平成２４年度

博士論文題目

# Asymptotic analysis of Bergman kernels for linear series and its application to Kähler geometry

# 線形系に対するベルグマン核の漸近解析とケーラー幾何への応用について

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# 謝辞

何よりまずはじめに、修士・博士課程の五年間を通じて私の研究を指導し、励まし続けてくださった、高山茂晴先生に最大の感謝を捧げます。私が大学院修士課程に入学したとき、高山先生はまず J. P. Demailly 氏の書かれたレクチャーノートを紹介してくださり、$L^2$ 評価の方法や Monge–Ampère 方程式といった、現在では私の研究の素地となっている道具を身につけるよう指導してくださいました。次第にセミナーで論文を紹介しながら研究テーマを探すようになりましたが、先生は自力で問題を見出すよう強調されながらも、私の興味と関連する、あるいは研究に役立ちそうな論文を折に触れて紹介してくださいました。その中に、博士論文の研究の起爆剤となるものが多く含まれていました。論文を書く段になると先生は私の拙い原稿を何度となく読んでくださり、数学的な内容から研究者としての常識や気構えといった範疇に及ぶことまで、丁寧に示唆してくださいました。

私が研究を進める過程で、Robert Berman 氏、Bo Berndtsson 氏、Sebastien Boucksom 氏、Dan Coman 氏、二木昭人先生、Vincent Guedj 氏、伊藤敦氏、Dano Kim 氏、大沢健夫先生、満渕俊樹先生、Mihai Paun 氏、辻元先生、David Witt Nyström 氏、Ahmed Zeriahi 氏をはじめとする研究者の方々にかけがえのない多くの助言と励ましの言葉をいただきました。この場を借りて改めて感謝申し上げます。特に Robert Berman 氏、Bo Berndtsson 氏、David Witt Nyström 氏は、Chalmers 及び Göteborg 大学を訪ねた私を暖かく迎えてくださり、約2ヶ月の滞在中常に貴重なアドバイスや議論の機会を与え続けてくださりました。この博士論文の主要な部分はこのスウェーデン滞在期間中に発展させられたものです。そしてそれぞれの論文を査読し助言をくださったレフェリーの方々に感謝します。ここに挙げました方々のうち一名でも欠ければ、今回の博士論文がこうして完成することは無かったと思います。

特に経済的な面、設備・経費といった側面においては、東京大学そして日本学術振興会様から多くの援助をいただきました。お世話になりました関係者の方々に改めて感謝申し上げます。

数学に関心を持ち勉強を進める上で、多くの素晴らしい先生方にめぐりあえたことはこの上もない幸運でした。特に私が数学を専攻しようと決心したきっかけを与えてくださった数理科学研究会の野口修先生、東京大学理学部数学科４年次のセミナーで指導を賜りました今野宏先生に心よりの感謝を捧げます。この場にお二人の名前を挙げることは私の学生時代の大きな目標でした。

長い大学院院生生活の中で、同期の友人や先輩・後輩方からは常に刺激を受け、励まされてきました。特に大川新之介さんと松本佳彦さんとは学部生時代からよく自主セミナーを行い、両氏の数学に対する誠実な姿勢に強い影響を受けました。また、大学院の外にあっても、多くの友人が私の慰めになってくれました。ここに名前を挙げられなかった中にも、あるいは私が無自覚でいるだけでお世話になった、ご迷惑をかけた方々が多くいらっしゃると思います。たとえば行きつけの喫茶店では私は長時間数学の論文を読んで居座っている嫌な客でありました。それでも毎回暖かく迎えてくださった店員の皆様方には頭が上がりません。

最後に、病弱であった私に健康と教育という名の光を授け、数学を志してからは絶えずこの身を励まし続けてくれた両親の忍耐と慈愛に感謝します。

# Introduction

## 1. Asymptotic analysis of Bergman kernel

Bergman kernel is classically defined as the reproducing kernel for the Hilbert space of square integrable holomorphic functions. More specifically, given a bounded domain $\Omega \subset \mathbb{C}^n$ the orthogonal projection to the space of $L^2$-holomorphic functions $P : L^2(\Omega) \to A^2(\Omega)$ has the integral representation:

$$(Pf)(z) = \int_{\Omega} K(z,w)f(w)dw,$$

and we call $K(z,w)$ Bergman kernel. Restriction to the diagonal $B(z) := K(z,z)$ is called the Bergman kernel function. Bergman kernel plays a role to control the space $A^2(\Omega)$ and it has been widely studied in the area of several complex variables. Asymptotic analysis of Bergman kernel in this classical setting focus on the boundary behavior of this kernel function. The celebrated result of Fefferman shows that there is the asymptotic expansion with respect to a boundary defining function $R(z)$:

$$B(z) = F(z)R(z)^{-n-1} + G(z)\log R(z).$$

In this paper we investigate the corresponding phenomenon in the compact manifold case. There is no holomorphic function over a compact complex manifold except constants. The next natural object for the function theory is line bundles over $X$ and their global sections. In fact the famous Kodaira embedding theorem tells that if $L$ admits a smooth Hermitian metric (with a local description $e^{-\varphi}$) having strictly positive curvature $dd^c\varphi$ then the section ring

$$R(L) := \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k})$$

recovers $X$. Given a Hermitian metric $e^{-\varphi}$ and a positive volume form $dV$, we may also define the Bergman kernel with respect to the norm

$$\|s\|^2 := \int_X |s|^2 e^{-k\varphi} dV.$$

Then the asymptotic analysis in this setting studies the behavior of Bergman kernels as $k \to \infty$. The results of Tian, Catlin, and Zelditch assure that these Bergman kernel functions admit the asymptotic expansion with respect to $k$:

$$B(z) = a_0(z)k^n + a_1(z)k^{n-1} + \cdots,$$

provided the metric of the line bundle has strictly positive curvature (see [5] for detail, especially on the relation with the domain case). More recently, the degenerated (in the sense of curvature positivity and smoothness of the metric) case was intensively studied by many authors.

In this paper we further aim to develop the above asymptotic analysis of Bergman kernel to the family of subspaces $W_k \subseteq H^0(X, L^{\otimes k})$ such that $\bigoplus_{k=0}^{\infty} W_k$ forms a subring of $R(L)$. In other words, we consider the graded linear series $W_k$ satisfying $W_k \cdot W_{k'} \subseteq W_{k+k'}$ for any $k, k' \geqslant 1$. Our subspace-generalization is motivated by many geometric applications. Let us show some examples.

**Example 1.** (Examples of graded linear series)

(1) Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal sheaf. The family of subspaces

$$W_k := H^0(X, L^{\otimes k} \otimes \mathfrak{a}^k)$$

defines a graded linear series.

(2) Assume that $X$ is a closed subvariety of a compact manifold $Y$ and $L$ is defined over $Y$. Then the family of subspaces defined by

$$W_k := \operatorname{Im}[\ H^0(Y, L^{\otimes k}) \to H^0(X, L^{\otimes k}|_X)\ ]$$

forms a graded linear series.

(3) Let $(\mathcal{X}, \mathcal{L})$ be a test configuration of a polarized manifold $(X, L)$ and $t$ be a parameter for the base $\mathbb{C}$. For each $s \in H^0(X, L^{\otimes k})$ we denote the unique meromorphic extension of $s$ by $\bar{s}$. Then for each $\lambda \in \mathbb{R}$

$$W_{\lambda,k} := \left\{ s \in H^0(X, L^{\otimes k}) \;\middle|\; t^{-\lceil \lambda k \rceil} \bar{s} \in H^0(\mathcal{X}, \mathcal{L}^{\otimes k}) \right\}.$$

defines a graded linear series of $(X, L)$.

In Part 1 (which corresponds to the paper [9]), we give an integral representation of the volume function

$$\operatorname{vol}(W) := \limsup_{k \to \infty} \frac{\dim W_k}{k^n/n!},$$

specifying the leading term of the asymptotic expansion for general graded linear series. This is given by the following equilibrium type metric whose definition is originated from Siciak.

**Definition 2.** For a given graded linear series $W \subseteq R(L)$ and a smooth metric $e^{-\varphi}$ on $L$, we define the equilibrium weight by

$$P_W\varphi := \sup \left\{ \frac{1}{k} \log |\sigma|^2 \;\middle|\; k \geqslant 1,\ \sigma \in W_k,\ \sup_X |\sigma|^2 e^{-k\varphi} \leqslant 1. \right\}.$$

Then $e^{-P_W\varphi}$ defines a singular Hermitian metric on $L$ with the curvature current $dd^c P_W\varphi$ semipositive.

Monge–Ampère product $(dd^c P_W\varphi)^n$ is well-defined on the bounded locus of $P_W\varphi$ and we denote the zero-extension to $X$ by $\operatorname{MA}(P_W\varphi)$.

**Theorem 3.** *Let $X$ be a smooth complex projective variety, $L$ a holomorphic line bundle, $W \subseteq R(L)$ be a graded subring such that $X \dashrightarrow \mathbb{P}W_k^*$ defines a birational map onto its image. Then we have*

$$\operatorname{vol}(W) = \int_X \operatorname{MA}(P_W\varphi).$$

This completely generalizes the results of [4] and [2] where they treated with the case $W = R(L)$. Our proof is based on the Fujita type approximative Zariski decomposition for graded linear series, which was exploited in [12], [13], and [7]. For the restricted case (Example (2)), we may give a much detail study for the convergence of the Bergman kernels thanks to the $L^2$-extension theorems for subvarieties. This is the main part of the author's master thesis [8]. See Part 4.

## 2. Existense problem of constant scalar curvature Kähler metric

The main part of the present paper is Part 2, which corresponds to the preprint [11]. In this part we give a geometric application of the above study for subspace version of Bergman kernel. We apply Theorem 3 to the family of graded linear series associated to a test configuration of a polarized manifold (appeared in Example (3)).

Let $(X, L)$ be a polarization of a smooth complex projective variety. As an analogue of Kobayashi–Hitchin correspondence for vector bundles it has been conjectured that the existence of a constant scalar curvature Kähler metric (cscK metric in short) in the first Chern class of $L$ is equivalent to certain stability of $(X, L)$ in the sense of geometric invariant theory. The expected notion of stability was first explored by Tian and later Donaldson gave the purely algebraic definition of *K-stability*. K-stability is detected by the following datum of degeneration. We call it a test configuration and denote by $\mathcal{T}$.

(1) A flat family of schemes with relatively ample $\mathbb{Q}$-line bundle $\pi \colon (\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ such that $(\mathcal{X}_1, \mathcal{L}_1) \simeq (X, L)$ holds.
(2) A $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ which makes $\pi$ equivariant, with respect to the canonical action of $\mathbb{C}^*$ on the target space $\mathbb{C}$.

For each $k \geqslant 1$ test configuration induces the $\mathbb{C}^*$-action $\rho : \mathbb{C}^* \to \mathrm{Aut}(H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}))$ which decomposes the vector space as $H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}) = \bigoplus_\lambda V_\lambda$ such that $\rho(\tau)v = \tau^\lambda v$ holds for any $v \in V_\lambda$ and $\tau \in \mathbb{C}^*$. By the equivariant Riemann–Roch Theorem, the total weight $w(k) = \sum_\lambda \lambda \dim V_\lambda$ is a polynomial of degree $n + 1$. The Donaldson–Futaki invariant of given a test configuration is defined to be the subleading term of

$$\frac{w(k)}{k \dim H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k})} = F_0 + F_1 k^{-1} + O(k^{-2}).$$

**Definition 4.** A polarization $(X, L)$ is said to be *K-stable* (*resp. K-semistable*) if $F_1 < 0$ (*resp.* $F_1 \leqslant 0$) holds for any non-trivial test configuration. We say $(X, L)$ is *K-polystable* if it is K-semistable and $F_1 = 0$ holds only for product test configurations.

**Conjecture 5.** (*Yau–Tian–Donaldson*) *A polarized manifold $(X, L)$ admits a cscK metric if and only if it is K-polystable.*

The stability part was recently proved by Donaldson, Stoppa, and Mabuchi. That is, the existence of cscK metric implies K-polystability of the polarized manifold. This is based on Donaldson's lower bound estimate of the Calabi functional, which was first proved in [6]:

$$\left(\int_X (S_\varphi - \hat{S})^2 \frac{(dd^c\varphi)^n}{n!}\right)^{\frac{1}{2}} \geqslant \frac{F_1}{\|\mathcal{T}\|},$$

where $dd^c\varphi$ is an arbitrary Kähler metric and $S_\varphi$ is its scalar curvature with mean value $\hat{S}$. The norm $\|\mathcal{T}\|$ is also introduced by Donaldson and defined as the leading term of the asymptotic expansion:

$$\sum_\lambda (\lambda - \hat{\lambda})^2 \dim V_\lambda = \|\mathcal{T}\|^2 k^{n+2} + O(k^{n+1}).$$

It is immediate from the inequality that the existence of cscK metric $dd^c\varphi$ (such that $S_\varphi = \hat{S}$) implies K-semistability.

One of the guiding principle to solve the existence part is variational principle over the space of Kähler metrics. Let us identify a positive curvature metric $e^{-\varphi}$ on $L$ with its curvature Kähler metric $\omega := dd^c\varphi$ on $X$. Then the space of Kähler metrics $\mathcal{H}$ consists of all positive curvature metric $h = e^{-\varphi}$ on $L$, endowed with the canonical Riemannian metric

$$\|u\|_2 := \left(\int_X u^2 \frac{(dd^c\varphi)^n}{n!}\right)^{\frac{1}{2}}$$

which is defined for any tangent vector $u$ at $\varphi$. There is the canonical K-energy functional $\mathcal{M} : \mathcal{H} \to \mathbb{R}$ defined by its difference:

$$\delta\mathcal{M}(\varphi) := -\int_X (S_\varphi - \hat{S})\delta\varphi \frac{(dd^c\varphi)^n}{n!},$$

such that any cscK metric is characterized as a critical point of this energy. This K-energy is known to be convex along any smooth geodesic ray in $\mathcal{H}$ hence it is important to investigate the gradient of the energy at infinity along a given geodesic ray $\varphi_t$ ($t \in [0, +\infty)$). On the other hand, Phong and Sturm ([14]) established that any test configuration $\mathcal{T}$ with fixed metric $\varphi$ canonically defines a *weak* geodesic ray $\varphi_t$ emanating from $\varphi$. They further conjectured that the Donaldson–Futaki invariant should be given by $\lim_{t\to\infty} \frac{d}{dt}\mathcal{M}(\varphi_t)$ if the latter one is properly defined for the weak geodesic ray. Thus we obtain a picture to prove the existence of a cscK metric.

Notice that in the above picture the asymptotic *moments* of eigenvalues for $\mathbb{C}^*$-action $(F_0, F_1, \|\mathcal{T}\|, \dots)$ played a central role. From this point of view, in the present paper we further study asymptotic *distribution* of eigenvalues

$$\sum_\lambda \delta_\lambda \dim V_\lambda.$$

Here $\delta_\lambda$ denote the Dirac delta function centered at $\lambda \in \mathbb{R}$. The point is that given test configuration one can relate the family of graded linear series

$$W_\lambda = \bigoplus_{k=0}^{\infty} W_{\lambda,k} \subseteq \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k})$$

as in Example (3). Just as $R(L)$ recovers $X$, $W_\lambda$ recovers $(\mathcal{X}, \mathcal{L})$ and the parameter $\lambda$ describe the degeneration to the central fiber. In fact one can show that the limit of spectral measures is given by the Lebesgue–Stieltjes measure of $\mathrm{vol}(W_\lambda)$ as follows:

$$\frac{n!}{k^n}\sum_\lambda \delta_\lambda \dim V_\lambda \to -d\,\mathrm{vol}(W_\lambda).$$

This is still an algebraic formula, however, applying Theorem 3 we obtain an analytic quantity on the right-hand side. Moreover, by [15] the Legendre transformation translate the family of equilibrium metric $P_{W_\lambda}\varphi$ to the weak geodesic ray $\varphi_t$ such that the canonical Duistermatt–Heckmann measure finally appears.

**Theorem 6.** *Let us fix a test configuration $\mathcal{T}$ and a smooth metric $e^{-\varphi}$ of $L$. Denote the associated weak geodesic ray by $\varphi_t$ and identify its tangent vector with the right derivative $\dot{\varphi}_t : X \to \mathbb{R}$. Then the sequence of spectral measure converges to the push-forward measure $(\dot{\varphi}_t)_*\,\mathrm{MA}(\varphi_t)$. That is,*

$$\frac{n!}{k^n}\sum_\lambda \delta_\lambda \dim V_\lambda \to (\dot{\varphi}_t)_*\,\mathrm{MA}(\varphi_t) \qquad (k\to\infty)$$

*holds in the sense of measure convergence. In particular the right-hand side is independent of $t$ and $\varphi$.*

The theorem was first conjectured and proved for the product test configurations by Witt Nyström ([16]). It can be also regarded as a variant of equivariant Chern–Weil theory ([1]) for the polarized family whose central fiber possibly be very singular. Letting $t\to\infty$ corresponds to the degeneration of manifolds to the central fiber and from this reason we may regard the right-hand side as the canonical Duistermatt–Heckmann measure. Taking the second moment of the measures we immediately obtain the analytic interpretation of the Donaldson's norm.

**Theorem 7.**

$$\|\mathcal{T}\| = \left(\int_X (\dot{\varphi}_t - F_0)^2 \frac{\mathrm{MA}(\varphi_t)}{n!}\right)^{\frac{1}{2}}.$$

From this, we may give a natural energy theoretic explanation for the Donaldson's lower bound of the Calabi functional. Moreover, one can define any $p$ norm $\|\mathcal{T}\|_p$ by the $L^p$ norm of the tangent vector and expect that the similar inequality also holds. In fact in the Fano case combining Theorem 6 with [3] we obtain:

**Theorem 8.** *Let $X$ be a Fano manifold. For any Kähler metric $\omega \in c_1(-K_X)$, take a function $h$ such that $\mathrm{Ric}(\omega) - \omega = dd^c h$ and $\int_X (e^h - 1)\omega^n = 0$ hold. Then for any conjugate exponent $p, q$ with $1/p + 1/q = 1$ we have*

$$\left(\int_X \left|e^h - 1\right|^q \frac{\omega^n}{n!}\right)^{\frac{1}{q}} \geqslant \frac{F_1}{\|\mathcal{T}\|_p}.$$

Our approach using Bergman kernels of graded linear series is itself new and should be studied more in the future. Applications to another type of graded linear series appearing in geometry are also expected. In Part 3 (corresponds to [10]), we treat with the restricted case (Example (2)) and give some application to the extension problem of semipositive curvature metric.

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# Part 1

On the volume of graded linear series and Monge–Ampère mass

# ON THE VOLUME OF GRADED LINEAR SERIES AND MONGE–AMPÈRE MASS

TOMOYUKI HISAMOTO

ABSTRACT. We give an analytic description of the volume of a graded linear series, as the Monge–Ampère mass of a certain equilibrium metric associated to any smooth Hermitian metric on the line bundle. We also show the continuity of this equilibrium metric on some Zariski open subset, under a geometric assumption.

## 1. INTRODUCTION

Let $L$ be a holomorphic line bundle on a smooth projective variety $X$. A *graded linear series* of $L$ is a graded subalgebra $W$ of the section ring $R = \bigoplus_{k\geqslant 0} H^0(X, L^{\otimes k})$. In this paper we study the following invariant which plays a fundamental role in the asymptotic analysis of graded linear series.

**Definition 1.1.** We denote the dimension of $X$ by $n$. The *volume* of a graded linear series $W$ is the nonnegative real number defined by

$$\mathrm{vol}(W) := \limsup_{k\to\infty} \frac{\dim W_k}{k^n/n!}.$$

This is known to be finite. The limit of supremum is in fact limit for sufficiently divisible $k$ from the result of [KK09]. In case $W$ is complete, *i.e.* $W = R$, $\mathrm{vol}(R)$ is nothing but the volume of the line bundle $\mathrm{vol}(L)$ which is a birational invariant of $L$ and widely studied. We refer to Chapter 2 of [Laz04] for the basic facts. Analytic studies of the volume of line bundles were initiated by [Bou02] and [Ber09]. For a general graded linear series, [KK09] and [LM09] originally systematically studied properties of the volume by relating it with the *Okounkov body* of $W$. Proper subalgebras of $R$ naturally arise in many interesting situations of algebraic geometry (see *e.g.* Example 3.3 or 3.4 below) and it is necessary to develop asymptotic analysis of graded linear series.

In this paper we study the volume for a general graded linear series from the analytic point of view, succeeding to the spirit of previous work of Boucksom, Berman and many other authors. We work over the complex number field $\mathbb{C}$. The main result is an integral representation of the volume via the Monge–Ampère product of a certain singular Hermitian metric on $L$, called the *equilibrium metric*, which is determined by $W$ with any fixed smooth Hermitian metric on $L$. In what follows we identify a singular Hermitian metric $h$ and its local weight function $\varphi$. They are related by the identity $h = e^{-\varphi}$ which holds in each local trivialization patch of $L$. We also identify

2000 *Mathematics Subject Classification.* Primary 32J25, Secondary 32W20, 32A25, 14C20.

*Key words and phrases.* graded linear series, volume, Bergman kernel, Monge–Ampère operator.

a holomorphic section $\sigma \in H^0(X, L^{\otimes k})$ with the corresponding function on each local trivialization patch and denote by $|\sigma|$ the absolute value taken for the function. For detail, see Section 2.

**Definition 1.2.** Let $h = e^{-\varphi}$ be a smooth Hermitian metric on $L$. For each $k \geqslant 1$, we define a singular Hermitian metric $h_k = e^{-\varphi_k}$ by

$$\varphi_k := \sup \left\{ \frac{1}{k} \log |\sigma|^2 \;\middle|\; \sigma \in W_k, \ \sup_X |\sigma|^2 e^{-k\varphi} \leqslant 1. \right\}.$$

The *equilibrium metric* of $h$ with respect to $W$ is the (possibly) singular Hermitian metric defined by its weight:

$$P_W\varphi := (\sup_k \varphi_k)^*.$$

Here we denote the upper-semicontinuous envelope of a function $f$ by $f^*(x) := \limsup_{y \to x} f(y)$.

The formulation of equilibrium metric here is originated from Siciak ([Sic62]) and [Ber09] introduced the corresponding idea for line bundles to investigate the asymptotic of related Bergman kernels. Roughly speaking, $P_W\varphi$'s epigraph is the $W$-polynomially convex hull of $\varphi$'s epigraph. $P_W\varphi$ is a plurisubharmonic function on each trivialization patch provided $W$ is non-trivial. That is, the curvature current $dd^cP_W\varphi$ defines a closed positive current on $X$. On the bounded locus of $P_W\varphi$ one can define the Monge–Ampère product $(dd^cP_W\varphi)^n$ in the manner of Bedford–Taylor. Further, the trivial extension of $(dd^cP_W\varphi)^n$ defines a positive measure $\mathrm{MA}(P_W\varphi)$ which has no mass on any pluripolar subset of $X$. This kind of measures is called *non-pluripolar Monge–Ampère product* and studied by [BEGZ10]. In particular, it was proved that $\mathrm{MA}(P_W\varphi)$ has finite mass over $X$. The following is our main theorem.

**Theorem 1.3.** *Let $X$ be a smooth projective variety and $L$ a holomorphic line bundle on $X$. Let $W$ be a graded linear series of $L$ such that the associated map $X \dashrightarrow \mathbb{P}W_k^*$ is birational onto its image for any sufficiently divisible $k$. Then, for any smooth Hermitian metric $h = e^{-\varphi}$ on $L$, the Monge–Ampère mass of the equilibrium metric $P_W\varphi$ gives the volume of $W$. That is,*

$$\mathrm{vol}(W) = \int_X \mathrm{MA}(P_W\varphi)$$

*holds.*

This formula enables us to investigate the positivity of $W$ locally in $X$ and it might be helpful for lower bound estimate of $\mathrm{vol}(W)$ in the future. Theorem 1.3 is a natural generalization of the results in [Bou02], [Ber09], [BB10], [BEGZ10], and [His12]. Note that the birationality assumption is necessary for general $W \neq R$ (see Remark 4.3). Moreover, thanks to the currently available technology, the proof in the present paper get much simpler than the previous ones, even in the case of complete graded linear series. On the other hand, our approach is rather algebraic relying on the result of [KK09], [LM09], [Jow10], and [DBP12] and does not give much information about Bergman kernel asymptotics as [Ber09] or [His12]. In particular, the following regularity problem is open.

**Question 1.4.** *In the situation of Theorem 1.3, $P_W\varphi$ has Lipschitz continuous derivatives on some non-empty Zariski open subset of $X$?*

In the complete case $W = R$ this is proved by [Ber09]. It also holds for the *restricted* linear series, by [His12]. See also [BD09]. Such a regularity is the key to analyze the asymptotic of Bergman kernels. For a general graded linear series, we may show the following at the present.

**Theorem 1.5.** *In the situation of Theorem 1.3 let us further assume that $W$ is finitely generated and* Proj $W$*, which gives the image of the associated rational map $X \dashrightarrow \mathbb{P}W_k^*$ for $k$ sufficiently divisible, is normal. Then $P_W\varphi$ is continuous on some non-empty Zariski open subset of $X$.*

The organization of this paper is as follows. In Section 2 and 3 we will give some preliminary materials, from the analytic and the algebraic viewpoints. We prove Theorem 1.3 in Section 4. Section 5 will be devoted to the proof of Theorem 1.5.

## 2. Monge–Ampère operator

In this section, we briefly review the definition and basic properties of the Monge–Ampère operator.

Let $L$ be a holomorphic line bundle on a projective manifold $X$. We usually fix a family of local trivialization patches $U_\alpha$ which cover $X$. A singular Hermitian metric $h$ on $L$ is by definition a family of functions $h_\alpha = e^{-\varphi_\alpha}$ which are defined on corresponding $U_\alpha$ and satisfy the transition rule: $\varphi_\beta = \varphi_\alpha - \log|g_{\alpha\beta}|^2$ on $U_\alpha \cap U_\beta$. Here $g_{\alpha\beta}$ are the transition functions of $L$ with respect to the indices $\alpha$ and $\beta$. *The weight functions* $\varphi_\alpha$ are assumed to be locally integrable. If $\varphi_\alpha$ are smooth, $\{e^{-\varphi_\alpha}\}_\alpha$ defines a smooth Hermitian metric on $L$. We usually denote the family $\{\varphi_\alpha\}_\alpha$ by $\varphi$ and omit the indices of local trivializations. Notice that each $\varphi = \varphi_\alpha$ is only a local function and not globally defined, but the curvature current $\Theta_h = dd^c\varphi$ is globally defined and is semipositive if and only if each $\varphi$ is plurisubharmonic (*psh* for short). Here we denote by $d^c$ the real differential operator $\frac{\partial-\bar\partial}{4\pi\sqrt{-1}}$. We call such a weight a *psh weight.* The most important example is those of the form $k^{-1}\log(|\sigma_1|^2 + \cdots + |\sigma_N|^2)$, defined by some holomorphic sections $\sigma_1, \cdots, \sigma_N \in H^0(X, L^{\otimes k})$. Here $|\sigma_i|$ $(1 \leqslant i \leqslant N)$ denotes the absolute value of the corresponding function of each $\sigma_i$ on $U_\alpha$. We call such weights *algebraic singular.* More generally, a psh weight $\varphi$ is said to have a *small unbounded locus* if it is locally bounded outside a closed complete pluripolar subset $S \subset X$. A singular Hermitian metric $h = e^{-\varphi}$ is said to have strictly positive curvature if $dd^c\varphi \geqslant \omega$ holds for some Kähler form $\omega$.

Let $n$ be the dimension of $X$. The Monge–Ampère operator is defined by

$$\varphi \mapsto \mathrm{MA}(\varphi) := (dd^c\varphi)^n$$

when $\varphi$ is smooth. On the other hand it does not make sense for general $\varphi$. The celebrated result of Bedford–Taylor [BT76] tells us that the right hand side can be defined as a current if $\varphi$ is at least in the class $L^\infty \cap \mathrm{PSH}(U_\alpha)$. Specifically, by induction on the exponent $q = 1, 2, ..., n$, it can be defined as:

$$\int_{U_\alpha} (dd^c\varphi)^q \wedge \eta := \int_{U_\alpha} \varphi(dd^c\varphi)^{q-1} \wedge dd^c\eta$$

for each test form $\eta$. Here $\int$ denotes the canonical pairing of currents and test forms. This is indeed well-defined and defines a closed positive current, because $\varphi$ is a bounded Borel function and $(dd^c\varphi)^{q-1}$ has measure coefficients by the induction hypothesis. Notice the fact that any closed positive current has measure coefficients. Bedford–Taylor's Monge–Ampère products have useful continuity properties:

**Theorem 2.1** ([Ko05], Theorem 1.11, Proposition 1.12, Theorem 1.15.). *For any sequence of bounded psh weights, the convergence of Monge–Ampère products*

$$(dd^c\varphi_k)^n \to (dd^c\varphi)^n$$

*holds in the sense of currents if it satisfies one of the following conditions.*

(1) *$\varphi_k$ is non-increasing and converges to $\varphi$ pointwise in $X$.*
(2) *$\varphi_k$ is non-decreasing and converges to $\varphi$ almost everywhere in $X$.*
(3) *$\varphi_k$ converges to $\varphi$ uniformly on any compact subset of $X$.*

It is also necessary to consider unbounded psh weights. On the other hand, for our purpose, it is enough to deal with weights with small unbounded loci.

**Definition 2.2.** Let $\varphi$ be a psh weight of a singular metric on $L$. If $\varphi$ has a small unbounded locus contained in an algebraic subset $S$, we define a positive measure $\mathrm{MA}(\varphi)$ on $X$ by

$$\mathrm{MA}(\varphi) := \text{ the zero extension of } (dd^c\varphi)^n.$$

Note that the coefficient of $(dd^c\varphi)^n$ is well-defined as a measure on $X \setminus S$. □

Actually $(dd^c\varphi)^n$ has a finite mass so that $\mathrm{MA}(\varphi)$ defines a closed positive current on $X$. For a proof, see [BEGZ10], Section 1.

*Remark* 2.3. In [BEGZ10], the *non-pluripolar* Monge–Ampère product was defined in fact for general psh weights on a compact Kähler manifold. Note that this definition of the Monge–Ampère operator makes the measure $\mathrm{MA}(\varphi)$ to have no mass on any pluripolar set. Roughly speaking, $\mathrm{MA}(\varphi)$ ignores the mass which comes from the singularities of $\varphi$. For this reason, as a measure-valued function in $\varphi$, $\mathrm{MA}(\varphi)$ is no longer continuous. This also applies to $\mathrm{MA}(P_W\varphi)$ and one of the technical point for us to prove Theorem 1.3. □

We recall the fundamental fact established in [BEGZ10] which states that the less singular psh weight has the larger Monge–Ampère mass. Recall that given two psh weight $\varphi$ and $\varphi'$ on $L$, $\varphi$ is said to be less singular than $\varphi'$ if there exists a constant $C > 0$ such that $\varphi' \leqslant \varphi + C$ holds on $X$. We say that a psh weight is *minimal singular* if it is minimal with respect to this partial order. When $\varphi$ is less singular than $\varphi'$ and $\varphi'$ is less singular than $\varphi$, we say that the two functions have the equivalent singularities. This defines a equivalence relation.

**Theorem 2.4** ([BEGZ10], Theorem 1.16.). *Let $\varphi$ and $\varphi'$ be psh weights with small unbounded loci such that $\varphi$ is less singular than $\varphi'$. Then*

$$\int_X \mathrm{MA}(\varphi') \leqslant \int_X \mathrm{MA}(\varphi)$$

*holds.*

*Remark* 2.5. It is unknown whether Theorem 2.4 holds for general psh weights. □

## 3. The volume of a graded linear series

First we introduce the suitable class of graded linear series to handle with the volume. Let $W = \bigoplus_{k\geqslant 0} W_k$ be a graded linear series. For each $k$, $W_k$ defines a rational map $f_k\colon X \dashrightarrow \mathbb{P}W_k^*$ unless $W_k = \{0\}$. Here $W_k^*$ denotes the dual vector space of $W_k$. Notice that the image of a rational map is defined to be the projection of the graph.

**Definition 3.1.** We call a graded linear series $W$ *birational* if the associated map $f_k\colon X \dashrightarrow \mathbb{P}W_k^*$ is birational onto its image for any sufficiently divisible $k$.

This notion is introduced by [LM09] (see Definition 2.5). If the line bundle $L$ is big, the graded linear series $R = \bigoplus_{k\geqslant 0} H^0(X, L^{\otimes k})$ is complete and by definition is birational. Thus the notion of birational graded linear series is a natural generalization of the complete linear series of a big line bundle. A line bundle $L$ is known to be big if and only if its volume is positive, *i.e.* $\mathrm{vol}(L) > 0$. This is why we concentrate on the class of big line bundles in the study of the volume. It is also true that the volume of a birational graded linear series is positive (see [LM09], Lemma 2.6 and Theorem 2.13), but the converse does not hold.

