

博 士 論 文

**MONOMIAL DEFORMATIONS OF CERTAIN HYPERSURFACES  
AND TWO HYPERGEOMETRIC FUNCTIONS**

和訳：ある種の超曲面の単項的変形と2種の超幾何関数

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# MONOMIAL DEFORMATIONS OF CERTAIN HYPERSURFACES AND TWO HYPERGEOMETRIC FUNCTIONS

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**ABSTRACT.** In this article, we give a certain class of one-parameter families of Calabi–Yau hypersurfaces of a projective space over a finite field that includes the Dwork family, and investigate the relationship of these families with hypergeometric series and with hypergeometric function over finite fields. The former is related via a formula for unit-root, and the latter is related via a factorization of the zeta function.

## 0. INTRODUCTION.

**0.1. Hypergeometric functions over finite fields.** As an analogy of the classical (generalized) hypergeometric function

$${}_{n+1}F_n \left( \begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix}; x \right) = \sum_{k=0}^{\infty} \frac{(A_1)_k \cdots (A_{n+1})_k}{(B_1)_k \cdots (B_n)_k (1)_k} x^k,$$

$$A_1, \dots, A_{n+1} \in \mathbb{C}, \quad B_1, \dots, B_n \in \mathbb{C} \setminus (-\mathbb{N}), \quad (C)_k := \frac{\Gamma(C+k)}{\Gamma(C)},$$

Greene [Gr] introduced a  $\overline{\mathbb{Q}}$ -valued function on  $\mathbb{F}_q$ , which he calls “Gaussian hypergeometric function”. Among several variants existing, we employ the following definition by McCarthy [MC]:

$${}_{n+1}F_n \left( \begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix}; x \right)_{\mathbb{F}_q} = \frac{1}{q-1} \sum_{\chi \in \mathbb{F}_q^\times} \prod_{i=1}^{n+1} \frac{G(A_i \chi)}{G(A_i)} \prod_{i=1}^n \frac{G(\overline{B_i} \chi)}{G(\overline{B_i})} G(\chi) \chi(-1)^{n+1} \chi(x) \quad (x \in \mathbb{F}_q^\times);$$

here,  $A_i$ ’s and  $B_i$ ’s are characters on  $\mathbb{F}_q^\times$  and  $G$  denotes the Gauss sum for a fixed non-trivial additive character (detailed explanation is given in Section 3). This function enjoys a lot of formulae, concerning transformations and summations, analogous to those for classical one. Another remarkable fact proven by Katz [Ka1], [Ka2] is that this function can be expressed as a trace function of a pure middle extension  $\overline{\mathbb{Q}_\ell}$ -sheaf on  $\mathbb{G}_m$ , which is smooth on  $\mathbb{G}_m \setminus \{1\}$ .

Some one-parameter families of smooth varieties have relations to usual hypergeometric functions with specific parameters; interestingly, some of these families are also related to hypergeometric functions over finite fields with “similar” parameters. Let us recall two examples of this phenomenon.

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**0.2. Example: Legendre family of elliptic curves.** The Legendre family of elliptic curves is defined by

$$E_\lambda: Y^2 Z = X(X-Z)(X-\lambda Z), \quad \lambda \neq 0, 1.$$

Firstly, this family over  $\mathbb{C}$  is related to the Gauss hypergeometric series

$$(0.2.1) \quad {}_2F_1 \left( \begin{matrix} 1/2, 1/2 \\ 1 \end{matrix}; \lambda \right);$$

in fact, the generator  $\omega = dx/y$  of  $\text{Fil}^1 \subset H_{\text{dR}}^1(E_\lambda)$  satisfies the differential equation  $\nabla(L)\omega = 0$ , where  $\nabla$  is the Gauss–Manin connection on  $H_{\text{dR}}^1(E_\lambda)$  and

$$L = \left( \theta - \frac{1}{2} \right)^2 - \theta \frac{d}{d\lambda}, \quad \theta = \lambda \frac{d}{d\lambda}.$$

Secondly, the Legendre family over  $\mathbb{F}_q$  with odd  $q = p^r$  is related to the hypergeometric series considered “ $p$ -adically”. In fact, let  $F(x)$  be the formal power series defined by (0.2.1) viewed as a series over  $W(\mathbb{F}_q)$ , let  $F_{p-1}(x)$  be the truncation of  $F(x)$  up to degree  $p-1$  and let  $f(x) := F(x)/F(x^p)$ . Then, if  $F_{p-1}(\lambda) \neq 0$  in  $\mathbb{F}_q$ , the elliptic curve  $E_\lambda$  is ordinary, and its unit root equals

$$(-1)^{(q-1)/2} \prod_{i=0}^{\log_p q - 1} f(\tilde{\lambda}^{p^i}).$$

The Legendre family over  $\mathbb{F}_q$  has a connection also with the hypergeometric function over  $\mathbb{F}_q$ ; the trace of the Frobenius map on the first étale cohomology  $H_{\text{ét}}^1(X_\lambda \times \overline{\mathbb{F}_q}, \mathbb{Q}_l)$  equals

$${}_2F_1 \left( \begin{matrix} \varphi_2, \varphi_2 \\ \epsilon \end{matrix}; \lambda \right)_{\mathbb{F}_q},$$

where  $\varphi_2$  denotes the quadratic character on  $\mathbb{F}_q^\times$  and  $\epsilon$  denotes the trivial character on  $\mathbb{F}_q^\times$  [Ko], [O].

**0.3. Example: Dwork family of Calabi–Yau varieties.** Let  $n$  be a natural number greater than or equal to 2. The Dwork family of Calabi–Yau  $(n-1)$ -folds is defined by

$$X_\lambda: T_1^{n+1} + \cdots + T_{n+1}^{n+1} - (n+1)\lambda T_1 \cdots T_{n+1} = 0, \quad \lambda^{n+1} \neq 0, 1.$$

in the projective space  $\mathbb{P}^n$ ; Dwork [Dw] investigated this family in detail in case  $n = 3, 4$ .

Firstly, this family over  $\mathbb{C}$  is related to the hypergeometric function

$${}_nF_{n-1} \left( \begin{matrix} 1/(n+1), 2/(n+1), \dots, n/(n+1) \\ 1, 1, \dots, 1 \end{matrix}; \lambda^{-(n+1)} \right),$$

via the fact that a differential form  $\omega$  on it satisfies the differential equation  $\nabla(L)\omega = 0$ , where  $\nabla$  is the Gauss–Manin connection on  $H_{\text{dR}}^1(X_\lambda)$  and where

$$L = \theta^n - \lambda^{-(n+1)} \prod_{i=1}^n \left( \theta + \frac{i}{n+1} \right), \quad \theta = \lambda^{-(n+1)} \frac{d}{d\lambda^{-(n+1)}}.$$

Secondly, the Dwork family over  $\mathbb{F}_q$  also is related to the hypergeometric series above viewed as a series over  $W(\mathbb{F}_q)$ ; this observation is made by Yu [Yu]. We omit the precise statement because it is similar to the statement for the Legendre family, except that ordinarity property should be replaced by the property that the first slope of Newton polygon of the crystalline cohomology at the middle degree is zero.

The relation to the hypergeometric function over finite fields is investigated by Katz and Goutet independently (although Goutet does not use the language of hypergeometric function over finite fields, his work is closely related to it). Assume that  $q \equiv 1 \pmod{n+1}$ . Then, Katz [Ka2] gives a factorization of the cohomology  $H_{\text{ét}}^{n-1}(X_\lambda, \mathbb{Q}_l)$  respecting Frobenius, and a consequence of it is that the polynomial

$$P(T) := \{\zeta(X_\lambda, T)(1-T)(1-qT)\dots(1-q^{n-1}T)\}^{(-1)^n}$$

is divisible by the “zeta function” (Definition 3.1.2) defined by the function

$$(0.3.1) \quad \mathbb{Z}_{>0} \rightarrow \overline{\mathbb{Q}}; \quad r \mapsto \eta^r {}_nF_{n-1} \left( \begin{matrix} \varphi_{n+1,r}, \varphi_{n+1,r}^2, \dots, \varphi_{n+1,r}^n \\ \epsilon, \epsilon, \dots, \epsilon \end{matrix}; \lambda^{-(n+1)} \right)_{\mathbb{F}_{q^r}},$$

where  $\varphi_{n+1,r}$  denotes a fixed character of order  $n+1$  on  $\mathbb{F}_{q^r}$  chosen so that  $\varphi_{n+1,r} = \varphi_{n+1,1} \circ \text{Norm}_{\mathbb{F}_{q^r}/\mathbb{F}_q}$ , and  $\eta$  is either  $-1$  or  $1$ . A series of works of Goutet [Go2], [Go3], [Go1] shows that, if  $n+1$  is a prime number other than 2 or 3, then  $P(X)$  is divisible by the zeta function defined as above with  $\eta = 1$ .

**0.4. The aim of this article.** Our aim is to generalize the results on the Dwork family over  $\mathbb{F}_q$  to more general family of smooth Calabi–Yau hypersurfaces of projective space, whose principal part is not necessarily symmetric. Our families are of the form

$$F_0(T) - \lambda T_1 \dots T_{n+1} = 0,$$

where  $F_0(T)$  is a sum of  $(n+1)$  monomials with some conditions; precise conditions are stated in the beginning of Section 1.

First, we generalize the result of Yu to our family, basically following his idea to use Stienstra–Beuker’s method on formal group laws. By doing so, we relate these families (over  $\mathbb{F}_q$ ) to a usual hypergeometric series, viewed  $p$ -adically, through the formula of unit-root.

Second, we give a factorization of the zeta function, one of whose factors is decided by the hypergeometric function over  $\mathbb{F}_q$  with “similar” parameters as the one appeared in the formula of unit-root. The basic strategy is the combination of point-counting and the purity result of Katz. This method has an advantage that we can explicitly write the factors of zeta function without ambiguity caused by the characters. In particular, in case of Dwork family with arbitrary integer  $n \geq 2$ , we see that the “zeta function” associated to the function (0.3.1) with  $\eta = 1$  divides the zeta function of Dwork family.

**0.5. Notation.** Throughout this article, we fix an odd prime number  $p$ , a power  $q$  of  $p$  and an integer  $n$  greater than or equal to 2. The finite field with  $q^r$  elements (for a positive integer  $r$ ) is denoted by  $\mathbb{F}_{q^r}$ .

We denote the group of characters  $\mathbb{F}_q^\times \rightarrow \overline{\mathbb{Q}}^\times$  by  $\widehat{\mathbb{F}_q^\times}$ , and the trivial character by  $\epsilon$ . A character  $\chi \in \widehat{\mathbb{F}_q^\times}$  is also considered as a function  $\mathbb{F}_q \rightarrow \overline{\mathbb{Q}}$  by setting  $\chi(0) = 0$ . For a positive integer  $m$  that divides  $q-1$ , we fix a character  $\varphi_m \in \widehat{\mathbb{F}_q^\times}$  of order  $m$ ; although this choice does not affect our result, we need one anyway.

By default, elements in  $A^m$  for an abelian group  $A$  and a natural number  $m$  are considered as column vectors. For such a vector  $b = {}^t(b_1, \dots, b_m)$ , we write  $|b| := b_1 + \dots + b_m$ .

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## 1. SETTINGS AND STATEMENTS OF THE RESULTS.

**1.1. Families of hypersurfaces considered.** Let  $X_0$  be the smooth hypersurface of  $\mathbb{P}_{\mathbb{F}_q}^n$  defined by the polynomial

$$F_0(T) = c_1 T^{a_1} + \cdots + c_{n+1} T^{a_{n+1}} \in \mathbb{F}_q[T_1, \dots, T_{n+1}],$$

where  $c_1, \dots, c_{n+1} \in \mathbb{F}_q^\times$  and where  $a_1, \dots, a_{n+1} \in \mathbb{N}^{n+1}$  with  $|a_i| = n+1$  ( $i = 1, \dots, n+1$ ), none of  $a_i$ 's being equal to  $(1, 1, \dots, 1)$ . Here, for  $a_i = {}^t(a_{1i}, \dots, a_{n+1,i})$ , the notation  $T^{a_i}$  denotes the monomial  $T_1^{a_{1i}} \cdots T_{n+1}^{a_{n+1,i}}$ .

In this article, we investigate the monomial deformation  $X_\lambda$  of  $X_0$  defined by the polynomial

$$F_\lambda(T) = c_1 T^{a_1} + \cdots + c_{n+1} T^{a_{n+1}} - \lambda T_1 \cdots T_{n+1},$$

where  $\lambda$  moves  $\mathbb{F}_q$ ; mainly, we restrict our attention to  $\lambda$  such that  $X_\lambda$  is smooth.

The assumptions on  $X_0$  restricts the possibility for vectors  $a_1, \dots, a_{n+1}$  as follows.

