Discretization and ultradiscretization of differential equations preserving characters of their solutions

和訳 微分方程式の解の特徴を保った離散化及び超離散化

松家　敬介

Chapter 1

Introduction

Differential equations are often used to describe various phenomena. For example, we use differential equation to describe the motion of the dynamics and solve them to determine the orbits. To derive differential equations, we take continuous limits, however it is not always necessary to use differential equations. We can use difference equations instead of differential equations. Hence, there are two ways to describe a phenomenon by differential equations and by difference equations. The author believes that phenomena which are described by differential equations can also be described by difference equations.

Discretization is a procedure to get difference equations from given differential equations. The difference equations change to the differential equations with limits of some parameters. Discretization is often used when one computes differential equations numerically. Properties of solutions for the difference equations might be different from those of solutions for the differential equations. If we discretize differential equations properly, some properties of differential equations are preserved. One of the aims of this thesis is to obtain the difference equations which have similar properties to those of given differential equations.

In this thesis, we study discretizations of the following partial differential equations:

$$\frac{\partial u}{\partial t} = D_u \Delta u + f(u), \tag{1.1}$$

$$\frac{\partial^2 v}{\partial t^2} = D_v \Delta v + g(v), \tag{1.2}$$

where $u := u(t, \vec{x})$, $v := v(t, \vec{x})$ $(t \geq 0,\ \vec{x} \in \mathbb{R}^d)$, $D_u,\ D_v > 0$, $f(u), g(v)$ are function of u, v and Δ is a d-dimensional Laplacian.

Now we provide one of the ways of discretization of (1.1) and (1.2). First, we consider discretizations of the following ordinary difference equations:

$$\frac{du}{dt} = f(u),$$

$$\frac{d^2 v}{dt^2} = g(v).$$

The following ordinary difference equations are discretizations of these ordinary differential equations.

$$u^{s+1} = f_{\mathrm{d}}(u^s, \delta) \ (s \in \mathbb{Z}_{\geq 0}),$$
$$v^{s+1} + v^{s-1} = g_{\mathrm{d}}(v^s, \delta) \ (s \in \mathbb{Z}_{\geq 0}),$$

where $\delta > 0$ and

$$f_{\mathrm{d}}(u, \delta) = u + \delta f(u) + o(\delta^2) \ (\delta \to 0),$$
$$g_{\mathrm{d}}(v, \delta) = 2v + \delta^2 g(v) + o(\delta^3) \ (\delta \to 0)$$

are held. If there exists the smooth functions $u(t), v(t)$ $(t \geq 0)$ that satisfies $u(\delta s) = u^s$, $v(\delta s) = v^s$, since the difference equations are formulated to

$$\frac{u(t+\delta) - u(t)}{\delta} = f(u(t)) + o(\delta) \ (\delta \to 0),$$
$$\frac{v(t+\delta) - 2v(t) + v(t-\delta)}{\delta^2} = g(v(t)) + o(\delta) \ (\delta \to 0),$$

these difference equations are discretization of the differential equations above.

Then, consider the following partial difference equations:

$$u_{\vec{n}}^{s+1} = f_{\mathrm{d}}(m(u_{\vec{n}}^s), \delta) \ (s \in \mathbb{Z}_{\geq 0}, \ \vec{n} \in \mathbb{Z}^d), \tag{1.3}$$

$$u_{\vec{n}}^{s+1} + u_{\vec{n}}^{s-1} = g_{\mathrm{d}}(m(u_{\vec{n}}^s), \delta) \ (s \in \mathbb{Z}_{\geq 0}, \ \vec{n} \in \mathbb{Z}^d), \tag{1.4}$$

where $u_{\vec{n}}^s := u(s, \vec{n})$, $v_{\vec{n}}^s := v(s, \vec{n})$ and

$$m(u_{\vec{n}}^s) := \sum_{k=1}^{d} \frac{u_{\vec{n}+\vec{e}_k}^s + u_{\vec{n}-\vec{e}_k}^s}{2d}$$

where $\vec{e}_k \in \mathbb{Z}^d$ is a unit vector whose k-th component is 1 and others are 0. (1.3) and (1.4) are discretization of (1.1) and (1.2).

Let $\xi_u := \sqrt{2dD_u\delta}$, $\xi_v := \sqrt{dD_v}\delta$ and there exists a smooth function $u(t, \vec{x})$ and $v(t, \vec{x})$ that satisfies $u_{\vec{n}}^s = u(\delta s, \xi_u \vec{n})$ and $v_{\vec{n}}^s = v(\delta s, \xi_v \vec{n})$. Using the assumptions, we obtain

$$\begin{aligned}
&\frac{u(t+\delta, \vec{x}) - u(t, \vec{x})}{\delta} \\
&\qquad = D_u \sum_{k=1}^{d} \frac{u(t, \vec{x} + \xi_u \vec{e}_k) - 2u(t, \vec{x}) + u(t, \vec{x} - \xi_u \vec{e}_k)}{\xi^2} \\
&\qquad\qquad + f(u(t, \vec{x})) + o(\delta) \ (\delta \to 0), \\
&\frac{v(t+\delta, \vec{x}) - 2v(t, \vec{x}) + v(t-\delta, \vec{x})}{\delta^2} \\
&\qquad = D_v \sum_{k=1}^{d} \frac{v(t, \vec{x} + \xi_v \vec{e}_k) - 2v(t, \vec{x}) + v(t, \vec{x} - \xi_v \vec{e}_k)}{\xi^2} \\
&\qquad\qquad + g(v(t, \vec{x})) + o(\delta) \ (\delta \to 0)
\end{aligned}$$

from (1.3) and (1.4). Taking a limit $\delta \to 0$, (1.1) and (1.2) are derived.

Ultradiscretization [20] is a limiting procedure transforming given difference equations into other difference equations which consists of addition, subtraction and maximum including cellular automata. Many soliton cellular automatons were constructed by this procedure. They preserve the essential properties of the original soliton equations, such as the structure of exact solutions [10, 13]. In this procedure, a dependent variable u_n in a given equation is replaced by

$$u_n = \exp\left(\frac{U_n}{\varepsilon}\right), \tag{1.5}$$

where ε is a positive parameter. Then, we apply $\varepsilon \log$ to both sides of (1.5) and take the limit $\varepsilon \to +0$. Using identity

$$\lim_{\varepsilon \to +0} \varepsilon \log \left(e^{U/\varepsilon} + e^{V/\varepsilon}\right) = \max\left(U, V\right)$$

and exponential laws, we find that multiplication, division and addition for the original variables are replaced by addition, subtraction and maximum for the new ones, respectively. In this way, the original difference equation is approximated by a piecewise linear equation.

Difference equations and ultradiscrete equations which is obtained from differential equations are often studied in the case of integrable equations such as soliton equations. Solutions for those equations inherit the similar properties of the solution for the differential equations. There are few cases that discretization and ultradiscretization whose solutions inherit similar properties of solutions for differential equations which is not integrable equations.

In this thesis, we investigate discretization of partial differential equations which have blow-up solutions in chapter 2. In section 1 of chapter 2, Dirichlet problem of the difference equation, which is a discretization of a semilinear heat equation is investigated. The difference equation in this section is proposed in [11]. In section 2 of chapter 2, we discretize a semilinear wave equation and prove that solutions for the difference equation have the similar properties of the solutions for the semilinear wave equation. In chapter 3, we investigate a discretization and a ultradiscretization of partial differential equations whose solutions reveal spatial patterns. In this chapter, we propose a discretization and a ultradiscretization of Gray-Scott model which is a two component reaction diffusion system and investigate solutions of the difference equation and the ultradiscrete equation. This work is joint work with Dr. Mikio Murata.

Chapter 2

Blow-up of solutions for difference equations

2.1 Discrete semilinear heat equation

2.1.1 Introduction

In this section, we consider the following partial difference equation with prescribed initial and boundary conditions:

$$\begin{cases} f_{\vec{n}}^{s+1} = \dfrac{g_{\vec{n}}^{s}}{\{1-\alpha\delta(g_{\vec{n}}^{s})^{\alpha}\}^{1/\alpha}} \ (s \in \mathbb{Z}_{\geq 0},\ \vec{n} \in \Omega_{\mathrm{D}}^{\circ}), \\ f_{\vec{n}}^{0} = a_{\vec{n}} \geq 0, \not\equiv 0 \ (\vec{n} \in \Omega_{\mathrm{D}}), \\ f_{\vec{n}}^{s} = 0 \ (s \in \mathbb{Z}_{\geq 0}\ \vec{n} \in \partial\Omega_{\mathrm{D}}), \end{cases} \tag{2.1}$$

Ω_{D} is a bounded subset of $\mathbb{Z}^d$, $\partial\Omega_{\mathrm{D}}$ is the boundary of Ω_{D}, $\Omega_{\mathrm{D}}^{\circ}$ is the interior of Ω_{D}, (namely $\Omega_{\mathrm{D}}^{\circ} := \Omega_{\mathrm{D}} \setminus \partial\Omega_{\mathrm{D}}$), $f_{\vec{n}}^{s} := f(s, \vec{n})$, $s \in \mathbb{Z}_{\geq 0}$, $\vec{n} \in \Omega_{\mathrm{D}}$. Moreover, we take $\alpha, \delta > 0$ and $g_{\vec{n}}^{s}$ define as:

$$g_{\vec{n}}^{s} := \sum_{k=1}^{d} \frac{f_{\vec{n}+\vec{e}_k}^{s} + f_{\vec{n}-\vec{e}_k}^{s}}{2d},$$

where $\vec{e}_k$ is the unit vector whose k-th component is 1 and the others are 0.

The difference equation in(2.1) is investigated [11] as a discretization of the following semi-linear heat equation:

$$\frac{\partial f}{\partial t} = \Delta f + f^{1+\alpha}, \tag{2.2}$$

where $f := f(t, \vec{x})$, $t \geq 0$, $\vec{x} \in \Omega_{\mathrm{C}} \subset \mathbb{R}^d$ and Δ is a d-dimensional Laplacian.

Solutions of (2.2) are not necessarily bounded for all $t \geq 0$. In general, if there exists a finite time $T > 0$ for which the solution of (2.2) in $(t, \vec{x}) \in [0, T) \times \Omega_{\mathrm{C}}$

satisfies

$$\limsup_{t \to T-0} \|f(t,\cdot)\|_{L^\infty} = \infty,$$

where

$$\|f(t,\cdot)\|_{L^\infty} := \sup_{\vec{x} \in \Omega_C} |f(t,\vec{x})|,$$

then we say that the solution of (2.2) blows up at time T.

The Cauchy problem for (2.2) has been studied and a critical exponent which characterises the blow-up of the solutions for (2.2) has been discovered and studied by Fujita and et al.[1, 6, 9, 21]
In fact, the difference equation (2.1) has similar characteristics to the critical exponent known from the continuous case.

Considering (2.2) on $[0,T) \times \Omega_C$ with the following initial and boundary conditions

$$\begin{cases} f(0,\vec{x}) = a(\vec{x}) \geq 0, \not\equiv 0 \ (\vec{x} \in \Omega_C), \\ f(t,\vec{x}) = 0 \ (t \geq 0,\ \vec{x} \in \partial\Omega_C), \end{cases} \tag{2.3}$$

where Ω_C is a bounded subset of $\mathbb{R}^d$, the following theorem can be shown to hold.

Theorem 2.1.1 *The solution of* (2.2) *with initial and boundary conditions* (2.3) *does not blow up at any finite time for sufficiently small initial conditions* $a(\vec{x})$.

In this section, we show that (2.1) has a property similar to theorem 1. In subsection 2.1.2, we define the blow-up of solutions for (2.1) and state the main theorem which is a discrete analogue of theorem 1. This theorem is proved in subsection 2.1.3.

