博士論文

論文題目: The Stokes semigroup on non-decaying spaces

（非減衰空間上のストークス半群）

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The Stokes semigroup on non-decaying spaces

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Contents

Preface

The propose of this thesis is to study the Stokes semigroup on spaces of bounded functions. It is well known that the solution operator of the linear Stokes equations, called the Stokes semigroup, is an analytic semigroup on L^r-solenoidal space, $r \in (1, \infty)$, for various kinds of domains including bounded domains with smooth boundaries [19], [7]. However, it had been a long-standing open problem whether or not the Stokes semigroup is an analytic semigroup on L^∞-type spaces even for smoothly bounded domains. For a half space, the Stokes semigroup is an analytic semigroup on L^∞-type spaces since explicit solution formulas are available [5], [20], [13]. It is the aim of the thesis to give an affirmative answer to this problem for bounded domains, and moreover, for a large class of domains including exterior domains and perturbed half spaces based on works [1], [2], [3].

For the Laplace operator or general elliptic operators, it is well known that the corresponding semigroup is analytic on L^∞-type spaces. K. Masuda was the first to prove the analyticity of the semigroup associated to general elliptic operators on a space of continuous functions in the whole space (including the case of higher orders) [14], [15], [16]. This result was then extended by H. B. Stewart to the case for the Dirichlet problem [21] and more general boundary conditions [22]. We refer to a book by A. Lunardi [12, Chapter 3] for this Masuda-Stewart method which applies to many other situations. However, it seems that their localization argument does not directly apply to the Stokes equations because of the presence of pressure.

In the sequel, we introduce a new a priori estimate for pressure in terms of velocity on L^∞ which plays a key role for the analyticity of the Stokes semigroup on L^∞. The new pressure estimate presented is available for merely bounded velocity while L^r-pressure bounds through the Helmholtz projection do not hold for $r = \infty$. The pressure estimate on L^∞ is a key in proving an a priori L^∞-estimate for solutions of the Stokes equations, which in particular implies that the Stokes semigroup is an analytic semigroup on L^∞.

The thesis is consist of 5 chapters. From Chapter 1 to Chapter 4, we prove an a priori L^∞-estimate for solutions of the non-stationary Stokes equations by a contradiction argument. Furthermore, in Chapter 5, we give a direct proof for the analyticity of the Stokes semigroup on L^∞ by a resolvent approach. We establish a corresponding resolvent estimate directly by the Maduda-Stewart technique. The former is the original proof based on a heuristic observation which implies a stronger estimate for higher derivatives than that of the resolvent. The latter is rather involved, but we are able to prove the maximum angle of the analytic semigroup on L^∞ which does not follow from a contradiction argument.

From Chapter 1 to Chapter 3, we study the a priori L^∞-estimate for solutions (v, q) of the non-stationary Stokes equations (subject to the Dirichlet boundary condition) in the domain Ω,

$$\sup_{0<t\leq T_0} \left\| N(v, q) \right\|_{L^\infty(\Omega)}(t) \leq C\|v_0\|_{L^\infty(\Omega)}, \tag{0.1.1}$$

where v_0 denotes the initial velocity and $N(v, q)(x, t)$ denotes the the scale invariant norm for solutions (v, q) up to second orders,

$$N(v, q)(x, t) = \left|v(x, t)\right| + t^{1/2}\left|\nabla v(x, t)\right| + t\left|\nabla^2 v(x, t)\right| + t\left|v_t(x, t)\right| + t\left|\nabla q(x, t)\right|. \tag{0.1.2}$$

The a priori estimate (0.1.1) implies that the Stokes semigroup is extendable to a C_0-analytic semigroup on the continuous solenoidal space $C_{0,\sigma}(\Omega)$. We prove the a priori L^∞-estimate (0.1.1) by *a blow-up argument*. A blow-up argument reduces the proof for the a priori L^∞-estimate (0.1.1) to the ”compactness” of a blow-up sequence and to the ”uniqueness” of a blow-up limit. By rescaling around a blow-up point, a limit problem is either the whole space or a half space. If the problem is the heat equation, it is easy to realize this argument. However, for the Stokes equations, both compactness and uniqueness are highly non-trivial problems because of the presence of pressure.

A blow-up argument was first introduced by E. De Giorgi [4] to study regularity of a minimal surface. B. Gidas and J. Spruck [6] adjusted a blow-up argument to derive an a priori bound for solutions of a semilinear elliptic problem. Y. Giga [8] applied it to the semilinear parabolic problem. The method has been further developed in recent years to obtain several a priori bounds, e.g., [18], [17]. However, it is quite recent to apply it to the Navier–Stokes equations [11], [9].

In Chapter 1, we study the uniqueness of the Stokes equations in a half space, which is used later in order to conclude that a blow-up limit is trivial. The uniqueness of the Stokes equations is well known for decaying velocity at infinity in spatial variables, but without assuming such a decay condition, the uniqueness results is less known. The L^∞-type uniqueness result was proved by V. A. Solonnikov [20], where a decay condition of pressure gradient to the normal direction is assumed. We give a short proof for his uniqueness result by using the L^1-estimate for spatial derivatives of the Stokes semigroup.

In order to solve both compactness of a blow-up sequence and uniqueness of a blow-up limit, a key is an estimate for pressure in terms of velocity called *the harmonic-pressure gradient estimate*,

$$\sup_{x\in\Omega} d_\Omega(x)\left|\nabla q(x, t)\right| \leq C_\Omega \left\| W(v) \right\|_{L^\infty(\partial\Omega)}(t), \tag{0.1.3}$$

where $d_\Omega(x)$ denotes the distance from $x \in \Omega$ to the boundary $\partial\Omega$ and $W(v) = -(\nabla v - \nabla^T v)n_\Omega$. When $n = 3$, $W(v)$ is nothing but the tangential component of vorticity, i.e., $-\mathrm{curl}\, v \times n_\Omega$. Here, n_Ω denotes the unit outward normal vector field on $\partial\Omega$. The harmonic-pressure gradient estimate (0.1.3) is a special case of an estimate for solutions of the homogeneous Neumann problem. A key observation is that the Neumann data of the pressure

q is transformed into the surface divergence of vorticity, i.e., $\Delta v \cdot n_\Omega = \mathrm{div}_{\partial\Omega}\, W(v)$. So the estimate (0.1.3) follows from an a priori estimate for solutions of the homogeneous Neumann problem:

$$\Delta q = 0 \text{ in } \Omega, \quad \frac{\partial q}{\partial n_\Omega} = \mathrm{div}_{\partial\Omega} W \text{ on } \partial\Omega. \tag{0.1.4}$$

The estimate (0.1.3) may not hold for general domains, so we call Ω *strictly admissible* if the a priori estimate (0.1.3) holds for the Neumann problem (0.1.4). Of course, a half space is strictly admissible. In Chapter 2, we give typical examples of strictly admissible domains: bounded domains, exterior domains and perturbed half spaces by showing the a priori estimate (0.1.3) by a blow-up argument. Recently, it turned out that the estimate (0.1.3) was also found by C. E. Kenig, F. Lin, and Z. Shen [10], independently of the works [1], [2]. Although they directly proved the estimate (0.1.3) for bounded domains by estimating the Green function, exterior domains and perturbed half spaces are not included there. Our proof by a blow-up argument is based on the uniqueness for the Neumann problem (0.1.4) and applicable to prove the estimate (0.1.3) without appealing to the Green function.

In Chapter 3, we establish the local Hölder estimates for the Stokes equations by using the harmonic-pressure gradient estimate (0.1.3), which is used to get a necessary compactness of a blow-up sequence for the Stokes equations. Using the results proved in Chapters 1 and 2, we prove the a priori L^∞-estimate (0.1.1) by a blow-up argument.

Chapter 4 is the goal. We extend the Stokes semigroup to the non-decaying type solenoidal space L^∞_σ. Note that for non-decaying initial data, the existence of solutions is non-trivial. We pointwise approximate elements of L^∞_σ by compactly supported smooth solenoidal vector fields and extend the Stokes semigroup to a non-C_0-analytic semigroup on L^∞_σ together with the L^∞-estimate (0.1.1).

Chapter 5 is devoted to the resolvent approach. We establish an a priori L^∞-estimate for the resolvent Stokes equations corresponding to (0.1.1) by the Masuda-Stewart technique, which in particular implies that the maximum angle of the analytic semigroup on L^∞ is $\pi/2$. Furthermore, the resolvent approach applies to different boundary conditions, e.g., to the Robin-type boundary condition, where a partial slip of velocity on the boundary is taken into account.

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Chapter 1

Uniqueness in a half space

In this chapter, we study the uniqueness of the Stokes equations in a half space in a space of bounded functions, which will be used later in Chapter 3 in order to prove the a priori L^∞-estimate for the non-stationary Stokes equations. The uniqueness of the Stokes equations is well known for decaying velocity at infinity in spatial variables, e.g., $v(\cdot,t) \in L^p$, $p \in (1,\infty)$. However, for merely bounded velocity, the uniqueness results is less known even for a half space. We prove the uniqueness of the Stokes equations for bounded velocity with assuming the decay condition for the tangential component of the pressure gradient, i.e., $\nabla_{\tan} q \to 0$ as $x_n \to \infty$. Such the decay condition is necessary since there exist non-trivial Poiseuille flow-type solutions. The proof is by a duality argument based on the L^1-estimate for spatial derivatives of the Stokes semigroup.

1.1 Introduction

We study the uniqueness of the Stokes equations in a half space $\mathbf{R}^n_+$, $n \geq 2$:

$$v_t - \Delta v + \nabla q = 0 \quad \text{in} \quad \mathbf{R}^n_+ \times (0,T), \tag{1.1.1}$$

$$\operatorname{div} v = 0 \quad \text{in} \quad \mathbf{R}^n_+ \times (0,T), \tag{1.1.2}$$

$$v = 0 \quad \text{on} \quad \partial\mathbf{R}^n_+ \times (0,T), \tag{1.1.3}$$

$$v(x,0) = v_0 \quad \text{on} \quad \mathbf{R}^n_+ \times \{t=0\}. \tag{1.1.4}$$

The uniqueness of the Stokes equations (1.1.1)–(1.1.4) is well known for decaying velocity at the infinity in spatial variables, e.g., $v(\cdot,t) \in L^p$ for $p \in (1,\infty)$. However, without assuming such the decay condition, the uniqueness results is less known even for a half space. The L^∞-type uniqueness was proved by V. A. Solonnikov [7, Theorem 1.1] for continuous velocity at $t=0$. We give a short proof for his uniqueness result based on [1]. The goal of this chapter is to prove:

Theorem 1.1.1. *Let $v \in C^{2,1}(\overline{\mathbf{R}^n_+} \times (0,T])$ and $\nabla q \in C(\overline{\mathbf{R}^n_+} \times (0,T])$ satisfy the Stokes equations (1.1.1)–(1.1.3). Assume that*

$$\sup_{0<t<T} \|v\|_{L^\infty(\mathbf{R}^n_+)}(t) < \infty, \tag{1.1.5}$$

and ∇v, $\nabla^2 v$, v_t, ∇q are bounded in $\mathbf{R}^n_+ \times [\delta, T]$ for each $\delta > 0$. Assume that $v \to 0$ weakly-$*$ on $L^\infty(\mathbf{R}^n_+)$ as $t \downarrow 0$. Assume in addition that

$$\nabla_{tan} q(x,t) \to 0 \quad as\ x_n \to \infty, \tag{1.1.6}$$

for $x' \in \mathbf{R}^{n-1}$, $t \in (0,T)$. Then, $v \equiv 0$ and $\nabla q \equiv 0$.

Remark 1.1.2. If we drop the condition (1.1.6), the statement of Theorem 1.1.1 does not hold since there exists a *Poiseuille flow-type solution*, which is a non-trivial solution satisfying the Stokes (Navier–Stokes) equations (1.1.1)–(1.1.4). We say a solution $(v, \nabla q)$ is Poiseuille flow-type in the sense that there is a function $a(t) = (a_{\tan}(t), 0)$ such that $(v, \nabla q)$ is represented by

$$v(x_n, t) = (v_{\tan}(x_n, t), 0), \quad \nabla q(t) = a(t), \tag{1.1.7}$$

where $v_{\tan}(x_n, t)$ and $a_{\tan}(t)$ respectively denote the tangential component of $v(x_n, t)$ and $a(t)$. Note that each component of velocity $v^i(x_n, t)$ solves the heat equation in a half line,

$$\partial_t v^i(x_n, t) - \partial^2_{x_n} v^i(x_n, t) = -a^i(t), \tag{1.1.8}$$

and satisfies the Dirichlet and initial conditions $v^i = 0$ on $\{x_n = 0\}$ and $\{t = 0\}$. The assumption (1.1.6) says that a Poiseuille flow-type solution must be zero. In fact, the condition (1.1.6) implies $a(t) \equiv 0$ so $v \equiv 0$ follows from the uniqueness of the heat equation.

Let us sketch the proof of Theorem 1.1.1. We apply a duality argument to the tangential derivatives of the velocity $\partial_{\tan} v$ instead of v, where $\partial_{\tan}$ indiscriminately denotes tangential derivatives $\partial_j v$ for $j \in \{1, \cdots, n-1\}$. We prove $\partial_{\tan} v \equiv 0$ by invoking the L^1-estimate of the Stokes semigroup,

$$\|\nabla S(t) v_0\|_{L^1(\mathbf{R}^n_+)} \leq C/t^{1/2} \|v_0\|_{L^1(\mathbf{R}^n_+)} \quad \text{for } t > 0, \tag{1.1.9}$$

while the solution $S(t)v_0$ itself does not belong to $L^1(\mathbf{R}^n_+)$ in general, i.e., $\|S(t)v_0\|_{L^1(\mathbf{R}^n_+)} \not\leq C\|v_0\|_{L^1(\mathbf{R}^n_+)}$ [3], [2], [5]. Once we have $\partial_{\tan} v \equiv 0$, i.e., $\partial_j v^i \equiv 0$ for $i \in \{1, \cdots, n\}$, $j \in \{1, \cdots, n-1\}$, by (1.1.2), it follows that

$$\frac{\partial v^n}{\partial x_n} = \sum_{j=1}^{n-1} \frac{\partial v^j}{\partial x_j} = 0,$$

so $v^n \equiv 0$ and $\partial q/\partial x_n \equiv 0$ by (1.1.1) and (1.1.3). Thus, $(v, \nabla q)$ is a Poiseuille flow-type solution (1.1.7). We are able to prove $\partial_{\tan} v \equiv 0$ without using the condition (1.1.6). On the contrary, this means the following:

Lemma 1.1.3. *Under the assumptions of Theorem 1.1.1 except (1.1.6), a non-trivial solution* $(v, \nabla q)$ *must be a Poiseuille flow-type solution (1.1.7).*

Proof of Theorem 1.1.1. By Lemma 1.1.3 and (1.1.6), it follows that $(v, \nabla q)$ is a Poiseuille flow-type solution (1.1.7) for $a(t) \equiv 0$. Then, each component $v^i(x_n, t)$ solves the heat equation in a half line and $v^i(x_n, t) \to 0$ weakly-$*$ on $L^\infty(0, \infty)$ as $t \downarrow 0$. By multiplying $\phi \in C_c^{2,1}([0, \infty) \times [0, T))$ satisfying $\phi = 0$ on $\{x_n = 0\} \times (0, T)$ to $v^i(x_n, t)$ and integrating by parts, it follows that

$$\int_0^T \int_0^\infty v^i(x_n, t)(\phi_t(x_n, t) + \partial_n^2 \phi(x_n, t)) \mathrm{d}x_n \mathrm{d}t = 0.$$

Then, by a duality argument to the heat equation, $v^i \equiv 0$ follows. □

This chapter is organized as follows. In Section 2, we prove Lemma 1.1.3 by a duality argument. In Section 3, we estimate L^1-norms for solutions of the dual problem based on the fundamental solutions to (1.1.1)–(1.1.4).

1.2 Duality arguments to tangential derivatives of velocity

We prove Lemma 1.1.3 by applying a duality argument to $\partial_{\tan} v$. We choose test functions by compactly supported solenoidal vector fields in order to estimate L^1-norms of solutions for the dual problem via the L^1-estimate of the Stokes semigroup (1.1.9). To state a result, let $C_{c,\sigma}^\infty(\mathbf{R}_+^n)$ be the space of all smooth solenoidal vector fields with compact support in $\mathbf{R}_+^n$. Let $C_{c,\sigma}^\infty(\mathbf{R}_+^n \times (0, T))$ be the space of all functions $f \in C_c^\infty(\mathbf{R}_+^n \times (0, T))$ such that $f(\cdot, t) \in C_{c,\sigma}^\infty(\mathbf{R}_+^n)$ for each $t \in (0, T)$. The goal of this section is to prove:

Proposition 1.2.1. *Under the assumption of Theorem 1.1.1 except (1.1.6), we have*

$$\int_0^T \int_{\mathbf{R}_+^n} v(x, t) \cdot \partial_{tan} f(x, t) dx dt = 0 \tag{1.2.1}$$

for all $f \in C_{c,\sigma}^\infty(\mathbf{R}_+^n \times (0, T))$.

Remark 1.2.2. Actually, we are able to prove (1.2.1) for all functions $f \in C_c^\infty(\mathbf{R}_+^n \times (0, T))$. In fact, in the original proof [7], $\partial_{\tan} v \equiv 0$ is directly proved by estimating L^1-norms of solutions for the dual problem:

$$-\partial_t \varphi - \Delta \varphi + \nabla \pi = \mathbf{P} \partial_{\tan} f \quad \text{in } \mathbf{R}_+^n \times (0, T),$$

and div $\varphi = 0$ in $\mathbf{R}_+^n \times (0, T)$, with the Dirichlet and terminal conditions for φ. Although we restrict test functions to $f \in C_{c,\sigma}^\infty(\mathbf{R}_+^n \times (0, T))$, our proof is rather simpler. Since $\mathbf{P} \partial_{\tan} f = \partial_{\tan} f$ and

$$S(t) \partial_{\tan} f = \partial_{\tan} S(t) f, \tag{1.2.2}$$

for $f \in C^\infty_{c,\sigma}(\mathbf{R}^n_+)$, $S(t)\partial_{\tan} f \in L^1(\mathbf{R}^n_+)$ directly follows from the L^1-estimate (1.1.9). This implies an L^1-bound for solutions of the dual problem.

Our restriction $f \in C^\infty_{c,\sigma}(\mathbf{R}^n_+ \times (0,T))$ in (1.2.1) is sufficient in order to show $\partial_{\tan} v \equiv 0$. In fact, we have the following:

Proposition 1.2.3. *Let $v(\cdot,t) \in C^2(\overline{\mathbf{R}^n_+})$, $t \in (0,T)$, satisfy (1.2.1) for all $f \in C^\infty_{c,\sigma}(\mathbf{R}^n_+ \times (0,T))$. Assume that $\nabla v(\cdot,t)$ is bounded in $\mathbf{R}^n_+$, div $v = 0$ in $\mathbf{R}^n_+$ and $v = 0$ on $\partial\mathbf{R}^n_+$ for each $t \in (0,T)$. Then, $\partial_{tan} v \equiv 0$.*

Proof. By (1.2.1) and the de Rham's theory [4], [6, Theorem 1.1], there exists the potential functions Φ^j such that

$$\partial_j v = \nabla \Phi^j \quad \text{for } j \in \{1, \cdots, n-1\}.$$

Since $v \in C^2(\overline{\mathbf{R}^n_+})$ satisfies div $v = 0$ in $\mathbf{R}^n_+$ and $v = 0$ on $\partial\mathbf{R}^n_+$, $\Phi^j \in C^2(\overline{\mathbf{R}^n_+})$ satisfies $\Delta\Phi^j = 0$ in $\mathbf{R}^n_+$ and $\nabla\Phi^j = 0$ on $\partial\mathbf{R}^n_+$. In particular, $\partial\Phi^j/\partial x_n = 0$ on $\partial\mathbf{R}^n_+$.
Let $\tilde{\Phi}^j$ be the even extension of Φ^j to $\mathbf{R}^n$, i.e.,

$$\tilde{\Phi}^j(x', x_n) = \begin{cases} \Phi^j(x', x_n) & \text{for } x' \in \mathbf{R}^{n-1}, x_n \geq 0, \\ \Phi^j(x', -x_n) & \text{for } x' \in \mathbf{R}^{n-1}, x_n < 0. \end{cases}$$

Then, $\tilde{\Phi}^j \in C^2(\overline{\mathbf{R}^n_+})$ and $\Delta\tilde{\Phi}^j = 0$ in $\mathbf{R}^n$ by $\partial\Phi^j/\partial x_n = 0$ on $\partial\mathbf{R}^n_+$. Since $\partial_j v$ is bounded in $\mathbf{R}^n_+$, $\nabla\tilde{\Phi}^j$ is bounded in $\mathbf{R}^n$. We apply the Liouville theorem and conclude that $\nabla\tilde{\Phi}^j$ is constant. Since $\nabla\Phi^j = 0$ on $\partial\mathbf{R}^n_+$, $\nabla\Phi^j$ is zero. Thus, $\partial_{\tan} v \equiv 0$. □

Proof of Proposition 1.2.1. We prove (1.2.1) by a duality argument. The proof reduces to L^1-estimates for solutions of the dual problem:

$$-\partial_t\varphi - \Delta\varphi + \nabla\pi = \partial_{\tan} f \quad \text{in } \mathbf{R}^n_+ \times (0,T), \tag{1.2.3}$$

$$\text{div}\, \varphi = 0 \quad \text{in } \mathbf{R}^n_+ \times (0,T), \tag{1.2.4}$$

$$\varphi = 0 \quad \text{on } \partial\mathbf{R}^n_+ \times (0,T), \tag{1.2.5}$$

$$\varphi = 0 \quad \text{on } \mathbf{R}^n_+ \times \{t = T\}. \tag{1.2.6}$$

For $f \in C^\infty_{c,\sigma}(\mathbf{R}^n_+ \times (0,T))$, set $\varphi(\cdot,t) = \psi(\cdot,T-t)$ and $\nabla\pi(\cdot,t) = \nabla s(\cdot,T-t)$ by

$$\psi(\cdot,t) = \int_0^t S(t-s)\partial_{\tan} g(s)\mathrm{d}s, \quad \nabla s(\cdot,t) = \int_0^t \Pi(t-s)\partial_{\tan} g(s)\mathrm{d}s, \tag{1.2.7}$$

and $g(\cdot,t) = f(\cdot,T-t)$. Here, $\Pi(t)$ denotes the solution operator to the pressure gradient for (1.1.1)–(1.1.4). Since $(\psi, \nabla s)$ is a solution of the initial problem,

$$\begin{aligned} \partial_t\psi - \Delta\psi + \nabla s = \partial_{\tan} g \quad & \text{in } \mathbf{R}^n_+ \times (0,T), \\ \text{div}\, \psi = 0 \quad & \text{in } \mathbf{R}^n_+ \times (0,T), \\ \psi = 0 \quad & \text{on } \partial\mathbf{R}^n_+ \times (0,T), \\ \psi = 0 \quad & \text{on } \mathbf{R}^n_+ \times \{t = 0\}, \end{aligned}$$

$(\varphi, \nabla\pi)$ satisfies (1.2.3)–(1.2.6). By the L^1-estimate (1.1.9), observe that

$$\psi \in L^\infty(0, T; L^1(\mathbf{R}^n_+)). \tag{1.2.8}$$

Moreover, from explicit representations of $S(t)$ and $\Pi(t)$, $(\psi, \nabla s)$ satisfies

$$\psi, \nabla\psi, \nabla^2\psi, \partial_t\psi, \nabla s \in L^1(\mathbf{R}^n_+ \times (0, T)). \tag{1.2.9}$$

In fact, we apply Proposition 1.3.3 in the next section. Thus, $(\varphi, \nabla\pi)$ is also integrable in $\mathbf{R}^n_+ \times (0, T)$ up to second orders.

We now prove (1.2.1). Since $\nabla v, \nabla^2 v, v_t$, and ∇q are bounded in $\mathbf{R}^n_+ \times [\delta, T)$ for each $\delta > 0$ and $v \to 0$ weakly-$*$ as $t \downarrow 0$, it follows that

$$\begin{aligned}
\int_\delta^T \int_{\mathbf{R}^n_+} v \cdot \partial_{\tan} f \mathrm{d}x\mathrm{d}t &= \int_\delta^T \int_{\mathbf{R}^n_+} v \cdot (-\partial_t\varphi - \Delta\varphi + \nabla\pi)\mathrm{d}x\mathrm{d}t \\
&= -\int_\delta^T \int_{\mathbf{R}^n_+} \nabla q \cdot \varphi \mathrm{d}x\mathrm{d}t + \int_{\mathbf{R}^n_+} v(x, \delta) \cdot \varphi(x, \delta)\mathrm{d}x \\
&= \int_{\mathbf{R}^n_+} v(x, \delta) \cdot \varphi(x, \delta)\mathrm{d}x \to 0 \quad \text{as } \delta \downarrow 0.
\end{aligned}$$

Thus, we have proved (1.2.1). The proof is now complete. □

1.3 L^1-estimates for solutions of the dual problem

In this section, we estimate L^1-norms of solutions to (1.1.1)–(1.1.4) which implies the integrability of solutions for the dual problem (1.2.3)–(1.2.6) (Proposition 1.3.3). We recall the explicit representation for the Stokes semigroup $S(t)$ as well as the solution operator to pressure gradient $\Pi(t)$.

1.3.1 Estimates for spatial derivatives of the Stokes semigroup

Let $T(t)$ be the heat semigroup in $\mathbf{R}^n$ and $\Gamma(x, t)$ be the heat kernel, i.e., $T(t)f = \Gamma * f$ and $\Gamma(x, t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$. We write $T(t)f = (\Gamma * f_j)_{1\le j\le n}$ also for the $\mathbf{R}^n$-valued function $f = (f_j)_{1\le j\le n}$. By the solution formula [8, p.347], the remainder term $S(t)f - T(t)f$ is explicitly given by

$$(S(t) - T(t))f = \int_{\mathbf{R}^n_+} G^*(x, y, t) f(y)\mathrm{d}y, \tag{1.3.1}$$

with the kernel $G^* = (G^*_{ij})_{1\le i,j\le n}$ of the form,

$$G^*_{ij}(x, y, t) = -\delta_{ij}\Gamma(x - y^*, t) + 4(1 - \delta_{jn})\frac{\partial}{\partial x_j} \int_0^{x_n} \int_{\mathbf{R}^{n-1}} \frac{\partial E}{\partial x_i}(x - z)\Gamma(z - y^*, t)\mathrm{d}z.$$

Here, δ_{ij} denotes the Kronecker's delta and $y^* = (y', -y_n)$ denotes the reflection point of $y \in \mathbf{R}^n_+$ with respect to $\partial\mathbf{R}^n_+$. The function $E(x)$ denotes the fundamental solution of the Laplace equation, i.e., $E(x) = C_n/|x|^{(n-2)}$ for $n \geq 3$ and $E(x) = -1/2\pi \log|x|$ for $n = 2$ with the constant $C_n = (an(n-2))^{-1}$, where a denotes the volume of n-dimensional unit ball. Since the functions $E(x)$ and $\Gamma(x,t)$ are radially symmetric, $S(t)$ is commutative with tangential derivatives (1.2.2). We estimate the remainder term $(T(t) - S(t))f$ from the pointwise estimates of the kernel $G^* = (G^*_{ij})_{1\leq i,j\leq n}$, i.e.,

$$|\partial_t^s \partial_x^k \partial_y^m G^*_{ij}(x,y,t)| \leq C \frac{e^{-c{y_n}^2/t}}{t^{s+m_n/2}({x_n}^2+t)^{k_n/2}(|x-y^*|^2+t)^{(n+|k'|+|m'|)/2}}, \tag{1.3.2}$$

where $\partial_x^k = \partial_{x_1}^{k_1}\cdots\partial_{x_{n-1}}^{k_{n-1}}\partial_{x_n}^{k_n}$ and $|k'| = \sum_{j=1}^{n-1} k_j$ for the multi-index $k = (k', k_n)$, $k' = (k_1, \cdots, k_{n-1})$.

We estimate L^1-norms for spatial derivatives of the Stokes semigroup.

Proposition 1.3.1. *There exists constants C_1 and C_2 independent of $t > 0$ such that*

$$\|\nabla S(t)f\|_{L^1(\mathbf{R}^n_+)} \leq C_1/t^{1/2}\|f\|_{L^1(\mathbf{R}^n_+)}, \tag{1.3.3}$$

$$\|\partial_{x_n}^2 S(t)f\|_{L^1(\mathbf{R}^n_+)} \leq C_2/t^{1/2}\left(\|\partial_{x_n} f\|_{L^1(\mathbf{R}^n_+)} + \sup_{x_n>0}\|f\|_{L^1(\mathbf{R}^{n-1})}(x_n)\right) \tag{1.3.4}$$

hold for $f \in C_c^\infty(\mathbf{R}^n_+)$.

Proof. It is well known that the heat semigroup satisfies the L^1-estimate (1.3.3). Moreover, by integration by parts, we have $\|\partial_{x_n}^2 T(t)f\|_{L^1} \leq C/t^{1/2}\|\partial_{x_n} f\|_{L^1}$. Thus, $T(t)f$ satisfies (1.3.3) and (1.3.4). We shall show the estimate (1.3.3) and (1.3.4) for the remainder $\phi = (S(t) - T(t))f$, i.e.,

$$\|\nabla\phi\|_{L^1(\mathbf{R}^n_+)} \leq C_3/t^{1/2}\|f\|_{L^1(\mathbf{R}^n_+)}, \tag{1.3.5}$$

$$\|\partial_{x_n}^2\phi\|_{L^1(\mathbf{R}^n_+)} \leq C_4/t^{1/2}\sup_{x_n>0}\|f\|_{L^1(\mathbf{R}^{n-1})}(x_n). \tag{1.3.6}$$

We show (1.3.5) for tangential derivatives of ϕ. The normal derivative is estimated in a similar way. By the kernel estimate (1.3.2), it follows that

$$|\nabla_{\tan}\phi(x,t)| \leq C_5\int_{\mathbf{R}^n_+}\frac{|f(y)|}{(|x-y^*|^2+t)^{(n+1)/2}}\mathrm{d}y,$$

with the constant C_5 independent of $t > 0$. By integrating by tangential variables, we have

$$\|\nabla_{\tan}\phi\|_{L^1(\mathbf{R}^{n-1})}(x_n,t) \leq \frac{C_6}{(x_n^2+t)}\|f\|_{L^1(\mathbf{R}^n_+)},$$

with $C_6 = C_5C_7$ where

$$\int_{\mathbf{R}^{n-1}}\frac{\mathrm{d}x'}{(|x-y^*|^2+t)^{(n+1)/2}} = \frac{C_7}{((x_n+y_n)^2+t)} \tag{1.3.7}$$

is used. By integrating by the normal variable, we obtain (1.3.5) for $\nabla_{\tan}\phi$.
We next show (1.3.6). By (1.3.2), it follows that

$$|\partial_{x_n}^2\phi(x,t)| \leq C_8 \int_{\mathbf{R}_+^n} \frac{e^{-cy_n{}^2/t}|f(y)|}{(x_n^2+t)(|x-y^*|^2+t)^{n/2}}\mathrm{d}y.$$

Integrating by tangential variables, we have

$$\|\partial_{x_n}^2\phi\|_{L^1(\mathbf{R}^{n-1})}(x_n,t) \leq C_9 \frac{t^{1/2}}{(x_n^2+t)^{3/2}}\left(\sup_{x_n>0}\|f\|_{L^1(\mathbf{R}^{n-1})}(x_n)\right).$$

By integrating by the normal variable, we obtain (1.3.6). The proof is now complete. □

1.3.2 Estimates for second derivatives of pressure

We next estimate second derivatives of pressure. We define the solution operator for the pressure gradient $\Pi(t) : f \mapsto \Pi(t)f = \nabla q(\cdot,t)$ associated to the Stokes equations (1.1.1)–(1.1.4) by

$$(\Pi(t)f)(x) = \nabla \int_{\mathbf{R}_+^n} P(x,y,t)\cdot f(y)\mathrm{d}y, \tag{1.3.8}$$

with the kernel $P = (P_j)_{1\leq j\leq n}$ and

$$P_j(x,t) = 4(1-\delta_{jn})\frac{\partial}{\partial x_j}\Bigg(\int_{\mathbf{R}^{n-1}} \frac{\partial E}{\partial x_n}(x'-z',x_n)\Gamma(z'-y',y_n,t)\mathrm{d}z' + E(x'-z',x_n)\frac{\partial \Gamma}{\partial y_n}(z'-y',y_n,t)\mathrm{d}z'\Bigg).$$

The kernel $P = (P_j)_{1\leq j\leq n}$ satisfies the pointwise estimates [8, p.346],

$$|\partial_t^s\partial_x^k\partial_y^m P(x,y,t)| \leq C\frac{e^{-cy_n{}^2/t}}{t^{1+s+m_n/2}(|x-y^*|^2+t)^{(n-1+|k|+|m'|)/2}}. \tag{1.3.9}$$

From the explicit representation of the kernel $P = (P_j)_{1\leq j\leq n}$, we observe that the operator $\Pi(t)$ is also commutative with tangential derivatives, i.e., $\Pi(t)\partial_{\tan}f = \partial_{\tan}\Pi(t)f$.

We shall estimate second derivatives of pressure.

Proposition 1.3.2. *There exists a constant C_{10} independent of $t > 0$ such that*

$$\|\nabla\Pi(t)f\|_{L^1(\mathbf{R}_+^n)} \leq C_{10}/t^{1/2}\|f\|_{L^1(\mathbf{R}_+^n)} \tag{1.3.10}$$

holds for $f \in C_c^\infty(\mathbf{R}_+^n)$.

Proof. By (1.3.9), it follows that

$$|\nabla(\Pi(t)f)(x)| \leq C_{11} \int_{\mathbf{R}^n_+} \frac{|f(y)|}{t(|x-y^*|^2+t)^{(n+1)/2}} \mathrm{d}y.$$

By integrating by the tangential variables and (1.3.7), it follows that

$$\|\nabla\Pi(t)f\|_{L^1(\mathbf{R}^{n-1})}(x_n) \leq \frac{C_{12}}{t(x_n^2+t)} \|f\|_{L^1(\mathbf{R}^n_+)}$$

with $C_{12} = C_7 C_{11}$. By integrating by the normal variable, we obtain (1.3.10). □

Propositions 1.3.1 and 1.3.2 now imply:

Proposition 1.3.3. *For $g \in C^\infty_{c,\sigma}(\mathbf{R}^n_+ \times (0,T))$, the functions $(\psi, \nabla s)$ defined by (1.2.7) satisfy (1.2.9), i.e.,*

$$\psi, \nabla\psi, \nabla^2\psi, \partial_t\psi, \nabla s \in L^\infty(0,T; L^1(\mathbf{R}^n_+)).$$

Proof. Since $S(t)$ and $\Pi(t)$ are commutative with tangential derivatives, by (1.3.3) and (1.3.10), it follows that $\psi, \nabla\psi, \nabla\pi \in L^\infty(0,T; L^1(\mathbf{R}^n_+))$. By (1.3.3), (1.3.4) and the equation $\partial_t\psi = \Delta\psi - \nabla s + \partial_{\tan} g$, we obtain $\nabla^2\psi$ and $\partial_t\psi \in L^\infty(0,T; L^1(\mathbf{R}^n_+))$. □

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Chapter 2

Estimates for solutions of the Neumann problem

In this chapter, we study an a priori estimate for solutions of the homogeneous Neumann problem to get the harmonic-pressure gradient estimate for pressure of the Stokes equations (0.1.3). The harmonic-pressure gradient estimate holds for a large class of domains, but there is a domain where the estimate does not hold. We introduce the notion of *strictly admissible domain* which deduces the harmonic-pressure gradient estimate. As typical examples, we shall show that bounded domains, exterior domains and perturbed half spaces are indeed strictly admissible.

2.1 Introduction

In this chapter, we study the homogeneous Neumann problem of the form,

$$\Delta P = 0 \text{ in } \Omega, \quad \frac{\partial P}{\partial n_\Omega} = \mathrm{div}_{\partial\Omega} W \text{ on } \partial\Omega, \tag{2.1.1}$$

where $\mathrm{div}_{\partial\Omega}$ denotes the surface divergence and W denotes the tangential vector field on $\partial\Omega$. We call Ω *strictly admissible* if the a priori estimate

$$\sup_{x\in\Omega} d_\Omega(x)\left|\nabla P(x)\right| \leq C_\Omega \|W\|_{L^\infty(\partial\Omega)} \tag{2.1.2}$$

holds for all solutions of the Neumann problem (2.1.1). (We give a rigorous definition later in Section 2). As explained in the preface, the estimate (2.1.2) implies the harmonic-pressure gradient estimate (0.1.3) for pressure of the Stokes equations, which plays a key role in proving the a priori estimate (0.1.1) for solutions of the Stokes equations. A question is what kinds of domains are strictly admissible. Of course, a half space is strictly admissible. We prove the estimate (2.1.2) for a half space directly by estimating the Green function for the Neumann problem (2.1.1). Moreover, we shall show typical examples of strictly admissible domains (with non-trivial boundaries):

(I) bounded domains,
(II) exterior domains,
(III) perturbed half spaces ($n \geq 3$).

Here, we call Ω a perturbed half space in the sense that there exists $R_\Omega > 0$ such that $\Omega \backslash B_0(R_\Omega) = \mathbf{R}^n_+ \backslash B_0(R_\Omega)$. For domains (I)–(III), we assume the boundaries of class C^3.

We appeal to a blow-up argument to prove the a priori estimate (2.1.2). Let us give a heuristic idea in proving (2.1.2) for bounded domains. To argue by contradiction, suppose that there are a sequence of solutions of (2.1.1), $\{P_m\}_{m=1}^\infty$, and a sequence of points $\{x_m\}_{m=1}^\infty \subset \Omega$ such that

$$\frac{1}{2} \leq d_\Omega(x_m)\left|\nabla P_m(x_m)\right| \leq \sup_{x\in\Omega} d_\Omega(x)\left|\nabla P_m(x)\right| = 1, \tag{2.1.3}$$

and the boundary data W_m tends to zero uniformly on $\partial\Omega$. If a subsequence of $\{x_m\}_{m=1}^\infty$ converges to an interior point, the limit P solves the Neumann problem (2.1.1) under the bound

$$\sup_{x\in\Omega} d_\Omega(x)\left|\nabla P(x)\right| < \infty. \tag{2.1.4}$$

So if the solution of this problem is unique (i.e. $\nabla P \equiv 0$), then one gets a contradiction. Note that P_m is harmonic so compactness part is easy. If $\{x_m\}_{m=1}^\infty$ converges to a boundary point (by taking a subsequence), we rescale P_m around x_m by $d_m = d_\Omega(x_m)$ to get

$$Q_m(x) = P_m(x_m + d_m x) \quad \text{for} \quad x \in \Omega_m = \frac{\Omega_{x_m}}{d_m}. \tag{2.1.5}$$

Then, the rescaled domains Ω_m expands to a half space and the limit Q solves the Neumann problem (2.1.1) in a half space with an estimate inherited from (2.1.3). We prove its uniqueness by reducing the problem to the whole space via a reflection argument. The compactness part is easy since the distance between the origin for Q_m and the boundary $\partial\Omega_m$ is always one.

For general unbounded domains, this argument is difficult to apply since the sequence $\{x_m\}_{m=1}^\infty$ may diverge to infinity, i.e., $d_m \uparrow \infty$. However, we are able to prove the estimate (2.1.2) for exterior domains and perturbed half spaces. If the sequence $\{x_m\}_{m=1}^\infty$ diverges to infinity, we rescale P_m again by (2.1.5). When the domain is an exterior domain, the rescaled domain Ω_m approaches to the whole space and the boundary $\partial\Omega_m$ accumulates to a point in $\mathbf{R}^n$. We remove a singularity of the limit Q by using a bound inherited from (2.1.3) and conclude that the limit is trivial. When the domain is a perturbed half space, a curved part of the boundary $\partial\Omega_m$ accumulates to a point (or diverge to infinity) and the rescaled domain Ω_m approached to a half space.

The uniqueness of the Neumann problem (2.1.1) under the bound (2.1.4) is necessary condition for the strictly admissibility of the domain Ω (see Remark 2.2.5 (ii)). In fact, layer-type domains are not strictly admissible since linear functions are non-trivial solutions for the Neumann problem (2.1.1). For instance, $P = x^1$ is a non-trivial solution for the Neumann problem (2.1.1) in a layer $\Omega = \{a < x_n < b\}$. We conjecture that quasi-cylindrical domains are not strictly admissible. Here, the domain Ω is called quasi-cylindrical if $\overline{\lim}_{|x|\to\infty} d_\Omega(x) < \infty$ (see [3, 4, 6.32]).

This chapter is organized as follows. In Section 2, we define strictly admissible domains. In Section 3, we show that a half space is strictly admissible by using an explicit solution formula for the Neumann problem (2.1.1). In Section 4, we prove the uniqueness of the Neumann problem (2.1.1) for domains (I)–(III) by a duality argument. In Section 5, we prove the a priori estimate (2.1.2) for the domains (I)–(III). In Section 6, we give extension theorems for harmonic functions which are used in proving the estimate (2.1.2) for exterior domains and perturbed half spaces.

During the preparation of this thesis, the author was informed of the recent paper [11] by C. E. Kenig, F. Lin and Z. Shen, where the estimate (2.1.2) is essentially proved for a bounded domain with $C^{1,\gamma}$-boundary (independently of the work [1]) by estimating the Green functions. However, the estimates (2.1.2) for exterior domains and perturbed half spaces are not included there. Our proof by a blow-up argument is based on a uniqueness theorem for (2.1.1) and applicable to prove (2.1.2) without appealing to the Green function. We shall give a detailed comparison in Remark 2.5.2 (i).

2.2 Admissible and strictly admissible domain

In this section, we define the terms admissible domain and strictly admissible domain. Since the original definition of an admissible domain involves the Helmholtz projection operator, the harmonic-pressure gradient estimate (0.1.3) was restricted to spatially decaying solutions of the Stokes equations. We define strictly admissible domains without using the Helmholtz projection. We first define admissible domains.

Let Ω be a domain in $\mathbf{R}^n$ for $n \geq 2$ with $\partial\Omega \neq \emptyset$. An admissible domain is defined by the Helmholtz projection operator $\mathbf{P} = \mathbf{P}_r : L^r(\Omega) \to L^r_\sigma(\Omega)$ and $\mathbf{Q} = I - \mathbf{P}$ associated to the Helmholtz decomposition,

$$L^r(\Omega) = L^r_\sigma(\Omega) \oplus G^r(\Omega) \quad \text{for } r \in (1, \infty),$$

$L^r_\sigma(\Omega) = \overline{C^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_r}$ and $G^r(\Omega) = \{\nabla p \in L^r(\Omega) \mid p \in L^r_{\mathrm{loc}}(\Omega)\}$. Although this decomposition is known to hold (see, e.g., [8, III.1]) for various domains such as bounded or exterior domains with smooth boundaries, in general, there is a domain with (uniformly) smooth boundary such that the L^r-Helmholtz decomposition may not hold (cf. [4], [12]).

In [6] R. Farwig, H. Kozono, and H. Sohr introduced an $\tilde{L}^r$ space and proved that the Helmholtz decomposition is valid for any uniformly C^2-domain for $n = 3$. Later, it is generalized for arbitrary uniformly C^1-domain for $n \geq 2$ [7]. We set

$$\tilde{L}^r(\Omega) = \begin{cases} L^2(\Omega) \cap L^r(\Omega), & 2 \leq r < \infty, \\ L^2(\Omega) + L^r(\Omega), & 1 < r < 2, \end{cases}$$

and define $\tilde{L}^r_\sigma(\Omega)$ and $\tilde{G}^r(\Omega)$ in a similar way. The space $\tilde{L}^r(\Omega)$ for $r \geq 2$ is equipped with the norm $\|f\|_{\tilde{L}^r(\Omega)} = \max(\|f\|_{L^r(\Omega)}, \|f\|_{L^2(\Omega)})$. In order to define of an admissible domain, let us recall the definition of a uniformly C^k-domain for $k \geq 1$ (see, e.g., [16, I.3.2]).

Definition 2.2.1. (Uniformly C^k-domain) Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, with $\partial\Omega \neq \emptyset$. Assume that there exists $\alpha, \beta, K > 0$ such that for each $x_0 \in \partial\Omega$, there exists C^k-function h of $n-1$ variables y' such that $\sup_{|l| \leq k, |y'| < \alpha} |\partial^l_{y'} h(y')| \leq K$, $\nabla' h(0) = 0$, $h(0) = 0$ and denote a neighborhood of x_0 by $U_{\alpha,\beta,h}(x_0) = \{(y', y_n) \in \mathbf{R}^n \ |h(y') - \beta < y_n < h(y') + \beta, |y'| < \alpha\}$. Assume that up to rotation and translation, we have

$$U_{\alpha,\beta,h}(x_0) \cap \Omega = \{(y', y_n) \mid h(y') < y_n < h(y') + \beta, \ |y'| < \alpha\},$$

and $U_{\alpha,\beta,h}(x_0) \cap \partial\Omega = \{(y', y_n) \, |y_n = h(y'), |y'| < \alpha\}$. Then, we call Ω a uniformly C^k-domain of type α, β, K. Here, $\partial^l_x = \partial^{l_1}_{x_1} \cdots \partial^{l_n}_{x_n}$ with multi-index $l = (l_1, \ldots, l_n)$ and $\partial_{x_j} = \partial / \partial x_j$ as usual and ∇' denotes the gradient in $y' \in \mathbf{R}^{n-1}$.