**PROPOSITION 1.1.1.**— *Let  $A$  be the matrix  $A = (a_1, \dots, a_{n+1})$ . Then, after a suitable change of indices of  $a_i$ 's, each diagonal entry of  $A$  equals  $n+1$  or  $n$ , the other entries are either 0 or 1, and moreover there exists at most one 1 in each row.*

*Proof.* First, Let us prove the first property. It suffices to show that, for each index  $i = 1, \dots, n+1$ , there exists an index  $j$  with  $a_{ij} \geq n$ . Assume that there exists an  $i$  that satisfies  $a_{ij} \leq n-1$  ( $j = 1, \dots, n+1$ ). Fix such an  $i$ , and let  $P_i$  denote the point  $[0 : \cdots : 0 : 1 : 0 : \cdots : 0]$  in  $\mathbb{P}_{\mathbb{F}_q}^n$ , where 1 sits in the  $i$ -th entry. Then, for each  $j, k \in \{1, \dots, n+1\}$ ,

$$\frac{\partial T^{a_j}}{\partial T_k} = \begin{cases} a_{kj} T^{a_j - e_k} & \text{if } a_{kj} \geq 1, \\ 0 & \text{if } a_{kj} = 0. \end{cases}$$

The value of this partial derivative at  $P_i$  is zero unless  $T^{a_j - e_k} = T_i^n$ , which is impossible because of the assumption on  $i$ . Therefore, we have shown that

$$\frac{\partial F_0}{\partial T_k}(P_i) = 0 \quad (k = 1, \dots, n+1) \quad \text{and} \quad F_0(P_i) = 0,$$

and consequently  $P_i$  is a singular point of  $X_0$ , which contradicts the hypothesis.

The second property follows from the first and from the assumption  $|a_i| = n+1$  for all  $i$ .

In order to prove the third property, we assume the contrary. Then, after a change of coordinates, we may assume that  $a_{12} = a_{13} = 1$ , that is,  $T^{a_2} = T_1 T_2^n$  and  $T^{a_3} = T_1 T_3^n$ . For an element  $x$  of  $\overline{\mathbb{F}_q}$  that satisfies  $c_2 + c_3 x^n = 0$ , the point  $P = [0 : 1 : x : 0 : \cdots : 0]$ , which is actually an  $\overline{\mathbb{F}_q}$ -rational point of  $X_0$ , gives a singular point of  $X_0$ . In fact, the choice of  $x$  shows that  $\partial F_0 / \partial T_1(P) = 0$ , and it is straightforward to check that  $\partial F_0 / \partial T_i(P) = 0$  for all  $i \geq 2$ ; thus we have derived a contradiction.  $\square$

Throughout this article, we always take indices so that each diagonal entry of  $A$  is  $n+1$  or  $n$ .

**1.2. Results.** In this article, we prove two theorems; one is related to the usual hypergeometric series, and the other to the hypergeometric function over  $\mathbb{F}_q$ .

In order to write the parameters and the input of the hypergeometric functions, we introduce some notation.

**PROPOSITION 1.2.1.**— *The kernel of the homomorphism  $\mathbb{Z}^{n+1} \rightarrow \mathbb{Z}^{n+1}$  defined by the matrix  $A' = (a_{ij} - 1)_{i,j=1,\dots,n+1}$  is free of rank one and generated by a vector  ${}^t(\alpha_1, \dots, \alpha_{n+1})$  with all  $\alpha_i > 0$ .*

*Proof.* The matrix  $A'$  is not invertible since  $(1, 1, \dots, 1)A' = 0$ .

It, therefore, suffices to show that every entries of a non-zero vector  ${}^t(x_1, \dots, x_{n+1})$  in the kernel have the same sign. By multiplying every entries by  $-1$  if necessary, we may assume that at least two entries are non-negative. After a suitable change of indices, we may assume that  $x_1 \geq x_2 \geq \dots \geq x_{n+1}$ ; consequently,  $x_1 > 0$  and  $x_2 \geq 0$ .

Now, if an index  $i \in \{1, \dots, n\}$  satisfies  $x_i > 0$  and  $x_{i+1} \leq 0$ , then

$$(a_{11} - 1)x_1 + \sum_{j=i+1}^{n+1} (a_{1j} - 1)x_j = \sum_{j=2}^i -(a_{1j} - 1)x_j.$$

The left-hand side is greater than or equal to  $(n-1)x_1$  since  $(a_{1j} - 1)x_j \geq 0$  for  $j = i+1, \dots, n+1$ . Now,  $-(a_{1j} - 1)$  being 0 or 1 for  $j \neq 1$ , the right-hand side is less than or equal to  $(i-1)x_1$ . This shows that  $i = n$ , which implies the contradiction

$$0 = \sum_{j=1}^n (a_{n+1,j} - 1)x_j + (a_{n+1,n+1} - 1)x_{n+1} < 0$$

because  $a_{n+1,j} = 0$  for at least one  $j \in \{1, \dots, n\}$ . □

We freely use the notation  $\alpha_1, \dots, \alpha_{n+1}$ , whose choice is obviously unique, throughout this article. The sum  $\sum_{i=1}^{n+1} \alpha_i$  is denoted by  $\alpha$ . We always assume that  $q$  is relatively prime to all  $\alpha_i$ 's and to  $\alpha$ .

**DEFINITION 1.2.2.**— We define an element  $C \in \mathbb{F}_q^\times$  by

$$C = \alpha^\alpha \frac{c_1^{\alpha_1}}{\alpha_1^{\alpha_1}} \dots \frac{c_{n+1}^{\alpha_{n+1}}}{\alpha_{n+1}^{\alpha_{n+1}}}.$$

**EXAMPLE 1.2.3.**— (i) In the case of the Dwork family, that is, if  $F_0(T) = T_1^{n+1} + \dots + T_{n+1}^{n+1}$ , then  ${}^t(\alpha_1, \alpha_2, \dots, \alpha_{n+1}) = {}^t(1, 1, \dots, 1)$ . Since  $\alpha = n+1$ , we have  $C = (n+1)^{n+1}$ .

(ii) Let us consider the following example discussed by Yu and Yui [Yu-Yui, (4.8.1)]:  $n = 3$  and  $F_0(T) = T_1^4 + T_1 T_2^3 + T_3^4 + T_4^4$ . Then, we have  ${}^t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = {}^t(2, 4, 3, 3)$ . Since  $\alpha = 12$ , we have  $C = 2^{14} \cdot 3^6$ .

Before stating the first theorem, let us recall the definition of (generalized) hypergeometric function, which we treat as a formal power series; moreover, we treat hypergeometric functions with rational parameters. For  $(n+1)$  rational numbers  $A_1, \dots, A_{n+1}, B_1, \dots, B_n$ , we define the hypergeometric function by

$${}_{n+1}F_n \left( \begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix}; x \right) := \sum_{k=0}^{\infty} \frac{(A_1)_k \cdots (A_{n+1})_k}{(B_1)_k \cdots (B_n)_k (1)_k} x^k;$$

here, for a number  $c$  and a natural number  $k$ , the Pochhammer symbol  $(c)_k$  is defined to be the product  $c(c+1)\cdots(c+k-1)$ .

We can now state the first theorem, which is Theorem 2.3.3.

**THEOREM 1.2.4.**— *Let  $\lambda$  be an element of  $\mathbb{F}_q$  such that  $X_\lambda$  is smooth. Let  $\mathcal{F}(x)$  denote the formal power series*

$$(1.2.1) \quad {}_{\alpha-1}F_{\alpha-2} \left( \begin{matrix} \frac{1}{\alpha}, \frac{2}{\alpha}, \dots, \frac{\alpha-1}{\alpha} \\ \frac{1}{\alpha_1}, \dots, \frac{\alpha_1-1}{\alpha_1}, \frac{1}{\alpha_2}, \dots, \frac{\alpha_2-1}{\alpha_2}, \dots, \frac{1}{\alpha_{n+1}}, \dots, \frac{\alpha_{n+1}-1}{\alpha_{n+1}}, 1, \dots, 1 \end{matrix}; \tilde{C}x \right) \in W(\mathbb{F}_q)[[x]],$$

where the tilde means the Teichmüller lift, and where there are  $(n-1)$  “1”s in the lower parameters (note that some of the upper parameters can appear in the lower parameters). We denote the truncation of  $\mathcal{F}(x)$  up to degree  $p-1$  by  $\mathcal{F}_{p-1}(x) \in W(\mathbb{F}_q)[x]$ .

Then, the first slope of Newton polygon of  $H_{\text{cris}}^{n-1}(X_\lambda/W(\mathbb{F}_q))$  is zero if and only if the element  $\mathcal{F}_{p-1}(\lambda^{-\alpha})$  of  $\mathbb{F}_q$  is non-zero. If it is the case, the formal power series  $f(x) := \mathcal{F}(x)/\sigma(\mathcal{F}(x))$  converges at  $\tilde{\lambda}$ , where  $\sigma$  denotes the endomorphism on  $W(\mathbb{F}_q)[[x]]$  which acts on coefficients by the Frobenius of  $W(\mathbb{F}_q)$  and to  $x$  by  $\sigma(x) = x^p$ , and the unit root equals

$$\prod_{i=0}^{\log_p q - 1} \sigma^i(f(\tilde{\lambda}^{-\alpha})).$$

Now, we state the second theorem. Here, we assume that  $\log_p q$  is sufficiently large; in particular, we assume that  $q$  is congruent to 1 modulo all  $\alpha_i$ 's and modulo  $\alpha$  (it is not sufficient; we state a precise condition for  $q$  in Subsection 3.4).

Let us fix a non-trivial additive character  $\theta: \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}^\times$ . Then we have the notion of Gauss sum  $G(\chi)$  for each character  $\chi \in \widehat{\mathbb{F}_q}^\times$  (Definition 3.2.2).

The second theorem uses the language of hypergeometric function with reduced parameters over  $\mathbb{F}_q$  and that of zeta function. We do not explain them here, and develop them in the first three subsections of Section 3. The following is the second theorem, which is a corollary of a stronger statement (Theorem 3.4.5).

**THEOREM 1.2.5.**— *Let  $\lambda$  be an element of  $\mathbb{F}_q^\times$  such that  $X_\lambda$  is smooth and  $\lambda^\alpha \neq C$ . Define the polynomial  $P(x) \in \mathbb{Z}[x]$  by*

$$\zeta(X_\lambda, x) = \frac{P(x)^{(-1)^n}}{(1-x)(1-qx)\cdots(1-q^{n-1}x)}.$$

Then,  $P(x)$  is divisible by the zeta function associated to the hypergeometric function with reduced parameters over  $\mathbb{F}_q$

$$r \mapsto {}_{\alpha-1}F_{\alpha-2}\text{Red}\left(\begin{matrix} \varphi_\alpha, \varphi_\alpha^2, \dots, \varphi_\alpha^{\alpha-1} \\ \varphi_{\alpha_1}, \dots, \varphi_{\alpha_1}^{\alpha_1-1}, \dots, \varphi_{\alpha_{n+1}}, \dots, \varphi_{\alpha_{n+1}}^{\alpha_{n+1}-1}, \epsilon, \dots, \epsilon \end{matrix}; C\lambda^{-\alpha}\right)_{\mathbb{F}_{q^r}},$$

where there are  $(n-1)$   $\epsilon$ 's in the lower parameters (the zeta function of this function is known to be a polynomial (Proposition 3.3.2)).

Notice the similarity occurring in each hypergeometric function in the two theorems.

## 2. FORMAL GROUP LAWS AND UNIT ROOT.

**2.1. Review on the formal group laws.** In this subsection, we recall some facts on formal group laws and a special case of Artin–Mazur functors. For the basic language of formal group laws, the reader may consult Hazewinkel's book [Haz].

Let  $R$  be a ring. A one-dimensional commutative formal group law over  $R$ , which we simply say “a formal group law over  $R$ ” in this article, is a formal power series  $G(X, Y) \in R[[X, Y]]$  that satisfies some conditions corresponding to group axioms [Haz, 1.1]. A logarithm of the formal group law  $G$  is a formal power series  $l(\tau) \in R[[\tau]]$  that satisfies  $l(\tau) \equiv \tau \pmod{\deg 2}$  and

$$G(X, Y) = l^{-1}(l(X) + l(Y)).$$

Equivalently, a logarithm of  $G$  is a strict isomorphism of  $G$  to the additive group  $\widehat{\mathbb{G}}_a$  defined by  $\widehat{\mathbb{F}}_a(X, Y) = X + Y$ . A logarithm of a formal group law is unique if the ring  $R$  is of characteristic zero, and it exists if  $R$  contains  $\mathbb{Q}$ . If  $R$  is of characteristic zero, we also say that  $l(\tau) \in R \otimes_{\mathbb{Z}} \mathbb{Q}[[\tau]]$  is a logarithm of the formal group  $G(X, Y)$  over  $R$ , if it is the logarithm of  $G(X, Y)$  viewed as a formal group law over  $R \otimes_{\mathbb{Z}} \mathbb{Q}$ .