2.1.2 Main theorem

First, we define the blow-up of solutions for (2.1). Because of the term $\{1 - \alpha\delta(g^s_{\vec{n}})\}^{1/\alpha}$, when $g^s_{\vec{n}} \to (\alpha\delta)^{-1/\alpha} - 0$, then $f^{s+1}_{\vec{n}} \to +\infty$. This behaviour may be regarded as an analogue of the blow up of solutions for the semi-linear heat equation. Thus we define a global solution of (2.1) as follows.

Definition 2.1.1 *Let* $f^s_{\vec{n}}$ *be a solution to* (2.1).

When there exists an $s_0 \in \mathbb{Z}_{\geq 0}$ *such that* $g^s_{\vec{n}} \leq (\alpha\delta)^{-1/\alpha}$ *for all* $s < s_0$ *and* $\vec{n} \in \Omega_D$, *and when there exists* $\vec{n}_0 \in \Omega_D$ *such that* $g^{s_0}_{\vec{n}_0} \geq (\alpha\delta)^{-1/\alpha}$, *then we say that the solution* $f^s_{\vec{n}}$ *blows up at time* s_0.

The following theorem is the main theorem of this section.

Theorem 2.1.2 *For* $\Omega_D = \{\vec{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d | 0 \leq n_k \leq N_k \ (k = 1, \cdots, d)\}$, *the solution of* (2.1) *does not blow up at any finite time with sufficiently small initial condition* $a_{\vec{n}}$.

2.1.3 Proof of the theorem

To prove the theorem, we make use of a comparison theorem.

First, to simplify the equations, we take the scaling $(\alpha\delta)^{1/\alpha} f^s_{\vec{n}} \to f^s_{\vec{n}}$ which changes the difference equation in (2.1) to

$$f^{s+1}_{\vec{n}} = \frac{g^s_{\vec{n}}}{\{1-(g^s_{\vec{n}})^\alpha\}^{1/\alpha}}$$

Now, we construct a majorant solution. Let

$$m(h_{\vec{n}}) := \frac{1}{2d}\sum_{k=1}^{d}(h_{\vec{n}+\vec{e}_k} + h_{\vec{n}-\vec{e}_k}). \tag{2.4}$$

We denote by $h^s_{\vec{n}}$ the solution to the initial and boundary condition problem of the linear partial difference equation

$$\begin{cases} h^{s+1}_{\vec{n}} = m(h^s_{\vec{n}}) \ (s \in \mathbb{Z}_{\geq 0},\ \vec{n} \in \Omega^\circ_{\mathrm{D}}) \\ h^0_{\vec{n}} = a_{\vec{n}} \ (\vec{n} \in \Omega_{\mathrm{D}}), \\ h^s_{\vec{n}} = 0 \ (s \in \mathbb{Z}_{\geq 0},\ \vec{n} \in \partial\Omega_{\mathrm{D}}). \end{cases} \tag{2.5}$$

The majorant solution is $\bar{f}^s_{\vec{n}}$ defined as follow:

$$\bar{f}^s_{\vec{n}} := \frac{h^s_{\vec{n}}}{\left\{1 - \sum_{k=0}^{s}|m_k|^\alpha\right\}^{1/\alpha}}, \tag{2.6}$$

where m_s is defined in terms of (2.5) as

$$m_s := \max_{\vec{n}\in\Omega^\circ_{\mathrm{D}}} h^s_{\vec{n}}. \tag{2.7}$$

Lemma 2.1.1 *When $\bar{f}^s_{\vec{n}}$ exists at s, for all $\vec{n} \in \mathbb{Z}^d$, namely when*

$$1 - \sum_{k=0}^{s}|m_k|^\alpha > 0$$

holds, the solution of (2.1) *does not blow up at any time s and moreover satisfies*

$$\bar{f}^s_{\vec{n}} \geq f^s_{\vec{n}}. \tag{2.8}$$

Proof *We precede by induction on s. When $s = 0$, by the definition of the initial and boundary condition problem, $f^0_{\vec{n}}$ exists and* (2.8) *holds because*

$$\bar{f}^0_{\vec{n}} = \frac{h^0_{\vec{n}}}{\{1-|m_0|^\alpha\}^{1/\alpha}} \geq h^0_{\vec{n}} = f^0_{\vec{n}}.$$

Suppose that the statement is true up to $s = s_0$ and that $\bar{f}_{\vec{n}}^{s_0+1}$ exists. When $\bar{f}_{\vec{n}}^{s_0+1} = 0$, we have that

$$
\begin{aligned}
\bar{f}_{\vec{n}}^{s_0+1} = 0 \quad &\Longleftrightarrow \quad h_{\vec{n}}^{s_0+1} = 0 \\
&\Longleftrightarrow \quad m(h_{\vec{n}}^{s_0}) = 0 \\
&\Longleftrightarrow \quad h_{\vec{n}\pm\vec{e}_k}^{s_0} = 0 \quad (k = 1, 2, \dots, d) \\
&\Longleftrightarrow \quad \bar{f}_{\vec{n}\pm\vec{e}_k}^{s_0} = 0 \quad (k = 1, 2, \dots, d) \\
&\Longrightarrow \quad f_{\vec{n}\pm\vec{e}_k}^{s_0} = 0 \quad (k = 1, 2, \dots, d) \\
&\Longleftrightarrow \quad g_{\vec{n}}^{s_0} = 0 \\
&\Longleftrightarrow \quad f_{\vec{n}}^{s_0+1} = 0.
\end{aligned}
$$

Hence (2.8) *holds.*

When $\bar{f}_{\vec{n}}^{s_0+1} > 0$, if $g_{\vec{n}}^{s_0} = 0$, then $f_{\vec{n}}^{s_0+1} = 0$ and the statement is true. Otherwise

$$
\begin{aligned}
0 < (\bar{f}_{\vec{n}}^{s_0+1})^{-\alpha} \quad &= \quad \frac{1 - \sum_{k=0}^{s_0+1} |m_k|^\alpha}{(h_{\vec{n}}^{s_0+1})^\alpha} = \frac{1 - \sum_{k=0}^{s_0} |m_k|^\alpha}{(h_{\vec{n}}^{s_0+1})^\alpha} - \left| \frac{m_{s_0+1}}{h_{\vec{n}}^{s_0+1}} \right|^\alpha \\
&\le \quad \frac{1 - \sum_{k=0}^{s_0} |m_k|^\alpha}{\left\{m(h_{\vec{n}}^{s_0})\right\}^\alpha} - 1 = \frac{1}{\left\{m(\bar{f}_{\vec{n}}^{s_0})\right\}^\alpha} - 1 \\
&\le \quad (g_{\vec{n}}^{s_0})^{-\alpha} - 1.
\end{aligned}
$$

From (2.4), *$(g_{\vec{n}}^{s_0})^{-\alpha} - 1 = (f_{\vec{n}}^{s_0+1})^{-\alpha}$ and we find $(\bar{f}_{\vec{n}}^{s_0+1})^{-\alpha} \le (f_{\vec{n}}^{s_0+1})^{-\alpha}$, i.e. $f_{\vec{n}}^{s_0+1} \le \bar{f}_{\vec{n}}^{s_0+1}$. Thus, from the induction hypothesis, the statement is true for any non-negative integer s.*

From this lemma, by proving that $1 - \sum_{k=0}^{s} |m_k|^\alpha > 0$ for all $s \in \mathbb{Z}_{\ge 0}$ with sufficiently small initial condition in (2.1), one can complete the proof of the main theorem.

The solution of (2.5) is

$$
h_{\vec{n}}^s = \sum_{\vec{n}' \in \Omega_{\mathrm{D}}^\circ} \left\{ B_{\vec{n}'} (c_{\vec{n}'})^s \prod_{k=1}^{d} \sin\left(\frac{n_k' \pi}{N_k} n_k \right) \right\},
$$

where $\vec{n} := (n_1, \cdots, n_d)$, $\vec{n}' := (n_1', \cdots, n_d')$, $c_{\vec{n}'} := \sum_{k=1}^{d} \frac{1}{d} \cos(n_k' \pi / N_k)$ and $B_{\vec{n}'}$ are constants that satisfy $h_{\vec{n}}^0 = a_{\vec{n}}$. The following proposition concerning $B_{\vec{n}}$ can be proven.

Proposition 2.1.1 *If the initial condition of* (2.5) *$a_{\vec{n}}$ is fixed, $B_{\vec{n}}$ are determined uniquely.*

Proof *This property is proved by induction on d.*

When $d=1$, put $N := N_1$. Solving $N-1$ linear equations with $N-1$ unknowns: $a_{n'} = \sum_{n=1}^{N-1} B_n \sin\left(\frac{n\pi}{N}n'\right)$ $(n' = 1, \cdots, N-1)$, the B_n are determined. If $N-1$ vectors $\left(\sin\frac{n\pi}{N}, \cdots, \sin\frac{n(N-1)\pi}{N}\right)$ $(n = 1, \cdots, N-1)$ are linearly independent, then the B_n are determined uniquely. On the other hand, these $N-1$ vectors are eigenvectors of the following $(N-1)\times(N-1)$ matrix:

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

All eigenvector are linearly independent so that the B_n are determined uniquely.

Suppose that the statement is true up to $d = d_0 - 1$. Now we consider the case of $d = d_0$.

$$\begin{aligned} a_{\vec{n}'} &= \sum_{\vec{n}\in\Omega_D^\circ} B_{\vec{n}} \prod_{k=1}^{d_0} \sin\left(\frac{n_k\pi}{N_k}n'_k\right) \\ &= \sum_{n_{d_0}=1}^{N_{d_0}-1} \sin\left(\frac{n_{d_0}\pi}{N_{d_0}}n'_{d_0}\right) \sum B_{\vec{n}} \prod_{k=1}^{d_0-1} \sin\left(\frac{n_k\pi}{N_k}n'_k\right) \end{aligned}$$

If $n_1, \cdots, n_{d_0-1}$ are fixed, then each $\sum B_{\vec{n}} \prod_{k=1}^{d_0-1} \sin\left(\frac{n_k\pi}{N_k}n_k\right)$ is determined uniquely from the case of $d=1$. Because of the induction hypothesis, the $B_{\vec{n}}$ are also determined uniquely. Thus, the statement is true for any d.

Now we estimate the infinite series $\sum_{k=0}^{\infty}|m_k|^\alpha$. Take $B := \max_{\vec{n}}|B_{\vec{n}}|$. If one lets $\max_{\vec{n}}|a_{\vec{n}}|$ be small, B also becomes small. We consider three cases $\alpha \le 1$, $\alpha > 1$.

When $\alpha \le 1$, we obtain

$$\begin{aligned} \sum_{k=0}^{\infty}|m_k|^\alpha &\le \sum_{k=0}^{\infty}\left(B\sum_{\vec{n}\in\Omega_D^\circ}|c_{\vec{n}}|^k\right)^\alpha \\ &\le \sum_{k=0}^{\infty} B^\alpha \sum_{\vec{n}\in\Omega_D^\circ}|c_{\vec{n}}|^{k\alpha} \\ &= B^\alpha \sum_{\vec{n}\in\Omega_D^\circ}\frac{1}{1-|c_{\vec{n}}|^\alpha} < \infty. \end{aligned}$$

We used the inequality $(x+y)^\alpha \le x^\alpha + y^\alpha$ $(x, y \ge 0)$ in the second line. The inequality above implies that $\sum_{k=0}^{\infty}|m_k|^\alpha$ can take an arbitrarily small value, if one lets the value of B small. Thus, $\sum_{k=0}^{\infty}|m_k|^\alpha < 1$ with sufficiently small initial condition in (2.5) and the statement of theorem 2 holds by lemma 3.1.