If the solution $(v, \nabla q)$ of the linear Stokes equations is defined on L^r, the pressure gradient ∇q is represented by the velocity v through the Helmholtz projection operator, i.e., $\nabla q = \mathbf{Q}[\Delta v]$ so the harmonic-pressure gradient estimate (0.1.3) can be viewed as an L^∞-type estimate for the Helmholtz projection. In the sequel, we define a strictly admissible domain without using the Helmholtz projection operator.

Definition 2.2.2. Let Ω be a uniformly C^1-domain in $\mathbf{R}^n, n \geq 2$, with $\partial\Omega \neq \emptyset$. We call Ω *admissible* if there exists $r \geq n$ and a constant $C = C_\Omega > 0$ such that

$$\sup_{x \in \Omega} d_\Omega(x) \left| \mathbf{Q}[\nabla \cdot f](x) \right| \leq C_\Omega \|f\|_{L^\infty(\partial\Omega)} \tag{2.2.1}$$

holds for all matrix-valued function $f = (f_{ij})_{1 \leq i,j \leq n} \in C^1(\bar{\Omega})$ satisfying $\nabla \cdot f = (\sum_{j=1}^n \partial_j f_{ij})_{1 \leq i \leq n} \in \tilde{L}^r(\Omega)$,

$$\operatorname{tr} f = 0 \quad \text{and} \quad \partial_l f_{ij} = \partial_j f_{il} \tag{2.2.2}$$

for $i, j, l \in \{1, \cdots, n\}$, where $\partial_j = \partial / \partial x_j$.

We define a strictly admissible domain by the a priori estimate (2.1.2) for solutions to the Neumann problem (2.1.1). We recall the Gauss–Green formula on a surface and understand the boundary condition in (2.1.1) in terms of an appropriate weak form.

Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, with C^1-boundary. We define the surface gradient on $\partial\Omega$ for a (scalar-valued) C^1-function φ in $\bar{\Omega}$ by a tangential component of $\nabla\varphi$, i.e.,

$$\nabla_{\partial\Omega}\varphi = \nabla\varphi - n_\Omega(\partial\varphi/\partial n_\Omega).$$

We also define the surface divergence on $\partial\Omega$ by $\mathrm{div}_{\partial\Omega}h = \mathrm{tr}\,\nabla_{\partial\Omega}h$ for a vector-valued C^1-function h, where $\nabla_{\partial\Omega}h = (\nabla_{\partial\Omega}h^1, \cdots, \nabla_{\partial\Omega}h^n)$. If a support of φh is compact on $\partial\Omega$, the Gauss-Green formula on $\partial\Omega$ holds (e.g. [9], [15]):

$$\int_{\partial\Omega} h \cdot \nabla_{\partial\Omega}\varphi \mathrm{d}\mathcal{H}^{n-1}(x) = -\int_{\partial\Omega} (\mathrm{div}_{\partial\Omega}h + \kappa h \cdot n_\Omega)\varphi \mathrm{d}\mathcal{H}^{n-1}(x), \tag{2.2.3}$$

where $\kappa = \kappa(x)$ denotes the mean curvature of $\partial\Omega$ and $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure.

We define the space $L^\infty_{\mathrm{tan}}(\partial\Omega)$ and $L^\infty_d(\Omega)$. Let $L^\infty(\partial\Omega)$ be the space of all essentially bounded functions on $\partial\Omega$ with respect to $\mathcal{H}^{n-1}$.The space $L^\infty(\partial\Omega)$ is equipped with the norm $\|\cdot\|_{L^\infty(\partial\Omega)} = \|\cdot\|_{\infty,\partial\Omega}$. The space $L^\infty_{\mathrm{tan}}(\partial\Omega)$ denotes the closed subspace of all tangential vector fields on $L^\infty(\partial\Omega)$. We say h is tangential if $h \cdot n_\Omega = 0$ on $\partial\Omega$. The space $L^\infty_d(\Omega)$ denotes the space of all locally integrable functions f such that $d_\Omega f$ is essentially bounded in Ω. The space $L^\infty_d(\Omega)$ is equipped with the norm

$$|f|_{\infty,d} = \sup_{x\in\Omega} d_\Omega(x)\,|f(x)|\,.$$

Note that $\nabla P \in L^\infty_d(\Omega)$ implies $P \in L^r_{\mathrm{loc}}(\bar{\Omega})$ for $r \in [1,\infty)$.

Definition 2.2.3 (Weak solution)**.** Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, with C^1-boundary. Let $P \in L^1_{\mathrm{loc}}(\bar{\Omega})$ satisfy

$$\int_\Omega P\Delta\varphi \mathrm{d}x = \int_{\partial\Omega} W \cdot \nabla_{\partial\Omega}\varphi \mathrm{d}\mathcal{H}^{n-1}(x) \tag{2.2.4}$$

for $W \in L^\infty_{\mathrm{tan}}(\partial\Omega)$ and all $\varphi \in C^2_c(\bar{\Omega})$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. If $\nabla P \in L^\infty_d(\Omega)$, we call P *weak solution* of (2.1.1).

We now define the term strictly admissible domain.

Definition 2.2.4 (Strictly admissible domain)**.** Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, with C^1-boundary. We call Ω *strictly admissible* if there exists a constant $C = C_\Omega > 0$ such that the a priori estimate

$$|\nabla P|_{\infty,d} \leq C_\Omega \|W\|_{\infty,\partial\Omega} \tag{2.2.5}$$

holds for all weak solutions $\nabla P \in L^\infty_d(\Omega)$ for $W \in L^\infty_{\mathrm{tan}}(\partial\Omega)$.

Remarks 2.2.5. (i) The constant C_Ω in (2.2.5) is invariant of dilation and translation of Ω, i.e., $C_{\lambda\Omega+x_0} = C_\Omega$ for $\lambda > 0$ and $x_0 \in \Omega$.
(ii) If Ω is strictly admissible, a weak solution for (2.1.1) is unique. In fact, if $\nabla P \in L^\infty_d(\Omega)$ satisfies (2.2.4) for $W = 0$, $\nabla P = 0$ follows from (2.2.5).

A strictly admissible domain is indeed admissible.

Proposition 2.2.6. *Let Ω be a strictly admissible domain in $\mathbf{R}^n$, $n \geq 2$, with uniformly C^1-boundary. Then, Ω is admissible.*

Proof. Let $f = (f_{ij})_{1\leq i,j\leq n} \in C^1(\bar{\Omega})$ be a matrix-valued function satisfying $\nabla \cdot f \in \tilde{L}^r(\Omega)$ for $r \geq n$ and (2.2.2). Set $\nabla P = \mathbf{Q}[\nabla \cdot f]$ and $W = -(f - f^T) \cdot n_\Omega$. Then, $W \in L^\infty_{\tan}(\partial\Omega)$ since $W \cdot n_\Omega = -\sum_{i,j=1}^n (f_{ij} - f_{ji}) n^j_\Omega n^i_\Omega = 0$. We show that P satisfies (2.2.4) for W. Let $\varphi \in C^2_c(\bar{\Omega})$ satisfy $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. By multiplying $\nabla\varphi$ to $\nabla P = \mathbf{Q}[\nabla \cdot f]$, it follows that

$$\int_\Omega \nabla P \cdot \nabla\varphi \mathrm{d}x = \sum_{i,j=1}^n \int_\Omega \partial_j f_{ij} \partial_i \varphi \mathrm{d}x.$$

The left-hand-side is $-\int_\Omega P\Delta\varphi dx$ since $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. By integration by parts, it follows that

$$\begin{aligned}\int_\Omega \partial_j f_{ij} \partial_i \varphi \mathrm{d}x &= -\int_\Omega f_{ij} \partial_j \partial_i \varphi \mathrm{d}x + \int_{\partial\Omega} f_{ij} \partial_i \varphi n^j_\Omega \mathrm{d}\mathcal{H}^{n-1}(x) \\ &= \int_\Omega \partial_i f_{ij} \partial_j \varphi \mathrm{d}x + \int_{\partial\Omega} f_{ij} (\partial_i \varphi n^j_\Omega - \partial_j \varphi n^i_\Omega) \mathrm{d}\mathcal{H}^{n-1}(x),\end{aligned}$$

where the symbol of summation is suppressed. By (2.2.2), the first term vanishes. Since $\sum_{i,j} f_{ij} \partial_i \varphi n^j_\Omega = \sum_{i,j} f_{ji} \partial_j \varphi n^i_\Omega$, the second term is $-\int_{\partial\Omega} W \cdot \nabla_{\partial\Omega} \varphi d\mathcal{H}^{n-1}(x)$. Thus P satisfies (2.2.4) for $W = -(f - f^T) \cdot n_\Omega$.
It remains to show $\nabla P \in L^\infty_d(\Omega)$. We shall show that $\nabla P \in \tilde{L}^r(\Omega)$ for $r \geq n$ implies $\nabla P \in L^\infty_d(\Omega)$ for the harmonic function P. By the mean value formula, it follows that

$$\nabla P(x) = \fint_{B_x(\tau)} \nabla P(y) \mathrm{d}\mathcal{H}^{n-1}(y) \quad \text{for } x \in \Omega \text{ and } \tau = d_\Omega(x).$$

Apply the Hölder inequality to get $|\nabla P(x)| \leq C_s/\tau^{n/s} \|\nabla P\|_{L^s(\Omega)}$ for $s \in (1, \infty)$, with the constant C_s independent of $\tau = d_\Omega(x)$. If $d_\Omega(x) \leq 1$ take $s = r \geq n$. If $d_\Omega(x) > 1$, take $s = 2$. Since $\mathbf{Q}$ is bounded on $\tilde{L}^r(\Omega)$, it follows that

$$|\nabla P|_{\infty,d} \leq C_r \|\nabla \cdot f\|_{\tilde{L}^r(\Omega)} \tag{2.2.6}$$

for the constant C_r depending on r. Thus, P is a weak solution of (2.1.1). If Ω is strictly admissible, (2.2.1) follows from (2.2.5). The proof is now complete. □

2.3 Examples

In this section, we prove that a half space is strictly admissible by an explicit solution formula for the Neumann problem (2.1.1). We then give non-trivial examples of strictly admissible domains. The proofs for the non-trivial examples are given in the subsequent sections.

Theorem 2.3.1. *A half space is strictly admissible.*

For a half space, we are able to show the estimate (2.1.2) directly by estimating the Green function for the Neumann problem (2.1.1). To represent weak solutions for (2.1.1) by the Green function, we first prove the uniqueness.

Lemma 2.3.2. *A weak solution of (2.1.1) on $\mathbf{R}^n_+$ is unique up to an additive constant.*

Proof. The proof is reduced to the whole space. Let P be a weak solution of (2.1.1) for $W = 0$ on $\mathbf{R}^n_+$. Let $\tilde{P}$ be the even extension of P to $\mathbf{R}^n$, i.e., $\tilde{P}(x', x_n) = P(x', x_n)$, $x_n \geq 0$ and $\tilde{P}(x', x_n) = P(x', -x_n)$, $x_n < 0$. For $\varphi \in C_c^\infty(\mathbf{R}^n)$, it follows from (2.2.4) that

$$\begin{aligned}\int_{\mathbf{R}^n} \tilde{P}\Delta\varphi \mathrm{d}x &= \int_0^\infty \int_{\mathbf{R}^{n-1}} P(x', x_n)\Delta\varphi(x', x_n)\mathrm{d}x'\mathrm{d}x_n + \int_{-\infty}^0 \int_{\mathbf{R}^{n-1}} P(x', x_n)\Delta\varphi(x', x_n)\mathrm{d}x'\mathrm{d}x_n \\ &= \int_{\mathbf{R}^n_+} P(x)\Delta(\varphi(x', x_n) + \varphi(x', -x_n))\mathrm{d}x'\mathrm{d}x_n \\ &= 0.\end{aligned}$$

Thus, $\tilde{P} \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ is weakly harmonic in $\mathbf{R}^n$. Set $\tilde{P}_\varepsilon = \tilde{P} * \eta_\varepsilon$ by the radially symmetric mollifier η_ε, $\varepsilon > 0$. Then, $\tilde{P}_\varepsilon \in C^\infty(\mathbf{R}^n)$ is harmonic in $\mathbf{R}^n$. By the mean value formula, it follows that

$$\tilde{P}_\varepsilon(x) = \fint_{\partial B_x(r)} \tilde{P}_\varepsilon(y)\mathrm{d}\mathcal{H}^{n-1}(y) \quad \text{for } x \in \mathbf{R}^n, r > 0.$$

By

$$\sup_{x\in\mathbf{R}^n} |x_n||\nabla\tilde{P}(x)| < \infty,$$

$\tilde{P}_\varepsilon$ is uniformly bounded by $|\log|x_n||$ near $\partial\mathbf{R}^n_+$. Since $\tilde{P}_\varepsilon \to \tilde{P}$ a.e. in $\mathbf{R}^n$ as $\varepsilon \downarrow 0$, by letting $\varepsilon \downarrow 0$, we have

$$\tilde{P}(x) = \fint_{\partial B_x(r)} \tilde{P}(y)\mathrm{d}\mathcal{H}^{n-1}(y) \quad \text{a.e. } x \in \mathbf{R}^n, r > 0.$$

Since η_ε is radially symmetric, eventually, $\tilde{P}_\varepsilon = \tilde{P} \in C^\infty(\mathbf{R}^n)$.
The function $\tilde{P}(x)$ may increase as $|x| \to \infty$, but increasing rates are at most polynomial orders. In fact, by integrating $\tilde{P}$ from $x \in \mathbf{R}^n_+$ to $x_0 = (0, \cdots, 0, 1)$, it follows that

$$|P(x)| \leq C_1|x||\log x_n| + C_2 \quad \text{for } x \in \mathbf{R}^n_+,$$

with $C_1 = |\nabla P|_{\infty,d}$ and $C_2 = |P(x_0)|$. Similarly, we are able to estimate $\tilde{P}(x)$ for $x_n < 0$. Applying the Liouville theorem implies that $\tilde{P}$ is a polynomial of degree two. By $\nabla P \in L^\infty_d(\mathbf{R}^n_+)$, $\nabla P \to 0$ as $x_n \to \infty$. Thus, $\nabla P \equiv 0$ follows. The proof is now complete. □

Proof of Theorem 2.3.1. For $W \in L^\infty_{\tan}(\partial\mathbf{R}^n_+)$, we set

$$P(x) = \sum_{i=1}^{n-1} \int_{\partial\mathbf{R}^n_+} \partial_{x_i} E(x-y')W^i(y)\mathrm{d}y'. \tag{2.3.1}$$

Then, it follows that

$$\begin{aligned} |\nabla P(x)| &\le C_1 \int_{\partial\mathbf{R}^n_+} \frac{\mathrm{d}y'}{(|y'|^2 + x_n{}^2)^{n/2}} \|W\|_{L^\infty(\partial\mathbf{R}^n_+)} \\ &\le \frac{C_2}{x_n} \|W\|_{L^\infty(\partial\mathbf{R}^n_+)}. \end{aligned}$$

Thus, $\nabla P \in L^\infty_d(\mathbf{R}^n_+)$ satisfies the estimate (2.1.2). Let $\varphi \in C^2_c(\mathbf{R}^n_+)$ satisfy $\partial\varphi/\partial x_n = 0$ on $\partial\mathbf{R}^n_+$. By multiplying $\nabla\varphi$ to ∇P and integration by parts, it follows that

$$\begin{aligned} -\int_{\mathbf{R}^n_+} P\Delta\varphi \mathrm{d}x &= \sum_{i=1}^{n-1} \int_{\partial\mathbf{R}^n_+} W^i(y')\mathrm{d}y' \int_{\mathbf{R}^n_+} \nabla\partial_{x_i} E(x-y') \cdot \nabla\varphi(x)\mathrm{d}x \\ &= -\sum_{i=1}^{n-1} \int_{\partial\mathbf{R}^n_+} W^i(y')\mathrm{d}y' \int_{\mathbf{R}^n_+} \nabla E(x-y') \cdot \nabla\partial_{x_i}\varphi(x)\mathrm{d}x \\ &= -\int_{\partial\mathbf{R}^n_+} W^i(y') \cdot \nabla_{\partial\mathbf{R}^n_+}\varphi(y')\mathrm{d}y', \end{aligned}$$

since

$$\begin{aligned} \int_{\mathbf{R}^n_+} \nabla E(x-y') \cdot \nabla\partial_{x_i}\varphi(x)\mathrm{d}x &= \int_{\partial\mathbf{R}^n_+} \partial_{x_n} E(x'-y')\partial_{x_i}\varphi(x')\mathrm{d}x' \\ &= \partial_{y_i}\varphi(y'). \end{aligned}$$

Thus, P is a weak solution of (2.1.1). By Lemma 2.3.2, weak solutions for $W \in L^\infty_{\tan}(\partial\mathbf{R}^n_+)$ are represented by (2.3.1). Thus, a half space is strictly admissible. □

For general domains, solution formulas for the Neumann problem (2.1.1) are not available, but we are able to prove the a priori estimate (2.2.5) for domains (I)–(III) by a blow-up argument.

Theorem 2.3.3. *The domains (I)–(III) with C^3-boundaries are strictly admissible.*

We prove the a priori estimate (2.1.2) by a blow-up argument later in Section 5. For this purpose, we prove the uniqueness of the Neumann problem (2.1.1) on the domains (I)–(III) in the next section. The uniqueness of weak solutions is important in order to know whether (2.2.5) holds as noted in Remark 2.2.5 (ii). In fact, in a layer domain $\Omega = \{a < x_n < b\}$, $P = x^1$ is a non-trivial weak solution for $W = 0$. Thus, layer domains and cylindrical domains are not strictly admissible. We conjecture that quasi-cylindrical domains are not strictly admissible. Here, the domain Ω is called quasi-cylindrical if $\overline{\lim}_{|x|\to\infty} d_\Omega(x) < \infty$ (see [3, 6.32]).

2.4 Uniqueness of the Neumann problem

In this section, we prove the uniqueness of the Neumann problem (2.1.1) on the domains (I)–(III) by a duality argument. We find a solution of the dual problem by using the Helmholtz projection. Note that $\nabla P \in L^\infty_d(\Omega)$ does not imply decay for $P(x)$ as $|x| \to \infty$. We give pointwise estimates for $P(x)$ as $|x| \to \infty$, and apply a duality argument.

2.4.1 Uniqueness on an exterior domain

We begin with a bounded domain.

Lemma 2.4.1. *Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geq 2$, with C^3-boundary. Then, a weak solution of (2.1.1) is unique up to an additive constant.*

Proof. Let P be a weak solution of (2.1.1) for $W = 0$ in Ω. Set $\nabla\varphi = \mathbf{Q}[g]$ for $g \in C^\infty_c(\Omega)$. Then, φ solves the weak Neumann problem: $\Delta\varphi = \operatorname{div} g$ in Ω, $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. Since $\partial\Omega$ is C^3, by the elliptic regularity theory, φ is in $W^{3,r}(\Omega)$ for $r \in (1,\infty)$. By the Sobolev embedding for $r > n$, observe that φ is a C^2-function in $\bar{\Omega}$. By substituting φ into (2.2.4), it follows that

$$\int_\Omega P \operatorname{div} g \, dx = 0.$$

Thus, $\nabla P \equiv 0$, i.e., P is constant. □

We next prove the uniqueness of weak solutions on an exterior domain.

Lemma 2.4.2. *Let Ω be an exterior domain in $\mathbf{R}^n$, $n \geq 2$, with C^3-boundary. A weak solution of (2.1.1) is unique up to an additive constant.*

In order to prove Lemma 2.4.2 by a duality argument, for $\nabla P \in L^\infty_d(\Omega)$, we estimate $P(x)$ as $|x| \to \infty$.

Proposition 2.4.3. *Let Ω be an exterior domain in $\mathbf{R}^n$, $n \geq 2$. Let $0 \in \Omega^c$ and $R_0 >$ diam Ω^c. For $\nabla P \in L^\infty_d(\Omega)$, there exists constants C_1 and C_2 such that*

$$|P(x)| \leq C_1 \log|x| + C_2 \quad \text{for } |x| \geq 2R_0. \tag{2.4.1}$$

Proof. For $x \in \Omega$ satisfying $|x| \geq 2R_0$, there is some $z \in \partial\Omega$ such that $d_\Omega(x) = |z-x|$. Since $|x| \leq d_\Omega(x) + R_0$, it follows that $|x| \leq 2d_\Omega(x)$. Thus, we estimate

$$\sup_{|x|\geq 2R_0} |x||\nabla P(x)| \leq 2|\nabla P|_{\infty,d}.$$

Set $y = 2R_0 x/|x|$ for $|x| > 2R_0$. Then, it follows that

$$\begin{aligned}|P(x) - P(y)| &\leq \int_0^1 \left|\frac{d}{dt}P(tx + (1-t)y)\right| \mathrm{d}t \\ &\leq |x-y| \left(\int_0^1 \frac{\mathrm{d}t}{|y| + t|x-y|}\right) \sup_{|z|\geq 2R_0} |z||\nabla P(z)| \\ &\leq 2(\log|x| - \log 2R_0)|\nabla P|_{\infty,d}.\end{aligned}$$

Thus, (2.4.1) holds with $C_1 = 2|\nabla P|_{\infty,d}$ and $C_2 = -2\log 2R_0|\nabla P|_{\infty,d} + \sup_{|y|=2R_0}|P(y)|$. □

Proof of Lemma 2.4.2. Let $\nabla P \in L^\infty_d(\Omega)$ be a weak solution of (2.1.1) for $W = 0$. We shall show that

$$\int_\Omega P \mathrm{div}\, g \mathrm{d}x = 0 \tag{2.4.2}$$

for all $g \in C_c^\infty(\Omega)$. Set $\nabla\varphi = \mathbf{Q}[g] \in L^r(\Omega)$ for $g \in C_c^\infty(\Omega)$. Then, φ satisfies the weak Neumann problem: $\Delta\varphi = \mathrm{div}\, g$ in Ω and $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. Let θ be a smooth function in $[0,\infty)$ satisfying $\theta \equiv 1$ in $[0, 1/2]$ and $\theta \equiv 0$ in $[1,\infty)$. Set $\theta_R(x) = \theta(|x|/R)$ for $R > 2R_0$ with $R_0 > \mathrm{diam}\,\Omega^c$. Since $\partial\Omega$ is C^3, by the elliptic regularity theory [10, Lemma 2.3], φ is a C^2-function in $\bar{\Omega}$. Then, $\varphi_R = \varphi\theta_R \in C_c^2(\bar{\Omega})$ and $\partial\varphi_R/\partial n_\Omega = 0$ on $\partial\Omega$. From the definition of weak solutions (2.2.4), it follows that

$$\int_\Omega P(\mathrm{div}\, g\ \theta_R + 2\nabla\varphi\cdot\nabla\theta_R + \varphi\Delta\theta_R)\,\mathrm{d}x = 0. \tag{2.4.3}$$

We show that the last two terms vanish as $R \to \infty$. By Proposition 2.4.3, it follows that

$$\begin{aligned}\left|\int_\Omega P\nabla\varphi\cdot\nabla\theta_R dx\right| \leq& \frac{(C_1\log R + C_2)}{R}\|\nabla\theta\|_{L^\infty(\mathbf{R})}\int_{R/2<|x|<R}|\nabla\varphi|\mathrm{d}x \\ \leq& \frac{(C_1\log R + C_2)}{R^{1-n+n/r}}C_n^{1-1/r}\|\nabla\theta\|_{L^\infty(\mathbf{R})}\|\nabla\varphi\|_{L^r(\Omega)}\end{aligned}$$

for $R > 4R_0$ where C_n denotes the volume of n dimensional unit ball. For $r \in (1, n/(n-1))$ the right-hand side vanishes as $R \to \infty$.
It remains to show that the last term of (2.4.3) vanishes as $R \to \infty$. Since P is harmonic in Ω and the support of $\Delta\theta_R$ is in $\overline{D_R}$ for $D_R = B_0(R)\backslash\overline{B_0(R/2)}$, we are able to shift φ by a constant. We replace φ to $\tilde{\varphi} = \varphi - \fint_{D_R}\varphi dx$. By the Poincaré inequality [5, 5.8.1], we

estimate $||\tilde{\varphi}||_{L^r(D_R)} \leq C_0 R ||\nabla\varphi||_{L^r(D_R)}$ with the constant C_0 independent of R. By Proposition 2.4.3, it follows that

$$\begin{aligned}\left|\int_\Omega P\tilde{\varphi}\Delta\theta_R dx\right| \leq & \frac{(C_1 \log R + C_2)}{R^2} ||\Delta\theta||_{L^\infty(\mathbf{R})} \int_{D_R} |\tilde{\varphi}| \mathrm{d}x \\ \leq & \frac{(C_1 \log R + C_2)}{R^{1-n+n/r}} ||\Delta\theta||_{L^\infty(\mathbf{R})} C_0 C_n^{1-1/r} ||\nabla\varphi||_{L^r(\Omega)} \to 0 \quad \text{as } R \to \infty.\end{aligned}$$

We proved (2.4.2) for all $g \in C_c^\infty(\Omega)$ so $\nabla P \equiv 0$. The proof is now complete. □

2.4.2 Uniqueness on a perturbed half space

We next prove the uniqueness of weak solutions on a perturbed half space Ω for $n \geq 2$. As stated below (Proposition 2.4.5), on a perturbed half space, $\nabla P \in L^\infty_d(\Omega)$ does not imply a logarithmic increasing order for $P(x)$ as $|x| \to \infty$. So the same duality argument for exterior domains does not directly apply to prove the uniqueness. We shall show $\nabla^2 P \equiv 0$ by taking test functions in differentiated forms so that solutions of the dual problem is L^1-integrable in Ω.

Lemma 2.4.4. *Let Ω be a perturbed half space in $\mathbf{R}^n$, $n \geq 2$, with C^3-boundary. Then, a weak solution of (2.1.1) is unique up to an additive constant.*

For $\nabla P \in L^\infty_d(\Omega)$, we estimate $P(x)$ as $|x| \to \infty$. Note that $P(x)$ may not be bounded near the boundary. To state a result, let $C_0(R)$ be the cylinder centered at the origin with hight $2R > 0$, i.e., $C_0(R) = B_0^{n-1}(R) \times (-R, R)$, where $B_0^{n-1}(R)$ denotes the $n-1$-dimensional ball with radius $R > 0$.

Proposition 2.4.5. *Let Ω be a perturbed half space in $\mathbf{R}^n$, $n \geq 2$. Let R_Ω be a positive constant such that $\Omega \backslash C_0(R_\Omega) = \mathbf{R}^n_+ \backslash C_0(R_\Omega)$. For $\nabla P \in L^\infty_d(\Omega)$, there exists constants $C_1 - C_4$ such that*

$$|P(x)| \leq C_1|x| + C_2 \quad \text{for } x' \in \mathbf{R}^{n-1},\ x_n \geq R_\Omega, \tag{2.4.4}$$

$$|P(x)| \leq C_3|\log x_n| + C_1|x| + C_4 \quad \text{for } |x'| \geq 4R_\Omega,\ x_n \leq R_\Omega. \tag{2.4.5}$$

Proof. By taking R_Ω large if necessary, we may assume $d_{R_\Omega} = \inf\{d_\Omega(x) \mid x = (x', x_n) \in \Omega,\ x_n = R_\Omega\} > 0$. For $x = (x', x_n) \in \Omega$ satisfying $x_n \geq R_\Omega$ and $x_0 = (0, \cdots, 0, R_\Omega)$, it follows that

$$\begin{aligned}|P(x) - P(x_0)| &= \left|\int_0^1 (x - x_0) \cdot \nabla P(tx + (1-t)x_0) \mathrm{d}t\right| \\ &\leq |x - x_0| {d_{R_\Omega}}^{-1} |\nabla P|_{\infty,d}.\end{aligned}$$

Thus, (2.4.4) holds with $C_1 = {d_{R_\Omega}}^{-1}|\nabla P|_{\infty,d}$ and $C_2 = C_1 R_\Omega + |P(x_0)|$.
We shall show (2.4.5). Observe that $d_\Omega(z) = z_n$ for $z = (z', z_n)$ such that $|z'| \geq 4R_\Omega$ and

$z_n \leq 2R_\Omega$ since $B_z(z_n) \cap C_0(R_\Omega) = \emptyset$. For $x = (x', x_n)$ such that $|x'| \geq 4R_\Omega$ and $x_n \leq R_\Omega$, set $x_{R_\Omega} = x + x_0$. It follows that

$$\begin{aligned}|P(x) - P(x_{R_\Omega})| &= \left|\int_0^1 (x - x_{R_\Omega}) \cdot \nabla P(tx + (1-t)x_{R_\Omega})\mathrm{d}t\right| \\ &\leq R_\Omega |\nabla P|_{\infty,d} \int_0^1 \frac{\mathrm{d}t}{d_\Omega(tx + (1-t)x_{R_\Omega})} \\ &= R_\Omega |\nabla P|_{\infty,d} \int_0^1 \frac{\mathrm{d}t}{tR_\Omega + x_n} \\ &\leq (|\log x_n| + |\log 2R_\Omega|)|\nabla P|_{\infty,d}.\end{aligned}$$

Since $|P(x_{R_\Omega})| \leq C_1|x| + C_1 R_\Omega + C_2$ by (2.4.4), (2.4.5) holds with $C_3 = |\nabla P|_{\infty,d}$ and $C_4 = |\log 2R_\Omega||\nabla P|_{\infty,d} + C_1 R_\Omega + C_2$. The proof is now complete. □

Proof of Lemma 2.4.4 . Let $\nabla P \in L^\infty_d(\Omega)$ be a weak solution for $W = 0$ on Ω. We shall show

$$\int_\Omega P \mathrm{div}\, \partial_x g \mathrm{d}x = 0$$

for all $g \in C^\infty_c(\Omega)$, where ∂_x indiscriminately denotes ∂_{x_i} for $i \in \{1, \cdots, n\}$. This implies that P is a polynomial of degree one. Since $\nabla P \in L^\infty_d(\Omega)$ implies $\nabla P \to 0$ as $d_\Omega(x) \to \infty$ so $\nabla P \equiv 0$ follows.

Since Ω is a perturbed half space, there exists $R_\Omega > 0$ such that $\Omega \backslash C_0(R_\Omega) = \mathbf{R}^n_+ \backslash C_0(R_\Omega)$. Set $\nabla\varphi = \mathbf{Q}[\partial_x g] \in L^p(\Omega)$ for $g \in C^\infty_c(\Omega)$. Then, φ solves the weak Neumann problem: $\Delta\varphi = \mathrm{div}\, \partial_x g$ in Ω and $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. By the elliptic regularity theory, $\varphi \in C^2(\bar{\Omega})$. Moreover, $\nabla\varphi \in L^1(\Omega)$ since $\partial_x g$ is a differentiated function. (In fact, we apply Lemma 2.4.6 below). Let $\theta \in C^\infty_c[0, \infty)$ be a smooth cutoff function such that $\theta \equiv 1$ in $[0, 1]$ and $\theta \equiv 0$ in $[2, \infty)$. Set $\tilde{\theta}_R(x) = \tilde{\theta}_R(|x'|)\tilde{\theta}_R(|x_n|)$ by $\theta_R(s) = \theta(s/R)$. Then, $\tilde{\theta}_R(x) \equiv 1$ in $C_0(R)_+$ and $\tilde{\theta}_R(x) \equiv 0$ in $\mathbf{R}^n_+ \backslash \overline{C_0(2R)_+}$, where $C_0(R)_+ = B^{n-1}(R) \times (0, R)$. By multiplying the cut-off function $\tilde{\theta}_R(x)$ to φ, we have $\varphi_R = \varphi\tilde{\theta}_R \in C^2_c(\bar{\Omega})$ satisfying $\partial\varphi_R/\partial n_\Omega = 0$ on $\partial\Omega$ for $R > R_\Omega$. We substitute φ_R into (2.2.4) for $W = 0$ to get

$$0 = \int_\Omega P(\mathrm{div}\, \partial_x g \tilde{\theta}_R + 2\nabla\varphi \cdot \nabla\tilde{\theta}_R + \varphi\Delta\tilde{\theta}_R)\mathrm{d}x.$$

It suffices to show that the last two terms vanishes as $R \to \infty$. We shall show that the second term vanishes as $R \to \infty$. By a similar way, we are able to show the last term also vanishes. We divide the second term into two terms,

$$\begin{aligned}\int_\Omega P\nabla\varphi \cdot \nabla\tilde{\theta}_R \mathrm{d}x &= \int_{\mathcal{D}_0(R)\cap\{x_n \geq R_\Omega\}} P\nabla\varphi \cdot \nabla\tilde{\theta}_R \mathrm{d}x + \int_{\mathcal{D}_0(R)\cap\{0 < x_n < R_\Omega\}} P\nabla\varphi \cdot \nabla\tilde{\theta}_R \mathrm{d}x \\ &= I_R + II_R,\end{aligned}$$

where $\mathcal{D}_0(R) = C_0(2R)\backslash\overline{C_0(R)}$. By (2.4.4), it follows that $|I_R| \leq C\|\nabla\varphi\|_{L^1(\mathcal{D}_0(R))} \to 0$ as $R \to \infty$. By (2.4.5), we estimate

$$|II_R| \leq \frac{C}{R}\int_{\mathcal{D}_0(R)\cap\{0<x_n<R_\Omega\}} (|\log x_n| + R)\,|\nabla\varphi|\mathrm{d}x.$$

The second term vanishes as $R \to \infty$. Applying the Hölder inequality implies that

$$\begin{aligned}\frac{1}{R}\int_{\mathcal{D}_0(R)\cap\{0<x_n<R_\Omega\}} |\log x_n||\nabla\varphi|\mathrm{d}x &\leq \frac{1}{R}\left(\int_0^{R_\Omega} |\log x_n|^{p'}\mathrm{d}x_n\right)^{1/p'}\int_{\{R<|x'|<2R\}} \|\nabla\varphi\|_{L^p(\{0<x_n<R_\Omega\})}(x')\mathrm{d}x' \\ &\leq \frac{C}{R^{1+(n-1)/p-(n-1)}}\|\nabla\varphi\|^{1/p}_{L^p(\mathcal{D}_0(R))}.\end{aligned}$$

For $n \geq 3$, take $p \in (1, (n-1)/(n-2)]$. For $n = 2$, take $p \in (1, \infty)$. Then the right-hand side converges to zero as $R \to \infty$. Thus $II_R \to 0$ as $R \to \infty$. The proof is now complete. □

We shall show the L^1-bound for $\nabla\varphi = \mathbf{Q}[\partial_x g]$, $g \in C^2_c(\Omega)$. We appeal to pointwise kernel estimates for solutions of the (weak) Neumann problem in a half space.

Lemma 2.4.6. *Let Ω be a perturbed half space in $\mathbf{R}^n$, $n \geq 2$, with C^1-boundary. For $g \in C^2_c(\Omega)$, set $\nabla\varphi = \mathbf{Q}[\partial_x g]$. Then, $\nabla\varphi \in L^1(\Omega)$.*

We first prove Lemma 2.4.6 for $\Omega = \mathbf{R}^n_+$.

Proposition 2.4.7. *The statement of Lemma 2.4.6 is valid when $\Omega = \mathbf{R}^n_+$.*

Proof. Let $E(x)$ be the fundamental solution of the Laplace equation, i.e., $E(x) = C_n/|x|^{(n-2)}$ for $n \geq 3$ and $E(x) = -1/2\pi \log|x|$ for $n = 2$ with the constant $C_n = (an(n-2))^{-1}$, where a denotes the volume of n-dimensional unit ball. Set $N(x, y) = E(x - y) + E(x - y^*)$ for $x, y \in \mathbf{R}^n_+$ and $y^* = (y', -y_n)$. Since solutions of the weak Neumann problem, i.e., $\Delta\varphi = \operatorname{div} \partial_x g$ in $\mathbf{R}^n_+$, $\partial\varphi/\partial n_\Omega = 0$ on $\partial\mathbf{R}^n_+$, are unique under the bound $\nabla\varphi \in L^p(\mathbf{R}^n_+)$, $p \in (1, \infty)$, the function $\nabla\varphi = \mathbf{Q}[\partial_x g]$ is expressed by the kernel $N(x, y)$, i.e.,

$$\nabla\varphi(x) = \nabla\int_{\mathbf{R}^n_+} \nabla_y N(x, y)\cdot\partial_y g(y)\mathrm{d}y.$$

Take a positive constant $R > 0$ such that spt $g \subset B_0(R)$. Since $|\partial^k_x E(x)| \leq C/|x|^{n-2+|k|}$ for $|k| \geq 1$, it follows that

$$|\nabla\varphi(x)| \leq C\int_{\mathbf{R}^n_+}\left(\frac{1}{|x-y|^{n+1}} + \frac{1}{|x-y^*|^{n+1}}\right)|g(y)|\mathrm{d}y.$$

For $|x| \geq 2R$ and $|y| \leq R$, we observe that

$$\begin{aligned}|x-y| &\geq \big||x| - |y|\big| \\ &\geq |x| - R \\ &\geq |x|/2.\end{aligned}$$

By the same way, it follows that $|x-y^*| \geq |x|/2$ for $|y^*| \leq R$. Thus, we have

$$|\nabla\varphi(x)| \leq \frac{C}{|x|^{n+1}} \quad \text{for } |x| \geq 2R,$$

with the constant $C' = 2^{n+2}C\|g\|_{L^1(\mathbf{R}^n_+)}$. Since $\nabla\varphi$ is L^p-integrable in $\mathbf{R}^n_+$, $\nabla\varphi \in L^1_{\mathrm{loc}}(\overline{\mathbf{R}^n_+})$. Thus, $\nabla\varphi \in L^1(\mathbf{R}^n_+)$. □

Proof of Lemma 2.4.6. Let R_Ω be a positive constant such that $\Omega\backslash B_0(R_\Omega) = \mathbf{R}^n_+\backslash B_0(R_\Omega)$. Let $\theta \in C^\infty_c[0,\infty)$ be a smooth cutoff function such that $\theta \equiv 1$ in $[0,1)$ and $\theta \equiv 0$ in $[2,\infty)$. Set $\theta_R(x) = \theta(|x|/R)$ for $R > R_\Omega$. Then, $\theta_R \equiv 1$ in $B_0(R)$ and $\theta_R \equiv 0$ in $B_0(2R)^c$. We divide φ into two terms $\varphi = \varphi\theta_R + \varphi(1-\theta_R)$. Observe that $\nabla(\varphi\theta_R) = \nabla\varphi\theta_R + \varphi\nabla\theta_R \in L^1(\Omega)$ since $\nabla\varphi = \mathbf{Q}[\partial_x g] \in L^p(\Omega)$, $p \in (1,\infty)$. It suffices to show that $\nabla\varphi_R \in L^1(\Omega)$ for $\varphi_R = \varphi(1-\theta_R)$. Set $g_R = g\theta_R$. Since the function φ satisfies the weak Neumann problem: $\Delta\varphi = \operatorname{div}\partial_x g$ in Ω, $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$, φ_R satisfies

$$\Delta\varphi_R = \operatorname{div}\partial_x g_R + f_R \quad \text{in } \Omega, \quad \frac{\partial\varphi_R}{\partial n_\Omega} = 0 \quad \text{on } \partial\Omega,$$

where

$$f_R = -2\nabla\varphi\cdot\nabla\theta_R - \varphi\Delta\theta_R + \partial_x g\cdot\nabla\theta_R + \operatorname{div} g\partial_x\theta_R + g\cdot\nabla\partial_x\theta_R.$$

We identify φ_R and its zero extension to $\mathbf{R}^n_+\backslash\Omega$. Since $\varphi_R = 0$ in $B_0(R)_+ = B_0(R)\cap\mathbf{R}^n_+$, φ_R satisfies

$$\Delta\varphi_R = \operatorname{div}\partial_x g_R + f_R \quad \text{in } \mathbf{R}^n_+, \quad \frac{\partial\varphi_R}{\partial x_n} = 0 \quad \text{on } \partial\mathbf{R}^n_+.$$

We observe that $\operatorname{spt} f_R \subset B_0(2R)_+$. Let φ^1_R be a solution of the Laplace equation in $B_0(2R)_+$,

$$\begin{aligned} \Delta\varphi^1_R &= f_R \quad \text{in } B_0(2R)_+, \\ \varphi^1_R &= 0 \quad \text{on } \partial B_0(2R)_+\cap\mathbf{R}^n_+, \\ \frac{\partial\varphi^1_R}{\partial x_n} &= 0 \quad \text{on } \partial\mathbf{R}^n_+\cap B_0(2R)_+. \end{aligned}$$

Since $f_R \in L^p(B_0(2R)_+)$, $\varphi^1_R \in W^{2,p}(B_0(2R)_+)$. (The existence of the solution φ^1_R follows from the L^p-theory for the Dirichlet problem in $B_0(2R)$ via a reflection argument). Denoting the zero extension of φ^1_R to $\mathbf{R}^n_+\backslash B_0(R)$ by $\overline{\varphi}^1_R$, we set $\varphi^2_R = \varphi_R - \overline{\varphi}^1_R$. Then φ^2_R satisfies

$$\Delta\varphi^2_R = \operatorname{div}\partial_x g_R \quad \text{in } \mathbf{R}^n_+, \quad \frac{\partial\varphi^2_R}{\partial x_n} = 0 \quad \text{on } \partial\mathbf{R}^n_+.$$

We apply Proposition 2.4.7 and observe that $\nabla\varphi^2_R \in L^1(\mathbf{R}^n_+)$. Thus, $\nabla\varphi_R \in L^1(\mathbf{R}^n_+)$. Since $\varphi_R = 0$ in $B_0(R)$, $\nabla\varphi_R \in L^1(\Omega)$. The proof is now complete. □

2.5 Blow-up arguments

Now, we prove Theorem 2.3.3 by a blow-up argument. A blow-up argument reduces the proof of the a priori estimate (2.2.5) to the uniqueness of weak solutions for the Neumann problem (2.1.3) . If blow-up points stay inside of the domain Ω, the proof is reduced to the uniqueness of weak solutions on Ω. If blow-up points accumulate to the boundary, it is reduced to the uniqueness on $\mathbf{R}^n_+$. Note that blow-up points may diverge to infinity when Ω is unbounded. We start from a bounded domain.

Lemma 2.5.1. *A bounded domain in $\mathbf{R}^n$, $n \geq 2$, with C^3-boundary is strictly admissible.*

Proof. We argue by contradiction. Suppose that (2.2.5) were false for any choice of the constant C. Then, there would exist a sequence of weak solutions $\{P_m\}_{m=1}^{\infty}$ such that

$$|\nabla P_m|_{\infty,d} > m\|W_m\|_{\infty,\partial\Omega},$$

where $\nabla P_m \in L^\infty_d(\Omega)$ satisfies (2.2.4) for $W_m \in L^\infty_{\tan}(\partial\Omega)$, i.e.,

$$\int_\Omega P_m \Delta\varphi \mathrm{d}x = \int_{\partial\Omega} W_m \cdot \nabla_{\partial\Omega}\varphi \mathrm{d}\mathcal{H}^{n-1}(x),$$

for all $\varphi \in C^2_c(\overline{\Omega})$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. We take a point $x_m \in \Omega$ such that $d_\Omega(x_m)|\nabla P_m(x_m)| \geq M_m/2$ for $M_m = |\nabla P_m|_{\infty,d}$ and normalize P_m by dividing by M_m to get $\tilde{P}_m = P_m/M_m$ and $\tilde{W}_m = W_m/M_m$ such that

$$|\nabla \tilde{P}_m|_{\infty,d} = 1, \tag{2.5.1}$$

$$\|\tilde{W}_m\|_{\infty,\partial\Omega} < 1/m, \tag{2.5.2}$$

$$d_\Omega(x_m)\left|\nabla \tilde{P}_m(x_m)\right| \geq 1/2. \tag{2.5.3}$$

Since $\overline{\lim}_{m\to\infty} d_m < \infty$, we may assume $x_m \to x_\infty \in \overline{\Omega}$ as $m \to \infty$. Then, the proof is divided into two cases depending on whether $x_\infty \in \Omega$ or $x_\infty \in \partial\Omega$.
(i) $x_\infty \in \Omega$. The proof reduces to the uniqueness of the Neumann problem (2.1.3) on Ω. Since $\tilde{P}_m$ is harmonic in Ω and $|\nabla \tilde{P}_m|_{\infty,d} = 1$, $\tilde{P}_m$ subsequently converges to a limit P locally uniformly in Ω together with its all derivatives. Moreover, by $|\nabla \tilde{P}_m|_{\infty,d} = 1$, $\tilde{P}_m$ converges to P weakly on $L^r(\Omega)$ for $r \in [1,\infty)$. Thus, the limit P satisfies

$$\int_\Omega P\Delta\varphi \mathrm{d}x = 0.$$

We apply Lemma 2.4.1 and conclude that $\nabla P \equiv 0$. This contradicts $d_\Omega(x_\infty)|\nabla P(x_\infty)| \geq 1/2$ so (i) $x_\infty \in \Omega$ does not occur.
(ii) $x_\infty \in \partial\Omega$. The proof reduces to the uniqueness on a half space. By rotation and

translation of Ω, we may assume $x_m = (0, d_m)$ and $d_m \downarrow 0$ as $m \to \infty$. Since $\partial\Omega$ is C^3, there exists constants α, β, K and a C^3-function h such that

$$\Omega \supset \Omega^{\mathrm{loc}} = \{x \in \mathbf{R}^n \mid h(x') < x_n < h(x') + \beta,\ |x'| < \alpha\},$$

$\|h\|_{C^3(B^{n-1}(\alpha))} \le K$ and $h(0) = 0$. We rescale $\tilde{P}_m$ around x_m to get the blow-up sequence,

$$Q_m(x) = \tilde{P}_m(x_m + d_m x) \quad \text{for } x \in \Omega_m = \frac{\Omega_{x_m}}{d_m},$$

and $G_m(x) = \tilde{W}_m(x_m + d_m x)$. Then, the estimates (2.5.1)–(2.5.3) are inherited to

$$\sup_{x\in\Omega_m} d_{\Omega_m}(x)|\nabla Q_m(x)| = 1,$$
$$\|G_m\|_{L^\infty(\partial\Omega_m)} < 1/m,$$
$$|\nabla Q_m(0)| \ge 1/2.$$

Note that the distance from the origin to the boundary is always one, i.e., $d_{\Omega_m}(0) = 1$. The rescaled domain Ω_m expands to the half space $\mathbf{R}^n_{+,-1} = \{x \in \mathbf{R}^n \mid x = (x', x_n)\ x_n > -1\}$. In fact,

$$\Omega_m \supset \Omega_m^{\mathrm{loc}} = \{x \in \mathbf{R}^n \mid h_m(x') < x_n < h_m(x') + \beta/d_m,\ |x'| < \alpha/d_m\}$$

for $h_m(x') = h(d_m x') - 1$, so Ω_m^{loc} converges to $\mathbf{R}^n_{+,-1}$. Note that this rescaling procedure keeps C^3-regularity of the boundary $\partial\Omega_m$, i.e., the C^3-norm of h_m in $B^{n-1}_{x_0}(\alpha/d_m)$, $x_0 = (0, \cdots, 0, -1)$, is uniformly bounded for $m \ge 1$. Moreover, the boundary $\partial\Omega_m$ converges to $\partial\mathbf{R}^n_+$, i.e., $h_m \to -1$ and $\partial_x^k h_m \to 0$ as $m \to \infty$ locally uniformly in $\mathbf{R}^{n-1}$ for $1 \le |k| \le 3$. Since Q_m is harmonic in Ω_m and $\sup_{x\in\Omega_m} d_{\Omega_m}(x)|\nabla Q_m(x)| \le 1$, Q_m subsequently converges to a limit Q locally uniformly in $\mathbf{R}^n_{+,-1}$ together with its all derivatives and weakly on $L^r_{\mathrm{loc}}(\overline{\mathbf{R}^n_{+,-1}})$ for $r \in [1, \infty)$.