The following lemma describes the condition for a power series to be a logarithm of a formal group law, with a restriction on  $R$  which is satisfied by, for example, the Witt ring  $W(\mathbb{F}_q)$  of  $\mathbb{F}_q$ . The proof can be found in the appendix of an article by Stienstra–Beukers [S-B, (A.8)]; it is nothing but the equivalence of (i) and (v) there.

**LEMMA 2.1.1.**— *Let  $R$  be a ring of characteristic zero that is  $p$ -adically separated and complete, and let  $\sigma: R \rightarrow R$  be a lift of  $p$ -th power morphism on  $R/pR$ . Then, a formal power series*

$$l(\tau) = \sum_{n=1}^{\infty} b_n \frac{\tau^n}{n} \in R \otimes_{\mathbb{Z}} \mathbb{Q}[[\tau]] \quad (b_n \in R, b_1 = 1, b_p \in R^\times)$$

*is the logarithm of a formal group law over  $R$  if and only if there exists an element  $a$  of  $R$  that satisfies*

$$b_{mp^r} \equiv a\sigma(b_{mp^{r-1}}) \pmod{p^r} \quad (\forall m, r \geq 1).$$

In this context, we can give a sufficient condition for two formal group laws to be strictly isomorphic.



LEMMA 2.1.2.— *Let  $R$  and  $\sigma$  be as in the lemma above. Let  $G_1(X, Y)$  (resp.  $G_2(X, Y)$ ) be two formal group laws over  $R$  that has logarithm  $l_1(\tau) = \sum_{n=1}^{\infty} b_n \tau^n / n$  (resp.  $l_2(\tau) = \sum_{n=1}^{\infty} c_n \tau^n / n$ ) (here, we do not assume that  $b_p$  and  $c_p$  are units). Then,  $G_1$  and  $G_2$  are strictly isomorphic if there exists an element  $a$  of  $R$  that satisfies*

$$b_{mp^r} \equiv a\sigma(b_{mp^{r-1}}) \quad \text{and} \quad c_{mp^r} \equiv a\sigma(c_{mp^{r-1}}) \pmod{p^r} \quad (\forall m, r \geq 1)$$

*simultaneously.*

*Proof.* This is also a corollary of the implication of (i) from (iii) of a theorem of Stienstra–Beukers [S-B, (A.9)].  $\square$

Next, we recall the height of a formal group law over a field of characteristic  $p$ . For a formal group law  $G(X, Y)$  over a ring  $R$  (not necessarily of characteristic  $p$ ) and a positive integer  $m$ , define a formal power series  $[m]_G(X)$  inductively by

$$[1]_G(X) = X, \quad [m]_G(X) = G(X, [m-1]_G(X)) \quad (m \geq 2).$$

Now, let  $G(X, Y)$  be a formal group law over a field  $k$  of characteristic  $p$ . If  $[p]_G(X)$  is non-zero, then the lowest term of  $[p]_G(X)$  is of degree  $p^h$  for some positive integer  $h$ . We call this  $h$  the height of  $G(X, Y)$ ; if  $[p]_G(X)$  is zero, the height is defined to be infinity.

The lemma below is often used to tell whether a formal group law over  $k$  is of height one.

LEMMA 2.1.3.— *Let  $R$  be a ring of characteristic zero whose reduction modulo  $p$  is a field. Let  $G(X, Y)$  be a formal group law over  $R$  whose logarithm is of the form  $l(\tau) = \sum_{s=1}^{\infty} a_s \tau^{p^s} / p^s$  ( $a_s \in R$ ,  $a_1 = 1$ ). Then, the formal group law  $\overline{G}(X, Y) := G(X, Y) \pmod{p}$  over  $k$  is of height one if and only if  $a_1 \not\equiv 0 \pmod{p}$ .*

*Proof.* Let  $u$  denote the coefficient of  $X^p$  in the formal power series  $[p]_G(X)$ . Then,  $[p]_G(X)$  is of the form

$$(2.1.1) \quad [p]_G(X) = pX + v_2 X^2 + \cdots + v_{p-1} X^{p-1} + uX^p + (\text{degree} \geq p+1)$$

where  $v_2, \dots, v_{p-1} \equiv 0 \pmod{p}$ ; the formal group law  $\overline{G}$  is of height one if and only if  $u \not\equiv 0 \pmod{p}$ . The definition of the logarithm implies the equation

$$l([p]_G(X)) = pl(X) = pX + a_1 X^p + a_2 \frac{X^{p^2}}{p} + \dots$$

By substituting (2.1.1), the coefficient of  $X^p$  in the left-hand side equals  $u$  modulo  $p$ , and the lemma follows.  $\square$

Before explaining the Artin–Mazur functor [A-Ma], let us recall the formal group.

Let  $\mathbf{NilAlg}_R$  denote the category of nilpotent  $R$ -algebras. For a natural number  $n$ , the functor  $\widehat{A}^n$  is defined to be

$$\widehat{A}^n: \mathbf{NilAlg}_R \rightarrow \mathbf{Set}; \quad N \rightarrow N^n.$$

Now, an  $n$ -dimensional formal group is a functor

$$G: \mathbf{NilAlg}_R \rightarrow \mathbf{Ab}$$

whose composition with the forgetful functor  $\mathbf{Ab} \rightarrow \mathbf{Set}$  is isomorphic to  $\widehat{\mathbf{A}}^n$ . The category of (one-dimensional commutative) formal group laws and that of one-dimensional formal groups are equivalent.

Let  $X$  be a scheme over  $R$  and  $i$  a natural number. Then, we define the *Artin–Mazur functor* by

$$H^i(X, \widehat{\mathbb{G}_m}): \mathbf{NilAlg}_R \rightarrow \mathbf{Ab}; \quad A \mapsto H^i(X, \widehat{\mathbb{G}_m}(\mathcal{O}_X \otimes_R A)).$$

Although this may or may not be a formal group, the following proposition states that it actually is in the case of our interest; in fact, the Artin–Mazur functor is deeply related to the Witt cohomology of  $X$  as in the following proposition [A-Ma, II (4.2), (4.3)].

**PROPOSITION 2.1.4.**— *Assume that  $R$  is a perfect field  $k$  of characteristic  $p$ , and let  $X$  be a complete intersection of dimension  $d \geq 2$ . Then,  $H^d(X, \widehat{\mathbb{G}_m})$  is a formal group of dimension  $\dim_k H^d(X, \mathcal{O}_X)$ , and the Cartier–Dieudonné module of it is isomorphic to  $H^d(X, W\mathcal{O}_X)$ .*

If, moreover,  $X$  is proper and smooth over  $k$ , the Witt cohomology is, after the base extension to  $K := \text{Frac}(W(k))$ , isomorphic to the maximal subspace  $H_{\text{cris}}^d(X/W(k))_K^{<1}$  of the crystalline cohomology  $H_{\text{cris}}^d(X/W(k))_K$  on which the Frobenius acts with slope  $< 1$ . Now, assume that  $k = \mathbb{F}_q$ , that  $H^d(X, \mathcal{O}_X)$  is one-dimensional, and that the height  $h$  of the formal group law  $H^d(X, \widehat{\mathbb{G}_m}) \bmod p$  is finite. Then, the dimension of the space  $H_{\text{cris}}^d(X/W(k))_K^{<1}$  equals  $h$  since, in fact, both equal the rank of Cartier–Dieudonné module associated to the formal group law [I, II, Remarques 2.15 (a)], [S-B, (A.13)]. The slope of Frobenius equals  $(h-1)/h$ , which shows that the first slope of Newton polygon of the crystalline cohomology is zero if and only if  $h = 1$  [De]. In this case, the unique eigenvalue of the Frobenius on  $H_{\text{cris}}^d(X/W(k))_K$  that is a  $p$ -adic unit is called unit-root of  $X_\lambda$ .

The following proposition, which is a very special case of a theorem of Stienstra [S, Theorem 1] (note that we have shifted the index of  $\beta_m$  by one), gives how to calculate the logarithm of Artin–Mazur functor in a restricted situation.

**PROPOSITION 2.1.5.**— *Let  $R$  be a noetherian ring which is flat over  $\mathbb{Z}$ , and let  $X$  be the flat hypersurface of  $\mathbb{P}_R^N$  defined by a homogeneous polynomial  $F(T) \in R[T_1, \dots, T_{N+1}]$  of degree  $N+1$ . Then,  $H^{N-1}(X, \widehat{\mathbb{G}_m})$  is a one-dimensional formal group whose logarithm  $l(\tau)$  is given by*

$$l(\tau) = \sum_{m=0}^{\infty} \beta_m \frac{\tau^{m+1}}{m+1},$$

where the  $\beta_m$  is the coefficient of  $T_1^m \dots T_{N+1}^m$  of  $F(T)^m$ .

**2.2. Formal group laws associated to the family.** For  $x \in \mathbb{F}_q$ , we denote the Teichmüller lift of  $x$  by  $\tilde{x} \in W(\mathbb{F}_q)$ . In this subsection, we fix a ring  $R$  that is flat over  $\mathbb{Z}$  containing  $W(\mathbb{F}_q)$ . For an element  $\Lambda$  in  $R$ , let  $\widetilde{X}_\Lambda$  be the hypersurface of  $\mathbb{P}_R^n$  defined by

$$\widetilde{F}_\Lambda(T) := \widetilde{c}_1 T^{a_1} + \dots + \widetilde{c}_{n+1} T^{a_{n+1}} - \Lambda T_1 \dots T_{n+1} \in R[T_1, \dots, T_{n+1}].$$

LEMMA 2.2.1.— *Let  $\Lambda$  be an element of  $R$ . Then, for each positive integer  $m$ , the coefficient of  $T_1^m \dots T_{n+1}^m$  of  $\widetilde{F}_\Lambda(T)^m$  equals*

$$(2.2.1) \quad (-\Lambda)^m {}_\alpha F_{\alpha-1} \left( \frac{-m}{\alpha}, \frac{-m+1}{\alpha}, \dots, \frac{-m+\alpha-1}{\alpha}; \frac{1}{\alpha_1}, \dots, \frac{\alpha_1-1}{\alpha_1}, 1, \frac{1}{\alpha_2}, \dots, \frac{\alpha_2-1}{\alpha_2}, 1, \dots, \frac{1}{\alpha_{n+1}}, \dots, \frac{\alpha_{n+1}-1}{\alpha_{n+1}}; \widetilde{C}\Lambda^{-\alpha} \right).$$

(Note that, in the lower parameters of  ${}_a F_{a-1}$ , the last 1 is dropped. The element  $C$  is the one defined in Definition 1.2.2.)

*Proof.* Looking at the binomial expansion

$$\widetilde{F}_\Lambda(T)^m = \sum_{m_1+\dots+m_{n+1}+m'=m} \frac{m!}{m_1! \dots m_{n+1}! m'!} \prod_{i=1}^{n+1} \widetilde{c}_i^{m_i} T^{m_i a_i} (-\Lambda T_1 \dots T_{n+1})^{m'},$$

we notice that the coefficient of  $T_1^m \dots T_{n+1}^m$  is the sum

$$\sum_{(m_1, \dots, m_{n+1}, m')} \frac{m!}{m_1! \dots m_{n+1}! m'!} \cdot \widetilde{c}_1^{m_1} \dots \widetilde{c}_{n+1}^{m_{n+1}} (-\Lambda)^{m'},$$

where the index runs so that

$$\begin{aligned} m_1 a_{i1} + \dots + m_{n+1} a_{i,n+1} + m' &= m \quad (i = 1, \dots, n+1), \\ m_1 + \dots + m_{n+1} + m' &= m. \end{aligned}$$

By Proposition 1.2.1, an  $(n+2)$ -tuple  $(m_1, \dots, m_{n+1}, m)$  satisfies this condition if and only if the equation

$$(m_1, \dots, m_{n+1}) = k(\alpha_1, \dots, \alpha_{n+1}), \quad m' = m - k\alpha$$

holds for a natural number  $k$ . This shows that the coefficient equals

$$(2.2.2) \quad \sum_{k \geq 0, m \geq k\alpha} \frac{m!}{(k\alpha_1)! \dots (k\alpha_{n+1})! (m - k\alpha)!} \widetilde{c}_1^{k\alpha_1} \dots \widetilde{c}_{n+1}^{k\alpha_{n+1}} (-\Lambda)^{m - k\alpha}.$$

Now, we show that the two numbers (2.2.1) and (2.2.2) coincide. By definition of hypergeometric function, (2.2.1) equals

$$\begin{aligned} & (-\Lambda)^m \sum_{k=0}^{\infty} \frac{(-m/\alpha)_k (-m+1/\alpha)_k \dots (-m+\alpha-1/\alpha)_k}{(1/\alpha_1)_k \dots (\alpha_1/\alpha_1)_k \dots (1/\alpha_{n+1})_k \dots (\alpha_{n+1}/\alpha_{n+1})_k} \left( \frac{\alpha^\alpha \widetilde{c}_1^{\alpha_1}}{\Lambda^\alpha \alpha_1^{\alpha_1}} \dots \frac{\widetilde{c}_{n+1}^{\alpha_{n+1}}}{\alpha_{n+1}^{\alpha_{n+1}}} \right)^k \\ &= \sum_{k=0}^{\infty} \frac{\alpha^{k\alpha} (-m/\alpha)_k (-m+1/\alpha)_k \dots (-m+k\alpha-1/\alpha)_k}{A_{1,k} \dots A_{n+1,k}} \widetilde{C}'_k (-\Lambda)^{m - k\alpha}, \end{aligned}$$

where

$$A_{i,k} = \alpha_i^{k\alpha_i} (1/\alpha_i)_k \dots ((\alpha_i-1)/\alpha_i)_k (\alpha_i/\alpha_i)_k \quad (i = 1, \dots, n+1)$$

and  $C'_k = c_1^{k\alpha_1} \dots c_{n+1}^{k\alpha_{n+1}}$ . Moreover, each summand is 0 if  $k - m/\alpha \geq 0$ . We, in fact, have  $A_{i,k} = (k\alpha_i)!$  and

$$\begin{aligned} & (-1)^{k\alpha} \alpha^{k\alpha} (-m/\alpha)_k (-m+1/\alpha)_k \dots (-m+k\alpha-1/\alpha)_k \\ &= (-1)^{k\alpha} (-m)(-m+1) \dots (-m+k\alpha-1) \\ &= \frac{m!}{(m - k\alpha)!}, \end{aligned}$$

which shows that the two numbers coincide.  $\square$

Now, the theorem below follows from this lemma and Proposition 2.1.5.