When $\alpha > 1$, since $|c_{\vec{n}}| < 1$ ($\vec{n} \in \Omega_{\mathrm{D}}^{\circ}$), $|c_{\vec{n}}|^{s} \to 0$ ($s \to \infty$) for all $\vec{n} \in \Omega_{\mathrm{D}}^{\circ}$. Thus, there exists $s_0 \in \mathbb{Z}_{\geq 0}$ such that $\sum_{\vec{n}\in\Omega_{\mathrm{D}}^{\circ}} |c_{\vec{n}}|^{s} < 1$ ($s \geq s_0$). Now we get

$$\begin{aligned}
\sum_{k=0}^{\infty} |m_k|^{\alpha} &= \sum_{k=0}^{s_0-1} |m_k|^{\alpha} + \sum_{k=s_0}^{\infty} |m_k|^{\alpha} \\
&\leq \sum_{k=0}^{s_0-1} |m_k|^{\alpha} + \sum_{k=s_0}^{\infty} B^{\alpha} \left(\sum_{\vec{n}\in\Omega_{\mathrm{D}}^{\circ}} |c_{\vec{n}}|^{k} \right)^{\alpha} \\
&\leq \sum_{k=0}^{s_0-1} |m_k|^{\alpha} + \sum_{k=s_0}^{\infty} B^{\alpha} \sum_{\vec{n}\in\Omega_{\mathrm{D}}^{\circ}} |c_{\vec{n}}|^{k} \\
&= \sum_{k=0}^{s_0-1} |m_k|^{\alpha} + \sum_{\vec{n}\in\Omega_{\mathrm{D}}^{\circ}} B^{\alpha} \frac{|c_{\vec{n}}|^{s_0}}{1-|c_{\vec{n}}|} < \infty.
\end{aligned}$$

$\sum\limits_{k=0}^{s_0-1} |m_k|^{\alpha}$ can take an arbitrarily small value, if one let the value of $\max_{\vec{n}\in\Omega_{\mathrm{D}}} a_{\vec{n}}$ be small so that the inequality above implies that $\sum_{k=0}^{\infty} |m_k|^{\alpha}$ can take an arbitrarily small value. (if B is sufficiently small.) Thus, $\sum_{k=0}^{\infty} |m_k|^{\alpha} < 1$ with sufficiently small initial condition in (2.5) and the statement of theorem 2.1.2 holds by lemma 2.1.1. This completes the proof of the main theorem.

2.2 Discrete semilinear wave equation

2.2.1 Introduction

Consider the Cauchy problem for the semilinear wave equation

$$\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} = \Delta u + |u|^{p} \ (p > 1), \\
u(0, \vec{x}) = f(\vec{x}), \\
\dfrac{\partial u}{\partial t}(0, \vec{x}) = g(\vec{x}),
\end{cases} \tag{2.9}$$

where $u := u(t, \vec{x})$ ($t \geq 0$, $\vec{x} := (x_1, \cdots, x_d) \in \mathbb{R}^d$) and Δ is the d-dimensional Laplacian $\Delta := \sum\limits_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$. When the initial conditions $f(\vec{x})$, $g(\vec{x})$ are continuous and unifomly bounded, there is a smooth solution for $t > 0$ whenever the solution is bounded. However, it is well known that the solutions of this problem are not necessarily bounded. For instance, this fact can be easily understood when one considers the spatially uniform initial condition, $f(\vec{x}) \equiv 0$, $g(\vec{x}) \equiv g > 0$.

In this case, $u(t,\vec{x}) = u(t)$ and (2.9) becomes an ordinary differential equation,

$$\begin{cases} \dfrac{d^2u}{dt^2} = |u|^p, \\ u(0) = 0, \\ \dfrac{du}{dt}(0) = g > 0. \end{cases} \tag{2.10}$$

Because of the initial condition, the solution of (2.10) is non negative if it is bounded, so that $|u|^p = u^p$ is obtained. Multiplying both sides of the equation by $\frac{du}{dt}$ and integrating from 0 to t, one obtains

$$\left(\frac{du}{dt}\right)^2 = \frac{2}{p+1}u^{p+1} + g^2.$$

Since $\frac{d^2u}{dt^2} \geq 0$ and $\frac{du}{dt}(0) = g > 0$, one has $\frac{du}{dt} \geq 0$ $(t \geq 0)$ and the differential inequality,

$$\frac{du}{dt} > \sqrt{\frac{2}{p+1}}u^{(p+1)/2} \tag{2.11}$$

follows. Since there exists a positive time ε such that $u(\varepsilon) > 0$, the solution of (2.11) is

$$u(t) > \frac{(\alpha C)^{-1/\alpha}}{\left\{(\alpha C)^{-1} u(\varepsilon)^{-\alpha} + \varepsilon - t\right\}^{1/\alpha}} \quad (t > \varepsilon)$$

where $\alpha = (p-1)/2$ and $C = \sqrt{2/(p+1)}$. As the right hand side diverges as $t \to \alpha^{-1}C^{-1}u(\varepsilon)^{-\alpha} + \varepsilon - 0$, the solution of (2.10) is clearly not bounded for all $t \geq 0$. In general, if there exists a finite time $T \in \mathbb{R}_{>0}$ and if the solution of (2.9) at $(t,\vec{x}) \in [0,T) \times \mathbb{R}^d$ satisfies

$$\limsup_{t \to T-0} \|u(t,\cdot)\|_{L^\infty} = \infty,$$

where

$$\|u(t,\cdot)\|_{L^\infty} := \sup_{\vec{x} \in \mathbb{R}^d} |u(t,\vec{x})|,$$

then we say that the solution of (2.9) blows up at time T. If such T does not exist for a solution of (2.9) then we call it a global solution.

The critical exponent $p_c(d) := \frac{d+1+\sqrt{d^2+10d-7}}{2(d-1)}$ $(d \geq 2)$ which characterises the blow up of the solutions for (2.9) was studied by many researchers. [2, 3, 4, 7, 18, 19, 23]. F. John[7] proved small data blow up for $1 < p < p_c(3)$ and small data global existence when $p_c(3) < p$. R.T. Glassey[3, 4] proved small data blow up for $1 < p < p_c(2)$ and small data global existence when $p_c(2) < p$. J. Schaeffer[18] proved small data blow up at $p = p_c(d)$ where $d = 2, 3$. T. Sideris[19] proved small data blow up when $1 < p < p_c(d)$ where $d \geq 4$. V. Georgiev, H. Lindblad and C. Sogge[2] proved small data global existence when $p_{c(d)} < p$ where $d \geq 4$. B. Yordanov and Q.S. Zhang[23] proved small data blow up for $p = p_{c(d)}$ where $d \geq 4$. Furthermore, Kato [8] proved the following theorem

Theorem 2.2.1 *Let u be a generalized solution of*

$$\frac{\partial^2 u}{\partial t^2} - \sum_{j,k=1}^{d} \frac{\partial}{\partial x_j} a_{jk}(t,\vec{x}) \frac{\partial}{\partial x_k} u - \sum_{j=1}^{d} \frac{\partial}{\partial x_j} a_j(t,\vec{x}) u = f(t,\vec{x},u)$$

$$(t \geq 0,\ \vec{x} \in \mathbb{R}^d)$$

on a time interval $0 \leq t < T \leq \infty$, which is supported on a forward cone

$$K_R = \{(t,\vec{x}); t \geq 0,\ |\vec{x}| \leq t + R\}\ (R > 0).$$

Assume that f satisfies

$$f(t,\vec{x},s) \geq \begin{cases} b|s|^{p_0} & (|s| \leq 1), \\ b|s|^p & (|s| \geq 1), \end{cases}$$

where $b > 0$ and $1 < p \leq p_0 = (d+1)/(d-1)$.
(If $d = 1$, p_0 may be any number greater than or equal to p.)
Moreover, assume that, for $w(t) = \int_{\mathbb{R}^d} u(t,\vec{x})d\vec{x}$, either (a) $\frac{dw}{dt}(0) > 0$, or (b) $\frac{dw}{dt}(0) = 0$ and $w(0) = 0$.
Then one must have $T < \infty$.

From this theorem, we obtain

Corollary 2.2.1 *Let u be the solution of* (2.9). *Assume that f and g in* (2.9) *satisfy $supp(f) \bigcup supp(g) \subset \{\vec{x} \in \mathbb{R}^d; |\vec{x}| \leq K\}$ $(K > 0)$ and $\int_{\mathbb{R}^d} g d\vec{x} > 0$. Moreover, assume $1 < p \leq (d+1)/(d-1)$ $(d \geq 2)$.*
(If $d = 1$, any assumption on p besides $p > 1$ is unnecessary.)
Then u blows up at some finite time.

In numerical computation of (2.9), one has to discretize it and consider a partial difference equation. A naive discretization would be to replace the t-differential and the Laplacian with central differences such that (2.9) turns into

$$\frac{u_{\vec{n}}^{s+1} - 2u_{\vec{n}}^{s} + u_{\vec{n}}^{s-1}}{\delta^2} = \sum_{k=1}^{d} \frac{u_{\vec{n}+\vec{e}_k}^{s} - 2u_{\vec{n}}^{s} + u_{\vec{n}-\vec{e}_k}^{s}}{\xi^2} + |u_{\vec{n}}^{s}|^p,$$

where $u(s,\vec{n})(=: u_{\vec{n}}^s) : \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d \to \mathbb{R}$, for positive constants δ and ξ, and where $\vec{e}_k \in \mathbb{Z}^d$ is the unit vector whose kth component is 1 and whose other components are 0. Putting $\lambda := \delta^2/\xi^2$, we obtain

$$u_{\vec{n}}^{s+1} = 2d\lambda m(u_{\vec{n}}^s) + (2 - 2d\lambda)u_{\vec{n}}^s - u_{\vec{n}}^{s-1} + \delta^2 |u_{\vec{n}}^s|^p \quad (p > 1). \tag{2.12}$$

Here

$$m(V_{\vec{n}}) := \frac{1}{2d} \sum_{k=1}^{d} (V_{\vec{n}+\vec{e}_k} + V_{\vec{n}-\vec{e}_k}). \tag{2.13}$$

For a spatially uniform initial condition, (2.12) becomes an ordinary difference equation

$$u^{s+1} = 2u^s - u^{s-1} + \delta|u^s|^p.$$

The above equation is a discretization of (2.10), but the features of its solutions are quite different. In fact, u^s will never blow up at finite time steps. Hence, (2.12) does not preserve the global nature of the original semilinear wave equation (2.9).

In this section, we propose and investigate a discrete analogue of (2.9) which does keep the characteristics of corollary 2.2.1.
In subsection 2.2.2, we present a partial difference equation with a parameter p whose continuous limit equals (2.9), and state the main theorem which shows that this difference equation has exactly the same properties as (2.9) with respect to p. This theorem is proved in subsection 2.2.3.

2.2.2 Discretization of the semilinear wave equation

We consider the following initial value problem for the partial difference equation

$$u_{\vec{n}}^{s+1} + u_{\vec{n}}^{s-1} = \frac{4v_{\vec{n}}^s}{2 - \delta^2 v_{\vec{n}}^s |v_{\vec{n}}^s|^{p-2}}, \quad (s \in \mathbb{Z}_{>0},\ \vec{n} \in \mathbb{Z}^d) \tag{2.14}$$

where $p > 1$ and $\delta > 0$ are parameters and $v_{\vec{n}}^s$ is defined by means of m (2.13) as

$$v_{\vec{n}}^s := m(u_{\vec{n}}^s).$$

If there exists a smooth function $u(t, \vec{x})$ $(t \in \mathbb{R}_{\geq 0},\ \vec{x} \in \mathbb{R}^d)$ that satisfies $u(s\delta, \xi\vec{n}) = u_{\vec{n}}^s$ with $\xi := \sqrt{d}\delta$, we find

$$u(t+\delta, \vec{x}) + u(t-\delta, \vec{x}) = v(t, \vec{x})(2 + \delta^2 v(t, \vec{x})|v(t, \vec{x})|^{p-2}) + o(\delta^4),$$

with

$$v(t, \vec{x}) := \frac{1}{2d}\sum_{k=1}^{d}(u(t, \vec{x} + \xi\vec{e}_k) + u(t, \vec{x} - \xi\vec{e}_k)),$$

or

$$\frac{u(t+\delta, \vec{x}) - 2u(t, \vec{x}) + u(t-\delta, \vec{x})}{\delta^2} = \sum_{k=1}^{d}\frac{u(t, \vec{x} + \xi\vec{e}_k) - 2u(t, \vec{x}) + u(t, \vec{x} - \xi\vec{e}_k)}{\xi^2} + |u(t, \vec{x})|^p + o(\delta^2).$$

Taking the limit $\delta \to +0$, we obtain the semilinear wave equation (2.9)

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + |u|^p.$$

Thus (2.14) can be regarded as a discrete analogue of (2.9).