**Example 3.2.** Let $f\colon X \to Y$ be a finite morphism to a projective variety and fix an embedding $Y \hookrightarrow \mathbb{P}^N$. Then $W_k := f^*H^0(Y, \mathcal{O}_Y(k))$ defines a graded linear series with positive volume. But this is not birational unless $\deg f = 1$.

The difference seems subtle at first glance, however in fact the class of graded linear series with positive volume is much harder to treat than the class of birational graded linear series. See Remark 3.8. To emphasize the significance of the generalization to non-complete linear series, let us describe some examples of birational graded linear series.

**Example 3.3.** Let $Y$ be a smooth projective variety and $L$ a holomorphic line bundle defined over $Y$. We assume that $X$ is a closed subvariety of $Y$. The family of subspaces defined by

$$W_k := \mathrm{Im}\big[\ H^0(Y, L^{\otimes k}) \to H^0(X, L^{\otimes k}|_X)\ \big]$$

is called the *restricted linear series.* The volume $\mathrm{vol}_{Y|X}(L) := \mathrm{vol}(W)$ is called the *restricted volume.* If $X$ is not contained in the augmented base locus $\mathbb{B}_+(L)$ of $Y$ (for the definition, see [Laz04], Definition 10.3.2), $W$ defines a birational graded linear series. For restricted linear series, the author investigated the asymptotic of related Bergman kernels in [His12] and obtained Theorem 1.3 as a corollary. The point is that in this special case we can prove Question 1.4 thanks to an $L^2$-extension theorem for the subvariety $X$.

**Example 3.4.** Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal sheaf. The family of subspaces

$$W_k := H^0(X, L^{\otimes k} \otimes \overline{\mathfrak{a}^k})$$

defines a graded linear series. Here $\overline{\mathfrak{a}^k}$ denotes the integral closure of $\mathfrak{a}^k$. There exist a modification $\mu\colon X' \to X$ and an effective divisor $F$ on $X'$ such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ and by definition $\overline{\mathfrak{a}} = \mu_* \mathcal{O}_{X'}(-F)$ hold. Hence we have $W_k = \mu_* H^0(X', \mu^* L^{\otimes k} \otimes \mathcal{O}_{X'}(-kF))$. In particular, $W$ is birational if and only if $\mu^* L \otimes \mathcal{O}_{X'}(-F)$ is big. If we assume further $\mu^* L \otimes \mathcal{O}_{X'}(-F)$ is ample, Theorem 1.3 follows from the results of [Ber07], Section 4. It seems not so hard to extend these results to the case when $\mu^* L \otimes \mathcal{O}_{X'}(-F)$ is big, along the same line as [Ber07].

**Example 3.5.** Let $X := \mathbb{P}^2$, $L := \mathcal{O}(1)$, and $W := \mathbb{C}[X, Y, YZ, YZ^2, \dots, YZ^i, \dots]$ $(i \geqslant 1)$ be the graded subalgebra of the homogeneous coordinate ring $\mathbb{C}[X, Y, Z]$. It is then easy to see that $W_k$ is base point free and the natural map $X \to \mathbb{P}W_k^*$ is birational onto its image for each $k$. However, $W$ is not finitely generated $\mathbb{C}$-algebra.

Study of the volume of birational graded linear series was first taken by [KK09] and [LM09]. They used the theory of Okounkov bodies to derive the log-concavity of the volume and the following type of Fujita's approximate Zariski decomposition. It motivates our study and will play the central role in the algebraic part of the proof of Theorem 1.3. See also [DBP12] and [Jow10].

**Theorem 3.6** ([KK09], Theorem 5. See also [LM09], Theorem 3.5 and [DBP12], Theorem 3.14.). *Let $W$ be a graded linear series with* $\mathrm{vol}(W) > 0$. *Then for any $\varepsilon > 0$ there exists a number $\ell_0$ such that*

$$\lim_{k \to \infty} \frac{\dim \mathrm{Im}\left[S^k W_\ell \to W_{k\ell}\right]}{k^n \ell^n / n!} \geqslant \mathrm{vol}(W) - \varepsilon$$

*holds for any $\ell \geqslant \ell_0$.*

We will use the following consequence which is equivalent to Theorem C of [Jow10], to prove Theorem 1.3.

**Proposition 3.7.** *Let $W$ be a birational graded linear series. For each $k$ let $\mu_k\colon X_k \to X$ be a resolution of the base ideal $\mathfrak{b}(W_k)$ such that $\mu_k^{-1}\mathfrak{b}(W_k) = \mathcal{O}(-F_k)$ holds for some effective divisor $F_k$ on $X_k$. Define the line bundle on $X_k$ by $M_k := \mu_k^* L^{\otimes k} \otimes \mathcal{O}(-F_k)$ and denote the self-intersection number of the globally generated line bundle $M_k$ by $M_k^n$. Then it holds that*

$$\mathrm{vol}(W) = \lim_{k \to \infty} \frac{M_k^n}{k^n}.$$

Here $k$ runs through sufficiently divisible numbers.

*Proof.* Let us first assume that $W$ is finitely generated. In that case there exists some $\ell$ such that for each $k$ the natural map $S^k W_\ell \to W_{k\ell}$ is surjective. Then composing with inverse of the Segre embedding to the image, the induced $\mathbb{P}W_{k\ell}^* \to \mathbb{P}S^k W_\ell^*$ maps the image of $X \dashrightarrow \mathbb{P}W_{k\ell}^*$ onto the image of $X \dashrightarrow \mathbb{P}W_\ell^*$ isomorphically. Let us fix such $\ell$, and denote the image of the natural map $f\colon X \dashrightarrow \mathbb{P}W_\ell^*$ by $Y$. Notice that $Y$ is isomorphic to $\mathrm{Proj}\, W$, which is well-defined since $W$ is finitely generated. We also denote the image of the natural map $g_k\colon X \dashrightarrow \mathbb{P}H^0(X_k, M_k)^*$ by $Z_k$. Then for any $k$ divided by $\ell$ the linear projection $\mathbb{P}H^0(X_k, M_k)^* \dashrightarrow \mathbb{P}W_k^*$ composed with the above isomorphism between the images induces the morphism $\pi_k\colon Z_k \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} X_k \arrow[r, "g_k"] \arrow[d, "\mu_k"'] & Z_k \arrow[d, "\pi_k"] \\ X \arrow[r, dashed, "f"'] & Y \end{tikzcd}$$

Here $\pi_k \circ g_k$ gives a resolution of $f$ for $X_k$. By the definition of $\pi_k$, we have $\pi_k^*\mathcal{O}_Y(\frac{k}{\ell}) = \mathcal{O}_{Z_k}(1)$. Taking $\ell$ sufficiently divisible, we may further assume $f$ is birational so that $\pi_k$ is also birational. Therefore we obtain

$$\mathrm{vol}(\mathcal{O}_Y(\frac{k}{\ell})) = \mathrm{vol}(\pi_k^*\mathcal{O}_Y(\frac{k}{\ell})) = \mathrm{vol}(\mathcal{O}_{Z_k}(1)).$$

The left hand side gives $k^n \,\mathrm{vol}(W)$ and $\mathrm{vol}(\mathcal{O}_{Z_k}(1)) = M_k^n$ so we conclude $\mathrm{vol}(W) = k^{-n}M_k^n$ in this case.

In general, Theorem 3.6 reduces the proof to the case where $W$ is finitely generated. Let us explain it. For each $\ell$ set $W_k^{(\ell)} := \mathrm{Im}[S^{\frac{k}{\ell}}W_\ell \to W_k]$ if $k$ is devided by $\ell$ and otherwise set $W_k^{(\ell)} := \{0\}$. Then $W^{(\ell)}$ defines a finitely generated graded linear series of $L$. Applying the above argument in the finitely generated case, we obtain $\mathrm{vol}(W^{(\ell)}) = \ell^{-n}M_\ell^n$. Note that $\mu_\ell$ gives a resolution of $\mathfrak{b}(W_k^{(\ell)})$ for any $k$ so that $M_\ell^{\otimes\frac{k}{\ell}} = \mu_\ell^* L^{\otimes k} \otimes \mathcal{O}(-\frac{k}{\ell}F_\ell)$ corresponds to $W_k^{(\ell)}$. Then for any $\varepsilon > 0$ sufficiently divisible $\ell$ assures

$$\mathrm{vol}(W) \geqslant \mathrm{vol}(W^{(\ell)}) = \ell^{-n}M_\ell^n \geqslant \mathrm{vol}(W) - \varepsilon.$$

This ends the proof. □

The reduction to the finitely generated case is the critical step to prove Theorem 1.3 and the idea comes down to the proof of Theorem 3.6. See also [Ito12] for a similar argument.

*Remark* 3.8. In the above proof, the assumption $W$ is birational is crucial. For general $W$, $\pi_k$ possibly has degree greater than one and Proposiion 3.7 does not hold. In fact in Example 3.2, the right hand side in the proposition gives $\deg f$ times of $\mathrm{vol}(W)$.

## 4. Proof of Theorem 1.3

Let us prove the main theorem of this paper. In the sequel we fix the notation as in the statement of Theorem 1.3. Recall that $\varphi_k$ is a singular metric on $L$ defined by

$$\varphi_k := \sup\left\{\frac{1}{k}\log|\sigma|^2 \;\middle|\; \sigma \in W_k,\ \sup_X |\sigma|^2 e^{-k\varphi} \leqslant 1.\right\}.$$

By compactness of the unit ball in $W_k$, this defines a psh weight. We claim that $\varphi_k$ has algebraic singularities described by $\mathfrak{b}(W_k)$, the base ideal of $W_k$. To see this we will compare $\varphi_k$ with the corresponding Bergman kernel weight which is defined as follows.

Fix a smooth volume form $dV$ on $X$. Set the $L^2$-norm by $\|\sigma\|^2_{k\varphi} := \int_X |\sigma|^2 e^{-k\varphi} dV$ and introduce the Bergman kernel weight $u_k$ as

$$u_k(x) := \sup \left\{ \frac{1}{k} \log |\sigma(x)|^2 \;\middle|\; \sigma \in W_k, \text{ and } \|\sigma\|^2_{k\varphi} \leqslant 1. \right\},$$

for any $x \in X$. It is then easy to see that

$$(\limsup_{k\to\infty} u_k)^* = P_W\varphi$$

holds. In fact, we obviously have

$$\|\sigma\|^2_{k\varphi} \leqslant \int_X dV \cdot \sup_X |\sigma|^2 e^{-k\varphi}$$

and by the mean value inequality

$$|\sigma(x)|^2 \leqslant C_r \sup_{B(x;r)} e^{k\varphi} \|\sigma\|^2_{k\varphi}$$

holds for any sufficiently small ball $B(x;r)$ in a local coordinate.

If one fix any orthonormal basis of $W_k$ with respect to the above $L^2$-norm, say $\{\sigma_1, \dots \sigma_N\}$, then it is easy to see that $u_k = k^{-1} \log(|\sigma_1|^2 + \cdots + |\sigma_N|^2)$ holds. Thus $\varphi_k$ also has algebraic singularities described by $\mathfrak{b}(W_k)$.

Now by the definition of $F_k$ there exists a smooth semipositive $(1,1)$-form $\gamma \in c_1(M_k)$ such that

$$\begin{aligned} dd^c \mu_k^* u_k &= dd^c \mu_k^* k^{-1} \log(|\sigma_1|^2 + \cdots + |\sigma_N|^2) \\ &= k^{-1}(\gamma + [F_k]) \end{aligned}$$

hold. Here $[F_k]$ stands for the current defined by the divisor $F_k$. Since $M_k$ is a globally generated line bundle we have $M_k^n = \int_X \gamma^n$ and the non-pluripolarity yields $k^{-n}\gamma^n = \mathrm{MA}(\mu_k^* u_k)$ on $X$. Therefore by Theorem 2.4 we obtain

$$k^{-n} M_k^n = \int_{X_k} \mathrm{MA}(\mu_k^* u_k) = \int_X \mathrm{MA}(\varphi_k). \tag{4.1}$$

On the other hand, we have $P_W\varphi = (\sup_k \varphi_k)^*$, and the sequence $\varphi_k$ is essentially increasing, in the sense that $\varphi_k \leqslant \varphi_\ell$ if $k$ divides $\ell$. As a consequence, the sequence of $\psi_k := \varphi_{2^k}$ is non-decreasing, and it converges in $L^1$-topology to $P_W\varphi$. We conclude using:

**Proposition 4.2.** *Let $\psi_k$ be an non-decreasing sequence of psh weights with small unbounded loci on a big line bundle $L$, and assume that $\psi_k \to \psi$ in the $L^1$ topology,* i.e. *$\psi = (\sup_k \psi_k)^*$. Then*

$$\int_X \mathrm{MA}(\psi_k) \to \int_X \mathrm{MA}(\psi).$$

*Proof.* We have $\sup_k \int_X \mathrm{MA}(\psi_k) \leqslant \int_X \mathrm{MA}(\psi)$ by Theorem 2.4. On the other hand, if $S$ is a proper algebraic subset of $X$ outside which $\psi_k$ is locally bounded, then $(dd^c\psi_k)^n \to (dd^c\psi)^n$ weakly on $X \setminus S$ by Theorem 2.1, hence

$$\int_X \mathrm{MA}(\psi) \leqslant \liminf_{k\to\infty} \int_X \mathrm{MA}(\psi_k).$$

□

Theorem 1.3 is now concluded by Proposition 3.7 together with Proposition 4.2.

*Remark* 4.3. Theorem 1.3 does not hold for general $W$ with $\mathrm{vol}(W) > 0$. To be precise, taking a resolution of the base ideal of $W_k$, the right hand side in Theorem 1.3 is given by $\lim_{k\to\infty} k^{-n} M_k^n$ just as (4.1). As it was explained in Remark 3.8, however, Proposition 3.7 is not true in general so that the equilibrium mass does not give $\mathrm{vol}(W)$.

## 5. Estimate of the Bergman kernels

In this section we give a lower bound estimate of the Bergman kernels to prove Theorem 1.5. Let us fix the notation as in Section 4. For simplicity, we denote $P_W\varphi$ by $P\varphi$. The following is the required one.

**Proposition 5.1.** *Fix a smooth volume form $dV$ on $X$. Set $\|\sigma\|_{k\varphi}^2 := \int_X |\sigma|^2 e^{-k\varphi} dV$ and*

$$u_k(x) := \sup\left\{ \frac{1}{k}\log|\sigma(x)|^2 \;\middle|\; \sigma \in W_k, \text{ and } \|\sigma\|_{k\varphi}^2 \leqslant 1. \right\}.$$

*for every $x \in X$. Assume that $W$ is finitely generated, birational, and further* Proj $W$, *which is isomorphic to the image of the associated birational map $f_k\colon X \dashrightarrow \mathbb{P}W_k^*$ for any sufficiently divisible $k$, is normal. Then there exist a proper algebraic subset $S$ such that a subsequence of $u_k$ converges to $P\varphi$ uniformly on any compact set of $X \setminus S$. In particular, $P\varphi$ is continuous on a non-empty Zariski open set of $X$.*

*Proof.* This has been already known in the complete case by [Ber09] and can be extended to the present case by pushing the $L^2$-estimate to the image of the map $f_k\colon X \dashrightarrow \mathbb{P}W_k^*$. We claim for any compact set $K \Subset X \setminus S$ there exist a constant $C_K$ and a positive integer $\ell$ such that

$$P\varphi - \frac{C_K}{k} \leqslant u_k \leqslant P\varphi + \frac{C_K}{k} \tag{5.2}$$

holds for any $k$ divisible by $\ell$. The right hand side inequality is obvious. The proof of the left hand side is essential and it needs some $L^2$-estimate for the solution of a $\bar\partial$-equation.

Denote the image of the map $f_k\colon X \dashrightarrow \mathbb{P}W_k^*$ by $Y_k$. By the finite generation, the canonical map $S^kW_\ell \to W_{k\ell}$ is surjective for any $k$ and sufficiently large $\ell$. The restriction of $\mathbb{P}W_{k\ell}^* \hookrightarrow \mathbb{P}S^kW_\ell^*$ to $Y_{k\ell}$ and the restriction of Segre embedding $\mathbb{P}W_\ell^* \to \mathbb{P}S^kW_\ell^*$ to $Y_\ell$ are isomorphic. Thus we have $Y_{k\ell} \simeq Y_\ell$ for every $k \geqslant 1$. Let us fix $\ell$ and denote $Y_\ell$ by $Y$. Denote by $\mathcal{O}_Y(1)$ the restriction of $\mathcal{O}_{\mathbb{P}W_\ell^*}(1)$ to $Y$. Note that by the Kodaira vanishing theorem there exists a number $k_0$ such that $S^kW_\ell = H^0(\mathbb{P}W_\ell^*, \mathcal{O}(k)) \to H^0(Y, \mathcal{O}_Y(k))$ is surjective for any $k \geqslant k_0$. We take a modification $\mu\colon X_\ell \to X$ such that

$X_\ell$ is smooth and there exist a decomposition $\mu^* |W_\ell| = |V| + E$ with a free linear system $V$ and an effective divisor $E$. Then the map $h\colon X_\ell \to Y$ defined by $V$ equals to $\mu \circ f$. Fix a standard section of $E$ and denote it by $s_E$. Given a section $s \in H^0(Y, \mathcal{O}_Y(k))$, one can push forward $h^*s \otimes s_E^k$ by $\mu$, thanks to the normality of $X$. We denote it by $f^*s$. This defines a map $f^*\colon H^0(Y, \mathcal{O}_Y(k)) \to W_{k\ell}$. Moreover, since $Y$ is normal by the assumption, given any $\sigma \in W_{k\ell}$ one can push-forward $\mu^*\sigma \otimes s_E^{-k}$ by the birational map $h$. We denote it by $h_*\sigma$. Thus we obtain an isomorphism $W_{k\ell} \simeq H^0(Y, \mathcal{O}_Y(k))$ for any $k$.

Taking a resolution $\pi\colon \tilde{Y} \to Y$ we have a nef and big line bundle $\pi^*\mathcal{O}_Y(1)$ and an induced birational map $\tilde{f}\colon X \dashrightarrow \tilde{Y}$. The normality of $Y$ deduces $W_{k\ell} \simeq H^0(Y, \mathcal{O}_Y(k)) \simeq H^0(\tilde{Y}, \pi^*\mathcal{O}_Y(k))$. Here one can also pull-back or push-forward sections by the birational map $\tilde{f}$, taking a modification of the source space as above. Further, we can pull-back psh weights. That is, given a psh weight $\psi$ of a singular metric on $\pi^*\mathcal{O}_Y(1)$, we may push-forward $\tilde{h}^*\psi + \log|s_E|^2$ by $\mu$ thanks to the normality of $X$. It defines a psh weight of a metric on $L^{\otimes \ell}$ and we denote it by $\tilde{f}^*\psi$. Finally, given a smooth weight $\varphi$ the push-forward $\tilde{f}_*P\varphi$ can be defined as follows:

$$(\tilde{f}_*P\varphi)(y) := \sup{}^*\left\{ \frac{1}{k\ell}\log\left|(\tilde{f}_*\sigma)(y)\right|^2 \;\middle|\; k \geqslant 1,\ \sigma \in W_{k\ell},\ \text{and } |\sigma|^2 e^{-k\ell\varphi} \leqslant 1 \text{ on } X \right\}.$$

This actually defines a psh weight of a singular metric on the $\mathbb{Q}$-line bundle $\ell^{-1}\pi^*\mathcal{O}_Y(1)$ and $\ell\tilde{f}_*P\varphi$ is a psh weight for the genuine line bundle $\pi^*\mathcal{O}_Y(1)$.

Fix a Kähler form $\omega$ on $\tilde{Y}$ and a psh weight $\psi$ of a metric on $\pi^*\mathcal{O}_Y(1)$, such that $\psi$ has algebraic singularity, $\tilde{f}^*\psi \leqslant \ell\varphi$, and $dd^c\psi \geqslant \omega$ hold. Let $S$ be a proper algebraic subset such that $\tilde{f}|_{X\setminus S}$ is isomorphic and $\psi$ is smooth outside $\tilde{f}(S)$.

We claim that there exist sufficiently large $\ell$, $C$ and a section $\sigma_k \in W_{k\ell}$ for each $k \geqslant 1$ such that

(1) $|\sigma_k(x)|^2 e^{-k\ell P\varphi} \geqslant C^{-1}$ *for any* $x \in K$, and
(2) $\|\sigma_k\|^2_{k\ell\varphi} \leqslant C$.

In fact, this implies

$$e^{k\ell u_{k\ell}} \geqslant \frac{|\sigma_k(x)|^2}{\|\sigma_k\|^2_{k\ell\varphi}} \geqslant C^{-2}e^{k\ell P\varphi}.$$

It then yields the inequality $P\varphi - \frac{C_K}{k\ell} \leqslant u_{k\ell}$ which is nothing but the left-hand side of (5.2).

Let us fix $x \in K$ and set $y := \tilde{f}(x)$. Then applying the Ohsawa–Takegoshi $L^2$-extension theorem, for any $a \in \mathbb{C}$ one can get a holomorphic function $g$ on a small ball $B(y; r)$ such that $g(y) = a$ and

$$\int_{B(y;r)} |g|^2 e^{-(k-1)\ell\tilde{f}_*P\varphi - \psi} dV_\omega \leqslant C\,|a|^2 e^{-(k-1)\ell\tilde{f}_*P\varphi - \psi}$$

hold. Let $\rho$ be a cut-off function supported on $B(y; r)$. Solving the equation $\bar{\partial}v = \bar{\partial}(\rho g)$ by Hörmandar's $L^2$-method, we get a solution $v$ with

$$\int_{\tilde{Y}} |v|^2 e^{-(k-1)\ell\tilde{f}_*P\varphi - \psi - \rho(n+2)\log|z|^2} dV_\omega \leqslant C\,|a|^2 e^{-(k-1)\ell\tilde{f}_*P\varphi - \psi}.$$

Here we take a local coordinate $z = (z_1, \dots z_n)$ around $y$ and $\ell$ sufficiently large to ensure the positivity of the curvature of the total weight. Set $s := \rho g - v \in H^0(\tilde{Y}, \pi^* \mathcal{O}_Y(k))$. Then it yields $s(y) = a$ and

$$\int_{\tilde{Y}} |s|^2 e^{-(k-1)\ell \tilde{f}_* P\varphi - \psi} dV_\omega \leqslant C |a|^2 e^{-(k-1)\ell \tilde{f}_* P\varphi - \psi}.$$

We may choose suitable $a$ so that the right hand side equals to $C$. Finally we set $\sigma_k := \tilde{f}^* s$. Then it yields:

(1) $|\sigma_k(x)|^2 e^{-(k-1)\ell P\varphi - \tilde{f}^*\psi} = 1$,

(2) $\|\sigma_k\|^2_{(k-1)\ell P\varphi + \tilde{f}^*\psi} \leqslant C$.

These inequalities respectively correspond to (1) and (2) of the claim. We indeed infer

$$\|\sigma_k\|^2_{k\ell\varphi} \leqslant \|s\|^2_{(k-1)\ell P\varphi + \tilde{f}^*\psi} \leqslant C.$$

On the other hand we may assume $e^{(\ell P\varphi - \tilde{f}^*\psi)(x)} \leqslant C$ by the smoothness of $\psi$ around $y$ so that

$$1 = |\sigma_k(x)|^2 e^{-(k-1)\ell P\varphi - \tilde{f}^*\psi} \leqslant C |\sigma_k(x)|^2 e^{-k\ell P\varphi}$$

holds. Here $C$ depends on only $\ell$ and $K$. Therefore the claim has been shown to conclude the theorem. □

**Acknowledgments.** The author would like to express his gratitude to his advisor Professor Shigeharu Takayama for his warm encouragements, suggestions and reading the drafts. The author is grateful to Atsushi Ito for several helpful comments on the algebraic reduction arguments in this paper. The author greatly indebted to the referee for his many kind advices on many mathematical or linguistic points of improvement. In particular he indicated to the author a rather straightforward proof of Theorem 1.3 after his reading the first version of this paper. In the first version the author used the uniform convergence in Proposition 5.1 to derive Theorem 1.3 but such a Bergman kernel estimate turned out to be unnecessary if one notes that $\varphi_k$ is essentially non-decreasing. This research is supported by JSPS Research Fellowships for Young Scientists (22-6742).

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# Part 2

On the limit of spectral measures associated to a test configuration of a polarized Kähler manifold

# ON THE LIMIT OF SPECTRAL MEASURES ASSOCIATED TO A TEST CONFIGURATION OF A POLARIZED KÄHLER MANIFOLD

TOMOYUKI HISAMOTO

ABSTRACT. We apply the main result of [His12] to the family of graded linear series constructed from any test configuration. This solves the conjecture raised by [WN10] so that the sequence of spectral measures for the induced $\mathbb{C}^*$-action on the central fiber converges to the canonical Duistermatt–Heckman measure defined by the associated weak geodesic ray. As a consequence, we show that the algebraic $p$-norm of the test configuration equals to the $L^p$-norm of tangent vectors. Using this result, We may give a natural energy theoretic explanation for the lower bound estimate on the Calabi functional by [Don05], extending the statement to any $p$-norm ($p \geqslant 1$), and prove the analogous result for the Kähler–Einstein metric.

## 1. INTRODUCTION

Let $X$ be an $n$-dimensional smooth projective variety and $L$ an ample line bundle over $X$. In the sequel we also fix a smooth Hermitian metric $h$ on $L$, which has strictly positive curvature over $X$. The curvature form defines a Kähler metric in the first Chern class $c_1(L)$. Conversely, any Kähler metric $\omega$ in $c_1(L)$ has a Kähler potential $\varphi$ in each local trivialization neighborhood such that the correction of $e^{-\varphi}$ defines a Hermitian metric with the curvature form $\omega = dd^c\varphi$, uniquely up to multiplication of a constant. We identify $h$ with the correction of weights $\varphi$ and with $\omega$. The space of Kähler metric $\mathcal{H}$ is the set of all $h = e^{-\varphi}$, endowed with the canonical Riemannian metric

$$\|u\|_2 := \left( \int_X u^2 \frac{(dd^c\varphi)^n}{n!} \right)^{\frac{1}{2}}$$

which is defined for any tangent vector $u$ at $\varphi$. There is the canonical K-energy functional $\mathcal{M} \colon \mathcal{H} \to \mathbb{R}$ such that any constant scalar curvature Kähler metric is characterized as a critical point of this energy. This K-energy is known to be convex along any smooth geodesic ray in $\mathcal{H}$ and it is important to investigate the gradient of the energy at infinity along a given geodesic ray $\varphi_t$ ($t \in [0, +\infty)$).

A geodesic ray in $\mathcal{H}$ corresponds to a special kind of degeneration of $(X, L)$ in algebraic geometry. A flat family of polarized schemes $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ with $(\mathcal{X}_1, \mathcal{L}_1) = (X, L)$ and an equivariant $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ is called a test configuration. We denote the datum by $\mathcal{T}$. For each $k \geqslant 1$ let $H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}) = \bigoplus_\lambda V_\lambda$ be the eigenspace decomposition of the induced $\mathbb{C}^*$-action $\rho : \mathbb{C}^* \to \mathrm{Aut}(H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}))$ on the central fiber such

2010 *Mathematics Subject Classification.* Primary 32Q26, Secondary 32W20, 32Q20.

*Key words and phrases.* stability, constant scalar curvature Kähler metric, graded linear series, Monge–Ampère operator.

that $\rho(\tau)v = \tau^\lambda v$ holds for every $\tau \in \mathbb{C}^*$ and $v \in V_\lambda$. Then we have the asymptotic expansion

$$\frac{\sum_\lambda \lambda \dim V_\lambda}{k \sum_\lambda \dim V_\lambda} = F_0 + F_1 k^{-1} + O(k^{-2}).$$

We call the coefficient $F_1$ in the subleading term as the Donaldson–Futaki invariant of $\mathcal{T}$. It was first established by [PS07] that any test configuration $\mathcal{T}$ with fixed metric $\varphi$ canonically defines a *weak* geodesic ray $\varphi_t$ emanating from $\varphi$, in the space of Kähler metric. (Here for the proof of the main theorem we adopt the construction of [RWN11] so that $\varphi_t - F_0$ gives the geodesic ray in [PS07].) In this situation it is now conjectured that the Donaldson–Futaki invariant corresponds to $\lim_{t\to\infty} \frac{d}{dt}\mathcal{M}(\varphi_t)$ if the latter one is properly defined for the weak geodesic ray. In this paper we further relate the asymptotic *distribution* of eigenvalues to $\varphi_t$ and give some application to the estimate for $F_1$. Our main theorem claims that the associated sequence of spectral measures converges to the canonical Duistermatt–Heckman measure defined by $\varphi_t$. The Monge–Ampère (or Liouville) measure $\mathrm{MA}(\varphi_t)$ is defined for each singular $\varphi_t$ and equals to $(dd^c\varphi_t)^n$ if $\varphi_t$ is smooth (see subsection 2.1).

**Theorem 1.1.** *Let $\mathcal{T}$ be a test configuration with normal $\mathcal{X}$. Then the weak limit of the normalized distribution of eigenvalues is given by the push-forward of the Monge–Ampère measure $\mathrm{MA}(\varphi_t)$ to the real line by the tangent vector $\dot{\varphi}_t$. That is, for any $t \geqslant 0$ we have*

$$\lim_{k\to\infty} \frac{n!}{k^n} \sum_\lambda \delta_{\frac{\lambda}{k}} \dim V_\lambda = (\dot{\varphi}_t)_* \mathrm{MA}(\varphi_t).$$

*Here $\delta_{\frac{\lambda}{k}}$ denotes the delta function for $\frac{\lambda}{k} \in \mathbb{R}$. In particular, the right hand side measure is independent not only of $t$ but also of $\varphi$, and defines the canonical Duistermatt–Heckman measure.*

Theorem 1.1 was first conjectured by [WN10] and proved for product test configurations in the same paper. The analogous result for geodesic segments was obtained by [Bern09] in a different approach. Recall that the above definition of $F_1$ was motivated by the equivariant Riemann–Roch formula of [AB84], which can be applied to the product test configuration and in that case one has the Duistermatt–Heckman measure on the central fiber in the usual way. In word of geodesic the central fiber corresponds to $t = \infty$ and our canonical Duistermatt–Heckman measure which is independent of $t$ gives the right generalization to any test configuration. Then Theorem 1.1 can be seen as a part of the *ideal* index theorem for an equivariant family which admits very singular fiber over the fixed point $0 \in \mathbb{C}$. Taking the $p$-th moment of the above measure, we may extend the definition of algebraic norm in [Don05] to any $p \geqslant 1$ and relate it to the $L^p$-norm of tangent vectors on the weak geodesic ray.

**Theorem 1.2.** *Let us define the trace-free part of each eigenvalue $\lambda$ as*

$$\bar{\lambda} := \lambda - \frac{\sum \lambda \dim V_\lambda}{\sum \dim V_\lambda}$$

*and for each $p \geqslant 1$ define the p-norm $\|\mathcal{T}\|_p$ by*

$$\|\mathcal{T}\|_p := \left(\lim_{k\to\infty}\frac{1}{k^n}\sum_{\lambda}\left|\frac{\lambda}{k}\right|^p \dim V_\lambda\right)^{\frac{1}{p}}.$$

*Then the limit exists and*

$$\|\mathcal{T}\|_p = \left(\int_X |\dot{\varphi}_t - F_0|^p \frac{\mathrm{MA}(\varphi_t)}{n!}\right)^{\frac{1}{p}}$$

*holds.*

Using Theorem 1.2, we may give an energy theoretic explanation for the [Don05]'s lower bound estimate on the Calabi functional, extending the result to any $p$-norm ($p \geqslant 1$). In particular in the Fano case we obtain the following. Note that when $L = -K_X$ any metric $h = e^{-\varphi}$ can be identified with the positive measure which is described as $e^{-\varphi}\bigwedge_{i=1}^n \frac{\sqrt{-1}}{2}dz_i \wedge d\bar{z}_i$ in each local coordinate. A metric $e^{-\varphi}$ is called a Kähler–Einstein metric if it satisfies the identity of the measures: $(dd^c\varphi)^n = n!e^{-\varphi}$.