Now, we observe that the limit Q is a trivial limit, i.e., $\nabla Q \equiv 0$. Let $\varphi \in C^2_c(\overline{\mathbf{R}^n_{+,-1}})$ satisfy $\partial\varphi/\partial x_n = 0$ on $\{x_n = -1\}$. We extend φ to $\mathbf{R}^n$ by the even extension of φ, which is still denoted by $\varphi \in C^2_c(\mathbf{R}^n)$. Set $\varphi_m(X) = \varphi(X_m^{-1}(X))$ for $X \in \Omega_m^{\mathrm{loc}}$ by the map $X_m : \mathbf{R}^n_{+,-1} \to \Omega_m^{\mathrm{loc}}$, i.e.,

$$X_m(x', x_n) = x_{\partial\Omega_m} - (x_n + 1)n_{\Omega_m}(x_{\partial\Omega_m}) \quad \text{for } x_{\partial\Omega_m} = (x', h_m(x')),\ x = (x', x_n) \in \mathbf{R}^n_{+,-1}.$$

This X_m is well-defined for sufficiently large m. Since $\partial\Omega_m$ is C^3, $\varphi_m \in C^2_c(\overline{\Omega_m})$. Moreover, $\partial\varphi_m/\partial n_{\Omega_m} = 0$ on $\partial\Omega_m$ since $n_{\Omega_m} = -\nabla d_m$. Since Q_m and G_m satisfies (2.2.4) for φ_m, by letting $m \to \infty$, we have

$$\int_{\mathbf{R}^n_{+,-1}} Q\Delta\varphi \mathrm{d}x = 0.$$

We apply Lemma 2.3.2 and conclude that $\nabla Q \equiv 0$. This contradicts $|\nabla Q(0)| \ge 1/2$. Thus (ii) does not occur. We reached a contradiction. The proof is now complete. □

We shall prove the estimate (2.2.5) for exterior domains and perturbed half spaces. To argue by contradiction, suppose that there are a sequence of weak solutions of (2.1.1), $\{P_m\}_{m=1}^{\infty}$, and a sequence of points $\{x_m\}_{m=1}^{\infty} \subset \Omega$ such that $|\nabla P_m|_{\infty,d} = 1$, $\|W_m\|_{\infty,\partial\Omega} \to 0$ as $m \to \infty$ and

$$d_\Omega(x_m)\big|\nabla P_m(x_m)| \geq \frac{1}{2}.$$

Set $d_m = d_\Omega(x_m)$. If $\overline{\lim}_{m\to\infty} d_m < \infty$, the proof reduces to the uniqueness of weak solutions on $\mathbf{R}^n_+$ and the domain Ω (Lemmas 2.4.2 and 2.4.4) as we proved (2.2.5) for bounded domains. The crucial case is when $\{x_m\}_{m=1}^{\infty}$ diverges to infinity, i.e., $d_m \uparrow \infty$ as $m \to \infty$. When Ω is an exterior domain, the problem is reduced to the whole space. When Ω is a perturbed half space, it is reduced to a half space.
Let us give ideas in proving (2.2.5) for an exterior domain. We rescale the solution P_m around x_m to get

$$Q_m(x) = P_m(x_m + d_m x) \quad \text{for } x \in \Omega_m = \frac{\Omega_{x_m}}{d_m}.$$

Then, Ω_m^c accumulates to the point $a \in \mathbf{R}^n$ ($a \neq 0$) and Ω_m approaches to $\mathbf{R}^n \backslash \{a\}$. We show that the limit Q is extendable to a harmonic function in $\mathbf{R}^n$ by using

$$\sup_{x \in \mathbf{R}^n \backslash \{a\}} |x - a||\nabla Q(x)| \leq 1,$$

and conclude that $\nabla Q \equiv 0$. This contradicts $|\nabla Q(0)| \geq 1/2$. For $n = 2$, above bound for Q is not enough in order to show Q is harmonic in $\mathbf{R}^2$, but in this case, we use the fact that the mean value of Q around $x = a$ on a surface of a ball is independent of the radius of a ball.

This idea works also for perturbed half space. When Ω is a perturbed half space, we rescale P_m around the point $x_m \in \Omega$ by replacing d_m to $\tilde{d}_m = (x_m)_n$ to get

$$Q_m(x) = \tilde{P}_m(x_m + \tilde{d}_m x) \quad \text{for } x \in \Omega_m = \frac{\Omega_{x_m}}{\tilde{d}_m}.$$

Then, the rescaled domain Ω_m approaches to a half space $\mathbf{R}^n_{+,-1}$ and a curbed part of the boundary $\partial\Omega_m$ accumulates to the point $a \in \partial\mathbf{R}^n_{+,-1}$ (or diverges to infinity). We remove the singularity at $x = a$ for the limit Q by using the bound,

$$\sup_{x \in \mathbf{R}^n_{+,-1}} (x_n + 1)|\nabla Q(x)| \leq 1,$$

and conclude that $\nabla Q \equiv 0$ for $n \geq 3$. It is likely that the estimate (2.2.5) holds also for $n = 2$, but the above bound does not exclude the singularity at $x = a$. In fact, $Q(x) = \log|x - a|$ is harmonic in $\mathbf{R}^n_{+,-1}$ and satisfies the bound.

Proof of Theorem 2.3.3. We argue by contradiction. Suppose that (2.2.5) were false for any choice of constants. Then, there would exist the weak solutions $\tilde{P}_m$ for $\tilde{W}_m$ and the

points $\{x_m\}_{m=1}^{\infty} \subset \Omega$ satisfying (2.5.1)–(2.5.3). Set $d_m = d_\Omega(x_m)$. If $\overline{\lim}_{m\to\infty} d_m < \infty$, by the same way as we proved Lemma 2.5, the proof reduces to the uniqueness of weak solutions in $\mathbf{R}^n_+$ and Ω (Lemmas 2.3.2, 2.4.2 and 2.4.4). We may assume $d_m \uparrow \infty$ as $m \to \infty$.
(a) *The case Ω is an exterior domain.* We rescale $\tilde{P}_m$ around x_m to get

$$Q_m(x) = \tilde{P}_m(x_m + d_m x) \quad \text{for } x \in \Omega_m,$$

where $\Omega_m = \{x \in \mathbf{R}^n \mid x = (y - x_m)/d_m, y \in \Omega\}$. Then, Q_m satisfies $\Delta Q_m = 0$ in Ω_m,

$$\sup_{x\in\Omega_m} d_{\Omega_m}(x)|\nabla Q_m(x)| = 1,$$

$$\left|\nabla Q_m(0)\right| \geq 1/2.$$

Take $y_m \in \partial\Omega$ such that $d_m = |y_m - x_m|$. Then $z_m = (y_m - x_m)/d_m \in \partial\Omega_m$ satisfies $|z_m| = 1$. We may assume $z_m \to a \in \mathbf{R}^n$ as $m \to \infty$. Since Ω_m^c accumulates to the point $a \in \mathbf{R}^n$, Ω_m approaches to $\mathbf{R}^n\backslash\{a\}$. Set $\phi_m(r) = \fint_{\partial B_a(r)} Q_m(x)\mathrm{d}\mathcal{H}^{n-1}(x)$ for $r > \operatorname{diam}\Omega_m^c$. Then, by (2.2.4), it follows that $d\phi_m(r)/dr \equiv 0$ for $r > \operatorname{diam}\Omega_m^c$ (One should apply Proposition 2.6.3).
Now, we observe a limit of Q_m is a trivial limit. Since ∇Q_m is uniformly bounded by $\sup_{x\in\Omega_m} d_{\Omega_m}(x)|\nabla Q_m(x)| = 1$ and Q_m is harmonic in Ω_m, Q_m subsequently converges to a limit Q locally uniformly in $\mathbf{R}^n\backslash\{a\}$. Thus, the limit Q satisfies $\Delta Q = 0$ in $\mathbf{R}^n\backslash\{a\}$,

$$\sup_{x\in\mathbf{R}^n\backslash\{a\}} |x-a||\nabla Q(x)| \leq 1, \tag{2.5.4}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}r}(r) \equiv 0 \quad \text{for } r > 0, \tag{2.5.5}$$

where $\phi(r) = \fint_{\partial B_a(r)} Q(x)\mathrm{d}\mathcal{H}^{n-1}(x)$. The estimate (2.5.4) and condition (2.5.5) imply that Q is extendable to a harmonic function in $\mathbf{R}^n$ for all $n \geq 2$; see Lemma 2.6.1. We apply the Liouville theorem and conclude that $\nabla Q \equiv 0$. This contradicts $|\nabla Q(0)| \geq 1/2$. We reached a contradiction. Thus, we proved (2.2.5) for exterior domains.
(b) *The case Ω is a perturbed half space.* Let $R = R_\Omega$ be a positive constant such that $\Omega\backslash B_0(R) = \mathbf{R}^n_+\backslash B_0(R)$. Let $\tilde{d}_m$ be the normal component of the point x_m. If $B_{x_m}(\tilde{d}_m)\cap\Omega = \emptyset$, then $\tilde{d}_m = d_m$. If $B_{x_m}(\tilde{d}_m) \cap \Omega \neq \emptyset$, then $\tilde{d}_m \geq d_m$. Thus, $d_m \leq \tilde{d}_m \to \infty$ as $m \to \infty$. We rescale $\tilde{P}_m$ around the point x_m to get

$$Q_m(x) = \tilde{P}_m(x_m + \tilde{d}_m x) \quad \text{for } x \in \Omega_m = \frac{\Omega_{x_m}}{\tilde{d}_m},$$

and $G_m(x) = \tilde{W}_m(x_m + \tilde{d}_m x)$. Then, the estimates (2.5.1)–(2.5.3) are inherited to

$$\sup_{x\in\Omega_m} d_{\Omega_m}(x)|\nabla Q_m(x)| = 1,$$

$$\|G_m\|_{\infty,\partial\Omega_m} < 1/m$$

$$|\nabla Q_m(0)| \geq \frac{\tilde{d}_m}{2d_m} \geq \frac{1}{2}.$$

The rescaled domain Ω_m satisfies $\Omega_m \backslash B_{\tilde{x}_m}(R_m) = \mathbf{R}^n_{+,-1} \backslash B_{\tilde{x}_m}(R_m)$ for $\tilde{x}_m = -x_m/\tilde{d}_m = (-x'_m/\tilde{d}_m, -1)$ and $R_m = R/\tilde{d}_m$. If $\overline{\lim}_{m\to\infty}|\tilde{x}_m| = \infty$, a curved part of $\partial\Omega_m$ diverges to infinity so the limit domain of Ω_m is $\mathbf{R}^n_{+,-1}$. We may assume $\tilde{x}_m \to a \in \partial\mathbf{R}^n_{+,-1}$ as $m \to \infty$. Then, $B_{\tilde{x}_m}(R_m)$ accumulates to the point a and Ω_m approaches to $\mathbf{R}^n_{+,-1}$. Since Q_m is harmonic in Ω_m and $\sup_{x\in\Omega_m} d_{\Omega_m}(x)|Q_m(x)| \leq 1$, Q_m subsequently converges to a limit Q locally uniformly in $\mathbf{R}^n_{+,-1}$ together with its all derivatives. Moreover, Q_m converges to Q weakly on $L^r_{\mathrm{loc}}(\overline{\mathbf{R}^n_{+,-1}})$ for $r \in [1,\infty)$.
We now observe that the limit Q is trivial, i.e., $\nabla Q \equiv 0$. Let $\varphi \in C^2_c(\overline{\mathbf{R}^n_{+,-1}})$ satisfy $\partial\varphi/\partial x_n = 0$ on $\partial\mathbf{R}^n_{+,-1}$ and $\mathrm{spt}\,\varphi \cap \{a\} = \emptyset$. Since Q_m satisfies (2.2.4) for G_m in Ω_m, by letting $m \to \infty$, it follows that

$$\int_{\mathbf{R}^n_{+,-1}} Q\Delta\varphi \mathrm{d}x = 0. \tag{2.5.6}$$

We remove the restrictive condition spy $\varphi \cap \{a\} = \emptyset$ for $n \geq 3$. Since the limit Q satisfies

$$\sup_{x\in\mathbf{R}^n_{+,-1}} (x_n + 1)|\nabla Q(x)| \leq 1,$$

applying Proposition 2.6.4 implies (2.5.6) for all $\varphi \in C^2_c(\overline{\mathbf{R}^n_{+,-1}})$ satisfying $\partial\varphi/\partial x_n = 0$ on $\partial\mathbf{R}^n_{+,-1}$ for $n \geq 3$. We apply Lemma 2.3.2 and conclude that $\nabla Q \equiv 0$. This contradicts $|\nabla Q(0)| \geq 1/2$. We reached a contradiction. Thus, we proved (2.2.5) for perturbed half spaces. The proof is now complete. □

Remarks 2.5.2. (i) For a bounded domain, the estimate (2.1.2) was also proved by C. E. Kenig, F. Lin and Z. Shen [11], independently of our previous work [1], where a slightly different version was proved. In [11], they studied the Neumann problem,

$$\Delta P = 0 \quad \text{in } \Omega, \qquad \frac{\partial P}{\partial n_\Omega} = \sum_{i,j}\left(n^i_\Omega \frac{\partial}{\partial x_j} - n^j_\Omega \frac{\partial}{\partial x_i}\right) g_{ij} \quad \text{on } \partial\Omega \tag{2.5.7}$$

for tensor-valued functions $g = (g_{ij})_{1\leq i,j\leq n}$, and proved the estimate

$$\sup_{x\in\Omega} d_\Omega(x)|\nabla P(x)| \leq C \sum_{i,j} \|g_{ij}\|_{L^\infty(\partial\Omega)} \tag{2.5.8}$$

for a bounded domain with $C^{1,\gamma}$-boundary. The boundary data in (2.5.7) can be written as the surface divergence (2.5.8) of a tangential vector field. In fact, set $W = -(g - g^T)n_\Omega$ for $g = (g_{ij})_{1\leq i,j\leq n}$. Then, W satisfies $W \cdot n_\Omega = 0$ and

$$\mathrm{div}_{\partial\Omega} W = \sum_{i,j}\left(n^i_\Omega \frac{\partial}{\partial x_j} - n^j_\Omega \frac{\partial}{\partial x_i}\right) g_{ij} \tag{2.5.9}$$

since $\partial_i n^j = \partial_j n^i$, $i, j \in \{1, \cdots, n\}$ and $\mathrm{div}_{\partial\Omega} W = \mathrm{div} W$ by extending n_Ω outside of $\partial\Omega$ as a gradient of $-d_\Omega$. Thus, our estimate (2.1.2) immediately implies (2.5.8). Moreover, we are able to estimate $d_\Omega \nabla P$ by the antisymmetric part $(g - g^T)n_\Omega$, i.e.,

$$\sup_{x\in\Omega} d_\Omega(x)|\nabla P(x)| \leq C\|(g - g^T)n_\Omega\|_{L^\infty(\partial\Omega)}.$$

On the other hand, for a tangential vector field $W = (W^i)_{1\leq i\leq n}$ and $g_{ij} = -W^i n^j_\Omega$, (2.5.9) holds so the estimate (2.5.8) implies our estimate (2.1.2).

Actually, in the paper [11, Lemma 6.2] the estimate (2.5.8) is discussed for the Neumann problem, $\mathcal{L}P = 0$ in Ω, $\partial P/\partial n_\Omega = \sum_{i,j}(n^i_\Omega \partial_j - n^j_\Omega \partial_i)g_{ij}$ on $\partial\Omega$, associated to the elliptic operator $\mathcal{L} = -\mathrm{div}(a(x)\nabla)$ and $a(x) = (a_{ij}(x))_{1\leq i,j\leq n}$. It is proved that the estimate (2.5.8) holds if $\mathcal{L}$ is symmetric and uniformly elliptic, i.e., $a_{ij}(x) = a_{ji}(x)$, $i, j \in \{1, \cdots, n\}$ and $\mu|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \mu^{-1}|\xi|^2$, $x \in \mathbf{R}^n$ and $\xi \in \mathbf{R}^n$ for some $\mu > 0$. The assumptions on the operator $\mathcal{L}$ are interesting, but for our purpose of estimating the pressure (0.1.3), the Laplace equation (2.1.1) is sufficient.

(ii) On an exterior domain, a unique weak solution $\nabla P \in L^\infty_d(\Omega)$ exists for $W \in L^\infty_{\tan}(\partial\Omega)$. We call the solution operator $\mathbf{K} : W \mapsto \nabla P$ *the harmonic-pressure operator*, which is a bounded operator from $L^\infty_{\tan}(\partial\Omega)$ to $L^\infty_d(\Omega)$. Although the representation by the Helmholtz projection, i.e., $\nabla q = \mathbf{Q}[\Delta v]$, may not hold for non-decaying solutions to the Stokes equations, the harmonic-pressure operator $\mathbf{K}$ recovers the pressure gradient from the velocity, i.e., $\nabla q = \mathbf{K}[W(v)]$ for $W(v) = -(\nabla v - \nabla^T v)n_\Omega$.

The existence of weak solutions immediately follows from the L^p-theory for the Neumann problem if one impose a suitable regularity for the Neumann data $W \in L^\infty_{\tan}(\partial\Omega)$. For instance, if $W \in L^\infty_{\tan}(\partial\Omega)$ satisfies $g = \mathrm{div}_{\partial\Omega} W \in W^{-1/p,p}(\partial\Omega)$, by taking a compactly supported extension $f \in L^p(\Omega)$ such that $\mathrm{div}\, f = 0$ in Ω, $f \cdot n_\Omega = g$ on $\partial\Omega$ (see, e.g., [13]), $\nabla P = \mathbf{Q}[f]$ is a weak solution for $W \in L^\infty_{\tan}(\partial\Omega)$. Here, $W^{-1/p,p}(\partial\Omega)$ denotes the dual space of the Sobolev space $W^{1-1/p',p'}(\partial\Omega)$, $p' = p/(p-1)$. For general $W \in L^\infty_{\tan}(\partial\Omega)$, approximating W by changing a coordinate to $\partial\mathbf{R}^n_+$, we take $W_m \in L^\infty_{\tan}(\partial\Omega) \cap W^{-1/p,p}(\partial\Omega)$ satisfying $\|W_m\|_{\infty,\partial\Omega} \leq C\|W\|_{\infty,\partial\Omega}$ and $W_m \to W$ a.e. on $\partial\Omega$ as $m \to \infty$, with the constant C independent of $m \geq 1$. Then, the a priori estimate (2.2.5) implies that $\nabla P_m = \mathbf{K}[W_m]$ is uniformly bounded on $L^\infty_d(\Omega)$. Thus, the limit ∇P is in $L^\infty_d(\Omega)$ and a weak solution for $W \in L^\infty_{\tan}(\partial\Omega)$. Note that the limit ∇P is independent of the choices of extension and approximation for $W \in L^\infty_{\tan}(\partial\Omega)$ since weak solutions are unique.

2.6 Extensions for harmonic functions

In this section, we give extension theorems for harmonic functions in $\mathbf{R}^n$ and $\mathbf{R}^n_+$, which are used in the proof of Theorem 2.3.3. We first state an extension theorem in $\mathbf{R}^n$. Although our assumption can be weakened, we give the statement in a simple form in order to apply it in the proof of Theorem 2.3.3.

Lemma 2.6.1. *Let P be a harmonic function in $\mathbf{R}^n\setminus\{0\}$ for $n \geq 2$. Assume that*

$$\sup_{|x|\leq 1} |x||\nabla P(x)| < \infty. \tag{2.6.1}$$

Then, for $n \geq 3$, P is extendable to a harmonic function in $\mathbf{R}^n$. For $n = 2$, assume in addition that

$$\frac{d}{dr} \fint_{\partial B_0(r)} P(x) d\mathcal{H}^1(x) \equiv 0 \quad \textit{for } \ r < 1. \tag{2.6.2}$$

Then, P is extendable to a harmonic function in $\mathbf{R}^2$.

Proof. By (2.6.1), P is locally integrable in $\mathbf{R}^n$. We prove the assertion by showing

$$\int_{\mathbf{R}^n} P\Delta\varphi \mathrm{d}x = 0 \tag{2.6.3}$$

for all $\varphi \in C_c^\infty(\mathbf{R}^n)$. Let η_ε be a radially symmetric mollifier, i.e., $\eta_\varepsilon(x) = \eta_\varepsilon(|x|)$, spt $\eta_\varepsilon \subset \overline{B_0(\varepsilon)}$ and $\int_{B_0(\varepsilon)} \eta_\varepsilon dx = 1$. Set $P_\varepsilon = P * \eta_\varepsilon$. Then ,$P_\varepsilon \in C^\infty(\mathbf{R}^n)$ is harmonic in $\mathbf{R}^n$ by (2.6.3). By the mean value formula, P_ε satisfies

$$P_\varepsilon(x) = \fint_{\partial B_x(r)} P_\varepsilon(y) d\mathcal{H}^{n-1}(y) \quad \text{for } x \in \mathbf{R}^n, \ r > 0.$$

Letting $\varepsilon \downarrow 0$, we obtain $P(x) = \fint_{\partial B_x(r)} P(y) d\mathcal{H}^{n-1}(y)$ for $x \in \mathbf{R}^n\setminus\{0\}$, $r > 0$. Since the right-hand side is continuous in $\mathbf{R}^n$, by setting $\bar{P}(x) = P(x)$ for $x \in \mathbf{R}^n\setminus\{0\}$ and $\bar{P}(0) = \fint_{\partial B_0(r)} P(y) d\mathcal{H}^{n-1}(y)$, $\bar{P} \in C(\mathbf{R}^n)$ and

$$\bar{P}(x) = \fint_{\partial B_x(r)} \bar{P}(y) d\mathcal{H}^{n-1}(y) \quad \text{for } x \in \mathbf{R}^n, \ r > 0.$$

Note that $\bar{P}(0)$ is independent of $r > 0$ since $\fint_{\partial B_x(r_1)} P(y) d\mathcal{H}^{n-1}(y) = \fint_{\partial B_x(r_2)} P(y) d\mathcal{H}^{n-1}(y)$ for $x \in \mathbf{R}^n\setminus\{0\}$, $r_1, r_2 > 0$ and $\fint_{\partial B_x(r_1)} P(y) d\mathcal{H}^{n-1}(y) \to \bar{P}(0)$ as $|x| \downarrow 0$. Since η_ε is radially symmetric mollifier, $\bar{P}_\varepsilon = P * \eta_\varepsilon \in C^\infty(\mathbf{R}^n)$ agrees with $\bar{P}$, i.e., $\bar{P}_\varepsilon = \bar{P}$. Thus, $\bar{P} \in C^\infty(\mathbf{R}^n)$ is harmonic in $\mathbf{R}^n$.

Now, we prove (2.6.3). We may assume spt $\varphi \subset B_0(1)$. For $x \in B_0(1)\setminus\{0\}$ and $y = x/|x|$, it follows that

$$\begin{aligned} |P(x) - P(y)| &\leq |x - y| \int_0^1 |\nabla P(\tau x + (1 - \tau)y)| \,\mathrm{d}\tau \\ &\leq C_1(1 - |x|) \int_0^1 \frac{\mathrm{d}\tau}{1 - \tau(1 - |x|)} \\ &= -C_1 \log |x| \end{aligned}$$

with the constant C_1 larger than (2.6.1). Thus, we have

$$|P(x)| \leq C_1 \log|x| + C_2 \quad \text{for } x \in B_0(1)\backslash\{0\} \tag{2.6.4}$$

with $C_2 = \sup_{|y|=1} |P(y)|$. Since P is harmonic in $B_0(1)\backslash\{0\}$, by integration by parts, it follows that

$$\int_{\varepsilon<|x|<1} P\Delta\varphi \mathrm{d}x = \int_{|x|=\varepsilon} \left(P\frac{\partial\varphi}{\partial n_{B_0(\varepsilon)}} - \frac{\partial P}{\partial n_{B_0(\varepsilon)}}\varphi \right) \mathrm{d}\mathcal{H}^{n-1}(x)$$

for $\varepsilon > 0$. By (2.6.5), the first term vanishes as $\varepsilon \downarrow 0$. We estimate the second term. By (2.6.1), it follows that

$$\left| \int_{|x|=\varepsilon} \frac{\partial P}{\partial n_{B_0(\varepsilon)}} \varphi \mathrm{d}\mathcal{H}^{n-1}(x) \right| \leq C\varepsilon^{n-2} \|\varphi\|_{L^\infty(\mathbf{R}^n)} \tag{2.6.5}$$

with the constant C independent of $\varepsilon > 0$. For $n \geq 3$, the right-hand side vanishes as $\varepsilon \downarrow 0$. Thus, we proved (2.6.3) for $n \geq 3$.
It remains to show (2.6.3) for $n = 2$. By (2.6.2), it follows that

$$0 = \frac{\mathrm{d}}{\mathrm{d}r} \fint_{\partial B_0(r)} P(x)\mathrm{d}\mathcal{H}^1(x) = \fint_{\partial B_0(r)} \frac{\partial P(x)}{\partial n_{B_0(r)}} \mathrm{d}\mathcal{H}^1(x) \quad \text{for } r < 1.$$

Thus, in (2.6.5), we are able to shift φ by a constant. We replace φ to $\varphi - \varphi(0)$ in (2.6.5). Since $|\varphi(x) - \varphi(0)| \leq |x| \|\nabla\varphi\|_{L^\infty(\mathbf{R}^n)}$, it follows that

$$\left| \int_{|x|=\varepsilon} \frac{\partial P}{\partial n_{B_0(\varepsilon)}} \varphi \mathrm{d}\mathcal{H}^{n-1}(x) \right| = \left| \int_{|x|=\varepsilon} \frac{\partial P}{\partial n_{B_0(\varepsilon)}} (\varphi - \varphi(0)) \mathrm{d}\mathcal{H}^1(x) \right| \leq C_n \varepsilon \|\nabla\varphi\|_{L^\infty(\mathbf{R}^n)}.$$

By letting $\varepsilon \downarrow 0$, (2.6.3) follows. The proof is now complete. □

Remarks 2.6.2. (i) If we drop (2.6.2) for $n = 2$, the statement of Lemma 2.6.1 does not hold. For example, $\log|x|$ is harmonic in $\mathbf{R}^2\backslash\{0\}$ and satisfies (2.6.1).
(ii) The condition (2.6.1) can be replaced by $\sup_{|x|\leq 1} |x|^{1+\alpha}|\nabla P(x)| < \infty$ for some $\alpha \in (0, 1)$ without modifying the proof.
(iii) We state Lemma 2.6.1 in a simple form in order to apply it in the proof of Theorem 2.3.3. If we assume (2.6.2) for all $n \geq 2$, we are able to replace (2.6.1) to $|P(x)| = O(|\log|x|||x|^{2-n})$ as $|x| \to 0$ (see, e.g., [14, Chapter I, Theorem 3.2]).

In an exterior domain Ω, the mean value of a weak solution on $\partial B_0(r)$ is independent of $r > \operatorname{diam} \Omega^c$. The following Proposition 2.6.3 is used in the proof of Theorem 2.3.3 in order to apply Lemma 2.6.1.

Proposition 2.6.3. *Let Ω be an exterior domain with C^1-boundary and $0 \in \Omega^c$. Let P be a weak solution of (2.1.1). Then,*

$$\frac{d}{dr} \fint_{\partial B_0(r)} P(x) d\mathcal{H}^{n-1}(x) \equiv 0 \quad \textit{for } r > \operatorname{diam} \Omega^c. \tag{2.6.6}$$

Proof. Since

$$\frac{\mathrm{d}}{\mathrm{d}r} \fint_{\partial B_0(r)} P(x)\mathrm{d}\mathcal{H}^{n-1}(x) = \fint_{\partial B_0(r)} \frac{\partial P}{\partial n_{B_0(r)}}(x)\mathrm{d}\mathcal{H}^{n-1}(x),$$

it suffices to show that the right-hand side is zero. For each $r > \operatorname{diam} \Omega^c$ we take a smooth function φ such that $\varphi \equiv 1$ for $|x| \leq r$ and $\varphi \equiv 0$ for $|x| \geq 2r$. Since P is a weak solution of (2.1.8), by (2.2.4) it follows that

$$\begin{aligned} -\int_{r<|x|<2r} \nabla P \cdot \nabla \varphi \mathrm{d}x &= \int_{\partial\Omega} W \cdot \nabla_{\partial\Omega} \varphi \mathrm{d}\mathcal{H}^{n-1}(x) \\ &= 0. \end{aligned}$$

Since P is harmonic in Ω and $\varphi(x) = 0$ on $|x| = 2r$, $\varphi(x) = 1$ on $|x| = r$, by integration by parts, it follows that

$$\begin{aligned} \int_{r<|x|<2r} \nabla P \cdot \nabla \varphi \mathrm{d}x &= \int_{\partial B_0(r)} \frac{\partial P}{\partial n_{B_0(r)}} \varphi \mathrm{d}\mathcal{H}^{n-1}(x) + \int_{\partial B_0(2r)} \frac{\partial P}{\partial n_{B_0(2r)}} \varphi \mathrm{d}\mathcal{H}^{n-1}(x) \\ &= \int_{\partial B_0(r)} \frac{\partial P}{\partial n_{B_0(r)}} \mathrm{d}\mathcal{H}^{n-1}(x). \end{aligned}$$

Thus, we obtain (2.3.6). The proof is now complete. □

We next state an extension theorem for harmonic functions in a half space corresponding to Lemma 2.6.1. In the proof of Theorem 2.3.3 for a perturbed half space by a blow-up argument, a curbed part of a perturbed half space accumulates to a point on $\partial\mathbf{R}^n_{+,-1}$. The following Proposition 2.4.4 implies that the limit of the rescaled solution Q_m does not have a singularity on $\partial\mathbf{R}^n_{+,-1}$ for $n \geq 3$.

Proposition 2.6.4. *Let $P \in L^1_{loc}(\overline{\mathbf{R}^n_+})$ satisfy*

$$\int_{\mathbf{R}^n_+} P \Delta \varphi dx = 0 \tag{2.6.7}$$

for $\varphi \in C^2_c(\overline{\mathbf{R}^n_+})$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\mathbf{R}^n_+$ and

$$spt\ \varphi \cap \{0\} = \emptyset. \tag{2.6.8}$$

Assume that

$$\sup_{|x|\leq 1} x_n |\nabla P(x)| < \infty.$$

Then, for $n \geq 3$, (2.6.7) holds without imposing the restrictive condition (2.6.8) for φ.

Proof. Let $\theta \in C_c^\infty[0,\infty)$ be a smooth cutoff function satisfying $\theta \equiv 1$ in $[0,1]$ and $\theta \equiv 0$ in $[2,\infty)$. Set $\psi_\varepsilon(x) = 1 - \tilde{\theta}_\varepsilon(x)$ by $\tilde{\theta}_\varepsilon(x) = \theta_\varepsilon(|x'|)\theta_\varepsilon(|x_n|)$ and $\theta_\varepsilon(s) = \theta(s/\varepsilon)$. Then, $\psi_\varepsilon \in C_c^\infty(\overline{\mathbf{R}^n_+})$ satisfies $\psi_\varepsilon \in C_0(\varepsilon)_+$, $\psi_\varepsilon \equiv 1$ in $\mathbf{R}^n_+\backslash\overline{C_0(2\varepsilon)_+}$ and $\partial\psi_\varepsilon/\partial x_n = 0$ on $\partial\mathbf{R}^n_+$. Here, $C_0(\varepsilon)_+ = B_0^{n-1}(\varepsilon)\times(0,\varepsilon)$. For $\varphi \in C_c^2(\overline{\mathbf{R}^n_+})$ satisfying $\partial\varphi/\partial x_n = 0$ on $\partial\mathbf{R}^n_+$, set $\varphi_\varepsilon = \varphi\psi_\varepsilon$. Then $\varphi_\varepsilon \in C_c^2(\overline{\mathbf{R}^n_+})$ satisfies $\partial\varphi_\varepsilon/\partial x_n = 0$ on $\partial\mathbf{R}^n_+$ and $\operatorname{spt}\varphi_\varepsilon \cap \{0\} = \emptyset$. By substituting φ_ε into (2.6.7), it follows that

$$0 = \int_{\mathbf{R}^n_+} P(\Delta\varphi\psi_\varepsilon + 2\nabla\varphi\cdot\nabla\psi_\varepsilon + \varphi\Delta\psi_\varepsilon)\mathrm{d}x. \tag{2.6.9}$$

The first term converges to $\int_{\mathbf{R}^n_+} P\Delta\varphi \mathrm{d}x$ as $\varepsilon \downarrow 0$. We shall show that the last two terms vanish as $\varepsilon \downarrow 0$. By connecting $x \in C_0(1/2)_+$ and $x_0 = (0,\cdots,0,1)$, it follows that

$$\begin{aligned}
|P(x) - P(x_0)| &\le \left|\int_0^1 (x-x_0)\cdot\nabla P(tx+(1-t)x_0)\mathrm{d}t\right| \\
&\le |x-x_0|\left(\sup_{|z|\le 1} z_n|\nabla P(z)|\right)\int_0^1 \frac{\mathrm{d}t}{tx_n+(1-t)} \\
&\le 4|\log x_n|\left(\sup_{|z|\le 1} z_n|\nabla P(z)|\right).
\end{aligned}$$

Thus, we have

$$|P(x)| \le C_1|\log x_n| + C_2 \quad \text{for } x \in C_0(1/2)_+ \tag{2.6.10}$$

with $C_1 = 4\sup_{|z|\le 1} z_n|\nabla P(z)|$ and $C_2 = |P(x_0)|$. By using the estimate (2.6.10), we estimate the last two terms in (2.6.9). Since $\operatorname{spt}\psi_\varepsilon \subset C_0(2\varepsilon)_+$, by (2.6.10), it follows that

$$\left|\int_{C_0(2\varepsilon)_+} P\nabla\varphi\cdot\nabla\psi_\varepsilon \mathrm{d}x\right| \le C\varepsilon^{n-1}(\log\varepsilon + 1)\|\nabla\varphi\|_{L^\infty(\mathbf{R}^n_+)}$$

for $\varepsilon < 1/4$. The right-hand side converges to zero as $\varepsilon \downarrow 0$ for $n \ge 2$. By the same way, we have

$$\left|\int_{C_0(2\varepsilon)_+} P\varphi\Delta\psi_\varepsilon \mathrm{d}x\right| \le C\varepsilon^{n-2}(\log\varepsilon + 1)\|\varphi\|_{L^\infty(\mathbf{R}^n_+)}.$$

The right-hand side converges to zero as $\varepsilon \downarrow 0$ for $n \ge 3$. Thus, P satisfies (2.6.7) for all $\varphi \in C_c^2(\mathbf{R}^n_+)$ satisfying $\partial\varphi/\partial x_n = 0$ on $\partial\mathbf{R}^n_+$. The proof is now complete. □

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Chapter 3

Analyticity and estimates for second derivatives

The goal of this chapter is to prove the a priori L^∞-estimate (0.1.1) for solutions of the non-stationary Stokes equations by a blow-up argument. The a priori L^∞-estimate (0.1.1) implies that the Stokes semigroup is uniquely extendable to an analytic semigroup on the (decaying) continuous solenoidal space $C_{0,\sigma}$. By using the harmonic-pressure gradient estimate (0.1.3), we establish local Hölder estimates for the Stokes equations both interior and up to boundary which implies a necessary compactness for a blow-up sequence.

3.1 Introduction

We consider the initial-boundary problem for the Stokes equations,

$$
\begin{aligned}
v_t - \Delta v + \nabla q &= 0 \quad \text{in} \quad \Omega \times (0,T), && (3.1.1)\\
\operatorname{div} v &= 0 \quad \text{in} \quad \Omega \times (0,T), && (3.1.2)\\
v &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), && (3.1.3)\\
v(x,0) &= v_0 \quad \text{on} \quad \Omega \times \{t=0\}, && (3.1.4)
\end{aligned}
$$

in the domain $\Omega \subset \mathbf{R}^n$, $n \geq 2$. It is well known that the solution operator $S(t) : v_0 \longmapsto v(\cdot,t)$ forms an analytic semigroup on the solenoidal L^r space, $L^r_\sigma(\Omega)$ for $r \in (1,\infty)$, for various kind of domains Ω including smoothly bounded domains [56], [28]. However, it had been a long-standing open problem whether or not the Stokes semigroup $\{S(t)\}_{t\geq 0}$ is analytic on L^∞-type spaces even if Ω is bounded. When Ω is a half space, it is known that the Stokes semigroup $\{S(t)\}_{t\geq 0}$ is analytic on L^∞-type spaces since explicit solution formulas are available [14], [44], [59].

The goal of this chapter is to give an affirmative answer to this open problem at least when Ω is bounded as a typical example. For a precise statement, let $C_{0,\sigma}(\Omega)$ denote the L^∞-closure of $C^\infty_{c,\sigma}(\Omega)$, the space of all smooth solenoidal vector fields with compact support in Ω. When Ω is bounded, $C_{0,\sigma}(\Omega)$ agrees with the space of all solenoidal vector fields continuous in $\bar{\Omega}$ vanishing on $\partial\Omega$ [43]. One of our main results is:

Theorem 3.1.1 (Analyticity on $C_{0,\sigma}$). *Let Ω be a bounded domain in $\mathbf{R}^n$ with C^3-boundary. Then, the solution operator (the Stokes semigroup) $S(t) : v_0 \mapsto v(\cdot, t)$ is a C_0-analytic semigroup on $C_{0,\sigma}(\Omega)$.*

Our approach to prove the analyticity is completely different from conventional approaches. We appeal to a blow-up argument which is often used in a study of nonlinear elliptic and parabolic equations. Let us give a heuristic idea of our argument. Our goal is to establish a bound for

$$N(v,q)(x,t) = \left|v(x,t)\right| + t^{\frac{1}{2}}\left|\nabla v(x,t)\right| + t\left|\nabla^2 v(x,t)\right| + t\left|\partial_t v(x,t)\right| + t\left|\nabla q(x,t)\right| \tag{3.1.5}$$

of the form

$$\sup_{0<t<T_0} \left\|N(v,q)\right\|_\infty(t) \leq C\|v_0\|_\infty \tag{3.1.6}$$

for some $T_0 > 0$ and C depending only on the domain Ω, where $\|v_0\|_\infty = \|v_0\|_{L^\infty(\Omega)}$ denotes the sup-norm of $|v_0|$ in Ω.

We argue by contradiction. Suppose that (3.1.6) were false for any choice of T_0 and C. Then, there would exist a sequence $\{(v_m, q_m)\}_{m=1}^\infty$ of solutions of (3.1.1)–(3.1.4) with $v_0 = v_{0m}$ and a sequence $\tau_m \downarrow 0$ such that$\|N(v_m, q_m)\|_\infty(\tau_m) > m\|v_{0m}\|_\infty$. There is $t_m \in (0, \tau_m)$ such that

$$\left\|N(v_m,q_m)\right\|_\infty(t_m) \geq \frac{1}{2}M_m, \quad M_m = \sup_{0<t<\tau_m} \left\|N(v_m,q_m)\right\|_\infty(t).$$

We normalize v_m, q_m by dividing M_m to observe that

$$\sup_{0<t<t_m} \left\|N(\tilde{v}_m,\tilde{q}_m)\right\|_\infty(t) \leq 1, \tag{3.1.7}$$

$$\left\|N(\tilde{v}_m,\tilde{q}_m)\right\|_\infty(t_m) \geq 1/2, \tag{3.1.8}$$

$$\|\tilde{v}_{0m}\|_\infty < 1/m, \tag{3.1.9}$$

with $\tilde{v}_m = v_m/M_m$, $\tilde{q}_m = q_m/M_m$. We rescale $(\tilde{v}_m, \tilde{q}_m)$ around a point $x_m \in \Omega$ satisfying

$$N(\tilde{v}_m,\tilde{q}_m)(x_m,t_m) \geq 1/4, \tag{3.1.10}$$

to get the blow-up sequence,

$$u_m(x,t) = \tilde{v}_m(x_m + t_m^{\frac{1}{2}}x, t_m t), \;\; p_m(x,t) = t_m^{\frac{1}{2}}\tilde{q}_m(x_m + t_m^{\frac{1}{2}}x, t_m t).$$

(Such an x_m exists because of (3.1.8)). Because of the scaling invariance of the equations (3.1.1) and (3.1.2), the rescaled function (u_m, p_m) solves (3.1.1) and (3.1.2) in the rescaled domain $\Omega_m \times (0,1]$. Note that the time interval is normalized to a unit interval and Ω_m tends to either a half space or the whole space $\mathbf{R}^n$ as $m \to \infty$.

The basic strategy is to prove that the blow-up sequence $\{(u_m, p_m)\}_{m=1}^{\infty}$ (subsequently) converges to a solution (u, p) of (3.1.1)–(3.1.4) with zero initial data. If the convergence is strong enough, (3.1.10) implies that $N(u, p)(0, 1) \geq 1/4$. If the limit (u, p) is unique, it is natural to expect $u \equiv 0$, $\nabla p \equiv 0$. This evidently yields a contradiction to $N(u, p)(0, 1) \geq 1/4$. The first part corresponds to "compactness" of a blow-up sequence and the second part corresponds to "uniqueness" of a blow-up limit. When the problem is the heat equation, this strategy is easy to realize. However, for the Stokes equations it turns out that this procedure is highly nontrivial because of the presence of the pressure.

In order to solve both compactness of a blow-up sequence and uniqueness of its limit, we appeal to the harmonic-pressure gradient estimate for solutions to the Stokes equations (3.1.1)–(3.1.4),

$$\sup_{x\in\Omega} d_\Omega(x)\left|\nabla q(x,t)\right| \leq C_\Omega \left\|W(v)\right\|_{L^\infty(\partial\Omega)}(t) \tag{3.1.11}$$

for $W(v) = -(\nabla v - \nabla^T v)n_\Omega$. Actually, the estimate (3.1.11) is a special case of an estimate for solutions of the homogeneous Neumann problem,

$$\Delta P = 0 \text{ in } \Omega, \quad \frac{\partial P}{\partial n_\Omega} = \operatorname{div}_{\partial\Omega} W \text{ on } \partial\Omega. \tag{3.1.12}$$

In fact, the pressure q is harmonic in Ω and $\partial q/\partial n_\Omega = \Delta v \cdot n_\Omega$ on $\partial\Omega$. The divergence-free condition implies $\Delta v \cdot n_\Omega = \operatorname{div}_{\partial\Omega} W(v)$. (We give a proof in Section 2). We call Ω *strictly admissible* if the a priori estimate

$$\sup_{x\in\Omega} d_\Omega(x)\left|\nabla P(x)\right| \leq C_\Omega \left\|W\right\|_{L^\infty(\partial\Omega)} \tag{3.1.13}$$

holds for all solutions of the Neumann problem (3.1.12). As we showed in Chapter 2, the estimate (3.1.13) holds not only on bounded domains but on exterior domains and perturbed half spaces.

We now study compactness of the blow-up sequence $\{(u_m, p_m)\}_{m=1}^{\infty}$. The proof is divided into two cases depending on whether the limit of Ω_m is a half space or the whole space. Let us consider the case when the limit is the whole space. We would like to prove that $N(u_m, p_m)$ converges to $N(u, p)$ near $(0, 1) \in \mathbf{R}^n \times (0, 1]$ uniformly by taking a subsequence. For this purpose, it is enough to prove that the local space-time Hölder norm in $\mathbf{R}^n \times (0, 1]$ near $(0, 1)$ for u_m, ∇u_m, $\nabla^2 u_m$, ∇p_m is bounded as $m \to \infty$. We are tempted to derive such as interior regularity estimate from (3.1.7) by localizing the problem. This idea works for the heat equation, but for the Stokes equations, it does not work (Remark 3.3.3 (i)). In fact, if we consider a solution of (3.1.1)–(3.1.2) of the form $v = g(t), q = -g'(t) \cdot x$ for $g \in C^1[0, 1]$, we do not expect the (local) Hölder continuity in time for ∇q and v_t although $N(v, q)$ is bounded in $\mathbf{R}^n \times (0, 1]$. We invoke the strictly admissibility of Ω and derive a uniform time Hölder estimate for $d_{\Omega_m}(x)\nabla p_m$ in $\Omega_m \times (\delta, 1] (\delta > 0)$ from (3.1.12).