Because of the term $2-\delta^2 v^s_{\vec{n}}|v^s_{\vec{n}}|^{p-2}$, if $v^s_{\vec{n}} \to (2\delta^{-2})^{1/(p-1)}$, then $u^{s+1}_{\vec{n}} \to +\infty$. This behaviour may be regarded as an analogue of th blow up of solutions for the semilinear wave equation. Thus we define a blow up of solution for (2.14) as follow.

Definition 2.2.1 *Let $u^s_{\vec{n}}$ be a solution of* (2.14).

When there exists $s_0 \in \mathbb{Z}_{\geq 0}$ such that $v^s_{\vec{n}} \leq (2\delta^{-2})^{1/(p-1)}$ for all $s < s_0$ and $\vec{n} \in \mathbb{Z}^d$, and there exists $\vec{n}_0 \in \mathbb{Z}^d$ such that $v^{s_0}_{\vec{n}_0} \geq (2\delta^{-2})^{1/(p-1)}$, then we say that the solution $u^s_{\vec{n}}$ blows up at time s_0.

An example of blow-up solutions for (2.14) is easily obtained. If one considers the spatially uniform initial condition $u^0_{\vec{n}} \equiv 0$, $u^1_{\vec{n}} \equiv g > 0$, $u^s_{\vec{n}} = u^s$ and (2.14) becomes an ordinary difference equation,

$$\begin{cases} u^{s+1} + u^{s-1} = \dfrac{4u^s}{2-\delta^2 u^s|u^s|^{p-2}}, \\ u^0 = 0, \\ u^1 = g > 0. \end{cases} \tag{2.15}$$

This is the discrete analogue of (2.10). One can see that the solution of (2.15) blows up in the following way:

If the solution of (2.15) does not blow up at any s,

i.e., $u^s < (2\delta^{-2})^{1/(p-1)}$ ($^\forall s \in \mathbb{Z}_{\geq 0}$), then it follows that

$$u^{s+1} - 2u^s + u^{s-1} = \frac{2\delta^2|u^s|^p}{2-\delta^2 u^s|u^s|^{p-2}} > 0$$

and here the difference inequality $u^{s+1} - 2u^s + u^{s-1} > 0$ holds. Solving this inequality for the above initial values, one obtains $u^s > gs$. This inequality means that u^s is arbitrarily large for large $s \in \mathbb{Z}_{>0}$. This statement contradicts $u^s < (2\delta^{-2})^{1/(p-1)}$ ($^\forall s \in \mathbb{Z}_{\geq 0}$) and hence one concludes that the solution of (2.15) blows up at some finite time.

Furthermore, (2.14) inherits quite similar properties to those of (2.9). The following theorem is the main result in this section.

Theorem 2.2.2 *Let $u^s_{\vec{n}}$ be the solution for* (2.14). *Assume that*

(A1) $\{\vec{n} \in \mathbb{Z}^d; u^j_{\vec{n}} \neq 0\} \subset \{\vec{n} \in \mathbb{Z}^d; \|\vec{n}\| \leq K\}$ $(j = 0, 1\ K > 0)$,

(A2) $\sum_{\vec{n}} u^1_{\vec{n}} > \sum_{\vec{n}} u^0_{\vec{n}}$,

where $\|\vec{n}\| := |n_1| + \cdots + |n_d|$ $(\vec{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d)$.
Moreover assume $1 < p \leq (d+1)/(d-1)$ $(d \geq 2)$.
(If $d = 1$, any assumption on p besides $p > 1$ is unnecessary.)
Then $u^s_{\vec{n}}$ blows up at some finite time.

Remark *The summations in (A2) may seem to be infinite series, but because of condition (A1) both summations are actually finite series.*
The author believes that (2.14) *preserves the characteristics of the critical exponent $p_{c(d)}$ known from the continuous case.*

2.2.3 Proof of the theorem

The idea of the proof is similar to that adopted by Kato [8].

First, to simplify the equations, one takes the scaling $(\delta^2/2)^{1/(p-1)}u^s_{\vec{n}} \to u^s_{\vec{n}}$ which changes (2.14) to

$$u^{s+1}_{\vec{n}} + u^{s-1}_{\vec{n}} = \frac{2v^s_{\vec{n}}}{1 - v^s_{\vec{n}}|v^s_{\vec{n}}|^{p-2}}. \tag{2.16}$$

One can deduce a contradiction by assuming that $u^s_{\vec{n}}$ does not blow up at any finite time, i.e., $v^s_{\vec{n}} < 1$ $({}^\forall(s,\vec{n}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d)$.

Put

$$U^s := \sum_{\vec{n}} u^s_{\vec{n}}. \tag{2.17}$$

Because of (A1), $\{\vec{n} \in \mathbb{Z}^d; u^s_{\vec{n}} \neq 0\} \subset \{\vec{n} \in \mathbb{Z}^d; \|\vec{n}\| \leq K + s - 1\}$ such that the summation of (2.17) is well-defined. Moreover, from $\{\vec{n} \in \mathbb{Z}^d; v^s_{\vec{n}} \neq 0\} \subset \{\vec{n} \in \mathbb{Z}^d; \|\vec{n}\| \leq K + s\}$, $U^s = \sum_{\vec{n}} v^s_{\vec{n}}$ and $v^s_{\vec{n}} < 1$, it follows that

$$U^s < T^s, \tag{2.18}$$

where $T^s := \#\{\vec{n} \in \mathbb{Z}^d; \|\vec{n}\| \leq K + s\}$. From (2.16), one has

$$\sum_{\vec{n}} (u^{s+1}_{\vec{n}} - 2v^s_{\vec{n}} + u^{s-1}_{\vec{n}}) = \sum_{\vec{n}} \frac{2|v^s_{\vec{n}}|}{1 - v^s_{\vec{n}}|v^s_{\vec{n}}|^{p-2}}. \tag{2.19}$$

The left hand side of (2.19) is nothing but $U^{s+1} - 2U^s + U^{s-1}$ and since the right hand side is clearly nonnegative, one has $U^{s+1} - 2U^s + U^{s-1} \geq 0$. From this inequality, it follows that there exists some positive number C_0 which satisfies the inequality

$$U^s \geq C_0 s \tag{2.20}$$

for sufficiently large $s \in \mathbb{Z}_{>0}$.
Note that $U^s \geq 0$ for sufficiently large $s \in \mathbb{Z}_{>0}$.

The next lemma yields another inequality for U^s.

Lemma 2.2.1 *Put*

$$h(x) = \frac{2|x|^p}{1 - x|x|^{p-2}} \quad (x < 1).$$

Let $0 \leq x_0 < 1$, $x_{j-1} \leq x_j$ $(j = 1, \cdots, s)$ *and* $\lambda_j \geq 0$ $(j = 0, \cdots, s)$, $\lambda_0 + \cdots + \lambda_s = 1$.
If $\lambda_0 x_0 + \cdots + \lambda_s x_s \geq 0$ *then the inequality*

$$\lambda_0 h(x_0) + \cdots + \lambda_s h(x_s) \geq h(\lambda_0 x_0 + \cdots + \lambda_s x_s)$$

is satisfied.

Proof

$$\frac{\partial}{\partial x_0}(\lambda_0 h(x_0) + \cdots + \lambda_s h(x_s) - h(\lambda_0 x_0 + \cdots + \lambda_s x_s))$$
$$= \lambda_0(h'(x_0) - h(\lambda_0 x_0 + \cdots + \lambda_s x_s)), \qquad (2.21)$$

where $h'(x) := \dfrac{dh}{dx}(x)$.
Since $h(x)$ *is convex on* $[0,1)$, $h'(x)$ *increases monotonically on the interval* $[0,1)$. *On the other hand,* $0 \le \lambda_0 x_0 + \cdots + \lambda_s x_s \le x_0 < 1$ *from the definitions. Hence* (2.21) *is nonnegative and*

$$\begin{aligned} &\lambda_0 h(x_0) + \cdots + \lambda_s h(x_s) - h(\lambda_0 x_0 + \cdots + \lambda_s x_s) \\ \ge\ &\lambda_0 h(-(\lambda_1 x_1 + \cdots \lambda_s x_s)/\lambda_0) + \lambda_1 h(x_1) + \cdots + \lambda_s h(x_s) - h(0) \\ \ge\ &0 \end{aligned}$$

is obtained. ■

Since $\{\vec{n} \in \mathbb{Z}^d; v^s_{\vec{n}} \neq 0\} \subset \{\vec{n} \in \mathbb{Z}^d; \|\vec{n}\| \le K+s\}$ and $U^s = \sum_{\vec{n}} v^s_{\vec{n}}$ is nonnegative for sufficiently large $s \in \mathbb{Z}_{\ge 0}$, this lemma can be adopted to the right hand side of (2.19) in the following way:

$$\begin{aligned} \frac{2|v^s_{\vec{n}}|^p}{1 - v^s_{\vec{n}}|v^s_{\vec{n}}|^{p-2}} &\ge T^s \frac{2|\frac{1}{T^s}\sum_{\vec{n}} v^s_{\vec{n}}|^p}{1 - \frac{1}{T^s}\sum_{\vec{n}} v^s_{\vec{n}}|\frac{1}{T^s}\sum_{\vec{n}} v^s_{\vec{n}}|^{p-2}} \\ &= \frac{2(T^s)^{1-p}(U^s)^p}{1-(T^s)^{1-p}(U^s)^{p-1}}, \end{aligned}$$

for $\lambda_j = 1/T^s$ $(j = 1, \cdots, T^s)$.
Note that there always exists a positive number C_T which satisfies $T^s < C_T s^d$ for sufficiently large $s \in \mathbb{Z}_{>0}$. Thus, from $U^s < T^s$ it follows that

$$U^{s+1} - 2U^s + U^{s-1} \ge 2C_T^{1-p} s^{-d(p-1)} (U^s)^p$$

for sufficiently large $s \in \mathbb{Z}_{>0}$.
Since $1 < p \le (d+1)/(d-1)$, i.e., $-d(p-1) \ge -(p+1)$,

$$U^{s+1} - 2U^s + U^{s-1} \ge C_2 s^{-(p+1)} (U^s)^p, \qquad (2{,}22)$$

where $C_2 := 2C_T^{1-p}$.
Moreover, using (2.11), it also follows that

$$U^{s+1} - 2U^s + U^{s-1} \ge C_2 C_0^{1-p} s^{-1}$$

for sufficiently large $s \in \mathbb{Z}_{>0}$.
Solving this difference inequality one finds that U^s increases monotonically and that there exists some positive number C_1' which satisfies the inequality

$$U^s \ge C_1' s \log s, \qquad (2.23)$$

for sufficiently large $s \in \mathbb{Z}_{>0}$.
Next, consider:

$$E^s := (U^{s+1} - U^s)^2 - \frac{C_2}{p+1} s^{-(p+1)} (U^s)^{p+1}.$$

As U^s is monotonically increasing, (2.22) yields

$$\begin{aligned}
&E^{s+1} - E^s \\
&\quad = \{(U^{s+1} - U^s)^2 - (U^s - U^{s-1})^2\} \\
&\qquad\qquad - \frac{C_2}{p+1}\{s^{-(p+1)}(U^s)^{p+1} - (s-1)^{-(p+1)}(U^{s-1})^{p+1}\} \\
&\quad \geq 2s^{-(p+1)}(U^s)^p(U^{s+1} - U^{s-1}) - \frac{C_2}{p+1} s^{-(p+1)}\{(U^s)^{p+1} - (U^{s-1})^{p+1}\} \\
&\quad \geq C_2 s^{-(p+1)}(U^s)^{p+1}\left\{1 - \frac{U^{s-1}}{U^s} - \frac{1}{p+1} + \frac{1}{p+1}\left(\frac{U^{s-1}}{U^s}\right)^{p+1}\right\}.
\end{aligned}$$