**Theorem 1.3.** *Let $X$ be a Fano manifold and $\mathcal{T}$ a test configuration of $(X, -K_X)$, whose total space $\mathcal{X}$ is normal. Then for any smooth Hermitian metric $h = e^{-\varphi}$ on $-K_X$ and exponents $1 \leqslant p, q \leqslant +\infty$ with $1/p + 1/q = 1$ we have*

$$\left\|\frac{n!e^{-\varphi}}{(dd^c\varphi)^n} - 1\right\|_q \geqslant \frac{F_1}{\|\mathcal{T}\|_p}.$$

*In other word, the difference from Kähler–Einstein metric is bounded from below by the Donaldson–Futaki invariant.*

Let us briefly explain the outline of our proof of Theorem 1.1. The proof is based on the analytic study for graded linear series, which was exploited in [His12]. We apply it to [WN10]'s family of graded subalgebras

$$W_\lambda = \bigoplus_{k=0}^{\infty} W_{\lambda,k} \subseteq \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}),$$

which is constructed from $\mathcal{T}$ and each $\lambda \in \mathbb{R}$ as follows. For a given section $s \in H^0(X, L^{\otimes k})$, let us denote its unique invariant extension which is at least meromorphic over $\mathcal{X}$ by $\bar{s}$. We define $W_{\lambda,k}$ as the set of sections $s$ whose invariant extensions $\bar{s}$ have poles along the central fiber $\mathcal{X}_0 = \{t = 0\}$ at most $-\lceil \lambda k \rceil$ order. In other words,

$$W_{\lambda,k} := \left\{ s \in H^0(X, L^{\otimes k}) \;\middle|\; t^{-\lceil\lambda k\rceil}\bar{s} \in H^0(\mathcal{X}, \mathcal{L}^{\otimes k}) \right\}.$$

Then it can be proved algebraically that the limit of spectral measures is given by the Lebesgue–Stieltjes measure of the volume function $\mathrm{vol}(W_\lambda)$ in $\lambda$. The main theorem of [His12] interpret each volume into the Monge–Ampère measure of associated equilibrium metric $P_{W_\lambda}\varphi$. The Legendre transformation of this family of equilibrium metrics is nothing but the weak geodesic ray $\varphi_t$ so that we may complete the proof by the recent developed techniques of pluripotential theory. This new approach via the family of graded linear series seems itself interesting and we hope it should be studied more in the future.

## 2. Analytic description of the volume

**2.1. Monge–Ampère operator.** In this section, we briefly review the definition and basic properties of the Monge–Ampère operator. Let $L$ be a holomorphic line bundle on a projective manifold $X$. We usually fix a family of local trivialization patches $U_\alpha$ which cover $X$. A singular Hermitian metric $h$ on $L$ is by definition a family of functions $h_\alpha = e^{-\varphi_\alpha}$ which are defined on corresponding $U_\alpha$ and satisfy the transition rule: $\varphi_\beta = \varphi_\alpha - \log|g_{\alpha\beta}|^2$ on $U_\alpha \cap U_\beta$. Here $g_{\alpha\beta}$ are the transition functions of $L$ with respect to the indices $\alpha$ and $\beta$. *The weight functions* $\varphi_\alpha$ are assumed to be locally integrable. If $\varphi_\alpha$ are smooth, $\{e^{-\varphi_\alpha}\}_\alpha$ defines a smooth Hermitian metric on $L$. We usually denote the family $\{\varphi_\alpha\}_\alpha$ by $\varphi$ and omit the indices of local trivializations. Notice that each $\varphi = \varphi_\alpha$ is only a local function and not globally defined, but the curvature current $\Theta_h = dd^c\varphi$ is globally defined and is semipositive if and only if each $\varphi$ is plurisubharmonic (*psh* for short). Here we denote by $d^c$ the real differential operator $\frac{\partial-\bar\partial}{4\pi\sqrt{-1}}$. We call such a weight a *psh weight.* The most important example is those of the form $k^{-1}\log(|s_1|^2 + \cdots + |s_N|^2)$, defined by some holomorphic sections $s_1, \cdots, s_N \in H^0(X, L^{\otimes k})$. Here $|s_i|$ $(1 \leqslant i \leqslant N)$ denotes the absolute value of the corresponding function of each $s_i$ on $U_\alpha$. We call such weights *algebraic singular.* More generally, a psh weight $\varphi$ is said to have a *small unbounded locus* if the pluripolar set $\varphi^{-1}(-\infty)$ is contained in some closed complete pluripolar subset $S \subset X$ (*e.g.* a proper algebraic subset).

Let $n$ be the dimension of $X$. The Monge–Ampère operator is defined by

$$\varphi \mapsto (dd^c\varphi)^n$$

when $\varphi$ is smooth. On the other hand it does not make sense for general $\varphi$. The celebrated result of Bedford–Taylor [BT76] tells us that the right hand side can be defined as a current if $\varphi$ is at least in the class $L^\infty \cap \mathrm{PSH}(U_\alpha)$. Specifically, by induction on the exponent $q = 1, 2, ..., n$, it can be defined as:

$$\int_{U_\alpha} (dd^c\varphi)^q \wedge \eta := \int_{U_\alpha} \varphi(dd^c\varphi)^{q-1} \wedge dd^c\eta$$

for each test form $\eta$. Here $\int$ denotes the canonical pairing of currents and test forms. This is indeed well-defined and defines a closed positive current, because $\varphi$ is a bounded Borel function and $(dd^c\varphi)^{q-1}$ has measure coefficients by the induction hypothesis. Notice the fact that any closed positive current has measure coefficients.

It is also necessary to consider unbounded psh weights. On the other hand, for our purpose, it is enough to deal with weights with small unbounded loci.

**Definition 2.1.** Let $\varphi$ be a psh weight of a singular metric on $L$. If $\varphi$ has a small unbounded locus contained in an algebraic subset $S$, we define a positive measure $\mathrm{MA}(\varphi)$ on $X$ by

$$\mathrm{MA}(\varphi) := \text{ the zero extension of } (dd^c\varphi)^n.$$

Note that the coefficient of $(dd^c\varphi)^n$ is well-defined as a measure on $X \setminus S$.

Actually $(dd^c\varphi)^n$ has a finite mass so that $\mathrm{MA}(\varphi)$ defines a closed positive current on $X$. For a proof, see [BEGZ10], Section 1.

*Remark* 2.2. In [BEGZ10], the *non-pluripolar* Monge–Ampère product was defined in fact for general psh weights on a compact Kähler manifold. Note that this definition of the Monge–Ampère operator makes the measure $\mathrm{MA}(\varphi)$ to have no mass on any pluripolar set. In other words, $\mathrm{MA}(\varphi)$ ignores the mass which comes from the singularities of $\psi$. For this reason, as a measure-valued function in $\varphi$, $\mathrm{MA}(\varphi)$ no longer has the continuous property.

We recall the fundamental fact established in [BEGZ10] which states that the less singular psh weight has the larger Monge–Ampère mass. Recall that given two psh weight $\varphi$ and $\varphi'$ on $L$, $\varphi$ is said to be less singular than $\varphi'$ if there exists a constant $C > 0$ such that $\varphi' \leqslant \varphi + C$ holds on $X$. We say that a psh weight is *minimal singular* if it is minimal with respect to this partial order. When $\varphi$ is less singular than $\varphi'$ and $\varphi'$ is less singular than $\varphi$, we say that the two functions have the equivalent singularities. This defines a equivalence relation.

**Theorem 2.3** ([BEGZ10], Theorem 1.16.). *Let $\varphi$ and $\varphi'$ be psh weights with small unbounded loci such that $\varphi$ is less singular than $\varphi'$. Then*

$$\int_X \mathrm{MA}(\varphi') \leqslant \int_X \mathrm{MA}(\varphi)$$

*holds.*

2.2. **Analytic representation of volume.** Let $X$ be a $n$-dimensional smooth complex projective variety and $L$ a holomorphic line bundle on $X$. Graded linear series is by definition a graded $\mathbb{C}$-subalgebra of the section ring

$$W = \bigoplus_{k=0}^{\infty} W_k \subseteq \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}).$$

They appear in many geometric situations. In fact in the present paper we give an application of the analysis of such proper subalgebras to the problem of constant scalar curvature Kähler metric. The volume of graded linear series is the nonnegative real number which measures the size of the graded linear series as follows:

$$\mathrm{vol}(W) := \limsup_{k\to\infty} \frac{\dim W_k}{k^n/n!}.$$

This is finite and in fact the limit of supremum is limit by the result of [KK09]. The main result of [His12] gives an analytic description of the volume. The analytic counterpart of the volume is the following generalized equilibrium metric, which is originated from [Berm07].

**Definition 2.4.** Let $W$ be a graded linear series of a line bundle $L$. Fix a smooth Hermitian metric of $L$ and denote it by $h = e^{-\varphi}$, where $\varphi$ is the weight function defined on a fixed local trivialization neighborhood. We define the equilibrium weight associated to $W$ and $\varphi$ by

$$P_W\varphi := \sup{}^*\left\{ \frac{1}{k}\log|s|^2 \;\middle|\; k \geqslant 1,\ s \in W_k \text{ such that } |s|^2 e^{-k\varphi} \leqslant 1. \right\}.$$

Here $*$ denotes taking the upper semicontinuous regularization of the function. The equilibrium weight $P_W\varphi$ on each local trivialization neighborhood patches together and define the singular Hermitian metric on $L$. We call it an equilibrium metric.

As in the subsection 2.1, we define the Monge–Ampère measure $\mathrm{MA}(P_W\varphi)$ on $X$.

**Theorem 2.5** (The main theorem of [His12]). *Let $W$ be a graded linear series of a line bundle $L$, such that the natural map $X \dashrightarrow \mathbb{P}W_k^*$ is birational onto its image for any sufficiently large $k$. Then for any fixed smooth Hermitian metric $h = e^{-\varphi}$ we have*

$$\mathrm{vol}(W) = \int_X \mathrm{MA}(P_W\varphi).$$

Note that Theorem 2.5 is valid for general line bundle which is possibly not ample. We will apply this general formula to the special graded linear series associated to a test configuration of a polarized manifold.

*Remark* 2.6. With no change of the proof in [His12], Theorem 2.5 can also be proved under the assumption $W_k$ is birational onto its image for any sufficiently *divisible* $k$. For non complete linear series, the condition $\mathrm{vol}(W) > 0$ does not imply that $X \dashrightarrow \mathbb{P}W_k^*$ is birational onto its image for sufficiently large $k$. For example, when $W$ is defined as the pull-back of $H^0(Y, \mathcal{O}(k))$ by a finite morphism $X \to Y \subseteq \mathbb{P}^N$, $\mathrm{vol}(W) > 0$ holds but $W_k$ never defines birational map onto its image for any $k$. For this reason, neither does Theorem 2.5 hold for general $W$ with $\mathrm{vol}(W) > 0$. To be precise, taking a resolution $\mu_k$ of the base ideal of $W_k$ and denoting the fixed component of $\mu_k^* W_k$ by $F_k$, the right hand side in Theorem 2.5 is given by the limit of self-intersection number of line bundles $M_k := \mu_k^* L^{\otimes k} \otimes \mathcal{O}(-F_k)$.

## 3. Test configuration and associated family of graded linear series

In this section we explain the construction of the family of graded linear series $W_\lambda$ ($\lambda \in \mathbb{R}$) from fixed test configuration $(\mathcal{X}, \mathcal{L})$, following the recipe of Witt-Nyström's paper [WN10]. First we introduce the notion of K-stability.

### 3.1. K-stability.

**Definition 3.1** (The definition of test configuration by [Don02]). Let $(X, L)$ be a polarized manifold. We call the following datum *a test configuration* (*resp.* semi test configuration) for $(X, L)$.

(1) A flat family of schemes with relatively ample (*resp.* semiample and big) $\mathbb{Q}$-line bundle $\pi\colon (\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ such that $(\mathcal{X}_1, \mathcal{L}_1) \simeq (X, L)$ holds.
(2) A $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ which makes $\pi$ equivariant, with respect to the canonical action of $\mathbb{C}^*$ on the target space $\mathbb{C}$.

*Remark* 3.2. As it was pointed out by [LX11], the above original definition by Donaldson should be a bit modified. For example, if one further assume $\mathcal{X}$ is normal, then the pathological example in [LX11] can be removed. On the other hand, recent paper [Sz11] proposed to consider the class of test configurations whose *norms* $\|\mathcal{T}\|$ are non-zero and this condition seems more natural and appropriate for our viewpoint. In fact Theorem 1.2 gives one evidence. See also [RWN11]. We assume, however, the normality of $\mathcal{X}$ in proving Theorem 1.1 and 1.2.

By the flatness of $\pi$ Hilbert polynomials of $(\mathcal{X}_t, \mathcal{L}_t)$ are independent of $t \in \mathbb{C}$. The $\mathbb{C}^*$-equivariance yields an isomorphism $(\mathcal{X}_t, \mathcal{L}_t) \simeq (X, L)$ for any $t \in \mathbb{C} \setminus \{0\}$. Note that the central fiber $(\mathcal{X}_0, \mathcal{L}_0)$ can be very singular. It is even not normal in general. A test configuration is said to be *product* if $\mathcal{X} \simeq X \times \mathbb{C}$ and *trivial* if further the action of $\mathbb{C}^*$ on $X \times \mathbb{C}$ is trivial. A test configuration $(\mathcal{X}, \mathcal{L})$ induces the $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k})$ for each $k \geqslant 1$. This action $\rho : \mathbb{C}^* \to \mathrm{Aut}(H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}))$ decomposes the vector space as $H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}) = \bigoplus_\lambda V_\lambda$ such that $\rho(\tau)v = \tau^\lambda v$ holds for any $v \in V_\lambda$ and $\tau \in \mathbb{C}^*$. By the equivariant Riemann–Roch Theorem, the total weight $w(k) = \sum_\lambda \lambda \dim V_\lambda$ is a polynomial of degree $n+1$. Let us denote the coefficients by

$$w(k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}). \tag{3.3}$$

We also write the Hilbert polynomial of $(X, L)$ by

$$N_k := \dim H^0(X, L^{\otimes k}) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}).$$

The Donaldson–Futaki invariant of given test configuration is defined to be the subleading term of

$$\frac{w(k)}{kN_k} = F_0 + F_1 k^{-1} + O(k^{-2}). \tag{3.4}$$

In other word,

$$F_1 := \frac{a_0 b_1 - a_1 b_0}{a_0^2}. \tag{3.5}$$

**Definition 3.6.** A polarization $(X, L)$ is said to be *K-stable* (*resp. K-semistable*) if $F_1 < 0$ (*resp.* $F_1 \leqslant 0$) holds for any non-trivial test configuration. We say $(X, L)$ is *K-polystable* if it is K-semistable and $F_1 = 0$ holds only for product test configuration.

This notion of K-stability was first introduced by [Tia97]. The above algebraic definition was given by [Don02]. Note that K-stability is unchanged if one replaces $L$ to $L^{\otimes k}$ since $F_1$ is so. The equivalence of certain GIT-stability and existence of special metric originates from Kobayashi–Hitchin correspondence for vector bundles. In the polarized manifolds case, we have the following conjecture.

**Conjecture 3.7.** (*Yau–Tian–Donaldson*) *A polarized manifold $(X, L)$ admits a cscK metric if and only if it is K-polystable.*

One direction of the above conjecture was proved by [Don05], [Sto09], [Mab08], and [Mab09]. That is, the existence of cscK metric implies K-polystability of the polarized manifold. See also [Berm12] for the detail study in the Kähler–Einstein case.

The stability of vector bundle is defined by *slope* of subbundles and to pursue the analogy to the vector bundle case, [RT06] studied the special type of test configurations which are defined by subschemes of $X$, and introduced the slope of a subscheme.

**Example 3.8.** A pair of an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ and $c \in \mathbb{Q}$ defines a test configuration as follows. We call such test configuration as *deformation to the normal cone* with respect to $(\mathcal{J}, c)$: Let $\mathcal{X}$ be the blow-up of $X \times \mathbb{C}$ along $\mathcal{J}$ and $P$ be the exceptional divisor. The action of $\mathbb{C}^*$ on $X \times \mathbb{C}$ fix $V(\mathcal{J})$ so that it induces actions on $\mathcal{X}$ and $P$. We denote the composition of the blow-down $\mathcal{X} \to X \times \mathbb{C}$ and the projection to $X$ by $p : \mathcal{X} \to X$. Let us define the $\mathbb{Q}$-line bundle $\mathcal{L}_c$ on $X$ by $\mathcal{L}_c := p^*L \otimes \mathcal{O}(-cP)$. When $V = V(\mathcal{J})$ is smooth, $P$ is a compactification of the normal bundle $N_{V/X}$. This is why we call $(\mathcal{X}, \mathcal{L}_c)$ the deformation to the normal cone. Let us denote the blow-up along $\mathcal{J}$ by $\mu : X' \to X$ and the exceptional divisor by $E$. The Seshadri constant of $L$ along $\mathcal{J}$ is defined by

$$\varepsilon(L, \mathcal{J}) := \sup \left\{ c \;\middle|\; \mu^* L \otimes \mathcal{O}(-cE) \text{ is ample} \right\}.$$

Then we have the following lemma so that $(\mathcal{X}, \mathcal{L}_c)$ actually defines a test configuration for any sufficiently small $c$.

**Lemma 3.9** ([RT06], Lemma 4.1). *For any $0 < c < \varepsilon(L, \mathcal{J})$, $\mathcal{L}_c$ is a $\pi$-ample $\mathbb{Q}$-line bundle.*

The slop theory of [RT06] was further developed by [Oda09]. We will use it in the next subsection to compute the associated graded linear series of $\mathcal{T}$. Consider a flag of ideal sheaves $\mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_{N-1} \subseteq \mathcal{O}_X$ and fix $c \in \mathbb{Q}_{>0}$. Let us take the blow up $\mathcal{X}$ of $X \times \mathbb{C}$ along the $\mathbb{C}^*$-invariant ideal sheaf

$$\mathcal{J} := \mathcal{J}_0 + t\mathcal{J}_1 + \cdot + t^{N-1}\mathcal{J}_{N-1} + (t^N)$$

and denote the exceptional divisor by $P$ and the projection map by $p : \mathcal{X} \to X$. Then $\mathcal{X}$ naturally admits a line bundle $\mathcal{L} := p^*L \otimes \mathcal{O}(-cP)$. In his paper [Oda09] Odaka derived intersection number formula of the Donaldson–Futaki invariant for this type of semi test configuration defined by flag ideals. The point is that any test configuration can be dominated by the above type of *semi* test configuration.

**Proposition 3.10** (Proposition 3.10 of [Oda09]). *For an arbitrary normal test configuration $\mathcal{T}$, there exist a flag of ideal sheaves $\mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_{N-1} \subseteq \mathcal{O}_X$ and $c \in \mathbb{Q}_{>0}$ such that $\mathcal{T}' = (\mathcal{X}', \mathcal{L}')$ defined by the flag is a semi test configuration which dominates $\mathcal{T}$ by a morphism $f \colon \mathcal{X}' \to \mathcal{X}$ with $\mathcal{L}' = f^*\mathcal{L}$. Moreover, $F_1(\mathcal{T}') = F_1(\mathcal{T})$ holds.*

3.2. **The associated family of graded linear series.** Let us denote the $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ by $\rho : \mathbb{C}^* \to \mathrm{Aut}(\mathcal{X}, \mathcal{L})$. For any $s \in H^0(X, L^{\otimes k})$, it naturally defines an invariant section $\bar{s} \in H^0(\mathcal{X}_{t\neq 0}, \mathcal{L}^{\otimes k})$ by $\bar{s}(\rho(\tau)x) := \rho(\tau)s(x)$ $(\tau \in \mathbb{C}^*, x \in \mathcal{X}_{t\neq 0})$.

**Lemma 3.11** ([WN10], Lemma 6.1.). *Let $t$ be the parameter of underlying space $\mathbb{C}$. For any $\lambda \in \mathbb{Z}, t^{-\lambda}\bar{s}$ defines a meromorphic section of $\mathcal{L}^{\otimes k}$ over $\mathcal{X}$.*

We then introduce the following filtration to measure the order of these meromorphic sections along the central fiber.

**Definition 3.12.** Fix a test configuration $(\mathcal{X}, \mathcal{L})$. For each $\lambda \in \mathbb{R}$, we define the subspace of $H^0(X, L^{\otimes k})$ by

$$\mathcal{F}_\lambda H^0(X, L^{\otimes k}) := \left\{ s \in H^0(X, L^{\otimes k}) \;\middle|\; t^{-\lceil \lambda \rceil} \bar{s} \in H^0(\mathcal{X}, \mathcal{L}) \right\}. \tag{3.13}$$

By definition $(\rho(\tau)s)(x) := \rho(\tau)s(\rho^{-1}(\tau)(x))$ so it holds $(\rho(\tau)\bar{s})(x) = \rho(\tau)\bar{s}(\rho^{-1}(\tau))(x) = \bar{s}(x)$ *i.e.* $\bar{s}$ is invariant under the $\mathbb{C}^*$-action. On the other hand, regarding $t$ as the section of $\mathcal{O}$ we have $(\rho(\tau)t)(x) = \rho(\tau)t(\rho^{-1}(\tau)x) = \rho(\tau)(\tau^{-1}t(x)) = \tau^{-1}t(x)$. Therefore $t^{-\lceil \lambda \rceil}\bar{s}$ is an eigenvector of weight $\lceil \lambda \rceil$ with respect to the $\mathbb{C}^*$-action. Note that the filtration is multiplicative, *i.e.*

$$\mathcal{F}_\lambda H^0(X, L^{\otimes k}) \cdot \mathcal{F}_{\lambda'} H^0(X, L^{k'}) \subset \mathcal{F}_{\lambda+\lambda'} H^0(X, L^{k+k'})$$

holds for any $\lambda, \lambda' \in \mathbb{R}$ and $k, k' \geqslant 0$. The relation to the weight of the action on the central fiber is given by the following proposition.

**Proposition 3.14.** *Let us denotes the weight decomposition of the* $\mathbb{C}^*$*-action by* $H^0(\mathcal{X}_0, \mathcal{L}_{\mathcal{X}_0}) = \bigoplus_\lambda V_\lambda$. *Then, for any* $\lambda \in \mathbb{R}$ *we have*

$$\dim \mathcal{F}_\lambda H^0(X, L^{\otimes k}) = \sum_{\lambda' \geqslant \lambda} \dim V_{\lambda'}. \tag{3.15}$$

Note that every weight is actually an integer so that each side of (3.15) is unchanged if one replaces $\lambda$ to $\lceil \lambda \rceil$. The fundamental fact established in [PS07] is that this filtration is actually linearly bounded in the following sense.

**Lemma 3.16** ([PS07], Lemma 4). *For any test configuration* $(\mathcal{X}, \mathcal{L})$ *there exists a constant* $C > 0$ *such that for any* $k \geqslant 1$ *and* $\lambda$ *with* $\dim V_\lambda > 0$,

$$|\lambda| \leqslant Ck$$

*holds.*

In other words, there exists a constant $C > 0$ such that

$$\mathcal{F}_{-Ck} H^0(X, L^{\otimes k}) = H^0(X, L^{\otimes k}) \quad \text{and} \quad \mathcal{F}_{Ck} H^0(X, L^{\otimes k}) = \{0\}$$

hold for every $k \geqslant 1$.

**Definition 3.17.** We set

$$\lambda_0 := \sup\left\{\lambda \mid \mathcal{F}_{\lambda k} H^0(X, L^{\otimes k}) = H^0(X, L^{\otimes k}) \text{ for any } k \geqslant 1\right\} \text{ and}$$
$$\lambda_c := \inf\left\{\lambda \mid \mathcal{F}_{\lambda k} H^0(X, L^{\otimes k}) = \{0\} \text{ for any } k \geqslant 1\right\}.$$

By Lemma 3.16, $\lambda_0$ and $\lambda_c$ are both finite. Lemma 3.16 indicate us to consider the graded linear series

$$W_\lambda = \bigoplus_{k=0}^{\infty} W_{\lambda,k} := \bigoplus_{k=0}^{\infty} \mathcal{F}_{\lambda k} H^0(X, L^{\otimes k}) \tag{3.18}$$

For each $\lambda \in \mathbb{R}$. It was shown by [Sz11] that this family has a sufficient information of original test configuration. A result of [WN10] in fact gives the explicit formula for $b_0$.

**Theorem 3.19** (A reformulation of [WN10], Corollary 6.6). *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration. Then the quantity $b_0$ is obtained by the Lebesgue–Stieltjes integral of $\lambda$ with respect to* $\operatorname{vol}(W_\lambda)$. *That is,*

$$n!b_0 = -\int_{-\infty}^{\infty} \lambda d(\operatorname{vol}(W_\lambda)).$$

Theorem 3.19 actually follows from Corollary 6.6 of [WN10] by change of variables in integration. Note that the concave function $G[\mathcal{T}]$ on the Okounkov body $\Delta(L)$ in [WN10] is determined by the property: $G[\mathcal{T}]^{-1}([\lambda, \infty)) = \Delta(W_\lambda)$ where $\Delta(W_\lambda) \subset \mathbb{R}^n$ is the Okounkov body of $W_\lambda$ in the sense of [LM09], Definition 1.15 and that $n!$ times the Euculidian volume $\operatorname{vol}(\Delta(W_\lambda))$ equals to $\operatorname{vol}(W_\lambda)$. Here however we give a self-contained proof of the above theorem. Our proof is rather simple than that of [WN10] which used the method of Okounkov body.

Set the counting function of weights as

$$f(\lambda) = f_k(\lambda) := \sum_{\lambda' \geqslant \lambda} \dim V_{\lambda'} = \dim \mathcal{F}_\lambda H^0(X, L^{\otimes k}). \tag{3.20}$$

It is easy to show that $f_k(\lambda)$ is actually left-continuous and non-increasing function. Hence the Lebesgue–Stieltjes integral makes sense and

$$\begin{aligned} w(k) &:= \sum_\lambda \lambda \dim V_\lambda = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}) \\ &= -\int_{-\infty}^{\infty} \lambda df(\lambda) = -\int_{-\infty}^{\infty} k\lambda df(k\lambda) \end{aligned}$$

hold for any $k$. For any small $\varepsilon > 0$ Integration by part yields

$$-\int_{-\infty}^{\infty} k\lambda df(k\lambda) = -\Big[k\lambda f(k\lambda)\Big]_{\lambda_0-\varepsilon}^{\infty} + \int_{\lambda_0-\varepsilon}^{\infty} kf(k\lambda)d\lambda.$$

By the definition of the volume we have

$$\limsup_{k\to\infty} \frac{f(k\lambda)}{k^n/n!} = \operatorname{vol}(W_\lambda).$$

If $\operatorname{vol}(W_\lambda) > 0$, the limit of supremum is in fact limit for $k$ sufficiently divisible, by Theorem 4 of [KK09] (or by the proof of Theorem 3.10, Corollary 3.11, and Lemma 3.2 of [DBP12]). Therefore the dominate convergence theorem concludes

$$n!b_0 = (\lambda_0 - \varepsilon)L^n + \int_{\lambda_0-\varepsilon}^{\infty} \operatorname{vol}(W_\lambda)d\lambda.$$

Thus we obtain Theorem 3.19.

We remark that one of the advantage to consider such graded linear series is to avoid the difficulty comes from the singularity of the central fiber $\mathcal{X}_0$. On the other hand, we have to treat with the difficulty comes from the non-completeness of linear series in this setting.

**Example 3.21.** Let $(\mathcal{X}, \mathcal{L})$ be the test configuration defined by an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ and $c \in \mathbb{Q}$ as in Example 3.8. Then the associated $W_\lambda$ are computed to be:

$$W_{\lambda,k} = \begin{cases} H^0(X, L^{\otimes k}) & (\lambda \leqslant -c) \\ H^0(X, L^{\otimes k} \otimes \mathcal{J}^{\lceil \lambda k \rceil + ck}) & (-c < \lambda \leqslant 0) \\ \{0\} & (\lambda > 0) \end{cases}$$

for any $k$. As a result, we have

$$n! b_0 = -cL^n + \int_{-c}^{0} \left(\mu^* L \otimes \mathcal{O}(-(\lambda + c)E)\right)^n d\lambda.$$

In fact thanks to Proposition 3.10 one can compute $W_\lambda$ for any test configuration.

**Proposition 3.22.** *For any test configuration $\mathcal{T}$, there exist a flag of ideals $\mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_{N-1} \subseteq \mathcal{O}_X$ and $c \in \mathbb{Q}_{>0}$ such that $W_{\lambda,k}$ can be computed for any $\lambda \in \mathbb{Q}_{>0}$ and $k$ sufficiently divisible, as follows:*

$$W_{\lambda,k} = H^0(X, L^{\otimes k}) \quad \text{if} \quad \lambda \leqslant -Nc,$$

$$W_{\lambda,k} = \{0\} \quad \text{if} \quad \lambda > \frac{N(N-1)}{2} c, \quad \text{and}$$

$$W_{\lambda,k} = H^0(X, L^{\otimes k} \otimes \mathcal{J}_{N-1}^{ck} \mathcal{J}_{N-2}^{ck} \cdots \mathcal{J}_{N-j+1}^{ck} \mathcal{J}_{N-j}^{\frac{\lceil \lambda k \rceil + (N - \frac{j(j-1)}{2})ck}{j}})$$

*if $-(N - \frac{j(j-1)}{2})c < \lambda \leqslant -(N - \frac{j(j+1)}{2})c$ holds for some $1 \leqslant j \leqslant N$.*

*Proof.* Let us take $f \colon \mathcal{X}' \to \mathcal{X}$ as Proposition 3.10. Note that $f$ is isomorphic except on the locus contained in $\mathcal{X}_0$, whose codimension is greater than 2 in $\mathcal{X}$. Then it is easy to see that $W_{\lambda,k}$ for $\mathcal{X}$ is naturally isomorphic to that for $\mathcal{X}'$. Therefore we may assume that $\mathcal{X}$ is defined by a flag $\mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_{N-1} \subseteq \mathcal{O}_X$, without loss of generality. Then we have the decomposition

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes k}) = H^0(X \times \mathbb{C}, p_1^* L^{\otimes k} \otimes (\mathcal{J}_0 + t\mathcal{J}_1 + \cdots + t^{N-1}\mathcal{J}_{N-1} + (t^N))^{ck})$$

$$= \left( \bigoplus_{i_0 + i_1 + \cdots + i_N = ck} t^{i_1 + 2i_2 + \cdots + Ni_N} H^0(X, L^{\otimes k} \otimes \mathcal{J}_0^{i_0} \mathcal{J}_1^{i_1} \cdots \mathcal{J}_{N-1}^{i_{N-1}}) \right) \oplus t^{Nck} \mathbb{C}[t] H^0(X, L^{\otimes k}).$$

To compute $\lceil \lambda k \rceil$-component of $H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k})$, we should solve

$$\begin{cases} i_1 + 2i_2 + \cdots + Ni_N = -\lceil \lambda k \rceil \\ i_0 + i_1 + \cdots + i_N = ck \end{cases}$$

$$\Leftrightarrow Ni_0 + (N-1)i_1 + \cdots + i_{N-1} = \lceil \lambda k \rceil + Nck.$$

The above equation for $(i_0, \ldots, i_{N-1})$ has many solutions but if two solutions satisfy $(i_0, \ldots, i_{N-1}) \prec (j_0, \ldots, j_{N-1})$ in the lexicographic order

$$H^0(X, L^{\otimes k} \otimes \mathcal{J}_0^{i_0} \mathcal{J}_1^{i_1} \cdots \mathcal{J}_{N-1}^{i_{N-1}}) \subseteq H^0(X, L^{\otimes k} \otimes \mathcal{J}_0^{j_0} \mathcal{J}_1^{j_1} \cdots \mathcal{J}_{N-1}^{j_{N-1}})$$

so that $W_{\lambda,k} = H^0(X, L^{\otimes k} \otimes \mathcal{J}_0^{i_0} \mathcal{J}_1^{i_1} \cdots \mathcal{J}_{N-1}^{i_{N-1}})$ holds for the maximal solution $(i_0, \ldots, i_{N-1})$. $\square$

If one take a resolution $\mu\colon X' \to X$ of $\mathcal{J}_0, \dots, \mathcal{J}_{N-1}$ and divisors $E_j$ $(1 \leqslant i \leqslant N-1)$ on $X'$ such that $\mathcal{J}\mathcal{O}_{X'} = \mathcal{O}_{X'}(-E_i)$ and $E_i \geqslant E_j$ $(i \leqslant j)$ hold, it holds that

$$W_{\lambda,k} \simeq H^0(X', \mu^* L^{\otimes k} \otimes \mathcal{O}(-kE_\lambda))$$

where

$$E_\lambda := cE_{N-1} + \cdots + cE_{N-j+1} + \frac{\lambda + (N - \frac{j(j-1)}{2})c}{j} E_{N-j}.$$

The above computation is not so practical but here we obtain the following observation so that we may apply Theorem 2.5 to $W_\lambda$.

