb{R}))$ is an analytic subgroup of $G_{\mathbb{C}}$ corresponding to the semisimple subalgebra $\phi(\mathfrak{sl}(2,\mathbb{R}))$ of $\mathfrak{g}$. In particular, $\Phi_{\mathbb{C}}(SL(2,\mathbb{R}))$ is a closed subgroup of G. Therefore, we can lift ϕ to $\Phi_{\mathbb{C}}|_{SL(2,\mathbb{R})}$. □

Let H be a reductive subgroup of G (see Definition 1.3.1) and denote by $\mathfrak{h}$ the Lie algebra of H. To prove Proposition 1.4.2, it remains to show the following corollary to Theorem 1.4.1:

Corollary 1.5.5. *Let $\Phi : SL(2,\mathbb{R}) \to G$ be a Lie group homomorphism, and denote its differential by $\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$. We put*

$$A_\phi := \phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{g}.$$

Then $SL(2,\mathbb{R})$ acts on G/H properly via Φ if and only if the real adjoint orbit through A_ϕ in $\mathfrak{g}$ does not meet $\mathfrak{h}$.

Proof of Corollary 1.5.5. Since $\mathfrak{sl}(2,\mathbb{R})$ is simple, we can assume that $\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$ is injective. We put

$$L := \Phi(SL(2,\mathbb{R})), \quad \mathfrak{l} := \phi(\mathfrak{sl}(2,\mathbb{R})).$$

Then L is a reductive subgroup of G (see Example 1.3.3). Since ϕ is injective and the center of $SL(2,\mathbb{R})$ is finite, the kernel $\operatorname{Ker}\Phi$ is also finite. Therefore, the action of $SL(2,\mathbb{R})$ on G/H via Φ is proper if and only if the action of L on G/H is proper. By Theorem 1.4.1, the action of L on G/H is proper if and only if there does not exist a real hyperbolic orbit in $\mathfrak{g}$ meeting both $\mathfrak{h}$ and $\mathfrak{l}$ apart from the zero-orbit. Here, we take $\mathfrak{a}_\mathfrak{l} := \mathbb{R}A_\phi$ as a maximally split abelian subspace of $\mathfrak{l}$. Then, by Lemma 1.5.2, for any real hyperbolic orbits in $\mathfrak{g}$, if it meets $\mathfrak{l}$ then also meets $\mathfrak{a}_\mathfrak{l}$. Therefore, the action of $SL(2,\mathbb{R})$ on G/H via Φ is proper if and only if the real adjoint orbit through A_ϕ in $\mathfrak{g}$ does not meet $\mathfrak{h}$. □

1.6 Weighted Dynkin diagrams of complex adjoint orbits

Let $\mathfrak{g}_\mathbb{C}$ be a complex semisimple Lie algebra. In this section, we recall some well-known facts for weighted Dynkin diagrams of complex hyperbolic orbits and complex nilpotent orbits in $\mathfrak{g}_\mathbb{C}$. We also prove Lemma 1.4.4, and determine weighted Dynkin diagrams of complex antipodal hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$.

1.6.1 Weighted Dynkin diagrams of complex hyperbolic orbits

In this subsection, we recall a parameterization of complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ by weighted Dynkin diagrams.

Fix a Cartan subalgebra $\mathfrak{j}_\mathbb{C}$ of $\mathfrak{g}_\mathbb{C}$. Let us denote by $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, and define the real form $\mathfrak{j}$ of $\mathfrak{j}_\mathbb{C}$ by

$$\mathfrak{j} := \{ A \in \mathfrak{j}_\mathbb{C} \mid \alpha(A) \in \mathbb{R} \text{ for any } \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \}.$$

Then $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ can be regarded as a subset of $\mathfrak{j}^*$. We fix a positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ of the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$. Then a closed Weyl chamber

$$\mathfrak{j}_+ := \{ A \in \mathfrak{j} \mid \alpha(A) \geq 0 \text{ for any } \alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \}$$

is a fundamental domain of $\mathfrak{j}$ for the action of the Weyl group $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$.

In this setting, the next fact for complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ is well known.

Fact 1.6.1. *Any complex hyperbolic orbit $\mathcal{O}_{hyp}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{j}$, and the intersection $\mathcal{O}_{hyp}^{G_\mathbb{C}} \cap \mathfrak{j}$ is a single $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$-orbit in $\mathfrak{j}$. In particular, we have one-to-one correspondences below:*

$$\mathcal{H}/G_\mathbb{C} \stackrel{1:1}{\longleftrightarrow} \mathfrak{j}/W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \stackrel{1:1}{\longleftrightarrow} \mathfrak{j}_+,$$

where $\mathcal{H}/G_\mathbb{C}$ is the set of complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{j}/W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ the set of $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$-orbits in $\mathfrak{j}$.

Let Π denote the fundamental system of $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$. Then, for any $A \in \mathfrak{j}$, we can define a map

$$\Psi_A : \Pi \to \mathbb{R}, \ \alpha \mapsto \alpha(A).$$

We call Ψ_A the weighted Dynkin diagram corresponding to $A \in \mathfrak{j}$, and $\alpha(A)$ the weight on a node $\alpha \in \Pi$ of the weighted Dynkin diagram. Since Π is a basis of $\mathfrak{j}^*$, the correspondence

$$\Psi : \mathfrak{j} \to \mathrm{Map}(\Pi, \mathbb{R}), \ A \mapsto \Psi_A \tag{1.6.1}$$

is a linear isomorphism between real vector spaces. In particular, Ψ is bijective. Furthermore,

$$\Psi|_{\mathfrak{j}_+} : \mathfrak{j}_+ \to \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}), \ A \mapsto \Psi_A$$

is also bijective. We say that a weighted Dynkin diagram is trivial if all weights are zero. Namely, the trivial diagram corresponds to the zero of $\mathfrak{j}$ by Ψ.

The weighted Dynkin diagram of a complex hyperbolic orbit $\mathcal{O}_{\mathrm{hyp}}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ is defined as the weighted Dynkin diagram corresponding to the unique element $A_\mathcal{O}$ in $\mathcal{O}_{\mathrm{hyp}}^{G_\mathbb{C}} \cap \mathfrak{j}_+$ (see Fact 1.6.1). Combining Fact 1.6.1 with the bijection $\Psi|_{\mathfrak{j}_+}$, the map

$$\mathcal{H}/G_\mathbb{C} \to \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}), \quad \mathcal{O}_{\mathrm{hyp}}^{G_\mathbb{C}} \mapsto \Psi_{A_\mathcal{O}}$$

is also bijective.

1.6.2 Weighted Dynkin diagrams of complex antipodal hyperbolic orbits

In this subsection, we determine complex antipodal hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ (see Definition 1.2.3) by describing the weighted Dynkin diagrams.

We consider the same setting as in Section 1.6.1. Let us denote by $w_0^\mathbb{C}$ the longest element of $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ corresponding to the positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$. Then, by the action of $w_0^\mathbb{C}$, every element in $\mathfrak{j}_+$ moves to $-\mathfrak{j}_+ := \{-A \mid A \in \mathfrak{j}_+\}$. In particular,

$$-w_0^\mathbb{C} : \mathfrak{j} \to \mathfrak{j}, \quad A \mapsto -(w_0^\mathbb{C} \cdot A)$$

is an involutive automorphism on $\mathfrak{j}$ preserving $\mathfrak{j}_+$. We put

$$\mathfrak{j}^{-w_0^\mathbb{C}} := \{\, A \in \mathfrak{j} \mid -w_0^\mathbb{C} \cdot A = A \,\}, \quad \mathfrak{j}_+^{-w_0^\mathbb{C}} := \mathfrak{j}_+ \cap \mathfrak{j}^{-w_0^\mathbb{C}}.$$

We recall that any complex hyperbolic orbit $\mathcal{O}_{\mathrm{hyp}}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{j}_+$ with a unique element $A_\mathcal{O}$ in $\mathcal{O}_{\mathrm{hyp}}^{G_\mathbb{C}} \cap \mathfrak{j}_+$ (see Fact 1.6.1). Then the lemma below holds:

Lemma 1.6.2. *A complex hyperbolic orbit $\mathcal{O}_{hyp}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ is antipodal if and only if the corresponding element $A_\mathcal{O}$ is in $\mathfrak{j}_+^{-w_0^\mathbb{C}}$. In particular, we have a one-to-one correspondence*

$$\mathcal{H}^a/G_\mathbb{C} \overset{1:1}{\longleftrightarrow} \mathfrak{j}_+^{-w_0^\mathbb{C}},$$

where $\mathcal{H}^a/G_\mathbb{C}$ is the set of complex antipodal hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$.

Proof of Lemma 1.6.2. The proof parallels to that of Lemma 1.5.3. □

Recall that the map

$$\Psi : \mathfrak{j} \to \mathrm{Map}(\Pi, \mathbb{R}), \quad A \mapsto \Phi_A$$

is a linear isomorphism (see Section 1.6.1). Thus $-w_0^\mathbb{C}$ induces an involutive endomorphism on $\mathrm{Map}(\Pi, \mathbb{R})$. By using this endomorphism, the following theorem gives a classification of complex antipodal hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$.

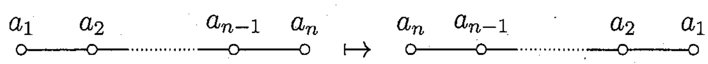
Theorem 1.6.3. *Let ι denote the involutive endomorphism on* $\mathrm{Map}(\Pi, \mathbb{R})$ *induced by $-w_0^\mathbb{C}$. Then the following holds:*

(*i*) *A complex hyperbolic orbit* $\mathcal{O}_{hyp}^{G_{\mathbb{C}}}$ *in* $\mathfrak{g}_{\mathbb{C}}$ *is antipodal if and only if the weighted Dynkin diagram of* $\mathcal{O}_{hyp}^{G_{\mathbb{C}}}$ *(see Section* 1.6.1 *for the notation) is held invariant by* ι*. In particular, we have a one-to-one correspondence*

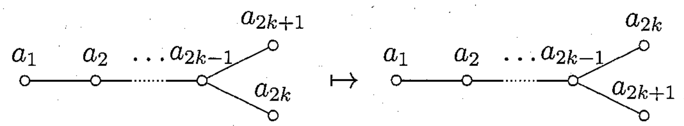
$$\mathcal{H}^a/G_{\mathbb{C}} \overset{1:1}{\longleftrightarrow} \{\, \Psi_A \in \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}) \mid \Psi_A \textit{ is held invariant by } \iota \,\}.$$

(*ii*) *Suppose* $\mathfrak{g}_{\mathbb{C}}$ *is simple. Then the endomorphism* ι *is non-trivial if and only if* $\mathfrak{g}_{\mathbb{C}}$ *is of type* A_n, D_{2k+1} *or* E_6 $(n \geq 2,\ k \geq 2)$*. In such cases, the forms of* ι*are :*

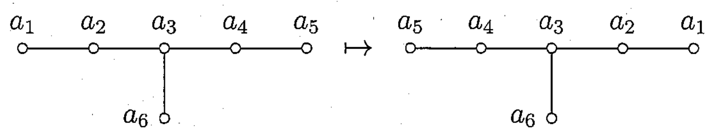
For type A_n $(n \geq 2,\ \mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{sl}(n+1, \mathbb{C}))$



For type D_{2k+1} $(k \geq 2,\ \mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{so}(4k+2, \mathbb{C}))$



For type E_6 $(\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{e}_{6,\mathbb{C}})$



It should be noted that for the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type D_{2k} $(k \geq 2)$, the involution ι on weighted Dynkin diagrams is trivial although the Dynkin diagram of type D_{2k} admits some involutive automorphisms.

Proof of Theorem 1.6.3. The first claim of the theorem follows from Lemma 1.6.2. One can easily show that the involutive endomorphism ι on $\mathrm{Map}(\Pi, \mathbb{R})$ is induced by the opposition involution on the Dynkin diagram with nodes Π, which is defined by

$$\Pi \to \Pi, \quad \alpha \mapsto -(w_0^{\mathbb{C}})^* \cdot \alpha.$$

Suppose that $\mathfrak{g}_\mathbb{C}$ is simple. Then the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ is irreducible. It is known that the opposition involution is non-trivial if and only if $\mathfrak{g}_\mathbb{C}$ is of type A_n, D_{2k+1} or E_6 ($n \geq 2$, $k \geq 2$) (see J. Tits [42, Section 1.5.1]), and the proof is complete. $\square$

As a corollary to Theorem 1.6.3, we have the following:

Corollary 1.6.4. *If the complex semisimple Lie algebra $\mathfrak{g}_\mathbb{C}$ has no simple factor of type A_n, D_{2k+1} or E_6 ($n \geq 2$, $k \geq 2$), then any complex hyperbolic orbit in $\mathfrak{g}_\mathbb{C}$ is antipodal. Namely, $\mathcal{H}/G_\mathbb{C} = \mathcal{H}^a/G_\mathbb{C}$.*

By Corollary 1.6.4, in Setting 1.2.1, if $\mathfrak{g}_\mathbb{C}$ has no simple factor of type A_n, D_{2k+1} or E_6 ($n \geq 2$, $k \geq 2$), then the condition (viii) in Theorem 1.2.2 and the condition (vii) in Fact 1.2.6 are equivalent.

1.6.3 Weighted Dynkin diagrams of complex nilpotent orbits

We consider the setting in Section 1.6.1, and use the notation $\mathcal{H}^n$ and $\mathcal{H}^n/G_\mathbb{C}$ as in Section 1.4.3. In this subsection, we prove Lemma 1.4.4, and recall weighted Dynkin diagrams of complex nilpotent orbits in $\mathfrak{g}_\mathbb{C}$.

First, we prove Lemma 1.4.4, which claims that $\mathcal{H}^n \subset \mathcal{H}^a$, as follows:

Proof of Lemma 1.4.4. For any $\mathfrak{sl}_2$-triple (A, X, Y) in $\mathfrak{g}_\mathbb{C}$, it is well known that $\mathrm{ad}_{\mathfrak{g}_\mathbb{C}}(A) \in \mathrm{End}(\mathfrak{g}_\mathbb{C})$ is diagonalizable with only real integral numbers. Hence, A is hyperbolic in $\mathfrak{g}_\mathbb{C}$. We shall prove that the orbit $\mathcal{O}_A^{G_\mathbb{C}} := \mathrm{Ad}(G_\mathbb{C})\cdot A$ is antipodal. We can easily check that the elements

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ in } \mathfrak{sl}(2, \mathbb{C})$$

are conjugate under the adjoint action of $SL(2, \mathbb{C})$. Then, for a Lie algebra homomorphism $\phi_\mathbb{C} : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}_\mathbb{C}$ with

$$\phi_\mathbb{C} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A,$$

the elements A and $-A$ are conjugate under the adjoint action of the analytic subgroup of $G_\mathbb{C}$ corresponding to $\phi_\mathbb{C}(\mathfrak{sl}(2, \mathbb{C}))$. Hence, the orbit $\mathcal{O}_A^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$ is antipodal. $\square$

Let $\mathcal{N}$ be the set of nilpotent elements in $\mathfrak{g}_{\mathbb{C}}$ and $\mathcal{N}/G_{\mathbb{C}}$ the set of nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$. For any $\mathfrak{sl}_2$-triple (A, X, Y) in $\mathfrak{g}_{\mathbb{C}}$, the element A is in $\mathcal{H}^n (\subset \mathcal{H}^a)$ and the elements X, Y are both in $\mathcal{N}$. Let us consider the map from the conjugacy classes of $\mathfrak{sl}_2$-triples in $\mathfrak{g}_{\mathbb{C}}$ by inner automorphisms of $\mathfrak{g}_{\mathbb{C}}$ to $\mathcal{N}/G_{\mathbb{C}}$ defined by

$$[(A, X, Y)] \mapsto \mathcal{O}_X^{G_{\mathbb{C}}}$$

where $[(A, X, Y)]$ is the conjugacy class of an $\mathfrak{sl}_2$-triple (A, X, Y) in $\mathfrak{g}_{\mathbb{C}}$ and $\mathcal{O}_X^{G_{\mathbb{C}}}$ the complex adjoint orbit through X in $\mathfrak{g}_{\mathbb{C}}$. Then, by the Jacobson–Morozov theorem, with a result in B. Kostant [25], the map is bijective. On the other hand, by A. I. Malcev [29], the map from the conjugacy classes of $\mathfrak{sl}_2$-triples in $\mathfrak{g}_{\mathbb{C}}$ by inner automorphisms of $\mathfrak{g}_{\mathbb{C}}$ to $\mathcal{H}^n/G_{\mathbb{C}}$ defined by

$$[(A, X, Y)] \mapsto \mathcal{O}_A^{G_{\mathbb{C}}}$$

is also bijective, where $\mathcal{O}_A^{G_{\mathbb{C}}}$ is the complex adjoint orbit through A in $\mathfrak{g}_{\mathbb{C}}$. Therefore, we have a one-to-one correspondence

$$\mathcal{N}/G_{\mathbb{C}} \overset{1:1}{\longleftrightarrow} \mathcal{H}^n/G_{\mathbb{C}}.$$

In particular, by combining the argument above with Fact 1.6.1, we also obtain a bijection:

$$\mathcal{N}/G_{\mathbb{C}} \to \mathfrak{j}_+ \cap \mathcal{H}^n, \quad \mathcal{O}_{\mathrm{nilp}}^{G_{\mathbb{C}}} \mapsto A_{\mathcal{O}},$$

where $A_{\mathcal{O}}$ is the unique element of $\mathfrak{j}_+$ with the property: there exist $X, Y \in \mathcal{O}_{\mathrm{nilp}}^{G_{\mathbb{C}}}$ such that $(A_{\mathcal{O}}, X, Y)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{C}}$.

Remark 1.6.5. *It is known that the Jacobson–Morozov theorem and the result of Kostant in* [25] *also hold for any real semisimple Lie algebra* $\mathfrak{g}$. *Therefore, we have a surjective map from the set of real nilpotent orbits in* $\mathfrak{g}$ *to* $\mathcal{H}^n(\mathfrak{g})/G$, *where* $\mathcal{H}^n(\mathfrak{g})/G$ *is the notation in Section* 1.4.3. *However, in general, the map is not injective.*

The weighted Dynkin diagram of a complex nilpotent orbit $\mathcal{O}_{\mathrm{nilp}}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$ is defined as the weighted Dynkin diagram corresponding to $A_{\mathcal{O}} \in \mathfrak{j}_+ \cap \mathcal{H}^n$. Obviously, the weighted Dynkin diagram of $\mathcal{O}_{\mathrm{nilp}}^{G_{\mathbb{C}}}$ is the same as the weighted Dynkin diagram of the corresponding orbit in $\mathcal{H}^n/G_{\mathbb{C}}$.

E. B. Dynkin [10] proved that any weight of a weighted Dynkin diagram of any complex adjoint orbit in $\mathcal{H}^n/G_{\mathbb{C}}$ is 0, 1 or 2. Hence, $\mathcal{H}^n/G_{\mathbb{C}}$

is (and therefore $\mathcal{N}/G_{\mathbb{C}}$ is) finite. Dynkin [10] gave a list of the weighted Dynkin diagrams of $\mathcal{H}^n/G_{\mathbb{C}}$ as the classification of $\mathfrak{sl}_2$-triples in $\mathfrak{g}_{\mathbb{C}}$. This also gives a classification of complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ (see Bala–Cater [4] or Collingwood–McGovern [8, Section 3] for more details).

We remark that by combining Theorem 1.6.3 with Lemma 1.4.4, if $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}(n+1,\mathbb{C})$, $\mathfrak{so}(4k+2,\mathbb{C})$ or $\mathfrak{e}_{6,\mathbb{C}}$ ($n \geq 2$, $k \geq 2$), then the weighted Dynkin diagram of any complex adjoint orbit in $\mathcal{H}^n/G_{\mathbb{C}}$ (and therefore the weighted Dynkin diagram of any complex nilpotent orbit) is invariant under the non-trivial involution ι.

Example 1.6.6. *It is known that there exists a bijection between complex nilpotent orbits in $\mathfrak{sl}(n,\mathbb{C})$ and partitions of n (see [8, Section3.1and3.6]). Here is the list of weighted Dynkin diagrams of complex nilpotent orbits in $\mathfrak{sl}(6,\mathbb{C})$ (i.e. the list of weighted Dynkin diagrams corresponding to $\mathfrak{j}_+ \cap \mathcal{H}^n$ for the case where $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(6,\mathbb{C})$):*

Partition	*Weighted Dynkin diagram*
$[6]$	2 — 2 — 2 — 2 — 2
$[5,1]$	2 — 2 — 0 — 2 — 2
$[4,2]$	2 — 0 — 2 — 0 — 2
$[4,1^2]$	2 — 1 — 0 — 1 — 2
$[3^2]$	0 — 2 — 0 — 2 — 0
$[3,2,1]$	1 — 1 — 0 — 1 — 1
$[3,1^3]$	2 — 0 — 0 — 0 — 2
$[2^3]$	0 — 0 — 2 — 0 — 0
$[2^2,1^2]$	0 — 1 — 0 — 1 — 0
$[2,1^4]$	1 — 0 — 0 — 0 — 1
$[1^6]$	0 — 0 — 0 — 0 — 0

Table 1.3: Classification of complex nilpotent orbits in $\mathfrak{sl}(6,\mathbb{C})$

1.7 Complex adjoint orbits and real forms

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra, and $\mathfrak{g}$ a real form of $\mathfrak{g}_{\mathbb{C}}$. Recall that, in Section 1.6, we have a parameterization of complex hyperbolic [resp. antipodal hyperbolic, nilpotent] orbits in $\mathfrak{g}_{\mathbb{C}}$ by weighted Dynkin diagrams. In this section, we also determine complex hyperbolic [resp. antipodal hyperbolic, nilpotent] orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$. For this, we give an algorithm to check whether or not a given complex hyperbolic [resp. nilpotent] orbit in $\mathfrak{g}_{\mathbb{C}}$ meets $\mathfrak{g}$. We also prove Proposition 1.4.5 and Proposition 1.4.8 (i) in this section.

1.7.1 Complex hyperbolic orbits and real forms

We give a proof of Proposition 1.4.5 (i) in this subsection.

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$, and use the following convention:

Definition 1.7.1. *We say that a Cartan subalgebra* $\mathfrak{j}_{\mathfrak{g}}$ *of* $\mathfrak{g}$ *is split if* $\mathfrak{a} := \mathfrak{j}_{\mathfrak{g}} \cap \mathfrak{p}$ *is a maximal abelian subspace of* $\mathfrak{p}$ *(i.e.* $\mathfrak{a}$ *is a maximally split abelian subspace of* $\mathfrak{g}$*).*

Note that such $\mathfrak{j}_{\mathfrak{g}}$ is unique up to the adjoint action of K, where K is the analytic subgroup of G corresponding to $\mathfrak{k}$.

Take a split Cartan subalgebra $\mathfrak{j}_{\mathfrak{g}}$ of $\mathfrak{g}$ in Definition 1.7.1. Then $\mathfrak{j}_{\mathfrak{g}}$ can be written as $\mathfrak{j}_{\mathfrak{g}} = \mathfrak{t} + \mathfrak{a}$ for a maximal abelian subspace $\mathfrak{t}$ of the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Let us denote by $\mathfrak{j}_{\mathbb{C}} := \mathfrak{j}_{\mathfrak{g}} + \sqrt{-1}\mathfrak{j}_{\mathfrak{g}}$ and $\mathfrak{j} := \sqrt{-1}\mathfrak{t} + \mathfrak{a}$. Then $\mathfrak{j}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{j}$ is a real form of it, with

$$\mathfrak{j} = \{A \in \mathfrak{j}_{\mathbb{C}} \mid \alpha(A) \in \mathbb{R} \text{ for any } \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})\},$$

where $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ is the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. We put

$$\Sigma(\mathfrak{g}, \mathfrak{a}) := \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})\} \setminus \{0\} \subset \mathfrak{a}^*$$

to the restricted root system of $(\mathfrak{g}, \mathfrak{a})$. Then we can take a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ such that the subset

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) := \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})\} \setminus \{0\}.$$

of $\Sigma(\mathfrak{g}, \mathfrak{a})$ becomes a positive system. In fact, if we take an ordering on $\mathfrak{a}$ and extend it to $\mathfrak{j}$, then the corresponding positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ satisfies the condition above. Let us denote by $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, $W(\mathfrak{g}, \mathfrak{a})$ the Weyl groups of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, $\Sigma(\mathfrak{g}, \mathfrak{a})$, respectively. We put the closed Weyl chambers

$$\mathfrak{j}_+ := \{ A \in \mathfrak{j} \mid \alpha(A) \geq 0 \text{ for any } \alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \},$$
$$\mathfrak{a}_+ := \{ A \in \mathfrak{a} \mid \xi(A) \geq 0 \text{ for any } \xi \in \Sigma^+(\mathfrak{g}, \mathfrak{a}) \}.$$

Then $\mathfrak{j}_+$ and $\mathfrak{a}_+$ are fundamental domains of $\mathfrak{j}$, $\mathfrak{a}$ for the actions of $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ and $W(\mathfrak{g}, \mathfrak{a})$, respectively. By the definition of $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ and $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, we have $\mathfrak{a}_+ = \mathfrak{j}_+ \cap \mathfrak{a}$.

We recall that any complex hyperbolic orbit $\mathcal{O}^{G_\mathbb{C}}_{\text{hyp}}$ in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{j}_+$ with a unique element $A_\mathcal{O}$ in $\mathcal{O}^{G_\mathbb{C}}_{\text{hyp}} \cap \mathfrak{j}_+$ (see Fact 1.6.1). Then the lemma below holds:

Lemma 1.7.2. *A complex hyperbolic orbit $\mathcal{O}^{G_\mathbb{C}}_{hyp}$ in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{g}$ if and only if the corresponding element $A_\mathcal{O}$ is in $\mathfrak{a}_+$. In particular, we have a one-to-one correspondence*

$$\mathcal{H}_\mathfrak{g}/G_\mathbb{C} \overset{1:1}{\longleftrightarrow} \mathfrak{a}_+,$$

where $\mathcal{H}_\mathfrak{g}/G_\mathbb{C}$ is the set of complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ meeting $\mathfrak{g}$.

Lemma 1.7.2 will be used in Section 1.7.2 to prove Theorem 1.7.4. We now prove Proposition 1.4.5 (i) and Lemma 1.7.2 simultaneously.

Proof of Proposition 1.4.5 (i) *and Lemma* 1.7.2. We show that for a complex hyperbolic orbit $\mathcal{O}^{G_\mathbb{C}}_{\text{hyp}}$ in $\mathfrak{g}_\mathbb{C}$, the element $A_\mathcal{O}$ is in $\mathfrak{a}_+$ if $\mathcal{O}^{G_\mathbb{C}}_{\text{hyp}}$ meets $\mathfrak{g}$. Note that an element of $\mathfrak{g}$ is hyperbolic in $\mathfrak{g}$ (see Definition 1.2.3) if and only if hyperbolic in $\mathfrak{g}_\mathbb{C}$. Thus any real adjoint orbit $\mathcal{O}'$ contained in $\mathcal{O}^{G_\mathbb{C}}_{\text{hyp}} \cap \mathfrak{g}$ is hyperbolic, and hence $\mathcal{O}'$ meets $\mathfrak{a}_+$ with a unique element $A_0 \in \mathcal{O}' \cap \mathfrak{a}_+$ by Fact 1.5.1. Since $\mathfrak{a}_+$ is contained in $\mathfrak{j}_+$, the element A_0 is in $\mathcal{O}^{G_\mathbb{C}}_{\text{hyp}} \cap \mathfrak{j}_+$. Thus, $A_0 = A_\mathcal{O}$. Therefore, we obtain that $A_\mathcal{O}$ is in $\mathfrak{a}_+$ for any $\mathcal{O}^{G_\mathbb{C}}_{\text{hyp}} \in \mathcal{H}_\mathfrak{g}/G_\mathbb{C}$, which completes the proof of Lemma 1.7.2.

To prove Proposition 1.4.5 (i), it suffices to show that the intersection $\mathcal{O}^{G_\mathbb{C}}_{\text{hyp}} \cap \mathfrak{g}$ becomes a single adjoint orbit. By the argument above, we have

$$\operatorname{Ad}(G) \cdot A_\mathcal{O} = \mathcal{O}^{G_\mathbb{C}}_{\text{hyp}} \cap \mathfrak{g},$$

and hence Proposition 1.4.5 (i) follows. □

1.7.2 Weighted Dynkin diagrams and Satake diagrams

Let us consider the setting in Section 1.7.1. In this subsection, we determine complex hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$ by using the Satake diagram of $\mathfrak{g}$.

First, we recall briefly the definition of the Satake diagram of the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ (see [2, 36] for more details). Let us denote by Π the fundamental system of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. Then

$$\overline{\Pi} := \{ \alpha|_{\mathfrak{a}} \mid \alpha \in \Pi \} \setminus \{0\}$$

is the fundamental system of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. We write Π_0 for the set of all simple roots in Π whose restriction to $\mathfrak{a}$ is zero. The Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ consists of the following data: the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ with nodes Π; black nodes Π_0 in S; and arrows joining $\alpha \in \Pi \setminus \Pi_0$ and $\beta \in \Pi \setminus \Pi_0$ in S whose restrictions to $\mathfrak{a}$ are the same.

Second, we give the definition of weighted Dynkin diagrams *matching* the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ as follows:

Definition 1.7.3. *Let $\Psi_A \in \operatorname{Map}(\Pi, \mathbb{R})$ be a weighted Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ (see Section 1.6.1 for the notation) and $S_{\mathfrak{g}}$ the Satake diagram of $\mathfrak{g}$ with nodes Π. We say that Ψ_A matches $S_{\mathfrak{g}}$ if all the weights on black nodes in Π_0 are zero and any pair of nodes joined by an arrow have the same weights.*

Then the following theorem holds:

Theorem 1.7.4. *The weighted Dynkin diagram of a complex hyperbolic orbit $\mathcal{O}^{G_{\mathbb{C}}}_{hyp}$ in $\mathfrak{g}_{\mathbb{C}}$ matches the Satake diagram of $\mathfrak{g}$ if and only if $\mathcal{O}^{G_{\mathbb{C}}}_{hyp}$ meets $\mathfrak{g}$. In particular, we have a one-to-one correspondence*

$$\mathcal{H}_{\mathfrak{g}}/G_{\mathbb{C}} \overset{1:1}{\longleftrightarrow} \{ \Psi_A \in \operatorname{Map}(\Pi, \mathbb{R}_{\geq 0}) \mid \Psi_A \textit{ matches } S_{\mathfrak{g}} \}.$$

Recall that Ψ is a linear isomorphism from $\mathfrak{j}$ to $\operatorname{Map}(\Pi, \mathbb{R})$ (see (1.6.1) in Section 1.6.1 for the notation), and there exists a one-to-one correspondence between $\mathcal{H}_{\mathfrak{g}}/G_{\mathbb{C}}$ and $\mathfrak{a}_+$ (see Lemma 1.7.2). Therefore, to prove Theorem 1.7.4, it suffices to show the next lemma:

Lemma 1.7.5. *The linear isomorphism $\Psi : \mathfrak{j} \to \operatorname{Map}(\Pi, \mathbb{R})$ induces a linear isomorphism*

$$\mathfrak{a} \to \{ \Psi_A \in \operatorname{Map}(\Pi, \mathbb{R}) \mid \Psi_A \textit{ matches } S_{\mathfrak{g}} \}, \quad A \mapsto \Psi_A.$$

Proof of Lemma 1.7.5. Let $A \in \mathfrak{j}$. By Definition 1.7.3, the weighted Dynkin diagram Ψ_A matches the Satake diagram of $\mathfrak{g}$ if and only if A satisfies the following condition $(\star)$:

$$(\star) \begin{cases} \alpha(A) = 0 & (\text{for any } \alpha \in \Pi_0), \\ \alpha(A) = \beta(A) & (\text{for any } \alpha, \beta \in \Pi \setminus \Pi_0 \text{ with } \alpha|_{\mathfrak{a}} = \beta|_{\mathfrak{a}}). \end{cases}$$

Thus, it suffices to show that the subspace

$$\mathfrak{a}' := \{ A \in \mathfrak{j} \mid A \text{ satisfies the condition } (\star) \}$$

of $\mathfrak{j}$ coincides with $\mathfrak{a}$. It is easy to check that $\mathfrak{a} \subset \mathfrak{a}'$. We now prove that $\dim_{\mathbb{R}} \mathfrak{a} = \dim_{\mathbb{R}} \mathfrak{a}'$. Recall that $\overline{\Pi}$ is a fundamental system of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. In particular, $\overline{\Pi}$ is a basis of $\mathfrak{a}^*$. Thus, $\dim_{\mathbb{R}} \mathfrak{a} = \sharp\overline{\Pi}$. We define the element A'_ξ of $\mathfrak{a}'$ for each $\xi \in \overline{\Pi}$ by

$$\alpha(A'_\xi) = \begin{cases} 1 & (\text{if } \alpha|_{\mathfrak{a}} = \xi), \\ 0 & (\text{if } \alpha|_{\mathfrak{a}} \neq \xi), \end{cases}$$

for any $\alpha \in \Pi$. Then $\{ A'_\xi \mid \xi \in \overline{\Pi} \}$ is a basis of $\mathfrak{a}'$ since

$$\overline{\Pi} = \{ \alpha|_{\mathfrak{a}} \mid \alpha \in \Pi \} \setminus \{0\}.$$

Thus, $\dim_{\mathbb{R}} \mathfrak{a}' = \sharp\overline{\Pi}$, and hence $\mathfrak{a} = \mathfrak{a}'$. □

1.7.3 Complex antipodal hyperbolic orbits and real forms

We consider the setting in Section 1.7.1 and 1.7.2. In this subsection, the proof of Proposition 1.4.5 (ii) is given. Concerning to the proof of Proposition 1.4.6 (i), which will be given in Section 1.7.5, we also determine the subset $\mathfrak{b}$ of $\mathfrak{a}$ (see Section 1.3.2 for the notation) by describing the weighted Dynkin diagrams in this subsection.

First, we prove Proposition 1.4.5 (ii), which gives a bijection between complex antipodal hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$ and real antipodal hyperbolic orbits in $\mathfrak{g}$, as follows:

Proof of Proposition 1.4.5 (ii). Note that Proposition 1.4.5 (i) has been already proved in Section 1.7.1. Therefore, to prove Proposition 1.4.5 (ii), it remains to show that for any $\mathcal{O}^{G_\mathbb{C}} \in \mathcal{H}^a_\mathfrak{g}/G_\mathbb{C}$ and any element A of $\mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{g}$, the element $-A$ is also in $\mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{g}$. Since $\mathcal{O}^{G_\mathbb{C}}$ is antipodal, the element $-A$ is also in $\mathcal{O}^{G_\mathbb{C}}$. Hence, we have $-A \in \mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{g}$. □

Recall that we have bijections between $\mathcal{H}^a/G_\mathbb{C}$ and $\mathfrak{j}_+^{-w_0^\mathbb{C}}$ (see Lemma 1.6.2) and between $\mathcal{H}^a(\mathfrak{g})/G$ and $\mathfrak{b}_+$ (see Lemma 1.5.3). By Proposition 1.4.5 (ii), which has been proved above, we have one-to-one correspondences

$$\mathfrak{b}_+ \overset{1:1}{\longleftrightarrow} \mathcal{H}^a(\mathfrak{g})/G \overset{1:1}{\longleftrightarrow} \mathcal{H}^a_\mathfrak{g}/G_\mathbb{C},$$

where $\mathcal{H}^a_\mathfrak{g}/G_\mathbb{C}$ is the set of complex antipodal hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ meeting $\mathfrak{g}$.

To explain the relation between $\mathfrak{j}_+^{-w_0^\mathbb{C}}$ and $\mathfrak{b}_+$, we show the following lemma:

Lemma 1.7.6. *Let $w_0^\mathbb{C}$, w_0 be the longest elements of $W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, $W(\mathfrak{g}, \mathfrak{a})$ with respect to the positive systems $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, respectively. Then:*

$$\mathfrak{b} = \mathfrak{j}^{-w_0^\mathbb{C}} \cap \mathfrak{a}, \quad \mathfrak{b}_+ = \mathfrak{j}_+^{-w_0^\mathbb{C}} \cap \mathfrak{a},$$

where $\mathfrak{b} = \{ A \in \mathfrak{a} \mid -w_0 \cdot A = A \}$ *and* $\mathfrak{j}^{-w_0^\mathbb{C}} = \{ A \in \mathfrak{j} \mid -w_0^\mathbb{C} \cdot A = A \}$.

Proof of Lemma 1.7.6. We only need to show that $w_0^\mathbb{C}$ preserves $\mathfrak{a}$ and the action on $\mathfrak{a}$ is same as w_0. Let us put τ to the complex conjugation on $\mathfrak{g}_\mathbb{C}$ with respect to the real form $\mathfrak{g}$. Then we can easily check that both Π and $-\Pi$ are τ-fundamental systems of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ in the sense of [36, Section 1.1]. Since $(w_0^\mathbb{C})^* \cdot \Pi = -\Pi$, the endomorphism $w_0^\mathbb{C}$ is commutative with τ on $\mathfrak{j}$, and $w_0^\mathbb{C}$ induces on $\mathfrak{a}$ an element w_0' of $W(\mathfrak{g}, \mathfrak{a})$ by [36, Proposition A]. Recall that $\overline{\Pi} = \{ \alpha|_\mathfrak{a} \mid \alpha \in \Pi \}$. Then we have $(w_0')^* \cdot \overline{\Pi} = -\overline{\Pi}$, and hence $w_0' = w_0$. □

Recall that we have a bijection between $\mathfrak{a}$ and the set of weighted Dynkin diagrams matching the Satake diagram of $\mathfrak{g}$ (see Lemma 1.7.5). Combining with Lemma 1.7.6, we have a linear isomorphism

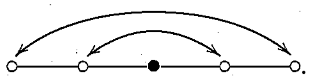
$$\begin{aligned} &\mathfrak{b} \to \{\Psi_A \in \mathrm{Map}(\Pi, \mathbb{R}) \mid \Psi_A \text{ is held invariant by } \iota \text{ and matches } S_\mathfrak{g}\}, \\ &A \mapsto \Psi_A, \end{aligned}$$

where ι is the involutive endomorphism on $\mathrm{Map}(\Pi, \mathbb{R})$ defined in Section 1.6.2. Therefore, we can determine the subsets $\mathfrak{b}$ and $\mathfrak{b}_+$ of $\mathfrak{a}$. Here is an example of the isomorphism for the case where $\mathfrak{g} = \mathfrak{su}(4, 2)$.

Example 1.7.7. *Let* $\mathfrak{g} = \mathfrak{su}(4,2)$. *Then the complexification of* $\mathfrak{su}(4,2)$ *is* $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(6,\mathbb{C})$, *and the involutive endomorphism* ι *on weighted Dynkin diagrams is described by*

$$\overset{a}{\circ}\!-\!\overset{b}{\circ}\!-\!\overset{c}{\circ}\!-\!\overset{d}{\circ}\!-\!\overset{e}{\circ} \;\mapsto\; \overset{e}{\circ}\!-\!\overset{d}{\circ}\!-\!\overset{c}{\circ}\!-\!\overset{b}{\circ}\!-\!\overset{a}{\circ}\,.$$

The Satake diagram of $\mathfrak{g} = \mathfrak{su}(4,2)$ *is here:*

$S_{\mathfrak{su}(4,2)}$: 

Therefore, we have a linear isomorphism

$$\mathfrak{b} \xrightarrow{\sim} \left\{ \overset{a}{\circ}\!-\!\overset{b}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{b}{\circ}\!-\!\overset{a}{\circ} \;\middle|\; a,b \in \mathbb{R} \right\}.$$

In particular, we have one-to-one correspondences below:

$$\mathcal{H}^a_{\mathfrak{g}}/G_{\mathbb{C}} \overset{1:1}{\longleftrightarrow} \mathfrak{b}_+ \overset{1:1}{\longleftrightarrow} \left\{ \overset{a}{\circ}\!-\!\overset{b}{\circ}\!-\!\overset{0}{\circ}\!-\!\overset{b}{\circ}\!-\!\overset{a}{\circ} \;\middle|\; a,b \in \mathbb{R}_{\geq 0} \right\}.$$

1.7.4 Complex nilpotent orbits and real forms

Let us consider the setting in Section 1.7.1 and 1.7.2. In this subsection, we introduce an algorithm to check whether or not a given complex nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ meets the real form $\mathfrak{g}$. In this subsection, we also prove Proposition 1.4.5 (iii).

First, we show the next proposition:

Proposition 1.7.8 (Corollary to J. Sekiguchi [38, Proposition 1.11]). *Let* (A, X, Y) *be an* $\mathfrak{sl}_2$*-triple in* $\mathfrak{g}_{\mathbb{C}}$. *Then the following conditions on* (A, X, Y) *are equivalent:*

(i) The complex adjoint orbit through X *meets* $\mathfrak{g}$.

(ii) The complex adjoint orbit through A *meets* $\mathfrak{g}$.

(iii) The complex adjoint orbit through X *meets* $\mathfrak{p}_{\mathbb{C}}$, *where* $\mathfrak{p}_{\mathbb{C}}$ *is the complexification of* $\mathfrak{p}$.

(iv) *The complex adjoint orbit through* A *meets* $\mathfrak{p}_{\mathbb{C}}$.

(v) *There exists an* $\mathfrak{sl}_2$*-triple* (A', X', Y') *in* $\mathfrak{g}$ *such that* A' *is in the complex adjoint orbit through* A.

(vi) *The weighted Dynkin diagram of the complex adjoint orbit through* X *matches the Satake diagram of* $\mathfrak{g}$.

Proof of Proposition 1.7.8. The equivalences between (i), (iii) and (iv) were proved by [38, Proposition 1.11]. The equivalence (iv) $\Leftrightarrow$ (ii) is obtained by the fact that $\mathcal{H}_{\mathfrak{g}} = \mathcal{H}_{\mathfrak{a}} = \mathcal{H}_{\mathfrak{p}_{\mathbb{C}}}$ (cf. Lemma 1.7.2 and the proof of [38, Proposition 1.11]). The equivalence (ii) $\Leftrightarrow$ (vi) is obtained by combining Theorem 1.7.4 with the observation that the weighted Dynkin diagrams of the complex adjoint orbit through X is same as the weighted Dynkin diagram of the complex adjoint orbit through A (see Section 1.6.3). The implication (ii) $\Rightarrow$ (v) can be obtained by the lemma below. □

Lemma 1.7.9. *Let* (A, X, Y) *be an* $\mathfrak{sl}_2$*-triple in* $\mathfrak{g}_{\mathbb{C}}$. *Then the following holds:*

(i) *If* A *is in* $\mathfrak{g}$, *then there exists* $g \in G_{\mathbb{C}}$ *such that* $\mathrm{Ad}(g) \cdot A = A$ *and* $\mathrm{Ad}(g) \cdot X$ *is in* $\mathfrak{g}$.

(ii) *If both* A *and* X *are in* $\mathfrak{g}$, *then* Y *is automatically in* $\mathfrak{g}$.

Proof of Lemma 1.7.9. (i): See the proof of [38, Proposition 1.11]. (ii): Easy. □

Here is a proof of Proposition 1.4.5 (iii), which gives a bijection between $\mathcal{H}^n_{\mathfrak{g}}/G_{\mathbb{C}}$ and $\mathcal{H}^n(\mathfrak{g})/G$ (see Section 1.4.3 for the notation):

Proof of Proposition 1.4.5 (iii). We recall that Proposition 1.4.5 (i) has been proved already in Section 1.7.1. Then Proposition 1.4.5 (iii) follows from the implication (ii) $\Rightarrow$ (v) in Proposition 1.7.8. □

Recall that we have the one-to-one correspondence

$$\mathfrak{j}_+ \cap \mathcal{H}^n \overset{1:1}{\longleftrightarrow} \mathcal{N}/G_{\mathbb{C}},$$

where $\mathcal{N}/G_{\mathbb{C}}$ is the set of complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ (see Section 1.6.3). Combining Lemma 1.7.2 with Proposition 1.7.8, we also obtain

$$\mathfrak{a}_+ \cap \mathcal{H}^n(\mathfrak{g}) = (\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a} \overset{1:1}{\longleftrightarrow} \mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}},$$

where $\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}$ is the set of complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$. Therefore, by Lemma 1.7.5, we obtain the theorem below:

Theorem 1.7.10. *Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra, and $\mathfrak{g}$ a real form of $\mathfrak{g}_{\mathbb{C}}$. Then for a complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}_{nilp}$ in $\mathfrak{g}_{\mathbb{C}}$, the following two conditions are equivalent:*

(*i*) $\mathcal{O}^{G_{\mathbb{C}}}_{nilp} \cap \mathfrak{g} \neq \emptyset$ (*i.e.* $\mathcal{O}^{G_{\mathbb{C}}}_{nilp} \in \mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}$).

(*ii*) *The weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}_{nilp}$ matches the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ (see Section 1.7.2 for the notation).*

Remark 1.7.11. (1) *The same concept as Definition 1.7.3 appeared earlier as "weighted Satake diagrams" in D. Z. Djokovic [9] and as the condition described in J. Sekiguchi [37, Proposition 1.16]. We call it "match".*

(2) *J. Sekiguchi [38, Proposition 1.13] showed the implication* (ii) $\Rightarrow$ (i) *in Theorem* 1.7.10. *Our theorem claims that* (i) $\Rightarrow$ (ii) *is also true.*

We give three examples of Theorem 1.7.10:

Example 1.7.12. *Let $\mathfrak{g}$ be a split real form of $\mathfrak{g}_{\mathbb{C}}$. Then all nodes of the Satake diagram $S_{\mathfrak{g}}$ are white with no arrow. Thus, all weighted Dynkin diagrams match the Satake diagram of $\mathfrak{g}$. Therefore, all complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ meet $\mathfrak{g}$.*

Example 1.7.13. *Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}_{\mathbb{C}}$. Then all nodes of the Satake diagram $S_{\mathfrak{u}}$ are black. Thus, no weighted Dynkin diagram matches the Satake diagram of $\mathfrak{u}$ except for the trivial one. Therefore, no complex nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ meets $\mathfrak{u}$ except for the zero-orbit.*

Example 1.7.14. *Let* $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}) = (\mathfrak{sl}(6,\mathbb{C}), \mathfrak{su}(4,2))$. *The Satake diagram of* $\mathfrak{su}(4,2)$ *was given in Example* 1.7.7. *Then, by combining with Example* 1.6.6, *we obtain the list of complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$ (i.e. the list of $(\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}$) as follows:*

$$\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}} \stackrel{1:1}{\longleftrightarrow} \{\, [5,1],\ [4,1^2],\ [3^2],\ [3,2,1],\ [3,1^3],\ [2^2,1^2],\ [2,1^4],\ [1^6] \,\}.$$

1.7.5 Proof of Proposition 1.4.8 (i)

In this subsection, we first explain the strategy of the proof of Proposition 1.4.8 (i), and then illustrate actual computations by an example.

By Lemma 1.5.3, we have

$$\mathfrak{b}_+ \supset \mathfrak{a}_+ \cap \mathcal{H}^n(\mathfrak{g}).$$

Furthermore, in Section 1.7.4, we also obtained

$$\mathfrak{a}_+ \cap \mathcal{H}^n(\mathfrak{g}) = (\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}.$$

Therefore, the proof of Proposition 1.4.8 (i) is reduced to the showing

$$\mathfrak{b} \subset \mathbb{R}\text{-span}((\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}) \tag{1.7.1}$$

for all simple Lie algebras $\mathfrak{g}$.

In order to show (1.7.1), we recall that the Dynkin–Kostant classification of weighted Dynkin diagrams corresponding to elements of $\mathfrak{j}_+ \cap \mathcal{H}^n$ (which gives a classification of complex nilpotent orbits in $\mathfrak{g}_\mathbb{C}$; see Section 1.6.3) As its subset, we can classify the weighted Dynkin diagrams corresponding to elements in $(\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}$ by using the Satake diagram of $\mathfrak{g}$ (cf. Example 1.7.14). What we need to prove for (1.7.1) is that this subset contains sufficiently many in the sense that the $\mathbb{R}$-span of the weighted Dynkin diagrams corresponding to this subset is coincide with the space of weighted Dynkin diagrams corresponding to elements in $\mathfrak{b}$. Recall that we can also determine such space corresponding to $\mathfrak{b}$ by the involution ι on weighted Dynkin diagrams (see Section 1.6.2 for the notation) with the Satake diagram of $\mathfrak{g}$ (cf. Example 1.7.7).

We illustrate this strategy by the following example:

Example 1.7.15. *We give a proof of Proposition* 1.4.8 (i) *for the case where* $\mathfrak{g} = \mathfrak{su}(4, 2)$, *with its complexification* $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(6, \mathbb{C})$.

By Example 1.7.14, *we have the list of weighted Dynkin diagrams corresponding to elements of* $(\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}$ *for* $\mathfrak{g} = \mathfrak{su}(4, 2)$. *Here is a part of it:*

Partition	*Weighted Dynkin diagram*
$[2^2, 1^2]$	0 —— 1 —— 0 —— 1 —— 0
$[2, 1^4]$	1 —— 0 —— 0 —— 0 —— 1

Table 1.4: A part of $(\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}$ *for* $\mathfrak{g} = \mathfrak{su}(4, 2)$

By Example 1.7.7, *we also have a linear isomorphism*

$$\mathfrak{b} \xrightarrow{\sim} \left\{ \overset{a}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{b}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{b}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{a}{\circ} \mid a, b \in \mathbb{R} \right\}.$$

Hence, we can observe that

$$\mathfrak{b} \subset \mathbb{R}\text{-span}((\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}).$$

This completes the proof of Proposition 1.4.8 (i) *for the case where* $\mathfrak{g} = \mathfrak{su}(4,2)$.

For the other simple Lie algebras $\mathfrak{g}$, we can find the Satake diagram of $\mathfrak{g}$ in [2] or [12, Chapter X, Section 6] and the classification of weighted Dynkin diagrams of complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ in [4]. Then we can verify (1.7.1) in the spirit of case-by-case computations for other real simple Lie algebras. Detailed computations will be given in Chapter 2.

1.8 Symmetric pairs

In this section, we prove Proposition 1.4.6 and Lemma 1.4.9.

Let $(\mathfrak{g}, \mathfrak{h})$ be a semisimple symmetric pair and write σ for the involution on $\mathfrak{g}$ corresponding to $\mathfrak{h}$. First, we give Cartan decompositions on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{g}^c$ (see (1.2.1) in Section 1.2 for the notation), simultaneously.

Recall that we can find a Cartan involution θ on $\mathfrak{g}$ with $\sigma\theta = \theta\sigma$ (cf. [6]). Let us denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{h} = \mathfrak{k}(\mathfrak{h}) + \mathfrak{p}(\mathfrak{h})$ the Cartan decompositions of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. We set $\mathfrak{u} := \mathfrak{k} + \sqrt{-1}\mathfrak{p}$. Then $\mathfrak{u}$ becomes a compact real form of $\mathfrak{g}_{\mathbb{C}}$. We write τ, τ^c for the complex conjugations on $\mathfrak{g}_{\mathbb{C}}$ with respect to the real forms $\mathfrak{g}$, $\mathfrak{g}^c$, respectively. Then τ^c is the anti $\mathbb{C}$-linear extension of σ on $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{C}}$, and hence τ and τ^c are commutative. The compact real form $\mathfrak{u}$ of $\mathfrak{g}_{\mathbb{C}}$ is stable under both τ and τ^c. We denote by $\overline{\theta}$ the complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ corresponding to $\mathfrak{u}$, i.e. $\overline{\theta}$ is anti $\mathbb{C}$-linear extension of θ. Then the restriction $\overline{\theta}|_{\mathfrak{g}^c}$ is a Cartan involution on $\mathfrak{g}^c$. We write

$$\mathfrak{g}^c = \mathfrak{k}^c + \mathfrak{p}^c \tag{1.8.1}$$

for the Cartan decomposition of $\mathfrak{g}^c$ with respect to $\overline{\theta}|_{\mathfrak{g}^c}$.

Let us fix a maximal abelian subspace $\mathfrak{a}_{\mathfrak{h}}$ of $\mathfrak{p}(\mathfrak{h})$, and extend it to a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ [resp. a maximal abelian subspace $\mathfrak{a}^c$ of $\mathfrak{p}^c$]. Obviously, $\mathfrak{a}_{\mathfrak{h}} = \mathfrak{a} \cap \mathfrak{a}^c$. We show the next lemma below:

Lemma 1.8.1. $[\mathfrak{a}, \mathfrak{a}^c] = \{0\}$.

The next proposition gives a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ which contains split Cartan subalgebras of $\mathfrak{g}$, $\mathfrak{g}^c$ and $\mathfrak{h}$ with respect to the Cartan decompositions.

Proposition 1.8.2. *There exists a Cartan subalgebra* $\mathfrak{j}_{\mathbb{C}}$ *of* $\mathfrak{g}_{\mathbb{C}}$ *with the following properties*:

- $\mathfrak{j}_{\mathfrak{g}} := \mathfrak{j}_{\mathbb{C}} \cap \mathfrak{g}$ *is a split Cartan subalgebra of* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ *(see Definition* 1.7.1 *for the notation) with* $\mathfrak{j}_{\mathfrak{g}} \cap \mathfrak{p} = \mathfrak{a}$.
- $\mathfrak{j}_{\mathfrak{g}^c} := \mathfrak{j}_{\mathbb{C}} \cap \mathfrak{g}^c$ *is a split Cartan subalgebra of* $\mathfrak{g}^c = \mathfrak{k}^c + \mathfrak{p}^c$ *with* $\mathfrak{j}_{\mathfrak{g}^c} \cap \mathfrak{p}^c = \mathfrak{a}^c$.
- $\mathfrak{j}_{\mathfrak{h}} := \mathfrak{j}_{\mathbb{C}} \cap \mathfrak{h}$ *is a split Cartan subalgebra of* $\mathfrak{h} = \mathfrak{k}(\mathfrak{h}) + \mathfrak{p}(\mathfrak{h})$ *with* $\mathfrak{j}_{\mathfrak{h}} \cap \mathfrak{p}(\mathfrak{h}) = \mathfrak{a}_{\mathfrak{h}}$.