Obviously, for $0 \leq \lambda \leq 1$, $\frac{1}{p+1}\lambda^{p+1} - \lambda + 1 - \frac{1}{p+1} > 0$ and hence $E^{s+1} - E^s > 0$ for sufficiently large $s \in \mathbb{Z}_{>0}$.
Since $U^s/s \geq C_1' \log s$ by (2.23), there exists some positive number C_3 which satisfies

$$(U^{s+1} - U^s)^2 \geq C_3 s^{-(p+1)}(U^s)^{p+1}$$

for sufficiently large $s \in \mathbb{Z}_{>0}$.
Because of (2.23), one has

$$\begin{aligned}
U^{s+1} - U^s &\geq C_3 \left(\frac{U^s}{s}\right)^{(p-1)/2} \frac{U^s}{s} \\
&\geq C_3 C_1'^{(p-1)/2} (\log s)^{(p-1)/2} \frac{U^s}{s},
\end{aligned}$$

for sufficiently large $s \in \mathbb{Z}_{>0}$.
Since $(\log s)^{(p-1)/2}$ is arbitrarily large for large $s \in \mathbb{Z}_{>0}$, the following linear difference inequality holds

$$U^{s+1} - U^s \geq C\frac{U^s}{s}$$

for any positive number C and $s \geq {}^{\exists}s_0$, where s_0 depends on C.
Solving this difference inequality, one has

$$U^s \geq \prod_{s=s_0}^{s-1} \frac{s+C}{s} U^{s_0} \ (s \geq s_0 + 1)$$

and if $C > d+1$, then

$$U^s \geq U^{s_0} \prod_{k=0}^{d} \frac{s+k}{s_0+k} \ (s \geq s_0 + 1).$$

This inequality means that there exists some positive number C' which satisfies the inequality $U^s \geq C' s^{d+1}$ for sufficiently large $s \in \mathbb{Z}_{>0}$ but this statement contradicts the assumption $U^s < T^s$.
This concludes the proof of the theorem.

Chapter 3

Spatial patterns of solutions for difference and ultradiscrete equations

3.1 Introduction

Gray-Scott model[5] is a variant of the autocatalytic model. Basically it considers the reactions

$$\begin{cases} \mathrm{U} + 2\mathrm{V} \to 3\mathrm{V}, \\ \mathrm{V} \to \mathrm{P}, \end{cases}$$

in an open flow reactor where U is continuously supplied, and the product P removed.

A mathematical model of the reactions bellow is the following system of partial differential equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - uv^2 + a(1-u), \\ \dfrac{\partial v}{\partial t} = D_v \Delta v + uv^2 - bv, \end{cases} \tag{3.1}$$

where $u := u(t, \vec{x})$, $v := v(t, \vec{x})$, $t \geq 0$, $\vec{x} \in \mathbb{R}^d$ and D_v, a and b are positive constants. Δ is d-dimensional Laplacian. The solutions of this system represent spatial patterns. Changing not only an initial condition but also parameters, various patterns are observed[12, 16, 17].

Considering (3.1) with a spatially uniform initial condition, we get

$$\begin{cases} \dfrac{du}{dt} = -uv^2 + a(1-u), \\ \dfrac{dv}{dt} = uv^2 - bv. \end{cases} \tag{3.2}$$

Solving simultaneous equations, we get equilibrium points of (3.2) as follow:

$$\begin{aligned} P_{c,0} : (u_{c,0}, v_{c,0}) &= (1,0), \\ P_{c,\pm} : (u_{c,\pm}, v_{c,\pm}) &= \left(\frac{1}{2}\left(1 \mp \sqrt{1-\frac{4b^2}{a}}\right), \frac{a}{2b}\left(1 \pm \sqrt{1-\frac{4b^2}{a}}\right) \right). \end{aligned}$$

$P_{c,+}$ and $P_{c,-}$ emerge when $a - 4b^2 \geq 0$ is held. $P_{c,0}$ is asymptotically stable. $P_{c,-}$ is unstable. $P_{c,+}$ is asymptotically stable if the following inequality

$$b - \frac{a^2}{2b^2}\left(1 + \sqrt{1-\frac{4b^2}{a}}\right) < 0 \tag{3.3}$$

is held.

In numerical computation of (3.1), we have to discretize it and consider a system of partial difference equations. A naive discretization would be to replace t-differentials with forward differences and Laplacians with central differences such that (3.1) turns into

$$\begin{cases} \dfrac{u_{\vec{n}}^{s+1} - u_{\vec{n}}^{s}}{\delta} = \displaystyle\sum_{k=1}^{d} \frac{u_{\vec{n}+\vec{e}_k}^{s} - 2u_{\vec{n}}^{s} + u_{\vec{n}-\vec{e}_k}^{s}}{\xi^2} - u_{\vec{n}}^{s}(v_{\vec{n}}^{s})^2 + a(1 - u_{\vec{n}}^{s}), \\ \dfrac{v_{\vec{n}}^{s+1} - v_{\vec{n}}^{s}}{\delta} = D_v \displaystyle\sum_{k=1}^{d} \frac{v_{\vec{n}+\vec{e}_k}^{s} - 2v_{\vec{n}}^{s} + v_{\vec{n}-\vec{e}_k}^{s}}{\xi^2} + u_{\vec{n}}^{s}(v_{\vec{n}}^{s})^2 - bv_{\vec{n}}^{s}, \end{cases} \tag{3.4}$$

where $u(s,\vec{n})(=: u_{\vec{n}}^{s})$, $v(s,\vec{n})(=: v_{\vec{n}}^{s}) : \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d \to \mathbb{R}$, for positive constants δ and ξ, and where $\vec{e}_k \in \mathbb{Z}^d$ is a unit vector whose kth component is 1 and whose other components are 0.

Considering (3.4) with a spatial uniform initial condition, we get a system of difference equations. We get similar equilibrium points of (3.2) but the stability of the equilibrium point $(1,0)$ is different. If the parameter δ is sufficiently large, $(1,0)$ is unstable. This case is different from the case of (3.1).

Since there are subtractions in (3.4), we cannot ultradiscretize (3.4). Indeed, following limit

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{U/\varepsilon} - e^{V/\varepsilon})$$

does not always exist. We can transform (3.4) without subtractions and ultradiscretize the equations, but the obtained equations are not evolution equations. This situation is inconvenient to investigate if solutions represent spatial patterns.

In this chapter, we propose discretization of (3.1) which can be ultradiscretized and investigate solutions of discretization and ultradiscretization of (3.1).

In section 2, we present a system of partial difference equations whose continuous limit equals (3.1), and consider the solutions of the discretization. In section 3, we present the ultradiscretization of the system of partial difference equations treated in section 2 and consider the solutions of the ultradiscretization. Finally concluding remarks are given in Section 4.

3.2 Discrete Gray-Scott model

In this section, we discretize (3.1) and investigate solutions.

3.2.1 Discretization of Gray-Scott model

Since it is more convenient to consider the ultradiscretization, we take the scaling $w := v + 1$ which changes (3.1) to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - u(w-1)^2 + a(1-u), \\ \dfrac{\partial w}{\partial t} = D_v \Delta w + u(w-1)^2 - b(w-1). \end{cases} \tag{3.5}$$

First we consider the discretization of following system of ordinary differential equations:

$$\begin{cases} \dfrac{du}{dt} = -u(w-1)^2 + a(1-u), \\ \dfrac{dw}{dt} = u(w-1)^2 - b(w-1). \end{cases} \tag{3.6}$$

We consider the following system of difference equations:

$$\begin{cases} u^{s+1} = \dfrac{u^s + \delta(2u^s w^{s+1} + a)}{1 + \delta\{(w^{s+1})^2 + a + 1\}}, \\ w^{s+1} = \dfrac{w^s + \delta[u^s\{(w^s)^2 + 1\} + b]}{1 + \delta(2u^s + b)}, \end{cases} \tag{3.7}$$

where $s \in \mathbb{Z}_{\geq 0}$, $\delta > 0$. The method of discretization is same to that used in [14, 15]

If there exists smooth functions $u(t)$, $w(t)$ $(t \geq 0)$ that satisfy $u(\delta s) = u^s$, $w(\delta s) = w^s$, we find

$$\begin{cases} \dfrac{u(t+\delta) - u(t)}{\delta} = -u(t)w(t+\delta)^2 + a(1 - u(t)) + O(\delta), \\ \dfrac{w(t+\delta) - w(t)}{\delta} = u(t)w(t)^2 - b(w(t) - 1) + O(\delta). \end{cases}$$

Taking the limit $\delta \to +0$, we obtain the system of differential equations (3.6). Thus, (3.7) can be regarded as a discretization of (3.6). Using (3.6), we can construct a system of partial difference equations:

$$\begin{cases} u_{\vec{n}}^{s+1} = \dfrac{m_p(u_{\vec{n}}^s) + \delta(2m_p(u_{\vec{n}}^s)w_{\vec{n}}^{s+1} + a)}{1 + \delta\{(w_{\vec{n}}^{s+1})^2 + a + 1\}}, \\ w_{\vec{n}}^{s+1} = \dfrac{m_q(w_{\vec{n}}^s) + \delta\{m_p(u_{\vec{n}}^s)(m_q(w_{\vec{n}}^s)^2 + 1) + b\}}{1 + \delta(2m_p(u_{\vec{n}}^s) + b)}, \end{cases} \tag{3.8}$$

where $s \in \mathbb{Z}_{\geq 0}$, $\vec{n} \in \mathbb{Z}^d$ and

$$m_p(f_{\vec{n}}) := \sum_{k=1}^{d} \frac{f_{\vec{n}+p\vec{e}_k} + f_{\vec{n}-p\vec{e}_k}}{2d} \quad (p \in \mathbb{Z}_{>0}).$$

Since (3.8) is equivalent to

$$\begin{cases} \dfrac{u_{\vec{n}}^{s+1} - u_{\vec{n}}^{s}}{\delta} = \displaystyle\sum_{k=1}^{d} \dfrac{u_{\vec{n}+p\vec{e}_k}^{s} - 2u_{\vec{n}}^{s} + u_{\vec{n}-p\vec{e}_k}^{s}}{(p\xi)^2} - m_p(u_{\vec{n}}^{s})(w_{\vec{n}}^{s+1})^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad + a(1 - m_p(u_{\vec{n}}^{s})) + o(\delta), \\ \dfrac{w_{\vec{n}}^{s+1} - w_{\vec{n}}^{s}}{\delta} = \dfrac{q^2}{p^2} \displaystyle\sum_{k=1}^{d} \dfrac{w_{\vec{n}+q\vec{e}_k}^{s} - 2w_{\vec{n}}^{s} + w_{\vec{n}-q\vec{e}_k}^{s}}{(q\xi)^2} + m_p(u_{\vec{n}}^{s}) m_q(w_{\vec{n}}^{s})^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad - b(m_q(w_{\vec{n}}^{s}) - 1) + o(\delta), \end{cases}$$

where $\xi := \sqrt{2d\delta}/p$, if there exists smooth functions $u(t, \vec{x})$, $w(t, \vec{x})$ ($t \geq 0$, $\vec{x} \in \mathbb{R}^d$) that satisfy $u(\delta s, \xi\vec{n}) = u_{\vec{n}}^{s}$ and $w(\delta s, \xi\vec{n}) = w_{\vec{n}}^{s}$, we obtain (3.5) where $D_v = (q/p)^2$ with the limit $\delta \to 0$. Solving simultaneous equations, we get equilibrium points of (3.7):

$$\begin{aligned} P_{d,0} : (u_{d,0}, w_{d,0}) &= (1, 1), \\ P_{d,\pm} : (u_{d,\pm}, w_{d,\pm}) &= \left(\frac{1}{2}\left(1 \mp \sqrt{1 - \frac{4b^2}{a}}\right), 1 + \frac{a}{2b}\left(1 \pm \sqrt{1 - \frac{4b^2}{a}}\right) \right). \end{aligned}$$

$P_{d,0}$ is asymptotically stable. $P_{d,-}$ is unstable. $P_{d,+}$ is asymptotically stable if the following inequality

$$b - \frac{a^2}{2b^2}\left(1 + \sqrt{1 - \frac{4b^2}{a}}\right) + \delta\left[(2b - a)\left\{1 + \frac{a}{2b}\left(1 + \sqrt{1 - \frac{4b^2}{a}}\right)\right\} - a\right] < 0$$

is held. The coefficient of δ^0 in left hand side is same to the left hand side of (3.3). These equilibrium points are same as those of continuous case (3.2) regardless of the value of δ.