**THEOREM 2.2.2.**— *Let  $\Lambda$  be an element of  $R$  such that  $\widetilde{X}_\Lambda$  is flat over  $R$ . Under the notation in the lemma above, the Artin–Mazur functor  $H^{n-1}(\widetilde{X}_\Lambda, \widehat{\mathbb{G}_m})$  is a formal group the logarithm  $l(\tau)$  of whose formal group law is given by*

$$\sum_{m=0}^{\infty} (-\Lambda)^m {}_\alpha F_{\alpha-1} \left( \frac{-m}{\alpha}, \frac{-m+1}{\alpha}, \dots, \frac{-m+\alpha-1}{\alpha}; \frac{1}{\alpha_1}, \dots, \frac{\alpha_1-1}{\alpha_1}, 1, \frac{1}{\alpha_2}, \dots, \frac{\alpha_2-1}{\alpha_2}, 1, \dots, \frac{1}{\alpha_{n+1}}, \dots, \frac{\alpha_{n+1}-1}{\alpha_{n+1}}; \widetilde{C}\Lambda^{-\alpha} \right) \frac{\tau^{m+1}}{m+1}.$$

**2.3. Unit root.** As in Section 1, let  $\mathcal{F}(x)$  denote the formal power series

$$(2.3.1) \quad {}_\alpha F_{\alpha-1} \left( \frac{1}{\alpha}, \frac{2}{\alpha}, \dots, \frac{\alpha-1}{\alpha}, 1; \frac{1}{\alpha_1}, \dots, \frac{\alpha_1-1}{\alpha_1}, 1, \frac{1}{\alpha_2}, \dots, \frac{\alpha_2-1}{\alpha_2}, 1, \dots, \frac{1}{\alpha_{n+1}}, \dots, \frac{\alpha_{n+1}-1}{\alpha_{n+1}}; \widetilde{C}x \right) \in W(\mathbb{F}_q)[[x]].$$

In this subsection, by a method of Stienstra and Beukers [S-B], we prove that  $\mathcal{F}(x)$  gives us information on unit root of the crystalline cohomology of  $X_\lambda$ . The argument in this subsection follows that of Yu [Yu, Section 5].

For positive integers  $m$  and  $s$ , let  $\mathcal{F}_{m,s}(x)$  denote the polynomial obtained by truncating  $\mathcal{F}(x)$  up to degree  $mp^s - 1$ . Let  $\mathcal{R}$  denote the  $p$ -adic completion of the ring

$$W(\mathbb{F}_q)[x, (x\mathcal{F}_{1,1}(x))^{-1}],$$

to which the Frobenius endomorphism  $\sigma$  on  $W(\mathbb{F}_q)$  extends by  $\sigma(x) = x^p$ .

**PROPOSITION 2.3.1.**— *The formal power series*

$$f(x) := \frac{\mathcal{F}(x)}{\sigma(\mathcal{F}(x))} \in W(\mathbb{F}_q)[[x]]$$

*is actually an element of  $\mathcal{R}$ .*

*Proof.* Define a polynomial  $G_{\mu,s}(x)$ , for positive integers  $\mu$  and  $s$ , by

$$G_{\mu,s}(x) = {}_\alpha F_{\alpha-1} \left( \frac{-\mu p^s + 1}{\alpha}, \frac{-\mu p^s + 2}{\alpha}, \dots, \frac{-\mu p^s + \alpha}{\alpha}; \frac{1}{\alpha_1}, \dots, \frac{\alpha_1-1}{\alpha_1}, 1, \frac{1}{\alpha_2}, \dots, \frac{\alpha_2-1}{\alpha_2}, 1, \dots, \frac{1}{\alpha_{n+1}}, \dots, \frac{\alpha_{n+1}-1}{\alpha_{n+1}}; \widetilde{C}x \right).$$

Let  $G'_{\mu,s}(t)$  be the polynomial defined by

$$G'_{\mu,s}(t) = (-t)^{\mu p^s - 1} G_{\mu,s}(t^{-\alpha}).$$

This is the coefficient of  $\tau^{\mu p^s} / \mu p^s$  in the logarithm  $l(\tau)$  in Theorem 2.2.2, applied with  $R$  being the  $p$ -adic completion  $\mathcal{S}$  of the ring

$$W(\mathbb{F}_q) \left[ t, (t\mathcal{F}_{1,1}(t^{-\alpha}))^{-1} \right]$$

and with  $\Lambda = t$ . Now, since we have  $G'_{1,1}(t) \equiv t^{p-1} \mathcal{F}_{1,1}(t^{-\alpha}) \pmod{p}$  by a straightforward observation, Lemma 2.1.1 shows that

$$G'_{\mu,s+1}(t) \equiv g'(t) \cdot \sigma(G'_{\mu,s}(t)) \pmod{p^{s+1}} \quad (\mu, s \geq 1)$$

for an element  $g'$  of  $\mathcal{S}$ , independent of  $\mu$  and  $s$ . Thus, we have

$$t^{p-1} G_{\mu,s+1}(t^{-\alpha}) \equiv g'(t) \sigma(G_{\mu,s}(t^{-\alpha})) \pmod{p^{s+1}}.$$

Since the polynomial  $G_{\mu,s}(x)$  converges  $p$ -adically to  $\mathcal{F}(x)$  as  $s \rightarrow \infty$ , the power series  $f(x)$  equals  $t^{-(p-1)}g'(t)$ , with  $x = t^{-\alpha}$ . This element, therefore, lies in the intersection of  $W(\mathbb{F}_q)[[x]]$  and the ring  $\mathcal{S}$ , which equals  $\mathcal{R}$ .  $\square$

LEMMA 2.3.2.— *Let  $\lambda$  be an element of  $\mathbb{F}_q^\times$ , and denote by  $\widetilde{X}_\lambda$  the variety  $\widetilde{X}_\Lambda$  with  $R = W(\mathbb{F}_q)$  and  $\Lambda = \widetilde{\lambda}$ . Put  $a := \widetilde{\lambda}^{p-1} f(\widetilde{\lambda}^{-\alpha})$  in the notation of the proposition above, put  $a_s := a\sigma(a) \dots \sigma^s(a)$  ( $s \geq 0$ ), and let  $G'$  be the formal group law over  $W(\mathbb{F}_q)$  with logarithm*

$$l'(\tau) = \sum_{s=0}^{\infty} a_s \frac{\tau^{p^s}}{p^s}.$$

*Then, the formal group law realizing  $H^{n-1}(\widetilde{X}_\lambda, \widehat{\mathbb{G}}_m)$  is strictly isomorphic to  $G'$ .*

*Proof.* This is a consequence of Lemma 2.1.2 because the element  $a$  makes the two formal group laws satisfy the necessary condition stated there.  $\square$

THEOREM 2.3.3.— *Let  $\lambda$  be an element of  $\mathbb{F}_q^\times$  such that  $X_\lambda$  is smooth. Then, the first slope of the Newton polygon of  $H_{\text{cris}}^{n-1}(X_\lambda/W(\mathbb{F}_q))$  is zero if and only if  $\mathcal{F}_{1,1}(\lambda^{-\alpha}) \neq 0$  in  $\mathbb{F}_q$ . In this case, the unit root of  $X_\lambda$  equals*

$$(2.3.2) \quad \prod_{i=0}^{r-1} \sigma^i(f(\widetilde{\lambda}^{-\alpha})),$$

where  $q = p^r$ .

*Proof.* Put  $a := \widetilde{\lambda}^{p-1} f(\widetilde{\lambda}^{-\alpha})$  as in the previous lemma. Then the first half follows from Lemma 2.1.3 because  $a \equiv \mathcal{F}_{1,1}(\lambda^{-\alpha}) \pmod{p}$ .

Now, assume that  $a$  is a  $p$ -adic unit. Then, the argument of Stienstra–Beukers [S-B, (A.13)] shows that the Cartier–Dieudonné module of  $G'$ , which is rank one (as is explained just after Proposition 2.1.4), has a basis  $\omega$  on which the Frobenius acts by  $\omega \mapsto a\omega$ ; Now, the Frobenius endomorphism of the Cartier–Dieudonné module of  $G$  equals

$$a\sigma(a) \dots \sigma^{r-1}(a),$$

which equals the number (2.3.2). Now, Proposition 2.1.4 shows the statement.  $\square$

### 3. FACTORIZATION OF ZETA FUNCTION.

3.1.  **$q$ -Weil functions.** In this subsection, we introduce a concept of  $q$ -Weil functions.

DEFINITION 3.1.1.— Let  $f: \mathbb{Z}_{>0} \rightarrow \overline{\mathbb{Q}}$  be a function and  $k$  an integer. We say that  $f$  is a *pure  $q$ -Weil function of weight  $k$*  if there exist some  $q$ -Weil numbers  $\alpha_1, \dots, \alpha_m$  of weight  $k$  satisfying

$$f(r) = \sum_{i=1}^m \alpha_i^r. \quad (\forall r \in \mathbb{Z}_{>0}).$$

We say that  $f$  is a  *$q$ -Weil function* if there exist some pure  $q$ -Weil functions  $f_i$  of pure weight  $k_i$  ( $i = 1, \dots, m$ ) and a number  $\varepsilon_1, \dots, \varepsilon_m \in \{1, -1\}$  satisfying

$$f = \sum_{i=1}^m \varepsilon_i f_i.$$

$f$  is a  $q$ -Weil function of weight  $\leq k$  if all  $k_i$ 's can be taken to be less than or equal to  $k$ .

DEFINITION 3.1.2.— The zeta function  $\zeta(f)(T)$  of a function  $f: \mathbb{Z}_{>0} \rightarrow \overline{\mathbb{Q}}$  is a formal power series

$$\zeta(f)(T) = \exp\left(-\sum_{r=1}^{\infty} f(r) \frac{T^r}{r}\right) \in 1 + T\overline{\mathbb{Q}}[[T]].$$

The following fact is standard.

PROPOSITION 3.1.3.— A function  $f: \mathbb{Z}_{>0} \rightarrow \overline{\mathbb{Q}}$  is a pure  $q$ -Weil function of weight  $k$  if and only if the inverse of zeta function  $1/\zeta(f)(T)$  is a polynomial in  $\overline{\mathbb{Q}}[T]$  all of whose reciprocal roots are a  $q$ -Weil number of weight  $k$ .

**3.2. Preliminaries on Gauss sum.** In this subsection, we recall the definition and properties of Gauss sum. In the rest of this article, we fix a non-trivial additive character  $\theta: \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}^\times$ .

The following lemma is a standard fact on characters.

LEMMA 3.2.1.— (i) For all  $x \in \mathbb{F}_q$ ,

$$\sum_{w \in \mathbb{F}_q} \theta(wx) = \begin{cases} 0 & \text{if } x \neq 0, \\ q & \text{if } x = 0. \end{cases}$$

(ii) For all  $\chi \in \widehat{\mathbb{F}_q^\times}$ ,

$$\sum_{x \in \mathbb{F}_q^\times} \chi(x) = \begin{cases} 0 & \text{if } \chi \neq \epsilon, \\ q-1 & \text{if } \chi = \epsilon. \end{cases}$$

Now, let us recall the definition of Gauss sum.