Proof of Lemma 1.8.1 *and Proposition* 1.8.2. We put

$$\mathfrak{h}^a := \{ X \in \mathfrak{g} \mid \theta\sigma X = X \}, \quad \mathfrak{q}^a := \{ X \in \mathfrak{g} \mid \theta\sigma X = -X \}.$$

Then $(\mathfrak{g}, \mathfrak{h}^a)$ is the associated symmetric pair of $(\mathfrak{g}, \mathfrak{h})$ (see [35, Section 1] for the notation). Note that $\mathfrak{q}^a = \mathfrak{p}^c \cap \mathfrak{g} + \sqrt{-1}(\mathfrak{p}^c \cap \sqrt{-1}\mathfrak{g})$ and $\mathfrak{p} \cap \mathfrak{q}^a = \mathfrak{p}(\mathfrak{h})$. Let us apply [35, Lemma 2.4 (i)] to the symmetric pair $(\mathfrak{g}, \mathfrak{h}^a)$. Then we have $[\mathfrak{a}, \mathfrak{a}^c] = \{0\}$, since the complexification of $\mathfrak{a}_c$ is a maximal abelian subspace of the complexification of $\mathfrak{q}^a$ containing $\mathfrak{a}_{\mathfrak{h}}$. This completes the proof of Lemma 1.8.1. Furthermore, let us extend $\mathfrak{a} + \mathfrak{a}^c$ to a Cartan subalgebra $\mathfrak{j}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$. Then $\mathfrak{j}_{\mathbb{C}}$ satisfies the properties in Proposition 1.8.2. □

We fix such a Cartan subalgebra $\mathfrak{j}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, and put

$$\mathfrak{j} := \mathfrak{j}_{\mathbb{C}} \cap \sqrt{-1}\mathfrak{u}.$$

Throughout this subsection, we denote the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ briefly by Δ, which is realized in $\mathfrak{j}^*$. Let us denote by Σ, Σ^c the restricted root systems of $(\mathfrak{g}, \mathfrak{a})$, $(\mathfrak{g}^c, \mathfrak{a}^c)$, respectively. Namely, we put

$$\Sigma := \{ \alpha|_{\mathfrak{a}} \mid \alpha \in \Delta \} \setminus \{0\} \subset \mathfrak{a}^*,$$
$$\Sigma^c := \{ \alpha|_{\mathfrak{a}^c} \mid \alpha \in \Delta \} \setminus \{0\} \subset (\mathfrak{a}^c)^*.$$

Then we can choose a positive system Δ^+ of Δ with the properties below:

- $\Sigma^+ := \{ \alpha|_{\mathfrak{a}} \mid \alpha \in \Delta^+ \} \setminus \{0\}$ is a positive system of Σ.
- $(\Sigma^c)^+ := \{ \alpha|_{\mathfrak{a}^c} \mid \alpha \in \Delta^+ \} \setminus \{0\}$ is a positive system of Σ^c.

In fact, if we take an ordering on $\mathfrak{a}_{\mathfrak{h}}$ and extend it stepwise to $\mathfrak{a}$, to $\mathfrak{a} + \mathfrak{a}^c$ and to $\mathfrak{j}$, then the corresponding positive system Δ^+ satisfies the properties above (see [35, Section 3] for more detail). Let us denote by

$$\begin{aligned} \mathfrak{j}_+ &:= \{ A \in \mathfrak{j} \mid \alpha(A) \geq 0 \text{ for any } \alpha \in \Delta^+ \}, \\ \mathfrak{a}_+ &:= \{ A \in \mathfrak{a} \mid \xi(A) \geq 0 \text{ for any } \xi \in \Sigma^+ \}, \\ \mathfrak{a}^c_+ &:= \{ A \in \mathfrak{a}^c \mid \xi^c(A) \geq 0 \text{ for any } \xi^c \in (\Sigma^c)^+ \}, \end{aligned}$$

the closed Weyl chambers in $\mathfrak{j}$, $\mathfrak{a}$ and $\mathfrak{a}^c$ with respect to Δ^+, Σ^+ and $(\Sigma^c)^+$, respectively.

Combining Fact 1.6.1 with Lemma 1.7.2, we obtain the lemma below:

Lemma 1.8.3. *Let $\mathcal{O}^{G_{\mathbb{C}}}_{hyp}$ be a complex hyperbolic orbit in $\mathfrak{g}_{\mathbb{C}}$. Then the following holds:*

(*i*) *There exists a unique element $A_{\mathcal{O}}$ in $\mathcal{O}^{G_{\mathbb{C}}}_{hyp} \cap \mathfrak{j}_+$.*

(*ii*) *$\mathcal{O}^{G_{\mathbb{C}}}_{hyp}$ meets $\mathfrak{g}$ if and only if $A_{\mathcal{O}}$ is in $\mathfrak{a}_+$.*

(*iii*) *$\mathcal{O}^{G_{\mathbb{C}}}_{hyp}$ meets $\mathfrak{g}^c$ if and only if $A_{\mathcal{O}}$ is in $\mathfrak{a}^c_+$.*

We now prove Proposition 1.4.6 by using Lemma 1.8.3.

Proof of Proposition 1.4.6. Let $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{hyp}}$ be a complex hyperbolic orbit in $\mathfrak{g}_{\mathbb{C}}$ meeting both $\mathfrak{g}$ and $\mathfrak{g}^c$. We shall prove that $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{hyp}}$ also meets $\mathfrak{h} = \mathfrak{g} \cap \mathfrak{g}^c$. By Lemma 1.8.3, there exists a unique element $A_{\mathcal{O}} \in \mathfrak{a}_+ \cap \mathfrak{a}^c_+$ with $A_{\mathcal{O}} \in \mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{hyp}}$, and hence our claim follows. ☐

Lemma 1.4.9 is proved by using Lemma 1.8.3 as follows.

Proof of Lemma 1.4.9. Let us take $A \in \mathfrak{a}_+$ such that $\mathcal{O}^G_A$ meets $\mathfrak{h}$, where $\mathcal{O}^G_A$ is the adjoint orbit in $\mathfrak{g}$ through A. To prove our claim, we only need to show that A is in $\mathfrak{a}_{\mathfrak{h}}$. We denote by $\mathcal{O}^{G_{\mathbb{C}}}_A$ the complexification of $\mathcal{O}^G_A$. Then $\mathcal{O}^{G_{\mathbb{C}}}_A$ is a complex hyperbolic orbit in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{h} = \mathfrak{g} \cap \mathfrak{g}^c$. Let us extend $\mathfrak{a}_{\mathfrak{h}}$ to a maximal abelian subspace $\mathfrak{a}^c$ of $\mathfrak{p}^c$ (see (1.8.1) for the notation of $\mathfrak{p}^c$) and take a Cartan subalgebra $\mathfrak{j}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ in Proposition 1.8.2. We also extend the ordering on $\mathfrak{a}$ stepwise to $\mathfrak{a} + \mathfrak{a}^c$ and to $\mathfrak{j}$. Then by Lemma 1.8.3, the orbit $\mathcal{O}^{G_{\mathbb{C}}}_A$ intersects $\mathfrak{j}_+$ with a unique element $A_{\mathcal{O}}$, and $A_{\mathcal{O}}$ is in $\mathfrak{a}_+ \cap \mathfrak{a}^c_+ \subset \mathfrak{a}_{\mathfrak{h}}$. Since A is also in $\mathcal{O}^{G_{\mathbb{C}}}_A \cap \mathfrak{j}_+$, we have $A = A_{\mathcal{O}}$. Hence A is in $\mathfrak{a}_{\mathfrak{h}}$. ☐

1.9 Algorithm for classification

Let $(\mathfrak{g}, \mathfrak{h})$ be a semisimple symmetric pair (see Setting 1.2.1). In this section, we describe an algorithm to check whether or not $(\mathfrak{g}, \mathfrak{h})$ satisfies the condition (viii) in Theorem 1.2.2, which coincides with the condition (v) in Theorem 1.1.3. More precisely, we give an algorithm to classify complex antipodal hyperbolic orbits $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$ such that $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}} \cap \mathfrak{g} \neq \emptyset$ and $\mathcal{O}_{\mathrm{hyp}}^{G_{\mathbb{C}}} \cap \mathfrak{g}^c = \emptyset$.

Recall that for any complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, we can determine the set of complex antipodal hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$, which is denoted by $\mathcal{H}^a/G_{\mathbb{C}}$, as ι-invariant weighted Dynkin diagrams by Theorem 1.6.3. Further, for any real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$, we can classify complex antipodal hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$ by using the Satake diagram of $\mathfrak{g}$ (see Section 1.7.3).

For a semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$, we can specify another real form $\mathfrak{g}^c$ of $\mathfrak{g}_{\mathbb{C}}$ (see (1.2.1) in Section 1.2 for the notation) by the list of [35, Section 1], since the symmetric pair $(\mathfrak{g}^c, \mathfrak{h})$ is same as $(\mathfrak{g}, \mathfrak{h})^{\mathrm{ada}}$. The Satake diagram of the real form $\mathfrak{g}$ [resp. $\mathfrak{g}^c$] of $\mathfrak{g}_{\mathbb{C}}$ can be found in [2] or [12, Chapter X, Section 6]. Therefore, we can classify the set of complex antipodal hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$ [resp. $\mathfrak{g}^c$], which is denoted by $\mathcal{H}^a_{\mathfrak{g}}/G_{\mathbb{C}}$ [resp. $\mathcal{H}^a_{\mathfrak{g}^c}/G_{\mathbb{C}}$]. This provides an algorithm to check whether the condition (viii) in Theorem 1.2.2 holds or not on $(\mathfrak{g}, \mathfrak{h})$.

Here, we give examples for the cases where $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(4,2), \mathfrak{sp}(2,1))$ or $(\mathfrak{su}^*(6), \mathfrak{sp}(2,1))$.

Example 1.9.1. *Let* $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(4,2), \mathfrak{sp}(2,1))$. *Then* $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(6, \mathbb{C})$ *and* $\mathfrak{g}^c = \mathfrak{su}^*(6)$. *We shall determine both* $\mathcal{H}^a_{\mathfrak{g}}/G_{\mathbb{C}}$ *and* $\mathcal{H}^a_{\mathfrak{g}^c}/G_{\mathbb{C}}$, *and prove that* $(\mathfrak{g}, \mathfrak{h})$ *satisfies the condition* (viii) *in Theorem* 1.2.2.

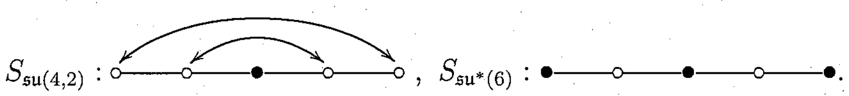
The involutive endomorphism ι *on weighted Dynkin diagrams of* $\mathfrak{sl}(6, \mathbb{C})$ *(see Section* 1.6.2 *for the notation) is given by*

$$\overset{a}{\circ}-\overset{b}{\circ}-\overset{c}{\circ}-\overset{d}{\circ}-\overset{e}{\circ} \mapsto \overset{e}{\circ}-\overset{d}{\circ}-\overset{c}{\circ}-\overset{b}{\circ}-\overset{a}{\circ}\ .$$

Thus, by Theorem 1.6.3, *we have the bijection below:*

$$\mathcal{H}^a/G_{\mathbb{C}} \overset{1:1}{\longleftrightarrow} \left\{ \overset{a}{\circ}-\overset{b}{\circ}-\overset{c}{\circ}-\overset{b}{\circ}-\overset{a}{\circ} \;\middle|\; a, b, c \in \mathbb{R}_{\geq 0} \right\}.$$

Here are the Satake diagrams of $\mathfrak{g} = \mathfrak{su}(4,2)$ *and* $\mathfrak{g}^c = \mathfrak{su}^*(6)$:



Thus, by Theorem 1.7.4, *we obtain the following one-to-one correspondences*:

$$\mathcal{H}^a_{\mathfrak{g}}/G_{\mathbb{C}} \overset{1:1}{\longleftrightarrow} \left\{ \overset{a}{\circ}\!-\!\!-\!\overset{b}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{b}{\circ}\!-\!\!-\!\overset{a}{\circ} \;\middle|\; a,b \in \mathbb{R}_{\geq 0} \right\},$$

$$\mathcal{H}^a_{\mathfrak{g}^c}/G_{\mathbb{C}} \overset{1:1}{\longleftrightarrow} \left\{ \overset{0}{\circ}\!-\!\!-\!\overset{b}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{b}{\circ}\!-\!\!-\!\overset{0}{\circ} \;\middle|\; b \in \mathbb{R}_{\geq 0} \right\}.$$

Therefore, the condition (viii) *in Theorem* 1.2.2 *holds on the symmetric pair* $(\mathfrak{su}(4,2), \mathfrak{sp}(2,1))$.

Example 1.9.2. *Let* $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}^*(6), \mathfrak{sp}(2,1))$. *Then* $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(6,\mathbb{C})$ *and* $\mathfrak{g}^c = \mathfrak{su}(4,2)$. *Thus, by the argument in Example* 1.9.1, *we have*

$$\mathcal{H}^a_{\mathfrak{g}}/G_{\mathbb{C}} \overset{1:1}{\longleftrightarrow} \left\{ \overset{0}{\circ}\!-\!\!-\!\overset{b}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{b}{\circ}\!-\!\!-\!\overset{0}{\circ} \;\middle|\; b \in \mathbb{R}_{\geq 0} \right\},$$

$$\mathcal{H}^a_{\mathfrak{g}^c}/G_{\mathbb{C}} \overset{1:1}{\longleftrightarrow} \left\{ \overset{a}{\circ}\!-\!\!-\!\overset{b}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{b}{\circ}\!-\!\!-\!\overset{a}{\circ} \;\middle|\; a,b \in \mathbb{R}_{\geq 0} \right\}.$$

Therefore, the condition (viii) *in Theorem* 1.2.2 *does not hold on the symmetric pair* $(\mathfrak{su}^*(6), \mathfrak{sp}(2,1))$. *However, if we take a complex hyperbolic orbit* $\mathcal{O}'$ *in* $\mathfrak{sl}(6,\mathbb{C})$ *corresponding to*

$$\overset{0}{\circ}\!-\!\!-\!\overset{b}{\circ}\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\overset{b'}{\circ}\!-\!\!-\!\overset{0}{\circ} \quad (\textit{for some } b, b' \in \mathbb{R}_{\geq 0},\ b \neq b'),$$

then $\mathcal{O}'$ *meets* $\mathfrak{g}$ *but does not meet* $\mathfrak{g}^c$. *Note that* $\mathcal{O}'$ *is not antipodal. Thus the condition* (vii) *in Fact* 1.2.6 *holds on the symmetric pair* $(\mathfrak{su}^*(6), \mathfrak{sp}(2,1))$. *In particular,* $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} > \operatorname{rank}_{\mathbb{R}} \mathfrak{h}$.

Combining our algorithm with Berger's classification [6], we obtain Table 1.2 in Section 1.2. Concerning this, if $\mathfrak{g}_{\mathbb{C}}$ has no simple factor of type A_n,

D_{2k+1} or E_6 ($n \geq 2$, $k \geq 2$), then the symmetric pair $(\mathfrak{g}, \mathfrak{h})$ satisfies the condition (viii) in Theorem 1.2.2 if and only if $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} > \operatorname{rank}_{\mathbb{R}} \mathfrak{h}$ (see Corollary 1.6.4 and Fact 1.2.6). Thus we need only consider the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type A_n, D_{2k+1} or E_6.

We also remark that for a given semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$, by using the Dynkin–Kostant classification [10] and Theorem 1.7.10, we can check whether the condition (vi) in Theorem 1.2.2 holds or not on $(\mathfrak{g}, \mathfrak{h})$, directly (see also Section 1.10).

1.10 Proper actions of $SL(2,\mathbb{R})$ and real nilpotent orbits

In this section, we describe a refinement of the equivalence (i) $\Leftrightarrow$ (vi) in Theorem 1.2.2, which provides an algorithm to classify proper $SL(2,\mathbb{R})$-actions on a given semisimple symmetric space G/H.

Let G be a connected linear semisimple Lie group and write $\mathfrak{g}$ for its Lie algebra. By the Jacobson-Morozov theorem and Lemma 1.5.4, we have a one-to-one correspondence between Lie group homomorphisms $\Phi : SL(2,\mathbb{R}) \to G$ up to inner automorphisms of G and real nilpotent orbits in $\mathfrak{g}$. We denote by $\mathcal{O}^G_\Phi$ the real nilpotent orbit corresponding to $\Phi : SL(2,\mathbb{R}) \to G$. Then, by combining Proposition 1.4.2, Proposition 1.4.6 with Lemma 1.4.7, we obtain the next theorem:

Theorem 1.10.1. *In Setting* 1.2.1*, the following conditions on a Lie group homomorphism* $\Phi : SL(2,\mathbb{R}) \to G$ *are equivalent:*

- *(i)* $SL(2,\mathbb{R})$ *acts on* G/H *properly via* Φ.
- *(ii) The complex nilpotent orbit* $\operatorname{Ad}(G_{\mathbb{C}}) \cdot \mathcal{O}^G_\Phi$ *in* $\mathfrak{g}_{\mathbb{C}}$ *does not meet* $\mathfrak{g}^c$*, where* $\mathfrak{g}^c$ *is the c-dual of the symmetric pair* $(\mathfrak{g}, \mathfrak{h})$ *(see* (1.2.1) *after Setting* 1.2.1*).*

In particular, we have the one-to-one correspondence

$$\{\, \Phi : SL(2,\mathbb{R}) \to G \mid SL(2,\mathbb{R}) \text{ acts on } G/H \text{ properly via } \Phi \,\}/G$$
$$\overset{1:1}{\longleftrightarrow} \{\, \text{Real nilpotent orbits } \mathcal{O}^G \text{ in } \mathfrak{g} \mid (\operatorname{Ad}(G_{\mathbb{C}}) \cdot \mathcal{O}^G) \cap \mathfrak{g}^c = \emptyset \,\}.$$

Here is an example concerning Theorem 1.10.1:

Example 1.10.2. *Let* $(G, H) = (SU(4,2), Sp(2,1))$. *Then* $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}, \mathfrak{g}^c) = (\mathfrak{sl}(6,\mathbb{C}), \mathfrak{su}(4,2), \mathfrak{su}^*(6))$. *Let us classify the following set:*

$$\{ \text{Real nilpotent orbits } \mathcal{O}^G \text{ in } \mathfrak{su}(4,2) \mid \text{the complexifications of } \mathcal{O}^G \text{ do not meet } \mathfrak{su}^*(6) \} \quad (1.10.1)$$

Recall that complex nilpotent orbits in $\mathfrak{sl}(6,\mathbb{C})$ *are parameterized by partitions of* 6 *and these weighted Dynkin diagrams are listed in Example* 1.6.6. *By Theorem* 1.7.10, *we can classify the complex nilpotent orbits in* $\mathfrak{sl}(6,\mathbb{C})$ *that meet* $\mathfrak{su}(4,2)$ *but not* $\mathfrak{su}^*(6)$, *by using these Satake diagrams (see Example* 1.9.1 *for Satake diagrams of* $\mathfrak{su}(4,2)$ *and* $\mathfrak{su}^*(6)$), *as follows:*

$$\{ \text{Complex nilpotent orbits } \mathcal{O}^{G_{\mathbb{C}}} \text{ in } \mathfrak{sl}(6,\mathbb{C}) \mid \mathcal{O}^{G_{\mathbb{C}}} \cap \mathfrak{su}(4,2) \neq \emptyset \text{ and } \mathcal{O}^{G_{\mathbb{C}}} \cap \mathfrak{su}^*(6) = \emptyset \} \stackrel{1:1}{\longleftrightarrow} \{ [5,1],\ [4,1^2],\ [3,2,1],\ [3,1^3],\ [2,1^4] \}.$$

It is known that real nilpotent orbits in $\mathfrak{su}(4,2)$ *are parameterized by signed Young diagrams of signature* $(4,2)$, *and the shape of the signed Young diagram corresponding to a real nilpotent orbit* $\mathcal{O}^G$ *in* $\mathfrak{su}(4,2)$ *is the partition corresponding to the complexification of* $\mathcal{O}^G$ *(see Theorem 9.3.3 and a remark after Theorem 9.3.5 in [8] for more details). Therefore, we have a classification of* (1.10.1) *as follows:*

Partition	*Signed Young diagram of signeture* $(4,2)$
$[5,1]$	$+ - + - +$ $+$
$[4,1^2]$	$+ - + -$ $+$ $+$, $- + - +$ $+$ $+$
$[3,2,1]$	$+ - +$ $+ -$ $+$, $+ - +$ $- +$ $+$
$[3,1^3]$	$+ - +$ $+$ $+$ $-$, $- + -$ $+$ $+$ $+$

$[2, 1^4]$	$\begin{array}{\|c\|c\|}\hline + & - \\ \hline + \\ \cline{1-1} + \\ \cline{1-1} + \\ \cline{1-1} - \\ \cline{1-1}\end{array}$,	$\begin{array}{\|c\|c\|}\hline - & + \\ \hline + \\ \cline{1-1} + \\ \cline{1-1} + \\ \cline{1-1} - \\ \cline{1-1}\end{array}$

Table 1.5: Classification of (1.10.1)

In particular, by Theorem 1.10.1, *there are nine kinds of Lie group homomorphisms* $\Phi : SL(2,\mathbb{R}) \to SU(4,2)$ *(up to inner automorphisms of* $SU(4,2)$*) for which the* $SL(2,\mathbb{R})$*-actions on* $SU(4,2)/Sp(2,1)$ *via* Φ *are proper.*

Appendix 1.A Classification

Here is a complete list of symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ with the following property:

$$\mathfrak{g} \text{ is simple, } (\mathfrak{g}, \mathfrak{h}) \text{ is a symmetric pair satisfying one of (therefore, all of) the conditions in Theorem 1.2.2.} \tag{1.A.1}$$

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sl}(2k, \mathbb{R})$	$\mathfrak{sl}(k, \mathbb{C}) \oplus \mathfrak{so}(2)$
$\mathfrak{sl}(n, \mathbb{R})$	$\mathfrak{so}(n-i, i)$ $(2i < n)$
$\mathfrak{su}^*(2k)$	$\mathfrak{sp}(k-i, i)$ $(2i < k-1)$
$\mathfrak{su}(2p, 2q)$	$\mathfrak{sp}(p, q)$
$\mathfrak{su}(2m-1, 2m-1)$	$\mathfrak{so}^*(4m-2)$
$\mathfrak{su}(p, q)$	$\mathfrak{su}(i, j) \oplus \mathfrak{su}(p-i, q-j) \oplus \mathfrak{so}(2)$ $(\min\{p, q\} > \min\{i, j\} + \min\{p-i, q-j\})$
$\mathfrak{so}(p, q)$ ($p+q$ is odd)	$\mathfrak{so}(i, j) \oplus \mathfrak{so}(p-i, q-j)$ $(\min\{p, q\} > \min\{i, j\} + \min\{p-i, q-j\})$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{su}(n-i, i) \oplus \mathfrak{so}(2)$

$\mathfrak{sp}(2k, \mathbb{R})$	$\mathfrak{sp}(k, \mathbb{C})$
$\mathfrak{sp}(p, q)$	$\mathfrak{sp}(i, j) \oplus \mathfrak{sp}(p - i, q - j)$ $(\min\{p, q\} > \min\{i, j\} + \min\{p - i, q - j\})$
$\mathfrak{so}(p, q)$ ($p + q$ is even)	$\mathfrak{so}(i, j) \oplus \mathfrak{so}(p - i, q - j)$ ($\min\{p, q\} > \min\{i, j\} + \min\{p - i, q - j\}$, unless $p = q = 2m + 1$ and $\lvert i - j \rvert = 1$)
$\mathfrak{so}(2p, 2q)$	$\mathfrak{su}(p, q) \oplus \mathfrak{so}(2)$
$\mathfrak{so}^*(2k)$	$\mathfrak{su}(k - i, i) \oplus \mathfrak{so}(2)$ $(2i < k - 1)$
$\mathfrak{so}(k, k)$	$\mathfrak{so}(2k, \mathbb{C}) \oplus \mathfrak{so}(2)$
$\mathfrak{so}^*(4m)$	$\mathfrak{so}^*(4m - 4i + 2) \oplus \mathfrak{so}^*(4i - 2)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(2, 2)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{su}^*(6) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{6(2)}$	$\mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$
$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(4, 2) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{6(2)}$	$\mathfrak{sp}(3, 1)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2)$
$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(4, 4)$
$\mathfrak{e}_{7(7)}$	$\mathfrak{so}^*(12) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{7(7)}$	$\mathfrak{su}^*(8)$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{su}(6, 2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{su}(6, 2)$
$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{8(8)}$	$\mathfrak{so}^*(16)$
$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(2, 1) \oplus \mathfrak{su}(2)$
$\mathfrak{sl}(2k, \mathbb{C})$	$\mathfrak{su}^*(2k)$
$\mathfrak{sl}(n, \mathbb{C})$	$\mathfrak{su}(n - i, i)$ $(2i < n)$
$\mathfrak{so}(2k + 1, \mathbb{C})$	$\mathfrak{so}(2k + 1 - i, i)$ $(i < k)$
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n - i, i)$
$\mathfrak{so}(2k, \mathbb{C})$	$\mathfrak{so}(2k - i, i)$ ($i < k$ unless $k = i + 1 = 2m + 1$)

$\mathfrak{so}(4m, \mathbb{C})$	$\mathfrak{so}(4m - 2i + 1, \mathbb{C}) \oplus \mathfrak{so}(2i - 1, \mathbb{C})$
$\mathfrak{so}(2k, \mathbb{C})$	$\mathfrak{so}^*(2k)$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(-14)}$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(-26)}$
$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{e}_{7(-5)}$
$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{e}_{7(-25)}$
$\mathfrak{e}_{8,\mathbb{C}}$	$\mathfrak{e}_{8(-24)}$
$\mathfrak{f}_{4,\mathbb{C}}$	$\mathfrak{f}_{4(-20)}$

Table 1.3: Classification of $(\mathfrak{g}, \mathfrak{h})$ satisfying (1.A.1)

Here, $k \geq 1$, $m \geq 1$, $n \geq 2$, $p, q \geq 1$ and $i, j \geq 0$. Note that $\mathfrak{so}(p, q)$ is simple if and only if $p + q \geq 3$ with $(p, q) \neq (2, 2)$, and $\mathfrak{so}(2k, \mathbb{C})$ is simple if and only if $k \geq 3$.

Appendix 1.B The Calabi–Markus phenomenon and hyperbolic orbits

Here is a proof of the equivalence among (v), (vi) and (vii) in Fact 1.2.6:

Proof of (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) *in Fact* 1.2.6. We take $\mathfrak{a}$ and $\mathfrak{a}_{\mathfrak{h}}$ in Section 1.3.1. The condition (v) means that $\mathfrak{a} \neq W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}_{\mathfrak{h}}$. By Fact 1.5.1 and Lemma 1.5.2, we have a bijection between the following two sets:

- The set of $W(\mathfrak{g}, \mathfrak{a})$-orbits in $\mathfrak{a}$ that do not meet $\mathfrak{a}_{\mathfrak{h}}$.
- The set of real hyperbolic orbits in $\mathfrak{g}$ that do not meet $\mathfrak{h}$.

Then the equivalence (v) $\Leftrightarrow$ (vi) holds. Further, (vi) $\Leftrightarrow$ (vii) follows from Proposition 1.4.5 (i) and Propositon 1.4.6. □

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Chapter 2

On $\mathfrak{sl}_2$-triples in real simple Lie algebras

We complete the proof of the claim in Section 1.7.5 *in Chapter* 1.

2.1 The purpose of this chapter

In Section 1.7.5 in the previous chapter, we claimed that for any simple Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, the following holds:

$$\mathfrak{b} = \mathbb{R}\text{-span}((\mathfrak{j}_+ \cap \mathcal{H}^n) \cap \mathfrak{a}). \tag{2.1.1}$$

However, the proof in Chapter 1 is sketchy. The purpose of this chapter is to complete the proof of (2.1.1).

We recall the notation as follows. Let $\mathfrak{g}_\mathbb{C}$ be a complex semisimple Lie algebra and $\mathfrak{g}$ a real form of $\mathfrak{g}_\mathbb{C}$. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, and use the following convention:

Definition 2.1.1. *We say that a Cartan subalgebra* $\mathfrak{j}$ *of* $\mathfrak{g}$ *is split if* $\mathfrak{a} := \mathfrak{j} \cap \mathfrak{p}$ *is a maximal abelian subspace of* $\mathfrak{p}$ *(i.e.* $\mathfrak{a}$ *is a maximally split abelian subspace of* $\mathfrak{g}$*).*

Note that split Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{g}$ is unique up to the adjoint action of K, where K is the analytic subgroup of G corresponding to $\mathfrak{k}$, and G is the inner-automorphism group of $\mathfrak{g}$.

Take a split Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{g}$ in Definition 2.1.1. Then $\mathfrak{j}$ can be written by $\mathfrak{j} = \mathfrak{t} + \mathfrak{a}$ for a maximal abelian subspace $\mathfrak{t}$ of the centralizer of

$\mathfrak{a}$ in $\mathfrak{k}$. Let us denote by $\mathfrak{j}_\mathbb{C} := \mathfrak{j} + \sqrt{-1}\mathfrak{j}$ and $\mathfrak{j} := \sqrt{-1}\mathfrak{t} + \mathfrak{a}$. Then $\mathfrak{j}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{j}$ is a real form of it with

$$\mathfrak{j} = \{A \in \mathfrak{j}_\mathbb{C} \mid \alpha(A) \in \mathbb{R} \text{ for any } \alpha \in \Delta\},$$

where Δ is the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$. We put

$$\Sigma := \{\alpha|_\mathfrak{a} \mid \alpha \in \Delta\} \setminus \{0\} \subset \mathfrak{a}^*$$

to the restricted root system of $(\mathfrak{g}, \mathfrak{a})$. Then we can take a positive system Δ^+ of Δ such that the subset

$$\Sigma^+ := \{\alpha|_\mathfrak{a} \mid \alpha \in \Delta^+\} \setminus \{0\}.$$

of Σ becomes a positive system. In fact, if we take an ordering on $\mathfrak{a}$ and extend it to $\mathfrak{j}$, then the corresponding positive system Δ^+ satisfies the condition above. Let us denote by $W^\mathbb{C}$, W the Weyl groups of Δ, Σ, respectively. We put the closed Weyl chambers

$$\mathfrak{j}_+ := \{\, A \in \mathfrak{j} \mid \alpha(A) \geq 0 \quad \text{for any } \alpha \in \Delta^+ \,\},$$
$$\mathfrak{a}_+ := \{\, A \in \mathfrak{a} \mid \xi(A) \geq 0 \quad \text{for any } \xi \in \Sigma^+ \,\}.$$

Then $\mathfrak{j}_+$ and $\mathfrak{a}_+$ are fundamental domains of $\mathfrak{j}$, $\mathfrak{a}$ for the actions of $W^\mathbb{C}$ and W, respectively. By the definition of Δ^+ and Σ^+, we have $\mathfrak{a}_+ = \mathfrak{j}_+ \cap \mathfrak{a}$.

Let us put w_0 to the longest element of W with respect to the positive system Σ^+. Then the linear transform $x \mapsto -w_0 \cdot x$ on $\mathfrak{a}$ leaves the closed Weyl chamber $\mathfrak{a}_+$ invariant. Here, we put

$$\mathfrak{b} := \{\, A \in \mathfrak{a} \mid -w_0 \cdot A = A \,\}.$$

The definition of $\mathfrak{b}$ is introduced by Benoist [3].

A triple (A, X, Y) is called an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_\mathbb{C}$ if $A, X, Y \in \mathfrak{g}_\mathbb{C}$ with

$$[A, X] = 2X,\ [A, Y] = -2Y,\ [X, Y] = A.$$

Here, we denote by

$$\mathcal{H}^n := \{\, A \in \mathfrak{g}_\mathbb{C} \mid \text{There exists } X, Y \in \mathfrak{g}_\mathbb{C} \text{ such that } (A, X, Y) \text{ is an } \mathfrak{sl}_2\text{-triple in } \mathfrak{g}_\mathbb{C} \,\}.$$

We should remark that $\mathfrak{j}_+ \cap \mathcal{H}^n$ does not depend on the real form $\mathfrak{g}$, and Dynkin [5] gave the classification of $\mathfrak{j}_+ \cap \mathcal{H}^n$ for each $\mathfrak{g}_\mathbb{C}$ (See Section 2.2.3 for more details).

The purpose of this chapter is to prove the next theorem:

Theorem 2.1.2 (The claim in Section 1.7.5). *For any semisimple Lie algebra* $\mathfrak{g}$ *over* $\mathbb{R}$*, the following holds:*

$$\mathfrak{b} = \mathbb{R}\text{-}span(\mathfrak{a}_+ \cap \mathcal{H}^n) \tag{2.1.2}$$

Remark 2.1.3. *Since* $\mathfrak{a}_+ = \mathfrak{j}_+ \cap \mathfrak{a}$*, the right hand side in* (2.1.2) *is equals to the right hand side in* (2.1.1).

We also denote by

$$\mathcal{H}^n(\mathfrak{g}) := \{\, A \in \mathfrak{g} \mid \text{There exists } X, Y \in \mathfrak{g} \text{ such that } (A, X, Y) \text{ is an } \mathfrak{sl}_2\text{-triple in } \mathfrak{g} \,\}.$$

Remark 2.1.4. *By the results of Kobayashi* [6]*, one can observe that the set* $\mathcal{H}^n(\mathfrak{g})$ *is important to find a proper* $SL(2,\mathbb{R})$*-actions on some homogeneous space of* G *(see Section* 1.5.2 *for more details).*

Then by using the results of Sekiguchi [10, Proposition 1.11], one can prove that

$$\mathfrak{a}_+ \cap \mathcal{H}^n = \mathfrak{a}_+ \cap \mathcal{H}^n(\mathfrak{g})$$

(see Section 1.7.4 for more details). Therefore Theorem 2.1.2 implies the next corollary:

Corollary 2.1.5 (Proposition 1.4.8 (*i*) in Chapter 1). *For any semisimple Lie algebra* $\mathfrak{g}$*, the following holds:*

$$\mathfrak{b} = \mathbb{R}\text{-}span(\mathfrak{a}_+ \cap \mathcal{H}^n(\mathfrak{g})).$$

Remark 2.1.6. *Corollary* 2.1.5 *plays an impotant role in the proof of the implication (iii)* $\Rightarrow$ *(i) in Theorem* 1.1.3 *in Chapter* 1.

2.2 Preliminary

Throughout this section, let us consider the same setting in Section 2.1. We recall some facts to compute $\mathfrak{b}$ and $\mathfrak{a}_+ \cap \mathcal{H}$ by using weighted Dynkin diagrams of $\mathfrak{g}_\mathbb{C}$ and Satake diagrams of $\mathfrak{g}$ in this section.

2.2.1 Satake diagrams and maximally split abelian subspaces

Let us denote by Π the fundamental system of Δ^+. Then

$$\overline{\Pi} := \{\, \alpha|_{\mathfrak{a}} \mid \alpha \in \Pi \,\} \setminus \{0\}$$

is the fundamental system of Σ^+. The Satake diagram $S_{\mathfrak{g}}$ of a semisimple Lie algebra $\mathfrak{g}$ consists of the following three data: the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ with nodes Π; black nodes Π_0 in S; and arrows joining $\alpha \in \Pi \setminus \Pi_0$ and $\beta \in \Pi \setminus \Pi_0$ in S whose restrictions to $\mathfrak{a}$ are the same (see [1, 9] for more details).

For any $A \in \mathfrak{j}$, we can define a map

$$\Psi_A : \Pi \to \mathbb{R}, \ \ \alpha \mapsto \alpha(A).$$

We call Ψ_A the weighted Dynkin diagram corresponding to $A \in \mathfrak{j}$, and $\alpha(A)$ the weight on a node $\alpha \in \Pi$ of the weighted Dynkin diagram. Since Π is a basis of $\mathfrak{j}^*$, the correspondence

$$\Psi : \mathfrak{j} \to \mathrm{Map}(\Pi, \mathbb{R}), \ \ A \mapsto \Psi_A \tag{2.2.1}$$

is a linear isomorphism between real vector spaces. In particular, Ψ is bijective. Furthermore,

$$\Psi|_{\mathfrak{j}_+} : \mathfrak{j}_+ \to \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}), \ \ A \mapsto \Psi_A$$

is also bijective. We say that a weighted Dynkin diagram is trivial if all weights are zero. Namely, the trivial diagram corresponds to the zero of $\mathfrak{j}$ by Ψ.

Here, we recall the definition of weighted Dynkin diagrams *matching* the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ as follows:

Definition 2.2.1 (Definition 1.7.3 in Chapter 1). *Let $\Psi_A \in \mathrm{Map}(\Pi, \mathbb{R})$ be a weighted Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ and $S_{\mathfrak{g}}$ the Satake diagram of $\mathfrak{g}$ with nodes Π. We say that Ψ_A matches $S_{\mathfrak{g}}$ if all the weights on black nodes in Π_0 are zero and any pair of nodes joined by an arrow have the same weights.*

Then the following lemma holds:

Lemma 2.2.2 (Lemma 1.7.5 in Chapter 1). *The linear isomorphism $\Psi : \mathfrak{j} \to \mathrm{Map}(\Pi, \mathbb{R})$ induces a linear isomorphism*

$$\mathfrak{a} \to \{\, \Psi_A \in \mathrm{Map}(\Pi, \mathbb{R}) \mid \Psi_A \text{ matches } S_{\mathfrak{g}} \,\}, \quad A \mapsto \Psi_A.$$

In particular, by this linear isomorphism, we have

$$\mathfrak{a}_+ \overset{1:1}{\longleftrightarrow} \{\, \Psi_A \in \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}) \mid \Psi_A \text{ matches } S_{\mathfrak{g}} \,\}.$$

2.2.2 Tits involution and $\mathfrak{b}$

Let us denote by $w_0^{\mathbb{C}}$ the longest element of $W^{\mathbb{C}}$ corresponding to the positive system Δ^+. Then, by the action of $w_0^{\mathbb{C}}$, every element in $\mathfrak{j}_+$ moves to $-\mathfrak{j}_+ := \{-A \mid A \in \mathfrak{j}_+\}$. In particular,

$$-w_0^{\mathbb{C}} : \mathfrak{j} \to \mathfrak{j}, \quad A \mapsto -(w_0^{\mathbb{C}} \cdot A)$$

is an involutive automorphism on $\mathfrak{j}$ preserving $\mathfrak{j}_+$. We put

$$\mathfrak{j}^{-w_0^{\mathbb{C}}} := \{\, A \in \mathfrak{j} \mid -w_0^{\mathbb{C}} \cdot A = A \,\}.$$

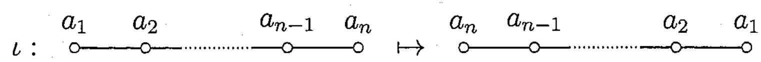
Recall that the map $\Psi : \mathfrak{j} \to \mathrm{Map}(\Pi, \mathbb{R})$ in Section 2.2.1 is a linear isomorphism. Thus $-w_0^{\mathbb{C}}$ induces an involutive endomorphism on $\mathrm{Map}(\Pi, \mathbb{R})$, which will be denoted by ι. In particular, we have

$$\Psi(\mathfrak{j}^{-w_0^{\mathbb{C}}}) = \mathrm{Map}(\Pi, \mathbb{R})^{\iota},$$

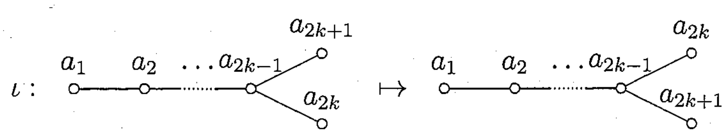
where $\mathrm{Map}(\Pi, \mathbb{R})^{\iota}$ denotes the set of all weighted Dynkin diagrams held invariant by ι. For each complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, we determine ι as follows:

Proposition 2.2.3 (Theorem 1.6.3 (*ii*) in Chapter 1). *Suppose that $\mathfrak{g}_{\mathbb{C}}$ is simple. The involution ι is not identity if and only if the complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is of type A_n, D_{2k+1} or E_6 ($n \geq 2$, $k \geq 2$). In other words, this is the complete list of simple $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{j}^{-w_0^{\mathbb{C}}} \neq \mathfrak{j}$. In such cases, the forms of ι are the following:*

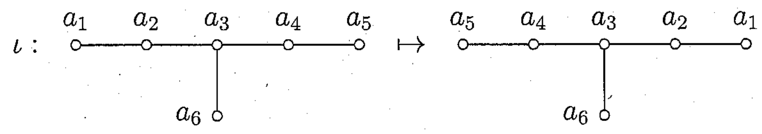
For type A_n ($n \geq 2$, $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{sl}(n+1, \mathbb{C})$):



For type D_{2k+1} ($k \geq 2$, $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{so}(4k+2, \mathbb{C})$):



For type E_6 ($\mathfrak{g}_\mathbb{C} \simeq \mathfrak{e}_{6,\mathbb{C}}$):



It should be noted that for the cases where $\mathfrak{g}_\mathbb{C}$ is of type D_{2k} ($k \geq 2$), the involution $-w_0^\mathbb{C}$ on $\mathfrak{j}$ is trivial although the Dynkin diagram of type D_{2k} admits some involutive automorphisms.

Here, the following lemma holds:

Lemma 2.2.4 (Lemma 1.7.6 in Chapter 1). $\mathfrak{b} = \mathfrak{j}^{-w_0^\mathbb{C}} \cap \mathfrak{a}$.

Thus, by combining Lemma 2.2.2 with Lemma 2.2.4, we obtain that

$$\Psi(\mathfrak{b}) = \{\, \Psi_A \in \mathrm{Map}(\Pi, \mathbb{R})^\iota \mid \Psi_A \text{ matches } S_\mathfrak{g} \,\}. \tag{2.2.2}$$

where $\mathrm{Map}(\Pi, \mathbb{R})^\iota$ denotes the set of all weighted Dynkin diagrams held invariant by ι, and S_g is the Satake diagram of $\mathfrak{g}$ (see Section 2.2.1).

2.2.3 Classification of $\mathfrak{a}_+ \cap \mathcal{H}^n$

Let us denote by $\mathrm{Hom}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g}_\mathbb{C})$ the set of all complex Lie algebra homomorphisms from $\mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{g}_\mathbb{C}$. For $\rho, \rho' \in \mathrm{Hom}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g}_\mathbb{C})$, we also write $\rho \sim \rho'$ if there exists $g \in G_\mathbb{C}$ and $l \in SL(2,\mathbb{C})$ such that

$$\rho' = \mathrm{Ad}_{\mathfrak{g}_\mathbb{C}}(g) \circ \rho \circ \mathrm{Ad}_{\mathfrak{sl}(2,\mathbb{C})}(l),$$

where $G_\mathbb{C}$ is a connected complex Lie group with $\mathrm{Lie}\, G_\mathbb{C} = \mathfrak{g}_\mathbb{C}$. Then $\sim$ defines an equivalent relation on $\mathrm{Hom}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g}_\mathbb{C})$. In fact, we can omit the adjoint action of $SL(2,\mathbb{C})$ on $\mathfrak{sl}(2,\mathbb{C})$, since for any $\rho \in \mathrm{Hom}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g}_\mathbb{C})$ and any $l \in SL(2,\mathbb{C})$, there exists $g_l \in G_\mathbb{C}$ such that

$$\rho \circ \mathrm{Ad}_{\mathfrak{sl}(2,\mathbb{C})}(l) = \mathrm{Ad}_{\mathfrak{g}_\mathbb{C}}(g_l) \circ \rho.$$

For $\rho \in \mathrm{Hom}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g}_\mathbb{C})$, let us put A_ρ to the unique element in $\mathfrak{j}_+$ which is conjugate to

$$\rho \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{g}_\mathbb{C}$$

under the adjoint action on $\mathfrak{g}_{\mathbb{C}}$. Then the correspondence $[\rho] \mapsto A_\rho$ gives a map from $\mathrm{Hom}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g}_{\mathbb{C}})/\sim$ to $\mathfrak{j}_+ \cap \mathcal{H}^n$. Malcev [8] proved that the map is bijective. That is,

$$\mathrm{Hom}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g}_{\mathbb{C}})/\sim \overset{1:1}{\longleftrightarrow} \mathfrak{j}_+ \cap \mathcal{H}^n.$$

Recall that Ψ in Section 2.2.1 induces a bijection between $\mathfrak{j}_+$ and $\mathrm{Map}(\Pi, \mathbb{R}_{\geq 0})$. Thus, a classification of $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ gives a classification of $\mathrm{Hom}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g}_{\mathbb{C}})/\sim$.

Dynkin [5] proved that any weight of an weighted Dynkin diagram in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ is given by 0, 1 or 2. Hence, $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ (and therefore $\mathrm{Hom}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g}_{\mathbb{C}})/\sim$ is) finite. Dynkin [5] also gave a complete list of the weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ for each simple $\mathfrak{g}_{\mathbb{C}}$.

Remark 2.2.5. *By combining the Jacobson–Morozov theorem with the results of Kostant [7], we also obtain a bijection between* $\mathrm{Hom}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g}_{\mathbb{C}})/\sim$ *and the set of complex adjoint nilpotent orbits in* $\mathfrak{g}_{\mathbb{C}}$*. Thus, the classification of* $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$*, done by Dynkin [5], gives a classification of complex adjoint nilpotent orbits in* $\mathfrak{g}_{\mathbb{C}}$ *(see Bala–Cater [2] or Collingwood–McGovern [4, Section 3] for more details).*

By Lemma 2.2.2, we have that

$$\Psi(\mathfrak{a}_+ \cap \mathcal{H}^n) = \{\Psi_A \in \Psi(\mathfrak{j}_+ \cap \mathcal{H}^n) \mid \Psi_A \text{ matches } S_{\mathfrak{g}}\},$$

where $S_{\mathfrak{g}}$ is the Satake diagram of $\mathfrak{g}$ (see Section 2.2.1 for the notation). Therefore, for each $\mathfrak{g}$, by using the classification of $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ and the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$, we obtain a classification of $\Psi(\mathfrak{a}_+ \cap \mathcal{H}^n)$.

2.3 Proof of Theorem 2.1.2

First, we show the following:

Lemma 2.3.1. $\mathfrak{a}_+ \cap \mathcal{H}^n \subset \mathfrak{b}$.

Proof of Lemma 2.3.1. One can observe that for any $A \in \mathcal{H}^n$, the element $-A \in \mathfrak{g}_{\mathbb{C}}$ is conjugate to A under the adjoint action on $\mathfrak{g}_{\mathbb{C}}$ (see the proof of Lemma 1.4.7 in Section 1.6.3 for more details). On the other hand, for any $A \in \mathfrak{a}_+ \subset \mathfrak{g}$, one can observe that

$$(\mathrm{Ad}(G_{\mathbb{C}}) \cdot A) \cap \mathfrak{a} = (\mathrm{Ad}(G) \cdot A) \cap \mathfrak{a} = W \cdot A$$

(see the proof of Proposition 1.4.5 (i) in Section 1.7.1 for more details). Hence, $-A$ is conjugate to A under the adjoint action on $\mathfrak{g}_{\mathbb{C}}$ if and only if A is in $\mathfrak{b}$. This completes the proof. □

Recall that $\Psi : \mathfrak{j} \to \mathrm{Map}(\Pi, \mathbb{R})$ in Section 2.2.1 is a linear isomorphism. Therefore, to prove Theorem 2.1.2, we only need to show that

$$\Psi(\mathfrak{b}) \subset \mathbb{R}\text{-span}\Psi(\mathfrak{a}_+ \cap \mathcal{H}^n) \tag{2.3.1}$$

To prove our claim for each $\mathfrak{g}$, we only need to find some weighted Dynkin diagrams $\Psi_1, \dots, \Psi_n$ in $\Psi(\mathfrak{j}_+ \cap \mathfrak{a})$ such that $\{\Psi_1, \dots, \Psi_n\}$ becomes a basis of $\Psi(\mathfrak{b})$.

We remark that our claim for $\mathfrak{g}$ is reduced to that on each simple factor of $\mathfrak{g}$. Furthermore, in the cases where $\mathfrak{g}$ is a complex simple Lie algebra, our claim is reduced to the cases where $\mathfrak{g}'$ is a split real form of $\mathfrak{g}$. Thus, it is enough to show (2.3.1) for the cases where $\mathfrak{g}_{\mathbb{C}}$ is simple and $\mathfrak{g}$ is non-compact real form of $\mathfrak{g}_{\mathbb{C}}$.

In the rest of this section, for each simple $\mathfrak{g}_{\mathbb{C}}$ we give an explicit form of $\mathrm{Map}(\Pi, \mathbb{R})^\iota$ (see Section 2.2.2 for the definition of ι). Furthermore, we give some examples from the list of $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ given by Dynkin [5] (we will refer [4] for classifications of weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$). Then for each non-compact real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$, we give the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$, which can be found in [1], and the explicit form of $\Psi(\mathfrak{b})$ by using (2.2.2) in Section 2.2.2. Finally, for each $\mathfrak{g}$, we give an example of a basis $\Psi_1, \dots, \Psi_n$ of $\Psi(\mathfrak{b})$ with $\Psi_i \in \Psi(\mathfrak{j}_+ \cap \mathfrak{a})$. Then the proof of Theorem 2.1.2 will be completed.

Remark 2.3.2. *As in the following subsections, one can find such $\Psi_1, \dots, \Psi_n$ as even weighted Dynkin diagrams, where even means any weight is* 0 *or* 2.