Then, one can use usual parabolic interior regularity theory [41] to derive necessary interior regularity estimate. Note that the constant in (3.1.12) is independent of the rescaling procedure so our Hölder estimate is uniform in m.

The case when Ω_m tends to a half space is more involved. We still use the admissibility of Ω to derive necessary Hölder estimates for p_m. Then, instead of using conventional parabolic local Hölder estimate, we are forced to use the Schauder estimates for the Stokes equations and Helmholtz decomposition for Hölder spaces developed by V. A. Solonnikov [61] since the boundary value problem for the Stokes equations cannot be reduced to usual parabolic theory.

We also invoke the strictly admissibility of Ω to derive the uniqueness of the blow-up limit (u, p). If Ω_m tends to the whole space, by (3.1.11), we observe that ∇p_m tends to zero locally uniformly in $\mathbf{R}^n \times (0, 1]$. This reduces the problem to the uniqueness result for the heat equation. If Ω_m tends to a half space, we use a uniqueness result for spatially non-decaying velocity in the half space $\mathbf{R}^n_+ = \{(x', x_n)|\ x_n > 0,\ x' \in \mathbf{R}^{n-1}\}$, which is essentially due to V. A. Solonnikov [59] as we proved in Chapter 2. Note that to assert the uniqueness of solutions (u, p) of the Stokes equations (3.1.1)-(3.1.4) with zero initial data and a bound for $\|N(u, p)\|_\infty(t)$, we need to assume some decay for ∇p, otherwise there is a counterexample (Remark 3.4.2). In fact, it suffices to assume that $\nabla p \to 0$ for $x_n \to \infty$. In our setting since (3.1.12) is a scaling invariant, this estimate is inherited to (u_m, p_m). Since $x_n = d_{\mathbf{R}^n_+}(x)$, we are able to conclude that $t^{\frac{1}{2}} x_n |\nabla p(x, t)|$ is bounded in $\mathbf{R}^n_+ \times (0, 1)$, which is enough to apply this available uniqueness result. Note that in the above uniqueness result, we do not assume any spatial decay condition for velocity fields at infinity.

As we have seen above, a blow-up argument yields a key estimate (3.1.6) for solutions of the Stokes equations (3.1.1)–(3.1.4) provided that $\|N(v, q)\|_\infty(t)$ (see (3.1.5)) is finite for $t > 0$ as far as Ω is strictly admissible not necessarily bounded. We prove the a priori estimate (3.1.16) for all solutions (v, q) such that

$$\sup_{0<t<T} \left\|N(v, q)\right\|_\infty(t) < \infty,$$

and (3.1.11) holds. We call such a solution L^∞*-solution* (see Definition 3.2.1). A question is whether or not an L^∞-solution actually exists. It is by now well known [24] that if a uniformly C^3-domain admits the Helmholtz decomposition in L^r, there exists an L^r-solution and the Stokes semigroup $S(t)$ is analytic in L^r_σ, the closure of $C^\infty_{c,\sigma}(\Omega)$ in L^r. However, in general, it is also known that the Helmholtz decomposition in L^r space may not hold (see [11], [48]), unless $r = 2$. Fortunately, R. Farwig, H. Kozono, and H. Sohr [16], [17], [18] established an $\tilde{L}^r$-theory with $\tilde{L}^r_\sigma = L^r_\sigma \cap L^2_\sigma$ for $r \geq 2$ for any uniformly C^2-domain for (3.1.1)–(3.1.4). In particular, they showed that the Stokes semigroup is analytic on $\tilde{L}^r_\sigma$ space. It turns out that their solution (called an $\tilde{L}^r$-solution) is an L^∞-solution provided that $r > n$ and v_0 is sufficiently regular, e.g., $v_0 \in C^\infty_{c,\sigma}(\Omega)$. Here is our main result.

Theorem 3.1.2 (A priori L^∞-estimates)**.** *Let Ω be a strictly admissible, uniformly C^3-domain in $\mathbf{R}^n$. Then, there exists positive constants C and T_0 depending only on Ω such* *that (3.1.6), i.e.,*

$$\sup_{0<t<T_0} \left\| N(v,q) \right\|_\infty (t) \leq C \|v_0\|_\infty \tag{3.1.15}$$

holds for all L^∞-solutions $(v, \nabla q)$ of (3.1.1)–(3,1.4) and $v_0 \in C^\infty_{c,\sigma}(\Omega)$.

By a density argument with (3.1.15), we are able to construct a solution semigroup $S(t)$ for (3.1.1)–(3,1.4) on $C_{0,\sigma}(\Omega)$. In particular, the estimate

$$\sup_{0<t<T_0} t\|v_t\|_\infty(t) \leq C\|v_0\|_\infty$$

from (3.1.15) shows that this semigroup is analytic on $C_{0,\sigma}(\Omega)$. Let us give a precise form of our result which includes Theorem 3.1.1 as a particular example.

Theorem 3.1.3 (Analyticity for a general domain)**.** *Let Ω be a strictly admissible, uniformly C^3-domain in $\mathbf{R}^n$. Then, the Stokes semigroup $S(t)$ is uniquely extendable to a C_0-analytic semigroup on $C_{0,\sigma}(\Omega)$. Moreover, the estimate (3.1.15) holds with some $C > 0$ and $T_0 > 0$ for $v = S(t)v_0$, $v_0 \in C_{0,\sigma}(\Omega)$ with a suitable choice of pressure q.*

Although there are several results on analyticity of $S(t)$ on L^r_σ for various domains such as a half space, a bounded domain [28], [56], an exterior domain [12], [36], an aperture domain [20], a layer domain [3], a perturbed half space [19], the result corresponding to Theorem 3.1.3 is available only for a half space [14], [44], [59] (and the whole space, where the Stokes semigroup agrees with the heat semigroup).

This chapter is organized as follows. In Section 2, we define L^∞-solutions and prove the harmonic-pressure gradient estimate (3.1.11) for all L^∞-solutions. In Section 3, we derive local Hölder estimates both interior and up to boundary which are key to derive necessary compactness for a blow-up sequence. In Section 4, we prove the a priori estimates (Theorem 3.1.2) by a blow-up argument. As an application we prove Theorem 3.1.3 (and Theorem 3.1.1 as a particular example).

3.2 L^∞-solutions and the harmonic-pressure gradient estimate

In this section, we define L^∞-solutions and prove that the harmonic-pressure gradient estimate (3.1.11) for all L^∞-solutions in a strictly admissible domain. As discussed later in Section 4, if initial data is sufficiently regular, an $\tilde{L}^r$-solution agrees with an L^∞-solution.

In order to define an L^∞-solution, we recall the space $L^\infty_\sigma(\Omega)$ defined by

$$L^\infty_\sigma(\Omega) = \left\{ f \in L^\infty(\Omega) \;\middle|\; \int_\Omega f \cdot \nabla\varphi \mathrm{d}x = 0 \quad \text{for all} \quad \varphi \in \hat{W}^{1,1}(\Omega) \right\},$$

where $\hat{W}^{1,1}(\Omega)$ denotes the homogeneous Sobolev space $\hat{W}^{1,1}(\Omega) = \{\, \varphi \in L^1_{\mathrm{loc}}(\Omega) \mid \nabla\varphi \in L^1(\Omega) \,\}$. Note that $L^\infty_\sigma(\Omega)$ is larger than $C_{0,\sigma}(\Omega)$ and includes non-decaying functions. Although the existence of L^∞-solutions is unknown in general, L^∞-solutions uniquely exist on an exterior domain and a perturbed half space as discussed later in Chapter 4.

Definition 3.2.1 (L^∞-solution)**.** Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, with $\partial\Omega \neq \emptyset$. Let $(v, \nabla q) \in C^{2,1}(\bar{\Omega} \times (0,T]) \times C(\bar{\Omega} \times (0,T])$ satisfy (3.1.1)–(3.1.3) and (3.1.4) for $v_0 \in L^\infty_\sigma(\Omega)$ in the sense that $v(\cdot, t) \to v_0$ weakly-$*$ on $L^\infty(\Omega)$ as $t \downarrow 0$. We call $(v, \nabla q)$ L^∞-*solution* if (3.1.5) and

$$t^{1/2} d_\Omega(x) |\nabla q(x,t)| \tag{3.2.1}$$

are bounded in $\Omega \times (0,T)$.

The bound for (3.2.1) implies that the pressure $q(\cdot, t)$ is a weak solution of (3.1.12). Thus, the harmonic-pressure gradient estimate (3.1.11) holds for all L^∞-solutions provided that Ω is strictly admissible.

Lemma 3.2.2 (Harmonic-pressure gradient estimate)**.** *Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, with C^2-boundary. Let $(v, \nabla q)$ be an L^∞-solution for (3.1.1)–(3.1.4) in $\Omega \times (0,T)$. Then, $q(\cdot,t)$ is a weak solution of (3.1.12) for $W(v) = -(\nabla v - \nabla^T v) n_\Omega$. Assume that Ω is strictly admissible. Then, the estimate*

$$|\nabla q|_{\infty,d}(t) \leq C_\Omega \|W(v)\|_{\infty,\partial\Omega}(t) \tag{3.2.2}$$

holds for each $t \in (0,T)$. The constant C_Ω is invariant of translation and dilation of Ω.

We shall show that the pressure q solves the Neumann problem (3.1.12) for $W(v) = -(\nabla v - \nabla^T v) n_\Omega$. We prepare the following:

Proposition 3.2.3. *Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$, with C^2-boundary. Set*

$$W = -(g - g^T) n_\Omega \tag{3.2.3}$$

for a tensor-valued function $g = (g_{ij})_{1 \leq i,j \leq n} \in C^1(\bar{\Omega})$. Then $W \cdot n_\Omega = 0$ on $\partial\Omega$ and

$$\mathrm{div}_{\partial\Omega} W = \sum_{i,j} (n^i_\Omega \partial_j - n^j_\Omega \partial_i) g_{ij}. \tag{3.2.4}$$

Proof. Since $\partial\Omega$ is C^2, $-\nabla d_\Omega$ is a C^1-function near $\partial\Omega$ and agrees with n_Ω on $\partial\Omega$. We extend n_Ω by $-\nabla d_\Omega$. We may assume W is a C^1-function near $\partial\Omega$. By multiplying n_Ω to W, it follows that

$$W \cdot n_\Omega = -\sum_{i,j} g_{ij} n^i_\Omega n^j_\Omega + \sum_{i,j} g_{ij} n^i_\Omega n^j_\Omega = 0.$$

We shall show (3.2.4). Since $\nabla_{\partial\Omega} W^i = \nabla W - n_\Omega(\nabla W \cdot n_\Omega)$, it follows that

$$\mathrm{div}_{\partial\Omega} W = \mathrm{div} W - \sum_{i,j} \partial_j W^i n^i_\Omega n^j_\Omega.$$

We show that

$$\mathrm{div} W = \sum_{i,j} (n_\Omega^i \partial_j - n_\Omega^j \partial_i) g_{ij}, \tag{3.2.5}$$

$$\sum_{i,j} \partial_j W^i n_\Omega^i n_\Omega^j = 0. \tag{3.2.6}$$

The equalities (3.2.5) and (3.2.6) imply (3.2.4). We first show (3.2.5). Take the divergence to W in (3.2.3) to get

$$\begin{aligned}
\mathrm{div} W &= -\sum_{i,j} (\partial_i g_{ij} - \partial_i g_{ji}) n_\Omega^j - \sum_{i,j} (g_{ij} - g_{ji}) \partial_i n_\Omega^j \\
&= -\sum_{i,j} (n_\Omega^i \partial_j - n_\Omega^j \partial_i) g_{ij} - \sum_{i,j} (g_{ij} - g_{ji}) \partial_i n_\Omega^j.
\end{aligned}$$

The second term vanishes since $\partial_j n_\Omega^i = \partial_i n_\Omega^j$ by $n_\Omega = -\nabla d_\Omega$. In fact, it follows that

$$\begin{aligned}
\sum_{i,j} (g_{ij} - g_{ji}) \partial_i n_\Omega^j &= \sum_{i,j} (g_{ij} - g_{ji}) \partial_j n_\Omega^i \\
&= -\sum_{i,j} (g_{ij} - g_{ji}) \partial_i n_\Omega^j.
\end{aligned}$$

Thus, (3.2.5) holds. It remains to show (3.2.6). Since

$$\partial_i W^j = -\sum_k (\partial_i g_{jk} - \partial_i g_{kj}) n_\Omega^k - \sum_k (g_{jk} - g_{kj}) \partial_i n_\Omega^k$$

for $i, j \in \{1, \cdots, n\}$. By multiplying $n_\Omega^i n_\Omega^j$ to the both sides and summing up with respect to i and j, we have

$$\begin{aligned}
\sum_{i,j} \partial_i W^j n_\Omega^j n_\Omega^i &= -\sum_{i,j,k} (\partial_i g_{jk} - \partial_i g_{kj}) n_\Omega^k n_\Omega^i n_\Omega^j - \sum_{i,j,k} (g_{jk} - g_{kj}) \partial_i n_\Omega^k n_\Omega^i n_\Omega^j \\
&= -\sum_{i,j,k} (\partial_i g_{jk} - \partial_i g_{kj}) n_\Omega^k n_\Omega^i n_\Omega^j.
\end{aligned}$$

Here, we use $\sum_i \partial_k n_\Omega^i n_\Omega^i = \partial_k |n|/2 = 0$ since $|n| = 1$ near $\partial\Omega$. By replacing k and j, we have

$$\begin{aligned}
\sum_{i,j,k} (\partial_i g_{jk} - \partial_i g_{kj}) n_\Omega^k n_\Omega^i n_\Omega^j &= \sum_{i,j,k} (\partial_i g_{kj} - \partial_i g_{jk}) n_\Omega^j n_\Omega^i n_\Omega^k \\
&= -\sum_{i,j,k} (\partial_i g_{jk} - \partial_i g_{kj}) n_\Omega^k n_\Omega^i n_\Omega^j.
\end{aligned}$$

Thus, (3.2.5) holds. The proof is now complete. □

Proof of Lemma 3.2.2. By taking the divergence to (3.1.1) and multiplying n_Ω to (3.1.1), we observe that the pressure q solves the Neumann problem,

$$\Delta q = 0 \quad \text{in } \Omega, \quad \partial q/\partial n_\Omega = \Delta v \cdot n_\Omega \quad \text{on } \partial\Omega,$$

where $v \cdot n_\Omega = 0$ on $\partial\Omega$ is used. Since div $v = 0$ on Ω, it follows that

$$\begin{aligned}
\Delta v \cdot n_\Omega &= \sum_{j\neq i} \partial_j^2 v^i n_\Omega^i + \sum_i \partial_i^2 v^i n_\Omega^i \\
&= \sum_{j\neq i} \partial_j^2 v^i n_\Omega^i - \sum_{j\neq i} \partial_j \partial_i v^j n_\Omega^i \\
&= \sum_{i,j} (n_\Omega^j \partial_i - n_\Omega^i \partial_j) \partial_i v^j.
\end{aligned}$$

By Proposition 3.2.3, the right-hand side agrees with $\mathrm{div}_{\partial\Omega} W$ for $W(v) = -(\nabla v - \nabla^T v) n_\Omega$. Since $\nabla q(\cdot, t) \in L^\infty_d(\Omega)$ by the definition of an L^∞-solution, $q(\cdot, t)$ is a weak solution of the Neumann problem (3.1.12) for $W(v)$. Since Ω is strictly admissible, (3.2.2) holds with the dilation invariant constant C_Ω. The proof is now complete. □

Remark 3.2.4. (i) The estimate (3.2.2) also holds for the Robin-type boundary condition, i.e., $v \cdot n_\Omega = 0$ on $\partial\Omega$ and

$$\alpha v_{\tan} + (D(v) n_\Omega)_{\tan} = h \quad \text{on } \partial\Omega$$

for a tangential vector field h and $\alpha \geq 0$, where $D(v) = (\nabla v + \nabla^T v)/2$ denotes the deformation tensor and $f_{\tan}$ denotes the tangential component of the vector field f (see, e.g., [53], [51] for the Robin-type boundary conditions).
(ii) Actually, the statement of Lemma 3.2.2 holds with C^1-boundary since by integration by parts, we are able to prove that pressure of an L^∞-solution is a weak solution for the Neumann problem (3.1.12) (without applying Proposition 3.2.3).

3.3 Local Hölder estimates for the Stokes equations

The goal of this section is to establish local Hölder estimates for second spatial derivatives and a time derivative of velocity solving the Stokes equations both interior and up to boundary. This procedure is a key to derive necessary compactness for blow-up sequences. Unlike the heat equation the result is not completely local even interior case since we need a uniform Hölder estimates in time for pressure gradients. For this purpose, we invoke the strictly admissibility of domains.

3.3.1 Interior Hölder estimates for pressure gradients

We use conventional notation [41] for Hölder (semi)norms for space-time functions. Let $f = f(x,t)$ be a real-valued or an $\mathbf{R}^n$-valued function defined in $Q = \Omega \times (0,T]$, where Ω is a domain in $\mathbf{R}^n$. For $\mu \in (0,1)$ we set several Hölder semi-norms

$$[f]^{(\mu)}_{(0,T]}(x) = \sup\left\{ \left| f(x,t) - f(x,s) \right| / |t-s|^\mu \;\middle|\; t,s \in (0,T], t \neq s \right\},$$
$$[f]^{(\mu)}_{\Omega}(t) = \sup\left\{ \left| f(x,t) - f(y,t) \right| / |x-y|^\mu \;\middle|\; x,y \in \Omega, x \neq y \right\},$$

and

$$[f]^{(\mu)}_{t,Q} = \sup_{x\in\Omega} [f]^{(\mu)}_{(0,T]}(x),\ [f]^{(\mu)}_{x,Q} = \sup_{t} [f]^{\mu}_{\Omega}(t).$$

In the parabolic scale, for $\gamma \in (0,1)$, we set

$$[f]^{(\gamma,\gamma/2)}_{Q} = [f]^{(\gamma/2)}_{t,Q} + [f]^{(\gamma)}_{x,Q}\,.$$

For later convenience, we also define the case $\gamma = 1$ so that

$$[f]^{(1,1/2)}_{Q} = \|\nabla f\|_{L^\infty(Q)} + [f]^{(1/2)}_{t,Q}\,.$$

If $l = [l] + \gamma$ where $[l]$ is a nonnegative integer and $\gamma \in (0,1)$, we set

$$[f]^{(l,l/2)}_{Q} = \sum_{|\alpha|+2\beta=[l]} [\partial^\alpha_x\, \partial^\beta_t f]^{(\gamma,\gamma/2)}_{Q}$$

and the parabolic Hölder norm

$$|f|^{(l,l/2)}_{Q} = \sum_{|\alpha|+2\beta\leq[l]} \|\partial^\alpha_x\, \partial^\beta_t\, f\|_{L^\infty(Q)} + [f]^{(l,l/2)}_{Q}\,.$$

When f is time-independent, we simply write $[f]^{(\mu)}_{x,Q}$ by $[f]^{(\mu)}_{\Omega}$.

Lemma 3.3.1. *Let Ω be a strictly admissible, uniformly C^2-domain in $\mathbf{R}^n$. Then there exists a constant $M(\Omega) > 0$ such that*

$$[d_\Omega(x)\nabla q]^{(1/2)}_{t,Q_\delta} \leq \frac{M}{\delta} \sup\left\{ (\|v_t\|_\infty(t) + \|\nabla^2 v\|_\infty(t))t \;\middle|\; \delta \leq t \leq T \right\}$$

holds for all L^∞-solutions $(v, \nabla q)$ of (3.1.1)–(3.1.4) and all $\delta \in (0,T)$, where $Q_\delta = \Omega \times (\delta, T)$. The constant M can be taken uniform with respect to translation and dilation, i.e., $M(\lambda\Omega + x_0) = M(\Omega)$ for all $\lambda > 0$ and $x_0 \in \Omega$.

Proof. By an interpolation inequality (e.g. [65], [40, 3.2]), there is a dilation invariant constant C such that for any $\varepsilon > 0$ the estimate

$$\|\nabla v\|_\infty(t) \leq \varepsilon\|\nabla^2 v\|_\infty(t) + (C/\varepsilon)\|v\|_\infty(t)$$

holds. Since our solution is an L^∞-solution, $q(\cdot,t) - q(\cdot,s)$ solves the Neumann problem (3.1.12) for $W(v(\cdot,t) - v(\cdot,s))$. Since Ω is strictly admissible, we have

$$\begin{aligned} d_\Omega(x)\left|\nabla q(x,t) - \nabla q(x,s)\right| &\le C(\Omega)\left\|\nabla(v(\cdot,t) - v(\cdot,s))\right\|_\infty \\ &\le C(\Omega)[\varepsilon \max\left(\|\nabla^2 v\|_\infty(t),\ \|\nabla^2 v\|_\infty(s)\right) + (C/\varepsilon)\left\|v(\cdot,t) - v(\cdot,s)\right\|_\infty]. \end{aligned}$$

Since

$$\begin{aligned} \left\|v(\cdot,t) - v(\cdot,s)\right\|_\infty &\le |t-s|\ \sup\{\|v_t\|_\infty(\tau) \mid \tau \text{ is between } t \text{ and } s\}, \\ &\le |t-s|\ \frac{1}{\delta}\ \sup\{\tau\|v_t\|_\infty(\tau) \mid \delta \le \tau \le T\} \end{aligned}$$

for $t, s \ge \delta$, the desired inequality follows by taking $\varepsilon = |t-s|^{1/2}$. Since C_Ω is also dilation and translation invariant, so is $M(\Omega)$. □

Proposition 3.3.2 (Interior Hölder estimates)**.** *Let Ω be a strictly admissible, uniformly C^2-domain in $\mathbf{R}^n$. Take $\gamma \in (0,1)$, $\delta > 0$, $T > 0$, $R > 0$. Then, there exists a constant $C = C(M(\Omega), \delta, R, d, \gamma, T)$ such that the estimate*

$$[\nabla^2 v]_{Q'}^{(\gamma,\gamma/2)} + [v_t]_{Q'}^{(\gamma,\gamma/2)} + [\nabla q]_{Q'}^{(\gamma,\gamma/2)} \le CN_T \tag{4.3.1}$$

holds for all L^∞-solutions $(v, \nabla q)$ of (3.1.1)–(3.1.4) provided that $B_{x_0}(R) \subset \Omega$ and $x_0 \in \Omega$, where $Q' = B_{x_0}(R) \times (\delta, T]$ and d denotes the distance of $B_x(R)$ and $\partial\Omega$. Here,

$$N_T = \sup_{0<t<T}\left\|N(v,q)\right\|_\infty(t) < \infty, \tag{3.3.2}$$

and $M(\Omega)$ is the constant in Lemma 3.3.1.

Proof. Since ∇q is harmonic in Ω, the Cauchy type estimate implies

$$\sup_{x \in B_{x_0}(R+d/2)}\left|\nabla^2 q(x,t)\right| \le \frac{C_0}{d}\|\nabla q\|_{L^\infty(\Omega)}(t),\ B_{x_0}(R+d/2) \subset \Omega,$$

where C_0 depends only on n. This together with Lemma 3.3.1 implies

$$[\nabla q]_{Q''}^{(1,1/2)} \le \Big(\frac{C_0 R'}{d} + M\Big)\frac{1}{\delta}N_T, \quad R' = R + d/2$$

for any $x_0 \in \Omega$, $R > 0$, $\delta > 0$, where $Q'' = B_{x_0}(R+d/2) \times (\delta/2, T]$. By the standard local Hölder estimate for the heat equation, i.e.,

$$v_t - \Delta v = -\nabla q \quad \text{in} \quad Q'',$$

this pressure gradient estimate implies estimates for $\nabla^2 v$, v_t for Q' [41, Chapter IV, Theorem 10.1]. □

Remarks 3.3.3. (i) We are tempted to claim that if (v, q) solves the Stokes system (3.1.1)–(3.1.2) without boundary and initial condition, then the desired interior Hölder estimate would be valid. Such a type estimate is in fact true for the heat equation [41, Chapter IV, Theorem 10.1]. However, for the Stokes equations, this is no longer true. In fact, if we take $v(x,t) = g(t)$ and $p(x,t) = -g'(t) \cdot x$ with $g \in C^1[0,\infty)$, this is always a solution of (3.1.1)–(3.1.2) satisfying $N_{T_1} < \infty$ for any $T_1 > 0$. However, evidently, v_t may not be Hölder continuous in time unless ∇p is Hölder continuous in time. This is why we use a global setting with the strictly admissibility of a domain.
(ii) In the constant C the dependence of Ω is through $M(\Omega)$ so it is invariant under a dilation provided that d and R are taken independent of a dilation.
(iii) The local Hölder estimate (4.3.1) says that L^∞-solutions are Hölder continuous both interior an up to the boundary of Ω and $t > 0$, i.e., $v \in C^{2+\gamma,1+\gamma/2}(\overline{\Omega} \times [\delta, T])$, $\nabla q \in C^{\gamma,\gamma/2}(\overline{\Omega} \times [\delta, T])$ for each $\delta > 0$; see Theorem 3.3.4 below for the estimate (4.3.1) up to the boundary.

3.3.2 Local Hölder estimates up to the boundary

The regularity up to boundary is more involved. We begin with the statement and give a proof in subsequent sections.

Theorem 3.3.4 (Estimates near the boundary)**.** *Let Ω be a strictly admissible, uniformly C^3-domain of type (α,β,K) in $\mathbf{R}^n$. Then, there exists $R_0 = R_0(\alpha,\beta,K) > 0$ such that for any $\gamma \in (0,1)$, $\delta \in (0,T)$ and $R \leq R_0/2$ there exists a constant*

$$C = C(M(\Omega), \delta, \gamma, T, R, \alpha, \beta, K)$$

such that (3.3.1) is valid for all L^∞-solution $(v, \nabla q)$ of (3.1.1)-(3.1.4) with

$$Q' = Q'_{x_0,R,\delta} = \Omega_{x_0,R} \times (\delta, T],\ \Omega_{x_0,R} = B_{x_0}(R) \cap \Omega$$

provided that $x_0 \in \partial\Omega$.

The proof is more involved. We first localize the Stokes equations near the boundary by using cutoff technique and the Bogovskiĭ operator [22, III.3] to recover divergence free property. Then, we apply the global Schauder estimate for the Stokes equations in a localized domain. As in the interior case, we use the strictly admissibility of a domain to obtain the Hölder estimate for pressure in time.

We begin with Hölder estimates for q in time since we are not able to control the Hölder norm of ∇q up to the boundary.

Lemma 3.3.5. *Assume the same hypotheses of Lemma 3.3.1. Then, there exists $R_0 = R_0(\alpha,\beta,K) > 0$ such that for $\nu \in (0,1)$ and $R \in (0, R_0]$, there exists a constant $C_0 = C_0(M(\Omega), \nu, \alpha, R, \beta, K)$ such that*

$$[q]^{(\nu,\nu/2)}_{Q'} \leq C_0 N_T/\delta \tag{3.3.3}$$

is valid for all L^∞-solutions $(v, \nabla q)$ of (3.1.1)–(3.1.4) and $Q' = Q'_{x_0,R,\delta}$ for $x_0 \in \partial\Omega$.

In order to show (3.4.3), we prepare basic facts for a distance function.

Proposition 3.3.6. *Let Ω be a uniformly C^2-domain of type (α,β,K).*
(i) There is a constant $R_* = R_*(\alpha,\beta,K) > 0$ such that $x \in \Gamma_{\Omega,R_*} = \{x \in \Omega \mid d_\Omega(x) < R_*\}$ has the unique projection $x_p \in \partial\Omega$ (i.e., $|x - x_p| = d_\Omega(x)$) and x is represented by $x = x_p - dn_\Omega(x_p)$ with $d = d_\Omega(x)$. The mapping $x \mapsto (x_p, d)$ is C^1 in Γ_{Ω,R_*}.
(ii) There is a positive constant $R_1 = R_1(\alpha,\beta,K) \le R_*$ such that $\Omega_{x_0,R_1} \subset U_{\alpha,\beta,h}(x_0)$ and the projection x_p of $x \in \Omega_{x_0,R_1}$ is on $x_0 + \mathrm{graph}\ h$.
(iii) For each $R \in (0,R_1)$ and $\nu \in [0,1)$, there is a constant $C = C(\alpha,\beta,K,R,\nu)$ such that

$$\left|\tilde q(x) - \tilde q(y)\right| \le C\|d_\Omega^\nu \nabla\tilde q\|_{L^\infty(\Omega)}\left\{\left|d_\Omega(y)^{1-\nu} - d_\Omega(x)^{1-\nu}\right| + |x_p - y_p| \,/\, \max(d_\Omega(x)^\nu, d_\Omega(y)^\nu)\right\} \quad \text{for } x, y \in \Omega_{x_0,R},$$

for all $\tilde q \in C^1(\Omega)$ and $x_0 \in \partial\Omega$.

Proof. (i) This is nontrivial but well known. See, e.g., [26] or [39, 4.4].
(ii) This is easy by taking R smaller. The smallness depends on a bound for the second fundamental form of $\partial\Omega$.
(iii) For $x \in \Omega_{x_0,R}$ ($R \le R_1$), we consider its normal coordinate (x_p, d). Since $\Omega_{x_0,R} \subset U_{\alpha,\beta,h}(x_0)$, there is unique $x_p' \in \mathbf{R}^{n-1}$ such that $x_p = (x_p', h(x_p'))$. Moreover, we are able to use (x_p', d) as a coordinate system. For $x, y \in \Omega_{x_0,R}$ with $x = (x_p', d_\Omega(x))$, $y = (y_p', d_\Omega(y))$ with $d_\Omega(y) > d_\Omega(x)$, we estimate

$$\left|\tilde q(x) - \tilde q(y)\right| \le \left|\tilde q(x) - \tilde q(z)\right| + \left|\tilde q(z) - \tilde q(y)\right|$$

with $z = (x_p', d_\Omega(y))$. Thus we connect x and z by a straight line which parallels to $n_\Omega(x_p)$ and observe that

$$\begin{aligned}\left|\tilde q(x) - \tilde q(z)\right| &\le |z - x| \int_0^1 \frac{1}{d_\Omega^\nu(x_\tau)}\left|d_\Omega^\nu \nabla\tilde q(x_\tau)\right| \mathrm{d}\tau, \ x_\tau = x(1-\tau) + \tau z \quad (0 \le \tau \le 1)\\ &\le \int_{d_\Omega(x)}^{d_\Omega(y)} \frac{1}{s^\nu} \mathrm{d}s \|d_\Omega^\nu \nabla\tilde q\|_{L^\infty(\Omega)}\\ &\le (d_\Omega(z)^{1-\nu} - d_\Omega(x)^{1-\nu})\|d_\Omega^\nu \nabla\tilde q\|_{L^\infty(\Omega)}(1-\nu)^{-1}.\end{aligned}$$

It remains to estimate $|\tilde q(z) - \tilde q(y)|$. We connect z and y by a curve $C_{z,y}$ of the form,

$$C_{z,y} = \left\{x(\tau) \,\middle|\, 0 \le \tau \le 1,\ x_p'(\tau) = x_p'(1-\tau) + \tau y_p',\ d_\Omega(x(\tau)) = d_\Omega(y)\right\},$$

so that the projection in $\mathbf{R}^{n-1}$ is a straight line connecting x_p' and y_p'. We now estimate

$$\begin{aligned}\left|\tilde q(z) - \tilde q(y)\right| &\le \int_{C_{z,y}} \frac{1}{d_\Omega(y)^\nu}\, d_\Omega^\nu(y)|\nabla\tilde q|(x)\, \mathrm{d}\mathcal{H}^1(x)\\ &= \frac{1}{d_\Omega(y)^\nu}\mathcal{H}^1(C_{z,y})\|d_\Omega^\nu \nabla\tilde q\|_{L^\infty(\Omega)}.\end{aligned}$$

Since $\mathcal{H}^1(C_{z,y}) \le C|x_p - y_p|$, the proof is now complete. □

Proof of Lemma 3.3.5. We take $R_1 > 0$ as in Proposition 3.3.6. For $x_0 \in \partial\Omega$, we take $\tilde{x}_0 = x_0 - \frac{R_1}{2} n_\Omega(x_0)$. We may assume that $q(\tilde{x}_0, t) = 0$ for all $t \in (0, T)$. Since

$$[d_\Omega(x)^\nu \nabla q]^{(\nu/2)}_{t,Q_\delta} \leq \left([d_\Omega(x)\nabla q]^{(1/2)}_{t,Q_\delta}\right)^\nu (2\|\nabla q\|_{L^\infty(Q_\delta)})^{1-\nu},$$

Lemma 3.3.1 implies that

$$\left\|d_\Omega(x)^\nu \nabla \tilde{q}(x, \cdot)\right\|_{L^\infty(\Omega)}(t, s) \leq \frac{M^\nu N_T 2^{1-\nu}}{\delta} |t - s|^{\nu/2} \quad \text{for} \quad t, s \in (\delta, T],$$

with $\tilde{q}(x, t, s) = q(x, t) - q(x, s)$. We now apply Proposition 3.3.6 (iii) with $y = \tilde{x}_0$ to get

$$\left|q(x, t) - q(x, s)\right| \leq C(d_\Omega(\tilde{x}_0)^{1-\nu} + |x_p - x_0| d_\Omega(\tilde{x}_0)^{-\nu}) \frac{M^\nu N_T 2^{1-\nu}}{\delta} |t - s|^{\nu/2}$$

for $t, s \in (\delta, T]$ and all $x \in \Omega_{x_0,R}$, $R \leq R_0 = R_1/4$. Since $d_\Omega(\tilde{x}_0) = 2R_0$ and $|x_p - x_0| < R$, the above inequality implies

$$[q]^{(\nu/2)}_{t,Q'} \leq C_0 N_T/\delta, \quad C_0 = C((2R_0)^{1-\nu} + R(2R_0)^{-\nu}) M^\nu 2^{1-\nu}.$$

For the Hölder estimate in space, we simply apply Proposition 3.3.6 (iii) with $\nu = 0$ to get

$$\begin{aligned} \left|q(x, t) - q(y, t)\right| &\leq C\|\nabla q\|_{L^\infty(\Omega)}(t)\left(\left|d_\Omega(y) - d_\Omega(x)\right| + |x_p - y_p|\right) \\ &\leq C\|\nabla q\|_{L^\infty(\Omega)}(t)|x - y|, \ x, y \in \Omega_{x_0,R}, \ R \leq R_0, \ t \in (0, T). \end{aligned}$$

This implies

$$[q]^{(\nu)}_{x,Q'} \leq C_0 N_T/\delta,$$

so the proof is now complete. □

3.3.3 Helmholtz decomposition and the Stokes equations in Hölder spaces

To prove local Hölder estimates up to boundary (Theorem 3.3.4), we recall several known Hölder estimates for the Helmholtz decomposition and the Stokes equations established by [56], [61] via potential theoretic approach. We recall notions for the spaces of Hölder continuous functions. By $C^\gamma(\bar{\Omega})$ with $\gamma \in (0, 1)$, we mean the space of all continuous functions in $\bar{\Omega}$ with $[f]^{(\gamma)}_\Omega < \infty$. Similarly, we use $C^{\gamma,\gamma/2}(\bar{Q})$ for the space of all continuous functions in $\bar{Q}$ with $[f]^{(\gamma,\gamma/2)}_Q < \infty$.

Proposition 3.3.7 (Helmholtz decomposition)**.** *Let Ω be a bounded $C^{2+\gamma}$-domain in $\mathbf{R}^n$ with $\gamma \in (0, 1)$.*
(i) For $f \in C^\gamma(\bar{\Omega})$ there is a (unique) decomposition $f = f_0 + \nabla\Phi$ with f_0, $\nabla\Phi \in C^\gamma(\bar{\Omega})$ such that

$$\int_\Omega f_0 \cdot \nabla\varphi \mathrm{d}x = 0 \quad \text{for all} \quad \varphi \in C^\infty(\bar{\Omega}). \tag{3.3.4}$$

(ii) There is a constant $C_H > 0$ depending only on γ and Ω only through its $C^{2+\gamma}$ regularity such that

$$|f_0|_\Omega^{(\gamma)} + |\nabla\Phi|_\Omega^{(\gamma)} \le C_H\, |f|_\Omega^{(\gamma)} \text{ for all } f \in C^\gamma(\bar{\Omega}). \tag{3.3.5}$$

(iii) For each $\varepsilon \in (0, 1-\gamma)$ there is a constant $C'_H > 0$ depending only on γ,ε and Ω only through its $C^{2+\gamma}$ regularity such that

$$|f_0|_Q^{(\gamma,\gamma/2)} + |\nabla\Phi|_Q^{(\gamma,\gamma/2)} \le C'_H\, |f|_Q^{(\gamma+\varepsilon,\frac{\gamma+\varepsilon}{2})} \textit{ for all } f \in C^{\gamma+\varepsilon,(\gamma+\varepsilon)/2}(\bar{Q}). \tag{3.3.6}$$

Proof. The part (i) and (ii) are established in [56], [61]; the dependence of the constant is not explicit but it is observed from the proof.

In [61, Corollary on p.175], it is proved that the left hand side of (3.3.6) is dominated by a (similar type) constant multiple of

$$|f|_Q^{(\gamma,\gamma/2)} + \sup_{\substack{x,y\in\Omega \\ t,s\in(0,T]}} \frac{\left|(f(x,t)-f(x,s)) - (f(y,t)-f(y,s))\right|}{|x-y|^\mu \cdot |t-s|^{\frac{\gamma}{2}}} \tag{3.3.7}$$

for arbitrary $\mu \in (0,1)$. By the Young inequality, we observe to get

$$\frac{1}{|x-y|^\varepsilon |t-s|^{\gamma/2}} \le \frac{\varepsilon}{\gamma+\varepsilon}\frac{1}{|x-y|^{\gamma+\varepsilon}} + \frac{\gamma}{\gamma+\varepsilon}\frac{1}{|t-s|^{\frac{\gamma+\varepsilon}{2}}}.$$

Thus, we take $\mu = \varepsilon$ to see that the second term of (3.3.7) is dominated by

$$\frac{2\varepsilon}{\gamma+\varepsilon} \sup_{t\in(0,T]} [f]_\Omega^{(\gamma+\varepsilon)}(t) + \frac{2\gamma}{\gamma+\varepsilon} \sup_{x\in\Omega} [f]_{(0,T]}^{(\frac{\gamma+\varepsilon}{2})}(x).$$

Thus, the estimate (3.3.6) follows and (iii) is proved. □

Remark 3.3.8. The operator $f \mapsto f_0$ is essentially the Helmholtz projection P for Hölder vector fields since (3.3.4) implies that $\operatorname{div} f = 0$ in Ω and $f \cdot n_\Omega = 0$ on $\partial\Omega$. The estimate (3.3.5) shows the continuity of P in the Hölder space $C^\gamma(\bar{\Omega})$. However, it is mentioned in [61] (without a proof) that P is not continuous in $C^{\gamma,\gamma/2}(\bar{Q})$. In other words, one cannot take $\varepsilon = 0$ in the estimate (3.3.6).

We next recall Schauder type estimates for the Stokes system:

$$v_t - \Delta v + \nabla q = f_0 \quad \text{in} \quad \Omega\times(0,T), \tag{3.3.8}$$

$$\operatorname{div} v = 0 \quad \text{in} \quad \Omega\times(0,T), \tag{3.3.9}$$

$$v = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{3.3.10}$$

$$v = 0 \quad \text{on} \quad \Omega\times\{t=0\}. \tag{3.3.11}$$

Proposition 3.3.9. *Let Ω be a bounded $C^{2+\gamma}$-domain in $\mathbf{R}^n$ with $\gamma \in (0,1)$ and $T > 0$. Then, for each $f_0 \in C^{\gamma,\gamma/2}(\bar{Q})$ satisfying (3.3.4) there is a unique solution $(v, \nabla q) \in C^{2+\gamma,1+\gamma/2}(\bar{Q}) \times C^{\gamma,\gamma/2}(\bar{Q})$ (up to an additive constant for q) of (3.3.8)–(3.3.11). Moreover, there is a constant C_S depending only on γ, T and Ω only through its $C^{2+\gamma}$-regularly such that*

$$|v|_Q^{(2+\gamma,1+\gamma/2)} + |\nabla q|_Q^{(\gamma,\gamma/2)} \leq C_S |f_0|_Q^{(\gamma,\gamma/2)} \tag{3.3.12}$$

Remarks 3.3.10. (i) This result is a special case of a very general result [61, Theorem 1.1], where the viscosity constant in front of Δ in (3.3.8) depends on space and time and the boundary and initial data are inhomogeneous. Note that the divergence free condition (3.3.4) for f_0 is assumed to establish (3.3.12).
(ii) If the domain is a bounded C^3-domain, clearly, it is a uniformly C^3-domain of type (α,β,K) with some (α,β,K). The constants C_H, C'_H and C_S in Propositions 3.3.7 and 3.3.9 depends on Ω only through this (α,β,K) when Ω is a bounded C^3-domain (which is of course a $C^{2+\gamma}$-domain for all $\gamma \in (0,1)$).

3.3.4 Localization procedure

We shall prove Theorem 3.3.4 by Lemma 3.3.5 and a localization procedure with necessary Hölder estimates (Propositions 3.3.7 and 3.3.9). We first recall the Bogovskiĭ operator B_E in [10]. Let E be a bounded subdomain in Ω with a Lipschitz boundary. The Bogovskiĭ operator B_E is a rather explicit operator, but here we only need a few properties. This linear operator B_E is well-defined for average-zero functions, i.e., $\int_E g\mathrm{d}x = 0$. Moreover, $\mathrm{div} B_E(g) = g$ in E and if the support spt $g \subset E$, then, spt $B_E(g) \subset E$.

The operator B_E fulfills estimates

$$\left\|B_E(g)\right\|_{W^{1,p}(E)} \leq C_B \|g\|_{L^p(E)} \text{ for } g \in L^p(E) \text{ satisfying } \int_E g\mathrm{d}x = 0, \tag{3.3.13}$$

$$\left\|B_E(g)\right\|_{L^p(E)} \leq C_B \|g\|_{W_0^{-1,p}(E)} \text{ for } h \in W_0^{-1,p}(E) = (W^{1,p'}(E))^*, \tag{3.3.14}$$

with some constant C_B independent of g, where $1/p' + 1/p = 1$ with $1 < p < \infty$. In particular, B_E is bounded from $L^p_{av}(E) = \{g \in L^p(E)\} | \int_E g\mathrm{d}x = 0\}$ to the Sobolev space $W^{1,p}(E)$. The result (3.3.14) is a special case of that of [23, Theorem 2.5] which asserts that B_E is bounded from $W_0^{s,p}(\Omega)$ to $W_0^{s+1,p}(\Omega)$ for $s > -2 + 1/p$. The bound C_B depends on p, but its dependence on E is through Lipschitz regularity constant of ∂E.

Proof of Theorem 3.3.4. We take R_0 as in Lemma 3.3.5 and take $R \leq R_0/2$. For $x_0 \in \partial\Omega$, we take a bounded C^3-domain Ω' such that $\Omega_{x_0,3R/2} \subset \Omega' \subset \Omega_{x_0,2R}$. Evidently, $\partial\Omega_{x_0,R} \cap \partial\Omega$ is strictly included in $\partial\Omega' \cap \partial\Omega$. Moreover, one can arrange that Ω' is of type (α',β',K') such that (α',β',K) depends on (α,β,K) and R. Such Ω' is constructed, for example, by considering $\Omega'' = \Omega_{x_0,7R/4}$ and mollifying near the set of intersection $\partial B_{x_0}(7R/4)$ and $\partial\Omega$ in a suitable way to get Ω'.