3.2.2 Solutions for the discrete Gray-Scott model

Now, let $d = 1,\ p = 3,\ q = 1,\ \delta = 0.1$ and

$$u_n^0 = \begin{cases} 1 - 0.3\cos\left(\dfrac{\pi n}{50}\right) & |n| \leq 25, \\ 1 & |n| > 25, \end{cases}$$

$$w_n^0 = \begin{cases} 1 + 0.5\cos\left(\dfrac{\pi n}{50}\right) & |n| \leq 25, \\ 1 & |n| > 25. \end{cases}$$

If one plots the solutions of (3.8) with a periodic boundary condition, following patterns are observed. The horizontal axis is for space variable n. The vertical axis is for time variable s. The height means the value of w_n^s.

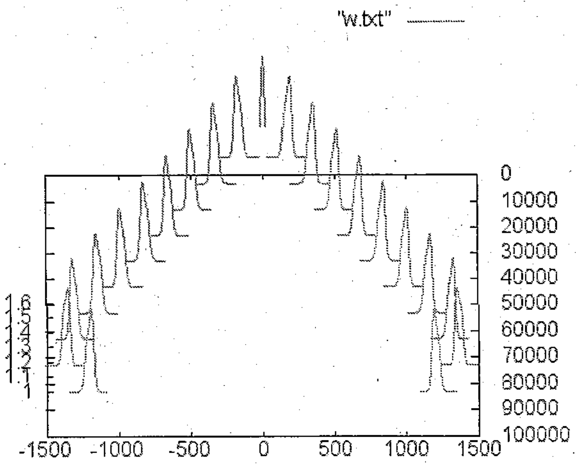


Figure 3.1: $a = 0.03,\ b = 0.10$

In Figure 3.1, a peak split into two peaks and two peaks move opposite side. We took a periodic boundary condition so that it is observed that two peaks pass each other.

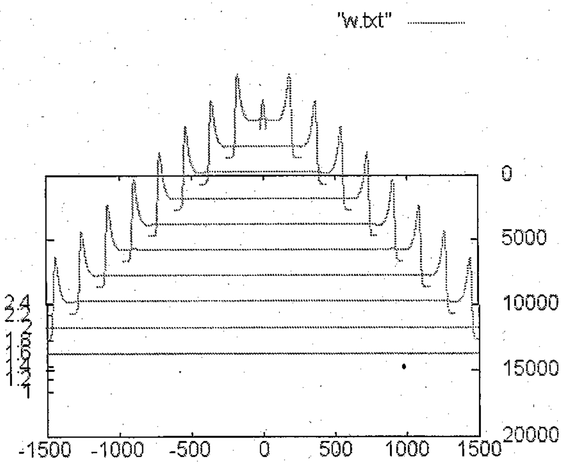


Figure 3.2: $a = 0.04,\ b = 0.06$

In Figure 3.2, a similar situation of Figure 3.1 is observed. Between two peaks, values of (u, w) converge to the stable equilibrium point $P_{d,+}$. Moreover, two peaks vanish, when they collide.

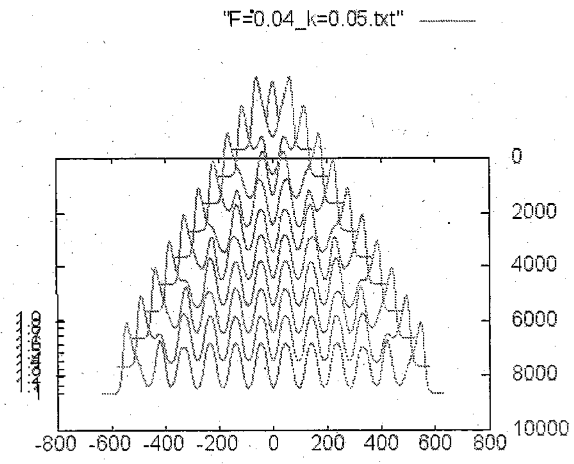


Figure 3.3: $a = 0.04,\ b = 0.09$

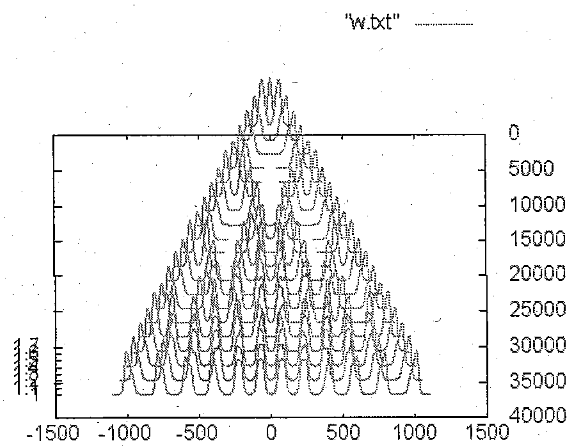


Figure 3.4: $a = 0.04,\ b = 0.11$

In Figure 3.3 and 3.4, a peak split into a two peaks several times and a self-replicating pattern is observed.

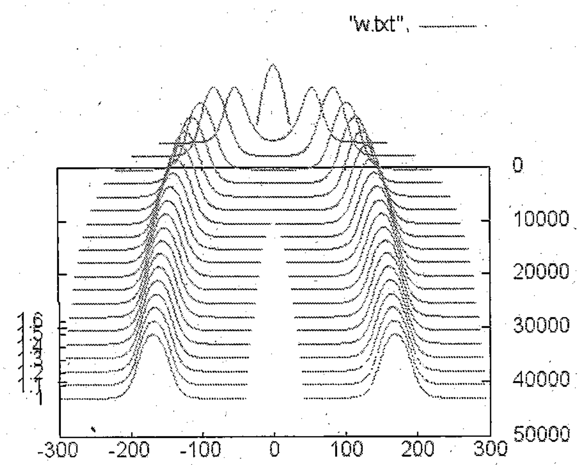


Figure 3.5: $a = 0.02,\ b = 0.09$

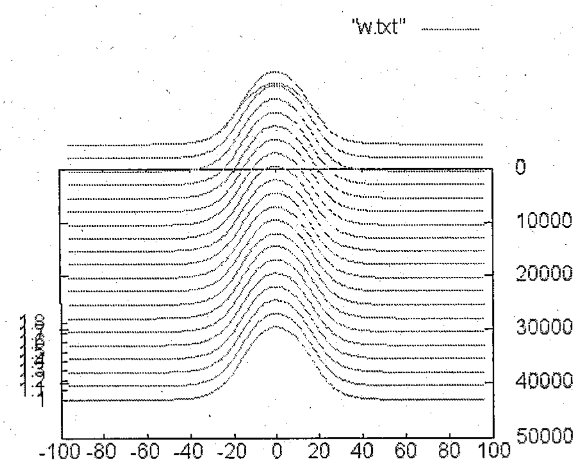


Figure 3.6: $a = 0.08,\ b = 0.17$

In Figure 3.5 and Figure 3.6, two types of steady state is observed.

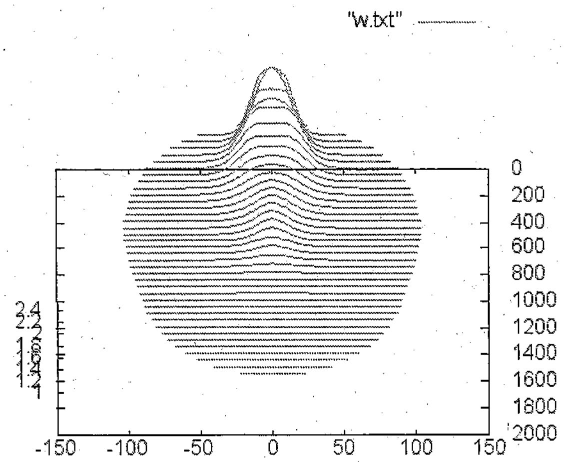


Figure 3.7: $a = 0.05,\ b = 0.15$

In Figure 3.7, values of (u, w) converge to the stable equilibrium point $P_{d,0}$.

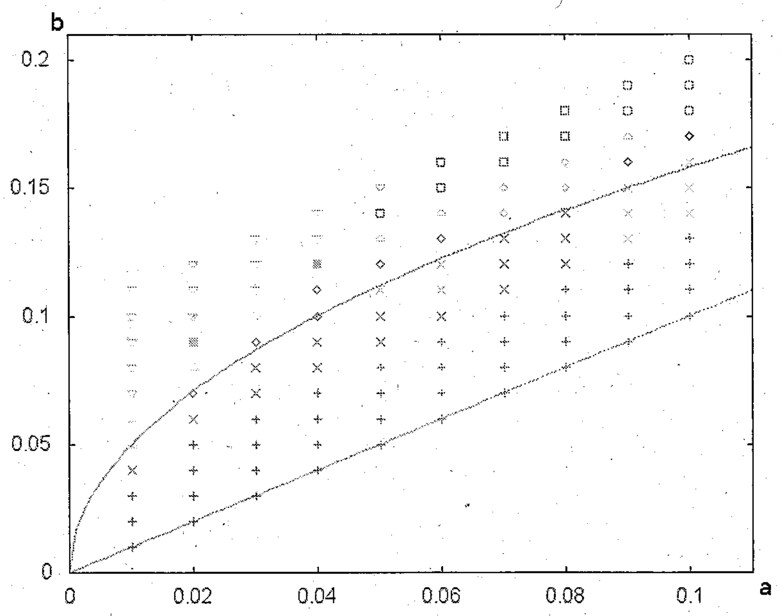


Figure 3.8:

Figure 3.8 reveals that which patterns are seen with the parameters (a, b). The horizontal axis is for a and the vertical axis for b. The upper curve is $a = 4b^2$ and the lower line is $b = a$. $\triangle$: Figure 3.1; $+$: Figure 3.2; $\times$: Figure 3.3;$\diamond$: Figure 3.4; $\blacksquare$: Figure 3.5; $\square$: Figure 3.6; ∇: Figure 3.7.