2.3.1 For the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type A_l

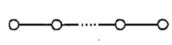
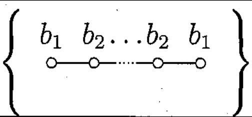
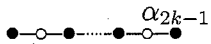
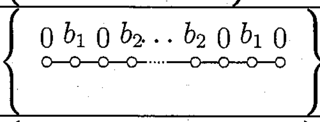
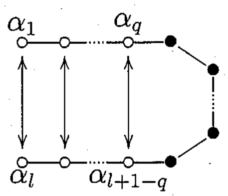
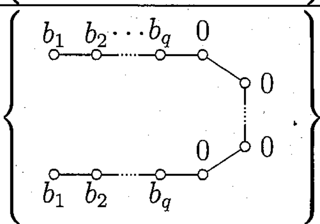
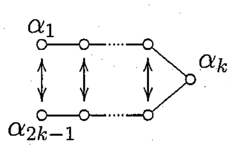
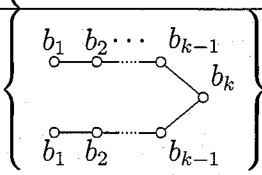
Let us consider the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type A_l for $l \geq 1$, that is, $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{sl}(l+1, \mathbb{C})$. Then we have

$$\mathrm{Map}(\Pi, \mathbb{R})^\iota = \left\{ \overset{a_1}{\circ} - \overset{a_2}{\circ} - \cdots - \overset{a_{l-1}}{\circ} - \overset{a_l}{\circ} \;\middle|\; a_i = a_{l+1-i} \text{ for } i = 1, \dots, l \right\}$$

By [4, Chapter 3.6], we can find some examples of weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

Symbol	Weighted Dynkin diagram in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$
$[l+1]$	$2\ \ 2 \cdots 2\ \ 2$
$[2s+1, 1^{l-2s}]$	$2 \cdots 2\ \ 2\ \ 0 \cdots 0\ \ 2\ \ 2 \cdots 2$ (α_s, α_{l+1-s})
$[(2s+1)^2, 1^{l-4s-1}]$	$0202 \cdots 0200 \cdots 0020 \cdots 2020$ (α_{2s}, α_{l+1-2s})

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. Then the Satake diagram $S_{\mathfrak{g}}$ and $\Psi(\mathfrak{b})$ are given as follows:

$\mathfrak{g}$	$S_{\mathfrak{g}}$	$\Psi(\mathfrak{b})$
$\mathfrak{sl}(l+1, \mathbb{R})$		
$\mathfrak{su}^*(2k)$ $(2k = l+1)$		
$\mathfrak{su}(p,q)$ $(p+q = l+1)$ $(p > q)$		
$\mathfrak{su}(k,k)$ $(2k = l+1)$		

Therefore, for each $\mathfrak{g}$, we can find a basis of $\Psi(\mathfrak{b})$ by taking some weighted

Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

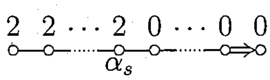
$\mathfrak{g}$	Example of basis of $\Psi(\mathfrak{b})$
$\mathfrak{sl}(2k,\mathbb{R})$ $(2k = l+1)$	$[3, 1^{2k-3}], [5, 1^{2k-5}], \dots, [2k-1, 1], [2k]$
$\mathfrak{sl}(2k+1,\mathbb{R})$ $(2k = l)$	$[3, 1^{2k-2}], [5, 1^{2k-4}], \dots, [2k+1]$
$\mathfrak{su}^*(4m)$ $(4m = l+1)$	$[3^2, 1^{4m-6}], [5^2, 1^{4m-10}], \dots, [(2m-1)^2, 1^2], [(2m)^2]$
$\mathfrak{su}^*(4m+2)$ $(4m = l-1)$	$[3^2, 1^{4m-4}], [5^2, 1^{4m-8}], \dots, [(2m+1)^2]$
$\mathfrak{su}(p,q)$ $(p+q = l+1, p > q)$	$[3, 1^{l-2}], [5, 1^{l-4}], \dots, [2q+1, 1^{l-2q}]$
$\mathfrak{su}(k,k)$ $(2k = l+1)$	$[3, 1^{2k-3}], [5, 1^{2k-5}], \dots, [2k-1, 1], [2k]$

2.3.2 For the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type B_l

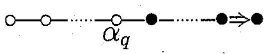
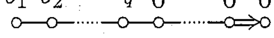
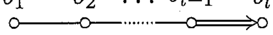
Let us consider the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type B_l for $l \geq 1$, that is, $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{so}(2l+1,\mathbb{C})$. Then we have

$$\mathrm{Map}(\Pi,\mathbb{R})^{\iota} = \mathrm{Map}(\Pi,\mathbb{R}) = \left\{ \quad \right\}$$

By [2, Chapter 5.3], we can find some examples of weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

Symbol	Weighted Dynkin diagram in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$
$[2l+1]$	
$[2s+1, 1^{2l-2s}]$	

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. Then the Satake diagram $S_{\mathfrak{g}}$ and $\Psi(\mathfrak{b})$ are given as follows:

$\mathfrak{g}$	$S_{\mathfrak{g}}$	$\Psi(\mathfrak{b})$
$\mathfrak{so}(p,q)$ $(p+q=2l+1)$ $(p>q+1)$	 (α_q)	$\left\{ \right.$  $b_1\ b_2\ \ b_q\ 0\ \ 0\ 0$ $\left. \right\}$
$\mathfrak{so}(l+1,l)$		$\left\{ \right.$  $b_1\ \ b_2\ \ldots\ b_{l-1}\ \ b_l$ $\left. \right\}$

Therefore, for each $\mathfrak{g}$, we can find a basis of $\Psi(\mathfrak{b})$ by taking some weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

$\mathfrak{g}$	Example of basis of $\Psi(\mathfrak{b})$
$\mathfrak{so}(p,q)$ $(p+q=2l+1, p>q+1)$	$[3,1^{2l-2}],[5,1^{2l-4}],\ldots,[2q+1,1^{2l-2q}]$
$\mathfrak{so}(l+1,l)$	$[3,1^{2l-2}],[5,1^{2l-4}],\ldots,[2l+1]$

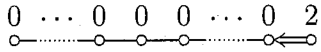
2.3.3 For the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type C_l

Let us consider the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type C_l for $l \geq 1$, that is, $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{sp}(l,\mathbb{C})$. Then we have

$$\mathrm{Map}(\Pi,\mathbb{R})^{\iota} = \mathrm{Map}(\Pi,\mathbb{R}) = \left\{ \right.$$

$$\left. \right\}$$

By [2, Chapter 5.3], we can find some examples of weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

Symbol	Weighted Dynkin diagram in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$
$[2^l]$	 $0\ \cdots\ 0\ \ 0\ \ 0\ \cdots\ 0\ \ 2$

$[2s+2, 2^{l-s}]$	$2 \cdots 2\ \ 2\ \ 0 \cdots 0\ \ 2$ (α_s)
$[(2s+1)^2, 1^{2l-4s-2}]$	$0\,2\,0\,2 \cdots 0\,2\,0\,0 \cdots 0\,0\ \ 0$ (α_{2s})
$[(2k)^2]$ $(2k = l)$	$0\,2\,0\,2 \cdots 0\,2\,0\,2 \cdots 2\,0\ \ 2$

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. Then the Satake diagram $S_{\mathfrak{g}}$ and $\Psi(\mathfrak{b})$ are given as follows:

$\mathfrak{g}$	$S_{\mathfrak{g}}$	$\Psi(\mathfrak{b})$
$\mathfrak{sp}(l, \mathbb{R})$		$\left\{ b_1\ b_2\ \ b_{l-1}\ b_l \right\}$
$\mathfrak{sp}(p, q)$ $(p+q=l)$ $(p>q)$	α_{2q}	$\left\{ 0\ b_1 0 \ldots b_q 0\ 0 \cdots 0\ 0 \right\}$
$\mathfrak{sp}(k, k)$ $(2k = l)$	α_{2k}	$\left\{ 0\ \ b_1\ 0 \ldots b_{k-1} 0\ \ b_k \right\}$

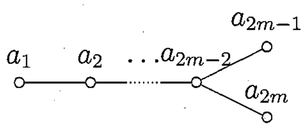
Therefore, for each $\mathfrak{g}$, we can find a basis of $\Psi(\mathfrak{b})$ by taking some weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

$\mathfrak{g}$	Example of basis of $\Psi(\mathfrak{b})$
$\mathfrak{sp}(l, \mathbb{R})$	$[2^l], [4, 2^{l-2}], [6, 2^{l-3}], \ldots, [2l]$
$\mathfrak{sp}(p, q)$ $(p+q=l, p>q)$	$[3^2, 1^{2l-6}], [5^2, 1^{2l-10}], \ldots, [(2q+1)^2, 1^{2l-2q-1}]$
$\mathfrak{sp}(k, k)$ $(2k = l)$	$[3^2, 1^{4k-6}], [5^2, 1^{4k-10}], \ldots, [(2k-1)^2, 1^2], [(2k)^2]$

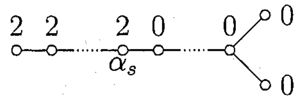
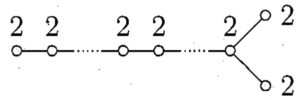
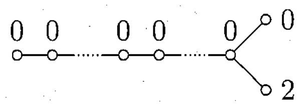
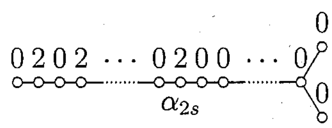
2.3.4 For the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type D_{2m}

Let us consider the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type D_{2m} for $m \geq 2$, that is, $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{so}(4m, \mathbb{C})$. Then we have

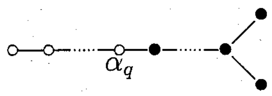
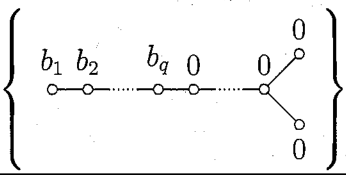
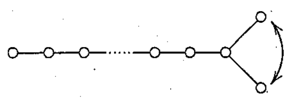
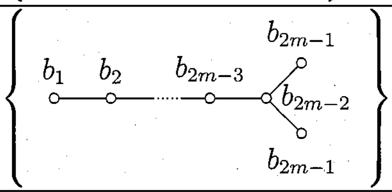
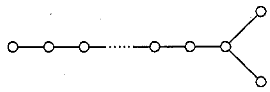
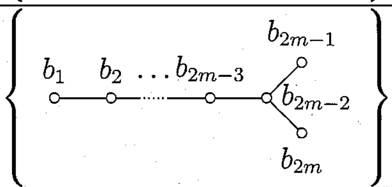
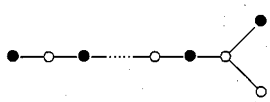
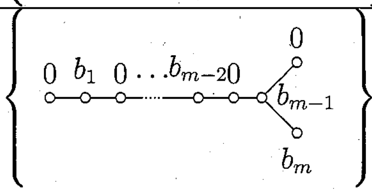
$$\mathrm{Map}(\Pi, \mathbb{R})^{\iota} = \mathrm{Map}(\Pi, \mathbb{R}) = \left\{ \quad \right\}$$



By [2, Chapter 5.3], we can find some examples of weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

Symbol	Weighted Dynkin diagram in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$
$[2s+1, 1^{4m-2s-1}]$	
$[4m-1, 1]$	
$[2^{2m}]_{\mathrm{I}}$	
$[(2s+1)^2, 1^{4m-4s-2}]$	

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. Then the Satake diagram $S_{\mathfrak{g}}$ and $\Psi(\mathfrak{b})$ are given as follows:

$\mathfrak{g}$	$S_{\mathfrak{g}}$	$\Psi(\mathfrak{b})$
$\mathfrak{so}(p,q)$ $(p+q=4m)$ $(p>q+2)$		
$\mathfrak{so}(2m+1,2m-1)$		
$\mathfrak{so}(2m,2m)$		
$\mathfrak{so}^*(4m)$		

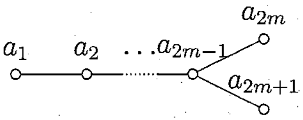
Therefore, for each $\mathfrak{g}$, we can find a basis of $\Psi(\mathfrak{b})$ by taking some weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

$\mathfrak{g}$	Example of basis of $\Psi(\mathfrak{b})$
$\mathfrak{so}(p,q)$ $(p+q=4m, p>q+2)$	$[3,1^{4m-3}],[5,1^{4m-5}],\ldots,[2q+1,1^{4m-2q-1}]$
$\mathfrak{so}(2m+1,2m-1)$	$[3,1^{4m-3}],[5,1^{4m-5}],\ldots,[4m-1,1]$
$\mathfrak{so}(2m,2m)$	$[3,1^{4m-1}],[5,1^{4m-3}],\ldots,[4m-1,1],[2^{2m}]_{\mathrm{I}}$
$\mathfrak{so}^*(4m)$	$[3^2,1^{4m-6}],[5^2,1^{4m-10}],\ldots,[(2m-1)^2,1^2],[2^{2m}]_{\mathrm{I}}$

2.3.5 For the cases where $\mathfrak{g}_\mathbb{C}$ is of type D_{2m+1}

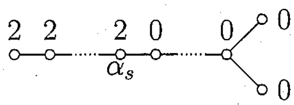
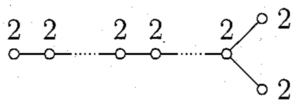
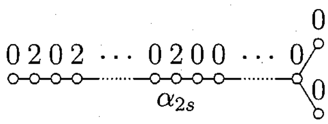
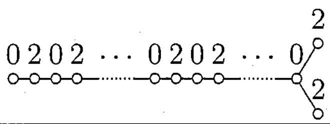
Let us consider the cases where $\mathfrak{g}_\mathbb{C}$ is of type D_{2m+1} for $m \geq 1$, that is, $\mathfrak{g}_\mathbb{C} \simeq \mathfrak{so}(4m+2, \mathbb{C})$. Then we have

$$\mathrm{Map}(\Pi, \mathbb{R})^\iota = \left\{ \right.$$

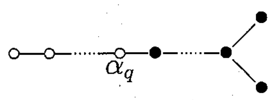
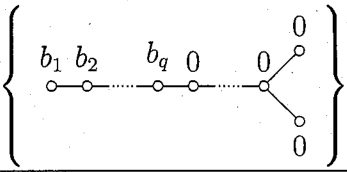
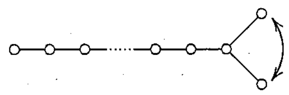
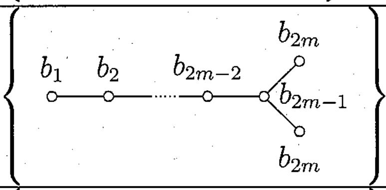
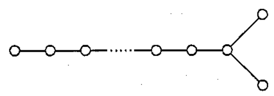
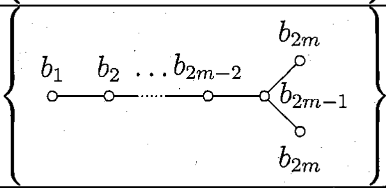
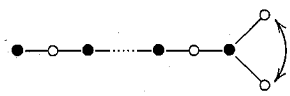
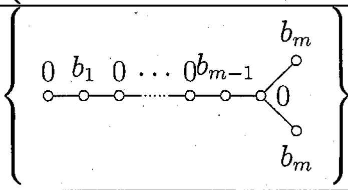


$$\left. \mid a_{2m} = a_{2m+1} \right\}$$

By [2, Chapter 5.3], we can find some examples of weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

Symbol	Weighted Dynkin diagram in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$
$[2s+1, 1^{4m-2s+1}]$	
$[4m+1, 1]$	
$[(2s+1)^2, 1^{4m-4s}]$	
$[(2m+1)^2]$	

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_\mathbb{C}$. Then the Satake diagram $S_\mathfrak{g}$ and $\Psi(\mathfrak{b})$ are given as follows:

$\mathfrak{g}$	$S_{\mathfrak{g}}$	$\Psi(\mathfrak{b})$
$\mathfrak{so}(p,q)$ $(p+q=4m+2)$ $(p>q+2)$		
$\mathfrak{so}(2m+2,2m)$		
$\mathfrak{so}(2m+1,2m+1)$		
$\mathfrak{so}^*(4m+2)$		

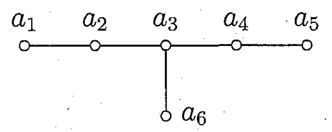
Therefore, for each $\mathfrak{g}$, we can find a basis of $\Psi(\mathfrak{b})$ by taking some weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

$\mathfrak{g}$	Example of basis
$\mathfrak{so}(p,q)$ $(p+q=4m+2, p>q+2)$	$[3,1^{4m-1}],[5,1^{4m-3}],\ldots,[2q+1,1^{4m-2q+1}]$
$\mathfrak{so}(2m+2,2m)$	$[3,1^{4m-1}],[5,1^{4m-3}],\ldots,[4m+1,1]$
$\mathfrak{so}(2m+1,2m+1)$	$[3,1^{4m-1}],[5,1^{4m-3}],\ldots,[4m+1,1]$
$\mathfrak{so}^*(4m+2)$	$[3^2,1^{4m-4}],[5^2,1^{4m-8}],\ldots,[(2m+1)^2]$

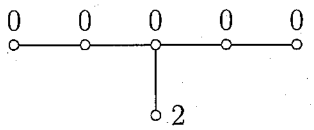
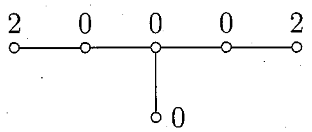
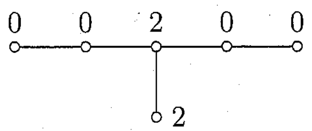
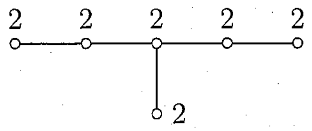
2.3.6 For the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type E_6

Let us consider the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type E_6, that is, $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{e}_{6,\mathbb{C}}$. Then we have

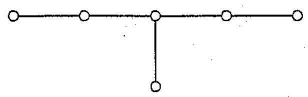
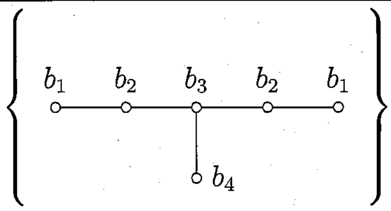
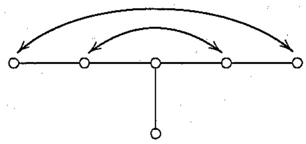
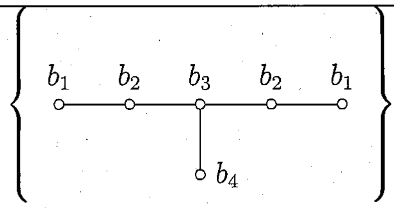
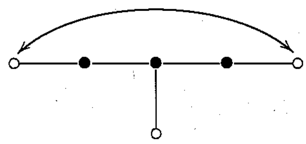
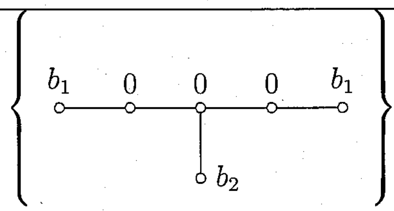
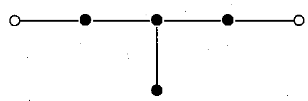
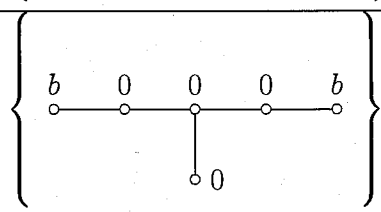
$$\mathrm{Map}(\Pi, \mathbb{R})^{\iota} = \left\{ \quad \mid a_1 = a_5,\ a_2 = a_4 \right\}$$



In [4, Chapter 8.4], we can find some examples of weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

Symbol	Weighted Dynkin diagram in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$
A_2	
$2A_2$	
D_4	
E_6	

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. Then the Satake diagram $S_{\mathfrak{g}}$ and $\Psi(\mathfrak{b})$ are given as follows:

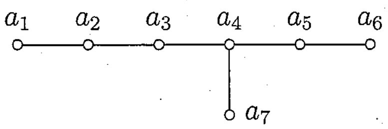
$\mathfrak{g}$	$S_{\mathfrak{g}}$	$\Psi(\mathfrak{b})$
$\mathfrak{e}_{6(6)}$		
$\mathfrak{e}_{6(2)}$		
$\mathfrak{e}_{6(-14)}$		
$\mathfrak{e}_{6(-26)}$		

Therefore, for each $\mathfrak{g}$, we can find a basis of $\Psi(\mathfrak{b})$ by taking some weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

$\mathfrak{g}$	Example of basis of $\Psi(\mathfrak{b})$
$\mathfrak{e}_{6(6)}$	$A_2, 2A_2, D_4, E_6$
$\mathfrak{e}_{6(2)}$	$A_2, 2A_2, D_4, E_6$
$\mathfrak{e}_{6(-14)}$	$A_2, 2A_2$
$\mathfrak{e}_{6(-26)}$	$2A_2$

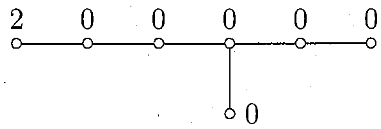
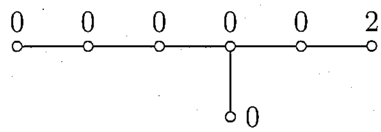
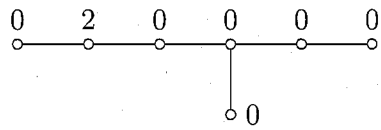
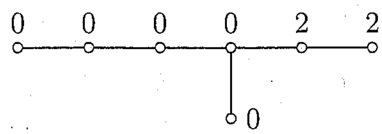
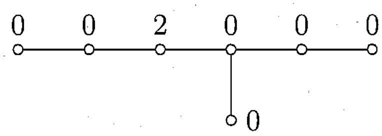
2.3.7 For the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type E_7

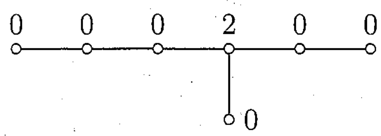
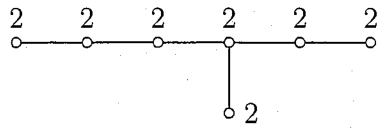
Let us consider the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type E_7, that is, $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{e}_{7,\mathbb{C}}$. Then we have

$$\mathrm{Map}(\Pi, \mathbb{R})^{\iota} = \mathrm{Map}(\Pi, \mathbb{R}) = \left\{ \right.$$


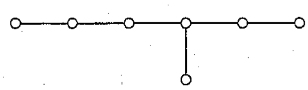
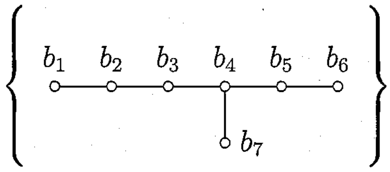
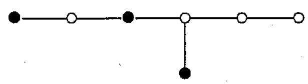
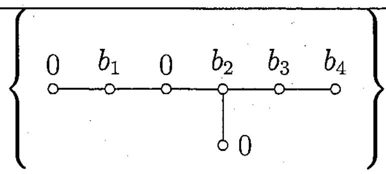
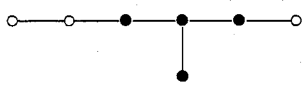
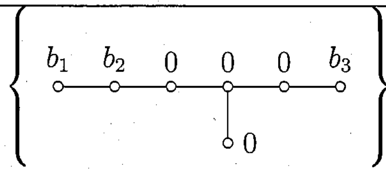
$$\left. \right\}$$

In [4, Chapter 8.4], we can find some examples of weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

Symbol	Weighted Dynkin diagram in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$
$(3A_1)''$	
A_2	
$2A_2$	
D_4	
$A_3 + A_2 + A_1$	

$A_4 + A_2$	
E_7	

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. Then the Satake diagram $S_{\mathfrak{g}}$ and $\Psi(\mathfrak{b})$ are given as follows:

$\mathfrak{g}$	$S_{\mathfrak{g}}$	$\Psi(\mathfrak{b})$
$\mathfrak{e}_{7(7)}$		
$\mathfrak{e}_{7(-5)}$		
$\mathfrak{e}_{7(-25)}$		

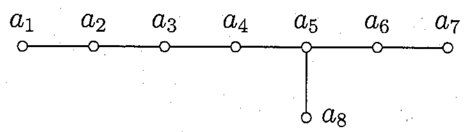
Therefore, for each $\mathfrak{g}$, we can find a basis of $\Psi(\mathfrak{b})$ by taking some weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

$\mathfrak{g}$	Example of basis of $\Psi(\mathfrak{b})$
$\mathfrak{e}_{7(7)}$	$3A_1'', A_2, 2A_2, D_4, A_3+A_2+A_1, A_4+A_2, E_7$
$\mathfrak{e}_{7(-5)}$	$A_2, 2A_2, D_4, A_4+A_2$
$\mathfrak{e}_{7(-25)}$	$3A_1'', A_2, 2A_2$

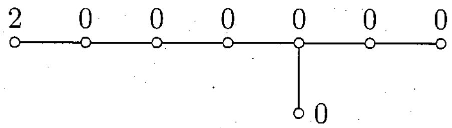
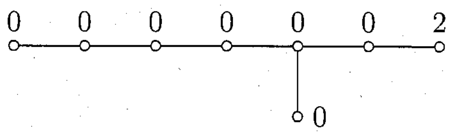
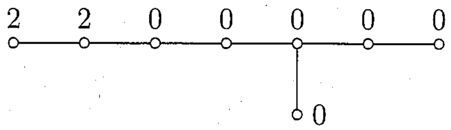
2.3.8 For the cases where $\mathfrak{g}_\mathbb{C}$ is of type E_8

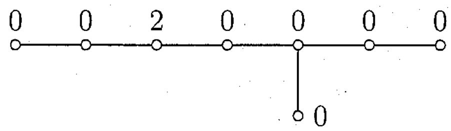
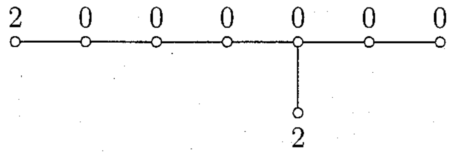
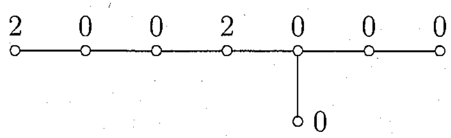
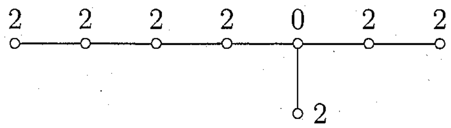
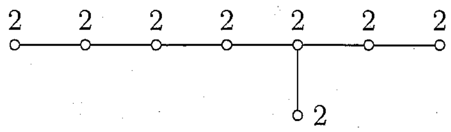
Let us consider the cases where $\mathfrak{g}_\mathbb{C}$ is of type E_8, that is, $\mathfrak{g}_\mathbb{C} \simeq \mathfrak{e}_{8,\mathbb{C}}$. Then we have

$$\mathrm{Map}(\Pi, \mathbb{R})^\iota = \mathrm{Map}(\Pi, \mathbb{R}) = \left\{ \quad \right\}$$

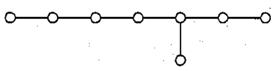
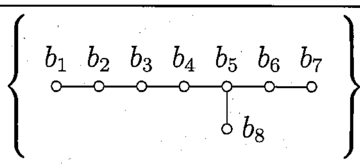
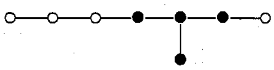
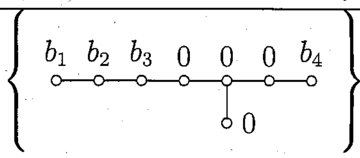


In [4, Chapter 8.4], we can find some examples of weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

Symbol	Weighted Dynkin diagram in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$
A_2	
$2A_2$	
D_4	

$A_4 + A_2$	
$D_4 + A_2$	
$D_5 + A_2$	
$E_8(a_1)$	
E_8	

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. Then the Satake diagram $S_{\mathfrak{g}}$ and $\Psi(\mathfrak{b})$ are given as follows:

$\mathfrak{g}$	$S_{\mathfrak{g}}$	$\Psi(\mathfrak{b})$
$\mathfrak{e}_{8(8)}$		
$\mathfrak{e}_{8(-24)}$		

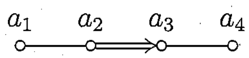
Therefore, for each $\mathfrak{g}$, we can find a basis of $\Psi(\mathfrak{b})$ by taking some weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

$\mathfrak{g}$	Example of basis of $\Psi(\mathfrak{b})$
$\mathfrak{e}_{8(8)}$	$A_2, 2A_2, D_4, A_4 + A_2, D_4 + A_2, D_5 + A_2, E_8(a_1), E_8$
$\mathfrak{e}_{8(-24)}$	$A_2, 2A_2, D_4, A_4 + A_2$

2.3.9 For the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type F_4

Let us consider the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type F_4, that is, $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{f}_{4,\mathbb{C}}$. Then we have

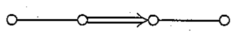
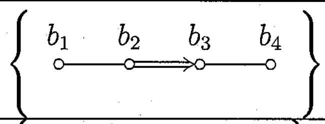
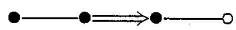
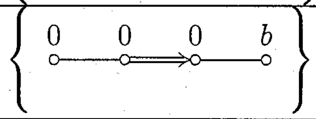
$$\mathrm{Map}(\Pi, \mathbb{R})^{\iota} = \mathrm{Map}(\Pi, \mathbb{R}) = \left\{ \quad \right\}$$



In [4, Chapter 8.4], we can find some examples of weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

Symbol	Weighted Dynkin diagram in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$
A_2	2 0 0 0
$\tilde{A}_2$	0 0 0 2
B_3	2 2 0 0
F_4	2 2 2 2

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. Then the Satake diagram $S_{\mathfrak{g}}$ and $\Psi(\mathfrak{b})$ are given as follows:

$\mathfrak{g}$	$S_{\mathfrak{g}}$	$\Psi(\mathfrak{b})$
$\mathfrak{f}_{4(4)}$		
$\mathfrak{f}_{4(-20)}$		

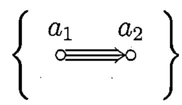
Therefore, for each $\mathfrak{g}$, we can find a basis of $\Psi(\mathfrak{b})$ by taking some weighted Dynkin diagrams in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ as follows:

$\mathfrak{g}$	Example of basis of $\Psi(\mathfrak{b})$
$\mathfrak{f}_{4(4)}$	$A_2, \tilde{A}_2, B_3, F_4$
$\mathfrak{f}_{4(-20)}$	$\tilde{A}_2$

2.3.10 For the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type G_2

Let us consider the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type G_2, that is, $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{g}_{2,\mathbb{C}}$. Then we have

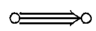
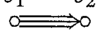
$$\mathrm{Map}(\Pi, \mathbb{R})^{\iota} = \mathrm{Map}(\Pi, \mathbb{R}) = $$



In [2, Section 6], we can find some examples of weighted Dynkin diagrams in $\mathfrak{j}_+ \cap \mathcal{H}^n$ as follows:

Symbol	Weighted Dynkin diagram in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$
$G_2(a_1)$	
G_2	

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. Then $\mathfrak{g}$ is a split real form of $\mathfrak{g}_{\mathbb{C}}$. In particular, the Satake diagram $S_{\mathfrak{g}}$ and $\Psi(\mathfrak{b})$ of $\mathfrak{g}$ are the following:

$\mathfrak{g}$	$S_{\mathfrak{g}}$	$\Psi(\mathfrak{b})$
$\mathfrak{g}_{2(2)}$		$\left\{ b_1 \quad b_2 \right\}$ 

Therefore, the weighted Dynkin diagrams $G_2(a_1)$ and G_2 in $\Psi(\mathfrak{j}_+ \cap \mathcal{H}^n)$ give a basis of $\Psi(\mathfrak{b})$.

Acknowledgements.

The author would like to express my gratitude to Professor Hiroyuki Ochiai, whose comments were invaluable to notice Remark 2.3.2

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Chapter 3

Homogeneous space with non virtually abelian discontinuous groups without any proper $SL(2,\mathbb{R})$-action

In Chapter 1, we proved that for a semisimple symmetric pair (G, H), if G/H admits a discontinuous group which is not virtually abelian, then G/H admits a proper $SL(2,\mathbb{R})$-action. In this chapter, we give an example of a non-symmetric reductive pair (G, H) such that G/H admits a discontinuous group which is not virtually abelian but does not admits proper $SL(2,\mathbb{R})$-actions.

3.1 Introduction and statement of main results.

For semisimple symmetric spaces G/H, we have proved in Theorem 1.1.3 in Chapter 1 that G/H admit a non virtually abelian properly discontinuous group if and only if G/H admit a proper action of certain subgroup L of G which is locally isomorphic to $SL(2,\mathbb{R})$.

In general, the letter condition implies the former condition because $SL(2,\mathbb{R})$ contains non virtually abelian discrete subgroups. We may ask if these two conditions are equivalent for reductive homogeneous spaces in general.

The purpose of this note is to show that this is not always true. In fact we give an example that admits a non virtually abelian discontinuous group but does not admit proper $SL(2,\mathbb{R})$-actions.

Our main theorem is here:

Theorem 3.1.1. *There exists a 3-dimensional split abelian subgroup H of $SL(5,\mathbb{R})$ satisfying the following:*

(*i*) *There exists a non virtually abelian discrete subgroup Γ of $SL(5,\mathbb{R})$ such that the Γ-action on the homogeneous space $SL(5,\mathbb{R})/H$ is properly discontinuous.*

(*ii*) *For any Lie group homomorphism $\Phi : SL(2,\mathbb{R}) \to SL(5,\mathbb{R})$, the action of $SL(2,\mathbb{R})$ on the homogeneous space $SL(5,\mathbb{R})/H$ via Φ is not proper.*

3.2 Criterion of proper actions

In this section, we recall results of T. Kobayashi [3] and Y. Benoist [1] in a form that we shall need.

Let G be a linear reductive Lie group, namely G is a real form of a connected complex reductive Lie group $G_{\mathbb{C}}$, and H a reductive subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. Let us take maximally split abelian subspaces $\mathfrak{a}$ of $\mathfrak{g}$. We denote the Weyl group of $(\mathfrak{g},\mathfrak{a})$ by W. Then any maximally split abelian subspace of $\mathfrak{h}$ can be transformed into a subspace of $\mathfrak{a}$ by inner-automorphisms. We denote this subspace by $\mathfrak{a}_{\mathfrak{h}}$, which is uniquely determined up to the Weyl group W. An analogous notation will be applied to another reductive subgroup L of G.

Let us denote the restricted root system of $(\mathfrak{g},\mathfrak{a})$ by Σ. We fix a simple system Π of Σ. We write $\mathfrak{a}_+$ and w_0 for the closure of the dominant Weyl chamber and the longest element in W corresponding to the simple system Π, respectively. Then the linear transform $x \mapsto -w_0 \cdot x$ on $\mathfrak{a}$ leaves the closed Weyl chamber $\mathfrak{a}_+$ invariant. Here, we put

$$\mathfrak{b}_+ := \{ A \in \mathfrak{a}_+ \mid -w_0 \cdot A = A \}.$$

Example 3.2.1. *For the cases where $G = SL(5,\mathbb{R})$, we can take $\mathfrak{a}$, $\mathfrak{a}_+$ and $\mathfrak{b}_+$ as*

$$\mathfrak{a} = \left\{ \operatorname{diag}(a_1, \ldots, a_5) \mid a_1, \ldots, a_5 \in \mathbb{R}, \sum_{i=1}^{5} a_i = 0 \right\},$$
$$\mathfrak{a}_+ = \{\operatorname{diag}(a_1, \ldots, a_5) \in \mathfrak{a} \mid a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5\},$$
$$\mathfrak{b}_+ = \{\operatorname{diag}(b_1, b_2, 0, -b_2, -b_1) \in \mathfrak{a}_+ \mid b_1, b_2 \in \mathbb{R}, b_1 \geq b_2\}.$$

The Weyl group W is isomorphic to the symmetric group $\mathfrak{S}_5$ and acts on $\mathfrak{a}$ as permutations of $a_1, \ldots, a_5$.

A continuous action of a locally compact group G on a locally compact topological space X is called proper if $\{ g \in G \mid gS \cap S \neq \emptyset \}$ is compact for any compact subset S of X. Furthermore, it is properly discontinuous if L is discrete.

The next fact is proved by T. Kobayashi in [3]:

Fact 3.2.2 (Theorem 4.1 in T. Kobayashi [3]). *Let H, L be reductive subgroups of a reductive Lie group G. Then the following two conditions on (G, H, L) are equivalent:*

(*i*) *The L-action on G/H is proper.*

(*ii*) $\mathfrak{a}_{\mathfrak{h}} \cap W \cdot \mathfrak{a}_{\mathfrak{l}} \neq \{0\}$.

Let ϕ be a Lie algebra homomorphism $\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$. We denote by $\mathcal{O}_\phi^{\mathrm{hyp}}$ the adjoint orbit through $\phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\mathfrak{g}$. Then it is known that $\mathcal{O}_\phi^{\mathrm{hyp}}$ and $\mathfrak{a}_+$ intersect in one point.

Definition 3.2.3. *For a Lie algebra homomorphism $\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$, We denote by A_ϕ the unique element in $\mathfrak{a}_+ \cap \mathcal{O}_\phi$.*

The next fact for the proper actions of $SL(2,\mathbb{R})$ follows from Fact 3.2.2:

Fact 3.2.4 (Corollary to Fact 3.2.2). *Let G be a reductive Lie group, H a reductive subgroup of G, and $\Phi : SL(2,\mathbb{R}) \to G$ a Lie group homomorphism. We denote by $\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}$ the differential of Φ, and take the element A_ϕ in $\mathfrak{a}_+$ described above. Then the following conditions on Φ are equivalent:*

(*i*) *The $SL(2,\mathbb{R})$-action on G/H via Φ is proper.*

(*ii*) *The element* A_ϕ *is not in* $W \cdot \mathfrak{a}_\mathfrak{h}$.

Let us denote by $\mathrm{Hom}(\mathfrak{sl}(2,\mathbb{R}),\mathfrak{g})$ the set of all Lie algebra homomorphisms from $\mathfrak{sl}(2,\mathbb{R})$ to $\mathfrak{g}$. By Fact 3.2.4, if the subset $\{ A_\phi \mid \phi \in \mathrm{Hom}(\mathfrak{sl}(2,\mathbb{R}),\mathfrak{g}) \}$ of $\mathfrak{a}_+$ is contained in $W \cdot \mathfrak{a}_\mathfrak{h}$, then for any Lie group homomorphism $\Phi : SL(2,\mathbb{R}) \to G$, the $SL(2,\mathbb{R})$-action on G/H via Φ is not proper.

Remark 3.2.5. *Let us define an equivalent relation on* $\mathrm{Hom}(\mathfrak{sl}(2,\mathbb{R}),\mathfrak{g})$ *by* $\phi \sim \phi'$ *if there exists* $l \in SL(2,\mathbb{R})$ *and* $g \in G$ *such that* $\phi' = \mathrm{Ad}_\mathfrak{g}(g) \circ \phi \circ \mathrm{Ad}_{\mathfrak{sl}(2,\mathbb{R})}(l)$ *(in fact, we can omit the adjoint action of* $SL(2,\mathbb{R})$ *on* $\mathfrak{sl}(2,\mathbb{R})$ *in this definition). Then we have a natural surjection from* $\mathrm{Hom}(\mathfrak{sl}(2,\mathbb{R}),\mathfrak{g})/\sim$ *to the set* $\{ A_\phi \mid \phi \in \mathrm{Hom}(\mathfrak{sl}(2,\mathbb{R}),\mathfrak{g}) \}$. *We remark that the map is not injective for general* $\mathfrak{g}$ *(See [2, Chapter 9] for more details). However, by Proposition 1.4.5 (iii) in Chapter 1, there exists a bijection below:*

$$\{ A_\phi \mid \phi \in \mathrm{Hom}(\mathfrak{sl}(2,\mathbb{R}),\mathfrak{g}) \}$$
$$\overset{1:1}{\longleftrightarrow} \{\textit{Complex nilpotent orbits } \mathcal{O} \textit{ in } \mathfrak{g}_\mathbb{C} \textit{ such that } \mathcal{O} \cap \mathfrak{g} \neq \emptyset\}$$

The next fact for the existence of properly discontinuous actions of a non virtually abelian discrete group is proved by Y. Benoist in [1]:

Fact 3.2.6 (Theorem 1.1 in Y. Benoist [1]). *Let* G *be a reductive Lie group,* H *a reductive subgroup of* G. *The following conditions on* (G,H) *are equivalent:*

(*i*) *There exists a non virtually abelian discrete subgroup* Γ *of* G *such that the* Γ*-action on* G/H *is properly discontinuous.*

(*ii*) $\mathfrak{b}_+ \not\subset W \cdot \mathfrak{a}_\mathfrak{h}$.

3.3 An example of $SL(5,\mathbb{R})$-spaces

Let $G = SL(5,\mathbb{R})$ and take $\mathfrak{a}$, $\mathfrak{a}_+$ and $\mathfrak{b}_+$ as in Example 3.2.1. We write $\langle\ ,\ \rangle$ for the inner product on $\mathfrak{a}$ induced by the Killing form on $\mathfrak{g}$. Namely,

$$\langle \mathrm{diag}(a_1,\dots,a_5), \mathrm{diag}(a'_1,\dots,a'_5)\rangle := \sum_{i=1}^{5} a_i \cdot a'_i \in \mathbb{R}$$

for any $\mathrm{diag}(a_1,\dots,a_5), \mathrm{diag}(a'_1,\dots,a'_5) \in \mathfrak{a}$.

Let us take

$$\mathfrak{h} := \{ \operatorname{diag}(a_1, \dots, a_5) \in \mathfrak{a} \mid 6a_1 + 6a_2 + a_3 - 4a_4 - 9a_5 = 0 \} \subset \mathfrak{a}.$$

That is, $\mathfrak{h}$ is the orthogonal complement subspace of $\mathfrak{a}$ for $\operatorname{diag}(6, 6, 1, -4, -9)$. Then

$$H := \exp \mathfrak{h} \subset SL(5, \mathbb{R})$$

is a split abelian subgroup of $SL(5, \mathbb{R})$ with $H \simeq \mathbb{R}^3$. In this case, we take $\mathfrak{a}_{\mathfrak{h}}$ as $\mathfrak{h}$ itself (see for the notation in Section 3.2). By using Fact 3.2.4 and Fact 3.2.6, to prove Theorem 3.1.1, we only need to show the following:

Claim A: $\mathfrak{b}_+ \not\subset W \cdot \mathfrak{a}_{\mathfrak{h}}$.

Claim B: For any Lie algebra homomorphism $\phi : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(5, \mathbb{R})$, the element A_ϕ (see Definition 3.2.3 for the notation) is in $W \cdot \mathfrak{a}_{\mathfrak{h}}$.

First, we shall show Claim A as follows:

Proof of Claim A. Let us take $r_1, r_2 \in \mathbb{R}$ with $r_1 > r_2 > 0$ such that r_1 and r_2 are linearly independent over $\mathbb{Z}$. For example, we can take $(r_1, r_2) = (\sqrt{2}, 1)$. We shall prove that the element $\operatorname{diag}(r_1, r_2, 0, -r_2, -r_1)$ of $\mathfrak{b}_+$ is not in $W \cdot \mathfrak{a}_{\mathfrak{h}}$. If

$$\langle \operatorname{diag}(r_1, r_2, 0, -r_2, -r_1), \operatorname{diag}(a_1, a_2, a_3, a_4, a_5) \rangle = 0$$

for some $a_1, \dots, a_5 \in \mathbb{Z}$, then we have $a_1 = a_5$ and $a_2 = a_4$ since r_1 and r_2 are linearly independent over $\mathbb{Z}$. Hence

$$\langle \operatorname{diag}(r_1, r_2, 0, -r_2, -r_1), \sigma \operatorname{diag}(6, 6, 1, -4, -9) \rangle \neq 0$$

for any $\sigma \in \mathfrak{S}_5 = W$. Therefore, we obtain that

$$\operatorname{diag}(r_1, r_2, 0, -r_2, -r_1) \notin W \cdot \mathfrak{a}_{\mathfrak{h}}.$$

□

To describe the proof of Claim A, we use the next fact for the set $\{ A_\phi \mid \phi \in \operatorname{Hom}(\mathfrak{sl}(2, \mathbb{R}), \mathfrak{sl}(5, \mathbb{R})) \}$, where $\operatorname{Hom}(\mathfrak{sl}(2, \mathbb{R}), \mathfrak{sl}(5, \mathbb{R}))$ is the set of all Lie algebra homomorphisms from $\mathfrak{sl}(2, \mathbb{R})$ to $\mathfrak{sl}(5, \mathbb{R})$:

Fact 3.3.1. *The set* $\{ A_\phi \mid \phi \in \operatorname{Hom}(\mathfrak{sl}(2, \mathbb{R}), \mathfrak{sl}(5, \mathbb{R})) \}$ *(see Definition* 3.2.3 *for the notation) is parametrised by partitions of* 5 *as follows:*

Partition of 5	A_ϕ
$[5]$	$\mathrm{diag}(4,2,0,-2,-4)$
$[4,1]$	$\mathrm{diag}(3,1,0,-1,-3)$
$[3,2]$	$\mathrm{diag}(2,1,0,-1,-2)$
$[3,1^2]$	$\mathrm{diag}(2,0,0,0,-2)$
$[2^2,1]$	$\mathrm{diag}(1,1,0,-1,-1)$
$[2,1^3]$	$\mathrm{diag}(1,0,0,0,-1)$
$[1^5]$	$\mathrm{diag}(0,0,0,0,0)$

Fact 3.3.1 can be obtained by combining [2, Chapter 3.6] with [2, Theorem 9.3.3 and remarks after Theorem 9.3.5].

Proof of Claim A. By Fact 3.3.1, we can observe that A_ϕ can be written by a scalar multiple of $\mathrm{diag}(3,1,0,-1,-3)$, $\mathrm{diag}(2,1,0,-1,-2)$, $\mathrm{diag}(1,1,0,-1,-1)$ or $\mathrm{diag}(1,0,0,0,-1)$ for any $\phi \in \mathrm{Hom}(\mathfrak{sl}(2,\mathbb{R}),\mathfrak{sl}(2,\mathbb{R}))$. Here, we compute the inner-products that

$$\begin{aligned}
\langle \mathrm{diag}(3,1,0,-1,-3), \mathrm{diag}(6,-9,-4,6,1)\rangle &= 0,\\
\langle \mathrm{diag}(2,1,0,-1,-2), \mathrm{diag}(6,-4,-9,6,1)\rangle &= 0,\\
\langle \mathrm{diag}(1,1,0,-1,-1), \mathrm{diag}(6,-9,6,-4,1)\rangle &= 0,\\
\langle \mathrm{diag}(1,0,0,0,-1), \mathrm{diag}(6,-9,-4,1,6)\rangle &= 0.
\end{aligned}$$

This means that these are in $W \cdot \mathfrak{a}_{\mathfrak{h}}$. This completes the proof. ☐

Acknowledgements.

To discover this example, the author owes match to Professor Yves Benoist for helpful discussions.

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Chapter 4

Products of two simple symmetric spaces with diagonal proper actions

Let G be a simple Lie group and take two symmetric pairs (G, H_1) and (G, H_2). We classify (G, H_1, H_2) such that the diagonal action of G on the product of the symmetric spaces $G/H_1 \times G/H_2$ is proper.

4.1 Introduction and statement of main results

Let G be a Lie group. We define a relation $\pitchfork$ on the power set of G as follows:

Definition 4.1.1 (T. Kobayashi [10]). *For two subset H_1 and H_2 of G, we write*

$$H_1 \pitchfork H_2 \text{ in } G$$

if for any compact subset S of G, the subset $H_1 \cap S H_2 S^{-1}$ is relatively compact in G.

Through this chapter, we only consider the cases where H_1 and H_2 are both closed subgroups of G.

Then one can obtain the next observation:

Observation 4.1.2 (cf. [8, Lemma 1] and [10, Observation 2.1.3]). *Let H_1 and H_2 be closed subgroups of G. Then the following conditions on (G, H_1, H_2) are equivalent:*

- $H_1 \pitchfork H_2$ *in* G.
- $H_2 \pitchfork H_1$ *in* G.
- *The diagonal G-action on $G/H_1 \times G/H_2$ is proper*
- *The H_1-action on G/H_2 is proper.*
- *The H_2-action on G/H_1 is proper.*

Remark 4.1.3. *By [7, Lemma 2.3], for any uniform lattice Γ of G, the diagonal G-action on $G/H_1 \times G/H_2$ is proper if and only if the diagonal Γ-action on $G/H_1 \times G/H_2$ is properly discontinuous.*

By the definiteion of $\pitchfork$, if at least one of G, L and H is compact, then $L \pitchfork H$ in G holds. In particular, in the cases where (G, H) is a Riemannian symmetric pair, i.e. H is a maximal compact subgroup of G, then any closed subgroup L of G acts on G/H properly. However, in the cases where all of G, L and H are non-compact, it is difficult to check whether $L \pitchfork H$ in G holds or not in general.

Let us conseder the cases where G is linear reductive and L, H are both reductive subgroups of G, Kobayashi [7, Theorem 4.1] gave a useful criterion to check whether $L \pitchfork H$ in G holds or not in this setting (see also Theorem 4.2.1 in Section 4.2). In particular, he gave a necessary and sufficient condition for Calabi–Markus phenomena ([7, Corollary 4.4]).

Throughout this chapter, we shall work on the following:

Setting 4.1.4. *G is a real form of a connected complex semisimple Lie group $G_{\mathbb{C}}$. σ_1 and σ_2 are involutive automorphisms on the Lie group G (possibly non-commutative from each other). H_1 and H_2 are open subgroups of $G^{\sigma_1} := \{ g \in G \mid \sigma_1(g) = g \}$ and $G^{\sigma_2} := \{ g \in G \mid \sigma_2(g) = g \}$, respectively.*

We write $\mathfrak{g}$, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ for the Lie algebras of G, H_1 and H_2, respectively. The differential action of σ_1 [resp. σ_2] on $\mathfrak{g}$ will be denoted by the same letter σ_1 [resp. σ_2]. Then for each $i = 1, 2$, we have $\mathfrak{h}_i = \{ X \in \mathfrak{g} \mid \sigma_i(X) = X \}$. Now we obtain two semisimple symmetric pairs $(\mathfrak{g}, \mathfrak{h}_1, \sigma_1)$ and $(\mathfrak{g}, \mathfrak{h}_2, \sigma_2)$. For each $i = 1, 2$, we put $\mathfrak{q}_i := \{ X \in \mathfrak{g} \mid \sigma_i(X) = -X \}$ Then $\mathfrak{g} = \mathfrak{h}_i \oplus \mathfrak{q}_i$ as a vector space. Let us define the c-dual $\mathfrak{g}_i^c$ of the symmetric pair $(\mathfrak{g}, \mathfrak{h}_i, \sigma_i)$ for each $i = 1, 2$ by

$$\mathfrak{g}_i^c := \mathfrak{h}_i \oplus \sqrt{-1}\mathfrak{q}_i. \tag{4.1.1}$$

We remark that $\mathfrak{g}_i^c$ is a real form of $\mathfrak{g}_{\mathbb{C}}$, and the complex conjugation corresponding to $\mathfrak{g}_i^c$ on $\mathfrak{g}_{\mathbb{C}}$ is the anti $\mathbb{C}$-linear extension of σ_i on $\mathfrak{g}_{\mathbb{C}}$.

Here is our first main result:

Theorem 4.1.5. *In Setting* 4.1.4*, the following conditions on* (G, H_1, H_2) *are equivalent:*

(*i*) $H_1 \pitchfork H_2$ *in* G.

(*ii*) *Without the zero-orbit, there does not exist a complex hyperbolic orbit* $\mathcal{O}$ *in* $\mathfrak{g}_{\mathbb{C}}$ *(defined below) satisfying that* $\mathfrak{g} \cap \mathcal{O} \neq \emptyset$, $\mathfrak{g}_1^c \cap \mathcal{O} \neq \emptyset$ *and* $\mathfrak{g}_2^c \cap \mathcal{O} \neq \emptyset$.

The definitions of *complex hyperbolic* orbits in $\mathfrak{g}_{\mathbb{C}}$ given here:

Definition 4.1.6. *Let* $\mathfrak{g}_{\mathbb{C}}$ *be a complex semisimple Lie algebra. An element* X *of* $\mathfrak{g}_{\mathbb{C}}$ *is said to be hyperbolic if the endomorphism* $\mathrm{ad}_{\mathfrak{g}_{\mathbb{C}}}(X) \in \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$ *is diagonalizable with only real eigenvalues. We say that an adjoint orbit* $\mathcal{O}$ *in* $\mathfrak{g}_{\mathbb{C}}$ *is complex hyperbolic if any (or some) element in* $\mathcal{O}$ *is hyperbolic.*

A proof of Theorem 4.1.5 will be given in Section 4.2.

Remark 4.1.7. *In Setting* 4.1.4*, we define the involutions* σ_1 *and* σ_2 *on* $\mathfrak{g}$ *as the differential of involutive automorphisms on* G*, respectively. However, Theorem* 4.1.5 *holds even if the involutions* σ_1 *and* σ_2 *can not be lifted to automorphisms on* G*. That is; Let* G *be a connected linear semisimple Lie group,* H_1 *and* H_2 *closed subgroups of* G*; Assume that there exists a (possibly non-commutative) pair* (σ_1, σ_2) *of involutions on* $\mathfrak{g} := \mathrm{Lie}(G)$ *such that* $\mathrm{Lie}(H_i) = \mathfrak{g}_i^{\sigma} := \{ X \in \mathfrak{g} \mid \sigma_i(X) = X \}$ *for each* $i = 1, 2$*; Then the two conditions on* (G, H_1, H_2) *described in Theorem* 4.1.5 *are equivalent.*

We remark that the second condition in Theorem 4.1.5 can be defined for a general semisimple Lie algebra $\mathfrak{g}$ and pair of two symmetric pairs $(\mathfrak{g}, \mathfrak{h}_1, \sigma_1)$ and $(\mathfrak{g}, \mathfrak{h}_2, \sigma_2)$. In Section 4.3, we will give a criterion for checking whether a given such $(\mathfrak{g}, \sigma_1, \sigma_2)$ satisfies the second condition of Theorem 4.1.5 or not.

Furthermore, we will give a classification of $(\mathfrak{g}, \sigma_1, \sigma_2)$ with $\mathfrak{g}$ satisfying that the second condition of Theorem 4.1.5 in Section 4.4.

Let G be a Lie group and H a closed subgroup of G. If there exists a discrete subgroup Γ of G such that the Γ-action on G/H is properly discontinuously and freely with compact quotient $\Gamma\backslash G/H$, then such the compact manifold $\Gamma\backslash G/H$ is called a compact Clifford–Klein form, and we say that G/H admits a compact Clifford–Klein form.

The existence problem for compact Clifford–Klein forms has been actively studied in the last two decades since [7]. The properness criterion of Kobayashi [7] yield necessary conditions on (G, H) for this. See also [2, 9, 8, 13, 14, 16, 17, 20, 23, 24] for some other methods and results on the existence problem of compact Clifford–Klein forms. The recent developments on this topic can be found in [11, 12, 15, 18].

Let (G, H_1, H_2) be a triple in Setting 4.1.4 with $H_1 \pitchfork H_2$ in G, and assume that the double coset space $H_1\backslash G/H_2$ is compact. Then both of G/H_1 and G/H_2 admit compact Clifford–Klein forms. For example, if we take a torsion-free cocompact lattice Γ of H_1, then $\Gamma\backslash G/H_2$ is a compact Clifford–Klein form (the existence of a cocompact lattice of a reductive Lie group was proved by Borel [4]).

By the classification in Section 4.4, we will obtain the next theorem:

Theorem 4.1.8. *In Setting 4.1.4, suppose that G is simple and all of G, H_1 and H_2 are non-compact. Then the following conditions on (G, H_1, H_2) are equivalent:*

(i) $H_1 \pitchfork H_2$ in G and the double coset space $H_1\backslash G/H_1$ is compact.

(ii) The triple $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$ is in Table 4.1 below, up to ordering of $\mathfrak{h}_1$ and $\mathfrak{h}_2$.

In particular, if (G, H_1, H_2) satisfies the equivalent conditions above, then both of G/H_1 and G/H_2 admit compact Clifford–Klein forms.

$\mathfrak{g}$	$\mathfrak{h}_1$	$\mathfrak{h}_2$
$\mathfrak{su}(2k-2, 2)$	$\mathfrak{u}(2k-2, 1)$	$\mathfrak{sp}(k-1, 1)$
$\mathfrak{so}(2k-2, 2)$	$\mathfrak{so}(2k-2, 1)$	$\mathfrak{u}(k-1, 1)$
$\mathfrak{so}(4, 4)$	$\mathfrak{so}(4, 3)$	$\mathfrak{so}(4, 1) \oplus \mathfrak{so}(3)$
$\mathfrak{so}(6, 2)$	$\mathfrak{so}(6, 1)$	$\mathfrak{su}(3, 1) \oplus \mathfrak{so}(2)$
$\mathfrak{so}(8, \mathbb{C})$	$\mathfrak{so}(7, \mathbb{C})$	$\mathfrak{so}(7, 1)$

Table 4.1: Classification of $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$ satisfying (i) in Theorem 4.1.8

We remark that any example of symmetric space G/H admitting a Clifford–Klein form obtained by Table 4.1 was already known by Kobayashi–Yoshino [14, Table 2.2 in Section 2.2].

4.2 Proof of Theorem 4.1.5

In this section, we give a proof of Theorem 4.1.5.

First, as a corollary to the results of [7], we obtain the next theorem:

Theorem 4.2.1 (Corollary to T. Kobayashi [7, Theorem 4.1]). *Let G be a real form of a connected complex semisimple Lie group $G_{\mathbb{C}}$, and H, L are both reductive subgroups of G. We write $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ for the Lie algebras of G, H and L, respectively. Then the following conditions on (G, H, L) are equivalent:*

(i) *$L \pitchfork H$ in G.*

(ii) *Without the zero-orbit, there does not exist a real hyperbolic orbit $\mathcal{O}^0$ in $\mathfrak{g}$ (defined below) such that $\mathcal{O}^0$ meets both $\mathfrak{h}$ and $\mathfrak{l}$, respectively.*

The definitions of *real hyperbolic* orbits in a real Lie algebra $\mathfrak{g}$ given here:

Definition 4.2.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$. An element X of $\mathfrak{g}$ is said to be hyperbolic if the endomorphism $\mathrm{ad}_{\mathfrak{g}}(X) \in \mathrm{End}(\mathfrak{g})$ is diagonalizable with only real eigenvalues. We say that an adjoint orbit $\mathcal{O}^0$ in $\mathfrak{g}$ is real hyperbolic if any (or some) element in $\mathcal{O}^0$ is hyperbolic.*

We remark that an element X of $\mathfrak{g}$ is hyperbolic in $\mathfrak{g}$ if and only if X is hyperbolic in $\mathfrak{g}_{\mathbb{C}}$ in the sense of Definition 4.1.6 in Section 4.1.