Let θ be a smooth cutoff function of $[0,1]$ supported in $[0,3/2)$, i.e., $\theta \in C^\infty[0,\infty)$ such that $\theta \equiv 1$ on $[0,1]$ and $0 \leq \theta \leq 1$ with $\operatorname{spt}\theta \subset [0,3/2)$. We set $\theta_R(x) = \theta(|x-x_0|/R)$ which is a cut-off function of $\Omega_{x_0,R}$ supported in Ω'. Because of construction, its derivatives depend only on R. We also take a cutoff function ρ_δ in time variable. Let $\rho \in C^\infty[0,\infty)$ satisfies $\rho \equiv 1$ on $[1,\infty)$ and $\rho = 0$ on $[0,1/2)$ with $0 \leq \rho \leq 1$. For $\delta > 0$ we set $\rho_\delta(t) = \rho(t/\delta)$. We set $\xi = \theta_R\rho_\delta$ and observe that $u = v\xi$ and $p = q\xi$ solves

$$u_t - \Delta u + \nabla p = f, \ \operatorname{div} u = g,$$

in $U = \Omega' \times (0,T)$ with

$$f = v\xi_t - 2\nabla v \cdot \nabla\xi - v\Delta\xi + q\nabla\xi, \ g = v\nabla\xi \ (= \operatorname{div}(v\xi)).$$

We next use the Bogovskiĭ operator $B_{\Omega'}$ so that the vector field is solenoidal. We set $u^* = B_{\Omega'}(g)$ and $\tilde{u} = u - u^*$. Then, $(\tilde{u}, p)$ solves

$$\tilde{u}_t - \Delta\tilde{u} + \nabla p = \tilde{f}, \ \operatorname{div}\tilde{u} = 0 \text{ in } U,$$

with $\tilde{f} = f + u_t^* - \Delta u^*$. We shall fix Ω' so that C'_H in (3.3.6) and C_S in (3.3.12) depends on Ω' only through (α,β,K) and R. If we know $\tilde{f} \in C^{\gamma+\varepsilon,\frac{\gamma+\varepsilon}{2}}(\bar{U})$ with $\varepsilon \in (0, 1-\gamma)$, then, by the Helmholtz decomposition in Hölder spaces (Proposition 3.3.7), one finds $\tilde{f} = f_0 + \nabla\Phi$ with $f_0 \in C^{\gamma,\gamma/2}(\bar{U})$ satisfying (3.3.4) and

$$|f_0|_{(\gamma)} + |\nabla\Phi|_{(\gamma)} \leq C'_H|\tilde{f}|_{(\gamma+\varepsilon)}, \tag{3.3.15}$$

where we use a short hand notation $|f|_{(\gamma)} = |f|_U^{(\gamma,\gamma/2)}$. If we set $\tilde{p} = p - \Phi$, then $(\tilde{u},\tilde{p})$ solves (3.3.8)–(3.3.11) with $\Omega = \Omega'$, where f_0 satisfies the solenoidal condition (3.3.4). Applying the Schauder estimate (3.3.12) yields

$$|\tilde{u}|_{(2+\gamma)} + |\nabla\tilde{p}|_{(\gamma)} \leq C_S|f_0|_{(\gamma)}. \tag{3.3.16}$$

By definition of $\tilde{f}$, we observe that

$$\begin{aligned}
|\tilde{f}|_{(\gamma+\varepsilon)} &\leq |f|_{(\gamma+\varepsilon)} + |u_t^*|_{(\gamma+\varepsilon)} + |\Delta u^*|_{(\gamma+\varepsilon)} \\
&\leq c_0\Big(|v|_{\Omega'\times(\frac{\delta}{2},T]}^{(\gamma+\varepsilon,\frac{\gamma+\varepsilon}{2})} + |\nabla v|_{\Omega'\times(\frac{\delta}{2},T]}^{(\gamma+\varepsilon,\frac{\gamma+\varepsilon}{2})} + |q|_{\Omega'\times(\frac{\delta}{2},T]}^{(\gamma+\varepsilon,\frac{\gamma+\varepsilon}{2})}\Big) + |u^*|_{(2+\gamma+\varepsilon)}
\end{aligned}$$

with c_0 depends only on R, T, δ and $\gamma+\varepsilon$. Since N_T in (3.3.2) is finite, by an interpolation inequality as in the proof of Lemma 3.3.1, we have $|\nabla v|_{t,Q_\delta}^{(1/2)} \leq CN_T/\delta$ with C depending only on (α,β,K). We now apply this estimate together with estimate (3.3.3) for q in Lemma 3.3.5 to get

$$|\tilde{f}|_{(\gamma+\varepsilon)} \leq CN_T + |u^*|_{(2+\gamma+\varepsilon)}, \tag{3.3.17}$$

with a constant $C = C(M(\Omega), \gamma+\varepsilon, \alpha, \beta, K, R, \delta)$. Since

$$\begin{aligned}
|v|_{Q'}^{(2+\gamma,1+\gamma/2)} &\leq |u|_{(2+\gamma)} \leq |\tilde{u}|_{(2+\gamma)} + |u^*|_{(2+\gamma)} \\
|\nabla q|_{Q'}^{(\gamma,\gamma/2)} &\leq |\nabla\tilde{p}|_{(\gamma)} + |\nabla\Phi|_{(\gamma)},
\end{aligned}$$

the desired estimates follow from (3.3.15)-(3.3.17) once we have established that

$$|u^*|_{(2+\gamma+\varepsilon)} \le CN_T.$$

with $C = C(M(\Omega), \gamma + \varepsilon, \alpha, \beta, K, R, \delta)$.

We shall present a proof for

$$[u_t^*]_{t,U}^{(\mu/2)} \le CN_T, \tag{3.3.18}$$

for $\mu \in (0, 1)$ since other quantities can be estimated in a similar way and even easier. By (3.3.13) and (3.3.14), we have

$$\|u_t^*\|_{L^p(\Omega')} \le C_B \|\mathrm{div}\ u_t\|_{W_0^{-1,p}(\Omega')}, \tag{3.3.19}$$

$$\|u_t^*\|_{W^{1,p}(\Omega')} \le C_B \|\mathrm{div}\ u_t\|_{L^p(\Omega')}. \tag{3.3.20}$$

To estimate $\|\mathrm{div}\ u_t\|_{W_0^{-1,p}(\Omega')}$, we use the equations $v_t - \Delta v + \nabla q = 0$ and $\mathrm{div}\ v = 0$. For an arbitrary $\varphi \in W^{1,p'}(\Omega')$, we have

$$\begin{aligned}
\int_{\Omega'} \varphi\ \mathrm{div}\ u_t\ \mathrm{d}x &= \int_{\Omega'} (\varphi\ v_t \cdot \nabla\xi + \varphi\ \nabla\xi_t \cdot v)\ \mathrm{d}x \\
&= \int_{\Omega'} (\varphi\ \nabla\xi \cdot (\Delta v - \nabla q) + \varphi\ \nabla\xi_t \cdot v)\ \mathrm{d}x \\
&= \int_{\Omega'} \Big\{ -\sum_{i=1}^{n} \partial_{x_i}(\varphi\nabla\xi) \cdot \partial_{x_i} v + q\ \mathrm{div}(\varphi\nabla\xi) + \varphi\nabla\xi_t \cdot v \Big\}\ \mathrm{d}x \\
&\quad + \int_{\partial\Omega'} \{\varphi\ \nabla\xi \cdot \partial v/\partial n_{\Omega'} - q\varphi\ \partial\xi/\partial n_{\Omega'}\}\ \mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}$$

This implies

$$\Big| \int_{\Omega'} \varphi\ \mathrm{div}\ u_t \mathrm{d}x \Big| \le C_\xi \Big\{ \|\nabla v\|_\infty + \|q\|_\infty + \|v\|_\infty \Big\} \Big(\|\varphi\|_{W^{1,1}(\Omega')} + \|\varphi\|_{L^1(\partial\Omega')} \Big) \tag{3.3.21}$$

with C_ξ depending only on R and δ (independent of t), where L^∞-norms are taken on Ω'. By a trace theorem (e.g. [15, 5.5, Theorem 1]), there is a constant C (depending only on Lipschitz regularity of the domain) such that

$$\|\varphi\|_{L^1(\partial\Omega')} \le C \|\varphi\|_{W^{1,1}(\Omega')}.$$

By the Hölder inequality $\|\varphi\|_{W^{1,1}(\Omega')} \le C' \|\varphi\|_{W^{1,p}(\Omega')}$ with C' depending on the volume of Ω'. Thus, (3.3.21) yields

$$\|\mathrm{div}\ u_t\|_{W_0^{-1,p}(\Omega')} \le C_0 (\|\nabla v\|_\infty + \|q\|_\infty + \|v\|_\infty)$$

with C_0 depending only on δ, R and Ω' through its (α, β, K). By (3.3.19), this yields

$$\|u_t^*\|_{L^p(\Omega')} \le C_B C_0 (\|\nabla v\|_\infty + \|q\|_\infty + \|v\|_\infty). \tag{3.3.22}$$

We next estimate $\|u_t^*\|_{W^{1,p}}$. By (3.3.20), a direct computation shows that

$$\|u_t^*\|_{W^{1,p}(\Omega')} \le C_0 C_B(\|v\|_\infty + \|v_t\|_\infty) \tag{3.3.23}$$

since div u_t = div $\partial_t(\xi v) = \partial_t(\nabla\xi \cdot v)$ by div $v = 0$.

We now apply the Gagliardo-Nirenberg inequality (e.g. [27]):

$$\|u_t^*\|_\infty \le c\|u_t^*\|_{L^p(\Omega')}^{1-\sigma}\|u_t^*\|_{W^{1,p}(\Omega')}^{\sigma}, \ \sigma = n/p,$$

to (3.3.22) and (3.3.23) to get

$$\|u_t^*\|_\infty \le C_1 C_B(\|v\|_\infty + \|v_t\|_\infty)^\sigma(\|\nabla v\|_\infty + \|v\|_\infty + \|q\|_\infty)^{1-\sigma}$$

with C_1 depending only on δ, R and Ω' through its (α,β,K). We replace u^* by $u^*(\cdot,t) - u^*(\cdot,s)$ and observe that

$$\begin{aligned}\left\|u_t^*(\cdot,t) - u_t^*(\cdot,s)\right\|_\infty \le C_1 C_B\big(&\left\|\nabla v(\cdot,t) - \nabla v(\cdot,s)\right\|_\infty + \left\|q(\cdot,t) - q(\cdot,s)\right\|_\infty \\ &+ \left\|v(\cdot,t) - v(\cdot,s)\right\|_\infty\big)^{1-\sigma}(2N_T/t\wedge s))^\sigma, \ t,s>0, \end{aligned} \tag{3.3.24}$$

where $t \wedge s = \min(t,s)$. As observed in the end of the proof of Lemma 3.3.1, we have

$$[\nabla v]_{t,Q_\delta}^{(1/2)} \le CN_T/\delta.$$

By (3.3.3), we now conclude that

$$\sup_{x\in\Omega'}[\nabla v]_{t,\Omega'\times(\frac{\delta}{2},T]}^{(\mu')} + \sup_{x\in\Omega'}[q]_{t,\Omega'\times(\frac{\delta}{2},T]}^{(\mu')} \le CN_T/\delta, \ \mu' = \frac{\mu}{2(1-\sigma)}$$

provided that $\mu' < 1/2$ (i.e. $p > n/(1-\mu)$). Dividing both sides of (3.3.24) by $|t-s|^{\mu/2}$ and take the supremum for $s,t \ge \delta/2$ to get (3.3.18) since $u^* = 0$ for $t \le \delta/2$. □

3.4 Blow-up arguments - a priori L^∞ estimates

In this section, we shall prove Theorem 3.1.2 by a blow-up argument. We then derive Theorem 3.1.3 which deduces Theorem 3.1.1 since a bounded domain is strictly admissible.

3.4.1 A priori estimates for L^∞-solutions

Proof of Theorem 3.1.2. We argue by contradiction. Suppose that (3.1.15) were false for any choice of T_0 and C. Then, there would exist an L^∞-solution (v_m, q_m) of (3.1.1)-(3.1.4) for $v_0 = v_{0m} \in C_{c,\sigma}^\infty(\Omega)$ and the sequence $\tau_m \downarrow 0$ (as $m \to \infty$) such that $\|N(v_m,q_m)\|_\infty(\tau_m) > m\|v_{0m}\|_\infty$. There is $t_m \in (0,\tau_m)$ such that

$$\left\|N(v_m,q_n)\right\|_\infty(t_m) \ge \frac{1}{2}M_m, \ M_m = \sup_{0<t<\tau_m}\left\|N(v_n,q_m)\right\|_\infty(t).$$

Note that M_m is finite for the L^∞-solution $(v_m, \nabla q_m)$. We normalize (v_m, q_m) by defining $\tilde{v}_m = v_m/M_m$, $\tilde{q}_m = q_m/M_m$. Then, $(\tilde{v}_m, \tilde{q}_m)$ enjoys estimates (3.1.7)–(3.1.9). Since Ω is strictly admissible, (3.1.7) implies that there is a dilation and translation invariant constant C_Ω independent of m such that

$$\sup\{t^{1/2} d_\Omega(x) |\nabla \tilde{q}_m(x,t)| \;\big|\; x \in \Omega_m, t \in (0, t_m)\} \le C_\Omega. \tag{3.4.1}$$

We rescale $(\tilde{v}_m, \tilde{q}_m)$ around the point $x_m \in \Omega$ satisfying (3.1.10) to get the blow-up sequence (u_m, p_m) of the form,

$$u_m(x,t) = \tilde{v}_m(x_m + t_m^{\frac{1}{2}} x, t_m t), \;\; p_m(x,t) = t_m^{\frac{1}{2}} \tilde{q}_m(x_m + t_m^{\frac{1}{2}} x, t_m t).$$

By the scaling invariance of the Stokes equations (3.1.1)–(3.1.2), this (u_m, p_m) solves the Stokes equations in a rescaled domain $\Omega_m \times (0, 1]$, where

$$\Omega_m = \{x \in \mathbf{R}^n \;\big|\; x = (y - x_m) \,/\, t_m^{1/2}, \; y \in \Omega\}.$$

It follows from (3.1.7), (3.4.1) and (3.1.10) that

$$\sup_{0<t<1} \left\| N(u_m, p_m) \right\|_{L^\infty(\Omega_m)} \le 1, \tag{3.4.2}$$

$$\sup\{t^{1/2} d_{\Omega_m}(x) |\nabla p_m(x,t)| \;\big|\; x \in \Omega_m, 0 < t < 1\} \le C_\Omega, \tag{3.4.3}$$

$$N(u_m, p_m)(0, 1) \ge 1/4. \tag{3.4.4}$$

Moreover, for initial data v_{0m}, the condition (3.1.9) implies $\|u_{0m}\|_{L^\infty(\Omega_m)} \to 0$ (as $m \to \infty$). The proof is divided into two cases depending on whether or not

$$c_m = d_\Omega(x_m)/t_m^{1/2}$$

tends to infinity as $m \to \infty$. This c_m is the distance from zero to $\partial\Omega_m$, i.e., $c_m = d_{\Omega_m}(0)$.

Case 1. $\overline{\lim}_{m\to\infty} c_m = \infty$. We may assume that $\lim_{m\to\infty} c_m = \infty$ by taking a subsequence. In this case, the rescaled domain Ω_m expands to $\mathbf{R}^n$. Thus, for any $\varphi \in C_c^\infty(\mathbf{R}^n \times [0, 1))$, the blow-up sequence (u_m, p_m) satisfies

$$\int_0^1 \int_{\mathbf{R}^n} \{u_m \cdot (\varphi_t + \Delta\varphi) - \nabla p_m \cdot \varphi\} \, \mathrm{d}x\mathrm{d}t = -\int_{\mathbf{R}^n} u_m(x, 0) \cdot \varphi(x, 0) \mathrm{d}x$$

for sufficiently large $m > 0$. By (3.4.2) and Proposition 3.3.2, we have a necessary compactness to conclude that there exists a subsequence of solutions still denoted by (u_m, p_m) such that (u_m, p_m) converges to some (u, p) locally uniformly in $\mathbf{R}^n \times (0, 1]$ together with ∇u_m, $\nabla^2 u_m$, $\partial_t u_m$, ∇p_m. (Note that the constant C in (3.3.1) is invariant under dilation and translation of Ω so (3.3.1) for (u_m, p_m) gives equi-continuity of $\nabla^2 u_m$, u_{mt} and ∇p_m.) Since for each $R > 0$,

$$\inf\{d_{\Omega_m}(x) \;\big|\; |x| \le R\} \to \infty \quad \text{as} \quad m \to \infty,$$

the estimate (3.4.3) implies that $\nabla p \equiv 0$. Thus, the limit $u \in C(\mathbf{R}^n \times (0, 1])$ solves

$$\int_0^1 \int_{\mathbf{R}^n} u \cdot (\varphi_t + \Delta\varphi)\, \mathrm{d}x\mathrm{d}t = 0$$

for all $\varphi \in C_c^\infty(\mathbf{R}^n \times [0, 1))$ since $\|u_{0m}\|_{L^\infty(\Omega_m)} \to 0$ as $m \to \infty$. Since u is bounded by (3.4.2), by the uniqueness of the heat equation, we conclude that $u \equiv 0$. However, (3.4.4) implies $N(u, p)(0, 1) \geq 1/4$ which is a contradiction so Case 1 does not occur.

Case 2. $\overline{\lim}_{m\to\infty} c_m < \infty$. By taking a subsequence, we may asuume that c_m converges to some $c_0 \geq 0$. We may also assume that x_m converges to a boundary point $\hat{x} \in \partial\Omega$. By rotation and translation of coordinates, we may assume that $\hat{x} = 0$ and that exterior normal $n_\Omega(\hat{x}) = (0, \dots, 0, -1)$. Since Ω is a uniformly C^3-domain of type (α, β, K), the domain Ω is represented locally near $\hat{x}$ of the form,

$$\Omega_{\mathrm{loc}} = \{(x', x_n) \in \mathbf{R}^n \mid h(x') < x_n < h(x') + \beta,\ |x'| < \alpha\},$$

with a C^3-function h such that $\nabla' h(0) = 0$, $h(0) = 0$, where derivatives up to third order of h is bounded by K. If one rescales with respect to x_m, Ω_{loc} is expanded as

$$\Omega_{m\ \mathrm{loc}} = \{(y', y_n) \in \mathbf{R}^n \mid h(t_m^{1/2} y' + x'_m) < t_m^{1/2} y_n + (x_m)_n < h(t_m^{1/2} y + x'_m) + \beta,\ |t_m^{1/2} y'| < \alpha\}.$$

Since $d_\Omega(x_m)/(x_m)_n \to 1$ as $m \to \infty$ and $x'_m \to 0$, this domain $\Omega_{m\ loc}$ converges to

$$\mathbf{R}^n_{+,-c_0} = \{(x', x_n) \in \mathbf{R}^n \mid x_n > -c_0\}.$$

In fact, if one expresses

$$\Omega_{m\ \mathrm{loc}} = \{(y', y_n) \in \mathbf{R}^n \mid h_m(y') < y_m < \beta_m + h_m(y'),\ |y'| < \alpha_m\},$$

with $\alpha_m = \alpha/t_m^{1/2}$, $\beta_m = \beta/t_m^{1/2}$, $h_m(y') = h(t_m^{1/2} y' + x'_m)/t_m^{1/2} - (x_m)_n/t_m^{1/2}$, then $h_m \to -c_0$ locally uniformly up to third derivatives and $\alpha_m, \beta_m \to \infty$. Note that $|\partial_x^\mu h_m|$ for μ, $1 \leq |\mu| \leq 3$ is uniformly bounded by K.

Thus, (u_m, p_m) solves (3.1.1)–(3.1.4) in $\Omega_{m\ \mathrm{loc}} \times (0, 1]$. By (3.4.2) and Theorem 3.3.4 we have a necessary compactness to conclude that there exists a subsequence (u_m, p_m) converges to some (u, p) locally uniformly in $\bar{\mathbf{R}}^n_{+,-c_0} \times (0, 1]$ together with ∇u_m, $\nabla^2 u_m$, u_{mt}, ∇p_m as interior case. (Note that Ω_m is still of type (α, β, K) which is uniform in m).

Now, we observe that the limit (u, p) solves the Stokes equations (3.1.1)-(3.1.4) in a half space with zero initial data in a weak sense. In fact, since (u_m, p_m) fulfills

$$\int_0^1 \int_{\mathbf{R}^n_{+,-c_0}} \{u_m \cdot (\varphi_t + \Delta\varphi) - \varphi \cdot \nabla p_m\}\, \mathrm{d}x\mathrm{d}t = -\int_{\mathbf{R}^n_{+,-c_0}} u_m(x, 0) \cdot \varphi(x, 0)\mathrm{d}x$$

for all $\varphi \in C_c^\infty(\mathbf{R}^n_{+,-c_0} \times (0,1))$. We note that (3.5.2) and (3.5.3) are inherited to (u,p), in particular,

$$\sup\Big\{t^{1/2}(x_n + c_0)\big|\nabla p(x,t)\big| \;\Big|\; x' \in \mathbf{R}^{n-1},\ x_n > -c_0,\ t \in (0,1)\Big\} \le C_\Omega.$$

Since the convergence of u_m is up to boundary, the boundary condition is also preserved. We thus apply the uniqueness to the Stokes equations in a half space (Theorem 1.1.1 in Chapter 1) to conclude $u \equiv 0$ and $\nabla p \equiv 0$.

However, (3.4.4) implies $N(u,p)(0,1) \ge 1/4$ which is a contradiction so Case 2 does not occur neither.

We have thus proved (3.1.15). □

Remarks 3.4.1. (i) Actually, the a priori estimate (3.1.15) holds for $v_0 \in L^\infty_\sigma(\Omega)$, i.e.,

$$\sup_{0<t<T_0} \big\|N(v,q)\big\|_\infty(t) \le C\|v_0\|_\infty$$

holds for all L^∞-solutions $(v,\nabla q)$ and $v_0 \in L^\infty_\sigma(\Omega)$ in a strictly admissible, uniformly C^3-domain. The a priori L^∞-estimate, in particular, implies the uniqueness of L^∞-solutions since (3.1.15) for $v_0 = 0$ implies $v \equiv 0$ and $\nabla q \equiv 0$. Note that in general, the existence of L^∞-solutions is unknown. In Chapter 4, we prove the unique existence of L^∞-solutions for exterior domains and perturbed half spaces by approximating $v_0 \in L^\infty_\sigma(\Omega)$ by elements of $C^\infty_{c,\sigma}(\Omega)$.
(ii) Once we know the existence of the time $T_0 > 0$ such that (3.1.15) holds, we are able to extend T_0 up to an arbitrary time. In fact, we are able to estimate L^∞-solutions $(v,\nabla q)$ between T_0 and $2T_0$ by applying the a priori estimate (3.1.15) again, i.e., the estimate

$$\sup_{T_0<t\le 2T_0} \big\|N(v,q)\big\|_\infty(t) \le C\|v(T_0)\|_\infty$$

holds, where $C = C(T_0) \ge 1$ is the constant in (3.1.15). Thus, we have

$$\sup_{0<t\le 2T_0} \|v\|_\infty(t) \le C^2\|v_0\|_\infty.$$

In a similar way, we are able to estimate other terms $\nabla v, \nabla^2 v, \partial_t v, \nabla q$. By an iteration argument, we are able to extend the time T_0 up to an arbitrary time.

3.4.2 Regularity for $\tilde{L}^r$-solutions

We shall prove that an $\tilde{L}^r$-solution is indeed an L^∞-solution for sufficiently regular initial data.

Proposition 3.4.2. *Let Ω be a uniformly C^3-domain in $\mathbf{R}^n$. Let $(v, \nabla q)$ be an $\tilde{L}^r$-solution of (3.1.1)–(3.1.4) for $r > n$. Assume that $v_0 \in D(\tilde{A}_r)$, where $\tilde{A}_r$ is the Stokes operator in $\tilde{L}^r_\sigma(\Omega)$, i.e., $-\tilde{A}_r$ is the generator of the Stokes semigroup in $\tilde{L}^r_\sigma(\Omega)$. Then $(v, \nabla q) \in C^{2,1}(\bar{\Omega} \times (0, T]) \times C(\bar{\Omega} \times (0, T])$ and $t^{1/2} d_\Omega(x)|\nabla q(x, t)|$ is bounded in $\Omega \times (0, T)$. Moreover, for each $T > 0$ we have*

$$\sup_{0<t<T} \left\|N(v, q)\right\|_\infty(t) < \infty, \tag{3.4.5}$$

i.e., $(v, \nabla q)$ is an L^∞-solution.

Proof. We shall claim a stronger statement

$$\sup_{0<t<T} \Big\{ \|v\|_{W^{1,r}_{\mathrm{ul}}}(t) + t^{1/2} \|\nabla v\|_{W^{1,r}_{\mathrm{ul}}}(t) + t\big(\|\nabla^2 v\|_{W^{1,r}_{\mathrm{ul}}}(t) + \|\partial_t v\|_{W^{1,r}_{\mathrm{ul}}}(t) + \|\nabla q\|_{W^{1,r}_{\mathrm{ul}}}(t)\big) \Big\} \\ \leq C\|v_0\|_{D(\tilde{A}_r)} \tag{3.4.6}$$

with $C = C(T, \Omega, r)$. Here, $W^{1,r}_{\mathrm{ul}}$ is a uniformly local $W^{1,r}$ space defined by

$$W^{1,r}_{\mathrm{ul}}(\Omega) = \{f \in L^r_{\mathrm{ul}}(\Omega) \mid \nabla f \in L^r_{\mathrm{ul}}(\Omega)\}, \quad \|f\|_{W^{1,r}_{\mathrm{ul}}} = \|f\|_{L^r_{\mathrm{ul}}} + \|\nabla f\|_{L^r_{\mathrm{ul}}}$$

and

$$L^r_{\mathrm{ul}}(\Omega) = \left\{ f \in L^r_{\mathrm{loc}}(\Omega) \;\middle|\; \|f\|_{L^r_{\mathrm{ul}}} = \sup_{x\in\Omega} \left(\int_{\Omega_{x,R}} \left|f(y)\right|^r dy \right)^{1/r} \right\},$$

where $\Omega_{x,R} = B_x(R) \cap \Omega$ and R is a fixed positive number. The norm depends on R but the topology defined by the norm is independent of the choice of R. The norm of the domain $D(\tilde{A}_r)$ is defined by

$$\|u\|_{D(\tilde{A}_r)} = \|u\|_{\tilde{L}^r(\Omega)} + \|\tilde{A}_r u\|_{\tilde{L}^r(\Omega)}, \ \|u\|_{\tilde{L}^r(\Omega)} = \max(\|u\|_{L^r(\Omega)}, \|u\|_{L^2(\Omega)})$$

when $r \geq 2$. As proved in [16], [18], this norm is equivalent to the norm

$$\|u\|_{\tilde{W}^{2,r}(\Omega)} = \sum_{|l|\leq 2} \|\partial_x^l u\|_{\tilde{L}^r(\Omega)}.$$

Note that once we have proved (3.4.6), the bound (3.4.5) and $v \in C^{2,1}(\bar{\Omega} \times (0, T])$ (also $\nabla q \in C(\bar{\Omega} \times (0, T])$) follow from the Sobolev embedding.

We shall prove (3.4.6). We first observe that by the analyticity of the semigroup $S(t) = e^{-t\tilde{A}_r}$,

$$\sup_{0<t<T} t\|v_t\|_{D(\tilde{A}_r)}(t) \leq C_1 \|v_0\|_{D(\tilde{A}_r)}$$

since $\tilde{A}_r v_t = \tilde{A}_r e^{-t\tilde{A}_r} \tilde{A}_r v_0$. It is easy to see that

$$\sup_{0<t<T} \|v\|_{D(\tilde{A}_r)}(t) \leq C_2 \|v_0\|_{D(\tilde{A}_r)}, \tag{3.4.7}$$

with C_j depending only on T, Ω and r. Thus, we have proved that

$$\sup_{0<t<T} \left(\|v\|_{\tilde{W}^{1,r}(\Omega)}(t) + \|\nabla v\|_{\tilde{W}^{1,r}(\Omega)}(t) + t\|v_t\|_{\tilde{W}^{2,r}(\Omega)}(t)\right) \leq C_3\|v_0\|_{D(\tilde{A}_r)} \tag{3.4.8}$$

since $D(\tilde{A}_r)$-norm and $\tilde{W}^{2,r}$-norm is equivalent.

To show (3.4.6), it remains to prove that

$$\sup_{0<t<T} t(\|\nabla^2 v\|_{W^{1,r}_{\mathrm{ul}}}(t) + \|\nabla q\|_{W^{1,r}_{\mathrm{ul}}}(t)) \leq C_4\|v_0\|_{D(\tilde{A}_r)}. \tag{3.4.9}$$

We take R sufficiently small such that $\Omega_{x,3R} \subset U_{\alpha,\beta,h}(x_0)$ for any $x_0 \in \partial\Omega$. We normalize q by taking

$$\hat{q}(x) = q(x) - \frac{1}{|\Omega''|}\int_{\Omega''} q(x)dx,\ \Omega'' = \Omega_{x_0,3R}.$$

It follows from the Poincaré inequality [15, 5.8.1] that

$$\|\hat{q}\|_{L^r(\Omega'')} \leq c\|\nabla q\|_{L^r(\Omega'')}, \tag{3.4.10}$$

with c independent of x_0. Since Ω is C^3 and (v, q) solves

$$-\Delta v + \nabla q = -v_t,\ \operatorname{div} v = 0 \quad \text{in} \quad \Omega'',$$

with

$$v = 0 \quad \text{on} \quad \partial\Omega'' \cap \partial\Omega,$$

the local higher regularity theory for elliptic systems (see [22, V]) shows that

$$\|\nabla^3 v\|_{L^r(\Omega')} + \|\nabla^2 q\|_{L^r(\Omega')} \leq C(\|v_t\|_{W^{1,r}(\Omega'')} + \|v\|_{W^{1,r}(\Omega'')} + \|\hat{q}\|_{L^r(\Omega'')})$$

with $\Omega' = \Omega_{x_0,2R}$. Here the dependence with respect to t is suppressed. The last term is estimated by (3.4.10) so we observe that

$$\|\nabla^3 v\|_{L^r(\Omega')} + \|\nabla^2 q\|_{L^r(\Omega')} \leq C(\|v_t\|_{W^{1,r}(\Omega)} + \|v\|_{W^{1,r}(\Omega)} + \|\nabla q\|_{L^r(\Omega)}) \tag{3.4.11}$$

with C depending only on Ω, R and r but independent of $x_0 \in \partial\Omega$. If $x_0 \in \Omega$ is taken so that $B_{x_0}(2R) \subset \Omega$, then, interior higher regularity theory yields (3.4.11) with $\Omega' = B_{x_0}(R)$ (by taking $\Omega'' = B_{x_0}(2R)$). Since Ω is covered by $\Omega_{x_0,2R}$, $x_0 \in \partial\Omega$ and $B_{x_0}(R)$ with $x_0 \in \Omega$ such that $B_{x_0}(2R) \subset \Omega$, the estimate (3.4.11) implies that

$$\|\nabla^3 v\|_{L^r_{\mathrm{ul}}(\Omega)} + \|\nabla^2 q\|_{L^r_{\mathrm{ul}}(\Omega)} \leq C(\|v_t\|_{W^{1,r}(\Omega)} + \|v\|_{W^{1,r}(\Omega)} + \|\nabla q\|_{L^r(\Omega)}). \tag{3.4.12}$$

Since $\nabla q = Q[\Delta v]$ implies

$$\|\nabla q\|_{\tilde{L}^r(\Omega)} \leq C'\|\Delta v\|_{\tilde{L}^r(\Omega)},$$

with $C' = C'(\Omega, r)$, the estimate (3.4.12) together with (3.4.8) now yields (3.4.9).

It remains to show $\nabla q(x) = \nabla q(x,\cdot) \in L^\infty_d(\Omega)$. By the mean value formula, it follows that

$$\nabla q(x) = \fint_{B_x(\tau)} \nabla q(y) d\mathcal{H}^{n-1}(y) \quad \text{for } x \in \Omega \text{ and } \tau = d_\Omega(x).$$

Apply the Hölder inequality to get $|\nabla q(x)| \leq C_s/\tau^{n/s} \|\nabla q\|_{L^s(\Omega)}$ for $s \in (1,\infty)$, with the constant C_s independent of $\tau = d_\Omega(x)$. If $d_\Omega(x) \leq 1$ take $s = r \geq n$. If $d_\Omega(x) > 1$, take $s = 2$. Since $\mathbf{Q}$ is bounded on $\tilde{L}^r(\Omega)$, it follows that

$$|\nabla q|_{\infty,d}(t) \leq C_r \|\Delta v\|_{\tilde{L}^r(\Omega)}(t)$$

with the constant C_r. By (3.4.9), $t^{1/2}|\nabla q|_{\infty,d}(t)$ is bounded in $(0,T)$.

Since $D(\tilde{A}_r) \subset W^{2,r}(\Omega)$, by the Sobolev embedding, $v_0 \in D(\tilde{A}_r) \subset L^\infty_\sigma(\Omega)$. Since $v \to v_0$ a.e. in Ω as $t \downarrow 0$, $v \to v_0$ weakly-$*$ on $L^\infty(\Omega)$ as $t \downarrow 0$. Thus, $(v,\nabla q)$ is an L^∞-solution. The proof is now complete. □

Although we use $\tilde{L}^r$-theory in order to extend the Stokes semigroup $S(t)$ to $C_{0,\sigma}(\Omega)$ on a uniformly regular domain, we can use L^r-theory for domains where the Helmholtz projection $\mathbf{P}$ acts as a bounded operator on $L^r(\Omega)$, $r \in (1,\infty)$. In fact, in Chapter 4 we extend $S(t)$ to $L^\infty_\sigma(\Omega)$ for domains (I)–(III) by using the L^r-theory. For this purpose, we give the statement for L^r-solutions.

Proposition 3.4.3. *Let Ω be a uniformly C^3-domain, which admits the Helmholtz projection on $L^r(\Omega)$, $r \in (1,\infty)$. Let $(v,\nabla q)$ be an L^r-solution for $r > n$. Assume that $v_0 \in D(A_r)$, where A_r is the Stokes operator on $L^r_\sigma(\Omega)$. Then, $(v,\nabla q) \in C^{2,1}(\bar{\Omega}\times(0,T]) \times C(\bar{\Omega}\times(0,T])$ and (3.4.5) holds. If in addition $v_0 \in D(A_2)$, then $t^{1/2}d_\Omega(x)|\nabla q(x,t)|$ is bounded in $\Omega\times(0,T)$. In particular, $(v,\nabla q)$ is an L^∞-solution.*

Proof. The estimate (3.4.6) is valid by replacing $\tilde{L}^r$-norm to L^r-norm since the L^r-Helmholtz projection as well as the analyticity of $S(t)$ on $L^r_\sigma(\Omega)$ are valid. Thus, $(v,\nabla q) \in C^{2,1}(\bar{\Omega}\times(0,T]) \times C(\bar{\Omega}\times(0,T])$ and (3.4.5) hold for L^r-solutions $(v,\nabla q)$ for $r > n$. By (3.4.5) and $v \to v_0$ on $L^r(\Omega)$ as $t \downarrow 0$, $v \to v_0$ weakly-$*$ on $L^\infty(\Omega)$ as $t \downarrow 0$ follows.
We use the assumption $v_0 \in D(A_2)$ in order to show $t^{1/2}d_\Omega(x)|\nabla q(x,t)|$ is bounded in $\Omega\times(0,T)$. Since $\mathbf{Q}$ is bounded on $L^r \cap L^2$, it follows that

$$|\nabla q|_{\infty,d}(t) \leq C_r(\|\Delta v\|_{L^r(\Omega)}(t) + \|\Delta v\|_{L^2(\Omega)}(t)).$$

Since $\|v\|_{W^{2,r}(\Omega)}$ is estimated by $\|v\|_{D(A_r)}$ and $A_r v = e^{-tA_r}A_r v_0$, it follows that

$$\sup_{0<t<T} |\nabla q|_{\infty,d}(t) \leq C(\|v_0\|_{D(A_r)} + \|v_0\|_{D(A_2)}).$$

Thus, $t^{1/2}d_\Omega(x)|\nabla q(x,t)|$ is bounded in $\Omega\times(0,T)$. □

3.4.3 Analyticity of the Stokes semigroup on $C_{0,\sigma}$

We shall prove Theorem 3.1.3. To show C_0-property of the semigroup we prepare

Proposition 3.4.4. *Let Ω be a uniformly C^2-domain in $\mathbf{R}^n$. Let $(v, \nabla q)$ be an $\tilde{L}^r$-solution of (3.1.1)–(3.1.4) for $r > n$ and $v_0 \in D(\tilde{A}_r)$. Then,*

$$\lim_{t\downarrow 0} \left\| v(\cdot, t) - v_0 \right\|_\infty = 0. \tag{3.4.13}$$

In other words,

$$\lim_{t\downarrow 0} \| e^{-t\tilde{A}_r} v_0 - v_0 \|_\infty = 0.$$

Proof. By the Gagliardo-Nirenberg inequality, we have

$$\left\| v(t) - v_0 \right\|_{L^\infty(\Omega)} \leq C \left\| v(t) - v_0 \right\|_{L^r(\Omega)}^{1-\theta} \left\| v(t) - v_0 \right\|_{W^{1,r}(\Omega)}^{\theta} \tag{3.4.14}$$

with $\theta = 1 - n/r$, where $v(t) = v(\cdot, t)$. Since

$$\|f\|_{W^{1,r}(\Omega)} \leq \|f\|_{W^{2,r}(\Omega)} \leq \|f\|_{\tilde{W}^{2,r}(\Omega)} \leq C' \|f\|_{D(\tilde{A}_r)},$$

we have by (3.4.7) that

$$\left\| v(t) - v_0 \right\|_{W^{1,r}(\Omega)} \leq C' \left(\left\| v(t) \right\|_{D(\tilde{A}_r)} + \|v_0\|_{D(\tilde{A}_r)} \right) \leq C'' \|v_0\|_{D(\tilde{A}_r)}. \tag{3.4.15}$$

Since $e^{-t\tilde{A}^r}$ is strongly continuous on $\tilde{L}^r$, (3.4.14) with (3.4.15) yields (3.4.13). □

Proof of Theorem 3.1.3. By Proposition 3.4.2, an $\tilde{L}^r$-solution for $v_0 \in C_{c,\sigma}^\infty$ is an L^∞-solution. By a priori estimate (3.1.15), the operator $S(t)$ is uniquely extended to a bounded operator $\tilde{S}(t)$ on $C_{0,\sigma}$ at least for a small t, i.e., $t \in [0, T_0)$. Since $S(t)$ is a semigroup on $\tilde{L}^r$, we have

$$\tilde{S}(t_1)\tilde{S}(t_2) = \tilde{S}(t_1 + t_2) \quad \text{as far as} \quad t_1 + t_2 < T_0. \tag{3.4.16}$$

We extend $\tilde{S}(t)$ to $t \geq T_0$ by $\tilde{S}(t) = \tilde{S}(t_1) \cdots \tilde{S}(t_m)$ so that $t_i \in (0, T_0)$ and $t_1 + \cdots + t_m = t$. This is well-defined in the sense that $\tilde{S}(t)$ is independent of the division of t by the semigroup property (3.4.16). Thus, we are able to define the Stokes semigroup $\tilde{S}(t)$ for all $t \geq 0$ which we simply write by $S(t)$ (since it agrees with $S(t)$ on $C_{0,\sigma} \cap \tilde{L}^r$). Our estimate (3.1.15) is inherited to $S(t)$. Moreover, by the semigroup property, the estimate (3.1.15) yields $\|S(t)v_0\|_\infty \leq C_T \|v_0\|_\infty$ with C_T independent of $v_0 \in C_{0,\sigma}(\Omega)$ and $t \in (0, T)$ for arbitrary $T > 0$. Since $\mathrm{d}S(t)/\mathrm{dt} = S(t-s)\mathrm{d}S(s)/\mathrm{ds}$ for $s \in (0, t)$, the estimate (3.1.15) together with an L^∞ bound for $S(t)$ yields

$$\sup_{0<t<T} t \left\| \frac{\mathrm{d}}{\mathrm{dt}} S(t) v_0 \right\|_\infty \leq C'_T \|v_0\|_\infty,$$

with a constant C'_T independent of $v_0 \in C_{0,\sigma}(\Omega)$. This implies that $S(t)$ is an analytic semigroup on $C_{0,\sigma}(\Omega)$.

It remains to prove that $S(t)$ is a C_0-semigroup on $C_{0,\sigma}(\Omega)$. Since $C^\infty_{c,\sigma}(\Omega)$ is dense on $C_{0,\sigma}(\Omega)$, for each $v_0 \in C_{0,\sigma}(\Omega)$ there is $v_{0m} \in C^\infty_{c,\sigma}(\Omega)$ such that $v_{0m} \to v_0$ in $L^\infty(\Omega)$. Since $\|S(t)v_0\|_\infty \le C_T\|v_0\|_\infty$ for $0 < t < T$ we have

$$\begin{aligned}\left\|S(t)v_0 - v_0\right\|_\infty &\le \left\|S(t)v_0 - S(t)v_{0m}\right\|_\infty + \left\|S(t)v_{0m} - v_{0m}\right\|_\infty + \|v_{0m} - v_0\|_\infty \\ &\le (C_T + 1)\|v_{0m} - v_0\|_\infty + \left\|S(t)v_{0m} - v_{0m}\right\|_\infty.\end{aligned}$$

By Proposition 3.4.4, sending $t \downarrow 0$ yields

$$\overline{\lim_{t\downarrow 0}} \left\|S(t)v_0 - v_0\right\|_\infty \le (C_T + 1)\|v_{0m} - v_0\|_\infty.$$

Letting m to infinity, we conclude that $S(t)$ is a C_0-semigroup on $C_{0,\sigma}(\Omega)$. □

Since a bounded domain is strictly admissible, Theorem 3.1.3 yields Theorem 3.1.1. Moreover, $S(t)$ is analytic semigroup on $C_{0,\sigma}(\Omega)$ for exterior domains and perturbed half spaces since these domains are also strictly admissible as we proved in Chapter 2.

Remarks 3.4.5. (i) In general, we do not know whether or not $S(t)$ is a bounded analytic semigroup in the sense that

$$\left\|\frac{\mathrm{d}}{\mathrm{dt}}S(t)v_0\right\|_\infty \le \frac{C}{t}\|v_0\|_\infty \tag{3.4.17}$$

for some C independent of $t > 0$. When Ω is bounded, one can claim such boundedness. In fact, multiplying v with (3.1.1) and integrating by parts, we obtain an energy equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|v\|^2_{L^2}(t) + \|\nabla v\|^2_{L^2}(t) = 0.$$

Since Ω is bounded, the Poincaré inequality implies

$$\|\nabla v\|^2_{L^2} \ge \nu\|v\|^2_{L^2}$$

with some $\nu > 0$. Thus,

$$\left\|S(t)v_0\right\|^2_{L^2} \le e^{-2\nu t}\|v_0\|^2_{L^2}.$$

If Ω is sufficiently smooth, by the Sobolev inequality and the property of the Stokes semigroup in L^2 (see [55, III.2.1]), we have

$$\left\|S(t)v_0\right\|_{L^\infty} \le C_1\left\|S(t)v_0\right\|_{W^{2k,2}} \le C_2\left\|A_2^k S(t)v_0\right\|_{L^2}$$

for an integer $k > n/4$ with C_j $(j = 1, 2, \dots)$ independent of t and $v_0 \in L^2_\sigma(\Omega)$. Since $S(t)$ is analytic semigroup in L^2_σ, this yields

$$\left\|S(t)v_0\right\|_{L^\infty} \le C_3\left\|S(t-1)v_0\right\|_{L^2} \quad \text{for} \quad t \ge 1.$$

We have thus proved that

$$\left\|S(t)v_0\right\|_{L^\infty} \le C_4 e^{-\nu t}\|v_0\|_{L^2} \le C_5 e^{-\nu t}\|v_0\|_{L^\infty},\ t \ge 1. \tag{3.4.18}$$

Similarly,

$$\left\|\frac{\mathrm{d}}{\mathrm{dt}}S(t)v_0\right\|_{L^\infty} \le C_1\left\|\frac{\mathrm{d}}{\mathrm{dt}}S(t)v_0\right\|_{W^{2k,2}} \le C_2\left\|A_2^{k+1}S(t)v_0\right\|_{L^2} \le C_6 e^{-\nu t}\|v_0\|_{L^\infty} \quad \text{for} \quad t \ge 1.$$

Since

$$\left\|\frac{\mathrm{d}}{\mathrm{dt}}S(t)v_0\right\|_\infty \le \frac{C_7}{t}\|v_0\|_\infty \quad \text{for} \quad t \le 1,$$

this yields (3.4.17). Thus, $S(t)$ is a bounded analytic semigroup in $C_{0,\sigma}(\Omega)$ and $L^\infty_\sigma(\Omega)$ (see in next the section) when Ω is a smoothly bounded domain. If one uses the L^r-theory ($r > n$) instead of L^2-theory, the result is still valid for a bounded domain with C^3-boundary.

(ii) Since we have (3.4.18) for $t \ge T_0 > 0$, our a priori estimate (3.1.15) in particular implies that

$$\left\|S(t)v_0\right\|_\infty \le C\|v_0\|_\infty \quad \text{for all} \quad t > 0,\ v_0 \in C_{0,\sigma}(\Omega)$$

with C depending only on Ω when Ω is bounded. This type of results is often called a maximum modulus result which is available in the literature.

The maximum modulus theorem is first stated in [66] when Ω is a bounded, convex domain with smooth boundary for $v_0 \in C^\infty_{c,\sigma}(\Omega)$. Later, a full proof is given in [57]. It is extended by [58] for a general bounded domain with C^2-boundary. It is extended by [43] for $v_0 \in C_{0,\sigma}(\Omega)$ but $\partial\Omega$ is assumed to be $C^{2+\gamma}$ with $\gamma \in (0,1)$.

By our extension to L^∞_σ space in the next chapter, we conclude that

$$\left\|S(t)v_0\right\|_\infty \le C\|v_0\|_\infty,\ v_0 \in L^\infty_\sigma(\Omega)$$

for all $t > 0$ with C depending only on Ω when Ω is bounded and of C^3 boundary.

(iii) It is interesting to discuss whether or not our semigroup $S(t)$ is an analytic semigroup of angle $\pi/2$, i.e., it is extendable as a holomorphic semigroup in $\operatorname{Re} t > 0$. Our results say that $S(t)$ is angle ε for some $\varepsilon > 0$. If we are able to prove (3.1.6) for $\operatorname{Re} t \in (0, T_0)$ with $|\arg t| < \alpha$ for $\alpha \in (0, \pi/2)$ where analyticity is valid, then, we conclude that $S(t)$ is angle $\pi/2$. This idea would work provided that the Schauder type estimate for complex t with $|\arg t| < \varepsilon$ would be available. It is of course likely but there seems to be no explicit reference. In Chapter 5, we shall prove a necessary resolvent estimate to conclude that $S(t)$ is an analytic semigroup of angle $\pi/2$ (without proving (3.1.6) for complex parameter).

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Chapter 4

Semigroup on BUC_σ and L^∞_σ spaces

We now extend the Stokes semigroup to non-decaying type solenoidal spaces L^∞_σ (and BUC_σ). As we proved in Chapter 3, the Stokes equations is uniquely solvable for bounded and decaying initial data v_0 as $|x| \to \infty$, i.e., $v_0 \in C_{0,\sigma}$, but for merely bounded initial data, the existence of solutions is non-trivial. For this purpose, we pointwise approximate $v_0 \in L^\infty_\sigma$ by compactly supported solenoidal vector fields and prove the unique existence of solutions by using the a priori L^∞-estimate (0.1.1).