DEFINITION 3.2.2.— For a character  $\chi: \mathbb{F}_q^\times \rightarrow \overline{\mathbb{Q}}^\times$ , the Gauss sum with respect to  $\chi$  is defined to be

$$G_\theta(\chi) := \sum_{x \in \mathbb{F}_q^\times} \theta(x) \chi(x).$$

Although this number depends on the choice of additive character  $\theta$ , we also denote it simply by  $G(\chi)$ .

The following two propositions are also standard.

PROPOSITION 3.2.3.— For all character  $\chi: \mathbb{F}_q^\times \rightarrow \overline{\mathbb{Q}}^\times$ ,

$$G(\chi)G(\overline{\chi}) = \begin{cases} q\chi(-1) & \text{if } \chi \neq \epsilon, \\ q\chi(-1) - (q-1) & \text{if } \chi = \epsilon. \end{cases}$$

PROPOSITION 3.2.4.— *For all  $x \in \mathbb{F}_q^\times$ ,*

$$\theta(x) = \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} G(\overline{\chi}) \chi(x).$$

At last, we recall an important property of Gauss sum, known as Davenport–Hasse product formula [Da-Has, (0.9<sub>1</sub>)].

PROPOSITION 3.2.5.— *Let  $\chi$  be a character on  $\mathbb{F}_q^\times$  and  $m$  a positive integer that satisfies  $q \equiv 1 \pmod{m}$ . Then, the formula*

$$\prod_{b=0}^{m-1} G(\chi \varphi_m^b) = -G(\chi^m) \chi(m^{-m}) \prod_{b=1}^{m-1} G(\varphi_m^b)$$

*holds.*

DEFINITION 3.2.6.— Let  $r$  be a positive integer.

- (i) The additive character  $\theta_{q^r}$  on  $\mathbb{F}_{q^r}$  is defined by  $\theta_{q^r} := \theta \circ \text{Tr}_{\mathbb{F}_{q^r}/\mathbb{F}_q}$ .
- (ii) For a character  $\chi$  on  $\mathbb{F}_q^\times$ , the character  $\chi_{q^r}$  on  $\mathbb{F}_{q^r}^\times$  is defined by  $\chi_{q^r} := \chi \circ \text{Norm}_{\mathbb{F}_{q^r}/\mathbb{F}_q}$ .

For a character  $\chi'$  on  $\mathbb{F}_{q^r}$ , the Gauss sum  $G_{\theta_{q^r}}(\chi')$  is denoted simply by  $G(\chi')$ .

The following proposition is also a result of Davenport and Hasse [Da-Has, (0.8)].

PROPOSITION 3.2.7.— *Let  $\chi$  be a character on  $\mathbb{F}_q^\times$ . Then, for all positive integer  $r$ , we have*

$$-G(\chi_{q^r}) = (-G(\chi))^r.$$

*In particular, the function  $r \mapsto -G(\chi_{q^r})$  is a pure  $q$ -Weil function of weight 1 (resp. of weight 0) if  $\chi$  is non-trivial (resp. if  $\chi$  is trivial).*

**3.3. Preliminaries on hypergeometric function over finite fields.** In this subsection, we review a function over  $\mathbb{F}_q$  which we here call hypergeometric function over  $\mathbb{F}_q$ , and we prove some properties. The key ingredient is the construction of hypergeometric sheaf by Katz [Kal]. In this section,  $n$  is an arbitrary positive integer.

Let us recall the definition of hypergeometric function over finite fields; we use the definition made by McCarthy [MC, Definition 1.4].

DEFINITION 3.3.1.— Let  $A_1, \dots, A_{n+1}$  and  $B_1, \dots, B_n$  be characters of  $\mathbb{F}_q^\times$ . Then, we define the *hypergeometric function over  $\mathbb{F}_q$*  as

$${}_{n+1}F_n \left( \begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix} ; x \right)_{\mathbb{F}_q} = \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \prod_{i=1}^{n+1} \frac{G(A_i \chi)}{G(A_i)} \prod_{i=1}^n \frac{G(\overline{B_i \chi})}{G(\overline{B_i})} G(\overline{\chi}) \chi(-1)^{n+1} \chi(x).$$

As in the case of Gauss sum, for a positive integer  $r$ , we also define a variant as

$${}_{n+1}F_n \left( \begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix} ; x \right)_{\mathbb{F}_{q^r}} = \frac{1}{q^r-1} \sum_{\chi \in \widehat{\mathbb{F}_{q^r}^\times}} \prod_{i=1}^{n+1} \frac{G(A_{i,q^r} \chi)}{G(A_{i,q^r})} \prod_{i=1}^n \frac{G(\overline{B_{i,q^r} \chi})}{G(\overline{B_{i,q^r}})} G(\overline{\chi}) \chi(-1)^{n+1} \chi(x).$$

The following two propositions describe the “ $q$ -Weil properties” of this function.

PROPOSITION 3.3.2.— *Let  $A_1, \dots, A_{n+1}$  be non-trivial characters on  $\mathbb{F}_q^\times$ , let  $B_1, \dots, B_n$  be (possibly trivial) characters on  $\mathbb{F}_q^\times$ , and assume that  $\{A_1, \dots, A_{n+1}\} \cap \{B_1, \dots, B_n\} = \emptyset$ . Let  $m$  be the number of trivial characters among  $B_i$ ’s. Then, for all  $x \in \mathbb{F}_q^\times \setminus \{1\}$ , the function*

$$r \mapsto {}_{n+1}F_n \left( \begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix}; x \right)_{\mathbb{F}_{q^r}}$$

is a pure  $q$ -Weil function of weight  $m$ .

*Proof.* Katz [Ka1, 8.2] constructs an irreducible  $\overline{\mathbb{Q}_\ell}$ -sheaf

$$(3.3.1) \quad \mathcal{H}(!, \theta; A_1, \dots, A_{n+1}; B_1, \dots, B_n, \epsilon).$$

on  $\mathbb{G}_m$ , which is smooth on  $\mathbb{G}_m \setminus \{1\}$ . This is pure of weight  $2n+1$  [Ka1, Theorem 8.4.2 (4)], and its trace of Frobenius at  $x$  is

$$- \sum_{(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) \in V(x)} \theta \left( \sum_{i=1}^{n+1} x_i - \sum_{i=1}^{n+1} y_i \right) A_1(x_1) \dots A_{n+1}(x_{n+1}) B_1(y_1) \dots B_{n+1}(y_{n+1}),$$

where

$$V(x) = \left\{ (x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) \in (\mathbb{F}_q^\times)^{2(n+1)} \mid x_1 \dots x_{n+1} = x y_1 \dots y_{n+1} \right\}.$$

This is [MC, Proposition 2.6] equal to

$$- \prod_{i=1}^{n+1} G(A_i) \prod_{i=1}^n G(B_i) B_i(-1) \cdot {}_{n+1}F_n \left( \begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix}; x^{-1} \right)_{\mathbb{F}_q},$$

which shows that the function

$$r \mapsto - \prod_{i=1}^{n+1} G(A_{i,q^r}) \prod_{i=1}^n G(B_{i,q^r}) B_{i,q^r}(-1) \cdot {}_{n+1}F_n \left( \begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix}; x^{-1} \right)_{\mathbb{F}_{q^r}}$$

is a pure  $q$ -Weil function of weight  $2n+1$  since the base extension of the object (3.3.1) to  $\mathbb{F}_{q^r}$  equals [Ka1, (8.2.6)] the middle extension  $\overline{\mathbb{Q}_\ell}$ -sheaf

$$\mathcal{H}(!, \theta_{q^r}; A_{1,q^r}, \dots, A_{n+1,q^r}; B_{1,q^r}, \dots, B_{n,q^r}, \epsilon).$$

Now, Proposition 3.2.7 shows that the function

$$r \mapsto - \prod_{i=1}^{n+1} G(A_{i,q^r}) \prod_{i=1}^n G(B_{i,q^r})$$

is a pure  $q$ -Weil function of weight  $2n+1-m$ , and it is obvious that the function  $r \mapsto \prod_{i=1}^n B_{i,q^r}(-1)$  is a pure  $q$ -Weil function of weight 0, which shows the proposition.  $\square$

REMARK 3.3.3.— Under the situation above, the hypergeometric function over  $\mathbb{F}_q$  equals the trace function of

$$\mathcal{H}^{\text{can}}(\theta; A_1, \dots, A_{n+1}; B_1, \dots, B_n, \epsilon)$$

in the notation of Katz [Ka2, Section 4].

The following proposition describes how cancellation of the parameters works when the set of upper parameters and that of lower parameters have non-empty intersection.



PROPOSITION 3.3.4.— Let  $A_1, \dots, A_{n+1}$  be non-trivial characters on  $\mathbb{F}_q^\times$ , let  $B_1, \dots, B_n$  be characters of  $\mathbb{F}_q^\times$  and assume that two multisets  $\{A_{n'+2}, \dots, A_{n+1}\}$  and  $\{B_{n'+1}, \dots, B_n\}$  coincide. Let  $m$  be the number of trivial characters among  $B_i$ 's. Then, for all  $x \in \mathbb{F}_q^\times$ , the function

$$r \mapsto {}_{n+1}F_n \left( \begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix}; x \right)_{\mathbb{F}_{q^r}} - {}_{n'+1}F_{n'} \left( \begin{matrix} A_1, \dots, A_{n'+1} \\ B_1, \dots, B_{n'} \end{matrix}; x \right)_{\mathbb{F}_{q^r}}$$

is a  $q$ -Weil function of weight  $\leq m-1$ .

*Proof.* It suffices to show that, for non-trivial characters  $A_1, \dots, A_n, C$  and characters  $B_1, \dots, B_{n-1}$ , the function

$$r \mapsto {}_{n+1}F_n \left( \begin{matrix} A_1, \dots, A_n, C \\ B_1, \dots, B_{n-1}, C \end{matrix}; x \right)_{\mathbb{F}_{q^r}} - {}_nF_{n-1} \left( \begin{matrix} A_1, \dots, A_n \\ B_1, \dots, B_{n-1} \end{matrix}; x \right)_{\mathbb{F}_{q^r}}$$

is a  $q$ -Weil function of weight  $\leq m-1$ .

We have

$$\begin{aligned} (3.3.2) \quad & {}_{n+1}F_n \left( \begin{matrix} A_1, \dots, A_n, C \\ B_1, \dots, B_{n-1}, C \end{matrix}; x \right)_{\mathbb{F}_q} \\ &= \frac{1}{q-1} \sum_{\chi \in \mathbb{F}_q^\times} \frac{G(A_1\chi) \dots G(A_n\chi) G(C\chi)}{G(A_1) \dots G(A_n) G(C)} \frac{G(\overline{B_1\chi}) \dots G(\overline{B_{n-1}\chi}) G(\overline{C\chi})}{G(\overline{B_1}) \dots G(\overline{B_{n-1}}) G(\overline{C})} G(\overline{\chi}) \chi((-1)^{n+1}) \chi(x). \end{aligned}$$

By Proposition 3.2.3,

$$\frac{G(C\chi) G(\overline{C\chi})}{G(C) G(\overline{C})} = \begin{cases} \frac{qC\chi(-1)}{qC(-1)} = \chi(-1) & \text{if } \chi \neq C^{-1}, \\ \frac{qC\chi(-1) - (q-1)}{qC(-1)} = \chi(-1) - \frac{q-1}{q} C(-1) & \text{if } \chi = C^{-1}, \end{cases}$$

which shows that (3.3.2) equals

$$\begin{aligned} & \frac{1}{q-1} \sum_{\chi \in \mathbb{F}_q^\times} \frac{G(A_1\chi) \dots G(A_n\chi)}{G(A_1) \dots G(A_n)} \frac{G(\overline{B_1\chi}) \dots G(\overline{B_{n-1}\chi})}{G(\overline{B_1}) \dots G(\overline{B_{n-1}})} G(\overline{\chi}) \chi((-1)^n) \chi(x) \\ & - \frac{1}{q} \frac{G(A_1 C^{-1}) \dots G(A_n C^{-1})}{G(A_1) \dots G(A_n)} \frac{G(\overline{B_1 C^{-1}}) \dots G(\overline{B_{n-1} C^{-1}})}{G(\overline{B_1}) \dots G(\overline{B_{n-1}})} G(\overline{C^{-1}}) C((-1)^n) C^{-1}(x). \end{aligned}$$

The first term of this equals

$${}_nF_{n-1} \left( \begin{matrix} A_1, \dots, A_n \\ B_1, \dots, B_{n-1} \end{matrix}; x \right)_{\mathbb{F}_q}.$$

By Proposition 3.2.7, the second term with  $q$  replaced by  $q^r$  for various  $r$  gives a  $q$ -Weil function of weight  $\leq 2n - \{(2n-1) - m\} - 2 = m-1$ .  $\square$

DEFINITION 3.3.5.— Let  $A_1, \dots, A_{n+1}, B_1, \dots, B_n$  be characters on  $\mathbb{F}_q^\times$ , and put  $B_{n+1} = \epsilon$ . Let  $n'$  be the number of elements of the maximal multiset of characters which is a sub-multiset of both  $\{A_1, \dots, A_{n+1}\}$  and  $\{B_1, \dots, B_{n+1}\}$ . After a change of indices, we assume that the maximal multiset is  $\{A_1, \dots, A_{n'}\} = \{B_1, \dots, B_{n'}\}$  (note that  $B_{n+1}$  may no longer be

trivial since we have changed the indices). Now, we define the *hypergeometric function with reduced parameters over  $\mathbb{F}_q$*  by

$${}_{n+1}F_n\text{Red}\left(\begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix}; x\right)_{\mathbb{F}_q} := \frac{1}{q-1} \sum_{\chi \in \mathbb{F}_q^\times} \prod_{i=n'+1}^{n+1} \frac{G(A_i \chi)}{G(A_i)} \prod_{i=n'+1}^{n+1} \frac{G(\overline{B_i \chi})}{G(\overline{B_i})} \chi(-1)^{n-n'+1} \chi(x).$$

The one over  $\mathbb{F}_{q^r}$  is defined similarly for each positive integer  $r$ .