**Corollary 3.23.** *If $\lambda < \lambda_c$, the natural map $X \dashrightarrow \mathbb{P}W^*_{\lambda,k}$ is birational onto its image for any $k$ sufficiently divisible.*

*Proof.* Note that $\lambda_c \leqslant \frac{N(N-1)}{2}c$. Let us first see that $\mu^* L \otimes \mathcal{O}(-E_\lambda)$ is big for any $\lambda < \lambda_c$. It is enough to consider the case where there exists a $j$ such that $E_\lambda$ is big for $\lambda < -(N - \frac{j(j+1)}{2})c$ but not so for $\lambda = -(N - \frac{j(j+1)}{2})c$. Since $E_{N-j-1} \geqslant E_{N-j}$ this actually implies $E_\lambda$ is not pseudo-effective for $\lambda > -(N - \frac{j(j+1)}{2})c$. Let us now fix a rational number $\lambda'$ such that $\lambda < \lambda' < \lambda_c$ holds. Then $W_{\lambda,k}$ contains $W_{\lambda',k} \simeq H^0(X', \mu^* L^{\otimes k} \otimes \mathcal{O}(-kE_{\lambda'}))$ for sufficiently divisible $k$. Since $\mu^* L \otimes \mathcal{O}(-E_{\lambda'})$ is big, this concludes the corollary. □

## 4. Study of the weak geodesic ray

In this section we apply Theorem 2.5 to each $W_\lambda$ constructed from the test configuration to study the associated weak geodesic ray.

### 4.1. **Construction of weak geodesic.**

One of the guiding principles to the existence problem of constant scalar curvature Kähler metric is to study the Riemannian geometry on the space of Kähler metrics in the first Chern class of $L$. A result of Phong and Sturm (in [PS07]) gives a milestone in this direction. They showed that a test configuration canonically defines a weak geodesic ray emanating from any fixed point $\varphi$ in the space of Kähler metrics. This builds a bridge between the algebraic definition of K-stability and the analytic stage where the cscK metric lives. Later it was shown by [RWN11] that one can also define the same weak geodesic via the associated family of graded linear series $\{W_\lambda\}$. Let us now recall their construction. Throughout this subsection we fix a smooth strictly psh weight $\varphi$. It will be shown that $\varphi$ and the family of graded linear series $\{W_\lambda\}$ canonically define the weak geodesic emanating from $\varphi$.

Recall that a family of psh weights $\psi_t$ $(a < t < b)$ is called *weak geodesic* if $\Psi(x, \tau) := \psi_{-\log|\tau|}(x)$ $(\tau \in \mathbb{C}, e^{-b} < |\tau| < e^{-a})$ is plurisubharmonic and satisfies the Monge–Ampère equation

$$\mathrm{MA}(\Psi) = 0.$$

Here we consider $\Psi(x, \tau)$ as the function of $(n+1)$-variables and the Monge-Ampère operator is defined in subsection 2.1. When each $dd^c\varphi_t$ is a smooth Kähler metric, there is the canonical Riemannian metric which is defined for a tangent vector $u$ at $\varphi_t$ by

$$\|u\|^2 := \int_X u^2 \frac{\mathrm{MA}(\varphi_t)}{n!}.$$

By [Sem92], it is known that $\mathrm{MA}(\Psi) = 0$ if and only if the geodesic curvature for this metric is zero.

First note that given test configuration $(\mathcal{X}, \mathcal{L})$, the associated family $\{W_\lambda\}$ defines the family of equilibrium weight $P_{W_\lambda}\varphi$. Let us from now write as

$$\psi_\lambda := P_{W_\lambda}\varphi.$$

The first easy observation is that $\psi_\lambda$ is decreasing with respect to $\lambda$. As a consequence of the Bergman approximation argument by Demailly and Lemma 3.16, we have

$$\psi_\lambda = \varphi \ \text{ if } \lambda < \lambda_0 \quad \text{and} \quad \lambda_c = \inf\{\ \lambda \mid \psi_\lambda = -\infty\ \}.$$

Further by the multiplicativity of $\mathcal{F}_\lambda H^0(X, L^{\otimes k})$ one can see that $\psi_\lambda$ is concave with respect to $\lambda$. The main result of [RWN11] states that the Legendre transformation of $\psi_\lambda$ defines a weak geodesic ray.

**Theorem 4.1** ([RWN11], Theorem 1.1 and Theorem 1.2, Theorem 9.2.). *Set the Legendre transformation of $\psi_\lambda$ by*

$$\varphi_t := \sup{}^*\big\{\psi_\lambda + t\lambda \ \big|\ \lambda \in \mathbb{R}\big\} \quad \textit{for } \ t \in [0, +\infty). \tag{4.2}$$

*Then $\varphi_t$ defines a weak geodesic emanating from $\varphi$. Moreover, $\varphi_t - F_0$ coincide with the weak geodesic ray constructed in* [PS07].

It is immediate to show that $\varphi_t$ is a bounded psh weight emanating from $\varphi$ and that it is convex with respect to $t$. The geodecity is derived from the maximality of $P_{W_\lambda}\varphi$, that is,

$$\psi_\lambda = \varphi \ \text{ a.e. with respect to } \mathrm{MA}(\psi_\lambda). \tag{4.3}$$

And one of the technical point in [RWN11] is to show (4.3). Such property is caused by the fact that $\psi_\lambda$ is defined as the upper envelopes of sufficiently many algebraic weights.

Note that the inverse Legendre transform maps $\varphi_t$ to $\psi_\lambda$ by

$$\psi_\lambda = \inf_t \big\{\varphi_t - t\lambda\big\} \tag{4.4}$$

which holds on almost every point on $X$. Therefore the two curves have the equivalent information. Fix $t \in [0, \infty)$. By the convexity of $\varphi_t$ in $t$, the right derivative $\dot{\varphi}_t(x)$ is defined for every $x \in X$. We identify this right derivative with the tangent vector of the weak geodesic. Moreover, the gradient map relation

$$-\psi_\lambda(x) + \varphi_t(x) = t\lambda \tag{4.5}$$

holds almost everywhere if one set $\lambda := \dot{\varphi}_t(x)$.

4.2. **Proof of Theorem 1.1.** Now we prove Theorem 1.1. It was shown in [WN10] that the push-forward of the Lebesgue measure by the concave function $G[\mathcal{T}]$ on the Okounkov body $\Delta(L)$ gives the weak limit. That is,

$$\lim_{k\to\infty}\frac{n!}{k^n}\sum_{\lambda}\delta_{\frac{k}{\lambda}}\dim V_\lambda = n!G[\mathcal{T}]_*(d\lambda|_{\Delta(L)}). \tag{4.6}$$

Recall that $G[\mathcal{T}]$ is characterized by its property: $G[\mathcal{T}]^{-1}([\lambda,\infty)) = \Delta(W_\lambda)$ where $\Delta(W_\lambda) \subseteq \mathbb{R}^n$ is the Okounkov body of $W_\lambda$ in the sense of [LM09], Definition 1.15 and $n!$ times the Euculidian volume $\mathrm{vol}(\Delta(W_\lambda))$ gives $\mathrm{vol}(W_\lambda)$. Therefore it is easy to observe that the right hand side of (4.6) equals to $-d(\mathrm{vol}(W_\lambda))$. Then, by Theorem 2.5 with Corollary 3.23, we may reduce the proof of Theorem 1.1 to show

$$-d\int_X \mathrm{MA}(\psi_\lambda) = (\dot{\varphi}_t)_*\,\mathrm{MA}(\varphi_t). \tag{4.7}$$

Here we used the assumption $\mathcal{X}$ is normal, in order to apply Corollary 3.23. By the main result of [PS10] $\varphi_t$ has the $C^{1,\alpha}$-regularity so that we can apply Proposition 2.2 of [Bern09]. Then it can be seen that the right hand side of (4.7) is independent of $t$ and the proof is reduced to the case $t=0$. Then by basic measure theory we conclude Theorem 1.1 if for any $\lambda \in \mathbb{R}$

$$\int_X \mathrm{MA}(\psi_\lambda) = \int_{\{\dot{\varphi}_0 \geqslant \lambda\}} \mathrm{MA}(\varphi) \tag{4.8}$$

holds. Or it is sufficient to show

$$\int_{\{\dot{\varphi}_0 > \lambda\}} \mathrm{MA}(\varphi) \leqslant \int_X \mathrm{MA}(\psi_\lambda) \leqslant \int_{\{\dot{\varphi}_0 \geqslant \lambda\}} \mathrm{MA}(\varphi) \tag{4.9}$$

for any $\lambda \in \mathbb{R}$. The following lemma is directly deduced from the definition of $\varphi_t$.

**Lemma 4.10.** *For almost every point in $X$, $\dot{\varphi}_0 \geqslant \lambda$ holds if and only if $\psi_\lambda = \varphi$. In particular*

$$\int_{\{\dot{\varphi}_0 \geqslant \lambda\}} \mathrm{MA}(\varphi) = \int_{\{\psi_\lambda = \varphi\}} \mathrm{MA}(\varphi).$$

*holds.*

*Proof.* Let $x$ be a point of $X$. If $\psi_\lambda(x) = \varphi(x)$, then

$$\dot{\varphi}_0(x) := \inf_t \frac{\varphi_t(x) - \varphi(x)}{t} \geqslant \frac{\psi_\lambda(x) + t\lambda - \varphi(x)}{t} \geqslant \lambda.$$

On the other hand, by the Legendre relation (4.4), $\dot{\varphi}_0(x) \geqslant \lambda$ yields

$$\begin{aligned}\psi_\lambda(x) &= \inf_t \{\varphi_t(x) - t\lambda\} \\ &\geqslant \inf_t \{t\dot{\varphi}_0(x) + \varphi(x) - t\lambda\} \geqslant \varphi(x)\end{aligned}$$

for almost every $x \in X$.

□

In the case of Example 3.8, the result of [Berm07] yileds much stronger conclusion that $\psi_\lambda$ has $C^{1,1}$-regularity on the bounded locus and

$$\mathrm{MA}(\psi_\lambda) = 1_{\{\psi_\lambda=\varphi\}}\,\mathrm{MA}(\varphi)$$

holds. Here, however, we give a proof of (4.9) without the regularity of $\psi_\lambda$. Note that the set $\{\dot\varphi_0 > \lambda\}$ is open (thanks to the regularity result of [PS10]) and contained in $\{\psi_\lambda = \varphi\}$. It was shown by [BEGZ10] that the Monge–Ampère product is local in the plurifine topology. Therefore we have

$$\int_{\{\dot\varphi_0>\lambda\}} \mathrm{MA}(\psi_\lambda) = \int_{\{\dot\varphi_0>\lambda\}} \mathrm{MA}(\varphi).$$

Then we obtain the one side inequality of (4.9),

$$\int_X \mathrm{MA}(\psi_\lambda) \geqslant \int_{\{\dot\varphi_0>\lambda\}} \mathrm{MA}(\varphi).$$

Let us take any $\varepsilon > 0$ to prove the converse inequality. Thanks to the maximality (4.3) we have

$$\int_X \mathrm{MA}(\psi_\lambda) = \int_{\{\psi_\lambda>\varphi-\varepsilon\}} \mathrm{MA}(\psi_\lambda) = \int_{\{\psi_\lambda>\varphi-\varepsilon\}} \mathrm{MA}(\max\{\psi_\lambda, \varphi-\varepsilon\}).$$

Note that the set $\{\psi_\lambda > \varphi - \varepsilon\}$ is pluri-open. The right hand side equals to

$$L^n - \int_{\{\psi_\lambda\leqslant\varphi-\varepsilon\}} \mathrm{MA}(\max\{\psi_\lambda, \varphi-\varepsilon\})$$

by Theorem 2.3. Therefore we obtain

$$\begin{aligned}\int_X \mathrm{MA}(\psi_\lambda) &\leqslant L^n - \int_{\{\psi_\lambda<\varphi-\varepsilon\}} \mathrm{MA}(\max\{\psi_\lambda, \varphi-\varepsilon\}) \\ &= L^n - \int_{\{\psi_\lambda<\varphi-\varepsilon\}} \mathrm{MA}(\varphi).\end{aligned}$$

If $\varepsilon > 0$ tends to $0$ then the set $\{\psi_\lambda < \varphi - \varepsilon\}$ converges to $\{\dot\varphi_0 < \lambda\}$ hence

$$\int_X \mathrm{MA}(\psi_\lambda) \leqslant \int_{\{\dot\varphi_0\geqslant\lambda\}} \mathrm{MA}(\varphi).$$

This ends the proof.

*Remark* 4.11. Note that our Duistermatt–Heckmann measure for singular settings has continuous density in $\lambda < \lambda_c$. That is, there is a continuous function $f(\lambda)$ such that $-d\,\mathrm{vol}(W_\lambda) = f(\lambda)d\lambda$ holds for $\lambda < \lambda_c$. In fact Proposition 3.22 yields $\mathrm{vol}(W_\lambda) = \mathrm{vol}(\mu^*L \otimes \mathcal{O}(-E_\lambda))$ and by [LM09] or by [BFJ09], the left hand side has continuous derivative which is expressed by certain restricted volumes.

4.3. **Norms on the weak geodesic ray.** We conclude this paper by discussing some consequences of Theorem 1.1, which are concerned with the $p$-norm of test configuration.

**Definition 4.12.** Fix any test configuration $(\mathcal{X}, \mathcal{L})$ of a polarized manifold $L$. Let $H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}) = \bigoplus_\lambda V_\lambda$ be the weight decomposition of the induced $\mathbb{C}^*$-action. Define the trace-free part of each eigenvalue as

$$\bar{\lambda} := \lambda - \frac{1}{N_k} \sum_\lambda \lambda \dim V_\lambda$$

and introduce the $p$-norms ($p \in \mathbb{Z}_{\geqslant 0}$) of the test configuration by

$$Q_p := \lim_{k \to \infty} \frac{1}{k^n} \sum_\lambda \left(\frac{\lambda}{k}\right)^p \dim V_\lambda$$

and

$$N_p := \lim_{k \to \infty} \frac{1}{k^n} \sum_\lambda \left(\frac{\bar{\lambda}}{k}\right)^p \dim V_\lambda.$$

Especially in the case $p = 2$ we denote $Q_2$ and $N_2$ by $Q$ and $\|\mathcal{T}\|^2 = \|\mathcal{T}\|_2^2$. Note that the limits exist since the summations in the right-hand side can be thought as the appropriate Hilbert polynomial.

It is easy to see that $Q_1 = b_0$, $N_1 = 0$, $N_2 = Q_2 - \frac{{b_0}^2}{a_0}$, and

$$\frac{1}{N_k} \sum_\lambda \frac{\lambda}{k} \to F_0 = \frac{b_0}{a_0}.$$

These norms are introduced by [Don05] and played the important role in their result for the lower bound of the Calabi functional. We can obtain the geometric meanings of these norms in word of weak geodesic ray.

**Theorem 4.13.** *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration and $\varphi_t$ be the weak geodesic associated to $(\mathcal{X}, \mathcal{L})$. Then we have*

$$Q_p = \int_X (\dot{\varphi}_t)^p \frac{\mathrm{MA}(\varphi_t)}{n!}$$

*and*

$$N_p = \int_X \left(\dot{\varphi}_t - F_0\right)^p \frac{\mathrm{MA}(\varphi_t)}{n!}.$$

*Proof.* By the same argument in the proof of Theorem 3.19, we obtain

$$n! Q_p = - \int_{-\infty}^{\infty} \lambda^p d \operatorname{vol}(W_\lambda).$$

This can be also obtained from the result of [WN10] if one note the volume characterization of the concave function $G[\mathcal{T}]$ in [WN10]. Taking the $p$-th moment of the two measures in Theorem 1.1, we deduce the claim. The formulas for $N_p$ can be proved in the same way. □

Let us examine Theorem 4.13. in the case $p = 0$ it only states that $n!a_0 = \int_X \mathrm{MA}(\varphi_t)$ and this can be easily seen from the definition of the Bedford–Taylor's Monge–Ampère product. The case $p = 1$ yields

$$n!b_0 = \int_X \dot{\varphi}_t \,\mathrm{MA}(\varphi_t).$$

In other words, the Aubin–Mabuchi energy functional along the weak geodesic is given by

$$\mathcal{E}(\varphi_t, \varphi) := \int_0^1 dt \int_X \dot{\varphi}_t \,\mathrm{MA}(\varphi_t) = n!b_0 t.$$

(For the definition of the Aubin–Mabuchi energy of a singular Hermitian metric, see [BEGZ10].) This is a well-known result to the experts. For example, the proof of [Berm12] in the Fano case works exactly the same way to yield that along the weak geodesic $b_0$ gives the gradient of the Aubin–Mabuchi energy. We have reproved it in the viewpoint of the associated family of graded linear series. It is conjectured that the gradient of the K-energy at infinity corresponds to the (minus of) Donaldson–Futaki invariant. This gives the variational approach to the existence problem.

The most interesting case is $p = 2$ which yields a part of Theorem 1.2 and this might be a new result. In particular, we obtain the following.

**Corollary 4.14.** *For any test configuration, the norm $\|\mathcal{T}\|$ is zero if and only if the associated weak geodesic ray $\varphi_t$ is $\varphi + F_0 t$.*

Only the case where the exponent $p$ is even was treated in [Don05] to assure the positivity of the norm but now we may define the positive norm for odd $p$ integrating the function $|\lambda|^p$, in place of $\lambda^p$, by each measure. In particular we can see that the limit

$$\|\mathcal{T}\|_p^p := \lim_{k\to\infty} \frac{1}{k^n} \sum_{\lambda} \left(\frac{|\bar{\lambda}|}{k}\right)^p \dim V_\lambda,$$

which can not necessarily be described by a Hilbert polynomial, exists and coincide with the $L^p$ norm of the tangent vector. Thus Theorem 1.2 was proved. Letting $p \to +\infty$, we obtain

$$\|\mathcal{T}\|_\infty := \lim_{p\to\infty} \|\mathcal{T}\|_p = \sup_X |\dot{\varphi}_t - F_0|\,. \tag{4.15}$$

In particular the right hand side is independent of $t$ and $\varphi$.

Let us remark some relation with [Don05] and prove Theorem 1.3. Let us denote the scalar curvature of the Kähler metric $dd^c\varphi$ by $S_\varphi$ and denote its mean value by $\hat{S}$. The main result of [Don05] states that

$$(Q_p)^{\frac{1}{p}} \|S_\varphi\|_{L^q} \geqslant b_1$$

and

$$\|\mathcal{T}\|_p \cdot \left\|S_\varphi - \hat{S}\right\|_{L^q} \geqslant F_1 \tag{4.16}$$

hold for any even $p$ and the conjugate $q$ which satisfies $1/p + 1/q = 1$. As a result one can see that the existence of constant scalar curvature Kähler metric implies K-semistability.In view of (4.16), [Sz11] suggested the stronger notion of K-stability which implies

$$F_1 \leqslant -\delta \|\mathcal{T}\| \tag{4.17}$$

for some uniform constant $\delta > 0$. One of the motivation of this condition is that one has to consider some limit of test configurations to assure the existence of constant scalar curvature Kähler metric. The above condition also excludes the pathological example raised in [LX11]. Corollary 4.14 supports the validity of [Sz11]'s suggestion since the gradient of the K-energy along the trivial ray $\varphi + F_0 t$ is zero.

Let us give an energy theoretic explanation for (4.16). Thanks to Theorem 1.2, we can apply the Hölder inequality to obtain

(4.18)

$$\left(\int_X |\dot{\varphi}_0 - F_0|^p \frac{\mathrm{MA}(\varphi)}{n!}\right)^{\frac{1}{p}} \left(\int_X \left|S_\varphi - \hat{S}\right|^q \frac{\mathrm{MA}(\varphi)}{n!}\right)^{\frac{1}{q}} \geqslant \int_X (\dot{\varphi}_0 - F_0)(S_\varphi - \hat{S}) \frac{\mathrm{MA}(\varphi)}{n!}$$

for any pair $(p, q)$ with $1/p + 1/q = 1$. Then the right hand side is minus of the gradient of K-energy along the weak geodesic ray. The definition of the gradient for singular $\varphi_t$ is not so clear but if it was well-defined, it should be increasing with respect to $t$. Moreover the limit should be smaller just as much as the multiplicity of the central fiber than minus of the Donaldson–Futaki invariant. (See also [PT06], [PT09] and [PRS08].) Assuming these points we have

$$\int_X (\dot{\varphi}_0 - F_0)(S_\varphi - \hat{S}) \frac{\mathrm{MA}(\varphi)}{n!} \geqslant F_1. \tag{4.19}$$

Notice that (4.19) implies (4.16) for *any* $1 \leqslant p \leqslant +\infty$. One of the proof of (4.19) following the above line will be given in our preparing note in collaboration with Robert Berman and David Witt Nyström. In fact in the Fano case, we may replace the K-energy to the Ding functional to obtain the corresponding result. Convexity of the Ding functional along any weak geodesic ray was established in [Bern11] and the relation between the gradient of the Ding functional and $F_1$ was shown in [Berm12]. As a corollary of these results we obtain

$$\int_X (\dot{\varphi}_0 - F_0)\big(e^{-\varphi} - \frac{\mathrm{MA}(\varphi)}{n!}\big) \geqslant F_1 \tag{4.20}$$

with some appropriate normalization for $\varphi$. and then (4.18) yields

$$\|\mathcal{T}\|_p \left\| \frac{n! e^{-\varphi}}{\mathrm{MA}(\varphi)} - 1 \right\|_{L^q} \geqslant F_1 \tag{4.21}$$

for any $1 \leqslant p \leqslant +\infty$. This can be seen as the analogue of the Donaldson's result in the Fano case.

Finally we remark that the strong K-stability condition (4.17) follows from the analytic condition:

$$\int_X (\dot{\varphi}_0 - F_0)(S_{\varphi_t} - \hat{S}) \frac{\mathrm{MA}(\varphi_t)}{n!} \leqslant -\delta \|\dot{\varphi}_0 - F_0\|, \tag{4.22}$$

in case $S_{\varphi_t}$ is well-defined. It is interesting to ask whether this condition implies the properness of the K-energy.

**Acknowledgments.**The author would like to express his gratitude to his advisor Professor Shigeharu Takayama for his warm encouragements, suggestions and reading the drafts. The author also would like to thank Professor Sébastien Boucksom for his indicating the relation between the authors preprint [His12] and the paper of Julius Ross and David Witt-Nyström. The author wishes to thank Doctor David Witt-Nyström, Professor Robert Berman and Professor Bo Berndtsson for stimulating discussion and helpful comments during the author's stay in Gothenburg. In particular, the formulation of Theorem 1.1 and Theorem 1.2 are due to the suggestions of Doctor David Witt-Nyström. The proof of Theorem 1.1 is also indebted to his many helpful comments. Our work was carried out at several institutions including the Charmers University of Technology, the Gothenburg university, and the University of Tokyo. We gratefully acknowledge their support. The author is supported by JSPS Research Fellowships for Young Scientists (22-6742).

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# Part 3

Remarks on $L^2$-jet extension and extension of singular Hermitian metric with semipositive curvature

# REMARKS ON $L^2$-JET EXTENSION AND EXTENSION OF SINGULAR HERMITIAN METRIC WITH SEMIPOSITIVE CURVATURE

TOMOYUKI HISAMOTO

ABSTRACT. We give a new variant of $L^2$-extension theorem for the jets of holomorphic sections and discuss the relation between the extension problem of singular Hermitian metrics with semipositive curvature.

## 1. INTRODUCTION

In this paper we first give the following new variant of $L^2$-extension theorem for the jets of holomorphic sections.

**Theorem 1.1.** *Let $X$ be a smooth projective variety with a fixed Kähler form $\omega$, and $S \subseteq X$ a smooth closed subvariety. Then there exist constants $N = N(S, X, \omega)$ such that the following holds; Let $L \to X$ be a holomorphic line bundle with a* smooth *Hermitian metric $h$ whose curvature current $\Theta_h$ satisfies*

$$\Theta_h \geqslant N\omega.$$

*Then for any $m \geqslant 1$ and any section $f \in H^0(S, L^{\otimes m})$ with*

$$\int_S |f|_{h^m}\, dV_{\omega,S} < +\infty,$$

*there exists a section $F \in H^0(X, L^{\otimes m})$ such that $F|_S = f$ and the every other term in $(m-1)$-jet along $S$ vanishes. That is, $J^{m-1}F|_S = f$ holds if one denotes the $(m-1)$-jet of $F$ along $S$ by $J^{m-1}F|_S$. Moreover, there exists a constant $C = C(S, X, h) > 0$ such that the $L^2$-estimate*

$$\int_X |F|^2_{h^m}\, dV_{\omega,X} \leqslant C^{m^2} \int_S |f|^2_{h^m}\, dV_{\omega,S}$$

*holds.*

Theorem 1.1 originates from [OT87]. Compact manifold case was first treated by [Man93] and Ohsawa considered arbitrary closed submanifolds in [Ohs01]. For the jet-extension, there is [Pop05] which generalizes Manivel's result assuming that the subvariety $S$ is defined as a locus of some holomorphic section of a vector bundle. Theorem 1.1 not only generalizes [Pop05] to general submanifolds but also specifies how the coefficient in the $L^2$-estimate varies when one twists the line bundle. This is our new viewpoint. On the other hand, in the present paper we only treat with a very special type of jet (every term of $J^{m-1}F$ except the zeroth order term vanishes).

2000 *Mathematics Subject Classification.* Primary 32U05, Secondary 32C25, 32L10, 32A25.
*Key words and phrases.* singular Hermitian metric, extension theorem, Bergman kernel.

The formulation of Theorem 1.1 is strongly motivated by the extension problem for singular Hermitian metrics with semipositive curvature. Let $L$ be a holomorphic line bundle over a smooth complex projective variety $X$. We fix a closed subvariety $S$ of $X$ and a singular Hermitian metric $h_S$ on $L|_S$. We will suggest a new approach to generalize the following result due to Coman, Guedj, and Zeriahi.

**Theorem 1.2** (smooth subvariety case of Theorem B in [CGZ10]). *Let $X$ be a smooth complex projective variety, $L$ an ample line bundle over $X$, and $S \subset X$ a smooth closed subvariety. Then for any singular Hermitian metric $h_S$ with semipositive curvature on $L|_S$ there exists a singular Hermitian metric $h$ on $L$ over $X$ such that $h|_S = h_S$ holds.*

Note that [CGZ10] treated arbitrary singular subvariety. A finite family of global sections $\{F_i\}_i \subset H^0(X, L)$ naturally defines a singular Hermitian metric $1/\sum_i |F_i|^2$ on $L$ so that Proposition 1.4 can be seen as an analytic generalization of the Serre vanishing theorem for ample line bundles. In the local setting there were previous works by [Ric68], [Sad82] and [Col91]. Following the approach of [Col91], [CGZ10] obtained the above result as a consequence of the growth-control extension of plurisubharmonic functions from a closed subvariety in the complex Euclidian space. Their proof is hence rather pluripotential-theoritic. In this paper we study a new approach via $L^2$-extension theorem for holomorphic sections of the line bundle, to give a direct relation between the extendability of sections and that of metrics. Moreover, it enables us to expect a consistent proof in the general big line bundle case. We suggest the following problem.

**Problem 1.3.** *Let $X$ be a smooth complex projective variety, $L$ a big line bundle over $X$, and $S \subset X$ a smooth closed subvariety. Let us take a singular Hermitian metric $h_S$ with semipositive curvature on $L|_S$, and a singular Hermitian metric $h_0$ with strictly positive curvature on $L$, so that $h_S \geqslant h_0|_S$ holds. Then does there exist a singular Hermitian metric $h$ on $L$ over $X$ such that $h|_S = h_S$ holds?.*

When $h_0$ is smooth the above problem is obviously reduced to Theorem 1.2. For the non-ample line bundle, is not always possible to extend arbitrary singular metrics from $S$. We will discuss this point in subsection 4.3. We show that Problem 1.3 comes down to the further refinement of Theorem 1.1.

**Proposition 1.4.** *If one can replace the $L^2$-coefficient $C^{m^2}$ in Theorem 1.1 to $C^m$ $(m \geqslant 1)$, then we obtain the $L^2$-theoritic proof of Theorem 1.2. If further $h$ in Theorem 1.1 can be taken singular, the same line solves Problem 1.3 affirmatively.*

We remark that in the proof of Theorem 1.1 the constant $C$ in fact depends on the modulus of continuity of $h$ so that the smoothness assumption of $h$ can not be removed so far.

Let us briefly explain the proof of Proposition 1.4. First of all, one can approximate the metric $h_S$ on $S$ by a sequence of singular metric $h_{S,m}$ defined by holomorphic sections of $L^{\otimes m} (m \geqslant 1)$. Then we can apply $L^2$-extension theorem to extend each section to $X$ so that they produce a sequence of algebraic singular metric $h_m$ on $L$ over $X$, which approximates $h_S$ on $S$. Further, one can get a convergent subsequence thanks to the $L^2$-estimate. The limit $h$ is naively to be desired extension of $h_S$. However, we unfortunately have $h|_S \neq h_S$ in general. To make $h_S$ dominate $h_{S,m}$ around $S$, one have to control the jet of the extended holomorphic sections and here we need the formulation of Theorem 1.1.

## 2. Jet extension

Let us first fix notations. Let $E$ be a holomorphic Hermitian line bundle on a smooth projective variety $X$. If a Kähler metric $\omega$ is fixed, the Chern curvature $\Theta(E)$ defines a Hermitian form on $(\bigwedge^{p.q} T^*_{X,x}) \otimes E_x$ as follows:

$$\theta(\alpha,\beta) := \big([\Theta(E),\Lambda]\alpha|\beta\big) \quad \text{for} \quad \alpha,\beta \in \big(\bigwedge\nolimits^{p,q} T^*_{X,x}\big) \otimes E_x \qquad (x \in X),$$

where $\Lambda$ denotes the formal adjoint operator of the multiplication by $\omega$. It is known that if $p = n$ and $\Theta(E)$ is semipositive, $\theta$ defines a semipositive Hermitian form. We will use the following norm:

$$|\alpha|^2_\theta = \inf\left\{ M \geqslant 0 \;\middle|\; \begin{array}{c} |(\alpha|\beta)|^2 \leqslant M \cdot \theta(\beta,\beta) \\ \text{for any } \;\beta \in (\bigwedge^{n,q} T^*_{X,x}) \otimes E_x \end{array} \right\} \in [0,+\infty]$$

for $\alpha \in (\bigwedge^{n,q} T^*_{X,x}) \otimes E_x$.

We will obtain Theorem 1.1 as a corollary of the following result.

**Theorem 2.1.** *Let $S$ be a $p$-codimensional closed submanifold of a $n$-dimensional projective manifold $X$ with Kähler form $\omega$. Then there exists a constant $N = N(S,X,\omega) > 0$ such that the following holds.*

*Fix a positive integer $m \geqslant 1$ , $0 \leqslant j \leqslant m-1$ and a holomorphic line bundle $L \to X$ with a smooth Hermitian metric $h$ whose Chern curvature satisfies*

$$\Theta_h \geqslant N\omega \qquad \text{on } X,$$

*and $f \in H^0(S, L^{\otimes m})$. Then there exist a constant $C = C(S,X,h)$ and a section $F_j \in H^0(X, L^{\otimes m})$ such that $J^j F_j = f$ and*

$$\int_X |F_j|^2_{h^m}\, dV_{\omega,X} \leqslant C^{(j+1)^2} \int_S |f|^2_{h^m}\, dV_{\omega,S}$$

*hold.*

*Proof.* The proof is based on an induction on $j$. When $j = 0$, Theorem 2.1 is the known result (see [Ohs01]). We assume that $F_{j-1}$ was obtained and will construct $F_j$ from $F_{j-1}$ by solving $\bar\partial$-equations. The constants $N$ and $C$ will be specified in the induction procedure below.

Let us first fix a finite system of local coordinates

$$\{s_{\alpha,1},\ldots,s_{\alpha,n-p},z_{\alpha,1},\ldots,z_{\alpha,p}\}_\alpha$$

so that

$$S \cap U_\alpha = \{z_{\alpha,1} = \cdots = z_{\alpha,p} = 0\}$$

hold. There exists some $\tilde f \in C^\infty(X, L^{\otimes m})$ such that

$$\tilde f(s,z) = f(s) + O(|z|^m).$$

This is easily seen gluing local $C^\infty$-extension by a partition of unity. Fix a smooth cut-off function $\rho : \mathbb{R} \to [0,1]$ satisfying

$$\rho(t) := \begin{cases} 1 & (t \leqslant \frac{1}{2}) \\ 0 & (t \geqslant 1) \end{cases} \qquad |\rho'| \leqslant 3.$$

Then we set as follows:

$$G_\varepsilon^{(j-1)} := \rho\left(\frac{e^\psi}{\varepsilon}\right) \cdot (\tilde{f} - F_{j-1})$$

$$g_\varepsilon := \bar\partial G_\varepsilon^{(j-1)} = \underbrace{\left(1 + \frac{e^\psi}{\varepsilon}\right)\rho'\left(\frac{e^\psi}{\varepsilon}\right)\bar\partial\psi_\varepsilon \wedge (\tilde{f} - F_{j-1})}_{g_\varepsilon^{(1)}} + \underbrace{\rho\left(\frac{e^\psi}{\varepsilon}\right)\bar\partial(\tilde{f} - F_{j-1})}_{g_\varepsilon^{(2)}},$$

where

$$\psi_\varepsilon := \log(\varepsilon + e^\psi) \qquad \left(\Leftrightarrow 1 + \frac{e^\psi}{\varepsilon} = \frac{e^{\psi_\varepsilon}}{\varepsilon}\right)$$

$$\psi := \log \sum_\alpha \chi_\alpha^2 \sum_{i=1}^p |z_{\alpha,i}|^2, \qquad \varepsilon > 0.$$

Here we choose a smooth function $\chi_\alpha$ so that the following hold.

$$\operatorname{Supp}\chi_\alpha \subseteq U_\alpha, \quad \sum_\alpha \chi_\alpha^2 > 0, \quad \text{and} \quad \sum_\alpha \chi_\alpha^2 \sum_{i=1}^p |z_{\alpha,i}|^2 < e^{-1} \quad \text{in } X.$$

This $\psi$ satisfies the following condition (see [Dem82], Proposition 1.4).