3.3 Ultradiscrete Gray-Scott model

In this section, we ultradiscretize (3.8) and investigate the solutions.
Let

$$u_{\vec{n}}^s = \exp\left(\frac{U_{\vec{n}}^s}{\varepsilon}\right), \; w_{\vec{n}}^s = \exp\left(\frac{W_{\vec{n}}^s}{\varepsilon}\right),$$
$$\delta = \exp\left(\frac{D}{\varepsilon}\right), \; a = \exp\left(\frac{A}{\varepsilon}\right), \; b = \exp\left(\frac{B}{\varepsilon}\right)$$

and take the limit $\varepsilon \to 0$, then we have

$$\begin{cases} U_{\vec{n}}^{s+1} = \max\left[M_p(U_{\vec{n}}^s), D + \max\left[M_p(U_{\vec{n}}^s) + W_{\vec{n}}^{s+1}, A\right]\right] \\ \qquad\qquad - \max\left[0, D + \max\left[2W_{\vec{n}}^{s+1}, A, 0\right]\right], \\ W_{\vec{n}}^{s+1} = \max\left[M_q(W_{\vec{n}}^s), D + \max\left[M_p(U_{\vec{n}}^s) + \max\left[2M_q(W_{\vec{n}}^s), 0\right], B\right]\right] \\ \qquad\qquad - \max\left[0, D + \max\left[M_p(U_{\vec{n}}^s), B\right]\right], \end{cases} \tag{3.9}$$

where

$$M_p(F_{\vec{n}}) := \max_{k=1,\dots,d}\left[F_{\vec{n}+p\vec{e}_k}, F_{\vec{n}-p\vec{e}_k}\right].$$

Taking a limit $D \to \infty$ and assuming $W_{\vec{n}}^s \geq 0$, then we get

$$\begin{cases} U_{\vec{n}}^{s+1} = \max\left[M_p(U_{\vec{n}}^s) + W_{\vec{n}}^{s+1}, A\right] - \max\left[2W_{\vec{n}}^{s+1}, A\right], \\ W_{\vec{n}}^{s+1} = \max\left[M_p(U_{\vec{n}}^s) + 2M_q(W_{\vec{n}}^s), B\right] - \max\left[M_p(U_{\vec{n}}^s), B\right]. \end{cases} \tag{3.10}$$

Let $d = 1$ and initial data of (3.10) $-U_n^0 \in \{0,1\}, W_n^0 \in \{0,1\}$. Taking some conditions to parameters A and B, the solution of (3.10) becomes to a cellular automaton. There are several types of conditions for A and B as follow:

Type I	Type II	Type III	Type IV	Type V
$A \leq -1$	$0 \leq A \leq 1$	$A \geq 2$	$A \leq -1$	$A \geq 0$
$B = 1$	$B = 1$	$B = 1$	$B \geq 2$	$B \geq 2$

Type I: The rule for $A \leq -1, B = 1$:

$-M_p(U_n^s), M_q(W_n^s)$	1,1	1,0	0,1	0,0
$-U_n^{s+1}, W_n^{s+1}$	1,0	1,0	1,1	0,0

In this case, moving pulses are observed. If two pulses collide, each pulse is disappeared.
Values of W_n^s is represent as follow: 0 (white) and 1 (black).

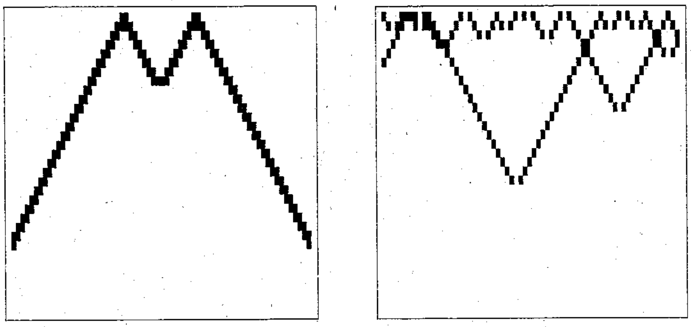


Figure 3.9: W_n^s with $A = -1$, $B = 1$ and $p = q = 1$.

Type II: The rule for $0 \leq A \leq 1, B = 1$:

$-M_p(U_n^s), M_q(W_n^s)$	1, 1	1, 0	0, 1	0, 0
$-U_n^{s+1}, W_n^{s+1}$	0, 0	0, 0	1, 1	0, 0

In this case, $U_n^{s+1} = -W_n^{s+1}$. Since this relation is held, W_n^s satisfies a single equation. Moreover, taking $p = q = 1$, the equation is same as ECA rule 90, which is well known for fractal design:

$W_{n+1}^s W_n^s W_{n-1}^s$	111	110	101	100	011	010	001	000
W_n^{s+1}	0	1	0	1	1	0	1	0

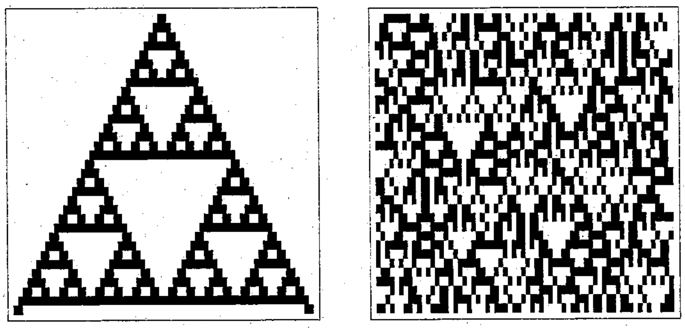


Figure 3.10: W_n^s with $A = 0$, $B = 1$ and $p = q = 1$.

Type III: The rule for $A \geq 2, B = 1$:

$-M_p(U_n^s), M_q(W_n^s)$	1, 1	1, 0	0, 1	0, 0
$-U_n^{s+1}, W_n^{s+1}$	0, 0	0, 0	0, 1	0, 0

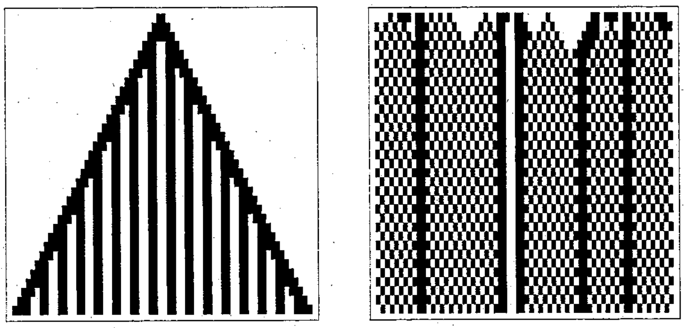


Figure 3.11: W_n^s with $A = 0$, $B = 1$, $p = 2$ and $q = 1$

In this case, $U_n^{s+1} = 0$ so that W_n^s satisfies $W_n^{s+1} = M_q(W_n^s)$.

Type IV: The rule of $A \leq -1, B \geq 2$:

$-M_p(U_n^s), M_q(W_n^s)$	1, 1	1, 0	0, 1	0, 0
$-U_n^{s+1}, W_n^{s+1}$	1, 0	1, 0	0, 0	0, 0

In this case, $W_n^{s+1} = 0$ so that U_n^s satisfies $U_n^{s+1} = M_p(U_n^s)$.

Type V: The rule of $A \geq 0, B \geq 2$:

$-M_p(U_n^s), M_q(W_n^s)$	1, 1	1, 0	0, 1	0, 0
$-U_n^{s+1}, W_n^{s+1}$	0, 0	0, 0	0, 0	0, 0

In this case, $U_n^{s+1} = W_n^{s+1} = 0$ so that U and W vanish immediately.

If one take $B \geq L$, $-U_n^s \in \{0, 1, \ldots, L\}$ and $W_n^s \in \{0, 1, \ldots, L\}$, the solution of (3.10) becomes to a cellular automaton whose dependent variable can have $L + 1$ values. The more L is large, the more the number of rule for evolution increases. In this case, the spatial pattern is also classified five types as follow:

Type I	Type II	Type III	Type IV	Type V
$A \leq -1$	$0 \leq A \leq 2L - 1$	$A \geq 2L$	$A \leq -1$	$A \geq 0$
$B = L$	$B = L$	$B = L$	$B \geq L + 1$	$B \geq L + 1$

In the case of type II, taking $L = 2$ and $p = q = 1$, a following Sierpinski gasket with shadow can be seen.
Values of W_n^s is represent as follow: 0 (white), 1 (gray) and 2 (black).

Moreover, taking $p = 2$, $q = 1$, we can see the following patterns.

Now, let $d = 2$. We also take similar condition to the initial condition of (3.10) in the case of $d = 1$: $W_{\bar{n}}^0 \in \{0, 1\}, -U_{\bar{n}}^0 \in \{0, 1\}$. We can separate spatial patterns to five types as similar to the case of $d = 1$. If $A \leq -1, B = 1$ then the rule of the evolution is as follow:

$-M_p(U_{\bar{n}}^s), M_q(W_{\bar{n}}^s)$	1, 1	1, 0	0, 1	0, 0
$-U_{\bar{n}}^{s+1}, W_{\bar{n}}^{s+1}$	1, 0	1, 0	1, 1	0, 0