REMARK 3.3.6.— We remark some easy facts on the hypergeometric function with reduced parameters. We follow the notation in Definition 3.3.5. Let  $m$  denote the number of trivial characters among  $n$  characters  $B_1, \dots, B_n$ ,

(i) If all  $A_i$ 's are non-trivial, then the hypergeometric function with reduced parameters over  $\mathbb{F}_q$  equals

$${}_{n-n'+1}F_{n-n'}\left(\begin{matrix} A_{n'+1}, \dots, A_{n+1} \\ B_{n'+1}, \dots, B_n \end{matrix}; x\right)_{\mathbb{F}_q}.$$

(ii) The hypergeometric function with reduced parameters gives a pure  $q$ -Weil function. Its weight is  $m$  if all  $A_i$ 's are non-trivial, and  $m-1$  if exactly one of  $A_i$ 's is trivial; the former half is nothing but Proposition 3.3.2, and the latter half is proved in the same way.

(iii) Proposition 3.3.4 shows that, if all  $A_i$ 's are non-trivial, then the function

$$r \mapsto {}_{n+1}F_n\left(\begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix}; x\right)_{\mathbb{F}_{q^r}} - {}_{n+1}F_n\text{Red}\left(\begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix}; x\right)_{\mathbb{F}_{q^r}}$$

is a  $q$ -Weil function of weight  $\leq m-1$ . By using the fact that for  $C = \epsilon$  we have

$$\frac{G(C\chi)G(\overline{C\chi})}{G(C)G(\overline{C})} = \begin{cases} q\chi(-1) & \text{if } \chi \neq \epsilon, \\ q\chi(-1) - (q-1)C(-1) & \text{if } \chi = \epsilon, \end{cases}$$

we can prove that, if exactly one of  $A_i$ 's is trivial, then the function

$$r \mapsto {}_{n+1}F_n\left(\begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix}; x\right)_{\mathbb{F}_{q^r}} - q {}_{n+1}F_n\text{Red}\left(\begin{matrix} A_1, \dots, A_{n+1} \\ B_1, \dots, B_n \end{matrix}; x\right)_{\mathbb{F}_{q^r}}$$

is also a  $q$ -Weil function of weight  $\leq m-1$ .

We finish this subsection by giving a calculation used in the last subsection.

PROPOSITION 3.3.7.— Let  $\alpha_1, \dots, \alpha_{n+1}$  be positive integers, put  $\alpha := \alpha_1 + \dots + \alpha_{n+1}$ , and assume that  $q$  is congruent to 1 modulo all  $\alpha_i$ 's and modulo  $\alpha$ . Recall the definition of  $\varphi_{\alpha_i}$ 's and  $\varphi_\alpha$  in Subsection 0.5.

Let  $A_1, \dots, A_{n+1}, B$  be characters on  $\mathbb{F}_q^\times$  with  $A_{n+1} = \epsilon$ . Then, we have the equation

$$\begin{aligned} & \sum_{\chi \in \mathbb{F}_q^\times} \prod_{i=1}^{n+1} G(\overline{A_i \chi}^{\alpha_i}) G((B\chi)^\alpha) \chi((-1)^\alpha x) \\ &= (-1)^n (q-1) \prod_{i=1}^{n+1} A_i(\alpha_i^{-\alpha_i}) \cdot B(\alpha^\alpha) \prod_{i=1}^{n+1} \left\{ G(\overline{A_i}) \prod_{b_i=1}^{\alpha_i-1} \frac{G(\overline{A_i \varphi_{\alpha_i}^{b_i}})}{G(\varphi_{\alpha_i}^{b_i})} \right\} \cdot G(B) \prod_{b=1}^{\alpha-1} \frac{G(B \varphi_\alpha^b)}{G(\varphi_\alpha^b)} \\ & \quad \times {}_\alpha F_{\alpha-1} \left( \begin{matrix} B, B\varphi_\alpha, B\varphi_\alpha^2, \dots, B\varphi_\alpha^{\alpha-1} \\ A_1, A_1\varphi_{\alpha_1}, \dots, A_1\varphi_{\alpha_1}^{\alpha_1-1}, \dots, A_{n+1}\varphi_{\alpha_{n+1}}, \dots, A_{n+1}\varphi_{\alpha_{n+1}}^{\alpha_{n+1}-1} \end{matrix}; \frac{\alpha^\alpha}{\alpha_1^{\alpha_1} \dots \alpha_{n+1}^{\alpha_{n+1}}} x \right). \end{aligned}$$

*Proof.* By Proposition 3.2.5, we have

$$G(\overline{A_i \chi}^{\alpha_i}) = -G(\overline{A_i \chi}) \prod_{s_i=1}^{\alpha_i-1} \frac{G(\overline{A_i \chi} \varphi_{\alpha_i}^{s_i})}{G(\varphi_{\alpha_i}^{s_i})} (A_i \chi) (\alpha_i^{-\alpha_i})$$

$$G((B \chi)^\alpha) = -G(B \chi) \prod_{s=1}^{\alpha-1} \frac{G(B \chi \varphi_\alpha^s)}{G(\varphi_\alpha^s)} (B \chi) (\alpha^\alpha),$$

which shows that the left-hand side of the equation in the statement equals

$$(-1)^n \prod_{i=1}^{n+1} A_i (\alpha_i^{-\alpha_i}) \cdot B(\alpha^\alpha)$$

$$\times \sum_{\chi \in \overline{\mathbb{F}_q}^\times} \prod_{i=1}^{n+1} G(\overline{A_i \chi}) \cdot G(B \chi) \prod_{i=1}^{n+1} \prod_{b_i=1}^{\alpha_i-1} \frac{G(\overline{A_i \chi} \varphi_{\alpha_i}^{b_i})}{G(\varphi_{\alpha_i}^{b_i})} \cdot \prod_{b=1}^{\alpha-1} \frac{G(B \chi \varphi_\alpha^b)}{G(\varphi_\alpha^b)} \chi \left( (-1)^\alpha \frac{\alpha^\alpha}{\alpha_1^{\alpha_1} \dots \alpha_{n+1}^{\alpha_{n+1}}} x \right).$$

This shows the proposition.  $\square$

**3.4. Precise statement of the main result.** In this subsection, we state the main theorem of this section, that is, the precise factorisation of the zeta function of  $X_\lambda$ . In order to construct the factor, we need the following lemma.

LEMMA 3.4.1.— *For a positive integer  $N$ , the endomorphism of  $(\mathbb{Z}/N\mathbb{Z})^{n+1}$  defined by the matrix  $A' \bmod N$  induces*

$$f_N: \frac{(\mathbb{Z}/N\mathbb{Z})^{n+1}}{\mathfrak{t}(\alpha_1, \dots, \alpha_{n+1})} \rightarrow (\mathbb{Z}/N\mathbb{Z})^{n+1}$$

*Now, let  $N$  be a positive integer that divides all  $\alpha_i$ 's,  $\alpha$  and all non-zero elementary divisors of  $A'$ . Moreover, let  $N'$  be a positive integer divisible by  $N$ . Then, the multiplication by  $N'/N$  induces the isomorphism*

$$\text{Ker}(f_N) \rightarrow \text{Ker}(f_{N'})$$

*Proof.* The cyclic subgroup of  $(\mathbb{Z}/N\mathbb{Z})^{n+1}$  generated by  $\mathfrak{t}(\alpha_1, \dots, \alpha_{n+1})$  is of order  $N$  because the greatest common divisor of all  $\alpha_i$ 's equals 1 by definition (Proposition 1.2.1). This fact shows that the homomorphism in question is injective. Now we are reduced to showing that the two groups are of the same order, and it therefore suffices to show that the image of  $f_N$  defined by  $A' \bmod N$  has exactly  $N^n d$  elements for an integer  $d$  independent of  $N$ .

Let  $d_1, \dots, d_n, 0$  be the elementary divisors of  $A'$  and put  $d := d_1 \dots d_n$ . After a base change of  $\mathbb{Z}^{n+1}$ , it suffices to show the claim for the endomorphism of  $(\mathbb{Z}/N\mathbb{Z})^{n+1}$  induced by the diagonal matrix  $\text{diag}(d_1, \dots, d_n, 0)$ . In this case, we have

$$\text{Image}(f_N) = \bigoplus_{i=1}^n d_i \mathbb{Z}/N\mathbb{Z},$$

and the claim is obvious by the hypothesis on  $N$ .  $\square$

In the rest of this section, we fix  $d$  vectors  $s_0 = 0, s_1, \dots, s_{d-1} \in \mathbb{N}^{n+1}$  that represent  $\text{Ker}(f_{q-1})$  so that all entries of these vectors are in  $\{0, 1, \dots, q-2\}$ . Let us write  $s_j$  as  ${}^t(s_{1j}, \dots, s_{n+1,j})$ . Replacing  $q$  by a power  $q^r$  invokes the multiplication on vectors  $s_j$  by  $(q^r - 1)/(q - 1)$ . Therefore, if we replace  $q$  by a sufficiently large power of  $q$ , each  $s_{ij}$  is divisible by  $\alpha_i$ , and all  $|s_j|$ 's are divisible by  $\alpha$ . *From now on, we assume that  $q$  satisfies this condition.*

We also assume that  $t_{n+1,j} = s_{n+1,j} = 0$  for all  $j$ , since the condition above allows it. Now, we put  $t_{ij} := s_{ij}/\alpha_i$  and  $t_j := |s_j|/\alpha$  (note that  $t_j$  is not the sum of  $t_{ij}$ 's). Since  $|s_j| < n(q-1) < \alpha(q-1)$ , we have  $t_j \in \{0, 1, \dots, q-2\}$ , and  $t_j = 0$  if and only if  $j = 0$ .

Besides the condition on  $q$  above, we assume the following congruence condition. Let  $J = \{j_1, \dots, j_t\}$  be a subset of  $\{1, 2, \dots, n+1\}$  with distinct  $t \geq (n+1)/2$  elements. *In the rest of this section, we assume the following congruence for all such  $J$ :* for the list  $i_1, \dots, i_s$  of all indices  $i = 1, \dots, n+1$  such that  $a_{ji} = 0$  for all  $j \notin J$ , and we assume that all elementary divisors of

$$\begin{pmatrix} a_{j_1, i_1} & \dots & a_{j_1, i_s} \\ \vdots & & \vdots \\ a_{j_t, i_1} & \dots & a_{j_t, i_s} \\ 1 & \dots & 1 \end{pmatrix} \in M_{t+1, s}(\mathbb{Z})$$

divide  $q-1$ . Here, note that this matrix is of rank  $s$  and therefore all elementary divisors are non-zero. In fact, since  $\{i_1, \dots, i_s\}$  is a subset of  $J$ , the matrix above contains

$$\begin{pmatrix} a_{i_1, i_1} & \dots & a_{i_1, i_s} \\ \vdots & & \vdots \\ a_{i_s, i_1} & \dots & a_{i_s, i_s} \\ 1 & \dots & 1 \end{pmatrix}$$

as a minor matrix. Then, for each element of the kernel of the homomorphism defined by this matrix, all the coefficients have the same sign as proved in Proposition 1.2.1, which forces the element to be zero since all the entries of the lowest row are 1.