A proof of Theorem 4.2.1 can be found in Section 1.5.1 as the proof of Theorem 1.4.1 in Section 1.4.1.

Remark 4.2.3. *In Chapter 1, we assume that G is connected. However, the proof of Theorem 1.4.1 given in Section 1.5.1 work even if G is a general real form of a connected complex semisimple Lie group $G_{\mathbb{C}}$.*

Let us fix a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, its real form $\mathfrak{g}$, and a connected Lie group G_0 with $\mathrm{Lie}\, G_0 = \mathfrak{g}$.

We will use the following proposition for hyperbolic orbits in $\mathfrak{g}$:

Lemma 4.2.4 (Proposition 1.4.5 (i) in Chapter 1). *For a complex hyperbolic orbit $\mathcal{O}$ in $\mathfrak{g}_{\mathbb{C}}$ if $\mathcal{O}$ meets $\mathfrak{g}$, then $\mathcal{O} \cap \mathfrak{g}$ becomes a single adjoint orbit in $\mathfrak{g}$, i.e. G_0 acts on $\mathcal{O} \cap \mathfrak{g}$ transitively.*

Remark 4.2.5. *In Lemma 4.2.4, we can not replace "hyperbolic" to "nilpotent". That is, for a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and its real form $\mathfrak{g}$, the intersection $\mathfrak{g} \cap \mathcal{O}^{nilp}$ of $\mathfrak{g}$ and a complex nilpotent (adjoint) orbit in $\mathfrak{g}_{\mathbb{C}}$ split into finitely many real nilpotent (adjoint) orbits in $\mathfrak{g}$, in general.*

Let us fix a symmetric pair $(\mathfrak{g}, \mathfrak{h}, \sigma)$, and denote by $\mathfrak{g}^c$ for the c-dual of the symmetric pair $(\mathfrak{g}, \mathfrak{h}, \sigma)$ (see (4.1.1) in Section 4.1 for the definition of c-duals of semisimple symmetric pairs).

We also use the next proposition for a relation between complex hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ and the c-dual of a semisimple symmetric pair:

Lemma 4.2.6. *For a complex hyperbolic orbit $\mathcal{O}$ in $\mathfrak{g}_{\mathbb{C}}$, the following two conditions are equivalent:*

(i) $\mathcal{O}$ meets both of $\mathfrak{g}$ and $\mathfrak{g}^c$.

(ii) $\mathcal{O}$ meets $\mathfrak{g} \cap \mathfrak{g}^c = \mathfrak{h}$.

We are ready to prove Theorem 4.1.5:

Proof of Theorem 4.1.5. By Theorem 4.2.1, we only need to show that the following two conditions on (G, H_1, H_2) are equivalent:

- Without the zero-orbit, there does not exist a real hyperbolic orbit $\mathcal{O}^0$ in $\mathfrak{g}$ such that $\mathcal{O}^0$ meets both of $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively.
- Without the zero-orbit, there does not exists a complex hyperbolic orbit $\mathcal{O}$ in $\mathfrak{g}_{\mathbb{C}}$ such that $\mathcal{O}$ meets all of $\mathfrak{g}$, $\mathfrak{g}_1^c$ and $\mathfrak{g}_2^c$, respectively.

By Lemma 4.2.4, for a complex hyperbolic orbit $\mathcal{O}$ in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$, the intersection $\mathcal{O} \cap \mathfrak{g}$ becomes a real hyperbolic orbit in $\mathfrak{g}$. Thus, we obtain a bijection between the set of complex hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{g}$ and the set of real hyperbolic orbits in $\mathfrak{g}$. Furthermore, by Lemma 4.2.6, in the bijection given above, a complex hyperbolic orbit $\mathcal{O}$ in $\mathfrak{g}_{\mathbb{C}}$ meets $\mathfrak{g}_1^c$ [resp. $\mathfrak{g}_2^c$] if and only if the corresponding real hyperbolic orbit $\mathcal{O} \cap \mathfrak{g}$ in $\mathfrak{g}$ meets $\mathfrak{h}_1$ [resp. $\mathfrak{h}_2$]. Therefore, we obtain a bijection between

$$\{\text{Non-zero complex hyperbolic orbits } \mathcal{O} \text{ in } \mathfrak{g}_{\mathbb{C}} \text{ meeting all of } \mathfrak{g},\ \mathfrak{g}_1^c \text{ and } \mathfrak{g}_2^c\}$$

and

$$\{\text{Non-zero real hyperbolic orbits } \mathcal{O}^0 \text{ in } \mathfrak{g} \text{ meeting both of } \mathfrak{h}_1 \text{ and } \mathfrak{h}_2\}.$$

This completes the proof. □

4.3 Algorithm for classification

Recall that the condition (ii) in Theorem 4.1.5 can be defined for a triple $(\mathfrak{g}, \sigma_1, \sigma_2)$, where $\mathfrak{g}$ is a semisimple Lie algebra and σ_1, σ_2 are both involutive automorphisms on $\mathfrak{g}$. We also remark that σ_1 and σ_2 may be non-commutative from each other. In this section, for such a triple $(\mathfrak{g}, \sigma_1, \sigma_2)$, we give a criterion to check whether the condition (ii) in Theorem 4.1.5 holds or not (see Corollary 4.3.6).

4.3.1 Complex hyperbolic orbits and weighted Dynkin diagrams

Let $\mathfrak{g}_\mathbb{C}$ be a complex semisimple Lie algebra. In this section, we recall some well-known facts for weighted Dynkin diagrams of complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$.

Fix a Cartan subalgebra $\mathfrak{j}_\mathbb{C}$ of $\mathfrak{g}_\mathbb{C}$. Let us denote by Δ the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, and define the real form $\mathfrak{j}$ of $\mathfrak{j}_\mathbb{C}$ by

$$\mathfrak{j} := \{A \in \mathfrak{j}_\mathbb{C} \mid \alpha(A) \in \mathbb{R} \text{ for any } \alpha \in \Delta\}.$$

Then Δ can be regarded as a subset of the dual space $\mathfrak{j}^*$ of $\mathfrak{j}$. We fix a positive system Δ^+ of the root system Δ. Then a closed Weyl chamber

$$\mathfrak{j}_+ := \{A \in \mathfrak{j} \mid \alpha(A) \geq 0 \text{ for any } \alpha \in \Delta^+\}$$

is a fundamental domain of $\mathfrak{j}$ for the action of the Weyl group W of Δ.

In this setting, the next fact for complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ is well known:

Fact 4.3.1. *Any complex hyperbolic orbit $\mathcal{O}$ in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{j}$, and the intersection $\mathcal{O} \cap \mathfrak{j}$ is a single W-orbit in $\mathfrak{j}$. In particular, we have one-to-one correspondences below:*

$$\mathcal{H}/G_\mathbb{C} \overset{1:1}{\longleftrightarrow} \mathfrak{j}/W \overset{1:1}{\longleftrightarrow} \mathfrak{j}_+,$$

where $\mathcal{H}/G_\mathbb{C}$ is the set of complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{j}/W$ the set of W-orbits in $\mathfrak{j}$.

Let Π denote the fundamental system of Δ^+. Then, for any $A \in \mathfrak{j}$, we can define a map

$$\Psi_A : \Pi \to \mathbb{R}, \ \alpha \mapsto \alpha(A).$$

We call Ψ_A the weighted Dynkin diagram corresponding to $A \in \mathfrak{j}$, and $\alpha(A)$ the weight on a node $\alpha \in \Pi$ of the weighted Dynkin diagram. Since Π is a basis of $\mathfrak{j}^*$, the correspondence

$$\Psi : \mathfrak{j} \to \mathrm{Map}(\Pi, \mathbb{R}), \ A \mapsto \Psi_A \tag{4.3.1}$$

is a linear isomorphism between real vector spaces. In particular, Ψ is bijective. Furthermore,

$$\Psi|_{\mathfrak{j}_+} : \mathfrak{j}_+ \to \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}), \ A \mapsto \Psi_A$$

is also bijective. We say that a weighted Dynkin diagram is trivial if all weights are zero. Namely, the trivial diagram corresponds to the zero of $\mathfrak{j}$ by Ψ.

The weighted Dynkin diagram of a complex hyperbolic orbit $\mathcal{O}$ in $\mathfrak{g}_\mathbb{C}$ is defined as the weighted Dynkin diagram corresponding to the unique element $A_\mathcal{O}$ in $\mathcal{O} \cap \mathfrak{j}_+$ (see Fact 4.3.1). Combining Fact 4.3.1 with the bijection $\Psi|_{\mathfrak{j}_+}$, the map

$$\mathcal{H}/G_\mathbb{C} \to \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}), \quad \mathcal{O} \mapsto \Psi_{A_\mathcal{O}}$$

is also bijective. This gives a classification of complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$.

Let us take other Cartan subalgebra $\mathfrak{j}'_\mathbb{C}$ of $\mathfrak{g}_\mathbb{C}$ with a positive system $(\Delta')^+$ of the root system Δ' for $(\mathfrak{g}_\mathbb{C}, \mathfrak{j}'_\mathbb{C})$. Then we can also define weighted Dynkin diagrams as an element in $\mathrm{Map}(\Pi', \mathbb{R})$, where Π' is the simple system for $(\Delta')^+$. It is well known that there uniquely exists a bijection $\varphi : \Pi' \simeq \Pi$ induced by an inner-automorphism $g_\mathbb{C}$ on $\mathfrak{g}_\mathbb{C}$ such that $g_\mathbb{C}(\mathfrak{j}'_\mathbb{C}) = \mathfrak{j}_\mathbb{C}$. We will identify $\mathrm{Map}(\Pi, \mathbb{R})$ with $\mathrm{Map}(\Pi', \mathbb{R})$ by φ

Remark 4.3.2. *In Dynkin–Kostant classification, which is the classification of complex adjoint nilpotent orbits in $\mathfrak{g}_\mathbb{C}$, we use some weighted Dynkin diagrams such that any weight is given by* 0, 1 *or* 2 *(see Dynkin [5]). However, in this chapter, we have to consider weighted Dynkin diagrams with any weights to parameterize complex hyperbolic orbits in $\mathfrak{g}_\mathbb{C}$.*

4.3.2 Real forms and Satake diagrams

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra and $\mathfrak{g}$ be a real form of $\mathfrak{g}_{\mathbb{C}}$. In this subsection, by using the Satake diagram of $\mathfrak{g}$, we give an algorithm to classify complex hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting the real form $\mathfrak{g}$.

First, we recall briefly the definition of the Satake diagram of the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ (see [1, 22] for more details).

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$, and use the following convention: We say that a Cartan subalgebra $\mathfrak{j}_{\mathfrak{g}}$ of $\mathfrak{g}$ is split if $\mathfrak{a} := \mathfrak{j}_{\mathfrak{g}} \cap \mathfrak{p}$ is a maximal abelian subspace of $\mathfrak{p}$ (i.e. $\mathfrak{a}$ is a maximally split abelian subspace of $\mathfrak{g}$). Note that such $\mathfrak{j}_{\mathfrak{g}}$ is unique up to the adjoint action of K, where K is analytic subgroup corresponding to $\mathfrak{k}$ of the inner-automorphism group on $\mathfrak{g}$.

Take a split Cartan subalgebra $\mathfrak{j}_{\mathfrak{g}}$ of $\mathfrak{g}$ defined above. Then $\mathfrak{j}_{\mathfrak{g}}$ can be written as $\mathfrak{j}_{\mathfrak{g}} = \mathfrak{t} + \mathfrak{a}$ for a maximal abelian subspace $\mathfrak{t}$ of the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Let us denote by $\mathfrak{j}_{\mathbb{C}} := \mathfrak{j}_{\mathfrak{g}} + \sqrt{-1}\mathfrak{j}_{\mathfrak{g}}$ and $\mathfrak{j} := \sqrt{-1}\mathfrak{t} + \mathfrak{a}$. Then $\mathfrak{j}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{j}$ is a real form of it, with

$$\mathfrak{j} = \{A \in \mathfrak{j}_{\mathbb{C}} \mid \alpha(A) \in \mathbb{R} \text{ for any } \alpha \in \Delta\},$$

where Δ is the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. We put

$$\Sigma := \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Delta\} \setminus \{0\} \subset \mathfrak{a}^*$$

to the restricted root system of $(\mathfrak{g}, \mathfrak{a})$. Then we can take a positive system Δ^+ of Δ such that the subset

$$\Sigma^+ := \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Delta^+\} \setminus \{0\}.$$

of Σ becomes a positive system. In fact, if we take an ordering on $\mathfrak{a}$ and extend it to $\mathfrak{j}$, then the corresponding positive system Δ^+ satisfies the condition above.

Let us denote by Π the fundamental system of Δ^+. Then

$$\overline{\Pi} := \{\, \alpha|_{\mathfrak{a}} \mid \alpha \in \Pi \,\} \setminus \{0\}$$

is the fundamental system of Σ^+. We write Π_0 for the set of all simple roots in Π whose restriction to $\mathfrak{a}$ is zero.

The Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ consists of the following three data: the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ with nodes Π; black nodes Π_0 in S; and arrows joining $\alpha \in \Pi \setminus \Pi_0$ and $\beta \in \Pi \setminus \Pi_0$ in S whose restrictions to $\mathfrak{a}$ are the same.

Second, we give the definition of weighted Dynkin diagrams *matching* the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ as follows:

Definition 4.3.3 (Definition 1.7.3 in Chapter 1). *Let $\Psi_A \in \mathrm{Map}(\Pi, \mathbb{R})$ be an weighted Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ (see Section 4.3.1 for the notation) and $S_{\mathfrak{g}}$ the Satake diagram of $\mathfrak{g}$ with nodes Π. We say that Ψ_A matches $S_{\mathfrak{g}}$ if all the weights on black nodes in Π_0 are zero and any pair of nodes joined by an arrow have the same weights.*

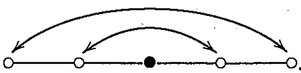
Then the following theorem holds:

Theorem 4.3.4 (Theorem 1.7.4 in Chapter 1). *The weighted Dynkin diagram of a complex hyperbolic orbit $\mathcal{O}$ in $\mathfrak{g}_{\mathbb{C}}$ matches the Satake diagram $S_{\mathfrak{g}}$ of the real form $\mathfrak{g}$ if and only if $\mathcal{O}$ meets $\mathfrak{g}$. In particular, we obtain a bijection between the set of complex hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting the real form $\mathfrak{g}$ and the set of weighted Dynkin diagrams in $\mathrm{Map}(\Pi, \mathbb{R}_{\geq 0})$ matching the Satake diagram $S_{\mathfrak{g}}$.*

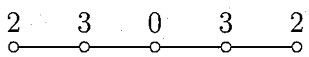
Theorem 4.3.4 gives a classification of complex hyperbolic orbits in $\mathfrak{g}_{\mathbb{C}}$ meeting a given real form $\mathfrak{g}$.

We give an example of Theorem 4.3.4 as follows:

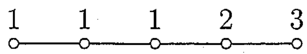
Example 4.3.5. *Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(6, \mathbb{C})$ and take a real form $\mathfrak{g} = \mathfrak{su}(4, 2)$ of $\mathfrak{sl}(6, \mathbb{C})$. By [1, Table in Section 5.11], the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g} = \mathfrak{su}(4, 2)$ is given as follows:*

$S_{\mathfrak{g}} =$ .

Let us consider two examples of weighted Dynkin diagrams of $\mathfrak{sl}(6, \mathbb{C})$ as

$\Psi_{A_1} :=$ 

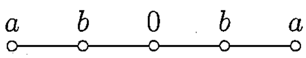
,

$\Psi_{A_2} :=$ 

.

Then one can observe that Ψ_{A_1} matches $S_{\mathfrak{g}}$ but Ψ_{A_2} does not matches $S_{\mathfrak{g}}$ (see Definition 4.3.3 for the definition of "match"). Therefore, by Theorem 4.3.4, the complex hyperbolic orbit $\mathcal{O}_1$ corresponding to Ψ_{A_1} meets $\mathfrak{g}$ but the complex hyperbolic orbit $\mathcal{O}_2$ corresponding to Ψ_{A_2} does not meet $\mathfrak{g}$.

We also obtain a bijection between the set of complex hyperbolic orbits in $\mathfrak{sl}(6, \mathbb{C})$ meeting $\mathfrak{su}(2, 4)$ and

$\Big\{$  $\mid a, b \in \mathbb{R}_{\geq 0} \Big\}$.

4.3.3 Criterion

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra and $\mathfrak{g}$ a real form of $\mathfrak{g}_{\mathbb{C}}$. We take a (possibly non-commutative) pair of involutive automorphisms σ_1 and σ_2 on $\mathfrak{g}$. Recall that the c-dual $\mathfrak{g}_i^c$ of $(\mathfrak{g}, \sigma_i)$ is a real form of $\mathfrak{g}_{\mathbb{C}}$ for $i = 1, 2$.

Then the next corollary follows from Theorem 4.3.4:

Corollary 4.3.6. *In the setting above, the following two conditions on* $(\mathfrak{g}, \sigma_1, \sigma_2)$ *are equivalent:*

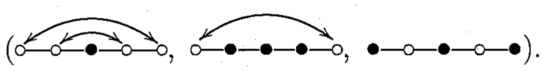
(*ii*) *Without the zero-orbit, there does not exist a complex hyperbolic adjoint orbit* $\mathcal{O}$ *in* $\mathfrak{g}_{\mathbb{C}}$ *satisfying that* $\mathfrak{g} \cap \mathcal{O} \neq \emptyset$, $\mathfrak{g}_1^c \cap \mathcal{O} \neq \emptyset$ *and* $\mathfrak{g}_2^c \cap \mathcal{O} \neq \emptyset$.

(*iii*) *There does not exist a non-zero weighted Dynkin diagram* Ψ_A *matching all of the Satake diagrams of* $\mathfrak{g}$, $\mathfrak{g}_1^c$ *and* $\mathfrak{g}_2^c$.

We give an example of Corollary 4.3.6 as follows:

Example 4.3.7. *Let* $\mathfrak{g} = \mathfrak{su}(4,2)$ *and consider the case where* σ_1 *and* σ_2 *defines* $\mathfrak{h}_1 = \mathfrak{su}(4,1) \oplus \mathfrak{so}(2)$, $\mathfrak{h}_2 = \mathfrak{sp}(2,1)$. *Then the complexification of* $\mathfrak{sl}(4,2)$ *is* $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(6,\mathbb{C})$. *The c-dual of the symmetric pair* $(\mathfrak{g}, \mathfrak{h}_1, \sigma_1)$ [*resp.* $(\mathfrak{g}, \mathfrak{h}_2, \sigma_2)$] *is* $\mathfrak{g}_1^c = \mathfrak{su}(5,1)$ [*resp.* $\mathfrak{g}_2^c = \mathfrak{su}^*(6)$]. *The Satake diagrams of the three real forms* $\mathfrak{g}$, $\mathfrak{g}_1^c$ *and* $\mathfrak{g}_2^c$ *of* $\mathfrak{g}_{\mathbb{C}}$ *are the following:*

$(S_{\mathfrak{g}}, S_{\mathfrak{g}_1^c}, S_{\mathfrak{g}_2^c})$:



Then one can easily check that there does not exists a non-zero weighted Dynkin diagram on $\mathfrak{g}_{\mathbb{C}}$ *matching all of three Satake diagrams above. Therefore, by Corollary* 4.3.6, *in this triple* $(\mathfrak{g}, \sigma_1, \sigma_2)$ *satisfies the condition* (ii) *in Theorem* 4.1.5.

4.4 Classification

Throughout this section, we consider the following setting:

Setting 4.4.1. $\mathfrak{g}$ *is a simple Lie algebra over* $\mathbb{R}$. σ_1 *and* σ_2 *are (possibly non-commutative) involutive automorphisms on* $\mathfrak{g}$.

We denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$, and write $\mathfrak{g}_i^c$ for the c-dual of the symmetric pair $(\mathfrak{g}, \sigma_i)$ for $i = 1, 2$ (see (4.1.1) in Section 4.1 for the definition of the c-dual). Then all of $\mathfrak{g}$, $\mathfrak{g}_1^c$ and $\mathfrak{g}_2^c$ are real forms of $\mathfrak{g}_\mathbb{C}$. We also remark that $\mathfrak{g}_i^c$ is a compact real form if and only if $(\mathfrak{g}, \sigma)$ is a Riemannian symmetric pair, i.e. σ is a Cartan involution on $\mathfrak{g}$.

In this section, by using Corollary 4.3.6, we give a classification of triples $(\mathfrak{g}, \sigma_1, \sigma_2)$ satisfying the condition (ii) in Theorem 4.1.5. In particular, we give a classification of (G, H_1, H_2) in Setting 4.1.4 with simple G and $H_1 \pitchfork H_2$ in G, locally.

To give our classification of such $(\mathfrak{g}, \sigma_1, \sigma_2)$, we use the classification of Satake diagrams in [1, Table in Section 5.11], the classification of symmetric pair in [3] and the list of the c-dual of each symmetric pair in [21] and [6]. We remark that in [21] and [6], the c-dual of a symmetric pair $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is called the associate-dual $\mathfrak{g}^{ad}$.

First, we observe that for a compact real form $\mathfrak{u}$ of $\mathfrak{g}_\mathbb{C}$, any element X of $\mathfrak{u}$ is not hyperbolic since any eigen-value of $\mathrm{ad}_\mathfrak{u} X \in \mathrm{End}(\mathfrak{u})$ is pure-imaginary. Therefore, the next observation holds:

Observation 4.4.2. *If at least one of $\mathfrak{g}$, $\mathfrak{g}_1^c$ and $\mathfrak{g}_2^c$ is a compact real form of $\mathfrak{g}_\mathbb{C}$, then $(\mathfrak{g}, \sigma_1, \sigma_2)$ satisfies the condition* (ii) *in Theorem* 4.1.5.

In the rest of this section, we consider the cases where all of $\mathfrak{g}$, $\mathfrak{g}_1$ and $\mathfrak{g}_2^c$ are non-compact real form of $\mathfrak{g}_\mathbb{C}$. That is, our setting in the rest of this section is the following:

Setting 4.4.3. *In Setting* 4.4, *we also assume that $\mathfrak{g}$ is non-compact and neither of σ_1 and σ_2 is not a Cartan involution on $\mathfrak{g}$.*

We will give a classification of $(\mathfrak{g}, \sigma_1, \sigma_2)$ in Setting 4.4.3 satisfying the condition (ii) in Theorem 4.1.5 dividing into Proposition 4.4.4, Proposition 4.4.10, Proposition 4.4.12 and Proposition 4.4.13.

4.4.1 In the cases where $\mathfrak{g}_\mathbb{C}$ is not of type A_{2k+1} nor D_l

In this subsection, we give a proof of the next proposition:

Proposition 4.4.4. *In Setting* 4.4.3, *the following hold:*

- *Let us consider the cases where $\mathfrak{g}$ has no complex structure and $\mathfrak{g}_\mathbb{C}$ is not of type A_{2k+1} nor D_l for $k \geq 1$, $l \geq 4$. Then $(\mathfrak{g}, \sigma_1, \sigma_1)$ does not satisfies the condition* (ii) *in Theorem* 4.1.5.

- *Let us consider the cases where* $\mathfrak{g}$ *admits a complex structure, i.e.* $\mathfrak{g}$ *is a complex simple Lie algebra, but not of not of type* A_{2k+1} *nor* D_l *for* $k \geq 1$, $l \geq 4$. *In this case,* $\mathfrak{g}_{\mathbb{C}}$ *is isomorphic to* $\mathfrak{g} \oplus \mathfrak{g}$ *as a complex Lie algebra. Then* $(\mathfrak{g}, \sigma_1, \sigma_1)$ *does not satisfies the condition* (ii) *in Theorem* 4.1.5.

To prove Proposition 4.4.4, we show the next two lemmas:

Lemma 4.4.5. *Let* $\mathfrak{g}_{\mathbb{C}}$ *be a complex simple Lie algebra but not of type* A_{2k+1} *nor* D_l *for* $k \geq 1$, $l \geq 4$. *Then there exists a non-zero weighted Dynkin diagram* Ψ_A *of* $\mathfrak{g}_{\mathbb{C}}$ *such that for any non-compact real form* $\mathfrak{g}$, *the weighted Dynkin diagram* Ψ_A *matches the Satake diagram of* $\mathfrak{g}$.

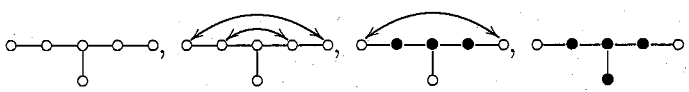
Remark 4.4.6. *Let* $\mathfrak{g}_{\mathbb{C}}$ *be a complex simple Lie algebra but not of type* A_{2k+1} *nor* D_l *for* $k \geq 1$, $l \geq 4$. *By combining Theorem* 4.3.4 *with Lemma* 4.4.5, *the complex hyperbolic orbit* $\mathcal{O}$ *corresponding to* Ψ_A, *defined in Section* 4.3.1, *meets any non-compact real form of* $\mathfrak{g}_{\mathbb{C}}$.

Lemma 4.4.7. *Let* $\mathfrak{g}$ *be a complex simple Lie algebra but not of type* A_{2k+1} *nor* D_l *for* $k \geq 1$, $l \geq 4$. *We denote by* $\mathfrak{g}_{\mathbb{C}}$ *the complexification of* $\mathfrak{g}$. *Then there exists a non-zero weighted Dynkin diagram* Ψ_A *of* $\mathfrak{g}_{\mathbb{C}}$ *matching* $S_{\mathfrak{g}}$ *satisfying that: For any involutive automorphism* σ *on* $\mathfrak{g}$ *without Cartan involutions on* $\mathfrak{g}$, Ψ_A *also matches the Satake diagram* $S_{\mathfrak{g}^c}$ *of* $\mathfrak{g}^c$, *where* $\mathfrak{g}^c$ *is the c-dual of* $(\mathfrak{g}, \sigma)$ *(see* (4.1.1) *in Section* 4.1 *for the definition of the c-dual).*

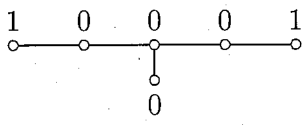
By combining Corollary 4.3.6 with Lemma 4.4.5 and Lemma 4.4.7, we obtain Proposition 4.4.4.

As an example, we give a proof of Lemma 4.4.5 in the cases where $\mathfrak{g}_{\mathbb{C}}$ is simple and of type E_6 as follows:

Example 4.4.8. *Let* $\mathfrak{g}_{\mathbb{C}}$ *be a complex simple Lie algebra of type* E_6. *Then by* [*1, Table in Section* 5.11], *the list of Satake diagrams of non-compact real forms of* $\mathfrak{g}_{\mathbb{C}}$ *are the following:*



Here we take the non-zero weighted Dynkin diagram Ψ_A *as*

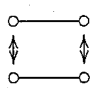
$\Psi_A =$  .

Then one can easily observe that Ψ_A *matches all of Satake diagrams given above.*

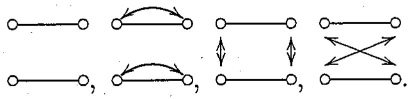
For other types in Lemma 4.4.5, by using the list of Satake diagrams in [1, Table in Section 5.11], we can prove the claim of Lemma 4.4.5.

We also give a proof of Lemma 4.4.7 in the cases there $\mathfrak{g}$ is a complex simple Lie algebra of type A_2 as follows:

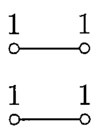
Example 4.4.9. *Let* $\mathfrak{g}$ *be a complex simple Lie algebra of type* A_2 *and put* $\mathfrak{g}_{\mathbb{C}}$ *to the complexification of* $\mathfrak{g}$. *Then the Satake diagram* $S_{\mathfrak{g}}$ *of* $\mathfrak{g}$ *is the following:*

$S_{\mathfrak{g}} =$ 

Let σ *be an involutive automorphism on* $\mathfrak{g}$ *and* σ *is not a Cartan involution on* $\mathfrak{g}$. *Then by using [6, Theorem 7.16] and [1, Table in Section 5.11], one can observe that the Satake diagram* $S_{\mathfrak{g}^c}$ *of the c-dual* $\mathfrak{g}^c$ *is one of the following:*



Here we take the non-zero weighted Dynkin diagram Ψ_A *as*

$\Psi_A =$  .

Then one can easily observe that Ψ_A *matches all of Satake diagrams given above.*

For other types in Lemma 4.4.7, we can prove the claim of Lemma 4.4.5 by using [6, Theorem 7.16]. We omit the details here.

4.4.2 In the cases where $\mathfrak{g}_{\mathbb{C}}$ is of type A_{2k+1} or D_l

Let us consider $(\mathfrak{g}, \sigma_1, \sigma_2)$ in Setting 4.4.3 in the cases where $\mathfrak{g}_{\mathbb{C}}$ or $\mathfrak{g}$ is complex simple and of type A_{2k+1} or D_l for $k \geq 1$, $l \geq 4$. Since type D_4 cases are complicated, we first assume that $l \geq 5$. Then by combining Corollary 4.3.6 with lists in [1, Table in Section 5.11], [21, Table I in Section 1] and [6, Theorem 7.16] we obtain the following classification result:

Proposition 4.4.10. *Let us consider* $(\mathfrak{g}, \sigma_1, \sigma_2)$ *in Setting* 4.4.3. *Assume that* $\mathfrak{g}_{\mathbb{C}}$ *or* $\mathfrak{g}$ *is complex simple and of type* A_{2k+1} *or* D_l *for* $k \geq 1$, $l \geq 5$. *Then* $(\mathfrak{g}, \sigma_1, \sigma_2)$ *satisfies the condition* (ii) *in Theorem* 4.1.5 *if and only if* $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$ *is in the Table* 4.2 *below, up to ordering of* $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$, *where* $\mathfrak{h}_i := \{X \in \mathfrak{g} \mid \sigma_i(X) = X\}$ *for* $i = 1, 2$.

Remark 4.4.11. *In Setting* 4.1.4, *for each* $i = 1, 2$, *we can take a Cartan subalgebra* θ_u *on* $\mathfrak{g}$ *such that* $\theta\sigma_i = \sigma_i\theta$ *(proved by Matsuki [19]). Let us denote by*

$$\mathfrak{g} = \mathfrak{k}_i \oplus \mathfrak{p}_i, \quad \mathfrak{h}_i = \mathfrak{k}(\mathfrak{h}_i) \oplus \mathfrak{p}(\mathfrak{h}_i),$$

the Cartan decompositions of $\mathfrak{g}$, $\mathfrak{h}_i$ *with respect to* θ_i. *Here, we put*

$$\mathrm{d}(\mathfrak{g}) := \dim \mathfrak{p}_1 = \dim \mathfrak{p}_2$$

Let us assume that (G, H_1, H_2) *satisfies the equivalent conditions in Theorem* 4.1.5 *and*

$$\dim \mathfrak{p}(\mathfrak{h}_1) + \dim \mathfrak{p}(\mathfrak{h}_2) = \mathrm{d}(\mathfrak{g}). \tag{4.4.1}$$

Then the double coset space $H_1 \backslash G / H_2$ *becomes a compact Hausdorff space (proved by Kobayashi [7]). In Table* 4.2 *above and Table* 4.5 *below, we check that the cocompact condition (the equation* (4.4.1)*) holds or not for each* $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$, *(See also Theorem* 4.1.8 *in Section* 4.1.)

$\mathfrak{g}$	$\mathfrak{h}_1$	$\mathfrak{h}_2$	cocompact
$\mathfrak{sl}(2k, \mathbb{R})$	$\mathfrak{so}(2k-1, 1)$	$\mathfrak{sl}(k, \mathbb{C})$	No
$\mathfrak{su}(2k-2j, 2j)$ $(1 \leq j \leq k-1)$	$\mathfrak{su}(2k-2j, 1) \oplus \mathfrak{su}(2j-1) \oplus \mathfrak{so}(2)$	$\mathfrak{sp}(k-j, j)$	Yes if $j = 1$

$\mathfrak{so}(2k-2j,2j)$ $(5 \leq k, 1 \leq j \leq k-1)$	$\mathfrak{so}(2k-2j,1)$ $\oplus\mathfrak{so}(2j-1)$	$\mathfrak{su}(k-j,j)$ $\oplus\mathfrak{so}(2)$	Yes if $j=1$
$\mathfrak{so}(2m,2m)$ $(3 \leq m)$	$\mathfrak{so}(2m,1)$ $\oplus\mathfrak{so}(2m-1)$	$\mathfrak{so}(2m,\mathbb{C})$ $\oplus\mathfrak{so}(2)$	No
$\mathfrak{sl}(2k,\mathbb{C})$	$\mathfrak{su}(2k-1,1)$	$\mathfrak{su}^*(2k)$	No
$\mathfrak{so}(2k,\mathbb{C})$ $(5 \leq k)$	$\mathfrak{so}(2k-1,1)$	$\mathfrak{so}^*(2k)$	No

Table 4.2: Classification of $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$ satisfying (ii) in Theorem 4.1.5 without type D_4

Finally, we consider the cases where $\mathfrak{g}_{\mathbb{C}}$ or $\mathfrak{g}$ is a complex simple Lie algebra of type D_4. Then by combining Corollary 4.3.6 with lists in [1, Table in Section 5.11], [21, Table I in Section 1] and [6, Theorem 7.16] we obtain the following classification results:

Proposition 4.4.12. *Let us consider* $(\mathfrak{g}, \sigma_1, \sigma_2)$ *in Setting* 4.4.3. *Assume that* $\mathfrak{g}_{\mathbb{C}}$ *is simple and of type* D_4. *Then* $(\mathfrak{g}, \sigma_1, \sigma_2)$ *satisfies the condition* (ii) *in Theorem* 4.1.5 *if and only if the triple of Satake diagrams* $(S_{\mathfrak{g}}, S_{\mathfrak{g}_1^c}, S_{\mathfrak{g}_2^c})$ *is in the Table* 4.3 *below, up to ordering of* $S_{\mathfrak{g}_1^c}$ *and* $S_{\mathfrak{g}_2^c}$, *where* $\mathfrak{g}_i^c$ *is the c-dual of* $(\mathfrak{g}, \sigma_i)$ *for* $i = 1, 2$ *(see* (4.1.1) *in Section* 4.1 *for the definition of the c-dual).*

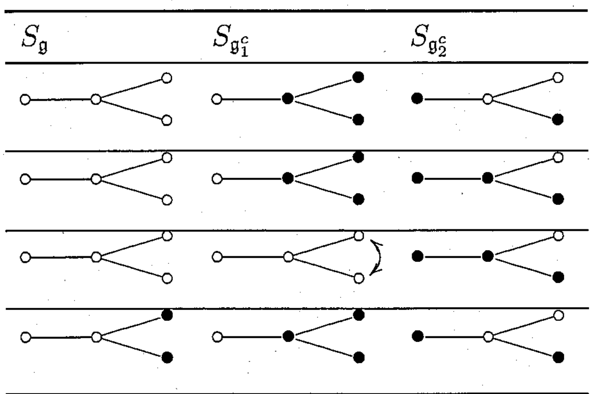
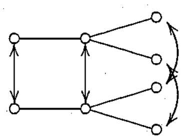


Table 4.3: Classification of $(\mathfrak{g}, \sigma_1, \sigma_2)$ with $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{so}(8,\mathbb{C})$ satisfying (ii) in Theorem 4.1.5

Proposition 4.4.13. *Let us consider* $(\mathfrak{g}, \sigma_1, \sigma_2)$ *in Setting* 4.4.3. *Assume that* $\mathfrak{g}$ *is complex simple and of type* D_4. *Then* $(\mathfrak{g}, \sigma_1, \sigma_2)$ *satisfies the condition* (ii) *in Theorem* 4.1.5 *if and only if the pair of Satake diagrams* $(S_{\mathfrak{g}_1^c}, S_{\mathfrak{g}_2^c})$ *is in the Table* 4.4 *below, up to ordering of* $S_{\mathfrak{g}_1^c}$ *and* $S_{\mathfrak{g}_2^c}$, *where* $\mathfrak{g}_i^c$ *is the c-dual of* $(\mathfrak{g}, \sigma_i)$ *for* $i = 1, 2$ (*see* (4.1.1) *in Section* 4.1 *for the definition of the c-dual*).

We remark that the Satake diagram of a complex simple Lie algebra $\mathfrak{g}$ of type D_4 is

$S_{\mathfrak{g}} =$ 

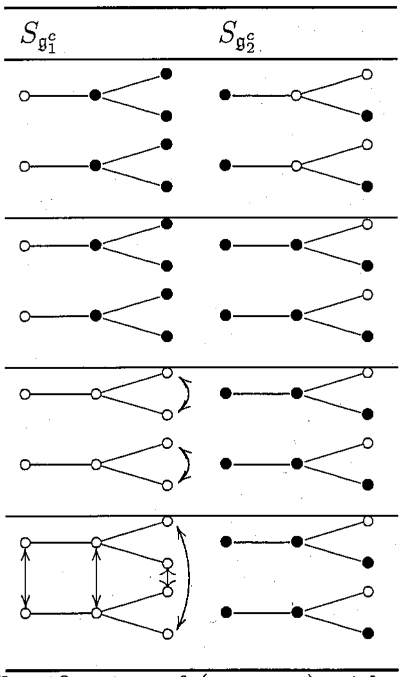


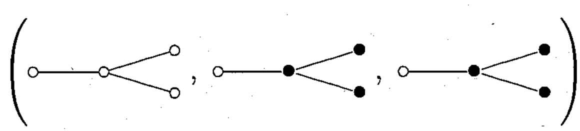
Table 4.4: Classification of $(\mathfrak{g}, \sigma_1, \sigma_2)$ with $\mathfrak{g} \simeq \mathfrak{so}(8, \mathbb{C})$ satisfying (ii) in Theorem 4.1.5

Let us consider $(\mathfrak{g}, \sigma_1, \sigma_2)$ in Setting 4.4.3 and assume that $\mathfrak{g}_{\mathbb{C}}$ or $\mathfrak{g}$ is complex simple and of type D_4. By Proposition 4.4.12 and Proposition 4.4.13, if such $(\mathfrak{g}, \sigma_1, \sigma_2)$ satisfies the condition (ii) in Theorem 4.1.5 then $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$ can be found in the Table 4.5 below, up to ordering of $\mathfrak{h}_1$ and $\mathfrak{h}_2$, where $\mathfrak{h}_i = \{ X \in \mathfrak{g} \mid \sigma_i(X) = X \}$.

$\mathfrak{g}$	$\mathfrak{h}_1$	$\mathfrak{h}_2$	cocompact
$\mathfrak{so}(4,4)$	$\mathfrak{so}(4,1) \oplus \mathfrak{so}(3)$	$\mathfrak{su}(2,2) \oplus \mathfrak{so}(2)$	No
		(or $\mathfrak{so}(4,\mathbb{C}) \oplus \mathfrak{so}(2)$)	No
$\mathfrak{so}(4,4)$	$\mathfrak{so}(4,1) \oplus \mathfrak{so}(3)$	$\mathfrak{so}(4,1) \oplus \mathfrak{so}(3)$	No
$\mathfrak{so}(4,4)$	$\mathfrak{so}(4,3)$	$\mathfrak{so}(4,1) \oplus \mathfrak{so}(3)$	Yes
	(or $\mathfrak{so}(3,2) \oplus \mathfrak{so}(1,2)$)		No
$\mathfrak{so}(6,2)$	$\mathfrak{so}(6,1)$	$\mathfrak{su}(3,1) \oplus \mathfrak{so}(2)$	Yes
	(or $\mathfrak{so}(5) \oplus \mathfrak{so}(1,2)$)		No
$\mathfrak{so}(8,\mathbb{C})$	$\mathfrak{so}(7,1)$	$\mathfrak{so}(6,2)$	No
$\mathfrak{so}(8,\mathbb{C})$	$\mathfrak{so}(7,1)$	$\mathfrak{so}(7,1)$	No
$\mathfrak{so}(8,\mathbb{C})$	$\mathfrak{so}(5,3)$	$\mathfrak{so}(7,1)$	No
$\mathfrak{so}(8,\mathbb{C})$	$\mathfrak{so}(7,\mathbb{C})$	$\mathfrak{so}(7,1)$	Yes
	(or $\mathfrak{so}(5,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})$)		No

Table 4.5: list of $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$ of type D_4 satisfying (ii) in Theorem 4.1.5

Remark 4.4.14. *It should be noted that we can NOT claim that "if* $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$ *is isomorphic to one of Table* 4.5, *then* $(\mathfrak{g}, \sigma_1, \sigma_2)$ *satisfies the condition* (ii) *in Theorem* 4.1.5*." For example, let us take* $\mathfrak{g} = \mathfrak{so}(4,4)$ *and an involutive automorphism* σ *on* $\mathfrak{g}$ *such that* $\mathfrak{h} := \mathfrak{g}^\sigma \simeq \mathfrak{su}(4,1) \oplus \mathfrak{so}(3)$. *Then* $(\mathfrak{g}, \mathfrak{h}, \mathfrak{h})$ *can be found in Table* 4.5. *However* $(\mathfrak{g}, \sigma, \sigma)$ *does not satisfies the condition* (ii) *in Theorem* 4.1.5, *since the Satake diagrams of* $(\mathfrak{g}, \mathfrak{g}^c, \mathfrak{g}^c)$, *where* $\mathfrak{g}^c$ *is the c-dual of* $(\mathfrak{g}, \sigma)$, *are given by*

$(S_{\mathfrak{g}}, S_{\mathfrak{g}^c}, S_{\mathfrak{g}^c}) =$ 

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Chapter 5

Smallest complex nilpotent orbits with real points

Let $\mathfrak{g}$ be a non-complex non-compact simple Lie algebra, and denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$. If the minimal complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ in $\mathfrak{g}_{\mathbb{C}}$ meets $\mathfrak{g}$, then the intersection $\mathcal{O}^{G_{\mathbb{C}}}_{\min} \cap \mathfrak{g}$ is the disjoint union of all real minimal nilpotent orbits in $\mathfrak{g}$. It is known that the minimal complex nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ does not meet $\mathfrak{g}$ if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{f}_{4(-20)}$ or $\mathfrak{e}_{6(-26)}$ ($k \geq 2$, $n \geq 5$, $p \geq q \geq 1$). In this chapter, in the cases where $\mathfrak{g}$ is one of the 5 types given above, we show that there uniquely exists a real minimal nilpotent orbit in $\mathfrak{g}$, and determine the complexification $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ of the real minimal nilpotent orbit in $\mathfrak{g}$ by describing the weighted Dynkin diagram of it. Note that in such cases, $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ is not minimal in $\mathfrak{g}_{\mathbb{C}}$.

5.1 Introduction and statement of main results

Let $\mathfrak{g}$ be a non-compact simple Lie algebra without complex structures. We write $\mathfrak{g}_{\mathbb{C}}$ for the complexification of $\mathfrak{g}$. Then, $\mathfrak{g}_{\mathbb{C}}$ is a complex simple Lie algebra. In this chapter, an adjoint nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ will be simply called a complex nilpotent orbit. We put

$$\mathcal{N}/G_{\mathbb{C}} := \{\,\text{Complex nilpotent orbits in } \mathfrak{g}_{\mathbb{C}}\,\},$$

and consider the closure ordering on $\mathcal{N}/G_{\mathbb{C}}$.

Let X_ϕ be a non-zero highest root vector of $\mathfrak{g}_\mathbb{C}$. Then, the complex nilpotent orbit

$$\mathcal{O}^{G_\mathbb{C}}_{\min} := G_\mathbb{C} \cdot X_\phi$$

is called the minimal complex nilpotent orbit in $\mathfrak{g}_\mathbb{C}$, where $G_\mathbb{C}$ is a connected Lie group with its Lie algebra $\mathfrak{g}_\mathbb{C}$. It is well known that $\mathcal{O}^{G_\mathbb{C}}_{\min}$ does not depend on a choice of X_ϕ, and $\mathcal{O}^{G_\mathbb{C}}_{\min}$ is minimum in $\mathcal{N}/G_\mathbb{C}$ without the zero-orbit. Namely, for any $\mathcal{O}' \in \mathcal{N}/G_\mathbb{C}$, the closure $\overline{\mathcal{O}'}$ contains $\mathcal{O}^{G_\mathbb{C}}_{\min}$ if $\mathcal{O}'$ is not the zero-orbit.

In this chapter, we set the subset $\mathcal{N}_\mathfrak{g}/G_\mathbb{C}$ of $\mathcal{N}/G_\mathbb{C}$ to

$$\begin{aligned}\mathcal{N}_\mathfrak{g}/G_\mathbb{C} : &= \{\text{ Complex nilpotent orbits in } \mathfrak{g}_\mathbb{C} \text{ which meets } \mathfrak{g}\,\} \\ &= \{\text{ Complex nilpotent orbits in } \mathfrak{g}_\mathbb{C} \text{ which meets } \mathfrak{p}_\mathbb{C}\,\}\end{aligned}$$

where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{p}_\mathbb{C}$ is the complexification of $\mathfrak{p}$.

Our first main result is here:

Theorem 5.1.1. *Let λ be a highest root of a restricted root system of $\mathfrak{g}$ and X_λ a non-zero highest root vector in $\mathfrak{g}_\lambda$. Then, the complex nilpotent orbit*

$$\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} := G_\mathbb{C} \cdot X_\lambda$$

in $\mathfrak{g}_\mathbb{C}$ does not depend on a choice of X_λ, and $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ is minimum in $\mathcal{N}_\mathfrak{g}/G_\mathbb{C}$ without the zero-orbit. Namely, for any $\mathcal{O}' \in \mathcal{N}_\mathfrak{g}/G_\mathbb{C}$, the closure $\overline{\mathcal{O}'}$ contains $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ in $\mathfrak{g}_\mathbb{C}$ if $\mathcal{O}'$ is not the zero-orbit.

Corollary 5.1.2. *The orbit $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ is a unique complex nilpotent orbit of minimum positive dimension of $\mathcal{N}_\mathfrak{g}/G_\mathbb{C}$.*

Let G be a connected Lie group with its Lie algebra $\mathfrak{g}$. Then G acts on $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} \cap \mathfrak{g}$. Let us also denote an adjoint nilpotent orbit in $\mathfrak{g}$ simply by a real nilpotent orbit. We say that a real nilpotent orbit $\mathcal{O}^G$ in $\mathfrak{g}$ is minimal if its closure $\overline{\mathcal{O}^G}$ is the union of $\mathcal{O}^G$ itself and the zero-orbit in $\mathfrak{g}$. In general, real minimal nilpotent orbits are not unique for real simple $\mathfrak{g}$.

By the next theorem, we obtain that

$$(\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} \cap \mathfrak{g})/G = \{\text{ Real minimal nilpotent orbits in } \mathfrak{g}\,\}.$$

Theorem 5.1.3. *Let $\mathcal{O}^G$ be an adjoint orbit in $\mathfrak{g}$. Then, the following two conditions on $\mathcal{O}^G$ are equivalent:*

(*i*) $\mathcal{O}^G$ *is a real minimal nilpotent orbit in* $\mathfrak{g}$.

(*ii*) $\mathcal{O}^G$ *is contained in* $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} \cap \mathfrak{g}$.

By Theorem 5.1.1, one can observe that the minimal complex nilpotent orbit $\mathcal{O}^{G_\mathbb{C}}_{\min}$ in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{g}$ if and only if $\mathcal{O}^{G_\mathbb{C}}_{\min} = \mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$. It is known that $\mathcal{O}^{G_\mathbb{C}}_{\min}$ meets $\mathfrak{g}$ if and only if $\mathfrak{g}$ is not isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{f}_{4(-20)}$ nor $\mathfrak{e}_{6(-26)}$ ($k \geq 2, n \geq 5, p \geq q \geq 1$). (see Brylinski [3, Theorem 4.1]). In particular, if $\mathfrak{g}$ is isomorphic to one of the 5 types given above, then $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} \neq \mathcal{O}^{G_\mathbb{C}}_{\min}$. We remark that in such cases, $(\mathfrak{g}, \mathfrak{k})$ is of non-Hermitian type, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$.

We determine $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ for such $\mathfrak{g}$ by describing the weighted Dynkin diagrams of $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ as follows:

Theorem 5.1.4. *Let* λ *be a highest root of a restricted root system of* $\mathfrak{g}$ *and denote by* $\mathfrak{g}_\lambda$ *the highest root space of* $\mathfrak{g}$. *Then the following hold:*

(*i*) *The following three conditions are equivalent:*

(*a*) $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} = \mathcal{O}^{G_\mathbb{C}}_{\min}$,

(*b*) $\mathcal{O}^{G_\mathbb{C}}_{\min} \cap \mathfrak{g} \neq \emptyset$,

(*c*) $\dim_\mathbb{R} \mathfrak{g}_\lambda = 1$.

(*ii*) *If* $\dim_\mathbb{R} \mathfrak{g}_\lambda \geq 2$, *then* $\mathfrak{g}$ *is isomorphic to one in the table below and* $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ *is described by the weighted Dynkin diagrams in the table:*

$\mathfrak{g}$	$\dim_\mathbb{C} \mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$	*Weighted Dynkin diagram of* $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$
$\mathfrak{su}^*(2k)$	$8k-8$	0 1 0 0 $\cdots$ 0 0 1 0 ($k \geq 3$)
		0 2 0 ($k = 2$)
$\mathfrak{so}(n-1,1)$	$2n-4$	2 0 0 $\cdots$ 0 0 (*n is odd*, $n \geq 5$)
		2 0 0 $\cdots$ 0 0 0 (*n is even*, $n \geq 6$)
$\mathfrak{sp}(k-q,q)$	$4k-2$	0 1 0 0 $\cdots$ 0 0 ($k \geq 3, \frac{k}{2} \geq q \geq 1$)
		0 2 ($k = 2, q = 1$)

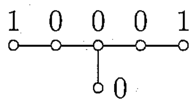
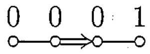
$\mathfrak{e}_{6(-26)}$	32	
$\mathfrak{f}_{4(-20)}$	22	

Table 5.1: List of $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min},\mathfrak{g}}$.

We will also prove the next theorem:

Theorem 5.1.5. *If $\mathfrak{g}$ is isomorphic to one in the table in Theorem* 5.1.4*, then there uniquely exists a real minimal nilpotent orbits in $\mathfrak{g}$.*

Theorem 5.1.5 follows from the next proposition:

Proposition 5.1.6. *Suppose that G is linear and $G_{\mathbb{C}}$ is a complexification of G. Let $\mathfrak{a}$ be a split maximal abelian subspace of $\mathfrak{g}$. We write MA for the centralizer of $\mathfrak{a}$ in G. Take λ for a highest root of the restricted root system $\Sigma(\mathfrak{g},\mathfrak{a})$ for $(\mathfrak{g},\mathfrak{a})$. Then, the following hold:*

(*i*) *The map*

$$\{\,\text{Non-zero } MA\text{-orbits in } \mathfrak{g}_\lambda\,\} \to \{\,G\text{-orbits in } \mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min},\mathfrak{g}} \cap \mathfrak{g}\,\}$$
$$\mathcal{O}^{MA} \mapsto G \cdot \mathcal{O}^{MA}$$

is bijective.

(*ii*) *If $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$, then $\mathfrak{g}_\lambda \setminus \{0\}$ becomes a single MA-orbit.*

Remark 5.1.7. *By combining Theorem* 5.1.5 *with some known facts for real minimal nilpotent orbits for the cases where the minimal complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min}}$ meets $\mathfrak{g}$, we have that*

$$\sharp\{\text{ Real minimal nilpotent orbits in } \mathfrak{g}\ \} = \begin{cases} 1 & \text{if } (\mathfrak{g},\mathfrak{k}) \text{ is of non-Hermitian type,} \\ 2 & \text{if } (\mathfrak{g},\mathfrak{k}) \text{ is of Hermitian type.} \end{cases}$$

This works motivated by recent works [8], by Joachim Hilgert, Toshiyuki Kobayashi and Jan Möllers, on the construction of an L^2-model of irreducible unitary representations of real reductive groups with smallest Gelfand-Kirillov dimension; and [9], by Toshiyuki Kobayashi and Yoshiki Oshima, on the classification of reductive symmetric pairs $(\mathfrak{g},\mathfrak{h})$ with a $(\mathfrak{g},K)$-module which is discretely decomposable as an $(\mathfrak{h}, H\cap K)$-module.

5.2 Preliminary results

5.2.1 Weighted Dynkin diagrams of complex nilpotent orbits

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra. In this subsection, we recall the definitions of weighted Dynkin diagrams of complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$.

Let us fix a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$. We denote by $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ the root system for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then, the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ becomes a subset of the dual space $\mathfrak{h}^*$ of

$$\mathfrak{h} := \{\, H \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(H) \in \mathbb{R} \text{ for any } \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \,\}.$$

We write $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ for the Weyl group of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ acting on $\mathfrak{h}$. Take a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then, a closed Weyl chamber

$$\mathfrak{h}_+ := \{\, H \in \mathfrak{h} \mid \alpha(H) \geq 0 \text{ for any } \alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \,\}$$

becomes a fundamental domain of $\mathfrak{h}$ for the action of $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

Let Π be the simple system of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then, for any $H \in \mathfrak{h}$, we can define a map

$$\Psi_H : \Pi \to \mathbb{R}, \ \alpha \mapsto \alpha(H).$$

We call Ψ_H the weighted Dynkin diagram corresponding to $H \in \mathfrak{h}$, and $\alpha(H)$ the weight on a node $\alpha \in \Pi$ of the weighted Dynkin diagram. Since Π is a basis of $\mathfrak{h}^*$, the map

$$\Psi : \mathfrak{h} \to \mathrm{Map}(\Pi, \mathbb{R}), \ H \mapsto \Psi_H$$

is bijective. Furthermore,

$$\mathfrak{h}_+ \to \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}), \ H \mapsto \Psi_H$$

is also bijective.

A triple (H, X, Y) is said to be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{C}}$ if

$$[H, X] = 2X, \ [H, Y] = -2Y, \ [X, Y] = H \quad (H, X, Y \in \mathfrak{g}_{\mathbb{C}}).$$

For any $\mathfrak{sl}_2$-triple (H, X, Y) in $\mathfrak{g}_{\mathbb{C}}$, the elements X and Y are nilpotent in $\mathfrak{g}_{\mathbb{C}}$, and H is hyperbolic in $\mathfrak{g}_{\mathbb{C}}$, i.e. $\mathrm{ad}_{\mathfrak{g}_{\mathbb{C}}} H \in \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$ is diagonalizable with only real eigenvalues.