4.1 Introduction

We consider the Stokes equations for $v_0 \in L^\infty_\sigma(\Omega)$ in the domain $\Omega \subset \mathbf{R}^n$, $n \geq 2$:

$$v_t - \Delta v + \nabla q = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{4.1.1}$$

$$\operatorname{div} v = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{4.1.2}$$

$$v = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{4.1.3}$$

$$v = v_0 \quad \text{on} \quad \Omega \times \{t = 0\}, \tag{4.1.4}$$

where $L^\infty_\sigma(\Omega)$ is the solenoidal L^∞ space defined by

$$L^\infty_\sigma(\Omega) = \left\{ f \in L^\infty(\Omega) \;\middle|\; \int_\Omega f \cdot \nabla\varphi \mathrm{d}x = 0 \quad \text{for all} \quad \varphi \in \hat{W}^{1,1}(\Omega) \right\},$$

and the homogeneous Sobolev space $\hat{W}^{1,1}(\Omega) = \{\varphi \in L^1_{\mathrm{loc}}(\Omega) \mid \nabla\varphi \in L^1(\Omega)\}$. When Ω is bounded, the Stokes semigroup $S(t)$ is defined on $L^\infty_\sigma(\Omega) \subset L^r_\sigma(\Omega)$, $r \in (1,\infty)$. We first state a result for bounded domains as a typical example.

Theorem 4.1.1 (Analyticity on L^∞_σ). *Let Ω be a bounded C^3-domain in $\mathbf{R}^n$, $n \geq 2$. Then the Stokes semigroup $S(t)$ is a (non-C_0-)analytic semigroup on $L^\infty_\sigma(\Omega)$.*

Since smooth functions are not dense in $L^\infty_\sigma(\Omega)$ and $S(t)v_0$ is smooth for $t>0$, $S(t)v_0 \to v_0$ as $t \downarrow 0$ in $L^\infty_\sigma(\Omega)$ does not hold for some $v_0 \in L^\infty_\sigma(\Omega)$. This means $S(t)$ is a non-C_0-semigroup.

Our approach for the extension to the space $L^\infty_\sigma(\Omega)$ is based on the a priori L^∞-estimate for solutions to the Stokes equations (4.1.1)–(4.1.4),

$$\sup_{0<t\leq T_0} \left\| N(v,q) \right\|_{L^\infty(\Omega)}(t) \leq C\|v_0\|_{L^\infty(\Omega)}, \tag{4.1.5}$$

where $N(v,q)(x,t)$ denotes the scale invariant norm for solutions up to second orders,

$$N(v,q)(x,t) = \left|v(x,t)\right| + t^{1/2}\left|\nabla v(x,t)\right| + t\left|\nabla^2 v(x,t)\right| + t\left|v_t(x,t)\right| + t\left|\nabla q(x,t)\right|. \tag{4.1.6}$$

The a priori L^∞-estimate (4.1.5) is available for sufficiently smooth initial data as we proved in Chapter 3 (Theorem 3.1.2 and Proposition 3.4.2). To extend $S(t)$ to $L^\infty_\sigma(\Omega)$, we approximate $v_0 \in L^\infty_\sigma(\Omega)$ by compactly supported functions $\{v_{0,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega)$ such that

$$\begin{aligned} &\|v_{0,m}\|_{L^\infty(\Omega)} \leq C\|v_0\|_{L^\infty(\Omega)}, \\ &v_{0,m} \to v_0 \quad \text{a.e. in } \Omega \quad \text{as } m \to \infty, \end{aligned} \tag{4.1.7}$$

with the constant C independent of $m \geq 1$. Note that $C^\infty_{c,\sigma}(\Omega)$ (or $C_{0,\sigma}(\Omega)$) is not dense in $L^\infty_\sigma(\Omega)$ so one cannot approximate v_0 by elements of $C^\infty_{c,\sigma}(\Omega)$ in a uniform topology. However, by a mollifying procedure keeping the divergence free condition, we are able to find the sequence $\{v_{0,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega)$ satisfying (4.1.7). This is very easy to prove when Ω is star-shaped while in general it is nontrivial. We localize the problem to reduce it to star-shaped case. Since Ω is bounded, $v_{0m} \to v$ in $L^r_\sigma(\Omega)$ so we extend the estimate (4.1.5) to $v = S(t)v_0$ with the associated pressure q for $v_0 \in L^\infty_\sigma(\Omega)$.

We prove the approximation (4.1.7) also for exterior domains and perturbed half spaces. By combining the approximation (4.1.7) for a bounded domain and that for the whole space (or a half space), we find the desired sequence. Once we have the sequence $\{v_{0,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega)$ satisfying (4.1.7), we are able to prove the existence of solutions for $v_0 \in L^\infty_\sigma(\Omega)$. We prove that the sequence of L^r-solutions $(v_m, \nabla q_m)$ for $v_{0,m} \in C^\infty_{c,\sigma}(\Omega)$ (subsquently) converges to the solution $(v, \nabla q)$ for $v_0 \in L^\infty_\sigma(\Omega)$. Then, the Stokes semigroup $S(t)$ is extended to $L^\infty_\sigma(\Omega)$ by the limit v, i.e., $S(t)v_0 = v$ for $v_0 \in L^\infty_\sigma(\Omega)$. The limit v is independent of the choice of approximation since an L^∞-solution is unique (Remarks 3.4.1 (i)). The main result of this chapter is the following:

Theorem 4.1.2. *Let Ω be an exterior domain in $\mathbf{R}^n$, $n \geq 2$, or a perturbed half space in $\mathbf{R}^n$, $n \geq 3$, with C^3-boundary.*
(i) (Unique existence of L^∞-solutions)
For $v_0 \in L^\infty_\sigma(\Omega)$, there exists a unique L^∞-solution $(v, \nabla q)$ satisfying (4.1.5) for any fixed T_0 with some constant C depending only on T_0 and Ω.
(ii) (Analyticity on L^∞_σ)
The Stokes semigroup $S(t)$ is uniquely extendable to a (non-C_0-)analytic semigroup on $L^\infty_\sigma(\Omega)$.

The L^∞-estimate (4.1.5) implies the analyticity of the Stokes semigroup $S(t)$ on $L^\infty_\sigma(\Omega)$. We call a semigroup $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ *analytic* if $t\|\mathrm{d}T(t)/\mathrm{d}t\|_{\mathcal{L}}$ is bounded for $t \in (0,1]$. Here, $\mathcal{L}= \mathcal{L}(X)$ denotes the space of all bounded linear operators from the Banach space X onto itself and is equipped with the operator norm $\|\cdot\|_{\mathcal{L}}$. Since $S(t)v_0 \to v_0$ as $t \downarrow 0$ on $L^\infty_\sigma(\Omega)$ may not hold for general $v_0 \in L^\infty_\sigma(\Omega)$, we call $S(t)$ a non C_0-analytic semigroup. We refer to [27] (also [21]) for properties of analytic semigroups generated by non-densely defined sectorial operators on $L^\infty(\Omega)$. It is natural to restrict $S(t)$ to the space of uniformly continuous functions $BUC_\sigma(\Omega)$ so that $S(t)$ is a C_0-analytic semigroup on $BUC_\sigma(\Omega)$. We discuss the continuity of $S(t)$ at time zero after we extend $S(t)$ to $L^\infty_\sigma(\Omega)$ (see Remark 4.1.3 (ii) below).

Remarks 4.1.3. (i) The statement of Theorem 4.1.2 is valid also for bounded domains with C^3-boundaries. In fact, we prove Theorem 4.1.2 for bounded domains together with exterior domains and perturbed half spaces, which deduces Theorem 4.1.1.
(ii) Let $BUC(\Omega)$ be the space of all uniformly continuous functions in $\bar{\Omega}$. Define the space $BUC_\sigma(\Omega)$ by

$$BUC_\sigma(\Omega) = \left\{ f \in BUC(\Omega) \,\middle|\, \operatorname{div} f = 0 \text{ in } \Omega,\ f = 0 \text{ on } \partial\Omega \right\}.$$

Then, $S(t)$ is a C_0-(analytic) semigroup on $BUC_\sigma(\Omega)$ for exterior domains Ω. We prove the continuity of $S(t)$ at $t = 0$ in Section 4 (Theorem 4.4.2). Note that $C_{0,\sigma}(\Omega) \subset BUC_\sigma(\Omega) \subset L^\infty_\sigma(\Omega)$. When Ω is bounded, the space $BUC_\sigma(\Omega)$ agrees with $C_{0,\sigma}(\Omega)$ [22], [1, Lemma 6.3] so we already know $S(t)$ is a C_0-analytic semigroup on $BUC_\sigma(\Omega)$ by Theorem 3.1.1 in Chapter 3.

It is well known that the Stokes semigroup $S(t)$ is a bounded analytic semigroup on $L^r_\sigma(\Omega)$ for exterior domains [7], [20], [8] in the sense that both $\|S(t)\|_{\mathcal{L}}$ and $t\|\mathrm{d}S(t)/\mathrm{d}t\|_{\mathcal{L}}$ are bounded in $(0,\infty)$, where $X = L^r_\sigma(\Omega)$ for $r \in (1,\infty)$. Recently, P. Maremonti [24] proved that $S(t)$ is a bounded semigroup on $L^\infty_\sigma(\Omega)$ for exterior domains based on our a priori L^∞-estimate (4.1.5). Note that it is unknown whether $t\|\mathrm{d}S(t)/\mathrm{d}t\|_{\mathcal{L}}$ is bounded in $(0,\infty)$.

The analyticity of the Stokes semigroup on L^∞ as well as (4.1.5) is fundamental to study the Navier–Stokes equations for non-decaying initial data on exterior domains. Although one can handle non-decaying Hölder initial data by reducing the initial problem to the boundary-value problem to the Navier–Stokes equations [16], a direct semigroup approach on $L^\infty_\sigma(\Omega)$ is still unknown. So far L^∞-type theory is only established when $\Omega = \mathbf{R}^n$ [18] (see also [19], [25]) and $\mathbf{R}^n_+$ [28], [5]. The analyticity of the Stokes semigroup on $L^\infty_\sigma(\Omega)$ is proved in [1] for bounded domains and in [2] for exterior domains. We extend the results for perturbed half space for $n \geq 3$, where the strictly admissibility is proved in Chapter 2.

This chapter is organized as follows. In Section 2, we prove the existence of L^∞-solutions for $v_0 \in L^\infty_\sigma(\Omega)$ by admitting the approximation (4.1.7). The proof of Theorem 4.1.2 (and also Theorem 4.1.1) is complete in Section 2. In Section 3, we prove the approximation (4.1.7) fir bounded domains, exterior domains and perturbed half spaces. In Section 4, we show that $S(t)$ is a C_0-semigroup on $BUC_\sigma(\Omega)$ for exterior domains.

4.2 Existence of L^∞-solutions

The goal of this section is to prove Theorem 4.1.2 (and Theorem 4.1.1). Since the approximation for initial data (4.1.7) is pointwise convergence in Ω, for the compactness of an approximate solution sequence, we apply the local Hölder estimates for the Stokes equations (Proposition 3.3.2 and Theorem 3.3.4 proved in Chapter 3). The proof for the approximation (4.1.7) is given in the next section.

Before starting to prove Theorem 4.1.2, we prepare the following Proposition 4.2.1. The local Hölder estimates (Proposition 3.3.2 and Theorem 3.3.4) imply that a limit of approximate solutions for $v_0 \in L^\infty_\sigma(\Omega)$ is (Hölder) continuous in $\overline{\Omega} \times (0,T]$. In order to prove that a limit solution converges to initial data weakly-$*$ on $L^\infty(\Omega)$ as $t \downarrow 0$, we understand initial data in terms of a weak form.

Proposition 4.2.1. *Let Ω be a domain in $\mathbf{R}^n$, $n \geq 2$ with $\partial\Omega \neq \emptyset$. Let $(v, \nabla q) \in C^{2,1}(\Omega \times (0,T]) \times C(\Omega \times (0,T])$ satisfy (4.1.1) and $\sup_{0<t\leq T}\{\|v\|_\infty(t) + t^{1/2}|\nabla q|_{\infty,d}(t)\} < \infty$. If $(v, \nabla q)$ satisfies*

$$\int_0^T \int_\Omega \{v \cdot (\varphi_t + \Delta\varphi) - \nabla q \cdot \varphi\}\, dxdt = -\int_\Omega v_0(x) \cdot \varphi(x,0) dx$$

for $v_0 \in L^\infty_\sigma(\Omega)$ and all $\varphi \in C^\infty_c(\Omega \times [0,T))$, then $v \to v_0$ weakly-$*$ on $L^\infty(\Omega)$ as $t \downarrow 0$. The converse also holds.

Proof. Since $(v, \nabla q)$ satisfies (4.1.1), by integration by parts it follows that

$$\int_\varepsilon^T \int_\Omega \{v \cdot (\varphi_t + \Delta\varphi) - \nabla q \cdot \varphi\}\, \mathrm{d}x\mathrm{d}t = -\int_\Omega v(x,\varepsilon) \cdot \varphi(x,\varepsilon)\mathrm{d}x$$

for all $\varphi \in C^\infty_c(\Omega \times [0,T))$ and $\varepsilon > 0$. By letting $\varepsilon \downarrow 0$, it follows that $\int_\Omega v(x,\varepsilon)\cdot\varphi(x,\varepsilon)\mathrm{d}x \to \int_\Omega v_0(x) \cdot \varphi(x,0)\mathrm{d}x$. Thus, $\int_\Omega v \cdot \psi \mathrm{d}x \to \int_\Omega v_0 \cdot \psi \mathrm{d}x$ as $t \downarrow 0$ for $\psi \in C^\infty_c(\Omega)$. Since $C^\infty_c(\Omega)$ is dense in $L^1(\Omega)$, $v \to v_0$ weakly-$*$ on $L^\infty(\Omega)$ as $t \downarrow 0$. The converse also holds. □

We now prove the existence of L^∞-solutions for $v_0 \in L^\infty_\sigma(\Omega)$. In order to apply (4.1.5) for L^r-solutions, we recall that an L^r-solution for $r > n$ is an L^∞-solution for sufficiently smooth and decaying initial data (Proposition 3.4.3).

Proof of Theorems 4.1.1 and 4.1.2. Let Ω be a bounded domain, an exterior domain or a perturbed half space ($n \geq 3$) with C^3-boundary. We prove the unique existence of L^∞-solutions for $v_0 \in L^\infty_\sigma(\Omega)$. We apply the approximation lemma (Lemmas 4.3.3, 4.3.5, and

4.3.10) to get a compactly supported smooth sequence $\{v_{0,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega)$ such that

$$\begin{aligned} &\|v_{0,m}\|_{L^\infty(\Omega)} \le C_\Omega \|v_0\|_{L^\infty(\Omega)}, \\ &v_{0,m} \to v_0 \quad \text{a.e. in } \Omega \text{ as } m \to \infty, \end{aligned} \tag{4.2.1}$$

with the constant C_Ω independent of $m \ge 1$. Let $(v_m, \nabla q_m)$ be an L^r-solution for $r > n$ and $v_{0,m}$. Let $T > 0$ be an arbitrary fixed time. By Remark 4.3.1 (ii), the L^∞-estimate (4.1.5) holds in (0,T). By integration by parts, it follows that

$$\int_0^T \int_\Omega \{v_m \cdot (\varphi_t + \Delta\varphi) - \nabla q_m \cdot \varphi\}\, \mathrm{d}x\mathrm{d}t = -\int_\Omega v_{0,m}(x) \cdot \varphi(x,0)\mathrm{d}x$$

for all $\varphi \in C^\infty_c(\Omega \times [0,T))$. Since $v_{0,m} \in D(A_r) \cap D(A_2)$, by Proposition 3.4.3, $(v_m, \nabla q_m)$ is an L^∞-solution. By Theorem 2.3.3, bounded domains, exterior domains and perturbed half spaces ($n \ge 3$) with C^3-boundaries are strictly admissible. We apply Lemma 3.3.2 to estimate

$$|\nabla q_m|_{\infty,d}(t) \le C \|W(v)\|_{L^\infty(\partial\Omega)}(t). \tag{4.2.2}$$

Combining the estimate (4.2.2), (4.1.5) and (4.2.1), it follows that

$$\sup_{0<t\le T} \{\|N(v_m, q_m)\|_{L^\infty(\Omega)}(t) + t^{1/2}|\nabla q_m|_{\infty,d}(t)\} \le C\|v_0\|_{L^\infty(\Omega)},$$

with the constant C independent of $m \ge 1$. We apply Proposition 3.3.2 and Theorem 3.3.4 to get a uniform local Hölder bound for $(v_m, \nabla q_m)$ in $\overline{\Omega} \times (0,T]$. Then, $(v_m, \nabla q_m)$ subsequently converges to a limit $(v, \nabla q)$ locally uniformly in $\bar{\Omega} \times (0,T]$ together with v_m, ∇v_m, $\nabla^2 v_m$, $\partial_t v_m$, ∇q_m. By letting $m \to \infty$, the limit $(v, \nabla q)$ satisfies

$$\int_0^T \int_\Omega \{v \cdot (\varphi_t + \Delta\varphi) - \nabla q \cdot \varphi\}\, \mathrm{d}x\mathrm{d}t = -\int_\Omega v_0(x) \cdot \varphi(x,0)\mathrm{d}x.$$

Since $v \to v_0$ weakly-$*$ on $L^\infty(\Omega)$ as $t \downarrow 0$ by Proposition 4.2.1, $(v, \nabla q)$ is an L^∞-solution for $v_0 \in L^\infty_\sigma(\Omega)$. By Remarks 3.4.1 (i), the L^∞-solution $(v, \nabla q)$ is unique. Since $T > 0$ is an arbitrary fixed time, for $v_0 \in L^\infty_\sigma(\Omega)$, a unique L^∞-solution $(v, \nabla q)$ exists in $\Omega \times (0, \infty)$. Thus, we proved the assertion (i).

We next extend $S(t)$ to $L^\infty_\sigma(\Omega)$ by the limit v for $v_0 \in L^\infty_\sigma(\Omega)$. We define $S(t)v_0 = v(\cdot, t)$ for $t > 0$ and $S(0) = I$. We shall show the semigroup property for $S(t)$ on $L^\infty_\sigma(\Omega)$, i.e., $S(t+s) = S(t)S(s)$ for $t, s \ge 0$. Since $S(0) = I$, we may assume $s > 0$. Let (v^1, q^1) be an L^∞-solution for $v_0 \in L^\infty_\sigma(\Omega)$. For each fixed $s > 0$, let $(v^2, \nabla q^2)$ be an L^∞-solution for initial data $v^1(\cdot, s)$. Then, by the uniqueness of L^∞-solutions, $(v^1, \nabla q^1) \equiv (v^2, \nabla q^2)$ for $t \ge s$. Thus, $S(t)v_0 = S(t-s)S(s)v_0$, $t \ge s$. By substituting $t = \tau + s$, $S(\tau + s) = S(\tau)S(s)$ for $\tau, s > 0$ follows so $S(t)$ satisfies the semigroup property on $L^\infty_\sigma(\Omega)$. The analyticity of $S(t)$ on $L^\infty_\sigma(\Omega)$ follows from (4.1.5). Thus, $S(t)$ is an analytic semigroup on $L^\infty_\sigma(\Omega)$.

This semigroup $S(t)$ is a non-C_0-semigroup on $L^\infty_\sigma(\Omega)$. Indeed, suppose the contrary to get

$$S(t)v_0 \to v_0 \quad \text{in} \quad L^\infty(\Omega) \quad \text{as } t \downarrow 0$$

for all $v_0 \in L^\infty_\sigma(\Omega)$. Our estimate for $\nabla^2 v$ implies that $S(t)v_0$ $(t > 0)$ is at least continuous in $\bar{\Omega}$. However, if $S(t)v_0$ converges uniformly, then v_0 must be continuous which is a contradiction. We have proved the assertion (ii). The proof is now complete. □

Remark 4.2.2. If the approximation (4.1.7) is known to hold, then we are able to prove the existence of L^∞-solutions for general strictly admissible domains by the same way. In order to approximate solutions in a uniformly regular domain, we appeal to $\tilde{L}^r$-theory [11], [12] although L^r-theory works for more general domains [13], [3], [14], [17].

4.3 Approximation for initial data

In this section, we prove the approximation (4.1.7) for bounded domains, exterior domains and perturbed half spaces. We first prove the approximation (4.1.7) for a bounded domain. We decompose a bounded domain and reduce the problem to star-shaped domains. In a star-shaped domain, we rescale a function so that whose support is compact in Ω and mollify it to get a compactly supported smooth sequence. By using the result for a bounded domain and the whole space (or a half space), we prove the approximation (4.1.7) for exterior domains and perturbed half spaces.

4.3.1 Reduction to star-shaped domains

We begin with an approximation result when Ω is star-shaped (with respect to some point $a \in \mathbf{R}^n$, i.e. $\lambda(\Omega - a) \subset \Omega - a$ for all $\lambda \in (0, 1)$).

Lemma 4.3.1 (Approximation)**.** *Let Ω be a bounded, star-shaped domain in $\mathbf{R}^n$. There exists a constant $C = C_\Omega$ such that for any $v \in L^\infty_\sigma(\Omega)$, there exists a sequence $\{v_m\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega)$ such that*

$$\|v_m\|_{L^\infty(\Omega)} \leq C\|v\|_{L^\infty(\Omega)}, \tag{4.3.1}$$

$$v_m \to v \quad \text{a.e. in } \Omega \quad \textit{as } m \to \infty. \tag{4.3.2}$$

If in addition $v \in C(\bar{\Omega})$, the convergence is locally uniform in Ω. If in addition $v = 0$ on $\partial\Omega$, the convergence is uniform in $\bar{\Omega}$.

Proof. Since Ω is star-shaped, we may assume that

$$\lambda\bar{\Omega} \subset \Omega \quad \text{for all} \quad \lambda \in [0, 1)$$

by a translation. We extend that $v \in L^\infty_\sigma(\Omega)$ by zero outside Ω and observe that the extension (still denoted by v) is in $L^\infty_\sigma(\mathbf{R}^n)$ with spt $v \subset \bar{\Omega}$. We set $v_\lambda(x) = v(x/\lambda)$ and observe that spt $v_\lambda \subset \lambda\bar{\Omega} \subset \Omega$. Since $v_\lambda \to v$ a.e. as $\lambda \uparrow 1$, it is easy to find the desired sequence by mollifying v_λ, i.e., $v_\lambda * \eta_\varepsilon$. Here, C in (4.3.1) can be taken 1. □

To establish the above approximation result for general bounded domains, we need a localization lemma.

Lemma 4.3.2 (Localization). *Let Ω be a bounded domain with Lipschitz boundary in $\mathbf{R}^n$. Let $\{G_k\}_{k=1}^N$ be an open covering of $\bar{\Omega}$ in $\mathbf{R}^n$ and $\Omega_k = G_k \cap \Omega$. Then, there exists a family of bounded linear operators $\{\pi_k\}_{k=1}^N$ from $L^\infty_\sigma(\Omega)$ into itself satisfying $u = \sum_{k=1}^N \pi_k u$ and for each $k = 1, \ldots, N$*

(i) $\pi_k u|_{\Omega_k} \in L^\infty_\sigma(\Omega_k)$, $\pi_k u|_{\Omega\backslash\Omega_k} = 0$ *for* $u \in L^\infty_\sigma(\Omega)$,

(ii) $\pi_k u \in C(\bar{\Omega}_k)$ *and* $\pi_k u|_{\partial\Omega_k\backslash\partial\Omega} = 0$ *for* $u \in C(\bar{\Omega}) \cap L^\infty_\sigma(\Omega)$,

(iii) $\pi_k u|_{\partial\Omega_k} = 0$ *if* $u|_{\partial\Omega} = 0$ *for* $u \in C(\bar{\Omega}) \cap L^\infty_\sigma(\Omega)$.

Proof. We shall prove by induction with respect to N. If $N = 1$, the result is trivial by taking π_1 as the identity.

Assume that the result is valid for N. We shall prove the assertion when the number of cover is $N + 1$. We set

$$D = \bigcup_{k=2}^{N+1} \Omega_k, \quad U = \bigcup_{k=2}^{N+1} G_k$$

and observe that $\Omega = \Omega_1 \cup D$ and $\{G_1, U\}$ is a covering of $\bar{\Omega}$.

Let $\{\xi_1, \xi_2\}$ be a partition of unity of Ω associated with $\{G, U\}$, i.e., $\xi_j \in C_c^\infty(\mathbf{R}^n)$ with $0 \le \xi_j \le 1$, spt $\xi_1 \subset G_1$, spt $\xi_2 \subset U$, $\xi_1 + \xi_2 = 1$ in $\bar{\Omega}$. For $E = \Omega_1 \cap D$, let B_E denotes the Bogovskiĭ operator. We set

$$\pi_1 u = \begin{cases} u\,\xi_1 - B_E(u \cdot \nabla\xi_1) & \text{in} \quad E, \\ u\,\xi_1 & \text{in} \quad \Omega_1\backslash D, \\ 0 & \text{in} \quad \Omega\backslash\Omega_1. \end{cases}$$

Since $u \in L^\infty_\sigma(\Omega)$ and $\xi_1 = 0$ in $\Omega\backslash\Omega_1$, $\nabla\xi_1 = 0$ in $\Omega_1\backslash D$, we see

$$\int_E u \cdot \nabla\xi_1 \mathrm{d}x = \int_\Omega u \cdot \nabla\xi_1 \mathrm{d}x = 0. \tag{4.3.3}$$

By the Sobolev inequality and (3.3.13), we observe that

$$\begin{aligned} \left\|B_E(u \cdot \nabla\xi_1)\right\|_{L^\infty(E)} &\le C\left\|B_E(u \cdot \nabla\xi_1)\right\|_{W^{1,p}(E)} \quad (p > n) \\ &\le CC_B\|u \cdot \nabla\xi_1\|_{L^p(E)} \le CC_B\|\nabla\xi_1\|_{L^p(E)}\|u\|_{L^\infty(\Omega)} \end{aligned}$$

with a constant C independent of u and ξ_1. We thus observe that

$$\|\pi_1 u\|_{L^\infty(\Omega_1)} \le C_1\|u\|_{L^\infty(\Omega)} \text{ for all } u \in L^\infty_\sigma(\Omega)$$

with C_1 independent of u.

By (4.3.3), we see $\operatorname{div} B_E(u\cdot\nabla\xi_1) = u\cdot\nabla\xi_1$ in E. Moreover, $B_E(u\cdot\nabla\xi_1) = 0$ on $\partial(\Omega_1\cap D)$. Thus for each $\varphi \in L^1_{loc}(\bar{\Omega}_1)$ with $\nabla\varphi \in L^1(\Omega_1)$, we have

$$\begin{aligned}\int_{\Omega_1} \pi_1 u\cdot\nabla\varphi \mathrm{d}x &= \int_{\Omega_1} u\,\xi_1\cdot\nabla\varphi\mathrm{d}x - \int_E B_E(u\cdot\nabla\xi_1)\cdot\nabla\varphi\mathrm{d}x\\ &= \int_{\Omega_1} u\,\xi_1\cdot\nabla\varphi\mathrm{d}x + \int_E (u\cdot\nabla\xi_1)\varphi\mathrm{d}x\\ &= \int_\Omega u\cdot\nabla(\xi_1\varphi)\mathrm{d}x = 0.\end{aligned}$$

By the Poincaré inequality if $\varphi \in \hat{W}^{1,1}(\Omega_1)$, then $\varphi \in L^1_{loc}(\bar{\Omega}_1)$ (not only $\varphi \in L^1_{loc}(\Omega_1)$). Thus, the above identity implies that $\pi_1 u|_{\Omega_1} \in L^\infty_\sigma(\Omega_1)$. By definition $\pi_1 u = 0$ in $\Omega\backslash\Omega_1$. If $u \in C(\bar{\Omega})$, it is easy to see that the term $B_E(u\cdot\nabla\xi_1)$ is always Hölder continuous by the Sobolev embeddings.

For $u \in L^\infty_\sigma(\Omega)$, we set

$$\pi_D u = \begin{cases} u\,\xi_2 - B_E(u\cdot\nabla\xi_2) & \text{in} \quad E,\\ u\,\xi_2 & \text{in} \quad D\backslash\Omega_1,\\ 0 & \text{in} \quad \Omega\backslash D.\end{cases}$$

By definition,

$$u = \pi_1 u + \pi_D u$$

and, as for π_1, this π_D satisfies all properties of π_k in (i), (ii), (iii) with Ω_k replaced by D. Since $\bar{D}$ is covered by $\{G_k\}_{k=2}^{N+1}$, by our induction assumption, there is a bounded linear operator $\{\hat{\pi}_k\}_{k=2}^{N+2}$ in $L^\infty_\sigma(D)$ satisfying $v = \sum_{k=2}^{N+1}\hat{\pi}_k v$ and (i), (ii), (iii) with u replaced by v and with π_k replaced by $\hat{\pi}_k$ for $k = 2,\dots,N+1$. If we set

$$\pi_1 = \pi_1,\ \pi_k = \hat{\pi}_k\cdot\pi_D \quad (k = 2,\dots,N+1),$$

then it is rather clear that this π_k satisfies all desired properties. □

Lemma 4.3.3 (Approximation)**.** *The assertion of Lemma 4.3.1 is still valid when Ω is a bounded domain with Lipschitz boundary in $\mathbf{R}^n$.*

Proof. If Ω is a bounded domain with Lipschitz boundary, then it is known that there is an open covering $\{G_k\}_{k=1}^N$ of $\bar{\Omega}$ such that $\Omega_k = G_k\cap\Omega$ is bounded, star-shaped with respect to an open ball $B_k(\bar{B}_k \subset \Omega)$ (i.e. star-shaped with respect to any point of B_k) and G_k has a Lipschitz boundary; see [15, III.3, Lemma 4.3]. In the sequel, we only need the property that G_k is bounded and star-shaped with respect to a point.

We apply Lemma 4.3.2 and set $u_k = \pi_k u$ to observe that $u_k|_{\Omega_k} \in L^\infty_\sigma(\Omega_k)$ and $u_k|_{\Omega\backslash\Omega_k} = 0$. Since Ω_k is star-shaped, by Lemma 4.3.1 there is $\{u_{k,j}\}_{j=1}^\infty \subset C^\infty_{c,\sigma}(\Omega_k)$ such that

$$\|u_{k,j}\|_{L^\infty(\Omega_k)} \le \|u_k\|_{L^\infty(\Omega_k)},\ u_{k,j} \to u_k \text{ a.e. in } \Omega.$$

(The constant C in (4.3.1) can be taken 1). We still denote the zero extension of $u_{k,j}$ on $\Omega\backslash\Omega_k$ by $u_{k,j}$.

If we set $u_m = \sum_{k=1}^N u_{k,m}$, by construction, $u_j \in C^\infty_{c,\sigma}(\Omega)$ and

$$u_m \to \sum_{k=1}^N u_k = u \quad \text{a.e. in } \Omega \quad \text{and,}$$

$$\|u_m\|_{L^\infty(\Omega)} \leq \sum_{k=1}^N \|u_{k,m}\|_{L^\infty(\Omega)} \leq \sum_{k=1}^N \|u_k\|_{L^\infty(\Omega)} \leq \Big(\sum_{k=1}^N \|\pi_k\|\Big)\|u\|_{L^\infty(\Omega)},$$

where $\|\pi_k\|$ denotes the operator norm of π_k in $L^\infty_\sigma(\Omega)$. We thus conclude that there is a desired approximate sequence $\{u_m\}_{m=1}^\infty$ for $u \in L^\infty_\sigma(\Omega)$.

If $u \in C(\bar{\Omega})$ $(\cap L^\infty_\sigma(\Omega))$, then $u_k \in C(\bar{\Omega}_k)$ and $u_k|_{\partial\Omega_k\backslash\partial\Omega} = 0$. Thus, for any compact set $K_k \subset \Omega_k$ such that $d_\Omega(K_k) = \inf_{x\in K_k} d_\Omega(x) > 0$, we see that $u_{k,m}$ converges to u_k uniformly in K_k by Lemma 4.3.1 as $m \to \infty$. Let K be a compact set in Ω. Then, $d(K_k) \geq d(K) > 0$ for $K_k = \bar{\Omega}_k \cap K$. Thus,

$$\begin{aligned}\|u - u_m\|_{L^\infty(K)} &\leq \sum_{k=1}^N \|u_k - u_{k,m}\|_{L^\infty(K)} \\ &= \sum_{k=1}^N \|u_k - u_{k,m}\|_{L^\infty(K_k)} \to 0 \text{ (as } m \to \infty).\end{aligned}$$

Thus, we have proved that u_m converges to u locally uniformly in Ω. If $u|_{\partial\Omega} = 0$ so that $u_k|_{\partial\Omega_k} = 0$, then $u_{k,m}$ converges to u_k uniformly in $\bar{\Omega}_k$ by Lemma 4.3.1. Arguing in the same way by replacing K by $\bar{\Omega}$, we conclude that u_m converges to u uniformly in $\bar{\Omega}$. □

Remarks 4.3.4. (i) This lemma in particular implies that

$$C_{0,\sigma}(\Omega) = \{v \in C(\bar{\Omega}) \mid \operatorname{div} v = 0 \text{ in } \Omega,\ v = 0 \text{ on } \partial\Omega\}$$

when Ω is bounded. This give an alternate and direct proof of a result of [22], where the maximum modulus result for the stationary problem is invoked.
(ii) For bounded domains, we are able to characterize the space $L^\infty_\sigma(\Omega)$ by

$$L^\infty_\sigma(\Omega) = \{v \in L^\infty(\Omega) \mid \operatorname{div} v = 0 \text{ in } \Omega,\ v \cdot n_\Omega = 0 \text{ on } \partial\Omega\}.$$

For $v \in L^\infty(\Omega) \subset L^r(\Omega)$ satisfying $\operatorname{div} v \in L^r(\Omega)$, $r \in (1, \infty)$, we understand the normal component $v \cdot n_\Omega$ as an element of the negative order Sobolev space; see, e.g., [26].

4.3.2 Approximation for $|x| \to \infty$

We next prove the approximation lemma for $L^\infty_\sigma(\Omega)$ for exterior domains. Recently, an approximate sequence for $L^\infty_\sigma(\Omega)$ is constructed in [24, Lemma 2.6] by smooth solenoidal vector fields in an exterior domain. Although a construction procedure is similar, we give an approximation by compactly supported functions in order to prove the existence of L^∞-solutions by applying the a priori estimate (4.1.5) for L^r-solutions.

Lemma 4.3.5 (Approximation in an exterior domain)**.** *Let Ω be an exterior domain with Lipschitz boundary. There exists a constant $C = C_\Omega$ such that for any $v \in L^\infty_\sigma(\Omega)$, there exist a sequence $\{v_m\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega)$ such that*

$$\|v_m\|_{L^\infty(\Omega)} \leq C\|v\|_{L^\infty(\Omega)}, \tag{4.3.4}$$

$$v_m \to v \quad \text{a.e. in } \Omega \quad \text{as } m \to \infty. \tag{4.3.5}$$

If in addition $v \in C(\bar{\Omega})$ and $v = 0$ on $\partial\Omega$, the above convergence can be replaced by locally uniform convergence in $\bar{\Omega}$. If in addition $v(x)$ vanishes at the space infinity, i.e., $\lim_{|x|\to\infty} v(x) = 0$, the convergence can be replaced by uniform convergence in $\bar{\Omega}$. In particular, $C_{0,\sigma}(\Omega)$ agrees with the space $\{v \in C(\bar{\Omega}) \mid \lim_{|x|\to\infty} v(x) = 0,\ \operatorname{div} v = 0 \text{ in } \Omega, v = 0 \text{ on } \partial\Omega\}$.

In order to prove Lemma 4.3.5, we recall the Bogovskiĭ operator [6], [15]. Let D be a bounded domain with Lipschitz boundary. The Bogovskiĭ operator B_D is a bounded operator from $L^r_{\mathrm{av}}(D)$ to the Sobolev space $W^{1,r}(D)$ for $r \in (1,\infty)$ such that div $B_D(g) = g$ in D, $B_D(g) = 0$ on ∂D and

$$\|B_D(g)\|_{W^{1,r}(D)} \leq C_D\|g\|_{L^r(D)} \tag{4.3.6}$$

for $g \in L^p_{\mathrm{av}}(D)$, where $L^r_{\mathrm{av}}(D)$ denotes the space of all average-zero functions in $L^r(D)$. The constant C_D depends on Lipschitz regularity of ∂D and is independent of g.

We first prove Lemma 4.3.5 for $\Omega = \mathbf{R}^n$.

Proposition 4.3.6. *The statement of Lemma 4.3.5 holds for $\Omega = \mathbf{R}^n$. If in addition $v \in C(\mathbf{R}^n)$, the convergence in (4.3.5) can be replaced by locally uniform convergence in $\mathbf{R}^n$. If in addition* $\lim_{|x|\to\infty} v(x) = 0$*, the convergence can be replaced by uniform convergence in $\mathbf{R}^n$.*

Proof. Let θ be a smooth cutoff function in $[0,\infty)$ satisfying $\theta \equiv 1$ in $[0,1/2]$, and $\theta \equiv 0$ in $[1,\infty)$. Set $\theta_m(x) = \theta(|x|/m)$ for $x \in \mathbf{R}^n$ and $m \geq 1$. Then, $\theta_m \equiv 1$ in $B_0(m/2)$ and $\theta_m \equiv 0$ in $B_0(m)^c$. For $v \in L^\infty_\sigma(\mathbf{R}^n)$, set $g_m = v \cdot \nabla\theta_m$. Then, $g_m \in L^r_{\mathrm{av}}(D_m)$ and spt $g_m \subset \overline{D_m}$ for $D_m = B_0(m)\backslash\overline{B_0(m/2)}$. Set $f_m(x) = g_m(mx)$ for $x \in D_1$ and apply the Bogovskiĭ operator for $f_m \in L^r_{\mathrm{av}}(D_1)$ to get $u^*_m = B_{D_1}(f_m)$ satisfying div $u^*_m = f_m$ in D_1, $u^*_m = 0$ on ∂D_1 and $\|u^*_m\|_{W^{1,r}(D_1)} \leq C_{D_1}\|f_m\|_{L^r(D_1)}$, where the constant C_{D_1} is independent of $m \geq 1$. By the Sobolev inequality, it follows that

$$\|u^*_m\|_{L^\infty(D_1)} \leq C_s\|f_m\|_{L^r(D_1)}.$$

for $r > n$ with the constant C_s, independent of $m \geq 1$. We set

$$v^*_m(x) = mu^*_m(x/m) \quad \text{for } x \in D_m.$$

Then, div $v^*_m = g_m$ in D_m, $v^*_m = 0$ on ∂D_m and $\|v^*_m\|_{L^\infty(D_1)} \leq mC_s\|f_m\|_{L^r(D_1)}$. Since $\|f_m\|_{L^r(D_1)} = m^{-n/r}\|g_m\|_{L^r(D_m)}$ and $\|\nabla\theta_m\|_{L^r(D_m)} = m^{n/r-1}\|\nabla\theta\|_{L^r(D_1)}$, it follows that

$$\begin{aligned}\|v^*_m\|_{L^\infty(D_m)} &\leq C_s m^{1-n/r}\|g_m\|_{L^r(D_m)} \\ &\leq C_s\|\nabla\theta\|_{L^r(D_1)}\|v\|_{L^\infty(D_m)}.\end{aligned}$$

Denoting the zero extension of v^*_m to $\mathbf{R}^n\backslash\overline{D}_m$ by $\bar{v}^*_m$, we set $\tilde{v}_m = v\theta_m - \bar{v}^*_m$. Then, $\tilde{v}_m \in L^\infty_\sigma(\mathbf{R}^n)$ and spt $\tilde{v}_m$ is compact in $\mathbf{R}^n$. By the standard mollifier η_ε, $\varepsilon > 0$, set $v_m = \tilde{v}_m * \eta_{1/m}$. Then, $v_m \in C^\infty_{c,\sigma}(\mathbf{R}^n)$ is desired sequence.
If $v \in C(\mathbf{R}^n)$, $v_m \to v$ locally uniformly in $\mathbf{R}^n$ as $m \to \infty$. If in addition $\lim_{|x|\to\infty} v(x) = 0$, $v_m \to v$ uniformly in $\mathbf{R}^n$ as $m \to \infty$. The proof is now complete. □

For sufficiently smooth $v \in L^\infty_\sigma(\mathbf{R}^n)$, Proposition 4.3.6 holds up to higher orders.

Corollary 4.3.7. *For $v \in W^{k,\infty}(\mathbf{R}^n) \cap L^\infty_\sigma(\mathbf{R}^n)$ and $k \geq 0$, (4.3.4) and (4.3.5) can be replaced to*

$$\|v_m\|_{W^{k,\infty}(\mathbf{R}^n)} \leq C\|v\|_{W^{k,\infty}(\mathbf{R}^n)}, \tag{4.3.7}$$

$$\partial^l_x v_m \to \partial^l_x v \quad \text{a.e. in } \mathbf{R}^n \text{ as } m \to \infty \text{ for } |l| \leq k. \tag{4.3.8}$$

Proof. We prove by induction with respect to k. For $k = 0$ the statement holds by Proposition 4.3.6. Assume that (4.3.7) and (4.3.8) hold for $k = k_0$. We shall show

$$\|\partial^l_x v_m\|_{L^\infty(\mathbf{R}^n)} \leq C\|v\|_{W^{k_0+1,\infty}(\mathbf{R}^n)}, \tag{4.3.9}$$

and (4.3.8) for $|l| = k_0 + 1$. Since the Bogovskiĭ operator is bounded from $W^{k_0+1,r}(D)$ to $W^{k_0+2,r}(D)$ [15], we have

$$\|u^*_m\|_{W^{k_0+2,r}(D_1)} \leq C\|f_m\|_{W^{k_0+1,r}(D_1)}.$$

By the Sobolev inequality, it follows that

$$\|\partial^l_x u^*_m\|_{L^\infty(D_1)} \leq C_s\|f_m\|_{W^{k_0+1,r}(D_1)}$$

for $r > n$ with the constant C_s independent of $m \geq 1$. Since $\|\partial^j_x v^*_m\|_{L^\infty(D_1)} = m^{1-|j|}\|\partial^j_x u^*_m\|_{L^\infty(D_m)}$ and $\|\partial^j_x f_m\|_{L^\infty(D_1)} = m^{|j|-n/r}\|\partial^j_x g_m\|_{L^\infty(D_m)}$ for $|j| \geq 0$, it follows that

$$\begin{aligned}\|\partial^l_x v^*_m\|_{L^\infty(D_m)} &\leq m^{1-|l|}\|f_m\|_{W^{k_0+1,r}(D_1)} \\ &\leq m^{1-n/r}\|g_m\|_{W^{k_0+1,r}(D_m)}.\end{aligned}$$

By $\|\nabla\theta_m\|_{L^\infty(D_m)} = m^{n/r-1}\|\nabla\theta\|_{L^r(D_1)}$, we estimate $\|g_m\|_{W^{k_0+1,r}(D_m)} \leq m^{n/r-1}C\|v\|_{W^{k_0+1,r}(D_m)}$ with the constant C independent of $m \geq 1$. Thus, we obtain

$$\|\partial_x^l v_m^*\|_{L^\infty(D_m)} \leq C\|v\|_{W^{k_0+1,\infty}(D_m)}.$$

Since $v_m = \tilde{v}_m * \eta_{1/m}$ and $\tilde{v}_m = v_m\theta_m - \bar{v}_m^*$, (4.3.9) and (4.3.8) hold for $k = k_0 + 1$. We proved (4.3.7) and (4.3.8) for all $k \geq 0$. The proof is now complete. □

Remark 4.3.8. The proof of Proposition 4.3.6 works for $v \in L^\infty(\mathbf{R}^n)$ satisfying div $v = 0$ in $\mathbf{R}^n$, i.e., there exists a sequence $\{v_{0,m}\}_{m=1}^\infty \subset C_{c,\sigma}^\infty(\mathbf{R}^n)$ satisfying (4.3.4) and (4.3.5). By the dominated convergence theorem, we have

$$0 = \lim_{m\to\infty} \int_{\mathbf{R}^n} v_m \cdot \nabla\varphi \mathrm{d}x = \int_{\mathbf{R}^n} v \cdot \nabla\varphi \mathrm{d}x \quad \text{for } \varphi \in \hat{\mathrm{W}}^{1,1}(\mathbf{R}^n)$$

so $v \in L_\sigma^\infty(\mathbf{R}^n)$. This implies $L_\sigma^\infty(\mathbf{R}^n) = \{v \in L^\infty(\mathbf{R}^n) \mid \text{div } v = 0 \text{ in } \mathbf{R}^n\}$.

We prove Lemma 4.3.5 for an exterior domain. By using the Bogovskiĭ operator, we divide a solenoidal vector field into two vector fields – one is compactly supported in Ω and the other is supported in $\mathbf{R}^n$ away from $\partial\Omega$. We reduce our problem to the case of $\mathbf{R}^n$ (Proposition 4.3.6) and a bounded domain. For a bounded domain, we already constructed the corresponding approximate sequence (Lemma 4.3.3).