EXAMPLE 3.4.2.— (i) Consider the Dwork family  $F_0(T) = T_1^{n+1} + \dots + T_{n+1}^{n+1}$ ; recall that  $\alpha = {}^t(1, 1, \dots, 1)$ . Then, the congruence relation we impose is  $q \equiv 1 \pmod{n+1}$ . In fact, first, the elementary divisors of  $A'$  is  $1, n+1, \dots, n+1, 0$ . Since  $\text{Ker}(f_{q-1})$  (under the notation in Lemma 3.4.1) is generated by  $n$  vectors  ${}^t((q-1)/(n+1), n(q-1)/(n+1), 0, \dots, 0)$ ,  ${}^t(0, (q-1)/(n+1), n(q-1)/(n+1), 0, \dots, 0)$ ,  $\dots$ ,  ${}^t(0, \dots, 0, (q-1)/(n+1), n(q-1)/(n+1))$  modulo  $q-1$ , there are no extra conditions concerning this vector. Moreover, all elementary divisors of the matrices above divide  $n+1$ .

(ii) Consider the example from Example 1.2.3 (ii), that is,  $F_0(T) = T_1^4 + T_1 T_2^3 + T_3^4 + T_4^4$ ; recall that  $\alpha = {}^t(2, 4, 3, 3)$ . Then, the congruence relation we impose is  $q \equiv 1 \pmod{24}$ . In fact, first, the elementary divisors of  $A'$  is  $1, 1, 4, 0$ . Since  $\text{Ker}(f_{q-1})$  is generated by the vector  ${}^t((q-1)/4, 0, 3(q-1)/4, 0)$ , we have to impose the condition “modulo 24”, not just “modulo 12”. The  $J$ 's to be taken care of is  $\{2, 3, 4\}$ ,  $\{1, 3, 4\}$ ,  $\{1, 2, 3\}$ ,  $\{1, 2\}$ ,  $\{2, 3\}$ ,  $\{1, 3\}$ ,  $\{3, 4\}$  and these sets with 3 and 4 reversed. In fact, all elementary divisors to be considered divide 4;

For example, the matrices above corresponding to  $J = \{2, 3, 4\}, \{1, 3, 4\}, \{3, 4\}$  are

$$\begin{pmatrix} 0 & 0 \\ 4 & 0 \\ 0 & 4 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 \\ 0 & 4 \\ 1 & 1 \end{pmatrix}$$

respectively.

DEFINITION 3.4.3. — Fix a generator  $\rho$  of  $\widehat{\mathbb{F}_q^\times}$ . For each  $j = 0, \dots, d-1$ , define the element  $\gamma(j)$  of  $\mathbb{F}_q$  by

$$\prod_{i=1}^{n+1} \rho^{s_{ij}} (\alpha_i^{-1} c_i) \cdot \rho^{s_j} (-\lambda)^{-1} \alpha \prod_{i=1}^{n+1} \left\{ G(\rho^{-t_{ij}}) \prod_{b_i=1}^{\alpha_i-1} \frac{G(\rho^{-t_{ij}} \varphi_{\alpha_i}^{b_i})}{G(\varphi_{\alpha_i}^{b_i})} \right\} G(\rho^{t_j}) \prod_{b=1}^{\alpha-1} \frac{G(\rho^{t_j} \varphi_{\alpha}^b)}{G(\varphi_{\alpha}^b)}.$$

For a positive integer  $r$ , the element  $\gamma_r(j)$  is similarly defined by replacing  $\rho$  by  $\rho_{q^r}$  and  $\varphi_{\beta}$ 's by  $\varphi_{\beta, q^r}$ 's for  $\beta \in \{\alpha_1, \dots, \alpha_{n+1}, \alpha\}$ .

Note that  $\gamma_r(0) = 1$  since  $s_{i0} = 0$  for all  $i$ . In fact, it gives a pure  $q$ -Weil function as follows.

LEMMA 3.4.4. — Let us fix an index  $j \in \{0, \dots, d-1\}$ . Then, for each positive integer  $r$ , we have

$$\gamma_r(j) = (\gamma(j))^r.$$

Moreover, if  $j \neq 0$ , then  $\gamma_r(j)$  is a pure  $q$ -Weil function of weight  $\#\{i \in \{1, 2, \dots, n+1\} \mid s_{ij} \neq 0\} + 1 - \delta_{|s_j|}$ , where

$$\delta_{|s_j|} := \begin{cases} 1 & \text{if } |s_j| \equiv 0 \pmod{q-1}, \\ 0 & \text{if } |s_j| \not\equiv 0 \pmod{q-1}. \end{cases}$$

*Proof.* By the definition of  $t_{ij}$ 's,  $\rho^{-t_{ij}} \varphi_{\alpha_i}^{b_i}$  ( $b_i = 1, \dots, \alpha_i - 1$ ) cannot be the trivial character unless  $t_{ij} \neq 0$ . If no  $\rho^{t_j} \varphi_{\alpha}^b$ 's are trivial, that is, if  $|s_j| \not\equiv 0 \pmod{q-1}$ , then the weight is  $\#\{i \in \{1, 2, \dots, n+1\} \mid s_{ij} \neq 0\} + 1$  by Proposition 3.2.7. If  $|s_j| \equiv 0 \pmod{q-1}$ , then exactly one  $G(\rho^{t_j} \varphi_{\alpha}^b)$  is of weight 0, and the claim follows.  $\square$

We can now state our main theorem.

THEOREM 3.4.5. — Let  $\lambda$  an element of  $\mathbb{F}_q^\times$  such that  $X_\lambda$  is smooth and  $C \neq \lambda^\alpha$ . Define the polynomial  $P(x) \in \mathbb{Z}[x]$  by

$$\zeta(X_\lambda, x) = \frac{P(x)^{(-1)^n}}{(1-x)(1-qx) \dots (1-q^{n-1}x)}.$$

Then,  $P(x)$  equals

$$(1 - q^{(n-1)/2} x)^D \prod_{j=0}^{d-1} \zeta(\gamma_r(j) F(j)_r, x)^{-1}.$$

Here, the function  $F(j)_r$  is defined by sending  $r$  to

$$\alpha_{-1} F_{\alpha-2} \text{Red} \left( \begin{matrix} \varphi_{\alpha}, \dots, \varphi_{\alpha}^{\alpha-1} \\ \varphi_{\alpha_1}, \dots, \varphi_{\alpha_1}^{\alpha_1-1}, \dots, \varphi_{\alpha_{n+1}}, \dots, \varphi_{\alpha_{n+1}}^{\alpha_{n+1}-1}, \epsilon, \dots, \epsilon \end{matrix} ; C\lambda^{-\alpha} \right)_{\mathbb{F}_{q^r}}$$

if  $j = 0$ , and to

$$q^{\delta_{|s_j|-1}} {}_\alpha F_{\alpha-1} \text{Red} \left( \begin{matrix} \rho^{t_j}, \rho^{t_j} \varphi_\alpha, \dots, \rho^{t_j} \varphi_\alpha^{\alpha-1} \\ \rho^{t_{1j}}, \rho^{t_{1j}} \varphi_{\alpha_1}, \dots, \rho^{t_{1j}} \varphi_{\alpha_1}^{\alpha_1-1}, \dots, \rho^{t_{n+1,j}} \varphi_{\alpha_{n+1}}, \dots, \rho^{t_{n+1,j}} \varphi_{\alpha_{n+1}}^{\alpha_{n+1}-1} \end{matrix}; C\lambda^{-\alpha} \right)_{\mathbb{F}_{q^r}}$$

if  $j \neq 0$ ; the corresponding zeta functions are polynomials. The number  $D$  is defined to be the number of subsets  $J \subset \{1, 2, \dots, n+1\}$  such that  $\#J = (n+1)/2$  and that for all  $i = 1, \dots, n+1$  there exists  $j \notin J$  such that  $a_{ji} \geq 1$  (in particular, it is zero if  $n$  is even).

In order to prove this theorem, it suffices to show the following proposition.

PROPOSITION 3.4.6.— *The number of  $\mathbb{F}_{q^r}$ -rational points of  $X_\lambda$  equals*

$$\sum_{i=0}^{n-1} (q^r)^i + D(q^r)^{(n-1)/2} + (-1)^n \sum_{j=0}^{d-1} \gamma_r(j) F(j)_r,$$

where  $F(j)_r$  is the function defined in the statement of the theorem.

*Proof of Theorem 3.4.5 assuming Proposition 3.4.6.* For  $j = 0$ , the function  $F(0)_r$  gives a pure  $q$ -Weil function of weight  $n-1$ .

Assume that  $j \neq 0$ . Then, the function  $F(j)_r$  gives a pure  $q$ -Weil function of weight  $\#\{i \in \{1, \dots, n+1\} \mid s_{ij} = 0\} - 1 - \delta_{|s_j|} + 2(-1 + \delta_{|s_j|})$  by Remark 3.3.6 (ii). Therefore we see that the function  $r \mapsto \gamma_r(j) F(j)_r$  is, taking Lemma 3.4.4 into account, a pure  $q$ -Weil function of weight  $n-1$ .  $\square$

The key point of the proof of this proposition is to calculate the number of  $\mathbb{F}_{q^r}$ -rational points of  $X_\lambda$  “modulo  $q$ -Weil function of weight  $\leq n-2$ ”; in fact, we prove the following assertion in the next subsection, from which Proposition 3.4.6 directly follows.

PROPOSITION 3.4.7.— *The function  $r \mapsto X_\lambda(\mathbb{F}_{q^r})$  is the sum of the following functions:*

(i) *the function*

$$r \mapsto \sum_{i=0}^{n-1} (q^r)^i + D(q^r)^{(n-1)/2},$$

(ii) *the function  $r \mapsto \gamma_r(j) F(j)_r$  for each  $j = 0, \dots, d-1$ ,*

(iii) *a  $q$ -Weil function of weight  $\leq n-2$ .*

**3.5. Proof: Counting rational points.** In this subsection, we fix an element  $\lambda$  of  $\mathbb{F}_q$  such that  $X_\lambda$  is smooth and  $C \neq \lambda^\alpha$ . Let  $X_\lambda(\mathbb{F}_q)_0$  be the set of  $\mathbb{F}_q$ -rational points  $[x_1 : \dots : x_{n+1}]$  of  $X_\lambda$  at least one of whose coordinates is zero, and set  $X_\lambda(\mathbb{F}_q)_* := X_\lambda(\mathbb{F}_q) \setminus X_\lambda(\mathbb{F}_q)_0$ .

PROPOSITION 3.5.1.— *The function  $r \mapsto \#X_\lambda(\mathbb{F}_{q^r})_0$  is the sum of the function*

$$r \mapsto \sum_{i=0}^{n-1} (q^r)^i + D(q^r)^{(n-1)/2} - \frac{(q^r - 1)^n}{q^r}$$

*and a  $q$ -Weil function of weight  $\leq n-2$ .*

*Proof.* For a proper non-empty subset  $J \subset \{1, \dots, n+1\}$  and a positive integer  $r$ , denote by  $s_r(J)$  the number of  $\mathbb{F}_{q^r}$ -rational points  $[x_1 : \dots : x_{n+1}]$  of  $X_\lambda$  such that  $x_j \neq 0$  if and only if  $j \in J$ . Then, we have

$$\#X_\lambda(\mathbb{F}_{q^r})_0 = \sum_{t=1}^n \sum_{\#J=t} s_r(J).$$

By Lemma 3.5.2, each  $s_r(J)$  is of the form

$$\frac{(q^r - 1)^{t-1}}{q^r} + N'_J(q^r)^{(n-1)/2} + (a \text{ } q\text{-Weil function of weight } \leq n-2),$$

where  $N'_J$  is the number given in the statement of Lemma 3.5.2. This shows that  $\#X_\lambda(\mathbb{F}_{q^r})_0$  is written as

$$(3.5.1) \quad \sum_{t=1}^n \binom{n+1}{t} \frac{(q^r - 1)^{t-1}}{q^r} + D(q^r)^{(n+1)/2} + (a \text{ } q\text{-Weil function of weight } \leq n-2).$$

Since

$$\sum_{t=0}^{n+1} \binom{n+1}{t} \frac{(q^r - 1)^t}{q^r} = (q^r)^n,$$

the first term of (3.5.1) equals

$$\frac{(q^r)^n}{q^r - 1} - \frac{1}{q^r(q^r - 1)} - \frac{(q^r - 1)^n}{q^r} = \frac{(q^r)^n - 1}{q^r - 1} - \frac{(q^r - 1)^n}{q^r} + \frac{1}{q^r}.$$

□

LEMMA 3.5.2.— *Let  $J$  be a subset of  $\{1, \dots, n+1\}$  with distinct  $t_j$  elements that satisfies  $1 \leq t_j \leq n$ . We define a number  $N'_J$  as follows; it is 1 if  $\#J = (n+1)/2$  and for each  $i \in \{1, 2, \dots, n+1\}$  there exists  $j \notin J$  with  $a_{ji} \geq 1$ ; it is 0 otherwise. Then, the function*

$$r \mapsto s_r(J) := \# \{ [x_1 : \dots : x_{n+1}] \in X_\lambda(\mathbb{F}_{q^r}) \mid x_j \neq 0 \text{ if and only if } j \in J \},$$

can be written as

$$\frac{(q^r - 1)^{t_J - 1}}{q^r} + N'_J(q^r)^{(n-1)/2} + (a \text{ } q\text{-Weil function of weight } \leq n-2),$$

*Proof.* If  $t_J \leq n/2$ , then  $s_r(J)$  can be considered as the number of  $\mathbb{F}_{q^r}$ -rational points of a closed subscheme of  $\mathbb{G}_{m, \mathbb{F}_q}^{t_J - 1}$  with  $t_J - 1 \leq n/2 - 1$ , and therefore each term of the function in the statement is a  $q$ -Weil function of weight  $\leq n-2$ . Therefore, we assume that  $t_J \geq (n+1)/2$ . If  $N'_J = 1$ , then the claim is also trivial since  $X_J$  is isomorphic to  $\mathbb{G}_{m, \mathbb{F}_q}^{(n+1)/2 - 1}$ ; now we assume that  $N'_J = 0$ . For simplicity, we change the coordinates so that  $J = \{1, \dots, t_J\}$ , and so that the following two conditions hold for an  $s \in \{1, 2, \dots, t_J\}$ :

- (i)  $a_{ij} = 0$  for all  $i \notin J$  and  $j = 1, \dots, s$  and
- (ii) for all  $j = s+1, \dots, n+1$ , there exists an  $i \notin J$  such that  $a_{ij} \geq 1$ .