Combining the Jacobson–Morozov theorem with the results of Kostant [11], for any complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}$, there uniquely exists an element $H_{\mathcal{O}}$ of $\mathfrak{h}_+$ with the following property: There exists $X, Y \in \mathcal{O}^{G_{\mathbb{C}}}$ such that $(H_{\mathcal{O}}, X, Y)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{C}}$. Furthermore, by the results of Malcev [12], the following map is injective:

$$\{\, \text{Complex nilpotent orbits in } \mathfrak{g}_{\mathbb{C}} \,\} \hookrightarrow \mathfrak{h}_+, \ \mathcal{O}^{G_{\mathbb{C}}} \mapsto H_{\mathcal{O}}.$$

For each complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}$, the weighted Dynkin diagram corresponding to $H_{\mathcal{O}}$ is called the weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}$. Dynkin [6] proved that for any complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}$, any weight of the weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}$ is given by 0, 1 or 2, and classified weighted Dynkin diagrams of complex nilpotent orbits (see Bala–Carter [2] for more details).

In the rest of this subsection, we suppose that $\mathfrak{g}_{\mathbb{C}}$ is simple. Let ϕ be the highest root of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then, the minimal complex nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ can be written by

$$\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min}} = G_{\mathbb{C}} \cdot (\mathfrak{g}_\phi \setminus \{0\}).$$

We define the element $H_{\phi^\vee}$ of $\mathfrak{h}$ by

$$\alpha(H_{\phi^\vee}) = \frac{2\langle \alpha, \phi \rangle}{\langle \phi, \phi \rangle} \quad \text{for any } \alpha \in \mathfrak{h}^*,$$

where $\langle\ ,\ \rangle$ is the inner product on $\mathfrak{h}^*$ induced by the Killing form on $\mathfrak{g}_{\mathbb{C}}$. In other words, $H_{\phi^\vee}$ is the element of $\mathfrak{h}$ corresponding to the coroot $\phi^\vee$ of ϕ. Since ϕ is dominant, $H_{\phi^\vee}$ is in $\mathfrak{h}_+$. Furthermore, $H_{\phi^\vee}$ is the hyperbolic element corresponding to $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min}}$ since we can find $X_\phi \in \mathfrak{g}_\phi$, $Y_\phi \in \mathfrak{g}_{-\phi}$ such that $(H_{\phi^\vee}, X_\phi, Y_\phi)$ is an $\mathfrak{sl}_2$-triple. The list of weighted Dynkin diagrams of $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min}}$ for each simple $\mathfrak{g}_{\mathbb{C}}$ can be found in [4, Chapter 5.4 and 8.4] (see also Table 5.2 in Section 5.4.2).

5.2.2 Sekiguchi–Kostant bijection

In this subsection, we recall the Sekiguchi–Kostant bijection.

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra and $\mathfrak{g}$ a real form of $\mathfrak{g}_{\mathbb{C}}$ with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We put $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{p}_{\mathbb{C}}$ for the complexifications of $\mathfrak{k}$, $\mathfrak{p}$, respectively. We take $G_{\mathbb{C}}$, G, $K_{\mathbb{C}}$ as a connected Lie group with its Lie algebra $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}$, $\mathfrak{k}_{\mathbb{C}}$, respectively.

Then, the next fact holds:

Fact 5.2.1 (Sekiguchi [16, Proposition 1.11]). *Let (H, X, Y) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_\mathbb{C}$. Then, the following three conditions on (H, X, Y) are equivalent:*

(*i*) $G_\mathbb{C} \cdot X$ *meets* $\mathfrak{p}_\mathbb{C}$.

(*ii*) $G_\mathbb{C} \cdot X$ *meets* $\mathfrak{g}$.

(*iii*) $G_\mathbb{C} \cdot H$ *meets* $\mathfrak{p}_\mathbb{C}$.

In particular, for any complex nilopotent orbit $\mathcal{O}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$, the orbit $\mathcal{O}^{G_\mathbb{C}}$ meets $\mathfrak{g}$ if and only if it meets $\mathfrak{p}_\mathbb{C}$. In such cases the intersection $\mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{g}$ [resp. $\mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{p}_\mathbb{C}$] split into finitely many G-orbits [resp. $K_\mathbb{C}$-orbits]. For each G-orbit $\mathcal{O}^G$ in $\mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{g}$ and each $K_\mathbb{C}$-orbit $\mathcal{O}^{K_\mathbb{C}}$ in $\mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{p}_\mathbb{C}$, the dimension can be written by

$$\dim_\mathbb{R} \mathcal{O}^G = \dim_\mathbb{C} \mathcal{O}^{G_\mathbb{C}}, \quad \dim_\mathbb{C} \mathcal{O}^{K_\mathbb{C}} = \frac{1}{2} \dim_\mathbb{C} \mathcal{O}^{G_\mathbb{C}}.$$

The following fact is well known:

Fact 5.2.2 (Sekiguchi–Kostant bijection [16]). *For any complex nilpotent orbit $\mathcal{O}^{G_\mathbb{C}}$ in $\mathfrak{g}_\mathbb{C}$, there exists a bijection*

$$\{\, G\text{-orbits in } \mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{g} \,\} \overset{1:1}{\longleftrightarrow} \{\, K_\mathbb{C}\text{-orbits in } \mathcal{O}^{G_\mathbb{C}} \cap \mathfrak{p}_\mathbb{C} \,\}.$$

In particular, there exists a bijection

$$\{\, \text{Nilpotent } G\text{-orbits in } \mathfrak{g} \,\} \overset{1:1}{\longleftrightarrow} \{\, \text{Nilpotent } K_\mathbb{C}\text{-orbits in } \mathfrak{p}_\mathbb{C} \,\}.$$

5.3 Well-definedness of $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$

In this section, we prove Theorem 5.1.1 and Theorem 5.1.3.

5.3.1 Some properties of a highest root of a restricted root system

To prove Theorem 5.1.1, we show some lemmas for a highest root of a restricted root system of in this subsection. Proofs of the lemmas will be described in the last of this subsection.

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra and $\mathfrak{g}$ a non-compact real form of $\mathfrak{g}$ with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, which is called a maximally split abelian subspace of $\mathfrak{g}$, and write $\Sigma(\mathfrak{g}, \mathfrak{a})$ for the restricted root system for $(\mathfrak{g}, \mathfrak{a})$. For any restricted root ξ of $\Sigma(\mathfrak{g}, \mathfrak{a})$, we define $A_{\xi^\vee} \in \mathfrak{a}$ by

$$\eta(A_{\xi^\vee}) = \frac{2\langle \xi, \eta \rangle}{\langle \xi, \xi \rangle} \quad \text{for any } \eta \in \mathfrak{a}^*,$$

where $\langle\ ,\ \rangle$ is the inner product on $\mathfrak{a}^*$ induced by the Killing form B on $\mathfrak{g}$. The lemma below will be play an important role through this chapter:

Lemma 5.3.1. *For any restricted root ξ of $\Sigma(\mathfrak{g}, \mathfrak{a})$ and any non-zero root vector X_ξ in $\mathfrak{g}_\xi$, there exists $Y_\xi \in \mathfrak{g}_{-\xi}$ such that $(A_{\xi^\vee}, X_\xi, Y_\xi)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$.*

We fix an ordering on $\mathfrak{a}$ and write $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ for the positive system of $\Sigma(\mathfrak{g}, \mathfrak{a})$ corresponding to the ordering on $\mathfrak{a}$. We denote by λ the highest root of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ with respect to the ordering on $\mathfrak{a}$. Next Lemma claims that the highest root λ depend only on the positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ but not on the ordering on $\mathfrak{a}$:

Lemma 5.3.2. *The highest root λ of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ is a unique dominant longest root of $\Sigma(\mathfrak{g}, \mathfrak{a})$.*

The following lemma gives a charactorization of the highest root λ of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$:

Lemma 5.3.3. *Let ξ be a root of $\Sigma(\mathfrak{g}, \mathfrak{a})$. If ξ is not highest, then for any non-zero root vector X_ξ in $\mathfrak{g}_\xi$, there exists a positive root η in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ and a root vector $X_\eta \in \mathfrak{g}_\eta$ such that $[X_\xi, X_\eta] \neq 0$. In particular, ξ is highest if and only if $\xi + \eta \in \mathfrak{a}^*$ is not a root of $\Sigma(\mathfrak{g}, \mathfrak{a})$ for any $\eta \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$.*

Proof of Lemma 5.3.1. By the definition of $A_{\xi^\vee}$, we have

$$\xi(A_{\xi^\vee}) = \frac{2\langle \xi, \xi \rangle}{\langle \xi, \xi \rangle} = 2.$$

Thus, we only need to show that for any $X_\xi \in \mathfrak{g}_\xi \setminus \{0\}$, there exists $Y_\xi \in \mathfrak{g}_{-\xi}$ such that

$$[X_\xi, Y_\xi] = A_{\xi^\vee}.$$

We write θ for the Cartan involution of $\mathfrak{g}$ corresponding to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then, $\theta X_\xi \in \mathfrak{g}_{-\xi}$ and $B(X_\xi, \theta X_\xi) < 0$, where B is the Killing form on $\mathfrak{g}$. We take $Y_\xi \in \mathfrak{g}_{-\xi}$ for

$$Y_\xi := \frac{2}{\langle \xi, \xi \rangle B(X_\xi, \theta X_\xi)} \theta X_\xi \in \mathfrak{g}_{-\xi}.$$

Then, $[X_\xi, Y_\xi] \in \mathfrak{a}$ since $[X_\xi, \theta X_\xi] \in \mathfrak{a}$. Furthermore, for any $A \in \mathfrak{a}$, we have

$$\begin{aligned} B([X_\xi, Y_\xi], A) &= B(X_\xi, [Y_\xi, A]) \\ &= \xi(A) B(X_\xi, Y_\xi) \\ &= \frac{2\xi(A)}{\langle \xi, \xi \rangle}. \end{aligned}$$

Hence, $[X_\xi, Y_\xi] = A_{\xi^\vee}$. □

Proof of Lemma 5.3.2. First, we prove that λ is dominant. Let ξ be a positive root of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. Then, there exists an $\mathfrak{sl}_2$-triple $(A_{\xi^\vee}, X_\xi, Y_\xi)$ in $\mathfrak{g}$ with $X_\xi \in \mathfrak{g}_\xi$ and $Y_\xi \in \mathfrak{g}_{-\xi}$ by Lemma 5.3.1. Then, by the definition of $A_{\lambda^\vee}$, we have

$$[A_{\xi^\vee}, X_\lambda] = \lambda(A_{\xi^\vee}) X_\lambda = \frac{2\langle \lambda, \xi \rangle}{\langle \xi, \xi \rangle} X_\lambda.$$

Since λ is the highest root, we also have $[X_\xi, X_\lambda] = 0$. Thus, X_λ is an eigenvector of $\mathrm{ad}_\mathfrak{g}(A_{\xi^\vee})$ and in $\mathrm{Ker}\,\mathrm{ad}_\mathfrak{g}(X_\xi)$. Recall the theory of representations of $\mathfrak{sl}(2, \mathbb{C})$, we can show that $\lambda(A_{\xi^\vee}) \geq 0$. In particular, $\langle \xi, \lambda \rangle \geq 0$ for any $\xi \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$. Hence, λ is dominant.

Second, for any dominant root λ' of $\Sigma(\mathfrak{g}, \mathfrak{a})$, we shall prove that $|\lambda| \geq |\lambda'|$ and the equality holds if and only if $\lambda = \lambda'$. We take a maximal abelian subalgebra $\mathfrak{t}$ in $Z_\mathfrak{k}(\mathfrak{a})$. Then, the complexification $\mathfrak{h}_\mathbb{C}$ of $\mathfrak{a} \oplus \mathfrak{t}$ becomes a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{h} := \mathfrak{a} \oplus \sqrt{-1}\mathfrak{t}$ is the real form of $\mathfrak{h}_\mathbb{C}$ with

$$\mathfrak{h} = \{ H \in \mathfrak{h}_\mathbb{C} \mid \alpha(H) \in \mathbb{R} \text{ for any } \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \}.$$

In particular, the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ for $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ becomes a subset of $\mathfrak{h}^*$, and the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ can be written by

$$\Sigma(\mathfrak{g}, \mathfrak{a}) = \{ \alpha|_\mathfrak{a} \in \mathfrak{a}^* \mid \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \} \setminus \{0\}.$$

We extend an ordering on $\mathfrak{a}$ to an ordering on $\mathfrak{h} = \mathfrak{a} \oplus \sqrt{-1}\mathfrak{t}$ and write $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ for the positive system of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ corresponding to the ordering on $\mathfrak{h}$. Let us denote by

$$\Pi = \{\alpha_1, \dots, \alpha_k\}, \quad \overline{\Pi} = \{ \xi_1, \dots, \xi_l \}$$

the simple system of $\Delta^+(\mathfrak{g},\mathfrak{a})$, $\Sigma^+(\mathfrak{g},\mathfrak{a})$, respectively. Then, $\overline{\Pi}$ can be written by

$$\overline{\Pi} = \{\,\alpha|_{\mathfrak{a}} \mid \alpha \in \Pi\,\} \setminus \{0\}.$$

We denote by ϕ the highest root of $\Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$. Then, for any root ϕ' in $\Delta(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$, we can find non-negative integers n_i $(i=1,\dots,k)$ such that

$$\phi = \phi' + \sum_{i=1}^{k} n_i\alpha_i.$$

The highest root λ in $\Sigma^+(\mathfrak{g},\mathfrak{a})$ is written by $\lambda = \phi|_{\mathfrak{a}}$ since the ordering on $\mathfrak{h}$ is an extension of the ordering on $\mathfrak{a}$. For any restricted root λ' in $\Sigma(\mathfrak{g},\mathfrak{a})$, there exists a root ϕ' of $\Delta(\mathfrak{g},\mathfrak{a})$ such that $\phi'|_{\mathfrak{a}} = \lambda'$. Hence, λ also can be written by

$$\lambda = \lambda' + \sum_{j=1}^{l} m_j\xi_j$$

where m_j is the sum of the elements in $\{\,n_i \mid \alpha_i|_{\mathfrak{a}} = \xi_j\,\}$ for each j. Note that m_j is non-negative. Therefore, for any dominant root λ' of $\Sigma(\mathfrak{g},\mathfrak{a})$,

$$\begin{aligned}|\lambda|^2 &= |\lambda'|^2 + 2\sum_{j=1}^{l} m_j\langle\lambda',\xi_j\rangle + |\sum_{j=1}^{l} m_j\xi_j|^2\\ &\geq |\lambda'|^2 + |\sum_{j=1}^{l} m_j\xi_j|^2 \quad (\because \lambda' \text{ is dominant}).\end{aligned}$$

Thus, $|\lambda| \geq |\lambda'|$ and the equality holds if and only if $\lambda = \lambda'$. □

Proof of Lemma 5.3.3. Let us put λ to the highest root of $\Sigma(\mathfrak{g},\mathfrak{a})$. Suppose that $[X_\xi,\mathfrak{g}_\eta] = \{0\}$ for any $\eta \in \Sigma^+(\mathfrak{g},\mathfrak{a})$, and we shall show that $\xi = \lambda$. We put

$$\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a}),\ \mathfrak{n} := \bigoplus_{\xi\in\Sigma^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\xi \text{ and } \mathfrak{n}^- := \bigoplus_{\xi\in\Sigma^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_{-\xi}.$$

Then,

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Let us denote by $\mathfrak{a}_{\mathbb{C}}$, $\mathfrak{m}_{\mathbb{C}}$, $\mathfrak{n}_{\mathbb{C}}$, $\mathfrak{n}^-_{\mathbb{C}}$ the complexification of $\mathfrak{a}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{n}^-$, respectively. By the Poincare–Birkhoff–Witt theorem, the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$ can be decomposed as

$$U(\mathfrak{g}_{\mathbb{C}}) = U(\mathfrak{n}^-_{\mathbb{C}})U(\mathfrak{m}_{\mathbb{C}})U(\mathfrak{a}_{\mathbb{C}})U(\mathfrak{n}_{\mathbb{C}}).$$

Since we are assuming that $U(\mathfrak{n}_{\mathbb{C}})X_\xi = \mathbb{C}X_\xi$, the ideal $U(\mathfrak{g}_{\mathbb{C}})X_\xi$ can be written by

$$U(\mathfrak{g}_{\mathbb{C}})X_\xi = U(\mathfrak{n}^-)U(\mathfrak{m}_{\mathbb{C}})\mathbb{C}X_\xi.$$

Thus, if $\xi \neq \lambda$, then we obtain that

$$\mathfrak{g}_\lambda \not\subset U(\mathfrak{g}_{\mathbb{C}})X_\xi.$$

In particular, $U(\mathfrak{g}_{\mathbb{C}})X_\xi$ is a non-trivial ideal of $\mathfrak{g}_{\mathbb{C}}$. This contradicts the simplicity of $\mathfrak{g}_{\mathbb{C}}$. Therefore, $\xi = \lambda$. □

5.3.2 Proofs of Theorem 5.1.1 and Theorem 5.1.3

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra, and $\mathfrak{g}$ a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. We use same notation in Section 5.3.1. We also fix connected Lie groups $G_{\mathbb{C}}$ and G with its Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}$, respectively.

In this subsection, we give proofs of Theorem 5.1.1 and Theorem 5.1.3. To this, we show the next two lemmas:

Lemma 5.3.4. *Let $\mathcal{O}'_0$ be a non-zero real nilpotent orbit in $\mathfrak{g}$. Then, there exists a non-zero highest root vector X_λ in $\mathfrak{g}_\lambda$ such that X_λ is in the closure of $\mathcal{O}'_0$.*

Lemma 5.3.5. *For any two highest root vectors X_λ, X'_λ in $\mathfrak{g}_\lambda$, there exists $g_{\mathbb{C}} \in G_{\mathbb{C}}$ such that $g_{\mathbb{C}}X_\lambda = X'_\lambda$.*

Theorem 5.1.1 follows from Lemma 5.3.4 and Lemma 5.3.5 immediately.

Proof of Lemma 5.3.4. There is no loss of generality in assuming that the ordering on $\mathfrak{a}$ is lexicographic. Let us put $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. Then, $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\xi \in \Sigma(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\xi.$$

For each $X' \in \mathfrak{g}$, we denote by

$$X' = X'_{\mathfrak{m}} + X'_{\mathfrak{a}} + \sum_{\xi \in \Sigma(\mathfrak{g},\mathfrak{a})} X'_\xi \quad (X'_{\mathfrak{m}} \in \mathfrak{m},\ X'_{\mathfrak{a}} \in \mathfrak{a},\ X'_\xi \in \mathfrak{g}_\xi).$$

For a fixed $X' \in \overline{\mathcal{O}'_0}$, we denote by λ' the highest root of

$$\Sigma_{X'} := \{\, \xi \in \Sigma(\mathfrak{g},\mathfrak{a}) \mid X'_\xi \neq 0 \,\}$$

with respect to the ordering on $\mathfrak{a}$. Here we remark that if $X' \neq 0$, then $\Sigma_{X'}$ is not empty since X' is nilpotent element in $\mathfrak{g}$. As a first step of the proof, we shall prove that for any $X' \in \overline{\mathcal{O}'_0}$, the root vector $X'_{\lambda'}$ is also in $\overline{\mathcal{O}'_0}$. We take $A' \in \mathfrak{a}$ satisfying that

$$0 < \lambda'(A') \text{ and } \xi(A') < \lambda'(A') \quad \text{for any } \xi \in \Sigma_{X'} \setminus \{\lambda'\}.$$

Note that such A' exists since λ' is the highest root of $\Sigma_{X'}$ with respect to the lexicographic ordering on $\mathfrak{a}$. Let us put

$$X'_k := \frac{1}{e^{k\lambda'(A')}} \exp(\mathrm{ad}_{\mathfrak{g}} kA')X' \quad \text{for each } k \in \mathbb{N}.$$

Then, X'_k is in $\overline{\mathcal{O}'_0}$ for any k since $\overline{\mathcal{O}'_0}$ is stable by positive scalars. Furthermore,

$$\lim_{k\to\infty} X'_k = \lim_{k\to\infty} \sum_{\xi \in \Sigma_{X'}} e^{k(\xi(A') - \lambda'(A'))} X'_\xi = X'_{\lambda'}.$$

This means that $X'_{\lambda'}$ is in $\overline{\mathcal{O}'_0}$. To complete the proof, we only need to show that there exists $X' \in \overline{\mathcal{O}'_0}$ such that $\lambda' = \lambda$, where λ' is the highest root of $\Sigma_{X'}$. Let us put

$$\begin{aligned}\Sigma_{\overline{\mathcal{O}'_0}} &:= \{\, \xi \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \text{ there exists } X' \in \overline{\mathcal{O}'_0} \text{ such that } X'_\xi \neq 0 \,\} \\ &= \bigcup_{X' \in \overline{\mathcal{O}'_0}} \Sigma_{X'}.\end{aligned}$$

We denote by λ_0 the highest root of $\Sigma_{\overline{\mathcal{O}'_0}}$. Then, we can find a root vector X'_{λ_0} in $\mathfrak{g}_{\lambda_0} \cap \overline{\mathcal{O}'_0}$ by using the our first claim proved above. We assume that $\lambda_0 \neq \lambda$. Then, by Lemma 5.3.3, we can find $\eta \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$ and $X_\eta \in \mathfrak{g}_\eta$ such that $[X'_{\lambda_0}, X_\eta] \neq 0$. Thus, for the element $X'' := \exp(\mathrm{ad}_{\mathfrak{g}} X_\eta) X'_{\lambda_0} \in \overline{\mathcal{O}'_0}$, we obtain that

$$\lambda_0 + \eta \in \Sigma_{X''} \subset \Sigma_{\overline{\mathcal{O}'_0}}.$$

This contradicts the definition of λ_0. Thus, $\lambda_0 = \lambda$. □

Proof of Lemma 5.3.5. Fix a non-zero highest root vector X_λ. Let $A_{\lambda^\vee}$ be the element of $\mathfrak{a}$ defined in Section 5.3.1. We put

$$(\mathfrak{g}_\mathbb{C})_2 = \{\, X \in \mathfrak{g}_\mathbb{C} \mid [A_{\lambda^\vee}, X] = 2X \,\}.$$

By Lemma 5.3.1, we can find $Y_\lambda \in \mathfrak{g}_\mathbb{C}$ such that $(A_{\lambda^\vee}, X_\lambda, Y_\lambda)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_\mathbb{C}$. Note that $X_\lambda \in (\mathfrak{g}_\mathbb{C})_2$. Thus, by applying Malcev's theorem, for any non-zero vector X' in $(\mathfrak{g}_\mathbb{C})_2$, there exists $g_\mathbb{C} \in G_\mathbb{C}$ such that $g_\mathbb{C} X_\lambda = X'$. Since $\mathfrak{g}_\lambda \subset (\mathfrak{g}_\mathbb{C})_2$, the proof is completed. □

The next proposition is also followed by Lemma 5.3.4.

Proposition 5.3.6. *For any real nilpotent orbit $\mathcal{O}'_0$ in $\mathfrak{g}$, the closure $\overline{\mathcal{O}'_0}$ in $\mathfrak{g}$ contains some real nilpotent orbits in $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} \cap \mathfrak{g}$.*

Theorem 5.1.3 follows from Proposition 5.3.6.

5.4 Complex nilpotent orbits and real forms

Let $\mathfrak{g}_\mathbb{C}$ be a complex simple Lie algebra and $\mathfrak{g}$ a non-compact real form of $\mathfrak{g}_\mathbb{C}$. In this section, we will give a necessary and sufficient condition of $\mathfrak{g}$ for $\mathcal{O}^{G_\mathbb{C}}_{\min} = \mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$, i.e. we prove the first half of Theorem 5.1.4 in this section.

We fix G, $G_\mathbb{C}$ for the connected Lie group with its Lie algebra $\mathfrak{g}$, $\mathfrak{g}_\mathbb{C}$, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and its ordering. Let λ be a highest root of a restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ for $(\mathfrak{g}, \mathfrak{a})$ with respect to the ordering on $\mathfrak{a}$. Then, by Theorem 5.1.1, the complex nilpotent orbit

$$\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} = G_\mathbb{C} \cdot (\mathfrak{g}_\lambda \setminus \{0\})$$

is minimum in $\mathcal{N}_\mathfrak{g}/G_\mathbb{C}$ without the zero orbit.

Our purpose in this section is to show the next proposition:

Proposition 5.4.1. *The following conditions on $\mathfrak{g}$ are equivalent (the last condition will be explained in Section 5.4.1):*

(*i*) $\mathcal{O}^{G_\mathbb{C}}_{\min} = \mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$.

(*ii*) $\mathcal{O}^{G_\mathbb{C}}_{\min} \cap \mathfrak{g} \neq \emptyset$.

(*iii*) $\mathcal{O}^{G_\mathbb{C}}_{\min} \cap \mathfrak{p}_\mathbb{C} \neq \emptyset$ (*where $\mathfrak{p}_\mathbb{C}$ is the complexification of $\mathfrak{p}$*).

(*iv*) $\dim_\mathbb{R} \mathfrak{g}_\lambda = 1$.

(*v*) *The weighted Dynkin diagram of $\mathcal{O}^{G_\mathbb{C}}_{\min}$ matches the Satake diagram of $\mathfrak{g}$.*

By the classification of simple Lie algebra $\mathfrak{g}$ and the dimension of its highest root space $\mathfrak{g}_\lambda$, we obtain the classification of $\mathfrak{g}$ such that $\mathcal{O}^{G_\mathbb{C}}_{\min} \neq \mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ as follows:

Corollary 5.4.2. *The minimal complex nilpotent orbit* $\mathcal{O}^{G_\mathbb{C}}_{\min}$ *does* **not** *coincide with* $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ *if and only if* $\mathfrak{g}$ *is isomorphic to*

$$\mathfrak{su}^*(2k),\ \mathfrak{so}(n,1),\ \mathfrak{sp}(p,q),\ \mathfrak{e}_{6(-26)} \text{ or } \mathfrak{f}_{4(-20)},$$

for $k \geq 2$, $n \geq 5$ *and* $p \geq q \geq 1$.

Observation 5.4.3. *If* $\dim_\mathbb{R} \mathfrak{g}_\lambda \geq 2$, *i.e.* $\mathcal{O}^{G_\mathbb{C}}_{\min} \neq \mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$, *then* $(\mathfrak{g}, \mathfrak{k})$ *is of non-Hermitian type.*

Remark 5.4.4. *By using Theorem* 5.1.1, *one can observe that* $\mathcal{O}^{G_\mathbb{C}}_{\min} = \mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ *if and only if* $\mathcal{O}^{G_\mathbb{C}}_{\min}$ *meets* $\mathfrak{g}$. *The classification of* $\mathfrak{g}$ *such that* $\mathcal{O}^{G_\mathbb{C}}_{\min} \cap \mathfrak{g} \neq \emptyset$ *can be found in [3] without proofs. Thus, Corollary* 5.4.2 *was already known in this sense.*

5.4.1 Match

First, we recall briefly the definition of Satake diagram of a real form $\mathfrak{g}$ of $\mathfrak{g}_\mathbb{C}$. All facts will be used for the definition of Satake diagrams are showed in Araki [1] or Satake [14]. Through Section 5.4.1 and Section 5.4.2, $\mathfrak{g}_\mathbb{C}$ can be general complex semisimple Lie algebra and $\mathfrak{g}$ can be a general (possibly compact) real form $\mathfrak{g}_\mathbb{C}$.

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$. Take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$, and extend it to a maximal abelian subspace $\mathfrak{h} = \sqrt{-1}\mathfrak{t} \oplus \mathfrak{a}$ in $\sqrt{-1}\mathfrak{k} \oplus \mathfrak{p}$. Then, the complexification, denoted by $\mathfrak{h}_\mathbb{C}$, of $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$, and $\mathfrak{h}$ coincide with the real form

$$\{X \in \mathfrak{h}_\mathbb{C} \mid \alpha(X) \in \mathbb{R} \text{ for any } \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})\}$$

of $\mathfrak{h}_\mathbb{C}$ where $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ is the reduced root system for $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Let us denote by

$$\Sigma(\mathfrak{g}, \mathfrak{a}) := \{\alpha|_\mathfrak{a} \mid \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})\} \setminus \{0\} \subset \mathfrak{a}^*$$

the restricted root system for $(\mathfrak{g}, \mathfrak{a})$. We will denote by $W(\mathfrak{g}, \mathfrak{a})$, $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ the Weyl group of $\Sigma(\mathfrak{g}, \mathfrak{a})$, $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, respectively. Fix an ordering on $\mathfrak{a}$ and extend it to an ordering on $\mathfrak{h}$. We write $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ for the positive system of $\Sigma(\mathfrak{g}, \mathfrak{a})$, $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ corresponding to the ordering on $\mathfrak{a}$, $\mathfrak{h}$, respectively. Then, $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ can be written by

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})\} \setminus \{0\}.$$

We denote by Π the fundamental system of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then,

$$\overline{\Pi} = \{\, \alpha|_{\mathfrak{a}} \mid \alpha \in \Pi \,\} \setminus \{0\}$$

becomes the simple system of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. Let Π_0 be the set of all simple roots in Π whose restriction to $\mathfrak{a}$ is zero.

The Satake diagram S of $\mathfrak{g}$ consists of the following three data: the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ with nodes Π, black nodes Π_0 in S, and arrows joining $\alpha \in \Pi \setminus \Pi_0$ and $\beta \in \Pi \setminus \Pi_0$ in S whose restrictions to $\mathfrak{a}$ are the same.

Second, we define that a weighted Dynkin diagram $\Psi_H \in \mathrm{Map}(\Pi, \mathbb{R})$ "matches" the Satake diagram S of $\mathfrak{g}$ as follows:

Definition 5.4.5. *Let $\Psi_H \in \mathrm{Map}(\Pi, \mathbb{R})$ be a weighted Dynkin diagram (see Section* 5.2.1 *for the definition) and S the Satake diagram of $\mathfrak{g}$ with nodes Π defined above. We say that Ψ_H matches S if all the weights on black nodes are zero and any pair of nodes joined by an arrow has the same weights.*

Remark 5.4.6. *The concept of "match" defined above is same as "weighted Satake diagrams" in Djocovic [5] and the condition described in Sekiguchi [15, Proposition* 1.16].

Recall that Ψ is a bijection from $\mathfrak{h}$ to $\mathrm{Map}(\Pi, \mathbb{R})$ (see Section 5.2.1). The next proposition claims that the subspace $\mathfrak{a}$ of $\mathfrak{h}$ corresponds to set of weighted Dynkin diagrams matching the Satake diagram of $\mathfrak{g}$ by the bijection Ψ.

Proposition 5.4.7. *The bijection Ψ between $\mathfrak{h}$ and* $\mathrm{Map}(\Pi, \mathbb{R})$ *defined in Section* 5.2.1 *induces a bijection below:*

$$\mathfrak{a} \overset{1:1}{\longleftrightarrow} \{\, \Psi_H \in \mathrm{Map}(\Pi, \mathbb{R}) \mid \Psi_H \text{ matches } S \,\}.$$

Proof. See the proof of Lemma 1.7.5 in Section 1.7.2. □

The following Lemma will be used in Section 5.5.2.

Lemma 5.4.8. *For each simple root α in Π, we define the element $H_{\alpha^\vee}$ of $\mathfrak{h}$ by*

$$\gamma(H_{\alpha^\vee}) = \frac{2\langle\gamma,\alpha\rangle}{\langle\alpha,\alpha\rangle} \quad \textit{for any } \gamma \in \mathfrak{h}^*.$$

Then, the set

$$\{\, H_{\alpha^\vee} \mid \alpha \textit{ is black in } S\,\} \sqcup \{\, H_{\alpha^\vee} - H_{\beta^\vee} \mid \alpha \textit{ and } \beta \textit{ are joined by an arrow in } S\,\}$$

becomes a basis of $\sqrt{-1}\mathfrak{t}$.

Proof. We denote by

$$\Omega = \{\, H_{\alpha^\vee} \mid \alpha \text{ is black in } S\,\} \sqcup \{\, H_{\alpha^\vee} - H_{\beta^\vee} \mid \alpha \text{ and } \beta \text{ are joined by an arrow in } S\,\}.$$

It is known that there are no triple $\{\alpha,\beta,\gamma\}$ in $\Pi \setminus \Pi_0$ such that $\alpha|_{\mathfrak{a}} = \beta|_{\mathfrak{a}} = \gamma|_{\mathfrak{a}}$ (this fact can be found in [1, Section 2.8]). Thus, Ω is linearly independent and

$$\sharp\Omega = \sharp\Pi - \sharp\overline{\Pi}.$$

By the proof of Proposition 5.4.7, $\dim_{\mathbb{R}} \mathfrak{a} = \sharp\overline{\Pi}$. Since $\dim_{\mathbb{R}} \mathfrak{h} = \sharp\Pi$ and $\sqrt{-1}\mathfrak{t}$ is the the orthogonal-complement space of $\mathfrak{a}$ in $\mathfrak{h}$ for the Killing form $B_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$, it remains to prove that

$$B_{\mathbb{C}}(H', A) = 0 \quad \text{for any } H' \in \Omega,\ A \in \mathfrak{a}.$$

For any $\alpha \in \Pi_0$, i.e. α is black in S, since $\alpha|_{\mathfrak{a}} = 0$, we have

$$B_{\mathbb{C}}(H_{\alpha^\vee}, A) = \frac{2\alpha(A)}{\langle\alpha,\alpha\rangle} = 0 \quad \text{for any } A \in \mathfrak{a}.$$

Furthermore, by [7, Lemma 2.10], there exists an involution σ^* of $\mathfrak{h}^*$ such that $\sigma^*\alpha = \beta$ for all pair $\alpha, \beta \in \Pi \setminus \Pi_0$ such that $\alpha|_{\mathfrak{a}} = \beta|_{\mathfrak{a}}$, i.e. α and β is joined by an arrow in S. In particular $|\alpha| = |\beta|$ for such pair. Thus, for any $A \in \mathfrak{a}$, we have

$$\begin{aligned} B_{\mathbb{C}}(H_{\alpha^\vee} - H_{\beta^\vee}, A) &= \frac{2\alpha(A)}{\langle\alpha,\alpha\rangle} - \frac{2\beta(A)}{\langle\beta,\beta\rangle} \\ &= 0 \quad (\because \alpha|_{\mathfrak{a}} = \beta|_{\mathfrak{a}} \text{ and } |\alpha| = |\beta|). \end{aligned}$$

This complete the proof. □

5.4.2 Complex nilpotent orbits and real forms

We prove the following theorem in this subsection:

Theorem 5.4.9. *Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra and $\mathfrak{g}$ a real form of $\mathfrak{g}_{\mathbb{C}}$. For a complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$, the orbit $\mathcal{O}^{G_{\mathbb{C}}}$ meets $\mathfrak{g}$ if and only if the weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}$ matches the Satake diagram of $\mathfrak{g}$ (see Section* 5.4.1 *for the notation).*

Remark 5.4.10. *Sekiguchi [16, Proposition1.13] showed that if $\mathcal{O}^{G_{\mathbb{C}}}$ meets $\mathfrak{g}$, then the weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}$ matches the Satake diagram of $\mathfrak{g}$. Our theorem claims that its converse is also true.*

By Proposition 5.4.7, the proof of Theorem 5.4.9 will be completed by showing the next lemma:

Lemma 5.4.11. *For a complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$, the orbit $\mathcal{O}^{G_{\mathbb{C}}}$ meets $\mathfrak{g}$ if and only if the hyperbolic element $H_{\mathcal{O}}$ corresponding to $\mathcal{O}^{G_{\mathbb{C}}}$ is in $\mathfrak{a}$ (see Section* 5.2.1 *for the notation).*

Proof. We take an $\mathfrak{sl}_2$-triple $(H_{\mathcal{O}}, X, Y)$ in $\mathfrak{g}_{\mathbb{C}}$ with $X, Y \in \mathcal{O}^{G_{\mathbb{C}}}$ (see Section 5.2.1 for the notation). By Fact 5.2.1, we obtain that the orbit $\mathcal{O}^{G_{\mathbb{C}}} = G_{\mathbb{C}} \cdot X$ meets $\mathfrak{g}$ if and only if $G_{\mathbb{C}} \cdot H_{\mathcal{O}}$ meets $\mathfrak{p}_{\mathbb{C}}$. Thus, it is sufficient to show that $H_{\mathcal{O}} \in \mathfrak{a}$ under the assumption that $G_{\mathbb{C}} \cdot H_{\mathcal{O}}$ meets $\mathfrak{p}_{\mathbb{C}}$. By [10, Theorem 1], we have

$$\mathfrak{p}_{\mathbb{C}} = \bigcup_{k \in K_{\mathbb{C}}} k \cdot \mathfrak{a}_{\mathbb{C}}$$

where $\mathfrak{a}_{\mathbb{C}}$ is the complexification of $\mathfrak{a}$. Since $H_{\mathcal{O}}$ is hyperbolic, i.e. $\operatorname{ad}_{\mathfrak{g}_{\mathbb{C}}} H_{\mathcal{O}} \in \operatorname{End}(\mathfrak{g}_{\mathbb{C}})$ is diagonalizable with only real eigenvalues, and any hyperbolic element in $\mathfrak{a}_{\mathbb{C}}$ is in $\mathfrak{a}$, we can assume that $G_{\mathbb{C}} \cdot H_{\mathcal{O}}$ meets $\mathfrak{a}$. Here we denote by

$$\mathfrak{a}_+ := \{ A \in \mathfrak{a} \mid \xi(A) \geq 0 \text{ for any } \xi \in \Sigma^+(\mathfrak{g}, \mathfrak{a}) \}.$$

Then $\mathfrak{a}_+$ is a fundamental domain of $W(\mathfrak{g}, \mathfrak{a})$-action on $\mathfrak{a}$. Thus, $G_{\mathbb{C}} \cdot H_{\mathcal{O}}$ meets $\mathfrak{a}_+$ with just one element $A_0 \in (G_{\mathbb{C}} \cdot H_{\mathcal{O}}) \cap \mathfrak{a}_+$. Since

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})\} \setminus \{0\},$$

we obtain that $\mathfrak{a}_+ = \mathfrak{h}_+ \cap \mathfrak{a}$. Thus, A_0 is in $\mathfrak{h}_+$ and $G_{\mathbb{C}}$-conjugate to $H_{\mathcal{O}} \in \mathfrak{h}_+$. Therefore, $A_0 = H_{\mathcal{O}} \in \mathfrak{a}$. □

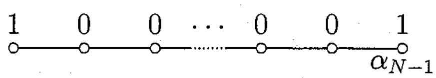
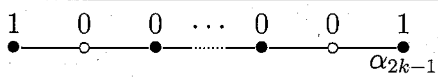
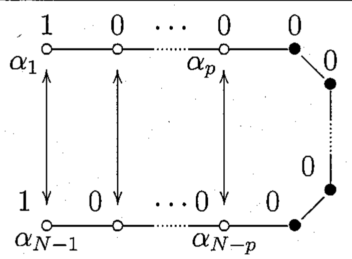
We give examples for Theorem 5.4.9 as follows:

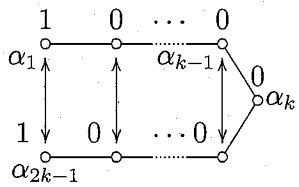
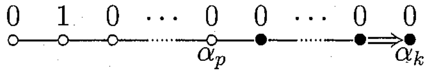
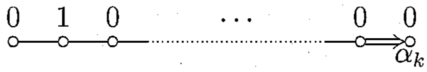
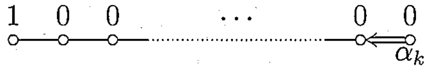
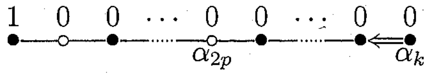
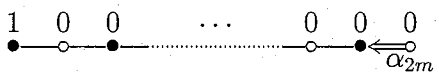
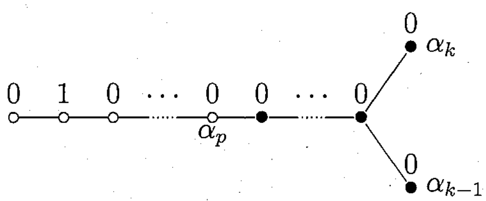
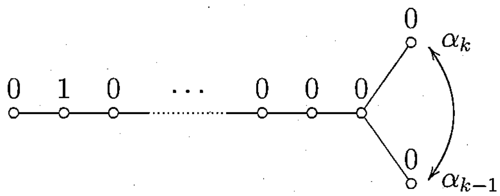
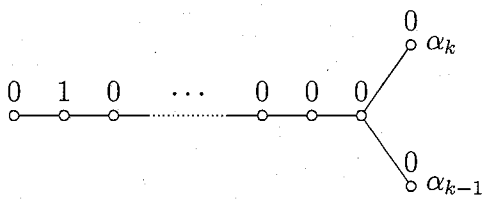
Example 5.4.12. *If $\mathfrak{g}$ is a split real form of $\mathfrak{g}_{\mathbb{C}}$, then all nodes of the Satake diagram of $\mathfrak{g}$ are white with no arrows. Thus, all complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ meets $\mathfrak{g}$ since all weighted Dynkin diagram matches the Satake diagram of $\mathfrak{g}$.*

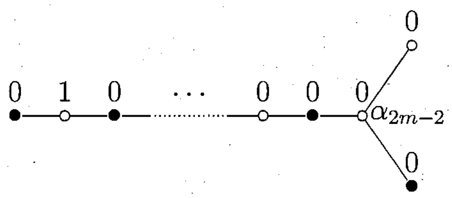
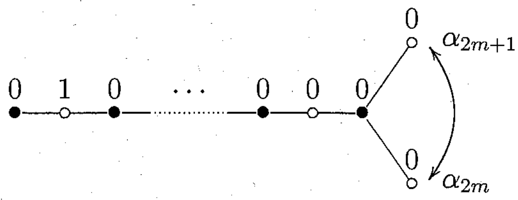
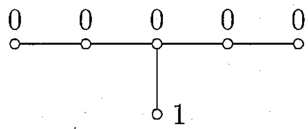
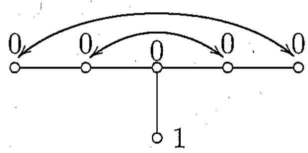
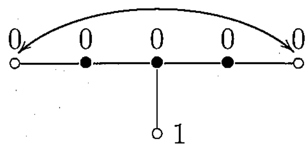
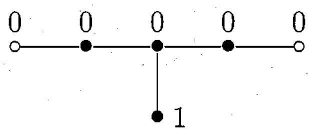
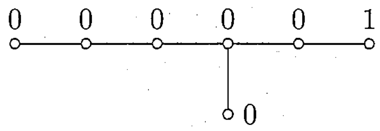
Example 5.4.13. *If $\mathfrak{u}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$, then all nodes of the Satake diagram of $\mathfrak{u}$ are black. Thus, any non-zero complex nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ does not meet $\mathfrak{u}$ since any non-zero weighted Dynkin diagram does not matches the Satake diagram of $\mathfrak{u}$.*

By the list of the weighted Dynkin diagrams of the minimal complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ for simple $\mathfrak{g}_{\mathbb{C}}$ (cf. Collingwood–McGovern [4, Chapter 5.4 and 8.4]) and the list of Satake diagrams of non-compact real forms $\mathfrak{g}$, we can easily check that $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ meets $\mathfrak{g}$ or not as follows:

Example 5.4.14. *Here is the table checking wheter the minimal complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ in a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ meets a non-compact real form $\mathfrak{g}$ or not :*

$\mathfrak{g}$	*Weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ on the Satake diagram of $\mathfrak{g}$*	*$\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ meets $\mathfrak{g}$?*
$\mathfrak{sl}(N,\mathbb{R})$		*Yes*
$\mathfrak{su}^*(2k)$		*No*
$\mathfrak{su}(N-p,p)$		*Yes*

$\mathfrak{su}(k,k)$		*Yes*
$\mathfrak{so}(2k+1-p,p)$		*No if* $p=1$
$\mathfrak{so}(k+1,k)$		*Yes*
$\mathfrak{sp}(k,\mathbb{R})$		*Yes*
$\mathfrak{sp}(k-p,p)$		*No*
$\mathfrak{sp}(m,m)$		*No*
$\mathfrak{so}(2k-p,p)$		*No if* $p=1$
$\mathfrak{so}(k+1,k-1)$		*Yes*
$\mathfrak{so}(k,k)$		*Yes*

$\mathfrak{so}^*(4m)$		*Yes*
$\mathfrak{so}^*(4m+2)$		*Yes*
$\mathfrak{e}_{6(6)}$		*Yes*
$\mathfrak{e}_{6(2)}$		*Yes*
$\mathfrak{e}_{6(-14)}$		*Yes*
$\mathfrak{e}_{6(-26)}$		*No*
$\mathfrak{e}_{7(7)}$		*Yes*

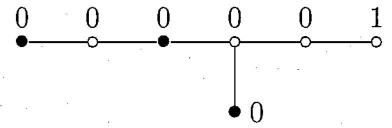
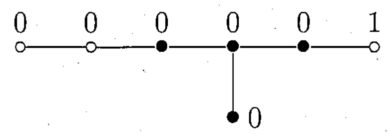
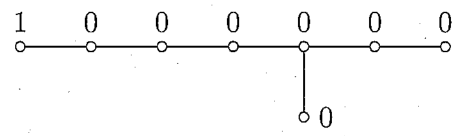
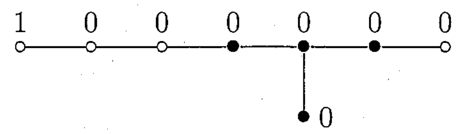
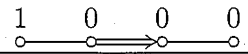
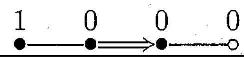
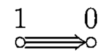
$\mathfrak{e}_{7(-5)}$		*Yes*
$\mathfrak{e}_{7(-25)}$		*Yes*
$\mathfrak{e}_{8(8)}$		*Yes*
$\mathfrak{e}_{8(-24)}$		*Yes*
$\mathfrak{f}_{4(4)}$		*Yes*
$\mathfrak{f}_{4(-20)}$		*No*
$\mathfrak{g}_{2(2)}$		*Yes*

Table 5.2: List of the weighted Dynkin diagram of $\mathcal{O}_{\mathrm{min}}^{G_{\mathbb{C}}}$ and the Satake diagram of $\mathfrak{g}$.

5.4.3 Proof of Proposition 5.4.1

We consider the same setting on Section 5.4.1 and suppose that $\mathfrak{g}_{\mathbb{C}}$ is simple and $\mathfrak{g}$ is not compact. In this subsection, we give a proof of Proposition 5.4.1.

In Proposition 5.4.1, the equivalence between (i) and (ii) can be proved by using Theorem 5.1.1; The equivalence between (ii) and (iii) is followed by Fact 5.2.1; The equivalence between (ii) and (v) is obtained by Theorem 5.4.9. To complete the proof of Proposition 5.4.1, we show the equivalence between (ii) and (iv) by using the following lemma.

Lemma 5.4.15. *In the setting of Section 5.4.1, suppose that $\mathfrak{g}_{\mathbb{C}}$ is simple and $\mathfrak{g}$ is not compact. Then, the highest root ϕ of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is a real root* (*i.e.* $\phi|_{\sqrt{-1}\mathfrak{t}} = 0$) *if and only if* $\dim_{\mathbb{R}} \mathfrak{g}_\lambda = 1$, *where* λ *is the highest root of* $\Sigma(\mathfrak{g}, \mathfrak{a})$.

Proof of Lemma 5.4.15. Recall that

$$\dim_{\mathbb{R}} \mathfrak{g}_\lambda = \sharp\{\, \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \mid \alpha|_{\mathfrak{a}} = \lambda \,\}.$$

If ϕ is a real root, then for any root $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ without ϕ, we have $\alpha|_{\mathfrak{a}} \neq \lambda\ (= \phi|_{\mathfrak{a}})$ since ϕ is the longest root of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Thus, $\dim_{\mathbb{R}} \mathfrak{g}_\lambda = 1$ in this case. Conversely, we assume that ϕ is not a real root. The anti $\mathbb{C}$-linear involution τ corresponding to $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}$, i.e. τ is the complex conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$, induces the involution τ^* on $\mathfrak{h}^*$, and it preserves $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Since $\phi|_{\sqrt{-1}\mathfrak{t}} \neq 0$, we obtain that $\tau^*\phi \neq \phi$ and $(\tau^*\phi)|_{\mathfrak{a}} = \phi|_{\mathfrak{a}} = \lambda$. Hence, $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$. □

Here is a proof of equivalence between (ii) and (iv) in Proposition 5.4.1.

Proof of equivalence between (ii) *and* (iv) *in Proposition* 5.4.1. Recall that $H_{\phi^\vee} \in \mathfrak{h}$ is the hyperbolic element corresponding to $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ (see Section 5.2.1 for the notation). Thus, by Lemma 5.4.11, $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ meets $\mathfrak{g}$ if and only if $H_{\phi^\vee}$ is in $\mathfrak{a}$. By the definition of $H_{\phi^\vee}$, the highest root ϕ is real if and only if $H_{\phi^\vee}$ is in $\mathfrak{a}$. Combining the claims above with Lemma 5.4.15, we obtain that $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ meets $\mathfrak{g}$ if and only if $\dim_{\mathbb{R}} \mathfrak{g}_\lambda = 1$. □

5.5 Weighted Dynkin diagrams of $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra and $\mathfrak{g}$ a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. In this section, we determine $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ for each $\mathfrak{g}$ by describing the weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$. Recall that Proposition 5.4.1 claims that $\mathcal{O}^{G_{\mathbb{C}}}_{\min} = \mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ if and only if $\dim_{\mathbb{R}} \mathfrak{g}_\lambda = 1$. Thus, our concern is in the cases where $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$ i.e. $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_{6(-26)}$ or $\mathfrak{f}_{4(-20)}$.

We use same notation in Section 5.4.1 (with simple $\mathfrak{g}_{\mathbb{C}}$ and non-compact $\mathfrak{g}$). Let us denote by

$$\mathfrak{a}_+ := \{\, A \in \mathfrak{a} \mid \xi(A) \geq 0 \text{ for any } \xi \in \Sigma^+(\mathfrak{g}, \mathfrak{a}) \,\}.$$

Then $\mathfrak{a}_+$ is a fundamental domain of $\mathfrak{a}$ for the action of $W(\mathfrak{g}, \mathfrak{a})$. Since

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{\, \alpha|_{\mathfrak{a}} \mid \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \,\} \setminus \{0\},$$

we have $\mathfrak{a}_+ = \mathfrak{h}_+ \cap \mathfrak{a}$.

Recall that λ is dominant by Lemma 5.3.2. Thus, $A_{\lambda^\vee}$ defined in Section 5.3.1 is in $\mathfrak{a}_+ (\subset \mathfrak{h}_+)$. Therefore, $A_{\lambda^\vee}$ is the hyperbolic element corresponding to $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ since we can find $X_\lambda \in \mathfrak{g}_\lambda$, $Y_\lambda \in \mathfrak{g}_{-\lambda}$ such that the triple $(A_{\lambda^\vee}, X_\lambda, Y_\lambda)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_\mathbb{C}$. Therefore, to determine the weighted Dynkin diagram of $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$, we shall compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$.

Our first purpose is to show the following proposition which claims that $A_{\lambda^\vee}$ can be written by $H_{\phi^\vee}$ which is the hyperbolic element corresponding to $\mathcal{O}^{G_\mathbb{C}}_{\min}$ (see Section 5.2.1).

Proposition 5.5.1. *We denote by τ the anti $\mathbb{C}$-linear involution corresponding to $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}$, i.e. τ is the complex conjugation of $\mathfrak{g}_\mathbb{C}$ with respect to the real form $\mathfrak{g}$. Then,*

$$A_{\lambda^\vee} = \begin{cases} H_{\phi^\vee} & \text{if } \dim_\mathbb{R} \mathfrak{g}_\lambda = 1, \\ H_{\phi^\vee} + \tau H_{\phi^\vee} & \text{if } \dim_\mathbb{R} \mathfrak{g}_\lambda \geq 2. \end{cases}$$

In particular, if $\dim_\mathbb{R} \mathfrak{g} \geq 2$, then the weighted Dynkin diagram of $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ can be computed by the sum of the weighted Dynkin diagrams corresponding to $H_{\phi^\vee}$, i.e. the weighted Dynkin diagram of $\mathcal{O}^{G_\mathbb{C}}_{\min}$, and that corresponding to $\tau H_{\phi^\vee}$.

We will compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$ for each $\mathfrak{g}$ with $\dim_\mathbb{R} \mathfrak{g}_\lambda \geq 2$ in Section 5.5.2.

5.5.1 Writing $A_{\lambda^\vee}$ by $H_{\phi^\vee}$

Recall that Lemma 5.4.15 claims that $\dim_\mathbb{R} \mathfrak{g}_\lambda = 1$ if and only if the highest root ϕ of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ is real, i.e. $\phi|_{\sqrt{-1}\mathfrak{t}} = 0$. We give a proof of Proposition 5.5.1 as a sequence of the following two lemmas:

Lemma 5.5.2.

$$A_{\lambda^\vee} = \frac{\langle \phi, \phi \rangle}{2\langle \lambda, \lambda \rangle}(H_{\phi^\vee} + \tau H_{\phi^\vee}).$$

In particular, if $\phi^\vee$ is real, then $A_{\lambda^\vee} = H_{\phi^\vee}$.