(1) $\psi \in C^\infty(X \setminus S) \cap L^1_{\mathrm{loc}}(X)$
$\psi < -1$ in $X$, $\psi \to -\infty$ around $S$.
(2) $e^{-p\psi}$ is *not* integrable around any point of $S$
(3) There exists a smooth real $(1,1)$-form $\gamma$ in $X$ such that
$\sqrt{-1}\partial\bar\partial\psi \geqslant \gamma$ holds in $X \setminus S$.

If the equation

$$\begin{cases} \bar\partial u_\varepsilon = \bar\partial G_\varepsilon^{(j-1)} & \text{in } \Omega \\ |u_\varepsilon|^2 e^{-(j+p)\psi} & \text{is locally integrable around } S \end{cases}$$

has been solved, $u_\varepsilon = O(|z|^{(j+1)})$ along $S$ holds by the above condition hence the sequence $\{G_\varepsilon^{(j-1)} - u_\varepsilon + F_{j-1}\}_\varepsilon$ is expected to converge to desired $F_j$. This is our strategy.

To solve $\bar\partial$-equations, we quote the following from [Dem00].

**Theorem 2.2** (Ohsawa's modified $L^2$-estimate. [Dem00], Proposition 3.1). *Let $X$ be a complete Kähler manifold with a Kähler metric $\omega$ ($\omega$ may not be necessarily complete),*

*$E$ a holomorphic Hermitian line bundle on $X$. Assume that there exist some smooth functions $a, b > 0$ and if we set*

$$\Theta'(E) := a \cdot \Theta(E) - \sqrt{-1}\partial\bar{\partial}a - \sqrt{-1}b^{-1}\partial a \wedge \bar{\partial}a$$
$$\theta'(\alpha,\beta) := \big([\Theta'(E),\Lambda]\alpha|\beta\big) \quad \textit{for} \quad \alpha,\beta \in (\textstyle\bigwedge^{n,q} T^*_{X,x}) \otimes E_x \qquad (x \in X),$$

*it holds that*

$$\theta' \geqslant 0 \quad \textit{on} \quad (\textstyle\bigwedge^{n,q} T_{X,x}) \otimes E_x \qquad \textit{for any } x \in X.$$

*Then we have the following.*

*For any $g \in L^2(X, (\bigwedge^{n,q} T^*_X) \otimes E)$ with $\bar{\partial}g = 0$ and*

$$\int_X |g|^2_{\theta'}\, dV_{\omega,X} < +\infty,$$

*there exists a section $u \in L^2(X, (\bigwedge^{n,q-1} T^*_X) \otimes E)$ with $\bar{\partial}u = g$ such that*

$$\int_X (a+b)^{-1}\, |u|^2\, dV_{\omega,X} \leqslant 2 \int_X |g|^2_{\theta'}\, dV_{\omega,X}.$$

We will apply Theorem 2.2 to $E := K_X^{-1} \otimes L^{\otimes m}$ and $q = 1$. Let us go back to the proof of Theorem 2.1. First, we are going to compute

$$(2.3)\quad \theta'_\varepsilon := \big[a_\varepsilon(\Theta(K_X^{-1} \otimes L^{\otimes m}) + (j+p)\sqrt{-1}\partial\bar{\partial}\psi) - \sqrt{-1}\partial\bar{\partial}a_\varepsilon - b_\varepsilon^{-1}\sqrt{-1}\partial a_\varepsilon \wedge \bar{\partial}a_\varepsilon, \Lambda\big].$$

($a_\varepsilon, b_\varepsilon$ will be defined in the following.) If we set

$$a_\varepsilon := \chi_\varepsilon(\psi_\varepsilon) > 0$$

for some smooth function $\chi_\varepsilon$, it can be computed as:

$$\begin{aligned} \partial a_\varepsilon &= \chi'_\varepsilon(\psi_\varepsilon)\partial\psi_\varepsilon, \\ \sqrt{-1}\partial\bar{\partial}a_\varepsilon &= \chi'_\varepsilon(\psi_\varepsilon)\sqrt{-1}\partial\bar{\partial}\psi_\varepsilon + \chi''_\varepsilon(\psi_\varepsilon)\sqrt{-1}\partial\psi_\varepsilon \wedge \bar{\partial}\psi_\varepsilon \\ &= \chi'_\varepsilon(\psi_\varepsilon)\sqrt{-1}\partial\bar{\partial}\psi_\varepsilon + \frac{\chi''_\varepsilon(\psi_\varepsilon)}{\chi'_\varepsilon(\psi_\varepsilon)^2}\sqrt{-1}\partial a_\varepsilon \wedge \bar{\partial}a_\varepsilon \end{aligned}$$

so comparing with (2.3), it is natural to set

$$b_\varepsilon := -\frac{\chi'_\varepsilon(\psi_\varepsilon)^2}{\chi''_\varepsilon(\psi_\varepsilon)} \quad (> 0).$$

And we finally define

$$\chi_\varepsilon(t) := \varepsilon - t + \log(1-t).$$

Then for sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned} &a_\varepsilon \;\geqslant\; \varepsilon - \log(\varepsilon + e^{-1}) \;\geqslant\; 1 \\ &\sqrt{-1}\partial\bar{\partial}a_\varepsilon + b_\varepsilon^{-1}\sqrt{-1}\partial a_\varepsilon \wedge \bar{\partial}a_\varepsilon = \chi'_\varepsilon(\psi_\varepsilon)\sqrt{-1}\partial\bar{\partial}\psi_\varepsilon \;\leqslant\; -\sqrt{-1}\partial\bar{\partial}\psi_\varepsilon \end{aligned}$$

hence

$$\theta'_\varepsilon \;\geqslant\; \big[\Theta(K_X^{-1} \otimes L^{\otimes m}) + (j+p)\sqrt{-1}\partial\bar{\partial}\psi + \sqrt{-1}\partial\bar{\partial}\psi_\varepsilon, \Lambda\big].$$

On the other hand, simple computations show:

$$\begin{aligned}
\partial\psi_\varepsilon &= \frac{e^\psi}{\varepsilon+e^\psi}\partial\psi,\\
\sqrt{-1}\partial\bar\partial\psi_\varepsilon &= \frac{e^\psi}{\varepsilon+e^\psi}\sqrt{-1}\partial\bar\partial\psi + \frac{e^\psi}{\varepsilon+e^\psi}\sqrt{-1}\partial\psi\wedge\bar\partial\psi - \frac{e^{2\psi}}{(\varepsilon+e^\psi)^2}\sqrt{-1}\partial\psi\wedge\bar\partial\psi\\
&= \frac{e^\psi}{\varepsilon+e^\psi}\sqrt{-1}\partial\bar\partial\psi + \frac{\varepsilon}{e^\psi}\sqrt{-1}\partial\psi_\varepsilon\wedge\bar\partial\psi_\varepsilon.
\end{aligned}$$

Therefore, by the compactness of $X$, there exists a constant $N(S,X,\omega)>0$ such that

$$\text{(2.4)}\qquad \Theta_h \geqslant N\omega \qquad \text{on } X$$

implies

$$\text{(2.5)}\qquad \theta'_\varepsilon \geqslant 0 \ \text{ on } \ (\textstyle\bigwedge^{n,1}T^*_{X,x})\otimes(K_X^{-1}\otimes L^{\otimes m})_x \qquad \text{for all } x\in X$$

and eigenvalues of $\theta'_\varepsilon$ are bounded from below by a positive constant (uniformly with respect to $\varepsilon$) near $S$.

Next we will estimate $\bar\partial\psi_\varepsilon$ by $|\cdot|_{\theta'_\varepsilon}$. Fix arbitrary $\alpha,\beta\in(\bigwedge^{n,1}T^*_{X,x})\otimes(K_X^{-1}\otimes L^{\otimes m})_x$. By definition,

$$\left|\bar\partial\psi_\varepsilon\wedge\alpha\right|^2_{\theta'_\varepsilon} = \inf\left\{ M\geqslant 0 \;\middle|\; \begin{array}{c} \left|(\bar\partial\psi_\varepsilon\wedge\alpha|\beta)\right|^2 \leqslant M\cdot\left([c'_\varepsilon(E)\Lambda]\beta|\beta\right) \\ \text{for any } \ \beta\in(\bigwedge^{n,1}T^*_{X,x})\otimes E_x \end{array}\right\}$$

so it is enough to estimate $\left|(\bar\partial\psi_\varepsilon\wedge\alpha|\beta)\right|^2$. This can be done as follows:

$$\begin{aligned}
&\left|(\bar\partial\psi_\varepsilon\wedge\alpha|\beta)\right|^2 = \left|(\alpha|(\bar\partial\psi_\epsilon)^\sharp\beta)\right|^2\\
&\leqslant |\alpha|^2\cdot\left|(\bar\partial\psi_\epsilon)^\sharp\beta\right|^2 = |\alpha|^2\left((\bar\partial\psi_\varepsilon)(\bar\partial\psi_\varepsilon)^\sharp\beta|\beta\right) = |\alpha|^2\left([\sqrt{-1}\partial\psi_\epsilon\wedge\bar\partial\psi_\epsilon,\Lambda]\beta|\beta\right)
\end{aligned}$$

by Shwartz' inequality ($\sharp$ denotes taking the formal adjoint of the multiplication operator), and the last term is bounded by

$$\begin{aligned}
&\frac{e^\psi}{\varepsilon}|\alpha|^2\left([\sqrt{-1}\partial\bar\partial\psi_\varepsilon - \frac{e^\psi}{\varepsilon+e^\psi}\sqrt{-1}\partial\bar\partial\psi,\Lambda]\beta|\beta\right)\\
&\leqslant \frac{e^\psi}{\varepsilon}|\alpha|^2\left([\Theta(K_X^{-1}\otimes L^{\otimes m}) + (j+p)\sqrt{-1}\partial\bar\partial\psi + \sqrt{-1}\partial\bar\partial\psi_\varepsilon,\Lambda]\beta|\beta\right)\\
&\leqslant \frac{e^\psi}{\varepsilon}|\alpha|^2\left([\Theta'_\varepsilon(K_X^{-1}\otimes L^{\otimes m}),\Lambda]\beta|\beta\right).
\end{aligned}$$

Thus we may get a desired estimate

$$\text{(2.6)}\qquad \left|\bar\partial\psi_\varepsilon\wedge\alpha\right|^2_{\theta'_\varepsilon} \leqslant \frac{e^\psi}{\varepsilon}|\alpha|^2.$$

This time we estimate $g_\varepsilon = g_\varepsilon^{(1)} + g_\varepsilon^{(2)}$ . By (2.6) and $\operatorname{Supp} g_\varepsilon^{(1)} \subseteq \{e^\psi<\varepsilon\}$, $g_\varepsilon^{(1)}$ can be estimated. Namely,

$$\int_{X\setminus S}\left|g_\varepsilon^{(1)}\right|^2_{\theta'_\varepsilon} e^{-(j+p)\psi}dV_{\omega,X} \leqslant 4\int_{X\setminus S}\left|\tilde f - F_{j-1}\right|^2_{h^m}\rho'\left(\frac{e^\psi}{\varepsilon}\right)^2 e^{-(j+p)\psi}dV_{\omega,X}$$

holds. Notice $e^{\psi} \sim \sum_{i=1}^{p} |z_{\alpha,i}|^2$ on $U_\alpha$ and $\rho'(t) = 0$ for $t < 1/2$. By the definition of $F_{j-1}$ we may assume that $U_\alpha$ admits a holomorphic local coordinate $z_1, \dots z_p$ in the normal direction along $S$ such that the Taylor expansion of $F_{j-1}$ along $S$ can be written as

$$F_{j-1}(x) = F_{j-1}(s,z) = f(s) + \sum_{k=j}^{\infty} \sum_{i_1+\cdots+i_p=k} a_{i_1,\dots i_p}(s) z^{i_1} \cdots z^{i_p} \tag{2.7}$$

for any $x = (s,z) \in U_\alpha$. The functions $a_{i_1,\dots i_p}(s)$ are holomorphic on $s \in S \cap U_\alpha$. Therefore by changing variable $z$ to $\varepsilon w$ we get

$$\limsup_{\varepsilon \to 0} \int_{U_\alpha \setminus S} \left| g_\varepsilon^{(1)} \right|^2_{\theta'_\varepsilon} e^{-(j+p)\psi} dV_{\omega,X} \leqslant A_m \int_{S \cap U_\alpha} \sum_{i_1+\cdots+i_p=j} \frac{\left| a_{i_1,\dots,i_p}(s) \right|^2_{h^m}}{\left| \bigwedge^p dz \right|^2} dV_{\omega,S} < +\infty.$$

Here the constant $A_m$ is determined by

$$\int_{w \in \mathbb{C}^p} \rho'(|w|^2)^2 \frac{\bigwedge_{i=1}^{p} \sqrt{-1} dw_i \wedge d\bar{w}_i}{|w|^{2(j+p)}}.$$

The point is to estimate the $L^2$-norm of $a_{i_1,\dots,i_p}(s)$. This is done by Cauchy's estimate but here we have to involve the metric $h^m$ so that the constant in the $L^2$-estimate must depends on $h$. The idea here originates from [Pop05] and in our situation we want to see how the coefficients depend on $j$. First we have

$$\left| a_{i_1,\dots,i_p}(s) \right|^2 \leqslant (2\pi)^{-p} R^{-2(j+p)} \int_{|z_i| \leqslant 2R} \left| F_{j-1}(s,z) \right|^2 dz_1 \cdots dz_p$$

for any small $R > 0$. See also subsection 4.2. From this the $L^2$-norm is bounded as follows:

$$\int_{S \cap U_\alpha} \left| a_{i_1,\dots,i_p}(s) \right|^2_{h^m} dV_{\omega,S} \leqslant (2\pi)^{-p} R^{-2(j+p)} \sup_{|z_i| \leqslant 2R, s \in U_\alpha} \frac{h^m(s,0)}{h^m(s,z)} \int_X |F_{j-1}|^2_{h^m} dV_{\omega,X}.$$

Now by the continuity of $h$, There exists a constant $R_0 = R_0(X,S,h)$ such that

$$\limsup_{\varepsilon \to 0} \int_{U_\alpha \setminus S} \left| g_\varepsilon^{(1)} \right|^2_{\theta'_\varepsilon} e^{-(j+p)\psi} dV_{\omega,X} \leqslant R^{-2(j+p)} \int_X |F_{j-1}|^2_{h^m} dV_{\omega,X}$$

holds for any $R \leqslant R_0$. Note that $R_0$ is determined by the modulus of continuity of $h$ and independent of $m$ or $j$.

We can also estimate $g_\varepsilon^{(2)}$. Note that eigenvalues of $\theta'_\varepsilon$ are bounded below. Then we get

$$\int_{\Omega \setminus S} \left| g_\varepsilon^{(2)} \right|^2_{\theta'_\varepsilon} e^{-(j+p)\psi} dV_{\omega,X} \leqslant O(\varepsilon) < +\infty$$

because we can see that $\left| g_\varepsilon^{(2)} \right|^2_{\theta'_\varepsilon} = O(|z|^m)$ holds in $\operatorname{Supp} g_\varepsilon^{(2)} \subseteq \{e^\psi < \varepsilon\}$, by $\bar{\partial} \tilde{f} = O(|z|^m)$ (using the Taylor expansion).

Now we can apply the modified $L^2$-estimate for each $\varepsilon$ in $X \setminus S$. Note that $X \setminus S$ is a complete Kähler manifold (see [Dem82], Theorem 1.5). There exists a sequence

$\{u_\varepsilon\} \subseteq L^2(X, L^{\otimes m})$ such that

$$\int_{X\setminus S} (a_\varepsilon + b_\varepsilon)^{-1} |u_\varepsilon|^2 e^{-(j+p)\psi} dV_{\omega,X} \leqslant 2 \int_{X\setminus S} |g_\varepsilon|^2_{\theta'_\varepsilon} e^{-(j+p)\psi} dV_{\omega,X} \ < +\infty$$

holds.

Let us estimate the left hand side of the inequality. It can be easily seen that

$$\begin{aligned} \psi_\varepsilon &\leqslant \log(\varepsilon + e^{-1}) \leqslant -1 + O(\varepsilon) \\ a_\varepsilon &\leqslant (1 + O(\varepsilon))\psi_\varepsilon^2 \\ b_\varepsilon &= (2 - \psi_\varepsilon)^2 \leqslant (9 + O(\varepsilon))\psi_\varepsilon^2 \\ a_\varepsilon + b_\varepsilon &\leqslant (10 + O(\varepsilon))\psi_\varepsilon^2 \leqslant (10 + O(\varepsilon))(-\log(\varepsilon + e^\psi))^2 \end{aligned}$$

and

$$\int_\Omega \frac{\left|G_\varepsilon^{(j-1)}\right|^2}{(\varepsilon + e^\psi)^p(-\log(\varepsilon + e^\psi))^2} dV_{\omega,X} \leqslant \frac{M}{(\log\varepsilon)^2}$$

hold for some constant $M$. Therefore, if we set $F_\varepsilon := G_\varepsilon^{(j-1)} - u_\varepsilon + F_{j-1}$, it follows:

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \int_{X\setminus S} \frac{|F_\varepsilon|^2}{(\varepsilon + e^\psi)^p(-\log(\varepsilon + e^\psi))^2} dV_{\omega,X} \\ &\leqslant 22 \limsup_{\varepsilon\to 0} \int_{X\setminus S} |g_\varepsilon|^2_{\theta'_\varepsilon} e^{-(j+p)\psi} dV_{\omega,X} + \int_X |F_{j-1}|^2_{h^m} dV_{\omega,X} \\ &\leqslant (1 + R^{-2(j+p)}) \int_X |F_{j-1}|^2_{h^m} dV_{\omega,X}. \end{aligned}$$

By construction, $\bar\partial F_\varepsilon = 0$ holds on $X \setminus S$ and in fact also in $X$, thanks to the Riemann extension theorem.

Finally, Let $\varepsilon \searrow 0$. Then after taking a weakly convergent subsequence, we get a $F_j \in L^2(X, L^{\otimes m})$ such that $\bar\partial F_j = 0$ in $X$ and

$$\int_X \left|F_j{}^2\right|_{h^m} dV_{\omega,X} \leqslant (1 + R^{-2(j+p)}) \int_X |F_{j-1}|^2_{h^m} dV_{\omega,X}.$$

Note that the jet condition $J^j F_\varepsilon = f$ is preserved under the weak convergence by Lemma 2.8 and Cauchy's integral formula. By induction we obtain

$$\int_X \left|F_j{}^2\right|_{h^m} dV_{\omega,X} \leqslant \Big(\prod_{k=0}^{j}(1 + R^{-2(k+p)})\Big) \int_S |f|^2_{h^m} dV_{\omega,S}.$$

$\square$

**Lemma 2.8.** *Let $f_k$, $f$ be holomorphic functions defined in a domain $\Omega \subseteq \mathbb{C}^n$. Assume that the sequence $\{f_k\}$ weakly $L^2$-converges to $f$. Then $\{f_k\}$ converges to $f$ pointwise in $\Omega$.*

*Proof.* Fix any point $x \in \Omega$. Taking $\chi \in C_0^\infty(\Omega)$ with $\chi \equiv 1$ near $x$, we have:

$$f_k(x) = \int_{\zeta\in\Omega} K_{\mathrm{BM}}^{n,0}(x,\zeta) \wedge \bar\partial\chi(\zeta) \wedge f_k(\zeta) \to \int_{\zeta\in\Omega} K_{\mathrm{BM}}^{n,0}(x,\zeta) \wedge \bar\partial\chi(\zeta) \wedge f(\zeta) = f(x)$$

by the Koppelman formula. Here $K^{p,q}_{\mathrm{BM}}$ denotes the $(p,q)$-part of the Bochner-Martinelli kernel. $\square$

## 3. Preliminary to section 4

3.1. **Psh weight.** Let us briefly review the notion of psh weights. Let $L$ be a holomorphic line bundle over a smooth complex projective variety $X$. We usually fix a family of local trivialization patches $U_\alpha$ which cover $X$. A singular Hermitian metric $h$ on $L$ is a family of functions $h_\alpha = e^{-\varphi_\alpha}$ which are defined on corresponding $U_\alpha$ and satisfy the transition rule: $\varphi_\beta = \varphi_\alpha - \log |g_{\alpha\beta}|^2$ on $U_\alpha \cap U_\beta$. Here $g_{\alpha\beta}$ are transition functions of $L$. *The weight functions* $\varphi_\alpha$ are assumed to be $L^1$. If $\varphi_\alpha$ are smooth, $\{e^{-\varphi_\alpha}\}_\alpha$ actually defines a smooth Hermitian metric on $L$. We usual denote the family $\{\varphi_\alpha\}_\alpha$ as $\varphi$ and omit the index of local trivializations. Notice that each $\varphi = \varphi_\alpha$ is only a local function and not globally defined. But the curvature current $\Theta_h = dd^c\varphi$ is globally defined and is semipositive if and only if each $\varphi$ is plurisubharmonic. Here we denote by $d^c$ the real differential operator $\frac{\partial - \bar\partial}{4\pi\sqrt{-1}}$. We call such a weight *psh weight* for short. The most important example is the type of weight $\frac{1}{m}\log(|F_1|^2 + \cdots + |F_N|^2)$, defined by some holomorphic sections $F_1 \cdots F_N \in H^0(X, L^{\otimes m})$. Here $|F_i|^2$ $(1 \leqslant i \leqslant N)$ denotes the absolute value taken for the local trivialization of each $F_i$ on $U_\alpha$. We call such weights *algebraic.* More generally, a psh weight $\varphi$ said to have a *small unbounded locus* if the pluripolar set $\varphi^{-1}(-\infty)$ is contained in some closed proper algebraic subset $S \subseteq X$. A singular Hermitian metric $h = e^{-\varphi}$ is said to have strictly positive curvature if $dd^c\varphi \geqslant \omega$ holds for some Kähler form $\omega$.

By the following theorem due to Demailly, one can approximate a psh weight by a sequence of algebraic weights.

**Proposition 3.1.** *Let $S$ be a smooth projective variety, $L$ a holomorphic line bundle on $S$, $h_S = e^{-\varphi_S}$ a singular Hermitian metric with semipotitive curvature, and $h_0 = e^{-\psi}$ a singular Hermitian metric with strictly positive curvature. Then there exist $m_0 = m_0(S,\psi) \geqslant 1$ and $C > 0$ such that the following holds: Fix $\{f_{m,j}\}_{j=1}^{N_m} \subseteq H^0(S, L^{\otimes m})$, an orthonormal basis with respect to an $L^2$-norm*

$$\int_S |f|^2_{h_S^{(m-m_0)} h_0^{m_0}} \, dV_{\omega,S} = \int_S |f|^2 \, e^{-(m-m_0)\varphi_S - m_0\psi} dV_{\omega,S}.$$

*Then psh weights $\varphi_{S,m} := \frac{1}{m}\log(\sum_{j=1}^{N_m} |f_{m,j}|^2)$ satisfies*

$$-\frac{\log C}{m} + \left(\frac{m-m_0}{m}\varphi_S(s_0) + \frac{m_0}{m}\psi(s_0)\right) \leqslant \varphi_{S,m}(s_0) \leqslant \frac{\log C_r}{m} + \sup_{s \in B(s_0;r)} \left(\frac{m-m_0}{m}\varphi_S(s) + \frac{m_0}{m}\psi(s)\right)$$

*for any $s_0 \in S$, small $r > 0$, and $m \geqslant m_0$. In particular, $\varphi_{S,m}$ converge to $\varphi_S$.*

Proof of the proposition is on the same line as Proposition 3.1 in [Dem92]. We omit it here. See also section 12 of [Dem96] for the related topics.

3.2. **Positivity of line bundle.** We recall the basic concepts of positivity for line bundles. For a recent account of the theory we refer [Laz04] in the algebraic course and [Dem96] for the analytic treatments. A line bundle $L$ is said to be ample (*resp.* semiample, big) if the associated rational map

$$\Phi_m : X \dashrightarrow \mathbb{P}(H^0(X, L^{\otimes m}))$$

is a closed embedding (*resp.* a holomorphic map, a birational map to the image) for any sufficiently large $m$. The classical result of Kodaira states that $L$ is ample if and only if it admits a smooth strictly positive curvature Hermitian metric $h_0 = e^{-\psi}$. Line bundles satisfy the latter condition usually called *positive.* This gives an analytic characterization of ample line bundles and indicates a general principle that positivity of a metric produces holomorphic sections of a line bundle. There also exists an analytic characterization of big line bundle due to Demailly.

**Proposition 3.2** (Demailly). *A line bundle $L$ is big if and only if it admits a singular Hermitian metric with strictly positive curvature.*

The proposition allows us to expect some analytic analogue between ample line bundle and big line bundle, and motivates our study for a generalization of a result of [CGZ10] to the big line bundle case.

## 4. Extension of singular metric with semipositive curvature

In this section, we prove Proposition 1.4.

4.1. **Construction.** Fix a $p$-dimensional closed submanifold $S \subset X$ and a singular metric $h_S = e^{-\varphi_S}$ on $L|_S$. Moreover, we fix $h_0 = e^{-\psi}$, a singular Hermitian metric on $L$ such that $h_0$ has the strictly positive curvature on $X$ and the inequality

$$\varphi_S \leqslant \psi|_S \tag{4.1}$$

holds on $S$. Such $h_0$ actually exists by the bigness of $L$. Denote a positive integer satisfying the condition in Proposition 3.1 by $m_0$ and the norm of each vector space $H^0(S, L^{\otimes m})$ $(m \geqslant m_0)$ by

$$\int_S |f|^2 e^{-(m-m_0)\varphi_S - m_0\psi} dV_{\omega,S}.$$

First of all, by Proposition 3.1, we have a sequence of orthonormal basis $\{f_{m,j}\}_{j=1}^{N_m} \subseteq H^0(S, L^{\otimes m})$ such that

$$\frac{1}{m} \log \sum_{j=1}^{N_m} |f_{m,j}|^2 \to \varphi_S \quad \text{on } S.$$

Note that $\varphi_S$ is not a priori defined on the ambient space $X$. For this reason, we will apply the $L^2$-extension theorem for $\psi$ and then compare the two norm each of which is defined by $\varphi_S$ and $\psi$. At this stage, the assumption $\varphi_S \leqslant \psi|_S$ will be crucially used. At any rate, we have

$$\int_S |f_{m,j}|^2 e^{-m\psi} dV_{S,\omega} < +\infty$$

by $\varphi_S \leqslant \psi|_S$. If a constant $N = N(S, X, \omega)$ (which appears in the assumption in Theorem 1.1) is given, we may assume $dd^c\psi \geqslant N\omega$ because we can replace $L$ and $\varphi_S$ by $L^{\otimes N}$ and $N\varphi_S$ to prove Proposition 1.4. So applying the jet-extension, there exists $F_{m,j} \in H^0(X, L^{\otimes m})$ such that $F_{m,j}|_S = f_{m,j}$ holds and every other term in $(m-1)$-jet along $S$ vanishes. That is, for any point $s_0 \in S$ there exists a neighborhood $U$ on $X$ and a holomorphic local coordinate $z_1, \ldots z_p$ in the normal direction along $S$, which centers at $s_0$, such that the Taylor expansion of $F_{m,j}$ along $S$ can be written as

$$F_{m,j}(x) = F_{m,j}(s,z) = f_{m,j}(s) + \sum_{k=m}^{\infty} \sum_{i_1+\cdots+i_p=k} a_{i_1,\ldots i_p}(s) z^{i_1} \cdots z^{i_p} \tag{4.2}$$

for any $x = (s, z) \in U$. Note that the functions $a_{i_1,\ldots i_p}(s)$ are holomorphic on $s \in U \cap S$. Moreover, if the coefficient of $L^2$-estimate is refined,

$$\int_X |F_{m,j}|^2 e^{-m\psi} dV_{X,\omega} \leqslant C_1^m \int_S |f_{m,j}| e^{-m\psi} dV_{S,\omega} \tag{4.3}$$

hold for each $m$ and $j$. Here the constant $C_1 = C_1(S, X, \psi)$ does not depend on $m$. Let us define

$$\varphi_m := \frac{1}{m} \log \sum_{j=1}^{N_m} |F_{m,j}|^2 . \tag{4.4}$$

This $\varphi_m$ actually defines an algebraic singular metric on $L$ over $X$. As a consequence of the above $L^2$-estimates one can derive an upper boundedness of $\varphi_m$. Actually by the mean value property of plurisubharmonic function one has

$$|F_{m,j}(x)|^2 \leqslant \frac{n!}{\pi^n r^{2n}} \int_{B(x,r)} |F_{m,j}|^2 \leqslant \frac{n!}{\pi^n r^{2n}} \sup_{B(x,r)} e^{m\psi} \int_X |F_{m,j}|^2 e^{-m\psi} dV_{X,\omega}$$

for any small ball $B(x, r) \subset X$. Then (4.3) yields

$$|F_{m,j}(x)|^2 \leqslant C_2{}^m \int_S |f_{m,j}|^2 e^{-m\psi} dV_{S,\omega}.$$

The constant $C_2$ here depends not only $S$ but also $\psi$. Let us exchange $\psi$ for $\varphi_S$ by the inequality

$$\int_S |f_{m,j}|^2 e^{-m\psi} dV_{S,\omega} \leqslant \sup_S e^{(m-m_0)(\varphi_S-\psi)} \int_S |f_{m,j}|^2 e^{-(m-m_0)\varphi_S - m_0\psi} dV_{S,\omega}.$$

Note that $\varphi_S - \psi$ is well-defined as a global function on $S$ and the integral on the right-hand side equals to 1 by the normality. So the assumption $\varphi_S \leqslant \psi|_S$ shows that the right-hand side is not greater than 1. Summarizing up, we have

$$|F_{m,j}(x)|^2 \leqslant C_2{}^m. \tag{4.5}$$

Then, by the fact $N_m = O(m^{n-p})$ derived from the Riemann-Roch formula, there exists a constant $C_3$ only depends on $S$ and $\psi$ such that

$$\varphi_m \leqslant C_3 \tag{4.6}$$

holds. This constant $C_3$ is independent of $m$ and therefore we get a subsequence of $\varphi_m$ , which converges to a psh weight $\varphi$. From a general theory of plurisubharmonic function, one may assume

$$\varphi(x) = \limsup_{m\to\infty}{}^{*} \varphi_m(x) := \limsup_{y\to x} \limsup_{m\to\infty} \varphi_m(y).$$

Here the symbol $*$ denotes the upper-semicontinuous envelop. This ends the construction of $\varphi$. By the definition, $\varphi|_S \geqslant \varphi_S$ is clear. Our remained task is to show the converse inequality.

4.2. **Upper bound estimate.** At this time we need the control of $(m-1)$-jet of each $F_{m,j}$. For a general convergence of plurisubharmonic functions, a value of the upper-semicontinuous envelop happens to jump at some point. But we may expect that if every $\sum_j F_{m,j}$ sufficiently tangents to $S$, this is not the case. Let us check it in our situation. We fix $s_0 \in S$ and its neighborhood $U$ to have (4.2). Note that we may concentrate on each point's neighborhood to get requiring upper bound estimate of $\varphi$, thanks to the compactness of $S$. Take $R > 0$ so that for any $1 \leqslant i \leqslant p$, $|z_i| < 3R$ holds on $U$.