Now, by Lemma 3.2.1 (i),

$$\begin{aligned}
 q(q-1)s_1(J) &= q\#\{(x_1, \dots, x_{t_j}) \in \mathbb{F}_q^\times \mid c_1x^{a_1} + \dots + c_sx^{a_s} = 0\} \\
 &= \sum_{w \in \mathbb{F}_q^\times} \sum_{x_1, \dots, x_{t_j} \in \mathbb{F}_q^\times} \theta(wc_1x^{a_1} + \dots + wc_sx^{a_s}) \\
 (3.5.2) \quad &= (q-1)^{t_j} + \sum_{w, x_1, \dots, x_{t_j} \in \mathbb{F}_q^\times} \prod_{i=1}^s \theta(wc_ix^{a_i}).
 \end{aligned}$$

Moreover, Proposition 3.2.4 shows that the second term equals

$$\begin{aligned}
 &\frac{1}{(q-1)^{t_j}} \sum_{w, x_1, \dots, x_{t_j} \in \mathbb{F}_q^\times} \prod_{i=1}^s \sum_{\chi_i \in \widehat{\mathbb{F}_q^\times}} G(\overline{\chi_i}) \chi_i(wc_ix^{a_i}) \\
 &= \frac{1}{(q-1)^{t_j}} \sum_{\chi_1, \dots, \chi_s \in \widehat{\mathbb{F}_q^\times}} \prod_{i=1}^s \chi_i(c_i) G(\overline{\chi_i}) \sum_{w \in \mathbb{F}_q^\times} \chi_1 \dots \chi_s(w) \prod_{m=1}^{t_j} \sum_{x_m \in \mathbb{F}_q^\times} \chi_1^{a_{m1}} \dots \chi_s^{a_{ms}}(x_m),
 \end{aligned}$$

and by Proposition 3.2.1 (ii), the number

$$\sum_{w \in \mathbb{F}_q^\times} \chi_1 \dots \chi_s(w) \prod_{m=1}^{t_j} \sum_{x_m \in \mathbb{F}_q^\times} \chi_1^{a_{m1}} \dots \chi_s^{a_{ms}}(x_m)$$

is zero unless  $\chi_1 \dots \chi_s = \epsilon$  and  $\chi_1^{a_{m1}} \dots \chi_s^{a_{ms}} = \epsilon$  for all  $m = 1, \dots, t_j$ , in which case it equals  $(q-1)^{t_j+1}$ . By writing  $\chi_i = \rho^{k_i}$ , the condition above is written as

$$\begin{aligned}
 k_1 + \dots + k_s &\equiv 0 \pmod{q-1}, \\
 a_{m1}k_1 + \dots + a_{ms}k_s &\equiv 0 \pmod{q-1} \quad (m = 1, \dots, t_j).
 \end{aligned}$$

Therefore, if we denote by  $d'_1, \dots, d'_s$  the elementary divisors of the  $(t_j+1) \times s$  matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1s} \\ \vdots & & \vdots \\ a_{t_j,1} & \dots & a_{t_j,s} \\ 1 & \dots & 1 \end{pmatrix},$$

then by hypothesis and an argument as in Lemma 3.4.1, the second term of (3.5.2) is  $(q-1)$  times the sum of  $d'_1 \dots d'_s$  numbers that are the products of  $s \leq t$  Gauss sums.

We may do this computation in the same way for various  $r$ , and therefore, the statement similar to Lemma 3.4.1 and Proposition 3.2.7 shows the claim.  $\square$

Now, let us investigate the function  $r \mapsto \#X_\lambda(\mathbb{F}_{q^r})_*$ . For the investigation, we prove the following fact.

**PROPOSITION 3.5.3.**— *The function  $r \mapsto \#X_\lambda(\mathbb{F}_{q^r})_*$  is the sum of the following functions:*

- (i)  $r \mapsto (q^r - 1)^n / q^r$ ,
- (ii)  $r \mapsto (-1)^n \gamma_r(0) F(0)_r$  for each  $j = 0, \dots, d-1$ , and
- (iii) a  $q$ -Weil function of weight  $\leq n-2$ .



*Proof.* In this proof, we only investigate the number  $X_\lambda(\mathbb{F}_{q^r})_*$  for  $r = 1$  because we may compute it for general  $r$  in exactly the same way.

By using Lemma 3.2.1 (i),

$$\begin{aligned}
 & q(q-1) \cdot \#X_\lambda(\mathbb{F}_q)_* \\
 &= q \# \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{F}_q^\times \mid F_\lambda(x_1, \dots, x_{n+1}) = 0 \right\} \\
 &= \sum_{w \in \mathbb{F}_q} \sum_{x_1, \dots, x_{n+1} \in \mathbb{F}_q^\times} \theta(w F_\lambda(x_1, \dots, x_{n+1})) \\
 (3.5.3) \quad &= (q-1)^{n+1} + \sum_{w, x_1, \dots, x_{n+1} \in \mathbb{F}_q^\times} \prod_{i=1}^{n+1} \theta(w c_i x^{a_i}) \cdot \theta(-w \lambda x_1 \dots x_{n+1}).
 \end{aligned}$$

Now, our task is the calculation of the second term.

By Proposition 3.2.4, it equals  $1/(q-1)^{n+2}$  times

$$\begin{aligned}
 & \sum_{w, x_1, \dots, x_{n+1} \in \mathbb{F}_q^\times} \prod_{i=1}^{n+1} \sum_{\chi_i \in \widehat{\mathbb{F}_q^\times}} G(\chi_i) \chi_i(w c_i x^{a_i}) \cdot \sum_{\mu \in \widehat{\mathbb{F}_q^\times}} G(\mu) \mu(-w \lambda x_1 \dots x_{n+1}) \\
 &= \sum_{w, x_1, \dots, x_{n+1} \in \mathbb{F}_q^\times} \sum_{\chi_1, \dots, \chi_{n+1}, \mu \in \widehat{\mathbb{F}_q^\times}} \prod_{i=1}^{n+1} G(\chi_i) \cdot G(\mu) \prod_{i=1}^{n+1} \chi_i(w c_i x^{a_i}) \cdot \mu(-w \lambda x_1 \dots x_{n+1}) \\
 &= \sum_{\chi_1, \dots, \chi_{n+1}, \mu \in \widehat{\mathbb{F}_q^\times}} \prod_{i=1}^{n+1} G(\chi_i) \cdot G(\mu) \prod_{i=1}^{n+1} \chi_i(c_i) \sum_{w \in \mathbb{F}_q^\times} \chi_1 \dots \chi_{n+1} \mu(w) \prod_{i=1}^{n+1} \sum_{x_i \in \mathbb{F}_q^\times} \chi_1^{a_{i1}} \dots \chi_{n+1}^{a_{i,n+1}} \mu(x_i) \cdot \mu(-\lambda).
 \end{aligned}$$

Now, fix a generator  $\rho$  of  $\widehat{\mathbb{F}_q^\times}$ , and write  $\chi_i = \rho^{k_i}$  and  $\mu = \rho^m$  for  $k_i, m = 0, \dots, q-2$ . Then, for an  $(n+2)$ -tuple  $(\chi_1, \dots, \chi_{n+1}, \mu)$ , the number

$$\sum_{w \in \mathbb{F}_q^\times} \chi_1 \dots \chi_{n+1} \mu(w) \prod_{i=1}^{n+1} \sum_{x_i \in \mathbb{F}_q^\times} \chi_1^{a_{i1}} \dots \chi_{n+1}^{a_{i,n+1}} (x_i)$$

is non-zero if and only if

$$\begin{aligned}
 & k_1 + \dots + k_{n+1} + m \equiv 0 \pmod{q-1}, \\
 & a_{11} k_1 + \dots + a_{1,n+1} k_{n+1} + m \equiv 0 \pmod{q-1}, \\
 & \vdots \\
 & a_{n+1,1} k_1 + \dots + a_{n+1,n+1} k_{n+1} + m \equiv 0 \pmod{q-1},
 \end{aligned}$$

in which case it equals  $(q-1)^{n+2}$  (Proposition 3.2.1 (ii)). By Lemma 3.4.1, an  $(n+2)$ -tuple  $(k_1, \dots, k_{n+1}, m)$  satisfies the relations above if and only if there exists an index  $j = 0, \dots, d-1$  and  $a = 0, \dots, q-2$  such that

$$k_i \equiv s_{ij} + a \alpha_i \quad (i = 1, \dots, n+1) \quad \text{and} \quad m \equiv -|s_j| - a \alpha;$$

the choice of  $j$  and  $a$  is unique. Now, we have shown that the second term of (3.5.3) is

$$\begin{aligned} & \sum_{j=0}^{d-1} \sum_{a=0}^{q-2} \left\{ \prod_{i=1}^{n+1} G(\rho^{-s_{ij}-aa_i}) \cdot G(\rho^{|s_j|+aa}) \prod_{i=1}^{n+1} \rho^{s_{ij}+aa_i}(c_i) \rho^{-|s_j|-aa}(-\lambda) \right\} \\ &= \sum_{j=0}^{d-1} \left\{ \prod_{i=1}^{n+1} \rho^{s_{ij}}(c_i) \cdot \rho^{-|s_j|}(-\lambda) \sum_{a=0}^{q-2} \prod_{i=1}^{n+1} G(\rho^{-s_{ij}-aa_i}) \cdot G(\rho^{|s_j|+aa}) \rho^a \left( (-1)^a \prod_{i=1}^{n+1} c_i^{\alpha_i} \cdot \lambda^{-\alpha} \right) \right\}. \end{aligned}$$

After fixing  $j$ , we apply Proposition 3.3.7 to get

$$\begin{aligned} & \sum_{a=0}^{q-2} \prod_{i=1}^{n+1} G(\rho^{-s_{ij}-aa_i}) \cdot G(\rho^{|s_j|+aa}) \rho^a \left( (-1)^a \prod_{i=1}^{n+1} c_i^{\alpha_i} \cdot \lambda^{-\alpha} \right) \\ &= (-1)^n (q-1) \prod_{i=1}^{n+1} \rho^{-t_{ij}}(\alpha_i^{-\alpha_i}) \cdot \rho^{t_j}(\alpha^\alpha) \prod_{i=1}^{n+1} \left\{ G(\rho^{-t_{ij}}) \prod_{b_i=1}^{\alpha_i-1} \frac{G(\rho^{-t_{ij}} \varphi_{\alpha_i}^{b_i})}{G(\varphi_{\alpha_i}^{b_i})} \right\} G(\rho^{t_j}) \prod_{b=1}^{\alpha-1} \frac{G(\rho^{t_j} \varphi_\alpha^b)}{G(\varphi_\alpha^b)} \\ & \quad \times {}_\alpha F_{\alpha-1} \left( \begin{matrix} \rho^{t_j}, \rho^{t_j} \varphi_\alpha, \dots, \rho^{t_j} \varphi_\alpha^{\alpha-1} \\ \rho^{t_{1j}}, \rho^{t_{1j}} \varphi_{\alpha_1}, \dots, \rho^{t_{1j}} \varphi_{\alpha_1}^{\alpha_1-1}, \dots, \rho^{t_{n+1,j}} \varphi_{\alpha_{n+1}}, \dots, \rho^{t_{n+1,j}} \varphi_{\alpha_{n+1}}^{\alpha_{n+1}-1} \end{matrix} ; C\lambda^{-\alpha} \right)_{\mathbb{F}_q}. \end{aligned}$$

Finally, the second term of (3.5.3) is the sum of the functions (ii) and (iii) because of Remark 3.3.6 (ii).  $\square$

*Proof of Proposition 3.4.7.* This is just a combination of Proposition 3.5.1 and 3.5.3.  $\square$

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