Lemma 5.5.3. *If ϕ is not real, then $\langle \phi, \phi \rangle = 2\langle \lambda, \lambda \rangle$.*

Proof of Lemma 5.5.2. We consider $\mathfrak{h}^*$ as $\mathfrak{a}^* \oplus \sqrt{-1}\mathfrak{t}^*$. Then, for any $\xi \in \mathfrak{a}^*$,

$$\begin{aligned}\xi(\frac{\langle\phi,\phi\rangle}{2\langle\lambda,\lambda\rangle}(H_{\phi^\vee}+\tau H_{\phi^\vee})) &= \frac{\langle\phi,\phi\rangle}{\langle\lambda,\lambda\rangle}\xi(H_{\phi^\vee}) \quad (\because \xi(H_{\phi^\vee}) = \xi(\tau H_{\phi^\vee}))\\ &= \frac{2\langle\xi,\phi\rangle}{\langle\lambda,\lambda\rangle} \quad \text{(by the definition of } H_{\phi^\vee})\\ &= \frac{2\langle\xi,\lambda\rangle}{\langle\lambda,\lambda\rangle} \quad (\because \phi|_{\mathfrak{a}} = \lambda).\end{aligned}$$

Thus, the element $\frac{\langle\phi,\phi\rangle}{2\langle\lambda,\lambda\rangle}(H_{\phi^\vee}+\tau H_{\phi^\vee})$ of $\mathfrak{a}$ is equals to $A_{\lambda^\vee}$. □

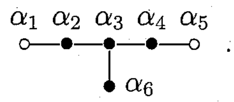
Proof of Lemma 5.5.3. We write τ^* for the involution on $\mathfrak{h}^*$ induced by τ. It is enough to show that $\langle\phi,\tau^*\phi\rangle = 0$ since $\lambda = \frac{1}{2}(\phi+\tau^*\phi)$. By [1, Proposition 1.3], τ^* is a normal involution of $\Delta(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$, i.e. for any root $\alpha \in \Delta(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$, the element $\alpha - \tau^*\alpha$ is not a root of $\Delta(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$. In particular, for any root $\alpha \in \Delta(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$ with $\tau^*\alpha \neq \alpha$, we have $\langle\alpha,\tau^*\alpha\rangle \leq 0$. Recall that we are assuming that ϕ is not real. Thus, $\phi \neq \tau^*\phi$, and then $\langle\phi,\tau^*\phi\rangle \leq 0$. The root $\tau^*\phi$ is in $\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$ since the ordering on $\mathfrak{h}$ is an extension of the ordering on $\mathfrak{a}$. Then, we also obtain that $\langle\phi,\tau^*\phi\rangle \geq 0$ since the highest root ϕ is dominant. Therefore, $\langle\phi,\tau^*\phi\rangle = 0$. □

5.5.2 Weighted Dynkin diagrams of $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$

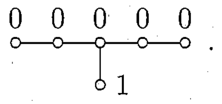
We now determine the weighted Dynkin diagram of $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ for each $\mathfrak{g}$ with $\dim_\mathbb{R}\mathfrak{g}_\lambda \geq 2$, i.e. $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_{6(-26)}$ or $\mathfrak{f}_{4(-20)}$. By Proposition 5.5.1, our purpose is to compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee} = H_{\phi^\vee} + \tau H_{\phi^\vee}$.

We only give the computation for the case $\mathfrak{g} = \mathfrak{e}_{6(-26)}$ below. For the other $\mathfrak{g}$ with $\dim_\mathbb{R}\mathfrak{g}_\lambda \geq 2$, we can compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$ by the same way.

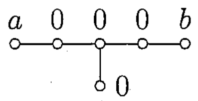
Example 5.5.4. *Let* $(\mathfrak{g}_\mathbb{C},\mathfrak{g}) = (\mathfrak{e}_{6,\mathbb{C}},\mathfrak{e}_{6(-26)})$. *We denote the Satake diagram of* $\mathfrak{e}_{6(-26)}$ *by*



By Table 5.2, *the weighted Dynkin diagram corresponding to* $H_{\phi^\vee}$ *is*



We now compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee} = H_{\phi^\vee} + \tau H_{\phi^\vee}$. *By Proposition* 5.4.7, *the weighted Dynkin diagram corresponding to* $A_{\lambda^\vee}$ *matches the Satake diagram of* $\mathfrak{e}_{6(-26)}$. *Thus, we can put the weighted Dynkin diagram corresponding to* $A_{\lambda^\vee}$ *as*

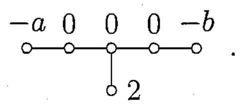


for $a, b \in \mathbb{R}$.

To determine $a, b \in \mathbb{R}$, *we also put*

$$H^{im}_{\phi^\vee} := H_{\phi^\vee} - \tau H_{\phi^\vee} \in \sqrt{-1}\mathfrak{t}.$$

Since $A_{\lambda^\vee} + H^{im}_{\phi^\vee} = 2H_{\phi^\vee}$, *the weighted Dynkin diagram corresponding to* $H^{im}_{\phi^\vee}$ *can be written by*



Namely, we have

$$\begin{aligned}
\alpha_1(H^{im}_{\phi^\vee}) &= -a,\\
\alpha_2(H^{im}_{\phi^\vee}) &= \alpha_3(H^{im}_{\phi^\vee}) = \alpha_4(H^{im}_{\phi^\vee}) = 0,\\
\alpha_5(H^{im}_{\phi^\vee}) &= -b,\\
\alpha_6(H^{im}_{\phi^\vee}) &= 2.
\end{aligned}$$

By Lemma 5.4.8, *the set* $\{ H_{\alpha_2^\vee}, H_{\alpha_3^\vee}, H_{\alpha_4^\vee}, H_{\alpha_6^\vee} \}$ *becomes a basis of* $\sqrt{-1}\mathfrak{t}$. *Thus,* $H^{im}_{\phi^\vee} \in \sqrt{-1}\mathfrak{t}$ *can be written by*

$$H^{im}_{\phi^\vee} = c_2 H_{\alpha_2^\vee} + c_3 H_{\alpha_3^\vee} + c_4 H_{\alpha_4^\vee} + c_6 H_{\alpha_6^\vee} \quad \text{for } c_2, c_3, c_4, c_6 \in \mathbb{R}.$$

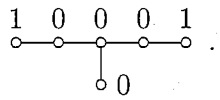
By comparing the Dynkin diagram of $\mathfrak{e}_{6,\mathbb{C}}$, *we can compute*

$$\alpha_i(H_{\alpha_j^\vee}) = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$$

for each i, j. Thus, we also have

$$\begin{aligned}
\alpha_1(H^{im}_{\phi^\vee}) &= -c_2, \\
\alpha_2(H^{im}_{\phi^\vee}) &= 2c_2 - c_3, \\
\alpha_3(H^{im}_{\phi^\vee}) &= -c_2 + 2c_3 - c_4 - c_6, \\
\alpha_4(H^{im}_{\phi^\vee}) &= -c_3 + 2c_4, \\
\alpha_5(H^{im}_{\phi^\vee}) &= -c_4, \\
\alpha_6(H^{im}_{\phi^\vee}) &= -c_3 + 2c_6.
\end{aligned}$$

Then, we obtain that $a = b = 1$. Therefore, the weighted Dynkin diagram of $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{e}_{6(-26)}$ is



The result of our computation for all $\mathfrak{g}$ with $\dim_\mathbb{R} \mathfrak{g}_\lambda \geq 2$ is in Table 5.1 of Theorem 5.1.4.

5.6 G-orbits in $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} \cap \mathfrak{g}$

Let $\mathfrak{g}_\mathbb{C}$ be a complex simple Lie algebra and $\mathfrak{g}$ a non-compact real form of $\mathfrak{g}_\mathbb{C}$. Through this section, we take G for the connected linear Lie group with its Lie algebra $\mathfrak{g}$ and $G_\mathbb{C}$ the complexification of G. We prove Thorem 5.1.5 in this section.

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$, and write K for the maximal compact subgroup of G with its Lie algebra $\mathfrak{k}$. Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and fix an ordering on $\mathfrak{a}$. Let λ be the highest root of $\Sigma(\mathfrak{g}, \mathfrak{a})$ with respect to the ordering on $\mathfrak{a}$. Let us denote by $M := Z_K(\mathfrak{a})$ and $A := \exp \mathfrak{a}$. Then, the closed subgroup MA of G is coincide with $Z_G(\mathfrak{a})$. Thus, MA acts on the highest root space $\mathfrak{g}_\lambda$ by the adjoint action.

Our first purpose in this section is to show the following two propositions (namely, Proposition 5.1.6 in Section 5.1.6):

Proposition 5.6.1. *Suppose that $\dim_\mathbb{R} \mathfrak{g}_\lambda \geq 2$. Then $\mathfrak{g}_\lambda \setminus \{0\}$ becomes a single MA-orbit.*

Proposition 5.6.2. *The map*

$$\{\text{ Non-zero } MA\text{-orbits in } \mathfrak{g}_\lambda \} \to \{ G\text{-orbits in } \mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}} \cap \mathfrak{g} \}$$
$$\mathcal{O}^{MA} \mapsto G \cdot \mathcal{O}^{MA}$$

is bijective.

Theorem 5.1.5 is followed by Proposition 5.6.1 and Proposition 5.6.2.

5.6.1 MA-orbits in $\mathfrak{g}_\lambda$ in the cases where $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$

In this subsection, we focus on the cases where $\dim_{\mathbb{R}} \mathfrak{g}_\lambda > 2$, i.e. $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_{6(-26)}$ or $\mathfrak{f}_{4(-20)}$.

We write M_0 for the identity component of M. Then, M_0, M_0A are the analytic subgroups of G with its Lie algebra $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, $\mathfrak{m} \oplus \mathfrak{a} = Z_{\mathfrak{g}}(\mathfrak{a})$, respectively.

We will use the next lemma to prove Proposition 5.6.1.

Lemma 5.6.3. *Suppose that* $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$ *and* $\mathfrak{g}$ *has real rank one, i.e.* $\dim_{\mathbb{R}} \mathfrak{a} = 1$. *Then* $\mathfrak{g}_\lambda \setminus \{0\}$ *becomes a single* M_0A*-orbit.*

Here is a proof of Proposition 5.6.1 by using Lemma 5.6.3:

Proof of Proposition 5.6.1. Let us put $\mathfrak{h}' := [\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] \subset \mathfrak{m} \oplus \mathfrak{a}$. Then $\mathfrak{g}' := \mathfrak{g}_{-\lambda} \oplus \mathfrak{h}' \oplus \mathfrak{g}_\lambda$ becomes a subalgebra of $\mathfrak{g}$ since $\pm 2\lambda$ is not a root. We shall prove that $\mathfrak{g}'$ is a simple Lie algebra of real rank one.

Let θ be the Cartan involution of $\mathfrak{g}$ corresponding to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then, $\mathfrak{h}'$ is θ-stable. Therefore, $\mathfrak{h}'$ can be written by $\mathfrak{h}' = \mathfrak{m}' \oplus \mathfrak{a}'$ with $\mathfrak{m}' \subset \mathfrak{m}$ and $\mathfrak{a}' \subset \mathfrak{a}$. For any $X_\lambda \in \mathfrak{g}_\lambda$, $X_{-\lambda} \in \mathfrak{g}_{-\lambda}$ and $A \in \mathfrak{a}$, we have

$$\begin{aligned} B([X_\lambda, X_{-\lambda}], A) &= B(X_\lambda, [X_{-\lambda}, A]) \\ &= \lambda(A) B(X_\lambda, X_{-\lambda}) \\ &= B(X_\lambda, X_{-\lambda}) \frac{\langle \lambda, \lambda \rangle}{2} B(A_{\lambda^\vee}, A). \end{aligned}$$

Thus, $\mathfrak{a}'$ can be written by $\mathfrak{a}' = \mathbb{R} A_{\lambda^\vee}$ since $B(\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}) = \mathbb{R}$, where $A_{\lambda^\vee}$ is the element of $\mathfrak{a}$ corresponding to λ defined in Section 5.3.1 and B is the Killing form on $\mathfrak{g}$. Let us fix any non-zero ideal $\mathfrak{I}$ of $\mathfrak{g}'$, and we shall prove that $\mathfrak{I} = \mathfrak{g}$. First, we prove that $\mathfrak{I} \cap \mathfrak{g}_{-\lambda} \neq \{0\}$. We take a non-zero element X in $\mathfrak{I}$. The element X can be written by

$$X = X_{\mathfrak{m}'} + cA_{\lambda^\vee} + X_\lambda + X_{-\lambda} \quad (X_{\mathfrak{m}'} \in \mathfrak{m}', c \in \mathbb{R}, X_\lambda \in \mathfrak{g}_\lambda, X_{-\lambda} \in \mathfrak{g}_{-\lambda}).$$

We now construct a non-zero element in $\mathfrak{I} \cap \mathfrak{g}_{-\lambda}$ dividing into the following cases:

The cases where $X_\lambda \neq 0$. In this case, by Lemma 5.3.1, there exists $Y_\lambda \in \mathfrak{g}_{-\lambda}$ such that $(A_{\lambda^\vee}, X_\lambda, Y_\lambda)$ becomes an $\mathfrak{sl}_2$-triple. Recall that -2λ is not a root of $\Sigma(\mathfrak{g}, \mathfrak{a})$. Thus we have

$$[Y_\lambda, [Y_\lambda, X]] = -2Y_\lambda,$$

and hence $Y_\lambda \in \mathfrak{I} \cap \mathfrak{g}_{-\lambda}$.

The cases where $X_\lambda = 0$ and $c \neq 0$. In this case, for any non-zero vector Y in $\mathfrak{g}_{-\lambda}$,

$$[Y, X] = [Y, X_{\mathfrak{m}'} + cA_{\lambda^\vee}] = [Y, X_{\mathfrak{m}'}] + 2cY \in \mathfrak{g}_{-\lambda}$$

is not zero since $\operatorname{ad}_{\mathfrak{g}} X_{\mathfrak{m}'}$ has no non-zero real eigen-value. Thus, $[Y, X]$ is a non-zero vector of $\mathfrak{I} \cap \mathfrak{g}_{-\lambda}$.

The cases where $X_\lambda = 0$, $c = 0$ and $X_{\mathfrak{m}'} \neq 0$. In this case, we shall show that $[\mathfrak{g}_{-\lambda}, X_{\mathfrak{m}'}] \neq \{0\}$, and then we can find $Y \in \mathfrak{g}_{-\lambda}$ such that $[Y, X] = [Y, X_{\mathfrak{m}'}]$ is a non-zero element of $\mathfrak{I} \cap \mathfrak{g}_{-\lambda}$. Since $X_{\mathfrak{m}'} \neq 0$, we have $\mathfrak{m}' \neq \{0\}$ in this case. We now assume that $[\mathfrak{g}_{-\lambda}, X_{\mathfrak{m}'}]$ is zero. Then,

$$B(\mathfrak{h}', X_{\mathfrak{m}'}) = B([\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}], X_{\mathfrak{m}'}) = B(\mathfrak{g}_\lambda, [\mathfrak{g}_{-\lambda}, X_{\mathfrak{m}'}]) = \{0\}.$$

In particular, $B(\mathfrak{m}', X_{\mathfrak{m}'}) = \{0\}$. This contradicts the non-degenerateness of B on $\mathfrak{k}$.

The cases where $X = X_{-\lambda}$. In this case, $X \in \mathfrak{I} \cap \mathfrak{g}_{-\lambda}$.

Thus, we obtain that $\mathfrak{I} \cap \mathfrak{g}_{-\lambda} \neq \{0\}$. We fix non-zero element Y_λ in $\mathfrak{I} \cap \mathfrak{g}_{-\lambda}$. Then, by using Lemma 5.3.1, we can find $X_\lambda \in \mathfrak{g}_\lambda$ such that $(A_{\lambda^\vee}, X_\lambda, Y_\lambda)$ becomes an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ (since we can find $X_\lambda \in \mathfrak{g}_\lambda$ such that $(-A_{\lambda^\vee}, Y_\lambda, X_\lambda)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ by Lemma 5.3.1 for $\xi = -\lambda$). Hence, $A_{\lambda^\vee} \in \mathfrak{I}$, and this implies that $\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda} \subset \mathfrak{I}$. Since $\mathfrak{h}' = [\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}]$, we have $\mathfrak{I} = \mathfrak{g}'$. This means that $\mathfrak{g}'$ is a simple Lie algebra. Since $\mathfrak{g}'$ is θ-stable, $\theta|_{\mathfrak{g}'}$ is a Cartan decomposition of $\mathfrak{g}'$ and $\mathfrak{a}' = \mathbb{R}A_{\lambda^\vee}$ is a maximally split abelian subspace of $\mathfrak{g}'$. In particular, $\mathfrak{g}' = \mathfrak{m}' \oplus \mathfrak{a}' \oplus \mathfrak{g}_{\pm\lambda}$ is a root space decomposition of $\mathfrak{g}'$. Therefore, $\mathfrak{g}'$ is a simple Lie algebra of real rank one with $\dim \mathfrak{g}'_\lambda \geq 2$. In particular, $\mathfrak{g}'$ has no complex structure or isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

We denote by M_0' the analytic subgroup of G with its Lie algebra $\mathfrak{m}'$ and put $A' = \operatorname{Exp} \mathbb{R}A_{\lambda^\vee}$. If $\mathfrak{g}'$ has no complex structure, then by Lemma 5.6.3, we obtain that $\mathfrak{g}_\lambda \setminus \{0\}$ becomes a single $M_0'A'$-orbit. If $\mathfrak{g}'$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. one can also observe that $\mathfrak{g}_\lambda \setminus \{0\}$ becomes a single $M_0'A'$-orbit directly. Since any adjoint $M_0'A'$-orbit is contained in an adjoint M_0A-orbit, $\mathfrak{g}_\lambda \setminus \{0\}$ also becomes a single M_0A-orbit. □

To complete the proof of Proposition 5.6.1, we shall prove Lemma 5.6.3.

Proof of Lemma 5.6.3. Let $A_{\lambda^\vee}$ be the element of $\mathfrak{a}$ defined in Section 5.3.1. Since $\mathfrak{g}$ has real rank one, $\mathfrak{a} = \mathbb{R}A_{\lambda^\vee}$ and $\mathfrak{g}$ can be written by

$$\mathfrak{g} = \mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_{-\frac{\lambda}{2}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_{\frac{\lambda}{2}} \oplus \mathfrak{g}_\lambda$$

(possibly $\mathfrak{g}_{\pm\frac{\lambda}{2}} = \{0\}$). Let us denote by $\mathfrak{g}_\mathbb{C}$, $\mathfrak{m}_\mathbb{C}$, $\mathfrak{a}_\mathbb{C}$, $(\mathfrak{g}_{\pm\lambda})_\mathbb{C}$, $(\mathfrak{g}_{\pm\frac{\lambda}{2}})_\mathbb{C}$ the complexification of $\mathfrak{g}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{g}_{\pm\lambda}$, $\mathfrak{g}_{\pm\frac{\lambda}{2}}$, respectively. We set

$$(\mathfrak{g}_\mathbb{C})_i = \{ X \in \mathfrak{g}_\mathbb{C} \mid [A_\lambda, X] = iX \} \quad \text{for each } i \in \mathbb{Z}.$$

Then,

$$(\mathfrak{g}_\mathbb{C})_0 = \mathfrak{m}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C},\ (\mathfrak{g}_\mathbb{C})_{\pm1} = (\mathfrak{g}_{\pm\frac{\lambda}{2}})_\mathbb{C},\ (\mathfrak{g}_\mathbb{C})_{\pm2} = (\mathfrak{g}_{\pm\lambda})_\mathbb{C}.$$

By Lemma 5.3.1, for any non-zero highest root vector X_λ in $\mathfrak{g}_\lambda$, there exists $Y_\lambda \in \mathfrak{g}_{-\lambda}$ such that $(A_{\lambda^\vee}, X_\lambda, Y_\lambda)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_\mathbb{C}$. By the theory of representations of $\mathfrak{sl}(2,\mathbb{C})$, we obtain that $[X_\lambda, (\mathfrak{g}_\mathbb{C})_0] = (\mathfrak{g}_\mathbb{C})_2$. In particular,

$$[\mathfrak{m} \oplus \mathfrak{a}, X_\lambda] = \mathfrak{g}_\lambda.$$

Therefore, for the M_0A-orbit $\mathcal{O}^{M_0A}(X_\lambda)$ in $\mathfrak{g}_\lambda$ through X_λ, we obtain that

$$\dim_\mathbb{R} \mathcal{O}^{M_0A}(X_\lambda) = \dim_\mathbb{R} \mathfrak{g}_\lambda.$$

This means that the M_0A-orbit $\mathcal{O}^{M_0A}(X_\lambda)$ is open in $\mathfrak{g}_\lambda$ for any $X_\lambda \in \mathfrak{g}_\lambda \setminus \{0\}$. Recall that $\dim_\mathbb{R} \mathfrak{g}_\lambda \geq 2$. Then, $\mathfrak{g}_\lambda \setminus \{0\}$ is connected. Therefore, $\mathfrak{g}_\lambda \setminus \{0\}$ becomes a single M_0A-orbit. □

5.6.2 Bijection between the set of non-zero MA-orbits in $\mathfrak{g}_\lambda$ and the set G-orbits in $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} \cap \mathfrak{g}$

We prove Proposition 5.6.2 in this subsection.

By Theorem 5.1.1, the orbit $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ can be written by

$$\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} = G_\mathbb{C} \cdot (\mathfrak{g}_\lambda \setminus \{0\}).$$

Thus, the map in Proposition 5.6.2 is well-defined and surjective. Thus, the proof of Proposition 5.6.1 is completed by showing that: For any $X_\lambda, X'_\lambda \in \mathfrak{g}_\lambda$, if there exists $g \in G$ such that $gX_\lambda = X'_\lambda$, then there exists $m \in M$ and $a \in A$ such that $maX_\lambda = X'_\lambda$.

We prove the claim above dividing into two lemmas below:

Lemma 5.6.4. *For any $X_\lambda, X'_\lambda \in \mathfrak{g}_\lambda$, if there exists $g \in G$ such that $gX_\lambda = X'_\lambda$, then there exists $m' \in N_K(\mathfrak{a})$ and $a \in A$ such that $m'aX_\lambda = X'_\lambda$.*

Lemma 5.6.5. *For any $X_\lambda, X'_\lambda \in \mathfrak{g}_\lambda$, if there exists $m' \in N_K(\mathfrak{a})$ such that $m'X_\lambda = X'_\lambda$, then there exists an element $m \in M$ $(= Z_K(\mathfrak{a}))$ such that $mX_\lambda = X'_\lambda$.*

Proof of Lemma 5.6.4. Since $N_G(\mathfrak{a}) = N_K(\mathfrak{a})A$, it is enough to find $g' \in G$ such that $g'X_\lambda = X'_\lambda$ and $g'\mathfrak{a} = \mathfrak{a}$. There is no loss of generality in assuming that X_λ and X'_λ are both non-zero elements in $\mathfrak{g}_\lambda$. Let $A_{\lambda^\vee}$ be the element of $\mathfrak{a}$ defined in Section 5.3.1. Then, by Lemma 5.3.1, there exists $Y_\lambda, Y'_\lambda \in \mathfrak{g}_{-\lambda}$ such that $(A_{\lambda^\vee}, X_\lambda, Y_\lambda)$ and $(A_{\lambda^\vee}, X'_\lambda, Y'_\lambda)$ are both $\mathfrak{sl}_2$-triples in $\mathfrak{g}$. Since g is an automorphism of $\mathfrak{g}$ and $gX_\lambda = X'_\lambda$, the triple $(gA_{\lambda^\vee},\ X'_\lambda,\ gY_\lambda)$ is also an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. In particular, $(A_{\lambda^\vee}, X'_\lambda, Y'_\lambda)$ and $(gA_{\lambda^\vee}, X'_\lambda, gY_\lambda)$ are both $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ with same nilpotent element. Therefore, by Kostant's theorem for $\mathfrak{sl}_2$-triples with same nilpotent element in a semisimple Lie algebra, there exists an element $g_1 \in G$ such that

$$g_1(gA_{\lambda^\vee}) = A_{\lambda^\vee},\ g_1X'_\lambda = X'_\lambda \text{ and } g_1(gY_\lambda) = Y'_\lambda.$$

Write $g_2 := g_1 \cdot g$. Then,

$$g_2A_{\lambda^\vee} = A_{\lambda^\vee},\ g_2X_\lambda = X'_\lambda \text{ and } g_2Y_\lambda = Y'_\lambda.$$

Recall that $\mathfrak{a} = \mathbb{R}A_{\lambda^\vee} \oplus \operatorname{Ker}\lambda$. If we find $g_3 \in G$ such that

$$g_3(g_2 \operatorname{Ker}\lambda) = \operatorname{Ker}\lambda,\ g_3A_{\lambda^\vee} = A_{\lambda^\vee} \text{ and } g_3X'_\lambda = X'_\lambda,$$

then we can take g' as $g_3 \cdot g_2$. We shall find such g_3. Let us denote by $\mathfrak{l}' = \mathbb{R}\text{-span}\langle A_{\lambda^\vee}, X'_\lambda, Y'_\lambda \rangle$ the subalgebra spaned by the $\mathfrak{sl}_2$-triple $(A_{\lambda^\vee}, X'_\lambda, Y'_\lambda)$. Then, there exists a Cartan involution θ' on $\mathfrak{g}$ preserving $\mathfrak{l}'$ by Mostow's theorem [13, Theorem 6]. We set

$$\begin{aligned}\mathfrak{g}_0 :&= Z_{\mathfrak{g}}(\mathfrak{l}') \\ &= \{ X \in \mathfrak{g} \mid [X, A_{\lambda^\vee}] = [X, X'_\lambda] = 0 \},\end{aligned}$$

where the second equation can be obtained by the representation theory of $\mathfrak{sl}(2, \mathbb{C})$. We note that $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$ since the Cartan involution θ' preserves $\mathfrak{g}_0$. Since $[\operatorname{Ker}\lambda, \mathfrak{l}'] = \{0\}$, the subspace $\operatorname{Ker}\lambda$ of $\mathfrak{a}$ is contained in $\mathfrak{g}_0$ and becomes a maximally split abelian subspace of $\mathfrak{g}_0$. Furthermore, we have

$$\begin{aligned}[g_2 \operatorname{Ker}\lambda, A_{\lambda^\vee}] &= g_2[\operatorname{Ker}\lambda, A_{\lambda^\vee}] = \{0\}, \\ [g_2 \operatorname{Ker}\lambda, X'_\lambda] &= g_2[\operatorname{Ker}\lambda, X_\lambda] = \{0\}.\end{aligned}$$

Thus, the subspace $g_2 \operatorname{Ker}\lambda$ of $g_2\mathfrak{a}$ is also contained in $\mathfrak{g}_0$ and becomes a maximally split abelian subspace of $\mathfrak{g}_0$. Let us write G_0 for the analytic subgroup of G with its Lie algebra $\mathfrak{g}_0$. Since any two maximally split subalgebras of $\mathfrak{g}_0$ are G_0-conjugate, there exists $g_3 \in G_0$ such that

$$g_3(g_2 \operatorname{Ker}\lambda) = \operatorname{Ker}\lambda.$$

Since g_3 is in G_0, we obtain that $g_3 A_{\lambda^\vee} = A_{\lambda^\vee}$ and $g_3 X'_\lambda = X'_\lambda$. □

To prove Lemma 5.6.5, we also show the following lemma for Weyl groups of root systems:

Lemma 5.6.6. *Let Σ be a root system realized in a vector space V with an inner product $\langle \ , \ \rangle$, and $W(\Sigma)$ the Weyl group of Σ acting on V. We fix a positive system Σ^+ of Σ, and write Π for the simple system of Σ^+. Let v be a dominant vector, i.e. $\langle \alpha, v \rangle \geq 0$ for any $\alpha \in \Sigma^+$, and $w \in W(\Sigma)$ with $w \cdot v = v$. Then, there exists a sequence $s_1, \dots, s_l$ of root reflections with $s_i \cdot v = v$ for any $i = 1, \dots, l$ such that*

$$w = s_1 s_2 \cdots s_l.$$

Proof of Lemma 5.6.6. Let $n := |\Sigma^+ \setminus w\Sigma^+|$. We prove our claim by the induction of n. If $n = 0$, then $\Sigma^+ = w\Sigma^+$. Thus, $w\Pi = \Pi$ and $w = \mathrm{id}_V$. We assume that $n \geq 1$. Then $\Pi \setminus w\Sigma^+ \neq \emptyset$. It suffice to show that any simple roots $\alpha \in \Pi \setminus w\Sigma^+$ satisfies that $\langle \alpha, v \rangle = 0$ and $|\Sigma^+ \setminus (s_\alpha w)\Sigma^+| \leq n-1$. Since $w^{-1}\alpha \notin \Sigma^+$, that is, $w^{-1}\alpha$ is a negative root, we obtain that

$$\langle w^{-1}\alpha, v \rangle \leq 0.$$

Combining $\langle \alpha, v \rangle \geq 0$ with $w \cdot v = v$, we obtain that

$$\langle \alpha, v \rangle = 0.$$

To complete the proof, we shall show the following:

- For any $\beta \in w\Sigma^+ \cap \Sigma^+$, the root $s_\alpha \beta$ is also in Σ^+.
- There exists $\gamma \in w\Sigma^+ \setminus \Sigma^+$ such that $s_\alpha \gamma$ is in Σ^+.

In general, for any positive root $\beta \in \Sigma^+$ without α or 2α, the root $s_\alpha\beta$ is also positive. Thus, for any $\beta \in w\Sigma^+ \cap \Sigma^+$, the root $s_\alpha\beta$ is in Σ^+ since α and 2α are both not in $w\Sigma^+$. Thus, the first one of our claims holds. We take $\gamma := -\alpha$. Then, γ is in $w\Sigma^+ \setminus \Sigma^+$ since α is in $\Sigma^+ \setminus w\Sigma^+$. Furthermore, $s_\alpha\gamma = \alpha$ is in Σ^+. Thus, the second one of our claims also holds. Combining the claims above, we obtain that

$$|\Sigma^+ \cap w\Sigma^+| < |\Sigma^+ \cap (s_\alpha w)\Sigma^+|,$$

and hence $|\Sigma^+ \setminus (s_\alpha w)\Sigma^+| \leq n-1$. □

Here is the proof of Lemma 5.6.5.

Proof of Lemma 5.6.5. There is no loss of generality in assuming that X_λ and X'_λ are both non-zero vectors in $\mathfrak{g}_\lambda$. Then, $m' \in N_K(\mathfrak{a})$ acts on $\mathfrak{a}$ as an element $w \in W(\mathfrak{g}, \mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ such that $w\lambda = \lambda$ since $m'\mathfrak{g}_\lambda = \mathfrak{g}_{w\lambda}$. By Lemma 5.6.6, w can be written by

$$w = s_1 s_2 \cdots s_l$$

where s_i are root reflections of $W(\mathfrak{g}, \mathfrak{a})$ with $s_i\lambda = \lambda$. We write ξ_i for the root of $\Sigma(\mathfrak{g}, \mathfrak{a})$ corresponding to s_i for each $i = 1, \ldots, l$. Let $\mathfrak{g}_i$ be the root space of ξ_i. Since $s_i\lambda = \lambda$, each ξ_i is orthogonal to λ in $\mathfrak{a}^*$. We can and do chose X_i be a non-zero root vector of $\mathfrak{g}_i$ such that

$$B(X_i, \theta X_i) = -\frac{2}{\langle \xi_i, \xi_i \rangle}$$

where θ is the Cartan involution of $\mathfrak{g}$ corresponding to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then, the element $k_i = \exp \frac{\pi}{2}(X_i + \theta X_i)$ in $N_K(\mathfrak{a})$ acts on $\mathfrak{a}$ as the reflection s_i. Thus, $m := m' k_l k_{l-1} \cdots k_1$ acts trivially on $\mathfrak{a}$. That is, $m \in Z_K(\mathfrak{a}) = M$. It remains to prove that $k_i X_\lambda = X_\lambda$. Since λ is longest root of $\Sigma(\mathfrak{g}, \mathfrak{a})$ by Lemma 5.3.2, and ξ_i is orthogonal to λ, the element $\xi_i \pm \lambda$ of $\mathfrak{a}^*$ is not a root of $\Sigma(\mathfrak{g}, \mathfrak{a})$. In particular, $[X_i, X_\lambda] = 0$ and $[\theta X_i, X_\lambda] = 0$. Hence, $k_i X_\lambda = X_\lambda$ for any i. Therefore, we obtain that $m X_\lambda = m' X_\lambda = X'_\lambda$. □

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Part II

Designs on compact homogeneous spaces

Chapter 6

Relation between spherical designs through a Hopf map

In this Chapter, for a given spherical t-design Y on S^2, we show an algorithm to make a spherical $2t$-design X on S^3 with $|X| = (2t+1)|Y|$ and $\pi(X) = Y$, where $\pi : S^3 \to S^2$ is a Hopf map. We also prove that if X is an antipodal spherical $(2t+1)$-design on S^3, then there exists $\sigma \in SO(4)$ such that $\pi(\sigma X)$ is a spherical t-design on S^2 with $|\pi(\sigma X)| = \frac{1}{2}|X|$. Moreover, our theorems are generalized to relations between designs on a compact Lie group G and that on a compact homogeneous space G/K.

6.1 Introduction

The purpose of this chapter is to show an algorithm to make designs on a compact group G from a given design on a closed subgroup K of G and that on the quotient space G/K. In particular, we give an algorithm to make a spherical $2t$-design on S^3 from a given spherical t-design on S^2.

6.1.1 Spherical designs

We write S^d for the unit sphere in the $(d+1)$-dimensional Euclidean space $\mathbb{R}^{d+1}$. The concept of spherical designs on S^d were introduced by Delsarte–Goethals–Seidel [8] in 1977 as follows: For a fixed $t \in \mathbb{N}$, a finite subset X of

S^d is called a spherical t-design on S^d if

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{|S^d|}\int_{S^d} f d\mu_{S^d} \tag{6.1.1}$$

for any polynomial f of degree at most t. Note that the left hand side and the right hand side in (6.1.1) are the averaging values of f on X and that on S^d, respectively. (see Definition 6.2.1 for more details). We also remark that any spherical $(t+1)$-design on S^d is also a spherical t-design on S^d. The development of spherical designs can be found in Bannai–Bannai [3].

A spherical t-design X on S^d is better if the cardinality $|X|$ is smaller. We motivate our work in one of the fundamental problems on spherical designs, stated below:

Problem 6.1.1. *What is the smallest cardinality of a spherical t-design on S^d?*

Let us denote by $N_{S^d}(t)$ the smallest cardinality of a spherical t-design on S^d. We remark that for the case where $d = 1$, one can prove that $N_{S^1}(t) = t+1$ for any t, by taking regular $(t+1)$-gon on S^1. Thus, our interesting is in the cases where $d \geq 2$.

Delsarte–Goethals–Seidel [8] gave lower bounds of cardinalities of spherical designs as follows:

Fact 6.1.2 (Delsarte–Goethals–Seidel [8]). *Any spherical t-design X on S^d satisfies*

$$|X| \geq \begin{cases} \binom{d+e}{e} + \binom{d+e-1}{e-1} & \text{if } t = 2e \text{ is even,} \\ 2\binom{d+e}{e} & \text{if } t = 2e+1 \text{ is odd.} \end{cases}$$

A spherical t-design X on S^d is said to be tight if the equality in Fact 6.1.2 holds. As an example of a spherical tight design, it is known that (normalized) 196560 minimal vectors of the Leech lattice gives a spherical tight 11-design on S^{23}. We remark that the uniqueness, up to the $O(24)$-action on S^{23}, of spherical tight 11-designs on S^{23} was proved by Bannai–Sloan [5].

Obviously, if a spherical tight t-design X on S^d exists, then $N_{S^d}(t) = |X|$. For example, we obtain $N_{S^{23}}(11) = 196560$. However, Bannai–Damerell [2] proved that if $d \geq 2$ and $t \geq 12$, then spherical tight t-design on S^d does not exists (see [3, Section 2] for more details). We should note that $N_{S^d}(t)$ are unknown for $d \geq 2$ with large t.

In this chapter, our interesting is in the asymptotic behavior of the function $N_{S^d}(t)$ for the cases where $d \geq 2$. Here, we remark that the existence of a spherical t-design on S^d was proved by Seymour–Zaslavsky [18] in 1984. Hence, $N_{S^d}(t) < \infty$ for any d and t.

One of available ways to give upper bounds of $N_{S^d}(t)$ is to construct interval t-designs defined below: Let us fix $d \geq 2$ and denote by $\omega_d(s) := \sqrt{(1-s^2)^{d-2}}$ the weight function on the interval $[-1,1]$. We say that a finite subset Y of $[-1,1]$ is an interval t-design with respect to ω_d if

$$\frac{1}{|Y|}\sum_{y \in Y} f(y) = \frac{1}{\int_{-1}^{1} \omega_d(s)ds} \int_{-1}^{1} f(s)\omega_d(s)ds$$

for any polynomial $f(s)$ of degree at most t. Wagner [20] and Rabau–Bajnok [14] proved the following theorem:

Fact 6.1.3 (See [14, Theorem 4.1] for more details). *If an interval t-design Y on $[-1,1]$ with $\pm 1 \notin Y$ and a spherical t-design Γ on S^{d-1} are given, then we can construct a spherical t-design on S^d such that $h(X) = Y$ and $|X| = |\Gamma| \cdot |Y|$, where $h : S^d \to [-1,1]$ is the hight function on S^d.*

Let us denote by $N^0_{[-1,1],\omega_d}(t)$ the smallest cardinality of an interval t-design Y on $[-1,1]$ with respect to the weight function ω_d such that $\pm 1 \notin Y$. Then by Fact 6.1.3, we have

$$N_{S^d}(t) \leq N_{S^{d-1}}(t) \cdot N^0_{[-1,1],\omega_d}(t). \tag{6.1.2}$$

Kuijlaars [11] proved that for a fixed $d \geq 2$, there exists a positive constant K_d such that

$$N^0_{[-1,1],\omega_d}(t) < K_d t^d \quad \text{for any } t.$$

In particular, by (6.1.2), there exists a positive constant C_d such that

$$N_{S^d}(t) \leq C_d t^{\frac{d(d+1)}{2}} \quad \text{for any } t > 0.$$

Yudin [21] mentioned that the conjecture

$$N_{S^d}(t) \ll t^d, \tag{6.1.3}$$

has been made by many authors, where $f(t) \ll g(t)$ means that there exists a positive constant C such that $f(t) < Cg(t)$ for any $t > 0$. This problem is still open for any $d \geq 2$.

Let us denote by $\pi : S^3 \to S^2$ a Hopf map. Then (S^3, S^2, π) is a principal S^1-bundle (see Section 6.2.2 for more details). In the first half part of this chapter, we focus our study a relation between $N_{S^2}(t)$ and $N_{S^3}(t)$ through the Hopf map $\pi : S^3 \to S^2$.

The first main theorem of this chapter is the following:

Theorem 6.1.4 (See Theorem 6.2.4). *Let Y be a spherical t-design on S^2. Then we can construct a spherical $2t$-design X [resp. $(2t+1)$-design X'] on S^3 with $|X| = (2t+1)|Y|$ and $\pi(X) = Y$ [resp. $|X'| = 2(t+1)|Y|$ and $\pi(X') = Y$]. In particular, for any t, we have*

$$N_{S^3}(2t) \leq (2t+1)N_{S^2}(t) \text{ and } N_{S^3}(2t+1) \leq 2(t+1)N_{S^2}(t).$$

By combining Theorem 6.1.4 with the result of Kuijlaars mentioned above for $d = 2$, we obtain

$$N_{S^3}(t) \ll t^4.$$

Furthermore, if the conjecture (6.1.3) for $d = 2$ is true, then that for $d = 3$ is also true.

In the cases where $t \leq 100$, Chen–Frommer–Lang [6] constructed spherical t-designs on S^2 with $(t+1)^2$ nodes for each $t \leq 100$. Thus, by Theorem 6.1.4, we also obtain

$$N_{S^3}(2t) \leq (2t+1)(t+1)^2 \text{ and } N_{S^3}(2t+1) \leq 2(t+1)^3 \quad \text{for } t \leq 100.$$

We also remark that Theorem 6.1.4 is constructive. That is, if we can construct a spherical t-design on S^2 explicitly, then we also obtain a spherical $2t$-design [resp. $(2t+1)$-design] on S^3 explicitly. (see Theorem 6.2.4 for more details). Kuperberg [12] gives an algorithm to construct explicitly an interval t-design Y' on $[-1, 1]$ with respect to ω_2 such that $\pm 1 \notin Y'$ by using roots of a certain polynomial of degree $\lfloor \frac{t}{2} \rfloor$. Thus, by Fact 6.1.3, we obtain a spherical t-design on S^2 explicitly, and hence we also obtain a spherical $2t$-design [resp. $(2t+1)$-design] on S^3 explicitly. This gives an algorithm to construct a spherical t-designs on S^3 for each t.

Theorem 6.1.4 gives an algorithm to make spherical $2t$-designs on S^3 from a given spherical t-design on S^2. As a kind of converse claim, we will prove the following theorem:

Theorem 6.1.5 (See Corollary 6.2.8). *Let X be a spherical $2t$-design on S^3 and fix $p \in \mathbb{N}$. If*

$$|X \cap \pi^{-1}(\pi(x))| = p \quad \textit{for any } x \in X, \tag{6.1.4}$$

then X maps to a spherical t-design on S^2 with $\frac{1}{p}|X|$ nodes by the Hopf map $\pi : S^3 \to S^2$, i.e. $Y := \pi(X)$ is a spherical t-design on S^2 with $|Y| = \frac{1}{p}|X|$.

Furthermore, we will also prove the following:

Theorem 6.1.6 (See Theorem 6.2.10). *If X' is an antipodal spherical $(2t+1)$-design on S^3, then there exists $\sigma \in SO(4)$ such that the antipodal spherical $(2t+1)$-design $\sigma X'$ on S^3 satisfies* (6.1.4) *for $p = 2$. In particular, $Y' := \pi(\sigma X')$ is a spherical t-design on S^2 with $|Y'| = \frac{1}{2}|X'|$.*

6.1.2 Designs on a compact homogeneous spaces

Throughout this subsection, let G be a compact Lie group and K a closed subgroup of G. We denote by $\pi : G \to G/K$ the quotient map.

The purpose of the last half part of this chapter is to generalize our results in Section 6.1.1 to relations between designs on a compact Lie group G and that on a compact homogeneous space G/K through the quotient map $\pi : G \to G/K$.

Let us consider a finite-dimensional representation (ρ, V) of G. For each $\Omega = G, K$ or G/K, we will define finite-dimensional functional spaces $\mathcal{H}^{\rho}_{\Omega}$ on Ω (see Section 6.5.1 for the definition of $\mathcal{H}^{\rho}_{\Omega}$). Then we define an $\mathcal{H}^{\rho}_{\Omega}$-design on Ω as follows: Let $\Omega = G, K$ or G/K. A finite subset X of Ω is called an $\mathcal{H}^{\rho}_{\Omega}$-design on Ω if

$$\frac{1}{|X|}\sum_{x \in X} f(x) = \frac{1}{|\Omega|}\int_{\Omega} f \, d\mu_{\Omega}$$

for any $f \in \mathcal{H}^{\rho}_{\Omega}$. In fact, we will give a more general definition of designs on a measure space in Section 6.3.

As a generalization of Theorem 6.1.4 and Theorem 6.1.5, we will prove the following two theorems:

Theorem 6.1.7 (See Theorem 6.5.8). *Let Y be an $\mathcal{H}^{\rho}_{G/K}$-design on G/K and Γ an $\mathcal{H}^{\rho}_{K}$-design on K. Then we can construct an $\mathcal{H}^{\rho}_{G}$-design X on G with $|X| = |\Gamma| \cdot |Y|$ and $\pi(X) = Y$.*

Theorem 6.1.8 (See Corollary 6.5.13). *Let X be an $\mathcal{H}^{\rho}_{G}$-design on G and fix $p \in \mathbb{N}$. If $|X \cap \pi^{-1}(\pi(x))| = p$ for any $x \in X$, then X maps to an $\mathcal{H}^{\rho}_{G/K}$-design on G/K with $\frac{1}{p}|X|$ nodes by the quotient map $\pi : G \to G/K$, i.e. $Y := \pi(X)$ is an $\mathcal{H}^{\rho}_{G/K}$-design on G/K with $|Y| = \frac{1}{p}|X|$.*

In the cases where (G, K) is a symmetric pair with some certain conditions, we will also prove a generalization of Theorem 6.1.6 (see Theorem 6.5.18).

It is well known that a Hopf map $S^3 \to S^2$ can be regarded as the quotient map $SU(2) \to SU(2)/T$, where

$$SU(2) := \{g \in SL(2, \mathbb{C}) \mid {}^t\overline{g} = g^{-1}\},$$
$$T := \{\, \mathrm{diag}(z, \overline{z}) \in SU(2) \mid z \in \mathbb{C} \text{ with } |z| = 1\,\}.$$

Here for each $l = 0, 1, 2, \dots,$ we put (ρ_l, V_l) to the unique irreducible $(l+1)$-dimensional representation of $SU(2)$, and denote by $\rho(t) = \bigoplus_{l=0}^{t} \rho_l$ for each t. Then we obtain that the functional spaces $\mathcal{H}^{\rho(t)}_{SU(2)}$ [resp. $\mathcal{H}^{\rho(t)}_{T}$] on $SU(2) \simeq S^3$ [resp. on $T \simeq S^1$] is a space of polynomial functions on S^3 [resp. on S^1] of degree at most t, and the functional space $\mathcal{H}^{\rho(t)}_{SU(2)/T}$ on $SU(2)/T \simeq S^2$ is a space of polynomial functions on S^2 of degree at most $\lfloor \frac{t}{2} \rfloor$. Recall that regular $(t+1)$-gon is a spherical t-design on $T \simeq S^1$. Therefore, the theorem in Section 6.1.1 follows from the theorems given above.

To prove the results in this subsection, we will use some well-known facts for the representation theory of compact Lie groups.

6.1.3 Organization of this chapter

This chapter is organized as follows. In Section 6.2, we set up notation and state our main theorems for relations between spherical designs on S^3 and that on S^2 through a Hopf map $\pi : S^3 \to S^2$. In Section 6.3, we give a definition of designs on a general measure space and prove some propositions for them as a preliminary. Main theorems in Section 6.2 will be proved in Section 6.4 by using propositions in Section 6.3. In Section 6.5, we give a statement of a generalization of the results in Section 6.2 to relations between designs on compact Lie group G and that on a compact symmetric space G/K through the quotient map $G \to G/K$. The results in Section 6.5 will be proved in Section 6.6.

6.2 Main results for spherical designs on S^3

6.2.1 Notation for spherical designs

We fix terminology for spherical designs as follows.

Let us denote by S^d the unit sphere in the $(d+1)$-dimensional Euclidean space $\mathbb{R}^{d+1}$, and denote by μ_{S^d} the spherical measure on S^d. We put $|S^d| := \mu_{S^d}(S^d)$. For each $t \in \mathbb{N}$, we write

$$P_t(\mathbb{R}^{d+1}) := \{ f \mid f \text{ is a polynomial over } \mathbb{C} \text{ on } \mathbb{R}^{d+1} \text{ with } \deg f \leq t \}.$$

Any element in $P_t(\mathbb{R}^{d+1})$ can be regard as a $\mathbb{C}$-valued function on $\mathbb{R}^{d+1}$. Here, we put

$$P_t(S^d) := \{ f|_{S^d} \mid f \in P_t(\mathbb{R}^{d+1}) \}.$$

Then $P_t(S^d)$ is a finite-dimensional functional space on S^d. It is well known that

$$\dim_{\mathbb{C}} P_t(S^d) = \binom{t+d}{d} + \binom{t+d-1}{d-1}.$$

We define spherical designs on S^d as follows:

Definition 6.2.1 (Delsarte–Goethals–Seidel [8]). *A finite subset X of S^d is called a spherical t-design on S^d if for any $f \in P_t(S^d)$, the averaging value of f on X is equals to the averaging value of f on S^d, that is,*

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^d|} \int_{S^d} f d\mu_{S^d}.$$

Remark 6.2.2. *In Definition 6.2.1, we can replace polynomials over $\mathbb{C}$ to that over $\mathbb{R}$. In fact, the original definition of spherical designs in [8] considered polynomials over $\mathbb{R}$. In this chapter, we consider polynomials over $\mathbb{C}$ since the representation theory over $\mathbb{C}$ is easier then that over $\mathbb{R}$.*

In Definition 6.2.1, the design X is a finite subset of S^d. We also define spherical designs for a finite multi-sets on S^d as follows:

Definition 6.2.3. *Let us denote by $(S^d)^N$ the direct product of N-times copies of S^d. For $X = (x_1, \dots, x_N) \in (S^d)^N$, we say that X is a spherical multi-t-design if*

$$\frac{1}{N} \sum_{i=1}^{N} f(x_i) = \frac{1}{|S^d|} \int_{S^d} f d\mu_{S^d} \quad \textit{for any } f \in P_t(S^d).$$

Here, we denote by

$$N_{S^d}(t) := \min\{\, |X| \in \mathbb{N} \mid X \text{ is a spherical } t\text{-design on } S^d \,\},$$
$$N_{S^d}^{\text{multi}}(t) := \min\{\, N \in \mathbb{N} \mid \text{there exists a spherical multi-}t\text{-design in } (S^d)^N \,\}.$$

In Definition 6.2.3, if $x_1, \dots, x_N$ are distinct from each other, then X can be regard as a spherical t-design on S^d with N nodes. Hence, $N_{S^d}^{\text{multi}}(t) \leq N_{S^d}(t)$ in general.

6.2.2 Main results

Throughout this chapter, let us denote by

$$S^3 := \left\{\, (a,b) \mid a,b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \,\right\} \subset \mathbb{C}^2 \simeq \mathbb{R}^4,$$
$$S^2 := \left\{\, (\xi,\eta) \mid \xi \in \mathbb{R},\ \eta \in \mathbb{C},\ \xi^2 + |\eta|^2 = 1 \,\right\} \subset \mathbb{R} \times \mathbb{C} \simeq \mathbb{R}^3,$$
$$S^1 := \{z \in \mathbb{C} \mid |z| = 1\} \subset \mathbb{C} \simeq \mathbb{R}^2.$$

We fix a Hopf map as follows:

$$\pi : S^3 \to S^2, \quad (a,b) \mapsto (|a|^2 - |b|^2, 2ab).$$

Let us denote by

$$(a,b) \cdot z := (az, b\overline{z}) \quad \text{for any } (a,b) \in S^3 \text{ and } z \in S^1.$$

Then

$$S^3 \times S^1 \to S^3,\ (x,z) \mapsto x \cdot z$$

defines a right action of S^1 on S^3 with respect to the usual group structure on S^1. The Hopf map $\pi : S^3 \to S^2$ is a principal S^1-bundle with respect to the right S^1-action. In particular, S^1 acts simply-transitively on a fiber $\pi^{-1}(y)$ for each $y \in S^2$.

Here is our first main theorem:

Theorem 6.2.4. *Let $Y \subset S^2$ be a spherical t-design and $\Gamma \subset S^1$ a spherical $2t$-design. For each $y \in Y$, we fix a base point $s(y)$ on the fiber $\pi^{-1}(y)$. Then the finite subset*

$$X(Y,s,\Gamma) := \{\, s(y) \cdot \gamma \in S^3 \mid y \in Y,\ \gamma \in \Gamma \,\}$$

is a spherical $2t$-design on S^3 with $|X(Y,s,\Gamma)| = |Y| \cdot |\Gamma|$. Furthermore, if Γ is a spherical $(2t+1)$-design, then $X(Y,s,\Gamma)$ is a spherical $(2t+1)$-design on S^3.

Theorem 6.2.4 will be proved in Section 6.4.2.

Let us denote by

$$\Gamma_q := \{ z \in S^1 \mid z^q = 1 \} \subset S^1 \quad \text{for each } q = 1, 2, \ldots.$$

In other words, Γ_q is the finite cyclic subgroup of S^1 of order q. It is well known that Γ_q is a spherical t-design on S^1 if $q \geq t + 1$. Thus, we can take spherical $2t$-design on S^1 with $2t + 1$ nodes, explicitly. Furthermore, for a given $y \in S^2$, we can choose a base point $s(y)$ in $\pi^{-1}(y)$, explicitly (see Section 6.4.1 for details). Therefore, Theorem 6.2.4 gives an algorithm to make a spherical $2t$-design X on S^3 with $|X| = (2t + 1)|Y|$ from a given spherical t-design Y on S^2. Recall that the results of Kuperberg [12] gives a construction of spherical t-designs on S^2 for each t (see also Bannai–Bannai [3, Section 2.7] for more details). Thus, we obtain a construction of spherical $2t$-designs on S^3 for each t.

Remark 6.2.5. *Cohn–Conway–Elkies–Kumar* [7, *Section* 4] *described that one can construct a family of designs on* S^{2n-1} *from a design in* $\mathbb{CP}^{n-1}$, *where* $\mathbb{CP}^{n-1}$ *denotes the complex projective space, i.e. the space consisted of all complex* 1*-dimensional subspaces in* $\mathbb{C}^n$. *Theorem* 6.2.4 *is a kind of formularization of it for the case where* $n = 2$.

Theorem 6.2.4 holds even if Y is a multi-set on S^2 (see Theorem 6.4.4 in Section 6.4.3 for more details). By using this, we will prove the corollary below in Section 6.4.3:

Corollary 6.2.6. *For any* t, *we have*

$$N^{multi}_{S^3}(2t) \leq N_{S^3}(2t) \leq (2t+1)N^{multi}_{S^2}(t),$$
$$N^{multi}_{S^3}(2t+1) \leq N_{S^3}(2t+1) \leq 2(t+1)N^{multi}_{S^2}(t).$$

(*see Section* 6.2.1 *for the notation of* $N_{S^d}(t)$, $N^{multi}_{S^d}(t)$ *and the definition of multi-designs*).

As a kind of converse claim of Theorem 6.2.4, We will also prove the following theorems:

Theorem 6.2.7. *Let* $X = (x_1, \ldots, x_N) \in (S^3)^N$ *be a spherical multi-*$(2t)$*-design on* S^3. *Then* $Y := (\pi(x_1), \ldots, \pi(x_N)) \in (S^2)^N$ *is a spherical multi-*t*-design on* S^2.

By the definition of multi-designs in Section 6.2.1, we obtain the following corollary to Theorem 6.2.7:

Corollary 6.2.8. *Let X be a spherical $2t$-design on S^3 and fix $p \in \mathbb{N}$. If $|X \cap \pi^{-1}(y)| = p$ for any $y \in \pi(X)$, then $\pi(X)$ is a spherical t-design on S^2 with $|\pi(X)| = \frac{1}{p}|X|$.*

Let us consider the natural left $SO(4)$-action on S^3. Recall that if X is a spherical t-design on S^3, then $\sigma X := \{\, \sigma x \mid x \in X \,\}$ is also a spherical t-design on S^3 for any $\sigma \in SO(4)$. We should remark that the left $SO(4)$-action and the right S^1-action on S^3 are not commutative (in fact, for each $z \in S^1$, we can find $\sigma_z \in SO(4)$ such that $x \cdot z = \sigma_z x$ for any $x \in S^3$).