We first estimate $a_{i_1,\dots,i_p}$in (4.2). This is done by Cauchy's integral formula:

$$a_{i_1,\dots,i_p}(s) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^p \int_{|z_p|=r} \cdots \int_{|z_1|=r} \frac{F_{m,j}(s,z)}{z_1^{i_1+1}\cdots z_p^{i_p+1}} dz_1 \cdots dz_p.$$

In fact by Cauchy-Schwarz' inequality one can extract the $L^2$-norm as:

$$\begin{aligned}
\left|a_{i_1,\dots,i_p}(s)\right|^2 &\leqslant \left(\frac{1}{2\pi}\right)^{2p} \left(\int_0^{2\pi} \cdots \int_0^{2\pi} \frac{|F_{m,j}(s,z)|}{r^{i_1+\cdots i_p}} d\theta_1 \cdots d\theta_p\right)^2 \\
&\leqslant \left(\frac{1}{2\pi}\right)^{2p} \left(\int_0^{2\pi} \cdots \int_0^{2\pi} |F_{m,j}(s,z)|^2 d\theta_1 \cdots d\theta_p\right) \left(\int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d\theta_1 \cdots d\theta_p}{r^{2(i_1+\cdots i_p)}}\right) \\
&= \left(\frac{1}{2\pi}\right)^{p} \frac{1}{r^{2(i_1+\cdots i_p)}} \int_0^{2\pi} \cdots \int_0^{2\pi} |F_{m,j}(s,z)|^2 d\theta_1 \cdots d\theta_p.
\end{aligned}$$

Integrating the both side from $r_i = R$ to $r_i = 2R$ $(1 \leqslant i \leqslant p)$, we get

$$\begin{aligned}
&\int_R^{2R} \cdots \int_R^{2R} \left|a_{i_1,\dots,i_p}(s)\right|^2 dr_1 \cdots dr_p \\
&\qquad \leqslant \left(\frac{1}{2\pi}\right)^p R^{-2(i_1+\cdots i_p)-p} \int_{R\leqslant|z_i|\leqslant 2R} |F_{m,j}(s,z)|^2 dz_1 \cdots dz_p.
\end{aligned}$$

Thus it holds that

$$\left|a_{i_1,\dots,i_p}(s)\right| \leqslant \left(\frac{1}{2\pi}\right)^{\frac{p}{2}} \left(\int_{|z_i|\leqslant 2R} |F_{m,j}(s,z)|^2 dz_1 \cdots dz_p\right)^{\frac{1}{2}} R^{-(i_1+\cdots i_p)-p}.$$

Take $0 < \varepsilon < 1/2$. Then in the ball $\{|z| \leqslant p^{-1}\varepsilon R\} \subset \mathbb{C}^p$ (here we fix $s$ and move only $z$) we have the following estimates:

$$
\begin{aligned}
|F_{m,j}(s,z) - f_{m,j}(s)| &\leqslant \sum_{i_1+\cdots i_p \geqslant m} \left|a_{i_1,\ldots i_p}(s)\right| \cdot |z|^{i_1+\cdots+i_p} \\
&\leqslant \left(\frac{1}{2\pi}\right)^{\frac{p}{2}} \left(\int_{|z_i|\leqslant 2R} |F_{m,j}(s,z)|^2 \, dz_1 \cdots dz_p\right)^{\frac{1}{2}} \sum_{i_1+\cdots i_p \geqslant m} R^{-(i_1+\cdots i_p)-p} \left(\frac{\varepsilon R}{p}\right)^{i_1+\cdots i_p} \\
&\leqslant \left(\frac{1}{2\pi}\right)^{\frac{p}{2}} \left(\int_{|z_i|\leqslant 2R} |F_{m,j}(s,z)|^2 \, dz_1 \cdots dz_p\right)^{\frac{1}{2}} 2R^{-p}\varepsilon^m.
\end{aligned}
$$

From (4.5), the integral in the last term is bounded by a constant $C_2^m$. Thus we obtain:

$$
|F_{m,j}(s,z) - f_{m,j}(s)| \leqslant C_2^m \varepsilon^m. \tag{4.7}
$$

Here the constant $C_2$ depends only on $S$ and $\psi$. This is what we need.

Now let us estimate $\varphi_m$. By the triangle inequality

$$
\begin{aligned}
\varphi_m(s,z) &= \frac{2}{m} \log \left(\sum_{j=1}^{N_m} |F_{m,j}(s,z)|^2\right)^{\frac{1}{2}} \\
&\leqslant \frac{2}{m} \log \left[\left(\sum_{j=1}^{N_m} |f_{m,j}(s)|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{N_m} |F_{m,j}(s,z) - f_{m,j}(s)|^2\right)^{\frac{1}{2}}\right]
\end{aligned}
$$

hold. Thanks to (4.7), the second term in the logarithm is bounded from above by the square root of $N_m C_2^m \varepsilon^m$. The first term can be bounded from above by Proposition 3.1. Thus for any $s \in B(s_0; r)$ we have

$$
\varphi_m(s,z) \leqslant \frac{2}{m} \log \left[\left(\sup_{s' \in B(s_0;2r)} C_r e^{(m-m_0)\varphi_S(s') + m_0\psi(s')}\right)^{\frac{1}{2}} + \left(N_m C_2^m \varepsilon^m\right)^{\frac{1}{2}}\right].
$$

By the concavity of the logarithmic function

$$
\log(a+b) \leqslant \log a + \frac{b}{a}
$$

holds for any $a, b > 0$ so it yields

$$
\begin{aligned}
\varphi_m(s,z) \leqslant \frac{1}{m} \log C_r &+ \sup_{s' \in B(s_0;2r)} \left(\frac{(m-m_0)}{m}\varphi_S(s') + \frac{m_0}{m}\psi(s')\right) \\
&+ \frac{2}{m} \frac{\left(N_m C_2^m \varepsilon^m\right)^{\frac{1}{2}}}{\left(\sup_{s' \in B(s_0;2r)} e^{(m-m_0)\varphi_S(s') + m_0\psi(s')}\right)^{\frac{1}{2}}}.
\end{aligned}
$$

Note that the denominator of the third term on the right-hand side

$$
\left(\sup_{s' \in B(s_0;2r)} e^{(m-m_0)\varphi_S(s') + m_0\psi(s')}\right)^{\frac{1}{2}}
$$

is positive but may go to 0 when $r \to 0$. However, taking $\varepsilon$ sufficiently small for $r$, we can make the third term smaller than $1/m$. Hence letting $m \to \infty$ we get

$$\limsup_{(s,z)\to 0} \limsup_{m\to\infty} \varphi_m(s,z) \leqslant \varphi_S(s_0).$$

This completes the proof of Proposition 1.4.

4.3. **Remarks on restricted volumes.** Before concluding the paper, we discuss the assumption which is necessary for generalizing Theorem 1.2 to big line bundles. For the proof of Proposition 1.4, especially on the ample line bundle case, one can skip this subsection. First we briefly review an analytic representation of the volume of a line bundle along a subvariety.

**Definition 4.8.** Let $S$ be a $p$-codimensional subvariety on a $n$-dimensional smooth projective variety $X$. Denote by $H^0(X|S, L^{\otimes m})$ the image of the restriction map $H^0(X, L^{\otimes m}) \to H^0(S, L^{\otimes m})$. The restricted volume of a line bundle $L$ along $S$ is defined to be

$$\mathrm{vol}_{X|S}(L) := \limsup_{m\to\infty} \frac{\dim H^0(X|S, L^{\otimes m})}{m^{n-p}/(n-p)!}.$$

When $S = X$, we simply write it as $\mathrm{vol}_X(L)$.

It is easy to show that if $L$ is ample (more generally semiample), $\mathrm{vol}_{X|S}(L) = \mathrm{vol}_S(L|_S) = (S.L^{n-p})$ holds, *i.e.* the restricted volume equals to the intersection number. In general restricted volumes are invariant under birational transformations and a line bundle $L$ is big if and only if $\mathrm{vol}_X(L) > 0$. In some sense they measure the positivity of line bundles along subvarieties. An analytic description of volumes was first given in [His11]. For a singular Hermitian metric $h = e^{-\varphi}$ on $X$, let us denote by $\langle (dd^c\varphi|_S)^{n-p}\rangle$ the non-pluripolar Monge-Ampère product of $\varphi|_S$, a positive measure which coincides with usual Monge-Ampère product $(dd^c\varphi|_S)^{n-p}$ when $\varphi$ is smooth. For the precise definition, see [BEGZ10].

**Theorem 4.9** (Theorem 1.3 in [His11]). *Assume there exists a singular Hermitian metric $h_0 = e^{-\psi}$ with strictly positive curvature on $L$, and $S \subsetneq \psi^{-1}(-\infty)$ hold. Then it holds that*

$$\mathrm{vol}_{X|S}(L) = \max_{\varphi} \int_S \langle (dd^c\varphi|_S)^{n-p}\rangle, \tag{4.10}$$

*where $e^{-\varphi}$ runs through all the singular Hermitian metric on $L$ over $X$.*

We can relate the analytic study of volumes to the problem of metric extension.

**Corollary 4.11.** *In general* $\mathrm{vol}_{X|S}(L) \leqslant \mathrm{vol}_S(L|_S)$ *and if every semipositive curvature singular metric on $L|_S$ is extendable* $\mathrm{vol}_{X|S}(L) = \mathrm{vol}_S(L|_S)$ *holds.*

**Example 4.12.** Let $\mu : X \to \mathbb{P}^2$ be the one point blow up and $E$ an exceptional divisor. We take $L = \mu^*\mathcal{O}(1) \otimes \mathcal{O}(E)$ and $S$ to be the strict transform of a line which through the blown up point. Then it is easy to see that this example satisfies the assumption of Theorem 4.9 but $\mathrm{vol}_{X|S}(L) = 1$ and $\mathrm{vol}_S(L|_S) = 2$.

The next example shows that there exists a non-extendable metric even if we assume $L$ is semiample and big.

**Example 4.13.** Let $\mu : X \to \mathbb{P}^2$ be the one point blow up and $E$ an exceptional divisor as the above. Take $L = \mu^*\mathcal{O}(1)$ and $S$ to be the smooth strict transform of a nodal curve. Then $L$ has a singular metric with semipositive curvature, which can not be extended to $X$. In fact one can construct a semipositive curvature metric $h_S = e^{-\varphi_S}$ weight on the ample line bundle $L|_S$, which is $+\infty$ on a point in $E \cap S$ and smooth on another. If a psh extension $\varphi_S$ exists, it is constant along $E$ since $L|_E$ is trivial. This contradicts the definition of $h_S$.

These examples naturally indicate to us the assumption in Problem1.3.

**Acknowledgments.** The author would like to express his gratitude to his advisor Professor Shigeharu Takayama for his warm encouragements, many helpful suggestions and reading the drafts. The author specially wishes to thank Professor Dan Coman for his pointing out a critical mistake in the first version of the preprint. The author wishes to thank Professor Vincent Guedj and Professor Ahmed Zeriahi for several helpful comments concerning the approach of $L^2$-extension, during the conference *Complex and Riemannian Geometry* held at CIRM, Luminy. The author also would like to thank Shin-ichi Matsumura for several helpful comments, especially on Example 4.13. This research is supported by JSPS Research Fellowships for Young Scientists (22-6742).

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# Part 4

Restricted Bergman kernel asymptotics

# RESTRICTED BERGMAN KERNEL ASYMPTOTICS

TOMOYUKI HISAMOTO

ABSTRACT. In this paper, we investigate a *restricted* version of Bergman kernels for high powers of a big line bundle over a smooth projective variety. The geometric meaning of the leading term is specified. As a byproduct, we derive some integral representations for the restricted volume.

## 1. INTRODUCTION

The subjects discussed in this paper originate from the extension problem. Let $L$ be a big line bundle on a smooth complex projective variety $X$ and $Z \subseteq X$ a subvariety. Denote by $\iota : Z \hookrightarrow X$ the inclusion map. We always use this notation unless specifically noted. It is important to know how many sections of $L|_Z$ are extended to the ambient space $X$. We can expect to get more such sections taking high tensor powers of $L$, thus we are led to consider the spaces of sections

$$H^0(X|Z, \mathcal{O}(mL)) := \mathrm{Im}\big[\iota^* : H^0(X, \mathcal{O}(mL)) \to H^0(Z, \mathcal{O}(mL))\big].$$

The restricted volume

$$\mathrm{Vol}_{X|Z}(L) := \limsup_{m\to\infty} \frac{\dim H^0(X|Z, \mathcal{O}(mL))}{m^p/p!}$$

measures the asymptotic growth of these spaces. Here $p$ denotes the complex dimension of $Z$. The notion of the restricted volume first appeared in Tsuji's paper [Tsu06] (see also [HM06], [Tak06], [ELMNP09]). In this paper, we investigate a local version of the restricted volume.

**Definition 1.1.** Let $h_L$ be a smooth Hermitian metric on $L$, $\varphi \in C^\infty(X;\mathbb{R})$ a smooth weight, and $d\mu$ a volume form on $Z$. Then for any positive integer $m$, *the restricted Bergman kernel* of $(Z, mL, h_L^m e^{-m\varphi}, d\mu)$ is defined as follows:

$$B_{X|Z}(m\varphi) := |s_{m,1}|^2_{m\varphi} + \dots + |s_{m,N(m)}|^2_{m\varphi}.$$

Here $\{s_{m,1}, \dots, s_{m,N(m)}\}$ is a complete orthonormal system of $H^0(X|Z, \mathcal{O}(mL))$ with respect to the norm

$$\|s\|^2_{m\varphi} := \int_Z |s|^2_{m\varphi}\, d\mu,$$
$$|s|^2_{m\varphi} := \iota^* h_L^m(s,s) e^{-m\iota^*\varphi}.$$

□

2000 *Mathematics Subject Classification.* Primary 32A25, Secondary 32L10, 32W20.
*Key words and phrases.* Bergman kernel, extension theorem, Monge-Ampère operator.

By definition, $B_{X|Z}(m\varphi)$ is a smooth function on $Z$ and

$$\int_Z B_{X|Z}(m\varphi)d\mu = \dim H^0(X|Z, \mathcal{O}(mL)).$$

In fact $B_{X|Z}(m\varphi)$ tells not only the dimension but rather deeper information of the space of sections. The study of the asymptotic behavior of $B_{X|Z}(m\varphi)$ is itself an important problem in complex geometry.

In this paper we closely examine the leading term of $B_{X|Z}(m\varphi)$. If $L$ is ample and the metric $h_L e^{-\varphi}$ has the positive curvature, then by the Serre vanishing theorem the problem is reduced to the case $Z = X$. In this case, Tian's classical result ([Tian90]) gives the complete answer. For general $L$ and $h_L$, Berman first treated the case $Z = X$ in [Ber09]. And in that paper, he also mentioned the restricted case without proof. Without the assumption of curvature positivity, the effect of the subvariety can not be ignored. We give a complete picture in the restricted case and specify the limit of $m^{-p}B_{X|Z}(m\varphi)$. Our study can be seen as a local version of the restricted Fujita-type approximation (Theorem 3.19). As a result, a localization of $\mathrm{Vol}_{X|Z}(L)$ is given.

To state our results, we need some notion which arises in non-positive curvature case. First denote by $\mathbb{B}_+(L) \subsetneq X$ the augmented base locus (see [Laz04], Definition 10.3.2). This is actually an algebraic subset of $X$ and $L$ is ample precisely if $\mathbb{B}_+(L) = \emptyset$. Secondly, we denote by $P_{X|Z}\varphi$ the equilibrium weight associated to $\varphi$ (see Definition 3.1). Let $\theta = -dd^c \log h_L$ be the Chern curvature of $h_L$, then $\theta + dd^c P_{X|Z}\varphi$ defines a positive current on $Z$, and $P_{X|Z}\varphi = \varphi$ holds if $\theta + dd^c\varphi$ is positive. Roughly speaking, $P_{X|Z}\varphi$ is the best $\theta$-plurisubharmonic function on $Z$ approximating $\varphi$. Further, one can measure the rest of the positivity of $\theta + dd^c\varphi$ by $\mathrm{MA}(P_{X|Z}\varphi) := \langle(\theta + dd^c P_{X|Z}\varphi)^p\rangle$, the non-pluripolar Monge-Ampère product of $P_{X|Z}\varphi$ (see Definition 2.4).

**Theorem 1.2.** *Assume $Z$ is smooth and $Z \not\subseteq \mathbb{B}_+(L)$. Then the convergence*

$$\frac{B_{X|Z}(m\varphi)}{m^p/p!}d\mu \to \mathrm{MA}(P_{X|Z}\varphi)$$

*holds in the sense of currents.*

As byproducts of our investigation of restricted Bergman kernel asymptotics, we can get several integral representations of restricted volumes (discussed in section 4). For instance, we have the following.

**Theorem 1.3.** *In the situation of Theorem 1.2, the following holds.*

$$\begin{aligned}\mathrm{Vol}_{X|Z}(L) &= \int_Z \mathrm{MA}(P_{X|Z}\varphi) = \int_Z \mathrm{MA}(\iota^* P_X\varphi)\\ &= \sup_T \int_Z \langle(\iota^*T)^p\rangle = \int_Z \langle(\iota^*T_{\min})^p\rangle,\end{aligned}$$

*where $T$ runs through all the closed positive currents in $c_1(L)$, with small unbounded loci not contained in $\iota(Z)$. We denote by $T_{\min}$ a minimum singular closed positive current in $c_1(L)$.*

These formulas can be seen as generalizations of the main result of [Bou02]. If $L$ is ample, $\mathrm{Vol}_{X|Z}(L)$ equals to the intersection number $(L^p.Z)$ which plays an important role in many geometric questions. But for general line bundles, these intersection numbers do not work well to describe function-theoretic properties of $L$. Our results indicate that $\mathrm{Vol}_{X|Z}(L)$ is the natural generalization of $(L^p.Z)$ for general line bundles.

Let us explain the point of our proof of Theorem 1.2. We basically follows Berman's approach but there are two difficulties in the restricted case. First, to deal with general subvarieties, we need a variant of $L^2$-extension theorems. The desired extension theorem is the following.

**Theorem 1.4.** *Let $X$ be a smooth projective variety, $Z \subseteq X$ a smooth subvariety, $\omega$ a fixed Kähler form, and $E \to X$ a holomorphic vector bundle with a smooth Hermitian metric $h_E$. Then there exist constants $N = N(Z, X, h_E, \omega)$ and $C = C(Z, X) > 0$ such that the following holds.*

*Let $L \to X$ be a holomorphic line bundle with a singular Hermitian metric $h_L e^{-\varphi}$ such that its Chern curvature satisfies*

$$\theta + dd^c\varphi \geqslant N\omega.$$

*Then for any section $s \in H^0(Z, \mathcal{O}(E \otimes L))$ with*

$$\int_Z |s|^2 e^{-\varphi} dV_{\omega,Z} < +\infty,$$

*there exists a section $\widetilde{s} \in H^0(X, \mathcal{O}(E \otimes L))$ such that $\widetilde{s}|_Z = s$ and*

$$\int_X |\widetilde{s}|^2 e^{-\varphi} dV_{\omega,X} \leqslant C \int_Z |s|^2 e^{-\varphi} dV_{\omega,Z}$$

*holds.*

*Remark* 1.5. It is natural to expect that Theorem 1.4 holds even if $Z$ has some mild singularities. But it seems to be unknown. □

Theorem 1.4 can be derived from Theorem 4 of [Ohs01] by a standard approximation technique. See also [Kim10]. It seems most likely that a slight change of the proof of Theorem 4.2 of [Kim10] can yield Theorem 1.4. At any rate we give a self-contained proof in section 5, as a courtesy to the reader. Theorem 5.1 in the present paper corresponds to Theorem 4 in [Ohs01] (but the situations in the two theorems are slight different). Theorem 1.4 is used in the two critical steps. One step is to show the regularity of the restricted equilibrium weight (Theorem 3.5) and the other is to show a lower bound of restricted Bergman kernels (Theorem 3.15). Second, we only have a weak lower bound in the restricted case since it becomes harder to estimate the lower bound of the Bergman kernels precisely as [Ber09]. We avoid this difficulty by using a proof of the restricted version of the Fujita-type approximation theorem. Note that a part of this strategy already appeared in [BB10]. We elaborate this strategy using a weak lower bound and the comparison theorem for the Monge-Ampère operator. From this, one can first get an integral representation of the restricted volume and then deduce Theorem 1.2. Compared with [Ber09] in the case $Z = X$, our proof of Theorem 1.2 is rather geometric thanks to the Fujita-type approximation. On the other hand, the convergence result obtained in this paper is weaker than that of [Ber09]. It seems to be unknown whether convergence in a strict sense holds in the restricted case.

## 2. Monge-Ampère operator

We briefly review the definition of the Monge-Ampère operator in this section. Fix a closed real smooth $(1,1)$-form $\theta$ defined on $X$. An $L^1_{\mathrm{loc}}$-function $\psi$ in $X$ is called *$\theta$-plurisubharmonic* ($\theta$-psh for short) when the associated current $\theta + dd^c\psi$ is positive (in the sense of currents). A function which is $\theta$-psh for some $\theta$ is called *quasi-plurisubharmonic* (quasi-psh for short). It is known that $\psi$ automatically becomes upper-semicontinuous by this condition. We denote the set of all $\theta$-psh functions by $\mathrm{PSH}(X,\theta)$. In this paper we are mainly interested in $\theta$ defined as $\theta = -dd^c \log h_L$, but this notion is in fact valid for an arbitrary $\theta$.

Let $n$ be the dimension of $X$. The Monge-Ampère operator should be defined as:

$$\psi \mapsto \mathrm{MA}(\psi) := (\theta + dd^c\psi)^n,$$

but for general $\psi$, this is nonsense. The celebrated result of Bedford-Taylor ([BT76]) tells us that the right hand side can be defined as a current for $\psi$ at least in the class $L^\infty_{\mathrm{loc}} \cap \mathrm{PSH}(X,\theta)$. That is, by induction on the exponent $q = 1, 2, ..., n$, it can be defined as:

$$\int_X (\theta + dd^c\psi)^q \wedge \eta := \int_X (\theta + dd^c\psi)^{q-1} \wedge (\tau + \psi) dd^c\eta$$

for each test form $\eta \in C_0^\infty(X, \bigwedge^{n-q,n-q} T_X^*)$. Here $\int_X$ denotes the canonical pairing of currents and test forms, and $\tau$ denotes a local $dd^c$-potential of $\theta$. This is indeed well-defined and defines a closed positive current, because $\tau + \psi$ is a bounded Borel function and $(\theta + dd^c\psi)^{q-1}$ has measure coefficients by the induction hypothesis and by the fact that any closed positive current has measure coefficients. Bedford-Taylor's Monge-Ampère products have useful continuity properties:

**Proposition 2.1.**

$$(\theta + dd^c\psi_k)^n \to (\theta + dd^c\psi) \qquad \textit{in the sense of currents}$$

*for any sequence of $\theta$-psh functions which satisfies one of the following conditions.*

(1) *$\psi_k \searrow \psi$ pointwise in $X$.*
(2) *$\psi_k \nearrow \psi$ for almost every point in $X$.*
(3) *$\psi_k \to \psi$ uniformly in any compact subset of $X$.*

It is still necessary to consider unbounded $\theta$-psh functions. On the other hand, for our purpose to investigate asymptotic behaviors of Bergman kernels, it is sufficient to deal with some special class of unbounded $\theta$-psh functions and we can omit a part of the contribution of unbounded loci.

**Definition 2.2.** A $\theta$-psh function $\psi$ is said to have *a small unbounded locus* if the pluripolar set $\psi^{-1}(-\infty)$ is contained in some closed proper algebraic subset $S \subsetneq X$. □

A quasi-psh function $\psi$ on $X$ is said to have *algebraic singularities*, if it can be locally written as

$$\psi = c \cdot \log(|f_1|^2 + ... + |f_N|^2) + u \tag{2.3}$$

for some $c \in \mathbb{Q}_{\geqslant 0}$, non-zero regular functions $f_i$ $(1 \leqslant i \leqslant N)$, and a smooth function $u$. Every $\theta$-psh function with algebraic singularities has a small unbounded locus. If we assume the subvariety $Z$ is smooth, $\iota^*\varphi + m^{-1}\log B_{X|Z}(m\varphi)$ gives the typical example of $\iota^*\theta$-psh function with algebraic singularities.

**Definition 2.4.** For a $\theta$-psh function $\psi$ on $X$ with a small unbounded locus, $\mathrm{MA}(\psi)$ is defined to be

$$\langle(\theta + dd^c\psi)^n\rangle := \text{ the zero extension of } (\theta + dd^c\psi)^n.$$

Note that the coefficient of $(\theta + dd^c\psi)^n$ is well-defined as a measure on $X \setminus S$. □

$\mathrm{MA}(\psi)$ actually defines a closed positive current on $X$ by famous Skoda's extension theorem. In particular, it has a finite mass on $X$. For a proof, see [BEGZ08], section 1.

*Remark* 2.5. In that paper, the *non-pluripolar* Monge-Ampère product was defined in fact for general $\theta$-psh functions on a compact Kähler manifold. Note that this choice of ways to define the Monge-Ampère operator makes $\mathrm{MA}(\psi)$ to have no mass on any pluripolar set so ignores some of the singularities of $\psi$. For this reason, $\langle(\theta + dd^c\psi)^n\rangle$ no longer has continuity property with respect to $\psi$. □

We recall the fundamental fact established in [BEGZ08] which states that the less singular $\theta$-psh function has the larger Monge-Ampère mass. Recall that given two $\theta$-psh $\psi$ and $\psi'$, $\psi$ is said to be less singular than $\psi'$ if there exists a constant $C > 0$ such that $\psi' \leqslant \psi + C$ in $X$. We say that a $\theta$-psh function is *minimal singular* if it is minimal with respect to this partial order. When $\psi$ is less singular than $\psi'$ and $\psi'$ is less singular than $\psi$, we say that the two functions are equivalent with respect to singularities. This defines a equivalence relation in $\mathrm{PSH}(X, \theta)$. When $\theta \in c_1(L)$, any minimal singular $\theta$-psh function $\psi$ has a small unbounded locus. In fact, $\psi^{-1}(-\infty) \subseteq \mathbb{B}_+(L)$ holds.

**Theorem 2.6** ([BEGZ08], Theorem 1.16). *If $\psi, \psi'$ are $\theta$-psh functions with small unbounded loci such that $\psi$ is less singular than $\psi'$, then*

$$\int_X \mathrm{MA}(\psi') \leqslant \int_X \mathrm{MA}(\psi)$$

*holds.*

*Remark* 2.7. It is unknown that Theorem 2.6 holds for general $\theta$-psh functions. □

The notion of types of $\theta$-psh functions with respect to singularities, explained in this subsection, are in fact determined by the closed positive currents $T := \theta + dd^c\psi$. Namely, a closed positive $(1,1)$-current $T \in \alpha$ is said to have a small unbounded locus if it can be written: $T = \theta + dd^c\psi$ with some $\psi$ which has a small bounded locus. Closed positive $(1,1)$-currents with algebraic singularities and those with minimal singularities can be defined in the same manner.

## 3. Restricted Bergman kernel asymptotics

### 3.1. **Restricted equilibrium weight.** 
In this subsection, we introduce the notion of the restricted equilibrium weight and discuss its properties, which we will use later to study asymptotics of restricted Bergman kernels. Unless otherwise stated, we fix

a big line bundle $L$ on a smooth projective variety $X$ and a smooth metric $h_L$. Let $\theta := -dd^c \log h_L$ be the Chen curvature form. Given a subvariety $Z$ of $X$, there exists a canonical way to associate any smooth function to the $\theta$-psh function on $Z$.

**Definition 3.1.** For a smooth weight $\varphi \in C^\infty(X;\mathbb{R})$ and a subvariety $Z \subseteq X$, the restricted equilibrium weight $P_{X|Z}\varphi$ is a function on $Z$ defined as follows:

$$P_{X|Z}\varphi(z) := \sup{}^* \left\{ \iota^*\psi(z) \;\middle|\; \begin{array}{c} \psi \in \mathrm{PSH}(X,\theta) \\ \text{with } \iota^*\psi \leqslant \iota^*\varphi \text{ on } Z \end{array} \right\} \tag{3.2}$$

for $z \in Z$. Here $\iota : Z \hookrightarrow X$ denotes the inclusion map. If there is no $\psi$ as above, $P_{X|Z}\varphi \equiv -\infty$ by definition. □

In the special case when $Z = X$, we use the notation $P_X$ instead of $P_{X|X}$ as in [BB10]. The symbol $\sup^*$ appeared in the above definition means

$$\sup_\alpha{}^* f_\alpha(z) := \limsup_{w\to z} \big(\sup_\alpha f_\alpha(w)\big)$$

which is called *the regularized upper envelope* for a family of functions $\{f_\alpha\}_\alpha$. It is easily seen that $\iota^*P_X\varphi \leqslant P_{X|Z}\varphi \leqslant P_Z\iota^*\varphi$ holds. By a classical result of Choquet (see e.g. [Kli91], Lemma 2.3.4.) and by the definition of $P_{X|Z}\varphi$, we get the following.

**Lemma 3.3.** *Assume that $P_{X|Z}\varphi$ is not identically infinity on $Z$. Then there exists a countable non-decreasing family of $\theta$-psh functions $\{\psi_k\}_k$ ($k = 1, 2, ...$) such that $\iota^*\psi_k \nearrow P_{X|Z}\varphi$ a.e., otherwise $P_{X|Z}\varphi \equiv -\infty$. In particular, $P_{X|Z}\varphi \in \mathrm{PSH}(Z, \iota^*\theta)$ unless $P_{X|Z}\varphi \equiv -\infty$.*

Now assume that $Z$ is smooth and that $\iota(Z) \not\subseteq \mathbb{B}_+(L)$. Then $P_{X|Z}\varphi$ has a small unbounded locus contained in $\iota^{-1}(\mathbb{B}_+(L))$. Then it follows that the Monge-Ampère mass of $P_{X|Z}\varphi$ can be defined as:

$$\int_Z \mathrm{MA}(P_{X|Z}\varphi) := \int_Z \langle(\iota^*\theta + dd^c P_{X|Z}\varphi)^p\rangle = \int_{Z\setminus\iota^{-1}(\mathbb{B}_+(L))} (\iota^*\theta + dd^c P_{X|Z}\varphi)^p.$$

The following is a consequence of Theorem 2.6, and it enables us to substitute $\mathrm{MA}(\iota^*P_X\varphi)$ for $\mathrm{MA}(P_{X|Z}\varphi)$ to estimate the lower bound of the restricted Bergman kernels. This is a starting point of our strategy to prove Theorem 1.2.

**Theorem 3.4.** *Assume that $Z$ is smooth and that $\iota(Z) \not\subseteq \mathbb{B}_+(L)$. Then It holds that*

$$\int_Z \mathrm{MA}(P_{X|Z}\varphi) = \int_Z \mathrm{MA}(\iota^*P_X\varphi).$$

*Proof.* By Theorem 2.6, we only have to show the first equality. The one side inequality $\geqslant$ is also an immediate consequence of Theorem 2.6. Since we may take $\psi_k$ minimal singular in Lemma 3.3 (by exchanging $\psi_k$ by $\max\{\psi_k, P_X\varphi\}$),

$$\int_Z \mathrm{MA}(\iota^*P_X\varphi) = \int_Z \mathrm{MA}(\iota^*\psi_k)$$

holds by Theorem 2.6. On the other hand, since

$$(\iota^*\theta + dd^c\iota^*\psi_k)^p \to (\iota^*\theta + dd^c P_{X|Z}\varphi)^p \quad \text{on } Z\setminus\iota^{-1}(\mathbb{B}_+(L))$$

by the continuity property of the Monge-Ampère operator, we have

$$\liminf_{k\to\infty}\int_Z \mathrm{MA}(\iota^*\psi_k) \geqslant \int_Z \mathrm{MA}(P_{X|Z}\varphi).$$

Therefore

$$\int_Z \mathrm{MA}(\iota^* P_X\varphi) \geqslant \int_Z \mathrm{MA}(P_{X|Z}\varphi).$$

□

The next theorem is a key ingredient to represent $\mathrm{MA}(P_{X|Z}\varphi)$ explicitly by $\varphi$. It states that the gradient of $P_{X|Z}\varphi$ is locally Lipschitz on $Z \setminus \mathbb{B}_+(L)$.

**Theorem 3.5.** *Assume that $Z$ is smooth and $\iota(Z) \not\subseteq \mathbb{B}_+(L)$. Then $P_{X|Z}\varphi$ has Lipschitz continuous first derivatives outside of $\iota^{-1}(\mathbb{B}_+(L))$. Namely,*

$$P_{X|Z}\varphi \in C^{1,1}\big(Z \setminus \iota^{-1}(\mathbb{B}_+(L))\big).$$

*Moreover,*

$$(\iota^*\theta + dd^c P_{X|Z}\varphi)^p = (\iota^*\theta + dd^c\iota^*\varphi)^p \qquad \textit{in the set} \quad \{P_{X|Z}\varphi = \iota^*\varphi\} \setminus \iota^{-1}(\mathbb{B}_+(L))$$

*a.e. with respect to $d\mu$.*

*Proof.* The proof is almost the same as $Z = X$ case in [Ber09] except that in the restricted case we need the Ohsawa-Takegoshi-type $L^2$-extension theorem for an arbitrary smooth subvariety (Theorem 1.4). We sketch the proof and omit the detail.