Proof of Lemma 4.3.5. We may assume $0 \in \Omega^c$. Let θ be a smooth cutoff function in $[0,\infty)$ satisfying $\theta \equiv 1$ in $[0, 1/2]$ and $\theta \equiv 0$ in $[1,\infty)$. Set $\theta_R(x) = \theta(|x|/R)$ for $x \in \mathbf{R}^n$ and $R > \operatorname{diam} \Omega^c$. Then, $\theta_R \equiv 1$ in $B_0(R/2)$, $\theta_R \equiv 0$ in $B_0(R)^c$ and spt $\nabla\theta_R \subset \overline{D_R}$ for $D_R = B_0(R)\backslash\overline{B_0(R/2)}$. For $v \in L_\sigma^\infty(\Omega)$ set $g_R = v \cdot \nabla\theta_R$. Then, $g_R \in L_{\mathrm{av}}^r(D_R)$. We apply the Bogovskiĭ operator for $g_R \in L_{\mathrm{av}}^r(D_R)$ to get $v_R^* = B_{D_R}(g_R)$ such that div $v_R^* = g_R$ in D_R, $v_R^* = 0$ on ∂D_R and $\|v_R^*\|_{W^{1,r}(D_R)} \leq C_{D_R}\|g_R\|_{L^r(D_R)}$. By the Sobolev inequality, we estimate $\|v_R^*\|_{L^\infty(D_R)} \leq C_s\|g_R\|_{L^r(D_R)} \leq C_s\|\nabla\theta_R\|_{L^r(D_R)}\|v\|_{L^\infty(\Omega)}$ for $r > n$ with the constant C_s independent of v. Denoting the zero extension of v_R^* to $\mathbf{R}^n\backslash D_R$ by $\bar{v}_R^*$, we set

$$\begin{aligned} v_1 &= v\theta_R - \bar{v}_R^*, \\ v_2 &= v(1-\theta_R) + \bar{v}_R^*. \end{aligned} \tag{4.3.10}$$

Then, v_1 and v_2 are estimated by v, i.e.,

$$\|v_i\|_{L^\infty(\Omega)} \leq C\|v\|_{L^\infty(\Omega)} \quad \text{for} \quad i = 1, 2$$

with the constant C independent of v.

We find an approximation for v_1. Since v_1 satisfies div $v_1 = 0$ in Ω, $v_1 \cdot n_\Omega = 0$ on $\partial\Omega$, and spt $v_1 \subset \overline{\Omega_R}$ for $\Omega_R = B_0(R) \cap \Omega$, it follows that $v_1 \in L_\sigma^\infty(\Omega_R)$. We apply the approximation lemma for a bounded domain (Lemma 4.3.3) to get $v_{1,m} \in C_{c,\sigma}^\infty(\Omega_R)$ such that $\|v_{1,m}\|_{L^\infty(\Omega_R)} \leq C_R\|v_1\|_{L^\infty(\Omega_R)}$ and $v_{1,m} \to v_1$ a.e. in Ω as $m \to \infty$. The constant C_R is independent of $m \geq 1$. We do not distinguish $v_{1,m}$ and its zero extension to $\Omega\backslash B_0(R)$.

We next find an approximation for v_2. Let $\overline{v}_2$ be a zero extension of v^2 to $\mathbf{R}^n\backslash\overline{\Omega}$. Since v_2 satisfies div $v_2 = 0$ in Ω and spt $v_2 \subset \mathbf{R}^n\backslash B_0(R/2)$, $\overline{v}_2$ is in $L^\infty_\sigma(\mathbf{R}^n)$. We apply Proposition 4.3.6 to get a sequence of functions $\{\overline{v}_{2,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\mathbf{R}^n)$ such that $\|\overline{v}_{2,m}\|_{L^\infty(\mathbf{R}^n)} \leq C\|\overline{v}_2\|_{L^\infty(\mathbf{R}^n)}$ and $\overline{v}_{2,m} \to \overline{v}_2$ a.e. in $\mathbf{R}^n$ as $m \to \infty$. Since $\overline{v}_2 = 0$ in $B_0(R/2)$, by construction of $\overline{v}_{2,m}$, $\overline{v}_{2,m}$ also satisfies $\overline{v}_{2,m} = 0$ in $B_0(R/2)$. Then, the restriction of $\overline{v}_{2,m}$ to Ω denoted by $v_{2,m}$ is in $C^\infty_{c,\sigma}(\Omega)$.
We set $v_m = v_{1,m} + v_{2,m}$. Then, v_m satisfies (4.3.4) and (4.3.5). If $v \in C(\bar{\Omega})$ and $v = 0$ on $\partial\Omega$, $v_{1,m} \to v_1$ uniformly in $\overline{\Omega}$ and $v_{2,m} \to v_2$ locally uniformly in $\overline{\Omega}$ as $m \to \infty$. Thus, v_m converges to v locally uniformly in $\overline{\Omega}$. If in addition $\lim_{|x|\to\infty} v(x) = 0$, $v_{2,m} \to v_2$ uniformly in $\overline{\Omega}$ as $m \to \infty$. Thus, v_m converges to v uniformly in $\overline{\Omega}$. The proof is now complete. □

Remark 4.3.9. The characterization of $C_{0,\sigma}(\Omega)$ in Lemma 4.3.5 was proved in [22, Lemma 3.1] ([23, Lemma A.1]) for bounded and exterior domains Ω with $C^{1,\gamma}$-boundaries. The proof depends on the maximum modulus theorem of the stationary Stokes problem. Lemma 4.3.5 is a natural extension of that for bound domains (Lemma 4.3.3) and the proof is direct via the Bogovskiĭ operator without appealing the Stokes equations.

4.3.3 Approximation in a perturbed half space

We prove the approximation (4.1.7) for perturbed half spaces. The approach is essentially the same with that of exterior domains. The proof is reduced to the approximation (4.1.7) for a half space and a bounded domain (Lemma 4.3.3).

Lemma 4.3.10 (Approximation in a perturbed half space)**.** *Let Ω be a perturbed half space in $\mathbf{R}^n$, $n \geq 2$, with Lipschitz boundary. There exists a constant C such that for $v \in L^\infty_\sigma(\Omega)$ there exists a sequence $\{v_m\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega)$ such that*

$$\|v_m\|_{L^\infty(\Omega)} \leq C\|v\|_{L^\infty(\Omega)}, \tag{4.3.11}$$

$$v_m \to v \quad a.e.\ in\ \Omega \quad as\ m \to \infty. \tag{4.3.12}$$

Assume in addition $v \in C(\overline{\Omega})$ and $v = 0$ on $\partial\Omega$, then v_m converges to v locally uniformly in $\overline{\Omega}$. Assume in addition $\lim_{|x|\to\infty} v(x) = 0$, then v_m converges to v uniformly in $\overline{\Omega}$. In particular, $C_{0,\sigma}(\Omega) = \{v \in C(\overline{\Omega}) \mid div v = 0 \text{ in } \Omega, v = 0 \text{ on } \partial\Omega,\ \lim_{|x|\to\infty} v(x) = 0\ \}$.

We first prove Lemma 4.3.10 for $\Omega = \mathbf{R}^n_+$. As we proved the approximation (4.1.7) for the whole space (Proposition 4.3.6), we cut off a function and apply the Bogovskiĭ operator to get a compactly supported solenoidal vector field. But in this case, a support of a cutoff function may not have a Lipschitz boundary because of the presence of the boundary $\partial\mathbf{R}^n_+$. We use a cutoff function associated with the cylinder $C_0(r) = B_0^{n-1}(r) \times (-r, r)$, $r > 0$ in order to show the estimate (4.3.11) via the Bogovskiĭ operator. (The constant in (4.3.6) depends on the Lipschitz regularity of the boundary).

Proposition 4.3.11. *The statement of Lemma 4.3.10 holds for $\Omega = \mathbf{R}^n_+$.*

Proof. Let $\theta \in C_c^\infty[0,\infty)$ be a smooth cutoff function such that $\theta \equiv 1$ in $[0,1]$, $\theta \equiv 0$ in $[2,\infty)$ and $0 \le \theta \le 1$. Set $\tilde{\theta}_m = \tilde{\theta}(x/m)$ for $\tilde{\theta}(x) = \theta(|x'|)\theta(|x_n|)$. Then, $\tilde{\theta}_m \equiv 1$ in $C_0(m)$ and $\tilde{\theta}_m \equiv 0$ in $C_0(2m)^c$. For $v \in L^\infty_\sigma(\mathbf{R}^n_+)$, set $g_m = v\cdot\nabla\tilde{\theta}_m$. Then, $g_m \in L^r_{\mathrm{av}}(\mathcal{D}_m)$ and spt $g_m \subset \mathcal{D}_m$ for $\mathcal{D}_m = \mathcal{D}_0(m) \cap \mathbf{R}^n_+$ and $\mathcal{D}_0(m) = C_0(2m)\backslash\overline{C_0(m)}$. Set $f_m(x) = g_m(mx)$ for $x \in \mathcal{D} = \mathcal{D}_1$. Then, $f_m \in L^r_{\mathrm{av}}(\mathcal{D})$ and spt $f_m \subset \mathcal{D}$. We apply the Bogovskiĭ operator in the Lipschitz domain $\mathcal{D}$ to get $u^*_m = B_{\mathcal{D}}(f_m)$ such that div $u^*_m = \tilde{g}_m$ in $\mathcal{D}$, $u^*_m = 0$ on $\partial\mathcal{D}$ and

$$\|u^*_m\|_{W^{1,r}(\mathcal{D})} \le C_B\|f_m\|_{L^r(\mathcal{D})},$$

with the constant C_B, independent of $m \ge 1$. By the Sobolev inequality for $r > n$ in the Lipschitz domain $\mathcal{D}$ (e.g. [4, Theorem 4.12]), we estimate $\|u^*_m\|_{L^\infty(\mathcal{D})} \le C_S C_B\|f_m\|_{L^r(\mathcal{D})}$. Since $\|f_m\|_{L^r(\mathcal{D})} = m^{-n/r}\|g_m\|_{L^r(\mathcal{D}_m)}$ and $\|\nabla\tilde{\theta}\|_{L^r(\mathcal{D})} = m^{1-n/r}\|\nabla\tilde{\theta}_m\|_{L^r(\mathcal{D}_m)}$, it follows that

$$\|u^*_m\|_{L^\infty(\mathcal{D})} \le \frac{C_1}{m}\|v\|_{L^\infty(\mathcal{D}_m)}$$

with the constant C_1 depending on C_S, C_B, r and $\|\nabla\theta\|_\infty$, independent of $m \ge 1$. We set

$$v^*_m(x) = mu^*_m(x/m) \quad \text{for } x \in \mathcal{D}_m$$

and observe that div $v^*_m = g_m$ in $\mathcal{D}_m$, $v^*_m = 0$ on $\partial\mathcal{D}_m$, spt $v^*_m \subset \overline{\mathcal{D}_m}$ and

$$\|v^*_m\|_{L^\infty(\mathcal{D}_m)} \le C_1\|v\|_{L^\infty(\mathcal{D}_m)}. \tag{4.3.13}$$

Denoting the zero extension of v^*_m to $\mathbf{R}^n_+\backslash\overline{\mathcal{D}_m}$ by $\bar{v}^*_m$, we set $\hat{v}_m = v\tilde{\theta}_m - \bar{v}^*_m$. Then, $\hat{v}_m \in L^\infty(\mathbf{R}^n_+)$ satisfies div $\hat{v}_m = 0$ in $\mathbf{R}^n_+$, $\hat{v}^n_m = 0$ on $\partial\mathbf{R}^n_+$, spt $\hat{v}_m \subset \overline{\mathcal{D}_m}$ and

$$\|\hat{v}_m\|_{L^\infty(\mathbf{R}^n_+)} \le (1 + C_1)\|v\|_{L^\infty(\mathbf{R}^n_+)}.$$

Finally, we set

$$v_m(x', x_n) = \begin{cases} \hat{v}_m(x', x_n - 1/m) & \text{for } x' \in \mathbf{R}^{n-1}, x_n \ge 1/m, \\ 0 & \text{for } x' \in \mathbf{R}^{n-1}, 0 < x_n < 1/m, \end{cases}$$

so that spt $v_m \subset \mathbf{R}^n_+$. We obtain the desired sequence by approximating v_m with the standard mollifier.

If $v \in C(\overline{\mathbf{R}^n_+})$ and $v = 0$ on $\partial\mathbf{R}^n_+$, it is easy to see that v_m converges to v locally uniformly in $\overline{\mathbf{R}^n_+}$. If in addition $\lim_{|x|\to\infty} v(x) = 0$, from (4.3.13), $\hat{v}_m$ converges to v uniformly in $\overline{\mathbf{R}^n_+}$ as $m \to \infty$ so $v_m \to v$ uniformly in $\overline{\mathbf{R}^n_+}$. The proof is now complete. □

Proof of Lemma 4.3.10. Take $R_\Omega > 0$ such that $\Omega\backslash C_0(R_\Omega) = \mathbf{R}^n_+\backslash C_0(R_\Omega)$. Let $\theta \in C_c^\infty[0,\infty)$ be a smooth cutoff function such that $\theta \equiv 1$ in $[0,1]$, $\theta \equiv 0$ in $[2,\infty)$ and $0 \le \theta \le 1$. Set $\tilde{\theta}_R(x) = \theta(|x'|/R)\theta(|x_n|/R)$ for fixed $R > R_\Omega$. For $v \in L^\infty_\sigma(\Omega)$, by the Bogovskiĭ operator on $\mathcal{D}_R$, we set

$$v_1 = v\tilde{\theta}_R - B(v\cdot\nabla\tilde{\theta}_R),$$
$$v_2 = v(1 - \tilde{\theta}_R) + B(v\cdot\nabla\tilde{\theta}_R).$$

Since $\|B(v\cdot\nabla\tilde{\theta}_R)\|_{L^\infty(\mathcal{D}_R)} \leq C_R\|v\|_{L^\infty(\mathcal{D}_R)}$, it follows that

$$\|v_i\|_{L^\infty(\Omega)} \leq (1+C_R)\|v\|_{L^\infty(\Omega)} \quad \text{for } i=1,2.$$

The function v_1 satisfies div $v_1 = 0$ in Ω, $v_1\cdot n_\Omega = 0$ on $\partial\Omega$ and spt $v_1 \subset \overline{\Omega_R}$ for the bounded Lipschitz domain $\Omega_R = \Omega\cap C_0(2R)$. Thus, $v_1 \in L^\infty_\sigma(\Omega_R)$. We apply Lemma 4.3.3 to get a sequence $\{v_{1,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega_R)$ satisfying (4.3.1) and (4.3.2) for v^1 in Ω_R. We identify $v_{1,m}$ and its zero extension to $\Omega\backslash\overline{\Omega_R}$.
The function v_2 satisfies div $v_2 = 0$ in Ω, $v_2\cdot n_\Omega = 0$ on $\partial\Omega$ and spt $v_2\cap C_0(R) = \emptyset$. We set $\bar{v}_2 = v_2$ in $\mathbf{R}^n_+\backslash\overline{C_0(R)}$ and $\bar{v}_2 = 0$ in $\mathbf{R}^n_+\cap\overline{C_0(R)}$. Then, $\bar{v}_2 \in L^\infty_\sigma(\mathbf{R}^n_+)$. We apply Proposition 4.3.11 to get a sequence $\{\bar{v}_{2,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\mathbf{R}^n_+)$ satisfying (4.3.11) and (4.3.12) for $\bar{v}_2$ in $\mathbf{R}^n_+$. From the proof of Proposition 4.3.11, we observe that $\bar{v}_{2,m}$ also satisfies spt $\bar{v}_{2,m}\cap C_0(R) = \emptyset$. We set $v_{2,m} = \bar{v}_{2,m}$ in $\Omega\backslash\overline{C_0(R)}$ and $v_{2,m} = 0$ in $\Omega\cap\overline{C_0(R)}$. Then, $v_{2,m}\in C^\infty_{c,\sigma}(\Omega)$ satisfies (4.3.11) and (4.3.12) for v_2 in Ω.
Now, we set $v_m = v_{1,m} + v_{2,m}$ and observe that $v_m \in C^\infty_{c,\sigma}(\Omega)$ satisfies (4.3.11) and (4.3.12) for v in Ω. If $v\in C(\overline{\Omega})$ and $v=0$ on $\partial\Omega$, then $v_1\in C(\overline{\Omega_R})$, $v_1 = 0$ on $\partial\Omega_R$ and $v_2\in C(\overline{\mathbf{R}^n_+})$, $v_2 = 0$ on $\partial\mathbf{R}^n_+$. For $v_1 \in L^\infty_\sigma(\Omega_R)\cap C(\overline{\Omega}_R)$ satisfying $v_1 = 0$ on $\partial\Omega_R$, $v_{1,m}\in C^\infty_{c,\sigma}(\Omega_R)$ converges to v_1 uniformly in $\bar{\Omega}_R$. Thus, $v_{1,m}\to v_1$ uniformly in $\bar{\Omega}$. For $\bar{v}_2\in L^\infty_\sigma(\mathbf{R}^n_+)\cap C(\overline{\mathbf{R}^n_+})$ satisfying $\bar{v}_2 = 0$ on $\partial\mathbf{R}^n_+$, $\bar{v}_{2,m}$ converges to $\bar{v}_2$ locally uniformly in $\overline{\Omega}$ so $v_{2,m}\to v_2$ locally uniformly in $\overline{\Omega}$. Thus, v_m converges to v locally uniformly in $\overline{\Omega}$ as $m\to\infty$.
If in addition $\lim_{|x|\to\infty} v(x) = 0$, $v_{2,m}$ converges to v_2 uniformly in $\overline{\Omega}$ as $m\to\infty$. Thus, $v_m\to v$ uniformly in $\overline{\Omega}$ as $m\to\infty$. The proof is now complete. □

4.4 Continuity at time zero

In this section, we show that $S(t)$ is strongly continuous at $t=0$ on $BUC_\sigma(\Omega)$ for exterior domains. We divide a support of $v_0\in BUC_\sigma(\Omega)$ into two parts so that one is compactly supported in $\bar{\Omega}$ and the other is supported away from $\partial\Omega$. For compactly supported initial data, i.e., $v_0\in C_{0,\sigma}(\Omega)$ we already know $S(t)v_0\to v_0$ on $BUC_\sigma(\Omega)$ as $t\downarrow 0$ since $S(t)$ is a C_0-semigroup on $C_{0,\sigma}(\Omega)$ by Theorem 3.1.3 in Chapter 3. Thus, we shall show:

Proposition 4.4.1. *Let Ω be an exterior domain with C^3-boundary. Let $S(t)$ be the Stokes semigroup on $L^\infty_\sigma(\Omega)$. Then, $S(t)v_0\to v_0$ on $BUC_\sigma(\Omega)$ as $t\downarrow 0$ for all $v_0\in BUC_\sigma(\Omega)$ satisfying* dist(spt $v_0,\partial\Omega) > 0$.

Proof. Let $v_0\in BUC_\sigma(\Omega)$ satisfy dist(spt $v_0,\partial\Omega) > 0$. Denoting the standard mollifier by η_ε, set $v_0^\varepsilon = v_0 * \eta_\varepsilon$. Then, $v_0^\varepsilon\in C^\infty(\overline{\Omega})$ is supported away from $\partial\Omega$ and converges to v_0 uniformly in Ω. Thus, we may assume $v_0\in W^{2,\infty}(\Omega)\cap BUC_\sigma(\Omega)$.
Set $v = S(t)v_0$. We shall show

$$\partial_t v = S(t)\Delta v_0. \tag{4.4.1}$$

This implies that $||v(t) - v_0||_\infty \leq Ct||\Delta v_0||_\infty$ as $t \downarrow 0$. By Corollary 4.3.7, for $v_0 \in W^{2,\infty}(\Omega) \cap BUC_\sigma(\Omega)$, there exists $v_{0,m} \in C^\infty_{c,\sigma}(\Omega)$ such that $||v_{0,m}||_{W^{2,\infty}(\Omega)} \leq C||v_0||_{W^{2,\infty}(\Omega)}$ and $\partial^l_x v_{0,m} \to \partial^l_x v_0$ a.e. in Ω as $m \to \infty$ for $|l| \leq 2$. Here, we do not distinguish v_0 and its zero extension to $\mathbf{R}^n \backslash \overline{\Omega}$. Set $v_m = S(t)v_{0,m}$. As we proved Theorem 4.1.2, v_m subsequently converges to v locally uniformly in $\bar{\Omega} \times (0, T]$ together with $\partial_t v_m$. Since $-Av_{0,m} = \Delta v_{0,m}$ for $v_{0,m} \in C^\infty_{c,\sigma}(\Omega)$, it follows that

$$\partial_t v_m = S(t)\Delta v_{0,m}. \tag{4.4.2}$$

Since $\Delta v_{0,m} \to \Delta v_0$ a.e. in Ω as $m \to \infty$, $S(t)\Delta v_{0,m}$ subsequently converges to $S(t)\Delta v_0$ locally uniformly in $\overline{\Omega} \times (0, T]$. By letting $m \to \infty$ to (4.4.2), we obtain (4.4.1). The proof is now complete. □

Proposition 4.4.1 now implies:

Theorem 4.4.2. *Let Ω be an exterior domain in $\mathbf{R}^n$, $n \geq 2$, with C^3-boundary. The Stokes semigroup $S(t)$ is a C_0-(analytic) semigroup on $BUC_\sigma(\Omega)$.*

Proof. We may assume $0 \in \Omega^c$. By (4.3.10), we divide $v_0 \in BUC_\sigma(\Omega)$ into two terms $v_0 = v_{0,1} + v_{0,2}$ so that $v_{0,1}$ is compactly supported in $\overline{\Omega}$ and $v_{0,2}$ is supported away from $\partial\Omega$, i.e. $\mathrm{dist}(\partial\Omega, \mathrm{spt}\, v_{0,2}) > 0$. By Lemma 4.3.5, $v_{0,1}$ is in $C_{0,\sigma}(\Omega)$ so $S(t)v_{0,1} \to v_{0,1}$ on $BUC_\sigma(\Omega)$ as $t \downarrow 0$ by Theorem 3.1.3. By Proposition 4.4.1, $S(t)v_{2,0} \to v_{2,0}$ on $BUC_\sigma(\Omega)$ as $t \downarrow 0$. Thus, $S(t)v_0 \to v_0$ on $BUC_\sigma(\Omega)$ as $t \downarrow 0$. The proof is now complete. □

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Chapter 5

Resolvent approach

This chapter is devoted to the resolvent approach for the analyticity of the Stokes semigroup on L^∞. We present an a priori L^∞-estimate for solutions to the resolvent Stokes equations, which in particular implies that the angle of the analytic semigroup on L^∞ is $\pi/2$. The approach is inspired by the Masuda-Stewart technique for elliptic operators. Furthermore, the method presented applies also to different type of boundary conditions, e.g., to the Robin boundary condition. Note that the harmonic-pressure gradient estimate (0.1.3) is available also for the resolvent Stokes equation.

5.1 Introduction

We consider the resolvent Stokes equations in the domain $\Omega \subset \mathbf{R}^n$, $n \geq 2$. When $\Omega = \mathbf{R}^n_+$, the analyticity of the Stokes semigroup on L^∞-type spaces was proved in [9] (see also [36], [25]) based on explicit calculations for the solution operator $R(\lambda) : f \mapsto v = v_\lambda$ to the corresponding resolvent problem:

$$\lambda v - \Delta v + \nabla q = f \quad \text{in } \Omega, \tag{5.1.1}$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega, \tag{5.1.2}$$

$$v = 0 \quad \text{on } \partial\Omega. \tag{5.1.3}$$

We present a direct resolvent approach to the resolvent Stokes equations (5.1.1)–(5.1.3) and establish the a priori estimate of the form,

$$M_p(v,q)(x,\lambda) = |\lambda||v(x)| + |\lambda|^{1/2}|\nabla v(x)| + |\lambda|^{n/2p}\|\nabla^2 v\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})} + |\lambda|^{n/2p}\|\nabla q\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})},$$

for $p > n$ and

$$\sup_{\lambda \in \Sigma_{\vartheta,\delta}} \left\| M_p(v,q) \right\|_{L^\infty(\Omega)}(\lambda) \leq C\|f\|_{L^\infty(\Omega)} \tag{5.1.4}$$

for some constant $C > 0$ independent of f. Here, $\Omega_{x,r}$ denotes the intersection of Ω with an open ball $B_x(r)$ centered at $x \in \Omega$ with radius $r > 0$, i.e. $\Omega_{x,r} = B_x(r) \cap \Omega$ and $\Sigma_{\vartheta,\delta}$ denotes the sectorial region in the complex plane given by $\Sigma_{\vartheta,\delta} = \{\lambda \in \mathbf{C}\backslash\{0\} \mid |\arg\lambda| < \vartheta,\ |\lambda| > \delta\}$ for $\vartheta \in (\pi/2, \pi)$ and $\delta > 0$. Our approach is inspired by the corresponding approach for general elliptic operators. K. Masuda was the first to prove analyticity of the semigroup associated to general elliptic operators in $C_0(\mathbf{R}^n)$ including the case of higher orders [27], [28] ([29].) This result was then extended by H. B. Stewart to the case for the Dirichlet problem [37] and more general boundary condition [38]. This Masuda-Stewart method was applied to many other situations [5], [24], [21], [6]. However, its application to the resolvent Stokes equations (5.1.1)–(5.1.3) was unknown.

In the sequel, we prove the estimate (5.1.4) by invoking the L^p-estimates for the resolvent Stokes equations with inhomogeneous divergence-free condition [14], [15]. We invoke *the strictly admissibility* of a domain introduced in Chapter 2 which implies an estimate of pressure q in terms of the velocity by

$$\sup_{x\in\Omega} d_\Omega(x)\left|\nabla q(x)\right| \leq C_\Omega \|W(v)\|_{L^\infty(\partial\Omega)} \tag{5.1.5}$$

for $W(v) = -(\nabla v - \nabla^T v)n_\Omega$, where ∇f denotes $(\partial f_i/\partial x_j)_{1\leq i,j\leq n}$ and $\nabla^T f = (\nabla f)^T$ for the vector field $f = (f_i)_{1\leq i\leq n}$. The estimate (5.1.5) plays a key role in transferring results from the elliptic situation to the situation of the Stokes system. Here, n_Ω denotes the unit outward normal vector field on $\partial\Omega$ and d_Ω denotes the distance function from the boundary, $d_\Omega(x) = \inf_{y\in\partial\Omega}|x-y|$ for $x \in \Omega$. The estimate (5.1.5) can be viewed as a regularizing-type estimate for solutions to the Laplace equation $\Delta P = 0$ in Ω with the Neumann boundary condition $\partial P/\partial n_\Omega = \operatorname{div}_{\partial\Omega} W$ on $\partial\Omega$ for a tangential vector field W where $\operatorname{div}_{\partial\Omega} = \operatorname{tr}\nabla_{\partial\Omega}$ denotes the surface divergence and $\nabla_{\partial\Omega} = \nabla - n_\Omega(n_\Omega\cdot\nabla)$ is the gradient on $\partial\Omega$. As is proved in Chapter 3 (Lemma 3.2.2), the pressure $P = q$ solves this Neumann problem for $W = W(v)$ and the estimate (5.1.5) holds for bounded domains, exterior domains and perturbed half spaces ($n \geq 3$). When $n = 3$, $W(v)$ is nothing but the tangential trace of vorticity, i.e. $-\operatorname{curl} v \times n_\Omega$. We call Ω *strictly admissible* if there exists a constant $C = C_\Omega$ such that the a priori estimate

$$\|\nabla P\|_{L^\infty_d(\Omega)} \leq C\|W\|_{L^\infty(\partial\Omega)} \tag{5.1.6}$$

holds for all solutions P of the Neumann problem for tangential vector fields $W \in L^\infty(\partial\Omega)$, where $\|f\|_{L^\infty_d(\Omega)} = \sup_{x\in\Omega} d_\Omega(x)|f(x)|$ denotes the norm for $f \in L^\infty_d(\Omega)$

We are now in the position to formulate the main results of this chapter.

Theorem 5.1.1. *Let Ω be a strictly admissible, uniformly C^2-domain in $\mathbf{R}^n$ for $n \geq 2$. Let $p > n$. For $\vartheta \in (\pi/2, \pi)$, there exists constants δ and C such that the a priori estimate (5.1.4) holds for all solutions $(v, \nabla q) \in W^{2,p}_{loc}(\bar{\Omega}) \times (L^p_{loc}(\bar{\Omega}) \cap L^\infty_d(\Omega))$ of (5.1.1)–(5.1.3) for $f \in C_{0,\sigma}(\Omega)$ and $\lambda \in \Sigma_{\vartheta,\delta}$.*

The a priori estimate (5.1.4) implies the analyticity of the Stokes semigroup on L^∞-type spaces. Let us observe the generation of an analytic semigroup on $C_{0,\sigma}(\Omega)$. By invoking

$\tilde{L}^p$-theory [11], [12], [13], we verify the existence of solutions to (5.1.1)–(5.1.3), $(v, \nabla q) \in W^{2,p}_{\mathrm{loc}}(\bar{\Omega}) \times (L^p_{\mathrm{loc}}(\bar{\Omega}) \cap L^\infty_d(\Omega))$ for $f \in C^\infty_{c,\sigma}(\Omega)$ in a uniformly C^2-domain Ω. We extend the solution operator $R(\lambda)$ to $C_{0,\sigma}$ by a uniform approximation and the estimates (5.1.4). (The solution operator to the pressure gradient $f \mapsto \nabla q_\lambda$ is also uniquely extendable for $f \in C_{0,\sigma}$). We observe that $R(\lambda)$ is injective on $C_{0,\sigma}$ since the estimate (5.1.5) immediately implies that $f = 0$ for $f \in C_{0,\sigma}$ satisfying $v_\lambda = R(\lambda)f = 0$. The operator $R(\lambda)$ may be regarded as a surjective operator from $C_{0,\sigma}$ to the range of $R(\lambda)$. The open mapping theorem then implies the existence of the closed operator A such that $R(\lambda) = (\lambda - A)^{-1}$; see [7, Proposition B.6]. We call A *the Stokes operator* in $C_{0,\sigma}(\Omega)$. From Theorem 5.1.1 we have:

Theorem 5.1.2. *Let Ω be a strictly admissible, uniformly C^2-domain in $\mathbf{R}^n$. Then, the Stokes operator A generates a C_0-analytic semigroup on $C_{0,\sigma}(\Omega)$ of angle $\pi/2$.*

We next consider the space $L^\infty_\sigma(\Omega)$ defined by

$$L^\infty_\sigma(\Omega) = \left\{ f \in L^\infty(\Omega) \;\middle|\; \int_\Omega f \cdot \nabla \varphi dx = 0 \quad \text{for all } \varphi \in \hat{W}^{1,1}(\Omega) \right\},$$

where $\hat{W}^{1,1}(\Omega)$ denotes the homogeneous Sobolev space of the form $\hat{W}^{1,1}(\Omega) = \{\varphi \in L^1_{\mathrm{loc}}(\Omega) \mid \nabla\varphi \in L^1(\Omega)\}$. Note that $C_{0,\sigma}(\Omega) \subset L^\infty_\sigma(\Omega)$. When the domain Ω is unbounded, the space $L^\infty_\sigma(\Omega)$ includes non-decaying solenoidal vector fields at infinity. Actually, the a priori estimates (5.1.4) is also valid for $f \in L^\infty_\sigma$. In particular, (5.1.4) implies the uniqueness of solutions for $f \in L^\infty_\sigma$. We verify the existence of solutions by approximating $f \in L^\infty_\sigma$ with compactly supported smooth solenoidal vector fields $\{f_m\}_{m=1}^\infty \subset C^\infty_{c,\sigma}$. Note that one can not approximate $f \in L^\infty_\sigma$ in a uniform topology by an element of $C^\infty_{c,\sigma}$. We weaken the convergence, for example, to the pointwise convergence, i.e., $f_m \to f$ a.e. in Ω and $\|f_m\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}$ with some constant $C = C_\Omega$, independent of $m \geq 1$. Although this approximation is non-trivial for general domains, for bounded domains, exterior domains and perturbed half spaces, this approximation is valid as we proved in Chapter 4. In the following, we restrict our results to (I) bounded domains, (II) exterior domains and (III) perturbed half spaces ($n \geq 3$). By an approximation argument, we verify the existence of solutions to (5.1.1)–(5.1.3) for general $f \in L^\infty_\sigma$. We then define the Stokes operator on L^∞_σ by the same way as for $C_{0,\sigma}$. Since the domains (I)–(III) are strictly admissible provided that the boundary is C^3 (Theorem 2.3.3 in Chapter 2), we have:

Theorem 5.1.3. *Let Ω be one of the domains (I)–(III) with C^3-boundary. Then, the Stokes operator A generates a (non-C_0-)analytic semigroup on $L^\infty_\sigma(\Omega)$ of angle $\pi/2$.*

Remarks 5.1.4. (i) The direct resolvent approach clarifies the angle of the analytic semigroup e^{tA} on $C_{0,\sigma}$. Theorem 5.1.2 (and also Theorem 5.1.3) assert that e^{tA} is angle $\pi/2$ on $C_{0,\sigma}$ which does not follow from the a priori L^∞-estimate (0.1.1) for solutions to the non-stationary Stokes equations proved in Chapter 3.

(ii) We observe that our argument applies to other boundary conditions, for example, to the Robin boundary condition, i.e., $B(v) = 0$ and $v \cdot n_\Omega = 0$ on $\partial\Omega$ where

$$B(v) = \alpha v_{\tan} + (D(v)n_\Omega)_{\tan} \quad \text{for } \alpha \geq 0.$$

Here, $D(v) = (\nabla v + \nabla^T v)/2$ denotes the deformation tensor and $f_{\tan}$ denotes the tangential component of the vector field f on $\partial\Omega$. Note that the case $\alpha = \infty$ corresponds to the Dirichlet boundary condition (1.3); see [30] for generation results subject to the Robin boundary conditions on L^∞ for $\mathbf{R}^n_+$. The L^p-resolvent estimates for the Robin boundary condition was established in [20] for concerning analyticity and was later strengthened in [32] to non-divergence free vector fields. We shall use the generalized resolvent estimate in [32] to extend our result in spaces of bounded functions to the Robin boundary condition (Theorem 5.3.6). For a more detailed discussion, see Remark 5.3.5.
(iii) We observe that the domain of the Stokes operator $D(A)$ is dense in $C_{0,\sigma}$. In fact, by invoking $\tilde{L}^p$-theory and using (5.1.4), we have

$$\|\lambda v - f\|_{L^\infty(\Omega)} = \|\tilde{A}_p v\|_{L^\infty(\Omega)} \leq \frac{C}{|\lambda|}\|\tilde{A}_p f\|_{L^\infty(\Omega)} \to 0, \quad |\lambda| \to \infty$$

for $f \in C^\infty_{c,\sigma} \subset D(\tilde{A}_p)$, where $\tilde{A}_p$ is the Stokes operator in $\tilde{L}^p$. Thus, we conclude that $D(A)$ is dense in $C_{0,\sigma}$. On the other hand, smooth functions are not dense in L^∞ and $e^{tA}f$ is smooth for $t > 0$, $e^{tA}f \to f$ as $t \downarrow 0$ in L^∞_σ does not hold for some $f \in L^\infty_\sigma$. This means e^{tA} is a non-C_0-analytic semigroup. In other words, $D(A)$ is not dense in L^∞_σ. We refer to [34, 1.1.2] for properties of the analytic semigroup generated by non-densely defined sectorial operators; see also [7, Definition 3.2.5].
(iv) For a bounded domain Ω, $v(\cdot, t) = e^{tA}v_0$ and $\nabla q = (1 - \mathbf{P})[\Delta v]$ give a solution to the non-stationary Stokes equations, $v_t - \Delta v + \nabla q = 0$, $\operatorname{div} v = 0$ in $\Omega \times (0, \infty)$ with $v = 0$ on $\partial\Omega$ for initial data $v_0 \in L^\infty_\sigma(\Omega)$. Although the Helmholtz projection operator $\mathbf{P} : L^p(\Omega) \to L^p_\sigma(\Omega)$ is not bounded on L^∞, we are able to define the pressure $\nabla q = \mathbf{K}[W(v)]$ at least for exterior domains Ω by *the harmonic-pressure operator* $\mathbf{K} : L^\infty_{\tan}(\partial\Omega) \ni W \mapsto \nabla P \in L^\infty_d(\Omega)$ (Remarks 2.5.2 (ii) in Chapter 2). Here, $L^\infty_{\tan}(\partial\Omega)$ denotes the closed subspace of all tangential vector fields in $L^\infty(\partial\Omega)$.
(v) We observe that the Masuda-Stewart method does not imply a large time behavior for e^{tA}. For a bounded domain, the energy inequality implies that $v(\cdot, t) = e^{tA}v_0$ (and also v_t) exponentially decay as $t \to \infty$ as discussed in Chapter 3 (Remarks 3.4.4 (i)). In particular, e^{tA} is a bounded analytic semigroup on L^∞_σ. Recently, based on the L^∞-estimates [1, Theorem 1.2] it was shown in [26] that e^{tA} is bounded semigroup on L^∞_σ for exterior domains by appealing to the maximum modulus theorem for the boundary-value problem of the stationary Stokes equations. Note that it is unknown whether e^{tA} is a bounded analytic semigroup on L^∞_σ.

In the sequel, we sketch the proof for the a priori estimate (5.1.4). Our argument can be divided into the following three steps:

(i) (Localization) We first localize a solution (v, q) of the Stokes equations (5.1.1)–(5.1.3) in a domain $\Omega' = B_{x_0}((\eta + 1)r) \cap \Omega$ for $x_0 \in \Omega, r > 0$ and parameters $\eta \geq 1$ by setting $u = v\theta_0$ and $p = (q - q_c)\theta_0$ with a constant q_c and the smooth cutoff function θ_0 around $\Omega_{x_0,r}$ satisfying $\theta_0 \equiv 1$ in $B_{x_0}(r)$ and $\theta_0 \equiv 0$ in $B_{x_0}((\eta + 1)r)^c$. We then observe that (u, p) solves the resolvent Stokes equations with inhomogeneous divergence-free condition in the localized domain Ω'. Applying the L^p-estimates for the localized Stokes equations, we have

$$\begin{aligned} &|\lambda|\|u\|_{L^p(\Omega')} + |\lambda|^{1/2}\|\nabla u\|_{L^p(\Omega')} + \|\nabla^2 u\|_{L^p(\Omega')} + \|\nabla p\|_{L^p(\Omega')} \\ &\leq C_p \left(\|h\|_{L^p(\Omega')} + \|\nabla g\|_{L^p(\Omega')} + |\lambda|\|g\|_{W_0^{-1,p}(\Omega')}\right), \end{aligned} \tag{5.1.7}$$

where $W_0^{-1,p}(\Omega')$ denotes the dual space of the Sobolev space $W^{1,p'}(\Omega')$ with $1/p+1/p' = 1$. The external forces h and g contain error terms appearing in the cutoff procedure and are explicitly given by

$$h = f\theta_0 - 2\nabla v\nabla\theta_0 - v\Delta\theta_0 + (q - q_c)\nabla\theta_0, \quad g = v \cdot \nabla\theta_0. \tag{5.1.8}$$

(ii) (Error estimates) A key step is to estimate the error terms of the pressure such as $(q - q_c)\nabla\theta_0$. We here simplify the description by disregarding the terms related to g in order to describe the essence of the proof. We will give precise estimates for the terms related to g in Section 3. Now, the error terms related to h are estimated in the form

$$\|h\|_{L^p(\Omega')} \leq Cr^{n/p}\left((\eta + 1)^{n/p}\|f\|_{L^\infty(\Omega)} + (\eta + 1)^{-(1-n/p)}\left(r^{-2}\|v\|_{L^\infty(\Omega)} + r^{-1}\|\nabla v\|_{L^\infty(\Omega)}\right)\right). \tag{5.1.9}$$

If we disregard the term $(q - q_c)\nabla\theta_0$ in h, the estimates (1.8) easily follows by using the estimates of the cutoff function θ_0, i.e., $\|\theta_0\|_\infty + (\eta + 1)r\|\nabla\theta_0\|_\infty + (\eta + 1)^2r^2\|\nabla^2\theta_0\|_\infty \leq K$ with some constant K. We invoke the estimate (5.1.5) in order to handle the pressure term by velocity through the *Poincaré-Sobolev-type inequality*:

$$\|\varphi - (\varphi)\|_{L^p(\Omega_{x_0,s})} \leq Cs^{n/p}\|\nabla\varphi\|_{L^\infty_d(\Omega)} \quad \text{for all } \varphi \in \hat{W}^{1,\infty}_d(\Omega), \tag{1.10}$$

with some constant C independent of $s > 0$, where (φ) denotes the mean value of φ in $\Omega_{x_0,s}$ and $\hat{W}^{1,\infty}_d(\Omega) = \{\varphi \in L^1_{\mathrm{loc}}(\bar{\Omega}) \mid \nabla\varphi \in L^\infty_d(\Omega)\}$. We prove the inequality (5.1.10) in Section 2. By taking $q_c = (q)$ and applying (5.1.10) for $\varphi = q$ and $s = (\eta+1)r$, we obtain the estimate (5.1.9) via (5.1.5).

(iii) (Interpolation) Once we establish the error estimates for h and g, it is easy to obtain the estimate (5.1.4) by applying the interpolation inequality,

$$\|\varphi\|_{L^\infty(\Omega_{x_0,r})} \leq C_I r^{-n/p}\left(\|\varphi\|_{L^p(\Omega_{x_0,r})} + r\|\nabla\varphi\|_{L^p(\Omega_{x_0,r})}\right) \quad \text{for } \varphi \in W^{1,p}_{\mathrm{loc}}(\bar{\Omega}), \tag{5.1.11}$$

for $\varphi = u$ and ∇u. Now, taking $r = |\lambda|^{-1/2}$, we obtain the estimate for $M_p(v, q)(x_0, \lambda)$ with the parameters η of the form,

$$M_p(v, q)(x_0, \lambda) \leq C\left((\eta + 1)^{n/p}\|f\|_{L^\infty(\Omega)} + (\eta + 1)^{-(1-n/p)}\|M_p(v, q)\|_{L^\infty(\Omega)}(\lambda)\right), \tag{5.1.12}$$

for some constant C independent of η. The second term in the right-hand side is absorbed into the left-hand side by letting η sufficiently large provided $p > n$.

Actually, in the procedure (ii), we take q_c by the mean value of q in $\Omega_{x_0,(\eta+2)r}$ since we estimate $|\lambda|\|g\|_{W_0^{-1,p}}$. By using the equation (5.1.1), we reduce the estimate of $|\lambda|\|g\|_{W_0^{-1,p}}$ to the L^∞-estimate for the boundary value of $q - q_c$ on $\partial\Omega'$. In order to estimate $\|q - q_c\|_{L^\infty(\Omega')}$, we use a uniformly local L^p-norm bound for ∇q besides the sup-bound for ∇v. This is the reason why we need the norm $\|M_p(v, q)\|_{L^\infty(\Omega)}(\lambda)$ in the right-hand side of (5.1.12). For general elliptic operators, the estimate (5.1.12) is valid without invoking the uniformly local L^p-norm bound for second derivatives of a solution.

This chapter is organized as follows. In Section 2, we prove the inequality (5.1.10) for uniformly C^2-domains. More precisely, we prove stronger estimates than (5.1.10) both interior and up to boundary $\Omega_{x_0,s}$ of Ω. In Section 3, we first prepare the estimates for h and g and then prove the a priori estimate (5.1.4) (Theorem 5.1.1). After proving Theorem 5.1.1, we also note the estimates (5.1.4) subject to the Robin boundary condition.

During the preparation of this thesis, the author was informed on the recent paper by Kenig et al. [22], where the estimate (5.1.6) was directly proved for $C^{1,\gamma}$-bounded domains by estimating the Green function for the Neumann problem (0.1.4). From their result, we observe that it is possible in Theorem 5.1.3 to reduce the boundary regularity from C^3 to C^2 at least for bounded domains. For elliptic operators, the estimate corresponding to (5.1.4) holds with $C^{1,1}$-boundary. However, we use the C^2-regularity of the boundary in the proof of the inequality (5.1.10) although the L^p-estimate for the Stokes equations (5.1.7) is valid with $C^{1,1}$-boundary (e.g. [15]).

5.2 Poincaré-Sobolev-type inequality

In this section, we prove the inequality (5.1.10) in a uniformly C^2-domain. We start with the Poincaré-Sobolev-type inequality in a bounded domain D and observe the compactness of the embedding from $\hat{W}^{1,\infty}_d(D)$ to $L^p(D)$, which is the key in proving the inequality (5.1.10) by *reductio ad absurdum*.

5.2.1 Curvilinear coodinates

Let D be a bounded domain in $\mathbf{R}^n, n \geq 2$ and $p \in [1, \infty)$. We prove the inequality of the form,

$$\|\varphi - (\varphi)\|_{L^p(D)} \leq C\|\nabla\varphi\|_{L^\infty_d(D)} \quad \text{for } \varphi \in \hat{W}^{1,\infty}_d(D) \tag{5.2.1}$$

where (φ) denotes the mean value of φ in D, i.e., $(\varphi) = ⨍_D \varphi dx$. If we replace the norm $\|\nabla\varphi\|_{L^\infty_d(D)}$ by the L^p-norm $\|\nabla\varphi\|_{L^p(D)}$, the estimate (5.2.1) is nothing but the Poincaré inequality [10, 5.8.1]. We observe that the bound for $\|\nabla\varphi\|_{L^\infty_d(\Omega)}$ implies the L^p-integrability of φ in D even if $\nabla\varphi$ is not in $L^p(D)$. For example, when $D = B_0(1)$, $\varphi(x) = \log(1-|x|)$ is in L^p although $|\nabla\varphi(x)| = d_D(x)^{-1}$ is not for any $p \in [1,\infty)$. Since the space $\hat{W}_d^{1,\infty}$ is compactly embedded to the space $C(\bar{D}')$ for each subdomain D' of D with $\overline{D'} \subset D$, we shall show a pointwise upper bound for φ near $\partial D'$ by an L^p-integrable function to conclude that the space $\hat{W}_d^{1,\infty}(D)$ is compactly embedded to $L^p(D)$ by the dominated convergence theorem. We estimate $\varphi \in \hat{W}_d^{1,\infty}(D)$ near ∂D directly by using the curvilinear coordinates. Here, for a domain $\Omega, \partial\Omega \neq \emptyset$, we say that $\partial\Omega$ is C^k if for each $x_0 \in \partial\Omega$, there exists constants α, β and C^k-function h of $n-1$ variables y' such that (up to rotation and translation if necessary) we have

$$U(x_0) \cap \Omega = \{(y', y_n) \mid h(y') < y_n < h(y') + \beta,\ |y'| < \alpha\},$$
$$U(x_0) \cap \partial\Omega = \{(y', y_n) \mid y_n = h(y'), |y'| < \alpha\},$$
$$\sup_{|l|\leq k, |y'|<\alpha} \left|\partial_{y'}^l h(y')\right| \leq K,\ \nabla' h(0) = 0,\ h(0) = 0,$$

with the constant K and the neighborhood of x_0, $U(x_0) = U_{\alpha,\beta,h}(x_0)$, i.e.,

$$U_{\alpha,\beta,h}(x_0) = \{(y', y_n) \in \mathbf{R}^n \mid h(y') - \beta < y_n < h(y') + \beta, |y'| < \alpha\}.$$

Here, $\partial_x^l = \partial_{x_1}^{l_1} \cdots \partial_{x_n}^{l_n}$ for a multi-index $l = (l_1, \ldots, l_n)$ and $\partial_{x_j} = \partial/\partial x_j$ as usual and ∇' denotes the gradient in $\mathbf{R}^{n-1}$. Moreover, if we are able to take uniform constants α, β, K independent of each $x_0 \in \partial\Omega$, we call Ω uniformly C^k-domain of type (α, β, K) as defined in [33, I.3.2].