For a given spherical $2t$-design X on S^3, does there exist $\sigma \in SO(4)$ such that σX satisfies the condition in Theroem 6.2.10? To state our results, we setup notation of antipodal and antipodal-free as follows: We say that a finite subset X of S^3 is antipodal [resp. antipodal-free] if for each $x \in X$, the element $-x$ of S^3 is in X [resp. not in X]. The next lemma will be proved in Section 6.4.4:

Lemma 6.2.9. *Let X be a finite subset of S^3 and U an open neighgberhood of the unit of $SO(4)$. If X is antipodal-free, then there exists $\sigma \in U$ such that $|\sigma X \cap \pi^{-1}(y)| = 1$ for any $y \in \pi(\sigma X)$. If X is antipodal, then there exists $\sigma \in U$ such that $|\sigma X \cap \pi^{-1}(y)| = 2$ for any $y \in \pi(\sigma X)$.*

By combining Lemma 6.2.9 with Corollary 6.2.8, we obtain the next theorem:

Theorem 6.2.10. *Let X be a spherical $2t$-design on S^3 and U an open neighberhood of the unit of $SO(4)$. Then the following hold:*

(*i*) *If X is antipodal-free (i.e. for any $x \in X$, the element $-x \in S^3$ is not in X), then there exists $\sigma \in U$ such that $Y := \pi(\sigma(X))$ is a spherical t-design on S^2 with $|Y| = |X|$.*

(*ii*) *If X is antipodal (then X automatically becomes a spherical $(2t+1)$-design on S^3), then there exists $\sigma \in U$ such that $Y := \pi(\sigma(X))$ is a spherical t-design on S^2 with $|Y| = \frac{1}{2}|X|$.*

Remark 6.2.11. *Cohn–Conway–Elkies–Kumar [7, Section 4] observed that a certain antipodal spherical 7-design on S^3 maps to a spherical 3-design on S^2 by a Hopf map.*

By combining Theorem 6.2.4 with Theorem 6.2.10, we will also prove the next theorem in Section 6.4.4:

Theorem 6.2.12. *For any t, we have*

$$N_{S^2}(t) \leq (t+1)N_{S^2}^{multi}(t)$$

(see Section 6.2.1 *for the notation).*

It is well known that S^3 admits a compact Lie group structure and for a maximal torus T of S^3, the Hopf map $\pi : S^3 \to S^2$ can be regard as a quotient map from the Lie group S^3 to the quotient space S^3/T. In Section 6.5, we will give a generalization of the results described in this section to some relations between designs on a compact group G and that on a compact symmetric space G/K.

6.3 Preliminary

In this section, we define designs on a general measure space and prove some propositions for them. Main results of this chapter will be proved by using propositions in this section.

6.3.1 Designs on a general measure space

Let (Ω, μ) be a general (possibly infinite) measure space. We define (weighted) designs for a finite-dimensional vector space consisted of L^1-integrable functions on (Ω, μ) as follows:

Definition 6.3.1. *Let X be a finite subset of Ω and $\lambda : X \to \mathbb{R}_{>0}$ be a positive function on X. For an L^1-integrable function $f : \Omega \to \mathbb{C}$, we say that (X, λ) is an weighted f-design on (Ω, μ) if*

$$\sum_{x \in X} \lambda(x) f(x) = \int_{\Omega} f d\mu.$$

For a vector space $\mathcal{H}$ consisted of L^1-integrable functions on Ω, we say that (X, λ) is an weighted $\mathcal{H}$-design on (Ω, μ) if (X, λ) is an weighted f-design on (Ω, μ) for any $f \in \mathcal{H}$. Furthermore, if λ is constant on X, then X is said to be an $\mathcal{H}$-design on (Ω, μ) with respect to the constant λ.

Example 6.3.2. *Let $\Omega = S^d$, $\mu = \frac{1}{|S^d|}\mu_{S^d}$ and $\mathcal{H} = P_t(S^d)$. Then a finite subset X of Ω is an $\mathcal{H}$-design on (Ω, μ) with respect to the constant $\frac{1}{|X|}$ if and only if X is a spherical t-design on S^d.*

Let us consider the cases where $\mu(\Omega) = 1$, i.e. μ is a probability measure, and any constant function on Ω is in $\mathcal{H}$. Then for any weighted $\mathcal{H}$-design (X, λ) on (Ω, μ), we have $\sum_{x \in X} \lambda(x) = 1$. In particular, if X is an $\mathcal{H}$-design on (Ω, μ) with respect to a positive constant λ, then $\lambda = \frac{1}{|X|}$.

Remark 6.3.3. *The concept of $\mathcal{H}$-designs on (Ω, μ) is a generalization of that of averaging sets on a topological finite measure space (Ω, μ) (see [18] for the definition of averaging sets). In particular, by [18, Main Theorem], if (Ω, μ) is a topological finite measure space and Ω is path-connected, then for any finite-dimensional vector space $\mathcal{H}$ consisted of continuous functions on Ω, an $\mathcal{H}$-design on (Ω, μ) exists.*

We give two easy observations for designs on (Ω, μ) as follows:

Observation 6.3.4.

- *If $\mathcal{H}' \subset \mathcal{H}$, then any (weighted) $\mathcal{H}$-design on (Ω, μ) is also an (weighted) $\mathcal{H}'$-design on (Ω, μ).*
- *Let λ be a positive constant and X, X' are both $\mathcal{H}$-designs on (Ω, μ) with respect to λ. If $X \cap X' = \emptyset$, then $X \sqcup X'$ is also an $\mathcal{H}$-design on (Ω, μ) with respect to λ.*

We also define multi-designs on (Ω, μ) as follows. Let us denote by Ω^N the direct product of N-times copies of Ω as a set. For $X = (x_1, \dots, x_N) \in \Omega^N$ and a vector space $\mathcal{H}$ consisted of L^1-integrable functions on (Ω, μ), we say that X is a multi-$\mathcal{H}$-design on (Ω, μ) with respect to a positive constant λ if

$$\lambda \sum_{i=1}^{N} f(x_i) = \int_{\Omega} f d\mu \quad \text{for any } f \in \mathcal{H}.$$

We shall explain that multi-designs can be regard as weighted designs as follows. Let us denote by $\overline{X} = \{x_1, \dots, x_N\} \subset \Omega$. Note that $|\overline{X}| < N$ if $x_1, \dots x_N$ are not distinct. For each element $\overline{x} \in \overline{X}$, we put

$$m(\overline{x}) := |\{ i \mid x_i = \overline{x} \}|.$$

For any positive constant $\lambda > 0$, we define a positive function $\lambda_{\overline{X}}$ on $\overline{X}$ by

$$\lambda_{\overline{X}} : \overline{X} \to \mathbb{R}_{>0}, \quad \overline{x} \mapsto \lambda \cdot m(\overline{x}).$$

Then by the definition of multi-designs and weighted designs on (Ω, μ), we have the next proposition:

Proposition 6.3.5. *Let us fix $X \in \Omega^N$, a functional space $\mathcal{H}$ and a positive constant λ as above. Then the following conditions on $(X, \mathcal{H}, \lambda)$ are equivalent:*

(i) *X is a multi-$\mathcal{H}$-design on (Ω, μ) with respect to the constant λ.*

(ii) *$(\overline{X}, \lambda_{\overline{X}})$ is an weighted $\mathcal{H}$-design on (Ω, μ).*

6.3.2 Key propositions

Let (Ω_1, μ_1), (Ω_2, μ_2) be general measure spaces and $\pi : \Omega_1 \to \Omega_2$ a map. For each element $\omega \in \Omega_2$, we fix a measure μ_ω on the fiber $\pi^{-1}(\omega)$.

Let us take an L^1-integrable function $f : \Omega_1 \to \mathbb{C}$. We say that the function f satisfies the property (F) if the following hold:

- For any $\omega \in \Omega_2$, the restriction $f|_{\pi^{-1}(\omega)}$ is also an L^1-integrable function on $(\pi^{-1}(\omega), \mu_\omega)$.
- The function
$$I_\pi f : \Omega_2 \to \mathbb{C}, \quad \omega \mapsto \int_{\pi^{-1}(\omega)} f d\mu_\omega,$$
is also an L^1-integrable function on Ω_2 with
$$\int_{\Omega_1} f d\mu_1 = \int_{\Omega_2} (I_\pi f) d\mu_2.$$

Remark 6.3.6. *The property (F) for a function f means that we can apply "Fubini's theorem" for f.*

Example 6.3.7. *Let $(\Omega_1, \mu_1) = (S^3, \frac{1}{|S^3|}\mu_{S^3})$, $(\Omega_2, \mu_2) = (S^2, \frac{1}{|S^2|}\mu_{S^2})$ and $\pi : S^3 \to S^2$ be the Hopf map. For each $y \in S^2$, we put the S^1-invariant probability measure μ_y on the fiber $\pi^{-1}(y)$. Then any continuous function on $\Omega_1 = S^3$ satisfies the property (F). Furthermore, for any $f \in P_t(S^3)$, the restricted function $f|_{\pi^{-1}(y)}$ can be regard as in $P_t(S^1)$ for each $y \in S^2$, and we have $I_\pi f \in P_{\lfloor \frac{t}{2} \rfloor}(S^2)$ (see Lemma 6.4.2 and Lemma 6.4.3 in Section 6.4.2 for more details).*

Let Y be a finite subset of Ω_2 and λ_Y a positive function on Y. For each $y \in Y$, we take a finite subset Γ_y of $\pi^{-1}(y)$ and a positive function λ_{Γ_y} on Γ_y. We denote by

$$X(Y,\Gamma) := \bigsqcup_{y \in Y} \Gamma_y \tag{6.3.1}$$

and define a positive function on $X(Y,\Gamma)$ by

$$\lambda_X : X(Y,\Gamma) = \bigsqcup_{y \in Y} \Gamma_y \to \mathbb{R}_{>0}, \quad x \mapsto \lambda_Y(y) \cdot \lambda_{\Gamma_y}(x) \quad \text{if } x \in \Gamma_y.$$

Then the next lemma holds:

Lemma 6.3.8. *Let f be a L^1-integrable function on Ω with the property (F). Suppose that (Y, λ_Y) is an weighted $(I_\pi f)$-design on (Ω_2, μ_2) and $(\Gamma_y, \lambda_{\Gamma_y})$ is an weighted $(f|_{\pi^{-1}(y)})$-design on $(\pi^{-1}(y), \mu_y)$ for each $y \in Y$. Then $(X(Y,\Gamma), \lambda_X)$ is an weighted f-design on (Ω_1, μ_1).*

Proof of Lemma 6.3.8. Let us compute $\sum_{x \in X} \lambda_X(x) f(x)$ as follows:

$$\begin{aligned}
\sum_{x \in X} \lambda_X(x) f(x) &= \sum_{y \in Y} \sum_{\gamma_y \in \Gamma_y} \lambda_Y(y) \lambda_{\Gamma_y}(\gamma_y) f(\gamma_y) \\
&= \sum_{y \in Y} \lambda_Y(y) \left(\sum_{\gamma_y \in \Gamma_y} \lambda_{\Gamma_y}(\gamma_y) f(\gamma_y) \right) \\
&= \sum_{y \in Y} \lambda_Y(y) \int_{\pi^{-1}(y)} f \, d\mu_y \\
&= \sum_{y \in Y} \lambda_Y(y) (I_\pi f)(y) \\
&= \int_{\Omega_2} (I_\pi f) d\mu_2 \\
&= \int_{\Omega_1} f d\mu_1.
\end{aligned}$$

This completes the proof. □

Lemma 6.3.8 claims that if we have weighted designs on Ω_2 and that on some fibers, then we have an weighted design on Ω_1. We also consider the converse setting as follows. Let X be a finite subset of Ω_1 and λ_X a positive function on X. We also fix a positive function $\lambda_{\pi(X)}$ on $\pi(X)$. For each $y \in \pi(X)$, let us denote by $\Gamma_y := X \cap \pi^{-1}(y)$ and

$$\lambda_{\Gamma_y} : \Gamma_y \to \mathbb{R}_{>0}, \quad \gamma_y \mapsto \frac{1}{\lambda_{\pi(X)}(y)} \lambda_X(\gamma_y).$$

Then the next lemma holds:

Lemma 6.3.9. *Let f be an L^1-integrable function on Ω with the property (F). Suppose that (X, λ_X) is an weighted f-design on (Ω_1, μ_1) and $(\Gamma_y, \lambda_{\Gamma_y})$ is an weighted $(f|_{\pi^{-1}(y)})$-design on $(\pi^{-1}(y), \mu_y)$ for each $y \in \pi(X)$. Then $(\pi(X), \lambda_{\pi(X)})$ is an weighted $(I_\pi f)$-design on (Ω_2, μ_2).*

The following corollary follows from Lemma 6.3.9 immediately:

Corollary 6.3.10. *Let (X, λ_X) be an weighted f-design on (Ω_1, μ_1), and we put*

$$\lambda_{\pi(X)} : \pi(X) \to \mathbb{R}_{>0}, \quad y \mapsto \sum_{x \in \Gamma_y} \lambda_X(x).$$

Assume that $\mu_y(\pi^{-1}(y)) = 1$ for any $y \in \pi(X)$. Let f be an L^1-integrable function on Ω with the property (F), and suppose that $f|_{\pi^{-1}(y)}$ is constant for each $y \in \pi^{-1}(y)$. Then $(\pi(X), \lambda_{\pi(X)})$ is an weighted $(I_\pi f)$-design on (Ω_2, μ_2).

Proof of Lemma 6.3.9. Let us compute $\sum_{y \in \pi(X)} \lambda_{\pi(X)}(y)(I_\pi f)(y)$ as follows:

$$\begin{aligned}
\sum_{y \in \pi(X)} \lambda_{\pi(X)}(y)(I_\pi f)(y) &= \sum_{y \in \pi(X)} \lambda_{\pi(X)}(y) \int_{\pi^{-1}(y)} f d\mu_y \\
&= \sum_{y \in \pi(X)} \lambda_{\pi(X)}(y) \sum_{\gamma_y \in \Gamma_y} \lambda_{\Gamma_y}(\gamma_y) f(\gamma_y) \\
&= \sum_{y \in \pi(X)} \sum_{\gamma_y \in \Gamma_y} \lambda_X(\gamma_y) f(\gamma_y) \\
&= \sum_{y \in \pi(X)} \sum_{x \in X \cap \pi^{-1}(y)} \lambda_X(x) f(x) \\
&= \sum_{x \in X} \lambda_X(x) f(x) \\
&= \int_{\Omega_1} f d\mu_1 \\
&= \int_{\Omega_2} (I_\pi f) d\mu_2
\end{aligned}$$

This completes the proof. □

Let us take a finite-dimensional vector space $\mathcal{H}$ consisted of L^1-integrable functions on Ω with the property (F). Then,

$$I_\pi \mathcal{H} := \{\, I_\pi f \mid f \in \mathcal{H} \,\},$$
$$\mathcal{H}|_{\pi^{-1}(\omega)} := \{\, f|_{\pi^{-1}(\omega)} \mid f \in \mathcal{H} \,\} \quad \text{for each } \omega \in \Omega_2$$

are also finite-dimensional vector spaces consisted of L^1-integrable functions.

The next propositions will be used for proofs of our main results in Section 6.4.2.

Proposition 6.3.11. *Let (Y, λ_Y) be an weighted $(I_\pi \mathcal{H})$-design on (Ω_2, μ_2) and $(\Gamma_y, \lambda_{\Gamma_y})$ an weighted $\mathcal{H}|_{\pi^{-1}(y)}$-design on $(\pi^{-1}(y), \mu_y)$ for each $y \in Y$. Then $(X(Y,\Gamma), \lambda_X)$ defined as (6.3.1) is an weighted $\mathcal{H}$-design on (Ω_1, μ_1). In particular, suppose that Y is a $(I_\pi \mathcal{H})$-design on (Ω_2, μ_2) with respect to a positive constant λ_Y, and there exists a positive constant λ_Γ such that Γ_y is an $\mathcal{H}|_{\pi^{-1}(y)}$-design on $(\pi^{-1}(y), \mu_y)$ with respect to λ_Γ for any $y \in Y$. Then $X(Y,\Gamma)$ is an $\mathcal{H}$-design on (Ω_1, μ_1) with respect to the constant $\lambda_Y \cdot \lambda_\Gamma$.*

Proposition 6.3.12. *Assume that $\mu_y(\pi^{-1}(y)) = 1$ for any $y \in \Omega_2$ and $\pi^*(I_\pi f) \in \mathcal{H}$ for any $f \in \mathcal{H}$, where $\pi^*(I_\pi f)$ is the pull back of $I_\pi f$ by π. Let (X, λ_X) be an weighted $\mathcal{H}$-design on (Ω_1, μ_1), and put*

$$\lambda_{\pi(X)} : \pi(X) \to \mathbb{R}_{>0}, \quad y \mapsto \sum_{x \in \pi^{-1}(y)} \lambda_X(x).$$

Then, $(\pi(X), \lambda_{\pi(X)})$ is an weighted $(I_\pi \mathcal{H})$-design on (Ω_2, μ_2). In particular, if X is a $\mathcal{H}$-design on (Ω_1, μ_1) with respect to a constant λ_X, and $q = |X \cap \pi^{-1}(y)|$ is constant on $y \in \pi(X)$, then $\pi(X)$ is an $(I_\pi \mathcal{H})$-design on (Ω_2, μ_2) with respect to the constant $q \cdot \lambda_X$.

These propositions are followed by Lemma 6.3.8 and Corollary 6.3.10, respectively.

Remark 6.3.13. *Fact* 6.1.3 *in Section* 6.1.1 *can be obtained by combining Proposition* 6.3.11 *with some arguments for the hight function $h : S^d \to [-1, 1]$.*

By combining Proposition 6.3.5 with Proposition 6.3.11, we also obtain the next corollary for an algorithm to make designs on Ω_1 from a multi-design on Ω_2:

Corollary 6.3.14 (Corollary to Proposition 6.3.11). *Let $Y \in \Omega_2^N$ be a multi-$(I_\pi\mathcal{H})$-design on (Ω_2, μ_2) with respect to a positive constant λ_Y. We use the notation $\overline{Y}$ and $m(\overline{y})$ for each $\overline{y} \in \overline{Y}$ as in Section 6.3.1. Let us fix a positive constant λ_Γ. Assume that for each $\overline{y} \in \overline{Y}$, there exists an $\mathcal{H}|_{\pi^{-1}(\overline{y})}$-design $\Gamma_{\overline{y}}$ on $(\pi^{-1}(y), \mu_y)$ with respect to the constant $\frac{1}{m(\overline{y})}\lambda_\Gamma$. We take such $\Gamma_{\overline{y}}$ for each $\overline{y} \in \overline{Y}$. Then*

$$X(\overline{Y}, \Gamma) := \bigsqcup_{\overline{y} \in \overline{Y}} \Gamma_{\overline{y}}$$

is an $\mathcal{H}$-design on (Ω_1, μ_1) with respect to $\lambda_Y \cdot \lambda_\Gamma$.

6.4 Proofs of results in Section 6.2

In this section, we prove Theorem 6.2.4, Theorem 6.2.7, Lemma 6.2.9 and Theorem 6.2.12 by using the results in Section 6.3.2.

6.4.1 Local trivializations of the Hopf map

In this subsection, we recall local trivializations of the Hopf map $\pi : S^3 \to S^2$ defined in Section 6.2.

Let us take an open covering $\{U_+, U_-\}$ of $S^2 \subset \mathbb{R} \times \mathbb{C}$ as

$$U_+ = \{ (\xi, \eta) \in S^2 \mid \xi \neq -1 \}, \; U_- = \{ (\xi, \eta) \in S^2 \mid \xi \neq 1 \}.$$

Then we have local trivializations of the S^1-bundle $\pi : S^3 \to S^2$ as

$$U_+ \times S^1 \xrightarrow{\sim} \pi^{-1}(U_+), \quad ((\xi, \eta), z) \mapsto \left(\sqrt{\frac{1+\xi}{2}} z, \sqrt{\frac{1}{2(1+\xi)}} \eta \overline{z} \right),$$

$$U_- \times S^1 \xrightarrow{\sim} \pi^{-1}(U_-), \quad ((\xi, \eta), z) \mapsto \left(\sqrt{\frac{1}{2(1-\xi)}} \eta z, \sqrt{\frac{1-\xi}{2}} \overline{z} \right).$$

In particular, for any element $y = (\xi, \eta) \in U_+$, the fiber $\pi^{-1}(y)$ can be written by

$$\pi^{-1}(y) = \left\{ \left(\sqrt{\frac{1+\xi}{2}} z, \sqrt{\frac{1}{2(1+\xi)}} \eta \overline{z} \right) \mid z \in S^1 \right\} \subset S^3.$$

Similarly, for any element $y \in (\xi, \eta) \in U_-$, we have

$$\pi^{-1}(y) = \left\{ \left(\sqrt{\frac{1}{2(1-\xi)}} \eta z, \sqrt{\frac{1-\xi}{2}} \overline{z} \right) \mid z \in S^1 \right\} \subset S^3.$$

Remark 6.4.1. *In Theorem* 6.2.4, *we need to take a base point* $s(y)$ *on* $\pi^{-1}(y)$ *for a given* $y \in S^2$. *By using the explicit form of* $\pi^{-1}(y)$ *above, one can choose* $s(y)$ *explicitly.*

6.4.2 Proof of Theorem 6.2.4 and Theorem 6.2.7

Throughout this subsection, we denote by $\mu'_{S^d} := \frac{1}{|S^d|}\mu_{S^d}$. Then μ'_{S^d} is the $O(d+1)$-invariant Haar measure on S^d with $\mu'_{S^d}(S^d) = 1$.

Let $\pi : S^3 \to S^2$ be the Hopf map defined in Section 6.2.2. For simplicity, we fix a base point $s(y)$ on a fiber $\pi^{-1}(y)$ for each $y \in S^2$. Note that we do not assume that the map $s : S^2 \to S^3$ with $s \circ \pi = \mathrm{id}_{S^2}$ is continuous (in fact, such a continuous map does not exist). Then we have an isomorphism

$$\iota_y : S^1 \to \pi^{-1}(y), \quad z \mapsto s(y) \cdot z.$$

For each $y \in S^2$, we consider the induced measure μ'_y on $\pi^{-1}(y)$ by the normalized measure μ'_{S^1} on S^1. Such the probability measure μ'_y on $\pi^{-1}(y)$ does not depend on the choice of the base point $s(y)$ since μ'_{S^1} is invariant by the S^1-action.

To prove Theorem 6.2.4 and Theorem 6.2.7, we show the next two lemmas.

Lemma 6.4.2. *Any* L^1*-integrable function on* S^3 *satisfies the property* (F) *with respect to the Hopf map* $\pi : S^3 \to S^2$, *the normalized spherical measures* μ'_{S^3}, μ'_{S^2} *and the measure* μ'_y *on* $\pi^{-1}(y)$ *for each* $y \in S^2$ *defined above (see Section* 6.3.2 *for the definition of the property* (F)*).*

Lemma 6.4.3. *For any* $t \in \mathbb{N}$, *we have*

$$\iota_y^*(P_t(S^3)|_{\pi^{-1}(y)}) = P_t(S^1) \quad \text{for any } y \in S^2,$$
$$I_\pi(P_t(S^3)) = P_{\lfloor \frac{t}{2} \rfloor}(S^2) \text{ and } \pi^*(P_{\lfloor \frac{t}{2} \rfloor}(S^2)) \subset P_t(S^3)$$

(see Section 6.3.2 *for the definition of* I_π*).*

Theorem 6.2.4 follows from Proposition 6.3.11 and Lemma 6.4.3. Theroem 6.2.7 also follows from Proposition 6.3.12 and Lemma 6.4.3.

Proof of Lemma 6.4.2. Let us denote by

$$S^3 = \{\, ((\cos\varphi)e^{\sqrt{-1}\theta_1}, (\sin\varphi)e^{\sqrt{-1}\theta_2}) \mid 0 \le \varphi \le \frac{\pi}{2}, 0 \le \theta_1, \theta_2 < 2\pi \,\} \subset \mathbb{C}^2,$$
$$S^2 = \{\, (\cos\psi, (\sin\psi)e^{\sqrt{-1}\phi}) \mid 0 \le \psi \le \pi,\ 0 \le \phi < 2\pi \,\} \subset \mathbb{R} \times \mathbb{C},$$
$$S^1 = \{\, e^{\sqrt{-1}\theta} \mid 0 \le \theta < 2\pi \,\} \subset \mathbb{C}.$$

Then the volume forms corresponding to the normalized measures μ'_{S^d} ($d = 1, 2, 3$) can be written by

$$d\mu'_{S^3} = \frac{1}{4\pi^2}(\sin 2\varphi)d\varphi d\theta_1 d\theta_2 \quad (0 \le \varphi \le \frac{\pi}{2}, 0 \le \theta_1, \theta_2 < 2\pi),$$
$$d\mu'_{S^2} = \frac{1}{4\pi}(\sin\psi)d\psi d\phi, \quad (0 \le \psi \le \pi,\ 0 \le \phi < 2\pi)$$
$$d\mu'_{S^1} = \frac{1}{2\pi}d\theta \quad (0 \le \theta < 2\pi)$$

Here we put

$$U_+ = \{\, (\cos\psi, (\sin\psi)e^{\sqrt{-1}\phi}) \mid 0 \le \psi < \pi,\ 0 \le \phi < 2\pi \,\} \subset S^2,$$
$$\pi^{-1}(U_+) = \{\, ((\cos\varphi)e^{\sqrt{-1}\theta_1}, (\sin\varphi)e^{\sqrt{-1}\theta_2}) \mid 0 \le \varphi < \frac{\pi}{2}, 0 \le \theta_1, \theta_2 < 2\pi \,\} \subset S^3.$$

Then the isomorphism between $U_+ \times S^1$ and $\pi^{-1}(U_+)$ given in Section 6.4.1 can be written by

$$U_+ \times S^1 \to \pi^{-1}(U_+),$$
$$(\cos\psi, (\sin\psi)e^{\sqrt{-1}\phi}, e^{\sqrt{-1}\theta}) \mapsto ((\cos\frac{\psi}{2})e^{\sqrt{-1}\theta}, (\sin\frac{\psi}{2})e^{\sqrt{-1}(\phi-\theta)}).$$

Under this isomorphism, we have

$$\varphi = \frac{\psi}{2},\ \theta_1 = \theta,\ \theta_2 = \phi - \theta.$$

Thus,

$$\begin{aligned} d\mu'_{\pi^{-1}(U_+)} &= \frac{1}{4\pi^2}(\sin 2\varphi)d\varphi d\theta_1 d\theta_2 \\ &= \frac{1}{8\pi^2}(\sin\psi)d\psi d\phi d\theta \\ &= d\mu'_{U_+} d\mu'_{S^1}, \end{aligned}$$

where $d\mu'_{\pi^{-1}(U_+)} = d\mu'_{S^3}|_{\pi^{-1}(U_+)}$ and $d\mu'_{U_+} = d\mu'_{S^2}|_{U_+}$. Therefore, we can apply Fubini's theorem for

$$\pi^{-1}(U_+) \simeq U_+ \times S^1.$$

Here, one can observe that $\mu'_{S^2}(S^2 \setminus U_+) = 0$ and $\mu'_{S^3}(S^3 \setminus \pi^{-1}(U_+)) = 0$. In particular, for any L^1-integrable function f on S^3, we have

$$\begin{aligned}\int_{S^3} f d\mu'_{S^3} &= \int_{\pi^{-1}(U_+)} f d\mu'_{\pi^{-1}(U_+)} \\ &= \int_{(\xi,\eta)\in U_+} \int_{z\in S^1} f(\xi,\eta,z) d\mu'_{S^1}(z) d\mu'_{U_+}(\xi,\eta) \\ &= \int_{U_+} (I_\pi f) d\mu'_{U_+} \\ &= \int_{S^2} (I_\pi f) d\mu'_{S^2}.\end{aligned}$$

This completes the proof. □

Proof of Lemma 6.4.3. First, we shall prove that

$$\iota_y^*(P_t(S^3)|_{\pi^{-1}(y)}) = P_t(S^1) \quad \text{for any } y \in S^2, \tag{6.4.1}$$

$$I_\pi(P_t(S^3)) \subset P_{\lfloor \frac{t}{2} \rfloor}(S^2). \tag{6.4.2}$$

Let us fix any n and denote by $f_{i,j,k,l}(a,\overline{a},b,\overline{b}) := a^i\overline{a}^j b^k \overline{b}^l$ the monomial on $\mathbb{R}^4 \simeq \mathbb{C}^2$ of degree $n = i+j+k+l$. We also denote by the same letter $f_{i,j,k,l}$ the restricted function on S^3 of the monomial $f_{i,j,k,l}(a,\overline{a},b,\overline{b})$. By Lemma 6.4.2, the function $f_{i,j,k,l}$ on S^3 satisfies the property (F). To prove (6.4.1) and (6.4.2), we only need to show the following:

- $\iota_y^*(f_{i,j,k,l}|_{\pi^{-1}(y)}) \in P_n(S^1)$ for any $y \in S^2$,
- $I_\pi f_{i,j,k,l} \in P_{\lfloor \frac{n}{2} \rfloor}(S^2)$.

For each $y = (\xi, \eta) \in S^2$, by the explicit formula of the fiber $\pi^{-1}(y)$ given in Section 6.4.1, there exists a constant $c_{i,j,k,l}(y) \in \mathbb{C}$ such that

$$f_{i,j,k,l}(\iota_y(z)) = c_{i,j,k,l}(y) z^{i-j-k+l}.$$

Thus, $\iota_y^*(f_{i,j,k,l}|_{\pi^{-1}(y)}) \in P_n(S^1)$. In the cases where $n = i+j+k+l$ is odd, we have $i-j-k+l \neq 0$. Therefore,

$$(I_\pi f_{i,j,k,l})(y) = c_{i,j,k,l}(y) \int_{S^1} z^{i-j-k+l} d\mu'_{S^1} = 0 \quad \text{for any } y \in S^2.$$

Let us consider the cases where $n = i+j+k+l = 2m$ is even. Here for $0 \leq i, k \leq m$, we define a polynomial $F_{i,k,m}(\xi, \eta, \overline{\eta})$ on $\mathbb{R}^3 \simeq \mathbb{R} \times \mathbb{C}$ by

$$F_{i,k,m}(\xi,\eta,\overline{\eta}) = \begin{cases} \frac{1}{2^m}(1+\xi)^{i-k}\eta^k\overline{\eta}^{m-i} & \text{if } k \leq i \leq m, \\ \frac{1}{2^m}(1-\xi)^{k-i}\eta^i\overline{\eta}^{m-k} & \text{if } i \leq k \leq m. \end{cases}$$

Then $\deg F_{i,k,m} = m$. For each $y = (\xi, \eta) \in S^2$, by using $|\eta|^2 = 1 - \xi^2$, one can compute that

$$\int_{\pi^{-1}(y)} f_{i,j,k,l} d\mu_y = \begin{cases} F_{i,k,m}(y) & \text{if } i+l = m = j+k \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have $I_\pi f_{i,j,k,l} \in P_m(S^2) = P_{\frac{n}{2}}(S^2)$.

Let us take a monomial $h_{i,j,k}(\xi,\eta,\overline{\eta}) := \xi^i \eta^j \overline{\eta}^l$ on $\mathbb{R}^3 \simeq \mathbb{R} \times \mathbb{C}$ of degree $i+j+k = n$. We also denote by the same letter $h_{i,j,k}$ the restricted function on S^2 of the monomial $h_{i,j,k}(\xi,\eta,\overline{\eta})$. Since $\mu'_y(\pi^{-1}(y)) = 1$, we have that $I_\pi(\pi_* h_{i,j,k}) = h_{i,j,k}$. Therefore, to complete the proof of Lemma 6.4.3, we only need to prove that

$$\pi^* h_{i,j,k} \in P_{2n}(S^3).$$

The function $\pi^* h_{i,j,k}$ on S^3 can be written by

$$\begin{aligned} (\pi^* h_{i,j,k})(a,b) &= h_{i,j,k}(\pi(a,b)) \\ &= (|a|^2 - |b|^2)^i (2ab)^j (\overline{2ab})^k \\ &= 2^{j+k}(a\overline{a} - b\overline{b})^i a^j \overline{a}^k b^j \overline{b}^k. \end{aligned}$$

Hence, we have $\pi^* h_{i,j,k} \in P_{2n}(S^3)$. This completes the proof. □

6.4.3 Proof of Corollary 6.2.6

By combining Corollary 6.3.14 with Lemma 6.4.3, we have a genelalization of Theorem 6.2.4 as follows:

Theorem 6.4.4. *Let $Y = (y_1, \dots, y_N) \in (S^2)^N$ be a spherical multi-t-design, and $\Gamma \subset S^1$ a spherical 2t-design [resp. (2t + 1)-design]. For each $i = 1, \dots, N$, we fix a base point $s(y_i)$ on $\pi^{-1}(y_i)$, and denote by*

$$\Gamma_i := \{ s(y_i) \cdot \gamma \mid \gamma \in \Gamma \} \subset \pi^{-1}(y_i).$$

Suppose that $\Gamma_1, \dots, \Gamma_N$ are disjoint from each other in S^3. Then

$$X(Y, s, \Gamma) := \bigsqcup_{i=1}^{N} \Gamma_i \subset S^3$$

is a spherical 2t-design [resp. (2t+1)-design] on S^3 with $|X(Y, s, \Gamma)| = N \cdot |\Gamma|$.

We give a proof of Corollary 6.2.6 as follows:

Proof of Corollary 6.2.6. Let us fix a a spherical multi-t-design $Y := (y_1, \dots, y_N) \in (S^2)^N$. We only need to show that there exists a spherical $2t$-design [resp. $(2t+1)$-design] X on S^3 with $|X| = (2t + 1)N$ [resp. $|X| = 2(t + 1)N$]. Recall that the finite cyclic subgroup Γ_q of S^1 of order q is a spherical t-design on S^1 if $q \geq t + 1$. First, we remark that for any fixed q, one can choose $s_q(y_i) \in \pi^{-1}(y_i)$ for each $i = 1, \dots, N$ such that

$$(\Gamma_q)_i \cap (\Gamma_q)_j = \emptyset \quad \text{for any } i, j \text{ with } i \neq j,$$

where

$$(\Gamma_q)_i := \{ s_q(y_i) \cdot \gamma \mid \gamma \in \Gamma_q \} \subset \pi^{-1}(y_i).$$

Hence, by Theorem 6.4.4, we obtain a spherical $2t$-design $X(Y, s_{2t+1}, \Gamma_{2t+1})$ on S^3 [resp. $(2t + 1)$-design $X(Y, s_{2t+2}, \Gamma_{2t+2})$ on S^3] with $|X(Y, s_{2t+1}, \Gamma_{2t+1})| = (2t + 1)N$ [resp. $|X(Y, s_{2t+2}, \Gamma_{2t+2})| = 2(t + 1)N$]. □

6.4.4 Proof of Lemma 6.2.9 and Theorem 6.2.12

Recall that $\pi : S^3 \to S^2$ is an S^1-principal bundle. Therefore, for each $x_1, x_2 \in S^3$, $\pi(x_1) = \pi(x_2)$ if and only if there exists $z \in S^1$ such that $x_2 = x_1 \cdot z$. We also note that $x \cdot z = \pm x$ if and only if $z = \pm 1$ for each $x \in S^3$. Therefore, to prove Lemma 6.2.9, we only need to show the following lemma:

Lemma 6.4.5. *For any open neighberhood U of the unit of $SO(4)$, there exists $\sigma \in U$ such that $\pi(\sigma(x)) \neq \pi(\sigma(x \cdot z))$ for any $x \in S^3$ and $z \in S^1 \backslash \{\pm 1\}$.*

Proof of Lemma 6.4.5. For each $\theta \in \mathbb{R}$, we define $\sigma_\theta \in SO(4)$ by

$$\sigma_\theta(a,b) = (a\cos\theta - \sqrt{-1}b\sin\theta, -\sqrt{-1}a\sin\theta + b\cos\theta) \quad \text{for any } (a,b) \in S^3.$$

Note that $\lim_{\theta\to 0}\sigma_\theta = e$, where e is the unit of $SO(4)$. Then for any $\theta \in \mathbb{R}$, $(a,b) \in S^3$ and any $z \in S^1$, we have

$$\begin{aligned}
&\pi(\sigma_\theta(a,b)) - \pi(\sigma_\theta((a,b)\cdot z)) \\
&\quad = \pi(\sigma_\theta(a,b)) - \pi(\sigma_\theta(az, b\overline{z})) \\
&\quad = \sqrt{-1}\sin 2\theta((1-z^2)a\overline{b} - (1-\overline{z}^2)\overline{a}b, (z^2-1)a^2 + (\overline{z}^2-1)b^2) \in \mathbb{R}\times\mathbb{C}
\end{aligned}$$

Let us fix any θ with $0 < \theta < \frac{\pi}{2}$. Then $\sin 2\theta \neq 0$. Thus we have $\pi(\sigma_\theta(a,b)) \neq \pi(\sigma_\theta((a,b)\cdot z))$ if $z \neq \pm 1$. This completes the proof. □

Recall that Theorem 6.2.10 follows from Lemma 6.2.9. We give a proof of Theorem 6.2.12 by using Theorem 6.2.10 and Theorem 6.4.4.

Proof of Theorem 6.2.12. Let $Y = (y_1, \dots, y_N) \in (S^2)^N$ be a spherical multi-t-design on S^2. To prove Theorem 6.2.12, we show that there exists a spherical t-design Y' on S^2 with $|Y'| = (t+1)N$. By Theorem 6.4.4 and the proof of Corollary 6.2.6 in Section 6.4.3, we can find a spherical $(2t+1)$-design $X := X(Y, s_{2t+2}, \Gamma_{2t+2})$ on S^3 with $|X| = 2(t+1)N$. Note that X is antipodal on S^3 since Γ_{2t+2} is antipodal on S^1. Hence by Theorem 6.2.10, there exists $\sigma \in SO(4)$ such that $Y' := \pi(\sigma X)$ is a spherical t-design on S^2 with $|Y'| = \frac{1}{2}|X| = (t+1)N$. □

6.5 Generalization to compact symmetric spaces

In this section, we generalize our results in Section 6.2 to some relations between designs on a compact Lie group G and that on a compact homogeneous space G/K.

6.5.1 Designs on a compact homogeneous space

We consider the following setting:

Setting 6.5.1. *G is a compact Lie group. K is a closed subgroup of G.*

Let us denote by G/K the quotient space of G by K, and write

$$\pi : G \to G/K, \ g \mapsto gK$$

for the quotient map. It is well known that the subgroup K is also a compact Lie group and the quotient space G/K has a C^∞-manifold structure such that the map $\pi : G \to G/K$ is a C^∞-submersion. Furthermore, $\pi : G \to G/K$ admits a K-principal bundle structure with respect to the natural right K-action on G. Let us denote by μ_G, μ_K the two-sided Haar measure on G and K with $\mu_G(G) = \mu_K(K) = 1$, respectively. We also denote by $\mu_{G/K}$ the left G-invariant Haar measure on G/K with $\mu_{G/K}(G/K) = 1$. In other words, for any continuous function h on G/K, we put

$$\int_{G/K} h d\mu_{G/K} := \int_G (\pi^* h)\mu_G, \tag{6.5.1}$$

where $\pi^* f$ is the pull back of f by π.

Example 6.5.2. *Let $G = SU(2)$ and $K = T$ be a maximal torus of $SU(2)$. Then G, K and G/K are isomorphic to S^3, S^1 and S^2, respectively. Furthermore, the quotient map $\pi : G \to G/K$ can be regard as a Hopf map. The measures μ_G, μ_K and $\mu_{G/K}$ correspond to the normalized spherical measures μ'_{S^3}, μ'_{S^1} and μ'_{S^2} defined in Section 6.4.2, respectively (see Section 6.B for more details).*

Let $\Omega = G, K$ or G/K. We note that any continuous function on Ω is L^1-integrable since Ω is compact. Therefore, for a vector space $\mathcal{H}$ consisted of continuous functions on Ω, we can define (weighted, multi) $\mathcal{H}$-designs on (Ω, μ_Ω) as in Section 6.3.1.

Let us assume that $\dim \mathcal{H} < \infty$. If Ω is connected, the existence of an $\mathcal{H}$-design on Ω follows from [18, Main Theorem in Section 1]. Even if Ω is non-compact in our setting, the existence of an $\mathcal{H}$-design on Ω will be proved in Section 6.6.1 as the next proposition:

Proposition 6.5.3. *Let $\Omega := G, K$ or G/K in Setting 6.5.1, and $\mathcal{H}$ a finite-dimensional vector space consisted of continuous functions on Ω. Then there exists an $\mathcal{H}$-design on Ω.*

Let us put $C^0(G)$, $C^0(K)$ and $C^0(G/K)$ to the space of $\mathbb{C}$-valued continuous functions on G, K and G/K, respectively. To state our results, for a finite-dimensional complex representation (ρ, V) of G, we shall define subspaces $\mathcal{H}^\rho_G$, $\mathcal{H}^\rho_K$ and $\mathcal{H}^\rho_{G/K}$ of $C^0(G)$, $C^0(K)$ and $C^0(G/K)$, respectively as follows (cf. [19, Chapter I, §1]):

Definition of $\mathcal{H}^\rho_G$ Let us denote by $V^\vee$ the dual space of V, i.e. $V^\vee$ is the vector space consisted of all $\mathbb{C}$-linear maps from V to $\mathbb{C}$. We define a $\mathbb{C}$-linear map $\Phi : V \otimes V^\vee \to C^0(G)$ by

$$\Phi(v \otimes \varphi)(g) := \langle \rho(g^{-1})v, \varphi \rangle \quad \text{for } v \in V,\ \varphi \in V^\vee \text{ and } g \in G.$$

We put

$$\mathcal{H}^\rho_G := \Phi(V \otimes V^\vee).$$

Definition of $\mathcal{H}^\rho_K$ Let us denote by

$$\mathcal{H}^\rho_K := \{ f|_K \mid f \in \mathcal{H}^\rho_K \} \subset C^0(K).$$

Note that $\mathcal{H}^\rho_K$ depends only on the representation $\rho|_K$ of K.

Definition of $\mathcal{H}^\rho_{G/K}$ We write

$$(V^\vee)^K := \{ \varphi \in V^\vee \mid \varphi \circ (\rho(k)) = \varphi : V \to \mathbb{C} \text{ for any } k \in K \},$$

and define a $\mathbb{C}$-linear map $\widehat{\Phi} : V \otimes (V^\vee)^K \to C^0(G/K)$ by

$$\widehat{\Phi}(v \otimes \psi)(gK) := \langle \rho(g^{-1})v, \psi \rangle \quad \text{for } v \in V,\ \psi \in (V^\vee)^K \text{ and } g \in G.$$

One can observe that $\widehat{\Phi}$ is well-defined and we put

$$\mathcal{H}^\rho_{G/K} := \widehat{\Phi}(V \otimes (V^\vee)^K).$$

We give two easy observations for $\mathcal{H}^\rho_\Omega$ as follows:

Observation 6.5.4. • *For two finite-dimensional representations (ρ_1, V_1) and (ρ_2, V_2) of G, we have $\mathcal{H}^{\rho_1 \oplus \rho_2}_\Omega = \mathcal{H}^{\rho_1}_\Omega + \mathcal{H}^{\rho_2}_\Omega$ for each $\Omega = G, K$ and G/K.*

- *If (ρ_1, V_1) and (ρ_2, V_2) are isomorphic from each other, then $\mathcal{H}^{\rho_1}_\Omega = \mathcal{H}^{\rho_2}_\Omega$ for each $\Omega = G, K$ and G/K. In particular, $\mathcal{H}^{\rho_1 \oplus \rho_2}_\Omega = \mathcal{H}^{\rho_1}_\Omega$.*

In the rest of this section, we consider $\mathcal{H}^\rho_\Omega$-designs on $\Omega = G, K$ or G/K.

Example 6.5.5. *Let us take $(G, K) = (SU(2), T)$ as in Example* 6.5.2. *Then for any $l = 0, 1, 2, \ldots$, there uniquely exists an irreducible representation (ρ_l, V_l) of $SU(2)$ with $\dim_{\mathbb{C}} V_l = l + 1$, up to isomorphisms. For any t, let us consider the representation $\rho(t) := \bigoplus_{l=0}^{t} \rho_l$ of G. Then, under the isomorphisms $G \simeq S^3$, $K \simeq S^1$ and $G/K \simeq S^2$, we have $\mathcal{H}_G^{\rho(t)} = P_t(S^3)$, $\mathcal{H}_K^{\rho(t)} = P_t(S^1)$ and $\mathcal{H}_{G/K}^{\rho(t)} = P_{\lfloor \frac{t}{2} \rfloor}(S^2)$ (see Section 6.B for more details).*

Remark 6.5.6. *Let us take $(G, K) = (SO(d+1), SO(d))$. Then a spherical t-designs on S^d can be regard as an $\mathcal{H}_{S^d}^{\rho}$-designs on $S^d \simeq SO(d+1)/SO(d)$, by taking a suitable ρ. For any rank one compact symmetric space $\Omega = G/K$, one can find a suitable representation ρ of G, such that a t-designs on Ω can be regard as an $\mathcal{H}_{\Omega}^{\rho}$-designs on Ω (see [4] for the definition of designs on rank one compact symmtric space). For some higher rank compact symmetric spaces and some homogeneous spaces $\Omega = G/K$, definitions of designs on Ω were given by [1, 13, 15, 16, 17]. We also remark that each of them can be regard as $\mathcal{H}_{\Omega}^{\rho}$-designs on Ω for some ρ.*

The following fundamental properties of $\mathcal{H}_{\Omega}^{\rho}$-designs will be proved in Section 6.6.2:

Proposition 6.5.7. *Let (ρ, V) be a finite-dimensional unitary representation of G. Then the following hold:*

(i) If X is an $\mathcal{H}_G^{\rho}$-design on G, then for any $g_1, g_2 \in G$, the subset

$$g_1 X g_2 := \{\, g_1 x g_2 \mid x \in X \,\} \subset G$$

is also an $\mathcal{H}_G^{\rho}$-design on G.

(ii) If Γ is an $\mathcal{H}_K^{\rho}$-design on K, then for any $k_1, k_2 \in K$, the subset

$$k_1 \Gamma k_2 := \{\, k_1 \gamma k_2 \mid \gamma \in \Gamma \,\} \subset K$$

is also an $\mathcal{H}_K^{\rho}$-design on K.

(iii) If Y is an $\mathcal{H}_{G/K}^{\rho}$-design on G/K, then for any $g \in G$, the subset

$$gY := \{\, gy \mid y \in Y \,\} \subset G/K$$

is also an $\mathcal{H}_{G/K}^{\rho}$-design on G/K.

6.5.2 Results for designs on a compact homogeneous space

Throughout this subsection, let us fix a finite-dimensional complex representation (ρ, V) of G. Recall that we defined a functional spaces $\mathcal{H}^\rho_\Omega$ on Ω for each $\Omega = G, K$ or G/K in the previous subsection.

We give a generalization of Theorem 6.2.4 as follows:

Theorem 6.5.8. *Let Y be an $\mathcal{H}^\rho_{G/K}$-design on G/K, and Γ an $\mathcal{H}^\rho_K$-design on K. We fix a map $s : Y \to G$ such that $\pi \circ s = id_Y$. Let us put*

$$X(Y, s, \Gamma) := \{\, s(y)\gamma \mid y \in Y,\ \gamma \in \Gamma \,\} \subset G.$$

Then $X(Y, s, \Gamma)$ is an $\mathcal{H}^\rho_G$-design on G.

Remark 6.5.9. *Let G be a finite group, K a subgroup of G, and (ρ, V) a finite-dimensional complex representation of G. Then K itself is an $\mathcal{H}^\rho_K$-design on K. Thus, by Theorem 6.5.8, for any $\mathcal{H}^\rho_{G/K}$-design Y on G/K, the finite subset $X := \pi^{-1}(Y)$ of G is an $\mathcal{H}^\rho_G$-design on G. This fact was already proved by T. Ito [10].*

Theorem 6.5.8 will be proved in Section 6.6.3.

The next corollary followed from Theorem 6.5.8 immediately:

Corollary 6.5.10. *For a fixed finite-dimensional complex representation (ρ, V) of G,*

$$N_G(\rho) \leq N_K(\rho) \cdot N_{G/K}(\rho),$$

where $N_\Omega(\rho)$ denotes the smallest cardinality of an $\mathcal{H}^\rho_\Omega$-design on Ω.

We also denote by

$$N^{\mathrm{multi}}_\Omega(\rho) := \min\{\, N \in \mathbb{N} \mid \text{there exists a multi-}\mathcal{H}^\rho_\Omega\text{-design in } \Omega^N \,\}.$$

Then $N_\Omega(\rho) \geq N^{\mathrm{multi}}_\Omega(\rho)$ in general.

Let us consider Setting 6.5.1 and suppose $\dim K > 1$. In this case, we obtain an improvement of Corollary 6.5.10 as follows:

Theorem 6.5.11. *In Setting 6.5.1, we suppose that $\dim K > 1$. Then for any finite-dimensional complex representation ρ of G,*

$$N_G(\rho) \leq N_K(\rho) \cdot N^{multi}_{G/K}(\rho).$$

We will prove Theorem 6.5.11 in Section 6.6.4.

As a generalization of Theorem 6.2.7, we will also prove the following theorem in Section 6.6.3:

Theorem 6.5.12. *Let $X = (x_1, \dots, x_N) \in G^N$ be a multi-$\mathcal{H}^\rho_G$-design on G. Then $Y := (\pi(x_1), \dots, \pi(x_N)) \in (G/K)^N$ is a multi-$\mathcal{H}^\rho_{G/K}$-design on G/K.*

Hence, we obtain the following corollary, which gives an algorithm to make a $\mathcal{H}^\rho_{G/K}$-design on G/K from an $\mathcal{H}^\rho_G$-design on G with a certain condition:

Corollary 6.5.13. *Let X be an $\mathcal{H}^\rho_G$-design on G and fix $p \in \mathbb{N}$. If $|X \cap \pi^{-1}(\pi(x))| = p$ for any $x \in X$, then $\pi(X)$ is an $\mathcal{H}^\rho_{G/K}$-design on G/K with $|\pi(X)| = \frac{1}{p}|X|$.*

6.5.3 Results for designs on a compact symmetric space

To state a generalization of Theorem 6.2.10, we need more assumptions for (G, K).

Throughout this subsection, we consider the following setting:

Setting 6.5.14. *G is a connected compact semisimple Lie group. $\tau : G \to G$ is an involutive homeomorphism on G such that $\operatorname{Lie} G^\tau$ contains no simple factor of $\operatorname{Lie} G$, where $G^\tau := \{ g \in G \mid \tau(g) = g \}$. K is a closed subgroup of G^τ with $\operatorname{Lie}(K) = \operatorname{Lie}(G^\tau)$.*

Then G/K becomes a compact symmetric space with respect to the canonical affine connection on G/K. Note that a connected compact Lie group G is semisimple if and only if the center of $\operatorname{Lie} G$ is trivial. In this setting, we give a generalization of Theorem 6.2.10 and Theorem 6.2.12 below.

We denote the center of G by

$$Z_G := \{ g_0 \in G \mid g_0 g {g_0}^{-1} = g \text{ for any } g \in G \}.$$

Let us put

$$Z_K(G) := K \cap Z_G.$$

Since G is semisimple, Z_G and is finite, and hence $Z_K(G)$ too.

Definition 6.5.15. *Let X be a subset of G. For $p \in \mathbb{N}$, we say that X has p-multiplicity for $Z_K(G)$ if*

$$|X \cap x Z_K(G)| = p \quad \text{for any } x \in X.$$

Since $x \in xZ_G(K)$ for any $x \in S^3$, we have $1 \leq |X \cap xZ_K(G)| \leq |Z_K(G)|$ for any subset X of G. Hence, if X has a p-multiplicity for $Z_K(G)$ then $1 \leq p \leq |Z_G(K)|$.

Example 6.5.16. *Let us take $(G,K) = (SU(2),T)$ as in Example 6.5.2. Then (G,K) is in Setting 6.5.14 by taking a certain involution $\tau : G \to G$. In this case, $Z_K(G) = \pm I_2$, where I_2 is the unit of $G = SU(2)$. Thus, a finite subset X of $G \simeq S^3$ has 1-multiplicity [resp. 2-multiplicity] if X is antipodal-free [resp. antipodal] on S^3 (see Section 6.B for more details).*

As a generalization of Lemma 6.2.9, we will prove the next proposition in Section 6.6.5:

Proposition 6.5.17. *We consider a symmetric pair (G,K) in Setting 6.5.14. Let X be a finite subset of G with p-multiplicity for $Z_K(G)$. Then for any open neighberhood U of the unit of G, there exists $g \in U$ such that $|Xg \cap \pi^{-1}(y)| = p$ for any $y \in \pi(Xg)$.*

Recall that by Proposition 6.5.7, for any $\mathcal{H}^\rho_G$-design X on G and any element g of G, the finite subset Xg is also an $\mathcal{H}^\rho_G$-design on G. Therefore, by combining Corollary 6.5.13 with Proposition 6.5.17, we obtain the next theorem:

Theorem 6.5.18. *We consider a symmetric pair (G,K) in Setting 6.5.14 and fix a finite-dimensional complex representation ρ of G. Then for any $\mathcal{H}^\rho_G$-design X on G with p-multiplicity for $Z_K(G)$ and any open neighborhood U of the unit of G, there exists $g \in U$ such that $Y := \pi(Xg)$ is an $\mathcal{H}^\rho_{G/K}$-design on G/K with $|Y| = \frac{1}{p}|X|$.*

Recall that $Z_K(G)$ is closed in K. We denote by $K/Z_K(G)$ the quotient space. For a finite-dimensional complex representation (ρ, V) of K, we define the functional space $\mathcal{H}^\rho_{K/Z_K(G)}$ on $K/Z_K(G)$ in the sense of Section 6.5.1. In particular, for a finite-dimensional complex representation (ρ, V) of G, we consider the representation $\rho|_K$ of K and put $\mathcal{H}^\rho_K$ to the corresponding functional space on $K/Z_K(G)$ for simplicity.