Let $Y$ be the total space of the dual line bundle $L^*$, identifying the base $X$ with its embedding as the zero-section in $Y$, and $\pi : Y \to X$ be the projection map. Given $\psi \in \mathrm{PSH}(X, \theta)$, one can associate a psh function $\chi_\psi$ defined on $Y$, as follows:

$$\chi_\psi(x, w) := \log |w|^2_{h_L^{-1}} + \psi(x) \qquad (x \in X, w \in L_x).$$

Berman's original argument is modeled on the proof of Bedford-Taylor for $C^{1,1}$-regularity of the solution of the Dirichlet problem for the complex Monge-Ampère equation in the unit-ball in $\mathbb{C}^n$. As opposed to the unit ball, $X$ has no global holomorphic vector fields. But one can reduce the regularity problem of $P_X\varphi$ on $X$ to a problem of $\chi_{P_X\varphi}$ on $Y$, where enough many vector fields exist. This argument is still valid in the restricted case once one can construct the suitable vector fields on $\pi^{-1}(Z)$ extended to $Y$.

For a proof, it is enough to show the regularity of $\chi_{P_X\varphi}$ at any given point $y_0 \in \pi^{-1}(Z \setminus \mathbb{B}_+(L)) \setminus Z$. By Kodaira's lemma, there exists an effective divisor $E$ on $X$ such that $y_0 \notin \pi^{-1}(\mathrm{Supp}\, E)$ and $mL = A + E$ hold with some positive integer $m$ and ample $\mathbb{Z}$-divisor $A$. We may assume $m = 1$ for the proof of the Theorem 3.5 since $mP_{X|Z}\varphi = P_{X|Z}(m\varphi)$ holds. By this decomposition, we can construct some $\psi_0 = \psi_A + \psi_E$, $\theta_L = \theta_A + \theta_E$ such that $\theta_A + dd^c\psi_A > 0$ is smooth and $\theta_E + dd^c\psi_E \geqslant 0$ has singularities only on $E$. Indeed, it is enough to set $\theta_E := -dd^c \log h_{E,\alpha}$, $\psi_E := \log |f_\alpha|^2 + \log h_{E,\alpha}$ for some smooth metric $h_E$ and system of local equations $\{f_\alpha\}$.

**Lemma 3.6.** *There exist holomorphic vector fields $V_1, \ldots, V_{p+1}$ on $\pi^{-1}(Z)$ satisfying the following properties.*

(1) $V_1, ..., V_{p+1}$ *is linearly independent at* $y_0$.
(2) *There exist holomorphic vector fields* $\widetilde{V}_1, ..., \widetilde{V}_{p+1}$ *on* $Y$ *such that* $\widetilde{V}_i|_{\pi^{-1}(Z)} = V_i$ $(1 \leqslant i \leqslant p+1)$.
(3) *For any fixed* $k \in \mathbb{N}$, $\widetilde{V}_i$ $(1 \leqslant i \leqslant p+1)$ *can be chosen to have zeros of order at least* $k$ *along* $X$ *and* $\pi^{-1}(\operatorname{Supp} E)$. *To be precise,*

$$\left|\widetilde{V}_i\right| \leqslant C(k) \cdot |w|^k, \quad \left|\widetilde{V}_i\right| \leqslant C(k) \cdot |f_\alpha(z)|^k$$

*hold locally in the set* $\{\chi_{\psi_0} \leqslant 1\}$ *for some constant* $C(k)$ *depending on* $k$.

*Proof.* Let $\widehat{Y} := \mathbb{P}(\mathcal{O}(-L) \oplus \mathcal{O}))$ be the Zariski closure of $Y$. Consider the line bundle $\pi^* L^{k_0} \otimes H_{\mathbb{P}(\mathcal{O}(-L)\oplus\mathcal{O})}$ on $\widehat{Y}$ and its metric

$$h_{k_0,\alpha} := \pi^* h_{L,\alpha}^{k_0} + \log(1 + e^{\chi_\varphi})$$

with weight

$$\psi_{k_0} := \pi^* \big(k_0(\psi_A + (1 + k_0^{-1/2})\psi_E)\big).$$

Here $H_{\mathbb{P}(\mathcal{O}(-L)\oplus\mathcal{O})}$ denotes the fiberwise hyperplane bundle. For $w$-direction, $\log(1+e^{\chi_\varphi})$ has the strictly positive curvature and for $x$-direction, $\psi_A + (1 + k_0^{-1/2})\psi_E$ is $\theta_L$-strictly positive if we take $k_0$ sufficiently large. Thus $h_{k_0} e^{-\psi_{k_0}}$ has strictly positive curvature in $\widehat{Y}$. From this, taking sufficiently large $k_1$, we can use Theorem 1.4 to get holomorphic sections

$$V_1, ..., V_{p+1} \in H^0(\widehat{\pi^{-1}(Z)}, \mathcal{O}(T'_{\widehat{\pi^{-1}(Z)}} \otimes (\pi^* L^{k_0} \otimes H_{\mathbb{P}(\mathcal{O}(-L)\oplus\mathcal{O})})^{k_1}))$$

which correspond to some basis $V_{1,0}, ..., V_{p+1,0}$ of $T'_{\pi^{-1}(Z),y_0}$. If we use Theorem 1.4 once more and take further large $k_1$, it can be seen that $V_1, ..., V_{p+1}$ are restrictions of some

$$\widetilde{V}_1, ..., \widetilde{V}_{p+1} \in H^0(\widehat{Y}, \mathcal{O}(T'_{\widehat{Y}} \otimes (\pi^* L^{k_0} \otimes H_{\mathbb{P}(\mathcal{O}(-L)\oplus\mathcal{O})})^{k_1}))$$

which are integrable with respect to $(h_{k_0} e^{-\psi_{k_0}})^{k_1}$. Note that $H_{\mathbb{P}(\mathcal{O}(-L)\oplus\mathcal{O})}|_Y$ is trivial and that $\pi^* L = -[X]$ (the dual of the line bundle defined by the divisor $X \subseteq Y$). Therefore $\widetilde{V}_1, ..., \widetilde{V}_{p+1}$ can be identified with holomorphic vector fields over $Y$ having zeros of order at least $k_0 k_1$ along $X$. Further, by the integrability condition, we get

$$\left|\widetilde{V}_i(x, w_\alpha)\right| \leqslant C \cdot \bigg( |w_\alpha|^{k_0} \cdot |f_\alpha(x)|^{k_0(1+k_0^{-1/2})} \cdot |w_\alpha| \bigg)^{k_1}$$

hence

$$\left|\widetilde{V}_i(x, w_\alpha)\right| \leqslant C(k) \cdot \bigg( |w_\alpha| \cdot |f_\alpha(x)| \bigg)^{(k_0+1)k_1} \cdot |f_\alpha(x)|^k.$$

The boundedness of $|w_\alpha| \cdot |f_\alpha(x)|$ in $\{\chi_{\psi_0} \leqslant 1\}$ implies the conclusion. □

Actually, Lemma 3.6 assures the existence of desired vector fields $V_i$ $(1 \leqslant i \leqslant p)$ and one can repeat the proof of Theorem 3.4 in [Ber09]. □

On the other hand, the repeating the proof of the $Z = X$ case in Proposition 3.1 of [Ber09] gives the following.

**Lemma 3.7.** *In the situation of Theorem 3.5,*

(1)
$$P_{X|Z}\varphi = \iota^*\varphi \ \ a.e.\ with\ respect\ to\ \mathrm{MA}(P_{X|Z}\varphi).$$
(2)
$$P_{X|Z}\varphi(z_0) = \iota^*\varphi(z_0) \ \Rightarrow \ (\iota^*\theta + dd^c\iota^*\varphi)(z_0) \geqslant 0.$$

As a consequence of Theorem 3.5 and Lemma 3.7 (1), we obtained the desired representation formula for $\mathrm{MA}(P_{X|Z}\varphi)$ as in the case $Z = X$.

**Theorem 3.8.** *Assume that $Z$ is smooth and $\iota(Z) \not\subseteq \mathbb{B}_+(L)$. Then the identity*

$$\mathrm{MA}(P_{X|Z}\varphi) = \mathbf{1}_{\{P_{X|Z}\varphi=\iota^*\varphi\}} \cdot (\iota^*\theta + dd^c\iota^*\varphi)^p \tag{3.9}$$

*holds. Here $\mathbf{1}_{\{P_{X|Z}\varphi=\iota^*\varphi\}}$ denotes the characteristic function of the set $\{P_{X|Z}\varphi = \iota^*\varphi\}$. In particular, the measure $\mathrm{MA}(P_{X|Z}\varphi)$ has $L^\infty$-density with respect to $d\mu$.*

3.2. **Restricted Bergman kernel asymptotics.** From now on, we compare $P_{X|Z}\varphi$ with $B_{X|Z}(m\varphi)$ in detail. Fix notations as in the previous subsections. In this subsection we always assume that $Z$ is a smooth subvariety of $X$ and that $\iota(Z) \not\subseteq \mathbb{B}_+(L)$ holds. First we specify the upper bound of restricted Bergman kernels and show the half of our main result.

**Proposition 3.10.**
$$\limsup_{m\to\infty} \frac{B_{X|Z}(m\varphi)}{m^p/p!} d\mu \leqslant \mathrm{MA}(P_{X|Z}\varphi).$$

*Proof.* This is deduced from the two estimates about the upper bound of Bergman kernels. First, we show the so-called *"Berman's local holomorphic Morse inequality"* (see [Ber04], Theorem 1.1) in the restricted case. The proof in the case $Z = X$ is applicable with no change.

*Claim* (1):

$$\limsup_{m\to\infty} \frac{B_{X|Z}(m\varphi)}{m^p/p!} d\mu \leqslant \mathbf{1}_{\{(\iota^*\theta+dd^c\iota^*\varphi)\geqslant 0\}} \cdot (\iota^*\theta + dd^c\iota^*\varphi)^p. \tag{3.11}$$

*Proof of the claim* (1). Fix any $z_0 \in Z$. If we take an appropriate trivialization patch $U$ around $z_0$ with $h_{L,U}(z_0)e^{-\varphi(z_0)} = 1$ and denote the eigenvalues of $\iota^*\theta + dd^c\iota^*\varphi$ with respect to the form $\frac{\sqrt{-1}}{2}\sum_{i=1}^p dz_i \wedge d\overline{z}_i$ at $z_0$ by $\lambda_1, ..., \lambda_p$, then for an arbitrary section $s \in H^0(X|Z, \mathcal{O}(mL))$ with $\|s\|^2_{m\varphi} = 1$, we have

$$\frac{|s(z_0)|^2_{m\varphi}}{m^p/p!} = \frac{|s_U(z_0)|^2}{m^p/p!}$$
$$\leqslant \left(\int_{|z|\leqslant\frac{\log m}{\sqrt{m}}} |s_U|^2 e^{-m\sum\lambda_i|z_i|^2} d\lambda(z)\right)\left(\int_{|z|\leqslant\frac{\log m}{\sqrt{m}}} e^{-m\sum\lambda_i|z_i|^2} d\lambda(z)\cdot m^p/p!\right)^{-1}$$

by the mean value inequality for subharmonic functions. Here $d\lambda$ denotes the Lebesgue measure with respect to $z_i$. The $\limsup_{m\to\infty}$ of the numerator in the last side is bounded by $\det_{d\mu} d\lambda(z_0)$, and the denominator behaves as follows if we let $m \to \infty$:

$$\frac{1}{p!}\int_{|w|\leqslant\log m} e^{-\sum\lambda_i|w_i|^2} d\lambda(w) \to \begin{cases} \pi^p/(p!\lambda_1\lambda_2\cdots\lambda_p) & \text{if } \lambda_i \geqslant 0 \\ \infty & \text{otherwise.} \end{cases}$$

(Here we use $w = \sqrt{m}z$ as a new variable.) From this, one can deduce the claim.

The second claim is a direct consequence of the definition of $P_{X|Z}\varphi$, and motivates the definition as well.

*Claim* (2):

$$\frac{B_{X|Z}(m\varphi)}{m^p/p!} \leqslant e^{-m(\iota^*\varphi - P_{X|Z}\varphi)} \cdot \sup_Z \frac{B_{X|Z}(m\varphi)}{m^p/p!}. \tag{3.12}$$

*Proof of the claim* (2). Note that the supremum in the right hand side is finite by claim (1). Fix any $z_0 \in Z$ and take any $s \in H^0(X|Z, \mathcal{O}(mL))$ satisfying $|s(z_0)|^2_{m\varphi} = B_{X|Z}(m\varphi)(z_0)$ and $\|s\|^2_{m\varphi} = 1$. Since $|s(z)|^2_{m\varphi} \leqslant \sup_Z B_{X|Z}(m\varphi)$ for any $z \in Z$, we have

$$\frac{1}{m}\big(\log|s(z)|^2_{h_L^m} - \log\sup_Z B_{X|Z}(m\varphi)\big) \leqslant \iota^*\varphi \;\text{ in } Z.$$

Since the left hand side is the pull-back of a $\theta$-psh function on $X$ the above inequality implies

$$\frac{1}{m}\big(\log|s(z)|^2_{h_L^m} - \log\sup_Z B_{X|Z}(m\varphi)\big) \leqslant P_{X|Z}\varphi \;\text{ in } Z.$$

Thus the claim (2) is obtained.

Proposition 3.10 is now easily proved. Actually, claim (1) and Lemma 3.7 (2) imply

$$\limsup_{m\to\infty} \frac{B_{X|Z}(m\varphi)}{m^p/p!} d\mu \leqslant (\iota^*\theta + dd^c\iota^*\varphi)^p \quad \text{in } \{P_{X|Z}\varphi = \iota^*\varphi\}$$

and claim (2) implies the pointwise convergence

$$\frac{B_{X|Z}(m\varphi)}{m^p/p!} \to 0 \quad (m\to\infty) \;\text{ in } \{P_{X|Z}\varphi \neq \iota^*\varphi\} \tag{3.13}$$

so one can conclude Proposition 3.10 by Theorem 3.8. □

**Corollary 3.14.**

$$\mathrm{Vol}_{X|Z}(L) \leqslant \int_Z \mathrm{MA}(P_{X|Z}\varphi)$$

*Proof.* Since $\mathrm{MA}(P_{X|Z}\varphi)$ has $L^\infty$-density by Theorem 3.8, we can apply Fatou's lemma to (3.10). □

We can now derive the fundamental relation between $P_{X|Z}\varphi$ and $B_{X|Z}(m\varphi)$.

**Theorem 3.15.** *For every compact set $K \Subset Z \setminus \iota^{-1}(\mathbb{B}_+(L))$, there exist an integer $m_0$ and a positive constant $C \geqslant 0$ such that the inequality*

$$C^{-1} \cdot e^{-m(\iota^*\varphi - P_{X|Z}\varphi)} \leqslant B_{X|Z}(m\varphi) \leqslant C \cdot m^p e^{-m(\iota^*\varphi - P_{X|Z}\varphi)} \tag{3.16}$$

*holds.*

*Proof.* The right hand side inequality is a direct consequence of Proposition 3.10 and Theorem 3.8. We will show the left hand side. By the extremal property of the Bergman kernel, it is enough to show the following claim.

*Claim*: There exist some $m_0$ , $C$ and section $s_m \in H^0(X|Z, \mathcal{O}(mL))$ for each $m \geqslant m_0$ such that

(1) $|s_m(z)|^2_{m\psi_k} \geqslant C^{-1}$ *for any* $z \in K$, $k \in \mathbb{N}$,

(2) $\|s_m\|^2_{m\varphi} \leqslant C$.

Here $\psi_k \in \mathrm{PSH}(X, \theta)$ are taken to satisfy $\iota^*\psi_k \nearrow P_{X|Z}\varphi$ a.e. with respect to $d\mu$. Actually, this implies

$$B_{X|Z}(m\varphi) \geqslant \frac{|s_m(z)|^2_{m\varphi}}{\|s_m\|^2_{m\varphi}} \geqslant C^{-2}e^{-m(\iota^*\varphi - \iota^*\psi_k)}$$

so letting $k \to \infty$ we get the inequality.

*Proof of the claim.* Fix $z \in K$. By Kodaira's lemma, we may take some ample $\mathbb{Q}$-divisor $A$ and some effective $\mathbb{Q}$-divisor $E$ on $X$ satisfying $L = A + E$. From this decomposition, we may construct a $\theta$-psh function $\psi_0$ with $\psi_0^{-1}(-\infty) \subseteq \operatorname{Supp} E$, $\psi_0 \leqslant \varphi$. Then using Theorem 1.4 twice, we may find suitable $m_0, C$ and sections $s_m \in H^0(X|Z, \mathcal{O}(mL))$ for each $m \geqslant m_0$ such that

(1) $|s_m(z)|^2_{\psi_{m,k}} = 1$

(2) $\|s_m\|^2_{\psi_{m,k}} \leqslant C$ ,

where $\psi_{m,k} = (m - m_0)\psi_k + m_0\psi_0$. Then we infer

$$\|s_m\|^2_{m\varphi} \leqslant \|s_m\|^2_{\psi_{m,k}} \leqslant C$$

and since we may assume $e^{m_0(\varphi - \psi_0)(z)} \leqslant C$ by the smoothness of $\psi_0$ around $z$,

$$1 = |s_m(z)|^2_{\psi_{m,k}} \leqslant C\,|s_m(z)|^2_{m\varphi_k}.$$

Here $C$ depends on $m_0$ and $K$. □

As a consequence of the above results, the sequence of the Monge-Ampère mass of the following Fubini-Study like potential functions converges to the Monge-Ampère mass of the restricted equilibrium weight. This fact corresponds to the description of restricted volumes via moving intersection numbers (see Theorem 4.6), and has a key role for us to prove the local version of the restricted Fujita approximation in the next subsection. Let us define:

$$u_m := \iota^*\varphi + \frac{1}{m}\log B_{X|Z}(m\varphi). \tag{3.17}$$

**Theorem 3.18.**

$$u_m \to P_{X|Z}\varphi \qquad \textit{uniformly in any compact subset of } Z \setminus \iota^{-1}(\mathbb{B}_+(L)),$$

*and*

$$\mathrm{MA}(u_m) \to \mathrm{MA}(P_{X|Z}\varphi) \quad (m \to \infty)$$

*in the sense of currents.*

*Proof.* The inequality (3.16) is equivalent to

$$-\frac{\log C}{m} + P_{X|Z}\varphi \leqslant \iota^*\varphi + \frac{1}{m}\log B_{X|Z}(m\varphi) \leqslant \frac{\log C + p\log m}{m} + P_{X|Z}\varphi.$$

This estimate implies that, on any compact subset of $Z \setminus \iota^{-1}(\mathbb{B}_+(L))$, $u_m$ converges uniformly to $P_{X|Z}\varphi$. By the continuity property of the Monge-Ampère operator, we deduce

$$(\iota^*\theta + dd^c u_m)^p \to (\iota^*\theta + dd^c P_{X|Z}\varphi)^p \quad \text{in } Z \setminus \iota^{-1}(\mathbb{B}_+(L)).$$

In particular,

$$\liminf_{m\to\infty} \int_{Z\setminus\iota^{-1}(\mathbb{B}_+(L))} (\iota^*\theta + dd^c u_m)^p \geqslant \int_{Z\setminus\iota^{-1}(\mathbb{B}_+(L))} (\iota^*\theta + dd^c P_{X|Z}\varphi)^p$$

holds. Therefore we only have to show

$$\limsup_{m\to\infty} \int_{Z\setminus\iota^{-1}(\mathbb{B}_+(L))} (\iota^*\theta + dd^c u_m)^p \leqslant \int_{Z\setminus\iota^{-1}(\mathbb{B}_+(L))} (\iota^*\theta + dd^c P_{X|Z}\varphi)^p,$$

because we already have the current convergence in $Z \setminus \iota^{-1}(\mathbb{B}_+(L))$, but this is directly seen by Theorem 2.6 and Theorem 3.4. □

3.3. **Restricted Fujita-type approximation.** In this subsection, we first give a proof of the restricted Fujita approximation theorem and then finish the poof of Theorem 1.2.

**Theorem 3.19** ([Tak06], Theorem 3.1, [ELMNP09], Theorem 2.13). *Let $X$ be a smooth projective variety, $\iota : Z \hookrightarrow X$ a subvariety, and $L$ a big line bundle on $X$. Then for an arbitrary $\varepsilon > 0$, the following diagram is commutative, where $\pi_Z$, $\pi_X$ are modifications and $\widetilde{Z}, \widetilde{X}$ are smooth such that*

(1) *in the sense of linear equivalence between $\mathbb{Q}$-divisors, $\pi_X^* L = A + E$ holds for some semiample and big divisor $A$ and effective divisor $E$, and*
(2) $\mathrm{Vol}_{\widetilde{X}|\widetilde{Z}}(A) \leqslant \mathrm{Vol}_{X|Z}(L) \leqslant \mathrm{Vol}_{\widetilde{X}|\widetilde{Z}}(A) + \varepsilon$

*hold.*

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\widetilde{\iota}} & \widetilde{X} \\ {\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ Z & \xrightarrow[\iota]{} & X \end{array}$$

*Remark* 3.20. By the continuity of the restricted volume (see Theorem A in [ELMNP09] or (3.27)), the divisor $A$ in Theorem 3.19 can be taken ample. This is shown as follows. First we get a decomposition of $\mathbb{Q}$-divisor: $A = A_0 + E_0 = ((1-\delta)A + \delta A_0) + (\delta E_0)$ by Kodaira's lemma. Then $A_\delta := (1-\delta)A + \delta A_0$ is ample and letting $\delta \to 0$, $\mathrm{Vol}_{X|Z}(A_\delta)$ approximates $\mathrm{Vol}_{X|Z}(A)$. From this, it follows that

$$\mathrm{Vol}_{X|Z}(L) = \lim_{m\to\infty} \frac{\dim H^0(X|Z, \mathcal{O}(mL))}{m^p/p!} \tag{3.21}$$

holds. Indeed one can reduce this to the case when $L$ is ample. Since the Serre vanishing theorem forces $H^0(X|Z, \mathcal{O}(mL)) = H^0(Z, \mathcal{O}(mL))$ in this case, we may assume $Z = X$. Then (3.21) is obtained from the Riemann-Roch theorem. □

Although a proof of Theorem 3.19 is already obtained in [Tak06] or [ELMNP09], we have to reprove this to show the local version (Theorem 1.2) at the same time. Our proof of Theorem 3.19 is essentially the same as the proof in [Tak06] or [ELMNP09], but we need a more direct proof and do not use a characterization of restricted volumes via multiplier ideal sheaves. We need the following *"The uniformly globally generation theorem"*, which was first proved in [Siu98]. It can also be obtained as a corollary of Theorem 1.4.

**Proposition 3.22** ([Siu98], Proposition 1). *Given a smooth projective variety $X$, there exists a line bundle $G$ such that for any pseudo-effective line bundle $F$ on $X$ with a singular Hermitian metric $h_F e^{-\psi}$ whose Chern curvature current is positive, the sheaf $\mathcal{O}(F+G)\otimes\mathcal{I}(\psi)$ is globally generated.*

We also need the following lemma which can be shown by simple algebraic computations. For a proof, see e.g. 2.2.C of [Laz04].

**Lemma 3.23.** *For an arbitrary line bundle $G$ on $X$ and a positive number $\varepsilon>0$, there exist a subsequence $\{\ell_k\}(k=1,2,...)$ and an integer $m_0$ such that*

$$\frac{\dim H^0(X|Z,\mathcal{O}(\ell_k(mL-G)))}{{\ell_k}^p/p!}\geqslant m^p\big(\operatorname{Vol}_{X|Z}(L)-\varepsilon\big)$$

*for any $m\geqslant m_0$.*

*Proof of Theorem 3.19.* Throughout this proof, we fix some $G$ which appeared in the Theorem 3.22, and a smooth metric $h_G$ on $G$. For any fixed integer $m$, we define the weight of $h_L^m h_G^{-1}$ as follows:

$$u_m:=\varphi+\frac{1}{m}\log(|s_{m,1}|^2_{m\varphi}+...+|s_{m,N(m)}|^2_{m\varphi}),$$

where $\{s_{m,1},...,s_{m,N(m)}\}$ is a complete orthonormal system of $H^0(X,\mathcal{O}(mL-G))$ with respect to the norm

$$\|s\|^2_{m\varphi}:=\int_X|s|^2_{m\varphi}\,d\mu$$

$$|s|^2_{m\varphi}:=(h_L^m h_G^{-1})(s,s)e^{-m\varphi}.$$

This is essentially the same as $u_m$ in Theorem 3.18 ($\iota=\mathrm{id}$ case). In fact, as in subsection 3.2, we can get

$$\langle(\iota^*T_m)^p\rangle\to\langle(\iota^*T)^p\rangle, \tag{3.24}$$

where $T_m:=\theta+dd^c u_m$, $T:=\theta+dd^cP_X\varphi$ assuming that $\iota$ is a closed embedding and that $Z$ is smooth, $\iota(Z)\not\subseteq\mathbb{B}_+(L)$. Here the difference caused by $G$ does not matter, for $G$ has no contribution to the asymptotic behavior of $H^0(X,\mathcal{O}(mL-G))$ thanks to the bigness of $L$.

If we set $\mathcal{J}$ as the ideal sheaf generated locally by $s_{m,1},...,s_{m,N(m)}$, then $\mathcal{J}\subseteq\mathcal{I}(mu_m)$ holds. Therefore, by taking a log resolution we have the following commutative diagram, where $\pi_Z^*$ and $\pi_X^*$ are modifications from smooth projective varieties such that

$$\pi_X^*\mathcal{I}(mu_m)=\mathcal{O}_{\tilde{X}}(-E'),\ \pi_X^*\mathcal{J}=\mathcal{O}_{\tilde{X}}(-F'),\ \text{ and }\ E'\leqslant F'. \tag{3.25}$$

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\widetilde{\iota}} & \widetilde{X} \\ {\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ Z & \xrightarrow[\iota]{} & X \end{array}$$

Moreover, since $T_m$ has algebraic singularities, we may assume

$$\pi_X^* T_m = \gamma + [F], \tag{3.26}$$

where $\gamma$ is a smooth semipositive form and $F := F'/m$. $[F]$ denotes the closed positive $(1,1)$-current defined by $F$. We claim that this diagram actually satisfies the condition in Theorem 3.19 for a sufficiently large $m$.

By Proposition 3.22, $\mathcal{O}(mL) \otimes \mathcal{I}(mu_m)$ is globally generated. Therefore its pull-back $\mathcal{O}(m\pi_X^* L - E')$ is also globally generated. For this reason, we may have a semiample divisor $A'$ satisfying $m\pi_X^* L = A' + E'$. Then the subadditivity property of multiplier ideal sheaves (see [Laz04] 9.5.B) implies

$$\begin{aligned} &H^0(\widetilde{X}|\widetilde{Z}, \mathcal{O}(\ell A')) = H^0(\widetilde{X}|\widetilde{Z}, \mathcal{O}(\ell(m\pi_X^* L - E'))) \\ &= H^0(\widetilde{X}|\widetilde{Z}, \pi_X^*(\mathcal{O}(\ell mL) \otimes \mathcal{I}(mu_m)^\ell)) \supseteq H^0(\widetilde{X}|\widetilde{Z}, \pi_X^*(\mathcal{O}(\ell mL) \otimes \mathcal{I}(\ell mu_m))) \end{aligned}$$

and we get

$$\begin{aligned} H^0(\widetilde{X}|\widetilde{Z}, \pi_X^*(\mathcal{O}(\ell mL) \otimes \mathcal{I}(\ell mu_m))) &\supseteq H^0(X|Z, \mathcal{O}(\ell mL) \otimes \pi_{X*}\pi_X^*\mathcal{I}(\ell mu_m)) \\ &= H^0(X|Z, \mathcal{O}(\ell mL) \otimes \mathcal{I}(\ell mu_m)) \end{aligned}$$

by the integral closedness of $\mathcal{I}(\ell mu_m)$. Further,

$$\begin{aligned} H^0(X|Z, \mathcal{O}(\ell mL) \otimes \mathcal{I}(\ell mu_m)) &\supseteq H^0(X|Z, \mathcal{O}(\ell(mL - G)) \otimes \mathcal{I}(\ell mu_m)) \\ &= H^0(X|Z, \mathcal{O}(\ell(mL - G))) \end{aligned}$$

by the definition of $u_m$. Consequently, with Lemma 3.23, it can be seen that there exists a subsequence $\{\ell_k\}$ and a sufficiently large $m$ such that

$$\frac{\dim H^0(\widetilde{X}|\widetilde{Z}, \mathcal{O}(\ell_k A'))}{\ell_k{}^p/p!} \geqslant m^p(\mathrm{Vol}_{X|Z}(L) - \varepsilon).$$

Setting $A := A'/m$, $E := E'/m$, By homogeneity of restricted volume ([ELMNP09] Lemma 2.2),

$$\mathrm{Vol}_{\widetilde{X}|\widetilde{Z}}(A) \geqslant \mathrm{Vol}_{X|Z}(L) - \varepsilon$$

and $\pi_X^* L = A + E$ hold. From this estimate it is also possible to deduce that $A$ is big for a sufficiently large $m$, because the above diagram for $\iota$ is also valid for the identity map. The proof of the reversed inequality is not hard. $\square$

With the proof of Theorem 3.19, we finally get to our goal of this subsection. Observe that one can approximate $\mathrm{MA}(P_{X|Z}\varphi)$ and $\mathrm{Vol}_{X|Z}(L)$ at the same time taking takes suitable modifications.

*Proof of Theorem 1.2.* Since $\iota$ is a closed embedding and $\iota(Z) \not\subseteq \mathbb{B}_+(L)$, we may assume $\widetilde{\iota}$ is also an embedding and $\widetilde{\iota}(\widetilde{Z}) \not\subseteq \mathbb{B}_+(A')$. By the semiampleness and the bigness of $A'$, there exists a smooth semipositive form $\theta_A$ in $c_1(A)$ such that

$$\begin{aligned}\mathrm{Vol}_{\widetilde{X}|\widetilde{Z}}(A) = \mathrm{Vol}_{\widetilde{Z}}(\widetilde{\iota}^* A) &= \int_{\widetilde{Z}} (\widetilde{\iota}^* \theta_A)^p \\ &= \int_{\widetilde{Z}} \langle (\widetilde{\iota}^* (\theta_A + [E]))^p \rangle.\end{aligned}$$

The last equality is a consequence of the non-pluripolarity of the Monge-Ampère product. $\widetilde{\iota}^*(\theta_A + [E])$ and $\widetilde{\iota}^*(\gamma + [F])$ are in the same class so that one can apply Theorem 2.6 to (3.25), to deduce the following:

$$\begin{aligned}\int_{\widetilde{Z}} \langle (\widetilde{\iota}^* (\theta_A + [E]))^p \rangle &\geqslant \int_{\widetilde{Z}} \langle (\widetilde{\iota}^* (\gamma + [F]))^p \rangle \\ &= \int_{\widetilde{Z}} \langle (\pi_Z^* \iota^* T_m)^p \rangle = \int_Z \langle (\iota^* T_m)^p \rangle.\end{aligned}$$

For an arbitrary $\varepsilon > 0$, the proof of Theorem 3.18 shows

$$\int_Z \langle (\iota^* T_m)^p \rangle \geqslant \int_Z \langle (\iota^* T)^p \rangle - \varepsilon$$

if we take $m$ sufficiently large. This implies

$$\mathrm{Vol}_{X|Z}(L) \geqslant \mathrm{Vol}_{\widetilde{X}|\widetilde{Z}}(A) \geqslant \int_Z \mathrm{MA}(\iota^* P_X \varphi)$$

so combining this inequality with Theorem 3.4 and Corollary 3.14, we finally get the identity

$$\mathrm{Vol}_{X|Z}(L) = \int_Z \mathrm{MA}(P_{X|Z} \varphi). \tag{3.27}$$

With this identity and Proposition 3.10, Theorem 1.2 is now concluded from Lemma 2.2 in [Ber06] which is shown by basic measure theory. □

## 4. Integral representations for the restricted volume

In this section, we discuss several integral representation of the restricted volume.