We estimate $\varphi \in \hat{W}_d^{1,1}(\Omega)$ along the boundary using the curvilinear coordinates.

Proposition 5.2.1. *Let D be a bounded domain with C^k-boundary ($k \geq 2$). Let $\Gamma = \{x \in \partial D \mid x = (x', h(x')), |x'| < \alpha'\}$ be a neighborhood of $x_0 \in \partial D$.*
(i) There exists positive constants μ and α' such that $(\gamma, d) \mapsto X(\gamma, d) = \gamma + d n_D(\gamma)$ is a C^{k-1} diffeomorphism from $\Gamma \times (0,\mu)$ onto

$$\mathcal{N}^\mu(\Gamma) = \{X(\gamma, d) \in U(x_0) \mid (\gamma, d) \in \Gamma \times (0, \mu)\},$$

i.e., $x \in \mathcal{N}^\mu(\Gamma)$ has a unique projection to ∂D denoted by $\gamma(x) \in \partial D$ such that

$$(\gamma(x), d_D(x)) = X^{-1}(x) \quad \textit{for } x \in \mathcal{N}^\mu(\Gamma).$$

(ii) There exists a constant C_1 such that for any $x_1 \in \overline{\mathcal{N}^\mu(\Gamma)}$ and $r_1 > 0$ satisfying $D_{x_1,r_1} = B_{x_1}(r_1) \cap D \subset \mathcal{N}^\mu(\Gamma)$,

$$|\varphi(x) - \varphi(y)| \leq C_1 \left(\left| \log \frac{d_D(x)}{d_D(y)} \right| + \frac{|\gamma(x) - \gamma(y)|}{\max\{d_D(x), d_D(y)\}} \right) \sup_{z \in D_{x_1,r}} d_D(z) |\nabla\varphi(z)| \quad \textit{for } x,\ y \in D_{x_1,r_1}$$

holds for $\varphi \in \hat{W}_d^{1,\infty}(D)$.

Proof. The assertion (i) is based on the inverse function theorem [23, Lemma 4.4.7]. We shall prove the second assertion (ii). We take points $x, y \in D_{x_1,r_1}$ for $x_1 \in \overline{\mathcal{N}^\mu(\Gamma)}$ and $r_1 > 0$ satisfying $D_{x_1,r_1} \subset \mathcal{N}^\mu(\Gamma)$. We may assume $d_D(y) = d(y) > d(x)$. By setting $z = (\gamma(x), d(y))$, we estimate

$$|\varphi(x) - \varphi(y)| \le |\varphi(x) - \varphi(z)| + |\varphi(z) - \varphi(y)|.$$

We connect x and z by the straight line to estimate

$$\begin{aligned}
|\varphi(x) - \varphi(z)| &= \left| \int_0^1 \frac{d}{dt}\varphi(X(\gamma(x), td(x) + (1-t)d(y)))dt \right| \\
&= \left| \int_0^1 (d(y) - d(x))(\nabla\varphi)(X(\gamma(x), td(x) + (1-t)d(y)) \cdot n_D(\gamma(x))dt \right| \\
&\le (d(y) - d(x)) \int_0^1 \frac{dt}{t(d(x) - d(y)) + d(y)} \sup_{z \in D_{x_1,r}} d(z)|\nabla\varphi(z)| \\
&= \left| \log \frac{d(y)}{d(x)} \right| \sup_{z \in D_{x_1,r}} d(z)|\nabla\varphi(z)|.
\end{aligned}$$

It remains to estimate $|\varphi(z) - \varphi(y)|$. We connect z and y by the curve

$$C_{z,y} = \{X(\gamma(t), d(y)) \mid \gamma(t) = (\gamma'(t), h(\gamma'(t))),\ \gamma'(t) = t\gamma'(x) + (1-t)\gamma'(y),\ 0 \le t \le 1\},$$

where γ' denotes the $n-1$ variables of γ. We then estimate

$$\begin{aligned}
|\varphi(z) - \varphi(y)| &= \left| \int_0^1 \frac{d}{dt}\varphi(X(\gamma(t), d(y)))dt \right| \\
&= \left| \int_0^1 \frac{d\gamma}{dt}(t)(1 + d(y)\nabla_{\partial D} n_D(\gamma(t)))\nabla\varphi(X(\gamma(t), d(y)))dt \right| \\
&\le C(1 + \mu K)\frac{|\gamma(x) - \gamma(y)|}{d(y)} \sup_{z \in D_{x_1,r_1}} d(z)|\nabla\varphi(z)|,
\end{aligned}$$

since $|d\gamma(t)/dt| \le C|\gamma(x) - \gamma(y)|$ and $|\nabla_{\partial D} n_D| \le K$ with a constant C depending on K. The assertion (ii) thus follows. □

Remarks 5.2.2. (i) We observe from the second assertion that $\varphi \in \hat{W}_d^{1,\infty}(D)$ is bounded from above by an L^p-integrable function for all $p \in [1, \infty)$ near ∂D, i.e., for each fixed $y \in D_{x_1,r_1}$ such that $d_D(y) \ge \delta$, we have

$$|\varphi(x)| \le C_2(|\log d_D(x)| + 1)\left(\sup_{z \in D_{x_1,r_1}} d_D(z)|\nabla\varphi(z)| \right) + |\varphi(y)| \quad \text{for } x \in D_{x_1,r_1} \tag{5.2.2}$$

with a constant C_2 depending on μ, δ.

(ii) Note that Proposition 5.2.1 is also valid for a uniformly C^k-domain Ω of type (α,β,K), i.e., there exist constants μ,α', depending only on α,β,K, such that for each $x_0\in\partial\Omega$ the assertions (i) and (ii) hold. The above constants C_1 and C_2 are depending only on α,β,K and δ. In the sequel, we will apply Proposition 5.2.1 to a uniformly C^2-domain to prove the inequality (5.1.10).

The estimate (5.2.2) implies the compactness from $\hat{W}_d^{1,\infty}(D)$ to $L^p(D)$.

Lemma 5.2.3. *Let D be a bounded domain in $\mathbf{R}^n, n\geq 2$, with C^2-boundary. Then, there exists a constant C_D such that the estimate (5.2.1) holds for all $\varphi\in\hat{W}_d^{1,\infty}(D)$. Moreover, the space $\hat{W}_d^{1,\infty}(D)$ is compactly embedded into $L^p(D)$.*

Proof. We argue by contradiction. Suppose that the estimate (5.2.1) were false for any choice of the constant C. Then, there would exist a sequence of functions $\{\varphi_m\}_{m=1}^\infty\subset\hat{W}_d^{1,\infty}(D)$ such that

$$\|\varphi_m-(\varphi_m)\|_{L^p(D)}>m\|\nabla\varphi_m\|_{L_d^\infty(D)},\quad m\in\mathbf{N}.$$

We may assume $(\varphi_m)=0$ by replacing φ_m to $\varphi_m-(\varphi_m)$. We divide φ_m by $M_m=\|\varphi_m\|_{L^p(D)}$ to get a sequence of functions $\{\phi_m\}_{m=1}^\infty$, $\phi_m=\varphi_m/M_m$ such that

$$\begin{aligned}&\|\nabla\phi_m\|_{L_d^\infty(D)}<1/m,\\&\|\phi_m\|_{L^p(D)}=1\quad\text{with }(\phi_m)=0.\end{aligned}$$

We now prove the compactness of $\{\phi_m\}_{m=1}^\infty$ in $L^p(D)$. Since $\|\nabla\phi_m\|_{L_d^\infty(D)}$ is bounded, $\{\phi_m\}_{m=1}^\infty$ subsequently converges to a limit $\bar{\phi}$ locally uniformly in D. By Proposition 5.2.1, in particular, the estimate (5.2.2) implies that ϕ_m is uniformly bounded from above by an L^p-integrable function near ∂D. The dominated convergence theorem implies that

$$\phi_m\to\bar{\phi}\quad\text{in }L^p(D)\quad\text{as }\ m\to\infty.$$

Since $\nabla\phi_m(x)\to 0$ as $m\to\infty$ for each $x\in D$ and $\|\bar{\phi}\|_{L^p(D)}=1$, $\bar{\phi}$ is a non-zero constant which contradicts $(\bar{\phi})=0$. We reached a contradiction.
For the compactness of $\{\phi_m\}_{m=1}^\infty$ in $L^p(D)$, we here only invoke the bound for $\|\nabla\phi_m\|_{L_d^\infty(D)}$. This means that the embedding from $\hat{W}_d^{1,\infty}(D)$ into $L^p(D)$ is compact. The proof is now complete. □

5.2.2 Estimates near the boundary

We now prove the inequality (5.1.10) for uniformly C^2-domains Ω. When the ball $B_{x_0}(r)$ locates in the interior of Ω, i.e., $\Omega_{x_0,r}=B_{x_0}(r)$, applying (5.2.1) to $\varphi_r(x)=\varphi(x_0+rx)$ in $D=B_0(1)$ implies the estimate

$$\|\varphi-(\varphi)\|_{L^p(\Omega_{x_0,r})}\leq Cr^{n/p}\sup_{z\in\Omega_{x_0,r}}d_{\Omega_{x_0,r}}(z)|\nabla\varphi(z)|,\quad r>0. \tag{5.2.3}$$

Since $d_{\Omega_{x_0,r}}(x) \leq d_\Omega(x)$ for $x \in \Omega_{x_0,r}$, the inequality (5.1.10) follows. However, if $B_{x_0}(r)$ involves $\partial\Omega$, the boundary of $\Omega_{x_0,r}$ may not have C^1-regularity. We thus prove

$$||\varphi - (\varphi)||_{L^p(\Omega_{x_0,r})} \leq Cr^{n/p} \sup_{z\in\Omega_{x_0,r}} d_\Omega(z)|\nabla\varphi(z)| \quad \text{for } \varphi \in \hat{W}^{1,\infty}_d(\Omega) \tag{5.2.4}$$

for $x_0 \in \Omega$ and $r > 0$ satisfying $d_\Omega(x_0) < r$, which is weaker than (5.2.3).

Proposition 5.2.4. *Let Ω be a uniformly C^2-domain. There exists constants r_0 and C such that for $x_0 \in \Omega$ and $r < r_0$ satisfying $d_\Omega(x_0) < r$, the estimate (5.2.4) holds for all $\varphi \in \hat{W}^{1,\infty}_d(\Omega)$ with a constant C independent of x_0 and r.*

The inequality (5.1.10) easily follows from Proposition 5.2.4.

Lemma 5.2.5. *The inequality (5.1.10) holds for $\varphi \in \hat{W}^{1,\infty}_d(\Omega)$ for all $x_0 \in \Omega$ and $r < r_0$ with a constant C independent of x_0 and r.*

Proof. For $r < r_0$, combining (5.2.3) for $d_\Omega(x_0) > r$ with (2.4) for $d_\Omega(x_0) < r$, the assertion (5.1.10) follows. □

Proof of Proposition 5.2.4. We argue by contradiction. Suppose that the estimate (5.2.4) were false for any choice of constants r_0 and C. Then, there would exist a sequence of functions $\{\varphi\}_{m=1}^\infty \subset \hat{W}^{1,\infty}_d(\Omega)$ and a sequence of points $\{x_m\}_{m=1}^\infty \subset \Omega$ satisfying $d_\Omega(x_m) < r_m \downarrow 0$ such that

$$||\varphi_m - (\varphi_m)||_{L^p(\Omega_{x_m,r_m})} > m{r_m}^{n/p} \sup_{z\in\Omega_{x_m,r_m}} d_\Omega(z)|\nabla\varphi_m(z)|, \quad m \in \mathbf{N}.$$

Replacing φ_m by $\varphi_m - (\varphi_m)$ and dividing φ_m by $r_m^{-n/p}||\varphi_m||_{L^p(\Omega_{x_m,r_m})}$ (still denoted by φ_m), we observe that φ_m satisfies ${r_m}^{-n/p}||\varphi_m||_{L^p(\Omega_{x_m,r_m})} = 1$ with $(\varphi_m) = 0$ and $\sup_{z\in\Omega_{x_m,r_m}} d_\Omega(z)|\nabla\varphi_m(z)| < 1/m$. Since the points $\{x_m\}_{m=1}^\infty$ accumulates at the boundary $\partial\Omega$, we may assume, by rotation and translation of Ω, that $x_m = (0, d_m)$ with $d_m = d_\Omega(x_m)$, which subsequently converges to the origin located on the boundary $\partial\Omega$. Here, the neighborhood of the origin is denoted by $\Omega_{\mathrm{loc}} = U(0) \cap \Omega$ with constants α, β and C^2-function h, i.e.,

$$\Omega_{\mathrm{loc}} = \{(x', x_n) \in \mathbf{R}^n_+ \mid h(x') < x_n < h(x') + \beta,\ |x'| < \alpha\}.$$

We rescale φ_m around the point x_m by setting

$$\phi_m(x) = \varphi_m(x_m + r_m x) \quad \text{for } x \in \Omega^m,$$

where $\Omega^m = \{x \in \Omega \mid x = (y - x_m)/r_m, y \in \Omega\}$ is the rescaled domain. Since $c_m = d_m/r_m < 1$, by taking a subsequence, we may assume $\lim_{m\to\infty} c_m = c_0 \leq 1$. We then observe that the rescaled domain Ω^m expands to a half space $\mathbf{R}^n_{+,-c_0} = \{(x', x_n) \in \mathbf{R}^n \mid x_n > -c_0\}$. In fact, the neighborhood $\Omega_{\mathrm{loc}} \subset \Omega$ is rescaled to the domain,

$$\Omega^m_{\mathrm{loc}} = \left\{(x', x_n) \in \mathbf{R}^n \;\middle|\; \frac{1}{r_m} h(r_m x') - c_m < x_n < \frac{1}{r_m} h(r_m x') + \frac{\beta}{r_m},\ |x'| < \frac{\alpha}{r_m}\right\},$$

which converges to $\mathbf{R}^n_{+,-c_0}$ by letting $m \to \infty$. Note that constants of uniformly regularity of $\partial\Omega_m$ are uniformly bounded under this rescaling procedure. Moreover, for any constants μ and α', the curvilinear neighborhood of the origin $\mathcal{N}^\mu(\Gamma)$ is in Ω^m_{loc} for sufficiently large $m \geq 1$, where $\Gamma = \Gamma_{\alpha'}(0)$ is the neighborhood of the origin on $\partial\Omega^m$. Then, the estimates for φ_m are inherited to the estimates for ϕ_m, i.e.,

$$\sup_{z \in \Omega^m_{0,1}} d_{\Omega^m}(z) |\nabla \phi_m(z)| < 1/m, \quad m \in \mathbf{N},$$

$$\|\phi_m\|_{L^p(\Omega^m_{0,1})} = 1 \quad \text{with } (\phi_m) = \fint_{\Omega^m_{0,1}} \phi_m = 0,$$

where $\Omega^m_{0,1} = B_0(1) \cap \Omega^m$. From above bound for ϕ_m, the sequence $\{\phi_m\}_{m=1}^\infty$ subsequently converges to a limit $\bar{\phi}$ locally uniformly in $(\mathbf{R}^n_{+,-c_0})_{0,1} = \mathbf{R}^n_{+,-c_0} \cap B_0(1)$.
We now observe the compactness of the sequence $\{\phi_m\}_{m=1}^\infty$ in $L^p((\mathbf{R}^n_{+,-c_0})_{0,1})$. By Remarks 5.2.2 (ii), applying Proposition 5.2.1 to Ω^m, the estimate (5.2.2) with $x_1 = 0, r = 1$ and a fixed $y \in \Omega^m_{0,1}$ satisfying $d_{\Omega^m}(y) \geq \delta$ yield

$$|\varphi_m(x)| \leq C(|\log d_{\Omega_m}(x)| + 1) \left(\sup_{z \in \Omega^m_{0,1}} d_{\Omega_m}(z) |\nabla \phi_m(z)| \right) + |\phi_m(y)| \quad \text{for } x \in \Omega^m_{0,1},$$

for sufficiently large $m \geq 1$. Here, the constant C is independent of $m \geq 1$. Since ϕ_m is uniformly bounded from above by an L^p-integrable function in $\Omega^m_{0,1}$, the dominated convergence theorem implies that ϕ_m converges to a limit $\bar{\phi}$ in $L^p((\mathbf{R}^n_{+,-c_0})_{0,1})$. Since $\nabla \phi_m(x) \to 0$ as $m \to \infty$ for each $x \in (\mathbf{R}^n_{+,-c_0})_{0,1}$ and $\|\bar{\phi}\|_{L^p((\mathbf{R}^n_{+,-c_0})_{0,1})} = 1$, $\bar{\phi}$ is a non-zero constant which contradicts $(\bar{\phi}) = 0$. We reached a contradiction and the proof is now complete. □

5.3 A priori estimates for the Stokes equations

The goal of this section is to prove the a priori estimate (5.1.4) by using the inequality (5.1.10). A key step is to establish the estimates for h and g in the procedure (ii) as explained in the introduction. We first recall the L^p-estimates to the Stokes equations (5.1.7) and the interpolation inequality (5.1.11). Note that the constants C_p and C_I in (5.1.7) and (5.1.11) respectively are independent of the volume of domains Ω', $\Omega_{x_0,r}$.

5.3.1 L^p-estimates for localized equations

Let Ω' be a bounded domain with C^2-boundary. For the a priori estimate (5.1.4), we invoke the L^p-estimates (5.1.7) to the resolvent Stokes equations with inhomogeneous divergence-free condition,

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= h \quad \text{in } \Omega', && (5.3.1)\\ \operatorname{div} u &= g \quad \text{in } \Omega', && (5.3.2)\\ u &= 0 \quad \text{on } \partial\Omega', && (5.3.3) \end{aligned}$$

for $h \in L^p(\Omega')$, $g \in W^{1,p}(\Omega') \cap L^p_{\mathrm{av}}(\Omega')$, $\lambda \in \sum_{\vartheta,0}$ and $\vartheta \in (\pi/2, \pi)$. Here, $L^p_{\mathrm{av}}(\Omega')$ denotes the space of all average-zero functions g in $L^p(\Omega')$, i.e., $\int_{\Omega'} g dx = 0$. The estimate (5.1.7) is proved by a perturbation argument [14], [15] with the constant C_p independent of the volume of Ω'.

Proposition 5.3.1. ([14], [15]) *Let $\vartheta \in (\pi/2, \pi)$ and $\lambda \in \sum_{\vartheta,0}$. For $h \in L^p(\Omega')$ and $g \in W^{1,p}(\Omega') \cap L^p_{av}(\Omega')$, there exists a unique solution of (5.3.1)–(5.3.3) satisfying the estimates (5.1.7) with the constant C_p, independent of the volume of Ω' and depending on ϑ, p, n and the C^2-regularity of $\partial\Omega'$.*

We estimate the L^∞-norms of a solution up to first derivatives via the Sobolev embeddings and the L^p-estimates (5.1.7) for $p > n$. In order to estimate the L^∞-norms of a solution, we apply the interpolation inequality (5.1.11) [24, Chapter 3. Lemma 3.1.4] in $\Omega_{x_0,r} = B_{x_0}(r) \cap \Omega$ for $x_0 \in \bar{\Omega}$ and $r < r_0$ with a constant r_0. In what follows, we fix the constant r_0 by taking the same constant r_0 given by Lemma 5.2.5. The constant C_I is also independent of the radius r.

5.3.2 Estimates in the localization procedure

We prepare the estimates for h and g in the procedure (ii). The estimate for $|\lambda| \|g\|_{W_0^{-1,p}}$ is different from that of $\|h\|_{L^p}$. In order to estimate $|\lambda| \|g\|_{W_0^{-1,p}}$, we use the uniformly local L^p-norm bound for ∇q besides the sup-bound of ∇v as in (5.3.8). After establishing these estimates, we will put the procedures (i)–(iii) together in the next subsection.

Let Ω be a uniformly C^2-domain. Let θ be a smooth cutoff function satisfying $\theta \equiv 1$ in $[0, 1/2]$ and $\theta \equiv 0$ in $[1, \infty)$. For $x_0 \in \Omega$ and $r > 0$ we set $\theta_0(x) = \theta(|x - x_0|/(\eta + 1)r)$ with parameters $\eta \geq 1$ and observe that $\theta_0 \equiv 1$ in $B_{x_0}(r)$ and $\theta_0 \equiv 0$ in $B_{x_0}((\eta + 1)r)^c$. The cutoff function θ_0 is uniformly bounded by a constant K, i.e.,

$$\|\theta_0\|_\infty + (\eta + 1)r\|\nabla\theta_0\|_\infty + (\eta + 1)^2 r^2 \|\nabla^2\theta_0\|_\infty \leq K, \quad \eta \geq 1 \tag{5.3.4}$$

Let $(v, \nabla q) \in W^{2,p}_{\mathrm{loc}}(\bar{\Omega}) \times L^p_{\mathrm{loc}}(\bar{\Omega})$ be a solution of (5.1.1)–(5.1.3) for $f \in L^\infty_\sigma(\Omega)$ and $\lambda \in \Sigma_{\vartheta,0}$. We localize a solution $(v, \nabla q)$ in a domain $\Omega' = \Omega_{x_0,(\eta+1)r}$ by setting $u = v\theta_0$ and $p = \hat{q}\theta_0$

with $\hat{q} = q - q_c$ and a constant q_c. We then observe that $(u, \nabla p)$ solves the localized equation (5.3.1)–(5.3.3) in the domain Ω' with h and g given by (5.1.8). We shall show the following estimates for h and g,

$$\|\nabla g\|_{L^p(\Omega')} \le C_1 r^{n/p}(\eta+1)^{-(1-n/p)}\left(r^{-1}\|\nabla v\|_{L^\infty(\Omega)} + r^{-2}\|v\|_{L^\infty(\Omega)}\right), \tag{5.3.5}$$

$$\begin{aligned}\|h\|_{L^p(\Omega')} \le C_2 r^{n/p}\Bigg(&(\eta+1)^{n/p}\|f\|_{L^\infty(\Omega)} \\ &+ (\eta+1)^{-(1-n/p)}\Big(r^{-1}\|\nabla v\|_{L^\infty(\Omega)} + r^{-2}\|v\|_{L^\infty(\Omega)}\Big)\Bigg),\end{aligned} \tag{5.3.6}$$

$$\begin{aligned}|\lambda|\|g\|_{W_0^{-1,p}(\Omega')} \le C_3 r^{n/p}\Bigg(&(\eta+1)^{n/p}\|f\|_{L^\infty(\Omega)} \\ &+ (\eta+1)^{-(1-2n/p)}\Big(r^{-1}\|\nabla v\|_{L^\infty(\Omega)} + r^{-n/p}\sup_{z\in\Omega}\|\nabla q\|_{L^p(\Omega_{z,r})}\Big)\Bigg),\end{aligned} \tag{5.3.7}$$

with constants C_1, C_2 and C_3 independent of r and $\eta \ge 1$. For the estimates of the terms of f, v and ∇v, we use the estimates

$$\|f\theta_0\|_{L^p(\Omega')} \le K C_n^{1/p} r^{n/p}(\eta+1)^{n/p}\|f\|_{L^\infty(\Omega)}, \tag{5.3.8}$$

$$\|\nabla v \nabla\theta_0\|_{L^p(\Omega')} \le K C_n^{1/p} r^{n/p}(\eta+1)^{-(1-n/p)} r^{-1}\|\nabla v\|_{L^\infty(\Omega)}, \tag{5.3.9}$$

$$\|v\nabla^2\theta_0\|_{L^p(\Omega')} \le K C_n^{1/p} r^{n/p}(\eta+1)^{-(1-n/p)} r^{-2}\|v\|_{L^\infty(\Omega)}, \tag{5.3.10}$$

for all $r > 0$ and $\eta \ge 1$, where the constant C_n denotes the volume of the n-dimensional unit ball. Since $\nabla g = \nabla v \nabla\theta_0 + v\nabla^2\theta_0$ does not contain the pressure, the estimate (5.3.5) easily follows from the estimates (5.3.9) and (5.3.10).

For the estimates (5.3.6) and (5.3.7), we apply the inequality (5.1.10). We choose a constant q_c by the mean value of q in $\Omega_{x_0,(\eta+2)r}$, i.e.,

$$q_c = \fint_{\Omega_{x_0,(\eta+2)r}} q(x)dx. \tag{5.3.11}$$

We then observe that the inequality (5.1.10) implies the estimate

$$\|\hat{q}\|_{L^p(\Omega_{x_0,(\eta+2)r})} \le C r^{n/p}(\eta+2)^{n/p}\|\nabla q\|_{L^\infty_d(\Omega)} \tag{5.3.12}$$

for $r > 0$ and $\eta \ge 1$ satisfying $(\eta+2)r \le r_0$, where $\hat{q} = q - q_c$.

In order to estimate (5.3.7), we estimate the L^∞-norm of $\hat{q}$ on Ω' since by using the equation $\lambda v = f + \Delta v - \nabla q$ we reduce (5.3.7) to the estimate of the boundary value of $\hat{q}$ on $\partial\Omega'$. This is the reason why we take q_c by (5.3.11). We apply the inequality (5.1.11) in $\Omega_{x_1,r} \subset \Omega_{x_0,(\eta+2)r}$ for $x_1 \in \Omega'$ and $r > 0$ with $p > n$ to estimate

$$\begin{aligned}\|\hat{q}\|_{L^\infty(\Omega_{x_1,r})} &\le C_I r^{-n/p}\Big(\|\hat{q}\|_{L^p(\Omega_{x_1,r})} + r\|\nabla q\|_{L^p(\Omega_{x_1,r})}\Big) \\ &\le C_I r^{-n/p}\Big(\|\hat{q}\|_{L^p(\Omega_{x_0,(\eta+2)r})} + r\sup_{z\in\Omega}\|\nabla q\|_{L^p(\Omega_{z,r})}\Big).\end{aligned} \tag{5.3.13}$$

Combining the estimate (5.3.13) with (5.3.12) and taking a supremum for $x_1 \in \Omega'$, we have

$$\|\hat{q}\|_{L^\infty(\Omega')} \leq C\Big((\eta+2)^{n/p}\|\nabla q\|_{L^\infty_d(\Omega)} + r^{1-n/p}\sup_{z\in\Omega}\|\nabla q\|_{L^p(\Omega_{z,r})}\Big). \tag{5.3.14}$$

We now invoke the strictly admissibility of a domain Ω to estimate the norm $\|\nabla q\|_{L^\infty_d(\Omega)}$ by the sup-norm of ∇v in Ω via (5.1.5).

Proposition 5.3.2. *Let Ω be a uniformly C^2-domain. Assume that Ω is strictly admissible. Then, the estimate*

$$\|\hat{q}\|_{L^p(\Omega')} \leq C_4 r^{n/p}(\eta+2)^{n/p}\|\nabla v\|_{L^\infty(\Omega)} \tag{5.3.15}$$

holds for all $r > 0$ and $\eta \geq 1$ satisfying $(\eta+2)r \leq r_0$ and $p \in [1,\infty)$. If in addition $p > n$, then the estimate

$$\|\hat{q}\|_{L^\infty(\Omega')} \leq C_5\left((\eta+2)^{n/p}\|\nabla v\|_{L^\infty(\Omega)} + r^{1-n/p}\sup_{z\in\Omega}\|\nabla q\|_{L^p(\Omega_{z,r})}\right) \tag{5.3.16}$$

holds. The constants C_4 and C_5 are independent of r and η.

Proof. By (5.1.5), (5.3.12) and (5.3.14), the assertion follows. □

By using the estimates (5.3.15) and (5.3.16), we obtain the estimates (5.3.6) and (5.3.7).

Lemma 5.3.3. *Let Ω be a strictly admissible, uniformly C^2-domain. Let $(v,\nabla q) \in W^{2,p}_{loc}(\bar{\Omega})\times (L^p_{loc}(\bar{\Omega}) \cap L^\infty_d(\Omega))$ be a solution of (5.1.1)–(5.1.3) for $f \in L^\infty_\sigma(\Omega)$ and $\lambda \in \sum_{\vartheta,0}$ with $p > n$. Then, the estimates (5.3.5)–(5.3.7) hold for $\Omega' = B_{x_0}((\eta+1)r) \cap \Omega$ with $x_0 \in \Omega$, $r > 0$ and $\eta \geq 1$ satisfying $(\eta+2)r \leq r_0$ with the constants C_1, C_2 and C_3 independent of x_0, r and η.*

Proof. As mentioned before, (5.3.5) follows from (5.3.9) and (5.3.10). The estimate (5.3.6) follows from the estimates (5.3.8)–(5.3.10) and (5.3.15). We shall show the estimate (5.3.7). By using the equation $\lambda g = \lambda v \cdot \nabla\theta_0 = (f + \Delta v - \nabla q)\cdot\nabla\theta_0$, we estimate

$$|\lambda|\|g\|_{W^{-1,p}_0(\Omega')} \leq \|f\cdot\nabla\theta_0\|_{W^{-1,p}_0(\Omega')} + \|\Delta v\cdot\nabla\theta_0\|_{W^{-1,p}_0(\Omega')} + \|\nabla q\cdot\nabla\theta_0\|_{W^{-1,p}_0(\Omega')}.$$

Since $\|f\cdot\nabla\theta_0\|_{W^{-1,p}_0(\Omega')} = \|f\theta_0\|_{L^p(\Omega')}$ for $f \in L^\infty_\sigma(\Omega)$, it suffices to show the estimates

$$\|\Delta v\cdot\nabla\theta_0\|_{W^{-1,p}_0(\Omega')} \leq C_6 r^{n/p}(\eta+1)^{-(1-n/p)}r^{-1}\|\nabla v\|_{L^\infty(\Omega)}, \tag{5.3.17}$$

$$\|\nabla q\cdot\nabla\theta_0\|_{W^{-1,p}_0(\Omega')} \leq C_7 r^{n/p}(\eta+1)^{-(1-2n/p)}\left(r^{-1}\|\nabla v\|_{L^\infty(\Omega)} + r^{-n/p}\sup_{z\in\Omega}\|\nabla q\|_{L^p(\Omega_{z,r})}\right). \tag{5.3.18}$$

We first show (5.3.17). Take $\varphi \in W^{1,p'}(\Omega')$ satisfying $\|\varphi\|_{W^{1,p'}(\Omega')} \leq 1$. By using $\operatorname{div} v = 0$, integration by parts yields that

$$\sum_{i,j=1}^n \int_{\Omega'} \partial_j^2 v^i \partial_i\theta_0\varphi dx = \sum_{i,j=1}^n \int_{\Omega'} (\partial_j v^i - \partial_i v^j)\partial_i\theta_0\partial_i\varphi dx - \int_{\partial\Omega'} (\partial_j v^i - \partial_i v^j)\partial_i\theta_0\varphi n^i_\Omega d\mathcal{H}^{n-1}(x).$$

We estimate the second term in the right-hand side by the $W^{1,1}$-norm of φ in Ω' [10, 5.5 Theorem 1.1] to estimate

$$\|\varphi\|_{L^1(\partial\Omega)} \leq C_T\|\varphi\|_{W^{1,1}(\Omega')} \leq 2C_T|\Omega'|^{1/p}, \tag{5.3.19}$$

with the constant C_T depending on the C^1-regularity of the boundary $\partial\Omega$, but independent of $|\Omega'|$, the volume of Ω'. We thus obtain

$$\begin{aligned}\left|\int_{\Omega'} \partial_j^2 v^i \partial_i \theta_0 \varphi dx\right| &\leq (1+2C_T)\|(\partial_j v^i - \partial_i v^j)\partial_j\theta_0\|_{L^\infty(\Omega')}|\Omega'|^{1/p} \\ &\leq 2(1+2C_T)K{C_n}^{1/p} r^{n/p}(\eta+1)^{-(1-n/p)} r^{-1}\|\nabla v\|_{L^\infty(\Omega)}.\end{aligned}$$

Thus, the estimate (5.3.17) holds with the constant C_6 independent of r and η. It remains to show the estimate (5.3.18). Since $\nabla q = \nabla\hat{q}$, integration by parts yields that

$$\begin{aligned}\int_{\Omega'} \nabla q \cdot \nabla\theta_0\varphi dx &= -\int_{\Omega'} \hat{q}(\Delta\theta_0\varphi + \nabla\theta_0\cdot\nabla\varphi)dx + \int_{\partial\Omega'} \hat{q}\varphi\nabla\theta_0\cdot n_{\Omega'} d\mathcal{H}^{n-1}(x) \\ &= I + II + III.\end{aligned}$$

Combining (5.3.4), (5.3.19) with (5.3.16), we obtain

$$\begin{aligned} II + III &\leq (1+2C_T)\|\hat{q}\nabla\theta_0\|_{L^\infty(\Omega')}|\Omega'|^{1/p} \\ &\leq (1+2C_T)K{C_n}^{1/p} r^{n/p}(\eta+1)^{-(1-n/p)} r^{-1}\|\hat{q}\|_{L^\infty(\Omega')} \\ &\leq Cr^{n/p}(\eta+1)^{-(1-2n/p)}\left(r^{-1}\|\nabla v\|_{L^\infty(\Omega)} + r^{-n/p}\sup_{z\in\Omega}\|\nabla q\|_{L^p(\Omega_{z,r})}\right),\end{aligned}$$

with the constant C depending on C_T, K, C_n, p, C_4 and C_5, but independent of r and η. We complete the proof by showing the estimate for I. Applying the Hölder inequality, for $s, s' \in (1,\infty)$ with $1/s + 1/s' = 1$, we have

$$I \leq K(\eta+1)^{-2}r^{-2}\|\varphi\|_{L^s(\Omega')}\|\hat{q}\|_{L^{s'}(\Omega')}.$$

Since $p > n$, the conjugate exponent p' is strictly smaller than $n/(n-1)$ for $n \geq 2$. By setting $1/s = 1/p' + 1/n$, we apply the Sobolev inequality [10, 5.6 Theorem 2] to estimate $\|\varphi\|_{L^s(\Omega')} \leq C_S\|\varphi\|_{W^{1,p'}(\Omega')} \leq C_S$ with the constant C_s independent of $|\Omega'|$. Applying the estimate (5.3.15) to $\hat{q}$ yields

$$\begin{aligned} I &\leq Cr^{n/s'-2}(\eta+2)^{n/s'-2}\|\nabla v\|_{L^\infty(\Omega)} \\ &\leq Cr^{n/p}(\eta+2)^{-(1-n/p)}r^{-1}\|\nabla v\|_{L^\infty(\Omega)},\end{aligned}$$

since $1/s' = 1 - 1/s = 1/p + 1/n$. The constant C is independent of r and η. The proof is now complete. □

Remark 5.3.4. From the estimate (5.3.7), we observe that the exponent $-(1-2n/p)$ of $(\eta+1)$ in front of the term $(r^{-1}\|\nabla v\|_{L^\infty(\Omega)} + r^{-n/p}\sup_{z\in\Omega}\|\nabla q\|_{L^p(\Omega_{z,r})})$ is negative provided that $p > 2n$. We thus first prove the a priori estimate (5.1.4) for $p > 2n$. Once we obtain the estimate $|\lambda|\|v\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}$, it is easy to replace the estimate (5.3.7) to

$$|\lambda|\|g\|_{W_0^{-1,p}(\Omega')} \leq CKC_n{}^{1/n}r^{n/p}(\eta+1)^{n/p}\|f\|_{L^\infty(\Omega)}$$

for $p > n$ since

$$\begin{aligned}|\lambda|\|v\cdot\nabla\theta_0\|_{W_0^{-1,p}(\Omega')} &= |\lambda|\|v\theta_0\|_{L^p(\Omega)}\\ &\leq C\|\theta_0\|_{L^p(\Omega')}\|f\|_{L^\infty(\Omega)}\\ &\leq CKC_n{}^{1/p}r^{n/p}(\eta+1)^{n/p}\|f\|_{L^\infty(\Omega)}.\end{aligned}$$

5.3.3 Interpolation

We now prove the a priori estimate (5.1.4) for $p > n$. The size of the parameters η and the constant δ are determined only through the constants C_p, C_I and C_1–C_3. Although we eventually obtain the estimate (5.1.12) for all $p > n$, we firstly prove the case $p > 2n$ as observed by Remark 5.3.4. The case $p > 2n$ is enough for analyticity, but for the completeness, we prove the estimate (5.1.4) for all $p > n$.

Proof of Theorem 5.1.1. We set $\delta = \delta_\eta = (\eta+2)^2/r_0{}^2$ and now take $r = 1/|\lambda|^{1/2}$ for $\lambda \in \sum_{\vartheta,\delta}$. We then observe that $r = 1/|\lambda|^{1/2}$ and $\eta \geq 1$ automatically satisfy $r(\eta+2) \leq r_0$ for $\lambda \in \Sigma_{\vartheta,\delta}$. We may assume that the boundary of $\Omega' = B_{x_0}((\eta+1)r)\cap\Omega$ is C^2 because the localized equations (5.3.1)–(5.3.3) can be regarded as the equation in a subdomain Ω'' of Ω by taking Ω'' with C^2-boundary so that $\Omega' \subset \Omega''$ and Ω'' preserves an order of the volume of Ω', i.e., $|\Omega''|$ is bounded from above by $C(\eta+1)^n r^n$ with a constant C independent of $r > 0$ and $\eta \geq 1$. We first prove:

Case (I) $p > 2n$. By applying the L^p-estimates (5.1.7) to $u = v\theta_0$ and $p = \hat{q}\theta_0$ in Ω' and combining the estimates (5.3.5)–(5.3.7) with (5.1.7), we obtain

$$\begin{aligned}&|\lambda|\|u\|_{L^p(\Omega')} + |\lambda|^{1/2}\|\nabla u\|_{L^p(\Omega')} + \|\nabla^2 u\|_{L^p(\Omega')} + \|\nabla p\|_{L^p(\Omega')}\\ &\leq C_8|\lambda|^{-n/2p}\left((\eta+1)^{n/p}\|f\|_{L^\infty(\Omega)} + (\eta+1)^{-(1-2n/p)}\|M_p(v,q)\|_{L^\infty(\Omega)}(\lambda)\right),\end{aligned} \tag{5.3.20}$$

with the constant C_8 independent of $r = 1/|\lambda|^{1/2}$ and $\eta \geq 1$. We next estimate the L^∞-norms of u and ∇u in Ω by interpolation. Applying the interpolation inequality (5.1.11) for $\varphi = u$ and ∇u implies the estimates

$$\begin{aligned}\|u\|_{L^\infty(\Omega_{x_0,r})} &\leq C_I r^{-n/p}\Big(\|u\|_{L^p(\Omega_{x_0,r})} + r\|\nabla u\|_{L^p(\Omega_{x_0,r})}\Big),\\ \|\nabla u\|_{L^\infty(\Omega_{x_0,r})} &\leq C_I r^{-n/p}\Big(\|\nabla u\|_{L^p(\Omega_{x_0,r})} + r\|\nabla^2 u\|_{L^p(\Omega_{x_0,r})}\Big).\end{aligned}$$

Summing up these norms together with $|\lambda|^{n/2p}\|\nabla^2 u\|_{L^p(\Omega_{x_0,r})}$ and $|\lambda|^{n/2p}\|\nabla p\|_{L^p(\Omega_{x_0,r})}$, we have

$$\begin{aligned}&M_p(u,p)(x_0,\lambda)\\&\leq C_9 r^{-n/p}\left(|\lambda|\|u\|_{L^p(\Omega_{x_0,r})}+|\lambda|^{1/2}\|\nabla u\|_{L^p(\Omega_{x_0,r})}+\|\nabla^2 u\|_{L^p(\Omega_{x_0,r})}+\|\nabla p\|_{L^p(\Omega_{x_0,r})}\right),\end{aligned} \tag{5.3.21}$$

with the constant C_9 independent of r and $\eta \geq 1$. Since $(u, \nabla p)$ agrees with $(v, \nabla q)$ in $\Omega_{x_0,r}$, combining (5.3.20) with (5.3.21) yields

$$M_p(v,q)(x_0,\lambda) \leq C_{10}\left((\eta+1)^{n/p}\|f\|_{L^\infty(\Omega)}+(\eta+1)^{-(1-2n/p)}\|M_p(v,q)\|_{L^\infty(\Omega)}(\lambda)\right), \tag{5.3.22}$$

with $C_{10} = C_8C_9$. By taking a supremum for $x_0 \in \Omega$ and letting $\eta \geq 1$ large so that $C_{10}(\eta+1)^{-(1-2n/p)} < 1/2$, we obtain (5.1.4) with $p > 2n$.

We shall complete the proof by showing the uniformly local L^p-bound for second derivatives of (v, q) for all $p > n$.

Case (II) $p > n$. Since $|\lambda|\|g\|_{W_0^{-1,\tilde{p}}}$ is bounded for $\tilde{p} > 2n$, we may assume $(v, \nabla q) \in W^{2,\tilde{p}}_{\mathrm{loc}}(\bar{\Omega}) \times L^{\tilde{p}}_{\mathrm{loc}}(\bar{\Omega})$ with $\tilde{p} > 2n$. By using $|\lambda|\|v\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)}$ for $\lambda \in \Sigma_{\vartheta,\delta}$ with $\delta = \delta_{\tilde{p}}$, we replace the estimate (5.3.7) to

$$|\lambda|\|g\|_{W_0^{-1,p}(\Omega')} \leq CKC_n^{1/p}r^{n/p}(\eta+1)^{n/p}\|f\|_{L^\infty(\Omega)}$$

by Remark 5.3.4. Then, we are able to replace the estimate (5.3.22) to

$$\|M_p(v,q)\|_{L^\infty(\Omega)}(\lambda) \leq C_{11}\left((\eta+1)^{n/p}\|f\|_{L^\infty(\Omega)}+(\eta+1)^{-(1-n/p)}\|M_p(v,q)\|_{L^\infty(\Omega)}(\lambda)\right).$$

Letting $\eta \geq 1$ large so that $C_{11}(\eta+1)^{-(1-n/p)} < 1/2$, we obtain (5.1.4) for all $p > n$. The proof is now complete.

Remark 5.3.5. (Robin boundary condition) Concerning the Robin boundary condition, we replace the Dirichlet boundary condition for the localized equations (5.3.3) to the inhomogeneous boundary condition with a tangential vector field k,

$$B(u) = k, \quad u \cdot n_{\Omega'} = 0 \quad \text{on } \partial\Omega'.$$

Instead of the estimate (5.1.7), we apply the L^p-estiamte of the form,

$$\begin{aligned}&|\lambda|\|u\|_{L^p(\Omega')}+|\lambda|^{1/2}\|\nabla u\|_{L^p(\Omega')}+\|\nabla^2 u\|_{L^p(\Omega')}+\|\nabla p\|_{L^p(\Omega')}\\&\leq C(\|h\|_{L^p(\Omega')}+\|\nabla g\|_{L^p(\Omega')}+|\lambda|\|g\|_{W_0^{-1,p}(\Omega')}+|\lambda|^{1/2}\|k\|_{L^p(\Omega')}+\|\nabla k\|_{L^p(\Omega')}),\end{aligned}$$

where k is identified with its arbitrary extension to Ω'. Since $k = v_{\tan}\partial\theta_0/\partial n_{\Omega'}$ for $u = v\theta_0$ and $p = \hat{q}\theta_0$, we observe that the norms of k in the right-hand side are estimated by the same way with $\|\nabla g\|_{L^p}$ where $g = v \cdot \nabla\theta_0$. The L^p-estimates for the Robin boundary condition is proved by [32] for bounded and exterior domains by generalizing the perturbation argument to the Dirichlet boundary condition [15]. We thus observe that the constant C is also independent of the volume Ω'. After proving the a priori estimate (5.1.4) for $f \in L^\infty_\sigma$ subject to the Robin boundary condition, we verify the existence of solutions of (5.1.1) and (5.1.2). In particular, $v \in L^\infty_\sigma$ (not in $C_{0,\sigma}$.) Then, we are able to define the Stokes operator $A = A_R$ in L^∞_σ in the same way as we did for the Dirichlet boundary condition. Our observations may be summarized as following:

Theorem 5.3.6. *Assume that Ω is a bounded or an exterior domain with C^3-boundary in $\mathbf{R}^n$. Then, the Stokes operator $A = A_R$ subject to the Robin boundary condition generates an analytic semigroup on $L^\infty_\sigma(\Omega)$ of angle $\pi/2$.*

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