Then the next theorem holds:

Theorem 6.5.19. *We consider a symmetric pair (G,K) in Setting 6.5.14 and assume that $\dim K \geq 1$. Then for any finite-dimensional complex representation (ρ, V) of G,*

$$N_{G/K}(\rho) \leq N_{K/Z_K(G)}(\rho) \cdot N^{multi}_{G/K}(\rho),$$

where $N_\Omega(\rho)$ denotes the smallest cardinality of $\mathcal{H}^\rho_\Omega$-design on Ω and

$$N^{multi}_\Omega(\rho) := \{ N \in \mathbb{N} \mid \text{there exists a multi } \mathcal{H}^\rho_\Omega\text{-design in } \Omega^N \}.$$

Theorem 6.5.19 will be proved as a more general theorem in Section 6.5.4.

Example 6.5.20. *Let us take $(G, K) = (SU(2), T)$ as in Example 6.5.2. Then $K/Z_K(G) = S^1/\pm 1$. We consider $(\rho(2t), V(2t))$ as in Example 6.5.5. Then $N_{K/Z_K(G)}(\rho(2t))$ means the half of smallest cardinality of an antipodal 2t-design on S^1, and hence $N_{K/Z_K(G)}(\rho(2t)) = t+1$. Therefore, Theorem 6.5.19 is a generalization of Theorem 6.2.12 (see Section 6.B for more details).*

6.5.4 Generalization of Theorem 6.5.19

Throughout this subsection, we consider the following setting:

Setting 6.5.21. *G is a connected compact semisimple Lie group. K is a closed subgroup of G. (G, τ', K') is in Setting 6.5.14 in Section 6.5.3. Assume that $\dim K \geq 1$ and $Z_K(G) = Z_{K'}(G)$, where Z_G denotes the center of G and $Z_H(G) := Z_G \cap H$ for $H = K$ or K'.*

For simplicity, we denote by $\Omega := G/K$ [resp. $\Omega' := G/K'$] and put $\pi : G \to \Omega$ [resp. $\pi' : G \to \Omega'$] the quotient map. Note that we do not assume that $K \subset K'$ nor $K \subset K'$. Thus we do not take a canonical map between Ω and Ω'. Let us also put $Z := Z_K(G) = Z_{K'}(G)$. We remark that Z is finite, since G is semisimple.

In Section 6.6.6, we will prove the next theorem as a generalization of Theorem 6.5.19:

Theorem 6.5.22. *For any finite-dimensional complex representation (ρ, V) of G, we have*

$$N_{\Omega'}(\rho) \leq N_{K/Z}(\rho) N^{multi}_\Omega(\rho),$$

where $N_{\Omega'}(\rho)$ [resp. $N_{K/Z}(\rho)$] denotes the smallest cardinality of an $\mathcal{H}^\rho_{\Omega'}$-design on Ω [resp. an $\mathcal{H}^\rho_{K/Z}$-design on K/Z] and

$$N^{multi}_\Omega(\rho) := \{ N \in \mathbb{N} \mid \text{there exists a multi-}\mathcal{H}^\rho_\Omega\text{-design in } \Omega^N \}.$$

We give an example of Theorem 6.5.22:

Example 6.5.23. *Let us fix m and l with $m \geq l \geq 1$. We put $G = SO(2m+1)$ and*

$$K := \left\{ \begin{pmatrix} 1 & \\ & A \end{pmatrix} \mid A \in SO(2m) \right\},$$
$$K' := \left\{ \begin{pmatrix} B & \\ & B' \end{pmatrix} \mid B \in O(l), B' \in O(2m+1-l) \text{ with } \det B \cdot \det B' = 1 \right\}.$$

Then we have $G/K \simeq S^{2m}$ and $G/K' \simeq \mathcal{G}^{\mathbb{R}}_{2m+1,l}$, here $\mathcal{G}^{\mathbb{R}}_{2m+1,l}$ denotes the real Grassmannian manifolds of rank l, i.e. the space consisted of all real l-dimensional subspaces of $\mathbb{R}^{2m+1}$. Then we have

$$Z := Z_K(G) = Z_{K'}(G) = \{I_{2m+1}\}$$

where I_{2m+1} is the unit of $SO(2m+1)$. Let us fix a finite-dimensional complex representation (ρ, V) of $G = SO(2m+1)$. Then by Theorem 6.5.22, we have that

$$N_{\mathcal{G}^{\mathbb{R}}_{2m+1,l}}(\rho) \leq N_{SO(2m)}(\rho) \cdot N^{multi}_{S^{2m}}(\rho).$$

Remark 6.5.24. *In Example 6.5.23, if we take a suitable (ρ, V), then $\mathcal{H}^{\rho}_{\mathcal{G}^{\mathbb{R}}_{2m+1,l}}$-designs on $\mathcal{G}^{\mathbb{R}}_{2m+1,l}$ can be regard as t-designs on $\mathcal{G}^{\mathbb{R}}_{2m+1,l}$ in the sense of Bachoc–Coulangeon–Nebe [1]. We omit the details here.*

6.6 Proofs of results in Section 6.5

We prove Proposition 6.5.3, Proposition 6.5.7, Theorem 6.5.8, Theorem 6.5.11, Theorem 6.5.12, Proposition 6.5.17 and Theorem 6.5.22 in this section.

6.6.1 Existence of designs

By using the results of Seymour–Zaslavsky [18] and Proposition 6.3.11, we prove Proposition 6.5.3 as follows:

Proof of Proposition 6.5.3. Since Ω is a compact manifold, the number of connected components of Ω is finite. Let us denote by $\Omega_0, \dots, \Omega_{N-1}$ the connected components of $\Omega = \bigsqcup_{i=0}^{N-1} \Omega_i$. Note that each component Ω_i is path-connected. In particular, if $N = 1$, i.e. Ω is connected, then the existence of $\mathcal{H}$-designs on Ω is followed by [18, Main theorem]. Therefore, we consider the cases where $N \geq 2$. We put μ_i to the measure on Ω_i obtained by restricting μ_Ω on Ω to Ω_i for each $i = 0, \dots, N-1$. Then (Ω_i, μ_i) are isomorphic from each other as measure spaces. We put $\Gamma := \{0, 1, \dots, N-1\}$ and define a natural measure μ_Γ on Γ, i.e. $\mu_\Gamma(\Gamma') = |\Gamma'|$ for any subset Γ' of Γ. Let us fix an isomorphism between (Ω, μ) and $(\Omega_0 \times \Gamma, \mu_0 \times \mu_\Gamma)$ as measure spaces. We put $\pi : \Omega \simeq \Omega_0 \times \Gamma \to \Omega_0$ the projection and denote by $I_\pi \mathcal{H}$ the finite-dimensional functional space on Ω_0 induced by $\mathcal{H}$ and π as in Section 6.3. Because Ω_0 is path-connected, by [18, Main theorem], there exists an $(I_\pi \mathcal{H})$-design Y on Ω_0. Furthermore, since Γ is fintie, for each $y \in \Omega_0$, $\pi^{-1}(y)$ itself is an $\mathcal{H}|_{\pi^{-1}(y)}$-design on $\pi^{-1}(y)$. Then, by Proposition 6.3.11, the finite subset $\pi^{-1}(Y) = Y \times \Gamma$ of $\Omega_0 \times \Gamma \simeq \Omega$ is an $\mathcal{H}$-design on Ω. This completes the proof. □

6.6.2 Representations on functional spaces

Throughout this subsection, we consider (G, K) in Setting 6.5.1. Proposition 6.5.7 is proved in this subsection.

Let us denote by $G \times G$ and $K \times K$ the direct product group of G and K, respectively. For any $g \in G$, we define

$$L^G_g : C^0(G) \to C^0(G), \quad f \mapsto L^G_g f$$
$$R^G_g : C^0(G) \to C^0(G), \quad f \mapsto R^G_g f$$

by $(L^G_g f)(g') := f(g^{-1}g')$ and $(R^G_g f)(g') := f(gg')$ for any $g' \in G$. Note that for any element $k \in K$, both L^G_k and R^G_k induce the maps L^K_k and R^K_k from $C^0(K)$ to $C^0(K)$. These maps give the left $(G \times G)$-action [resp. $(K \times K)$-action] on $C^0(G)$ [resp. $C^0(K)$], which are denoted by $L^G \times R^G$ [resp. $L^K \times R^K$]. That is, for any $(g_1, g_2) \in G \times G$ and $f \in C^0(G)$, we put

$$(L^G \times R^G)_{(g_1, g_2)} f := (R^G_{g_2} \circ L^G_{g_1}) f \in C^0(G).$$

We also define

$$L^{G/K}_g : C^0(G/K) \to C^0(G/K), \quad f \mapsto L^{G/K}_g f$$

by $(L^{G/K}_g f)(g'K) := f(g^{-1}g'K)$ for any $g' \in G$. We denote by $L^{G/K}$ the induced left G-action on $C^0(G/K)$.

Let us fix a finite-dimensional complex representation (ρ, V) of G. Recall that $V^\vee$ is the dual space of V. Then $V^\vee$ can be regard as a representation of G as follows:

$$\rho^\vee(g) : V^\vee \to V^\vee, \quad \varphi \mapsto \varphi \circ (\rho(g^{-1})).$$

In particular, we obtain a representation $(\rho \boxtimes \rho^\vee, V \otimes V^\vee)$ of $G \times G$. Furthermore, for $m \in \mathbb{N}$, let us denote by $(\rho^{\oplus m}, V^{\oplus m})$ the direct sum of m-th copies of (ρ, V). Then $V \otimes (V^\vee)^K$ can be regard as a representation of G isomorphic to $(\rho^{\oplus \dim(V^\vee)^K}, V^{\oplus \dim(V^\vee)^K})$.

Recall that in Section 6.5.1, we defined a $\mathbb{C}$-linear map $\Phi : V \otimes V^\vee \to C^0(G)$ [resp. $\widehat{\Phi} : V \otimes (V^\vee)^K \to C^0(G)$] and put $\mathcal{H}^\rho_G$ [resp. $\mathcal{H}^\rho_{G/K}$] to the image of Φ [resp. $\widehat{\Phi}$]. One can observe that the next lemma holds:

Lemma 6.6.1 (cf. [19, Chapter I, §1]). *(i) $\Phi : V \otimes V^\vee \to C^0(G)$ is $(G \times G)$-intertwining.*

(ii) $\widehat{\Phi} : V \otimes (V^\vee)^K \to C^0(G/K)$ is G-intertwining.

The next corollary follows from Lemma 6.6.1 immediately:

Corollary 6.6.2. *For a finite-dimensional complex representation (ρ, V) of G, the following hold:*

(i) $G \times G$ acts on the subspace $\mathcal{H}^\rho_G$ of $C^0(G)$ by $L^G \times R^G$.

(ii) $K \times K$ acts on the subspace $\mathcal{H}^\rho_K$ of $C^0(K)$ by $L^K \times R^K$.

(iii) G acts on the subspace $\mathcal{H}^\rho_{G/K}$ of $C^0(G/K)$ by $L^{G/K}$.

We will give a formula of irreducible decompositions of representations $\mathcal{H}^\rho_\Omega$ by using the irreducible decomposition of (ρ, V) in Section 6.A.

By combining Corollary 6.6.2 with the definition of μ_G, μ_K and $\mu_{G/K}$, we obtain Proposition 6.5.7.

6.6.3 Proofs of Theorem 6.5.8 and Theorem 6.5.12

Throughout this subsection, we consider Setting 6.5.1. Recall that $\pi : G \to G/K$ is a K-principal bundle. For simplicity, we fix a base point $s(y)$ of the fiber $\pi^{-1}(y)$ for each $y \in G/K$. We should remark that the map $s : G/K \to G$, $y \mapsto s(y)$ is not continuous in general. For each $y \in G/K$, the map

$$\iota_y : K \to \pi^{-1}(y), \quad k \mapsto s(y)k$$

is an isomorphism between topological spaces. In particular, the normalized two-sided Haar measure μ_K induces a probability K-invariant measure μ_y on $\pi^{-1}(y)$ for each $y \in G/K$. Note that μ_y does not depend on the choice of the base point $s(y)$.

Let us fix a finite-dimensional complex representation (ρ, V) of G. To prove Theorem 6.5.8 and Theorem 6.5.12, we show the next two lemmas:

Lemma 6.6.3. *Any $f \in \mathcal{H}_G^\rho$ satisfies the property (F) with respect to the quotient map $\pi : G \to G/K$, the normalized measures μ_G, $\mu_{G/K}$ and μ_y $(y \in G/K)$ defined above (see Section 6.3.2 for the definition of the property (F)).*

Lemma 6.6.4. *The following hold:*

$$\iota_y^*(\mathcal{H}_G^\rho|_{\pi^{-1}(y)}) = \mathcal{H}_K^\rho \quad \text{for any } y \in G/K,$$
$$I_\pi(\mathcal{H}_G^\rho) = \mathcal{H}_{G/K}^\rho \text{ and } \pi^*(\mathcal{H}_{G/K}^\rho) \subset \mathcal{H}_G^\rho.$$

(see Section 6.3.2 for the definition of I_π).

Theorem 6.5.8 is followed by Lemma 6.6.3 and Proposition 6.3.11. Furthermore, Theorem 6.5.12 is followed by Lemma 6.6.4 and Proposition 6.3.12. Therefore, we shall prove Lemma 6.6.3 and Lemma 6.6.4.

It is well known that the complex vector space V admits a G-invariant Hermitian inner-product $(\ , \ )_V$. That is, the representaion (ρ, V) can be regard as a unitary representation of G with respect to the inner-product $(\ , \ )_V$. In the rest of this subsection, we fix such G-invariant inner-product on V.

Let us put

$$V^G := \{ v \in V \mid \rho(g)v = v \quad \text{for any } g \in G \},$$
$$V^K := \{ v \in V \mid \rho(k)v = v \quad \text{for any } k \in K \}.$$

Note that $V^G \subset V^K \subset V$. We write $p_G : V \to V^G$ and $p_K : V \to V^K$ for the orthogonal projections. We remark that $p_G \circ p_K = p_G$.

We will use the following well-known fact:

Fact 6.6.5. *Let H be a compact Lie group and μ_H the two-sided normalized Haar measure on H. We take a finite-dimensional unitary representation*

(ρ, V) of H with respect to the Hermitian inner-product $(\ ,\ )_V$ on V. Let us denote by $p_H : V \to V^H$ the orthogonal projection on V to the subspace

$$V^H := \{\, v \in V \mid \rho(h)v = v \,\} \subset V.$$

Then for each $v \in V$, the vector $p_H(v)$ in V^H is the unique one with

$$(v', p_H(v))_V = \int_{h \in H} (v', \rho(h)v)_V d\mu_H(h) \quad \textit{for any } v' \in V.$$

We are ready to prove Lemma 6.6.3 and Lemma 6.6.4:

Proof of Lemma 6.6.3 *and Lemma* 6.6.4. By the definition of Φ and $\widehat{\Phi}$, we have that

$$\pi^*(\widehat{\Phi}(v \otimes \psi)) = \Phi(v \otimes \psi) \in \mathcal{H}^\rho_G \tag{6.6.1}$$

for any $v \in V$ and any $\psi \in (V^\vee)^K$. In particular, we also obtain that $I_\pi(\Phi(v \otimes \psi)) = \widehat{\Phi}(v \otimes \psi)$ since $\mu_y(\pi^{-1}(y)) = 1$ for any $y \in G/K$. Thus we have proved that $\pi^*(\mathcal{H}^\rho_{G/K}) \subset \mathcal{H}^\rho_G$ and $\mathcal{H}^\rho_{G/K} \subset I_\pi(\mathcal{H}^\rho_G)$. Recall that we are considering the G-invariant inner-product $(\ ,\ )_V$ on V. For each $v \in V$, we define $\overline{v} \in V^\vee$ by

$$\overline{v} : V \to \mathbb{C}, \quad v' \mapsto (v', v)_V.$$

Then the map

$$V \to V^\vee, \quad v \mapsto \overline{v}$$

is an anti $\mathbb{C}$-linear isomorphism. Let us fix any $v_1, v_2 \in V$, and put $f := \Phi(v_1 \otimes \overline{v_2})$. To completes the proof of Lemma 6.6.3 and Lemma 6.6.4, we only need to show the following:

- $\iota_y^*(f|_{\pi^{-1}(y)})$ is in $\mathcal{H}^\rho_K$ for any $y \in G/K$,
- $I_\pi f$ is in $\mathcal{H}^\rho_{G/K}$ and

$$\int_G f d\mu_G = \int_{G/K} (I_\pi f) d\mu_{G/K}.$$

First, we remark that $f(g) = (v_1, \rho(g)v_2)_V$ for any $g \in G$. Therefore, for each $y \in G/K$ and $k \in K$, we have

$$\begin{aligned} \iota_y^*(f|_{\pi^{-1}(y)})(k) &= (v_1, \rho(s(y)k)v_2)_V \\ &= (\rho(s(y)^{-1})v_1, \rho(k)v_2)_V \\ &= \Phi(\rho(s(y)^{-1})v_1 \otimes \overline{v_2})(k) \end{aligned}$$

In particular, $\iota_y^*(f|_{\pi^{-1}(y)}) \in \mathcal{H}_K^\rho$ for any $y \in G/K$. Here we remark that $\overline{w}$ is in $(V^\vee)^K$ for any $w \in V^K$. Therefore, by using Fact 6.6.5 and $\pi(s(y)) = s(y)K = y$ for $y \in G/K$, we have

$$\begin{aligned}(I_\pi f)(y) &= \int_{\pi^{-1}(y)} f d\mu_y \\ &= \int_{k \in K} (\rho(s(y)^{-1})v_1, \rho(k)v_2)_V d\mu_K(k) \\ &= (\rho(s(y)^{-1})v_1, p_K(v_2))_V \\ &= \widehat{\Phi}(v_1 \otimes \overline{p_K(v_2)})(y).\end{aligned}$$

Hence $I_\pi f = \widehat{\Phi}(v_1 \otimes \overline{p_K(v_2)}) \in \mathcal{H}_{G/K}^\rho$. Finally, by using the property (6.5.1) of the measure $\mu_{G/K}$ in Section 6.5.1, Fact 6.6.5 and the equation (6.6.1) above, we obtain that

$$\begin{aligned}\int_{G/K} (I_\pi f) d\mu_{G/K} &= \int_{G/K} \widehat{\Phi}(v_1 \otimes \overline{p_K(v_2)}) d\mu_{G/K} \\ &= \int_G (\pi^* \widehat{\Phi}(v_1 \otimes \overline{p_K(v_2)})) d\mu_G \\ &= \int_G \Phi(v_1 \otimes \overline{p_K(v_2)})) d\mu_G \\ &= \int_{g \in G} (v_1, \rho(g) p_K(v_2))_V d\mu_G(g) \\ &= (v_1, (p_G \circ p_K)(v_2))_V \\ &= (v_1, p_G(v_2))_V \\ &= \int_{g \in G} (v_1, \rho(g) v_2)_V d\mu_G(g) \\ &= \int_G f d\mu_G.\end{aligned}$$

This completes the proof. □

6.6.4 Proof of Theorem 6.5.11

Let us consider (G, K) in Setting 6.5.1 and fix a finite-dimensional complex representation (ρ, V) of G. By combining Corollary 6.3.14 with Lemma 6.6.4, we have a generalization of Theorem 6.5.8 as follows:

Theorem 6.6.6. *Let $Y = (y_1, \ldots, y_N) \in (G/K)^N$ be a multi-$\mathcal{H}^\rho_{G/K}$-design, and $\Gamma \subset K$ an $\mathcal{H}^\rho_K$-design. For each $i = 1, \ldots, N$, we fix a base point $s(y_i)$ on $\pi^{-1}(y_i)$, and denote by*

$$\Gamma_i := \{ s(y_i)\gamma \mid \gamma \in \Gamma \} \subset \pi^{-1}(y_i).$$

Suppose that $\Gamma_1, \ldots, \Gamma_N$ are disjoint from each other in G. Then

$$X(Y, s, \Gamma) := \bigsqcup_{i=1}^{N} \Gamma_i \subset G$$

is an $\mathcal{H}^\rho_G$-design on G with $|X(Y, s, \Gamma)| = N \cdot |\Gamma|$.

To prove Theorem 6.5.11, we show the next lemma:

Lemma 6.6.7. *Let K be a Lie group with $\dim K \geq 1$. Then for any finite subset Γ of K and $m \in \mathbb{N}$, there exists $k_1, \ldots, k_m \in K$ such that $k_1\Gamma, \ldots, k_m\Gamma$ are disjoint from each other in K.*

Proof of Lemma 6.6.7. We only need to show that for any finite subset Γ' of K with $\Gamma \subset \Gamma'$, there exists $k \in K$ such that $k\Gamma \cap \Gamma' = \emptyset$. Since Γ and Γ' are finite, we can find an open neighborhood U of the unit of K such that $k\gamma \neq \gamma'$ for any $\gamma \in \Gamma$ and $\gamma' \in \Gamma'$ with $\gamma \neq \gamma'$. Since $\dim K \neq 0$, we can take $k \in U \setminus \{e\}$, where e is the unit of K. Then we have $k\Gamma \cap \Gamma' = \emptyset$. □

We are ready to prove Theorem 6.5.11:

Proof of Theorem 6.5.11. Let $Y \in (G/K)^N$ be a multi-$\mathcal{H}^\rho_{G/K}$-design, and Γ an $\mathcal{H}^\rho_K$-design on K. We only need to show that there exists an $\mathcal{H}^\rho_G$-design X on G with $|X| = N \cdot |\Gamma|$. By using Lemma 6.6.7, one can choose a base point $s_\Gamma(y_i)$ on $\pi^{-1}(y_i)$ for each y_i such that

$$\Gamma_i \cap \Gamma_j = \emptyset \quad \text{for any } i, j \text{ with } i \neq j,$$

where $\Gamma_i := \{ s_\Gamma(y_i)\gamma \mid \gamma \in \Gamma \}$. Hence, by Theorem 6.6.6, the finite subset

$$X := \bigsqcup_{i=1}^{N} \Gamma_i$$

is an $\mathcal{H}^\rho_G$-design on G with $|X| = N \cdot |\Gamma|$. □

6.6.5 Proof of Proposition 6.5.17

We consider (G, K) in the general setting 6.5.1 and suppose that G is connected. Let us fix a finite-dimensional complex representation (ρ, V) of G.

Let us define the closed subgroup I_K^G of K as follows:

$$\mathrm{I}_K^G := \{ k \in K \mid gkg^{-1} \in K \text{ for any } g \in G \}.$$

Note that I_K^G is a normal subgroup of G.

Definition 6.6.8. *Let X be a finite subset of G. For $p \in \mathbb{N}$, we say that X has p-multiplicity for I_K^G if*

$$|X \cap x\mathrm{I}_K^G| = p \quad \text{for any } x \in X.$$

Since I_K^G is a normal subgroup of G, if X has p-multiplicity for I_K^G, then the finite subset Xg of G also has p-multiplicity for I_K^G, for any $g \in G$.

Proposition 6.5.17 is followed by the three lemmas below:

Lemma 6.6.9. *We consider (G, K) in the general setting 6.5.1. Suppose that G is connected and (G, K) satisfies the following condition:*

Condition $(\star)$ *For any $k \in K \setminus \mathrm{I}_K^G$ and any open neighberhood U of the unit of G, there exists $g \in U$ such that $gkg^{-1} \notin K$.*

Then for any finite subset X of G with p-multiplicity for I_K^G and any open neighberhood U of the unit of G, there exists $g \in U$ such that

$$|Xg \cap xgK| = p$$

for any $x \in X$.

Lemma 6.6.10. *We consider (G, K) in the general setting 6.5.1. Suppose that G is connected and there exists an involutive homeomorphism τ on G such that K is a closed subgroup of G^τ with $\mathrm{Lie}\, K = \mathrm{Lie}\, G^\tau$, where $G^\tau := \{ g \in G \mid \tau(g) = g \}$. Then (G, K) satisfies the condition $(\star)$ in Lemma 6.6.9.*

Lemma 6.6.11. *In Setting 6.5.14, $\mathrm{I}_K^G = Z_K(G)$.*

Proof of Lemma 6.6.9. Let us fix any finite subset X of G and any open neighborhood U of the unit of G. We only need to show that there exists $g \in U$ such that

$$|Xg \cap xgK| = |X \cap x\mathrm{I}_K^G| \quad \text{for any } x \in X.$$

We denote by $\pi_0 : G \to G/\mathrm{I}_K^G$ for the quotient map. Since I_K^G is a normal subgroup of G, for any $g \in G$ and any $x, x' \in G$ with $\pi_0(x) = \pi_0(x')$, we have $\pi_0(xg) = \pi_0(x'g)$. Furthermore, since I_K^G is a subgroup of K, for any $x, x' \in G$ with $\pi_0(x) = \pi_0(x')$, we have $\pi(x) = \pi(x')$. Recall that both of the quotient map $\pi : G \to G/K$ and the action map

$$G \times G \to G, \quad (x, g) \mapsto xg$$

are continuous. Since X is finite in G, we can find an open neighborhood U_0 of the unit of G such that $\pi(xg_0) \neq \pi(x'g_0)$ for any $g_0 \in U_0$ and $x, x' \in X$ with $\pi(x) \neq \pi(x')$. Let us consider $x, x' \in X$ with $\pi(x) = \pi(x')$ but $\pi_0(x) \neq \pi_0(x')$. Then there exists $k \in K \setminus \mathrm{I}_K^G$ such that $x' = xk$. In general, for $x \in G$, $k \in K$ and $g \in G$, we have that $\pi(xg) \neq \pi(xkg)$ if and only if $g^{-1}kg \notin K$. Therefore, by the condition ($\star$) of (G, K), we can take $g \in U \cap U_0$ satisfying that $\pi(xg) \neq \pi(x'g)$ for any $x, x' \in X$ with $\pi_0(x) \neq \pi_0(x')$. That is, for $x, x' \in X$, we have that $\pi(xg) = \pi(x'g)$ if and only if $\pi_0(x) = \pi_0(x')$, and hence

$$|Xg \cap xgK| = |Xg \cap \pi^{-1}(\pi(xg))| = |(X \cap \pi_0{}^{-1}(\pi_0(x)))g| = |X \cap x\mathrm{I}_K^{G_0}|.$$

□

Proof of Lemma 6.6.10. Let us fix any $k \in K$. Suppose that there exists an open neighborhood U of the unit of G such that

$$gkg^{-1} \in K \quad \text{for any } g \in U.$$

Our claim is $k \in \mathrm{I}_K^G$. Since G is connected and K is union of some connected components of G^τ, we only need to prove that $gkg^{-1} \in G^\tau$ for any $g \in G$. Let us put

$$Z_G(k) := \{g \in G \mid gkg^{-1} = k\}.$$

Then $gkg^{-1} \in G^\tau$ if and only if $\tau(g)^{-1}g \in Z_G(k)$. Therefore, we have

$$\tau(g)^{-1}g \in Z_G(k) \quad \text{for any } g \in U.$$

Then by Lemma 6.6.12 below, we have $G = Z_G(k)G^\tau$. Hence, $gkg^{-1} \in G^\tau$ for any $g \in G$. □

To completes the proof of Lemma 6.6.10, we show the next lemma:

Lemma 6.6.12. *Let G be a connected compact Lie group, H a closed subgroup of G and τ an involutive homeomorpshim on G. Suppose that there exists an open neighberhood U of the unit of G such that*

$$\tau(g)^{-1}g \in H \quad \textit{for any } g \in U.$$

Then $G = HG^{\tau}$, where $G^{\tau} := \{\, g \in G \mid \tau(g) = g \,\}$.

Proof of Lemma 6.6.12. For simplicity, we denote by $K' := G^{\tau} = \{g \in G \mid \tau(g) = g\}$ and write $\pi' : G \to G/K'$ for the quotient map. Our claim means that the natural left H-action on G/K' is transitive. It is known that the H-orbit $\pi'(H)$ through $e_G K$ is closed in G/K', where e_G is the unit of G (see Helgason [9, Proposition 4.4 (b)]). Since G/K' is connected, we only need to show that the H-orbit $\pi'(H)$ is open in G/K'.

We write $\mathfrak{g}$ and $\mathfrak{k}$ for the Lie algebra of G and K', respectively. The differential action of τ on $\mathfrak{g}$ is denoted by the same letter τ. Then $\mathfrak{k} = \{X \in \mathfrak{g} \mid \tau(X) = X\}$. We also denote by $\mathfrak{q} = \{X \in \mathfrak{g} \mid \tau(X) = -X\}$. Then we have the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ as a real vector sapce. We denote by $p_{\mathfrak{q}} : \mathfrak{g} \to \mathfrak{q}$ the projection with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$. It is well known that $\mathfrak{g}$ [resp. $\mathfrak{q}$] can be regard as the tangent space of G at the unit $e_G \in G$ [resp. the tangent space of G/K' at $e_G K'$]. Then the differential of the quotient map $\pi' : G \to G/K'$ at the unit e_G can be considered as $p_{\mathfrak{p}} : \mathfrak{g} \to \mathfrak{q}$ (see Helgason [9, Theorem 3.3 (iii)]). Let us denote by

$$\varphi : G \to G, \quad g \mapsto \tau(g)^{-1}g.$$

Then, one can observe that the differential map $d\varphi : \mathfrak{g} \to \mathfrak{g}$ of φ is

$$d\varphi = 2p_{\mathfrak{q}}.$$

Thus, the differential of the map

$$\pi' \circ \varphi : G \to G/K', \quad g \mapsto \tau(g)^{-1}gK'$$

at the unit e_G is surjective. In particular, there exists an open neighborhood U' of the unit e_G such that $\pi(U')$ is open in G/K'. By the assumption, we have an open neighborhood U of e_G such that $\varphi(U) \subset H$. Therefore, $(\pi' \circ \varphi)(U \cap U')$ is open in G/K' and included in $\pi'(H)$. Hence, the H-orbit $\pi'(H)$ in G/K' is open. This completes the proof. $\square$

To prove Lemma 6.6.11, we show the next lemma:

Lemma 6.6.13. *Let G be a connected semisimple Lie group and M a closed normal subgroup such that* $\mathrm{Lie}\, M$ *contains no simple factor of* $\mathrm{Lie}\, G$*. Then M is contained in the center Z_G of G.*

Proof of Lemma 6.6.13. $\mathrm{Lie}\, M$ is an ideal of $\mathrm{Lie}\, G$ since M is normal in G. By the assumption that $\mathrm{Lie}\, M$ contains no simple factor of $\mathrm{Lie}\, G$, we have that $\mathrm{Lie}\, M = \{0\}$ and hence M is discrete. It is well known that any discrete normal subgroup of a connected Lie group G is contained in the center Z_G of G. Thus, $M \subset Z_G$. □

We are ready to prove Lemma 6.6.11:

Proof of Lemma 6.6.11. By the definition of I_K^G, one can observe that I_K^G is the maximum closed normal subgroup of G contained in K. Recall that we are assuming that $\mathrm{Lie}\, K$ containes no simple factor of $\mathrm{Lie}\, G$, and $Z_K(G) := K \cap Z_G$ is normal in G. Therefore, by Lemma 6.6.13, we have $\mathrm{I}_K^G = Z_K(G)$. □

6.6.6 Proof of Theorem 6.5.22

In this subsection, we prove Theorem 6.5.22. To this, let us denote by $\varpi : K \to K/Z$ and $\varpi' : K' \to K'/Z$ the quotient maps. Throughout this subsection, we fix a multi-$\mathcal{H}_\Omega^\rho$-design $Y \in \Omega^N$ and an $\mathcal{H}_{K/Z}^\rho$-design Ξ on K/Z. To prove Theorem 6.5.22, it suffices to show that there exists an $\mathcal{H}_{\Omega'}^\rho$-design Y' on Ω' with $|Y'| = N \cdot |\Xi|$.

First, we shall prove the next lemma:

Lemma 6.6.14. *For any $m \in \mathbb{N}$, there exists $k_1, \dots, k_m \in K$ such that $k_1\Xi, \dots, k_m\Xi$ are disjoint from each other in K/Z.*

Proof of Lemma 6.6.14. Since Z is a finite normal subgroup of K, the quotient space K/Z is also a compact Lie group with $\dim K/Z = \dim K \geq 1$. Thus, our claim follows from Lemma 6.6.7. □

Since Z is finite group, Z itself is an $\mathcal{H}_Z$-design on Z for any functional space $\mathcal{H}_Z$ on Z. We put $p_Z := |Z|$. By using the observation above, we prove the next lemma:

Lemma 6.6.15. *For any $m \in \mathbb{N}$, there exists an $\mathcal{H}_K^\rho$-design $\Gamma(m)$ on K with $|\Gamma(m)| = m \cdot p_Z \cdot |\Xi|$ such that $\gamma Z \subset \Gamma(m)$ for any $\gamma \in \Gamma(m)$.*

Proof of Lemma 6.6.15. Let us take $k_1, \dots, k_m \in K$ as in Lemma 6.6.14. Then by Observation 6.3.4,

$$\Xi(m) := \bigsqcup_{i=1}^{m} k_i \Xi$$

is an $\mathcal{H}^\rho_{K/Z}$-design on K/Z with $|\Xi(m)| = m|\Xi|$. Recall that Z itself is an $\mathcal{H}^\rho_Z$-design on Z. Thus by applying Theorem 6.5.8 for (K, Z), we obtain an $\mathcal{H}^\rho_K$-design

$$\Gamma(m) := \varpi^{-1}(\Xi(m))$$

on K with $|\Gamma(m)| = |\Xi(m)| \cdot |Z| = m \cdot p_Z \cdot |\Xi|$. By the definition of $\Gamma(m)$, one can observe that $\gamma Z \subset \Gamma(m)$ for any $\gamma \in \Gamma(m)$. □

We are ready to prove Theorem 6.5.22:

Proof of Theorem 6.5.22. We fix $Y \in \Omega^N$, $\Xi \subset K/Z$ as above. It suffices to show that there exists an $\mathcal{H}^\rho_{\Omega'}$-design Y' on Ω' with $|Y'| = N \cdot |\Xi|$. Here, we use the notation $\overline{Y}$ and $m(\overline{y})$ for $\overline{y} \in \overline{Y}$ as in Section 6.3.1. We put $p_Z := |Z|$. By Lemma 6.6.15, for each $\overline{y} \in \overline{Y}$, we have an $\mathcal{H}^\rho_K$-design $\Gamma(m(\overline{y}))$ on K with $|\Gamma(m(\overline{y}))| = m(\overline{y}) \cdot p_Z \cdot |\Xi|$ such that $\gamma Z \subset \Gamma(m(\overline{y}))$ for any $\gamma \in \Gamma(m(\overline{y}))$. For each $\overline{y} \in \overline{Y}$, we also fix a base point $s(\overline{y}) \in \pi^{-1}(\overline{y})$ and put

$$\Gamma_{\overline{y}} := \{\, s(\overline{y})\gamma \in \pi^{-1}(\overline{y}) \mid \gamma \in \Gamma(m(\overline{y}))\,\}.$$

Then by combining Corollary 6.3.14 with Lemma 6.6.4, the finite subset

$$X := \bigsqcup_{\overline{y} \in \overline{Y}} \Gamma_{\overline{y}}$$

of G is an $\mathcal{H}^\rho_G$-design on G with $|X| = p_Z \cdot N \cdot |\Xi|$. Recall that $\gamma Z \subset \Gamma(m(\overline{y}))$ for any $\gamma \in \Gamma(m(\overline{y}))$. Thus, $xZ \subset X$ for any $x \in X$. This means that X has p_Z-multiplicity for Z. Hence, by Theorem 6.5.18, there exists $g \in G$ such that $Y' := \pi(Xg)$ is an $\mathcal{H}^\rho_{\Omega'}$-design on Ω' with $|Y'| = \frac{1}{p_Z}|X| = N \cdot |\Xi|$. □

Appendix 6.A Irreducible decomposition of $\mathcal{H}^\rho_\Omega$

Let us consider (G, K) in the general setting 6.5.1, and fix a finite-dimensional complex representation (ρ, V) of G. In this section, we recall some relations between the irreducible decomposition of (ρ, V) and that of the corresponding functional space $\mathcal{H}_G^\rho$ [resp. $\mathcal{H}_{G/K}^\rho$] defined in Section 6.5.1 (see Proposition 6.A.4 below). Note that the results for the relation between representations (ρ, V) and $\mathcal{H}_G^\rho$ of G can be used for the relation between representations $(\rho|_K, V)$ and $\mathcal{H}_K^\rho$ of K.

By Lemma 6.6.1 in Section 6.6.2, the $\mathbb{C}$-lienear map $\Phi : V \otimes V^\vee \to C^0(G)$ is $(G \times G)$-intertwining and $\widehat{\Phi} : V \otimes (V^\vee)^K \to C^0(G/K)$ is G-intertwining. In the case where (ρ, V) is irreducible, the next fact is well known:

Fact 6.A.1 (cf. [19, Theorem 1.1 (1) and Theorem 1.3 (1)]). *Suppose that (ρ, V) is irreducible. Then, both $\Phi : V \otimes V^\vee \to C^0(G)$ and $\widehat{\Phi} : V \otimes (V^\vee)^K \to C^0(G/K)$ are injective. In particular, in this case, the following hold:*

(i) *As a representation of $G \times G$, the functional space $\mathcal{H}_G^\rho$ on G is isomorphic to the irreducible representation $(\rho \boxtimes \rho^\vee, V \otimes V^\vee)$.*

(ii) *As a representation of G, the functional space $\mathcal{H}_{G/K}^\rho$ on G/K is isomorphic to the direct sum of $(\dim V^K)$-copies of the irreducible representation (ρ, V).*

Remark 6.A.2. *In this case, we have* $\dim V^K = \dim(V^\vee)^K$.

Let us consider the Hermitian inner-product $(\ ,\ )_\Omega$ on $C^0(\Omega)$ for $\Omega = G$ or G/K as follows:

$$(f, f')_\Omega := \int_\Omega f \cdot \overline{f} d\mu_\Omega \quad \text{for } f, f' \in C^0(\Omega),$$

where $\overline{f'}$ is the complex conjugation of f'. Then $(\ ,\ )_G$ [resp. $(\ ,\ )_{G/K}$] is invariant by the $(G \times G)$-action [resp. G-action].

The following fact is also well known:

Fact 6.A.3 (cf. [19, Theorem 1.1 (1) and Theorem 1.3 (1)]). *Let (ρ_1, V_1) and (ρ_2, V_2) be irreducible finite-dimensional complex representatons of G. Suppose that (ρ_1, V_1) and (ρ_2, V_2) are not isomorphic from each other. Then for each $\Omega = G$ or G/K,*

$$\mathcal{H}_\Omega^{\rho_1} \perp \mathcal{H}_\Omega^{\rho_2}$$

with respect to the inner-product $(\ ,\ )_\Omega$ on $C^0(\Omega)$.

Let us fix a finite-dimensional representation (ρ, V) of G with its irreducible decomposition

$$\rho \simeq \rho_0^{\oplus m_0} \oplus \cdots \oplus \rho_t^{\oplus m_t},$$

where $(\rho_0, V_0), \ldots, (\rho_t, V_t)$ are irreducible representations of G not isomorphic from each other, and $\rho_l^{\oplus m_l}$ is the direct sum of m_l-th copies of irreducible representation ρ_l of G for each $l = 0, \ldots, t$ (where $m_l \geq 1$).

By combining Observation 6.5.4, Fact 6.A.1 with Fact 6.A.3, we have the next proposition:

Proposition 6.A.4. *For the representation (ρ, V) of G above, we have*

$$\mathcal{H}_G^\rho \simeq (\rho_0 \boxtimes \rho_0^\vee) \oplus \cdots \oplus (\rho_t \boxtimes \rho_t^\vee) \quad \text{as representations of } G \times G,$$
$$\mathcal{H}_{G/K}^\rho \simeq \rho_0^{\oplus \dim V_0^K} \oplus \cdots \oplus \rho_t^{\oplus \dim V_t^K} \quad \text{as representations of } G.$$

Finally, we give useful two lemmas below:

Lemma 6.A.5. *Let us take a finite-dimensional complex irreducible representation (ρ, V) and fix a finite-dimensional irreducible $(G\times G)$-subrepresentation $\mathcal{H}$ of $C^0(G)$. If $(\rho \boxtimes \rho^\vee, V \otimes V^\vee)$ is isomorphic to $((L^G \times R^G), \mathcal{H})$ as representation of $G \times G$, then $\mathcal{H}_G^\rho = \mathcal{H}$.*

Lemma 6.A.6. *Let us take a finite-dimensional complex irreducible representation (ρ, V) and fix a finite-dimensional irreducible G-subrepresentation $\mathcal{H}$ of $C^0(G/K)$. Suppose that $\dim V^K = 1$ and (ρ, V) is isomorphic to $(L^{G/K}, \mathcal{H})$ as a representation of G. Then $\mathcal{H}_{G/K}^\rho = \mathcal{H}$.*

These lemmas follows from Peter–Weyl's Theorem (cf. [19, Theorem 1.1 (2) and Theorem 1.3 (2)])

Appendix 6.B Hopf map as a quotient map

In this section, we explain that Main results in Section 6.2 is followed by results in Section 6.5.

6.B.1 Hopf map as a quotient map

Let us consider the group structure on S^3 by

$$(a,b)\cdot(a',b') = (aa' - b\overline{b'}, ab' + b\overline{a'}) \quad \text{for } (a,b),(a',b') \in S^3.$$

Then S^3 is a 3-dimensional connected compact simple Lie group. Here we put

$$SU(2) := \{g \in GL(2,\mathbb{C}) \mid gg^* = I_2,\ \det g = 1\}$$

where g^* is the conjugate transpose of $g \in GL(2,\mathbb{C})$. Then we can identify S^3 with $SU(2)$ by

$$S^3 \xrightarrow{\sim} SU(2),\ (a,b) \mapsto \begin{pmatrix} a & \sqrt{-1}b \\ \sqrt{-1}\overline{b} & \overline{a} \end{pmatrix}.$$

We remark that the center of the Lie group $SU(2) \simeq S^3$ can be written by

$$Z_{SU(2)} = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Let us consider the direct product group $SU(2) \times SU(2)$. Then $SU(2) \times SU(2)$ acts on $SU(2) \simeq S^3$ as follows:

$$(SU(2) \times SU(2)) \times SU(2) \to SU(2), \quad ((g_1,g_2),g) \mapsto g_1 g {g_2}^{-1}.$$

One can observe that the $(SU(2) \times SU(2))$-action on S^3 induces a Lie group homomorphism $\phi : SU(2) \times SU(2) \to SO(4)$ with

$$\operatorname{Ker}\phi = \left\{ (g_0,g_0) \in SU(2) \times SU(2) \mid g_0 = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

In particular, $\phi : SU(2) \times SU(2) \to SO(4)$ is double covering.

Recall that the spherical measure μ_{S^3} on S^3 is an $SO(4)$-invariant Haar measure on S^3. The normalized spherical measure $\mu'_{S^3} := \frac{1}{|S^3|}\mu_{S^3}$ can be regard as the normalized two-sided Haar measure $\mu_{SU(2)}$ on $SU(2) \simeq S^3$ since $\phi : SU(2) \times SU(2) \to SO(4)$ is a covering.

We denote the Lie algebra of $SU(2)$ by

$$\mathfrak{su}(2) := \{ A \in M(2,\mathbb{C}) \mid A + A^* = 0, \operatorname{Tr} A = 0 \},$$

and define the inner-product $\langle\ ,\ \rangle_{\mathfrak{su}(2)}$ on the 3-dimensional real vector space $\mathfrak{su}(2)$ by

$$\langle\ ,\ \rangle_{\mathfrak{su}(2)} : \mathfrak{su}(2) \times \mathfrak{su}(2) \to \mathbb{R}, \quad (A_1, A_2) \mapsto \mathrm{Tr}(A_1 A_2).$$

where A_1A_2 is the product of the matrices $A_1, A_2 \in M(2, \mathbb{C})$. We denote the adjoint action of $SU(2)$ on $\mathfrak{su}(2)$ by

$$SU(2) \times \mathfrak{su}(2) \to \mathfrak{su}(2), \quad (g, A) \mapsto gAg^{-1}.$$

Then it is well known that the adjoint action of $SU(2)$ on $\mathfrak{su}(2)$ preserves the inner-product $\langle\ ,\ \rangle_{\mathfrak{su}(2)}$ on $\mathfrak{su}(2)$. Here we give an isomorphism

$$S^2 \xrightarrow{\sim} \{\, A \in \mathfrak{su}(2) \mid \langle A, A\rangle_{\mathfrak{su}(2)} = 1 \,\}, \quad (\xi, \eta) \mapsto \begin{pmatrix} \sqrt{-1}\xi & \eta \\ \bar{\eta} & -\sqrt{-1}\xi \end{pmatrix}.$$

Then, $SU(2)$ acts on S^2 by the adjoint action. One can observe that the $SU(2)$-action on S^2 induces a Lie group homomorphism $\psi : SU(2) \to SO(3)$ with

$$\operatorname{Ker}\psi = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

In particular, $\psi : SU(2) \to SO(3)$ is double covering and $SU(2)$ acts on S^2 transitively.

Here, we put

$$T := \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \mid z \in \mathbb{C},\ |z| = 1 \right\} \subset SU(2).$$

Then T is the isotropy subgroup of $SU(2)$ at

$$\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \in S^2 \subset \mathfrak{su}(2).$$

In particular, we have an isomorphism

$$SU(2)/T \xrightarrow{\sim} S^2, \quad gT \mapsto g \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} g^{-1}.$$

Recall that the spherical measure μ_{S^2} on S^2 is an $SO(3)$-invariant Haar measure on S^2. The normalized spherical measure $\mu'_{S^2} := \frac{1}{|S^2|}\mu_{S^2}$ can be re-gard as the normalized $SU(2)$-invariant Haar measure $\mu_{SU(2)/T}$ on $SU(2)/T \simeq S^2$ since $\psi : SU(2) \to SO(3)$ is a covering. Furthermore, by the isomorphism

$$T \to S^1, \quad \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \mapsto z,$$

the normalized spherical measure $\mu'_{S^1} := \frac{1}{|S^1|}\mu_{S^1}$ can be regard as the normalized two-sided Haar measure μ_T on $T \simeq S^1$.

The natural right T-action on $SU(2)$ can be regard as the right S^1-action on S^3 defined in Section 6.2.2. Hence, the Hopf map $\pi : S^3 \to S^2$ can be regard as the quotient map $\pi : SU(2) \to SU(2)/T$.

Let us put

$$\tau : SU(2) \to SU(2),\ g \mapsto I_{1,1} g I_{1,1}$$

where

$$I_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $SU(2)^\tau = T$. Therefore, $(SU(2), T)$ is in Setting 6.5.14. Since $Z_{SU(2)}$ is contained in T, we have

$$Z_T(SU(2)) = Z_{SU(2)} = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Hence we obtain the next proposition:

Proposition 6.B.1. *(i) If a finite subset X of $S^3 \simeq SU(2)$ has p-multiplicity for $Z_T(SU(2))$, then $p = 1$ or 2.*

(ii) A finite subset X of $S^3 \simeq SU(2)$ has 1-multiplicity for $Z_T(SU(2))$ if and only if X is antipodal-free, i.e. for any $x \in X$, the element $-x \in S^3$ is not in X.

(iii) A finite subset X of $S^3 \simeq SU(2)$ has 2-multiplicity for $Z_T(SU(2))$ if and only if X is antipodal set on S^3.

6.B.2 Harmonic analysis

It is well known that for any $l = 0, 1, 2, \ldots$, there uniquely exists an irreducible complex representation (ρ_l, V_l) of $SU(2)$ with $\dim V_l = l + 1$.

Example 6.B.2. *For $l = 0, 1, 2, \ldots$, let us denote by $\mathrm{Hom}_l(u, v)$ the set of all homogeneous polynomial over $\mathbb{C}$ with variable u, v such that homogeneous degree is l. Then $\mathrm{Hom}_l(u, v)$ can be regard as an $(l+1)$-dimensional representation of $SU(2)$. One can check that this representation is irreducible. Therefore, we can take $V_l = \mathrm{Hom}_l(u, v)$.*

Recall that for $\Omega = SU(2), T$ or $SU(2)/T$, we defined functional spaces $\mathcal{H}_\Omega^{\rho_l}$ on Ω in Section 6.5.1. For $d = 1,2,3$, what is $\mathcal{H}_{S^d}^{\rho_l}$? To answer this question, we introduce some notation as follows. For $l = 0,1,2,\ldots$, let us denote by

$$\begin{aligned} \mathrm{Hom}_l(\mathbb{R}^{d+1}) &:= \{\, \text{Homogeneous polynomials over } \mathbb{C} \text{ on } \mathbb{R}^{d+1} \text{ with degree } l \,\}, \\ \mathrm{Harm}_l(\mathbb{R}^{d+1}) &:= \{\, f \in \mathrm{Hom}_l(\mathbb{R}^{d+1}) \mid \Delta f = 0 \,\}, \\ \mathrm{Harm}_l(S^d) &:= \{\, f|_{S^d} \mid f \in \mathrm{Harm}_l(\mathbb{R}^{d+1}) \,\}, \end{aligned}$$

where Δ is the Laplacian. We remark that it is well known that

$$\dim_{\mathbb{C}} \mathrm{Harm}_l(S^d) = \binom{d+l}{l} - \binom{d+l-2}{l-2}$$

and $\mathrm{Harm}_l(S^d)$ is irreducible representation of $SO(d+1)$ if $d \geq 2$.

Then the next lemma holds:

Lemma 6.B.3. *For each* $l = 0,1,2,\ldots,$ *we have*

$$\begin{aligned} \mathcal{H}_{S^3}^{\rho_l} &= \mathrm{Harm}_l(S^3), \\ \mathcal{H}_{S^1}^{\rho_l} &= \mathrm{Harm}_l(S^1) \oplus \mathrm{Harm}_{l-2}(S^1) \oplus \cdots \oplus \begin{cases} \mathrm{Harm}_0(S^1) & \text{if } l \text{ is even,} \\ \mathrm{Harm}_1(S^1) & \text{if } l \text{ is odd,} \end{cases} \\ \mathcal{H}_{S^2}^{\rho_l} &= \begin{cases} \mathrm{Harm}_{\frac{l}{2}}(S^2) & \text{if } l \text{ is even,} \\ \{0\} & \text{if } l \text{ is odd,} \end{cases} \end{aligned}$$

Sketch of the proof of Lemma 6.B.3. For each $n \in \mathbb{Z}$, let us denote by χ_n the 1-dimensional complex representation of $T \simeq S^1$ as

$$\chi_n : T \to \mathbb{C}^\times, \quad \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \mapsto z^n.$$

By Example 6.B.2, one can prove that $\mathcal{H}_{S^1}^{\chi_n \oplus \chi_{(-n)}} = \mathrm{Harm}_n(S^1)$ for each n. Here, one can also observe that the following holds: The restriction $(\rho_l)|_T$ of ρ to $T \simeq S^1$ can be decomposed by

$$(\rho_l)|_T \simeq \chi_l \oplus \chi_{(l-2)} \oplus \cdots \oplus \chi_{(-l+2)} \oplus \chi_{(-l)}.$$

In particular,

$$\dim V_l^T = \begin{cases} 1 & \text{if } l \text{ is even,} \\ 0 & \text{if } l \text{ is odd.} \end{cases}$$

Hence, we have

$$\mathcal{H}_{S^1}^{\rho_l} = \mathrm{Harm}_l(S^1) \oplus \mathrm{Harm}_{l-2}(S^1) \oplus \cdots \oplus \begin{cases} \mathrm{Harm}_0(S^1) & \text{if } l \text{ is even,} \\ \mathrm{Harm}_1(S^1) & \text{if } l \text{ is odd,} \end{cases}$$

$$\mathcal{H}_{S^2}^{\rho_l} = \{0\} \quad \text{if } l \text{ is odd.}$$

If l is even, then $\mathrm{Harm}_{\frac{l}{2}}(S^2)$ is an irreducible representation of $SO(3)$ with

$$\dim \mathrm{Harm}_{\frac{l}{2}}(S^2) = l + 1.$$

Recall that $SU(2)$-action on S^2 factors the double cover $SU(2) \to SO(3)$. Thus, $\mathrm{Harm}_{\frac{l}{2}}(S^2)$ is an irreducible representation of $SO(3)$ with its dimension $l+1$. Therefore, $\mathrm{Harm}_{\frac{l}{2}}(S^2) \simeq V_l$ as an irreducible representation of $SU(2)$. By Lemma 6.A.6, we have $\mathcal{H}_{S^2}^{\rho_l} = \mathrm{Harm}_{\frac{l}{2}}(S^2)$ if l is even.

Recall that for each $l = 0, 1, 2, \dots$, there uniquely exists an complex $(l+1)$-dimensional irreducible representation ρ_l of $SU(2)$, up to isomorphisms. By using Peter–Weyl's theorem (cf. [19, Theorem 1.1 (2)]), one can prove that $\mathrm{Harm}_l(S^3)$ is the unique irreducible $(l+1)^2$-dimensional $(SU(2) \times SU(2))$-subrepresentation of $C^0(S^3) \simeq C^0(SU(2))$. Hence, $\mathrm{Harm}_l(S^3) \simeq V_l \otimes V_l^\vee$ as an irreducible representation of $SU(2) \times SU(2)$. Thus, by Lemma 6.A.5, we also have $\mathcal{H}_{S^3}^{\rho_l} = \mathrm{Harm}_l(S^3)$. □

For each $t \in \mathbb{N}$, let us denote by $\rho(t) := \bigoplus_{l=0}^{t} \rho_l$. For simplicity, we put $\mathcal{H}_{S^d}^t := \mathcal{H}_{S^d}^{\rho(t)}$ for $d = 1, 2, 3$. By combining Lemma 6.B.3 with the well-known fact that

$$P_t(S^d) = \bigoplus_{l=0}^{t} \mathrm{Harm}_l(S^d) \quad \text{for any } d \text{ and } t,$$

we obtain the following proposition:

Proposition 6.B.4. *For any t, we have*

$$\mathcal{H}_{S^3}^t = P_t(S^3), \; \mathcal{H}_{S^1}^t = P_t(S^1) \textit{ and } \mathcal{H}_{S^2}^t = P_{\lfloor \frac{t}{2} \rfloor}(S^2).$$

Therefore, by Proposition 6.B.1 and Proposition 6.B.4, the results in Section 6.2 follows from the results in Section 6.5.

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