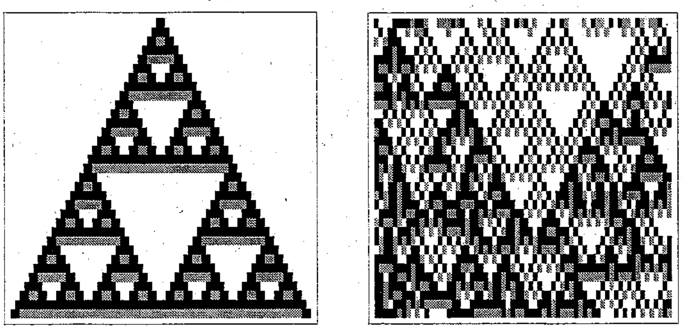


Figure 3.12: $W_n^s \in \{0,1,2\}$ with $A=3$, $B=2$ and $p=q=1$

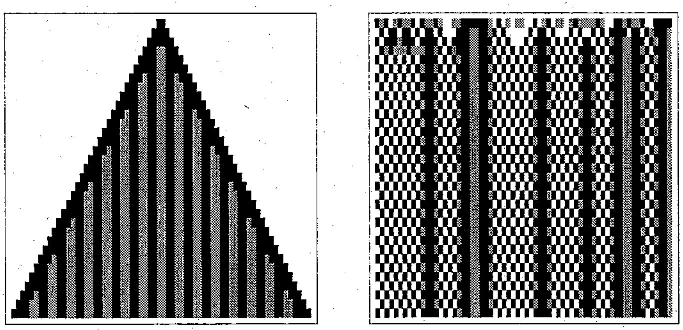


Figure 3.13: $W_n^s \in \{0,1,2\}$ with $A=3$, $B=2$, $p=2$ and $q=1$

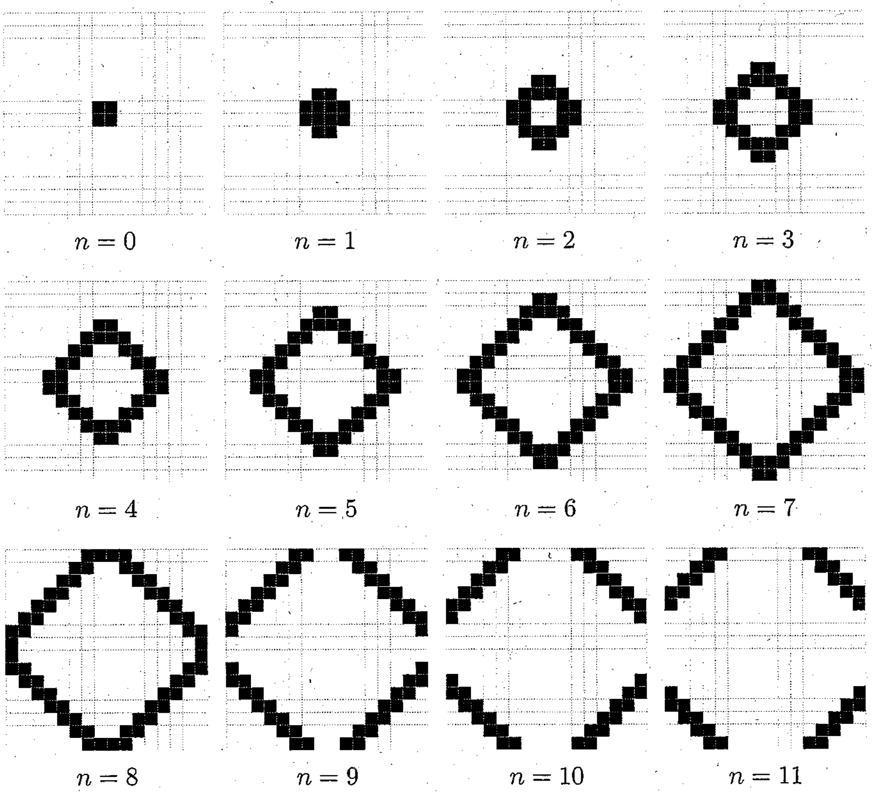


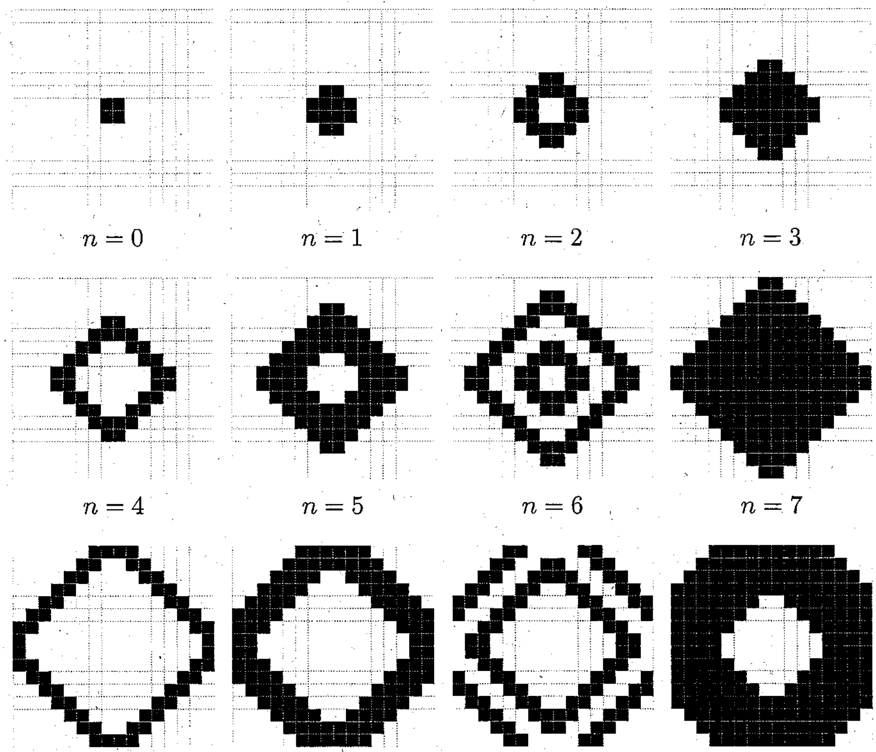
Figure 3.14: A ring pattern is spreading.

In this case, the following pattern is observed. Values of $W^s_{\vec{n}}$ is represent as follow: 0 (white) and 1 (black).

If $0 \leq A \leq 1, B = 1$, the rule of evolution is as follow:

$-M_p(U^s_{\vec{n}}), M_q(W^s_{\vec{n}})$	1,1	1,0	0,1	0,0
$-U^{s+1}_{\vec{n}}, W^{s+1}_{\vec{n}}$	0,0	0,0	1,1	0,0

In this case, we can see the following patterns:



$n = 0$ $n = 1$ $n = 2$ $n = 3$

$n = 4$ $n = 5$ $n = 6$ $n = 7$

$n = 8$ $n = 9$ $n = 10$ $n = 11$

Figure 3.15: Chaotic pattern. $W^s_{\vec{n}}$ with $p = q = 1$.

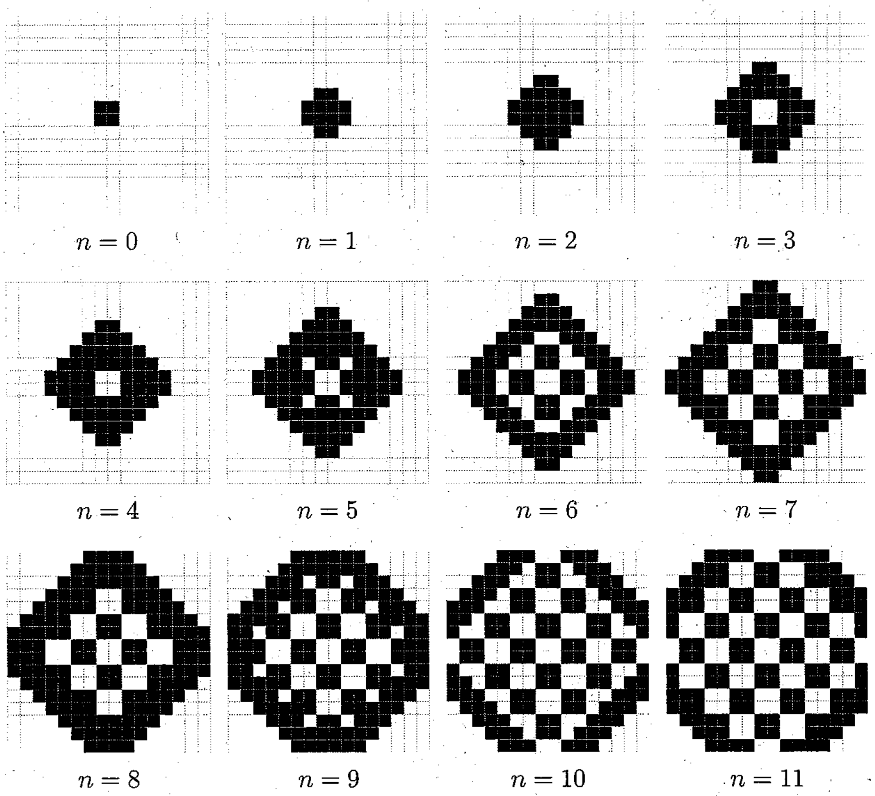


Figure 3.16: Self-replicating pattern. $W^s_{\vec{n}}$ with $p = 1$ and $q = 2$.

3.4 Concluding remarks

In this chapter we proposed and investigated discrete and ultradiscrete Gray-Scott model, which is a two component reaction diffusion system. We found that solutions of each equation reveal various spatial patterns. Moreover, there are solutions of the discrete equation and the ultradiscrete equation which correspond to each other. Indeed, the parameters (a, b) with which the spatial pattern Figure 3.3 is observed correspond to the parameters (A, B) with which the spatial pattern Figure 3.11 or Figure 3.13. The ultradiscrete equation we investigated has a solution which is an elementary cellular automaton and which reveals Sierpinski gasket. This is answer of the question "What is the correspondence between cellular automata and continuous systems?" in [22]. Discrete equations and ultradiscrete equations whose solutions inherit properties of differential equations are studied in case of integrable equations. We expect that more discretizations and ultradiscretizations which inherit properties of differential equations are studied and the various phenomena are made clear.

Chapter 4

Concluding remarks

In this thesis, we investigated discretizations and ultradiscretizations of differential equations preserving properties of solutions. Blow-up of the solutions and their spatial patterns are investigated. The author believes that discretizations and ultradiscretizations which preserves essential properties of differential equations do exist. In the future, the author would like to establish the general correspondence of differential equations, difference equations and ultradiscrete equations. It is expected that investigating one of the differential, difference and ultradiscrete equations may give information of other two equations.

Acknowledgements

The author is deeply grateful to Professor Tetsuji Tokihiro who provided helpful comments and suggestions. The author appreciates Dr. Mikio Murata with whom discussion inspired the author to complete this thesis. The author would also like to thank to Professor Ralph Willox and Dr. Shinsuke Iwao for useful comments and warm encouragements.

Bibliography

[1] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect. A Math.* **16**(1966), 109–124.

[2] V. Georgiev, H. Lindblad and C. D. Sogge, Weighted strichartz estimates and global existence for semilinear wave equations, *Amer. J. Math.* **119**(1997), 1291–1319.

[3] R. T. Glassey, Finite-time blow-up for solutions of nonlinear wave equations, *Math. Z.* **177**(1981), 323–340.

[4] R. T. Glassey, Existence in the large for $\Box u = F(u)$ in two space dimensions, *Math. Z.* **178**(1981), 233–261.

[5] P. Gray and S. K. Scott, Sustained oscillations and other exotic patterns of behaviour in isothermal reactions, *J. Phys. Chem.* **89**(1985), 22–32.

[6] K. Hayakawa, On nonexistence of global solutions of some semilinear parabolic equations, *Proc. Japan Acad.* **49**(1973), 503–505.

[7] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.* **28**(1980), 235–268.

[8] T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations, *Comm. Pure Appl. Math.* **32**(1980), 501–505.

[9] K. Kobayashi, T. Sirao and H. Tanaka, On the growing up problem for semilinear heat equations, *J. Math. Soc. Japan* **29**(1977), 407–424.

[10] J. Matsukidaira, J. Satsuma, D. Takahashi, T. Tokihiro and M. Torii, Toda-type cellular automaton and its N-soliton solution, *Phys. Lett. A* **225**(1997), 287–295.

[11] K. Matsuya and T. Tokihiro, Existence and non-existence of global solutions for a discrete semilinear heat equation, *Discrete Contin. Dyn. Syst.* **31**(2011), 209–220.

[12] W. Mazin, K. E. Rasmussen, E. Mosekilde, P. Borckmans and G. Dewel, Pattern formation in the bistable Gray-Scott model, *Math. Comput. Simul.* **40**(1996), 371–396.

[13] M. Murata, S. Isojima, A. Nobe and J. Satsuma, Exact solutions for discrete and ultradiscrete modified KdV equations and their relation to box-ball systems, *J. Phys. A* **39**(2006), L27–L34.

[14] M. Murata, J. Satsuma, A. Ramani and B. Grammaticos, How to discretize differential systems in a systematic way, *J. Phys. A* **43**(2010), 315203, 15pp.

[15] M. Murata, Tropical discretization: ultradiscrete Fisher-KPP equation and ultradiscrete Allen-Cahn equation, *J. Difference Equ. Appl.* available online Jul. 2012
DOI:10.1080/10236198.2012.705834.

[16] Y. Nishiura and D. Ueyama, A skeleton structure of self-replicating dynamics, *Phys. D* **130**(1999), 73–104.

[17] Y. Nishiura and D. Ueyama, Spatio-temporal chaos for the Gray-Scott model, *Phys. D* **150**(2001), 137–162.

[18] J. Schaeffer, The equation $\Box u = |u|^p$ for the critical value of p, *Proc. Roy. Soc. Edinburgh Sect. A* **101**(1985), 31–44.

[19] T. C. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions, *J. Differential Equations* **52**(1984), 378–406.

[20] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, From soliton equations to integrable cellular automata through a limiting procedure, *Phys. Rev. Lett.* **29**(1996), 3247–3250.

[21] F. B. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, *Israel J. Math.* **38**(1981), 29–40.

[22] S. Wolfram, Twenty Problems in the Theory of Cellular Automata, *Phys. Scripta* **1985**(1985), 170–183.

[23] B. T. Yordanov and Q. S. Zhang, Finite time blow up for critical wave equations in high dimensions, *J. Funct. Anal.* **231**(2006), 361–374.