**Theorem 4.1.** *Let $Z \subseteq X$ be a (possibly singular) subvariety of $X$ and assume $\iota(Z) \not\subseteq \mathbb{B}_+(L)$. Then the following holds.*

$$\begin{aligned}\mathrm{Vol}_{X|Z}(L) &= \int_{Z_{\mathrm{reg}}} \mathrm{MA}(P_{X|Z_{\mathrm{reg}}} \varphi) = \int_{Z_{\mathrm{reg}}} \mathrm{MA}((\iota|_{Z_{\mathrm{reg}}})^* P_X \varphi) \\ &= \sup_T \int_{Z_{\mathrm{reg}}} \langle ((\iota|_{Z_{\mathrm{reg}}})^* T)^p \rangle = \int_{Z_{\mathrm{reg}}} \langle ((\iota|_{Z_{\mathrm{reg}}})^* T_{\min})^p \rangle = \int_{X \setminus \mathbb{B}_+(L)} (T_{\min})^p \wedge [Z],\end{aligned}$$

*where $T$ runs through all the closed positive currents in $c_1(L)$, with small unbounded loci not contained in $\iota(Z)$. We denote by $T_{\min}$ a minimum singular current in $c_1(L)$ and denote by $Z_{\mathrm{reg}}$ the regular locus of $Z$. The last integrand is defined as a closed positive current on $X \setminus \mathbb{B}_+(L)$ in the manner of Bedford-Taylor, and $[Z]$ denotes the closed positive current defined by $Z$.*

*Proof.* First assume $Z$ is smooth. The first two identities are nothing but (3.27) and Theorem 3.4. The second two are consequences of Theorem 2.6. Let us prove the last identity. Note that the trivial extension of the current $(T_{\min})^p \wedge [Z]$ to $X$ is a closed positive current and has finite mass by Skoda's extension theorem. Fix a Borel function $\psi$ such that $T_{\min} = \theta + dd^c\psi$. By induction on $p$, we are going to prove that

$$\int_{X \setminus \mathbb{B}_+(L)} \rho(\theta + dd^c\psi)^p \wedge [Z] = \int_{Z \setminus \iota^{-1}(\mathbb{B}_+(L))} \iota^*\rho(\iota^*\theta + dd^c\psi)^p \tag{4.2}$$

for any Borel function $\rho$ on $X$. The case $p = 0$ is trivial. Assume this is true for $p-1$. First fix a *smooth* function $\rho$ on $Z$. Take some $\chi_k \in C_0^\infty(X \setminus \mathbb{B}_+(L))$ $(k = 1, 2, 3, \dots)$ such that $\chi_k \equiv 1$ outside of the $1/k$-neighborhood of $\mathbb{B}_+(L)$. Then

$$\begin{aligned}
&\int_{X \setminus \mathbb{B}_+(L)} \chi_k\rho(\theta + dd^c\psi) \wedge (\theta + dd^c\psi)^{p-1} \wedge [Z] \\
&= \int_{X \setminus \mathbb{B}_+(L)} \chi_k\rho\theta \wedge (\theta + dd^c\psi)^{p-1} \wedge [Z] + \psi dd^c(\chi_k\rho) \wedge (\theta + dd^c\psi)^{p-1} \wedge [Z] \\
&= \int_{Z \setminus \iota^{-1}(\mathbb{B}_+(L))} \iota^*(\chi_k\rho\theta) \wedge (\iota^*\theta + dd^c\iota^*\psi)^{p-1} + \iota^*\psi dd^c\iota^*(\chi_k\rho) \wedge (\iota^*\theta + dd^c\iota^*\psi)^{p-1}
\end{aligned}$$

by the induction hypothesis. This equals to

$$\int_{Z \setminus \iota^{-1}(\mathbb{B}_+(L))} \iota^*(\chi_k\rho\theta) \wedge (\iota^*\theta + dd^c\iota^*\psi)^{p-1} + \iota^*(\chi_k\rho) dd^c\iota^*\psi \wedge (\iota^*\theta + dd^c\iota^*\psi)^{p-1}.$$

Letting $k \to \infty$, we get (4.2) by the Lebesgue convergence theorem. The general case follows from the density.

The assumption $Z$ is smooth can be dropped if we consider a resolution of singularities, because the Monge-Ampère measure has no mass on any closed proper algebraic subset. For instance, let us prove the first identity. Definition of $P_{X|Z_{\mathrm{reg}}}\varphi$ is the same as (3.17). If we take a resolution of singularities, $\mathrm{Vol}_{X|Z}(L) = \mathrm{Vol}_{\widetilde{X}|\widetilde{Z}}(\pi_Z^* L)$ holds. It is enough to show $P_{\widetilde{X}|\widetilde{Z}}\pi_Z^*\varphi = \pi_Z^* P_{X|Z_{\mathrm{reg}}}\varphi$ in the regular locus of $\pi_Z$. $P_{\widetilde{X}|\widetilde{Z}}\pi_Z^*\varphi \leqslant \pi_Z^* P_{X|Z_{\mathrm{reg}}}\varphi$ is trivial and the converse inequality follows by the Riemann-type extension theorem for psh functions. Other identities above are shown in the same manner. □

*Remark* 4.3. The assumption that $T$ has a small unbounded locus can be dropped since we may define the *non-pluripolar* Monge-Ampère product for any $\theta$-psh function and approximate it by the sequence of minimal singular $\theta$-psh (As in the proof of Proposition 1.20 in [BEGZ08]). □

The last representation shows that $\mathrm{Vol}_{X|Z}(L)$ is independent of $L$ in the same first Chern class. This result was already proved in [ELMNP09] algebraically.

**Corollary 4.4.** $\mathrm{Vol}_{X|Z}(L)$ *is determined only by* $Z \subseteq X$ *and* $c_1(L)$.

Further, these representations of the restricted volume do not need sections of $L$ hence we can extend the definition of restricted volumes to any class.

**Definition 4.5.** For any big class $\alpha \in H^{1,1}(X;\mathbb{R})$ and subvariety $Z \subseteq X$, we define the restricted volume as follows.

$$\begin{aligned}\mathrm{Vol}_{X|Z}(\alpha) &:= \int_{Z_{\mathrm{reg}}} \mathrm{MA}(P_{X|Z_{\mathrm{reg}}}\varphi) = \int_{Z_{\mathrm{reg}}} \mathrm{MA}((\iota|_{Z_{\mathrm{reg}}})^* P_X\varphi) \\ &= \int_{Z_{\mathrm{reg}}} \langle ((\iota|_{Z_{\mathrm{reg}}})^* T_{\min})^p \rangle = \sup_T \int_{Z_{\mathrm{reg}}} \langle ((\iota|_{Z_{\mathrm{reg}}})^* T)^p \rangle = \int_{X\setminus \mathbb{B}_+(\alpha)} (T_{\min})^p \wedge [Z],\end{aligned}$$

where T runs through all the closed positive currents in $\alpha$, with small unbounded loci not contained in $\iota(Z)$. □

For the definitions of the bigness and the augmented base locus for an arbitrary class, see [BEGZ08]. Note that the regularity of $P_X\varphi$ for a general class $\alpha$ is already shown in [BD09]. We will prove the regularity of $P_{X|Z}\varphi$ for the class $c_1(L)$ in section 5. But for a general $\alpha$, the corresponding regularity result seems unknown. The second identity in the above definition is true since it is easily seen that $P_{X|Z}\varphi$ has a small unbounded locus even in this case and the proof of Theorem 3.4 is still valid. The another identities can be proved totally the same as in the case $\alpha = c_1(L)$.

In the end of this subsection, we give the representation of restricted volumes via so-called moving intersection number. By definition, the moving intersection number counts the number of points where $Z$ and a general divisor $D \in |mL|$ intersects outside of the base locus. We denote it by $\langle (mL)^p, Z\rangle$. It is already known that $\mathrm{Vol}_{X|Z}(L) = \lim_{m\to\infty} m^{-p}\langle (mL)^p, Z\rangle$ (see [ELMNP09], Theorem 2.13). The refinement of this result is now obtained.

**Theorem 4.6.** *In the situation of Theorem 4.1,*

$$\begin{aligned}\mathrm{Vol}_{X|Z}(L) &= \lim_{m\to\infty} \int_Z \mathrm{MA}(u_m) \\ &= \lim_{m\to\infty} \frac{\langle (mL)^p, Z\rangle}{m^p} =: \|L^p.Z\|.\end{aligned}$$

*Proof.* The first identity is a direct consequence of Theorem 3.18. The second is easily seen by taking a log resolution of $|mL|$. In fact the second identity holds before taking limit. Notation in the third identity follows [ELMNP09]. □

## 5. $L^2$-EXTENSION THEOREM FROM A SUBVARIETY

In this section, we state the desired $L^2$-extension theorem for our purpose and give a proof.

Let us first fix notations. Given a holomorphic Hermitian vector bundle $E$ with a metric $h_E$ on a Kähler manifold $X$, we denote its Chern curvature tensor by $c(E)$. That is, $c(E) := \sqrt{-1}D^2$ where $D$ denotes the exterior covariant derivative associated to the Chern connection of $(E, h_E)$. $c(E)$ is an $E^* \otimes E$-valued real $(1,1)$-form and defines a Hermitian form on $T_{X,x} \otimes E_x$ $(x \in X)$ as follows:

$$H(t_1 \otimes e_1, t_2 \otimes e_2) := \big(c(E)(t_1, \sqrt{-1}t_2)e_1|e_2\big) \qquad \text{for } t_1, t_2 \in T_{X,x},\ e_1, e_2 \in E_x.$$

Here $(\ |\ )$ is defined by $h_E$. Recall that $c(E)$ is said to be semipositive in the sense of Nakano if $H$ is semipositive everywhere in $X$. And we denote it by $c(E) \geqslant_{\mathrm{Nak}} 0$. If a Kähler metric $\omega$ is fixed, $c(E)$ also defines a Hermitian form on $(\bigwedge^{p,q} T^*_{X,x}) \otimes E_x$ as follows:

$$\theta(\alpha,\beta) := \big([c(E),\Lambda]\alpha|\beta\big) \quad \text{for} \quad \alpha,\beta \in (\bigwedge\nolimits^{p,q} T^*_{X,x}) \otimes E_x \qquad (x \in X),$$

where $\Lambda$ denotes the formal adjoint operator of the multiplication by $\omega$. It is known that if $p = n$ and $c(E)$ is semipositive in the sense of Nakano, $\theta$ defines a semipositive Hermitian form. We will use the following norm:

$$|\alpha|^2_\theta = \inf \left\{ M \geqslant 0 \;\middle|\; \begin{array}{c} |(\alpha|\beta)|^2 \leqslant M \cdot \theta(\beta,\beta) \\ \text{for any} \;\; \beta \in (\bigwedge^{n,q} T^*_{X,x}) \otimes E_x \end{array} \right\} \in [0,+\infty]$$

for $\alpha \in (\bigwedge^{n,q} T^*_{X,x}) \otimes E_x$.

**Theorem 5.1.** *Let $Z$ be a $p$-dimensional submanifold of a $n$-dimensional Kähler manifold $X$ with its Kähler form $\omega$, $K$ a compact subset of $X$. Then there exist constants $N = N(Z,K) > 0$ and $C = C(Z,K) > 0$ such that the following holds.*

*Fix any complete Kähler open set $\Omega \subseteq X$ contained in $K$, a holomorphic vector bundle $E \to X$ with a smooth Hermitian metric $h_E$ whose Chern curvature satisfying*

$$c(E) \geqslant_{\mathrm{Nak}} N \cdot \mathrm{id}_E \qquad \text{on } \Omega,$$

*and $f \in H^0(Z\cap\Omega, \mathcal{O}(K_X \otimes E))$. Then we have a section $F \in H^0(\Omega, \mathcal{O}(K_X \otimes E))$ which satisfies $F|_{Z\cap\Omega} = f$ and*

$$\int_\Omega |F|^2_{h_E}\, dV_{\omega,X} \leqslant C \int_{Z\cap\Omega} |f|^2_{h_E}\, dV_{\omega,Z}.$$

Although the following proof of this theorem is almost the same as the proof of *"the Ohsawa-Takegoshi-Manivel $L^2$-extension theorem"* in [Dem00], we describe it for account of the proof of Theorem 1.4. The difference from [Dem00] is that we deal with arbitrary submanifolds and general vector bundles while we give up sharp estimates.

*Proof.* There exists some $G \in C^\infty(\Omega, K_X \otimes E)$ such that

$$G|_{Z\cap\Omega} = f, \qquad (\bar\partial G)|_{Z\cap\Omega} = 0.$$

Fix a smooth cut-off function $\rho : \mathbb{R} \to [0,1]$ satisfying

$$\rho(t) := \begin{cases} 1 & (t \leqslant \frac{1}{2}) \\ 0 & (t \geqslant 1) \end{cases} \qquad |\rho'| \leqslant 3.$$

Then we set as follows:

$$G_\varepsilon := \rho\left(\frac{e^\psi}{\varepsilon}\right) \cdot G$$

$$g_\varepsilon := \bar\partial G_\varepsilon = \underbrace{\left(1 + \frac{e^\psi}{\varepsilon}\right)\rho'\left(\frac{e^\psi}{\varepsilon}\right)\bar\partial\psi_\varepsilon \wedge G}_{g_\varepsilon^{(1)}} + \underbrace{\rho\left(\frac{e^\psi}{\varepsilon}\right)\bar\partial G}_{g_\varepsilon^{(2)}},$$

where

$$\psi_\varepsilon := \log(\varepsilon + e^{\psi}) \qquad \left( \Leftrightarrow 1 + \frac{e^{\psi}}{\varepsilon} = \frac{e^{\psi_\varepsilon}}{\varepsilon} \right)$$

$$\psi := \log \sum_\alpha \chi_\alpha^2 \sum_{i=p+1}^{n} |z_{\alpha,i}|^2, \qquad \varepsilon > 0.$$

Here we choose a locally finite system of local coordinates $\{z_{\alpha,1}, ..., z_{\alpha,n}\}_\alpha$ so that

$$Z \cap U_\alpha = \{z_{\alpha,p+1} = \cdots = z_{\alpha,n} = 0\}$$

hold and choose a smooth function $\chi_\alpha$ so that the following hold.

$$\operatorname{Supp} \chi_\alpha \subseteq U_\alpha, \quad \sum_\alpha \chi_\alpha^2 > 0, \quad \text{and} \quad \sum_\alpha \chi_\alpha^2 \sum_{i=p+1}^{n} |z_{\alpha,i}|^2 < e^{-1} \quad \text{in } X.$$

This $\psi$ satisfies the following condition (see [Dem82], Proposition 1.4).

(1) $\psi \in C^\infty(X \setminus Z) \cap L^1_{\mathrm{loc}}(X)$
$\psi < -1$ in $X$, $\psi \to -\infty$ around $Z$.

(2) $e^{-(n-p)\psi}$ is *not* integrable around any point of $Z$

(3) There exists a smooth real $(1,1)$-form $\gamma$ in $X$ such that $\sqrt{-1}\partial\bar{\partial}\psi \geqslant \gamma$ holds in $X \setminus Z$.

If the equation

$$\begin{cases} \bar{\partial} u_\varepsilon = \bar{\partial} G_\varepsilon \ \text{ in } \ \Omega \\ |u_\varepsilon|^2 e^{-(n-p)\psi} \ \text{ is locally integrable around } \ Z \end{cases}$$

has been solved, $u_\varepsilon = 0$ on $Z$ holds by the above condition hence the sequence $\{G_\varepsilon - u_\varepsilon\}_\varepsilon$ is expected to converge to what we want. This is our strategy.

To solve $\bar{\partial}$-equations, we quote the following from [Dem00].

**Theorem 5.2** (Ohsawa's modified $L^2$-estimate. [Dem00], Proposition 3.1). *Let $X$ be a complete Kähler manifold with a Kähler metric $\omega$ ($\omega$ may not be necessarily complete), $E \to X$ a holomorphic Hermitian vector bundle. Assume that there exist some smooth functions $a, b > 0$ and if we set*

$$c'(E) := a \cdot c(E) - \sqrt{-1}\partial\bar{\partial} a - \sqrt{-1} b^{-1} \partial a \wedge \bar{\partial} a$$

$$\theta'(\alpha, \beta) := \big([c'(E), \Lambda]\alpha | \beta\big) \ \text{ for } \ \alpha, \beta \in (\textstyle\bigwedge^{n,q} T^*_{X,x}) \otimes E_x \qquad (x \in X),$$

*it holds that*

$$\theta' \geqslant 0 \ \text{ on } \ (\textstyle\bigwedge^{n,q} T_{X,x}) \otimes E_x \qquad \text{for any } x \in X.$$

*Then we have the following.*

*For any $g \in L^2(X, (\bigwedge^{n,q} T^*_X) \otimes E)$ with $\bar{\partial} g = 0$ and*

$$\int_X |g|^2_{\theta'} \, dV_{\omega,X} < +\infty,$$

*there exists a section* $u \in L^2(X, (\bigwedge^{n,q-1} T_X^*) \otimes E)$ *with* $\bar\partial u = g$ *such that*

$$\int_X (a+b)^{-1} |u|^2 \, dV_{\omega,X} \leqslant 2 \int_X |g|^2_{\theta'} \, dV_{\omega,X}.$$

Let us go back to the proof of Theorem 5.1. First, we are going to compute

$$\theta'_\varepsilon := \left[a_\varepsilon(c(E) + (n-p)\sqrt{-1}\partial\bar\partial\psi) - \sqrt{-1}\partial\bar\partial a_\varepsilon - b_\varepsilon^{-1}\sqrt{-1}\partial a_\varepsilon \wedge \bar\partial a_\varepsilon, \Lambda\right]. \tag{5.3}$$

($a_\varepsilon, b_\varepsilon$ will be defined in the following. ) If we set

$$a_\varepsilon := \chi_\varepsilon(\psi_\varepsilon) > 0$$

for some smooth function $\chi_\varepsilon$, it can be computed as:

$$\begin{aligned} \partial a_\varepsilon &= \chi'_\varepsilon(\psi_\varepsilon)\partial\psi_\varepsilon, \\ \sqrt{-1}\partial\bar\partial a_\varepsilon &= \chi'_\varepsilon(\psi_\varepsilon)\sqrt{-1}\partial\bar\partial\psi_\varepsilon + \chi''_\varepsilon(\psi_\varepsilon)\sqrt{-1}\partial\psi_\varepsilon \wedge \bar\partial\psi_\varepsilon \\ &= \chi'_\varepsilon(\psi_\varepsilon)\sqrt{-1}\partial\bar\partial\psi_\varepsilon + \frac{\chi''_\varepsilon(\psi_\varepsilon)}{\chi'_\varepsilon(\psi_\varepsilon)^2}\sqrt{-1}\partial a_\varepsilon \wedge \bar\partial a_\varepsilon \end{aligned}$$

so comparing with (5.3), it is natural to set

$$b_\varepsilon := -\frac{\chi'_\varepsilon(\psi_\varepsilon)^2}{\chi''_\varepsilon(\psi_\varepsilon)} \quad (>0).$$

And we finally define

$$\chi_\varepsilon(t) := \varepsilon - t + \log(1-t).$$

Then for sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned} &a_\varepsilon \geqslant \varepsilon - \log(\varepsilon + e^{-1}) \geqslant 1 \\ &\sqrt{-1}\partial\bar\partial a_\varepsilon + b_\varepsilon^{-1}\sqrt{-1}\partial a_\varepsilon \wedge \bar\partial a_\varepsilon = \chi'_\varepsilon(\psi_\varepsilon)\sqrt{-1}\partial\bar\partial\psi_\varepsilon \leqslant -\sqrt{-1}\partial\bar\partial\psi_\varepsilon \end{aligned}$$

hence

$$\theta'_\varepsilon \geqslant \left[c(E) + (n-p)\sqrt{-1}\partial\bar\partial\psi + \sqrt{-1}\partial\bar\partial\psi_\varepsilon, \Lambda\right].$$

On the other hand, simple computations show:

$$\begin{aligned} \partial\psi_\varepsilon &= \frac{e^\psi}{\varepsilon + e^\psi}\partial\psi, \\ \sqrt{-1}\partial\bar\partial\psi_\varepsilon &= \frac{e^\psi}{\varepsilon + e^\psi}\sqrt{-1}\partial\bar\partial\psi + \frac{e^\psi}{\varepsilon + e^\psi}\sqrt{-1}\partial\psi \wedge \bar\partial\psi - \frac{e^{2\psi}}{(\varepsilon + e^\psi)^2}\sqrt{-1}\partial\psi \wedge \bar\partial\psi \\ &= \frac{e^\psi}{\varepsilon + e^\psi}\sqrt{-1}\partial\bar\partial\psi + \frac{\varepsilon}{e^\psi}\sqrt{-1}\partial\psi_\varepsilon \wedge \bar\partial\psi_\varepsilon. \end{aligned}$$

Therefore, by the compactness of $K$, there exists a constant $N(Z,K) > 0$ such that

$$c(E) \geqslant_{\mathrm{Nak}} N \cdot \mathrm{id_E} \qquad \text{on } \Omega \tag{5.4}$$

implies

$$\theta'_\varepsilon \geqslant 0 \text{ on } (\textstyle\bigwedge^{n,1} T^*_{X,x}) \otimes E_x \qquad \text{for all } x \in \Omega \tag{5.5}$$

and eigenvalues of $\theta'_\varepsilon$ are bounded from below by a positive constant (uniformly with respect to $\varepsilon$) near $Z \cup \Omega$.

Next we will estimate $\bar\partial\psi_\varepsilon$ by $|\cdot|_{\theta'_\varepsilon}$. Fix arbitrary $\alpha, \beta \in (\bigwedge^{n,1} T^*_{X,x}) \otimes E_x$. By definition,

$$|\bar\partial\psi_\varepsilon \wedge \alpha|^2_{\theta'_\varepsilon} = \inf\left\{ M \geqslant 0 \;\middle|\; \begin{array}{c} |(\bar\partial\psi_\varepsilon \wedge \alpha|\beta)|^2 \leqslant M \cdot ([c'_\varepsilon(E)\Lambda]\beta|\beta) \\ \text{for any } \beta \in (\bigwedge^{n,1} T^*_{X,x}) \otimes E_x \end{array} \right\}$$

so it is enough to estimate $|(\bar\partial\psi_\varepsilon \wedge \alpha|\beta)|^2$. This can be done as follows:

$$\begin{aligned} |(\bar\partial\psi_\varepsilon \wedge \alpha|\beta)|^2 &= |(\alpha|(\bar\partial\psi_\epsilon)^\sharp\beta)|^2 \\ &\leqslant |\alpha|^2 \cdot |(\bar\partial\psi_\epsilon)^\sharp\beta|^2 = |\alpha|^2 ((\bar\partial\psi_\varepsilon)(\bar\partial\psi_\varepsilon)^\sharp\beta|\beta) = |\alpha|^2 ([\sqrt{-1}\partial\psi_\epsilon \wedge \bar\partial\psi_\epsilon, \Lambda]\beta|\beta) \end{aligned}$$

by Shwartz' inequality ($\sharp$ denotes taking the formal adjoint of the multiplication operator), and the last term is bounded by

$$\begin{aligned} &\frac{e^\psi}{\varepsilon}|\alpha|^2 ([\sqrt{-1}\partial\bar\partial\psi_\varepsilon - \frac{e^\psi}{\varepsilon + e^\psi}\sqrt{-1}\partial\bar\partial\psi, \Lambda]\beta|\beta) \\ \leqslant\ &\frac{e^\psi}{\varepsilon}|\alpha|^2 ([c(E) + (n-p)\sqrt{-1}\partial\bar\partial\psi + \sqrt{-1}\partial\bar\partial\psi_\varepsilon, \Lambda]\beta|\beta) \\ \leqslant\ &\frac{e^\psi}{\varepsilon}|\alpha|^2 ([c'_\varepsilon(E), \Lambda]\beta|\beta). \end{aligned}$$

The last inequality is a consequence of (5.4). Thus we may get a desired estimate

$$|\bar\partial\psi_\varepsilon \wedge \alpha|^2_{\theta'_\varepsilon} \leqslant \frac{e^\psi}{\varepsilon}|\alpha|^2. \tag{5.6}$$

This time we estimate $g_\varepsilon = g_\varepsilon^{(1)} + g_\varepsilon^{(2)}$ .
By (5.6) and $\operatorname{Supp} g_\varepsilon^{(1)} \subseteq \{e^\psi < \varepsilon\}$, $g_\varepsilon^{(1)}$ can be estimated. Namely,

$$\int_{\Omega\setminus Z} |g_\varepsilon^{(1)}|^2_{\theta'_\varepsilon} e^{-(n-p)\psi} dV_{\omega,X} \leqslant 4\int_{\Omega\setminus Z} |G|^2 \rho'\left(\frac{e^\psi}{\varepsilon}\right)^2 e^{-(n-p)\psi} dV_{\omega,X}$$

holds. Since $e^\psi \sim \sum_{i=p+1}^n |z_{\alpha,i}|^2$ on $U_\alpha$, thanks to the compactness of $K$ we get:

$$\limsup_{\varepsilon\to 0} \int_{\Omega\setminus Z} |g_\varepsilon^{(1)}|^2_{\theta'_\varepsilon} e^{-(n-p)\psi} dV_{\omega,X} \leqslant C \int_{Z\cap\Omega} |f|^2 dV_{\omega,Z} < +\infty.$$

We can also estimate $g_\varepsilon^{(2)}$. Note that eigenvalues of $\theta'_\varepsilon$ are bounded below. Then we get

$$\int_{\Omega\setminus Z} |g_\varepsilon^{(2)}|^2_{\theta'_\varepsilon} e^{-(n-p)\psi} dV_{\omega,X} \leqslant O(\varepsilon) < +\infty$$

because we can see that $|g_\varepsilon^{(2)}|^2_{\theta'_\varepsilon} = O(e^\psi)$ holds in $\operatorname{Supp} g_\varepsilon^{(2)} \subseteq \{e^\psi < \varepsilon\}$, by $\bar\partial G|_{Z\cap\Omega} = 0$ (using the Taylor expansion).

Now we can apply the modified $L^2$-estimate for each $\varepsilon$ in $\Omega \setminus Z$. Note that $\Omega \setminus Z$ is a complete Kähler manifold (see [Dem82], Theorem 1.5). There exists a sequence $\{u_\varepsilon\} \subseteq L^2(\Omega, K_X \otimes E)$ such that

$$\int_{\Omega\setminus Z} (a_\varepsilon + b_\varepsilon)^{-1} |u_\varepsilon|^2 e^{-(n-p)\psi} dV_{\omega,X} \leqslant 2\int_{\Omega\setminus Z} |g_\varepsilon|^2_{\theta'_\varepsilon} e^{-(n-p)\psi} dV_{\omega,X} < +\infty$$

holds.

Let us estimate the left hand side of the inequality. It can be easily seen that

$$\begin{aligned}
\psi_\varepsilon &\leqslant \log(\varepsilon + e^{-1}) \leqslant -1 + O(\varepsilon) \\
a_\varepsilon &\leqslant (1+O(\varepsilon))\psi_\varepsilon^2 \\
b_\varepsilon &= (2-\psi_\varepsilon)^2 \leqslant (9+O(\varepsilon))\psi_\varepsilon^2 \\
a_\varepsilon + b_\varepsilon &\leqslant (10+O(\varepsilon))\psi_\varepsilon^2 \leqslant (10+O(\varepsilon))(-\log(\varepsilon+e^\psi))^2
\end{aligned}$$

and

$$\int_\Omega \frac{|G_\varepsilon|^2}{(\varepsilon+e^\psi)^{(n-p)}(-\log(\varepsilon+e^\psi))^2} dV_{\omega,X} \leqslant \frac{M}{(\log\varepsilon)^2}$$

hold. Therefore, if we set $F_\varepsilon := G_\varepsilon - u_\varepsilon$, it follows:

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_{\Omega\setminus Z} \frac{|F_\varepsilon|^2}{(\varepsilon+e^\psi)^{(n-p)}(-\log(\varepsilon+e^\psi))^2} dV_{\omega,X} \\
&\leqslant \limsup_{\varepsilon\to 0} \left( 22\int_{\Omega\setminus Z} |g_\varepsilon|^2_{\theta'_\varepsilon} e^{-(n-p)\psi} dV_{\omega,X} + \frac{2M}{(\log\varepsilon)^2} \right) \leqslant C\int_{Z\cap\Omega} |f|^2\, dV_{\omega,Z} < +\infty.
\end{aligned}$$

By construction, $\bar\partial F_\varepsilon = 0$ holds on $\Omega\setminus Z$ and in fact also in $\Omega$, thanks to the Riemann extension theorem.

Finally, Let $\varepsilon \searrow 0$. Then after taking a weakly convergent subsequence, we get a $F \in L^2(\Omega, K_X\otimes E)$ such that $\bar\partial F = 0$ in $\Omega$ and

$$\int_\Omega \frac{|F|^2}{e^{(n-p)\psi}(-\psi)^2} dV_{\omega,X} \leqslant C\int_{Z\cap\Omega} |f|^2\, dV_{\omega,Z}.$$

By the compactness of $K$, we get the conclusion. □

*Proof of Theorem 1.4.* Since $X$ is projective, we may take a global meromorphic section $\sigma$ of $L$ and may assume $\mathrm{Supp}(\mathrm{div}(\sigma)) \cap Z \subsetneq Z$. Fix a hypersurface $H \subseteq X$ such that $X\setminus H$ is Stein, $\mathrm{Supp}(\mathrm{div}(\sigma)) \subseteq H$, and $H\cap Z \subsetneq Z$ hold. Then $L|_{X\setminus H}$ is trivial so that we may identify $\varphi$ as a psh function on $X\setminus H$.

Let $\psi$ be a smooth exhaustive strictly-psh function in $X\setminus H$ and set $\Omega_k := \{\psi < k\}$. Since $X\setminus H$ is Stein, there exists a sequence $\varphi_k \in \mathrm{PSH}(\Omega_k)$ satisfying $\varphi_k \searrow \varphi$ (pointwise convergence in $\Omega_k$). Note that $\varphi_k$ does not loss positivity.

We apply Theorem 5.1 to $\Omega' := \Omega_k$ and $E' := K_X^{-1}\otimes E\otimes L$ for each $k$. Then by assumption there are sections $\widetilde{s}_k \in H^0(\Omega_k, \mathcal{O}(K_X\otimes E'))$ such that $\widetilde{s}_k|_{Z\cap\Omega_k} = s$ and

$$\begin{aligned}
\int_{\Omega_k} |\widetilde{s}_k|^2 e^{-\varphi_k} dV_{\omega,X} &\leqslant C\int_{Z\cap\Omega_k} |s|^2 e^{-\varphi_k} dV_{\omega,Z} \\
&\leqslant C\int_Z |s|^2 e^{-\varphi} dV_{\omega,Z}
\end{aligned} \tag{5.7}$$

for a constant $C$. If we fix $l\in\mathbb{N}$, there exists some constant $c(l) \leqslant e^{-\varphi_l}$ in $\Omega_l$ hence we have

$$c(l)\cdot \int_{\Omega_l} |\widetilde{s}_k|^2\, dV_{\omega,X} \leqslant C\int_Z |s|^2 e^{-\varphi} dV_{\omega,Z}.$$

Using the diagonal process, we may find a subsequence: $\widetilde{s}_{k(i)} \to \widetilde{s}$ (weakly $L^2$-convergent on $X$). By Lemma 5.8, $\bar{\partial}\widetilde{s}_{k(i)} = 0$ implies that this is actually the pointwise convergence so that $\widetilde{s}|_{Z\cap(X\setminus H)} = s$ holds. We can deduce

$$\int_{X\setminus H} |\widetilde{s}|^2 e^{-\varphi} dV_{\omega,X} \leqslant C \int_Z |s|^2 e^{-\varphi} dV_{\omega,Z}$$

by (5.7) and by the lower-semicontinuity of $L^2$-norm. $\widetilde{s}$ can be extended to $X$ by the Riemann extension theorem and thus we conclude the theorem. □

**Lemma 5.8.** *Let $f_k$, $f$ be holomorphic functions defined in a domain $\Omega \subseteq \mathbb{C}^n$. Assume that the sequence $\{f_k\}$ weakly $L^2$-converges to $f$. Then $\{f_k\}$ converges to $f$ pointwise in $\Omega$.*

*Proof.* Fix any point $x \in \Omega$. Taking $\chi \in C_0^\infty(\Omega)$ with $\chi \equiv 1$ near $x$, we have:

$$f_k(x) = \int_{\zeta\in\Omega} K_{\mathrm{BM}}^{n,0}(x,\zeta) \wedge \bar{\partial}\chi(\zeta) \wedge f_k(\zeta) \to \int_{\zeta\in\Omega} K_{\mathrm{BM}}^{n,0}(x,\zeta) \wedge \bar{\partial}\chi(\zeta) \wedge f(\zeta) = f(x)$$

by the Koppelman formula. Here $K_{\mathrm{BM}}^{p,q}$ denotes the $(p,q)$-part of the Bochner-Martinelli kernel. □

**Acknowledgments.** The author would like to express his gratitude to his advisor Professor Shigeharu Takayama for his warm encouragements, suggestions and reading the drafts. The author also would like to thank Professor Takeo Ohsawa, Professor Mihai Paun, and Professor Dano Kim for several helpful comments concerning the $L^2$-extension theorem (Theorem 1.4). He is indebted to Doctor Shin-ichi Matsumura for valuable discussions in the seminars. This research is supported by JSPS Research Fellowships for Young Scientists (22-6742).

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